The Effect of Students' Physics Background on their Understanding of Linear Algebra

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master in the Teaching of Mathematics
Concordia University
Montréal, Québec, Canada

August 2000

0-612-54288-2

ABSTRACT

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In the hope of reducing the level of abstraction with the concepts of vectors, linear combinations and spanning, transformations and linear transformations in the teaching of Linear Algebra, a geometric approach was taken. Activities in the aforementioned areas were designed in a dynamic computer environment Cabri Geometry II that allowed students to interact with vectors in a two-dimensional coordinate free vector space. Some of these activities embodied the notion of vectors in physical situations such as systems of pulleys and weights being suspended from a string. Unfortunately these representations proved problematic for a group of students having a working knowledge of elementary physics. Therefore, the aim of this thesis is to understand the ways that experience with physics knowledge interferes with the acquisition of knowledge in Linear Algebra. It considered two groups of students, those with physics experience and those without such experience. The topics that gave difficulty to both groups of students were attributed to the complex nature of Linear Algebra and those that were different were considered to be specific to a background in physics.

ACKNOWLEDGMENTS

I would like to thank Dr. A. Sierpinska for her tremendous help in making this thesis possible. Her input, understanding, patience, and constant encouragement proved to be a source of inspiration.

I would also like to thank Dr. J. Hillel and Dr. A. Oktac for taking the time to review this thesis. Your constructive criticism was greatly appreciated.

I would also like to say thank you to Dr. R. Thompson, T. Zaki, A. Douglas, G. Grant, E. Saidah and M. Kingsley for lending me a listening ear when I needed one and for your technical support.

I would also like to say a very special thank you to my family. Your smiles and laughter kept me going. Thank you.

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Introduction

The aim of this thesis is to understand the ways that experience with physics knowledge affects the acquisition of skills and knowledge in linear algebra. Linear algebra has many complex abstract mathematical concepts (i.e. vectors, vector spaces, notion of solution, and spanning sets) that prove difficult to teach to students effectively. For this reason, it is important to understand the reasons for this difficulty, and the ways in which it can be circumvented. Thus, the thesis has important implications for the ways that linear algebra is taught to students, particularly students with relevant previous academic experiences.

In Chapter I, the reasons why students have problems with learning linear algebra will be discussed generally. As well, the specific ways that physics knowledge may potentially interfere with this acquisition will be examined. Under the best of conditions, it is frequently very difficult for students to obtain a generalized knowledge and skill with the use of linear algebra. The problems these students face appear to be, at least in part, inherent to the problem of learning this difficult set of concepts. There are also, however, ways in which previous experiences of the students may interfere with the acquisition of linear algebra knowledge and skills. In particular, past experience with physics may predispose students to specific problems grasping linear algebra concepts. This is primarily because students have acquired problem-solving skills that are useful in physics but frequently misleading or counterproductive when applied in the setting of linear algebra.

In Chapter II, the methods and results of the study are described. The study is described in detail; two groups of students attempting to learn linear algebra are compared. One group of students had previous experience with the concepts of vectors from physics. The other group had no such experience. The results of this study are also reported in Chapter II. Both sets of students exhibited a number of problems acquiring the linear algebra knowledge in question, suggesting that some problems in learning linear algebra are inherent to the process. There were, however, specific problems found only in the group of students with previous physics experience, suggesting that previous experience with physics can have a detrimental effect on learning, certain concepts in linear algebra.

Chapter III proposes some implications of this research for the teaching of linear algebra.

Chapter I

Difficulty Students Encounter in Learning Linear Algebra and the Concept of Vector

Introduction

Because of its numerous applications, the study of Linear Algebra has become a vital subject for undergraduate students majoring in social/pure sciences. Linear Algebra offers a powerful computational tool; matrix theory and vector space concept allow many complex problems to be modeled and discussed in linear terms. For example, static mechanics concerned with the equilibrium of rigid bodies use vectors as models of forces to determine some net force that may act on a particular body, and forest management problems use matrix model to find optimal solutions of a periodic harvest yield. As well, Linear Algebra is often studied as an abstract mathematical theory for its own value.

The aim of this chapter is to report some of the difficulty students encounter when trying to learn Linear Algebra, the concept of vectors as they relate to physics, and obstacles within the learning process that appear to be unavoidable. This chapter is separated into three section: (1) Students' difficulty in three areas of Linear Algebra, (2) the concept of vectors in physics, and (3) obstacles that impede the learning of new concepts.

The first section examines three areas that are essential to the learning of linear algebra: symbolic representations, generalization of linear systems, and the notion of vector space, and then reports difficulties experienced by some students when learning to become proficient in these areas.

The second section reports some of the beliefs that students develop about force and motion prior to formal instruction, and the difficulties that they encounter when working with vectors and their properties.

The third section takes a psychological approach to learning in general. It reviews four learning stages: acquisition, proficiency, maintenance, generalization, and also looks at obstacles such as affective, cognitive and didactic that some students encounter.

1.1 Students' Difficulty in Three Areas of Linear Algebra

The main topics covered in elementary linear algebra courses are systems of linear equations, matrices, determinants and vector spaces. Students normally exhibit very few difficulties in performing matrix operations on matrices with numerical entries, computing the numerical value of a determinant, or using the technique of Gaussian elimination to solve a system of linear equations. Difficulties arise, however, when students are asked to interpret the outcome of the Gaussian elimination process on a system of linear equations and generalize this experience to higher dimensions and an arbitrary system. Even greater difficulties are encountered in understanding the nature of vector spaces. For these reasons some college students consider linear algebra to be the most abstract and rigorous math course they take in their college careers.

1.1.1 *Difficulties with different symbolic representations*

Students enrolled in an elementary linear algebra course are often baffled by the many different modes of representations available in this domain (Dias & Artigue, 1995) and the simultaneous geometrical and algebraic settings in which the course is taught (Hillel, 1997). For example, a vector can be represented in several ways: by a directed line segment (see Figure 1.1a), as an ordered n-tuple (b), as n x 1 matrix (c), or as a linear combination of other vectors (d).

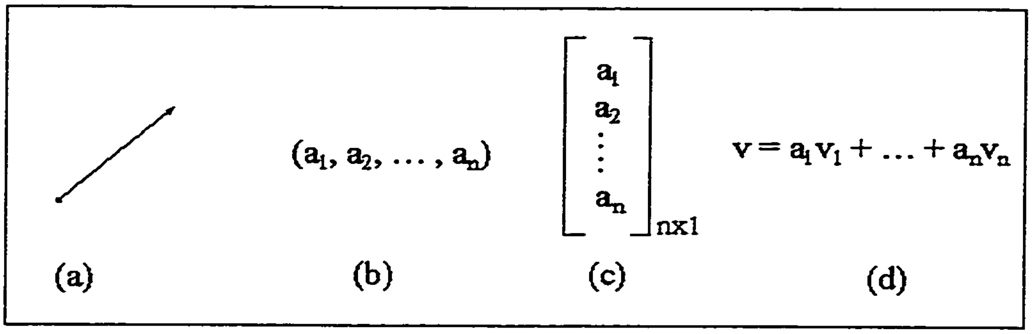

Figure 1.1. Different representations of vectors

Thus, if the vector is considered a mathematical object, it can be thought of geometrically as a directed line segment or a point in a coordinate system as well as analytically as an n-tuple. Thus, switching between the geometrical and algebraic domains is an extensive part of linear algebra. These domains are not equivalent: each can lead to a different formulation of a problem and different results, techniques, and the creation of different mathematical objects. Knowing when to switch from one domain to another and recognizing the similarities that exist between the domains is a major source of difficulty for students.

In considering the most popular equation in linear algebra, $Ax = b$, where A is an m x n matrix some students also experience difficulties recalling that it, too, can be represented in three different but equivalent ways (Lay, 1994). For example this equation can be viewed as a system of linear equations (see Figure 1.2a), a linear combination of vectors (2b), and as a matrix equation (2c). These equivalent representations of the equation $Ax = b$ allow us to conveniently change a given problem into a more suitable from. Students are often quite good in writing the matrix representation of a linear system and writing a system of linear equations from a linear combination of vectors.

However, they seem to experience difficulties in moving between the matrix equation and the vector equation.

(a) System of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

(b) Vector equation

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(c) Matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Figure 1.2. Equivalent representations of the equation $Ax = b$

In the course of studying linear equations, students are acquainted with matrix representations using the augmented and/or coefficient matrix. These matrices are represented symbolically as [A | b] and [A] respectively. It has been my experience that students have difficulties remembering the difference between these two matrices, which leads to erroneous solutions and mistaken interpretations of the row reduction process.

Some students pay little attention to notation. For example, they would see the symbol |A| as synonymous with [A] using the former in two meanings, as determinant and coefficient matrix, because it is easier to write. These students are under the impression that the determinant symbol is a shorthand notation of writing the coefficient matrix. This notational confusion leads to problems when the students are asked to calculate the determinant of a matrix, and the exercise is formulated as follows:

Calculate the determinant

$$\begin{vmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 0 & 7 & 0 \\ 4 & -6 & 7 & -3 \end{vmatrix}.$$

First year students of linear algebra sometimes respond to the problem by first reducing the determinant to row-echelon form by using elementary row operations. This reduces the amount of computations needed to calculate the determinant, since it introduces zeros in its equivalent system. The students then seem to forget that they are calculating the value of a determinant and start solving the following homogenous system of linear equations, by row reduction

row reduction $$\left[\begin{array}{cccc|c} -2 & 8 & 1 & 4 & 0 \\ 3 & 2 & 5 & 1 & 0 \\ 1 & 0 & 7 & 0 & 0 \\ 4 & -6 & 7 & -3 & 0 \end{array}\right].$$

A common problem at all levels of mathematics education is that some students only pay attention to numbers and procedures and do not read the text of the problem carefully.

1.1.2 *Difficulties with generalizing from small linear systems to solving any linear system*

Another difficulty students encounter in learning linear algebra is that they are expected to generalize from solving systems of linear equations in R^2 and R^3 to arbitrary systems without extensive preparation (Harel, 1989). For example, the solution set of an arbitrary 2 x 2 system of linear equations:

$$a_{11}x + a_{12}y = b_1 \quad \text{(where } a_{11} \text{ and } a_{12} \text{ are not both zero)}$$
$$a_{21}x + a_{22}y = b_2 \quad \text{(where } a_{21} \text{ and } a_{22} \text{ are not both zero)}$$

can be described geometrically. The graphs of these equations are lines, say L_1 and L_2, respectively. A solution that satisfies both equations simultaneously corresponds to points of intersection of L_1 and L_2. Thus, there are three possible configurations: the lines L_1 and L_2 are parallel and distinct thus resulting in no intersection point and no solution (see Figure 1.3a). The two lines intersect at exactly one common point thus yielding a unique solution (see Figure 1.3b). The lines L_1 and L_2 coincide, thus yielding many points of intersection with an infinite number of solutions (see Figure 1.3c) (Anton & Rorres, 1991).

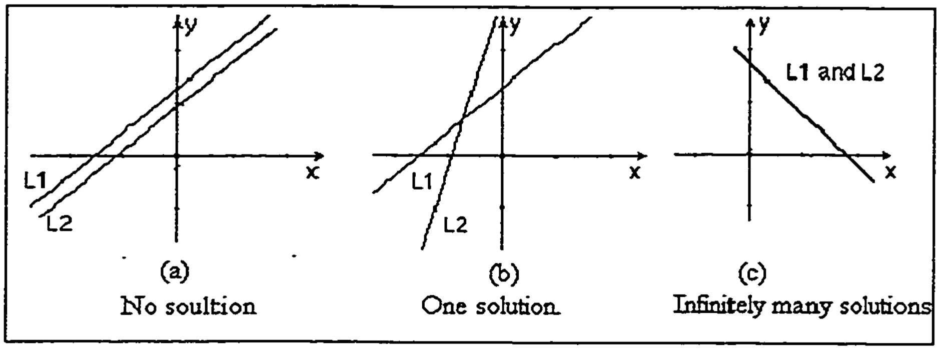

Figure 1.3. Geometrical interpretation of the solution set of a 2 x 2 system of linear equations

Similarly, the solution set of an arbitrary 3 x 3 system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad \text{(where } a_{11}, a_{12} \text{ and } a_{13} \text{ are not all zero)}$$
$$a_{21}x + a_{22}y + a_{23}z = b_2 \quad \text{(where } a_{21}, a_{22} \text{ and } a_{23} \text{ are not all zero)}$$
$$a_{31}x + a_{32}y + a_{33}z = b_3 \quad \text{(where } a_{31}, a_{32} \text{ and } a_{33} \text{ are not all zero)}$$

can be interpreted as common points of intersection of planes.

This visualization is no longer available in higher dimensions. However, it is still possible to transfer the idea of a solution set to higher dimensions. This relies on the ability to switch from the geometrical domain and its properties to the analytical or numerical domain where the properties of points and vectors can be easily interpreted.

But, students feel that they should be able to visualize the higher dimensions, and have difficulties accepting the abstraction.

Another problem encountered by some students in linear algebra is the question of how to translate a given problem and situation into a general setting (Harel, 1989). For example, the statements in Figure 1.4 are all equivalent for any square invertible n x n matrix A.

1) A is invertible
2) AX = 0 has only the trivial solution
3) A is row equivalent to $I_{n \times n}$
4) AX = B is consistent for every n x 1 matrix B
5) det(A) $\neq$ 0

Figure 1.4. Equivalent statements for an invertible matrix

Suppose that students are given a system of linear equations and told that the coefficient matrix is invertible and asked to make conclusions about the solution set. Some students would prefer to solve the system using Gauss-Jordan elimination rather than using the equivalence of the statements in Figure 4 to conclude that the system will always have a unique solution. This sort of procedural thinking can inhibit students' ability to make generalizations of results to different domains.

