Frames and Reproducing Kernels in a Hilbert space

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Abstract

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Let H be a Hilbert space. A set of vectors $\eta_x^i \in H$, $i = 1,2,\ldots,n$, $x \in X$, where X is a locally compact space with Borel measure ν on it, constitute a rank-n continuous frame, $F(\eta_x^i, A, n)$ if for each $x \in X$ the set $\{\eta_x^1, \eta_x^2, \ldots, \eta_x^n\}$ is linearly independent and there exists a positive operator $A \in GL(H)$ such that

$$\sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| d\nu(x) = A \qquad (*)$$

Further the frame becomes discrete if (*) is replaced by

$$\sum_{k \in K} |\eta_k\rangle\langle\eta_k| = A$$

We first study discrete frames and then move to the continuous case, where we develop a connection between frames and reproducing kernels and using this connection we categorize the frames into various kinds. Finally we give a recipe for the general construction of frames on an abstract Hilbert space H using reproducing kernel Hilbert spaces H_K on $\widetilde{H} = L^2(X, \nu, C^n)$.

To the Memory of

My father

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1.1 Introduction

Let $\{\phi_n\}_{n=1}^{r}$ be a basis for a Hilbert space H (we know that any separable Hilbert space possesses an orthonormal basis) then any $\phi \in H$ can be written in a unique way as

$$\phi = \sum_{n=1}^{r} a_n \phi_n \qquad (*)$$

Indeed, $a_n = \langle \phi | \phi_n \rangle$ if $\{\phi_n\}_{n=1}^{r}$ is an orthonormal basis. The uniqueness of (*) mostly cause problems in real applications, i.e. it is hard to impose any extra conditions needed to make the basis suitable to the problem, which we are working with. By relaxing the orthogonality condition (usually by choosing an over complete collection) we can make the problem easier. In this case also we can express any $\phi \in H$ by an expression similar to (*), but the expression will not be unique. This over complete collection could allow us to put extra conditions needed to make the decomposition suitable to our problem. Such an overcomplete set of vectors is known as frames, provided an additional condition (see (1.1.1)) is satisfied.

Frames were introduced by Duffin and Schaeffer (1952) in the context of non-harmonic Fourier analysis. Frames can be considered as an alternative to orthonormal basis. We present in this thesis a survey of the theory of frames.

In the present section, we will define frames and give some examples. In section (1.2), we will define frame operators and also give some useful theorems. Section (1.3) will differentiate frames from bases. In chapter 2 we will define reproducing kernels and also give some useful references. Reproducing kernels have close connection with frames, which will be described in chapter 3.

In section (3.1), we will define the concept of continuous frames. In section (3.2), we will see various equivalencies of frames in terms of reproducing kernels. Finally in chapter 4, we will construct frames in an abstract Hilbert space in a more general sense.

We assume everywhere that H is a separable Hilbert space with norm $\|\bullet\|$ and inner product $\langle \bullet | \bullet \rangle$. End of a proof will be marked by ■.

Definition(1.1.1):

A collection of vectors $\{\phi_n\}_{n \in N}$ in a Hilbert space H is called a discrete frame if there exist numbers $A, B > 0$ such that for all $\phi \in H$.

$$A\|\phi\|^2 \leq \sum_{n\in\aleph} |\langle\phi_n|\phi\rangle|^2 \leq B\|\phi\|^2 \tag{1.1.1}$$

The numbers A, B are called frame bounds. The frame is called tight if $A = B$. In a tight frame we have , for all $\phi \in H$,

$$\sum_{n\in\aleph} |\langle\phi_n|\phi\rangle|^2 = A\|\phi\|^2 \tag{1.1.2}$$

The frame is called exact if it fails to be a frame whenever any single element is deleted from the sequence, $\{\phi_n\}_{n\in\aleph}$.

Note:

(1) $\sum_{n\in\aleph} |\langle\phi_n|\phi\rangle|^2$ is a series of positive real numbers which converges absolutely and unconditionally.

(2) Every rearrangement of the sum also converges, and converges to the same value. Therefore, every rearrangement of a frame is also a frame.

(3) If $\phi \in H$ and $\langle\phi_n|\phi\rangle = 0 \ \forall n$, then

$$0 \leq A\|\phi\|^2 \leq \sum_{n\in\aleph} |\langle\phi_n|\phi\rangle|^2 = 0$$

$$\Rightarrow A\|\phi\|^2 = 0 \Rightarrow \|\phi\|^2 = 0 \Rightarrow \phi = 0 \text{ because } A > 0$$

So frames are complete.

Example(1.1.2):

Let $H = C^2$ and $\phi_1 = (0, 1), \phi_2 = (1, \frac{-1}{\sqrt{2}}), \phi_3 = (-1, \frac{-1}{\sqrt{2}})$ then for all $z = (z_1, z_2) \in H$,

$$\sum_{n=1}^{3} |\langle\phi_n|z\rangle|^2 = |\langle\phi_1|z\rangle|^2 + |\langle\phi_2|z\rangle|^2 + |\langle\phi_3|z\rangle|^2$$

$$= |z_2|^2 + \left|z_1 - \frac{1}{\sqrt{2}}z_2\right|^2 + \left|-z_1 - \frac{1}{\sqrt{2}}z_2\right|^2$$

$$= 2\left(|z_1|^2 + |z_2|^2\right) = 2\|z\|^2$$

Thus $\{\phi_1, \phi_2, \phi_3\}$ is a tight frame with frame bound $A = 2$.

Here notice that the set $\{\phi_1, \phi_2, \phi_3\}$ is not a basis because the vectors are not linearly independent. But these three vectors in C^2 constitute a tight frame.■

Theorem (1.1.3):

Given an orthonormal sequence $\{\phi_n\}$ in an infinite dimensional Hilbert space H, the following statements are equivalent

(i) $\{\phi_n\}$ is complete

(ii) $\sum_{n\in\aleph} |\langle\phi_n|\phi\rangle|^2 = \|\phi\|^2 \quad \forall\phi \in H$

(iii) $\phi = \sum_{n\in\aleph} \langle\phi_n|\phi\rangle\phi_n \quad \forall\phi \in H.$

Proof of this theorem can be found in elementary text books. The statement (ii) is referred to as the Plancherel theorem for orthonormal bases.

Note:

Plancherel theorem tels us that every orthonormal basis is a tight exact frame with $A = B = 1$. Now we will see the converse of this statement.

Theorem (1.1.4):

If $\{\phi_n\}_{n\in\aleph}$ is a tight frame with frame bound $A = 1$, and if $\|\phi_n\| = 1$ for all $n \in \aleph$ then $\{\phi_n\}_{n\in\aleph}$ constitute an orthonormal basis.

Proof:

Let $\{\phi_n\}_{n\in\aleph}$ be a tight frame in H. Then for any $\phi_m \in H$,

$$\|\phi_m\|^2 = \sum_{n\in\aleph} |\langle\phi_n|\phi_m\rangle|^2 \text{ because } A = 1.$$

$$= \|\phi_m\|^4 + \sum_{n\in\aleph, n\neq m} |\langle\phi_n|\phi_m\rangle|^2$$

But we have $\|\phi_m\| = 1$. Therefore,

$$\sum_{n\in\aleph, n\neq m} |\langle\phi_n|\phi_m\rangle|^2 = 0$$

Thus $\langle\phi_n|\phi_m\rangle = 0 \quad \forall n \neq m$ because $\langle\phi_n|\phi_m\rangle \geq 0$

Therefore, $\{\phi_n\}_{n\in\aleph}$ is orthogonal. Further, $\langle f|\phi_n\rangle = 0$ for all $n \in \aleph$ implies $f = 0$. Thus $\{\phi_n\}_{n\in\aleph}$ span all of H. Hence $\{\phi_n\}_{n\in\aleph}$ constitute an orthonormal basis.■

The following very simple examples explain the difference between tightness, exactness and bases.

Example(1.1.5):

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H.

(a) $\{\phi_1,\phi_1,\phi_2,\phi_2,\phi_3,\phi_3,.........\}$ is a tight inexact frame with bounds $A = B = 2$ but it is not an orthonormal basis.

(b) $\left\{\phi_1,\frac{\phi_2}{2},\frac{\phi_3}{3},.......\right\}$ is a complete orthogonal sequence, but not a frame.

(c) $\{2\phi_1,\phi_2,\phi_3,......\}$ is a non-tight exact frame with bounds $A = 1, B = 2$.

1.2 Frame operators and frame bounds.

In this section, we shall build an operator called frame operator and then we will define the frame condition, (1.1.1) in terms of this operator. Further, we shall also define and discuss the concept of duality of a frame.

Definition(1.2.1):

If $\{\phi_n\}_{n \in \aleph}$ is a frame in H, then T is the linear operator

$$T : H \to l^2(\aleph)$$

defined by

$$(T\phi)_n = \langle \phi | \phi_n \rangle \tag{1.2.1}$$

where

$$l^2(\aleph) = \left\{ c = \{c_n\}_{n \in \aleph} \mid \|c\|^2 = \sum_{n \in \aleph} |c_n|^2 < \infty \right\}$$

Then T^*T is called the frame operator, where T^* is the adjoint of T.

Note:

Let us compute the adjoint T^* of T. The adjoint T^* is a mapping from $l^2(\aleph)$ to H. So for some $c \in l^2(\aleph)$, consider

$$\begin{aligned} \langle T^* c | \phi \rangle &= \langle c | T\phi \rangle \\ &= \sum_{n \in \aleph} c_n \overline{\langle \phi | \phi_n \rangle} \\ &= \sum_{n \in \aleph} c_n \langle \phi_n | \phi \rangle \end{aligned}$$

which gives

$$T^* c = \sum_{n \in \aleph} c_n \phi_n \tag{1.2.2}$$

Since $\|T^*\| = \|T\|$, by (1.2.2) we have

$$\|T^{*}c\| \leq B^{\frac{1}{2}}\|c\|$$

Further the definition of T gives,

$$\sum_{n\in\aleph}|\langle\phi|\phi_n\rangle|^2 = \|T\phi\|^2 = \langle T^{*}T\phi|\phi\rangle = T^{*}T\|\phi\|^2 \qquad (1.2.3)$$

Now the frame condition (1.1.1) and the equation (1.2.3) together gives

$$AI \leq T^{*}T \leq BI \qquad (1.2.4)$$

The condition (1.2.4) is the equivalent version of (1.1.1) in the frame operator sense. From now on, the statement "frame condition" means either (1.1.1) or (1.2.4). Now let us put it as a theorem.

Theorem(1.2.2):

Given a sequence $\{\phi_n\}$ in a Hilbert space H, the following two statements are equivalent

(1) $\{\phi_n\}$ *is a frame with bounds A, B*

(2) $S\phi = \sum\langle\phi|\phi_n\rangle\phi_n$ *is a bounded linear operator with*

$$AI \leq S = T^{*}T \leq BI,$$

called the frame operator for $\{\phi_n\}$.

proof:

(2) $\Rightarrow$ (1) suppose (2) holds. Consider

$$S\phi = \sum_{n\in\aleph}\langle\phi|\phi_n\rangle\phi_n$$
$$\Rightarrow \langle S\phi|\phi\rangle = \sum_{n\in\aleph}|\langle\phi|\phi_n\rangle|^2$$

Further $\langle I\phi|\phi\rangle = \|\phi\|^2$ [$\because$ I is the identity operator on H]. Thus for any $\phi \in H$,

$$AI \leq T^{*}T \leq BI$$

gives

$$\langle AI\phi|\phi\rangle \leq \langle S\phi|\phi\rangle \leq \langle BI\phi|\phi\rangle$$
$$\Rightarrow A\|\phi\|^2 \leq \sum_{n\in\mathbb{N}} |\langle\phi|\phi_n\rangle|^2 \leq B\|\phi\|^2$$

Thus $\{\phi_n\}$ is a frame with bounds A, B.

(1) $\Rightarrow$ (2) suppose (1) holds. Therefore, for any $\phi \in H$, (1.1.1) gives

$$A\|\phi\|^2 \leq \sum_{n\in\mathbb{N}} |\langle\phi|\phi_n\rangle|^2 \leq B\|\phi\|^2$$

Let us fix $\phi \in H$ and set

$$S_N = \sum_{n\in\mathbb{N}} \langle\phi|\phi_n\rangle\phi_n$$

For $M \leq N$ we therefore have from the Cauchy Schwarz's inequality and the frame condition

$$\begin{aligned}
\|S_N - S_M\|^2 &= \sup_{\substack{\|\psi\| = 1 \\ \psi\in H}} |\langle S_N - S_M|\psi\rangle|^2 \\
&= \sup_{\|\psi\|=1} \left| \sum_{M<n\leq N} \langle\phi|\phi_n\rangle\langle\phi_n|\psi\rangle \right|^2 \\
&\leq \sup_{\|\psi\|=1} \left(\sum_{M<n\leq N} |\langle\phi|\phi_n\rangle|^2 \right)\left(\sum_{M<n\leq N} |\langle\phi_n|\psi\rangle|^2 \right) \\
&\leq \sup_{\|\psi\|=1} \left(\sum_{M<n\leq N} |\langle\phi|\phi_n\rangle|^2 \right) B\|\psi\|^2 \\
&= B\left(\sum_{M<n\leq N} |\langle\phi|\phi_n\rangle|^2 \right) \rightarrow 0 \text{ as } M, N \rightarrow \infty
\end{aligned}$$

Thus $\{S_N\}$ is a Cauchy sequence in H and H is a Hilbert space. So it must converge in H. Thus $S\phi$ is a well- defined element of H.Now by a similar argument as bellow, we can obtain $\|S\| \leq B$

$$\begin{aligned}
\|S\phi\|^2 &= \sup_{\|\psi\|=1} |\langle S\phi|\psi\rangle|^2 \\
&= \sup_{\|\psi\|=1} \left| \sum_{n\in\mathbb{N}} \langle\phi|\phi_n\rangle\langle\phi_n|\psi\rangle \right|^2
\end{aligned}$$

$$\leq \sup_{\|\psi\|=1}\left(\sum_{n\in\mathbb{N}}|\langle\phi|\phi_n\rangle|^2\right)\left(\sum_{n\in\mathbb{N}}|\langle\phi_n|\psi\rangle|^2\right)$$

$$\leq \sup_{\|\psi\|=1}\left(\sum_{n\in\mathbb{N}}|\langle\phi|\phi_n\rangle|^2\right)B\|\psi\|^2$$

$$= B\left(\sum_{n\in\mathbb{N}}|\langle\phi|\phi_n\rangle|^2\right)$$

$$= B^2\|\phi\|^2$$

Therefore

$$\|S\| \leq B$$

Hence S is bounded. Further, the frame condition gives

$$AI \leq S \leq BI \ \blacksquare \tag{1.2.5}$$

In order to define duality, let us prove the invertibility of the operator S in the following lemma.

