Conductance and magnetoconductance of parabolically confined quasi-one-dimensional channels

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Abstract

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Electrical conduction is studied along parabolically confined quasi-one dimensional channels in the framework of linear-response theory. In the absence of a magnetic field an expression for the conductance is obtained, that agrees with those in the previous literature on this subject, as well as with the limit of the conductance in the Born approximation. A similar but new expression is obtained in the presence of a magnetic field perpendicular to the channel. This expression is more general than those contained in previous literature as it accounts explicitly for the Hall field. Some particular cases are also discussed.

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Chapter 1

Introduction

Since the discovery of electricity, there has been a longtime interest in understanding and utilizing the electric properties of materials. Different materials can have very different electric properties. In electricity an essential characteristic is the resistance of materials. The conductance [23] is the inverse of the electric resistance. The expression for conductance is well known for a rectangular box conductor. It is simply equal to the Ohmic conductivity times a geometrical ratio (width over length). This expression of conductivity is based on experimental studies. The first theoretical understanding was developed by Drude in 1900 [13]. In the Drude model the electron is represented as a classical particle driven by an electric field. The electron is slowed when bouncing against the wall of the conductor and when hitting other electrons or obstacles.

This representation was refined by taking into account more physical effects and using more complete model. The classical theory was replaced by quantum theory around 1920. The new picture of a particle as a wave was added to the collisions of a classical particle. New concepts such as Bloch electron, band model, phonon vibrations etc., were used to explain the conductivity at the end of the 60's.

With new development in semiconductor manufacturing, it became possible to produce very small devices such as quantum wires and quantum dots of nanometer sizes. At this scale the electron evolves along a distance less than its mean free path. New phenomena occuring in electronic transport were observed at low temperatures, e.g, the conductance quantization. For a large sample, the Ohmic relation is still valid. If the length or width decreases, the conductance decreases too. However, in mesoscopic samples, the Ohmic relation is no longer valid. The conductance reaches a limit where its value equals a universal discrete quantity. Van Wees et al. [33] and Wharam et al. [36] in 1988 experimentally demonstrated that the conductance of electrons through a narrow channel is quantized in units of e^2/h.

These experimental results on nanostructures stimulated further theoretical understanding. Different forms were proposed to explain these new properties. The first one was by Landauer in 1957 [25] for one dimensional wire. He expressed the problem in terms of scattering. His expression was used to explain experimental results found by Van Wees and Wharam. Anderson [1] in the same year deduced the Landauer expression and was able to describe a large range of phenomena such as localisation in one-dimension and universal conductance fluctuations. In 1985 Gefen [16] solved the case where there are two resistances in parallel. The conductance generalization for multichannel propagation was proposed in 1981 [3] and in 1985 [7]. Simultaneously researchers tried to find a more rigorous foundation.

The linear response theory was proposed by Kubo in 1957. This theory was used to derive expressions for the conductance [15, 14, 32]. Slight variations between different results were a source of discussion. Crucial importance was given to conditions of measurement. It was established that four-probe measurements do not give the same answer as two-probe measurements [18]. In 1985 Buttiker [7] clarified the physical conditions necessary for the multichannel generalization. Following the Landauer approach a derivation from the linear response theory was given in 1988 [32].

The influence of a magnetic field was also studied. The two-probe formula and its generalization were still applicable in a magnetic field. They were derived again using linear response theory in 1988 [23, 4] and recently by several groups [29, 21, 24]. The Onsager's relation, relating the symmetry of the conductance upon changing the direction of the magnetic field, was verified. An apparent contradiction was expressed [30] using gauge transformation. The response clarified the different steps of calculation and showed a simpler derivation [28].

For the four-probe measurement with the magnetic field, it was realised that the conductance can be asymmetric under magnetic field reversal. Numerical simulation [31] and specific cases (with a small number of channel) [8] showed an asymmetric relation with the magnetic field. Instead of following the Onsager relation, an equivalent relation between the resistances [10] was shown.

In our work we propose to derive a conductance expression from linear-response theory. The case with the electric field is formulated. The conductivity is compared with that obtained by Verboven. The next situation is with the presence of an electric field and a magnetic field. An attempt is made to generalize the results of the literature by including the Hall field and the curvature of the confining potential, and by evaluating their approximate expressions.

Chapter 2

Mesoscopic systems and Landauer formulation

2.1 Mesoscopic systems

2.1.1 Physical properties

Mesoscopic systems lie between macroscopic systems, such as bulk semiconductors and metals, and microscopic systems, such as atoms and molecules. They have common properties with these two opposite scales. There are, like in solid state, a great number of electrons involved. We can use the Fermi level and velocity concepts. The presence of chemical (impurities) and crystalline defects gives to each sample its uniqueness because the electron mean free path is greater than the sample size. Similar to microscopic systems, electrons are in coherent quantum states. Their phases can be conserved during the propagation. We can have interference effects. The phase coherence is typical for mesoscopic systems and is the source of new phenomena. This is possible for low temperatures when the crystalline vibrations have weak effect on the phase coherence.

2.1.2 Quantum transport

In quantum transport we can have three different regimes. They depend on the Fermi wavelength of a free electron λ_F, the mean free path l_e , the typical size of the sample L, and the phase-relaxation length L_φ. The ballistic regime is when $\lambda_F \ll l_e$, $L \leq l_e$ and $L_\varphi > L$. It is observed in semiconductor heterojunctions where l_e can reach 10 μm (and $\lambda_F \approx 300$ angstroms). It is the situation when electrons suffer a very small number of collisions and hit (bounce on) few times the surface of the sample. The diffusive regime is when $\lambda_F \ll l_e \ll L$ and $L_\varphi > L$. It is the usual situation in metals where $\lambda_F \approx$ 1 angstroms and $l_e \approx 100$ angstroms. Electrons can suffer many collisions scatterings and follow a random trajectory. The classical theory can then be applied. The localisation regime is when the $L_\varphi < L$. Electrons are confined in small different regions and propagate by hoping mechanism. In our study,

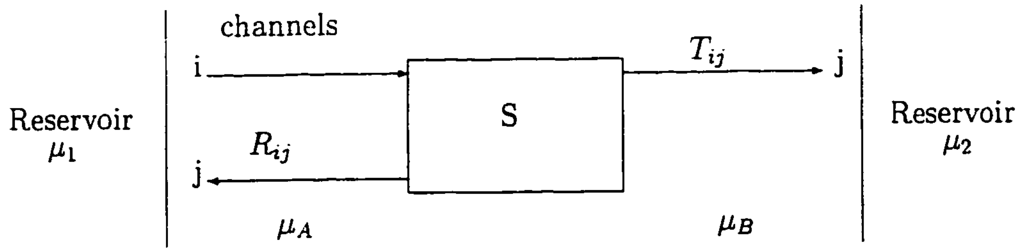

Figure 2.1: Multichannel propagation in the Landauer model

we will have a region of scattering (diffusive) between two perfect ballistic regions.

2.2 Multichannel Landauer formulation

2.2.1 Conductance between reservoirs

We consider the following model (see Figure 2.1) [19]. The scattering centres (S) constitute the inhomogenous part of the sample. They are represented as a potential barrier. The connections in the leads are ideal and represented by channels. The reservoirs are the perfect sources (or sinks) of electrons. Electrons coming from reservoir 1 propagate coherently in channel i. They pass through the sample where they are elastically scattered and reach reservoir 2. Each side of the leads has a finite cross-section A and $N_{\perp}$ transmission channels. Channels are the discrete energy coming from the quantization along the perpendicular axis of propagation. N longitudinal waves pass through channels. At temperatures close to zero, their energies are given by

$$E_{\perp i} + \hbar^2 k_i^2/2m = E_F,\ i = 1, .., N_\perp \tag{2.1}$$

Only channels with real k_i are considered. Those with imaginary k_i are evanescent waves and are ignored. The inhomogenous part of the sample behaves as potential barrier. An incoming wave coming from channel i, has a probability $T_{ij} = |t_{ij}|^2$ to be transmitted in the j channel of lead B and a probability $R_{ij} = |r_{ij}|^2$ to be reflected by the channel j. In the reservoirs the chemical potential is μ_1 on the left side and μ_2 on the right side with $\mu_1 > \mu_2$. The following assumptions are made :

1- The reservoir fills the channel until the population reaches the thermal equilibrium corresponding to the Fermi distribution.

2- There is no phase relation between channels.

3- The propagation in the channels are completely coherent and are not modified by changes in the geometry.

4- The difference $\mu_1 - \mu_2$ is weak enough to be in the linear response regime.

The current is the sum of the transmitted minus the reflected electrons. It is expressed in terms of energy for a channel i by

$$I_i = q \int n_i(E) v_i [T_i(E) f_1(E) - (1 - R_i) f_2(E)] dE, \tag{2.2}$$

where $T_i = \sum_j T_{ij}$ and $R_i = \sum_j R_{ij}$ are the total transmission and reflection probabilities and $f_{1(2)}(E)$ is the Fermi distribution on the left (right) side. The density of states n_i in the direction of propagation, in one dimension, is

$$n_i(E) = (2\pi\hbar v_i)^{-1}; \tag{2.3}$$

therefore the total current is

$$I_i = \frac{(\mu_1 - \mu_2)e^2}{2\pi\hbar} \int (\frac{\partial f}{\partial E}) \sum_i T_i(E) dE \tag{2.4}$$

where we used the relation $T_i(E) = (1 - R_i)$ and we linearize the expression $f(\mu_1 + \Delta\mu) - f(\mu_2) = \Delta\mu \frac{\partial f}{\partial E}$. This current is identical for each side. The measured conductance between reservoirs is given by

$$\begin{aligned} G_2 &= \frac{I}{\mu_1 - \mu_2} \\ &= \frac{e^2}{2\pi\hbar} \int_i dE (\frac{\partial f}{\partial E}) \sum_i T_i(E) dE \\ &= \lim_{T \to 0} \frac{e^2}{2\pi\hbar} \sum_{ij} T_{ij}(E_F) \\ &= \frac{e^2}{2\pi\hbar} tr\{tt^\dagger\} \end{aligned} \tag{2.5}$$

where $tr\{tt^\dagger\} = \sum_i (tt^\dagger)_{ii} = \sum_i \sum_j |t_{ij}|^2$ and t_{ij} is the transmission coefficient. This formula yields the conductance for a two-probe measurement. It represents a voltage measurement between the reservoirs and takes into account the metallic wires connecting the sample to the voltmeter.

Conductance in the presence of a magnetic field

When a magnetic field is present, Eq.(2.5) satisfies the Onsager's relation [8]

$$G_2(B) = G_2(-B) \tag{2.6}$$

This is verified in two steps. Firstly, if one changes the current direction G_{-I}, the current conservation is still valid. The conductance is conserved : $G_{-I} = G_I$. Therefore we have $G_{-I}(-B) = G(-B)$. Secondly, time-reversal invariance requires $S(B) = S^t(-B)$, (t for transpose) for the scattering matrices. This implies $G_{-I}(-B) = G(B)$. From theses two relations, we have $G(-B) = G_{-I}(-B) = G(B)$ and Onsager's relation follows.

2.2.2 Barrier Conductance

To calculate the conductance between the leads μ_A and μ_B, several hypotheses are available. It can be done directly upon assuming that the density of states does not change with measurement. The chemical potentials between the reservoirs and the nonhomogenous part of a sample are defined by the equality of the density of states $n_A = n_i$, $n_B = n_j$.

For $T = 0$, the right lead has

$$n_B = \frac{2}{h}\Delta\mu \sum_i \frac{1}{v_i} T_i \tag{2.7}$$

and the left lead has

$$n_A = \frac{2}{h}\Delta\mu \sum_i \frac{1}{v_i}(1 + R_i). \tag{2.8}$$

If we use the Einstein relation for a degenerate gas, we obtain

$$e\Delta V \left(\frac{\partial n}{\partial E}\right) = \Delta n, \tag{2.9}$$

here

$$\Delta n = n_A - n_B = \frac{2}{h}\sum_i \frac{1}{v_i}(1 + R_i - T_i) \tag{2.10}$$

and

$$\left(\frac{\partial n}{\partial E}\right) = \frac{2}{h}\sum_i \frac{1}{v_i} \tag{2.11}$$

is the total density of current in the channels. The potential difference is

$$e\Delta V = \frac{\sum_i \frac{1}{v_i}(1 + R_i - T_i)}{2\sum_i \frac{1}{v_i}}(\mu_1 - \mu_2) \tag{2.12}$$

and we can deduce conductance

$$G_4 = \frac{I}{\Delta V} = \frac{2e^2}{h}\frac{tr\{tt^\dagger\}2\sum_i \frac{1}{v_i}}{\sum_i \frac{1}{v_i}(1 + R_i - T_i)} \tag{2.13}$$

This is the conductance between the scatterings centers. Experimentally, it is found using a four-probe measurement. Physically it means that current and voltage are measured. In the presence of a magnetic field, this expression does not always follow the Onsager relation [8] [31]. This conclusion brought a general discussion about the symmetry relation for four-probe measurements.

Symmetry in the presence of a magnetic field

For a four-probe measurement, Buttiker clarified the symmetry involved [9]. The hypothesis of a perfect voltage measurement without influence on the current equilibrium was no longer applied. The current and the voltage were considered on the same level. Using the resistance matrix, the current in one channel is found in terms of chemical potentials of different channels. Relating the chemical potential to the measured potential, using current conservation and symmetry arguments, it was shown that there is an Onsager symmetry. Instead of being one for the conductance, it is between resistance elements.

Chapter 3

Electric field only

3.1 Expression for the conductivity

In Ref.[6] a general formula for the conductance is found in terms of the transmission and reflection coefficients in the absence of a magnetic field. The derivation is based on the Hamiltonian and von Neuman's equation. The model (see Figure 3.1) proposes two perfect leads with scattering centres in the middle. The electric field is applied only in the inhomogenous part.

