Asymptotic Analysis of Vibrations of Thin Cylindrical Shells

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ABSTRACT

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An algorithm for the asymptotic solution of boundary value problems involving vibrations of thin cylindrical shells by means of symbolic computation is presented. The algorithm is based on the method of asymptotic integration of the vibration equations of thin shells, developed by Goldenveizer, Lidsky and Tovstik. A linear shell theory of the Kirchhoff-Love type is employed. The equations describing the vibrations of thin shells contain several parameters, the main of which is the small parameter of the shell thickness. Formal asymptotic solutions in different domains of the space of the parameters are obtained by using a computational geometry approach. Computer algebra methods are employed to study the characteristic equation that involves the construction of the convex hull of a set of points.

The study is limited to the cases for which the asymptotic representation of the solution is the same in the entire domain of integration, and solutions are linearly independent (no turning points, no multiple roots). Axisymmetric as well as non-axisymmetric vibrations are considered. The constructed solutions are used for studying the free vibration spectra of the shells.

The numerical results obtained by applying this algorithm to the particular problem of low frequency vibrations of thin cylindrical shells are in good agreement with the results obtained by finite element analysis, as well as with asymptotic results obtained by authors using other solution techniques.

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Table of Contents

List of Figures x

List of Tables xii

List of Symbols xiii

Chapter 1

Introduction 1

Chapter 2

Shell Theory 5

2.1 Introduction 5

2.2 Historical Aspects 6

2.3 Two-dimensional Shell Theories 11

2.3.1 Geometry of the Neutral Surface and its Deformations 11

2.3.2 Equilibrium Equations and Elasticity Relations 14

2.3.3 Shallow Shell Equations 16

2.3.4 Shell Dynamics Equations 18

Chapter 3

Vibration Equations of Thin Cylindrical Shells 19

3.1 Introduction 19

3.2 General System of Vibration Equations of Thin Cylindrical Shells 20

3.3 System of Vibration Equations of Thin Cylindrical Shells of Thin Cylindrical Shells 23

3.4 Asymptotic Solution of Vibration Equations for Thin Shells 25

3.5 Asymptotic Solution. First Approximation 28

3.6 Axisymmetric Vibrations 29

Chapter 4

Methods of Investigation of the Characteristic Equation 30

4.1 Introduction 30

4.2 Introduction to Geometrical Approach 31

4.3 Historical Aspects 34

4.4 Construction of the Convex Hull in 2D 37

4.4.1 The Three-Coins Algorithm 37

4.4.2 Gift Wrapping Algorithm (Jarvis' March) 39

4.4.3 Throw-away Principle (Qull-Hull) 39

4.4.4 Divide-and-Conquer Algorithm for 2D 41

4.5 Construction of the Convex Hull in 3D 44

4.5.1 Gift Wrapping Algorithm for 3D 44

4.5.2 Divide-and-Conquer Algorithm for 3D 45

4.6 Construction of the Convex Hull in nD 49

4.7 Methods for Simplifying the Characteristic Equation 50

Chapter 5

Asymptotic Solution of the Vibration Equations of Thin Cylindrical Shells 52

5.1 Introduction 52

5.2 Axisymmetric Vibrations (m = 0) 53

5.2.1 Analysis for Different Domains κ 55

5.2.1.1 Convex Hull for Domains I 55

5.2.1.2 Convex Hull for Critical Point I 56

5.2.1.3 Roots and Amplitude Vectors for Domains III 57

5.2.1.4 Roots and Amplitude Vectors for Critical Point II 59

5.2.1.5 Roots and Amplitude Vectors for Domains II 61

5.2.2 Boundary Value Problem 63

5.2.2.1 Low Frequency (Domain III) for Simply Supported Edges.. 63

5.2.2.2 High Frequency (Domain II) for Simply Supported Edges.. 64

5.2.2.3 Low Frequency (Domain III) for Clamped Edges 65

5.2.2.4 High Frequency (Domain III) for Clamped Edges 66

5.3 Non-axisymmetric Vibrations 67

5.3.1 Separating Points for Non-axisymmetric Vibrations (4D) 69

5.3.2 Separating Lines for Non-axisymmetric Vibrations (4D) . 71

5.3.2.1 Separating Segments for Domain $0<\tau<1$ 71

5.3.2.2 Separating Segments for Domain $1<\tau<2$ 73

5.3.2.3 Separating Segments for Domain $-4<\kappa<0$ 74

5.3.2.4 Separating Segments for Domain $0<\kappa<-4$ 75

5.3.3 Short Forms of Equation in All Domains, on Separating Lines and at Separating Points 77

5.3.3.1 Short Forms of Equation in Domains 77

5.3.3.2 Short Forms of Equation on Separating Lines 80

5.3.3.3 Short Forms of Equation at Separating Points 83

5.4 Special Cases - $m \sim 1$ and $m = 1$ 85

5.5 Example of Solutions of the Boundary Value Problem 94

5.6 The Superlow Frequencies 98

Chapter 6

Numerical Analysis 104

6.1. Introduction 104

6.2 Numerical Solutions for Non-Axisymmetric Vibrations 104

Chapter 7

Summary, Conclusions and Future Work 110

7.1 Summary and Conclusions 110

7.2 Future Work 112

Bibliography 113

Appendices 118

Appendix 1: Mathematica 3.0 Code for Construction of Convex Hull in 2D

Appendix 2: Mathematica 3.0 Code for Construction of Convex Hull in 3D

Appendix 3: Mathematica 3.0 Code for Construction of Lower part of Convex Hull in 4D

Appendix 4: Mathematica 3.0 Code for Construction of Separating Lines

Appendix 5: Mathematica 3.0 Code for Construction of Short Form of Equation on Separating Line

Appendix 6: Mathematica 3.0 Code for Construction of Short Form of Equation

Appendix 7: Mathematica 3.0 Code for Analysis of Characteristic Equation for Different Boundary Conditions and Frequency Domains

List of Figures

2.1 Geometry of neutral surface G 12

3.1 Circular cylindrical shell 20

4.1 Graph for the proof of Theorem 1 32

4.2 Graph for the quick-hull algorithm 40

4.3 Graph for the divide-and-conquer algorithm in 2D 41

4.4. First step of the construction algorithm of CH in 2D 42

4.5 Fourth step of the construction algorithm of CH in 2D 43

4.6 Fifth step of the construction algorithm of CH in 2D 43

4.7 Sixth step of the construction algorithm of CH in 2D 44

4.8 Graph for divide-and-conquer algorithm in 3D 46

4.9 First step of the construction algorithm of CH in 3D 47

4.10 Second step of the construction algorithm of CH in 3D 48

4.11 Third step of the construction algorithm of CH in 3D 48

4.12 Fourth step of the construction algorithm of CH in 3D 49

5.1 3D convex hull for $m = 0$ 53

5.2 Domains of κ 55

5.3 Newton's diagram for $m = 0$, $\kappa = -5$ 55

5.4 Newton's diagram for $m = 0$, $\kappa = -4$ 56

5.5 Newton's diagram for $m = 0,\ \kappa = 1$ 57
5.6 Newton's diagram for $m = 0,\ \kappa = 0$ 59
5.7 Newton's diagram for $m = 0,\ \kappa = -1$ 61
5.8 τ versus κ for $0 < \tau < 1$ 72
5.9 τ versus κ for $1 < \tau < 2$ 73
5.10 τ versus κ for $0 < \tau < 2$ 74
5.11 τ versus κ for $-4 < \kappa < 0$ 75
5.12 τ versus κ for $0 < \kappa < 4$ 76
5.13 τ versus κ for $-4 < \kappa < 4$ and $0 < \tau < 2$ 76
5.14 3D convex hull for $m \sim 1$ 86
5.15 Newton's diagram for $m \sim 1,\ \kappa = -6$ 87
5.16 Newton's diagram for $m \sim 1,\ \kappa = -4$ 88
5.17 Newton's diagram for $m \sim 1,\ \kappa = -2$ 89
5.18 Newton's diagram for $m \sim 1,\ \kappa = 0$ 90
5.19 Newton's diagram for $m \sim 1,\ \kappa = 2$ 91
5.20 Newton's diagram for $m \sim 1,\ \kappa = 4$ 92
5.21 Newton's diagram for $m \sim 1,\ \kappa = 5$ 93
5.22 3D convex hull for $m = 1$ 94
5.23 3D convex hull for $m \sim h^{(-1/2)}$ 99
5.24 Newton's diagram for $m \sim h^{(-1/2)},\ \kappa = 1$ 101
5.25 Newton's diagram for $m \sim h^{(-1/2)},\ \kappa = 2$ 102
5.26 Newton's diagram for $m \sim h^{(-1/2)},\ \kappa = 10$ 103
6.1 Vibration mode (ANSYS) 107

List of Tables

5.1 Roots and eigenvectors for $m = 0$, $\kappa = 1$ 59

5.2 Roots and eigenvectors for $m = 0$, $\kappa = 0$ 60

5.3 Roots and eigenvectors for $m = 0$, $\kappa = -1$ 62

5.4 Facets for non-axisymmetrical case 70

6.1 Asymptotic versus FEM results for $t/R = 0.01$ 106

6.2 Asymptotic versus FEM results for $t/R = 0.001$ 108

6.3 Asymptotic versus FEM results for $t/R = 0.0005$ 109

List of Symbols

$a, b(p;\ h,\ \lambda,\ m)$	——	matrix entries
$A_1,\ A_2,\ A_3, A_4$	——	points having $\min\{x\}$, $\min\{y\}$, $\max\{x\}$, $\max\{y\}$
$A,\ B,\ C$	——	points of the 2D-algorithms
$A_1(\alpha_1, \beta_1), A_2(\alpha_2, \beta_2)$	—	points in the plane in Theorem 1
A_1, A_2	——	Lamé's coefficients
$B(s)$	——	distance between points on the shell and the axis of symmetry
$C_i, C,\ c,\ d$	——	constants
$c_{\alpha\beta}(s)$	——	weight of term $p^\alpha q^\beta$
$D(\lambda)$	——	determinant of characteristic equation
D	——	bending stiffness
$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$	——	local orthogonal coordinate system
E	——	Young's modulus
F_1, F_2, F_n	——	projections of distributed external load
G	——	neutral surface
h	——	thickness parameter
h	——	number of sides of convex hull

$H(L)$, $H(R)$	——	convex hulls of L and R
H_i, M_i	——	projections of stress couples of internal forces
k_i	——	shell curvatures
K	——	membrane stiffness
$l = n-m$	——	number of additional roots of perturbed system
l_1	——	number of additional boundary conditions on left edge
l_2	——	number of additional boundary conditions on right edge
L	——	length of cylindrical shell
L, R	——	two sets of representative points of equal size
L_μ, L	——	linear differential operators of the 8th and 4th order
$L_{ij}(\alpha_1, \alpha_2)$	——	components of linear differential operator
m	——	wave number in circumferential direction
m	——	order of unperturbed system
M	——	point on S
M_i	——	moment resultants
$M_i = \{k_i, \alpha_i, \beta_i\}$	——	representative points in (k, α, β) space
$M_i = \{k_i, \alpha_i + \beta_i\kappa\}$	——	representative points in (p, h) plane
n	——	number of points
n	——	order of perturbed system
$\mathbf{n}=\mathbf{e}_1\times\mathbf{e}_2$	——	unit normal vector
N_i	——	transverse shear resultants

p, q	——	small parameters
p_i	——	roots of characteristic equation
p_0, p_1, p_2	——	labels of "back", "center", and "front" vertices
$\boldsymbol{r} = \boldsymbol{r}(\alpha_1, \alpha_2)$	——	position vector
R	——	radius of cylindrical shell, characteristic shell length
R_i	——	radii of curvature
s_0, s_1, s_2	——	points in 2D-algorithm
(s, φ)	——	curvilinear coordinates
t	——	thickness of thin cylindrical shell
t/R	——	relative thickness
T_i, S_i	——	components of stress resultant
$U=(u, v, w)=(u_1, u_2, u_3)$		displacement vector
U	——	normal vector to hyperplane which goes through point V_i
U_0, V_0, W_0	——	amplitude vectors in the first approximation
V_i	——	set of demonstrative visible points of Linear Programming
W	——	centroid of representative points
Y^i_k	——	matrix of amplitude vectors
(x, y, z)	——	3D-Cartesian coordinates
(x, y)	——	2D-Cartesian coordinates
(x)	——	abscissa (power of first small parameter)

(y)	——	ordinate (power of first small parameter)
(z)	——	applicate (power of first small parameter)
z	——	distance along the normal to the neutral surface

Greek Symbols

α	——	constant depending on boundary conditions
(α, β)	——	powers of small parameters
(α_1, α_2)	——	orthogonal curvilinear coordinates
α_i	——	weights of polynomial terms
$(\varepsilon_1, \varepsilon_2, \omega)$	——	tangential (membrane) surface deformations in linear approximation
Φ	——	force function
κ_1, κ_2, τ	——	bending surface deformations
$\Gamma=\partial G$	——	boundary of domain G
κ_i	——	depends on order of p_i with respect to μ
$\{\kappa_i\}$	——	set of separating points in 1D
(κ, τ)	——	set of separating points in 2D
λ	——	frequency parameter, eigenvalue
λ_0	——	constant
μ	——	small parameter of shell thickness
ω	——	natural frequency
ρ	——	shell mass density

ν	——	Poisson's ratio
γ_1, γ_2	——	angles of rotation of normal $\boldsymbol{n}$ with respect to $\boldsymbol{e}_1, \boldsymbol{e}_2$
O, Ω, Θ	——	Landau's symbol

Superscripts

*	——	dimensional variables
(α_1, α_2)	——	power of small parameter p
(β_1, β_2)	——	power of small parameter q

Subscripts

(α, β)	——	label for weight $c_{\alpha\beta}(s)$
μ	——	moment terms

Abbreviations

SDOF	——	number of degrees of freedom of system
ODE	——	ordinary differential equation
FEM	——	finite element method
Qhull	——	code for computing convex hull in any dimension

LEDA	——	C++ library of computational geometry software
4D	——	four dimensions
3D	——	three dimensions
2D	——	two dimensions
nD	——	arbitrary dimensions
CH	——	convex hull
BC	——	boundary condition
□	——	Q.E.D., quod erat demonstrandum ("which was to be proved")

Chapter 1

Introduction

Shell structures play an important role in a wide variety of applications, ranging from very small structures visible only by microscope to large structural elements used in building constructions. Thin shells are susceptible to vibrations and buckling.

Only few of the shell dynamics problems arising in practice have exact solutions. Thus, one is usually forced to appeal to approximate solutions (Olver, 1974, Erdelyi, 1956, Evgrafov, 1961, Vainbeg, 1982, Verhulst, 1979, Vishik, 1992 and Wasow, 1965) among which one can distinguish the analytical and the numerical solutions.

At the beginning of the XXth century, the analytical solutions were occupying the main place, but later, especially during the last decades, due to the advent of powerful computers and well-developed numerical methods, in particular, the finite element method, the situation has changed completely. Nowadays, most problems can be solved by means of numerical methods. Consequently, the approximate analytical solutions (Murray, 1984) have been somehow underestimated.

The main advantage of the asymptotic methods is the final analytical formulae that are obtained, for example, for natural frequencies and modal vectors. Usually the order of the relative error of such formulae is known. The accuracy of the formulae increases as a small parameter, in our case the parameter of the shell thickness, converges to 0. The asymptotic formulae permit to study the effect of different shell parameters on the spectrum and modal vectors and to understand the physics of shell vibrations (Koiter, 1960). Among the disadvantages of the asymptotic methods one may list that they may be derived only for shells of relative simple geometry and that they are less accurate as the shell thickness increases.

Numerical methods produce results for a given set of shell parameters with good accuracy, but take computer time. Beside that, the stiffness matrix, which contains terms proportional to a small parameter and to its inverse, becomes ill-conditioned as the relative thickness of the thin shells decreases and, as a result, the accuracy of the calculations decreases. At the same time, asymptotic methods will give more accurate results when the main small parameter becomes smaller. Therefore, the analytical and numerical solutions have to complement each other.

In our study of freely vibrating circular cylindrical shells we adopt the following scheme. We start with the vibration equations of cylindrical shells in terms of displacements (Markus, 1988). We use the exponential representation for the solutions. In this work we limit our attention to the construction of the first approximation to the solution. The form of the solution depends on the roots of the characteristic equations for the powers of the exponents p_i. The characteristic equation contains a number of parameters. In different domains of the parameters, the main terms for the roots will be different. To determine the main terms, the geometrical method based on the construction of the convex hull for the point set was developed in this work. After constructing the formal asymptotic solutions, we use them to solve the boundary value problem for the given boundary conditions. This leads to the characteristic equation for the frequency parameter. Under some assumptions on p_i, this equation may be simplified. Then, the analytical expressions for the frequency parameters are obtained.

A brief description of the following chapters of the thesis is given in the next paragraphs.

In Chapter 2, the equations of two-dimentional shell theory are introduced. A brief history of vibration analysis is first presented. Then, the general system of equations of shell vibrations is introduced using curvilinear coordinates. This system of ODE's includes the relations for the neutral surface deformations, the equilibrium equations and the elasticity relations. Different shell theories including Donnell's and Timoshenko's theories are briefly discussed. The linear shell theory of Kirchhoff-Love type is selected for further study. Finally, the shell vibration equations in general form are obtained.

In Chapter 3, the vibration equations of cylindrical shells are presented. The general system of equations for arbitrary shells of revolution given in Chapter 2 is written here for the case of cylindrical shells. The displacements are the unknowns of this system. The specific form of the vibration equations of the axisymmetric case is discussed.

In Chapter 4, we use a geometrical approach to solve the characteristic equation describing the vibrations of thin circular cylindrical shells. This approach leads to the construction of a convex hull in the space of powers of the small parameters involved in the characteristic equation. The methods of construction of the convex hulls in 2D and 3D are presented. Special attention is devoted to the methods and algorithms used to construct the convex hull and to study the characteristic determinants. These are: standard algorithms of Mathematica 3.0 based on the Graham scan method for 2D cases, algorithms we developed based on the "gift wrapping" methods for 2D and 3D, and the Qhull algorithm based on the same principle for 4D. Methods for simplifying the characteristic equations are also discussed.

Chapter 5 is concerned with the application of the construction methods of the formal asymptotic solutions described in Chapter 4 to the vibration equations of thin cylindrical shells given in Chapter 3. The axisymmetric and non-axisymmetric cases are analyzed. The cases of superlow frequencies are of particular interest. These are frequencies proportional to the positive power of the small parameter of the shell thickness. These frequencies converge to zero simultaneously with the small parameter. The boundary eigenvalue problems are solved for all of the cases mentioned above. For non-axisymmetric vibrations, the system of equations depends on three parameters: h, m and λ. For axisymmetric vibrations we have only two of them, since m is equal to 0. The following boundary conditions are considered in our analysis: simply supported edges and clamped-clamped edges.

In Chapter 6, we compare the results obtained by the asymptotic methods of Chapter 5 with the FEM results we obtained by using ANSYS.

Conclusions and future work directions are presented in Chapter 7.

Selected Mathematica 3.0 codes for the construction of CH in 2 and 3D, for the construction of separating lines and points as well as for the analysis of the characteristic equation are given in the Appendices.

Chapter 2

Shell Theory

2.1 Introduction

In this chapter, the equations of two-dimentional shell theory are introduced. A brief history of vibration analysis is first presented. Then, the general system of equations of shell vibrations is derived in curvilinear coordinates (Koiter, 1960). This system of ODE's includes the relations for the neutral surface deformations, the equilibrium equations and the elasticity relations. Different shell theories including those of Donnell and Timoshenko types are briefly discussed. The linear shell theory of the Kirchhoff-Love type is selected for further study. Finally, the shell vibration equations in general form are obtained.

2.2 Historical Aspects

In this section we present a brief history of shell theories (Soedel, 1981, Bauer *et al.*, 1993).

Vibration analysis has its beginnings with Galileo Galilei (1564-1642), who solved by geometrical means the dependence of the natural frequency of a simple pendulum on the pendulum length. He proceeded to make experimental observations on the vibration behavior of strings and plates, but could not offer any analytical treatment. He was partially anticipated in his observations of strings by his contemporary, Marin Mersenne (1588-1648), a French priest. Mersenne recognized that the frequency of vibration is inversely proportional to the length of the string and directly proportional to the square root of the cross-sectional area. This approach was followed also by Joseph Sauveur (1653-1716), who coined the terminology "nodes" for zero displacement points on a string vibrating at its natural frequency, and also actually calculated an approximate value for the fundamental frequency as a function of the measured static sag at its center, similar to the way the natural frequency of a single degree of freedom spring-mass system can be calculated from its static deflection.

The foundation for a more precise treatment of the vibration of continuous systems was laid by Robert Hook (1635-1703), who established the basic law of elasticity, by Newton (1642-1727), who established that force was equal to mass times acceleration, and by Leibnitz (1646-1716), who established differential calculus. An approach similar to differential calculus called *fluxions* was developed independently by Newton, at the same time. In 1713 the English mathematician Brook Taylor (1685-1731) actually used the fluxion approach, together with Newton's second law applied to an element of the continuous string, to calculate the true value of the first natural frequency of a string. The approach was based on an assumed first mode shape. This is where work in vibration analysis stagnated in England since the fluxion method and especially its notation proved to be too clumsy to allow anything but the attack of simple problems. Because of the controversy between followers of Newton and Leibnitz as to the origin of differential calculus, patriotic Englishmen refused to use anything but fluxions and left the fruitful use of the Leibnitz notation and approach to the investigators on the continent. There, the mathematics of differential calculus prospered and paved the way for Jean Le Rond d'Alembert (1717-1783), who derived in 1747 the partial differential equation which today is referred to as the wave equation and who solved the traveling wave problem. He was ably assisted in this by Daniel Bernoulli (1700-1782) and Leonard Euler (1707-1783), both German speaking Swiss and friends, but did not give them due credit. It is still a controversial subject to decide who did actually what, especially since the participants were not too bashful to insult each other and claim credit right and left. However, it seems fairly clear that the principle of superposition of modes was first noted in 1747 by Daniel Bernoulli and proven in 1753 by Euler. These two must, therefore, be credited as being the fathers of the modal expansion technique or of eigenvalue expansion in general. The technique did not find immediate general acceptance. In 1822 Joseph Fourier (1768-1830) used it to solve certain problems in the theory of heat. The resulting Fourier series can be viewed as a special case of the use of orthogonal functions and might as well carry the name of Bernoulli. However, it is almost a rule in the history of science that people who are credited with an achievement do not completely deserve it.

Progress moves in small steps and it is often the individual who publishes at the right developmental step and at the right time who gets the public acclaim.

The longitudinal vibration of rods was investigated experimentally by Chladni and Biot. However, not until 1824 do we find the published analytical equation and solutions, done by Navier. This is interesting since the analogous problem of the longitudinal vibration of air columns was already done in 1727 by Euler.

The equation for the transverse vibration of flexible thin beams was derived in 1735 by Daniel Bernoulli and the first solutions for simply supported ends, clamped ends, and free ends where found by Euler and published in 1744.

The first torsional vibration solution, but not in a continuous sense, was given in 1784 by Coulomb. But not until 1827 do we find an attempt to derive the continuous torsional vibration equation. This was done by Cauchy in an approximate fashion. Poisson (1781-1840) is generally credited for having derived the one-dimensional torsional wave equation and the credit of giving some rigorous results belongs to Saint-Venant (1797-1886), who published this in 1849.

In membrane vibrations, Euler in 1766 published equations for a rectangular membrane that were incorrect for the general case but will reduce to the correct equation for the uniform tension case. It is interesting to note that the first membrane vibration case investigated analytically was not the circular membrane, even while the latter, in form of the drumhead, would have been the more obvious shape. The reason is that Euler was able to picture the rectangular membrane as a superposition of a number of crossing strings. In 1828 Poisson read a paper to the French Academy of Science on the special case of uniform tension and showed the circular membrane equation as well as its solution for the special case of axisymmetric vibrations. One year later, Pagani furnished the non-axisymmetric solution. In 1852 Lamé (1795-1870) published his lectures which summarize the work on rectangular and circular membranes and contain an investigation of triangular membranes.

Work on plate vibration analysis went on in parallel. Influenced by Euler's success in deriving the membrane equation by considering the superposition of strings, James Bernoulli, a nephew of Daniel Bernoulli, attempted to derive the plate equation by considering the superposition of beams. The resulting equation was wrong. James, in his 1788 presentation to the St. Petersburg Academy, acknowledged that he was stimulated in his attempt by the German experimentalist Chladni, who demonstrated the beautiful node lines of vibrating plates at the courts of Europe. A presentation by Chladni before Napoléon Bonaparte who was a trained military engineer and very interested in technology and science caused the latter to transfer money to the French Academy of Science for a prize to the person who would best explain the vibration behavior of plates. The prize was won, after several attempts, by a woman, Sophie Germain (1776-1831), in 1815. She gave an almost correct form of the plate equation. The bending stiffness and the density constants were not defined. Neither were the boundary conditions stated correctly. These errors are the reason that her name is not associated today with the equation, despite the brilliance of her approach. Contributing to this was Todhunter, who compiled a fine history of the theory of elasticity which was published posthumously in 1886, in which he is unreasonably critical of her work, demanding a standard of perfection that he does not apply to the works of the Bernoullis, Euler, Lagrange, and others, where he is quite willing to accept partial results. Also, Lagrange (1736-1813) entered into the act by correcting errors that Germaine made when first competing for the prize in 1811. Thus, indeed we do find the equation first stated in its modern form by Lagrange in 1811 in response to Germaine's submittal of her first competition paper.

What is even more interesting is that Sophie Germaine published in 1821 a very simplified equation for the vibration of a cylindrical shell. Unfortunately again it contained mistakes. This equation can be reduced to the current rectangular plate equation, but when it is reduced to the ring equation a mistake in sign is passed on. But for the sign difference in one of its terms, the ring equation is identical to one given by Euler.

The correct bending stiffness was first identified in 1829 by Poisson. Consistent boundary conditions were not developed until 1850 by Kirchoff (1824-1887), who also gave the correct solution for a circular plate example.

The problem of shell vibrations was first approached by Sophie Germaine before 1821, as already pointed out. She assumed that the tangential deflection of the neutral surface of a cylindrical shell was negligible. Her result contained errors. In 1874, Aron derived a set of five equations, which he showed to reduce to the plate equation when curvatures are set to zero. The equations are complicated because of his reluctance to employ simplifications. They are in curvilinear coordinate form and apply in general. The simplification that are logical extensions of the beam and plate equations both for transverse and tangential motions were introduced by Love (1863-1940) in 1888. In between Aron and Love, Lord Rayleigh (1842-1919) proposed in 1882 various simplifications that viewed the shell neutral surface as either extensional or inextensional. His simplified solutions are special cases of Love's general theory. Love's equations brought the basic development of the theory of vibration of continuous structures that have a thickness that is much less than any length or surface dimensions to a satisfying end. Subsequent development was concerned with higher order or complicating effects and will be discussed in this work when appropriate.

2.3 Two-dimensional Shell Theories

Since in this work, much of the attention is focused on the vibrations of thin circular cylindrical shells, we will present here the main points of the simplest two-dimensional variant of thin shell theory. The equations of shell theory may be found in great detail in several monographs (Novozilov, 1970, Soedel, 1981).

2.3.1 Geometry of the Neutral Surface and its Deformations

We introduce a system of orthogonal curvilinear coordinates α_1 and α_2, which coincide with the lines of curvature of the neutral surface S of the shell. Let a point M on S be determined by the position vector $\boldsymbol{r} = \boldsymbol{r}(\alpha_1, \alpha_2)$.

The shell fills the volume

$$(\alpha_1, \alpha_2) \in G, \qquad |z| \leq t/2, \tag{2.1}$$

where z is the coordinate of a point measured along the normal to the neutral surface and t is the shell thickness (see Figure 2.1).

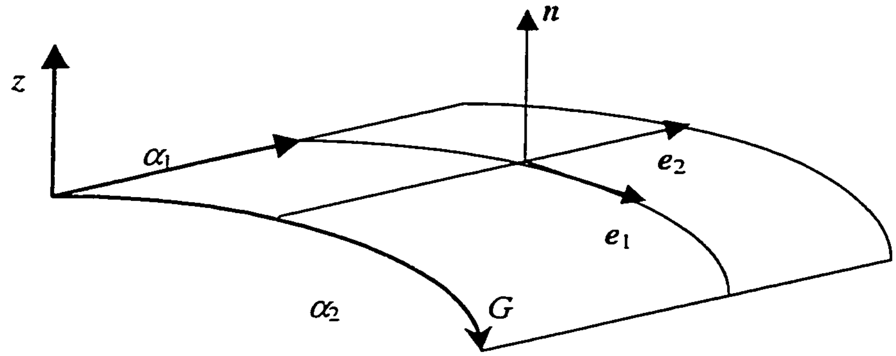

Figure 2.1. The geometry of the neutral surface G

Let $\Gamma = \partial G$ be the boundary of the domain G. The shell is said to be thin if its relative thickness t/R is small, where R is the characteristic shell length.

We introduce a local orthogonal system of coordinates by means of the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$, where

$$\mathbf{e}_1 = \frac{1}{A_1}\frac{\partial \mathbf{r}}{\partial \alpha_1}, \quad A_1 = \left|\frac{\partial \mathbf{r}}{\partial \alpha_1}\right|, \qquad (1\leftrightarrow 2),$$

$$\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2. \tag{2.2}$$

In this section, the notation $(1\leftrightarrow 2)$ indicates that the formula preceding it is valid with 1 and 2 interchanged.

The first and the second quadratic forms of the surface are

$$\begin{aligned} I &= ds^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2, \\ II &= \frac{A_1^2}{R_1} d\alpha_1^2 + \frac{A_2^2}{R_2} d\alpha_2^2, \end{aligned} \tag{2.3}$$

where ds is the arc length of a differential element on the surface, A_1 and A_2 are Lamé's coefficients, and R_1 and R_2 are the radii of curvature. We also use the notation k_i for the curvature $k_i = R_i^{-1}$.

To describe the deformation of the neutral surface, let u_1, u_2 and w be the projections of the displacements of a point M on the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$ before deformation. In the linear approximation, the tangential (membrane) surface deformations ε_1, ε_2 and ω, are

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_2 - \frac{w}{R_1}, \qquad (1\leftrightarrow 2) \qquad (2.4)$$

$$\omega_1 = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_1,$$

$$\omega = \omega_1 + \omega_2.$$

The angles of rotation γ_1 and γ_2 of the normal $\boldsymbol{n}$ with respect to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are equal to

$$\gamma_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u_1}{R_1}. \qquad (1\leftrightarrow 2) \qquad (2.5)$$

The bending surface deformations, κ_1, κ_2 and τ are given by

$$\kappa_1 = -\frac{1}{A_1}\frac{\partial \gamma_1}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_2, \qquad (1\leftrightarrow 2) \qquad (2.6)$$

$$\tau = -\frac{1}{A_2}\frac{\partial \gamma_1}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_2 + \frac{\omega_1}{R_2}.$$

2.3.2 Equilibrium Equations and Elasticity Relations

For the case of small deformations, the equations of equilibrium of an element of the neutral surface are

$$\begin{aligned}
&\frac{\partial(A_2T_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}T_2+\frac{\partial(A_1S_2)}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}S_1-\frac{A_1A_2}{R_1}N_1+A_1A_2F_1=0, \qquad (1\leftrightarrow 2)\\
&\frac{\partial(A_2N_1)}{\partial\alpha_1}+\frac{\partial(A_1N_2)}{\partial\alpha_2}+A_1A_2\left(\frac{T_1}{R_1}+\frac{T_2}{R_2}+F_n\right)=0,\\
&A_1A_2N_2+\frac{\partial(A_2H_1)}{\partial\alpha_1}+\frac{\partial(A_1M_2)}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}H_2=0,\\
&S_1-S_2+\frac{H_1}{R_1}-\frac{H_2}{R_2}=0,
\end{aligned} \tag{2.7}$$

where T_i and S_i are the projections of the stress resultant of the internal forces acting in the cross-section α_i = constant, on the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$, N_i are the transverse shear forces, H_i and M_i are the projections of the stress couples of the internal forces, and F_1, F_2 and F_n are the projections of the distributed external load.