Lemma(1.2.3):

If a positive bounded linear operator S on H is bounded bellow by a strictly positive constant α, then S is invertible and its inverse S^{-1} is bounded by α^{-1}.

Proof:

Claim: $\text{Ran}(S) = \{f \in H \mid f = Sg \text{ for some } g \in H\}$ is a closed subspace of H.

For. In order to prove this, it is enough to show that every Cauchy sequence in $\text{Ran}(S)$ converges in $\text{Ran}(S)$. That is

$$\|f_n - f_m\| \to 0 \text{ as } n, m \to \infty$$

Since $f_n \in \text{Ran}(S)$, there exists $g_n \in H$ such that

$$f_n = Sg_n \forall n$$

That implies

$$\|g_n - g_m\|^2 \leq \alpha^{-1}\langle S(g_n - g_m)|g_n - g_m\rangle [\because\ \alpha\langle h|h\rangle \leq \langle Sh|h\rangle]$$
$$\leq \alpha^{-1}\|S(g_n - g_m)\|\,\|g_n - g_m\|$$

That implies

$$\|g_n - g_m\| \le \alpha^{-1}\|S(g_n - g_m)\| = \alpha^{-1}\|f_n - f_m\| \to 0 \text{ as } m,n \to \infty$$

Therefore $\{g_n\}$ is a Cauchy sequence in H and H is a Hilbert space.

Therefore there exists $g \in H$ such that $g_n \to g$ as $n \to \infty$ in H. Further S is continuous, therefore we have

$$\lim_n f_n = \lim_n Sg_n = S\left(\lim_n g_n\right) = Sg$$

Therefore

$$\lim_n f_n \in \text{Ran}(S) \Rightarrow \text{Ran}(S) \text{ is closed}$$

Now we shall prove that the orthogonal complement of $\text{Ran}(S) = \{0\}$.

If $\langle f|Sg\rangle = 0 \forall g \in H$ then $\langle f|Sf\rangle = 0$. But $\|f\|^2 \le \langle Sf|f\rangle \Rightarrow \|f\| = 0 \Rightarrow f = 0 \Rightarrow$orthogonal complement of $\text{Ran}(S) = \{0\}$.We already proved that $\text{Ran}(S)$ is closed . Therefore $\text{Ran}(S) = H$. That implies S is invertible.Therefore for each $f \in H, \exists g \in H$ s.t. $f = Sg$. So we can define $g = S^{-1}f$

Further

$$\alpha\|S^{-1}f\| \le \alpha^{-1}\|f\|$$

Thus S^{-1} is bounded and its bound is α^{-1} as stated.■

The next theorem gives a definition to the dual frame of a frame and the associated frame operator.

Theorem(1.2.4):

(i) The family $\{\phi'_n\}$, with $\phi'_n = S^{-1}\phi_n$, constitute a frame with bounds B^{-1}, A^{-1} (where A and B are frame bounds of the frame $\{\phi_n\}$), called the dual frame of $\{\phi_n\}$.

(ii) Every $\phi \in H$ can be written as

$$\phi = \sum\langle\phi|\phi_n\rangle\phi'_n = \sum\langle\phi|\phi'_n\rangle\phi_n \qquad (1.2.6)$$

(iii) The associated frame operator $\tilde{T} = T(T^*T)^{-1}$ satisfies

$$\tilde{T}^*\tilde{T} = (T^*T)^{-1}, \quad \tilde{T}^*T = I = T^*\tilde{T}$$

where $\tilde{T}T^* = T\tilde{T}^*$ is the orthogonal projection operator in $l^2(\aleph)$ on the range of T.

Proof:

(i) By the above lemma we have

$$B^{-1}I \le S^{-1} \le A^{-1}I \tag{1.2.7}$$

since S^{-1} is positive and self-adjoint, we have

$$\langle \phi | \phi'_n \rangle = \langle \phi | S^{-1}\phi_n \rangle = \langle S^{-1}\phi | \phi_n \rangle$$

$$\Rightarrow \sum_{n\in\mathbb{N}} |\langle \phi | \phi'_n \rangle|^2 = \sum_{n\in\mathbb{N}} |\langle \phi | S^{-1}\phi_n \rangle|^2 = \| S(S^{-1}\phi) \| \quad \left[\because \text{ by the definition of frame operator}\right]$$

$$= \langle S^{-1}\phi | S S^{-1}\phi \rangle = \langle S^{-1}\phi | \phi \rangle \tag{1.2.8}$$

(1.2.6) and (1.2.7) together gives

$$B^{-1}\|\phi\|^2 \le \sum_{n\in\mathbb{N}} |\langle \phi | \phi'_n \rangle|^2 \le A^{-1}\|\phi\|^2 \tag{1.2.9}$$

Thus $\{\phi'_n\}$ constitute a frame with frame bounds B^{-1}, A^{-1}.

(ii) We have

$$\phi = S(S^{-1}\phi) = S^{-1}(S\phi)$$

$$\begin{aligned}
\phi &= \sum_{n\in\mathbb{N}} \langle \phi | \phi'_n \rangle \phi_n \\
&= \sum_{n\in\mathbb{N}} \langle \phi | S^{-1}\phi_n \rangle \phi_n \\
&= \sum_{n\in\mathbb{N}} \langle S^{-1}\phi | \phi_n \rangle \phi_n \\
&= S^{-1}\left(\sum_{n\in\mathbb{N}} \langle \phi | \phi_n \rangle \phi_n \right) \\
&= \sum_{n\in\mathbb{N}} \langle \phi | \phi_n \rangle S^{-1}\phi_n \\
&= \sum_{n\in\mathbb{N}} \langle \phi | \phi_n \rangle \phi'_n
\end{aligned}$$

(iii) By the definition of the ϕ_n, we have

$$
\begin{aligned}
(\tilde{T}\phi)_n &= \langle \phi | \phi_n' \rangle \\
&= \langle \phi | (T^*T)^{-1}\phi_n \rangle \\
&= \langle (T^*T)^{-1}\phi | \phi_n \rangle \\
&= \left(T(T^*T)^{-1}\phi\right)_n
\end{aligned}
$$

Therefore

$$\tilde{T} = T(T^*T)^{-1}$$

Now

$$\tilde{T}^*\tilde{T} = (T^*T)^{-1}T^*T(T^*T)^{-1} = (T^*T)^{-1}$$

And

$$
\begin{aligned}
\tilde{T}^*T &= (T^*T)^{-1}T^*T = T^{-1}(T^*)^{-1}T^*T = T^{-1}T = I \\
T^*\tilde{T} &= T^*T(T^*T)^{-1} = T^*TT^{-1}(T^*)^{-1} = T^*T^{*-1} = I
\end{aligned}
$$

Finally, it remains to prove $\tilde{T}T^* = T\tilde{T}^*$ is the orthogonal projection operator in $l^2(\aleph)$ on to the range of T. Since $T^*c = 0$ for any c orthogonal to the range of T, it is enough to prove that $T(T^*T)^{-1}T^*c = c$ for c in the range of T.Now if $c \in \mathrm{Ran}(T)$ then $c = Tf$ for some $f \in H$. That implies

$$T(T^*T)^{-1}T^*c = T(T^*T)^{-1}T^*Tf = Tf = c$$

Therefore $\tilde{T}T^* = T\tilde{T}^*$ is the stated projection operator.■

The operation $\{\phi_n\}_{n\in\aleph} \rightarrow \{\phi_n'\}_{n\in\aleph}$ defines, in a sense, a duality operation. The same procedure, applied to the frame $\{\phi_n'\}_{n\in\aleph}$ gives the original frame $\{\phi_n\}_{n\in\aleph}$ back again. We shall therefore call $\{\phi_n'\}_{n\in\aleph}$ the dual frame of $\{\phi_n\}_{n\in\aleph}$. The duality $\phi_n \leftrightarrow \phi_n'$ is expressed by

$$
\begin{aligned}
\tilde{T}^*\tilde{T} &= (T^*T)^{-1} \\
\tilde{T}^*T &= I = T^*\tilde{T}
\end{aligned}
$$

Further, for any $\phi,\psi \in H$, we have

$$\langle \phi | \psi \rangle = \sum_{n\in\aleph}\langle \phi | \phi_n' \rangle\langle \phi_n | \psi \rangle = \sum_{n\in\aleph}\langle \phi | \phi_n \rangle\langle \phi_n' | \psi \rangle$$

As we already denoted, here we write $T^{*}T = S$: $\tilde{T}^{*}\tilde{T} = S^{-1}$. In particular

$$S = \sum_{n\in\aleph} |\phi_n\rangle\langle\phi_n| \qquad (1.2.10)$$

and $\phi_n' = S^{-1}\phi_n$ yields

$$S^{-1} = \sum_{n\in\aleph} |\phi_n'\rangle\langle\phi_n'| \qquad (1.2.11)$$

Example(1.2.5):

R^2 is a real Hilbert space. Consider $\left\{\eta_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ which is a frame.The corresponding frame operator can be calculated as

$$\begin{aligned} A &= |\eta_1\rangle\langle\eta_1| + |\eta_2\rangle\langle\eta_2| \\ Ae_1 &= \langle\eta_1|e_1\rangle\eta_1 + \langle\eta_2|e_1\rangle\eta_2 \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= (2, 1) \end{aligned}$$

And

$$\begin{aligned} Ae_2 &= \langle\eta_1|e_2\rangle\eta_1 + \langle\eta_2|e_2\rangle\eta_2 \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= (1, 1) \end{aligned}$$

Now frame operator in matrix form is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Now

$$A^{-1}\eta_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta_1'$$

$$A^{-1}\eta_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \eta_2'$$

So $\{\eta_1', \eta_2'\}$ is the dual frame and the frame operator corresponding to this dual frame is A^{-1}.

1.3 Frames and bases

Definition(1.3.1):

A sequence $\{\phi_n\}$ in a Hilbert space H is a basis for H if for every $\phi \in H$ there exist unique scalars a_n, such that

$$\phi = \sum_{n \in \mathbb{N}} a_n \phi_n$$

The basis is bounded if

$$0 < \inf \|\phi_n\| \leq \sup \|\phi_n\| < \infty$$

It is unconditional if the series $\sum a_n \phi_n$ converges unconditionally for every $\phi \in H$, that is, every permutation of the series converges.

Recall that in theorem (1.1.4) we proved a tight frame with frame bound $A = 1$ is an orthonormal basis. Further, example (1.1.5) (a) is an evidence to the fact that an inexact frame cannot be a basis. In theorem (1.2.4) we have shown that frames provide decompositions of H, i.e., for every $\phi \in H$

$$\phi = \sum \langle \phi | \phi_n \rangle \phi_n' = \sum \langle \phi | \phi_n' \rangle \phi_n \qquad (1.3.1)$$

where $\{\phi_n'\}_{n \in \mathbb{N}}$ is the dual of $\{\phi_n\}_{n \in \mathbb{N}}$. This means that we have a reconstruction formula for ϕ from the $\langle \phi_n | \phi \rangle$. We now consider whether these representations are unique. We have mentioned before that frames, even tight frames, are generally not (orthonormal) bases because ϕ_n are typically not linearly independent.

This means that for a given ϕ, there exist many supperpositions of the ϕ_n which all add up to ϕ.

Example(1.3.2):

Let $H = C^2$ and $\phi_1 = (0,1)$, $\phi_2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\phi_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. It can be easily seen that $\{\phi_1, \phi_2, \phi_3\}$ is a frame with frame bounds $A = B = \frac{3}{2}$.

Than for any $\phi \in H$,

$$\phi = \frac{2}{3}\sum_{n=1}^{3}\langle\phi|\phi_n\rangle\phi_n \tag{1.3.2}$$

Since $\sum_{n=1}^{3}\phi_n = 0$ we can also write

$$\phi = \frac{2}{3}\sum_{n=1}^{3}[\langle\phi|\phi_n\rangle + \alpha]\phi_n \tag{1.3.3}$$

for any arbitrary $\alpha \in C$. Indeed, (1.3.3) gives all possible supperposition formulas valid for ϕ.

Theorem(1.3.3):

If $\phi = \sum_{n\in\aleph} c_n\phi_n$ *for some* $c = \{c_n\} \in l^2(\aleph)$ (here $l^2(\aleph)$ is as in definition (1.2.1)) , *and if not all* c_n *equal* $\langle\phi|\phi_n'\rangle$, *then*

$$\sum_{n\in\aleph}|c_n|^2 > \sum_{n\in\aleph}|\langle\phi|\phi_n'\rangle|^2 \tag{1.3.4}$$

Proof:

We know $\phi = \sum_{n\in\aleph} c_n\phi_n$ is equivalent to $\phi = T^*c$. Write $c = a + b$ where $a \in Ran(T) = Ran(\tilde{T})$ and $b \perp Ran(T) \Rightarrow a \perp b \Rightarrow \|c\|^2 = \|a\|^2 + \|b\|^2$. Since $a \in Ran(\tilde{T}) \Rightarrow \exists\psi \in H$ such that $a = \tilde{T}\psi \Rightarrow c = \tilde{T}\psi + b$. Now $\phi = T^*c = T^*\tilde{T}\psi + T^*b$ but $b \perp Ran(T)$.Therefore$T^*b = 0$ and $T^*\tilde{T} = I$. Therefore $\phi = \psi$. That implies

$$\begin{aligned} c &= \tilde{T}\phi + b. \\ &\Rightarrow \sum_{n\in\aleph}|c_n|^2 = \|c\|^2 \\ &= \left\|\tilde{T}\phi\right\|^2 + \|b\|^2 \\ &= \sum_{n\in\aleph}|\langle\phi|\phi_n'\rangle|^2 + \|b\|^2 \end{aligned}$$

So if $b \neq 0$, then .

$$\sum_{n\in\aleph}|c_n|^2 > \sum_{n\in\aleph}|\langle\phi|\phi_n'\rangle|^2$$

Hence the theorem.■

Theorem(1.3.4):

Given a frame $\{\phi_n\}$ and given $\phi \in H$. Let $a_n = \langle \phi | S^{-1}\phi_n \rangle$ so $\phi = \sum a_n \phi_n$. If it is possible to find other scalars c_n such that $\phi = \sum c_n \phi_n$ then

$$\sum_{n\in\mathbb{N}} |c_n|^2 = \sum_{n\in\mathbb{N}} |a_n|^2 + \sum_{n\in\mathbb{N}} |a_n - c_n|^2$$

Proof:

We have

$$\langle \phi_n | S^{-1}\phi \rangle = \langle S^{-1}\phi_n | \phi \rangle = \overline{a_n}$$

Substituting $\phi = \sum a_n \phi_n$ and $\phi = \sum c_n \phi_n$ into the first term of the inner product $\langle \phi | S^{-1}\phi \rangle$, we get

$$\sum_{n\in\mathbb{N}} |a_n|^2 = \sum_{n\in\mathbb{N}} \overline{a_n} c_n$$

$$\sum_{n\in\mathbb{N}} |a_n|^2 + \sum_{n\in\mathbb{N}} |a_n - c_n|^2 = \sum_{n\in\mathbb{N}} |a_n|^2 + \sum_{n\in\mathbb{N}} \left(|a_n|^2 - a_n\overline{c_n} - \overline{a_n}c_n + |c_n|^2 \right)$$

$$= \sum |c_n|^2. \blacksquare$$

Theorem (1.3.5):

The removal of a vector from a frame leaves either a frame or an incomplete set. In particular.