The action of procedural thinking in solving a system of linear equations can leave students void of any mathematical meaning within the procedure and relationships that exist between the initial and the final systems (Harel, 1989). An example of this would be when students use the technique of Gauss-Jordan elimination to row reduce a complex system to an easier system and not realize the equivalence of the two systems. If we consider the following concrete system of equations,

$$\begin{cases} -4x + y = 16 \\ -2x + y = 4 \end{cases}$$

and its equivalent system,

$$\begin{cases} x = -6 \\ y = -8 \end{cases}$$

We see that they both define the same set, namely {(-6,-8)}, and can be interpreted graphically as intersecting lines (see Figure 1.5)

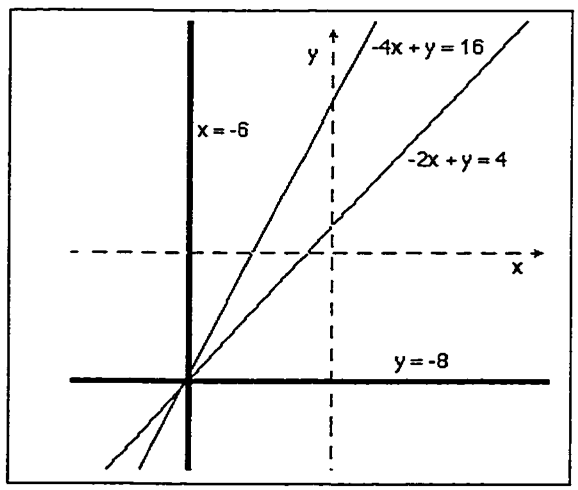

Figure 1.5. Graphical representation of an equivalent system

1.1.3 *Difficulties with vector spaces*

Vector spaces is the topic that perhaps gives most students taking a course in elementary linear algebra the greatest difficulty. This particular section relies on students being able to understand that the term "vector" can be applied to a variety of objects that are different from the familiar notions of directed segments, force, acceleration, momentum, and velocity that have both magnitude and direction. The term "vector" can also be used to describe mathematical objects such as matrices, polynomials, lists of numbers (vectors in R^n) and real-valued functions (Johnson, Riess and Arnold, 1993).

In every set of these "vectors" two operations can be defined, addition and scalar multiplication, satisfying certain basic properties. Linear algebra generalizes this structure to form the abstract system called a vector space. Now, for the first time, students are introduced to an abstract mathematical concept, which refers to systems that differ from the real number system with which they are familiar. In the aim of facilitating the understanding of vector spaces, instructors provide models that are supposed to be more "concrete" than the abstract concept. But, since these models are formulated in unfamiliar algebraic or geometrical terms, they add to rather than take away the difficulty (Harel, 1989).

But can some of the difficulties that arise from the abstractions in linear algebra be attributed to problems of conceptually understanding the real number system that students are expected to be familiar with? For example, in order to get a good understanding of the algebraic system called a vector space, it may be necessary for students to have a good understanding of the real numbers *as a system* and a mathematical structure. This includes knowing the different subsets that exist within the real number system and how the arithmetic operations of addition and multiplication, when applied to the objects within the set, transform the set into a particular mathematical structure. For example, if we consider the set of natural numbers, $N = \{1, 2, 3, \ldots\}$, and let n and m be any two numbers in N, then their sum $n + m$ and product $n \cdot m$ are also in N. It is important, at this point, that students realize that the operations of division and subtraction are not permissible since they could produce elements that are not contained in the set N. An example of this would be to divide 5 by 2 ($5/2 = 2.5$) and to subtract 5 from 2 ($2 - 5 = -3$) which both yield results which are not elements of the set N.

At this point some students could start to experience a conflict with the mathematical structure of the natural numbers which is closed under addition and multiplication. They could argue that the operation of division and subtraction should be allowed for the set of natural numbers since 10/2, 27/3, 10 - 2, 27 - 3 produce 5, 9, 8 and 24, respectively, which are all elements of N. Even though the former examples produced results that are contained in the set of natural numbers, the operations of division and subtractions only work for special elements in the set of natural numbers, thus not making the set closed under division and subtraction. The convention of mathematics is to make general statements and reach general conclusions about any set of elements (Dorier, 1995). This requires students to view the real numbers not only as individual objects but also as a mathematical structure with closure properties.

Another number system that students are familiar with is the set of integers $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. If we let n and m to be any two elements in Z then their sum $n + m$ and $n \cdot m$ are also elements in Z. Now if we compare the elements of the set N and Z, it is not difficult to see that the set Z contains some elements that are not in the set N. These different elements of set Z can be obtained from the operation of subtracting any two natural numbers, say n and m, where $n - m < 0$ when $m > n$. This example could provide students with a good concrete example of how a mathematical structure such as the natural numbers can be used to build the integer, rational and eventually the real number system by extending the range of operations under which a set is closed. However, the operations of addition, subtraction, multiplication, and division are not enough to build the real number system. Rather, in order to do this, the introduction of the limit operation on sequences of rational numbers or the axiom of continuity are necessary.

With students having gained a better understanding of numbers as mathematical systems, closed under certain operations, they would now be ready to explore the special mathematical system known as a vector space with fewer difficulties. First, however, some students would have to overcome their false notion of what a vector space is. For example, when the topic of vector spaces are introduced, some students immediately start thinking in terms of astronomy. They associate the word "space" with the universe and the objects that exist within it, such as galaxies, stars, planets and so on. Unfortunately this problem gets compounded when vectors in n-dimensional space are discussed. Some students try to imagine how objects may visually appear in the 4^{th} or 5^{th} dimension, and others believe that the n^{th} dimension is the infinite dimension.

It appears that students at the college level have certain connotations of these mathematical terms that are not necessarily compatible with the intended mathematical meaning. Indeed, one can draw a parallel between our universe and the mathematical abstraction of a vector space. For example, the universe consists of galaxies, stars, and planets, which can be considered as natural objects that are governed by the laws of physics. In considering a mathematical vector space, it, too, consists of objects, but of a mathematical nature. These objects are themselves governed by a set of axioms that must be satisfied in order for them to be a mathematical vector space.

1.2 The vector concept in physics

Physics, unlike mathematics, (which is an abstract science) is a natural science, which "examines the relationships of matter and energy" (Murphy & Smoot, 1977, p. 3). Since physics is a natural science, it is quite normal for students that are taking an elementary physics course for the first time to enter into the classroom with their own conceptions and beliefs about the physical world around them. These conceptions and beliefs are formed from their daily interactions within their environment. However, the beliefs, which are developed by students about force and motion, significantly differ from those that are intended by Newtonian mechanics during formal instruction (Aguirre & Erickson, 1984). For example, when a ball is thrown at an angle, its motion is not perceived by most students as two-dimensional motion, but, rather, as a one-dimensional motion.

1.2.1 *Students' difficulties with vector in physics*

Physical quantities such as length, speed, time, temperature, mass, displacement, velocity, acceleration, force, and momentum are the foundational concepts we use to express the laws of physics. However, these physical quantities can be classified as either scalar or vector quantities (see Figure 1.6).

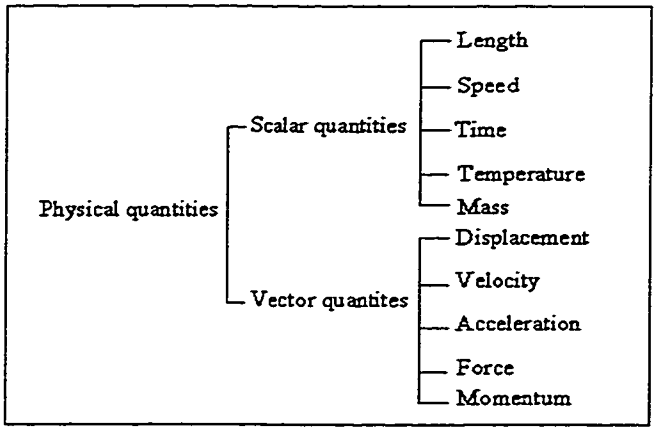

Figure 1.6. Classification of physical quantities as scalar and vector quantities

In studying physics, students frequently work with both scalar and vector quantities. A scalar quantity is described by its magnitude, which consists of a number and a unit (Murphy & Smoot, 1977). If we consider the scalar quantities listed in Figure 1.6, we see that they can also be written as, example, 35 m, 120 km/hr, 22 s, 100 °C, and 102 kg.

A vector quantity, on the other hand, is characterized by two features: its magnitude (which consist of a number and a unit) and direction. For example, the velocity of the wind is given to be blowing 45km/hr [E] ([E] = east), and a group of hikers' displacement is 5 km [S] ([S]= south) of their initial position.

The concept of a vector is an essential and fundamental part of physics as well as mathematics. Physics uses vectors to represent forces that act on objects in Newtonian mechanics. It also uses vectors as a means to track and locate the position of a moving particle. Therefore, students possessing an insufficient knowledge of vector quantities may experience difficulties in two areas: being able to solve problems that require vectorial reasoning, and understanding other related topics that encompass the vector notion (Chee, 1988).

Many physics educators, such as Kodratgev and Aguirre, clearly describe situations where it is evident that students experience difficulty with the notion of vectors. For example, Knight (1995), conducted a study on "the vector knowledge of beginning physics students", which consisted of a sample of 286 university students from engineering, architecture, science and mathematics. The goal of the study was to see if these students "possess the minimal basic knowledge of vectors that will allow them to proceed with a study, either qualitative or quantitative, of Newtonian mechanics." (Knight, 1995, p. 75). His results showed that approximately 35% of the sample would be able to read a physics text and solve basic vector problems. These students, however, would need some instruction on how to find vector directions. A reported 15% of the sample possessed some basic knowledge of vector properties, but would most likely not apply this knowledge to mathematics unless explicitly told. The remaining 50% of the sample had no useful knowledge of vectors at all.

Because half of most students enter into a beginners physics course with essentially no useful knowledge of vectors, a question that comes to mind is, what kind of knowledge are they bringing into the classroom about how objects move, and how does this inhibit them from understanding vectors? Champagne (1983), suggests that students' conceptualizations about how objects move are very resistant to change and seem to have traits which are similar to the Aristotelian view of motion which significantly differs from the accepted Newtonian mechanics. As Dijksterhuis (1961) wrote:

> *To this day every student of elementary physics has to struggle with the same errors and misconceptions which then had to be overcome, and on a reduced scale, in the teaching of this branch of knowledge in schools, history repeats itself every year. The reason is obvious: Aristotle merely formulated the most commonplace experiences in the matter of motion as universal scientific propositions, whereas classical mechanics, with its principle of inertia and its proportionality of force and acceleration, makes assertions which not only are never confirmed by everyday experience, but whose direct experimental verification is fundamentally impossible ... (p.30).*

This suggests that students' every day experiences are not quantified through Newtonian mechanics, thus leaving students with the impression that physics word problems are fictional, puzzle-like tasks that are unrelated to the real world. When it comes to solving problems that require the following kinematics equations,

$$v = v_o + at$$
$$x - x_o = v_0 t + (1/2)at^2$$
$$v^2 = v_o^2 + 2a(x - x_0)$$

students are content with just writing down the given information and matching the given quantity with the equations without using any vectorial reasoning at all (Resnick & Gelman, 1985; Champagne, 1983). For example, *"A car traveling at 44 m/sec is uniformly decelerated to a speed of 22 m/s over an 11-sec period. What distance does it travel during this time? (answer: 363 m)"* (Murphy & Smoot, 1977, p. 38).

Another difficulty that students confront when working with vectors is the question of how to use their properties when discussing problems that involve acceleration, force and velocity. As a result of this, some students often end up using vector notation without assigning any physical meaning to vectors (Chee, 1988). This problem is apparent when students are asked to find the acceleration of a block that is being pulled up an inclined plane (see Figure 1.7).

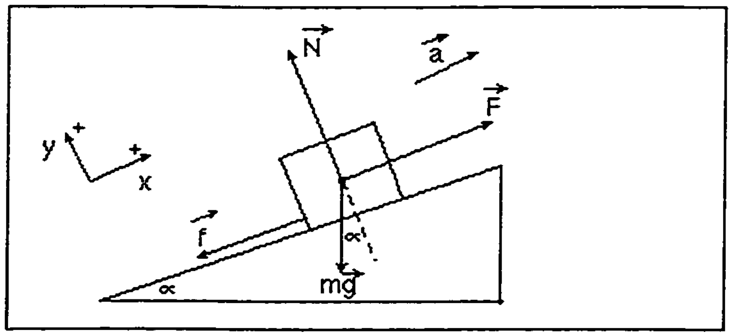

Figure 1.7. Free body diagram of a block being pulled up an inclined plane

In order to reduce the amount of arithmetic calculations needed to solve this problem, it is useful to turn to vectorial reasoning and choose the x-axis along the incline of the plane and the y-axis perpendicular to the inclined plane. This eliminates the need for decomposing forces, which are parallel to the axis, into their components. This line of reasoning is not intuitive and therefore difficult for some students, since their natural instinct would be to draw the x and y axis in the standard horizontal/vertical position thus undoubtedly increasing the difficulty of the problem.

In order to help students gain a better understanding of how the abstract mathematical notion of a vector can be made tangible in a physical situation, many of the contextual problems that involve vector addition and subtraction are performed graphically (by parallelogram, triangular, and polygon method) as well as analytically. For example, suppose a plane which is flying at 275 km/hr west is blown north at 75 km/hr. The true velocity of the plane is 285 km/hr 15° [NW] ([NW]= north of west), see Figure 1.8a. Now, if we consider the case where the same plane is travelling West (at the same speed) but now, encounters a head wind at 75 km/hr east, then the true velocity of the plane is 200 km/hr W (see Figure 1.8b).

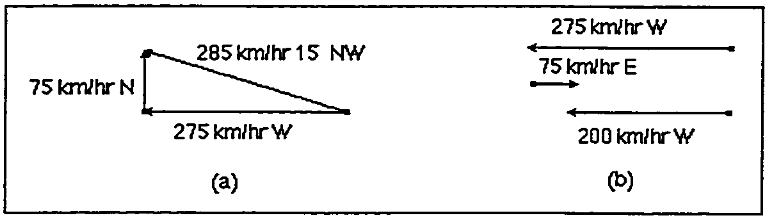

Figure 1.8. Contextual problems showing graphical method of adding vectors

These graphical methods, which are used to represent vectors, provide a useful way for students to translate words into vectorial diagrams, which have both physical and mathematical meaning, thus reducing the level of abstraction from the general theory to the particular situation at hand (Kondratgev & Sperry, 1994).

However, if we consider the situation in Figure 1.8a: The majority of students are able to determine that the resultant vector of 285km/hr 15^0 [NW] is going to be somewhere in between the two contributing vector components with a magnitude that is different from that of the vector components. But, as reported by Aguirre and Erickson (1984), some students think that these two vector components (75 km/hr [N] and 275 km/hr [W]) are not independent. They feel that the velocity of the planes' engine is altered as a result of the interaction with the wind.