The many-body Hamiltonian is

$$H_{tot}(t) = H_N + W(t) + H^I \tag{3.1}$$

H_N is the unpertubed Hamiltonian, W the pertubation, and H^I the interaction between electrons and impurities. H_N describes independent electrons

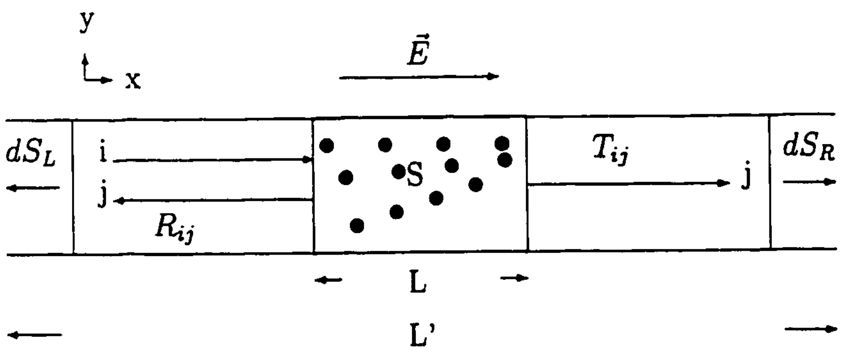

Figure 3.1: A quasi-one-dimensional conductor, connected to left (L) and right (R) reservoirs in the presence of a longitudinal the electric field. The length of the conductor is L'. The solid dots represent scattering centers.

and phonons. It is supposed that impurities are fixed, i.e., they do not move. There are no interactions between electrons. It is a gas of free electrons which interacts only with impurities. We can transform the many-body Hamiltonian Eq.(3.1) to a sum of one electron Hamiltonians at $T = 0$ (no phonons)

$$H_{tot} = \sum_i H_i + \sum_i w_i + h^I; \tag{3.2}$$

H_i is the one-electron Hamiltonian, h^I one-electron interaction with impurities, and w_i is related to the electric field by $e\mathbf{E} = -\nabla w(r)$. Then von Neuman's equation is used. For elastic scattering, it can be transformed into a similar one-body equation. In the linear response theory for a Fermi-Dirac distribution $f(h)$ it reads

$$\frac{\partial}{\partial t}\rho(t) + i\mathcal{L}\rho(t) = -\frac{i}{\hbar}[w(t), f(h)] \tag{3.3}$$

with $\mathcal{L}\bullet \equiv \frac{1}{\hbar}[H(t)_i, \bullet]$

The one-body density operator ρ is the sum of unpertubed operator $f(h)$ and of the pertubation operator $\tilde{\rho}(t)$

$$\rho(t) = f(h) + \tilde{\rho}(t). \tag{3.4}$$

Since $\mathcal{L}f(h) = 0$ (for N-body see [34]), the equation for $\tilde{\rho}(t)$ is

$$\frac{\partial}{\partial t}\tilde{\rho}(t) + i\mathcal{L}\tilde{\rho}(t) = -\frac{i}{\hbar}[\tilde{w}(t), f(h)] \tag{3.5}$$

with the initial condition

$$\tilde{\rho}(0) = 0 \tag{3.6}$$

The solution is found using Laplace transforms. In Laplace space (Eq.3.5) becomes

$$\tilde{\rho} = -\frac{i}{\hbar}\frac{1}{s + i\mathcal{L}}[\tilde{w}(s), f(h)] \tag{3.7}$$

This equation can be separated in two parts. In a representation in which h_i is diagonal, the operator $\tilde{\rho}$ has a diagonal ($\tilde{\rho}_d$) and nondiagonal ($\tilde{\rho}_{nd}$) part, $\tilde{\rho} = \tilde{\rho}_d + \tilde{\rho}_{nd}$. The diagonal equation is found using the diagonal projection superator $\mathcal{P}$. The equation is

$$s\tilde{\rho}_d(s) + i\mathcal{P}\mathcal{L}^1\tilde{\rho}_{nd}(s) = -\frac{i}{\hbar}\mathcal{P}[\tilde{w}(s), f(h)] \tag{3.8}$$

The nondiagonal equation is found using the nondiagonal projection superator $\mathcal{Q}$. The equation is

$$s\tilde{\rho}_{nd}(s) + i(\mathcal{L}^0 + \mathcal{Q}\mathcal{L}^1)\tilde{\rho}_{nd}(s) = -i\mathcal{L}^1\tilde{\rho}_{nd}(s) - \frac{i}{\hbar}\mathcal{Q}[\tilde{w}(s), f(h)] \tag{3.9}$$

The steady state solution is represented by the limit $t \to \infty$. In Laplace space this is equivalent to the limit $s \to 0+$. In this limit Eqs.(3.8) and (3.9) can be combined in one equation

The result obtained for the diagonal part ρ_d of the density operator, the only one pertinent to the conductance, is

$$\tilde{\rho}_d = -\frac{i}{\hbar}\bar{\Lambda}^{-1}\Gamma[w, f(h)] \tag{3.10}$$

where the *Master* $\tilde{\Lambda}$ and Γ superoperators are given by

$$\tilde{\Lambda} = \mathcal{P}\mathcal{L}^1 \frac{1}{i\mathcal{L} + 0^+} \mathcal{L}^1 \tag{3.11}$$

$$\tilde{\Gamma} = \mathcal{P}\left[1 - \mathcal{L}^1 \frac{1}{i\mathcal{L} + 0^+}\right] \tag{3.12}$$

$\mathcal{L}$ and $\mathcal{L}^1$ are defined by

$$\mathcal{L}\bullet \equiv \frac{1}{\hbar}[H, \bullet] \tag{3.13}$$

and

$$\mathcal{L}^1\bullet \equiv \frac{1}{\hbar}[V, \bullet], \tag{3.14}$$

In matrix notation we can write:

$$[w, f(h)] = \sum_{\alpha\beta} [w, f(h)]_{\alpha\beta} |\psi_\alpha\rangle\langle\psi_\beta| \tag{3.15}$$

with $H\psi_i = E_i\psi_i$. This gives

$$\begin{aligned}
\tilde{\rho}_d &= -\frac{i}{\hbar}\tilde{\Lambda}^{-1}\Gamma \sum_{\alpha\beta} [w, f(h)]_{\alpha\beta} |\psi_\alpha\rangle\langle\psi_\beta| \\
&= -\frac{i}{\hbar}\tilde{\Lambda}^{-1} \sum_{\alpha\beta} \Gamma [w, f(h)]_{\alpha\beta} |\psi_\alpha\rangle\langle\psi_\beta| \\
&= -\frac{i}{\hbar}\tilde{\Lambda}^{-1} \sum_{\alpha\beta} [w, f(h)]_{\alpha\beta} \Gamma |\psi_\alpha\rangle\langle\psi_\beta| \qquad (3.16)
\end{aligned}$$

The operator Γ doesn't affect the sum and the number $[w, f(h)]_{\alpha\beta}$. Using the relation [6]

$$\Gamma|\psi_\alpha\rangle\langle\psi_\beta| = |\varphi_\alpha\rangle\langle\varphi_\beta|\delta_{\alpha\beta}, \tag{3.17}$$

in terms of the unpertubed Hamiltonian h_0, we obtain

$$\tilde{\rho}_d = -\frac{i}{\hbar}\tilde{\Lambda}^{-1}\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}|\varphi_\alpha\rangle\langle\varphi_\beta|\delta_{\alpha\beta} \tag{3.18}$$

the matrix elements are

$$\begin{aligned}
\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle &= \langle\varphi_\theta| -\frac{i}{\hbar}\tilde{\Lambda}^{-1}\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}|\varphi_\alpha\rangle\langle\varphi_\beta|\delta_{\alpha\beta}|\varphi_\gamma\rangle \\
&= -\frac{i}{\hbar}\langle\varphi_\theta|\tilde{\Lambda}^{-1}\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}|\varphi_\alpha\rangle\langle\varphi_\beta|\varphi_\gamma\rangle\delta_{\alpha\beta} \\
&= -\frac{i}{\hbar}\langle\varphi_\theta|\tilde{\Lambda}^{-1}\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta} \\
&= -\frac{i}{\hbar}\sum_{\alpha\beta}[w, f(h)]_{\alpha\beta}\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta} \\
&= -\frac{i}{\hbar}\sum_{\alpha\beta}\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}\langle\psi_\alpha|[w, f(h)]|\psi_\beta\rangle
\end{aligned} \tag{3.19}$$

Using the relation [6]

$$\langle\psi_\alpha|[w, f(h)]|\psi_\beta\rangle = (-i\hbar)\left(\frac{f(\epsilon_\beta) - f(\epsilon_\alpha)}{\epsilon_\beta - \epsilon_\alpha}\right)\int_\Omega dr' E(r')\langle\psi_\alpha|j(r')|\psi_\beta\rangle, \tag{3.20}$$

where $f(h)\psi_i = f(\epsilon_i)$, we have

$$\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle = -\frac{i}{\hbar}\sum_{\alpha\beta}\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}(-i\hbar)\left(\frac{f(\epsilon_\beta) - f(\epsilon_\alpha)}{\epsilon_\beta - \epsilon_\alpha}\right)\int_\Omega dr' E(r')\langle\psi_\alpha|j(r')|\psi_\beta\rangle \tag{3.21}$$

With the help of the identity

$$\frac{f(\epsilon_\beta) - f(\epsilon_\alpha)}{\epsilon_\beta - \epsilon_\alpha}\delta_{\alpha\beta} \rightarrow f'(\epsilon_\alpha)\delta_{\alpha\beta} \tag{3.22}$$

Eq.(3.21) takes the form

$$\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle = -\sum_{\alpha\beta}\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}f'(\epsilon_\alpha)\delta_{\alpha\beta}\int_\Omega dr'\langle\psi_\alpha|j(r')|\psi_\beta\rangle E(r'). \tag{3.23}$$

The current density is

$$\begin{aligned} J(r) &= Tr\{j(r)\tilde{\rho}_d\} \\ &= \sum_{\gamma}\langle\varphi_\gamma|j(r)\tilde{\rho}_d|\varphi_\gamma\rangle \\ &= \sum_{\gamma\theta}\langle\varphi_\gamma|j(r)|\varphi_\theta\rangle\langle\varphi_\theta|\tilde{\rho}_d|\varphi_\gamma\rangle \\ &= \sum_{\gamma\theta} j_{\gamma\theta}(r)\tilde{\rho}_{d\theta\gamma} \end{aligned} \tag{3.24}$$

Using Eq.(3.23) we rewrite $J(r)$ as

$$J(r) = -\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\bar{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}f'(\epsilon_\alpha)\int_\Omega dr' E(r')\langle\psi_\alpha|j(r')|\psi_\beta\rangle \tag{3.25}$$

and we get

$$J(r) = -\int_\Omega dr' \left(\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\bar{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}f'(\epsilon_\alpha)\langle\psi_\alpha|j(r')|\psi_\beta\rangle\right) E(r'), \tag{3.26}$$

where $E(r')$ is the electric field current density and the electric field FROM

$$J(r) = \int_\Omega dr'\sigma(r,r')E(r') \tag{3.27}$$

we find the conductivity expression

$$\begin{aligned} \sigma^d(r,r') &= -\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\bar{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}f'(\epsilon_\alpha)\langle\psi_\alpha|j(r')|\psi_\beta\rangle \\ &= -\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\bar{\Lambda}^{-1}|\varphi_\alpha\rangle f'(\epsilon_\alpha)\langle\psi_\alpha|j(r')|\psi_\beta\rangle\delta_{\beta\gamma}\delta_{\alpha\beta} \end{aligned}$$

$$\begin{aligned}
\sigma^d(r,r') &= -\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle\delta_{\beta\gamma}\delta_{\alpha\beta}f'(\epsilon_\alpha)\langle\psi_\alpha|j(r')|\psi_\beta\rangle \\
&= -\sum_{\gamma\theta\alpha\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\alpha\rangle f'(\epsilon_\alpha)\langle\psi_\alpha|j(r')|\psi_\beta\rangle\delta_{\beta\gamma}\delta_{\alpha\beta} \\
&= -\sum_{\gamma\theta\beta} j_{\gamma\theta}(r)\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\beta\rangle f'(\epsilon_\beta)\langle\psi_\beta|j(r')|\psi_\beta\rangle\delta_{\beta\gamma} \\
&= -\sum_{\gamma\theta} j_{\gamma\theta}(r)\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\gamma\rangle f'(\epsilon_\gamma)\langle\psi_\gamma|j(r')|\psi_\gamma\rangle \qquad (3.28)
\end{aligned}$$

In summary, we find that the conductivity is given by

$$\sigma(r,r') = -\sum_{\gamma\theta} j_{\gamma\theta}(r)\langle\varphi_\theta|\tilde{\Lambda}^{-1}|\varphi_\gamma\rangle f'(\epsilon_\gamma)\langle\psi_\gamma|j(r')|\psi_\gamma\rangle. \qquad (3.29)$$

3.2 Conductivity according to Verboven

Verboven's expression [35] can be deduced from Eq.(3.29). The conductivity is

$$\sigma^d_{\mu\gamma}(r,r') = \sum_s \left(\frac{\partial f}{\partial \epsilon_s}\right)(\tilde{\Lambda}^{-1}j(r))_{\varphi_s} < \psi_s|j(r')|\psi_s > . \qquad (3.30)$$

If we use the relaxation-time approximation we can write $(\tilde{\Lambda}^{-1}j(r))_{kk} = \tau_k^{-1}j_{kk}$ and

$$\sigma^d_{\mu\gamma}(r,r') = \sum_k \left(\frac{\partial f}{\partial \epsilon_k}\right)\tau < \varphi_k|j(r)|\varphi_k >< \psi_s|j(r')|\psi_s > . \qquad (3.31)$$

The expression of τ is

$$\tau_k^{-1}j_{kk} = \sum_{k'} T_{kk'}(j_{kk} - j_{k'k'}) \qquad (3.32)$$

$$= \frac{2\pi}{\hbar}|W(k,k')|^2\delta(\epsilon_k - \epsilon_{k'}). \quad (3.34)$$

In following Verboven we choose a free electron along z. The following eigenfunctions φ_k are chosen

$$\varphi_k = \frac{1}{\sqrt{N}} e^{\pm ik_z z}\chi_n(x,y) \quad (3.35)$$

With a parabolic confinement along y the energy is

$$\epsilon_{k,n} = (n+\frac{1}{2})\hbar\omega + \frac{(\hbar k_z)^2}{2m} \quad (3.36)$$

where the confining potential is $V_y = m\omega^2 y^2/2$. The current density is

$$\begin{aligned} j_{kk} &= \frac{-ie\hbar}{2m}\int \varphi_k^* \overleftrightarrow{\nabla} \varphi_k dV \\ &= \frac{-ie\hbar}{2m}\int (\varphi_k^*\nabla\varphi_k - \varphi_k\nabla\varphi_k^*)dV \\ &= \frac{-ie\hbar}{2m}(2ik)\int \varphi_k^*\varphi_k dV \end{aligned} \quad (3.37)$$

Verboven's hypothesis is repeated. The first one is the spherical potential which allows to define a vector by its longitudinal components θ, $\vec{k} = |\vec{k}|cos\theta$. Taking into account normalization, it gives

$$\begin{aligned} j_{\vec{k}\vec{k}} - j_{\vec{k}'\vec{k}'} &= e\frac{|\vec{k}|\hbar}{m}\left(1 - \frac{|\vec{k}'|}{|\vec{k}|}cos\theta\right) \\ &= e\frac{|\vec{k}|\hbar}{m}(1-cos\theta) \end{aligned} \quad (3.38)$$

the last step of Eq.(3.38) is obtained by using Verboven's hypothesis of one band. In this case the indices are $n = n' = 1$. Due to the Dirac function in

Eq.(3.34), the states have the same energy and $|\vec{k}| = |\vec{k'}|$. Then the relaxation time has the form

$$\begin{aligned} \tau_k^{-1}\vec{j}_{kk} &= \sum_{k'} \frac{q\hbar}{m}|\vec{k}|(1 - cos\theta)\frac{2\pi}{\hbar}|W(k,k')|^2\delta(\epsilon_k - \epsilon_{k'}) \\ &= \sum_{k'} |\vec{j}_k|(1 - cos\theta)\frac{2\pi}{\hbar}|W(k,k')|^2\delta(\epsilon_k - \epsilon_{k'}) \end{aligned} \tag{3.39}$$

from that we can deduce τ_k^{-1} wich is similar to Verboven's. The final result for the conductivity is

$$\sigma^d_{\mu\gamma}(r,r') = \sum_k \left(\frac{\partial f}{\partial \epsilon_k}\right) \tau < \varphi_k|j(r)|\varphi_k >< \psi_s|j(r')|\psi_s > \tag{3.40}$$

and coincides with σ given by Verboven.