In this work, shells made of homogeneous and isotropic materials are considered.

Relations (2.1)-(2.7) are the same for all linear two-dimensional shell theories. The differences are in the formulae connecting forces to deformations, knows as elasticity relations (Zhilin, 1976). The Kirchhoff-Love hypotheses, which generally assume that a linear element normal to the neutral surface before deformation, preserves its length and remains straight and normal to the neutral surface after deformation, lead to the elasticity relations introduced by Novozhilov (1970):

$$T_1 = K(\varepsilon_1 + \nu\varepsilon_2), \qquad S_1 = \frac{K(1-\nu)}{2}\left(\omega + \frac{h^2\tau}{6R_2}\right), \qquad (1 \leftrightarrow 2),$$
$$M_1 = D(\kappa_1 + \nu\kappa_2), \qquad H_1 = H = D(1-\nu)\tau, \tag{2.8}$$
$$K = \frac{Et}{1-\nu^2}, \qquad D = \frac{Et^3}{12(1-\nu^2)},$$

where E is Young's modulus and ν is Poisson's ratio.

The small parameter μ is introduced into the system of equations through the elasticity relations (2.8) as follows:

$$\mu^4 = \frac{t^2}{12R^2}. \tag{2.9}$$

Indeed, K is proportional to the shell thickness t, and D is proportional to t^3. Assuming that the shell thickness t is small compared to the characteristic shell length R, after transition to non-dimensional variables, we get the small parameter μ connected to the ratio t/R by formula (2.9).

Simpler elasticity relations were proposed by Love (1944), where

$$S_1 = S_2 = S = \frac{1}{2}K(1-\nu)\,\omega. \tag{2.10}$$

But these relations have the disadvantage that the last of the equilibrium equations (2.7) is not accurately satisfied.

Elasticity relations more complex than (2.8) are given in Goldenveizer (1961).

Since the system of equations (2.7) is of the eighth order, we need to supply four boundary conditions at each edge of the shell. The simplest alternatives for the boundary conditions at the edge $\alpha_1 = \alpha_1^0$ are to impose the generalized displacements (u_1, u_2, w, γ_1) or the corresponding generalized forces, as follows:

$$
\begin{aligned}
&u_1 = u_1^0 && \text{or} \quad T_1 = T_1^0, \\
&u_2 = u_2^0 && \text{or} \quad S_1 + \frac{H}{R_2} = S_1^0 + \frac{H^0}{R_2}, \\
&w = w^0 && \text{or} \quad N_1 - \frac{1}{A_2}\frac{\partial H}{\partial \alpha_2} = N_1^0 - \frac{1}{A_2}\frac{\partial H^0}{\partial \alpha_2}, \\
&\gamma_1 = \gamma_1^0 && \text{or} \quad M_1 = M_1^0.
\end{aligned}
\tag{2.11}
$$

2.3.3 Shallow Shell Equations

As one can see, the system of equations of the general shell theory is rather complex. But, in some cases, these equations may be simplified. For example, if the deformations are accompanied by small waves (the sizes of which, at least in one direction, being small compared with the characteristic sizes of the shell), then the shallow shell equations can be applied. In this case, the equilibrium equations and the compatibility equations of the deformations reduce to the following pair of equations for the deflection function w, and the force function Φ (Donnell, 1976):

$$
\begin{aligned}
&D\Delta^2 w - \Delta_k \Phi + F_n = 0, \\
&\frac{1}{Eh}\Delta^2 \Phi + \Delta_k w = 0,
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
\Delta &= \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\right)\right], \\
\Delta_k &= \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\right)\right].
\end{aligned}
\tag{2.13}
$$

The stress resultants are connected with the force function Φ by the following relations:

$$T_1 = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\Phi}{\partial\alpha_2}\right) + \frac{1}{A_1^2 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\Phi}{\partial\alpha_1}, \qquad (1 \leftrightarrow 2)$$

$$S = -\frac{1}{2}\left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2^2}\frac{\partial\Phi}{\partial\alpha_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_1^2}\frac{\partial\Phi}{\partial\alpha_1}\right)\right].$$

The transverse shear forces N_1 and N_2 may be determined from equations (2.7).

If we do not adopt the Kirchhoff-Love hypotheses, we can introduce the transverse shift angles δ_1 and δ_2. Then, there are five main unknowns, i.e. $(u_1, u_2, w, \delta_1, \delta_2)$. In this case, we add two more equations to the elasticity relations of type (2.8), namely

$$N_1 = \frac{Eh}{2(1+\nu)}\delta_1, \qquad N_2 = \frac{Eh}{2(1+\nu)}\delta_2. \qquad (2.14)$$

Thus, the order of the system increases from 8 to 10. This theory is associated with the name of Timoshenko (Timoshenko *et al.*, 1959). Here, it is assumed that a linear element normal to the neutral surface before deformation, remains straight, preserves its length, and becomes inclined to the deformed neutral surface at angles δ_1 and δ_2 (measured with respect to α_1 respectively α_2, the orthogonal curvilinear coordinates on the neutral surface coinciding with the lines of curvature). However, there exist theories described by systems of the 12th order, which allow element elongation. The two-dimensional shell theories of higher order based on series expansions in powers of the thickness are not widely used (Bauer *et al.*, 1993).

Later in this work, the theory of Kirchhoff-Love type is used.

2.3.4 Shell Dynamics Equations

If we take relations (2.4)-(2.8) into account, the system of equations of shell theory may be reduced to a form in which only the displacements u_1, u_2 and w are unknown (Goldenveizer, 1961, Aslanyan *et al.*, 1974). Thus we have

$$\sum_{j=1}^{3}\left(\mu^4 N_{ij} + L_{ij}\right)u_j + F_i = 0, \qquad i = 1,2,3, \tag{2.15}$$

where $u_j(\alpha_1, \alpha_2)$ and $u_3 = w$ are the displacement projections $N_{ij}(\alpha_1, \alpha_2)$ and $L_{ij}(\alpha_1, \alpha_2)$ are linear differential operators (generally with variable coefficients in α_1 and α_2), the independent variables $(\alpha_1, \alpha_2) \in S$ are orthogonal curvilinear coordinates on the neutral surface coinciding with the lines of curvature, $\mu > 0$ is a shell thickness parameter and F_i are the load projections.

We consider problems in which F_i are proportional to the eigenvalue λ. In the case of vibrations,

$$F_i = -\lambda u_i, \qquad \omega^2 = \frac{E\lambda}{\rho R^2}, \tag{2.16}$$

where ρ is the density, R is the characteristic length and ω is the natural frequency.

Chapter 3

Vibration Equations of Thin Cylindrical Shells

3.1 Introduction

In this chapter the vibration equations of cylindrical shells are presented. The general system of equations for arbitrary shells of revolution given in Chapter 2 is written here for the case of cylindrical shells. The displacements are the unknowns of this system. The specific form of the vibration equations for the axisymmetric case is discussed.

3.2 General System of Vibration Equations of Thin Cylindrical Shells

We consider the vibration of thin circular cylindrical shells.

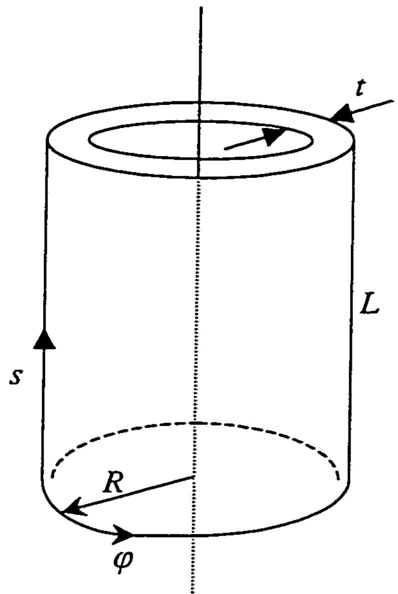

Figure 3.1. Circular cylindrical shell

A thin cylindrical shell of thickness t, length L and radius R is considered. We introduce a system of orthogonal coordinates $(\alpha_1, \alpha_2) = (s, \varphi)$ that defines the position of a point on the neutral surface of the shell, where s is the length of the generatrix ($0 \leq s \leq L$), and φ is the longitudinal angle ($0 \leq \varphi \leq 2\pi$). The shell is limited by two parallel planes $s = 0$ and $s = L$.

A cylindrical shell is said to be thin if its relative thickness t/R is small. We introduce a local orthogonal system of coordinates $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{n}$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors in the s and φ directions, respectively, and $\boldsymbol{n}$ is the normal unit vector (see (2.2)).

Let u, v, and w be the components of the displacement $\boldsymbol{U}$ in the directions $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$, respectively.

Here we use the thin shell equations given in Chapter 2. First, we introduce the relations between non-dimensional and dimensional (marked by *) variables :

$$(u,v,w,R_i,B,s)=\frac{1}{R}(u^*,v^*,w^*,R_i^*,B^*,s^*),$$

$$(\varepsilon_i,\omega,\gamma_i)=(\varepsilon_i^*,\omega^*,\gamma_i^*),$$

$$(\kappa,\tau)=R(\kappa^*,\tau^*), \tag{3.1}$$

$$(T_i,S_{ij},N_i)=\frac{(1-\nu^2)}{Eh^*}(T_i^*,S_{ij}^*,N_i^*),$$

$$(M_i,H)=\frac{(1-\nu^2)}{REh^*}(M_i^*,H^*).$$

Simplify the formulae (2.4)-(2.8) for the case of cylindrical shells by using the relations: $1/R_1=0$, $R_2=B=R$, $B'=0$, $A_1=1$, $A_2=B=1$.

Formulae (2.5) for shear strains versus displacements become:

$$\gamma_1=-\frac{\partial w}{\partial s}, \tag{3.2}$$

$$\gamma_2=-\frac{\partial w}{\partial s}-u.$$

Formulae (2.4) and (2.6) for strains versus displacements become:

$$\varepsilon_1=\frac{\partial u}{\partial s},$$

$$\varepsilon_2=\frac{\partial v}{\partial \varphi}-w,$$

$$\omega=\frac{\partial v}{\partial s}-\frac{\partial u}{\partial \varphi}, \tag{3.3}$$

$$\kappa_1 = -\frac{\partial \gamma_1}{\partial s},$$

$$\kappa_2 = -\frac{\partial \gamma_2}{\partial \varphi},$$

$$\tau = -\frac{\partial w}{\partial s} + \frac{\partial v}{\partial s}.$$

The shell equilibrium equations (2.7) become:

$$\begin{gathered}
\frac{\partial S_2}{\partial s} + \frac{\partial T}{\partial \varphi} - N_2 + \lambda v = 0, \\
T_2 + \frac{\partial N_1}{\partial s} + \frac{\partial N_2}{\partial \varphi} + \lambda w = 0, \\
\frac{\partial T_1}{\partial s} + \frac{\partial S_2}{\partial \varphi} + \lambda u = 0, \\
\frac{\partial M_1}{\partial s} + \frac{\partial H}{\partial \varphi} + N_1 = 0, \\
\frac{\partial H}{\partial s} + \frac{\partial M_2}{\partial \varphi} + N_2 = 0.
\end{gathered} \tag{3.4}$$

The elasticity relations (2.8) become:

$$\begin{gathered}
T_1 = \varepsilon_2 + \nu \varepsilon_1, \\
T_2 = \varepsilon_1 + \nu \varepsilon_2, \\
S_1 = \frac{1-\nu}{2}\left(\omega + 2\mu^2 \tau\right), \\
S_2 = \frac{1-\nu}{2}\omega, \\
M_1 = \mu^2(\kappa_1 + \nu \kappa_2), \\
M_2 = \mu^2(\kappa_2 + \nu \kappa_1).
\end{gathered} \tag{3.5}$$

The boundary conditions are:

$$
\begin{aligned}
&u_1 = u_1^0 \quad &&\text{or} \quad T_1 = T_1^0, \\
&u_2 = u_2^0 \quad &&\text{or} \quad S_1 + H = S_1^0 + H^0, \\
&w = w^0 \quad &&\text{or} \quad N_1 - \frac{\partial H}{\partial \varphi} = N_1^0 - \frac{\partial H^0}{\partial \varphi}, \\
&\gamma_1 = \gamma_1^0 \quad &&\text{or} \quad M_1 = M_1^0.
\end{aligned}
\tag{3.6}
$$

Thus, we obtain a system of 20 equations in the following 20 variables: u, v, w, T_1, T_2, S_1, S_2, N_1, N_2, M_1, M_2, H_{12}, ε_1, ω, ε_2, κ_1, τ, κ_2, γ_1, and γ_2.

3.3 System of Vibration Equations of Thin Cylindrical Shells in Terms of Displacements

Expressing the resultants and strains as functions of the displacements we obtain (Rimrott *et al.*, 1994) the equations describing the vibrations of thin cylindrical shells given by (3.2)-(3.5) in terms of displacements $\boldsymbol{U} = (u, v, w)$ as:

$$
-\frac{\partial^2 u}{\partial s^2} - \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \varphi^2} - \left(1-\nu^2\right)\lambda u - \frac{1+\nu}{2}\frac{\partial^2 v}{\partial s \partial \varphi} + \nu \frac{\partial w}{\partial s} = 0,
$$

$$
\begin{aligned}
&-\frac{1+\nu}{2}\frac{\partial^2 u}{\partial s \partial \varphi} - \frac{1-\nu}{2}\frac{\partial^2 v}{\partial s^2} - \frac{\partial^2 v}{\partial \varphi^2} + \mu^4\left(-2(1-\nu)\frac{\partial^2 v}{\partial s^2} - \frac{\partial^2 v}{\partial \varphi^2}\right) - \\
&\left(1-\nu^2\right)\lambda v + \frac{\partial w}{\partial \varphi} + \mu^4\left(-(2-\nu)\frac{\partial^3 w}{\partial s^2 \partial \varphi} - \frac{\partial^3 w}{\partial \varphi^3}\right) = 0,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&-\nu\frac{\partial u}{\partial s} - \frac{\partial v}{\partial \varphi} + \mu^4\left((2-\nu)\frac{\partial^3 v}{\partial s^2 \partial \varphi} + \frac{\partial^3 v}{\partial \varphi^3}\right) + w \\
&-\left(1-\nu^2\right)\lambda w + \mu^4\left(\frac{\partial^4 w}{\partial s^4} + 2\frac{\partial^4 w}{\partial s^2 \partial \varphi^2} + \frac{\partial^4 w}{\partial \varphi^4}\right) = 0.
\end{aligned}
$$

This system is of the form (2.15).

Separating the variables in s and φ in the expressions of the displacements

$$\begin{aligned} u\,(s,\,\varphi) &= U(s)\sin m\varphi, \\ v\,(s,\,\varphi) &= V(s)\cos m\varphi, \\ w\,(s,\,\varphi) &= W(s)\sin m\varphi, \end{aligned} \tag{3.8}$$

and substituting them in (3.7) we obtain a system of ordinary differential equations describing the vibrations of cylindrical shells:

$$\begin{aligned}
&-\frac{\partial^2 U}{\partial s^2}+\frac{1-\nu}{2}m^2U-\left(1-\nu^2\right)\lambda U+\frac{1+\nu}{2}m\frac{\partial V}{\partial s}+\nu\frac{\partial W}{\partial s}=0, \\
&-\frac{1+\nu}{2}m\frac{\partial U}{\partial s}-\frac{1-\nu}{2}\frac{\partial^2 V}{\partial s^2}+m^2V+\mu^4\left(-2(1-\nu)\frac{\partial^2 V}{\partial s^2}+m^2V\right)- \\
&\left(1-\nu^2\right)\lambda V+mW+\mu^4\left(-(2-\nu)m\frac{\partial^2 W}{\partial s^2}+m^3W\right)=0, \\
&-\nu\frac{\partial U}{\partial s}+mV+\mu^4\left(-(2-\nu)m\frac{\partial^2 V}{\partial s^2}+m^3V\right)+W \\
&-\left(1-\nu^2\right)\lambda W+\mu^4\left(\frac{\partial^4 W}{\partial s^4}+2m^2\frac{\partial^2 W}{\partial s^2}+m^4W\right)=0.
\end{aligned} \tag{3.9}$$

System (3.9) together with the boundary conditions (3.6) constitute the boundary value problem, the solution of which we wish to construct.

3.4 Asymptotic Solution of Vibration Equations of Thin Shells

System (3.9) has the form:

$$\mu^4 L_\mu(U,\mu,m) + L(U,\mu,m) + \lambda U = 0, \tag{3.10}$$

where L_μ and L are linear differential operators of the 8th and fourth order, respectively, and μ is a small parameter of the shell thickness. In this case, the boundary conditions must be formulated in terms of u, v, w and their derivatives. For example,

$$u' = w = w'' = v = 0 \tag{3.11a}$$

are the conditions for a simple-supported edge, and

$$u = w = w' = v = 0 \tag{3.11b}$$

are the conditions for a free edge.

To solve the boundary value problem (3.10)-(3.11) we apply the method of asymptotic solution described in Goldenveizer *et al.* (1978). For this we need to construct a formal asymptotic solution for equation (3.18) and then impose boundary conditions (3.11).

We seek the solution of equation (3.10) in the form

$$Y(s,\mu) = \sum_{i=1}^{8} \sum_{k=0}^{\infty} C_i Y_k^i \mu^{k\kappa_i} e^{p_i s}, \tag{3.12}$$

where, for each i, C_i is an arbitrary constant, Y^i_k is the matrix of the amplitude vectors, and κ_i depends on the order of p_i with respect to μ. For example, if $p_i \sim \mu^{-1}$, then $\kappa_i = 1$.

Substituting solution (3.12) into equation (3.10) we obtain the characteristic equation for p_i

$$|A(m,\mu,\lambda)-pI|=0, \tag{3.13}$$

where I is the identity matrix. In this work we consider only the cases where all p_i are simple roots of equation (3.13), i.e. when $p_i \neq p_j$ for all $i \neq j$. Under such assumptions we can use the formal asymptotic solution (3.12). Then, all solutions $e^{p_i s}$ are linearly independent, and their linear combination provides the general solution of the initial equation. In the next chapters, we shall limit attention to the construction of only the first term of the asymptotic expansion (3.12).

For different relations between the parameters, the solutions (3.12) have different forms. In this work we use symbolic computation to construct formal asymptotic solutions for different values of the parameters μ, λ and m.

The order of the function $|p|$ in μ is called the index of variation of the solution. The solution is exponentially increasing away from the edge $s = 0$, if $\Re p_i > 0$. Such solutions is called the edge effect integral near the end $s = L$. The solution is exponentially decreasing away from the edge $s = 0$, if $\Re p_i < 0$. Such solution is called the edge effect integral near the end $s = 0$. The solution is oscillating, if $\Re p_i = 0$ and $\Im p_i \neq 0$. If $p_i \equiv 0$, the solution is called slowly varying. In solving the boundary value problem with an error of order of e^{-c/μ^d}, where c and d are some positive constants, we may take the value of the edge effect integrals near one edge to be equal to zero at the other end.

After constructing formal asymptotic solutions, boundary conditions should be imposed to find the frequency parameter λ. Substituting (3.12) into (3.10) we obtain a system of linear equations in C_i that has nonzero solutions if its determinant vanishes

$$\Delta(\lambda,\mu)=0. \tag{3.14}$$

One can solve this eighth-degree equation numerically. In some cases this equation may be simplified.

Simultaneously with the problem for $\mu \neq 0$ (perturbed problem (Nayfeh, 1973, Nayfeh, 1981, Kevorkian *et al.*, 1981) we consider the same problem with $\mu = 0$ (unperturbed problem).

If all p_i are different from 0 and not pure imaginary, then

$$\lim_{\mu \to 0} \Delta(\lambda, \mu) = \Delta(\lambda, 0) \tag{3.15}$$

and

$$\lambda = \lambda_0 + \mu\lambda_1 + \ldots, \tag{3.16}$$

where λ_0 is the frequency for the unperturbed system, i.e. $\Delta(\lambda_0, 0) = 0$.

Of special interest are the cases of regular degeneracy (regular singular perturbation) (Vishik *et al.*, 1957, Trenogin, 1970, O'Malley, 1974, Kevorkian, 1996). Let the perturbed system have order n, the unperturbed system have order m. Let the perturbed system have $l = n - m$ additional roots such that l_1 of them have negative real parts and l_2 have positive real parts, where l_1 is the number of additional boundary conditions at the left edge and l_2 is the number of additional boundary conditions at the right edge. In this case the solution may be constructed using an iterative method.

The existence of pure imaginary roots makes the problem more difficult. As a rule in this case, the function $\Delta(\lambda,\mu)$ has a limit point at $\mu = 0$ and $\lim_{\mu \to 0}\Delta(\lambda, \mu) \neq \Delta(\lambda, 0)$.

3.5 Asymptotic Solution. First Approximation.

We seek a solution of system (3.9) in the form

$$U=U_0\, e^{ps},\ V=V_0\, e^{ps},\ W=W_0\, e^{ps}. \tag{3.17}$$

Substituting (3.17) in (3.9) we find a system of equations with respect to U_0, V_0, and W_0 in the form:

$$\begin{gathered}
-p^2U_0+\frac{1-\nu}{2}m^2U_0-(1-\nu^2)\lambda U_0+\frac{1+\lambda}{2}mpV_0+\nu pW_0=0, \\
-\frac{1+\nu}{2}mpU_0-\frac{1-\nu}{2}p^2V_0+m^2V_0+ \\
\mu^4\left(-2(1-\nu)p^2V_0+m^2V_0\right)- \\
(1-\nu^2)\lambda V_0+mW_0+\mu^4\left(-(2-\nu)mp^2W_0+m^3W_0\right)=0, \\
-\nu pU_0+mV_0+\mu^4\left(-(2-\nu)mp^2V_0+m^3V_0\right)+W_0- \\
(1-\nu^2)\lambda W_0+\mu^4\left(p^2-m^2\right)^2W_0=0.
\end{gathered} \tag{3.18}$$

System (3.18) has nontrivial solutions if its determinant is equal to zero. So, we have the eighth-order equation

$$D(p;h,\lambda,m)=\begin{vmatrix} -p^2+\frac{1-\nu}{2}m^2-(1-\nu^2)\lambda & \frac{1+\nu}{2}mp & \nu p \\ \frac{1+\nu}{2}mp & g(p,\mu,m)-(1-\nu^2)\lambda & f(p,\mu,m) \\ -\nu p & f(p,\mu,m) & 1-(1-\nu^2)\lambda+\mu^4(p^2-m^2)^2 \end{vmatrix}=0, \tag{3.19}$$

from which all eight roots p_i may be determined. Here

$$f(p,\mu,m)=m+\mu^4\left(-(2-\nu)mp^2+m^3\right)$$

and

$$g(p,\mu,m)=\mu^4\left(-2(1-\nu)p^2+m^2\right)-\frac{1-\nu}{2}p^2+m^2.$$

The method of analysis of equation (3.19) will be discussed in Chapter 5.

3.6 Axisymmetric Vibrations

The axisymmetric vibrations of cylindrical shells, where $m = 0$ are a special case, since the system of equations splits (Bauer *et al.* (1995), Bauer *et al.* (1997)). The set of the first and third equations in (3.9) defines the transverse-axial vibrations, and the second equation defines the torsional vibrations. We consider only the transverse-axial vibrations.

For such vibrations:

$$-\frac{\partial^2 u}{\partial s^2}+\nu\frac{\partial w}{\partial s}-\left(1-\nu^2\right)\lambda u=0, \tag{3.20}$$

$$-\nu\frac{\partial u}{\partial s}+w+\mu^4\frac{\partial^4 w}{\partial s^4}-\left(1-\nu^2\right)\lambda w=0.$$

System (3.18) becomes

$$-p^2U_0-\left(1-\nu^2\right)\lambda U_0+\nu pW_0=0,$$

$$-\frac{1-\nu}{2}p^2V_0+\mu^4\left(-2(1-\nu)p^2V_0\right)-\left(1-\nu^2\right)\lambda V_0=0, \tag{3.21}$$

$$-\nu pU_0+W_0-\left(1-\nu^2\right)\lambda W_0+\mu^4p^4W_0=0.$$

The characteristic equation is

$$\begin{vmatrix} -p^2-\left(1-\nu^2\right)\lambda & \nu p \\ -\nu p & 1-\left(1-\nu^2\right)\lambda+\mu^4p^4 \end{vmatrix}=0 \tag{3.22}$$

or

$$P\,(p;\,h,\,\lambda) = \lambda - \lambda^2 + \lambda^2\,\nu^2 + p^2 - \lambda\,p^2 + h^4\,\lambda p^4 - h^4\lambda\nu^2p^4 + h^4p^6 = 0, \tag{3.23}$$

where $h^4=\dfrac{\mu^4}{1-\nu^2}$.

Chapter 4

Methods of Investigation of the Characteristic Equation

4.1 Introduction

The geometrical approach is employed to solve the characteristic equation describing the vibrations of thin circular cylindrical shells. This approach leads to the construction of a convex hull in the space of powers of the small parameters involved in the characteristic equation. The methods of construction of convex hulls in 2D and 3D are presented. Special attention is devoted to the methods and algorithms (namely standard algorithms of Mathematica 3.0 based on the Graham scan method for 2D cases, algorithms developed by us based on the "gift wrapping" methods for 2D and 3D, and the Qhull algorithm based on the same principle for 4D) used in this work to construct the convex hull necessary to study the characteristic determinants. Methods for simplifying the characteristic equations are also discussed.

4.2 Introduction to Geometrical Approach

To find the roots p_i of equation (4.11) for different values of the small parameter $h \ll 1$ and λ, we write (4.11) in the form

$$P(p;h,\lambda)=\sum_{i}^{6} a_i p^{k_i} h^{\alpha_i} \lambda^{\beta_i} = 0, \tag{4.1}$$

where a_i are coefficients not depending on p, h and λ, and i is the number of terms in (4.11). We call the points $M_i = \{k_i, \alpha_i, \beta_i\}$ in the space $\{k, \alpha, \beta\}$ the representative points. Each point is associated with the coefficients a_i, that are later called the weights of the point.

Firstly, we consider the geometrical interpretation of the 2D-problem of finding the main terms of the polynomial

$$P(p,q,s) = \sum_{\alpha,\beta} c_{\alpha\beta}(s) p^{\alpha} q^{\beta}, \tag{4.2}$$

which depends on two small parameters, p and q, with the weight $c_{\alpha\beta}(s)$.

We consider α and β as arbitrary real numbers. In the case of thin shells of revolution, the weight functions $c_{\alpha\beta}(s)$ depend on the radii of curvature of the neutral surface $R_1(s)$, $R_2(s)$ and on the distance to the axis of rotation $B(s)$, as well as on B'.

These weight functions may be constant or may depend on s, but they must be $O(h^0)$ and be non zero. They will be considered as equal to +1 in the construction of the convex hulls in this chapter.

For each term of the polynomial $c_{\alpha\beta}(s)p^{\alpha}q^{\beta}$ we do a mapping to the pair of numbers α, β with the weight $c_{\alpha\beta}(s)$ in the 2D space of (α, β). Moreover, the terms with the same pair (α, β) map to one point associated with the total weight of those terms.

Next we represent this set of points in the plane $\{\alpha, \beta\}$. We choose the axes such that all the points lie in the first quadrant and the extremal points, i.e. the points with coordinates$\{\min\alpha(\beta), \max\alpha(\beta)\}$ lie on the coordinate axes. Algebraically, this operation is equivalent to taking out of the brackets the terms with the minimum powers of p and q.

<u>Definition</u>: A point is called ***invisible*** from the origin, if and only if, this point and the origin lie in different domains, defined by the line which goes through the points $A_1(\alpha_1, \beta_1)$ *and* $A_2(\alpha_2, \beta_2)$, and this point must lie in the interior of the sector, defined by the lines OA_1 and OA_2.

<u>Theorem 1</u>: For invisible points, the following inequalities are valid

$$p^{\alpha_1}q^{\beta_1} + p^{\alpha_2}q^{\beta_2} > p^{\alpha}q^{\beta} \text{ (*) for } 0 < p, q < 1. \tag{4.3}$$

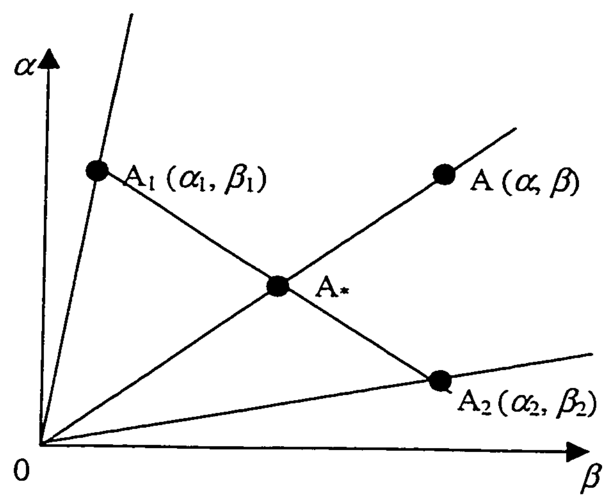

Fig.4.1 Graph for the proof of Theorem 1

Proof:

1. $p^{\alpha_*} > p^{\alpha}$ as $\alpha_* < \alpha \quad and \quad 0 < p < 1$ and

   $q^{\beta_*} > q^{\beta}$ as $\beta_* < \beta \quad and \quad 0 < q < 1$. Therefore, $p^{\alpha_*} q^{\beta_*} > p^{\alpha} q^{\beta}$. (4.4)

2. To complete the proof, we have to show that the function

$$f(k) = p^{\alpha_*} \cdot q^{\beta_*} = p^{k\alpha_1 + (1-k)\alpha_2} \cdot q^{k\beta_1 + (1-k)\beta_2} < p^{\alpha_1} q^{\beta_1} + p^{\alpha_2} q^{\beta_2} \tag{4.5}$$

(for $0 \le k \le 1$, $0 < p, q < 1$).

Next we find the location of the maximum of function $f(k)$:

$$f'(k) = [(\alpha_1 - \alpha_2)\ln p + (\beta_1 - \beta_2)\ln q] \cdot p^{k\alpha_1 + (1-k)\alpha_2} \cdot q^{k\beta_1 + (1-k)\beta_2} = 0. \tag{4.6}$$

As the expression in the brackets does not depend on k, and the other factors are greater than zero, the function $f(k)$ attains its maximum at one of the ends of the segment: 0 or 1. Thus,

$$\max f(k) = f(0) = p^{\alpha_*} \cdot q^{\beta_*} = p^{\alpha_2} \cdot q^{\beta_2} < p^{\alpha_1} q^{\beta_1} + p^{\alpha_2} q^{\beta_2} \text{ or}$$

$$\max f(k) = f(1) = p^{\alpha_*} \cdot q^{\beta_*} = p^{\alpha_1} \cdot q^{\beta_1} < p^{\alpha_1} q^{\beta_1} + p^{\alpha_2} q^{\beta_2}. \text{ Therefore,}$$

$$f(k) < \max\{f(k)\} = \max\{f(0), f(1)\} = \max\{p^{\alpha_1} \cdot q^{\beta_1}, p^{\alpha_2} \cdot q^{\beta_2}\} < p^{\alpha_1} q^{\beta_1} + p^{\alpha_2} q^{\beta_2}.$$

□

Therefore, for small parameters p and q, the term defined by the point A may be neglected compared to the sum of the terms defined by the points A_1 and A_2. As this criterion is applied for any three points, the main terms of the polynomial will be defined by points of the convex hull of the point set. Moreover, the whole convex hull defines the main terms of the polynomial for any values of the parameters, i.e. may be small, large or $O(1)$. If some parameters are small, then we have to consider only a part of the convex hull. If one of the parameters is small, we should consider only the lower part of the convex hull, i.e. those points and facets, which are visible from the point $(p, h) = (0, -\infty)$. For instance, in the case of a vibrating cylindrical shell, the relative thickness h is small. If two parameters are small, we will consider only the part of the convex hull, which is visible from the two points $(-\infty, 0)$ and $(0, -\infty)$, or, in other words, the part visible from the origin.