1.3 Obstacles Within The Learning Process

The complexities within the learning process have forced psychologists to develop many learning theories that try to account for the manner in which we learn. These learning theories can be divided into the following psychological categories: Developmental, Behavioral, and Cognitive. Developmental psychology (which includes theories of Piaget, Vygotsky, etc.) focuses on the development of intelligence in early childhood through adolescence, and operates on the premise that in order for an individual to learn, they need to attain a certain level of cognitive capacity (i.e. thinking, reasoning ability etc.). Behavioral psychology (which includes theories of Thorndike's connectionism, Pavlov's classical conditioning, and Skinner's operant conditioning etc.) focuses on observable behavior not specific to a particular age and considers aspects of the "behavioral unit" which consists of three steps. First is the antecedent event (or stimulus), followed by the target behavior (or the response), and finally the consequent event (or reinforcement). The main goal of a pedagogy based on behavioral psychology is to promote learning through association by means of a stimulus-response mechanism (Bower & Hilgard, 1981). Cognitive psychology (which includes Gestalt theories and Tolman's sign learning) focuses on non-observable mental mechanisms that are not specific to a particular age. Its main feature is to model the storage and flow of information within cognitive and memory structures to gain a better understanding of how people construct knowledge by linking new information to prior knowledge.

Therefore, these learning theories all require that the learner be able to perceive an initial piece of information and process it in a purposeful manner, regardless of individual learning styles (i.e. reflective, impulsive, active, passive etc.). There are many obstacles that can disrupt an individual's cognitive processes (since individuals have different experiences which may affect perception, memory and judgment) and hinder learning in subjects such as mathematics and physics per se. These obstacles may result from the beliefs individuals possess about the world around them, misconceptions about mathematics and physics, and irrelevant or misleading prior knowledge.

1.3.1 *A general cognitive approach to learning*

The three main components of cognition are perception, memory, and judgment, each of which plays an important role in the way one interprets, recalls, and makes conclusions about information that is being processed. Furthermore, because learning is sometimes defined as an improvement in performance (Median & Ross, 1992), it is difficult to say exactly when or how this improvement is achieved (i.e. is improvement a result of learning or is it a result of memory?). It is nevertheless reasonable to think that this improvement could result from an increase in an individual's cognitive ability (conceptualizing, thinking, abstracting, etc.) and cognitive skills (reading, problem-solving, computing, etc.). The process by which an individual learns new skills occurs in four main stages (acquisition, proficiency, maintenance, and generalization).

In the "*acquisition stage*" of learning, students are exposed to new knowledge. Some students receive this knowledge best through visual means, some by the auditory process of listening, while others respond best to new knowledge by performing (kinesthetic) what was done in class (Lerner, 2000). In addition, learning is dependent upon perceptual abilities, because most information is taken in through auditory and visual means. These perceptual systems are necessary to convey information to the brain so that relevant knowledge can be retrieved from memory. Thus, perception is defined as "*the process of recognizing and interpreting sensory information*" (Lerner, 2000, p. 265). This process is not a "passive" but rather an "active" one (Medin and Ross, 1992), and gives the ability to bring structure and organization to different levels of information. For example, in the course of studying physics students are introduced to vectors through the force metaphor. They are told that forces can be represented graphically as arrows in general and in content specific areas such as equilibrium problems, where two or more forces can act concurrently on an object producing a vector sum of zero (see Figure 1.9a). The students are then shown how to represent a mass in equilibrium suspended from strings by means of a free body force vector diagram (see Figure 1.9b) and how to decompose forces into their vertical and horizontal components (see Figure 1.9c). Therefore, in this stage the student has acquired a set of declarative and procedural knowledge that he or she tries to commit to memory by reciting it several times until it becomes integrated in their vocabulary (Anderson, 1985).

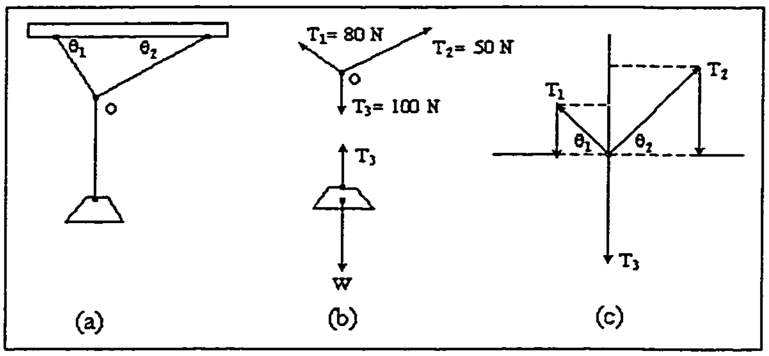

Figure: 1.9. a) A diagram of a system in equilibrium. b) A force vector free body diagram of a system in equilibrium. c) A force vector free body diagram of a system in equilibrium with its forces decomposed in vertical and horizontal components.
Problem taken form Concepts in Physics notes and laboratory manual p.145

In the "*proficiency stage*" of learning, students begin to sharpen their perceptive and memory abilities through practice. This is achieved by detecting and eliminating any errors and inconsistent thoughts that are present in the initial understanding of concepts (Anderson, 1985). This process can be extremely slow and frustrating because it requires the learner to integrate new ideas with old ones and resolve internal conflicts in thought. For instance, labeling the angles within the free body diagram (see Figure 1.9c) requires the student to be able to integrate their knowledge from geometry, which says, alternate interior angles are the same. Students also must resolve the conflicts which arise from seeing vector quantities as scalar quantities. In the equilibrium problem in Figure 1.9c, the arithmetic sum of the tension forces T_1 and T_2 in the upward ropes is 130 N (where N= Newton, is a unit of force), which is more than the weight being supported (**w** = 100 N); this might give the impression that the system is not in equilibrium. However, the vector sum of T_1 and T_2 is 100 N, acting upward, which cancels the force **w** of 100 N, acting downward and thus ensuring that the system is in equilibrium. With continual practice of routine problems (of a similar nature) students develop their procedural skills of drawing and interpreting free body diagrams, which allows them to heighten their perception and memory abilities.

In the "*maintenance stage*", the learners can now demonstrate the ability to recall key elements (formulas and diagrams) which will enable them to perform at a high level of efficiency. At this stage the students' declarative information is transformed into a set of procedural schemas, which encompass a reliable working knowledge of Newton's laws from memory and relies less on the instructor and other external sources in problem solving.

The final stage of the learning process is the "*generalization stage*". At this stage, students are able to use their heightened memory and perceptual abilities to make judgments about other situations from the knowledge that they have internalized. For example, students are able to apply their knowledge of free body diagrams to situations where objects are not in equilibrium and use Newton's laws to solve problems quantitatively.

This description, however, is simply an idealization of how a student would go about learning a particular subject in mathematics or physics. There are many obstacles that may disrupt this construction of knowledge in students. These obstacles will be discussed in the following subsections.

1.3.2 *Affective obstacles*

In the course of studying mathematics and physics many students fall victim to affective obstacles. An affective obstacle is defined here as an emotional feeling or belief that hinders or diminishes one's ability to perform a task. Anomalies in the cognitive process, which include affective obstacles like anxiety, self-doubt, and thought shift patterns can have a negative impact on a students performance (Klinger, 1996).

Irrational beliefs also represent an important set of affective obstacles. Many students who suffer from math anxiety do so as a result of their misconceptions about mathematics. These students enter into a mathematics classroom with strong preconceived ideas of what mathematics is about. For example, some students believe that the goal of mathematics is to use formulas to solve word problems in terms of numerical answers (category 1), while other students (category 2), think of mathematics as formulating equations with x and y's (Tobias, 1978). While these beliefs are not completely untrue, they can result in limiting students' views of mathematics in general, and thus increase math anxiety when beliefs are challenged. For example, the student who is in the first category might suffer from the belief that math problems have only one right answer and subsequently believe that there is only one way of obtaining it. Therefore, much of the students' time is devoted to practicing the same rote procedure on several problems of the same type, instead of developing or practicing different schemas to solve the same problems. An example of this is when students are asked to solve the following 3 x 3 system of linear equations

$$S = \begin{cases} 3x + 4y + 5z = 100 \\ x + y + z = 25 \\ 10x + 3y + 7z = 130 \end{cases}.$$

This could be done in several different ways by using Cramer's rule, Gaussian elimination, Matrix inversion, and Gauss-Jordan elimination. The preferred method by most students is to use Gauss-Jordan elimination even though the problem might specifically request that matrix inversion be used.

Students in the second category focus most of their attention on formulating word problems into equations (see Figure 1.10). A system of linear equations can be written from Figure 1.10 if we let x, y and z equal the number of Rattles, Flippers and Busters respectively.

> A toy company makes three kinds of toys, Rattle, Flipper and Buster. It takes 3 machine- hours and 1 safety-check hour to make one Rattle; 4 machine-hours and 1 safety-check hour to make one Flipper and 5 machine-hours and 1 safety-check hour to make one Buster. The company has 100 machine-hours and 25 safety-check hours a day. How many toys of each kind must be made each day to reap a daily profit of $130.00 if each toy is sold at a profit of $10, $3 and $7 respectively?
> Problem taken form Phull "assignment on linear algebra" p.5

Figure 1.10 Example of a word problem that can be written into a system of linear equations.

This system of linear equations is identical to the system of linear equations S. Students that lack the ability to restructure a problem so that it makes sense to them often suffer from high anxiety if a particular word problem does not fit a familiar systematic procedure that was practiced.

Students often feel that math is difficult because of the rigorous proofs done in class which require a certain level of reasoning to understand them. This leads some students to doubt their mathematical abilities and develop the idea that either one has a mathematical mind or one doesn't. This misconception is often confirmed when students watch the teacher present mathematical proofs and solve complex problems in a straightforward manner without any hesitation (i.e. no false starts or second-guessing). Therefore some students, unfortunately, develop the belief that if they can't solve problems or do proofs within a short span of time (i.e. 5 to 10 minutes) they won't get them if they keep on working (Mandl, Gruber and Renkl, 1993; Tobias, 1978; Schommer, 1989).

1.3.3 *Cognitive obstacles*

One objective of natural sciences such as physics is to account for the way that forces affect objects to cause motion, and to describe phenomena, which exist in our environment such as gravity, in a formal manner. Consequently, physics relies on diagrams that represent physical situations and exemplify how forces act on objects, and makes use of algebraic mathematical equations to quantify these forces. However, these diagrams and mathematical equations can sometimes be difficult to understand conceptually and seem incomprehensible to the learner at times. As discussed earlier in Chapter 1, the beliefs that students possess about forces and motion often differ significantly from what is intended by formal instruction. It is these differences that I believe constitute cognitive obstacles that will be the focus of this section. It is important to note that cognitive obstacles are inherent in the learning process and only become problematic when they prevent further learning of solutions of more complex problems (Alexander, 1992).

The first cognitive obstacle that will be discussed is the superposition of motion. The notion of superposition of motion refers to the idea that an object can move in different directions simultaneously with each of its contributing motions independent of the other. This is not an easily accepted concept. When this motion is viewed from a fixed reference point, it appears as one superposed motion. For example, a motor boat crossing a river moves diagonally relative to the bank. This superposed diagonal motion is composed of the moving water and the motion of the motor boat, which are perpendicular and have no influence on each other. The following study about a boat crossing a river was conducted by Aguirre and Erickson (1984). They reported that 80 percent of their sample of students felt that the current flow of the river would in some way have an effect on the magnitude of the velocity from the motor of the boat. The Tübingen research group in Germany also conducted a similar experiment on the superposition of motion and found that some university students possess incorrect cognitive concepts, which are a direct result of several misconceptions. One misconception that was noted by the Tübingen research group was the "Dominant motion misconception" where students assume that a motion is dominant over another. Another misconception is that of "Active/ Passive motions" in which students assume that active motions are dominated by passive motions (Mandl et al. 1993). In the previous example the passive motion would be considered the magnitude of the velocity from the motor of the boat and the active motion would be the current flow of the river.

1.3.4 *Didactic obstacles*

Models that are introduced to students with the intention of reducing mathematical abstractions can be considered didactic constructs. Examples of these constructs include a teacher introducing the mathematical operation of division as a sharing property and the idea that multiplication makes things larger. Although these familiar restricted settings are not totally untrue, they can, however, hinder later learning, when the operation of division also leads to an increase in quantity (i.e. dividing by a positive proper fraction), and multiplication can lead to a reduction in quantity (multiplication by a positive proper fraction).

In elementary physics students are told that displacement vectors, velocity vectors, force vectors etc., can be added respectively by the following graphical methods: Triangle, Parallelogram, and Polygon. It is quite common that when students are introduced to the Triangle method of vector addition in high school physics that the two vectors to be added (**u** and **v**) are both drawn to the same scale and placed tail to tip at a 90 degree angle. The resultant vector (or vector sum) is the vector drawn from the tail of one (say vector **u**) to the tip of the other (say vector **v**), with vectors **u** and **v** as two of the four sides of a rectangle. This method of vector addition is frequently dominant in elementary physics textbooks, and has a strong effect on how students view the resulting figure (as we will see in Chapter II) formed by the addition of two vectors in general.

Chapter II

Previous Experience with Physics as a Possible Source of Difficulty in Understanding Linear Algebra in a Geometric Setting

Introduction

Given its complex and diverse nature, it is natural that problems emerge when students attempt to learn linear algebra. Three basic classes of difficulties can emerge to interfere with this learning: difficulties with symbolic notation, problems generalizing from small systems to solving problems in general linear systems, and problems understanding vector space. The purpose of this study was to understand factors that contribute to problems understanding linear algebra, specifically the influence of prior experience with physics knowledge was examined. As well as the learning problems inherent in linear algebra, experience with physics may interfere with the acquisition of linear algebra material in specific ways. It is important to understand these problems so that the needs of these students can be better met. This chapter describes the methods used to examine the effects of experience with physics on linear algebra acquisition and the results obtained from such an examination.

This chapter includes a section on the Cabri Geometry II environment in which the students worked. It gives a step by step example of how to draw a vector in Cabri and explicitly states the difference between a drawn and constructed object. The chapter then reviews the first three sessions and the individual activities performed by each group of students. The remaining two sessions were summarized since their activities for the most part were not referred to in the result section with the exception of Session IV Part II Grid transformation task. Each session was composed of one or more topics that were explored in the individual activities. For example Session I looked at vectors, equality of vectors and operations on vectors. The first activity in this session that was preformed by the students highlighted two main features of vectors, their magnitude and direction. The second activity looked at the equality of vectors i.e. when they have the same magnitude and direction. The remaining activities for this session involved constructing new vectors using the operations of vector addition and scalar multiplication.

The chapter also summarizes the selection of the students that took part in the experiment and describes how the data were collected and analyzed. The results section reports the effects of incorporating physics in the learning of linear algebra. It also reports obstacles and difficulties encountered by the two groups of students centering around the notion of vector, operations on vectors, linear combination, spanning set and linear/ non-linear transformation and tries to analyze the possible sources of these difficulties.