3.3 Conductance

The conductance is given by the integral

$$G = -\int_{-\infty}^{+\infty} d\epsilon f'(\epsilon)G(\epsilon) \tag{3.41}$$

where

$$G(\epsilon) = \int_S \int_{S'} d\epsilon \sigma^\epsilon_d(r,r')dSdS'; \tag{3.42}$$

dS and dS' are two surfaces and

$$\sigma^\epsilon_d(r,r') \equiv \sum_k \delta(\epsilon - \epsilon_k)(\tilde{\Lambda}^{-1} j_{kk}(r)) j_{KK}(r') \tag{3.43}$$

The main idea now is to connect the scattering eigenfunctions ψ_i with the free-particle eigenfunctions φ_i using the transmission and reflection coefficients respectively $t_{aa'}$ and $r_{aa'}$. This is posible since we look far away from the scattering center of length L. In detail we have

$$\psi_{a+} = \sum_{a'}^{\epsilon} t^L_{aa'}(\epsilon)\varphi_{a'+}(r), \qquad x \gg L, \tag{3.44}$$

$$\psi_{a-} = \varphi_{a-}(r) + \sum_{a'}^{\epsilon} r^R_{aa'}(\epsilon)\varphi_{a'+}(r), \qquad x \gg L, \tag{3.45}$$

$$\psi_{a+} = \varphi_{a+}(r) + \sum_{a'}^{\epsilon} r^L_{aa'}(\epsilon)\varphi_{a'-}(r), \qquad x \ll 0, \tag{3.46}$$

$$\psi_{a-} = \sum_{a'}^{\epsilon} t^R_{aa'}(\epsilon)\varphi_{a'-}(r), \qquad x \ll 0, \tag{3.47}$$

We evaluate the matrices $J_{KK}(r')$ for different regions. To do so, we use new eigenfunctions $\bar{\varphi}$

$$\bar{\varphi}_{a\pm}(r) = \frac{1}{\sqrt{\theta_a}} e^{\pm i l_a z} \chi^{\pm}_a(x, y) \tag{3.48}$$

such as

$$\int_S j_{\bar{\varphi}}(r) \cdot dS = q \tag{3.49}$$

with

$$\theta_a = \frac{\hbar}{m}|l_a| \tag{3.50}$$

The difference from the normalized eigenfunction φ is simply a coefficient

$$\varphi_{a\pm} = \bar{\varphi}_{a\pm}\sqrt{\frac{\theta_a}{L'}} \tag{3.51}$$

The surface integral $\int_S j_{KK}(r) \cdot dS|_{(a\pm)}$ is calculated using these eigenfunctions. In terms of the transmission and reflection coefficients we obtain

$$\int_{S_d} j_{KK}(r) \cdot dS|_{(a+)} = e\,\frac{\theta_a}{L'} \sum_{a'} |t^L_{aa'}|^2, \qquad x \gg L, \tag{3.52}$$

$$\int_{S_g} j_{KK}(r) \cdot dS|_{(a+)} = e\frac{\theta_a}{L'}[1 - \sum_{a'} |r^L_{aa'}|^2], \qquad x \ll 0, \tag{3.53}$$

$$\int_{S_d} j_{KK}(r) \cdot dS|_{(a-)} = -e\frac{\theta_a}{L'} \sum_{a'} |t^R_{aa'}|^2, \qquad x \ll 0, \tag{3.54}$$

$$\int_{S_g} j_{KK}(r) \cdot dS|_{(a-)} = -e\frac{\theta_a}{L'}[1 - \sum_{a'} |r^R_{aa'}|^2]. \qquad x \gg L \tag{3.55}$$

Now we need to evaluate $\tilde{\Lambda}^{-1}$. Since we consider only elastic scattering, we can use the relaxation-time aproximation which gives

$$(\tilde{\Lambda} j(r))_{kk} = \tau_k^{-1} j_{kk}(r), \tag{3.56}$$

and

$$(\tilde{\Lambda}^{-1} j(r))_{kk} = \tau_k j_{kk}(r). \tag{3.57}$$

The calculation gives [6]

$$\int_{S_g} (\tilde{\Lambda} j_{kk}(r))_{kk} \cdot dS|_{(a\pm)} = \pm \frac{L'}{\hbar^2} \sum_{a'}^{\epsilon} \frac{1}{\theta_{a'}} \left\{ |U^{\pm\pm}_{aa'}|^2 \left(1 - \frac{\theta_{a'}}{\theta_a}\right) + |U^{\pm\mp}_{aa'} \left(1 + \frac{\theta_{a'}}{\theta_a}\right) \right\}, \tag{3.58}$$

with

$$U_{kk'} = \frac{\hbar^2}{2m} \int_S \varphi^*{}_k(r) \overleftrightarrow{\nabla} \psi_{k'}(r) \tag{3.59}$$

We evaluate the integral with the transmission and reflection coefficients and obtain

$$\tau_{a+}^{-1} = \frac{\theta_a}{L'}\left\{1 + \frac{1}{\theta_a}\sum_{a'}^{\epsilon}\frac{\theta_{a'}}{\theta_a}(|r_{aa'}^L|^2 - |t_{aa'}^R|^2)\right\}, \tag{3.60}$$

$$\tau_{a-}^{-1} = \frac{\theta_a}{L'}\left\{1 + \frac{1}{\theta_a}\sum_{a'}^{\epsilon}\frac{\theta_{a'}}{\theta_a}(|r_{aa'}^R|^2 - |t_{aa'}^L|^2)\right\}; \tag{3.61}$$

Here R and L refer to the right and left regions. Finally, we find the conductance in the form

$$G(\epsilon) = \frac{2e^2}{h}\sum_a^{\epsilon}\left[\frac{(t^L t^{L\dagger})_{aa}}{1 + \frac{1}{\theta_a}\sum_{a'}^{\epsilon}\theta_{a'}(|r_{aa'}^R|^2 - |t_{aa'}^L|^2)} + \frac{(t^R t^{R\dagger})_{aa}}{1 + \frac{1}{\theta_a}\sum_{a'}^{\epsilon}\theta_{a'}(|r_{aa'}^L|^2 - |t_{aa'}^R|^2)}\right] \tag{3.62}$$

with $(tt^\dagger)_{aa} = \sum_{a'}^{\epsilon}|t_{aa'}|^2$. To gain some insight, we further evaluate this conductance below in two different approximations.

3.4 Conductance with free-particle eigenfunctions

We start with the expression of $G(\epsilon)$ given by Eqs.(3.42) and (3.43)

$$G(\epsilon) = \frac{L'}{2\pi}\sum_a^{\epsilon}\frac{1}{\hbar\theta_a}\left(\int_S \tilde{\Lambda}^{-1} j_{kk}(r)dS\int_{S'} j_{KK}(r')dS'\right)_{a\pm}. \tag{3.63}$$

We aproximate the scattering eigenfunctions by the free-particle eigenfunctions

$$\bar{\psi}_{a\pm} \simeq \bar{\varphi}_{a\pm}. \tag{3.64}$$

Then the current density

$$\vec{j}_{\alpha\beta}(r) = \frac{-ie\hbar}{2m}(\psi_\alpha^* \vec{\nabla}\psi_\beta - \psi_\beta \vec{\nabla}\psi_\alpha^*) \tag{3.65}$$

translates to

$$j_{KK}(r) \simeq \frac{-ie\hbar\theta_a}{2mL'}\bar{\varphi}_{a\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{a\pm}. \tag{3.66}$$

Using the following properties, given in [6]

$$\int_S dS \cdot \bar{\varphi}_{a\pm}^*(r) \overleftrightarrow{\nabla} \bar{\varphi}_{a'\pm}(r) = \pm\frac{2mi}{\hbar}\delta_{a'a} \tag{3.67}$$

$$\int_S dS \cdot \bar{\varphi}_{a\mp}^*(r) \overleftrightarrow{\nabla} \bar{\varphi}_{a\pm}, = 0, \tag{3.68}$$

we obtain

$$\int_S j_{KK}(r) dS|_{(a\pm)} \simeq \pm\frac{e\theta_a}{L'}. \tag{3.69}$$

For the other integral we use the relaxation-time approximation

$$\begin{aligned} \int_S \bar{\Lambda}^{-1} j_{kk}(r) ds|_{(a\pm)} &= \int_S \tau_{kk} j_{kk}(r) ds|_{(a\pm)} \\ &= \tau_{kk} \int_S j_{kk}(r) ds|_{(a\pm)} \\ &= \tau_{a\pm}\frac{-ie\hbar\theta_a}{2mL'}\int_S \bar{\varphi}_{a\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{a\pm} ds \\ &= \tau_{a\pm}(\frac{\pm e\theta_a}{L'}). \end{aligned} \tag{3.70}$$

The expression of $\tau_{a\pm}$ is

$$\tau_{a\pm}^{-1} = \frac{L'}{\hbar^2}\sum_{a'}^{\epsilon}\frac{1}{\theta_{a'}}\left\{|U_{aa'}^{\pm\pm}|^2\left(1-\frac{\theta_{a'}}{\theta_a}\right)+|U_{aa'}^{\pm\mp}|^2\left(1+\frac{\theta_{a'}}{\theta_a}\right)\right\}. \tag{3.71}$$

With

$$\begin{aligned} U_{aa'}^{(\pm or\mp)} &= \frac{\hbar^2}{2m}\int_s ds\cdot\bar{\varphi}_{a\pm}^*(r)\overset{\leftrightarrow}{\nabla}\bar{\psi}_{a'(\pm or\mp)}(r) \\ &\simeq \frac{\hbar^2}{2mL'}\sqrt{\theta_a\theta_{a'}}\int_s ds\cdot\varphi_{a\pm}^*(r)\overset{\leftrightarrow}{\nabla}\varphi_{a'(\pm or\mp)}(r) \end{aligned} \tag{3.72}$$

we get

$$U_{aa'}^{(\pm or\mp)} = \begin{cases} \frac{i\theta_a\hbar}{L'}\delta_{aa'} & \text{for } a^+a'^+ \\ 0 & \text{for } a^+a'^- \\ -\frac{i\theta_a\hbar}{L'}\delta_{aa'} & \text{for } a^-a'^- \\ 0 & \text{for } a^-a'^+ \end{cases} \tag{3.73}$$

$$\begin{aligned} \tau_{a\pm}^{-1} &= \frac{mL'}{\hbar^3}\sum_{a'}^{\epsilon}\frac{1}{l_{a'}}\left\{\left|\pm\frac{i\theta_a\hbar}{L}\delta_{aa'}\right|^2\left(1-\frac{\theta_{a'}}{\theta_a}\right)+|0|^2\left(1+\frac{\theta_{a'}}{\theta_a}\right)\right\} \\ &\simeq \frac{mL'}{\hbar^3}\sum_{a'}^{\epsilon}\frac{1}{l_{a'}}\left\{(\frac{\theta_a\hbar}{L})^2\delta_{aa'}^2\left(1-\frac{\theta_{a'}}{\theta_a}\right)+0\right\} \\ &= 0. \end{aligned} \tag{3.74}$$

For the conductance we obtain

$$\begin{aligned} G(\epsilon) &= \frac{L'}{2\pi}\sum_a^{\epsilon}\frac{1}{\hbar\theta_a}\left(\int_S\bar{\Lambda}^{-1}j_{kk}(r)dS\int_{S'}j_{KK}(r')dS'\right)_{a\pm} \\ &\simeq \frac{L'}{2\pi}\sum_a^{\epsilon}\frac{1}{\hbar\theta_a}\left(\tau_{a\pm}(\frac{\pm e\theta_a}{L'})\right)\left(\pm\frac{e\theta_a}{L'}\right) \\ &= \infty \end{aligned} \tag{3.75}$$

In summary we find that for plane waves the resistance is zero. Clearly this shows that this approximation is not valid.

3.5 Transmission and reflection coefficient in the first-order pertubed eigenfunctions

We use the expression for the scattering eigenfunctions to the first order in the scattering potential H^1

$$|\psi_n\rangle = |\varphi_n\rangle + \sum_{p\neq n} \frac{\langle\varphi_p|H^1|\varphi_n\rangle}{E_n^0 - E_p^0}|\varphi_p\rangle + O(\lambda^2) \tag{3.76}$$

We write Eq.(3.76) in terms of the channel indices

$$\begin{aligned} |\bar{\psi}_{a\pm}\rangle &= |\bar{\varphi}_{a\pm}\rangle + \sum_{p\neq a} \frac{\langle\bar{\varphi}_{p\pm}|H^1|\bar{\varphi}_{a\pm}\rangle}{E_a^0 - E_p^0}|\bar{\varphi}_{p\pm}\rangle + O(\lambda^2) \\ &= |\bar{\varphi}_{a\pm}\rangle + \sum_{p\neq a} D_{ap}|\bar{\varphi}_{p\pm}\rangle. \end{aligned} \tag{3.77}$$

The current density is given by

$$\begin{aligned} j_{KK}(r) &= \frac{-ie\hbar}{2m}\psi_{a\pm}(r)^* \overleftrightarrow{\nabla} \psi_{a\pm}(r) \\ &= \frac{-ie\hbar\theta_a}{2mL'}\bar{\psi}_{a\pm}(r)^* \overleftrightarrow{\nabla} \bar{\psi}_{a\pm}(r) \\ &= c(\bar{\varphi}_{a\pm} + \sum_{p\neq a} D_{ap}\bar{\varphi}_{p\pm})^* \overleftrightarrow{\nabla} (\bar{\varphi}_{a\pm} + \sum_{b\neq a} D_{ab}\bar{\varphi}_{b\pm}) \\ &= c\bar{\varphi}_{a\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{a\pm} + c\sum_{b\neq a} D_{ab}\bar{\varphi}_{a\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{b\pm} + c\sum_{p\neq a} D_{ap}^*\bar{\varphi}_{p\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{a\pm} \\ &\quad + c\sum_{p\neq a}\sum_{b\neq a} D_{ap}^* D_{ab}\bar{\varphi}_{p\pm}^* \overleftrightarrow{\nabla} \bar{\varphi}_{b\pm} \end{aligned} \tag{3.78}$$

with $c = \frac{-ie\hbar\theta_a}{2mL'}$ and

$$\begin{aligned}
\int j_{KK}(r)_{a\pm} dS &= c\frac{\pm 2mi}{\hbar}\left(1 + \sum_{b\neq a} D_{ab}\delta_{a,b} + \sum_{p\neq a} D^*_{ap}\delta_{p,a} + \sum_{p\neq a}\sum_{b\neq a} D^*_{ap}D_{ab}\delta_{p,b}\right) \\
&= \frac{\pm e\theta_a}{L'}\left(1 + 0 + 0 + \sum_{p\neq a} D^*_{ap}D_{ap}\right) \\
&= \frac{\pm e\theta_a}{L'}\left(1 + \sum_{p\neq a}\left|\frac{\langle\bar{\varphi}_{p\pm}|H^1|\bar{\varphi}_{a\pm}\rangle}{E^0_a - E^0_p}\right|^2\right). \qquad (3.79)
\end{aligned}$$

If we use Eqs.(3.52) - (3.54) we have

$$\int j_{KK}(r)_{a+} dS = \frac{+e\theta_a}{L}\sum_{a'}|t^R|^2, \qquad (3.80)$$

$$\int j_{KK}(r)_{a-} dS = \frac{-e\theta_a}{L}\sum_{a'}|t^L|^2. \qquad (3.81)$$

we then see that

$$\sum_{a'}|t_{aa'}|^2 = 1 + \sum_{p\neq a}\left|\frac{\langle\bar{\varphi}_{p\pm}|H^1|\bar{\varphi}_{a\pm}\rangle}{E^0_a - E^0_p}\right|^2. \qquad (3.82)$$