$\langle \phi_m | S^{-1}\phi_m \rangle \neq 1 \Rightarrow \{\phi_n\}_{n\neq m}$ is a frame

$\langle \phi_m | S^{-1}\phi_m \rangle = 1 \Rightarrow \{\phi_n\}_{n\neq m}$ is incomplete.

Proof:

Fix m, and define

$$a_n = \langle \phi_m | S^{-1}\phi_n \rangle = \langle S^{-1}\phi_m | \phi_n \rangle$$

We know that $\phi_m = \sum a_n \phi_n$ [by (1.2.6)]

But we also have $\phi_m = \sum c_n \phi_n$ where $c_n = \delta_{nm}$. therefore by theorem (1.3.4), we can write.

$$1 = \sum_{n\in\mathbb{N}}|c_n|^2 = \sum_{n\in\mathbb{N}}|a_n|^2 + \sum_{n\in\mathbb{N}}|a_n - c_n|^2$$
$$= |a_m|^2 + \sum_{n\neq m}|a_n|^2 + |a_m - 1|^2 + \sum_{n\neq m}|a_n|^2.$$

Suppose now that $a_m = 1$. Then

$$\sum_{n\neq m}|a_n|^2 = 0.$$

Thus, $a_n = \langle S^{-1}\phi_m|\phi_n\rangle = 0 \quad \forall\ n \neq m$ which implies $S^{-1}\phi_m$ is orthogonal to $\phi_n \quad \forall\ n \neq m$. But $S^{-1}\phi_m \neq 0$ [$\because \langle S^{-1}\phi_m|\phi_m\rangle = a_m = 1$]. Therefore $\{\phi_n\}_{n\neq m}$ is incomplete in this case. On the other hand, if $a_m \neq 1$ then

$$\phi_m = \frac{1}{1-a_m}\sum_{n\neq m}a_n\phi_n.$$

Which implies for $\phi \in H$ we have

$$|\langle\phi|\phi_m\rangle|^2 = \left|\frac{1}{1-a_m}\sum_{n\neq m}a_n\langle\phi|\phi_n\rangle\right|^2 \leq C\sum_{n\neq m}|\langle\phi|\phi_n\rangle|^2 \text{ where } C = |1-a_m|^{-2}\sum_{n\neq m}|a_n|^2$$
$$\Rightarrow \sum_{n}|\langle\phi|\phi_n\rangle|^2 = |\langle\phi|\phi_m\rangle|^2 + \sum_{n\neq m}|\langle\phi|\phi_n\rangle|^2 \leq (1+C)\sum_{n\neq m}|\langle\phi|\phi_n\rangle|^2.$$

$\Rightarrow \{\phi_n\}_{n\neq m}$ is a frame with frame bounds $\frac{A}{1+C}$, B where A , B are the frame bounds of the original frame.■

Corollary (1.3.6):

If $\{\phi_n\}$ is an exact frame, then $\{\phi_n\}$ and $\{S^{-1}\phi_n\}$ are biorthonormal, that is

$$\langle\phi_m|S^{-1}\phi_n\rangle = \delta_{nm}$$

Proof:

Direct from the above theorem.

Lemma(1.3.7):

If $\{\phi_n\}$ is a bounded unconditional basis for a Hilbert space H, then there is an orthonormal basis $\{e_n\}$ and a topological isomorphism $U : H \to H$ such that $\phi_n = Ue_n \forall n$.

The above lemma helps us to prove the following characterization of exact frames. The proof of this lemma is irrelavant here and can be found in standard texts. So we omit the proof here.

Theorem(1.3.8):

A sequence $\{\phi_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a bounded unconditional basis for H.

Proof:

Assume $\{\phi_n\}$ is an exact frame with frame bounds A , B. Then by the corollary (1.3.6) $\{\phi_n\}$ and $\{S^{-1}\phi_n\}$ are biorthonormal. So for fixed m, we have

$$A\|S^{-1}\phi_m\|^2 \leq \sum_n |\langle S^{-1}\phi_m|\phi_n\rangle|^2 = |\langle S^{-1}\phi_m|\phi_m\rangle|^2 \leq \|S^{-1}\phi_m\|^2\|\phi_m\|^2$$

and

$$\|\phi_m\|^4 = |\langle\phi_m|\phi_m\rangle|^2 \leq \sum_n |\langle\phi_m|\phi_n\rangle|^2 \leq B\|\phi_m\|^2$$

Which implies

$$A \leq \|\phi_m\|^2 \leq B$$

So, $\{\phi_n\}$ is bounded in norm. By theorem (1.2.4)(ii) we have,

$$\phi = \sum_n \langle\phi|S^{-1}\phi_n\rangle\phi_n \quad \forall\phi \in H. \tag{1.3.5}$$

So, it is enough to show that the representation (1.3.5) is unique. If $\phi = \sum c_n\phi_n$ then

$$\langle\phi|S^{-1}\phi_m\rangle = \sum c_n\langle\phi_n|S^{-1}\phi_m\rangle = c_m$$

Therefore (1.3.5) is unique. Thus $\{\phi_n\}$ is a basis for H, and since the sum converges unconditionally we can conclude that the basis is unconditional.

Conversely, suppose that $\{\phi_n\}$ is a bounded unconditional basis for H. Then by lemma (1.3.7), there is an orthonormal basis $\{e_n\}$ and a topological isomorphism.

$$u : H \to H$$

such that $ue_n = \phi_n$ for all n. Therefore, given $\phi \in H$, we have

$$\sum_n |\langle \phi | \phi_n \rangle|^2 = \sum_n |\langle \phi | u e_n \rangle|^2 = \sum_n |\langle u^* \phi | e_n \rangle|^2 = \|u^* \phi\|^2.$$

But $\|(u^*)^{-1}\|^{-1} \|\phi\| \leq \|u^* \phi\| \leq \|u^*\| \|\phi\|$. Thus,

$$\left\{ \|(u^*)^{-1}\|^{-1} \right\}^2 \|\phi\|^2 \leq \sum_n |\langle \phi | \phi_n \rangle|^2 \leq \|u^*\|^2 \|\phi\|^2.$$

Therefore $\{\phi_n\}$ form a frame. Furthermore, $\{\phi_n\}$ is clearly exact because the removal of any vector from a basis leaves an incomplete set.■

In the next chapter we will discuss the theory of reproducing kernels and reproducing kernel Hilbert spaces. Reproducing kernels have close connection with frames which will be discussed in chapter 3. Even though, in chapter 3 we will discuss the concept of continuous frames in Hilbert spaces, we will give many discrete examples to visualize the concept.

Chapter 2

Reproducing Kernels

In this thesis we use reproducing kernels at several instances. Indeed, in chapter 3 we are going to develop the relation between reproducing kernels and frames. Further, in chapter 4 we will give a recipe for the general construction of frames, for this purpose we will depend heavily on reproducing kernel Hilbert spaces. Even though we use several results about this concept in this thesis we are not going to go deep into the theory of reproducing kernels because our aim is to develop the theory of frames not the theory of reproducing kernels.

Let $\widetilde{H} = L^2(X,\nu,C^n) = \left\{\phi : X \to C^n \mid \int_X \|\phi(x)\|^2 d\nu < \infty\right\}$ and let $M(C^n)$, the set of all linear maps from C^n to C^n. Here X is a locally compact space with Borel measure ν on it.

Definition:

A reproducing kernel K on $\widetilde{H}$ is a measurable map

$$K : X \times X \to M(C^n)$$

satisfying

(i) $K(x,x) \geq 0$

(ii) $K(x,y) = K(y,x)^*$, * *stands for the adjoint map*

(iii) *The integral operator P_K, defined on $\widetilde{H}$ by K,*

$$(P_K\Psi)(x) = \int_X K(x,y)\Psi(y)d\nu(y) \qquad \forall \Psi \in \widetilde{H}$$

is bounded

(iv) $\int_X \langle z|K(x,y)K(y,x')z'\rangle_n d\nu(y) = \langle z|K(x,x')z'\rangle_n \quad \forall z,z' \in C^n$, *where $\langle .|.\rangle_n$ is the usual scalar product on C^n.*

Note:

$H_K = P_K\widetilde{H}$, the subspace of $\widetilde{H}$ is called the reproducing kernel Hilbert space corresponding to the kernel K.

In the literature there are many papers published on the theory of reproducing kernels. Many of them are easily readable. In particular, for our purpose the following references are very useful [2],[5],[9]. From now on , whereever needed, we will use the results on reproducing kernels without giving details.

Chapter 3

Continuous Frames

In this chapter we will discuss continuous frames in detail. We will also give some discrete examples to describe the theory in an easy way. In section (3.1), we will define a continuous frame and then its dual frame. Then we will give some examples also. In section (3.2), we will see the connection between reproducing kernels and frames. Through this connection we will also establish some interesting equivalences between various types of frames.

3.1 Definition and Examples

As usual, let us start this section by setting up the notation which will be used through out this chapter. Let **H** be an abstract separable Hilbert space over the complex numbers, **C**. $GL(\mathbf{H})$ denote the group of all bounded linear operators on **H** which have bounded inverses. X stands for a locally compact space and let v be a regular Borel measure on X with support X. In the case X is discrete, the measure v is the counting measure on X.

Theorem (3.1.1):

Let X be a locally compact set with Borel measure v on it and let for each $x \in X$ the set $\{\eta_x^i | i = 1,\dots,n\}$ be a linearly independent set in H. We define an operator A by

$$\sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x) = A \qquad (3.1.1)$$

and we always assume $A \in GL(\mathbf{H})$. Then the operator A is positive and self-adjoint.

Proof:

Let $\phi \in \mathbf{H}$. Consider

$$\begin{aligned}\langle A\phi|\phi\rangle &= \left\langle \sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x)\phi|\phi \right\rangle \\ &= \sum_{i=1}^{n} \int_X \langle\eta_x^i|\phi\rangle\langle\phi|\eta_x^i\rangle dv(x)\end{aligned}$$

$$= \sum_{i=1}^{n} \int_X |\langle \eta_x^i | \phi \rangle|^2 d\nu(x) \geq 0.$$

That is,

$$\langle A\phi | \phi \rangle \geq 0 \ \forall \phi \in \mathbf{H}.$$

Therefore A is positive.

Now consider

$$\langle A\phi | \psi \rangle = \sum_{i=1}^{n} \int_X \langle \eta_x^i | \psi \rangle \langle \phi | \eta_x^i \rangle d\nu(x).$$

and

$$\langle \phi | A\psi \rangle = \sum_{i=1}^{n} \int_X \langle \eta_x^i | \psi \rangle \langle \phi | \eta_x^i \rangle d\nu(x).$$

Thus, $\langle A\phi | \psi \rangle = \langle \phi | A\psi \rangle \Rightarrow A$ is self-adjoint.■

Definition(3.1.2):

The resolvent set $\varrho(A)$ of A consists of all complex numbers λ for which $(\lambda I - A)^{-1}$ is a bounded operator. The spectrum, $\sigma(A)$ of A consists of the complement of $\varrho(A)$ in C.

Definition(3.1.3):

The upper bound of a self-adjoint operator A is the number

$$M = \sup_{\|\phi\|=1} \langle \phi | A\phi \rangle.$$

The lower bound of a self-adjoint operator A is the number

$$m = \inf_{\|\phi\|=1} \langle \phi | A\phi \rangle.$$

Theorem (3.1.4):

Suppose $A \in GL(\mathbf{H})$, where A is defined by (3.1.1). let $\sigma(A)$ be the spectrum of A, and

$$m(A) = \inf_{\|\phi\|=1} \langle \phi | A\phi \rangle \neq 0,$$

$$M(A) = \sup_{\|\phi\|=1} \langle \phi | A\phi \rangle \neq 0, \phi \in \mathbf{H}. \tag{3.1.2}$$

such that $m(A), M(A) \in \sigma(A)$ and $\sigma(A) \subset [m(A), M(A)]$ (This is possible because A is self-adjoint). Then, for $\phi \in H$

$$m(A)\|\phi\|^2 \leq \sum_{i=1}^{n} \int_X |\langle \eta_x^i | \phi \rangle|^2 d\nu(x) \leq M(A)\|\phi\|^2. \tag{3.1.3}$$

Proof:

By the construction it is clear that

$$m(A)I \leq A \leq M(A)I \tag{3.1.4}$$

That is,

$$m(A)I \leq \sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle \eta_x^i | d\nu(x) \leq M(A)I.$$

$$m(A)\phi \leq \sum_{i=1}^{n} \int_X \langle \eta_x^i | \phi \rangle \eta_x^i d\nu(x) \leq M(A)\phi.$$

$$m(A)\|\phi\|^2 \leq \sum_{i=1}^{n} \int_X |\langle \eta_x^i | \phi \rangle|^2 d\nu(x) \leq M(A)\|\phi\|^2.$$

which completes the proof ■

The inequality (3.1.3) gives us the frame condition for the set of vectors $\{\eta_x^i \in \mathbf{H} \mid i = 1, 2, \ldots, n\}$ with frame bounds $m(A)$ and $M(A)$. Now let us define a rank n frame.

Definition(frame)(3.1.5):

A set of vectors $\eta_x^i \in H$, $i = 1, 2, \ldots, n$, $x \in X$, constitute a rank-n frame, denoted $F(\eta_x^i, A, n)$ if

(i) for each $x \in X$, the vectors η_x^i, $i = 1, 2, \ldots, n$, are linearly independent.

(*ii*) *there exists a positive operator* $A \in GL(\mathbf{H})$ *such that*

$$\sum_{i=1}^{n}\int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x) = A$$

Note:

(*i*) From theorem (3.1.4) we see that the above definition obviously giving the frame condition(3.1.3)

(*ii*) In the above definition, if the set $X = J$, some discrete set and v a counting measure then the operator A becomes identical to the frame operator of the discrete frame,

$$\sum_{i=1}^{n}\sum_{j\in J} |\eta_j^i\rangle\langle\eta_j^i| dv(x) = A$$

or,

$$\sum_{k\in K} |\eta_k\rangle\langle\eta_k| dv(x) = A$$

where k is some other discrete set.