2.1 Description of the Cabri Geometry II environment

The computer software program Cabri Geometry II (or simply Cabri), that was used by the students allowed them to draw and construct geometrical figures and objects in a two dimensional space from the Cabri menus shown in (see Figure 2.1).

Once a figure was constructed, the Cabri dynamic environment allowed the students to interact with the figure by dragging any one of its elements. For example, vectors are represented by arrows in Cabri and can be drawn by selecting *Lines* from the *Cabri Toolbar* menu and then choosing *Vector* in the *Lines* menu. Once the user has selected *Vector* from the *Lines* menu they simply click and release the mouse button once to create the initial point of the vector. They then move the mouse until the desired length and direction of the vector is found. Then a second click of the mouse button is

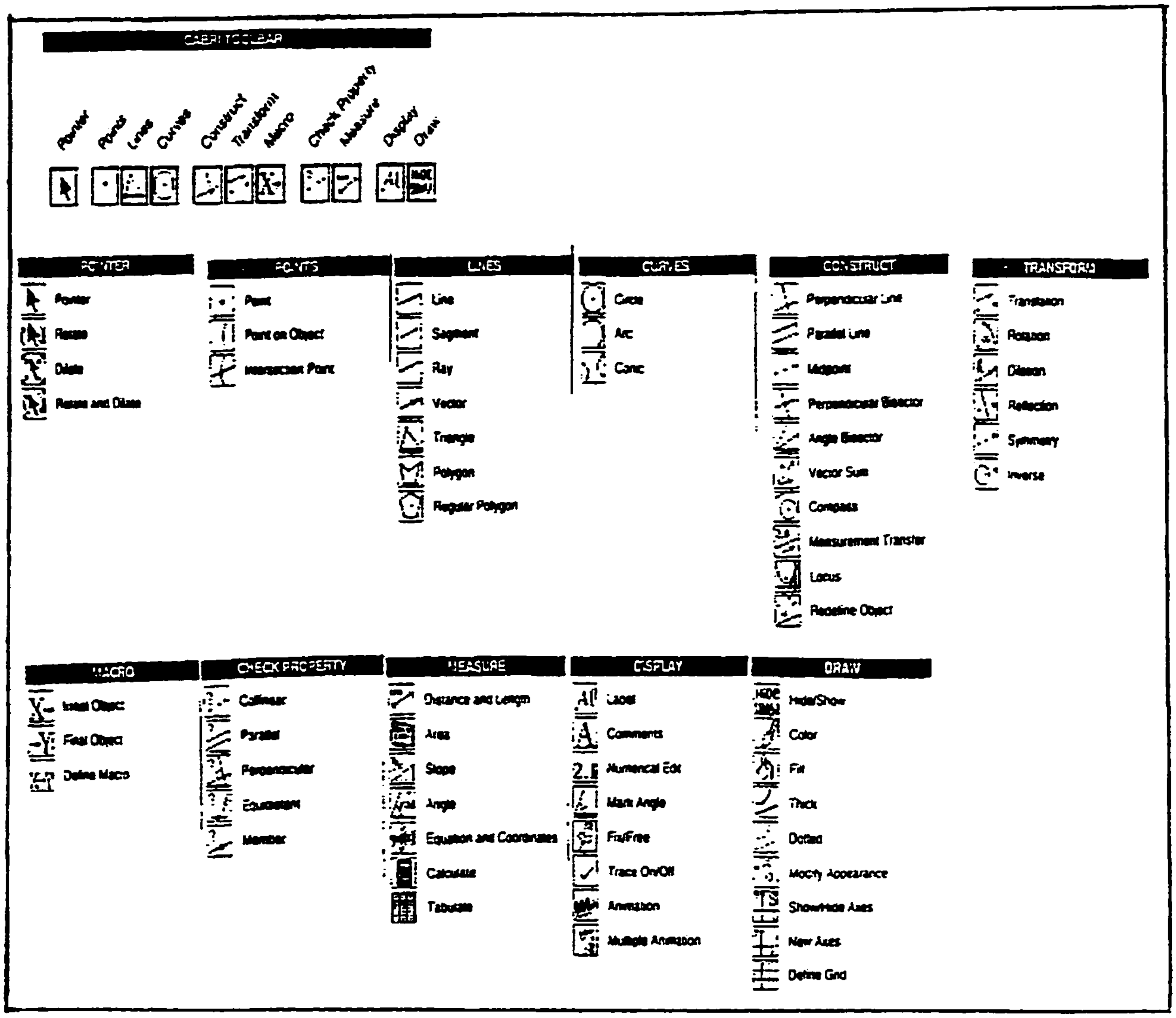

Figure 2.1. Cabri Toolbar Menu

necessary to disengage the vector command. The newly drawn vector **ov** (see Figure 2.2a.) is called a free or independent[1] vector.

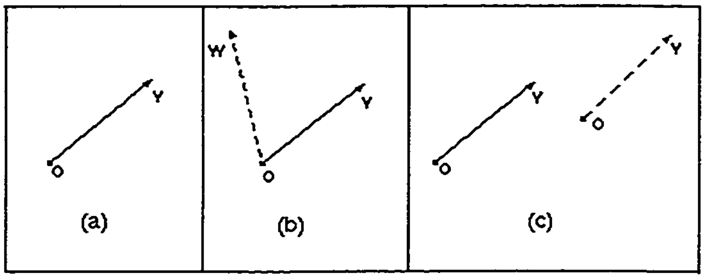

Figure 2.2. Drawing and moving vectors in Cabri

The students were able to interact with this free vector in two ways. First, the vector could be *dragged* by its free endpoint; this action made it possible for the user to vary the magnitude and direction of the vector thus creating a new vector altogether (vector **ow** see Figure 2.2b). The second way was to *move the vector as a vector*, this action did not change the direction and magnitude of the vector but allowed the user to translate the vector from one area of the screen to another (see Figure 2.2c).

Objects that are constructed (as opposed to just drawn) in Cabri are called dependent[2] objects and can only be interacted with indirectly. For example, in the figure below, the vector v_2 is constructed from vector v_1 by means of a rotation about the origin *O* by an angle of 75 degrees and is therefore dependent on v_1 (see Figure 2.3a). In order to interact with v_2 it is necessary to move v_1 (see Figure 2.3b).

---

[1] Independent objects: Objects drawn by the user by using Cabri commands found in the Lines or Curves Menus.
[2] Dependent objects: Objects constructed by the user from existing objects by using the commands found in the Transform/ Construction Menus.

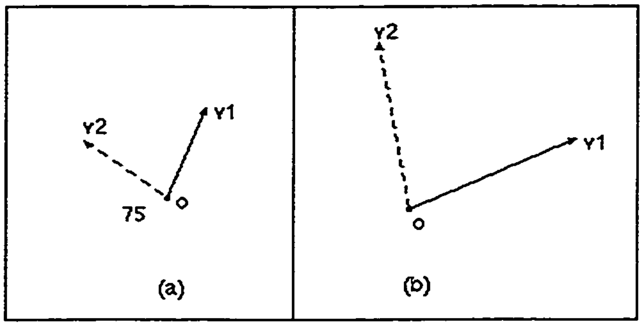

Figure 2.3. Interaction with independent vectors

2.2 Summary of the Experimental Sessions

The Geometric Approach to Teaching Linear Algebra Project 1998 (Sierpinska, Trgalova, Hillel, Dreyfus, 1999) consisted of five consecutive sessions, with each session requiring approximately two to three hours to administer. The inspiration for these sessions and their activities came from a previous experiment in 1997 (Sierpinska, Dreyfus, Hillel, 1999) which considered teaching the notion of vectors, vectors spaces, linear transformation, and eigenvectors in elementary linear algebra in a dynamic computer environment which unfortunately revealed some shortcomings in the design during the experimental stage. The experiment aimed at testing amendments introduced into the most flawed parts of the old design related, in particular, with the notions of vectors and linear transformation. The first session examined vectors, equality of vectors and operations on vectors. The second looked at linear combinations and spanning. The third session introduced the notion of transformations and linear transformations. The fourth session took a deeper look at linear transformations. For example, it examined how a 8 x 8 square grid is transformed into a parallelogram grid. It also considered how to define a linear transformation, which would turn a given square into a rectangle. The fifth and final session examined how to define a linear transformation on a basis.

2.2.1 *Session I*

Session I was divided into three sections Part I, II and III. In the first part, the students were introduced to Cabri Geometry II software program and acquainted themselves with the Cabri Toolbar menus through a series of activities which required the students to draw points, lines, lines segments, triangles, vectors, reflecting points, rotating points, and labeling figures.

The second part of Session I consisted of three activities that targeted specific features of vectors. The purpose of the first activity was to focus the students' attention on two features of, a vector, namely magnitude and direction. The objective of the second activity *Planes* was to demonstrate the equality of two vectors through a metaphor of forces. In order to do this, the students were shown the following situation (see Figure 2.4) in the Cabri Geometry computer environment and asked the following question[3] (amongst others):

> *Drag the blue vector around on the screen. Observe the effect on the blue plane.*
>
> *Can you make the blue plane overlap with the red plane? What can you say about the forces represented by the blue and red vectors now? What can you say about the lengths and the directions of these vectors?*

[3] The expected answers to the questions are: (1) Moving the blue vector as a vector has no effect on the blue plane since the direction and magnitude of the vector remains unchanged. (2) Yes. (3) They are the same and parallel. (4) The lengths are equal and their direction is the same.

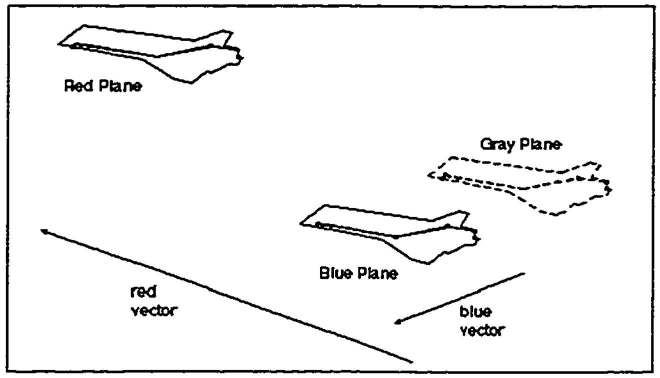

Figure 2.4. Planes

The third activity, *Origin*, had several goals. The main goal was to introduce the students to the convention of starting all vectors from a common point of reference on the Cabri screen that is called the origin *O*. It also reinforced the notion of the equality of two vectors from the previous activity with the planes. It did this by placing an arbitrary fixed vector on the screen and a point *O* that the students were unable to move. The students were then asked to construct or draw a vector starting from *O* that would be equal to the given vector.

Part III of Session I: *Operations on vectors*, consisted of four activities. The first of these activities Activity 4, *Vector addition 1*, looked at how to combine two non-collinear vectors to form a new vector by applying a macro[4] construction called *vector addition*. The main goal of this activity was to show that when two vectors in R^2 (say **v** and **w**) are added geometrically their vector sum (**v** + **w**) could be interpreted as the

[4] Macro constructions are special commands found in the macro menu that were made by the experiment designers to reduce the amount of steps when performing operations on vectors i.e. vector addition, scalar multiplication, linear combinations etc,.

diagonal of a parallelogram. Another goal was to realize that the vector sum (**v** + **w**) is always located in between the two vectors **v** and **w**.

In Activity 5, *Vector addition 2*, the students worked with vectors as forces in a contextual setting. They were given an object that was acted upon concurrently by two vectors. They were asked to translate the object by the vectors **v** and then **w**. Once the final position of the object was established the students were asked the following question[5]:

> *What do you think would happen if, instead of applying the forces v and w in a row to the ball, we applied the vector sum?*

The two remaining activities looked at the operation of scalar multiplication. In Activity 6, *Scalar multiplication 1*, the students multiplied a vector by a scalar quantity *k* and explored the effects of this operation by varying the scalar quantity *k*. This could be done by moving the shaded point on a number line (the non-shaded point represented the origin, see Figure 2.5). It was hoped that the students would realize the following relationships between the vector **v** and **kv**; the vectors **v** and **kv** always lie on the same line and the magnitude of **kv** is $|k|$ times the vector **v**. If the vector **v** is multiplied by a negative scalar *k*, then the product **kv** has the opposite orientation of **v**, and the same orientation as **v** if *k* is positive.

[5] The ball will have the same final position.

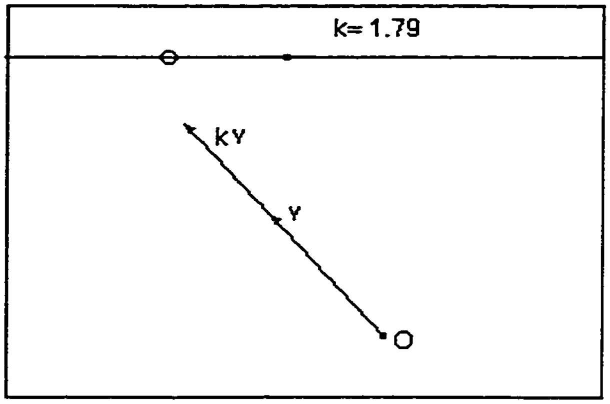

Figure 2.5. Introduction of scalar multiplication

In Activity 7, *Scalar multiplication 2*, the students considered the operation of scalar multiplication in a contextual setting. They were shown the initial figure below (Figure 2.6) which contained a variable triangular mass that was suspended from two anchors along with a vector which represented the gravitational force (W= mg). Where m is the mass of the object and g is the acceleration due to gravity 9.8m/s^2 acting on the triangular object.