For U we find

$$\begin{aligned}
U_{a\pm c\pm} &= \frac{\hbar^2}{2m}\int \varphi^*_{a\pm}\overleftrightarrow{\nabla}\psi_{c\pm} dS \\
&= \frac{\hbar^2}{2m}\sqrt{\frac{\theta_a}{L'}}\sqrt{\frac{\theta_c}{L'}}\int \bar{\varphi}^*_{a\pm}\overleftrightarrow{\nabla}\bar{\psi}_{c\pm} ds \\
&= \frac{\hbar^2}{2m}\sqrt{\frac{\theta_a}{L'}}\sqrt{\frac{\theta_c}{L'}}\int \bar{\varphi}^*_{a\pm}\overleftrightarrow{\nabla}\bar{\varphi}_{c\pm} + \sum_{p\neq c} D_{cp}\bar{\varphi}_{p\pm} ds \\
&= c_{ac}\int \bar{\varphi}^*_{a\pm}\overleftrightarrow{\nabla}\bar{\varphi}_{c\pm} ds + c_1\sum_{p\neq c} D_{cp}\int \bar{\varphi}^*_{a\pm}\bar{\varphi}_{p\pm} ds \\
&= c_{ac}\left(\frac{\pm 2mi}{\hbar}\right)\left(\delta_{ac} + \sum_{p\neq c} D_{cp}\delta_{ap}\right)
\end{aligned}$$

$$= \pm i\hbar \frac{\sqrt{\theta_a \theta_c}}{L'} \left(\delta_{ac} + \sum_{p \neq c} D_{cp} \delta_{ap} \right)$$

$$= \pm i\hbar \frac{\sqrt{\theta_a \theta_c}}{L'} \left(\delta_{ac} + \sum_{a \neq c} \frac{\langle \bar{\varphi}_{a\pm} | H^1 | \bar{\varphi}_{c\pm} \rangle}{E_c^0 - E_a^0} \right). \quad (3.83)$$

with $c_{ac} = \pm i\hbar \frac{\sqrt{\theta_a \theta_c}}{L}$ and

$$U_{a\pm c\mp} = c_{ac} \int \bar{\varphi}^*_{a\pm} \overleftrightarrow{\nabla} \bar{\varphi}_{c\mp} dS + c_{ac} \sum_{p \neq c} D_{cp} \int \bar{\varphi}^*_{a\pm} \bar{\varphi}_{p\mp} dS$$

$$= 0 \quad (3.84)$$

With U given in [6],

$$|U_{a\pm c\pm}|^2 = \frac{\hbar^2 \theta_a \theta_c}{L^2} |\delta_{ac} + t_{ac}^{R\ or L}|^2, \quad (3.85)$$

and

$$|U_{a\mp c\mp}|^2 = \frac{\hbar^2 \theta_a \theta_c}{L^2} |r_{ac}^{R\ or L}|^2, \quad (3.86)$$

we find

$$t_{ac} = \sum_{p \neq c} \frac{\langle \bar{\varphi}_{a\pm} | H^1 | \bar{\varphi}_{c\pm} \rangle}{E_c^0 - E_a^0} \quad (3.87)$$

and

$$r_{ac} = 0. \quad (3.88)$$

3.6 Total transmission and reflection coefficients

We have to solve the equation

$$\frac{d^2\psi}{dx^2} + \left(k^2 - \lambda U(x)\right)\psi = 0 \tag{3.89}$$

with the boundary conditions

$$\psi \rightarrow e^{ikx} + Re^{-ikx} \text{ for x} \rightarrow -\infty \tag{3.90}$$

$$\psi \rightarrow Te^{ikx} \text{ for x} \rightarrow +\infty \tag{3.91}$$

The Green's function

$$G(x|x_0) = (\frac{i}{2k})e^{ik|x-x_0|} \tag{3.92}$$

is a solution of

$$\frac{d^2G}{dx^2} + k^2G = -\delta(x - x_0), \tag{3.93}$$

Therefore, the solution of Eq.(3.92)

$$\psi(x) = e^{ikx} - \lambda\int_{-\infty}^{+\infty} G(x|x_0)U(x_0)\psi(x_0)dx_0 \tag{3.94}$$

becomes

$$\psi(x) = e^{ikx} + (\frac{\lambda}{2ik})\left[\int_{-\infty}^{x} e^{ik(x-x_0)}U(x_0)\psi(x_0)dx_0 + \int_{x}^{+\infty} e^{ik(x_0-x)}U(x_0)\psi(x_0)dx_0\right] \tag{3.95}$$

where λ is a parameter with value between 0 and 1. Its limiting behavior is

$$\psi(x) = e^{ikx} + e^{-ikx}\frac{\lambda}{2ik}\int_{-\infty}^{+\infty} e^{ikx_0}U(x_0)\psi(x_0)dx_0 \text{ for x} \rightarrow -\infty \tag{3.96}$$

$$\psi(x) = e^{ikx} + e^{ikx}\frac{\lambda}{2ik}\int_{-\infty}^{+\infty} e^{-ikx_0}U(x_0)\psi(x_0)dx_0 \text{ for x} \rightarrow +\infty \tag{3.97}$$

Comparing Eq.(3.90) and Eq.(3.91) with the boundary conditions (Eq.(3.96) and Eq.(3.54)) we obtain the total transmission and reflection coefficients

$$R = \frac{\lambda^2}{4k^2}\left|\int_{-\infty}^{+\infty} e^{ikx_0}U(x_0)\psi(x_0)dx_0\right|^2 \tag{3.98}$$

$$T = \left|1 + \frac{\lambda^2}{4k^2}\int_{-\infty}^{+\infty} e^{-ikx_0}U(x_0)\psi(x_0)dx_0\right|^2 \tag{3.99}$$

Thus, the Born approximation has the drawback that the transmission coefficient is larger than 1.

Chapter 4

Conductance in the presence of crossed electric and magnetic fields

4.1 Magnetoconductance, conductivity tensor, and Hall coefficient

Magnetoconductance

We discuss briefly the distribution of the electric field and the current in a magnetic field. Consider a two-dimensional system in the xy-plane with

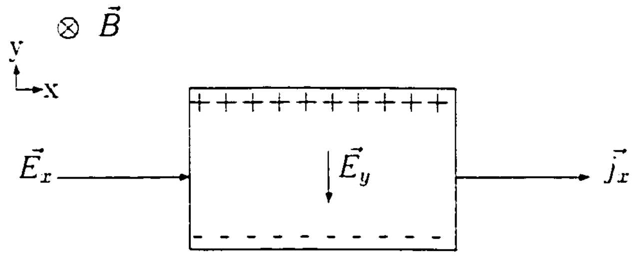

Figure 4.1: Schematic view a Hall bar.

a magnetic field along z (see Figure 4.1). If we apply a voltage along the conductor. there is a flow of current in the direction E_x of the electric field represented by the current density j_x. The voltage V_y perpendicular to the bar is zero. The situation changes if we apply a magnetic field in the direction -$\hat{z}$. As shown in Figure 4.1 the Lorentz force $\vec{f}(t) = q(\vec{E} + \vec{v} \times \vec{B})$ creates an accumulation of positive charges along one side of the bar and leaves the other one negative. This gives rise to an electric field E_y perpendicular to the current. The voltage V_y is no longer zero. The ratio E_x/j_x is the usual conductance. In the presence of a transverse magnetic field $B_\perp$ this conductance is called magnetoconductance.

Conductivity tensor

The classical calculation starts with the variation of the momentum during collisions

$$\frac{d\vec{p}(t)}{dt} = -\frac{\vec{p}(t)}{\tau} + \vec{f}(t) \tag{4.1}$$

We multiply these equations by $nq\tau/m$ and current density $\vec{j} = n\vec{p}/m$. It gives

$$\sigma_o E_x = j_x + \omega_c \tau j_y, \tag{4.2}$$

$$\sigma_o E_y = -\omega_c \tau j_x + j_y, \tag{4.3}$$

where $\sigma_o = nq^2\tau/m$ is the Drude conductivity and $\omega_c = q|\vec{B}|/m$. The expression of the current density in terms of the electric field is

$$j_x = \frac{\sigma_o}{1+\omega_c^2\tau^2}(E_x - \omega_c \tau E_y) \tag{4.4}$$

$$j_y = \frac{\sigma_o}{1+\omega_c^2\tau^2}(\omega_c \tau E_x + E_y) \tag{4.5}$$

The conductivity is a tensor

$$\vec{J} = \overleftrightarrow{\sigma}\, \vec{E} \tag{4.6}$$

For a material the electric field at one specific location can be different than the applied electric field. This is represented by the equation

$$\vec{J}(r) = \overleftrightarrow{\sigma}\,(r, r')\vec{E}(r') \tag{4.7}$$

In our model this situation is considered. The relation between the conductivity and the conductance is

$$G(r, r') = \int_A \int_{A'} dA \sigma(r, r') dA' \tag{4.8}$$

Hall coefficient

The Hall fied E_y is found using the requirement $j_y = 0$. From Eq. (4.3) we obtain

$$R_H = \frac{E_y}{j_x B} = \frac{1}{nq} \tag{4.9}$$

4.2 Schroedinger's equation

In the following section we will give the classical Hamiltonian for an electron in an electromagnetic field. Then from this Hamiltonian we will write Schroedinger's equation without pertubation.

4.2.1 Classical Hamiltonian

The electromagnetic field interacts with charged particles. Its effect is represented by the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{4.10}$$

The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields can be expressed in terms of the vector ($\vec{A}$) and scalar (ϕ) potentials as

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}, \tag{4.11}$$

$$\vec{B} = \nabla \times \vec{A}. \tag{4.12}$$

The Lorentz force is rewritten in term of theses two potentials. Using the identity

$$\vec{B} \times (\nabla \times \vec{C}) = \nabla(\vec{B} \cdot \vec{C}) - (\vec{B} \cdot \nabla)\vec{C} - (\vec{C} \cdot \nabla)\vec{B} - (\vec{C} \times (\nabla \times \vec{B})) \quad (4.13)$$

and the expression

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A} \quad (4.14)$$

to transform the triple product

$$\vec{v} \times (\nabla \times \vec{A}) = \nabla(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \nabla)\vec{A} \quad (4.15)$$

and the velocity not being an explicit function of the position. Eq.(4.10) becomes

$$\vec{F} = q\left[-\vec{\nabla}\phi + \vec{\nabla}(\vec{v} \cdot \vec{A}) - \frac{d\vec{A}}{dt}\right]. \quad (4.16)$$

In Lagrange's formulation, the generalized forces are related to the potential $U(x_i, \dot{x}_i)$ by

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt}(\frac{\partial U}{\partial \dot{x}_i}). \quad (4.17)$$

Using

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}\vec{\nabla}_{\vec{v}}(\vec{A} \cdot \vec{v}) \quad (4.18)$$

where the derivative is with respect to the velocity we get

$$F_i = -\frac{\partial}{\partial x_i}\left(q\phi - q\vec{v} \cdot \vec{A}\right) + \frac{d}{dt}\frac{\partial}{\partial v_i}\left(q\phi - q\vec{v} \cdot \vec{A}\right). \quad (4.19)$$

the scalar potential is added since it is independent of the velocity $v(\vec{r}, t)$. A comparison between Eq.(4.16) and Eq.(4.19) gives the generalized potential

$$U = q\phi - q\vec{v} \cdot \vec{A}. \tag{4.20}$$

The Lagrangian $L = T - U$ becomes

$$L = \frac{1}{2}m\vec{v}^2 - q\phi + q\vec{v} \cdot \vec{A}. \tag{4.21}$$

In classical mechanics the relation between the Lagrangian and Hamiltonian is given by

$$H(\vec{r}, \vec{p}) = \sum_i p_i \dot{x}_i - L(x_i, \dot{x}_i). \tag{4.22}$$

This leads to

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi. \tag{4.23}$$

This equation is the classical Hamiltonian for a free, charged particle in an electromagnetic field.

Quantum expression

The transition to non-relativistic quantum mechanics is done by replacing the momentum $\vec{p}$ with the operator $\hbar\vec{\nabla}/i$. Therefore we have

$$H_{op} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\phi \tag{4.24}$$

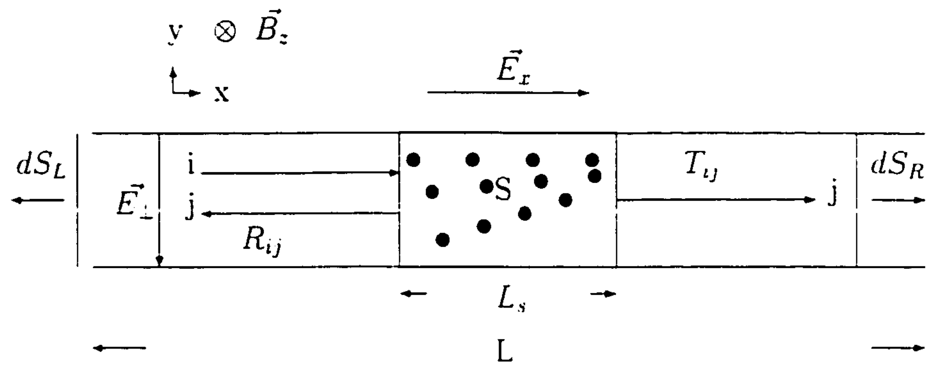

Figure 4.2: A quasi-one-dimensional conductor, connected to left (L) and right (R) reservoirs in the presence of a crossed electric and a magnetic fields. The length of the conductor is L. The solid dots represent scattering centers.

This is the quantum mechanical Hamiltonian for an electron in an electromagnetic field. Since the later is not quantized, it is a semiclassical hamiltonian.

Hamiltonian

We consider the model shown in Figure 4.2. We use the one-electron approximation i.e, no Coulomb interactions are present, and $\phi = 0$. The magnetic field **B** is constant along the z axis ($\vec{B} = -B\hat{z}$), the Hall field $E_\perp$ is opposite to the y axis [27], the confining potential is parabolic along the y axis.

$V_y = m\Omega^2 y^2/2$, and the vector potential is along the x axis (Landau gauge) $\vec{A} = By\hat{x}$. Then

$$\begin{aligned} H_0 &= \frac{1}{2m}\left(\vec{P} - q\vec{A}\right)^2 - qE_\perp y + \frac{1}{2}m\Omega^2 y^2 \\ &= \frac{1}{2m}(P_x^2 - P_x qBy - qByP_x + q^2B^2y^2) \\ &\quad + \frac{P_y^2}{2m} - qE_\perp y + \frac{m}{2}\Omega^2 y^2. \end{aligned} \tag{4.25}$$

4.2.2 Solution of Schroedinger's equation

Harmonic Oscillator

We attempt a solution of Eq.(4.25) in the form

$$\varphi(x, y) = \chi(y)\exp(ik_x x) \tag{4.26}$$

This gives

$$\frac{1}{2m}[\hbar^2 k_x^2 - 2\hbar k_x m\omega_C y - 2mqE_\perp y + m^2(\omega_C^2 + \Omega^2)y^2]\chi(y) - \frac{\hbar^2}{2m}\chi''(y) = \epsilon\chi(y) \tag{4.27}$$

with the cyclotron frequency $\omega_C = qB/m$. Introducing

$$\xi = \frac{\hbar k_x}{qB} + \frac{qE_\perp}{m\omega_C^2}, \tag{4.28}$$

writing $\omega_T^2 = \omega_c^2 + \Omega^2$, and completing the square, Eq.(4.27) is rewritten as

$$\frac{m\omega_T^2}{2}\left(y - \frac{\omega_C^2}{\omega_T^2}\xi\right)^2\chi(y) - \frac{\hbar^2}{2m}\chi''(y) = E\chi(y) \tag{4.29}$$

where

$$E = \epsilon - \frac{m\omega_C^2\Omega^2}{2\omega_T^2}Y_k^2 + \frac{\omega_C}{m\omega_T^2}\hbar k_x qE_\perp + \frac{1}{2m\omega_T^2}q^2E_\perp^2 \tag{4.30}$$

and $Y_k = \hbar k_x/qB$. Equation (4.29) describes a displaced harmonic oscillator, centered at $y = \omega_C^2\xi/\omega_T^2$.

Eigenfunctions, energy, and velocity

The solution of Eq.(4.29) is obtained by a power-series method or by using creation and anhilation operators. It is expressed in terms of the Hermite polynomials. The eigenfunctions are

$$\chi_n(y) = G_n\left(q - \frac{\omega_C^2}{\omega_T^2}q_\xi\right) \tag{4.31}$$

where $q = (m\omega_T/\hbar)^{1/2}y$ and $q_\xi = (m\omega_T/\hbar)^{\frac{1}{2}}\xi$ with

$$G_n(\zeta) = e^{-\zeta^2/2}H_n(\zeta), \tag{4.32}$$

where $H_n(\zeta)$ are the Hermite polynomials.