Lemma(3.1.6):

Under the notation and conditions of theorem (3.1.4) *we have,* $M(A)^{-1}$, $m(A)^{-1}$ *are the lower and upper bounds of* $\sigma(A^{-1})$ *and* $M(A)^{-1}, m(A)^{-1} \in \sigma(A^{-1})$ *with*
$\sigma(A^{-1}) \subset [M(A)^{-1}, m(A)^{-1}]$.

Proof:

By (3.1.2) it is clear that $M(A)^{-1}$and $m(A)^{-1}$ are the lower and upper bounds of $\sigma(A^{-1})$ respectively.

Now

$$Ax = M(A)x \quad [\because M(A) \in \sigma(A)]$$
$$\Rightarrow x = A^{-1}(M(A)x)$$

$$\Rightarrow x = M(A)A^{-1}x$$
$$\Rightarrow A^{-1}x = M(A)^{-1}x$$
$$\therefore M(A)^{-1} \in \sigma(A^{-1})$$

Similarly $m(A)^{-1} \in \sigma(A^{-1})$.

Now $\sigma(A^{-1}) \subset [M(A)^{-1}, m(A)^{-1}]$ is trivial.■

Theorem(3.1.7):

Let $\eta_x''^{i} = A^{-1}\eta_x'^{i}$, $i = 1, 2, \ldots, n$, $x \in X$. *Then*

$$\text{(i)} \quad \sum_{i=1}^{n} \int_X |\eta_x''^{i}\rangle\langle\eta_x''^{i}| d\nu(x) = A^{-1} \tag{3.1.5}$$

$$\text{(ii)} \quad \textit{For all } \phi \in H,\ M(A)^{-1}\|\phi\|^2 \le \sum_{i=1}^{n} \int_X |\langle\eta_x''^{i}|\phi\rangle|^2 d\nu(x) \le m(A)^{-1}\|\phi\|^2 \tag{3.1.6}$$

Proof:

Consider

$$\begin{aligned}
&\sum_{i=1}^{n} \int_X |\eta_x''^{i}\rangle\langle\eta_x''^{i}| d\nu(x) \\
&= \sum_{i=1}^{n} \int_X |A^{-1}\eta_x'^{i}\rangle\langle A^{-1}\eta_x'^{i}| d\nu(x) \\
&= A^{-1}\left(\sum_{i=1}^{n} \int_X |\eta_x'^{i}\rangle\langle\eta_x'^{i}| d\nu(x)\right)A^{-1} \\
&= A^{-1}AA^{-1} \\
&= A^{-1}.
\end{aligned}$$

Now $\sigma(A^{-1}) \subset [M(A)^{-1}, m(A)^{-1}]$. Thus, $M(A)^{-1}I \le A^{-1} \le m(A)^{-1}I$. Therefore, as in theorem (3.1.4) we get

$$M(A)^{-1}\|\phi\|^2 \le \sum_{i=1}^{n} \int_X |\langle\eta_x''^{i}|\phi\rangle|^2 d\nu(x) \le m(A)^{-1}\|\phi\|^2,\ \forall\, \phi \in \mathbf{H}.$$

Hence the theorem.■

Theorem (3.1.7), indeed, defines a frame $F(\eta_x'', A^{-1}, n)$.

Definition(3.1.8):

The frame $F(\eta_x'', A^{-1}, n)$ is said to be the dual frame of $F(\eta_x', A, n)$, where

$$\eta_x'' = A^{-1}\eta_x'. \tag{3.1.7}$$

Definition(3.1.9):

As in the discrete case, here also we say that a frame is tight if its frame bounds are equal.

Definition(3.1.10):

The quantity

$$w(F) = \frac{M(A) - m(A)}{M(A) + m(A)} \tag{3.1.8}$$

is called the width or snugness of the frame $F(\eta_x', A, n)$. It is clear that $0 \leq w(F) < 1$, and $w(F)$ measures the spectral width of the operator A.

Theorem(3.1.11):

If the snugness $w(F) = 0$ then the frame $F(\eta_x', A, n)$ is tight, that is $A = \lambda I$, where $\lambda > 0$ and I is the identity operator on H.

Proof:

Suppose $w(F) = 0$

$$\Rightarrow \frac{M(A) - m(A)}{M(A) + m(A)} = 0$$

$$\Rightarrow M(A) = m(A).$$

Therefore the frame is tight. Further if $M(A) = m(A)$ then (3.1.4) gives

$$A = M(A)I = m(A)I = \lambda I \text{ (say).} \blacksquare$$

Theorem(3.1.12):

A frame $F(\eta_x', A, n)$ and its dual $F(\eta_x'', A^{-1}, n)$ have the same width and a frame is self-dual if and only if $A = I$.

Proof:

If $M(A)$ and $m(A)$ are the frame bounds of $F(\eta_x^i, A, n)$ then $m(A)^{-1}$ and $M(A)^{-1}$ are the frame bounds of its dual. Therefore

$$w(\text{dual}) = \frac{m(A)^{-1} - M(A)^{-1}}{m(A)^{-1} + M(A)^{-1}} = \frac{M(A) - m(A)}{M(A) + m(A)} = w(\text{original})$$

If $F(\eta_x^i, A, n)$ is a self-dual then $A = A^{-1} \Leftrightarrow A = I$.■

Theorem(3.1.13):

Let $F(\eta_x^i, A, n)$ be a rank-n frame. Then there is a self-dual tight frame $F(\bar{\eta}_x^i, \bar{A}, n)$ associated to the given frame by $\bar{\eta}_x^i = A^{-\frac{1}{2}}\eta_x^i$, where $A^{-\frac{1}{2}}$ is the square root of the operator A.

Proof:

We have

$$\bar{A} = \sum_{i=1}^{n}\int_X |\bar{\eta}_x^i\rangle\langle\bar{\eta}_x^i| dv(x)$$

Then

$$\begin{aligned}
\bar{A}\phi &= \left(\sum_{i=1}^{n}\int_X \left|A^{-\frac{1}{2}}\eta_x^i\right\rangle\left\langle A^{-\frac{1}{2}}\eta_x^i\right| dv(x)\right)\phi \\
&= A^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\int_X \eta_x^i\left\langle A^{-\frac{1}{2}}\eta_x^i|\phi\right\rangle dv(x)\right) \\
&= A^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\int_X \eta_x^i\left\langle \eta_x^i|A^{-\frac{1}{2}}\phi\right\rangle dv(x)\right) && \left[\text{ because } A \text{ is self-adjoint therefore } A^{-\frac{1}{2}} \text{ is self-adjoint}\right] \\
&= A^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x)\right)A^{-\frac{1}{2}}\phi \\
&= A^{-\frac{1}{2}}AA^{-\frac{1}{2}}\phi \\
&= I\phi
\end{aligned}$$

Thus $\bar{A} = I$. Therefore by theorems (3.1.11) and (3.1.12), the frame $F(\bar{\eta}_x^i, I, n)$ is self-dual and tight.■

Theorem(3.1.14):

For any $T \in GL(H)$ and any rank-n frame $F(\eta_x^i, A, n)$ if we take $\tilde{\eta}_x^i = T\eta_x^i$ then there exists another frame $F(\tilde{\eta}_x^i, \tilde{A}, n)$ with $\tilde{A} = TAT^*$, where T^* is the adjoint of T.

Proof:

Let us consider, for $\phi \in H$

$$
\begin{aligned}
\tilde{A}\phi &= \left(\sum_{i=1}^{n} \int_X |\tilde{\eta}_x^i\rangle\langle\tilde{\eta}_x^i| dv(x)\right)\phi \\
&= \sum_{i=1}^{n} \int_X \tilde{\eta}_x^i \langle\tilde{\eta}_x^i|\phi\rangle dv(x) \\
&= \sum_{i=1}^{n} \int_X T\eta_x^i \langle T\eta_x^i|\phi\rangle dv(x) \\
&= T\left(\sum_{i=1}^{n} \int_X \eta_x^i \langle\eta_x^i|T^*\phi\rangle dv(x)\right) \\
&= T\left(\sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x)\right)T^*\phi \\
&= TAT^*\phi
\end{aligned}
$$

Therefore $\tilde{A} = TAT^*$ which proves the theorem.■

The new frame obtained in theorem (3.1.14) is interesting. In the case, T is unitary, it gives a new class of frames under an equivalent relation.

Definition(3.1.15):

If $T \in GL(H)$ is a unitary operator i.e. $TT^* = T^*T = I$ then the frames $F(\eta_x^i, A, n)$ and $F(\tilde{\eta}_x^i, \tilde{A}, n)$ are said to be unitary equivalent, where

$$\tilde{\eta}_x^i = T\eta_x^i \text{ and } \tilde{A} = TAT^* \tag{3.1.9}$$

Theorem(3.1.16):

If $F(\eta_x^i, A, n)$ and $F(\tilde{\eta}_x^i, \tilde{A}, n)$ are unitary equivalent frames then

(i) $\sigma(A) = \sigma(\tilde{A})$

(ii) $w(F(\eta_x^i, A, n)) = w(F(\tilde{\eta}_x^i, \tilde{A}, n))$

Where $\sigma(A)$ is the spectrum of A and $w(F(\eta_x^i, A, n))$, the width of the frame $F(\eta_x^i, A, n)$.

Proof:

Suppose $F(\eta'_x,A,n)$ and $F(\tilde{\eta}'_x,\tilde{A},n)$ are unitary equivalent frames then there exists a unitary operator $T \in GL(H)$ *such that*

$$\tilde{\eta}'_x = T\eta'_x \text{ and } \tilde{A} = TAT^*$$

Let $\lambda \in \sigma(A)$ then $A\phi = \lambda\phi$ for some ϕ

$$\Leftrightarrow TAT^*\phi = \lambda TT^*\phi = \lambda\phi$$
$$\Leftrightarrow \tilde{A}\phi = \lambda\phi$$
$$\Leftrightarrow \lambda \in \sigma(\tilde{A})$$

Thus $\sigma(A) = \sigma(\tilde{A})$. Further $w(F(\eta'_x,A,n)) = w(F(\tilde{\eta}'_x,\tilde{A},n))$ is obvious. Hence the theorem.■

In theorem (3.1.13) we obtained a self-dual tight frame $F(\tilde{\eta}'_x,I,n)$ associated to the frame $F(\eta'_x,A,n)$. Now the question is, is that the only way to obtain a self-dual tight frame from $F(\eta'_x,A,n)$? The following theorem is going to give the answer to this question.

Theorem(3.1.17):

Let $F(\eta'_x,A,n)$ be a rank-n frame. Then there is a self-dual tight frame $F(\tilde{\eta}'_x,\tilde{A},n)$, which is unitary equivalent to $F(\tilde{\eta}'_x,I,n)$, where $\tilde{\eta}'_x = A^{-\frac{1}{2}}\eta'_x$

Proof:

Let U be a unitary operator on H. then we can write

$$A = A^{-\frac{1}{2}}U^*UA^{-\frac{1}{2}}.$$

Set

$$\tilde{\eta}'_x = UA^{-\frac{1}{2}}\eta'_x. \qquad (3.1.10)$$

Consider

$$\tilde{A} = \sum_{i=1}^{n}\int_X |\tilde{\eta}'_x\rangle\langle\tilde{\eta}'_x| dv(x).$$

Then.

$$
\begin{aligned}
\tilde{A}\phi &= \sum_{i=1}^{n}\int_X UA^{-\frac{1}{2}}\eta_x^i\left\langle UA^{-\frac{1}{2}}\eta_x^i|\phi\right\rangle dv(x) \\
&= UA^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\int_X \eta_x^i\left\langle \eta_x^i|A^{-\frac{1}{2}}U^*\phi\right\rangle dv(x)\right)\left[\because A^{-\frac{1}{2}} \text{ self adjoint}\right] \\
&= UA^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x)\right)\left(A^{-\frac{1}{2}}U^*\right)\phi \\
&= UA^{-\frac{1}{2}}AA^{-\frac{1}{2}}U^*\phi \\
&= UU^*\phi \\
&= I\phi
\end{aligned}
$$

Thus, $\tilde{A} = I$. Therefore, by theorem(3.1.11) and (3.1.12) the frame $F\left(\tilde{\eta}_x^i,\tilde{A},n\right)$ is self-dual and tight. Furthermore, $I = UIU^*$ and $\tilde{\eta}_x^i = UA^{-\frac{1}{2}}\eta_x^i = U\eta_x^i$.Thus, by definition (3.1.15) the frames $F(\eta_x^i,I,n)$ and $F\left(\tilde{\eta}_x^i,I,n\right)$ are unitary equivalent.■

Now consider a frame $F(\eta_x^i,A,n)$. As we did earlier obtain the new frame $F\left(\tilde{\eta}_x^i,\tilde{A},n\right)$ by taking $\tilde{\eta}_x^i = T\eta_x^i$ and $\tilde{A} = TAT^*$ for some $T \in GL(H)$. Is there any equivalency between these two frames? In order to answer this question, let us introduce the positive operator

$$
F(x) = \sum_{i=1}^{n}|\eta_x^i\rangle\langle\eta_x^i| \tag{3.1.11}
$$

for each $x \in X$. Then the frame operator becomes

$$
\int_X F(x)dv(x) = A \tag{3.1.12}
$$

Now for each $x \in X$, there is more than one set of linearly independent vectors η_x^i for which (3.1.11) is satisfied. Indeed, for each $x \in X$, the choice of the basis $\{\eta_x^i\}_{i=1}^{n}$ is as large as $U(n)$, set of all $n x n$ unitary matrices. Let us see this through the following theorem.

Theorem(3.1.18):

$\{\tilde{\eta}_x^i\}_{i=1}^{n}$ is another linearly independent set of vectors for which

$$
F(x) = \sum_{i=1}^{n}|\tilde{\eta}_x^i\rangle\langle\tilde{\eta}_x^i|. \tag{3.1.13}
$$

iff there exists a unitary matrix $u(x) = (u_{ij}(x))_{nxn}$ such that

$$\bar{\eta}^{i}_{x} = \sum_{j=1}^{n} u_{ji}(x)\eta^{j}_{x} \quad , \; i = 1,2,......,n. \tag{3.1.14}$$

Proof:

Suppose there exists a unitary matrix $u(x) = (u_{ij}(x))_{n\times n}$ such that (3.1.14) holds.

Then

$$\begin{aligned}
\sum_{i=1}^{n} |\bar{\eta}^{i}_{x}\rangle\langle\bar{\eta}^{i}_{x}| &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} u_{ji}(x)\eta^{j}_{x} \right\rangle \left\langle \sum_{k=1}^{n} u_{ki}(x)\eta^{k}_{x} \right| \\
&= \sum_{i=1}^{n} |\eta^{i}_{x}\rangle\langle\eta^{i}_{x}| \quad \left[\begin{array}{l} \text{because } u \text{ is unitary} \Rightarrow \\ \sum_{i=1}^{n} \overline{u_{ij}(x)}u_{kj}(x) = \delta_{ik} \end{array} \right] \\
&= F(x)
\end{aligned}$$

Conversely, suppose that $\{\bar{\eta}^{i}_{x}\}_{i=1}^{n}$ is a linearly independent set and (3.1.13) holds.