The above considerations may be generalized to the cases of higher dimensions.

Next we consider the problem of constructing the convex hull for the set of points.

4.3 Historical Aspects

The construction of the convex hull for a finite point set can be used for a variety of problems in science. The convex hull is the most ubiquitous structure in computational geometry and it also represents something of a success story in this science.

Chand and Kapur (1970) described the algorithm for the construction of a CH for a space of arbitrary dimension. Their approach is based on the so-called "gift wrapping" principle. The basic idea of this principle consists in projecting the space-points into a 2D plane, constructing there the convex hull, and then, "wraping" the object. Our algorithm in 3D is based on this idea. This algorithm is $O(n^2)$, where n is the number of points considered.

The first algorithm for constructing the convex hull with an order less than n^2 was developed by Graham (1972) for the 2D case. The Graham method is based on the representation of points in polar coordinates, sorting them by angles and using the "3-coins" algorithm described below. This algorithm is $O(n \log n+Cn)$, where C is a constant defined by the conversion of Cartesian coordinates of the point set to the polar coordinate system.

Jarvis, using an idea similar to that in the Chand-Kapur method, introduced an algorithm of $O(nh)$ for the 2D case. This algorithm is described below (O'Rourke, 1998). Here h is the number of points on the convex hull.

The studies on the construction of convex hulls have been continued by Preparata and Shamos (1988), who developed several algorithms of $O(n \log n)$ for the 2D case using ideas from Shamos (1978).

For higher dimensions the construction of algorithms with an order less than n^2 was a problem for many years. Preparata and Hong (1977) proposed an algorithm of $O(n \log n)$ for the worst case. This 2D algorithm is considered as a variation of the algorithm for 3D and is based on the "divide and conquer" principle (O'Rourke, 1998). This principle states that the set of given points is divided in subsets of 3-4 points and then these subsets are merged together.

Since the statement that "the number of edges of a convex hull for n points is less or equal to n" is valid only for 2D and not for higher dimensions, we can conclude that for higher dimensions (three and more) the order of the algorithms based on the "edge" principle cannot be less than n^2.

On the other hand, for the special case of arranging the initial set of points, the algorithm proposed by Brown (1979) and Aggarwal et al. (1989) for nD can be used.

This so-called "spherical inversion" algorithm can be employed to construct the Voronoi diagram.

To define the visible points (not edges), a simpler approach can be used. For example, a point V_i is visible, if:

$$\begin{gathered} \exists U: \forall j: \\ U^T V_j \geq b \\ U^T V_i = b \end{gathered} \tag{4.7}$$

Here $b > 0$, V_j are the points, U is a normal vector to the line (hyperplane) which goes through the V_i and which separates the points from the origin. The system (4.7) can be solved by methods of linear programming (simplex method, for instance). Algorithms of these types are realized, for example, in *Mathematica* (*LinearProgramming* function) and *Maple* Software.

One can also use specialized geometrical software to construct a convex hull. The best sources for software links are the *Directory of Computational Geometry Software* and the *Stonybrook Algorithms Repository*. For the purpose of this work the following software packages deserve special attention:

`Qhull` (Barber *et al.*, 1996) which is a high-quality, robust, user-friendly code for computing the convex hull in any dimension,

LEDA (Mehlhorn *et al.*, 1998) which is a full C++ library of computational geometry software, including an extensive class library and robust primitives.

We used `Qhull` to construct the convex hull in 4D for the case of non-axisymmetric vibrations.

4.4 Construction of the Convex Hull in 2D

Firstly, we describe the most popular technique - the "three-coins algorithm". Most algorithms are based on this approach. This is one of the simplest algorithms used to find the convex hull of a simple polygon.

4.4.1 The Three-Coins Algorithm

The three-coins algorithm was developed independently by Graham and Sklansky (1972) to find convex hulls. Here is a short description of the *Graham's algorithm* known as the Graham Scan method.

Graham's algorithm

- Find an extremal point (for example, the point with the smallest y coordinate) and label it p_0.
- Sort the remaining n-1 points radially, using p_0 as the origin.

3. Place three coins on vertices p_0, p_1, p_2 and label them "back", "center", and "front" respectively. (They will form a right turn from "back" to "front").

- **Do** :

If the 3 coins form a right turn (or if the 3 coins lie on collinear vertices),

- Take "back", place it on the vertex ahead of "front".
- Relabel: "back" becomes "front", "front" becomes "center", "center" becomes "back".

Else (the 3 coins form a left hand turn)

- Take "center", place it on the vertex behind "back".

- Remove (or ignore hereafter) the vertex that "center" was on.

- Relabel : "center" becomes "back", "back" becomes "center".

Until "front" is on vertex p_0 (our start vertex) and the 3 coins form a right turn.

- Connect the remaining points in the order they were sorted at Step 2. This forms the convex hull of the original set of n points.

The three coins advance along the ordered vertices as long as they keep forming right-hand turns. If this were to continue till the end, the algorithm would have merely verified that the ordered vertices form a convex polygon.

Complexity:

Since a vertex is deleted every time we backtrack one step, it is apparent that there is a maximum of n backtracks. So, conceivably, we could get n loop iterations + n backtracks = $2n$ coin placements. Each coin placement requires a constant amount of work (locating next vertex, calculating angle, relabeling), so the running time of the three-coins loop is $O(n)$. It is well known that sorting is $O(n \log n)$ (and $\Omega(n \log n)$), so the overall run-time is dominated by sorting. The time complexity of the Graham Scan is the worst case optimal: $\Theta(n \log n)$.

Symbols Ω, O, and Θ are called the Landau symbols.

$g(n)= \Omega(f(n))$ means that $\exists c$ and n_0 such that $|g(n)| \geq cf(n)$ for $\forall\ n \geq n_0$.

$g(n)= O(f(n))$ means that $\exists c$ and n_0 such that $|g(n)| \leq cf(n)$ for $\forall\ n \geq n_0$.

$g(n)= \Theta(f(n))$ means both $O(f(n))$ and $\Omega(f(n))$ hold.

4.4.2 Gift Wrapping Algorithm (Jarvis' March)

This is perhaps one of the simplest algorithms for the construction of the convex hull, and yet in some cases it can be very fast. The basic idea is as follows:

- Start at some extremal point, which is guaranteed to be on the hull;
- At each step, test each of the points, and find one which corresponds to the largest right hand turn. That point has to be the next one on the hull.

Because this process marches around the hull in counter-clockwise order, like a ribbon wrapping itself around the points, this algorithm is also called the gift wrapping algorithm. Jarvis' march takes time proportional to nh, where n is the number of input points, and h is the number of points on the hull. In other words, Jarvis' march is output-sensitive. As we can see, this algorithm is not very fast. In fact if n points are arranged in a circle, Jarvis' march will take time proportional to n^2. Quick-hull, which we will describe next would probably be faster.

4.4.3 Throw-away Principle (Quick-hull)

Here is an algorithm that deserves its name. It is a fast way to compute the convex hull of a set of points on the plane. It is recursive and each recursive step partitions data into several groups.

The partitioning step does all the work. The basic idea is as follows:

1. We are given a point set, and a line segment AB which we know is a chord of the convex hull (i.e., its endpoints are known to be on the convex hull). A good chord to start the algorithm goes from the leftmost to the rightmost point in the set.
2. Among the given points, find the one which is farthest from AB. Let us call, this point C.
3. The points inside the triangle ABC cannot be on the hull. Put them in set s_0.
4. Put the points, which lie outside edge AC in set s_1, and points outside edge BC in set s_2.

Once the partitioning is done, we recursively invoke quick-hull on sets s_1 and s_2. The algorithm works fast on random sets of points because step 3 of the partition typically discards a large fraction of the points.

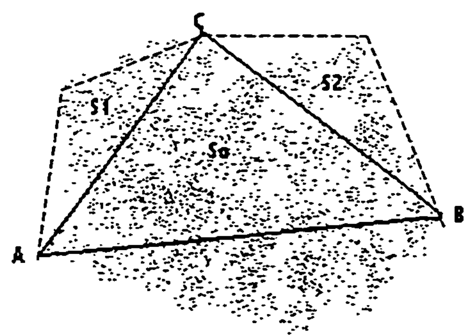

Figure 4.2 Graph for the quick-hull algorithm

It can be seen that if the set of points are arranged in a circle, then no points are discarded (s_i is always empty), so the algorithm runs more slowly. For this particular example, Graham's scan described above may be more efficient.

4.4.4 Divide-and-Conquer Algorithm for 2D

The divide-and-conquer algorithm is an algorithm for computing the convex hull of a set of points in two or more dimensions.

- First sort the points by x coordinate.
- Divide the points into two sets, L and R, L containing the left $\lceil n/2 \rceil$ points, and R the right $\lceil n/2 \rceil$ points.
- Compute the convex hulls of L = H(L) and R = H(R) recursively.
- Merge L and R: Compute H(L$\cup$R).

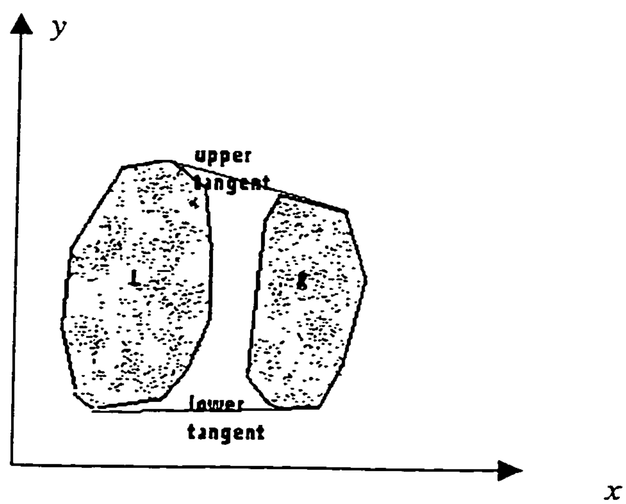

Figure 4.3 Graph for the divide-and-conquer algorithm

To merge the left and right hulls it is necessary to find the upper and lower common tangents. The upper common tangent can be found in linear time by scanning around the left hull in a clockwise direction, and around the right hull in a counterclockwise direction. The two tangents divide each hull into two pieces. The edges belonging to one of these pieces must be deleted.

Because the merge can be done in linear time, the total time is $O(n \log n)$.

Now, we describe the algorithm that we used to construct the convex hull in 2D. As it was mentioned above, this algorithm is based on the "gift wrapping" principle.

So, suppose we are given the set of points plotted in Figure 4.4.

1. Plot these points in the Cartesian coordinate system.

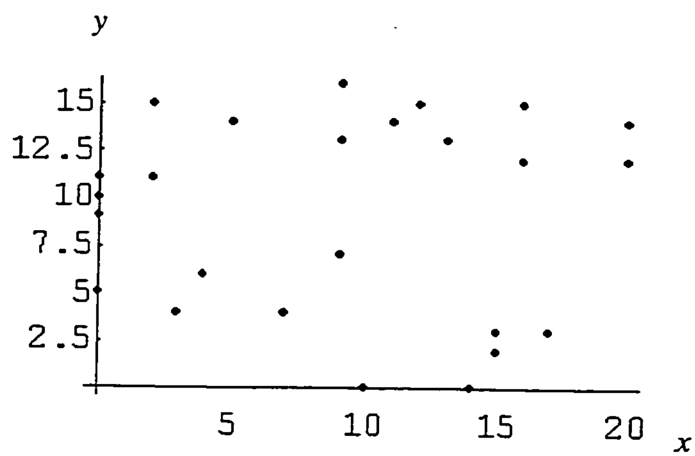

Figure 4.4 First step of the construction algorithm of CH in 2D

2. Delete the repeated points from the list.
3. Find the centroid of these points, W.
4. Sort the points by an increasing angle that is computed as an angle between the line joining W and the point with $\max\{x\}$, WA_2, and the line WA, where A is the current point (see Figure 4.5).

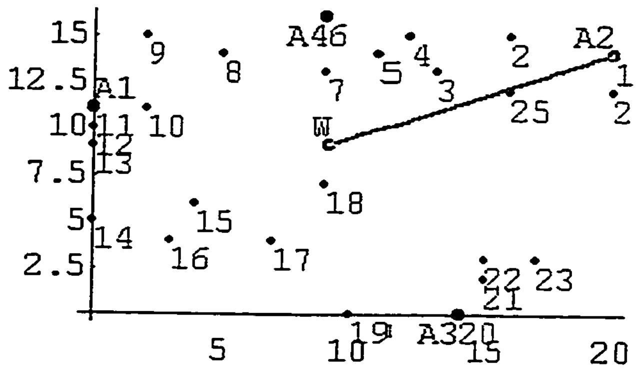

Figure 4.5 Fourth step of the construction algorithm of CH in 2D

5. Divide the list of points in four subsets, the boundaries of which are defined by the points A_1, A_2, A_3 and A_4 with $\min\{x\}$, $\min\{y\}$, $\max\{x\}$, $\max\{y\}$. These points must be included in the final convex set. Consider each subset separately, but in the same manner. In each subset join all the sorted points consecutively. Choose in each subset only those points which lie in different half spaces with W with respect to the facets of the current convex hull (see Figure 4.6).

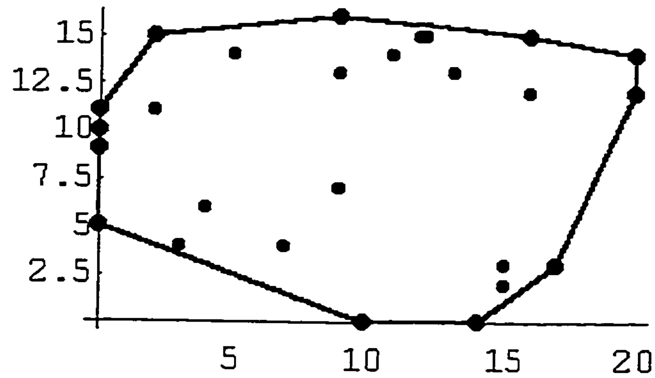

Figure 4.6 Fifth step of the construction algorithm of CH in 2D

6. Sometimes we need to construct only a part of the convex hull, for example, the facets of the convex hull that are visible from some point. For instance, one may wish to construct the facets visible from the origin. To achieve this, we draw the line through A_1 and A_3 (that is 1 and 6 on Figure 4.7), and select only the points of the convex hull that lie under this line. It will be the convex hull visible from the origin.

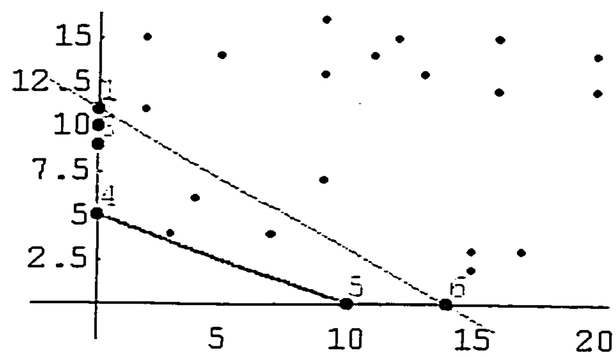

Figure 4.7 Sixth step of the construction algorithm of CH in 2D

4.5 Construction of the Convex Hull in 3D

4.5.1 Gift Wrapping Algorithm for 3D

As mentioned previously, the gift-wrapping algorithm was developed to work in arbitrary dimensions (Chand and Kapur, 1970). The three-dimensional version is a direct generalization of the two-dimensional algorithm. At any step, a connected portion of the hull is constructed. A face F on the boundary of this partial hull is selected, and an edge e of this face whose second adjacent face remains to be found is also selected. The plane π containing F is "bent" over e toward the set until the first point p is encountered. Then $\{p, e\}$ is a new triangular face of the hull, and the wrapping can continue. As in 2D, p can be characterized by the minimum turning angle from π. A careful implementation can achieve $O(n^2)$ time complexity: $O(n)$ work per face, and the number of faces is $O(n)$. And as in 2D, this algorithm has the advantage of being output-size sensitive: $O(nF)$ for a hull of F faces.

4.5.2 Divide-and-Conquer algorithm for 3D

Although several of the 2D algorithms extend (with complications) to 3D, the only one to achieve optimal $O(n \log n)$ time is the divide-and-conquer algorithm of Preparata and Hong (1977). It is, however, rather difficult to implement, and it is not used as frequently in practice as other asymptotically slower algorithms. The paradigm is the same as in 2D:

- First sort the points by x coordinate.
- Divide the points into two sets, L and R, L containing the left $\lceil n/2 \rceil$ points, and R the right $\lceil n/2 \rceil$ points.
- Compute the convex hulls of L = H(L) and R = H(R) recursively.
- Merge L and R: Compute H(L∪R).

The merge must be accomplished in $O(n)$ time to achieve the desired $O(n \log n)$ bound. All the work is in the merge, and we concentrate solely on this:

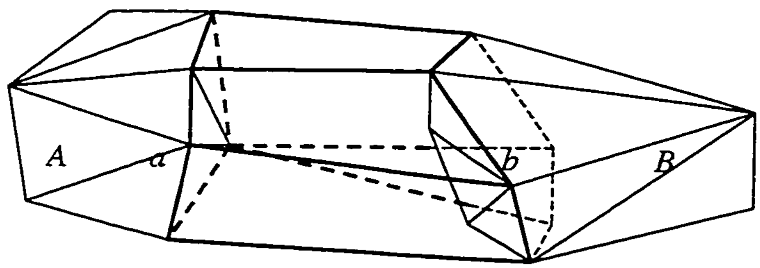

Figure 4.8 Graph for divide-and-conquer algorithm

Let A and B be two hulls to be merged. This can be achieved in two different ways:

- The hull $A \cup B$ will add a single "band" of facets with the topology of a cylinder without endcaps (see Figure 4.8). As the number of these faces will be linear in the size of the two polytopes, it is feasible to perform the merge in linear time, as long as faces can be added in constant time (on average). Let π be a plane that supports A and B from below, touching A at the vertex a and B at the vertex b. Then, π contains the line L determined by ab. Now "crease" the plane along L and rotate half of it about L until it bumps into point c on polytope A (say), then ac must be an edge of A. In other words, the first point c hit by π must be a neighbour of either a or b. This limits the vertices that need to be examined to determine the next to be bumped. Once π hits c, one triangular face of the merging band has been found: (a, b, c). Now the procedure is repeated, but this time around the line through cb (if $c \in A$). The wrapping stops when it closes upon itself.

- Project A and B on the coordinate plane and find A' and B' - their convex hulls in 2D. Then, compute a bridge of A' and B'. This line corresponds to the line in 3D that extends to the plane and rotates this plane as the "gift wrapping" algorithm does. To complete the procedure, remove the "inside" facets.

We can generalize this algorithm to any number of dimensions. However, the merge cannot be guaranteed to be completed in linear time, so the algorithm could take more time than $O(n \log n)$.

Now, we describe the algorithm we used to construct the convex hull in 3D. As it was mentioned above, this algorithm is based on the gift wrapping principle.

So, suppose we are given the following set of points (see Figure 4.9).

1. Represent these points in the Cartesian coordinate system.

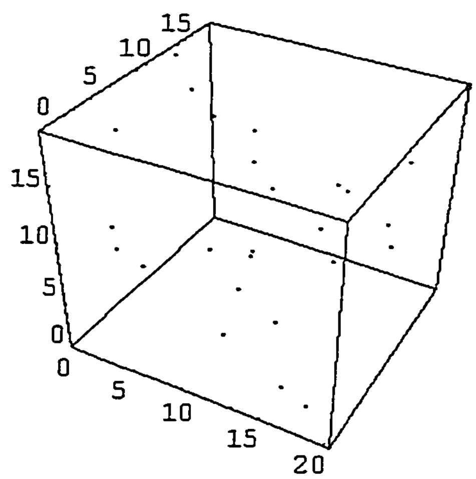

Figure 4.9 First step of the construction algorithm of CH in 3D

2. Project all the points on the plane $x + y + z = \text{const}$.

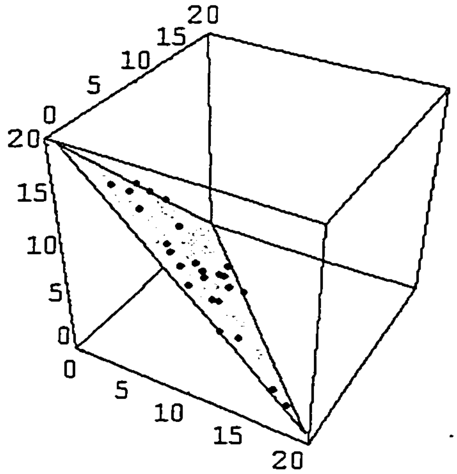

Figure 4.10 Second step of the construction algorithm of CH in 3D

3. Using the algorithm for 2D, find the space boundary of the 3D convex hull under construction by plotting the convex hull of the points on the plane.

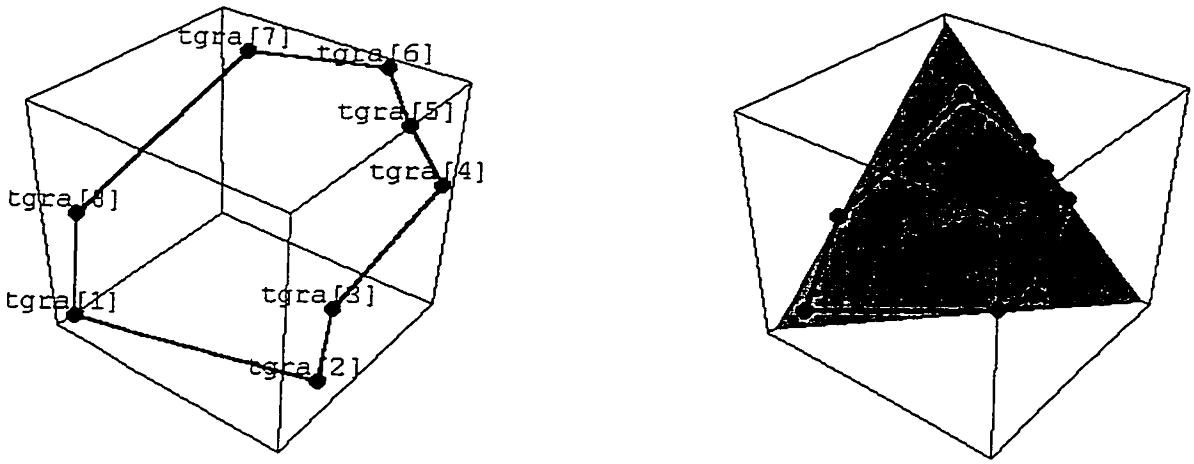

Figure 4.11 Third step of the construction algorithm of CH in 3D

4. Consecutively, for each edge construct the facet such that the rest of the points lie above this facet.

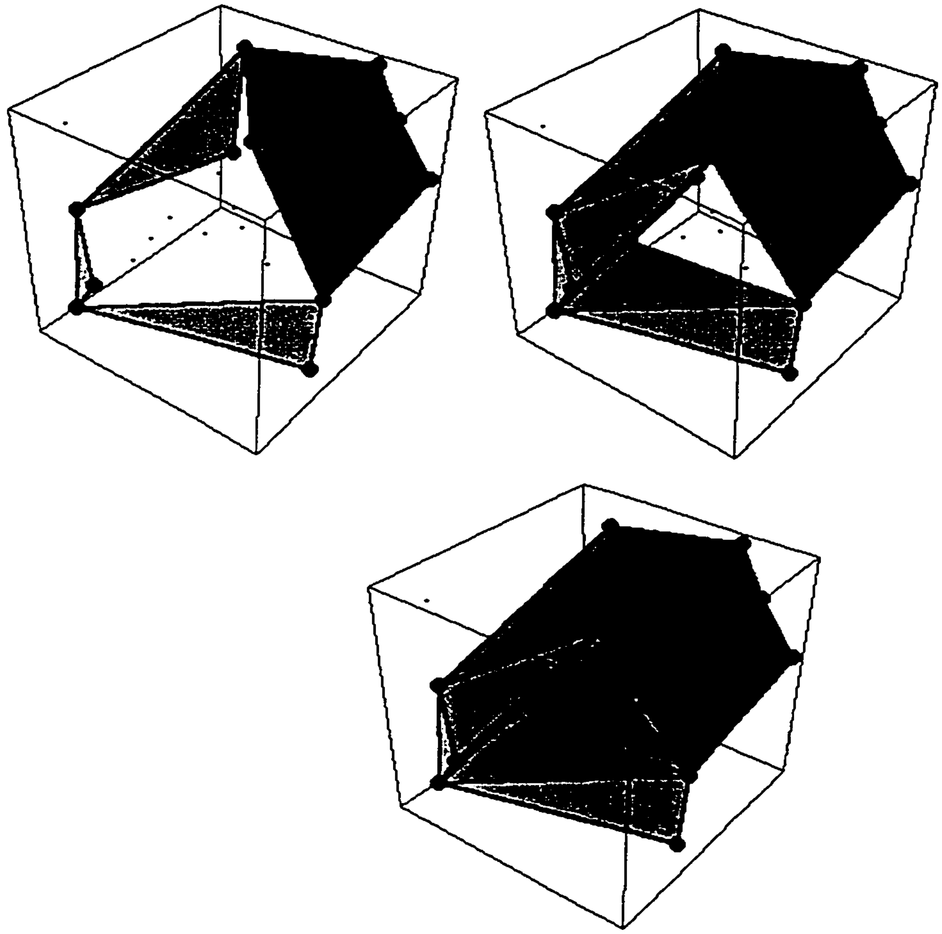

Figure 4.12 Fourth step of the construction algorithm of CH in 3D

To find the whole convex hull we can use any of the algorithms described previously.

4.6 Construction of the Convex Hull in nD

It is best to approach higher dimensions by analogy with lower dimensions. Unfortunately, there is a fundamental obstruction to obtaining efficient algorithms: the structure of the hull is so complicated that just printing it out sets a stiff lower bound on algorithms. Klee (1980) proved that the hull of n points in d dimensions could have

Ω ($n^{\lceil d/2 \rceil}$) facets. Hence in particular, the convex hull in 4D may have quadratic size, and no $O(n \log n)$ algorithm is possible under the circumstances: worst case $O(n \log n + n^{\lceil d/2 \rceil})$.

4.7 Methods for Simplifying the Characteristic Equation

Assume that we are given a polynomial $P(p; h, \lambda)$, where p is the variable and h and λ are parameters. As proven in Section 4.2, the terms in the polynomial defined by the facets of the convex hull are the main ones. For each facets the main terms will be different. Each facet defines the relation between the parameters λ and h. We represent these relations in the form $\lambda = \lambda_0 h^{\kappa_i}$, as we assumed that h is the main parameter. The sets of points κ_i will be called the separating points. We can construct the solution at each point κ_i.

To construct the solutions for an intermediate value of λ, for which $\kappa \neq \kappa_i$, let the points κ_i separate the entire range of the parameter λ into domains. For any λ inside a domain the structure of the convex hull and, therefore, the formulae for the roots and eigenvectors are similar. Thus, we can obtain the values of the roots and eigenvectors considering only one value of λ for each domain. Using this, we can construct solutions at all separating points and for all domains between the separating points. As soon as we know the order of parameter λ, the number of parameters of the characteristic equation is reduced by one.

As we will see, in the case of axisymmetric vibrations, the characteristic equation contains only one small parameter, h. To obtain the roots of such an equation Newton's diagram method (i.e. 2D convex hull) may be used. In this case the representative points lie in the (p, h) plane and have the form $M_i = \{k_i, \alpha_I + \beta_i\kappa\}$. The segments of the lower part of the convex hull of the set of points M_i, i. e. the segments that are visible from the point $(p, h) = (0, -\infty)$, define the terms of the characteristic equation that should be kept to determine the main terms of the roots, p_i .

Chapter 5

Asymptotic Solution of the Vibration Equations of Thin Cylindrical Shells

5.1 Introduction

This chapter is concerned with the application of the construction methods of the formal asymptotic solutions described in Chapter 4 to the vibration equations of thin cylindrical shells given in Chapters 2 and 3. The axisymmetric ($m = 0$) and non-axisymmetric ($m \neq 0$) cases for a shell of medium length are analyzed. The cases of superlow frequencies, i .e. when $\lambda = \lambda_0 h^{\kappa}$, $\kappa > 0$ are of particular interest. The boundary eigenvalue problems are solved for all these cases. For non-axisymmetric vibrations, the system of equations depends on three parameters: h, m and λ. For axisymmetric vibrations we have only two of them, since m is equal to 0. The following boundary conditions are considered in our analysis: simply supported edges and clamped-clamped edges.

5.2 Axisymmetric Vibrations (m = 0)

For equation (3.23) we have the following representative points M_i = {{1, {0, 0, 1}}, {{-1+v^2, {0, 0, 2}}, {{1, {2, 0, 0}}, {{-1, {2, 0, 1}}, {{1-v^2, {4, 4, 1}}, {{1, {6, 4, 0}}.

If we plot the representative points in 3D-space (p, h, λ), then, the 3D convex hull facets determine the separating cases (Landman *et al.*, 1999).