The oscillator energy is a multiple of the energy $\hbar\omega_T$ and n is the Landau level index. The total energy is

$$\epsilon(k_x, n) = \left(n + \frac{1}{2}\right)\hbar\omega_T + \frac{\hbar^2k_x^2}{2m}\frac{\Omega^2}{\omega_T^2} - \frac{\omega_C}{m\omega_T^2}\hbar k_x qE_\perp - \frac{q^2E_\perp^2}{2m\omega_T^2} \tag{4.33}$$

From this expression we obtain the velocity along the direction of propagation using $\vec{v} = \vec{\nabla}_{\vec{k}}\epsilon(\vec{k})/\hbar$. The result is

$$v_x = \frac{\hbar k_x}{m}\frac{\Omega^2}{\omega_T^2} - \frac{\omega_C}{m\omega_T^2}qE_\perp \tag{4.34}$$

4.3 Current Density in a magnetic field

4.3.1 Current density

The current density is found by combining Schroedinger's equation with the continuity equation. The Schroedinger's equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi; \tag{4.35}$$

taking the complex conjuguate, we have

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = H^*\Psi^* \tag{4.36}$$

We now multiply Eq.(4.35) by Ψ^* and Eq.(4.36) by Ψ. Substracting the results we obtain

$$i\hbar\frac{\partial(\Psi^*\Psi)}{\partial t} = \Psi^*(H\Psi) - (H^*\Psi^*)\Psi. \tag{4.37}$$

Using the Hamiltonian operator we have

$$\begin{aligned} i\hbar\frac{\partial(\Psi^*\Psi)}{\partial t} &= \frac{-\hbar^2}{2m}\left[\Psi^*(\nabla^2\Psi) - (\nabla^2\Psi^*)\Psi\right] \\ &+ \frac{i\hbar q}{2m}\left[\Psi^*(\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}) + \left(\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\right)\Psi^*\right]\Psi \end{aligned} \tag{4.38}$$

We write the first term on the right-hand side as

$$\Psi^*(\nabla^2\Psi) - (\nabla^2\Psi^*)\Psi = \vec{\nabla}\cdot\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right). \tag{4.39}$$

Since

$$\vec{\nabla}(a\vec{\nabla}b) = (\vec{\nabla}a)(\vec{\nabla}b) + a\nabla^2 b, \tag{4.40}$$

the second term on the right-hand side is simplified :

$$\begin{aligned} 2\Psi^*\Psi\vec{\nabla}\cdot\vec{A} + 2\vec{A}\cdot(\Psi^*\nabla\Psi + (\nabla\Psi^*)\Psi) &= 2\Psi^*\Psi\vec{\nabla}\cdot\vec{A} + 2\vec{A}\cdot\vec{\nabla}(\Psi^*\Psi) \\ &= 2\vec{\nabla}\cdot(\vec{A}\Psi^*\Psi). \end{aligned} \tag{4.41}$$

Given these modifications, we deduce the following equation

$$\begin{aligned} i\hbar\frac{\partial(\Psi^*\Psi)}{\partial t} &= \frac{-\hbar^2}{2m}\vec{\nabla}\cdot\left[\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right] \\ &+ \frac{i\hbar q}{m}\vec{\nabla}\cdot(\vec{A}\Psi^*\Psi). \end{aligned} \tag{4.42}$$

This equation has the form of the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \tag{4.43}$$

where ρ is the probability density

$$\rho(\vec{r},t) = \Psi^*(\vec{r},t)\Psi(\vec{r},t), \tag{4.44}$$

and $\vec{J}$ the current density

$$\vec{J} = \frac{-i\hbar}{2m}\left[\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right] - \frac{q}{m}\vec{A}\Psi^*\Psi. \tag{4.45}$$

4.3.2 Representation of the current density

The current density operator is expressed in terms of the eigenfunctions in matrix form as

$$\vec{J}_{\beta\alpha} = \frac{-iq\hbar}{2m}\left[\varphi_\beta^*(\vec{\nabla}\varphi_\alpha) - (\vec{\nabla}\varphi_\beta^*)\varphi_\alpha\right] - \frac{q}{m}\vec{A}\varphi_\beta^*\varphi_\alpha \tag{4.46}$$

We rewrite Eq.(4.46) with the gauge-invariant derivative $\vec{D} = \vec{\nabla} - iq\vec{A}/m$ using the notation $f \overleftrightarrow{D} g = f\vec{\nabla}g - g\vec{\nabla}^* f$:

$$\vec{J}_{\beta\alpha} = \frac{-iq\hbar}{2m}\varphi_\beta^* \overleftrightarrow{D} \varphi_\alpha. \tag{4.47}$$

The components along different axes are

$$j^x_{k_{x1},n;k_{x2},m}(x,y) = q\left[\frac{\hbar}{2m}(k_{x2}+k_{x1}) - yqB/m\right] e^{i(k_{x2}-k_{x1})x}\chi^*_{k_{x1},n}\chi_{k_{x2},m} \tag{4.48}$$

$$j^y_{k_{x1},n;k_{x2},m}(x,y) = \frac{q\hbar}{2im}\left[\chi^*_{kx1,n}\frac{\partial\chi_{kx2,m}}{\partial y} - \chi_{k_{x2},m}\frac{\partial\chi^*_{k_{x1},n}}{\partial y}\right] e^{i(k_{x2}-k_{x1})x}. \tag{4.49}$$

We notice some perticuliarities of the diagonal terms ($k_{x2} = k_{x1}, m = n$). The current density in the x direction depends only on the y position and vanishes along the y direction since $\chi^*(y) = \chi(y)$. Along the y direction the nondiagonal current density is

$$\int_{-\infty}^{+\infty} j^y_{ij}dx = F(\chi_{i,j}, \chi'_{i,j})\int_{-\infty}^{+\infty} e^{i(k_j-k_i)x}dx = 0 \tag{4.50}$$

It makes sense since in our model there is a confinement along the y axis. This confinement is represented by a parabolic potential. The consequence of this potential is to cancel the result of the integral of the current density along the y axis.

4.3.3 Probability density and normalized flux

Usually the solutions of Schroendinger's equation are normalized. In order to have a probability equal to one in the entire space, a normalization constant N_n is used.

$$\varphi_n(x,y) = \frac{1}{\sqrt{N_n}}\chi_n(y)e^{(ik_x x)} \tag{4.51}$$

with $\int |\varphi|_n^2 dv = 1$

$$\begin{aligned}\int |\varphi|_n^2 dv &= \frac{1}{N_n}\int\int e^{i(k_x x - k_x x)}\chi_n^2(y)dxdy \\ &= \frac{L}{N_n}\int e^{-\zeta^2}H_n^2(\zeta)dy,\end{aligned} \tag{4.52}$$

With the change of variable $y = (\gamma - d)/c$ ($c = \sqrt{m\omega_T/\hbar}$, $d = \omega_C^2 c\xi/\omega_T^2$) and $dy = d\gamma/c$, we have

$$\begin{aligned}\int |\varphi|_n^2 dv &= \frac{L}{cN_n}\int e^{-\gamma^2}H_n^2(\gamma)d\gamma \\ &= \frac{L}{cN_n}\sqrt{\pi}\cdot 2^n \cdot n!,\end{aligned} \tag{4.53}$$

the normalization constant N_n is

$$N_n = L\sqrt{\pi}2^n n!/c \tag{4.54}$$

But the model we used here is a scattering problem. In order to define a scattering matrix it is more convenient to normalize the flux using $\int \vec{J}\vec{ds}/q = 1$.

The new eigenfunction is

$$\overline{\varphi}_{\pm,a} = \frac{1}{\sqrt{\theta_a}} e^{\pm ik_{xa}x} \overline{\chi}_{n_a,\pm k_{xa}}(y) \tag{4.55}$$

where θ is a normalized flux constant and $\int \overline{\chi}^2 dy = 1$. We have

$$\begin{aligned}
\theta_{\pm a} &= \frac{\pm 1}{q} \int J^x_{\pm a} dy \\
&= \frac{\pm(-i\hbar)L}{2mN_n} \int dy (e^{\pm ik_a x}\chi)^* \overleftrightarrow{D} \cdot \hat{x} (e^{\pm ik_a x}\chi) \\
&= \frac{\pm(-i\hbar)L}{2mN_n} \int dy e^{\mp ik_a x} \chi \left[\overleftrightarrow{d/dx - qiBy/\hbar} \right] e^{\pm ik_a x} \chi \\
&= \frac{\pm(-i\hbar)L}{2mN_m} \int (\pm 2ik_a - 2\frac{qi}{\hbar} By)\chi^2 dy \\
&= \pm \frac{L}{N_n} \int (\frac{\pm k_a \hbar}{m} - \frac{qB}{m} y)\chi^2 dy.
\end{aligned} \tag{4.56}$$

If we repeat the change of variable $y = (\gamma - d)/c$ (see above) and apply the properties of even ($\int_{-l}^{+l} f(y)dy = 0$) and odd functions ($f(-y) = -f(y)$), we obtain

$$\begin{aligned}
\theta_{\pm a} &= \pm \frac{\pm k_a \hbar L}{mN_n c} \int \chi^2(\gamma) d\gamma - \frac{\pm qBdL}{mN_n c^2} \int \chi^2(\gamma) d\gamma - \frac{\pm qBL}{mN_n c^2} \int \gamma \chi^2(\gamma) d\gamma \\
&= \pm \left(\frac{\pm k_a \hbar}{m} - \frac{qBd}{mc} \right) \frac{L}{N_n c} \int \chi^2(\gamma) d\gamma \\
&= \left(\frac{|k_a|\hbar}{m} \mp \frac{qBd}{mc} \right)
\end{aligned}$$

$$= v_{\pm a}, \tag{4.57}$$

where

$$\begin{aligned} v_{\pm a} &= \left[\frac{|k_a|\hbar}{m} \mp \frac{qB}{m}\frac{\omega_C^2}{\omega_T^2}\xi\right] \\ &= \left[\frac{|k_a|\hbar}{m} - \frac{qB}{m}\frac{\omega_C^2}{\omega_T^2}\left(\frac{|k_a|\hbar}{qB} \pm \frac{qE_\perp}{m\omega_C^2}\right)\right] \end{aligned} \tag{4.58}$$

or after simplification

$$v_{\pm a} = \left[\frac{|k_a|\hbar}{m}\frac{\Omega^2}{\omega_T^2} \mp \frac{\omega_c qE_\perp}{m\omega_T^2}\right] \tag{4.59}$$

The relation between the two eigenfunctions is

$$\overline{\varphi}_n = \sqrt{\frac{L}{v_n}}\varphi_n \tag{4.60}$$

4.4 Properties

4.4.1 First relation

The gradient of the current density can be expressed in terms of an energy difference

$$\vec{\nabla}\vec{J}_{\beta\alpha} = \frac{iq}{\hbar}\epsilon_{\alpha\beta}\varphi_\beta^*\varphi_\alpha. \tag{4.61}$$

Explicitly we obtain

$$\vec{\nabla}\vec{J}_{\beta\alpha} = \frac{-iq\hbar}{2m}\{\varphi_\beta^*(\nabla^2\varphi_\alpha) - \varphi_\alpha(\nabla^2\varphi_\beta^*) - \frac{2iq}{\hbar}(\nabla A)\varphi_\beta^*\varphi_\alpha - \frac{2iq}{\hbar}A\nabla(\varphi_\beta^*\varphi_\alpha)\}$$

$$
\begin{aligned}
&= \frac{iq}{\hbar}\{\varphi_\beta^*(\frac{P^2}{2m} - \frac{W}{2m})\varphi_\alpha - \varphi_\alpha(\frac{P^2}{2m} - \frac{N}{2m}\varphi_\alpha)\} \\
&= \frac{iq}{\hbar}\{\varphi_\beta^*(\frac{P^2}{2m} - \frac{W}{2m} + q^2A^2 + U(y))\varphi_\alpha - \varphi_\alpha(\frac{P^2}{2m} - \frac{N}{2m} + q^2A^2 + U(y))\varphi_\alpha\} \\
&= \frac{iq}{\hbar}\{\varphi_\beta^*(H\varphi_\alpha) - \varphi_\alpha(H\varphi_\beta)^*\} \\
&= \frac{iq}{\hbar}(\epsilon_\alpha - \epsilon_\beta)\varphi_\alpha\varphi_\beta^* \qquad (4.62)
\end{aligned}
$$

where $N = qP^*A + qAP^*$ and $W = qPA + qAP$ are used.

4.4.2 Second relation

We deduce some properties from the first relation. With eigenfunctions of the same energy, the current density matrix elements are constant [4].

Since

$$\vec{\nabla}\vec{J}_{\beta\alpha} = \frac{ie}{\hbar}\epsilon_{\alpha\beta}\varphi_\beta^*\varphi_\alpha = \frac{ie}{\hbar L^2}\epsilon_{\alpha\beta}\sqrt{v_\beta v_\alpha}\overline{\varphi}_\beta^*\overline{\varphi}_\alpha, \qquad (4.63)$$

gives

$$\vec{\nabla}\vec{J}_{\beta\alpha} = 0 \qquad (4.64)$$

for $\epsilon_\alpha = \epsilon_\beta$, we have

$$I_{\beta\alpha} = \int \vec{J}_{\beta\alpha}d\vec{S} = \int \nabla\vec{J}_{\beta\alpha}dV = constant. \qquad (4.65)$$

There is no current along the y axis ; consequently

$$I_{\beta\alpha}(x) = \int < \varphi_\beta|j^x|\varphi_\alpha > dy$$

$$= \frac{\sqrt{v_\beta v_\alpha}}{L} \int < \overline{\varphi}_\beta | j^x | \overline{\varphi}_\alpha > dy \tag{4.66}$$

and

$$I_{\beta\alpha}(x) = constant. \tag{4.67}$$

The displacement of the homogenous system by $\Delta\vec{x}$ can be represented by a unitary operator. The displacement operator is $\hat{T}(\Delta\vec{x}) = e^{-i\Delta\vec{x}\cdot\vec{p}/\hbar}$. After displacing, operators associated with physical quantities are modified by $\hat{T}$. New operators are obtained by $\hat{A}' = \hat{T}\hat{A}\hat{T}^\dagger$. This modification is done on the current. We obtain

$$I_{\beta\alpha}(x + \Delta x) = \int \hat{x}\cdot < \varphi_\beta | \mathbf{J}_{op}(x + \Delta x) | \varphi_\alpha > dy \tag{4.68}$$

the result is

$$I_{\beta\alpha}(x + \Delta x) = \int \hat{x} < \varphi_\beta | e^{-i\Delta x p_x/\hbar} \mathbf{J}_{op}(x + \Delta x) e^{+i\Delta x p_x/\hbar} | \varphi_\alpha > dy \tag{4.69}$$

The basis is chosen such that the functions are eigenfunctions of the translation operator. The exponential acts on $|\varphi_i>$ and gives a phase factor

$$I_{\beta\alpha}(x + \Delta x) = e^{i(k_\beta - k_\alpha)\Delta x} I_{\beta\alpha}(x) \tag{4.70}$$

Equations (4.67) and (4.70) are verified for identical states $\varphi_{\pm\beta} = \varphi_{\pm\alpha}$. If we choose a constant equal to $\sqrt{v_\beta v_\alpha}/qL$ and use Eq.(4.47), we obtain

$$\int dy\, \overline{\varphi}^*_{\pm\beta} (\overleftrightarrow{D} \cdot \vec{x}) \overline{\varphi}_{\pm\alpha} = \frac{\pm 2mi}{\hbar} \delta_{\alpha\beta},\ \epsilon_\beta = \epsilon_\alpha \tag{4.71}$$

For two states $\varphi_{\mp\beta}$ and $\varphi_{\pm\alpha}$ with the same energy $\epsilon_\beta = \epsilon_\alpha$, the only possibility to satisfy the two equations (4.67) and (4.70) is to have a current equal to zero: $I_{\beta\alpha} = 0$. The resulting property is

$$\int dy\, \overline{\varphi}^{*}_{\mp\beta}(\overleftrightarrow{D} \cdot \vec{x})\overline{\varphi}_{\pm\alpha} = 0, \;\; \epsilon_\beta = \epsilon_\alpha. \tag{4.72}$$

Equations (4.71) and (4.72) are the main results of this section.