Let $\phi \in H$,then

$$\begin{aligned}
\langle\phi|F(x)\phi\rangle &= \langle\phi| \sum_{i=1}^{n} |\bar{\eta}^{i}_{x}\rangle\langle\bar{\eta}^{i}_{x}|\phi\rangle \\
&= \sum_{i=1}^{n} \langle\bar{\eta}^{i}_{x}|\phi\rangle\langle\phi|\bar{\eta}^{i}_{x}\rangle \\
&= \sum_{i=1}^{n} |\langle\bar{\eta}^{i}_{x}|\phi\rangle|^{2}.
\end{aligned}$$

and

$$\begin{aligned}
\langle\phi|F(x)\phi\rangle &= \langle\phi| \sum_{i=1}^{n} |\eta^{i}_{x}\rangle\langle\eta^{i}_{x}|\phi\rangle \\
&= \sum_{i=1}^{n} |\langle\eta^{i}_{x}|\phi\rangle|^{2}
\end{aligned}$$

$$\text{i.e. } \langle\phi|F(x)\phi\rangle = \sum_{i=1}^{n} |\langle\bar{\eta}^{i}_{x}|\phi\rangle|^{2} = \sum_{i=1}^{n} |\langle\eta^{i}_{x}|\phi\rangle|^{2}. \tag{3.1.15}$$

Set $z_i = \langle\eta^{i}_{x}|\phi\rangle \in \mathbf{C}$ and $z'_i = \langle\bar{\eta}^{i}_{x}|\phi\rangle \in \mathbf{C}$. Then, (3.1.15) becomes

$$\sum_{i=1}^{n}|z_i|^2 = \sum_{i=1}^{n}|z_i'|^2. \tag{3.1.16}$$

$F(x) : H \to H$. Let $F(x)H = \bar{H}$. Now let $P(x) : H \to \bar{H}$ be the projection operator. Therefore $P(x)H$ can be spanned by both $\{\eta_x^i\}_{i=1}^n$ and $\{\bar{\eta}_x^i\}_{i=1}^n$, i.e. $\{\eta_x^i\}_{i=1}^n$ and $\{\bar{\eta}_x^i\}_{i=1}^n$ are two different bases of $P(x)H$. Therefore, there exists an invertible matrix $u(x) = (u_{ij}(x))_{n\times n}$ such that

$$\bar{\eta}_x^i = \sum_{j=1}^{n} u_{ji}(x)\eta_x^j$$

Furthermore, for any $\phi \in H$,

$$\langle\bar{\eta}_x^i|\phi\rangle = \sum_{j=1}^{n} u_{ji}(x)\langle\eta_x^j|\phi\rangle$$

$$\Rightarrow z_i' = \sum_{j=1}^{n} u_{ji}(x)z_j$$

Therefore, by (3.1.16), $u(x)$ is unitary, which completes the proof.■

So far, in this thesis, we gave several examples of discrete frames and related concepts. The following example gives a continuous frame on the Hilbert space $L^2(\Re^+)$.

Example(3.1.19):

Let $\eta \in L^2(\Re^+)$ with

$$(i) \quad \int_0^{\infty}|\eta(x)|^2 dx < \infty,$$

and

$$(ii) \quad \int_0^{\infty}\frac{|\eta(x)|^2}{x}dx < \infty. \tag{3.1.17}$$

Take $\tau = \{(a,b) \mid a \in \Re_*^+, b \in \Re\}$. Define $\eta_{a,b}$ by

$$\eta_{a,b}(x)=\exp(ibx)\eta(ax)a^{\frac{1}{2}}. \tag{3.1.18}$$

Now we will prove $\{\eta_{a,b}\}_{(a,b)\in\tau}$ constitute a tight frame on $L^2(\Re^+)$. Indeed, we will show

$$(i) \qquad \eta_{a,b} \in L^2(\Re^+)$$
$$(ii) \qquad \|\eta_{a,b}\|^2 = \|\eta\|^2$$
$$(iii) \qquad \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{+\infty} db|\eta_{a,b}\rangle\langle\eta_{a,b}| = \lambda I \text{ for some } \lambda > 0$$

Proof:

(i) Consider

$$\begin{aligned}
\int_0^{\infty} |\eta_{a,b}(x)|^2 dx &= \int_0^{\infty} \left|e^{ibx}\eta(ax)a^{\frac{1}{2}}\right|^2 dx \\
&= \int_0^{\infty} |e^{ibx}|^2 |\eta(ax)|^2 \left|a^{\frac{1}{2}}\right|^2 dx \\
&= \int_0^{\infty} |\eta(ax)|^2 \, a \, dx \\
&= \int_0^{\infty} |\eta(x')|^2 \, a \, \frac{dx'}{a} \qquad \left[\text{Take } ax = x', dx = \frac{dx'}{a}\right] \\
&= \int_0^{\infty} |\eta(x')|^2 dx' < \infty.
\end{aligned}$$

Therefore, $\int_0^{\infty} |\eta_{a,b}(x)|^2 \, dx < \infty$. Thus, $\eta_{a,b} \in L^2(\Re^+)$

For (ii),

$$\begin{aligned}
\|\eta_{a,b}\|^2 &= \int_0^{\infty} |\eta_{a,b}(x)|^2 dx \\
&= \int_0^{\infty} |\eta(x')|^2 dx' \\
&= \|\eta\|^2
\end{aligned}$$

For (iii), let

$$A = \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{+\infty} db|\eta_{a,b}\rangle\langle\eta_{a,b}|$$

$$\langle A\phi|\psi\rangle = \int_0^{\infty} \int_{-\infty}^{+\infty} \langle\eta_{a,b}|\psi\rangle\langle\phi|\eta_{a,b}\rangle \frac{dadb}{a^{\frac{1}{2}}}. \qquad (3.1.19)$$

Now

$$\begin{aligned}
\langle\eta_{a,b}|\psi\rangle &= \int_0^{\infty} \overline{\eta_{a,b}(x')}\psi(x')dx' \\
&= \int_0^{\infty} e^{-ibx'}\overline{\eta(ax')}\,\overline{a^{\frac{1}{2}}}\,\psi(x')dx'
\end{aligned}$$

and

$$\langle \phi | \eta_{a,b} \rangle = \int_0^{\infty} \overline{\phi(x)} e^{ibx} \eta(ax) a^{\frac{1}{2}} dx.$$

(3.1.19)⇒

$$\begin{aligned}
\langle A\phi|\psi\rangle &= \int_0^{\infty}\int_{-\infty}^{+\infty} e^{ib(x-x')}\eta(ax)\overline{\eta(ax')}a\psi(x')\overline{\phi(x)}dxdx'\frac{dadb}{a^{\frac{1}{2}}} \\
&= 2\pi\int_0^{\infty}\delta(x-x')\eta(ax)\overline{\eta(ax')}a\psi(x')\overline{\phi(x)}dxdx'\frac{da}{a^{\frac{1}{2}}} \qquad \left[\because \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ib(x-x')}dx = \delta(x-x')\right] \\
&= 2\pi\int_0^{\infty}\eta(ax)\overline{\eta(ax')}a\psi(x')\overline{\phi(x)}\frac{da}{a}dx \qquad \left[\because \int_0^{\infty} f(x)\delta(x-x')dx = f(x')\right] \\
&= 2\pi\int_0^{\infty}|\eta(ax)|^2\frac{da}{a}\int_0^{\infty}\overline{\phi(x)}\psi(x)dx \\
&= 2\pi\int_0^{\infty}|\eta(ax)|^2\frac{dax}{ax}\int_0^{\infty}\overline{\phi(x)}\psi(x)dx \\
&= 2\pi\int_0^{\infty}\frac{|\eta(a')|^2}{a'}da'\int_0^{\infty}\overline{\phi(x)}\psi(x)dx \qquad [\because ax = a', d(ax) = da'] \\
&= 2\pi M\langle\phi|\psi\rangle
\end{aligned}$$

where $M = \int_0^{\infty}\frac{|\eta(a')|^2}{a'}da' < \infty$. Thus $A = 2\pi MI = \lambda I$ where $\lambda = 2\pi M$.

Hence by theorem (3.1.11) $\{\eta_{a,b}\}$ constitute a tight frame.■

3.2 Frames and Reproducing Kernels

In this section we will establish the connection between frames and reproducing kernels. Using this connection we will develop equivalences between various kinds of frames in the continuous frame sense. Before giving the proofs it is helpful to visualize the basic concept through some elementary examples. In this regard, we shall often demonstrate the concepts using discrete examples.

Let X be a locally compact space and let ν be a Borel measure on it. Define $\widetilde{H} = L^2(X,\nu,C^n)$. Before we define the connection between frames and reproducing kernels we shall establish the existence of reproducing kernels through the following few theorems.

Theorem(3.2.1):

Let H be a separable Hilbert space, and $\widetilde{H} = L^2(X,\nu,C^n)$. Then $W_\eta : H \to \widetilde{H}$ defined by

$$(W_\eta\phi)_i(x)=\langle\eta_x^i|\phi\rangle \tag{3.2.1}$$

is a bounded linear map, where $F(\eta_x^i,A,n)$ is a frame in H as before.

Proof:

Let $\phi_1,\phi_2 \in H$ and $\alpha,\beta \in C$. Then consider

$$\begin{aligned}(W_\eta(\alpha\phi_1+\beta\phi_2))_i(x) &= \langle\eta_x^i|\,\alpha\phi_1+\beta\phi_2\rangle\\ &= \alpha\langle\eta_x^i|\phi_1\rangle+\beta\langle\eta_x^i|\phi_2\rangle\\ &= \alpha(W_\eta\phi_1)_i(x)+\beta(W_\eta\phi_2)_i(x)\\ &= [\alpha(W_\eta\phi_1)_i+\beta(W_\eta\phi_2)_i](x)\end{aligned}$$

Therefore $W_\eta(\alpha\phi_1+\beta\phi_2) = \alpha(W_\eta\phi_1)+\beta(W_\eta\phi_2)$. Thus W_η is linear.

Now consider

$$\begin{aligned}\|W_\eta\phi\|_{\widetilde{H}}^2 &= \int_X |(W_\eta\phi)(x)|^2\,d\nu(x)\\ &= \int_X\sum_{i=1}^n\overline{(W_\eta\phi)_i(x)}(W_\eta\phi)_i(x)\,d\nu(x)\\ &= \int_X\sum_{i=1}^n\overline{\langle\eta_x^i|\phi\rangle}\langle\eta_x^i|\phi\rangle\,d\nu(x)\end{aligned}$$

$$= \int_X \sum_{i=1}^{n} |\langle \eta_x^i | \phi \rangle|^2 \, d\nu(x)$$

$$\leq M \, \|\phi\|^2,$$

where $M > 0$ is a frame bound. Thus $\| W_\eta \| \leq M$. Therefore W_η is bounded.■

Theorem(3.2.2):

Let $W_\eta : H \rightarrow \widetilde{H}$ be as before. Then the image $W_\eta(H)$ is a closed subspace of $\widetilde{H}$.

Thus, $W_\eta(H)$ =Ran(W_η) $= H_\eta$ is itself a Hilbert space. The inner product and norm on it are given by the following theorem.

Theorem(3.2.3):

Let $W_\eta : H \rightarrow \widetilde{H}$ be as before. Then, for any $\Phi, \Psi \in Ran(W_\eta)$

$$\langle \Phi | \Psi \rangle_\eta = \langle \Phi | A_\eta^{-1} \Psi \rangle_{\widetilde{H}} \qquad (3.2.2)$$

is an inner product on H_η, and if we take

$$\| \Phi \|_\eta = \langle \Phi | \Phi \rangle^{\frac{1}{2}} \qquad (3.2.3)$$

Then $\| \bullet \|_\eta$ is a norm on H_η, where $A_\eta = W_\eta A W_\eta^{-1}$.

Proof:

The conditions for inner product and norm can easily be verified. We omit the details.

Theorem(3.2.4):

Let $W_\eta : H \rightarrow \widetilde{H}$ be as before. Then,

(i) $W_\eta(H)$ =Ran(W_η) $= H_\eta$ is closed with respect to the norm $\| \bullet \|_\eta$;

(ii) $(H_\eta, \| \bullet \|_\eta)$ is a Hilbert space ;

(iii) $\| \bullet \|_\eta$ and $\| \bullet \|_{\widetilde{H}}$ are equivalent norms ;

(iv) $W_\eta : H \rightarrow H_\eta$ is an isometry.

Theorem(3.2.5):

Consider the map $K^\eta : X \times X \rightarrow M(C^n)$ defined by $K^\eta(x, y) = \left(K_{ij}^\eta\right)_{n \times n}$ where

$$K^{\eta}_{ij}(x,y)=\left\langle \eta^{i}_{x}|A^{-1}\eta^{j}_{y}\right\rangle \tag{3.2.3}$$

$i,j = 1,2,\ldots\ldots\ldots\ldots,n$. Then $K^{\eta}_{ij}(x,y)$ is a reproducing kernel on H_{η} and H_{η} is the corresponding reproducing kernel Hilbert space. η^{i}_{x} and A are as in theorem (3.2.1).