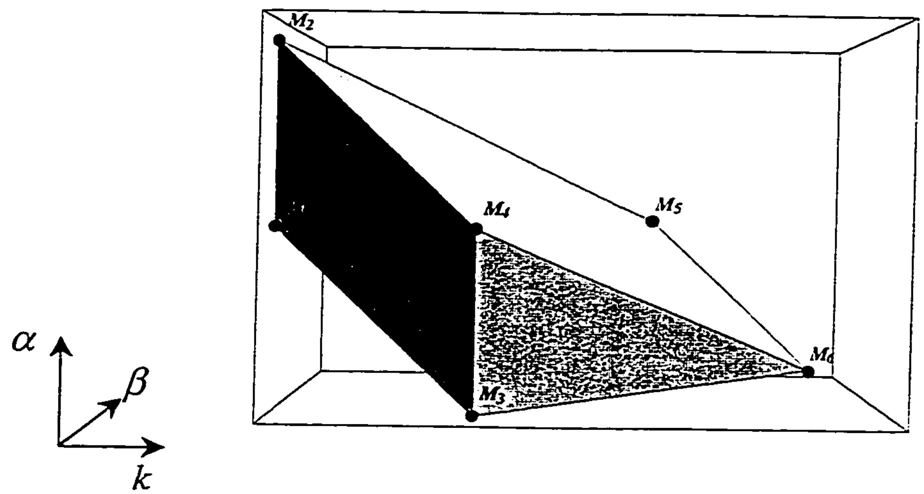

Figure 5.1 3D convex hull for $m = 0$

Since we consider only thin shells we assume that h is small. In this case, as proved in Chapter 4, we should keep only the facets of the convex hull that are visible from the point $h = -\infty$. For equation (3.23) the facets of the lower part of the 3D convex hull are plotted in Figure 5.1. This part of the 3D convex hull consists of 3 facets: 1: (M_1, M_2, M_3, M_4); 2: (M_3, M_4, M_6); 3: (M_2, M_4, M_5, M_6). Imposing that the orders of all terms corresponding to the points forming a facet be equal to each other, we find the relations from which the orders of λ for the separating cases may be determined:

$$\lambda \sim \lambda^2 \sim p^2 \sim \lambda p^2,$$
$$\lambda p^2 \sim p^2 \sim h^4 p^6, \tag{5.1}$$
$$\lambda^2 \sim \lambda p^2 \sim \lambda h^4 p^4 \sim h^4 p^6.$$

So, for the first and the second relations $\kappa = 0$, and for the third $\kappa = -4$. Therefore, the entire range of λ is divided into 3 domains where the 2D convex hulls are essentially different. Each of these domains defined as Domain I: $\kappa > 0$, Domain II: $0 > \kappa > -4$ and Domain III: $\kappa < -4$, as well as the separating cases A: $\kappa = 0$, and B: $\kappa = -4$ must be considered separately. As shown in Section 4.7, for any λ inside a domain the structure of the convex hull and, therefore, the formulae for the roots and eigenvectors are similar. Thus, we can obtain the values of the roots and eigenvectors by considering only one value of λ for each domain. Therefore, we know now the order of λ. Since λ is given, i.e. $\lambda = \lambda_0\, h^{\kappa}$, where $\lambda_0 \sim 1$ and κ is known, equation (3.23) contains only one small parameter, h. To obtain the roots of such an equation Newton's diagram method may be used (Goldenveizer *et al.*, 1978). In this case the representative points lie in the (p, h) plane and have the form $M_i = \{k_i, \alpha_i + \beta_i \kappa\}$. The segments of the lower part of the convex hull of the set of the points M_i, i. e. the segments that are visible from the point $(p, h) = (0, -\infty)$, define the terms of equation (3.23) that should be kept to determine the main terms of the roots, p_i.

5.2.1 Analysis for Different Domains κ

Using this approach, we arbitrary choose for the domains I, II and III $\kappa = 1$, $\kappa = -1$ and $\kappa = -5$, respectively, and we should also analyze the cases A: $\kappa = 0$ and B: $\kappa = -4$. Therefore, we are considering five cases here, where κ is equal to -5, -4, -1, 0 and 1, respectively.

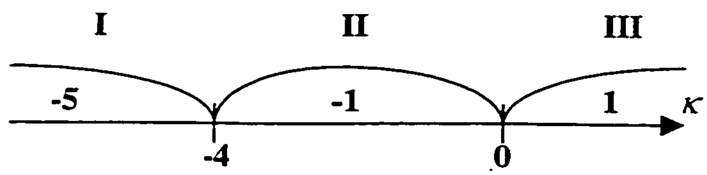

Figure 5.2 Domains of κ

5.2.1.1 Convex Hull for Domain I: $\kappa = -5$

We start with the case $\kappa = -5$ for which Newton's diagram is plotted in Figure 5.3.

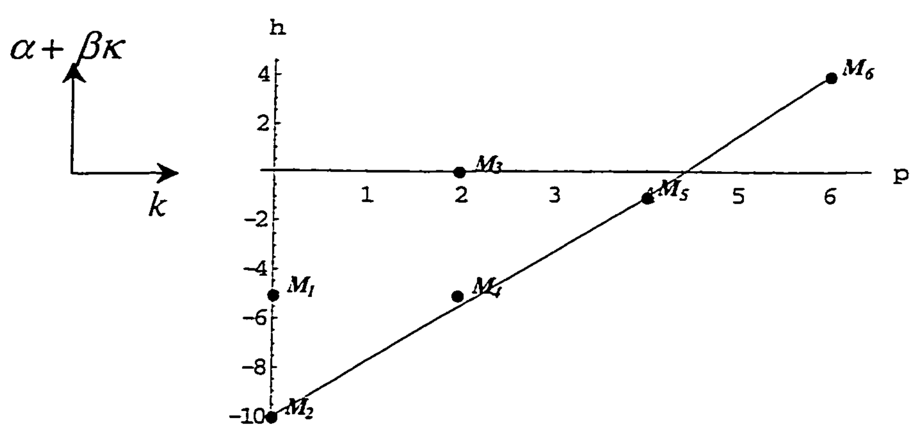

Figure 5.3 Newton's diagram for $m = 0$, $\kappa = -5$

In this case the representative points for equation (3.23) are $M_1 = (0, -5)$, $M_2 = (0, -10)$, $M_3 = (2, 0)$, $M_4 = (2, -5)$, $M_5 = (4, -1)$, $M_6 = (6, 4)$. Newton's diagram consists of 2 segments. The first segment is determined by points $M_1 = (0, -5)$, and $M_3 = (2, 0)$, and the second segment by points $M_3 = (2, 0)$ and $M_6 = (6, 4)$.

5.2.1.2 Convex Hull for Critical Point I: $\kappa = -4$

In the second case, $\kappa = -4$, the representative points for equation (3.23) are $M_1 = (0, -4)$, $M_2 = (0, -8)$, $M_3 = (2, 0)$, $M_4 = (2, -4)$, $M_5 = (4, 0)$, $M_6 = (6, 4)$. In this case Newton's diagram consists again of 2 segments (Figure 5.4).

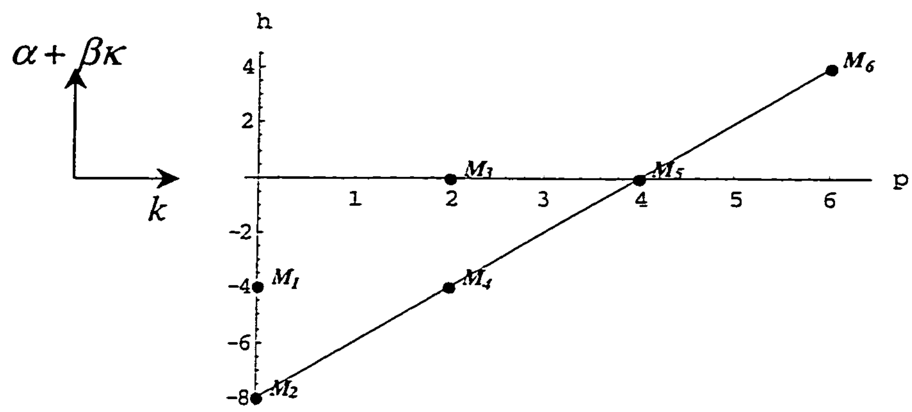

Figure 5.4 Newton's diagram for $m = 0$, $\kappa = -4$

According to the previous chapters, the initial equations are valid for $\lambda \ll h^{-4}$ and, therefore, cases III and B should be neglected in the following analysis. Case A is special since, in this case, the second term in the expansion for p is important (Goldenveizer *et al.*, 1978). Therefore, we consider only the solutions in Domains I, II and A and the previous cases were presented to show how the points are moving and how the structure is changing.

5.2.1.3 Roots and Amplitude Vectors for Domain III: $\kappa = 1$

We start our real mechanical analysis with the case $\kappa = 1$ for which Newton's diagram is plotted in Figure 5.5.

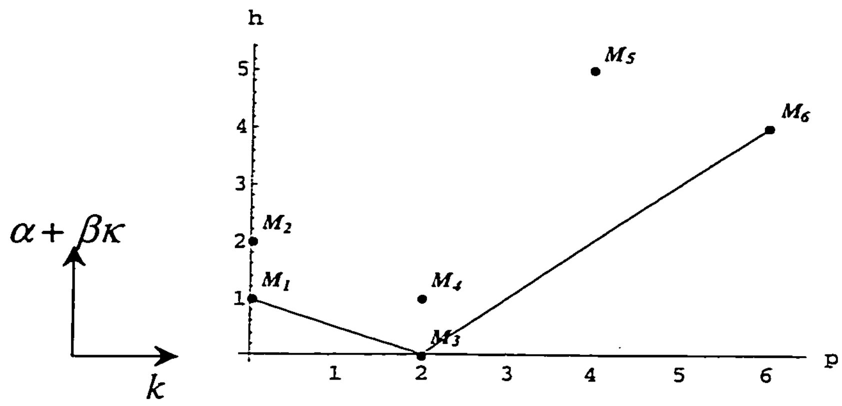

Figure 5.5 Newton's diagram for $m = 0$, $\kappa = 1$

In this case, the representative points for equation (3.23) are $M_1 = (0, 1)$, $M_2 = (0, 2)$, $M_3 = (2, 0)$, $M_4 = (2, 1)$, $M_5 = (4, 5)$, $M_6 = (6, 4)$. Newton's diagram consists of 2 segments. The first segment is determined by points $M_1 = (0, 1)$, and $M_3 = (2, 0)$, and the second segment by points M_3 and M_6. Therefore, equation (3.23) has 2 groups of roots, the first of which is defined by the equation

$$\lambda + p^2 = 0, \tag{5.2}$$

while the second one may be found from the equation

$$p^2 + h^4 p^6 = 0. \tag{5.3}$$

Hence, the roots are

$$\boxed{p_{1,2} = \pm\sqrt{\lambda}\, i,} \tag{5.4}$$

and

$$\boxed{p_j = \frac{\varepsilon_j}{h}, \qquad \varepsilon_j = \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}, \qquad j = 3,4,5,6.} \tag{5.5}$$

The orders of the variables p may be determined as the inclination angles between the segments and the p axis.

Having found the sets of roots p_i, we also can determine the relative orders of the eigenvectors U_0^i, W_0^i. To find them, we substitute the expressions for the roots p_i in equations (3.9). As we claimed in advance, we are limited to consider only the case when the multiplicity of the roots is equal to 1. Thus, we get a system of linearly dependent equations and, therefore, either the first or the third equation in (3.9) should be chosen. The only limitation for this choice is that the coefficients of U_0^i and W_0^i are nonzero. Finally, we get an equation of the following type:

$$a(p;h,\lambda,m)U_0^i + b(p;h,\lambda,m)W_0^i = 0. \tag{5.6}$$

It can be concluded from (5.6) that

$$\begin{aligned} U_0^i &= b(p;h,\lambda,m), \\ W_0^i &= -a(p;h,\lambda,m). \end{aligned} \tag{5.7}$$

To simplify the coefficients $a(p;\ h,\ \lambda,\ m)$ and $b(p;\ h,\ \lambda,\ m)$, we should keep only the main terms. For this purpose, the same "convex hull" algorithm can be applied, since these coefficients have the form of polynomials as well.

The main terms for U_0^i and W_0^i are given in Table 5.1.

Table 5.1 Roots and eigenvectors for $m = 0$, $\kappa = 1$

	1	2	3	4	5	6
p	$\sqrt{\lambda}i$	$-\sqrt{\lambda}i$	$\frac{\varepsilon_1}{h}$	$\frac{\varepsilon_2}{h}$	$\frac{\varepsilon_3}{h}$	$\frac{\varepsilon_4}{h}$
U_0	p_1	p_2	v	v	v	v
W_0	$-v\lambda$	$-v\lambda$	p_3	p_4	p_5	p_6

5.2.1.4 Roots and Amplitude Vectors for Critical Point II: $\kappa = 0$

In the second real case, $\kappa = 0$, the representative points for equation (3.23) are $M_1 = (0, 1)$, $M_2 = (0, 1)$, $M_3 = (2, 0)$, $M_4 = (2, 0)$, $M_5 = (4, 4)$, $M_6 = (6, 4)$. In this case, Newton's diagram consists of 2 segments (Figure 5.6).

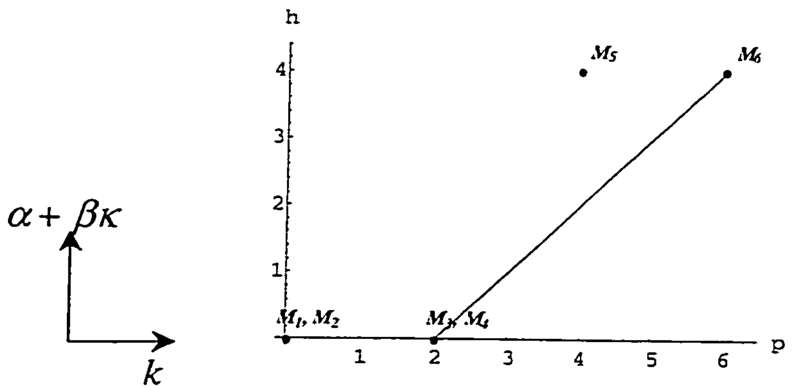

Figure 5.6 Newton's diagram for $m = 0$, $\kappa = 0$

The first segment is determined by M_1, M_2, M_3 and M_4 and the second one by M_3, M_4 and M_6. Therefore, equation (3.23) has 2 groups of roots, the first one being defined by equation

$$\lambda-\lambda^2+\lambda^2\nu^2+p^2-\lambda p^2=0, \tag{5.8}$$

and the second one by equation

$$p^2-\lambda p^2+h^4p^6=0. \tag{5.9}$$

Hence

$$\boxed{p_{1,2}=\pm F(\lambda), \qquad F(\lambda)=\sqrt{\frac{\lambda-(1-\nu^2)\lambda^2}{\lambda-1}},} \tag{5.10}$$

and

$$\boxed{p_{3,4,5,6}=\frac{(\lambda-1)^{1/4}}{h}.} \tag{5.11}$$

In this case, the roots and the eigenvectors are shown in Table 5.2.

Table 5.2 Roots and eigenvectors for $m=0$, $\kappa=0$

	1	2	3	4	5	6
p	$F(\lambda)$	$-F(\lambda)$	$\frac{(\lambda-1)^{1/4}}{h}$	$\frac{(\lambda-1)^{1/4}}{h}$	$\frac{(\lambda-1)^{1/4}i}{h}$	$-\frac{(\lambda-1)^{1/4}i}{h}$
U_0	p_1	p_2	ν	ν	ν	ν
W_0	$\frac{\lambda\nu}{\lambda-1}$	$\frac{\lambda\nu}{\lambda-1}$	p_3	p_4	p_5	p_6

Note that the above results for $\kappa=0$ are valid when λ is not too close to 1, otherwise the first negligible term for p has the same order as the main term.

5.2.1.5 Roots and Amplitude Vectors for Domain II: $\kappa = -1$

Finally, in the third real case, $\kappa = -1$ and the representative points are $M_1 =$ (0, -1), $M_2 =$ (0, -2), $M_3 =$ (2, 0), $M_4 =$ (2, -1), $M_5 =$ (4, 3) and $M_6 =$ (6, 4) (Figure 5.7).

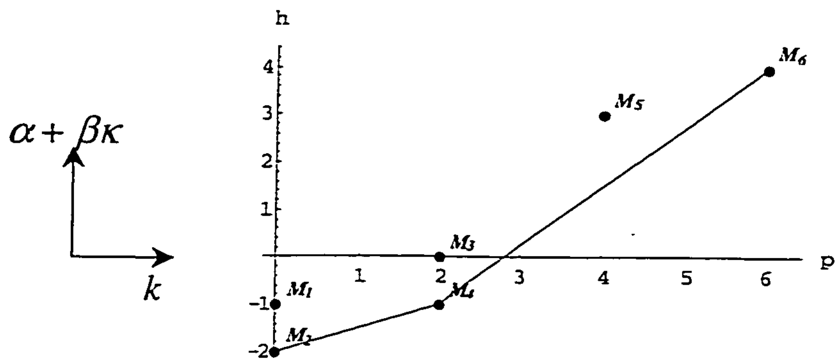

Figure 5.7 Newton's diagram for $m = 0$, $\kappa = -1$

In this case Newton's diagram consists again of 2 segments. The first segment is determined by M_2 and M_4, and the second one by M_4 and M_6. Therefore, equation (3.23) has two groups of roots, the first one being defined by equation

$$-\lambda^2 + \lambda^2 v^2 - \lambda p^2 = 0, \tag{5.12}$$

and the second one defined by

$$-\lambda p^2 + h^4 p^6 = 0. \tag{5.13}$$

In this case, the roots and the eigenvectors are given in Table 5.3.

Table 5.3 Roots and eigenvectors for $m = 0$, $\kappa = -1$

	1	2	3	4	5	6
p	$\sqrt{(1-\nu^2)\lambda i}$	$-\sqrt{(1-\nu^2)\lambda i}$	$\frac{\lambda^{1/4}}{h}$	$-\frac{\lambda^{1/4}}{h}$	$\frac{\lambda^{1/4} i}{h}$	$-\frac{\lambda^{1/4} i}{h}$
U_0	p_1	p_2	ν	ν	ν	ν
W_0	ν	ν	p_3	p_4	p_5	p_6

Note that for roots p_1 and p_2 the coefficient of U_0 in the first equation in (3.23) is equal to zero, and to determine U_0 and W_0 we must use the third equation in (3.23).

As we can see above, the representative points move in the (p, h) plane as κ changes. We are interested in the cases (called separating) when the convex hull changes. These occur when one of the interior points reaches the convex hull, or two or more segments form a straight line.

5.2.2 Boundary Value Problem

5.2.2.1 Low Frequency (Domain III) for Simply Supported Edges

The geometry of the point set and its convex hull for Domain III is given in Figure 5.5. In this case the solution may be written as

$$u=\sum_{i=0}^{6}U_i e^{p_i s}, \qquad w=\sum_{i=0}^{6}W_i e^{p_i s}, \tag{5.14}$$

where p_i, U_i and W_i are determined from Table 5.1.

We consider two types of boundary conditions: simply supported edges and clamped edges. For low frequency vibrations ($\lambda \ll 1$, $\kappa > 0$) of a cylindrical shell with simply supported edges, the boundary conditions have the form

$$u' = w = w'' = 0 \text{ at } s=0 \text{ and } s=L. \tag{5.15}$$

Substituting solution (3.38) into the boundary conditions (3.39) we get the characteristic equation from which the first approximation for the frequency parameter λ may be found

$$D(\lambda)=\begin{vmatrix} p_1U_1 & p_2U_2 & p_3U_3 & p_4U_4 & p_5U_5 & p_6U_6 \\ W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\ p_1^2U_1 & p_2^2U_2 & p_3^2U_3 & p_4^2U_4 & p_5^2U_5 & p_6^2U_6 \\ p_1U_1e^{p_1L} & p_2U_2e^{p_2L} & p_3U_3e^{p_3L} & p_4U_4e^{p_4L} & p_5U_5e^{p_5L} & p_6U_6e^{p_6L} \\ W_1e^{p_1L} & W_2e^{p_2L} & W_3e^{p_3L} & W_4e^{p_4L} & W_5e^{p_5L} & W_6e^{p_6L} \\ p_1^2U_1e^{p_1L} & p_2^2U_2e^{p_2L} & p_3^2U_3e^{p_3L} & p_4^2U_4e^{p_4L} & p_5^2U_5e^{p_5L} & p_6^2U_6e^{p_6L} \end{vmatrix} \tag{5.16}$$

The values of λ may be obtained numerically from this equation. However, we may also try to simplify this determinant. We neglect the values of the third and fourth terms on the left edge compared to those on the right edge, and the values of the fifth and the sixth integral on the right edge compared to those on the left edge. Then, after factorization, the determinant has the form

$$D(\lambda) = -4e^{(\sqrt{2}-ih\sqrt{\lambda})L/h}\left(-1+e^{2iL\sqrt{\lambda}}\right)\lambda^2\left(-1+\nu^2\right)^2/h^8 = 0 \tag{5.17}$$

or

$$D(\lambda) = \left(-1+e^{2iL\sqrt{\lambda}}\right) = 0.$$

Therefore, we obtain one series for the natural frequency parameter

$$\boxed{\lambda = \left(\frac{\pi k}{L}\right)^2.} \tag{5.18}$$

This frequency coincides with that for the unperturbed (momentless) system. In this case two additional roots have negative real parts, and two have positive parts. Since there are four additional boundary conditions (two on each edge) this is a case of regular degeneracy or regular singular perturbation (Vishik *et al.*, 1957) and the next corrections for λ may be constructed with an iterative method. Note that relation (5.18) is valid for $\lambda \ll 1$.

5.2.2.2 High Frequency (Domain II) for Simply Supported Edges

Similarly, for high frequency vibrations of a cylindrical shell with simply supported edges ($\lambda \gg 1$, $-4 < \kappa < 0$), we use the same equation (5.18), but now p_i, U_i and W_i are determined from Table 5.3.

As usual, we neglect the values of the edge effect solutions on the other edge. As a result, after simplification we get the following expression for $D(\lambda)$:

$$D(\lambda) = -\frac{1}{h^8} 4 e^{\frac{iL\lambda^{1/4}\left((1+i)+h\lambda^{1/4}\sqrt{1-\nu^2}\right)}{h}} \left(-1+e^{\frac{2iL\lambda^{1/4}}{h}}\right)\left(-1+e^{2iL\sqrt{\lambda(1-\nu^2)}}\right)\lambda^2\left(\nu^2+\lambda\left(1-\nu^2\right)\right)^2 = 0, \tag{5.19}$$

i.e.

$$D(\lambda) = \left(-1+e^{\frac{2iL\lambda^{1/4}}{h}}\right)\left(-1+e^{2iL\sqrt{\lambda(1-\nu^2)}}\right) = 0.$$

So, we obtain two series for the natural frequency parameter:

$$\boxed{\lambda = \frac{1}{1-\nu^2}\left(\frac{\pi k}{L}\right)^2,} \tag{5.20}$$

and

$$\boxed{\lambda = \left(\frac{\pi k}{L} h\right)^4.} \tag{5.21}$$

Here, there are four pure imaginary roots among the additional ones, and this is not a case of regular degeneracy. Expressions (5.20) and (5.21) are valid for $\lambda \gg 1$.

5.2.2.3 Low Frequency (Domain III) for Clamped Edges

For low frequency vibrations of a cylindrical shell with clamped edges ($\lambda \ll 1$, $\kappa > 0$) the boundary conditions have the form

$$u = w = w' = 0 \text{ at } s = 0 \text{ and } s = L. \tag{5.22}$$

So, we must solve the equation

$$D(\lambda)=\begin{vmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\ W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\ p_1W_1 & p_2W_2 & p_3W_3 & p_4W_4 & p_5W_5 & p_6W_6 \\ U_1e^{p_1L} & U_2e^{p_2L} & U_3e^{p_3L} & U_4e^{p_4L} & U_5e^{p_5L} & U_6e^{p_6L} \\ W_1e^{p_1L} & W_2e^{p_2L} & W_3e^{p_3L} & W_4e^{p_4L} & W_5e^{p_5L} & W_6e^{p_6L} \\ p_1W_1e^{p_1L} & p_2W_2e^{p_2L} & p_3W_3e^{p_3L} & p_4W_4e^{p_4L} & p_5W_5e^{p_5L} & p_6W_6e^{p_6L} \end{vmatrix}, \tag{5.23}$$

where p_i, U_i and W_i are determined from Table 5.1. After transformations we keep only the main terms and obtain

$$D(\lambda) = 2e^{(\sqrt{2}-ih\sqrt{\lambda})L/h}\lambda\left(-1+e^{2iL\sqrt{\lambda}}\right)+O(h)) = 0. \tag{5.24}$$

This equation has only the series of roots

$$\boxed{\lambda = \left(\frac{\pi k}{L}\right)^2.} \tag{5.25}$$

Again this is a case of regular degeneracy.

5.2.2.4 High Frequency (Domain III) for Clamped Edges

For higher frequency vibrations ($\lambda \gg 1$, $-4 < \kappa < 0$) the determinant (5.23) must be used, but p_i, U_i and W_i should be determined from Table 5.3.

After transformations, we keep only the main terms and obtain

$$\begin{aligned} D(\lambda) &= -\frac{1}{h^6}2e^{-\left(1+i+h\lambda^{1/4}\sqrt{1-\nu^2}i\right)L\lambda^{1/4}/h} \\ &\lambda i\left(\left(-1+e^{2iL\sqrt{\lambda}\sqrt{1-\nu^2}}\right)\left(1+e^{2iL\lambda^{1/4}/h}\right)+O(h)\right) = 0. \end{aligned} \tag{5.26}$$

This equation has two series of roots

$$\boxed{\begin{aligned}\lambda_1 &= \frac{1}{1-\nu^2}\left(\frac{\pi k}{L}\right)^2, \\ \lambda_2 &= \left(\frac{\pi(2k+1)h}{2L}\right)^4.\end{aligned}} \tag{5.27}$$

The second series has no analogue for the unperturbed (momentless) system. Again this is a case of nonregular degeneracy (singular perturbation).

Note, again that the expression (5.27) are obtained assuming $\lambda \gg 1$.

5.3 Non-axisymmetric Vibrations

The same approach may be used to study the non-axisymmetric vibrations of cylindrical shells. Equations (3.9) now should be analyzed for $m \neq 0$ (Landman et al., 2000) . In this case, the system does not split, and one has to find the roots of the characteristic equation of the eighth order (3.19):

$$P(p;h,\lambda) = \sum_i a_i p^{k_i} h^{\alpha_i} \lambda^{\beta_i} m^{l_i}. \tag{5.28}$$

The representative points have four coordinates $M_i = \{k_i, \alpha_i, \beta_i, l_i\}$ in the 4D space (p, h, λ, m). Thus, now $M_i^* =$

{{-1, {0, 0, 1, 2}}, {-1, {0, 0, 1, 4}},

{2(1 + ν), {0, 0, 2, 0}}, {-(-3 + ν)(1 + ν), {0, 0, 2, 2}},

{2(-1 + ν)(1 + ν)^2, {0, 0, 3, 0}}, {1, {0, 4, 0, 4}},

$\{-2, \{0, 4, 0, 6\}\}$, $\{1, \{0, 4, 0, 8\}\}$,

$\{-2(1+\nu), \{0, 4, 1, 2\}\}$, $\{(1+\nu)(3+\nu), \{0, 4, 1, 4\}\}$,

$\{(-3+\nu)(1+\nu), \{0, 4, 1, 6\}\}$, $\{-2(-1+\nu)(1+\nu)^2, \{0, 4, 2, 2\}\}$,

$\{-2(-1+\nu)(1+\nu)^2, \{0, 4, 2, 4\}\}$, $\{3+2\nu, \{2, 0, 1, 0\}\}$, $\{2, \{2, 0, 1, 2\}\}$,

$\{(-3+\nu)(1+\nu), \{2, 0, 2, 0\}\}$, $\{-4, \{2, 4, 0, 2\}\}$, $\{8, \{2, 4, 0, 4\}\}$,

$\{-4, \{2, 4, 0, 6\}\}$, $\{-4(-1+\nu)(1+\nu), \{2, 4, 1, 0\}\}$,

$\{2(1+\nu)(-2+\nu^2), \{2, 4, 1, 2\}\}$, $\{-3(-3+\nu)(1+\nu), \{2, 4, 1, 4\}\}$,

$\{-4(-1+\nu)^2(1+\nu)^2, \{2, 4, 2, 0\}\}$, $\{4(-1+\nu)(1+\nu)^2, \{2, 4, 2, 2\}\}$,

$\{1, \{4, 0, 0, 0\}\}$, $\{-1, \{4, 0, 1, 0\}\}$,

$\{-4(-1+\nu)(1+\nu), \{4, 4, 0, 0\}\}$, $\{2(-2+\nu)(2+\nu), \{4, 4, 0, 2\}\}$,

$\{6, \{4, 4, 0, 4\}\}$, $\{4(-1+\nu)(1+\nu), \{4, 4, 1, 0\}\}$, $\{3(-3+\nu)(1+\nu), \{4, 4, 1, 2\}\}$,

$\{-2(-1+\nu)(1+\nu)^2, \{4, 4, 2, 0\}\}$, $\{(-1+\nu)^2(1+\nu)^2, \{4, 8, 0, 4\}\}$,

$\{-2(-1+\nu)^2(1+\nu)^3, \{4, 8, 1, 2\}\}$, $\{-4, \{6, 4, 0, 2\}\}$, $\{-(-3+\nu)(1+\nu), \{6, 4, 1, 0\}\}$,

$\{4(-1+\nu)(1+\nu), \{6, 8, 0, 2\}\}$, $\{4(-1+\nu)^2(1+\nu)^2, \{6, 8, 1, 0\}\}$,

$\{1, \{8, 4, 0, 0\}\}$, $\{-4(-1+\nu)(1+\nu), \{8, 8, 0, 0\}\}\}$.

Similar to the previous (axisymmetric) case we must construct a convex hull in 4D, the facets of which determine the lines that divide the (λ, m)-plane into domains with different structures of the roots of the characteristic equation.

The algorithm is the same as in the case of axisymmetric vibrations:

- construct all solutions at the separating points, separating segments and in the domains between the separating segments;

- find the relative orders of the eigenvectors and substitute the solutions into the imposed boundary conditions;
- solve the characteristic equation numerically or analytically (if possible), to obtain the natural frequency parameter.

As we consider the general case of non-axisymmetric vibrations of the shell, the order of m is not given. In this case the coefficients in the characteristic equation (3.19) depend on three parameters: small h ($0 < h << 1$), and positive λ ($\lambda > 0$) and m ($m > 0$). The analysis of the roots of the characteristic equation for non-axisymmetric vibrations involves the construction of the 4D convex hull in the $(p,\ h,\ \lambda,\ m)$ space.

5.3.1 Separating Points for Non-Axisymmetric Vibrations (4D)

One assumes that $m = m_0\ h^{\tau}$ and $\lambda = \lambda_0\ \ h^{\kappa}$, where $m_0 \sim 1$ and $\lambda_0 \sim 1$.

The steps of the algorithm are the same as for the 3D case, but to construct the 4D convex hull the code *Qhull* has been used. Since only the cases where h is small are of interest, after constructing the 4D convex hull one should select only the facets on the "lower" part of the convex hull, i.e. the facets that are visible from the point $(p, h, \lambda, m) = (0, -\infty, 0, 0)$. Each facet is determined by 4 or more than 4 vertices. Assuming that the orders of the terms corresponding to the vertices of each facet are equal to each other, one finds the orders of λ and m, i.e. the separating points in the $(\kappa,\ \tau)$ plane.

Applying the *Qhull* code to the M_i^* for the non-axisymmetric case, we get the data in the form of Table 5.4.

Table 5.4 Facets for non-axisymmetrical case

Number of Facets	Number of points on Facet	Position-type-output of Points
1	7	31 12 25 1 7 4 38
2	6	24 2 25 1 4 0
3	5	24 25 1 7 38
4	5	5 24 1 7 0
5	9	11 5 2 12 8 1 7 4 0
6	6	11 22 2 8 19 4
7	8	33 39 31 12 37 7 32 38
8	7	33 11 22 31 12 37 4
9	6	33 11 22 37 8 19
10	7	33 11 5 12 8 7 32
11	7	26 5 24 2 8 19 0
12	11	26 39 24 22 31 2 25 37 19 4 38
13	8	26 33 39 5 37 8 19 32
14	7	26 39 5 24 7 32 38

From the first row of Table 5.4 we get the first relation between parameters with regard to the first facet of the convex hull in 4D:

$$p^4h^4\lambda^2 \sim h^4\lambda^2m^4 \sim p^4\lambda \sim \lambda m^4 \sim h^4m^8 \sim \lambda^3 \sim p^8h^4.$$

This leads to the first critical point: $(\beta, \alpha) = (-4, 2)$. Using the same approach, we can find the others critical points from the 3rd, 4th and 10th relations of Table 5.4: (0, 1), (0, 0) and (4, 0) respectively.