4.5 Diagonal and nondiagonal current density

Every transport theory must satisfy current conservation in the static limit ($\nabla < \vec{J} >= 0$). This property is verified for states with the same energy. More precisely, if we assume a density operator which does not depend on the position $\rho \neq f(x, y)$, we obtain

$$\begin{aligned}
\vec{\nabla} < \vec{J}_{op} > &= \vec{\nabla} Tr\{\rho \vec{J}_{op}\} \\
&= \vec{\nabla} \sum_{i,j} < i|\rho|j >< j|\vec{J}_{op}|i > \\
&= \sum_{i,j} < i|\rho|j > \vec{\nabla} < j|\vec{J}_{op}|i > \\
&= \sum_{i,j} \rho_{i,j} \vec{\nabla} \vec{J}_{j,i} = 0. \qquad (4.73)
\end{aligned}$$

The current can be separated in diagonal and nondiagonal components

$$\begin{aligned}
\vec{\nabla} < \vec{J}_{op} > &= \vec{\nabla} \sum_{i,j} \rho_{i,j} \vec{J}_{j,i} \\
&= \vec{\nabla} \left(\sum_{i,j} \rho_{i,j} \vec{J}_{j,i} \delta_{j,i} + \sum_{i \neq j} \rho_{i,j} \vec{J}_{j,i} \right) \\
&= \vec{\nabla} \left(< \vec{J}_D > + < \vec{J}_{ND} > \right) \\
&= \sum_{i,j} \rho_{i,j} \vec{\nabla} \vec{J}_{j,i} \delta_{j,i} + \sum_{i \neq j} \rho_{i,j} \vec{\nabla} \vec{J}_{j,i} \\
&= \vec{\nabla} < \vec{J}_D > + \vec{\nabla} < \vec{J}_{ND} >
\end{aligned} \tag{4.74}$$

Using the diagonal terms in Eqs.(4.48) and (4.49) we obtain $\vec{\nabla} \vec{J}_{i,i} = 0 \Rightarrow \vec{\nabla} < \vec{J}_D >= 0$. The diagonal part is conserved. From its conservation and the complete current density, we deduce the conservation of the nondiagonal term. Thus, conservation is verified for all parts of the current density involving states with the same energy.

4.6 Conductivity

The conductivity, in terms of summation over states, reads ([6])

$$\overleftrightarrow{\sigma}_d (r, r') = - \sum_s f'(\epsilon_s) \left(\tilde{\Lambda}^{-1} j(r) \right)_{sS} j(r')_{SS}; \tag{4.75}$$

here s labels the unperturbed states $\varphi_{n_s, k_{xs}}$ and S the scattering states $\psi_{n_S, k_{xS}}$. The energy is restricted to a specific value ϵ_p. The conductivity is

$$\overleftrightarrow{\sigma}_d(r,r') = -\int f'(\epsilon_p)\, \overleftrightarrow{\sigma}_d^{\epsilon_p}(r,r')d\epsilon_p, \tag{4.76}$$

where

$$\overleftrightarrow{\sigma}_d^{\epsilon_p}(r,r') = \sum_s \delta(\epsilon_p - \epsilon_{n_s,k_{x_s}})\left(\tilde{\Lambda}^{-1}j(r)\right)_{ss} j(r')_{SS}. \tag{4.77}$$

The Dirac function is rewritten in terms of k_x using the property

$$\delta\left(g(k_x)\right) = \sum_i \frac{1}{|g'(k_{x_i}|}\delta(k_x - k_{x_i}), \tag{4.78}$$

where g' is the derivative of $g(k_x)$ and k_{x_i} the root of $g(k_x) = 0$. The equation $g(k_{x_i}) = 0$ becomes

$$\frac{\hbar^2}{2m}\frac{\Omega^2}{\omega_T^2}k_{x_i}^2 - \frac{\omega_C \hbar q E_\perp}{m\omega_T^2}k_{x_i} + \left(n+\frac{1}{2}\right)\hbar\omega_T - \frac{q^2E_\perp^2}{2m\omega_T^2} - \epsilon_p = 0 \tag{4.79}$$

The roots k_{x_i} are $k_{x_{1,2}} = (-b \pm \sqrt{b^2-4ac})/2a$. They are real if $b^2 > 4ac$. If this condition is respected, the wavefunctions can propagate in different channels. For imaginary roots, the wavefunctions have negative exponentials and their amplitude decreases with propagation. If so, there is energy absorption. The waves become evanescent and negligible. The condition of propagation is given by

$$\frac{\omega_C^2 q^2 E_\perp^2}{m\omega_T^2} > 2\Omega^2\left[-\frac{q^2E_\perp^2}{2m\omega_T^2} + \left(n+\frac{1}{2}\right)\hbar\omega_T - \epsilon_p\right] \tag{4.80}$$

Propagation modes depend on confinement, magnetic field, Landau levels and electric field. For a given energy, Eq.(4.78) with $g'(k_{x_i}) = (\hbar\Omega^2/m\omega_T^2)\,k_{x_i} -$

$\omega_C \hbar q E_\perp / m\omega_T^2$, leads to

$$\overset{\leftrightarrow \epsilon_p}{\sigma_d}(r,r') = \sum_{n_s,k_{x_s}}^{\epsilon_p} \left[\frac{\delta(k_x - k_{x1})}{|g'(k_{x1})|} + \frac{\delta(k_x - k_{x2})}{|g'(k_{x2})|}\right] \left(\tilde{\Lambda}^{-1} j(r)\right)_{ss} j(r')ss \quad (4.81)$$

When the k_x components become continuous, i.e, in the limit $L \to \infty$, the sum over k_x is replaced by an integral

$$\sum_{k_x} \to \lim_{L\to\infty} \frac{L}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dk_x. \quad (4.82)$$

Then the conductivity becomes

$$\begin{aligned} \overset{\leftrightarrow \epsilon_p}{\sigma_d}(r,r') &= \sum_{n_s}^{\epsilon_p} \lim_{L\to\infty} \frac{L}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dk_x \left[\frac{\delta(k_x - k_{x1})}{|g'(k_{x1})|} + \frac{\delta(k_x - k_{x2})}{|g'(k_{x2})|}\right] \\ &\quad \times \left(\tilde{\Lambda}^{-1} j(r)\right)_{ss} j(r')ss \\ &= \frac{L}{2\pi} \sum_{n_s}^{\epsilon_p} \left[M_{k_{x_1}} + M_{k_{x_2}},\right] \end{aligned} \quad (4.83)$$

where

$$M_{k_{x_i}} = \frac{1}{|g'(k_{x_i})|} j(r')_{k_{x_i},ns;k_{x_i},ns} \left(\tilde{\Lambda}^{-1} j(r)\right)_{k_{x_i},n;k_{x_i},n_s}. \quad (4.84)$$

4.7 Conductance

The conductance is calculated by integrating the conductivity along any two perpendicular surfaces. The conductivity calculations are repeated. A specific energy is chosen for the conductance

$$G(\epsilon_p) = \int_A \int_{A'} d\vec{A} \overset{\leftrightarrow \epsilon_p}{\sigma_d}(r,r') d\vec{A}', \quad (4.85)$$

with the expression of the conductivity presented in Eq.(4.83) we obtain

$$G(\epsilon_p) = \frac{L}{2\pi} \sum_{n_s}^{\epsilon_p} \left(N_{k_{x_{i1}}} + N_{k_{x_{i2}}} \right), \tag{4.86}$$

where

$$N_{k_{x_i}} = \frac{1}{|g'(k_{x_i})|} \int j(r')_{K_{x_i},n_s;K_{x_i},n_s} dA' \int \left(\tilde{\Lambda}^{-1} j(r) \right)_{k_{x_i},n_s;k_{x_i},n_s} dA \cdot \tag{4.87}$$

4.8 Conductance in terms of transmission and reflection coefficients

4.8.1 General discussion

To clearly see every step involved in the calculations, the two values $k_{x_{i1/2}}$ are taken to be in opposite direction. That is, in the expression of Eq.(4.79) we start in supposing one positive and one negative solution. Then

$$G(\epsilon_p) = \frac{L}{2\pi} \sum_{n_s}^{\epsilon_p} \left(N_{+k_{x_{i1}}} + N_{-k_{x_{i2}}} \right) \tag{4.88}$$

where

$$N_{\pm k_{x_i}} = \frac{1}{|g'(\pm k_{x_i})|} \int dA' j(r')_{\pm k_{x_i},n_S;\pm k_{x_i},n_S} \int dA \left(\tilde{\Lambda}^{-1} j(r) \right)_{\pm k_{x_i},n_s;\pm k_{x_i},n_s} . \tag{4.89}$$

We now proceed with the evaluation of these two integrals that are related to transmission and reflection coefficients. By choosing two surfaces (A, A')

in an asymptotic region we can integrate. The choice of surface is arbitrary. It is not necessary to know the exact scattering states. It is sufficient to have their asymptotic expression in a region away from the scattering center. The scattering states are represented by a linear combination of eigenfunctions of the unpertubed Hamiltonian.

$$\overline{\psi}_{s+} = \sum_{n_{s'}}^{\epsilon_p} t^L_{ss'}(\epsilon)\overline{\varphi}_{s'+}(r) \qquad z \gg L_s \tag{4.90}$$

$$\overline{\psi}_{s+} = \overline{\varphi}_{s+}(r) + \sum_{n_{s'}}^{\epsilon_p} r^L_{ss'}(\epsilon_p)\overline{\varphi}_{s'-}(r) \qquad x \ll 0 \tag{4.91}$$

$$\overline{\psi}_{s-} = \overline{\varphi}_{s-}(r) + \sum_{n_{s'}}^{\epsilon_p} r^R_{ss'}(\epsilon_p)\overline{\varphi}_{s'+}(r) \qquad x \gg L_s \tag{4.92}$$

$$\overline{\psi}_{s-} = \sum_{n_{s'}}^{\epsilon_p} t^R_{ss'}(\epsilon_p)\overline{\varphi}_{s'-}(r) \qquad x \ll 0 \tag{4.93}$$

Using the normalisation of flux the current density is

$$\vec{J}_{\beta\alpha} = \frac{\sqrt{v_\beta v_\alpha}}{L}\frac{-iq\hbar}{2m}\overline{\psi}^*_\beta \overleftrightarrow{D} \overline{\psi}_\alpha \tag{4.94}$$

For the different regions we have

$$j_{\overline{\psi}_{s+}}(r') = \frac{-iq\hbar v_{s+}}{2mL}\sum_{n_{s'}}^{\epsilon_p}\sum_{n_{s''}}^{\epsilon_p} t^{L*}_{ss'}t^L_{ss''}\overline{\varphi}_{s'+} \overleftrightarrow{D} \overline{\varphi}_{s''+}, \qquad z \gg L_s, \tag{4.95}$$

$$\begin{aligned} j_{\overline{\psi}_{s+}}(r') &= \frac{-iq\hbar v_{s+}}{2mL}\{\overline{\varphi}^*_{s+} \overleftrightarrow{D} \overline{\varphi}_{s+} + \sum_{n_{s'}}^{\epsilon_p} r^{L*}_{ss'}\overline{\varphi}^*_{s'-} \overleftrightarrow{D} \overline{\varphi}_{s''+} \\ &+ \sum_{n_{s''}}^{\epsilon_p} r^L_{ss''}\overline{\varphi}^*_{s'+} \overleftrightarrow{D} \overline{\varphi}_{s''-} \\ &+ \sum_{n_{s'}}^{\epsilon_p}\sum_{n_{s''}}^{\epsilon_p} r^{L*}_{ss'}r^L_{ss''}\overline{\varphi}^*_{s'-} \overleftrightarrow{D} \overline{\varphi}_{s''-}\}, \qquad z \ll 0, \end{aligned} \tag{4.96}$$

$$j_{\overline{\psi}_{s-}}(r') = \frac{-iq\hbar v_{s-}}{2mL}\{\overline{\varphi}^*_{s-} \overleftrightarrow{D} \overline{\varphi}_{s-} + \sum_{n_{s'}}^{\epsilon_p} r^{R*}_{ss'}\overline{\varphi}^*_{s'+} \overleftrightarrow{D} \overline{\varphi}_{s''-}$$

$$
\begin{aligned}
& + \sum_{n_{s''}}^{\epsilon_p} r^R_{ss''}\overline{\varphi}^*_{s'-} \overleftrightarrow{D}\, \overline{\varphi}_{s''+} \\
& + \sum_{n_{s'}}^{\epsilon_p}\sum_{n_{s''}}^{\epsilon_p} r^{R*}_{ss'} r^R_{ss''}\overline{\varphi}^*_{s'+} \overleftrightarrow{D}\, \overline{\varphi}_{s''+}\} \qquad z \gg L_s, \qquad (4.97)
\end{aligned}
$$

$$
j_{\overline{\psi}_{s-}}(r') = \frac{-iq\hbar v_{s-}}{2mL} \sum_{n_{s'}}^{\epsilon_p}\sum_{n_{s''}}^{\epsilon_p} t^{R*}_{ss'} t^R_{ss''}\overline{\varphi}^*_{s'-}\overline{\varphi}_{s''-}, \qquad x \ll 0. \qquad (4.98)
$$

4.8.2 Evaluation of the first integral

Using Eqs.(4.71) and (4.72) we obtain

$$
\int j_{\overline{\psi}_{s+}}(r')dA' = \frac{qv_{s+}}{L}\sum_{n_{s'}}^{\epsilon_p} t^{L*}_{ss'} t^L_{ss'} \qquad z \gg L_s, \qquad (4.99)
$$

$$
\int j_{\overline{\psi}_{s+}}(r')dA' = \frac{qv_{s+}}{L}\{1 - \sum_{n_{s'}}^{\epsilon_p} r^{L*}_{ss'} r^L_{ss'}\}\ z \ll 0, \qquad (4.100)
$$

$$
\int j_{\overline{\psi}_{s-}}(r')dA' = -\frac{qv_{s-}}{L}\{1 - \sum_{n_{s'}}^{\epsilon_p} r^{R*}_{ss'} r^R_{ss'}\}\ z \gg L_s, \qquad (4.101)
$$

$$
\int j_{\overline{\psi}_{s-}}(r')dA' = -\frac{qv_{s-}}{L}\sum_{n_{s'}}^{\epsilon_p} t^{R*}_{ss'} t^R_{ss'} \qquad x \ll 0 \qquad (4.102)
$$

With current conservation ($1 = |r|^2 + |t|^2$) we obtain far away from each scattering region the same result

$$
\int j(r')_{\pm K_{x_i},n;\pm K_{x_i},n_s} dA' = \pm\left(\frac{qv_{x_i}}{L}\right)\sum_{n_{s''}} t^*_{ss''} t_{ss''} \qquad (4.103)
$$

4.8.3 Evaluation of the second integral

The second integral has the operator $\tilde{\Lambda}$. To easily calculate its inverse, we use the relaxation-time approximation which is valid only for elastic scattering.