Proof:

(i) Consider

$$\begin{aligned} K^{\eta}_{ij}(x,x) &= \langle \eta^{i}_{x}|A^{-1}\eta^{i}_{x}\rangle \\ &= \left\langle \eta^{i}_{x}\left|\left(\sum_{i=1}^{n}\int_X \left|\eta^{i}_{x}\right\rangle\left\langle\eta^{i}_{x}\right| dv(x)\right)\right.\eta^{i}_{x}\right\rangle \\ &= \sum_{i=1}^{n}\int_X \left\langle\eta^{i}_{x}|\eta^{i}_{x}\right\rangle\left\langle\eta^{i}_{x}|\eta^{i}_{x}\right\rangle dv(x) \end{aligned}$$

$$= \sum_{i=1}^{n}\int_X \left|\left\langle\eta^{i}_{x}|\eta^{i}_{x}\right\rangle\right|^{2} dv(x) > 0. \tag{3.2.4}$$

(ii)

$$\begin{aligned} K^{\eta}_{ij}(x,y) &= \left\langle \eta^{i}_{x}|A^{-1}\eta^{j}_{y}\right\rangle \\ &= \overline{\left\langle A^{-1}\eta^{j}_{y}|\eta^{i}_{x}\right\rangle} \\ &= \overline{\left\langle \eta^{j}_{y}|A^{-1}\eta^{i}_{x}\right\rangle} \quad \left[\because \text{ A is self-adjoint}\right] \end{aligned}$$

Thus,

$$K^{\eta}_{ij}(x,y) = \overline{K^{\eta}_{ji}(y,x)}. \tag{3.2.5}$$

(iii) Consider

$$\begin{aligned} &\sum_{i=1}^{n}\int_X K^{\eta}_{ik}(x,z)K^{\eta}_{kj}(z,y)\, dv(z) \\ &= \sum_{k=1}^{n}\int_X \langle\eta^{i}_{x}|A^{-1}\eta^{k}_{z}\rangle\left\langle\eta^{k}_{z}|A^{-1}\eta^{j}_{y}\right\rangle dv(z) \\ &= \langle\eta^{i}_{x}|\left(\sum_{k=1}^{n}\int_X |A^{-1}\eta^{k}_{z}\rangle\langle\eta^{k}_{z}|dv(z)\right)\left|A^{-1}\eta^{j}_{y}\right\rangle \end{aligned}$$

$$= \langle \eta_x^i | A^{-1} \sum_{k=1}^{n} \int_X |\eta_z^k\rangle\langle\eta_z^k| dv(z) \big| A^{-1}\eta_y^j \big\rangle$$
$$= \langle \eta_x^i | A^{-1} A \big| A^{-1}\eta_y^j \big\rangle$$
$$= \big\langle \eta_x^i | A^{-1}\eta_y^j \big\rangle.$$

Thus,

$$\sum_{i=1}^{n} \int_X K_{ik}^{\eta}(x,z) K_{kj}^{\eta}(z,y)\, dv(z) = K_{ij}^{\eta}(x,y). \qquad (3.2.6)$$

Thus, K^{η} is a reproducing kernel. The last property, equation (3.2.6) is said to be the reproducing property of the kernel.■

In order to make this concept more clear let us give an elementary numerical example.

Example(3.2.6):

Let $\eta_1 = (1,1,1)$, $\eta_2 = (1,1,0)$, $\eta_3 = (1,0,0)$. Here $X = \{1,2,3\}$ and $\widetilde{H} = L^2(X,\nu,C^3)$, where ν is counting measure. $\{\eta_1,\eta_2,\eta_3\}$ is clearly a frame.

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, the natural basis.

$$A = |\eta_1\rangle\langle\eta_1| + |\eta_2\rangle\langle\eta_2| + |\eta_3\rangle\langle\eta_3|$$
$$Ae_1 = \langle\eta_1|e_1\rangle\eta_1 + \langle\eta_2|e_1\rangle\eta_2 + \langle\eta_3|e_1\rangle\eta_3$$
$$= 1.(1,1,1) + 1.(1,1,0) + 1.(1,0,0)$$
$$= (3,2,1)$$

Similarly, $Ae_2 = (2,2,1)$, $Ae_3 = (1,1,1)$

Therefore

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

which is invertible and

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Now

$$K^{\eta} : X \times X \to M(C^3)$$

$$\begin{aligned} K_{11} &= \langle \eta_1 | A^{-1} \eta_1 \rangle \\ &= (1,1,1).\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= 1 \end{aligned}$$

Similarly

$$\begin{aligned} K_{12} &= \langle \eta_1 | A^{-1} \eta_2 \rangle = 0 \\ K_{13} &= \langle \eta_1 | A^{-1} \eta_3 \rangle = 0 \\ K_{21} &= \langle \eta_2 | A^{-1} \eta_1 \rangle = 0 \\ K_{22} &= \langle \eta_2 | A^{-1} \eta_2 \rangle = 1 \\ K_{23} &= \langle \eta_2 | A^{-1} \eta_3 \rangle = 0 \\ K_{31} &= \langle \eta_3 | A^{-1} \eta_1 \rangle = 0 \\ K_{32} &= \langle \eta_3 | A^{-1} \eta_2 \rangle = 0 \\ K_{33} &= \langle \eta_3 | A^{-1} \eta_3 \rangle = 1 \end{aligned}$$

Therefore

$$K^{\eta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Further, K^{η} obviously satisfies the reproducing kernel properties.

Theorem(3.2.7):

The map $E^{i}_{\eta}(x) : H_{\eta} \to C$ *given by*

$$E_\eta^i(x)\Phi = \sum_{j=1}^{n} \int_X K_{ij}^\eta(x,y)\, \Phi_j(y)\, dv(y) = \Phi_i(x) \tag{3.2.7}$$

is linear. Indeed, it is an evaluation map.

Proof:

Consider

$$\begin{aligned}
&E_\eta^i(x)(\alpha\Phi + \beta\Psi) \\
&= \sum_{j=1}^{n} \int_X K_{ij}^\eta(x,y)\, (\alpha\Phi + \beta\Psi)_j\, dv(y) \\
&= \alpha\left(\sum_{j=1}^{n} \int_X K_{ij}^\eta(x,y)\, \Phi_j(y)\, dv(y) \right) + \beta\left(\sum_{j=1}^{n} \int_X K_{ij}^\eta(x,y)\, \Psi_j(y)\, dv(y) \right) \\
&= \alpha\, \Phi_i(x) + \beta\, \Psi_i(x) \blacksquare
\end{aligned}$$

Notice that the reproducing property, (3.2.6) of the kernel has the effect of acting as the evaluation map for any vector $\Phi \in H_\eta$. We have established the existence of a reproducing kernel corresponding to a frame $F(\eta_x^i, A, n)$ and a reproducing kernel Hilbert space H_η. Let us write it down as a definition.

Definition(3.2.8):

Let $F(\eta_x^i, A, n)$ be a frame. Then the $n \times n$ matrix-valued function

$$K^\eta : X \times X \to M(C^n)$$

defined by

$$K^\eta(x,y) = \left(K_{ij}^\eta(x,y) \right)_{n \times n}$$

where

$$K_{ij}^\eta(x,y) = \left\langle \eta_x^i | A^{-1} \eta_y^j \right\rangle \tag{3.2.8}$$

is a reproducing kernel corresponding to the given frame, which we call the frame kernel.

In the next theorem we will establish the fact that a frame and its dual frame will have the same reproducing kernel. Before stating the theorem in the continuous frame sense, let us see this fact through an elementary discrete example.

Example(3.2.9):

Let $\eta_1 = (0,1)$, $\eta_2 = \left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$, $\eta_3 = \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$. It can be easily seen that $\{\eta_1,\eta_2,\eta_3\}$ is a frame in $H = C^2$. Here $e_1 = (1,0)$, $e_2 = (0,1)$, the natural basis. Now

$$A = |\eta_1\rangle\langle\eta_1| + |\eta_2\rangle\langle\eta_2| + |\eta_3\rangle\langle\eta_3|$$
$$Ae_1 = \langle\eta_1|e_1\rangle\eta_1 + \langle\eta_2|e_1\rangle\eta_2 + \langle\eta_3|e_1\rangle\eta_3$$
$$= (0,1)\bullet(1,0)(0,1) + \left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\bullet(1,0)\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\bullet(1,0)\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$
$$= \left(\frac{3}{2},0\right)$$

And

$$Ae_2 = \langle\eta_1|e_2\rangle\eta_1 + \langle\eta_2|e_2\rangle\eta_2 + \langle\eta_3|e_2\rangle\eta_3$$
$$= (0,1)\bullet(0,1)(0,1) + \left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\bullet(0,1)\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\bullet(0,1)\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$
$$= \left(0,\frac{3}{2}\right)$$

Therefore,

$$A = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

and

$$A^{-1} = \frac{4}{9}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

The reproducing kernel corresponding to the frame $\{\eta_1,\eta_2,\eta_3\}$

$$K_{11} = \langle\eta_1|A^{-1}\eta_1\rangle$$
$$= \begin{pmatrix} 0 & 1 \end{pmatrix}.\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \frac{2}{3}$$

Similarly

$$K_{12} = \langle \eta_1 | A^{-1} \eta_2 \rangle = -\frac{1}{3}$$
$$K_{13} = \langle \eta_1 | A^{-1} \eta_3 \rangle = -\frac{1}{3}$$
$$K_{21} = \langle \eta_2 | A^{-1} \eta_1 \rangle = -\frac{1}{3}$$
$$K_{22} = \langle \eta_2 | A^{-1} \eta_2 \rangle = \frac{2}{3}$$
$$K_{23} = \langle \eta_2 | A^{-1} \eta_3 \rangle = -\frac{1}{3}$$
$$K_{31} = \langle \eta_3 | A^{-1} \eta_1 \rangle = -\frac{1}{3}$$
$$K_{32} = \langle \eta_3 | A^{-1} \eta_2 \rangle = -\frac{1}{3}$$
$$K_{33} = \langle \eta_3 | A^{-1} \eta_3 \rangle = \frac{2}{3}$$

Therefore

$$K^{\eta} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Let

$$\eta_1' = A^{-1}\eta_1 = \left(0, \frac{2}{3}\right)$$
$$\eta_2' = A^{-1}\eta_2 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right)$$
$$\eta_3' = A^{-1}\eta_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{3}\right).$$

Then $\{\eta_1', \eta_2', \eta_3'\}$ is the dual frame of $\{\eta_1, \eta_2, \eta_3\}$. Now let us calculate the reproducing kernel corresponding to this dual frame.

$$\begin{aligned} K_{11}' &= \langle \eta_1' | A \eta_1' \rangle \\ &= \begin{pmatrix} 0 & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} \\ &= \frac{2}{3} \end{aligned}$$

Similarly

$$K'_{12} = \langle \eta'_1 | A\eta'_2 \rangle = -\frac{1}{3}$$
$$K'_{13} = \langle \eta'_1 | A\eta'_3 \rangle = -\frac{1}{3}$$
$$K'_{21} = \langle \eta'_2 | A\eta'_1 \rangle = -\frac{1}{3}$$
$$K'_{22} = \langle \eta'_2 | A\eta'_2 \rangle = \frac{2}{3}$$
$$K'_{23} = \langle \eta'_2 | A\eta'_3 \rangle = -\frac{1}{3}$$
$$K'_{31} = \langle \eta'_3 | A\eta'_1 \rangle = -\frac{1}{3}$$
$$K'_{32} = \langle \eta'_3 | A\eta'_2 \rangle = -\frac{1}{3}$$
$$K'_{33} = \langle \eta'_3 | A\eta'_3 \rangle = \frac{2}{3}$$

Therefore

$$K^{\eta'} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Clearly K^{η} and $K^{\eta'}$ satisfy the reproducing properties. Thus they are reproducing kernels. Further, notice that $K^{\eta} = K^{\eta'}$.

Theorem(3.2.10):

A frame and its dual frame have the same reproducing kernel.

Proof:

Let $F(\eta^i_x, A, n)$ be a given frame. As we know already its dual frame is $F(\eta'^i_x, A^{-1}, n)$, where $\eta'^i_x = A^{-1}\eta^i_x$

and

$$K^{\eta}_{ij}(x,y) = \left\langle \eta^i_x | A^{-1} \eta^j_y \right\rangle$$

is the reproducing kernel corresponding to the frame and

$$K^{\eta'}_{ij}(x,y) = \left\langle \eta'^i_x | (A^{-1})^{-1} \eta'^j_y \right\rangle$$

is the reproducing kernel corresponding to its dual frame. Now

$$\begin{aligned}
K^{\eta}_{ij}(x,y) &= \langle \eta^i_x | A^{-1}\eta^j_y \rangle \\
&= \langle AA^{-1}\eta^i_x | A^{-1}\eta^j_y \rangle \\
&= \langle A\eta'^i_x | \eta'^j_y \rangle \\
&= \langle \eta'^i_x | A\eta'^j_y \rangle \quad [\because A \text{ is self-adjoint}] \\
&= K^{\eta'}_{ij}(x,y)
\end{aligned}$$

Thus $K^{\eta'} = K^{\eta}$ ■

Theorem(3.2.11):

Let $F(\eta^i_x, A, n)$ be given and let $F\left(\tilde{\eta}^i_x, \widetilde{A}, n\right)$ be given by

$$\tilde{\eta}^i_x = \sum_{j=1}^{n} u_{ji}(x)\eta^j_x \quad ; i = 1,2,\dots\dots,n$$

where for each $x \in X$, $u(x) = (u_{ij}(x))_{n\times n}$ is a unitary matrix as in (3.1.14). *Then*

$$K^{\tilde{\eta}}(x,y) = u(x)^* K^{\eta}(x,y)u(y) \tag{3.2.9}$$

Proof:

We have

$$\widetilde{A} = \sum_{i=1}^{n} \int_X |\tilde{\eta}^i_x\rangle\langle\tilde{\eta}^i_x| \, d\nu(x)$$

Consider

$$\begin{aligned}
K^{\tilde{\eta}}_{ij}(x,y) &= \left\langle \tilde{\eta}^i_x | \widetilde{A}^{-1} \tilde{\eta}^j_y \right\rangle \\
&= \langle \tilde{\eta}^i_x | A^{-1}\tilde{\eta}^j_y \rangle \quad [\text{by theorem } (3.1.18)] \\
&= \left\langle \sum_{k=1}^{n} u_{ki}(x)\eta^k_x \mid A^{-1}\left(\sum_{l=1}^{n} u_{lj}(y)\eta^l_y\right)\right\rangle \\
&= \sum_{k,l=1}^{n} \overline{u_{ki}(x)} \langle \eta^k_x \mid A^{-1}\eta^l_y \rangle u_{lj}(y) \\
&= \sum_{k,l=1}^{n} \overline{u_{ki}(x)} K^{\eta}_{kl}(x,y) u_{lj}(y)
\end{aligned}$$

Thus $K^{\tilde{\eta}}(x,y) = u(x)^* K^{\eta}(x,y)u(y)$ ■

Definition(3.2.12):

Two frames $F(\eta_x^i, A, n)$ and $F\left(\tilde{\eta}_x^i, \tilde{A}, n\right)$ are said to be gauge equivalent if their reproducing kernels are related by (3.2.9) and $\tilde{A} = A$.

Theorem(3.2.13):

Let $F(\eta_x^i, A, n)$ and $F\left(\tilde{\eta}_x^i, \tilde{A}, n\right)$ be gauge equivalent frames. Then the reproducing kernel Hilbert space $H_{\tilde{\eta}}$ for the frame $F\left(\tilde{\eta}_x^i, A, n\right)$ consists of vectors of the type Ψ,

$$\Psi(x) = u(x)\Phi(x)\ ,\ x \in X \tag{3.2.10}$$

where $\Phi \in H_\eta$, the reproducing kernel Hilbert space for the frame $F(\eta_x^i, A, n)$.