5.3.2 Separating Lines for Non-Axisymmetric Vibrations (4D)

After finding the separating points κ_i and τ_i, one can construct the separating lines in the (κ, τ) plane by plotting the horizontal lines $\tau = \tau_i$ through the separating points (κ_i, τ_i). The separating points are $(\kappa_i, \tau_i) = \{(0, 0), (0, 1), (4, 0), (-4, 2)\}$, so the horizontal lines are $\tau_i = 0, 1, 2$. These lines divide the entire plane into three zones: $0<\tau<1$, $1<\tau<2$, $2<\tau$. For any fixed κ inside one zone the structures of the corresponding 3D convex hulls are similar. So, one may choose an arbitrary point inside each domain and obtain the relations between κ and τ, which determine the separating lines (Landman, 2000).

5.3.2.1 Separating Segments for Domain $0 < \tau < 1$

In the case under consideration, the domain $0 < \tau < 1$ is analyzed. Setting $\tau = ½$ arbitrarily, one can find the facets of the 3D convex hull. The facets are: $\{\{M_{11}, M_3, M_{25}\}, \{M_{11}, M_{25}, M_{24}, M_{26}\}, \{M_{11}, M_{13}, M_{15}, M_{24}, M_{26}, M_2\}, \{M_{25}, M_{39}, M_{26}\}, \{M_2, M_{26}, M_{39}, M_{15}, M_{35}, M_{29}\}\}$ (see Figure 5.8). Note that for any τ in the domain (0, 1) the 3D convex hull has such a form.

Imposing that the orders of all terms corresponding to the points forming a facet be equal to each other, we find the relations from which the orders of λ for the separating cases may be determined:

(1) $h^{(-4\alpha)}\lambda \sim h^{-4(-1+2\alpha)} \sim p^4 \Rightarrow \lambda \sim h^{(4-8\alpha+4\alpha)} \Rightarrow \beta = 4 - 4\alpha$

$\Rightarrow \alpha = 1 - \beta/4;$

(2) $h^{(-4\alpha)}\lambda \sim p^4 \sim \lambda p^4 \Rightarrow \lambda \sim 1 \Rightarrow \beta = 0;$

(3) $h^{(-4\alpha)}\lambda \sim \lambda p^4 \sim \lambda^3 \Rightarrow \alpha = -\beta/2;$ (5.29)

(4) $p^4 \sim h^4p^8 \sim \lambda p^4 \Rightarrow \beta = 0;$

(5) $\lambda^3 \sim \lambda p^4 \sim h^4p^8 \sim h^4\lambda^2p^4 \Rightarrow \beta = -4.$

For the domain analyzed, these segments are plotted in Figure 5.8.

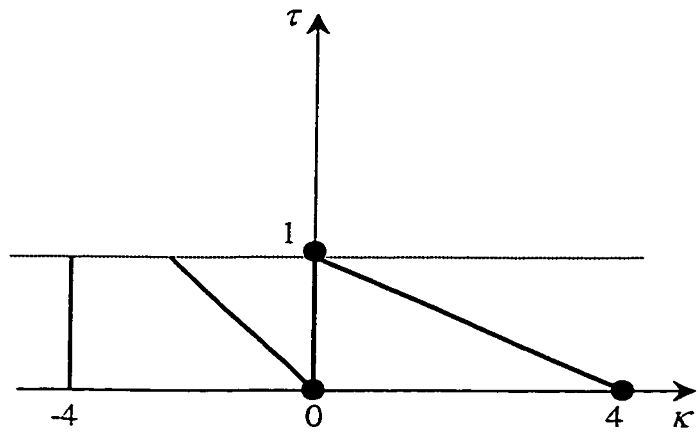

Figure 5.8 τ versus κ for $0 < \tau < 1$

For any point (κ, τ) inside one domain the structures of the corresponding 2D convex hulls are similar.

The other 2 domains are analyzed in the same manner. As the initial system was constructed with the restriction that it be valid only for $0 \leq \tau \leq 2$, we should consider the second zone to complete this figure.

5.3.2.2 Separating Segments for Domain $1 < \tau < 2$

Next, the domain $1 < \tau < 2$ is analyzed. Setting $\tau = 3/2$ arbitrarily, one can find the facets of the 3D convex hull. The facets are: $\{\{M_{11}, M_{26}, M_2\}, \{M_3, M_{39}, M_{26}, M_{11}\}, \{M_2, M_{26}, M_{39}, M_{29}\}$ (see Figure 5.8). Note that for any τ in the domain (1, 2) the 3D convex hull has such a form.

This leads to the following relations:

(1) This facet coincides with the third one in the case $\tau = 1/2$ and $h^{(-4\alpha)}\lambda \sim \lambda p^4 \sim \lambda^3 \Rightarrow \alpha = -\beta/2$;

(2) $h^{-4(2\alpha+1)} \sim h^4 p^8 \sim \lambda p^4 \sim h^{(-4\alpha)}\lambda \Rightarrow \lambda \sim h^{-4\alpha+4} \Rightarrow \beta = -4\alpha+4 \Rightarrow \alpha = 1-\beta/4;$ (5.30)

(3) This facet coincides with the fifth one in the case $\tau = 1/2$ and $\lambda^3 \sim \lambda p^4 \sim h^4 p^8 \sim h^4 \lambda^2 p^4 \Rightarrow \beta = -4$.

For the analyzed domain, these segments are plotted in Figure 5.9.

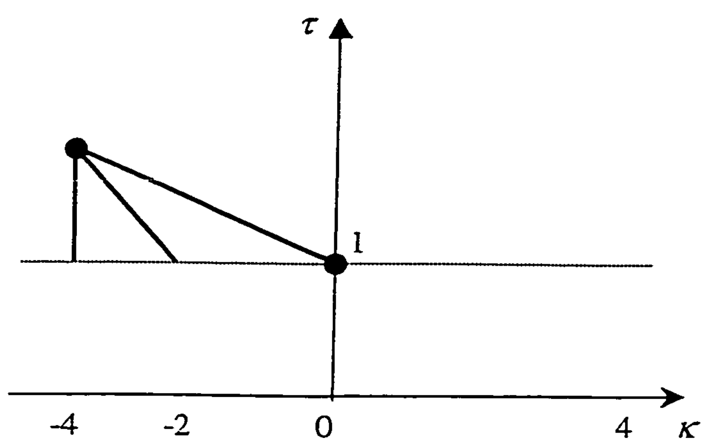

Figure 5.9 τ versus κ for $1 < \tau < 2$

For any point (κ; τ) inside one domain the structures of the corresponding 2D convex hulls are similar.

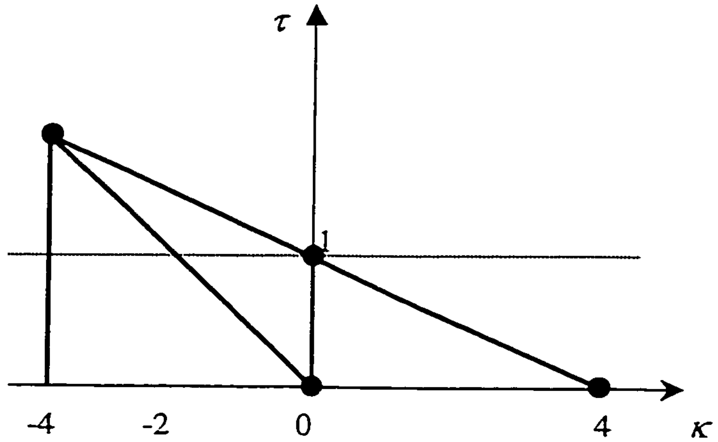

Figure 5.10 τ versus κ for $0 < \tau < 2$

As one can see, using this procedure, we cannot find the "horizontal" segments, α = const. To find them, we repeat the construction, but for the vertical zones formed by vertical lines thought critical points. To illustrate this, we consider two domains: (1): $-4 < \kappa < 0$ and (2): $0 < \kappa < 4$.

5.3.2.3 Separating Segments for Domains $-4 < \kappa < 0$

We analyze here the range $-4 < \kappa < 0$:

Arbitrarily setting $\kappa = -2$, one can find the facets of the 3D convex hull. The facets are: $\{\{M_5, M_8, M_{31}\}, \{M_3, M_5, M_{31}, M_{39}\}$. This leads to the following relations:

(1) $h^{\beta}m^4 \sim h^{3\beta} \sim p^4h^{4+\beta} \Rightarrow m^4 \sim h^{2\beta} \sim h^{-4\alpha} \Rightarrow \beta = -2\alpha;$

(2) $h^4m^8 \sim h^{\beta}m^4 \sim p^4h^{4+\beta} \sim p^8h^4 \Rightarrow m^4 \sim h^{\beta-4} \sim h^{-4\alpha} \Rightarrow \beta = 4 - 4\alpha.$ (5.31)

For the analyzed domain, these segments are plotted in Figure 5.11.

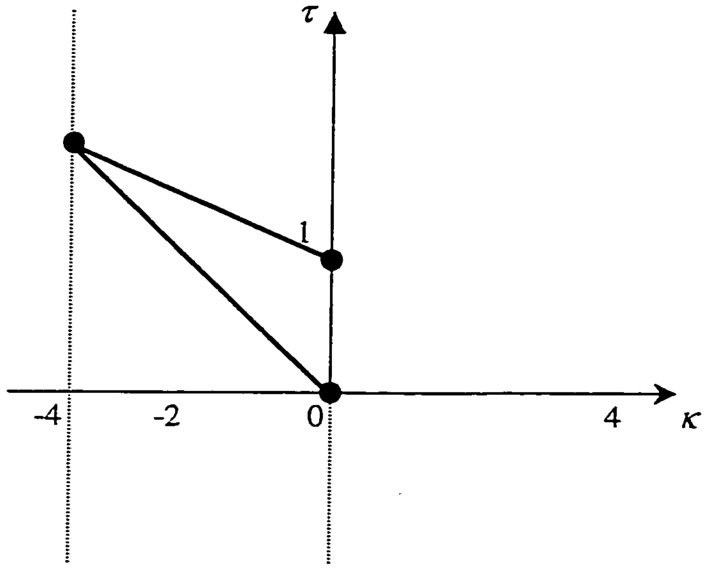

Figure 5.11 τ versus κ for $-4 < \kappa < 0$

5.3.2.4 Separating Segments for Domains $0 < \kappa < 4$

We analyze here the range $0 < \kappa < 4$:

Setting $\kappa = 2$ arbitrarily, one can find the facets of the 3D convex hull. The facets are: $\{\{M_3, M_5, M_{25}\}, \{M_3, M_{25}, M_{39}\}, \{M_4, M_6, M_{25}\}, \{M_4, M_{25}, M_5\}$. This leads to the following relations:

$$
\begin{aligned}
&(1) \quad h^4m^8 \sim h^\beta m^4 \sim p^4 \Rightarrow m^4 \sim h^{\beta-4} \sim h^{-4\alpha} \Rightarrow \beta = 4 - 4\alpha;\\
&(2) \quad h^4m^8 \sim p^4 \sim h^4p^8 \Rightarrow m^8 \sim h^{-8} \sim h^{-8\alpha} \Rightarrow \alpha = 1;\\
&(3) \quad h^\beta m^2 \sim h^{2\beta} \sim p^4 \Rightarrow m^2 \sim h^\beta \sim h^{-2\alpha} \Rightarrow \beta = -2\alpha;\\
&(4) \quad h^\beta m^2 \sim p^4 \sim h^\beta m^4 \Rightarrow m^2 \sim h^0 \sim h^{-2\alpha} \Rightarrow \alpha = 0.
\end{aligned}
\tag{5.32}
$$

For the analyzed domain, these segments are plotted in Figure 5.12.

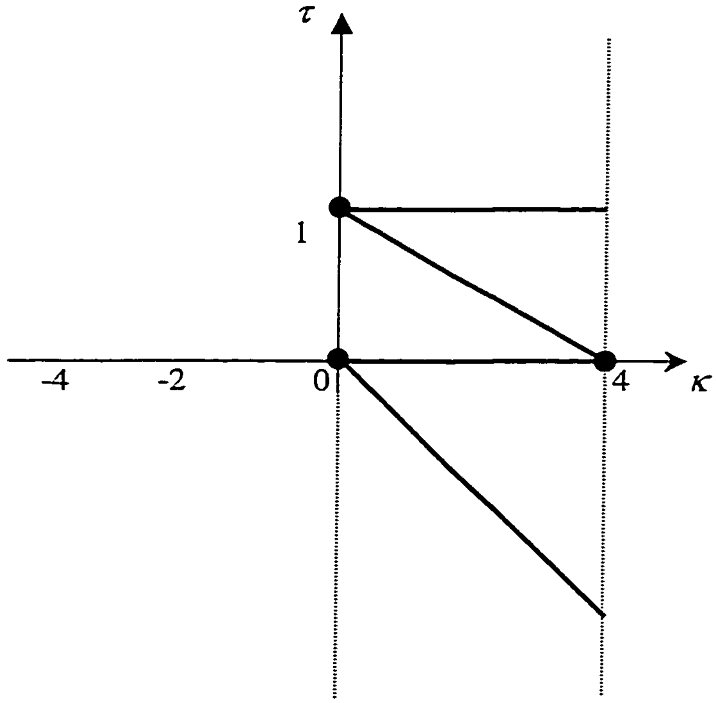

Figure 5.12 τ versus κ for $0 < \kappa < 4$

Now we can get the final graph:

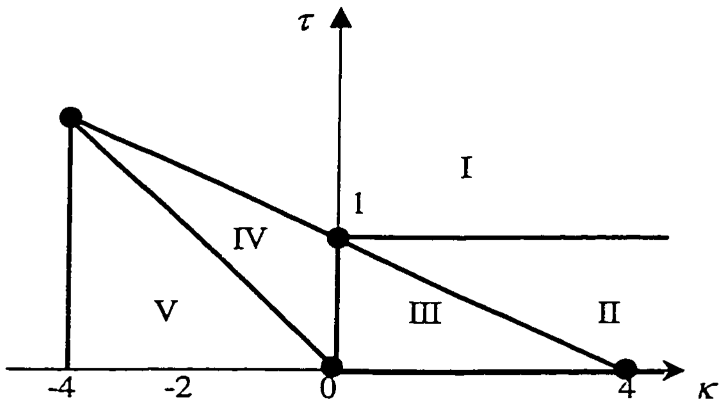

Figure 5.13 τ versus κ for $-4 < \kappa < 4$ and $0 < \tau < 2$

The final graph (Figure 5.13) representing the domains and separating lines is similar to that obtained in (Goldenveizer *et al.*, 1973). Since the initial equations describing the vibrations of shells are valid if the frequency is not too high and the wave number in the circumferential direction is not too large, the analysis is limited to the cases $-4 < \kappa$ and $-2 < \tau \leq 0$.

According to the analysis steps, next we have to find the roots of the short form of the equations.

5.3.3 Short Forms of Equation in all Domains, on Separating Lines and at Separating Points

5.3.3.1 Short Forms of Equation in all Domains

(I): The main terms in the characteristic equation, which we must keep in their short form for the first domain ($1 < \alpha < 2$ and $\beta < 4 - 4\alpha$), are defined by the points M_i, where $i = \{3, 18, 30, 37, 39\}$. The corresponding short equation is

$$m^8h^4 - 4m^6p^2h^4 + 6m^4p^4h^4 - 4m^2p^6h^4 + p^8h^4 = 0.$$

After simplifications, the eight roots can be found from

$$\left(p^2 - m^2\right)^4 = 0.$$

Therefore, we get two roots with multiplicity four:

$$\boxed{p_{1-4} = m \text{ and } p_{5-8} = -m.} \tag{5.33}$$

(II): The main terms in the characteristic equation, which we must keep in their short form for the first domain ($0 < \alpha < 1$ and $\beta > 4 - 4\alpha$), are defined by the points M_i, where $i = \{3, 25\}$ and $\{25, 39\}$. The corresponding short equations are

$$m^8 h^4 + p^4 = 0 \qquad \text{and} \qquad p^8 h^4 + p^4 = 0.$$

After simplifications, the eight roots can be found from:

$$m^8 h^4 + p^4 = 0 \qquad \text{and} \qquad 1 + h^4 p^4 = 0.$$

Therefore, we get two series of roots:

$$\boxed{\begin{gathered} p_1 = m^2 h \ , \ p_2 = -m^2 h,\ p_3 = im^2 h,\ p_4 = -im^2 h, \\ \text{and} \\ p_5 = \varepsilon_1 \frac{1}{h}, p_6 = \varepsilon_2 \frac{1}{h}, p_7 = \varepsilon_7 \frac{1}{h}, p_8 = \varepsilon_8 \frac{1}{h}. \end{gathered}} \tag{5.34}$$

(III): The main terms in the characteristic equation which we must keep in their short form for the first domain ($0 < \alpha < 1$ and $\beta < 4 - 4\alpha$), are defined by the points M_i, where $i = \{11, 25\}$ and $\{25, 39\}$. The corresponding short equations are

$$p^4 - \lambda m^4 = 0 \qquad \text{and} \qquad p^8 h^4 + p^4 = 0.$$

After simplifications, the eight roots can be found from

$$p^4 - \lambda m^4 = 0 \qquad \text{and} \qquad 1 + h^4 p^4 = 0.$$

Therefore, we get two series of roots:

$$\boxed{\begin{gathered} p_1 = m\sqrt[4]{\lambda} \ , \ p_2 = -m\sqrt[4]{\lambda},\ p_3 = im\sqrt[4]{\lambda},\ p_4 = -im\sqrt[4]{\lambda}, \\ \text{and} \\ p_5 = \varepsilon_1 \frac{1}{h}, p_6 = \varepsilon_2 \frac{1}{h}, p_7 = \varepsilon_7 \frac{1}{h}, p_8 = \varepsilon_8 \frac{1}{h}. \end{gathered}} \tag{5.35}$$

(IV): The main terms in the characteristic equation, which we must keep in their short form for the first domain ($0 > \beta = -2\alpha$ and $\beta < 4 - 4\alpha$), are defined by the points M_i, where $i = \{11, 24, 26\}$ and $\{26, 39\}$. The corresponding short equations are

$$-\lambda m^4 + 2m^2 p^2 \lambda - p^4 \lambda = 0 \qquad \text{and} \qquad p^8 h^4 - p^4 \lambda = 0.$$

After simplifications, the eight roots can be found from

$$\left(p^2 - m^2\right)^2 = 0 \qquad \text{and} \qquad -\lambda + h^4 p^4 = 0.$$

Therefore, we get two roots with multiplicity two:

$$\boxed{p_{1,2} = m \ \text{ and } \ p_{3,4} = -m \text{ ;and } p_5 = \frac{\sqrt[4]{\lambda}}{h}, p_6 = -\frac{\sqrt[4]{\lambda}}{h}, p_7 = i\frac{\sqrt[4]{\lambda}}{h}, p_8 = -i\frac{\sqrt[4]{\lambda}}{h}.} \tag{5.36}$$

(V): The main terms in the characteristic equation, which we must keep in their short form for the first domain ($0 < \beta = -2\alpha$ and $\alpha > 0$), are defined by the points M_i, where $i = \{2, 15, 26\}$ and $\{26, 39\}$. The corresponding short equations are

$$2(-1+\nu)(1+\nu)^2 \lambda^3 + (\nu - 3)(1+\nu) p^2 \lambda^2 - p^4 \lambda = 0 \qquad \text{and} \qquad p^8 h^4 - p^4 \lambda = 0.$$

After simplifications, the eight roots can be found from:

$$\left(p^2 + 2(1+\nu)\lambda\right)\left(p^2 + \left(1 - \nu^2\right)\lambda\right) = 0 \qquad \text{and} \qquad -\lambda + h^4 p^4 = 0.$$

Therefore, we get two series of roots:

$$\boxed{\begin{array}{l} p_{1,2} = \pm\sqrt{2(1+\nu)\lambda} \ \text{ and } \ p_{3,4} = \pm\sqrt{\left(1-\nu^2\right)\lambda}\,; \\ \text{and} \\ p_5 = \frac{\sqrt[4]{\lambda}}{h}, p_6 = -\frac{\sqrt[4]{\lambda}}{h}, p_7 = i\frac{\sqrt[4]{\lambda}}{h}, p_8 = -i\frac{\sqrt[4]{\lambda}}{h}. \end{array}} \tag{5.37}$$

5.3.3.2 Short Forms of Equation on Separating Lines

To find the short form of the characteristic equation on a separating line we can use one of the two approaches:

Firstly, we may consider a separating line as a set of the 3D separating points. The equations for these points have been constructed in Section 5.3.2. The main terms of the characteristic equation are the same for any point of the separating line.

On the other hand, the terms of the characteristic equation in a separating line between two domains are the disjunction of the terms of the characteristic equations for these domains.

In this section we list the positions of the main terms in the initial list of points (5.28) for each separating line. Then we construct the short equations for each point of the separating line.

- $\beta = -4$

The positions of the main terms in the initial list of points are $\{2, 15, 26, 29, 35, 39\}$. This gives the equation

$$-2\lambda^3 - 2\lambda^3 v + 2\lambda^3 v^2 + 2\lambda^3 v^3 - 3\lambda^2 p^2 - 2\lambda^2 v p^2 + \lambda^2 v^2 p^2 - \lambda p^4 + 2h^4\lambda^2 p^4 + 2h^4\lambda^2 v p^4 -$$
$$-2h^4\lambda^2 v^2 p^4 - 2h^4\lambda^2 v^3 p^4 + 3h^4\lambda p^6 + 2h^4\lambda v p^6 - h^4\lambda v^2 p^6 + h^4 p^8 = 0,$$

or, after simplifications:

$$\boxed{\begin{array}{l} 2\lambda^3(-1+v)(1+v)^2 - \lambda^2(1+v)\left(3 - v + 2h^4 p^2\left(v^2 - 1\right)\right)p^2 - \lambda p^4 \\ -\lambda h^4(v-3)(v+1)p^6 + h^4 p^8 = 0. \end{array}} \qquad (5.38)$$

From this equation we can find the set of the roots.

- $\beta = -2\alpha$

The positions of the main terms in the initial list of points are $\{11, 13, 2, 24, 15, 26\}$.

This gives the equation

$$-2\lambda^3 + 3\lambda^2 m^2 - \lambda m^4 - 2\lambda^3 v + 2\lambda^2 m^2 v + 2\lambda^3 v^2 - \lambda^2 m^2 v^2 + 2\lambda^3 v^3 - 3\lambda^2 p^2 + 2\lambda m^2 p^2 - \\ -2\lambda^2 v p^2 + \lambda^2 v^2 p^2 - \lambda p^4 = 0$$

and $\{26, 39\}$ gives

$$-\lambda p^4 + h^4 p^8 = 0,$$

or, after simplifications:

$$\boxed{\lambda\left(2\lambda^2(v-1)(1+v)^2 - \lambda(1+v)(v-3)(m^2-p^2) - (m^2-p^2)^2\right) = 0} \quad (5.39)$$

and

$$\boxed{-\lambda + h^4 p^4 = 0.} \quad (5.40)$$

From equations (5.39) and (5.40) two sets of the roots can be found.

- $\alpha = 1 - \dfrac{\beta}{4}$ Thus we have two different segments:

1. First segment:

The positions of the main terms in the initial list of points are $\{3, 11, 18, 24, 26, 30, 37, 39\}$. This gives the equation

$$m^8 h^4 - 4m^6 p^2 h^4 + 6m^4 p^4 h^4 - 4m^2 p^6 h^4 + p^8 h^4 - \lambda m^4 + 2m^2 p^2 \lambda - p^4 \lambda = 0.$$

After simplifications we obtain:

$$\boxed{(p^2 - m^2)^4 h^4 - \lambda(p^2 - m^2)^2 = 0.} \quad (5.41)$$

2. Second segment:

The positions of the main terms in the initial list of points are $\{3, 11, 25\}$. This gives the equation

$$\boxed{-\lambda m^4 + h^4 m^8 + p^4 = 0} \qquad (5.42)$$

and $\{25, 39\}$ gives

$$p^8 h^4 + p^4 = 0.$$

After simplifications, the other four roots can be found from

$$\boxed{1 + h^4 p^4 = 0.} \qquad (5.43)$$

- $\beta = 0$

The positions of the main terms in the initial list of points are $\{11, 24, 25, 26\}$. This gives the equation

$$-\lambda m^4 + 2\lambda m^2 p^2 + p^4 - \lambda p^4 = 0, \qquad (5.44)$$

and $\{25, 26, 39\}$ gives

$$p^4 - \lambda p^4 + h^4 p^8 = 0.$$

After simplifications we obtain:

$$\boxed{p^4 - \lambda(p^2 - m^2)^2 = 0,} \qquad (5.45)$$

and

$$\boxed{1 - \lambda + h^4 p^4 = 0.} \qquad (5.46)$$

- $\alpha = 0$

$$\boxed{-\lambda m^4 - \lambda m^2 + p^4 = 0,} \qquad (5.47)$$

and

$$\boxed{1+h^4p^4=0}. \tag{5.48}$$

- $\alpha=1$

$$\boxed{\begin{gathered}2\lambda^2(1+v)-\lambda m^2+p^2\lambda(3+2v)+p^4=0,\\ \text{and}\\ 1+h^4p^4=0.\end{gathered}}$$

5.3.3.3 Short Forms of Equation at Separating Points

First of all the main terms of the short equations have been obtained considering the 4D critical points (see Section 5.3.1). Besides that, the main terms at a critical point are the disjunction of the main terms on the separating lines and at the same time they are the disjunction of the main terms in the domains.

- Point (0, 1). The main terms in the characteristic equation, which we must keep in their short form for the first separating point, are defined by the points M_i, where i = {3, 11, 18, 24, 25, 26, 30, 37, 39}. Indeed, the main terms at the separating point are the disjunction of the main terms of the domains I, II, III and IV (or the separating lines 3a and 3b or 4 and 6). The corresponding short equation is

$$m^8h^4-4m^6p^2h^4+6m^4p^4h^4-4m^2p^6h^4+p^8h^4-\lambda m^4+2m^2p^2\lambda-p^4\lambda+p^4=0$$

After simplifications we obtain:

$$\boxed{(p^2-m^2)^4h^4-\lambda(p^2-m^2)^2+p^4=0.} \tag{5.49}$$

• Point (0, 0). The main terms in the characteristic equation, which we must keep in their short form for the first separating point, are defined by the points M_i, where i are listed below. Indeed, the main terms at the separating point are the disjunction of the main terms of the domains III, IV and V (or the separating lines 2, 4 and 5). The positions of the main terms in the initial list of points are {2, 15, 11, 24, 25, 26}. The corresponding short equation is

$$-p^4\lambda + 2(-1+\nu)(1+\nu)^2\lambda^3 + (\nu-3)(1+\nu)p^2\lambda^2 = 0.$$

After the simplifications, the eight roots can be found from:

$$\boxed{\left(p^2 + 2(1+\nu)\lambda\right)\left(p^2 + \left(1-\nu^2\right)\lambda\right) = 0} \qquad (5.50)$$

and {25, 26, 39} gives

$$p^4 - \lambda p^4 + h^4 p^8 = 0.$$

After simplifications, we obtain

$$\boxed{1 - \lambda + h^4 p^4 = 0.} \qquad (5.51)$$

5.4 Special Cases: m ~ 1 and m = 1

Now we consider only the cases for which the order of m is known. This permits to reduce the 4D problem to the 3D one discussed in the previous sections.

We consider the case, when $m = m_0 h^{\tau}$, $\tau = 0$, i.e. $M_i = \{k_i, \alpha_i, \beta_i\}$. Equation (3.52) in this case may be written as

$$P(p;h,\lambda)=\sum_{i=1}^{24} a_i p^{\kappa_i} h^{\alpha_i} \lambda^{\beta_i} \tag{5.52}$$

where the 24 representative points $M_i = \{a_i, \{\kappa_i, \alpha_i, \beta_i\}\}$, $i = 1, \ldots, 24$, (5.53)

with their weights a_i are listed below:

$\{-m^2 (1 + m^2), \{0, 0, 1\}\}$, $\{ - (1 + \nu) (-2 - 3 m^2 + m^2 \nu), \{0, 0, 2\}\}$,

$\{2 (-1 + \nu) (1 + \nu)$^2, $\{0, 0, 3\}\}$, $\{ (-1 + m)^2 m^4 (1 + m)^2, \{0, 4, 0\}\}$,

$\{ m^2 (1 + \nu)(-2 + 3 m^2 - 3 m^4 + m^2 \nu + m^4 \nu), \{0, 4, 1\}\}$,

$\{ -2 m^2 (1 + m^2) (-1 + \nu) (1 + \nu)^2, \{0, 4, 2\}\}$, $\{3 + 2 m^2 + 2 \nu, \{2, 0, 1\}\}$,

$\{ (-3 + \nu) (1 + \nu), \{2, 0, 2\}\}$, $\{-4 (-1 + m)^2 m^2 (1 + m)^2, \{2, 4, 0\}\}$,

$\{ - (1 + \nu) (-4 + 4 m^2 - 9 m^4 + 4 \nu + 3 m^4 \nu - 2 m^2 \nu^2), \{2, 4, 1\}\}$,

$\{ 4 (1 + m^2 - \nu) (-1 + \nu) (1 + \nu)^2, \{2, 4, 2\}\}$, $\{1, \{4, 0, 0\}\}$,

$\{ -1, \{4, 0, 1\}\}$, $\{ 2 (2 - 4 m^2 + 3 m^4 - 2 \nu^2 + m^2 \nu^2), \{4, 4, 0\}\}$,

$\{ (1 + \nu) (-4 - 9 m^2 + 4 \nu + 3 m^2 \nu), \{4, 4, 1\}\}$, $\{ -2 (-1 + \nu) (1 + \nu)^2, \{4, 4, 2\}\}$,

$\{m^4 (-1 + \nu)^2 (1 + \nu)^2, \{4, 8, 0\}\}$, $\{ -2 m^2 (-1 + \nu)^2 (1 + \nu)^3, \{4, 8, 1\}\}$,

$\{ -4 m^2, \{6, 4, 0\}\}$, $\{ - (-3 + \nu) (1 + \nu), \{6, 4, 1\}\}$, $\{ 4 m^2 (-1 + \nu) (1 + \nu), \{6, 8, 0\}\}$,

$\{ 4 (-1 + \nu)^2 (1 + \nu)^2, \{6, 8, 1\}\}$, $\{1, \{8, 4, 0\}\}$, $\{ -4 (-1 + \nu) (1 + \nu), \{8, 8, 0\}\}$.

The convex hull for these points is plotted in Figure 5.14.