It is very often an excellent approximation for weakly inelastic scattering. The approximation leads to

$$\int (\tilde{\Lambda} j(r))_{s;s} dA = \frac{1}{\tau_s} \int j_{ss} dA. \tag{4.104}$$

We deduce the value of τ_s as follows. We have

$$\begin{aligned} \beta_{\pm} &= \int j_{\pm s \pm s} dA \\ &= \sqrt{\frac{v_{\pm s}}{L}} \sqrt{\frac{v_{\pm s}}{L}} \frac{-iq\hbar}{2m} \int \overline{\varphi}_s \overset{\leftrightarrow}{D} \overline{\varphi}_s dA \\ &= \frac{v_{\pm s}}{L} \frac{-iq\hbar}{2m} \frac{\pm 2mi}{\hbar} \delta_{\pm s \pm s} \\ &= \pm \frac{q v_{\pm s}}{L}. \end{aligned} \tag{4.105}$$

In the integral on the left-hand size of Eq.(4.104), the operator $\tilde{\Lambda}$ is equal to [6]

$$\left(\tilde{\Lambda} j(r)\right)_{\pm s;\pm s} = \sum_{s'} \delta(\epsilon_p - \epsilon_{k'}) |T_{\pm s s'}|^2 \left(j_{\pm s \pm s} - j_{s's'}\right). \tag{4.106}$$

where $T_{\pm ss'} =< \varphi_{\pm s}|V|\psi_{\pm s'} >$ is the transition operator. In our model V represent the scattering potential. The Dirac function is rewritten in terms of the wavevector longitudinal components and of the two roots $k'_{i1/2}$; then the summation over k'_x is replaced by an integration, see Eq.(4.82). This leads to

$$\left(\tilde{\Lambda} j(r)\right)_{\pm s;\pm s} = \frac{L}{2\pi} \sum_{s'}^{\epsilon_p} \left[\frac{|T_{\pm s;n_{s'},+k'_{xi1}}|^2}{|g'(+k'_{xi1})|} \left(j_{ss} - j_{n_{s'},+k'_{xi1}}\right) + \frac{|T_{\pm s;n_{s'},-k'_{xi2}}|^2}{|g'(-k'_{xi2})|} \left(j_{ss} - j_{n_{s'},-k'_{xi2}}\right) \right] \tag{4.107}$$

To evaluate the integral on the right-hand side of Eq.(4.104) we use Eq.(4.105) and obtain

$$\begin{aligned}\int (\tilde{\Lambda} j(r))_{\pm s} dA &= \int (\tau j(r))_{\pm s \pm s} dA \\ &= \tau_{\pm s} \int j_{\pm s \pm s} dA \\ &= \tau_{\pm s} \left(\pm \frac{q v_{\pm s}}{L} \right) . \end{aligned} \tag{4.108}$$

With this result, Eq.(4.107), and Eq.(4.104) the relaxation-time becomes

$$\frac{1}{\tau_{\pm s}} = \frac{L}{\hbar} \sum_{n_{s'}}^{\epsilon_p} \left[\frac{|T_{\pm s; n_{s'}, +k'_{x_{i1}}}|^2}{|g'(+k'_{x_{i1}})|} \left(1 \mp \frac{\beta'_{+i1}}{\beta_{i_{\pm s}}} \right) + \frac{|T_{\pm s; n_{s'}, -k'_{x_{i2}}}|^2}{|g(-k'_{x_{i2}})|} \left(1 \pm \frac{\beta'_{-i2}}{\beta_{i_{\pm s}}} \right) \right] . \tag{4.109}$$

Using Eq.(4.103), Eq.(4.105), and Eq.(4.109), in Eq.(4.89) we get

$$N_{\pm k_{x_i}} = \frac{1}{|g'(\pm k_{x_i})|} \tau_{\pm s} \left(\pm q \frac{v_{\pm s}}{L} \right)^2 \sum_{n_{s''}} t^*_{ss''} t_{ss''} . \tag{4.110}$$

4.8.4 Calculation of T

Expression of T

The matrix element of the transition operator T between a state φ_s and a scattering state $\psi_{s'}$ is

$$T_{ss'} = < \varphi_s | V | \psi_{s'} > . \tag{4.111}$$

If we write $V = H - H_0$, we obtain

$$T_{ss'} = \epsilon_{s'} < \psi_s|\varphi_{s'} > - < \varphi_s|\,(H_0|\psi_{s'} >)\,. \tag{4.112}$$

We modify the second term on the right-hand side. The Hamiltonian will operate on the left element. In order to do so, we recall the expression

$$\begin{aligned}\int \varphi^* P_x \psi dv &= \frac{\hbar}{i}\int \frac{\partial}{\partial x}(\varphi^*\psi)dv - \frac{\hbar}{i}\int (\frac{\partial}{\partial x}\varphi^*)\psi)dv \\ &= \int P_x(\varphi^*\psi)dv + \int (P_x^*\varphi^*)\psi dv. \end{aligned}\tag{4.113}$$

We obtain

$$\begin{aligned}\int \varphi^* P_x(P_x\psi)dv &= \int P_x(\varphi^* P_x\psi)dv + \int (P_x^*\varphi^*)P_x\psi dv \\ &= \int P_x(\varphi^* P_x\psi)dv + \int P_x[(P_x^*\varphi^*)\psi]dv + \int (P^2\varphi^*)\psi dv \\ &= \int (P^2\varphi^*)\psi dv - \int \frac{\hbar^2\partial}{\partial x}\left[\varphi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\varphi^*\right]dv. \end{aligned}\tag{4.114}$$

and

$$\int \varphi^* P_x y\psi dv = \int P_x(\varphi^* y\psi)dv + \int (P_x^*\varphi^*)y\psi dv. \tag{4.115}$$

If we combine these results with the Hamiltonian given by Eq.(4.25), we obtain

$$\begin{aligned}< \varphi_s|(H_0|\psi_{s'} >) &= \frac{1}{2m}(< \varphi_s|P_x^2 + P_y^2)|\psi_{s'} > \\ &\quad -\frac{(\hbar)^2}{2m}\int \nabla[\varphi^*\nabla\psi - \psi\nabla\varphi^*]dv\end{aligned}$$

$$-\frac{qB}{m}\int (P_x^*\varphi^*)y\psi dv - \frac{qB}{m}\int P_x(\varphi^* y\psi)dv$$

$$= (<\varphi_s|H_0|)\psi_{s'}>$$

$$-\frac{\hbar^2}{2m}\int \nabla(\varphi^* \overleftrightarrow{\nabla} \psi)dv$$

$$-\frac{qB}{m}\int P_x(\varphi^* y\psi)dv. \tag{4.116}$$

If we combine this result with Green's theorem, we obtain

$$T_{ss'} = (\epsilon_s - \epsilon_{s'}) < \varphi_s|\psi_{s'} >) + \frac{\hbar^2}{2m}\int_A d\vec{A}(\varphi_s^* \overleftrightarrow{\nabla} \psi_{s'}) + \frac{qB}{m}\int P_x(\varphi_s^* y\psi_{s'})dv \tag{4.117}$$

The first term is zero if the energies are the same. If so, the remaining terms can be simplified. The result

$$T_{ss'} = \frac{\hbar^2}{2m}\int_s (\varphi_s^* \frac{\overleftrightarrow{d}}{dx} \psi_{s'})dy + \frac{2qB(-i\hbar)}{2m}\int \varphi_s^* y\psi_{s'}dy \tag{4.118}$$

is equal to

$$T_{ss'} = \frac{\hbar^2}{2m}\int_s \varphi_s^* \left(\frac{\overleftrightarrow{d}}{dx} - \frac{2qiBy}{\hbar}\right)\psi_{s'}dy; \tag{4.119}$$

thus

$$T_{ss'} = \frac{\hbar^2}{2m}\int_A d\vec{A}\cdot\hat{x}\varphi_s^*(\overleftrightarrow{D})\psi_{s'}. \tag{4.120}$$

Finally, if we write the results in terms of the normalized flux, we obtain

$$T_{ss'} = \frac{\sqrt{v_s v_{s'}}}{L}\frac{\hbar^2}{2m}\int_A d\vec{A}\cdot\hat{x}\bar{\varphi}_s^*(\overleftrightarrow{D})\bar{\psi}_{s'} \tag{4.121}$$

T_{ss} in terms of transmission and reflection coefficients

To evaluate the term $T_{s\pm s'+}$, we use equations (4.90) and (4.92) together with equations (4.71) and (4.72). For $z \gg L_s$ we obtain

$$T_{s+s'+} = \frac{i\hbar}{L}\sqrt{v_{s+}v_{s'+}}t^L_{s's}, \tag{4.122}$$

$$T_{s-s'+} = 0. \tag{4.123}$$

For $z \ll 0$ the results are

$$T_{s+s'+} = \frac{i\hbar}{L}\sqrt{v_{s+}v_{s'+}}\delta_{ss'}, \tag{4.124}$$

$$T_{s-s'+} = -\frac{i\hbar}{L}\sqrt{v_{s-}v_{s'+}}r^L_{s's}. \tag{4.125}$$

To evaluate the term $T_{s\pm s'-}$ we use equations (4.91) and Eq.(4.93) together with equations (4.71) and (4.72). For $z \gg L_s$ we obtain

$$T_{s+s'-} = \frac{i\hbar}{L}\sqrt{v_{s+}v_{s'-}}r^R_{s's}, \tag{4.126}$$

$$T_{s-s'-} = -\frac{i\hbar}{L}\sqrt{v_{s-}v_{s'-}}\delta_{ss'}. \tag{4.127}$$

and for $z \ll 0$

$$T_{s-s'-} = -\frac{i\hbar}{L}\sqrt{v_{s-}v_{s'-}}t^R_{s's}, \tag{4.128}$$

$$T_{s+s'-} = 0. \tag{4.129}$$

4.8.5 Relaxation time in terms of transmission and reflection coefficients

With the form of T and the relaxation time given by Eq. (4.109), the results for the various asymptotic regions are as follows.

For $z \gg L_s$ we have

$$\frac{1}{\tau_{+s}} = \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{\hbar^2}{L^2}v_{s+}v_{s'+}\frac{|t^L_{ss'}|^2}{|g'(+k'_{x_i})|}\left(1-\frac{\beta'_{+i}}{\beta_{i_s}}\right)+\frac{\hbar^2}{L^2}v_{s+}v_{s'-}\frac{|r^R_{ss'}|^2}{|g'(-k'_{x_i})|}\left(1+\frac{\beta'_{-i}}{\beta_{i_s}}\right)\right] \tag{4.130}$$

and

$$\begin{aligned}\frac{1}{\tau_{-s}} &= \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{|0|^2}{|g'(+k'_{x_i})|}\left(1+\frac{\beta'_{+i}}{\beta_{i_s}}\right)+\frac{\hbar^2}{L^2}v_{s-}v_{s'-}\frac{|\delta_{ss'}|^2}{|g'(-k'_{x_i})|}\left(1-\frac{\beta'_{-i}}{\beta_{i_s}}\right)\right]\\ &= 0\end{aligned} \tag{4.131}$$

For $z \ll 0$ the results are

$$\begin{aligned}\frac{1}{\tau_{+s}} &= \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{\hbar^2}{L^2}v_{s+}v_{s'+}\frac{|\delta_{ss'}|^2}{|g'(+k'_{x_i})|}\left(1-\frac{\beta'_{+i}}{\beta_{i_s}}\right)+\frac{|0|^2}{|g'(-k'_{x_i})|}\left(1+\frac{\beta'_{-i}}{\beta_{i_s}}\right)\right]\\ &= 0\end{aligned} \tag{4.132}$$

and

$$\frac{1}{\tau_{-s}} = \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{\hbar^2}{L^2}v_{s-}v_{s'}\frac{|r^L_{ss'}|^2}{|g'(+k'_{x_i})|}\left(1-\frac{\beta'_{+i}}{\beta_{i_s}}\right)+\frac{\hbar^2}{L^2}v_{s-}v_{s'-}\frac{|t^R_{ss'}|^2}{|g'(-k'_{x_i})|}\left(1+\frac{\beta'_{-i}}{\beta_{i_s}}\right)\right] \tag{4.133}$$

In summary, the result is

$$\tau_{\pm s}^{-1} = \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{|i\hbar\sqrt{v_{s\pm}v_{s'\pm}}t_{ss'}^{L(R)}|^2}{L^2|g'(+k'_{xi})|}\left(1-\frac{\beta'_i}{\beta_{\pm i_s}}\right)+\frac{|i\hbar\sqrt{v_{s\pm}v_{s'\mp}}r_{ss'}^{R(L)}|^2}{L^2|g'(-k'_{xi})|}\left(1+\frac{\beta'_i}{\beta_{\pm i_s}}\right)\right] \tag{4.134}$$

4.8.6 Conductance expression

Using the expression of Eqs.(4.110), (4.134) and (4.105) we obtain

$$N_{\pm k_{x_i}} = \frac{1}{L\hbar|g'(\pm k_{x_i})|}\frac{(\pm q v_{\pm x_i})^2\sum_{n_{s''}} t_{ss''}^* t_{ss''}}{\sum_{ns'}^{\epsilon_p}\left[\frac{v_{\pm xi}v_{\pm x'_{i1}}|t_{ss'}^{L(R)}|^2}{|g'(+k'_{xi1})|}\left(1-\frac{v_{+x'i1}}{v_{\pm xi}}\right)+\frac{v_{\pm xi}v_{\mp x'i2}|r_{ss'}^{R(L)}|^2}{|g'(-k'_{xi2})|}\left(1+\frac{v_{-x'i2}}{v_{\pm xi}}\right)\right]} \tag{4.135}$$

Noticing that

$$\begin{aligned} g'(\vec{k}) &= \vec{\nabla}_{\vec{k}}\epsilon(\vec{k}) \\ &= \hbar\vec{v} \end{aligned} \tag{4.136}$$

see Eqs. (4.77) and (4.78). We have

$$|g'(\pm k_{xi})| = \hbar v_{\pm xi} \tag{4.137}$$

this gives

$$N_{\pm k_{x_i}} = \frac{q^2\sum_{n_{s''}} t_{ss''}^* t_{ss''}}{L\hbar\sum_{ns'}^{\epsilon_p}\left[|t_{ss'}^{L(R)}|^2\left(1-\frac{v_{+x'i1}}{v_{\pm xi}}\right)+|r_{ss'}^{R(L)}|^2\left(1+\frac{v_{-x'i2}}{v_{\pm xi}}\right)\right]}. \tag{4.138}$$

Using current conservation $\sum_{n_{s'}}\left(|t^{L(R)}_{ss'}|^2+|r^{R(L)}_{ss'}|^2\right)=1$, we obtain

$$N_{\pm k_{x_i}}=\frac{q^2\sum_{n_{s''}}t^*_{ss''}t_{ss''}}{L\hbar\left[1+\sum^{\epsilon_p}_{ns'}\left(|r^{R(L)}_{ss'}|^2\frac{v_{-x'i2}}{v_{\pm xi}}-|t^{L(R)}_{ss'}|^2\frac{v_{+x'i1}}{v_{\pm xi}}\right)\right]}. \tag{4.139}$$

From this expression the conductance is found by Eq.(4.88)

$$\begin{aligned}G(\epsilon_p) &= \frac{q^2}{h}\sum_{n_s}^{\epsilon_p}\left[\frac{\sum_{n_{s''}}t^{L*}_{ss''}t^{L}_{ss''}}{1+\sum^{\epsilon_p}_{ns'}\left(|r^R_{ss'}|^2\frac{v_{-x'i2}}{v_{+xi1}}-|t^L_{ss'}|^2\frac{v_{+x'i1}}{v_{+xi1}}\right)}\right.\\ &+\left.\frac{\sum_{n_{s''}}t^{R*}_{ss''}t^{R}_{ss''}}{1+\sum^{\epsilon_p}_{ns'}\left(|r^L_{ss'}|^2\frac{v_{-x'i2}}{v_{-xi2}}-|t^R_{ss'}|^2\frac{v_{+x'i1}}{v_{-xi2}}\right)}\right]\end{aligned} \tag{4.140}$$

4.9 Limit for $E_\perp = 0$

If we neglect $E_\perp$ in Eq.(4.59), we obtain

$$v_{\pm x}=\frac{\hbar|k_x|}{m}\left(1-\frac{\omega_C^2}{\omega_T^2}\right)=\frac{\hbar|k_x|\Omega^2}{m\omega_T^2}. \tag{4.141}$$

The wavefunctions are the same with ξ given by

$$\xi=\frac{\hbar k_x}{qB}. \tag{4.142}$$

The energy is

$$\epsilon_{k_x,n}=\left(n+\frac{1}{2}\right)\hbar\omega_T+\frac{\hbar^2k_x^2\Omega^2}{2m\omega_T^2} \tag{4.143}$$

The equation $g(k_x)=0$ becomes

$$g(k_x)=\epsilon_p-\left(n+\frac{1}{2}\right)\hbar\omega_T-\frac{\hbar^2k_x^2}{2m}\frac{\Omega^2}{\omega_T^2}=0, \tag{4.144}$$

the two roots + and - are

$$k_{x1,2} = \pm\frac{2mw_T^2}{\hbar^2\Omega^2}\sqrt{\epsilon_p - (n+\frac{1}{2})\hbar\omega_T} \tag{4.145}$$

and

$$v_{\pm xi} = v_{xi}. \tag{4.146}$$

From these expressions the conductance is found to be

$$G(\epsilon_p) = \frac{q^2}{h}\sum_{n_s}^{\epsilon_p}\left[\frac{(t^L t^{L*})_{ss}}{1+\frac{1}{v_{xi}}\sum_{ns'}^{\epsilon_p} v_{x'i}\left(|r_{ss'}^R|^2 - |t_{ss'}^L|^2\right)} + \frac{(t^R t^{R*})_{ss}}{1+\frac{1}{v_{xi}}\sum_{ns'}^{\epsilon_p} v_{x'i}\left(|r_{ss'}^L|^2 - |t_{ss'}^R|^2\right)}\right] \tag{4.147}$$

This result is formally identical to the one given in [6] for $B = 0$. The differences are in the wavefunctions used, the " renormalized " energy levels, and the corresponding expressions for the transmission and reflection coefficients.