Proof:

We have isometries,

$$W_\eta : H \to H_\eta \text{ and } W_{\tilde{\eta}} : H \to H_{\tilde{\eta}}$$

Let $\Psi \in H_{\tilde{\eta}}$. Then, $\exists \phi \in H$ such that $W_{\tilde{\eta}}\phi = \Psi$. Furthermore $\phi \in H$ implies $W_\eta \phi = \Phi \in H_\eta$, which implies

$$\begin{aligned}
\Psi_i(x) &= (W_{\tilde{\eta}}\phi)_i(x) \\
&= \langle \tilde{\eta}_x^i | \phi \rangle \\
&= \left\langle \sum_{j=1}^{n} u_{ji}(x)\eta_x^j | \phi \right\rangle \\
&= \sum_{j=1}^{n} \overline{u_{ji}(x)} \langle \eta_x^j | \phi \rangle \\
&= \sum_{j=1}^{n} \overline{u_{ji}(x)} (W_\eta \phi)_j(x) \\
&= \sum_{j=1}^{n} \overline{u_{ji}(x)} \Phi_j(x)
\end{aligned}$$

That is $\Psi(x) = \overline{u(x)}\Phi(x)$. Thus $\Psi(x) = u(x)\Phi(x)$.■

Note :

(i) The relation (3.2.10) defines a unitary map between the two reproducing kernel Hilbert spaces H_η and $H_{\tilde{\eta}}$.

(ii) notice also that (3.2.10) does not gaurantee the existence of a unitary operator U on the Hilbert space H for which

$$\tilde{\eta}'_x = U\eta'_x \text{ and } A = UAU^*$$

(iii) From (ii) we can conclude that two frames can be gauge equivalent without being unitary equivalent.

In section (3.1), definition (3.1.15) defines the unitary equivalence of the two frames. In the next theorem we will establish the connection between reproducing kernels of two unitary equivalence frames.

Theorem (3.2.14):

Let $F\left(\tilde{\eta}'_x,\tilde{A},n\right)$ and $F(\eta'_x,A,n)$ be unitary equivalent frames. Then

(*i*) $K^\eta(x,y) = K^{\tilde{\eta}}(x,y)$

(*ii*) $H_{\tilde{\eta}} = H_\eta$

Proof:

Suppose $F\left(\tilde{\eta}'_x,\tilde{A},n\right)$ and $F(\eta'_x,A,n)$ are unitary equivalent frames. Then there exist a unitary map $U \in GL(H)$ such that

$$\tilde{\eta}'_x = U\eta'_x \quad \text{and } \tilde{A} = UAU^{-1}$$

consider

$$\begin{aligned} K^{\tilde{\eta}}_{\eta}(x,y) &= \left\langle \tilde{\eta}'_x | \tilde{A}^{-1} \tilde{\eta}'_y \right\rangle \\ &= \left\langle U\eta'_x | (U^{-1})^{-1} A^{-1} U^{-1} \eta'_y \right\rangle \\ &= \langle U\eta'_x | UA^{-1} \eta'_y \rangle \\ &= \langle U^{-1} U\eta'_x | A^{-1} \eta'_y \rangle \\ &= \langle \eta'_x | A^{-1} \eta'_y \rangle \\ &= K^{\eta}_{\eta}(x,y) \end{aligned}$$

Thus $K^\eta(x,y) = K^{\tilde{\eta}}(x,y)$. Now let us prove $H_{\tilde{\eta}} = H_\eta$. We have isometries

$$W_\eta : H \to H_\eta \quad \text{and} \quad W_{\tilde{\eta}} : H \to H_{\tilde{\eta}}$$

defined respectively by

$$(W_\eta\phi)_i(x) = \langle \eta_x^i | \phi\rangle \text{ and } \left(W_{\tilde\eta}\phi\right)_i(x) = \langle \tilde\eta_x^i | \phi\rangle$$

Let $\Phi \in H_{\tilde\eta}$, then

$$\begin{aligned}
\Phi_i(x) &= \left(W_{\tilde\eta}\phi\right)_i(x) \text{ for some } \phi \in H \\
&= \langle \tilde\eta_x^i | \phi\rangle \\
&= \langle U\eta_x^i | \phi\rangle \\
&= \langle \eta_x^i | U^{-1}\phi\rangle \\
&= \Psi_i(x) \text{ for some } \Psi \in H_\eta \text{ because } U^{-1}\phi \in H
\end{aligned}$$

Thus $\Phi \in H_\eta$, which proves $H_{\tilde\eta} \subset H_\eta$. Further U^* is invertible therefore we can establish $H_\eta \subset H_{\tilde\eta}$ in a similar fashion. Thus $H_{\tilde\eta} = H_\eta$■

The results of theorem (3.2.14) will hold even if we replace the unitary operator U by any other operator $T \in GL(H)$.

Theorem(3.2.15):

Suppose two frames $F\left(\tilde\eta_x^i, \widetilde{A}, n\right)$ *and* $F(\eta_x^i, A, n)$ *are related by* $\tilde\eta_x^i = T\eta_x^i$ *and* $\widetilde{A} = TAT^*$. *Then*

(*i*) $K^\eta(x,y) = K^{\tilde\eta}(x,y)$

(*ii*) $H_{\tilde\eta} = H_\eta$

This connection is called similarity relation.

Proof:

Similar to theorem (3.2.14).■

Definition(3.2.16):

Two frames are said to be kernel equivalent if their kernels are gauge related in the sense of (3.2.9).

Theorem(3.2.17):

Gauge equivalent frames are kernel equivalent.

Proof:

Obvious.■

From the previous discussion we can easily see that the converse need not be true. The following theorem also makes this clear.

Theorem(3.2.18):

Let $F\left(\tilde{\eta}^i_x,\tilde{A},n\right)$ and $F(\eta^i_x,A,n)$ be two frames for which

$$\tilde{\eta}^i_x=\sum_{j=1}^{n} u_{ij}(x)T\eta^i_x \text{ and } \tilde{A}= TAT^* \tag{3.2.11}$$

holds, where $u(x)$ is an $n\times n$ unitary matrix for each $x\in X$ and $T\in GL(H)$ if and only if the frames are kernel equivalent.

Proof:

Consider

$$\begin{aligned} K^{\tilde{\eta}}_{ij}(x,y) &= \left\langle \tilde{\eta}^i_x \middle| \tilde{A}^{-1}\tilde{\eta}^j_y\right\rangle \\ &= \left\langle \sum_{k=1}^{n} u_{ki}(x)T\eta^k_x \middle| (T^*)^{-1}A^{-1}T^{-1}\sum_{l=1}^{n} u_{lj}(y)T\eta^l_y\right\rangle \\ &= \sum_{k,l=1}^{n}\overline{u_{ki}(x)}\langle T^{-1}T\eta^k_x|A^{-1}\eta^l_y\rangle u_{lj}(y) \\ &= \sum_{k,l=1}^{n}\overline{u_{ki}(x)}\langle \eta^k_x|A^{-1}\eta^l_y\rangle u_{lj}(y) \\ &= \sum_{k,l=1}^{n}\overline{u_{ki}(x)}K^{\eta}_{kl}(x,y)u_{lj}(y) \end{aligned}$$

Thus $K^{\tilde{\eta}}(x,y) = u(x)^* K^{\eta}(x,y)u(y)$ i.e. $K^{\tilde{\eta}}$ and K^{η} are guage related. Thus the frames are kernel equivalent.

Conversly, suppose that the frames $F\left(\tilde{\eta}^i_x,\tilde{A},n\right)$ and $F(\eta^i_x,A,n)$ are kernel equivalent. Therefore their kernels are guage related, i.e, there exist an $n\times n$ unitary matrix $u(x)$ such that $K^{\tilde{\eta}}(x,y) = u(x)^* K^{\eta}(x,y)u(y)$ which implies

$$\left\langle \tilde{\eta}^i_x \middle| \tilde{A}^{-1}\tilde{\eta}^j_y\right\rangle = \sum_{k,l=1}^{n}\overline{u_{ki}(x)}u_{lj}(y)\langle \eta^k_x|A^{-1}\eta^l_y\rangle \tag{3.2.12}$$

Thus

$$\sum_{i=1}^{n}\int_X |\tilde{\eta}^i_x\rangle\left\langle \tilde{\eta}^i_x \middle| \tilde{A}^{-1}\tilde{\eta}^j_y\right\rangle d\nu(x) = \sum_{i,k,l=1}^{n}\int_X \overline{u_{ki}(x)}u_{lj}(y)|\tilde{\eta}^i_x\rangle\langle \eta^k_x|A^{-1}\eta^l_y\rangle d\nu(x)$$

But

$$\sum_{i=1}^{n}\int_X |\tilde{\eta}^i_x\rangle\langle\tilde{\eta}^i_x| d\nu(x) = \tilde{A}$$

Therefore

$$\tilde{\eta}^j_y = \sum_{l=1}^{n} u_{lj}(y) T\eta^l_y \tag{3.2.13}$$

where

$$T = \sum_{i,k,l=1}^{n}\int_X \overline{u_{ki}(x)}|\tilde{\eta}^i_x\rangle\langle\eta^k_x|A^{-1} d\nu(x)$$

which is clearly bounded. Further (3.2.13) gives $\tilde{A} = TAT^*$. Now it is enough to show that T is invertible. Since for all $\Phi, \Psi \in H_K$, reproducing kernel Hilbert space ($H_\eta = H_{\tilde{\eta}} = H_K$). set $W_\eta\phi = \Phi_\eta$, $W_\eta\psi = \Psi_\eta$

$$\langle\phi|T\psi\rangle = \sum_{i,k,=1}^{n}\int_X \overline{u_{ki}(x)}\langle\phi|\tilde{\eta}^i_x\rangle\langle\eta^k_x|A^{-1}\psi\rangle d\nu(x)$$

$$= \sum_{i,k,=1}^{n}\int_X \overline{u_{ki}(x)(W_{\tilde{\eta}}\phi)_i(x)}A^{-1}(W_\eta\psi)_i(x) d\nu(x) \text{ because of (3.2.1), } A \text{ is self adjoint.}$$

$$= \sum_{i,k,=1}^{n}\int_X \overline{\Phi_{\tilde{\eta}i}(x)u_{ki}(x)}(A_\eta^{-1}\Phi_\eta)_k(x) d\nu(x) \text{ because } A_\eta = W_\eta A W_\eta^{-1} \tag{3.2.14}$$

Since $\Phi_{\tilde{\eta}i}(x)$ and $\sum_{k=1}^{n} u_{ki}(x)(A_\eta^{-1}\Phi_\eta)_k(x)$, $i = 1,\ldots\ldots,n$ defines vectors in $L^2(X,\nu,C^n)$. the integral (3.2.14) always converges. Further $u(x)$ is unitary. therefore (3.2.12) can be written as.

$$\left\langle\eta^i_x|A^{-1}\eta^j_y\right\rangle = \sum_{k,l=1}^{n} u_{ik}(x)\overline{u_{lj}(y)}\left\langle\tilde{\eta}^k_x|\tilde{A}^{-1}\tilde{\eta}^l_y\right\rangle \tag{3.2.15}$$

By setting

$$T' = \sum_{i,k,=1}^{n}\int_X u_{ik}(x)|\eta^i_x\rangle\langle\tilde{\eta}^k_x|\tilde{A}^{-1} d\nu(x) \tag{3.2.16}$$

We can write from (3.2.15).

$$\eta_y^i = \sum_{l=1}^{n} \overline{u_{lj}(y)} T' \overline{\eta}_y^l \qquad (3.2.17)$$

Now $\{\eta_x^i | i = 1,\ldots\ldots, n, x \in X\}$ and $\{\overline{\eta}_x^i | i = 1,\ldots\ldots, n, x \in X\}$ are total in H. Therefore (3.2.13) and (3.2.17) gives $T' = T^{-1}$. Thus T is invertible.■

Now let $\Im$ is the set of all n-rank continuous frames in H. Define a relation ~on $\Im$ by

$$F_1 \sim F_2 \Leftrightarrow F_1 \text{ and } F_2 \text{ are kernel equivalent frames.}$$

Then ~is an equivalent relation and for a fixed $F_0 \in \Im$ we can define the equinalence class of F_0 as

$$[F_0] = \{F \in \Im | F \text{ and } F_0 \text{ are kernel equivalent}\}$$

Definition(3.2.19):

An equivalence class $[F_0]$ is said to be self-dual if given $F_1 \in [F_0]$ there exist $F_2 \in [F_0]$ such that F_2 is the dual of F_1.

Theorem (3.2.20):

Each class of kernel equivalent frame is self-dual

Proof:

Fix $F_0(\eta_x^i, A, n)$ and set $\eta_x'^i = A^{-1}\eta_x^i$ then $F(\eta_x'^i, A^{-1}, n) \in [F_0(\eta_x^i, A, n)]$. For,

$$\begin{aligned} K_{ij}^{\eta'}(x,y) &= \langle \eta_x'^i | (A^{-1})^{-1} \eta_y'^j \rangle \\ &= \langle A^{-1}\eta_x^i | A A^{-1} \eta_y^j \rangle \\ &= \langle \eta_x^i | A^{-1} \eta_y^j \rangle \text{ because } A \text{ is self-dual} \\ &= K_{ij}^{\eta}(x,y) \end{aligned}$$

i.e. $K^{\eta'}(x,y) = u(x)^* K^{\eta}(x,y) u(y)$ where $u(x) = u(y) = I_n$, the identity. Thus $F_0(\eta_x^i, A, n)$ and $F(\eta_x'^i, A^{-1}, n)$ are kernel equivalent and clearly $F(\eta_x'^i, A^{-1}, n)$ is the dual of $F_0(\eta_x^i, A, n)$. Therefore $[F_0(\eta_x^i, A, n)]$ is self-dual.■

Theorem(3.2.21):

Each kernel equivalent class contains a unique subclass of self-dual tight frames, which can be *generated by the joint action , on any fixed member of this subclass, of the groups $U(n)$ and $U(H)$, where $U(n)$, $n \times n$ complex unitary matrices and $U(H)$, unitary operators on H.*

Proof:

Fix $F(\eta_x^i, A, n)$, an n-rank frame. Consider the frame $F(\bar{\eta}_x^i, I, n)$ with $\bar{\eta}_x^i = A^{-\frac{1}{2}}\eta_x^i$. Then $F(\bar{\eta}_x^i, I, n)$ is a self-dual and tight frame(by theorem(3.1.13)) and $F(\bar{\eta}_x^i, I, n) \in [F(\eta_x^i, A, n)]$. Now

$$\begin{aligned} K_{ij}^{\bar{\eta}}(x,y) &= \langle \bar{\eta}_x^i | I^{-1} \bar{\eta}_y^j \rangle \\ &= \langle A^{-\frac{1}{2}}\eta_x^i | A^{-\frac{1}{2}}\eta_y^j \rangle \\ &= \langle \eta_x^i | A^{-1}\eta_y^j \rangle \text{ because } A \text{ is self-adjoint} \\ &= K_{ij}^{\eta}(x,y) \end{aligned}$$