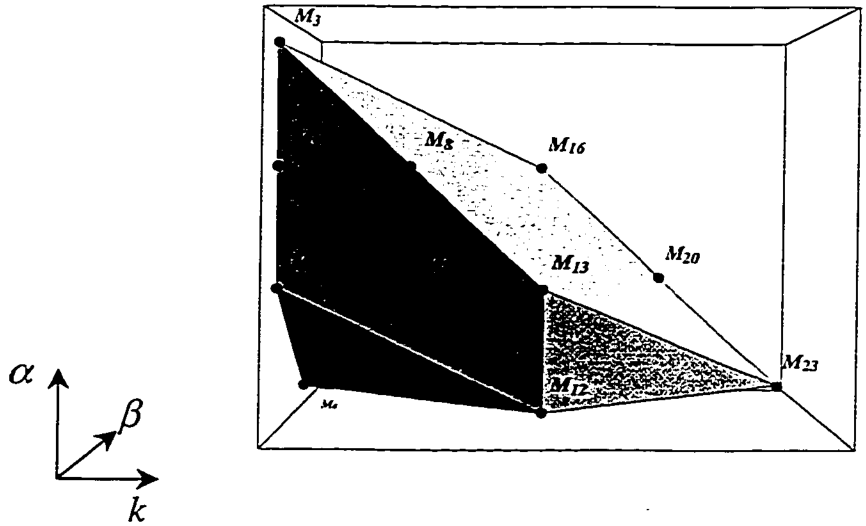

Figure 5.14 3D convex hull for $m \sim 1$

The facets of the convex hull determine the separating points $\kappa(\lambda \sim h^{\kappa})$. In the present case, the 3D convex hull consists of 4 facets: 1. (M_1, M_2, M_3, M_7, M_8, M_{12}, M_{13}); 2. (M_{12}, M_{13}, M_{23}); 3. (M_1, M_4, M_{12}); 4. (M_3, M_8, M_{13}, M_{16}, M_{20}, M_{23}). Imposing that the orders of all terms forming a facet be equal to each other, we find the relations from which the orders of λ for the separating cases may be determined:

$$\lambda \sim \lambda^2 \sim \lambda^3 \sim \lambda p^2 \sim \lambda^2 p^2 \sim p^4 \sim \lambda p^4 ,$$

$$p^4 \sim \lambda p^4 \sim h^4 p^8 ,$$

$$h^4 \sim \lambda \sim p^4 , \tag{5.54}$$

$$\lambda^3 \sim \lambda^2 p^2 \sim \lambda p^4 \sim \lambda^2 h^4 p^4 \sim \lambda h^4 p^6 \sim h^4 p^8 .$$

So, for the first and second relations $\kappa = 0$, for the third $\kappa = 4$, and for the fourth $\kappa = -4$. For any λ inside a domain the structure of the convex hull and, therefore, the roots and the eigenvectors are similar, and thus we can obtain the values of the roots and eigenvectors considering only one value of λ for each domain. We substitute for the case A: $\kappa = -4$, B: $\kappa = 0$ and C: $\kappa = 4$, and domains I, II, III and IV: $\kappa = -6$, $\kappa = -2$, $\kappa = 2$, and $\kappa = 5$ respectively. Therefore, we consider seven cases here, where κ is equal to -6, -4, -2, 0, 2, 4 and 5, respectively.

- **Case $\kappa = -6$**

  Newton's diagram is plotted in Figure 5.15.

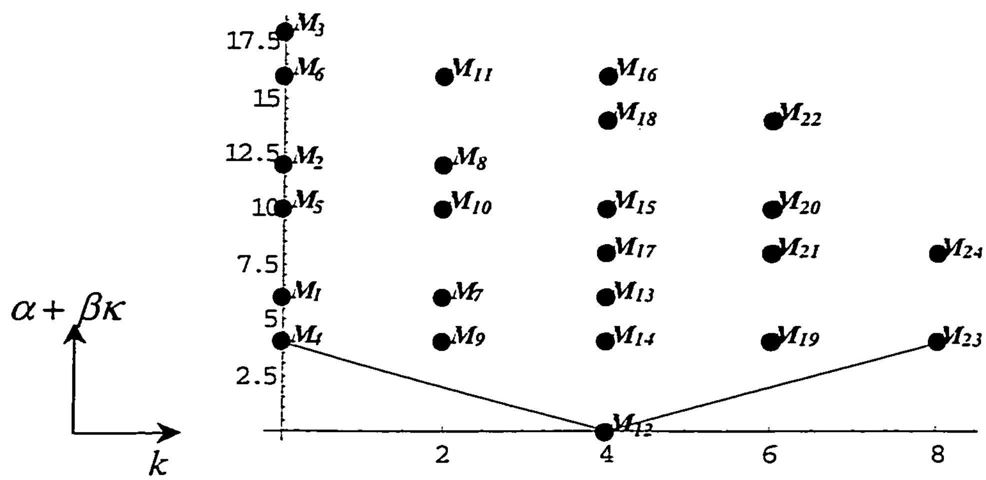

Figure 5.15 Newton's diagram for $m \sim 1$, $\kappa = -6$

In this case the representative points for equation (3.23) are $M_i = \{\{0, 6\}, \{0, 12\}, \{0, 18\}, \{0, 4\}, \{0, 10\}, \{0, 16\}, \{2, 6\}, \{2, 12\}, \{2, 4\}, \{2, 10\}, \{2, 16\}, \{4, 0\}, \{4, 6\}, \{4, 4\}, \{4, 10\}, \{4, 16\}, \{4, 8\}, \{4, 14\}, \{6, 4\}, \{6, 10\}, \{6, 8\}, \{6, 14\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, ..., 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_4 = (0, 4)$, and $M_{10} = (4, 0)$, and the second segment by the points $M_{10} = (4, 0)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = -4$**

Newton's diagram is plotted in Figure 5.16.

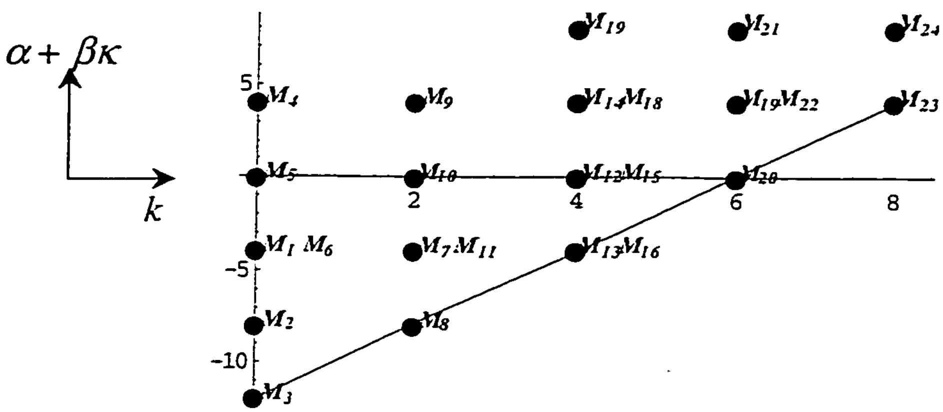

Figure 5.16 Newton's diagram for $m \sim 1$, $\kappa = -4$

In this case the representative points for equation (3.23) are $M_i = \{\{0, -4\}, \{0, -8\}, \{0, -12\}, \{0, 4\}, \{0, 0\}, \{0, -4\}, \{2, -4\}, \{2, -8\}, \{2, 4\}, \{2, 0\}, \{2, -4\}, \{4, 0\}, \{4, -4\}, \{4, 4\}, \{4, 0\}, \{4, -4\}, \{4, 8\}, \{4, 4\}, \{6, 4\}, \{6, 0\}, \{6, 8\}, \{6, 4\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, \ldots, 24$.

Newton's diagram consists of 1 segment. This segment is determined by the points $M_3 = (0, -12)$, $M_8 = (2, -8)$, $M_{13} = M_{16} = (4, -4)$, $M_{20} = (6, 0)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = -2$**

Newton's diagram is plotted in Figure 5.17.

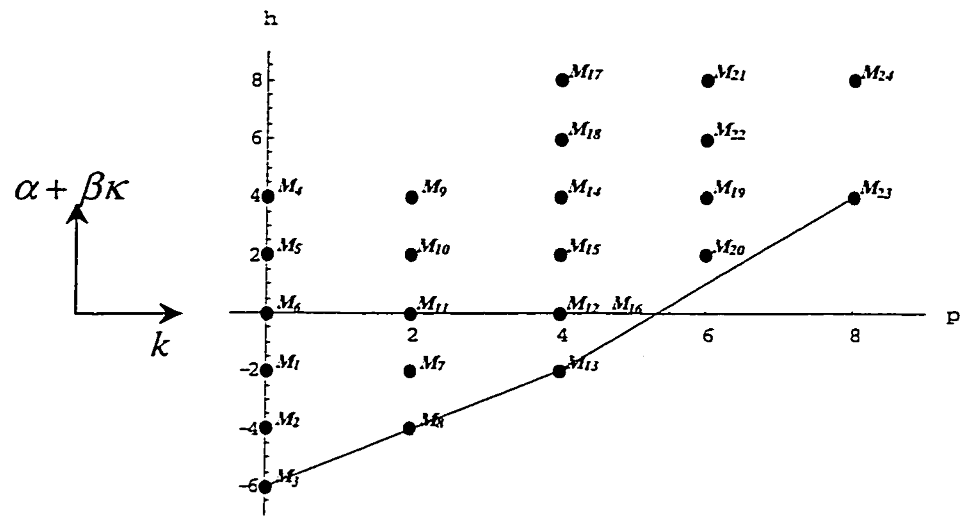

Figure 5.17 Newton's diagram for $m \sim 1$, $\kappa = -2$

In this case the representative points for equation (3.23) are $M_i = \{\{0, -2\}, \{0, -4\}, \{0, -6\}, \{0, 4\}, \{0, 2\}, \{0, 0\}, \{2, -2\}, \{2, -4\}, \{2, 4\}, \{2, 2\}, \{2, 0\}, \{4, 0\}, \{4, -2\}, \{4, 4\}, \{4, 2\}, \{4, 0\}, \{4, 8\}, \{4, 6\}, \{6, 4\}, \{6, 2\}, \{6, 8\}, \{6, 6\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, \ldots, 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_3 = (0, -6)$, $M_8 = (2, -4)$, $M_{13} = (4, -2)$, and the second segment by the points $M_{13} = (4, -2)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = 0$**

Newton's diagram is plotted in Figure 5.18.

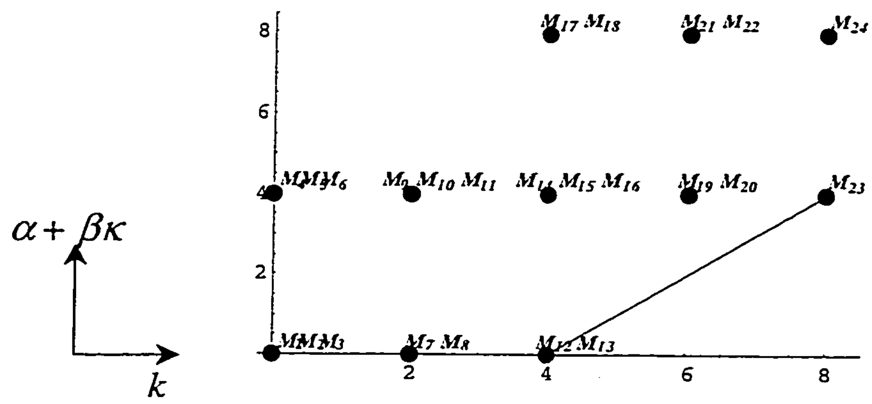

Figure 5.18 Newton's diagram for $m \sim 1$, $\kappa = 0$

In this case the representative points for equation (3.23) are $M_i = \{\{0, 0\}, \{0, 0\}, \{0, 0\}, \{0, 4\}, \{0, 4\}, \{0, 4\}, \{2, 0\}, \{2, 0\}, \{2, 4\}, \{2, 4\}, \{2, 4\}, \{4, 0\}, \{4, 0\}, \{4, 4\}, \{4, 4\}, \{4, 4\}, \{4, 8\}, \{4, 8\}, \{6, 4\}, \{6, 4\}, \{6, 8\}, \{6, 8\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, \ldots, 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_1 = M_2 = M_3 = (0, 0)$, $M_7 = M_8 = (2, 0)$ and $M_{12} = M_{13} = (4, 0)$, and the second segment by the points $M_{12} = M_3 = (4, 0)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = 2$**

Newton's diagram is plotted in Figure 5.19.

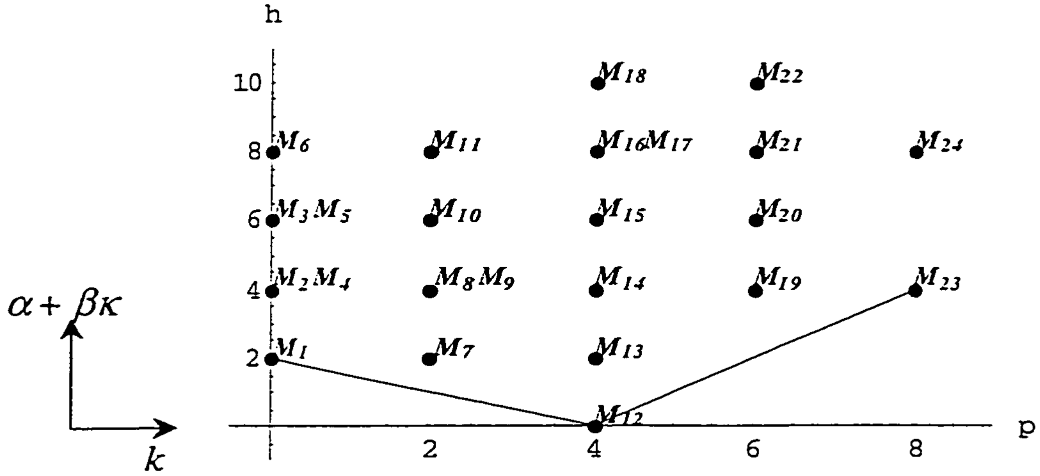

Figure 5.19 Newton's diagram for $m \sim 1$, $\kappa = 2$

In this case the representative points for equation (3.23) are $M_i = \{\{0, 2\}, \{0, 4\}, \{0, 6\}, \{0, 4\}, \{0, 6\}, \{0, 8\}, \{2, 2\}, \{2, 4\}, \{2, 4\}, \{2, 6\}, \{2, 8\}, \{4, 0\}, \{4, 2\}, \{4, 4\}, \{4, 6\}, \{4, 8\}, \{4, 8\}, \{4, 10\}, \{6, 4\}, \{6, 6\}, \{6, 8\}, \{6, 10\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, \ldots, 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_1 = (0, 2)$, $M_{72} = (4, 0)$, and the second segment by the points $M_{12} = (4, 0)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = 4$**

Newton's diagram is plotted in Figure 5.20.

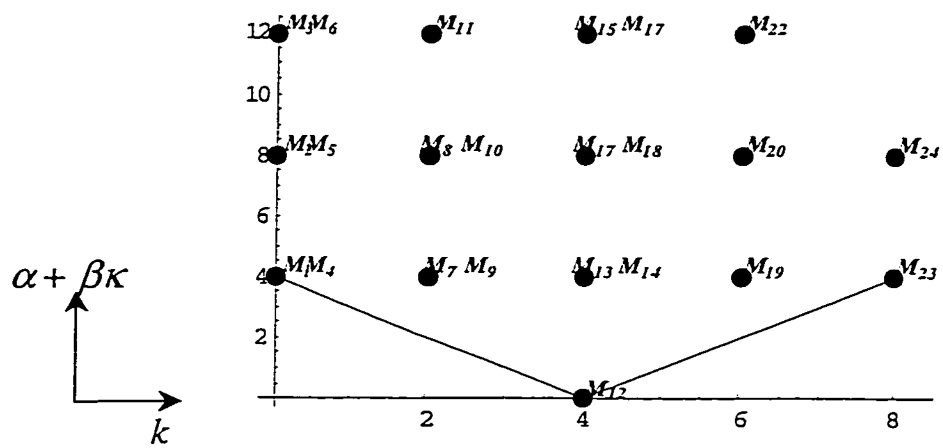

Figure 5.20 Newton's diagram for $m \sim 1$, $\kappa = 4$

In this case the representative points for equation (3.23) are $M_i = \{\{0, 4\}, \{0, 8\}, \{0, 12\}, \{0, 4\}, \{0, 8\}, \{0, 12\}, \{2, 4\}, \{2, 8\}, \{2, 4\}, \{2, 8\}, \{2, 12\}, \{4, 0\}, \{4, 4\}, \{4, 4\}, \{4, 8\}, \{4, 12\}, \{4, 8\}, \{4, 12\}, \{6, 4\}, \{6, 8\}, \{6, 8\}, \{6, 12\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, \ldots, 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_1 = M_4 = (0, 4)$ and $M_{72} = (4, 0)$, and the second segment by the points $M_{12} = (4, 0)$ and $M_{23} = (8, 4)$.

- **Case $\kappa = 5$**

Newton's diagram is plotted in Figure 5.21.

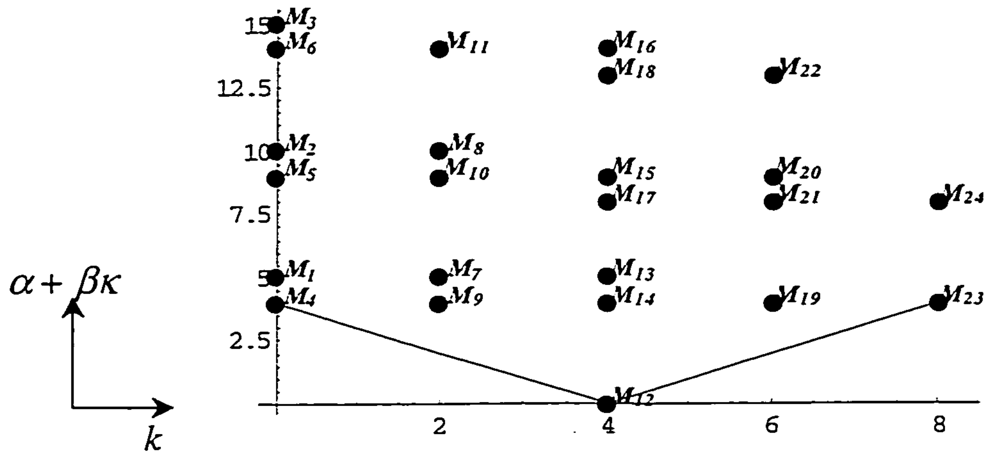

Figure 5.21 Newton's diagram for $m \sim 1$, $\kappa = 5$

In this case the representative points for equation (3.23) are $M_i = \{\{0, 5\}, \{0, 10\}, \{0, 15\}, \{0, 4\}, \{0, 9\}, \{0, 14\}, \{2, 5\}, \{2, 10\}, \{2, 4\}, \{2, 9\}, \{2, 14\}, \{4, 0\}, \{4, 5\}, \{4, 4\}, \{4, 9\}, \{4, 14\}, \{4, 8\}, \{4, 13\}, \{6, 4\}, \{6, 9\}, \{6, 8\}, \{6, 13\}, \{8, 4\}, \{8, 8\}\}$, where $i = 1, ..., 24$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_4 = (0, 4)$ and $M_{72} = (4, 0)$, and the second segment by the points $M_{12} = (4, 0)$ and $M_{23} = (8, 4)$.

Note that for $m = 1$, the representative points M_4 and M_9 are absent since their weights $a_i = 0$ (see Figure 5.22). For this specific case there is no facet 3 and, therefore no separating point $\kappa = 4$. This case is similar to the case $m = 0$.

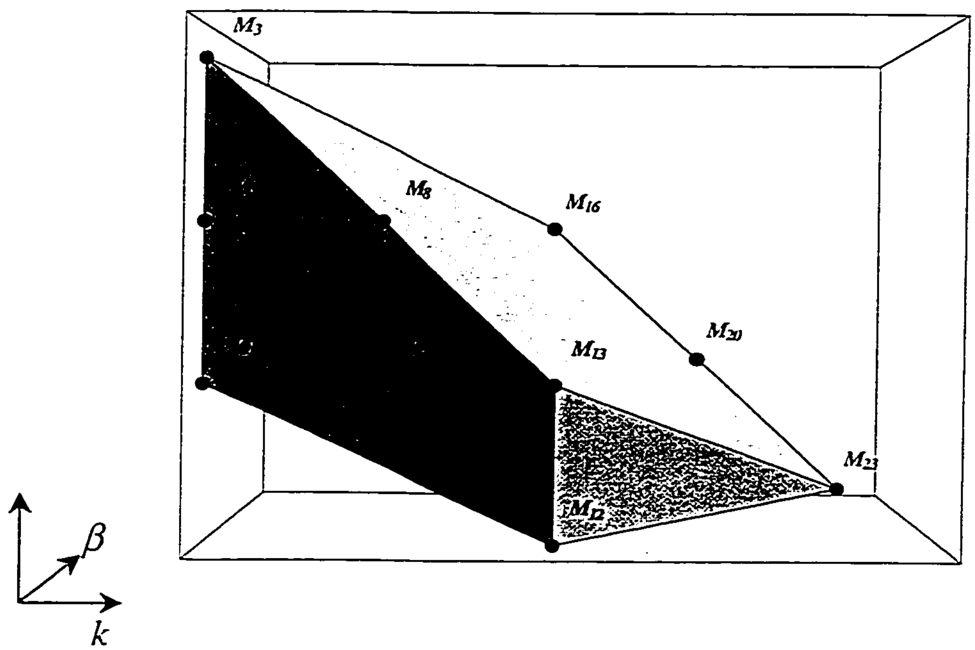

Figure 5.22 3D convex hull for $m = 1$

5.5 Example of Solutions of the Boundary Value Problem

Since the low frequency vibrations are the most important in practice, as an example, we consider the construction of the solutions on the boundary between domains II and III, i.e. on the line $4 - 4\alpha = \beta$. The roots of the characteristic equation (5.28) in this case are given by (5.42) and (5.43).

For the first series of roots the system for the amplitude vector (3.18), after simplification, has the form:

$$\begin{pmatrix} \frac{1-\nu}{2}m^2 & \frac{1+\nu}{2}mp \\ -\frac{1+\nu}{2}mp & m^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \nu p \\ m \end{pmatrix}w. \qquad (5.55)$$

To determine the main term for each coefficient, we construct the 2D convex hull. For example, on the line 4 - $4\alpha=\beta$ the main part of the term $-p^2+\frac{1-\nu}{2}m^2-(1-\nu^2)\lambda$ of determinant in (3.19) is $\frac{1-\nu}{2}m^2$.

Assuming w = 1, we find the amplitude vector for (5.55)

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{p}{m^2} \\ -\frac{1}{m} \\ 1 \end{pmatrix}. \qquad (5.56)$$

For the roots of the second series (5.43), we obtain in the same manner

$$\begin{pmatrix} -p^2 & \frac{1+\nu}{2}mp \\ -\frac{1+\nu}{2}mp & -\frac{1-\nu}{2}p^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \nu p \\ m \end{pmatrix}w \qquad (5.57)$$

and

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\nu}{p} \\ \frac{m}{p^2} \\ 1 \end{pmatrix}. \qquad (5.58)$$

We consider two types of boundary conditions: simply supported edges and clamped-clamped edges.

- **Simply supported edges**

The boundary conditions have the form

$$u' = w = w'' = v = 0 \text{ at } s = 0 \text{ and } s = L. \tag{5.59}$$

Substituting a linear combination of the solutions (5.56) and (5.58) into the boundary conditions (5.59) we get the characteristic equation of the eighth order, similar to (5.16), from which the first approximation for the frequency λ may be found. Again, we neglect the values of the third and fourth solutions (with $\Re(p_i) > 0$) on the left edge compared to those on the right edge; and the values of the fifth and the sixth solutions (with $\Re(p_i) < 0$) on the right edge compared to those on the left edge. Then, after factorization, the determinant has the form

$$D(\lambda)=\frac{16}{h^4 m^2} e^{\frac{\lambda\left(\sqrt{2}+(1-i)hm\left(\lambda-h^4m^4\right)^{1/4}\right)}{h}}\left(-1+e^{2i\lambda m\left(\lambda-h^4m^4\right)^{1/4}}\right)\left(-\lambda+h^4m^4\right)\left(-1+h^4m^4\nu\right)^2 \tag{5.60}$$

The condition $D(\lambda) = 0$ leads to equation:

$$-1+e^{2i\lambda m\left(\lambda-h^4m^4\right)^{1/4}}=0 \tag{5.61}$$

Hence, we obtain the natural frequency:

$$\lambda=\left(\frac{\pi k}{Lm}\right)^4+m^4h^4 \tag{5.62}$$

- **Clamped edges**

The boundary conditions have the form

$$u = w = v = w' = 0 \ \text{ at } \ s = 0 \text{ and } s = L. \tag{5.63}$$

In a similar manner we find

$$D(\lambda) = \frac{4i}{h^2 m^4} e^{\frac{\lambda\left(\sqrt{2}+(1-i)hm\left(\lambda-h^4m^4\right)^{1/4}\right)}{h}} \left(1 + e^{2i\lambda m\left(\lambda-h^4m^4\right)^{1/4}}\right)\sqrt{\lambda - h^4 m^4}\left(\left(1 - h^4 m^4 \nu + h^2 m^2 (1+\nu)\right)\right)^2 \tag{5.64}$$

The natural frequency is:

$$\boxed{\lambda = \left(\frac{\pi(2k+1)}{2Lm}\right)^4 + m^4 h^4} \tag{5.65}$$

Minimizing (5.65) with respect to m, we get

$$\lambda_{\min} = \frac{2\alpha^2}{L^2} h^2, \tag{5.66}$$

$$m_0 = \sqrt{\frac{\alpha}{L}\frac{1}{\sqrt{h}}},$$

where $\alpha = \dfrac{\pi(2\times 1+1)}{2} = \dfrac{3\pi}{2} = 4.73$. The same formulae with $\alpha = \pi$ are valid for the simply supported edges.

The frequency attains its minimum when the wave number in the axial direction is $\kappa = 1$ and for such m_0 that $\dfrac{d\lambda}{dm} = 0$.

5.6 The Superlow Frequencies

In the previous section we found the frequencies $\lambda \sim h^{\kappa}$, $\kappa > 0$. Such frequencies are called superlow, since they become infinitely small as the relative thickness h goes to 0. For thin shells the lower part of the frequency spectrum consists of superlow frequencies. The vibration quality of a construction is defined by the lower part of the spectrum.

To state general conditions leading to vibrations with superlow frequencies we consider the geometry of the shell, i.e. the shell neutral surface. The vibrations with superlow frequencies may occur if and only if the main deformations of the neutral surface of the shell are bending deformations or if the shell is not fixed strongly enough at the edges.

For the case of a cylindrical shell, i.e. a shell with one of the curvatures equal to zero, we find that, for any type of boundary conditions for the lower part of the frequency spectrum, $\lambda \sim h$. If the tangential boundary conditions do not permit bending, then, for the lowest frequency, $\lambda \sim h$, $m \sim h^{-1/2}$ (Goldenveizer *et al.*, 1973). We consider in detail the case of the frequencies, i.e. when $m = m_0 h^{\tau}$, $\tau = -1/2$. In this case equation (3.52) may be written as

$$P(p;h,\lambda) = \sum_{i=1}^{24} a_i p^{\kappa_i} h^{\alpha_i} \lambda^{\beta_i} \tag{5.67}$$

where the 24 representative points $M_i = \{a_i, \{\kappa_i, \alpha_i, \beta_i\}\}$, $i = 1, \ldots, 24$, (5.68)

with their weights a_i are listed below:

{{2(1 + ν), {0, 0, 2}}, {2(-1 + ν)$(1 + \nu)^2$, {0, 0, 3}}, {m^8, {0, 0, 0}}, {-2m^6, {0, 1, 0}}, {m^6(-3 + ν)(1 + ν), {0, 1, 1}}, {m^4, {0, 2, 0}}, {m^4(1 + ν)(3 + ν), {0, 2, 1}}, {-2m^4(-1 + ν)$(1 + \nu)^2$, {0, 2, 2}}, {-2m^2(1 + ν), {0, 3, 1}}, {-2m^2(-1 + ν)$(1 + \nu)^2$, {0, 3, 2}}, {-m^4, {0, -2, 1}}, {-m^2, {0, -1, 1}}, {-m^2(-3 + ν)(1 + ν), {0, -1, 2}}, {3 + 2ν, {2, 0, 1}}, {(-3 + ν)(1 + ν), {2, 0, 2}}, {-4(-1 + ν)(1 + ν), {2, 4, 1}}, {-4$(-1 + \nu)^2(1 + \nu)^2$, {2, 4, 2}}, {-4m^6, {2, 1, 0}}, {8m^4, {2, 2, 0}}, {-3m^4(-3 + ν)(1 + ν), {2, 2, 1}}, {-4m^2, {2, 3, 0}}, {2m^2(1 + ν)$(-2 + \nu^2)$, {2, 3, 1}}, {4m^2(-1 + ν)$(1 + \nu)^2$, {2, 3, 2}}, {2m^2, {2, -1, 1}}, {1, {4, 0, 0}}, {-1, {4, 0, 1}}, {-4(-1 + ν)(1 + ν), {4, 4, 0}}, {4(-1 + ν)(1 + ν), {4, 4, 1}}, {-2(-1 + ν)$(1 + \nu)^2$, {4, 4, 2}}, {6m^4, {4, 2, 0}}, {$m^4(-1 + \nu)^2(1 + \nu)^2$, {4, 6, 0}}, {2$m^2$(-2 + ν)(2 + ν), {4, 3, 0}}, {3m^2(-3 + ν)(1 + ν), {4, 3, 1}}, {-2$m^2(-1 + \nu)^2(1 + \nu)^3$, {4, 7, 1}}, {-(-3 + ν)(1 + ν), {6, 4, 1}}, {4$(-1 + \nu)^2(1 + \nu)^2$, {6, 8, 1}}, {-4m^2, {6, 3, 0}}, {4m^2(-1 + ν)(1 + ν), {6, 7, 0}}, {1, {8, 4, 0}}, {-4(-1 + ν)(1 + ν), {8, 8, 0}}}

The 3D convex hull in this case is plotted in Figure 5.23.

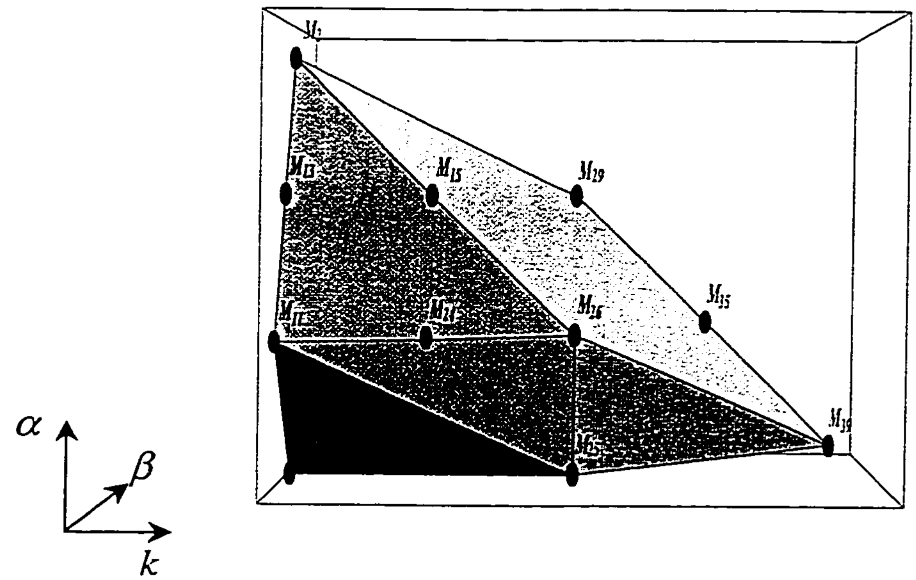

Figure 5.23 3D convex hull for $m \sim h^{(-1/2)}$

The facets of the convex hull determine the separating points $\kappa(\lambda \sim h^{\kappa})$. In this case, the 3D convex hull consists of 5 facets: 1. (M_{11}, M_3, M_{25}); 2. $(M_{11}, M_{124}, M_{26}, M_{25})$; 3. $(M_{11}, M_{13}, M_{15}, M_{24}, M_2, M_{26})$; 4. (M_{25}, M_{26}, M_{39}); 5. $(M_{15}, M_2, M_{26}, M_{29}, M_{35}, M_{39})$. Imposing that the orders of all terms forming a facet be equal to each other, we find the relations from which the orders of λ for the separating cases may be determined:

$$
\begin{gathered}
h^{-2}\lambda \sim 1 \sim p^4, \\
h^{-2}\lambda \sim h^{-1}\lambda^2 \sim \lambda^3 \sim p^2 h^{-1}\lambda \sim p^2\lambda^2 \sim p^4\lambda, \\
h^{-2}\lambda \sim p^2 h^{-1}\lambda \sim p^4\lambda \sim p^4, \\
p^4 \sim p^4\lambda \sim p^8 h^4, \\
\lambda^3 \sim p^2\lambda^2 \sim p^4\lambda \sim p^8 h^4 \sim p^6 h^4\lambda \sim p^4 h^4 \lambda^2.
\end{gathered}
\tag{5.69}
$$

So, for the second and the fourth relations $\kappa = 0$, for the first $\kappa = 2$, for the third $\kappa = -1$, and for the fifth $\kappa = -4$. For any λ inside a domain the structure of the convex hull and, therefore, the roots and the eigenvectors are similar, and thus, we can obtain the values of the roots and eigenvectors considering only one value of λ for each domain. We substitute for the four separating points A: $\kappa = -4$, B: $\kappa = -1$, C: $\kappa = 0$ and D: $\kappa = 2$ and five domains I, II, III, IV, and V: $\kappa = -6$, $\kappa = -3.5$, $\kappa = -1/4$, $\kappa = 1$ and $\kappa = 10$, respectively. Therefore, we should consider nine cases here, where κ is equal to -6, -4, -3.5, -1, -1/4, 0, 1, 2 and 10, respectively; but as we consider only superlow frequency, we can consider the cases where $\kappa > 0$, i.e. κ is equal to 1, 2 and 10, respectively.