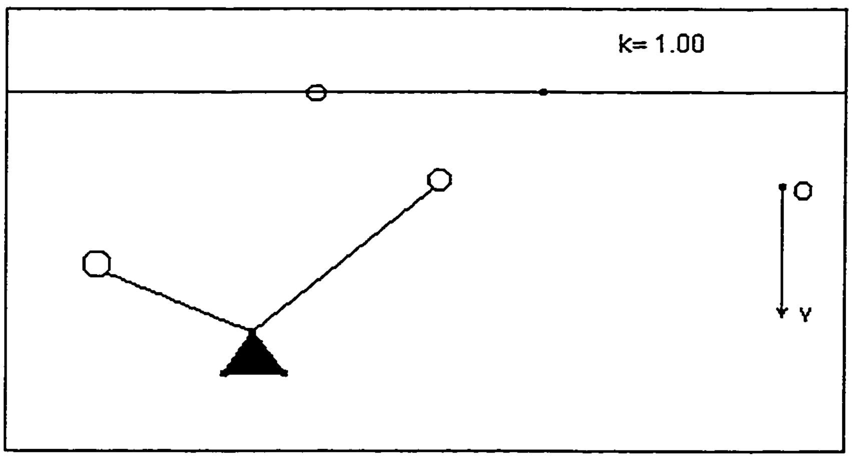

Figure 2.6. Representing the change of weight by scalar multiplication of a vector

The students were told that they could increase or decrease the mass of the triangular object by a factor of *k* simply by moving the shaded point on the number line. However, this action would alter the gravitational force proportionally that was acting on the object by a factor of *k*. The students were asked the following question[6]:

> *Could you represent the changes of this force by a vector, a variable vector in this figure here?*

2.2.2 Session II

Session II: *Linear combination and spanning*, was composed of four activities. In the first activity (Linco.1), the students were presented with two vectors (v_1 and v_2) starting from the origin *O*, along with two number lines and scalars k_1 and k_2. The students were asked to construct a vector $\mathbf{w} = k_1v_1 + k_2v_2$. This was done by multiplying the vector v_1 by scalar k_1 and v_2 by scalar k_2, and then summing the two new vectors (see Figure 2.7). Then an arbitrary vector u was placed on the screen and the students were asked to make the vector **w** equal to the vector **u**. As this was done, the vector **u** was expressed as a linear combination of the vectors v_1 and v_2. The next line of questioning[7] was to get students to realize what effect the scalars k_1 and k_2 had on vector **w** in the equation $\mathbf{w} = k_1v_1 + k_2v_2$.

> *Where is the vector **w** with relation to v_1 and v_2 if $k_1 = 0$?*
> *Where is the vector **w** with relation to v_1 and v_2 if $k_2 = 0$?*
> *Where is the vector **w** with relation to v_1 and v_2 if $k_1=k_2=1$?*
> *Where is the vector **w** given that vectors v_1 and v_2 are not collinear?*

---

[6] The question is answered by performing the operation of scalar multiplication of the vector v by the scalar on the number line.

[7] The answers to the question: (1) Vector w lies along v2. (2) Vector w lies along v1. (3) Vector w lies in between v1 and v2. (4) The vector w coincides with v1 and v2.

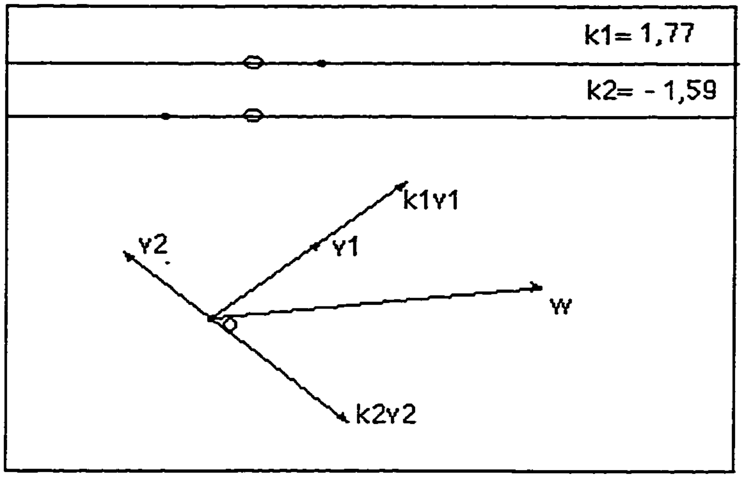

Figure 2.7. The vector **w** as a linear combination of k_1, v_1, k_2 and v_2

The main goal of Activity 2 was to stress the role that scalars play in the composition and decomposition of vectors. The students perform the first part of the activity in a paper-pencil environment. They were given two vectors **u** and **v** emanating form the origin *O* and forming an obtuse angle, and asked to draw of the following vectors: 2**u** + **v**, 1/2**v** –**u**, and –(**v** – 3**u**). The second part of this activity was also done in a paper-pencil environment that required the students to express the vector **w** as a linear combination of vectors **u** and **v** (see Figure 2.8).

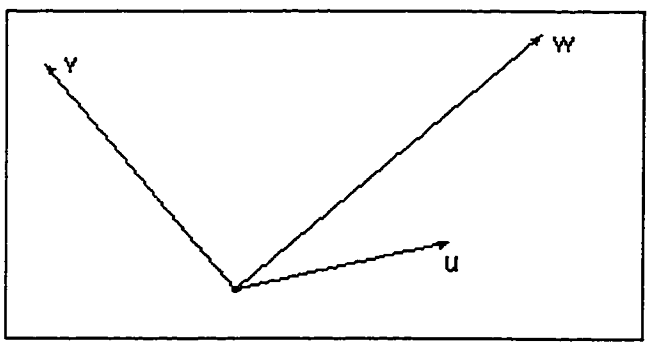

Figure 2.8. Vector **w** to be expressed as a linear combination of vectors **v** and **u**

On completing the former activity the students were asked to repeat the same activity in the Cabri environment. The purpose of this was to use the Cabri dynamic environment to make the students realize that any vector can be decomposed even though it does not lie between the vectors **u** and **v**. The last part of Activity 2 was also done in a paper-pencil environment. Its goal was to probe the students' understanding of the decomposition of a vector into a linear combination of two given vectors which include the use of negative scalar for the vector **u** (see Figure 2.9).

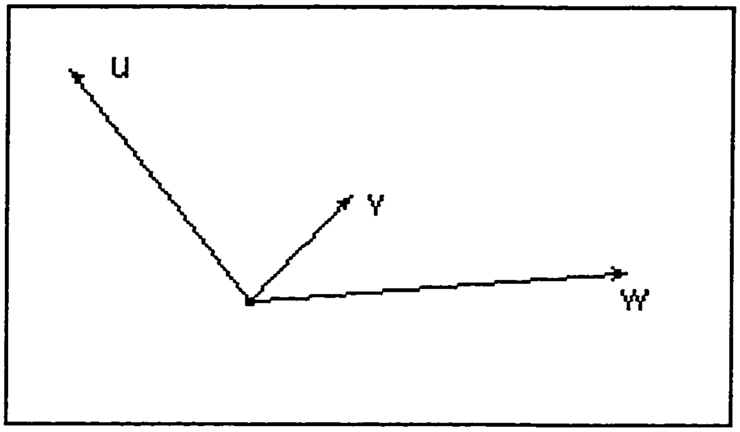

Figure 2.9. Configuration for the decomposition of vector **w** into its components along **v** and **u**

The main focus of Activity 3 was to get the students to understand the notion of a basis in two dimensions. The activity could be done with or without the computer and the questions[8] were as follows:

> *You are given one vector. What other vectors can you obtain from it, if you are allowed to do scalar multiplication, vector addition, and the combination of the two operations?*
>
> *Suppose you are given two non-collinear vectors. Same question.*

[8] Answer to the questions: (1) You can obtain all vectors that lie along that line. (2) You can obtain all vectors in R^2. (3) Yes, since two non-collinear vectors form a basis for R^2. (4) No, since the two vectors are dependent.

Suppose you are given two non-collinear vectors. Is it true that every vector can be obtained as a linear combination of these two? Explain.

Suppose you are given two collinear vectors. Is it true that every vector can be obtained as a linear combination of these two? Explain.

In Activity 4 the students were given a physical situation in which they could contextualize the notion of the zero sum of vectors. This was done by showing the students a system (see Figure 2.10) that contained three weights, which is not necessarily in equilibrium. In order to solve this problem, the students had to sum the vectors that were laying along the cables. Then the system was moved at point C until the vector sum and the vector pointing downwards had the same magnitude and opposite orientation.

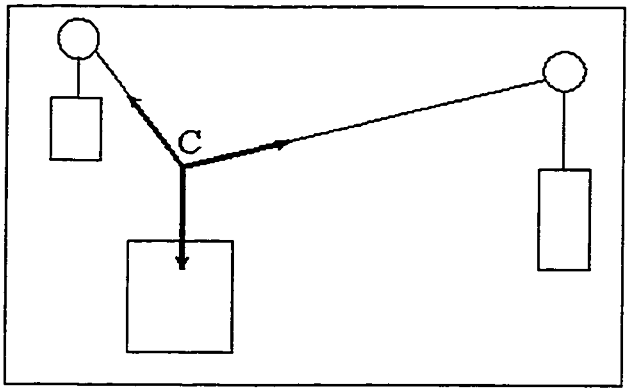

Figure 2.10. Looking for an equilibrium point

2.2.3 Session III

Session III: *Transformation and Linear Transformation,* consisted of two parts, the first part considered the general notion of transformations in four activities, while the second part of session III introduced the definition of the linear transformation as one that preserves linear combinations.

In Activity 1 (part I) *Transformations and relations between forces in physics*, the students manipulated the scalar on the number line and observed how rectilinear motion (represented by a horizontal blue vector that had a variable magnitude and a fixed direction) was transformed into circular motion. This idea was modeled through the mechanism of a steam engine. The horizontal blue vector represented the force of the steam that pushed on the piston, thus causing the linkage to rotate the wheel (see Figure 2.11). The students were asked the following questions[9]:

> *Focus on the thick blue vector and describe how it is changing.*
>
> *If you imagine the segment from the center of the wheel as a vector, can you describe how it is changing?*

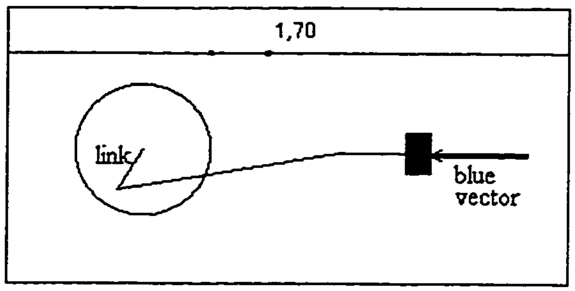

Figure 2.11. Transformation of forces in a steam engine

The second transformation, the Paucellier linkage, dealt with transforming circular motion into rectilinear motion. The last activity *Symmetric forces,* required the students to establish a relation between two vectors F_1 and F_2. The students were presented with a set of weights that was in equilibrium (see Figure 2.12).

---

[9] Answer to questions: (1) The vector has a variable magnitude and a constant direction. (2) The link has a variable magnitude and direction.

The ball that was suspended from the center of the rope indicates the equilibrium position. The square objects, that were suspended from both pulleys were identical and had variable mass, which could be changed by simply moving the scalar on the number line. When the mass of the square objects was increased the equilibrium position of the ball moved upwards. The students were encouraged to play with the system and asked the following question:

Can you describe how these two vectors are related to each other?

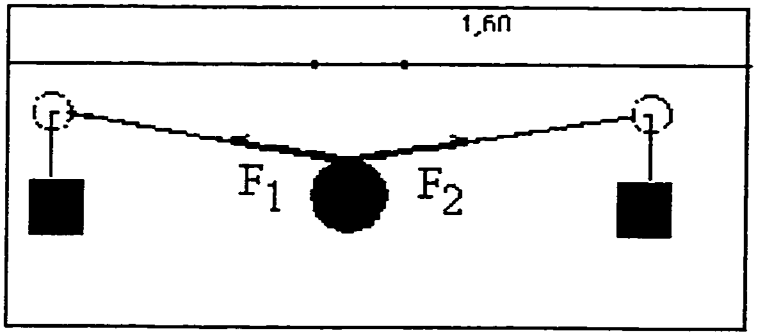

Figure 2.12. Symmetric forces

In the second activity, *Symmetric vectors*, the students were presented with two vectors, one blue and the other red. The blue vector was drawn as an independent Cabri object, while the red vector was constructed form the blue vector by means of a reflection transformation. The line of reflection was made invisible by the Cabri command *Hide/Show*. The students were encouraged to play with the blue vector and asked to figure out what was the transformation that linked the red vector to the blue vector.

In Activity 3, *Projection with dilation*, the students were once again presented with a blue and a red vector. The red vector was constructed from the blue vector by an orthogonal projection on a horizontal line (which was invisible on the screen) combined with a dilation by a factor of 2. The students were once again encouraged to play with the blue vector and asked if they could figure out how the red vector is related to the blue vector.

In Activity 4 (which was separated into two parts) the students were given, in each part, four vectors v_1, w_1, and v_2, w_2. Here, w_1 and w_2 were the image vectors of v_1 and v_2 respectively, under some transformations. They were told that v_1 was transformed by a transformation T_1 to obtain w_1, and similarly v_2 was transformed by a transformation T_2 to get w_2. The students were asked to figure out, if they could, whether the transformation that linked the vectors v_1 and w_1 was the same as the transformation that linked the vectors v_2 and w_2. In the first part of the activity, the two transformations turned out to be unequal, and equal in the second scenario.

The goal of Activity 5, *Parameters of a transformation*, was to let the students realize that when the parameters of a transformation are changed, a new transformation is obtained. The students were given two vectors **v** and **w**, where the vector **w** was constructed from **v** by a rotation of 60 degrees combined with a dilation by a factor of 2. They were instructed to change the angle of rotation and the dilation factor by means of the Cabri command *Numerical Edit* that was found in the Display menu. Once this was done the students were asked, "Has the relation between v and w changed"?

In Part II *Linear transformation*, of Session III Activity 6, the students were asked to verify whether a transformation Shear-11 was linear or not, by using the linearity equation $T(k_1v_1 + k_2v_2)= k_1T(v_1) + k_2T(v_2)$. This was done with the aid of a macro construction called linear combination. The macro construction was used as follows: in order to obtain a linear combination of $k_1v_1 + k_2v_2$, first select the macro linear combination and then click in the following order k_1, v_1, k_2, v_2 (this action produces a single vector). In the initial screen the students were given a line L, a point *O* on L, vector **v** from *O*, and a rotation angle of -90°, and dilation factor of 1.5. The students were told that:

> *Your task is to verify if the transformation* $Shear_{L,1.5}$ *is linear, which means that, for any two vectors* v_1 *and* v_2, *and any two scalars* k_1 *and* k_2,
> $$S(k_1v_1 + k_2v_2) = k_1S(v_1) + k_2S(v_2)$$
> *where S stands for* $Shear_{L,1.5}$.

2.2.4 Session IV

Session IV, *Linear transformations*, consisted of three parts. In Part I, *Examples and non-examples of linear transformations*, the students were given three transformations and asked to verify if they were linear or not. The first transformation was a rotation followed by dilation, which turned out to be linear. The second was a semi-linear transformation, which preserved scalar multiplication and not vector addition. The third transformation preserved neither scalar multiplication nor vector addition.