4.10 Limit for $B = 0$

If the magnetic field is absent the Hall field $E_\perp$ vanishes. Then

$$v_{\pm x} = \frac{\hbar|k_x|}{m} \tag{4.148}$$

and the wavefunctions have the form

$$\chi(y) = \chi\left(\left(\frac{m\Omega}{\hbar}\right)^{\frac{1}{2}} y\right) \tag{4.149}$$

The energy is given by

$$\epsilon_{k_x,n} = (n + \frac{1}{2})\hbar\Omega + \frac{\hbar^2 k_x^2}{2m} \tag{4.150}$$

The equation $g(k_x) = 0$ becomes

$$g(k_x) = \epsilon_p - \left(n + \frac{1}{2}\right)\hbar\Omega + \frac{\hbar^2 k_x^2}{2m} = 0, \tag{4.151}$$

the two roots + and - are

$$k_{xi} = \pm\frac{2m}{\hbar^2}\sqrt{\epsilon_p - (n + \frac{1}{2})\hbar\Omega}, \tag{4.152}$$

and

$$v_{\pm xi} = v_{xi}. \tag{4.153}$$

The conductance is

$$G(\epsilon_p) = \left[\frac{q^2}{h}\sum_{n_s}^{\epsilon_p} \frac{(t^L t^{L*})_{ss}}{1 + \frac{1}{v_{xi}}\sum_{ns'}^{\epsilon_p} v_{x'i}\left(|r_{ss'}^R|^2 - |t_{ss'}^L|^2\right)} + \frac{(t^R t^{R*})_{ss}}{1 + \frac{1}{v_{xi}}\sum_{ns'}^{\epsilon_p} v_{x'i}\left(|r_{ss'}^L|^2 - |t_{ss'}^R|^2\right)}\right] \tag{4.154}$$

The result is the same of Eq. (3.62) in chapter 3 and coincides with that of Ref.[6] with $\overleftrightarrow{D}\equiv\overleftrightarrow{\nabla}$.

4.11 Conductance generalization

We have found the conductance for two roots in opposite direction. Here we are going to obtain the conductance for the situation where the root can have any direction. We repeat the same procedure. The conductance is evaluated at a specific energy value.

$$G(\epsilon_p) = \int_A \int_{A'} d\vec{A}\, \overleftrightarrow{\sigma}_d^{\epsilon_p} (r, r') d\vec{A'} \tag{4.155}$$

Using Eq.(4.83) the conductivity, we obtain

$$G(\epsilon_p) = \frac{L}{2\pi} \sum_{n_s}^{\epsilon_p} \left[N_{k_{x_{i_1}}} + N_{k_{x_{i_2}}} \right] \tag{4.156}$$

with

$$N_{k_{x_i}} = \frac{1}{|g'(k_{x_i})|} \int \left(\tilde{\Lambda}^{-1} j(r) \right)_{k_{x_i}, n; k_{x_i}, n_s} dA \cdot \int j(r')_{K_{x_i}, n; K_{x_i}, n_s} dA' \tag{4.157}$$

The calculation of the second integral has been done, before, see Eqs (4.89)-(4.103). The result depends on the direction of the wave vector (positive for $+\hat{x}$ negative for $-\hat{x}$).

$$\int j(r')_{K_{x_i}, n; K_{x_i}, n_s} dA' = \pm \frac{q v_{x_i}}{L} \sum_{n_{s''}} t^*_{ss''} t_{ss''} \tag{4.158}$$

Concerning the first integral we have

$$\left(\tilde{\Lambda} j(r) \right)_{s;s} = \sum_{s'} \delta(\epsilon_p - \epsilon_{k'}) |T_{ss'}|^2 \left(j_{ss} - j_{s's'} \right). \tag{4.159}$$

The previous steps in the evaluation of the conductance, see Eqs (4.106)-(4.107), are repeated. The Dirac function is rewritten in terms of the longitudinal vector and the sum over k_x is replaced by an integral. The result is

$$\left(\tilde{\Lambda}j(r)\right)_{s;s} = \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{|T_{s;n_{s'},k'_{x_{i_1}}}|^2}{|g'(k'_{x_{i_1}})|}\left(j_{ss} - j_{n_{s'},k'_{x_{i_1}}}\right) + \frac{|T_{s;n_{s'},k'_{x_{i_2}}}|^2}{|g'(k'_{x_{i_2}})|}\left(j_{ss} - j_{n_{s'},k'_{x_{i_2}}}\right)\right] \tag{4.160}$$

This result is integrated and compared with the expression for the relaxation-time $\int(\tilde{\Lambda}j(r))_{s;s}ds = \int j_{ss}ds/\tau_s$. The expression for $1/\tau$ is

$$\frac{1}{\tau_s} = \frac{L}{\hbar}\sum_{n_{s'}}^{\epsilon_p}\left[\frac{|T_{s;n_{s'},k'_{x_{i_1}}}|^2}{|g'(k'_{x_{i_1}})|}\left(1 - \frac{\beta'_{i1}}{\beta_i}\right) + \frac{|T_{s;n_{s'},k'_{x_{i_2}}}|^2}{|g'(k'_{x_{i_2}})|}\left(1 - \frac{\beta'_{i2}}{\beta_i}\right)\right] \tag{4.161}$$

with $\beta_j = qv_{+xj}/L$ for $k_{xj} > 0$, $\beta_j = -qv_{-xj}/L$ for $k_{xj} < 0$, and

$$|g'(k'_{xi})| = v'_{xi} \tag{4.162}$$

see Eq.(4.137)

Then

$$N_{k_{x_i}} = \frac{1}{|g'(k_{x_i})|}\int (\tau j(r))_{k_{xi},n_s;k_{xi},n_s}dA \cdot \int j(r')_{K_{xi},n_s;K_{xi},n_s}dA' \tag{4.163}$$

Combining Eq.(4.158) and Eq.(4.161) we obtain

$$N_{k_{x_i}} = \frac{\hbar}{v_{x_i}}\frac{\beta_i^2\sum_{n_{s''}}t^*_{ss''}t_{ss''}}{\sum_{n_{s'}}\left[\frac{|T_{s;n_{s'},k'_{xi1}}|^2}{v'_{xi1}}\left(1 - \frac{\beta'_{i1}}{\beta_i}\right) + \frac{|T_{s;n_{s'},k'_{xi2}}|^2}{v'_{xi2}}\left(1 - \frac{\beta'_{i2}}{\beta_i}\right)\right]} \tag{4.164}$$

The difference with the expression (4.135) is that the transition operator T is not written explicitly and k_{xi} is not assumed positive or negative.

If we look at the case when the two roots have the same sign. For positive (negative) roots we have for $z \gg L_s$ ($z \ll 0$) the results

$$|T_{s\pm s'\pm}|^2 = \frac{\hbar^2}{L^2}v_{s\pm}v_{s'\pm}|t^{L(R)}_{s\pm s'\pm}|^2 \tag{4.165}$$

which gives

$$N_{k_{x_i}} = \frac{q^2}{L\hbar} \frac{\sum_{n_{s''}} t^*_{ss''} t_{ss''}}{\sum_{n_{s'}} \left[|t^{L(R)}_{s\pm s'1\pm}|^2 \left(1 - \frac{v'_{\pm i1}}{v_{\pm i}}\right) + |t^{L(R)}_{s\pm s'2\pm}|^2 \left(1 - \frac{v'_{\pm i2}}{v_{\pm i}}\right) \right]} \tag{4.166}$$

and

$$\begin{aligned} G(\epsilon) &= \frac{q^2}{h} \sum_{n_s}^{\epsilon_p} [\frac{\sum_{n_{s''}} t^*_{ss''} t_{ss''}}{\sum_{n_{s'}} \left[|t^{L(R)}_{s1\pm s'1\pm}|^2 \left(1 - \frac{v'_{\pm i1}}{v_{\pm i1}}\right) + |t^{L(R)}_{s1\pm s'2\pm}|^2 \left(1 - \frac{v'_{\pm i2}}{v_{\pm i1}}\right) \right]} \\ &+ \frac{\sum_{n_{s''}} t^*_{ss''} t_{ss''}}{\sum_{n_{s'}} \left[|t^{L(R)}_{s2\pm s'1\pm}|^2 \left(1 - \frac{v'_{\pm i1}}{v_{\pm i2}}\right) + |t^{L(R)}_{s2\pm s'2\pm}|^2 \left(1 - \frac{v'_{\pm i2}}{v_{\pm i2}}\right) \right]}] \end{aligned} \tag{4.167}$$

for $z \ll 0$ ($z \gg L_s$) the relaxation time is zero and there is no resistance.

4.12 Discussion

From our calculations we can deduce the result of Ref.[6]. In this reference it is explained how this formula, which coincides with the Eq.(4.154), can be used to deduce Buttiker's equation Eq.(2.13). The difference is that in Buttiker model the injected current in each channel is constant (independent of the channel indices). In our situation the current can be different for each channel. It is not restricted to a constant current. In this respect, the result of Ref.[6] and ours are more general.

The result of Eq.(4.139) is very general. It is not limited to two identical terminals. We can interchange the indices R and T without changing the expression. This means that the conductance does not depend on the direction of the current. We notice that the sum depends on channel indices in the numerator and denominator. This is interesting since, except from Ref.[6], this aspect is not present in the literature. The consequence is that it is possible to take into account the current depending of the channel [6]. It is still valid when a crossed electric and magnetic field are present.

This major result was not expected in Ref.[6] since in a magnetic field the eigenfunctions along two opposite directions would be separated. It was thought that the expression will change dramatically. This first picture made it difficult to imagine a similar expression in the presence of a magnetic field. To resolvele this, we chose to incorporate directly in the Hamiltonian the (perpendicular) Hall field, due to $\vec{B}$, and the magnetic field itself. In this way we had coherent eigenfunction. The exact relation between the electric and the magnetic fields is not specified since we suppose implicitly that we are in equilibrium situation. This choice of eigenfunction gives useful properties which allow us to calculate simply the conductance.

This expression for conductance is not the first one in the literature when there is a magnetic field see [4, 21, 23, 22, 28]. The most common general formula is obtained for a multiterminal [4] configuration. It is equal to

$$G_{mn}(\epsilon) = \frac{q^2}{h} \sum_{ac}^{\epsilon} |t_{mn,ac}|^2, \tag{4.168}$$

where $t_{mn,ac}$ is the transmission coefficient between channel a in terminal

m and channel c in terminal n. This conductance is found by using the Green's function of the total Hamiltonian. It is then expressed in terms of reflection and transmission coefficients. In chapter 2 this formula was valid for a measurement between reservoirs, see Eq. (2.5).

When there is no magnetic field, the conductance expression, given by Eq. (4.154), is similar to the one in Ref.[6] (see Eq. (3.62)). It is shown that from this expression we can obtain Eq. (2.13). This last equation applies for a four-probe measurement (between the barrier [19]). Since our result is a generalization, we can deduce that our result is valid for a four-probe measurement too. Thus we have found a general expression for conductance between barrier valid for a four-probe measurement in a steady state when there are a crossed electric and magnetic fields.

Chapter 5

Conclusion

Our goal was to generalize the conductance calculation when a magnetic field is present. We started our calculation from the diagonal von Neumann equation. A valid solution was found.

From this expression, a current density was obtained. We calculated the conductivity. The conductance was evaluated in an asymptotic region, away from the scattering region, using the reflection and transmission coefficients. The conductance expression in the presence of crossed electric and magnetic fields was found to be similar to the one in an electric field only. Physically, it is possible because the conductance expression uses very general transmission and reflection coefficients. The difference is in the summation, it is over a more convenient set of eigenfunctions.

From our calculation, we have demonstrated that we could find, from our expression, Verboven's conductivity formula and the Buttiker's conductance formula. It makes it possible to see the origin of their differences.

We have used a method very similar to the one used in the most recent literature. The most useful is the scattering formulation. Usually the density operator is approximated to first order only. In our situation it is exact. In the recent literature, the conductance expressions are similar to the one between reservoirs. They generalize the first formula. Our conductance result seems to be a new conductance expression between the barrier. Further studies are needed to explain the causes of their differences.

The role of the nondiagonal part was not considered in our study. It would be interesting to see if we could obtain the quantum Hall effect using this term. At the present time, there are still answers needed to give an exact and complete picture of mesoscopic physics.

Bibliography

[1] P. W. Anderson, D. J. Thouless, E. Abrahams et D. S. Fisher, Phys. Rev. Lett B **22**, 3519, (1980)

[2] P. W. Anderson, Phys. Rev. B **23**, 4828 (1981)

[3] M. Ya. Azbel, J. Phys. C: Solid State Phys. **14**, L225 (1981)

[4] H. U. Baranger and A. D. Stone, Phys. Rev. B **40**, 8169 (1989)

[5] C.W.J. Beenakker and al, J. Phys. C: Solid State Phys. **44**, 1 (1991)

[6] F. Benamira, Ph.D. dissertation, University of Montreal (1996)

[7] M. Buttiker, Y. Imry, R. Landauer and S. Pinhas, Phys. Rev. B **31**, 6207, (1985)

[8] M. Buttiker and Y. Imry, J. Phys. C: Solid State Phys. **18** L467 (1985)

[9] M. Buttiker, Physical Review B **33**, 3020 (1986)

[10] M. Buttiker, Phys. Rev. Lett. **57**, 1761 (1986)

[11] M. Buttiker, Phys. Rev. B **38**, 9375 (1988)

[12] H. B. J. Casimir, Rev. Mod. Phys. **17**, 343 (1945)

[13] P. Drude, Annalen der Physik **1**,566 and **3**, 369 (1900)

[14] E. N. Economou and C. M. Soukoulis, Phys. Rev. Lett. **46**, 618 (1981)

[15] D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981)

[16] Y. Gefen, Y. Imry, M. Ya Azbel Phys. Rev. Lett B **52**, 129, (1984)

[17] R. J. Haug and al, Phys. Rev. Lett. **61**, 2799 (1988)

[18] Y. Imry, in "Directions in Condensed Matter Physics",Vol.1 , World Scientific, Singapore, (1986)

[19] Y. Imry, "Introduction to Mesoscopic Physics", Oxford University Press, (1997)

[20] J. K. Jain, Phys. Rev. B **37**, 4276 (1988)

[21] M. Janben, Solid State Commun. **79**, 1073 (1991)

[22] J. Kucera, P. Streda and al, Phys. Rev. Lett. **17**, 1973 (1987)

[23] J. Kucera and P. Streda, J. Phys. C: Solid State **21**, 4357 (1987)

[24] J. Kucera, Czech. J. Phys. **41**, 749 (1991)

[25] R. Landauer, IBM J. Res. Dev. **1**, 223 (1957)

[26] D. C. Langreth, E. Abrahams, Phys. Rev. B **24**, 2978 (1981)

[27] A. H. Macdonald, T. M. Rice, and W. F. Brinkman, Phys. Rev. B **28**, 3648 (1983)

[28] J. U. Nockel and al, Phys. Rev. B **48**, 17569 (1993)

[29] K. Shepard, Phys. Rev. B **43**, 11623 (1991)

[30] F. Sols, Phys. Rev. Lett. **67**, 2874 (1991)

[31] A. D. Stone, Phys. Rev. Lett. **54**, 2692 (1985)

[32] A. D. Stone and A. Szafer, IBM J. Res. Dev. **32**, 384 (1988)

[33] B. J. Van Wees, H. Van Houten, C.W.J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. Van der Marel, and C. T. Foxon, Phys. Rev. Lett. **60**, 848 (1988).

[34] P. Vasilopoulos, C. M. Van Vliet, J. Math. Phys. **25**, 1391, (1984)

[35] E. Verboven, Physica **26**, 1091 (1960)

[36] D. A. Wharam, T. J. Thornton, R. Newbury, M. Pepper, H. Ahmed, J.E.F. Frost, D. G. Hasko, D. C. Peacock, D. A. Ritchie, and G.A.C. Jones, J. Phys. C: Solid State Phys. **21**, L209 (1988)