- **Case $\kappa = 1$**

Newton's diagram is plotted in Figure 5.24.

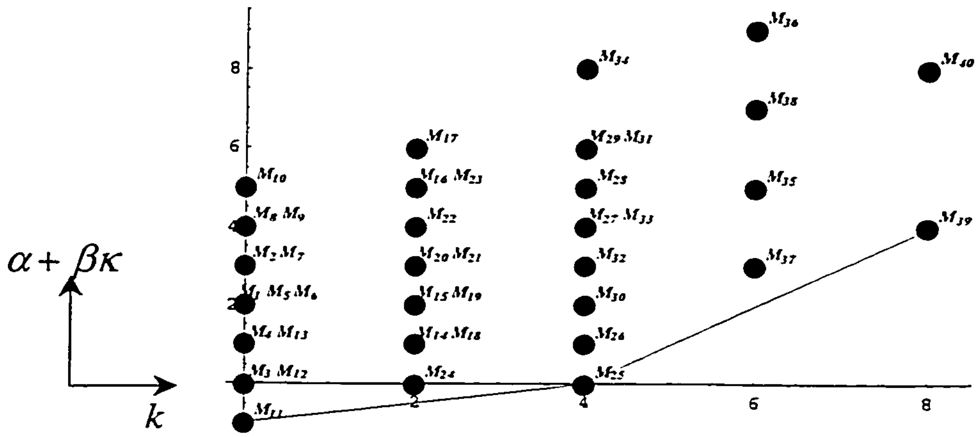

Figure 5.24 Newton's diagram for $m \sim h^{(-1/2)}$, $\kappa = 1$

The representative points for equation (3.19) are M_i = {{0, 2}, {0, 3}, {0, 0}, {0, 1}, {0, 2}, {0, 2}, {0, 3}, {0, 4}, {0, 4}, {0, 5}, {0, -1}, {0, 0}, {0, 1}, {2, 1}, {2, 2}, {2, 5}, {2, 6}, {2, 1}, {2, 2}, {2, 3}, {2, 3}, {2, 4}, {2, 5}, {2, 0}, {4, 0}, {4, 1}, {4, 4}, {4, 5}, {4, 6}, {4, 2}, {4, 6}, {4, 3}, {4, 4}, {4, 8}, {6, 5}, {6, 9}, {6, 3}, {6, 7}, {8, 4}, {8, 8}}, where $i = 1, \ldots, 40$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_{11} = (0, -1)$, $M_{25} = (4, 0)$, and the second segment is determined by $M_{25} = (4, 0)$ and $M_{39} = (8, 4)$.

- **Case $\kappa = 2$**

Newton's diagram is plotted in Figure 5.25.

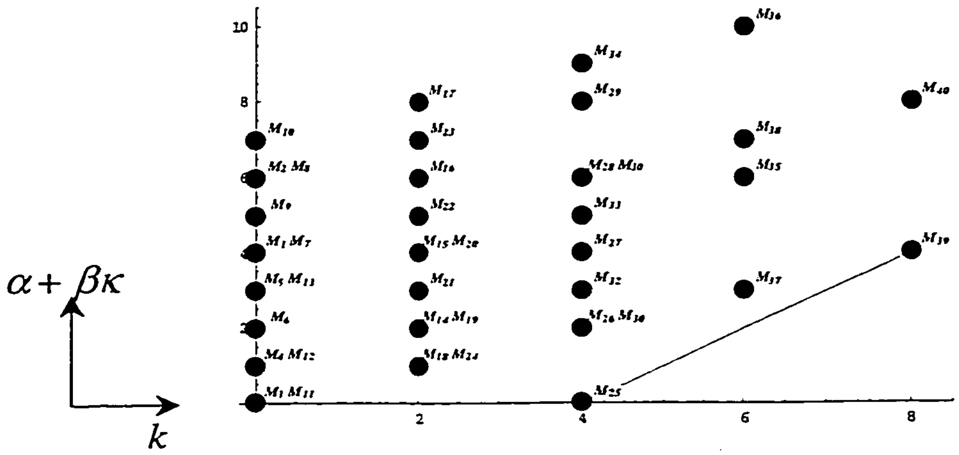

Figure 5.25 Newton's diagram for $m \sim h^{(-1/2)}$, $\kappa = 2$

The representative points for equation (3.19) are M_i = {{0, 4}, {0, 6}, {0, 0}, {0, 1}, {0, 3}, {0, 2}, {0, 4}, {0, 6}, {0, 5}, {0, 7}, {0, 0}, {0, 1}, {0, 3}, {2, 2}, {2, 4}, {2, 6}, {2, 8}, {2, 1}, {2, 2}, {2, 4}, {2, 3}, {2, 5}, {2, 7}, {2, 1}, {4, 0}, {4, 2}, {4, 4}, {4, 6}, {4, 8}, {4, 2}, {4, 6}, {4, 3}, {4, 5}, {4, 9}, {6, 6}, {6, 10}, {6, 3}, {6, 7}, {8, 4}, {8, 8}}, where $i = 1, \ldots, 40$.

Newton's diagram consists of 2 segments. The first segment is determined by the points $M_3 = M_{11} =$ (0, 0) and M_{25} = (4, 0), and the second segment is determined by M_{25} = (4, 0) and M_{39} = (8, 4).

- **Case $\kappa = 10$**

Newton's diagram is plotted in Figure 5.26.

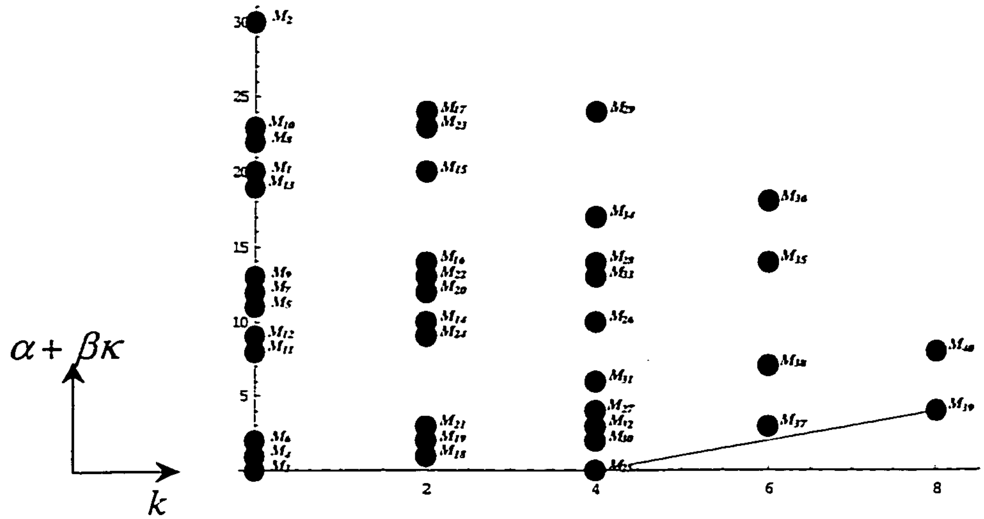

Figure 5.26 Newton's diagram for $m \sim h^{(-1/2)}$, $\kappa = 10$

The representative points for equation (3.19) are M_i = {{0, 20}, {0, 30}, {0, 0}, {0, 1}, {0, 11}, {0, 2}, {0, 12}, {0, 22}, {0, 13}, {0, 23}, {0, 8}, {0, 9}, {0, 19}, {2, 10}, {2, 20}, {2, 14}, {2, 24}, {2, 1}, {2, 2}, {2, 12}, {2, 3}, {2, 13}, {2, 23}, {2, 9}, {4, 0}, {4, 10}, {4, 4}, {4, 14}, {4, 24}, {4, 2}, {4, 6}, {4, 3}, {4, 13}, {4, 17}, {6, 14}, {6, 18}, {6, 3}, {6, 7}, {8, 4}, {8, 8}}, where i = 1, ..., 40.

Newton's diagram consists of 2 segments. The first segment is determined by the points M_3 = (0, 0) and M_{25} = (4, 0), and the second segment is determined by M_{26} = (4, 0) and M_{39} = (8, 4).

Chapter 6

Numerical Analysis

6.1 Introduction

In this chapter we compare the results obtained by using the asymptotic method presented in the previous section with results obtained by performing a finite element analysis with ANSYS, for some particular boundary value problem.

6.2 Numerical Solutions for Non-Axisymmetric Vibrations

In the previous chapter we obtained the asymptotic expression for the first approximation of the lower frequencies of a cylindrical shell with clamped-clamped and simply supported edges.

The formula given in Chapter 5 has the form:

$$\lambda = \frac{\alpha}{m^4}\left(\frac{R}{L}\right)^4 + m^4 h^4. \tag{7.1}$$

Here α depends on the type of boundary conditions (BC):

- If both edges are clamped, $\alpha = 3\pi/2$,
- If both edges are supported, $\alpha = \pi$.

This formula is valid for shells of medium length, for which the ratio R/L is of order $O(h^0)$. The formulae for the lowest frequency and for the wave number corresponding to the lowest frequency have also been found :

$$\lambda_{min} = \frac{2\alpha^2 h}{Rl^2\sqrt{3(1-\nu^2)}}, \tag{7.2}$$

$$m_0^4 = \frac{\sqrt{3(1-\nu^2)}R^3\alpha^2}{l^2 h}. \tag{7.3}$$

To obtain numerical results, we consider a cylindrical shell with the following parameters $R = 1m$, $L = 5m$, $E = 2.07\text{E}+11 N/m^2$, $\nu = 0.29$, $\rho = 7.85\text{E}+3 kg/m^3$.

We solve the boundary value problem for clamped-clamped edges and for the following thickness ratios: $t/R = 0.01$; 0.001 and 0.0005.

- **Thickness ratio $t/R = 0.01$**

The numerical results obtained for ω by asymptotic analysis and by finite element analysis for the first three modes are presented in Table 6.1.

Table 6.1: Asymptotic versus FEM results for $t/R = 0.01$

Mode number	Wave Number m	λ(asymptotic) [1/sec]	ω (asymptotic) [rad/sec]	ω (FEM) [rad/sec]	Error in [%]
1	4	$5.47 \cdot 10^{-3}$	379.9	337.9	11.05
2	5	$7.00 \cdot 10^{-3}$	429.8	406.4	5.44
3	3	$10.06 \cdot 10^{-3}$	529.4	410.1	22.53

The lowest vibration mode obtained with ANSYS is shown in Figure 6.1.

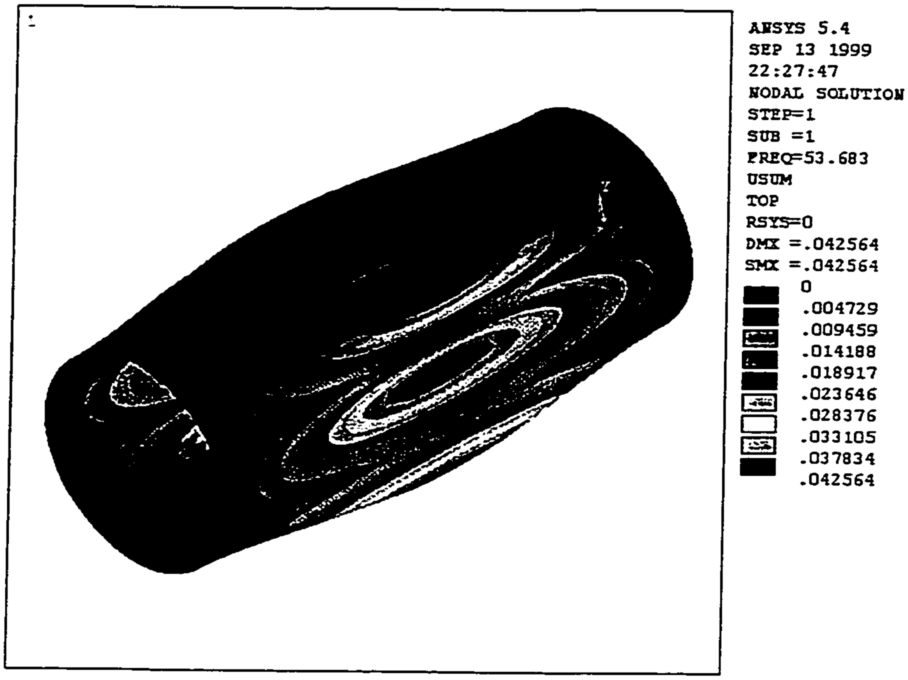

Figure 6.1 Vibration mode for the lowest frequencies (ANSYS)

From the asymptotic analysis we obtained $m_0 = 4.15$, whereas the finite element analysis gave $m_0 = 4$, the asymptotic result for m_0 being in good agreement with the FEM result. However, as can be seen from Table 6.1, the relative error for the frequencies is significant. This is due in part to the thickness ratio of the shell being not too small, as well as to the fact that this is a first approximation result. To obtain more accurate results for this thickness ratio the next approximation should be used.

- **Thickness ratio $t/R = 0.001$**

The results of the asymptotic and finite element analyses for the first three modes are shown in Table 6.2.

Table 6.2 Asymptotic versus FEM results for $t/R = 0.001$

Mode number	Wave Number m	λ(asymptotic) [1/sec]	ω (asymptotic) [rad/sec]	ω (FEM) [rad/sec]	Error in [%]
1	7	$5.52 \cdot 10^{-4}$	120.6	115.9	3.95
2	8	$5.68 \cdot 10^{-4}$	122.4	118.8	2.92
3	6	$7.36 \cdot 10^{-4}$	139.3	130.9	6.05

From the asymptotic analysis $m_0 = 7.38$, whereas from FEM analysis $m_0 = 7$, the asymptotic result being close to the FEM result.

- **Thickness ratio $t/R = 0.0005$**

The results for this case are shown in Table 6.3.

Table 6.3 Asymptotic versus FEM results for $t/R = 0.0005$

Mode number	Wave Number m	λ(asymptotic) [1/sec]	ω (asymptotic) [rad/sec]	ω (FEM) [rad/sec]	Error in [%]
1	9	$2.71 \cdot 10^{-4}$	84.5822	82.4417	2.5
2	8	$2.88 \cdot 10^{-4}$	87.2509	84.534	3.1
3	10	$3.07 \cdot 10^{-4}$	90.0551	-----	-----

In this case we obtained from the asymptotic analysis $m_0 = 8.77$, whereas FEM gave $m_0 = 9$, which again shows good agreement of asymptotic versus FEM results.

Comparing the results for these 3 cases, we see that the relative error of the asymptotic results decreases with the decrease in the ratio t/R. However, the relative error does not go to zero as t/R decreases further.

The residual error may be explained by the fact that the shell is not long enough to neglect the influence of the edge effect solutions on the other edge.

Also, the standard FEM does not give any results for very thin shells, especially for higher modes, as indicated in Table 6.3.

Chapter 7

Summary, Conclusions and Future Work

7.1 Summary and Conclusions

In this work we present an algorithm for the asymptotic solution of boundary value problems involving vibrations of thin cylindrical shells by means of symbolic computation. The algorithm is based on the method of asymptotic integration of the vibration equations of thin shells, developed by Goldenveizer, Lidsky and Tovstik. A linear shell theory of the Kirchhoff-Love type is employed. The equations describing the vibrations of thin shells contain several parameters, the main of which is the small parameter of the shell thickness. Formal asymptotic solutions in different domains of the space of the parameters are obtained by using a computational geometry approach. Computer algebra methods are employed to study the characteristic equation that involves the construction of the convex hull of a set of points.

The study is limited to the cases for which the asymptotic representation of the solution is the same in the entire domain of integration, and solutions are linearly independent (no turning points (Wasow, 1985), no multiple roots). Axisymmetric as well as non-axisymmetric vibrations are considered. The constructed solutions are used for studying the free vibration spectra of the shells.

The numerical results obtained by applying this algorithm to the particular problem of low frequency vibrations of thin cylindrical shells are in good agreement with the results obtained by finite element analysis, as well as with asymptotic results obtained by authors using other solution techniques.

The original contributions of this work consist of:

1. The application of the methods of computational geometry to the study of characteristic equations;
   - The characteristic polynomial for the power p arising in the exponential form of solution of the vibration equation for thin shells;
   - The characteristic equation for the natural frequency parameter λ arising in the boundary value problem;
   - For both characteristic equations the convex hulls were constructed to determine the main terms in these equations for different domains of the parameters λ, μ and h, and to find the first approximation for the roots of these equations;
2. The usage of computer algebra methods to construct formal asymptotic solutions and to solve the boundary value problem.

The practical applicability of the results of this work consists in the following: the analytical expressions derived for the natural frequencies may be used to validate results obtained with numerical techniques, such as the finite element method.

7.2. Future Work

Future work is intended in the following directions:

- The problems of cylindrical shells with a larger number of parameters will be considered. The analysis of vibrations of rotating shells is an example of particular interest for engineering applications (Tovstik, 1963 and Tovstik, 1966). In this case the characteristic equation for the determination of p_i contains an additional fourth parameter, which is the angular velocity of shell rotation. The convex hull in 5D will be constructed to find the roots of the characteristic equation. Another important example of large number of parameters is the vibration of long shells, with *R/L* being a new small parameter.
- To obtain more accurate results in the asymptotic analysis, the next approximation for the frequency parameters will be sought, as well as the next terms in the formal asymptotic solutions. For that the next approximation for p_i should be constructed. The use of the second terms permits us to consider the most complicated cases of vibrations of cylindrical shells, when the frequency parameter is close to 1.
- Finally, the study of vibrations of shells of other geometries will be considered. For other types of geometries, the construction of the formal asymptotic solution is more difficult, since the coefficients of the system of equations vary and there exist so-called turning points where $p_i = p_j$. The construction of the solutions near the turning points requires special consideration.

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Appendices

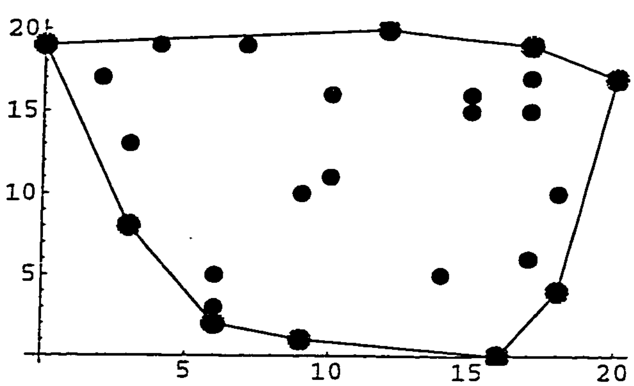
APPENDIX 1

Mathematica 3.0 Code for Construction of Convex Hull in 2D

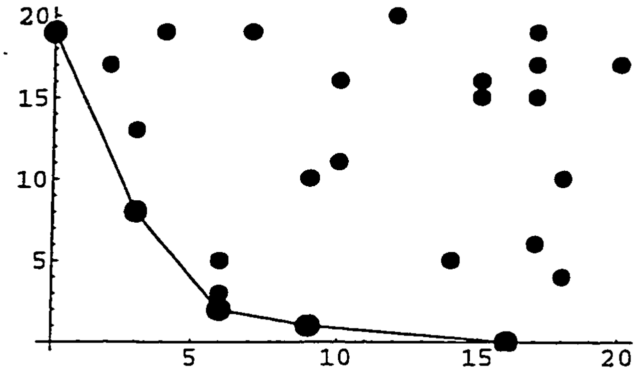
```
(* Programm for Computing the Convex Hull in 2 D using the Algorithm
  Described in Section 4.4 for the random points*)

(* Randomize twenty five points in a range from 0 to 20 *)
x := Random[Integer, {0, 20}];
te = Table[{x, x}, {25}];
g2 = ListPlot[te, DisplayFunction → Identity];
t = Union[te];

(* Find the centroid of this set *)
w = Plus@@t / Length[t];
XT = w[[1]];
YT = w[[2]];
n = Length[t];

(* Sort this set with respect to the angle (see step #3 in Algorithm described in Section) *)
z = (#1[[1]] + #1[[2]] I&)/@t;
z0 = XT + YT I;
z1 = z - z0;
z11 = z1[[n]];
Ira[z_] := If[N[Arg[z] - Arg[z11]] ≥ 0, Arg[z] - Arg[z11], 2π + Arg[z] - Arg[z11]];
Lan[x_, y_] := N[Ira[x] < Ira[y]];
z2 = Sort[z1, Lan];
z3 = z2 + z0;
t33 = ({Re[#1], Im[#1]}&)/@z3;
t32 = Join[t33, {t33[[1]]}];
p = Length[t32];
k = t32; p1 = 0; u = 0;

(* Illustration of the step #4 *)
While[p ≠ p1, p = Length[k]; A = Table[b, {i, 2, p}]; B = Table[c, {i, 2, p}]; l = 0;
  For[i = 2, i < p - l, i++, If[k[[i + 1, 1]] == k[[i - 1, 1]],
     A[[i]] = k[[i, 1]] - k[[i - 1, 1]]; B[[i]] = XT - k[[i - 1, 1]],
     A[[i]] = -k[[i, 2]] + k[[i - 1, 2]] + ((k[[i, 1]] - k[[i - 1, 1]]) (k[[i + 1, 2]] - k[[i - 1, 2]])) / (k[[i + 1, 1]] - k[[i - 1, 1]]);
     B[[i]] = -YT + k[[i - 1, 2]] + ((XT - k[[i - 1, 1]]) (k[[i + 1, 2]] - k[[i - 1, 2]])) / (k[[i + 1, 1]] - k[[i - 1, 1]])];
   If[A[[i]] B[[i]] > 0, k = Drop[k, {i, i}]; l++]]; p1 = Length[k]; Clear[A]; Clear[B]; u++]

s = Flatten[(Position[te, #1]&)/@k]
```

```
(* Draw the final convex hull *)
g1 = ListPlot[k, PlotJoined → True, AxesOrigin → {0, 0}, DisplayFunction → Identity];
t9 = Point/@k;
t10 = Point/@t;
g3 = Graphics[{PointSize[0.03], t10}];
h1 = Graphics[{RGBColor[1, 0, 0], PointSize[0.04], t9}];
Show[h1, g1, g3, AxesOrigin → {0, 0}, Axes → True, DisplayFunction → $DisplayFunction];

(* Choosing the part visible from the origin *)
tt = Transpose[t];
pp = Flatten[Position[tt[[1]], Min[tt[[1]]]]];
rr = t[[Last[pp]]];
pp2 = Flatten[Position[tt[[2]], Min[tt[[2]]]]];
rr2 = t[[Last[pp2]]];
kk1 = ((#1[[1]] - rr[[1]]) (rr2[[2]] - rr[[2]]) ≥ (#1[[2]] - rr[[2]]) (rr2[[1]] - rr[[1]])&)/@k;
ss = Flatten[Position[kk1, True]];
kk = k[[ss]]
g4 = ListPlot[kk, PlotJoined → True, AxesOrigin → {0, 0}, DisplayFunction → Identity];
t11 = Point/@kk;
g5 = Graphics[{PointSize[0.04], t11}];
Show[g3, g4, g5, AxesOrigin → {0, 0}, Axes → True, DisplayFunction → $DisplayFunction];
s1 = Flatten[(Position[te, #1]&)/@kk]
```

```
{24, 12, 9, 7, 19, 1, 22, 4, 21, 24}
```




```
{{0, 19}, {3, 8}, {6, 2}, {9, 1}, {16, 0}}
```



{7, 19, 1, 22, 4}

APPENDIX 2

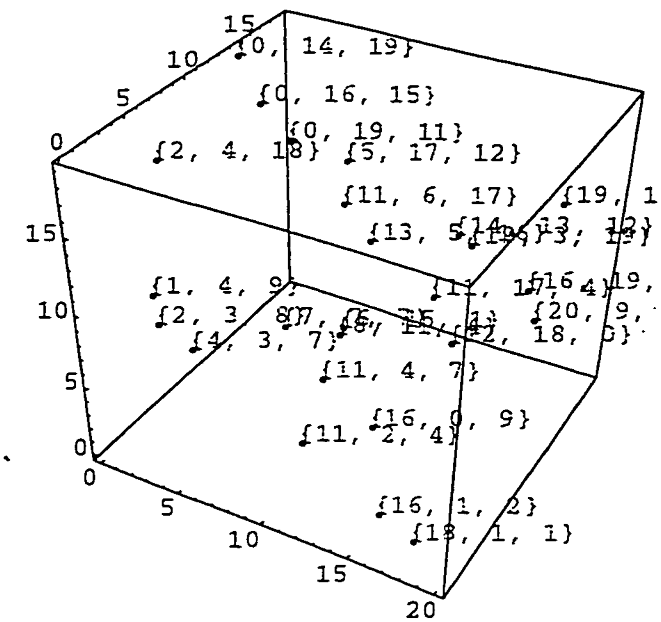
Mathematica 3.0 Code

for Construction of Convex Hull in 3D

```
(*Programm for computing the CH in 3D using the Aggorithm
describing in Chapter 5*)

(* Randomize te=wenty five points in a range from 0 to 20 *)
xr:=Random[Integer,{0,20}];
et=Table[{xr,xr,xr},{25}]
```

```
{{0, 19, 11}, {4, 3, 7}, {11, 2, 4}, {7, 7, 7}, {16, 19, 5},
 {14, 13, 12}, {11, 6, 17}, {1, 4, 9}, {19, 14, 15}, {8, 11, 4},
 {13, 5, 16}, {19, 3, 19}, {2, 4, 18}, {2, 3, 8}, {5, 17, 12},
 {0, 16, 15}, {11, 4, 7}, {16, 1, 2}, {16, 0, 9}, {0, 14, 19},
 {12, 18, 0}, {18, 1, 1}, {20, 9, 11}, {6, 15, 1}, {11, 17, 4}}
```

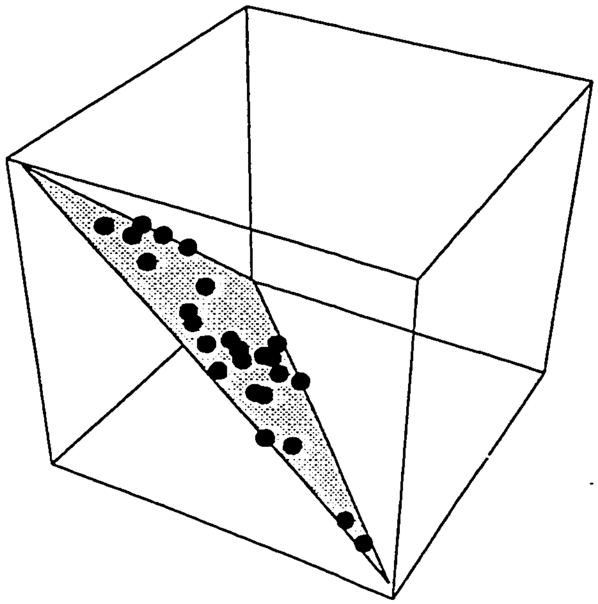
```
(* Plot this set of points *)
yy=Table[Text[et[[i]],et[[i]],{-1,-1}],{i,1,Length[et]}];
ff=Graphics3D[{RGBColor[0,0,1],Thickness[0.1],yy}];
g4=Show[ff,DisplayFunction->Identity];
tr= Point /@ et;
q1=Show[Graphics3D[tr],g4,Axes->True,
DisplayFunction->$DisplayFunction];
```




```
(* Project this set to the plane x+y+z=20 *)
AL[er_]:=Block[{e1,e2,e3,et0},et0={e1,e2,e3}/.
(Solve[{e1+e2+e3-20==0,
e1*(#[[2]])==e2*(#[[1]]),e2*(#[[3]])==e3*(#[[2]]),
e1*(#[[3]])==e3*(#[[1]])},{e1,e2,e3}])& /@er;
Flatten[et0,1] /. e3->0];
et1=AL[et];
```

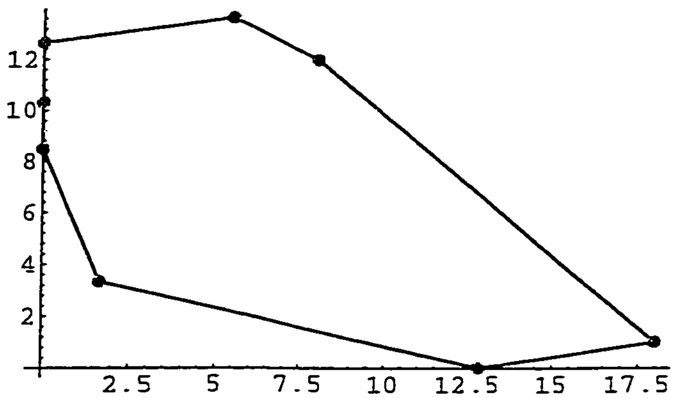
```
(* and plot it *)
tr1=Map[Point,et1];
```

```
f=Show[Graphics3D[Polygon[{{20,0,0},{0,20,0},{0,0,20}}]],DisplayFunction->Identity];
g=Show[Graphics3D[{PointSize[0.03],tr1}],DisplayFunction->Identity];
Show[f,g,DisplayFunction->$DisplayFunction];
```



```
(* Construct the CH in 2D in this plane, using the function LPLOT
AT[r_]:=Block[{we1},we1=AL[r];
Return[({#[[1]],#[[2]]})& /@ we1]];
et7=AT[et];
```

```
Remove[s];
s=LPLOT[et7]
```