In the second part of Session IV, the students were given a square grid with 8 labeled vertices, which was transformed into a parallelogram grid. The goal of the activity was to identify where the old labels got transformed on the new parallelogram grid.

In Part III, of Session IV, the students were given a square figure that was transformed into a rectangular figure and asked if they could construct the transformation that represents this situation.

2.2.5 *Session V*

Session V, *Linear transformations: definition on a basis*, consisted of three parts. In the first part of Session V the students worked together and try to define a linear transformation knowing its values on a pair of non-collinear vectors. In the second part, the students worked individually on the following question for approximately ten minutes:

> *If T is a linear transformation of the vectorial plane, v1 and v2 are two non-collinear vectors, and T(v1) and T(v2) are given, then, for any vector v, its image T(v) can be found. Justify your answer in writing.*

In Part III of Session V, the students worked individually on a problem using Cabri, which was similar to the last one in Session IV. Once again the students had to find a transformation that linked two figures.

2.3 Sample, Data Collection and Method of Data Analysis

This section discusses the background of the students that participated in the study, and describes the manner in which the data was collected and analyzed.

2.3.1 *Sample*

The sample under consideration consisted of 4 university students[10] (2 female and 2 male), who had not previously taken a course in linear algebra. The participants were selected from a group of volunteers and were then broken up into groups of two. The first group (Group I) consisted of the two male students Sam and Ben who each possessed a background in physics, having both completed an introduction college level course in physics. The second group (Group II) consisted of two female students (Jenn

[10] The names given to the students in each group are fictitious to protect their anonymity.

and Rachel), who had no formal background in physics. It was by chance, not design, that the two groups happened to be separated into groups of students who had completed a course in college physics or the equivalent and students without. It was also by chance that the two groups were separated by gender. After analyzing the protocols several times for both groups, it became interesting to look at the effect of these students' physics background on the understanding of Linear Algebra. Would both groups encounter the same obstacles or drastically different ones as a result of modeling vectors in the metaphor of forces? Would both groups achieve the same or drastically different levels of understanding?

2.3.2 *Data Collection*

In order to understand the effect of students' physics background on the understanding of Linear Algebra, data were collected from several levels and from two sources. The two data sources consisted of Group I and Group II. Neither group communicated with the other for the duration of the experiment and the groups had different instructors who will be referred to as "Tutor". Data were collected over a five day period from five consecutive sessions which ran for 2 to 3 hours, each of which was divided into several activities, each encompassing its own objective as stated previously in Section 2.2.

The sessions for both groups were recorded on audio and videotape and then transcribed into text. Field notes were taken during each session and copies of the students' group written work and computer work were saved for future reference. Copies of the students' individual written work and computer work were also saved.

2.3.3 *Method of Data Analysis*

The discourse of both groups was analyzed to find how the students understood the activities of the sessions. Then the field notes and transcribed data were read several times and reviewed with the videotape. Notes were made and analyzed for patterns of consistent and inconsistent thought in concepts from each of the students' responses to the activities. The focus of the analysis was on the students' difficulty. It was assumed that a student experiences a difficulty if:

(a) the student develops a pattern of argumentation which can lead to statements incompatible with the theory, or

(b) shows uneasiness with a statement or problem through tone and body language, or

(c) claims that he or she "does not understand" or "is confused", in spite of repeated coaching by the tutor, or

(d) fails to produce an answer to a problem.

As this behavior intensified and/or recurred in relation with specific mathematical themes, areas of difficulty were identified and an explanation of their appearance sought.

2.4 Results

This section reports the results of the analysis of the data in terms of the areas of difficulty that have been identified.

2.4.1 Area of difficulty 1: The concept of vector: <u>*Students believe that vectors are fully characterized by their magnitude*</u>

A recurring problem observed within group I was that the students predominately characterized a vector by its magnitude. For example, in Session I Activity 2, *Planes*, the students were asked to move the blue vector and observed its effect on the blue plane. Both students felt that moving the blue vector as a vector, i.e. translating it from one point on the screen to another, had no effect on the blue plane. When asked why, Sam replied:

> **Sam:** *The length didn't change…*

His response seemed to indicate that his focus was on the length of the vector and not on its direction.

Another example of a vector predominately being characterized by its magnitude comes from within the same activity, Planes, when the students were asked if it was possible to make the blue plane overlap with the red plane. At this point Ben was controlling the mouse and super-imposed the blue plane on the red plane. He did this by dragging the end point of the blue vector, thus changing its magnitude and direction until it was equal to that of the red vector. Ben goes on to make the following comment:

> **Ben:** *Now the length of the blue vector and red vector will be same.*

Ben's response also appeared to focus on the vector's length and not on its direction.

The last example of the students from group I characterizing vectors by their magnitude arises in Session III Activity 1. The students were presented with a physical situation of a Steam Engine. The instructor demonstrated how the linear motion of a blue vector gets converted into circular motion. The students were then asked to describe how the blue vector is changing. The following is an excerpt of how the students described the blue vector:

> **Sam**: *It's increasing up to a limit then it start to decrease.*
> **Ben:** *It comes to the same position. The length of the vector is changing and applying a force on the piston.*
> **Sam:** *The length of the vector.*

They continued to make reference to the length of the vector and not its direction.

In response to the second question from the same activity:

> *Imagine the segment from the center of the wheel as a vector, can you describe how it is changing?*

The students viewed the segment from the center of the circle as moving with circular motion. They observed that the segment would move clockwise or anti-clockwise depending on how the scalar is moved in relation to the origin. They also suggested that the segment was similar to the radius of a circle. Their excerpt reads as follows:

> **Ben:** *The motion is a circular motion. As you move to the right side [referring to moving the scalar away from the origin (right) motion is clockwise and towards the origin (left) the motion is anti-clockwise] it's clockwise and anti-clockwise. It depends on which side you move.*
> **Tutor**: *Okay but I am saying if you were just describing the vector itself [referring to the link]. Just this vector and again I am not looking on the screen so I have to understand. What can you tell me about this vector [referring to the link] as the force here changes [referring to the blue vector]? What is happening to this vector here [referring to the link]?*
> **Sam:** *It's like a part of the radius with the center that's moving in circular motion [referring to the link].*
> **Tutor:** *A vector moves in a circular motion [referring to the link].*
> **Sam:** *Ya! and its initial position is the center of this circle.*
> **Tutor:** *Is that enough for me to ah: Is that enough information for me to know what the changes are to this vector? I am just trying to ah maybe to tell you what I am looking for a little bit more. Maybe it ain't, I*

know vector by direction and length. These are the two things about the vectors. So if you are describing a vector to me and want it to make sense, I have to know something about those two aspects of the vector. So what would you say about the changes to this vector?

Ben: *This one has circular motion [referring to the link]*

Sam: *Circular motion describes ah, the length that is always the same.*

Tutor: *What changes?*

Sam: *Only the direction.*

The students' description of the motion of the link as a vector at the center of the circle only made reference to its length unless explicitly asked for by the tutor.

In the hope of discovering why these students mainly characterized vectors by their magnitude while omitting their direction the following hypothesis is proposed. Upon further examination of the problem of translation of Planes (Session I Activity 2), it appeared to center around the topic of dynamics (the study of forces that cause motion). In a typical physics textbook this section deals with "Newton's Three Laws". Its focus is to demonstrate the forces that act on an object by means of a free body diagram with the direction of these force vectors implicitly understood. Hence the activity *Planes* is reminiscent of the free body diagrams these students are accustomed to. Thus, recalling their prior knowledge in this area possibly led the students to characterize vectors only by their magnitude.

In reviewing the responses given by Group II, for Session I Activity 2, *Planes*, Jenn and Rachel appeared to be comfortable with describing vectors by their direction and magnitude. However, Rachel was uncertain about how vectors function in the physical setting of the activity on *Planes*, when asked to move the blue vector (as a vector) and observe its effect on the blue plane. She replied:

Rachel: *When you move the whole vector [referring to the blue vector], why doesn't it change [referring to the blue plane]?*

Even though Rachel appeared not to understand how vectors can be represented as forces to move objects she was still able to notice that the vectors' magnitude and direction did not change.

2.4.2 Area of difficulty 2: *Equality of vectors as equality of their lengths*

The focus of the students of group I, was on the magnitude of vectors and not their direction, led them to confusion when deciding whether two vectors were considered equal. For example, when Ben and Sam performed Activity 4 *Vector addition 1* from Session I, the group was asked to find a vector **w** so the vectors **v** and **v** + **w** are equal (see Figure 2.13.). This can be done by letting vector **w** equal to zero. The excerpt from the protocol reads as follows:

Ben: *Equal means their length or their direction?*
Tutor: *That is a good question. What does equal mean?*
Ben: *If the length in different places are equal, but ah ...*
Tutor: *But when are two vectors equal?*
Sam: *Excuse me!*
Tutor: *Pardon! Have you made now the vector v + w equal to the vector v?*
Sam: *I am not sure.*
Tutor: *Why?*
Sam: *Cause I am not sure that it has the same length.*

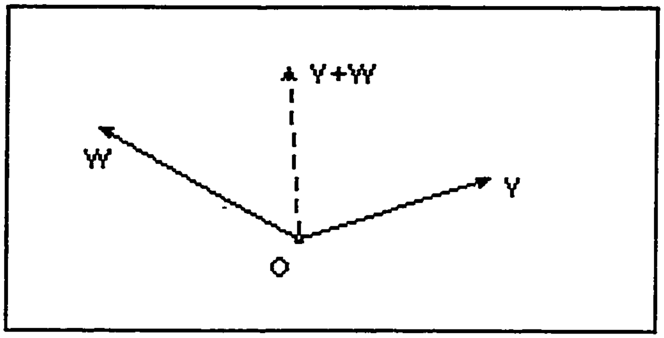

Figure 2.13. The sum of vector **v** and **w**

The excerpt suggests that the students were strongly under the influence of their prior knowledge from studying vectors from physics. This was in spite of the tutor telling them that when checking whether two vectors are equal it is not enough for them to have the same length.

The second group did not exhibit any noticeable problems when answering the questions from the same activity as group I. One possible reason for this was that Rachel and Jenn measured the length of each individual vector **v**, **w**, and **v** + **w**, and worked from a numerical viewpoint and not a conceptual one.

2.4.3 Area of difficulty 3: *The classification of vector quantities as being equal to scalar quantities*

In Session II Activity 1, *Linear combination 1* (see Figure 2.7) both groups of students were observed as classifying vector quantities as scalar quantities when asked to find values for scalars k_1 and k_2 such that the vector $\mathbf{w} = k_1v_1 + k_2v_2$ is equal to the zero vector. The vectors v_1 and v_2 were given to be non-collinear. In considering group I, the students were primarily characterizing vectors by their magnitude. This instinctively led them to incorrectly classify vector quantities as scalar quantities. Sam replied:

> **Sam:** *Ok we should make k_1v_1 and k_2v_2 equal put them in the opposite direction.*

Since Sam was representing vectors as forces the zero vector was perceived as having a zero force. Therefore, the vector equation, $\theta_{(vector)} = k_1v_1 + k_2v_2$, was reduced to an algebraic equation $0 = k_1v_1 + k_2v_2$ where, k_1v_1 and k_2v_2 were seen as scalar quantities and not as vector quantities.

In analyzing the second group's response to the same question, Rachel mentioned that k_1 and k_2 both have to equal zero which is subsequently the correct answer. But when the tutor probed her understanding by asking her if this was the only possibility, her response was as follows:

> **Rachel:** *So if they were reciprocals or... is that the right word? If one's five and one's negative five then they'd.*
> **Tutor:** *Well maybe.*
> **Jenn:** *Try it.*
> **Rachel:** *That depends on what the vectors are.*
> **Tutor:** *If the vector's were what?*
> **Rachel:** *If the vectors were equal that would work.*
> **Tutor:** *Equal?*
> **Rachel:** *If the vectors were exactly the same.*
> .
> .
> .
> **Rachel:** *Well, the... if you had k_1 such that k_1v_1 is the reciprocal... they're reciprocal... and k_2 such that k_2v_2 is the reciprocal... they're reciprocals of each other, then they'd cancel each other out.*

This excerpt suggested that Rachel might be interpreting the vector equation $\theta_{(vector)} = k_1v_1 + k_2v_2$ as the following algebraic equation $0= k_1v_1 + k_2v_2$ (with $v_1=v_2$, $k_1=5$ and $k_2=-5$). Therefore that equation was reduced to $0= 5v_1 - 5v_1$. This suggested that she was thinking about a vector solely in terms of its magnitude.

2.4.4 Area of difficulty 4:
The geometric representation of vectors as a source of visual obstacles

The students were able to perform operations on vectors geometrically with the aid of the Cabri macro construction *Vector addition* and *Scalar multiplication.* Once these operations were performed on the vectors, however, *some of the* resulting geometric diagrams presented themselves as visual obstacles for the students in group I. A first instance of this obstacle was noted in Session I Activity 4 *Vector addition 1.* The students were to construct the sum of two non-collinear vectors **v** and **w** (the vector sum is labeled **v** + **w**). They were, then asked a series of questions and instructed to draw line segments from **v** to **v** + **w** and from **v** + **w** to **w** (see Figure 2.14). The tutor then asked the following question to the group, "What kind of geometrical figure did you obtain?"

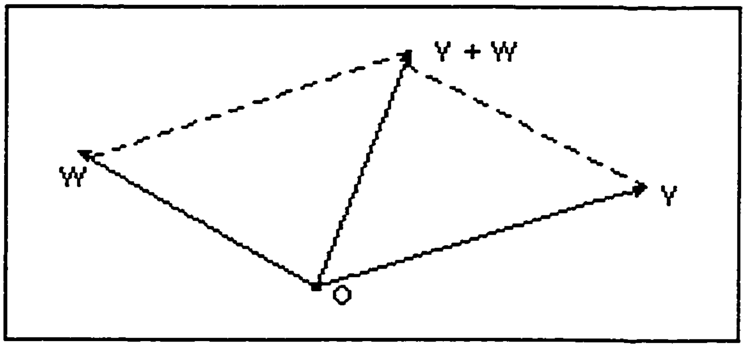

Figure 2.14. Classification of a parallelogram as a rectangle

The group was looking at a figure similar to the one shown in the figure above. Shortly after the students constructed the sum of vectors **v** and **w**, Ben had strong convictions that the vectors form a rectangle:

> **Ben:** *The sum of the vectors is acting like the two corners of the rectangle.*

Ben continued to call the figure a rectangle several more times even though the tutor told him that in a rectangle the measure of each interior angle was equal to 90 degrees.