Further

$$\bar{\bar{\eta}}_x^i = \sum_{j=1}^{n} u_{ji}(x) U \bar{\eta}_x^j$$

where $u(x) \in U(n)$ and $U \in U(H)$, generate a subclass consists of self-dual tight frame $F(\bar{\bar{\eta}}_x^i, I, n)$. i.e.

$$\mathfrak{I} = \left\{ F(\bar{\bar{\eta}}_x^i, I, n) | \bar{\bar{\eta}}_x^i = \sum_{j=1}^{n} u_{ji}(x) U \bar{\eta}_x^j,\ u(x) \in U(n) \text{ and } U \in U(H), x \in X \right\}$$

is a class of self-dual tight frames. Now let us show $\mathfrak{I} \subset [F(\eta_x^i, A, n)]$. Let $F(\bar{\bar{\eta}}_x^i, I, n) \in \mathfrak{I}$ then

$$\begin{aligned} K_{ij}^{\bar{\bar{\eta}}}(x,y) &= \langle \bar{\bar{\eta}}_x^i | I^{-1} \bar{\bar{\eta}}_y^j \rangle \\ &= \left\langle \sum_{k=1}^{n} u_{ki}(x) U \bar{\eta}_x^k \ | \ \sum_{l=1}^{n} u_{lj}(y) U \bar{\eta}_y^l \right\rangle \\ &= \sum_{k,l=1}^{n} \overline{u_{ki}(x)} \langle U\bar{\eta}_x^k \ | \ U\bar{\eta}_y^l \rangle u_{lj}(y) \\ &= \sum_{k,l=1}^{n} \overline{u_{ki}(x)} \langle \bar{\eta}_x^k \ | \ \bar{\eta}_y^l \rangle u_{lj}(y) \text{ because } U \in U(H) \text{ is unitary.} \\ &= \sum_{k,l=1}^{n} \overline{u_{ki}(x)} \langle \eta_x^k | \ A^{-1}\eta_y^l \rangle u_{lj}(y) \end{aligned}$$

Therefore, $K^{\bar{\bar{\eta}}} = u^* K^{\eta} u$ which means $K^{\bar{\bar{\eta}}}$ and K^{η} are guage related. Thus, $\mathfrak{I} \subset [F(\eta_x^i, A, n)]$ which completes the proof.■

Definition(3.2.22):

Two frames $F(\eta_x^i, A, n)$ and $F\left(\tilde{\eta}_x^i, \tilde{A}, n\right)$ are said to be boundle equivalent if there exist a rank-n operator $T(x)$, $x \in X$, on H such that

$$\tilde{\eta}_x^i = \sum_{j=1}^{n} u_{ji}(x) T(x) \eta_x^j \tag{3.2.18}$$

where $u(x)$ is an $n \times n$ unitary matrix.

Note that we do not impose any connection on A and $\tilde{A}$. So this kind of equivalence is different from kernel equivalence. Indeed, we do not necessarily have any connection between the corresponding reproducing kernels.

Theorem(3.2.24):

Two boundle equivalent frames $F(\eta_x^i, A, n)$ and $F\left(\tilde{\eta}_x^i, \tilde{A}, n\right)$ are kernel equivalent if there exists $T \in GL(H)$ such that

$$T(x) = TP(x) \text{ and } \tilde{A} = TAT^{*}, \tag{3.2.19}$$

where $P(x)$ is the projection operator onto the range of $F(x)$, the operator as in theorem (3.1.18).

Proof:

Consider

$$\begin{aligned} K_{ij}^{\tilde{\eta}}(x,y) &= \left\langle \tilde{\eta}_x^i | \tilde{A}^{-1} \tilde{\eta}_y^j \right\rangle \\ &= \left\langle \sum_{k=1}^{n} u_{ik}(x) T(x) \eta_x^k | (T^{*})^{-1} A^{-1} T^{-1} \sum_{l=1}^{n} u_{jl}(y) T(y) \eta_y^l \right\rangle \\ &= \sum_{k,l=1}^{n} \overline{u_{ik}(x)} u_{jl}(y) \left\langle T(x) \eta_x^k | (T^{*})^{-1} A^{-1} T^{-1} T(y) \eta_y^l \right\rangle \\ &= \sum_{k,l=1}^{n} \overline{u_{ik}(x)} u_{jl}(y) \left\langle TP(x) \eta_x^k | (T^{*})^{-1} A^{-1} T^{-1} TP(y) \eta_y^l \right\rangle \end{aligned}$$

$$= \sum_{k,l=1}^{n} \overline{u_{ik}(x)}u_{jl}(y)\langle T^{-1}TP(x)\eta_x^k | A^{-1}P(y)\eta_y^l\rangle$$

$$= \sum_{k,l=1}^{n} \overline{u_{ik}(x)}u_{jl}(y)\langle P(x)\eta_x^k | A^{-1}P(y)\eta_y^l\rangle$$

$$= \sum_{k,l=1}^{n} \overline{u_{ik}(x)}u_{jl}(y)\langle \eta_x^k | A^{-1}\eta_y^l\rangle$$

because $P(x) : H \to F(x)H$ is the projection operator and $\{\eta_x^k | k = 1, ..., n\}$ are linearly independent (see the proof of theorem(3.1.18)). Thus $K^{\tilde{\eta}} = u(x)^* K^{\eta} u(y)$, i.e. the kernels are guage related. Therefore the frames are kernel equivalent, which completes the proof.■

Note:

In order to summarize the equivalencies let us build the following diagram

Unitary equivalence

$\tilde{\eta}'_x = U\eta'_x$

$\tilde{A} = UAU^{-1}$

$\tilde{K}(x,y) = K(x,y)$

↘

Kernel equivalence

$\tilde{\eta}'_x = \sum_{j=1}^{n} u_{ji}(x)T\eta^j_x$

$\tilde{A} = TAT^*$

$\tilde{K}(x,y) = u(x)^* K(x,y)u(y)$

→

Boundle equivalence

$\tilde{\eta}'_x = \sum_{j=1}^{n} u_{ji}(x)T(x)\eta^j_x$

↗

Gauge equivalance

$\tilde{\eta}'_x = \sum_{j=1}^{n} u_{ji}(x)\eta^j_x$

$\tilde{A} = A$

$\tilde{K}(x,y) = u(x)^* K(x,y)u(y)$

Chapter-4

General construction of frames

In this chapter we will give a recipe for the general construction of all possible rank-n frames on an abstract Hilbert space H. In the previous chapter we saw how a given rank-n frame on an abstract Hilbert space H is associated to a reproducing kernel and its reproducing kernel Hilbert space. Here, in this chapter we will see that the general construction is heavily dependent on reproducing kernel Hilbert spaces. But , once again, the results on reproducing kernels, which we use for the purpose, will only be stated here. The details and the proofs of those results can be found in the references given in chapter-2. The notations such as $\widetilde{H} = L^2(X, \nu, C^n)$ will be as usual.

The following theorem tells us how a reproducing kernel Hilbert space can be constructed on $\widetilde{H}$.

Theorem (4.1.1):

If $H_K \subset \widetilde{H}$ is a reproducing kernel Hilbert space then every orthonormal basis $\{\Phi_k\}_{k=1}^{r}$ in H_K satisfies

$$(i)\ \sum_{k=1}^{r} \|\Phi_k(x)\|_n^2 = \|K(x,x)\| < \infty \quad \forall x \in X;$$

(ii) For each $x \in X$, the set of vectors $\{\Phi_k(x)\}_{k=1}^{r}$ spans C^n

Conversely, if $H_0 \subset \widetilde{H}$ is a closed subspace which has an orthonormal basis of functions $\Phi_k : X \to C^n$, $n = 1, 2, \ldots \infty$, satisfying

$$(i)\ \sum_{k=1}^{r} \|\Phi_k(x)\|_n^2 < \infty \quad \forall x \in X;$$

(ii) For each $x \in X$, the set of vectors $\{\Phi_k(x)\}_{k=1}^{\infty}$ spans C^n

Then H_0 is identifiable with a reproducing kernel Hilbert space consisting of functions $\Psi : X \to C^n$.

Now let us answer the following question. How do we construct all possible rank-n frames ? The following recipe is going to give the answer to constructing all possible rank-n frames in an abstract Hilbert space H.

STEP-1:Chose an arbitrary locally compact space X and a Borel measure ν on it with

$\text{supp}(\nu) = X$.

STEP-2:Construct the Hilbert space $\widetilde{H} = L^2(X,\nu,C^n)$ and make sure that $\dim\left(\widetilde{H}\right) \geq \dim(H)$

STEP-3:Choose a reproducing kernel subspace H_k of $\widetilde{H}$ with the property $\dim H_k = \dim H = N$ (say). (see theorem (4.1.1) or theorem (4.1.3)). That is, find an orthonormal basis Φ_k, $k = 1,2,........,N$ in $\widetilde{H}$ such that (i), (ii) of theorem (3.1.1) satisfied.

STEP-4:Take an arbitrary orthonormal basis $\{\phi_k\}_{k=1}^{N}$ in H

STEP-5:Take an arbitrary operator $T \in GL(H)$.

Now the following theorem tells us how frames arise.

Theorem(4.1.2):

The vectors

$$\eta_x^i = \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} T\phi_k \tag{4.1.1}$$

define a rank-n frame, $F(\eta_x^i, TT^*, n)$, *in H. Its frame kernel,*

$$K_{ij}^{\eta}(x,y) = \langle \eta_x^i | (TT^*)^{-1} \eta_y^j \rangle \tag{4.1.2}$$

is just the defining kernel K of H_k, *expressed in the canonical basis* $\{e_i\}_{i=1}^{n}$ *of* C^n *that is*

$$K_{ij}^{\eta}(x,y) = K_{ij}(x,y) = \sum_{k=1}^{N} \Phi_{k,i}(x) \overline{\Phi_{k,j}(y)} \tag{4.1.3}$$

Proof:

We have $\{\phi_k\}_{k=1}^{N}$ an orthonormal basis of H and $\Phi_{k,i}(x) \in C^n$ $\forall$ k,i and $x \in X$. So consider the vectors formed by the linear combinations of ϕ_k,

$$\eta_x^i = \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \phi_k$$

Then

$$\|\eta_x^i\|^2 = \left\langle \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \phi_k \;\middle|\; \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \phi_k \right\rangle$$

$$= \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)}\, \Phi_{k,i}(x)\, \langle \phi_k \mid \phi_k \rangle$$

$$= \sum_{k=1}^{N} |\Phi_{k,i}(x)|^2 \text{ because } \langle \phi_k \mid \phi_k \rangle = 1. \tag{4.1.4}$$

Further T is linear,so

$$T\eta_x^i = \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)}T\phi_k = \eta_x^i. \tag{4.1.5}$$

Now consider

$$\sum_{i=1}^{n} \int_X |\eta_x^i\rangle\langle\eta_x^i| dv(x) = A.$$

Thus,

$$\langle \phi_k \mid A\phi_l \rangle = \sum_{i=1}^{n} \int_X \langle \phi_k|\eta_x^i\rangle\langle\eta_x^i|\phi_l\rangle dv(x).$$

But

$$\langle \phi_k|\eta_x^i\rangle = \left\langle \phi_k \mid \sum_{j=1}^{N} \overline{\Phi_{j,i}(x)}\phi_j \right\rangle = \Phi_{k,i}(x)$$

$$\langle \eta_x^i|\phi_l\rangle = \left\langle \sum_{j=1}^{N} \overline{\Phi_{j,i}(x)}\phi_j \mid \phi_l \right\rangle = \overline{\Phi_{l,i}(x)}.$$

Therefore,

$$\langle \phi_k \mid A\phi_l \rangle = \sum_{i=1}^{n} \int_X \Phi_{k,i}(x)\overline{\Phi_{l,i}(x)} dv(x).$$

Furthermore the orthogonality of $\{\Phi_k\}_{k=1}^{N}$ gives $\langle \Phi_k|\Phi_l\rangle = \delta_{kl}$. Thus, $\langle \phi_k \mid A\phi_l \rangle = \delta_{kl}$,which gives us $A = I$, the identity. So we got a self-dual tight frame, $F(\eta_x^i, I, n)$. The corresponding kernel is,

$$K_{ij}^{\eta}(x,y) = \langle \eta_x^i | I^{-1} \eta_y^j \rangle = \langle \eta_x^i | \eta_y^j \rangle = \left\langle \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \phi_k | \sum_{k=1}^{N} \overline{\Phi_{k,j}(y)} \phi_k \right\rangle$$

$$= \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \Phi_{k,j}(y) \text{ because } \{\phi_k\} \text{ is orthonormal.} \qquad (4.1.6)$$

Now the reproducing kernel corresponding to the frame $F(\eta_x^i, TT^*, n)$ is

$$\begin{aligned}
K_{ij}^{\eta}(x,y) &= \langle \eta_x^i | (TT^*)^{-1} \eta_y^j \rangle \\
&= \langle T\eta_x^i | (TT^*)^{-1} T\eta_y^i \rangle \\
&= \langle T\eta_x^i | T^{*-1} T^{-1} T\eta_y^i \rangle \\
&= \langle T\eta_x^i | T^{*-1} \eta_y^i \rangle \\
&= \langle \eta_x^i | T^* T^{*-1} \eta_y^i \rangle \\
&= \langle \eta_x^i | \eta_y^i \rangle \\
&= \sum_{k=1}^{N} \overline{\Phi_{k,i}(x)} \Phi_{k,j}(y), \text{ by (4.1.6)}
\end{aligned}$$

which completes the proof.■

Note:

Theorems (3.2.20) and (3.2.21) guarantee that the above construction gives all frames in H which are kernel equivalent to $F(\eta_x^i, I, n)$.

The above construction completely depends on finding reproducing kernel subspaces H_K of $\widetilde{H}$. In order to find it one may not start with an orthonormal set of vectors $\{\Phi_k\}_{k=1}^{N}$ in $\widetilde{H}$. Instead we may use the following theorem.

Theorem (4.1.3) :

$H_0 \subset \widetilde{H}$ is a reproducing kernel Hilbert space if and only if there exist a basis $\{\Psi_k\}_{k=1}^{N}$ in it, consisting of functions $\Psi_k : X \to C^n$ such that

(i) $a\|\Psi\|_{\widetilde{H}}^2 \le \sum_{k=1}^{N} |\langle \Psi_k | \Psi \rangle|^2 \le b\|\Psi\|_{\widetilde{H}}^2 \quad \forall \Psi \in H_0,\ a,b > 0;$

(ii) $\sum_{k=1}^{N} \|\Psi_k(x)\|_n^2 < \infty \quad \forall x \in X;$

(iii) $\{\Psi_k(x)\}_{k=1}^{N}$ *spans* C^n $\forall x \in X$.

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