A second instance of the visual obstacle was observed in Session I Activity 6, *Scalar multiplication 1*. The students were asked to multiply a vector **v** by a variable scalar *k*, which was controlled by a point on a number line. Once the operation *scalar multiplication* was performed, the new vector was labeled **kv** (see Figure 2.15).

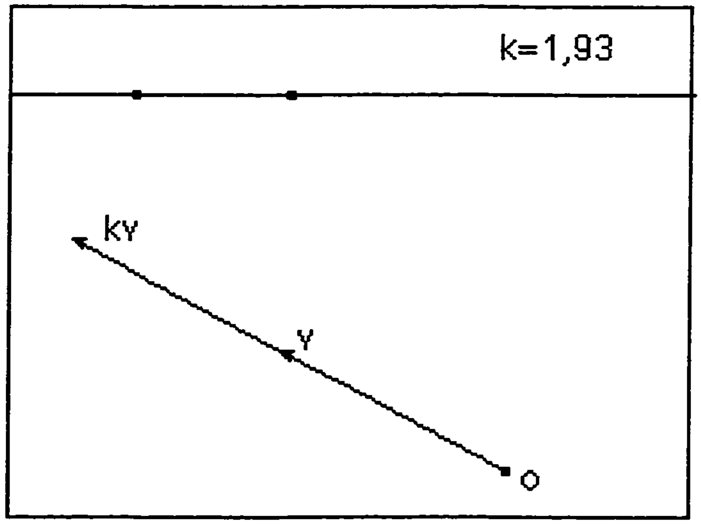

Figure 2.15. Scalar multiplication of vector v and scalar *k*

The students were then asked to describe the relation between **v** and **kv**. Both Ben and Sam appeared to have perceived the previous figure as vector addition of two forces, which were acting in the same direction. This obstacle was most likely the result of their physics background. The students gave evidence of this in the excerpts of their protocol:

Sam: *kv is the length of v + this distance which is plus the length of k.*
Ben: *It should be v + k.*

While Ben was responding to the question he was pointing at the vector **v** and the distance between the tip of **v** and **kv**. He appeared to be looking at the vector from the head of **v** to the head of **kv** as a vector which he identifies as vector *k*. He associated this vector *k* as being the same distance as the point *k* on the number line from the origin.

The tutor tried to use arithmetic to help the students realize that the diagram was representing scalar multiplication and not vector addition. It was his intention to use arithmetic because it produced numerical results that the students could verify. Therefore, the tutor instructed the students to measure the lengths of the vectors **v** and **kv**. He also instructed Ben and Sam to move the scalar *k* until its value is 3 in order to simplify the arithmetic (addition and multiplication). However, this demonstration did not seem to convince them. The excerpt reads as follows:

> **Tutor:** *What happens if k is 3, let's say to make is easy on arithmetic?*
> **Sam:** *k is 3*
> **Ben:** *move it to the right side.*
> **Sam:** *If k is 3 this is 5 [referring to the vector kv]*
> **Ben:** *.08 it should be 3.69, it's larger*
> **Sam:** *So kv is the sum.*
> **Tutor:** *The sum of ?*
> **Sam:** *Of k and v.*
> **Ben:** *of kv. is 5.08 the distance from o and kv is 5.08, but it should be 3 + 1.69. It should be 4.69 but it's larger.*

Another visual problem that was noticed within group I occurred in Session I Activity 7, *Scalar multiplication 2* (see Figure 2.6). The vector **v** represents the sum of the tension forces in the rope (not shown in the diagram) that acts on the object in equilibrium. The students, however, did not recognize this right away and Sam states that this was the source of confusion in the group.

> **Ben:** *The sum of the actual vector [He is referring to the vector that will represent the change in mass of the object but he does not associate it with the vector v which is already on the screen.] will depend on the weight of the object and the more the weight the greater the force will be.*

Tutor: *The sum of which vectors?*
Ben: *These two acting in different directions [referring to the tension forces in the rope made by the suspended object]. We want to create the sum of the vectors or something.*
Sam: *v is already the sum of these two forces.*
Tutor: *v is the force which acts on this point of suspension [referring to the point at which the mass it connected to the rope]*
Sam: *We are confused only because of these two, v is the sum of these two.*

In trying to understand the possible sources of these difficulties led me to the examination of some elementary physics textbooks. The explanation for Ben assuming that the addition of two vectors **v** and **w** would result in a rectangle could be as follows. There are three ways of adding vectors in physics geometrically: by means of parallelogram, triangle and polygon methods. In the parallelogram method of vector addition only two vectors can be added at a time. These vectors, say **u** and **v**, are connected by their tails from a common point and the resultant vector **u** + **v** is the diagonal of a parallelogram formed with **u** and **v** as two of its four sides. In the triangular method of vector addition only two vectors can be added at a time. The vectors say **u** and **v** are placed from the tail of one to the tip of the other, and the resultant vector **u** + **v** is drawn from the tail of the first to the tip the second. In the polygon method of vector addition several vectors can be added at once. The vectors are placed from tail to tip with the resultant vector drawn from the tail of the first to the tip of the last.

In high school physics, when adding two vectors (**v** and **w**) using the triangular method, the vectors are separated by a 90-degree angle. The choice of the 90-degree angle is most convenient since the equation $|u|^2 = |v|^2 + |w|^2 - 2|v||w| \cos U$ (where U is the angle between vectors **v** and **w**) simplifies to the Pythagorean Theorem ($|u|^2 = |v|^2 + |w|^2$).

This minimizes any calculation that might be required to solve a particular problem. Ben's previous exposure to vector addition from physics might have given him the impression that the figure is a possibly badly drawn rectangle.

Ben's obstacle was consistent with what Presmeg (1986), described as "*the one-case concreteness of an image or diagram (which) may tie thought to irrelevant details, or may even introduce false data* (p.44)". For example, some of their students assumed that lines were parallel if they looked parallel. This was the case with Ben. He perceived the angle that was formed by vectors **v** and **w** to be 90 degrees.

The inability to see a diagram in different ways can explain the difficulties observed by the students from group I when performing the operation of scalar multiplication.

When adding two vectors **v** and **w** for example which act in the same direction, while having the same or opposite orientation, "*their numerical sum is the same as their algebraic sum* (Murphy and Smoot p.61)" see Figure 2.16. The convention (in physics) is to join the vectors from tail to tip and add their lengths to obtain a new vector **v** + **w**. The following vector diagram illustrates this situation.

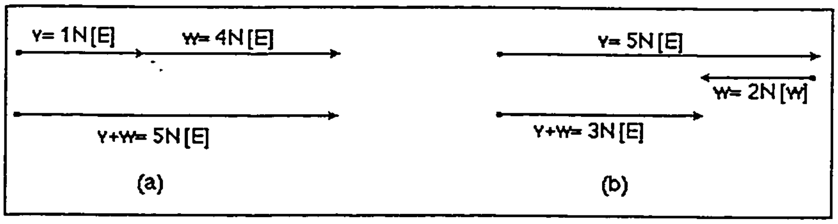

Figure 2.16. Diagram of vector addition resembling scalar multiplication

Here, the operations of scalar multiplication and vector addition share the same diagram for two different concepts. Therefore, Ben and Sam associated Figure 2.16a as vector addition and not scalar multiplication. Examples of scalar multiplication appear in many places in physics: force (F= ma), momentum (p= mv) and weight (w= mg) but are not called as such. The vector quantities **a**, **v** and **g** stand for acceleration, velocity and acceleration due to gravity respectively, while the scalar quantity *m* represents the mass of an object. Most of the problems, however, that deal with scalar multiplication in physics do not consider the case of a variable mass of an object.

The last visual obstacle encountered by group I was a result of a standard free body diagram of a system in equilibrium. The free body diagram for this system was presented in a non-standard way to the group and, as reported by Presmeg in 1986 "*an image of a standard figure (diagram) may induce inflexible thinking, which prevents the recognition of a concept in a non standard diagram*". The variation in this problem was that the sum of the tension forces t_1 and t_2 in the rope was equal to the vector **v** that was not placed at the point of equilibrium, but to the left of the system. In the construction of a free body diagram all the external forces are shown as emanating from the object.

A review of the same activities for Rachel and Jenn suggest, for the most part, that they were able to perform the intended operations without incident. However, in Session I Activity 6, *Scalar multiplication 1*, Jess had the same concerns as Ben and Sam about the resulting diagram that was formed as a result of performing the operation scalar multiplication on a vector, but she does not have too much convictions about it since she has no formal background in physics.

> **Jenn:** *So what does this mean from this point to this point is 1.4 [from tip of the old vector to the tip of the new vector] What is k, what is this?*

2.4.5 Area of difficulty 5:
The notion of scalar multiplication: Seen as extending preexisting vectors and not creating new ones

In Session II Activity 1 (see Figure 2.7), the construction of the linear combination $\mathbf{w} = k_1v_1 + k_2v_2$, revealed a misunderstanding that both groups had concerning scalar multiplication. It appears that the groups were uncertain about the function of scalars k_1 and k_2. One of the tasks in this activity required the groups to place an arbitrary vector **u** on the screen (starting from the origin) and to represent **u** as a linear combination of v_1 and v_2. In considering group I, Ben appeared unsure of how the scalars were to be manipulated to form new vectors form the operation of scalar multiplication, even though his partner Sam was trying to explain it to him, and replied:

Ben: *I'm getting confused here, but you can try.*

Ben relinquished control of the mouse to Sam who quickly adjusted the scalars on the number line, thus expressing vector **u** as a linear combination of v_1 and v_2. However, even though Sam was able to manipulate the scalars successfully, he was under the impression that it was the vectors v_1 and v_2 that were moving. This was seen in the following example.

The tutor immediately presented Ben and Sam with the following situation. He drew three vectors v_1, v_2, and **v** all starting from a common point on a sheet of paper, and told them that the vectors were fixed (i.e. they are not allowed to move them) see Figure 2.17. The tutor asked the group the following question. Can you find scalars k_1 and k_2 such that v will be a linear combination of v_1 and v_2?

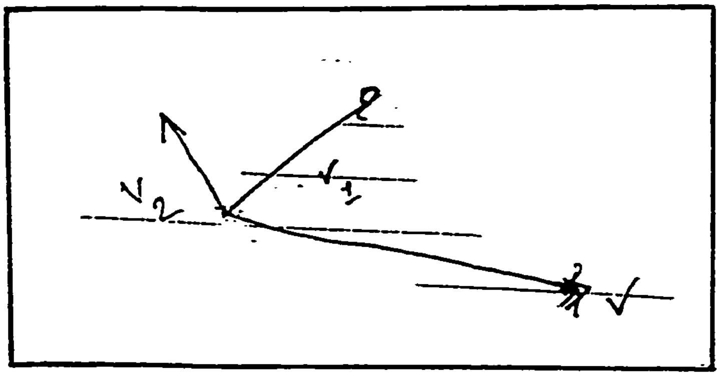

Figure 2.17. Diagram drawn by tutor

Sam tried to write vector **v** as a linear combination of v_1 and v_2 by rearranging the vectors (even though he was told that all the vectors are fixed) to the following configuration with v_1 and v_2 longer than originally given (see Figure 2.18).

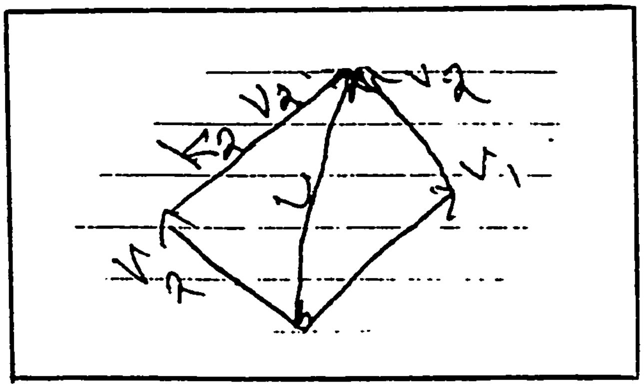

Figure 2.18. Diagram drawn by Sam

The tutor reminded Sam that the vectors were fixed and he was not allowed to move them. At this point, Sam's facial expression appeared to be one of uncertainty so the tutor asked him the following question:

Tutor: *You don't think it could be done?*
Sam: *No.*
Tutor: *But, you could do it here [referring to the previous problem that was done in Cabri.]. You did it here. You did it from here and what's the difference?*

Sam: *Cause we moved the vectors.*

The tutor proceeded to explain to Sam and Ben that when they moved the scalars on the number line they were in fact changing the length and direction of the vector **w** since it depends upon the length of k_1v_1 and k_2v_2. However, Sam had some difficulty accepting this explanation about the function of scalars in the paper-pencil environment since he did not see their existence:

> **Sam:** *But, what we care about is to change the direction*
> *[making reference to v1 and v2 of in figure].*
> **Tutor:** *Ah, so why can't you do it here?*
> **Sam:** *Because you don't have a scalar.*

The difficulties that Ben and Sam encountered could be explained from their physics background. Sam's first instinct was to rearrange the vectors, which is permitted in physics as long as the magnitude and direction remain unchanged. However, his knowledge of vectors was still primarily focused on magnitude and not direction. Note that in both situations the vector **v** has approximately the same length but different direction and is viewed as equal to the original.

The way how Sam viewed the non-existence of scalars in the paper-pencil environment was of convention. When we have the following equation $\mathbf{u} = k_1v_1 + k_2v_2$, where $k_1=k_2=1$, it is usually written as $v_1 + v_2$ and not $(1)v_1 + (1)v_2$ since the number 1 is implicitly understood, thus giving the false impression of absence of scalars.

The analysis of the protocols for Group II also revealed that these students had similar misconceptions about scalar multiplication. This operation was not viewed as creating a new vector but only as a way of extending the old vector. For example, when a vector v_2 was multiplied by scalar k_2, the new vector k_2v_2 was not seen as a new vector. It was only seen as an extension of v_2. Evidence of this line of reasoning was present in

Session 2 Activity 1 when the group was asked to draw any vector **u** starting from the origin and to find scalars k_1 and k_2 such that $k_1v_1 + k_2v_2 = \mathbf{u}$. Rachel replied:

Rachel: *Move k_2 and v_2 around until u is the diagonal.*

Her response seems to indicate that she was thinking that the scalar k_2 is controlling the vector v_2.

In the second activity the students were asked to represent the vector **w** as a linear combination of the vector **u** and **v** (see Figure 2.8). Their excerpt was:

Rachel: *So if we bring down v ... can we change v?*
Jenn: *Can't we just sort of move it around? How do you move a vector?*

Rachel and Jenn's responses seemed to indicate that they were uncertain about how the operation of scalar multiplication is used to create new vectors.

2.4.6 Area of difficulty 6: Notion of a basis: *Students believe that all vectors can be constructed from a single vector*

In Session II Activity 3 the students were introduced to the notion of a basis in two dimensions. In the first question, the students were told that they were given one vector and asked "What other vectors can you obtain from it if you are allowed to perform the operation of vector addition and scalar multiplication, or a combination of both these operations". In the first group, Ben's willingness to rearrange vectors led to complications in understanding the question; he replied:

Ben: *Suppose v is here and if you put here [making reference to any arbitrary vector different to v], you know, the vector, you can get the linear combination.*
it's possible to get any direction, it's possible to get the value.

Ben's response suggested that he was interpreting the question in the following manner. If someone places a vector v on the screen, followed by another vector w that is different from **v**, then he can always move the vector **v** so that it will coincide with the vector **w**. Therefore, all vectors can be expressed as a linear combination of a single vector **v**.

Once again the idea of moving the vector **v** may have come from Ben's physics background which unfortunately created confusion in understanding the notion of the span of a single vector **v**.

In the second part of the question Ben and Sam were given two non-collinear vectors and asked the following question, "What vectors can you get as linear combinations of these two and what vectors you can not get as linear combinations of these two?". The excerpts read as follows:

> **Sam:** *Ok you can get 3 of them*
> **Tutor:** *Uh huh*
> **Sam:** *Yah, and you get another 3 which will be 90 degrees each time from the first. Did you get it?*
> **Tutor:** *How do you mean that 90 degrees?*
> **Sam:** *I mean the first will be like this, between them [pointing between the vectors where their sum would be approximately (see figure 2.19 vector a).]*
> Tutor: *Yah*
> **Sam:** *Ok, the second will be 90 degrees from this [making reference to vector b in figure 2.19.]. And the third will be 180 degrees [making reference to vector c in figure2.19.]. The second will er, the last one will be 270 degrees [making reference to vector d in figure2.19].*
> **Tutor:** *So, which vectors are the vectors which are not linear combinations of these two?*
> **Sam:** *Everything else, except this one.*
> **Sam:** *These you can get. You can get 4 [referring to the 4 vectors a, b, c, and d that he described earlier].*

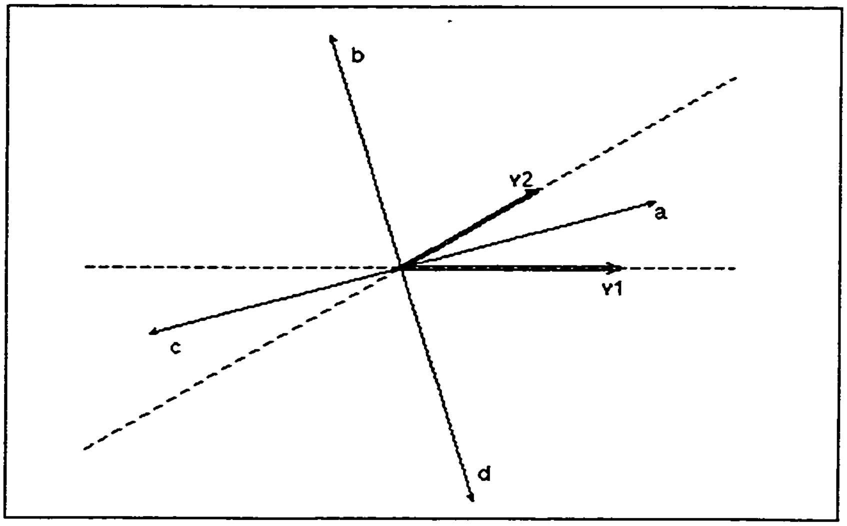

Figure 2.19. Notion of a basis for two non-collinear vectors

Sam understood from the previous activity that once the two non-collinear vectors were selected they could not be moved. With this in mind and recalling from previous activities that the sum of two non-collinear vectors lies in between the two vectors that were added, he replied that only four vectors could be formed. This response confirmed Sam's uncertainty about the function of the scalars and how they could be used to make new vectors. For example, the result of a linear combination of two non-collinear vectors need not lie between them. An example of this would be when one of the scalars is equal to zero.

The second group did not have that same difficulty when responding to the questions of Session II activity 3. In answering the question on span of a single vector Rachel replies that infinitely many vectors could be obtained, but they will all be collinear. In answering the question on span of two non-collinear she also replied that "*you can get vectors anywhere*".

2.4.7 Area of difficulty 7:
Physical embodiments of linear algebra concepts as a source of mental conflict

The students were introduced to several pre-made models of physical situation, which embodied the idea of transformations of vectors. They were asked to manipulate specific vectors within these models and to observe and describe its effect on another vector within the same system. The descriptions of the transformation for group I was expressed in physics terms because this has where the students were used to seeing them. Unfortunately, these descriptions, while valid, seemed to fall short of the expected result[11] by the researcher. For example, in Session III Activity 1 *symmetric forces* (see Figure 2.12) the students were asked to describe how the vectors F_1 and F_2 were related to each other. The students replied:

> **Ben**: *The angle is increasing.*
> **Sam**: *The angle of ah F_1 and F_2.*
> **Sam**: *When you increase the weight these vectors are increasing.*
> **Ben**: *And the angle.*
> **Sam:** *And the angle between them is increasing too.*
> **Ben**: *v_2 will be the mirror image of.*

Once the instructor realized that he was not getting the response that he desired, he tried to move the students' line of thought from the physical situation which led to an interesting result.

> **Tutor**: *Suppose I don't have a physical situation any more all I have is the vector [referring to the vectors F1 and F2 form figure 2.12]*
> **Sam**: *You can't separate it from the physical.*

Here we see Sam's strong attachment to the physical situation of pulleys and weights. His previous exposure of physics led him to some inflexible thinking. He

[11] The two vectors F1 and F2 have the same magnitude and different direction.

experienced difficulty in trying to separate the vectors from the physical situation in which they were presented. This difficulty could have been a result of his thinking of vectors as forces. He might be under the impression that vectors only exist in some physical situation.

The students in the second group were able to describe the motions of the vectors in terms of their length and direction without incident. However, the extent to which they understood the embodiments of the physical setting could not be determined.

2.4.8 Area of difficulty 8:
The axiomatic definition linear transformation

In comparing two unequal transformations (Session III Activity 4), there seemed to be some disagreement between the students from group I on the equality of the transformation. Ben said that the transformations did not coincide, whereas Sam thought the two transformations were the same. Sam based his decision on analyzing the motion of the vectors and, because they were moving in the same direction, he saw the transformations as being equal. He was able to overcome this difficulty by readjusting his definition of the word "same" and said, "when we always put v_1 the same length as v_2, w_1 should be always the same on w_2".

In analyzing the second group's response to the same activity Jenn was also looking at the motion of the vectors. She did not see T as being a function where the vectors v_1 and v_2 were like the domain and when acted upon by T got transformed into w_1 and w_2 in the range.

An analysis of both groups' work with the linearity equation

$T(k_1v_1+k_2v_2) = k_1T(v_1)+k_2T(v_2)$, revealed several things. The students were given three transformations and asked to verify whether they were linear or not. The first transformation was a rotation followed by dilation, which is linear. The second was the "semi linear" transformation, defined in an orthonormal system of coordinates as $T((0,0)) = (0,0)$ and if $v= (x,y) \neq (0,0)$ then $T((x,y))= c(-y,x)$ where $c= |x|/||v||$, which preserves scalar multiplication and not vector addition. The third transformation preserved neither scalar multiplication nor vector addition.

In analyzing the results for the first group, I will focus on the first activity because the last two were done well by this group. In verifying the first transformation, the students did not realize that once a vector **v** is transformed by a transformation T its image is dependent on the vector **v**. The students would create a free vector, rotate it, then label the rotated vector as v_2. The students also did not see rotation combined with dilation of a vector as a transformation.

> **Sam:** *But, what is the relation? Ok so what is the relation [referring to the transformation T].*
> **Ben:** *Don't know.*
> **Sam**: *Ok so, what is the relation? Dilation or rotation, what's it?*
> **Ben:** *I've got the T. We find T.*
> **Sam:** *Yah, what's the T?*

It appeared that Ben and Sam were uncertain about the nature of the linear operator T. They assumed that T stood for the Shear-11 which they verified in Session III Activity 6.

> **Sam:** *But, we don't know what's T, you know.*
> **Tutor:** *Why don't you know what T is?*
> **Sam:** *Yesterday you told us it was Shear-11 but today you didn't.*

The second group also showed the same misunderstanding of the transformation T. They assumed that T represented the Shear-11 transformation which they verified in

Session III Activity 6, and not the newly defined transformation rotation by 55 degrees and dilating by a factor of 1.6. The excerpt read as follows:

> **Rachel:** *So we will take that shear*
> **Tutor:** *Why shear, you've already checked shear, what you have to do is this*
> **Rachel:** *Yes, but you have to go and do this shear, w and see if it's equal to…*
> **Tutor:** *You've already done that. The transformation that you're checking for linearity today is this one, you do rotation and the…*
> **Jenn:** *Why can't we do this one, what's the difference?*

The students were uncertain about the nature of the linear operator T. After they had applied the correct transformation Rachel mentioned:

> **Rachel:** *Ok, I am not very clear about what we just did. "Angle" of rotation and dilation factor are these functions?*

In analyzing the results of how a square grid with 8 vertices gets transformed into a parallelogram grid (Session IV Part II), was solved through spatial orientation by both groups. They had to keep in mind that when looking at a transformation of the given grid, the points on a line remained on the same line before and after transformation. Parallel lines remain parallel before and after transformation and the ratio between the lengths of segments' change proportionally in the transformation. Even though they claimed that it was not useful to know that the transformation was linear, the students still used these properties of linearity in their reasoning to solve the problem.

> **Tutor:** *…How did you use linearity at all, was it useful to know that this figure was transformed because you had a linear transformation? was that*
> **Sam:** *it wasn't useful for me.*
> **Tutor:** *It wasn't? so, do you think if, if the transformation was linear , er.*
> **Sam:** *I don't know. I tried to put, to get something useful from knowing this, that this one is linear*

A summary of the difficulties encountered by both groups can be found in table 1.

Table 1: Summary of the difficulties encountered by the groups

Areas of Observed Difficulties	**Ben and Sam**	**Rachel and Jenn**
Difficulty 1: Characterizing a vector by its magnitude	Yes	No
Difficulty 2: Equality of vectors as equality of their lengths	Yes	No
Difficulty 3: Distinction between vector/scalar quantities	Yes	Yes
Difficulty 4: Visual obstacles	Yes	No
Difficulty 5: The notion of scalar multiplication	Yes	Yes
Difficulty 6: Notion of a basis	Yes	No
Difficulty 7: Difficulty with embodiments	Yes	No
Difficulty 8: The axiomatic definition of linear transformation	Yes	Yes

Chapter III

Conclusion

3.1 *Summary*

Some general issues that interfere with learning linear algebra in a geometric setting were discovered in this study. These are issues that gave difficulty to both the students with physics experience and those without such experience. Thus, these issues appear to be inherent to the process of learning linear algebra. These issues were:

1. Distinction between vector quantities and scalar quantities

Students in both groups had frequent difficulties understanding that vector quantities are distinct from scalar quantities. It appears that this is a common problem for students learning linear algebra. The conceptual distinctions between vector and scalar quantities must be made clearer to students, possibly in a preemptive fashion. This would eliminate much of the unnecessary errors and false starts students make when attempting to learn linear algebra.

2. The notion of scalar multiplication

Students in both groups had much problems grasping the notion that scalar multiplication is one way that new vectors are sometimes created. They tended to think that scalars simply modified old vectors (stretching or shrinking), rather than creating new ones. The way that new vectors are created by the operation of scalar multiplication, and the fact that they are new vectors, must be emphasized with these students.

3. The axiomatic definition of linear transformation

The students exhibited a number of difficulties using linear transformations. When comparing two unequal transformations one student from each group analyzed the motion of the vectors and used the fact that the vectors appear to coincide at special points to conclude that the transformations were equal transformation. However, the notion of equality of transformations has to be made explicitly distinct from the vague idea of similar. These students also did not regard "T" as an arbitrary function that transformed one vector into another vector. Both groups were under the impression that the linear operator T stood for a specific transformation they verified in a previous activity. It is essential to make it explicitly clear to students that the operator T represents a variable transformation which could stand for many transformations.

As well, there were a number of difficulties that emerged only for students with physics backgrounds. This might suggest that these issues are ones that are especially problematic for students with previous exposure to physics knowledge. As well, these are less likely to be inherent problems in the learning of algebra, and may indicate competing processes at work. The problems specific to experience with physics include:

1. <u>Characterizing a vector by its magnitude</u>

This was a recurring problem that was noted by Ben and Sam, which might have been influenced from their background in physics. These students were accustomed to dealing with vectors quantitatively, with their direction implicitly understood. Therefore it appears that they transferred this same reasoning when working with vectors geometrically as arrows. This created problems when deciding when two vectors are considered equal.

2. Equality of vectors as equality of their lengths

The characterization of a vector by its magnitude only, and not its magnitude and direction, created confusion for Sam and Ben when considering the equality of two vectors. They were not sure if two vectors were considered equal if they had the same magnitude or direction or both. It appears that direct emphasis is needed on two features of a vector: its magnitude and direction.

3. Visual obstacles

The geometric representation of vectors in the context of the operations of vector addition and scalar multiplication caused some confusion in Ben and Sam. Ben's classification of Figure 2.13 as a rectangle, and the group's interpretation of the diagram for scalar multiplication as synonymous with vector addition, gave the impression that these students' previous experience with physics was interfering with their acquisition of new knowledge. Thus the meaning of the diagrams cannot be taken for granted in teaching and it must be made explicit to the students so as not to create confusion with new concepts.

4. Notion of a basis

Students' experience from physics in rearranging vectors creates confusion in understanding the notion of span of a single and two non-collinear vectors. The idea of vectors being fixed is not easily accepted and must be made explicit.

5. Difficulties with embodiments

The way in which vectors were presented in physical situation of pulleys and weights led to the students having difficulty abstracting the mathematical notion of a vector. This suggests that we must be careful with the representations that we use to enhance students' understanding of a concept.

This study highlights the importance of being aware of the previous relevant knowledge of students when attempting to teach them a complex topic such as linear algebra. Being aware of such knowledge should help teachers anticipate areas of weakness, where "unlearning" may be necessary. At the very least, teachers will have a sense of the specific problems that they are dealing with, and the ways to address these problems. Ideally some of these solutions will be preemptive ones.

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