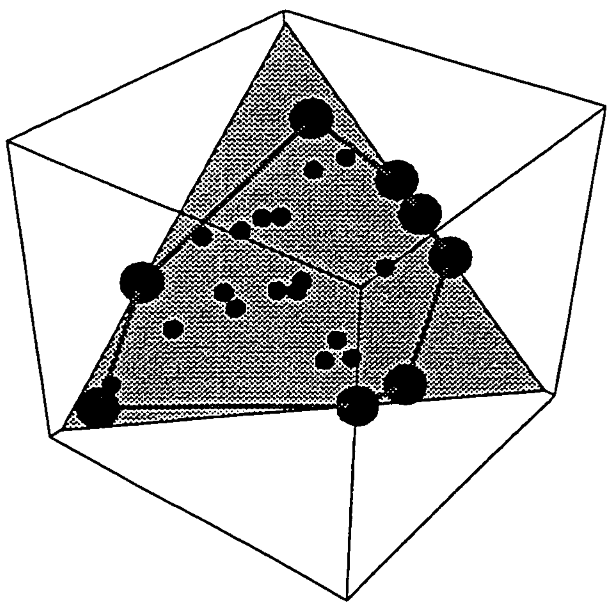
{22, 21, 24, 1, 16, 20, 13, 19, 22}

```
(* Plot 2D chosen points *)
et2=et1[[s]]
et9=et[[s]]
k=Table[j,{j,1,Length[et9]-1}];
edge={et9[[#]],et9[[#+1]]}&/@k
et10=Complement[et,et9];
r=Length[et9];
tr2=Map[Point,et2];
g1=Show[Graphics3D[{RGBColor[1,0,0],Thickness[0.008],Line[et1[[s]]],RGBColor[0,0,1],
PointSize[0.07],tr2}],DisplayFunction->Identity];
EE=Show[f,g,g1,ViewPoint->{2.170,1.656,2.000},DisplayFunction->$DisplayFunction];
```

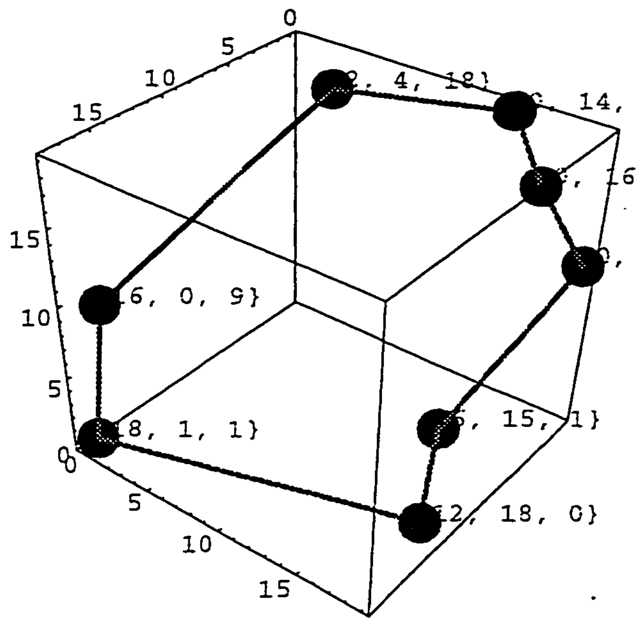
$\{\{18, 1, 1\}, \{8, 12, 0\}, \{\frac{60}{11}, \frac{150}{11}, \frac{10}{11}\}, \{0, \frac{38}{3}, \frac{22}{3}\}, \{0, \frac{320}{31}, \frac{300}{31}\}, \{0, \frac{280}{33}, \frac{380}{33}\}, \{\frac{5}{3}, \frac{10}{3}, 15\}, \{\frac{64}{5}, 0, \frac{36}{5}\}, \{18, 1, 1\}\}$

```
{{18, 1, 1}, {12, 18, 0}, {6, 15, 1}, {0, 19, 11}, {0, 16, 15},
 {0, 14, 19}, {2, 4, 18}, {16, 0, 9}, {18, 1, 1}}
```

```
{{{18, 1, 1}, {12, 18, 0}}, {{12, 18, 0}, {6, 15, 1}},
 {{6, 15, 1}, {0, 19, 11}}, {{0, 19, 11}, {0, 16, 15}},
 {{0, 16, 15}, {0, 14, 19}}, {{0, 14, 19}, {2, 4, 18}},
 {{2, 4, 18}, {16, 0, 9}}, {{16, 0, 9}, {18, 1, 1}}}
```



```
(* Plot 3D chosen points *)
yq=Table[Text[et9[[i1]],et9[[i1]],{-1,-1}],{i1,1,Length[et9]}];

ffq=Graphics3D[{RGBColor[0,0,1],Thickness[0.1],yq}];

g14=Show[ffq,DisplayFunction->Identity];


tr3=Map[Point,et9];
g2=Show[Graphics3D[{RGBColor[1,0,0],Thickness[0.008],Line[et9],RGBColor[0,0,1],
PointSize[0.07],tr3}],DisplayFunction->Identity];
FF=Show[g2,g14,Axes->True,ViewPoint->{2.170,1.656,2.000},
DisplayFunction->$DisplayFunction];
```

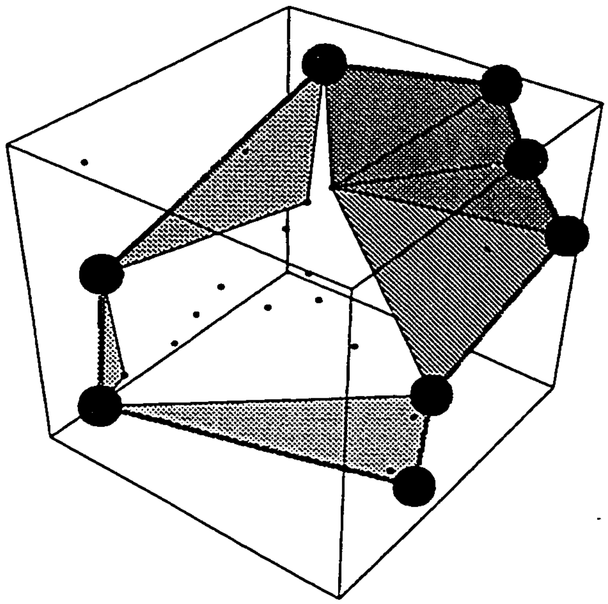


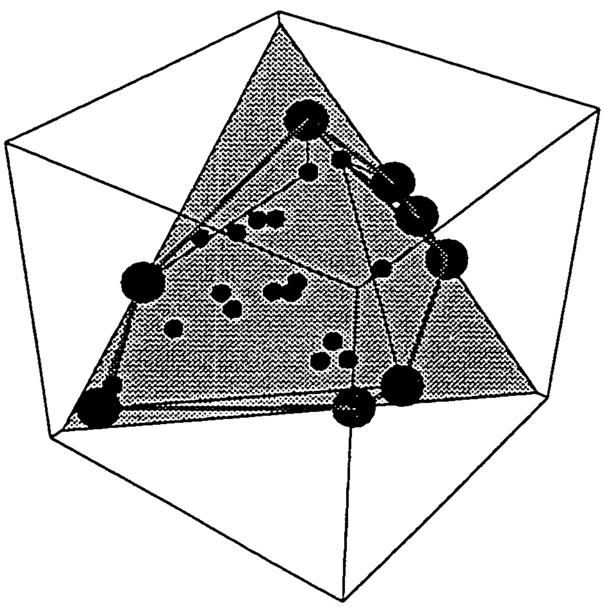

```
Sm[t11_,t22_,cc_]:=Block[{t1,t2,c,as1,ws,ed,po,dd},
t1=AT[{t11}]//Flatten;t2=AT[{t22}]//Flatten;c=AT[cc];
If[t2[[1]]==t1[[1]],as1={1,0},If[t1[[2]]==t2[[2]],as1={0,1},
as1={1/(t2[[1]]-t1[[1]]),1/(t1[[2]]-t2[[2]])}]];
ws=as1*(#-t1)& /@c;
If[as1[[1]]*as1[[2]]>=0,ed=#[[1]]+#[[2]]>=0& /@ws,ed=#[[1]]+#[[2]]<=0& /@ws];
po=Position[ed,True]//Flatten;
dd=cc[[po]]
]
```

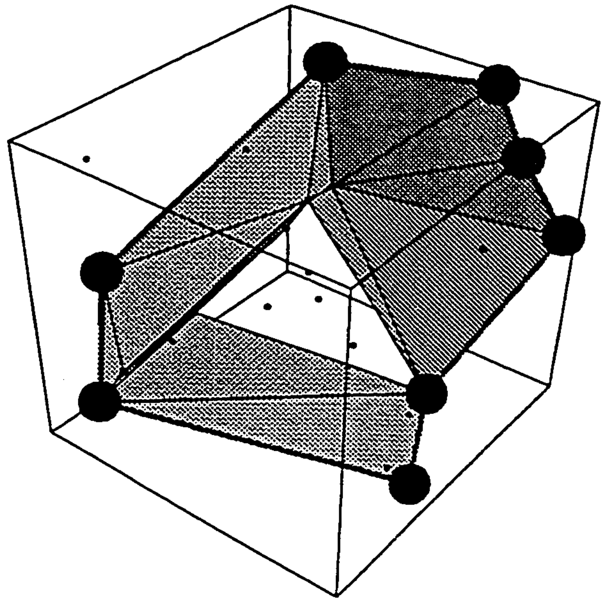
```
en1=Length[et9];
en2=Length[et10];Clear[n];
An[x_,y_,r_,r2_]:=Block[{ en=Length[r],en2=Length[r2]
                        },
J1=Table[(r[[i,2]]-x[[2]])(y[[3]]-x[[3]])-(y[[2]]-x[[2]])
(r[[i,3]]-x[[3]]),{i,en}];
J2=Table[(r[[i,3]]-x[[3]])(y[[1]]-x[[1]])-(r[[i,1]]-x[[1]])
(y[[3]]-x[[3]]),{i,en}];
J3=Table[(r[[i,1]]-x[[1]])(y[[2]]-x[[2]])-(r[[i,2]]-x[[2]])
(y[[1]]-x[[1]]),{i,en}];
J4=Table[-r[[i,1]]J1[[i]]-r[[i,2]]J2[[i]]-r[[i,3]]J3[[i]],
i,en}];
t41=Table[((-J1[[i]]#[[1]]-J2[[i]]#[[2]]-J4[[i]])>=
#[[3]]J3[[i]])& /@r2,{i,en}];
k=Table[j,{j,1,en}];
T=Table[True,{j,1,en2}];
R1=If[T==t41[[#]],True,False]& /@k;
op=Position[R1,True]//Flatten;
xx=r[[op]]

]




(* Main function *)
wic=edge;

While[Length[edge]>2,


ui=Length[et]-2;
yy=Length[edge]+1;
tw=Table[An[edge[[i,1]],edge[[i,2]],Complement[Sm[edge[[i,1]],
edge[[i,2]],et],{edge[[i,1]],edge[[i,2]]}],Complement[et,
{edge[[i,1]]},{edge[[i,2]]}]],{i,1,yy-1}];
to=Flatten[tw,1];


edge2=Table[a,{i,1,yy-1}];
Pol2=Table[b,{i,1,yy-1}];
```

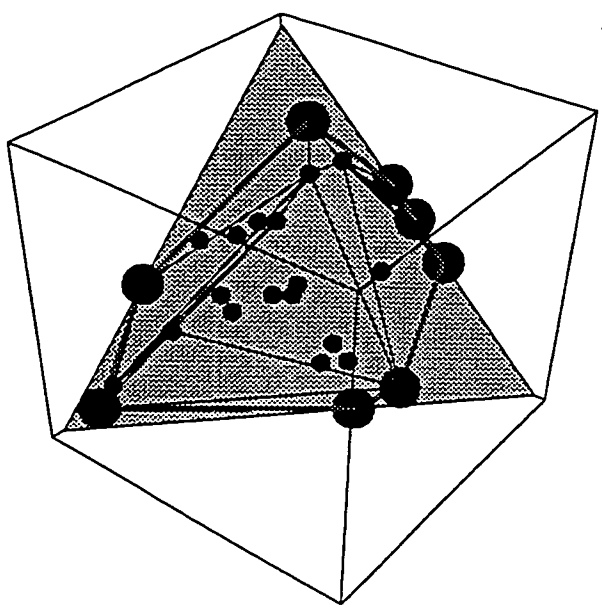
```
 For[i=1,i<yy,i++,If[Length[tw[[i]]]==1,
         edge2[[i]]={{{edge[[i,1]],tw[[i,1]]},
         {tw[[i,1]],edge[[i,2]]}}};
         Pol2[[i]]={{{edge[[i,1]],tw[[i,1]],edge[[i,2]]}}} ,
         edge2[[i]]=Table[{{edge[[i,1]],tw[[i,k]]},
         {tw[[i,k]],tw[[i,k+1]]},
         {tw[[i,k+1]],edge[[i,2]]}},{k,1,Length[tw[[i]]]-1}];
         Pol2[[i]]=Table[{{edge[[i,1]],tw[[i,k]],tw[[i,k+1]]},
         {tw[[i,k+1]],edge[[i,1]],edge[[i,2]]}},
         {k,1,Length[tw[[i]]]-1}]
   ]];

 edge3=Flatten[edge2,2];
 Pol3=Flatten[Pol2,2];

 yt=Table[Polygon[{Pol3[[i,1]],Pol3[[i,2]],Pol3[[i,3]]}],
 {i,1,Length[Pol3]}];
 f1=Graphics3D[yt];
 FF=Show[FF,q1,g2,f1,ViewPoint->{2.170,1.656,2.000},
 DisplayFunction->$DisplayFunction];
 po=AL[to];
 yt2=Table[Polygon[AL[{Pol3[[i,1]],Pol3[[i,2]],Pol3[[i,3]]}]],
 {i,1,Length[Pol3]}];
 f2=Graphics3D[yt2];
 EE=Show[EE,f,g,g1,f2,ViewPoint->{2.170,1.656,2.000},
 DisplayFunction->$DisplayFunction];
 l=edge3;
 eee[x_,y_]:=If[x[[1]]==y[[2]]&&x[[2]]==y[[1]],True,False];
 m=Join[edge3,edge];
 a1=Table[eee[m[[i]],m[[j]]],{i,1,Length[m]-1},{j,i+1,Length[m]}]
 b1=Position[a1,True];
 c1=Table[{b1[[i,1]],b1[[i,2]]+b1[[i,1]]},{i,1,Length[b1]}]
 //Flatten;
 e1=Union[c1];
 ge=Complement[m,m[[e1]]];

 edge=Complement[ge,edge];
 wic=Union[wic,edge3];


 ]
```

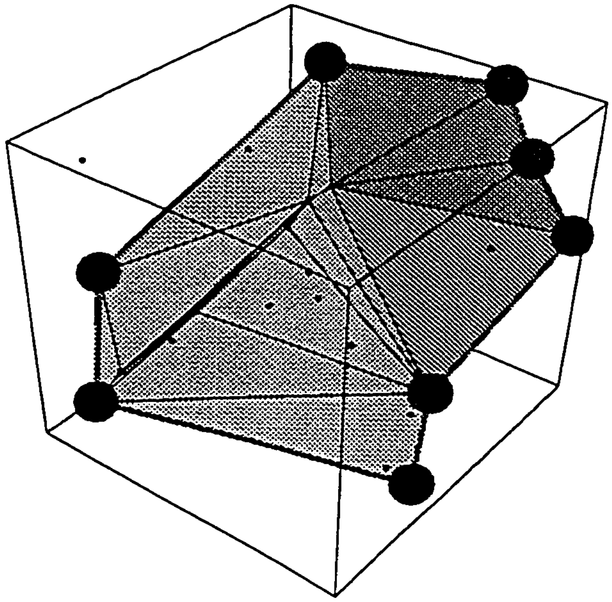
]

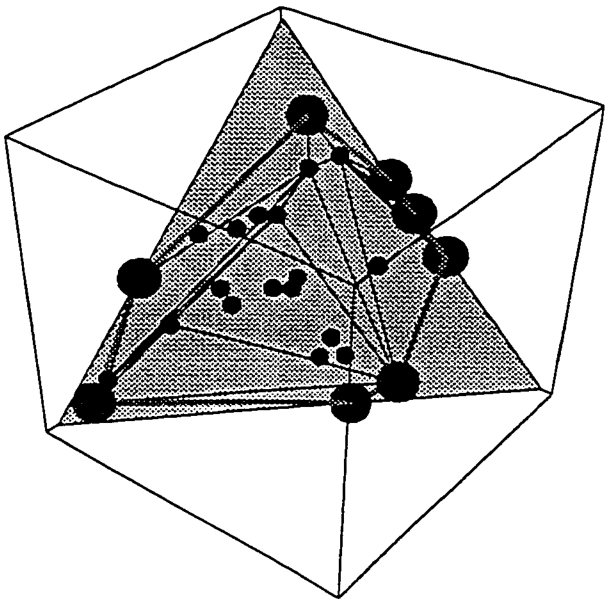




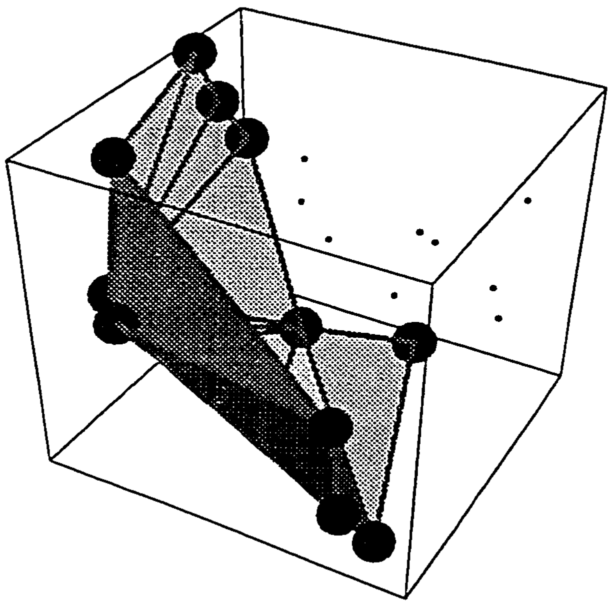


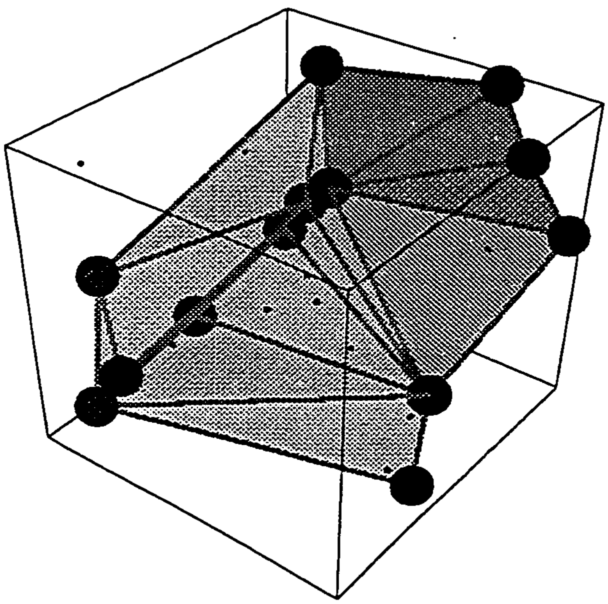






```
ea1=Flatten[wic,1];
ea=Union[Flatten[wic,1]];
tr4=Map[Point,ea];
eqq1=Table[Show[Graphics3D[{RGBColor[1,0,0],
Thickness[0.008],Line[{ea1[[i]],ea1[[i+1]]}],RGBColor[0,0,1],PointSize[0.07],tr4}],
DisplayFunction->Identity],{i,1,Length[ea1]-1,2}];
Show[eqq1,FF,ViewPoint->{1.300,-2.400,2.000},
DisplayFunction->$DisplayFunction];
Show[eqq1,FF,ViewPoint->{2.170,1.656,2.000},
DisplayFunction->$DisplayFunction];
```





```
wic
Union[Flatten[wic,1]]
```

```
{{{0, 14, 19}, {1, 4, 9}}, {{0, 14, 19}, {2, 4, 18}},
 {{0, 16, 15}, {0, 14, 19}}, {{0, 16, 15}, {1, 4, 9}},
 {{0, 19, 11}, {0, 16, 15}}, {{0, 19, 11}, {1, 4, 9}},
 {{1, 4, 9}, {0, 14, 19}}, {{1, 4, 9}, {0, 16, 15}},
 {{1, 4, 9}, {0, 19, 11}}, {{1, 4, 9}, {2, 3, 8}},
 {{1, 4, 9}, {2, 4, 18}}, {{2, 3, 8}, {1, 4, 9}},
 {{2, 3, 8}, {2, 4, 18}}, {{2, 3, 8}, {4, 3, 7}},
 {{2, 3, 8}, {6, 15, 1}}, {{2, 3, 8}, {16, 0, 9}},
 {{2, 3, 8}, {16, 1, 2}}, {{2, 4, 18}, {1, 4, 9}},
 {{2, 4, 18}, {2, 3, 8}}, {{2, 4, 18}, {16, 0, 9}},
 {{4, 3, 7}, {2, 3, 8}}, {{4, 3, 7}, {6, 15, 1}},
 {{4, 3, 7}, {11, 2, 4}}, {{4, 3, 7}, {18, 1, 1}},
 {{6, 15, 1}, {0, 19, 11}}, {{6, 15, 1}, {1, 4, 9}},
 {{6, 15, 1}, {2, 3, 8}}, {{6, 15, 1}, {4, 3, 7}},
 {{6, 15, 1}, {12, 18, 0}}, {{6, 15, 1}, {18, 1, 1}},
 {{11, 2, 4}, {4, 3, 7}}, {{11, 2, 4}, {6, 15, 1}},
 {{11, 2, 4}, {18, 1, 1}}, {{12, 18, 0}, {6, 15, 1}},
 {{12, 18, 0}, {18, 1, 1}}, {{16, 0, 9}, {2, 3, 8}},
 {{16, 0, 9}, {16, 1, 2}}, {{16, 0, 9}, {18, 1, 1}},
 {{16, 1, 2}, {2, 3, 8}}, {{16, 1, 2}, {4, 3, 7}},
 {{16, 1, 2}, {16, 0, 9}}, {{16, 1, 2}, {18, 1, 1}},
 {{18, 1, 1}, {4, 3, 7}}, {{18, 1, 1}, {6, 15, 1}},
 {{18, 1, 1}, {11, 2, 4}}, {{18, 1, 1}, {12, 18, 0}}}

{{0, 14, 19}, {0, 16, 15}, {0, 19, 11}, {1, 4, 9}, {2, 3, 8},
 {2, 4, 18}, {4, 3, 7}, {6, 15, 1}, {11, 2, 4}, {12, 18, 0},
 {16, 0, 9}, {16, 1, 2}, {18, 1, 1}}
```

APPENDIX 3

Mathematica 3.0 Code

for Construction of Lower Part of Convex Hull in 4D

```
t = {{{0, 0, 1}, {0, 4, 0}, {4, 0, 0}}, {{4, 0, 0}, {8, 4, 0}, {4, 0, 1}},
  {{0, 0, 1}, {0, 0, 3}, {0, 4, 2}, {0, 4, 0}}, {{0, 0, 1}, {4, 0, 0}, {4, 0, 1}, {0, 0, 3}},
  {{0, 0, 3}, {4, 0, 1}, {8, 4, 0}, {4, 4, 2}}, {{0, 4, 0}, {0, 4, 2}, {4, 8, 1}, {4, 8, 0}},
  {{4, 4, 2}, {8, 4, 0}, {8, 8, 0}, {6, 8, 1}}, {{4, 8, 0}, {4, 8, 1}, {6, 8, 1}, {8, 8, 0}},
  {{0, 0, 3}, {4, 4, 2}, {6, 8, 1}, {4, 8, 1}, {0, 4, 2}},
  {{0, 4, 0}, {4, 8, 0}, {8, 8, 0}, {8, 4, 0}, {4, 0, 0}}}

{{{0, 0, 1}, {0, 4, 0}, {4, 0, 0}}, {{4, 0, 0}, {8, 4, 0}, {4, 0, 1}},
 {{0, 0, 1}, {0, 0, 3}, {0, 4, 2}, {0, 4, 0}}, {{0, 0, 1}, {4, 0, 0}, {4, 0, 1}, {0, 0, 3}},
 {{0, 0, 3}, {4, 0, 1}, {8, 4, 0}, {4, 4, 2}}, {{0, 4, 0}, {0, 4, 2}, {4, 8, 1}, {4, 8, 0}},
 {{4, 4, 2}, {8, 4, 0}, {8, 8, 0}, {6, 8, 1}}, {{4, 8, 0}, {4, 8, 1}, {6, 8, 1}, {8, 8, 0}},
 {{0, 0, 3}, {4, 4, 2}, {6, 8, 1}, {4, 8, 1}, {0, 4, 2}},
 {{0, 4, 0}, {4, 8, 0}, {8, 8, 0}, {8, 4, 0}, {4, 0, 0}}}

t[[1, 2, 1]]
(t[[1, 2, 1]] - t[[1, 1, 1]]) (t[[1, 3, 2]] - t[[1, 1, 2]]) -
 (t[[1, 3, 1]] - t[[1, 1, 1]]) (t[[1, 2, 2]] - t[[1, 1, 2]])
J3 = Table[(t[[i, 2, 1]] - t[[i, 1, 1]]) (t[[i, 3, 3]] - t[[i, 1, 3]]) -
   (t[[i, 3, 1]] - t[[i, 1, 1]]) (t[[i, 2, 3]] - t[[i, 1, 3]]), {i, 1, Length[t]}]
J31 = Table[Det[{{(t[[i, 2, 1]] - t[[i, 1, 1]]), (t[[i, 2, 3]] - t[[i, 1, 3]])},
    {(t[[i, 3, 1]] - t[[i, 1, 1]]), (t[[i, 3, 3]] - t[[i, 1, 3]])}}], {i, 1, Length[t]}]
J4 = J3 // Positive
p = Position[J4, True] // Flatten
t[[p]]

0

-16

{4, 4, 0, 4, 4, -8, 0, -2, -2, 0}

{4, 4, 0, 4, 4, -8, 0, -2, -2, 0}

{True, True, False, True, True, False, False, False, False, False}

{1, 2, 4, 5}

{{{0, 0, 1}, {0, 4, 0}, {4, 0, 0}}, {{4, 0, 0}, {8, 4, 0}, {4, 0, 1}},
 {{0, 0, 1}, {4, 0, 0}, {4, 0, 1}, {0, 0, 3}}, {{0, 0, 3}, {4, 0, 1}, {8, 4, 0}, {4, 4, 2}}}
```

```
tt = {{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0},
  {0, 4, 0, 4}, {0, 4, 0, 6}, {0, 4, 0, 8}, {0, 4, 1, 2}, {0, 4, 1, 4}, {0, 4, 1, 6},
  {0, 4, 2, 2}, {0, 4, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 4, 0, 2},
  {2, 4, 0, 4}, {2, 4, 0, 6}, {2, 4, 1, 0}, {2, 4, 1, 2}, {2, 4, 1, 4}, {2, 4, 2, 0},
  {2, 4, 2, 2}, {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 4, 0, 0}, {4, 4, 0, 2}, {4, 4, 0, 4},
  {4, 4, 1, 0}, {4, 4, 1, 2}, {4, 4, 2, 0}, {4, 8, 0, 4}, {4, 8, 1, 2}, {6, 4, 0, 2},
  {6, 4, 1, 0}, {6, 8, 0, 2}, {6, 8, 1, 0}, {8, 4, 0, 0}, {8, 8, 0, 0}}

{{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0}, {0, 4, 0, 4},
 {0, 4, 0, 6}, {0, 4, 0, 8}, {0, 4, 1, 2}, {0, 4, 1, 4}, {0, 4, 1, 6}, {0, 4, 2, 2},
 {0, 4, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 4, 0, 2}, {2, 4, 0, 4},
 {2, 4, 0, 6}, {2, 4, 1, 0}, {2, 4, 1, 2}, {2, 4, 1, 4}, {2, 4, 2, 0}, {2, 4, 2, 2},
 {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 4, 0, 0}, {4, 4, 0, 2}, {4, 4, 0, 4}, {4, 4, 1, 0},
 {4, 4, 1, 2}, {4, 4, 2, 0}, {4, 8, 0, 4}, {4, 8, 1, 2}, {6, 4, 0, 2}, {6, 4, 1, 0},
 {6, 8, 0, 2}, {6, 8, 1, 0}, {8, 4, 0, 0}, {8, 8, 0, 0}}

tr = {{7, 25, 1, 31, 12, 4, 38},
{24, 2, 25, 1, 4, 0},
{24, 25, 1, 7, 38},
{5, 24, 1, 7, 0},
{11, 5, 2, 12, 8, 1, 7, 4, 0},
{11, 22, 2, 8, 19, 4},
{33, 39, 31, 12, 37, 7, 32, 38},
{33, 11, 22, 31, 12, 37, 4},
{33, 11, 22, 37, 8, 19},
{33, 11, 5, 12, 8, 7, 32},
{26, 5, 24, 2, 8, 19, 0},
{26, 39, 24, 22, 31, 2, 25, 37, 19, 4, 38},
{26, 33, 39, 5, 37, 8, 19, 32},
{26, 39, 5, 24, 7, 32, 38}}

{{7, 25, 1, 31, 12, 4, 38}, {24, 2, 25, 1, 4, 0}, {24, 25, 1, 7, 38}, {5, 24, 1, 7, 0},
 {11, 5, 2, 12, 8, 1, 7, 4, 0}, {11, 22, 2, 8, 19, 4}, {33, 39, 31, 12, 37, 7, 32, 38},
 {33, 11, 22, 31, 12, 37, 4}, {33, 11, 22, 37, 8, 19}, {33, 11, 5, 12, 8, 7, 32},
 {26, 5, 24, 2, 8, 19, 0}, {26, 39, 24, 22, 31, 2, 25, 37, 19, 4, 38},
 {26, 33, 39, 5, 37, 8, 19, 32}, {26, 39, 5, 24, 7, 32, 38}}

tre = tr + 1

{{8, 26, 2, 32, 13, 5, 39}, {25, 3, 26, 2, 5, 1}, {25, 26, 2, 8, 39}, {6, 25, 2, 8, 1},
 {12, 6, 3, 13, 9, 2, 8, 5, 1}, {12, 23, 3, 9, 20, 5}, {34, 40, 32, 13, 38, 8, 33, 39},
 {34, 12, 23, 32, 13, 38, 5}, {34, 12, 23, 38, 9, 20}, {34, 12, 6, 13, 9, 8, 33},
 {27, 6, 25, 3, 9, 20, 1}, {27, 40, 25, 23, 32, 3, 26, 38, 20, 5, 39},
 {27, 34, 40, 6, 38, 9, 20, 33}, {27, 40, 6, 25, 8, 33, 39}}
```

```
t = tt[[#]]&/@tre

{{{0, 4, 0, 8}, {4, 0, 1, 0},
  {0, 0, 1, 4}, {4, 4, 2, 0}, {0, 4, 2, 4}, {0, 0, 3, 0}, {8, 4, 0, 0}},
 {{4, 0, 0, 0}, {0, 0, 2, 0}, {4, 0, 1, 0}, {0, 0, 1, 4}, {0, 0, 3, 0}, {0, 0, 1, 2}},
 {{4, 0, 0, 0}, {4, 0, 1, 0}, {0, 0, 1, 4}, {0, 4, 0, 8}, {8, 4, 0, 0}},
 {{0, 4, 0, 4}, {4, 0, 0, 0}, {0, 0, 1, 4}, {0, 4, 0, 8}, {0, 0, 1, 2}},
 {{0, 4, 2, 2}, {0, 4, 0, 4}, {0, 0, 2, 0}, {0, 4, 2, 4}, {0, 4, 1, 2}, {0, 0, 1, 4},
  {0, 4, 0, 8}, {0, 0, 3, 0}, {0, 0, 1, 2}}, {{0, 4, 2, 2}, {2, 4, 2, 0}, {0, 0, 2, 0},
  {0, 4, 1, 2}, {2, 4, 1, 0}, {0, 0, 3, 0}}, {{4, 8, 1, 2}, {8, 8, 0, 0},
  {4, 4, 2, 0}, {0, 4, 2, 4}, {6, 8, 1, 0}, {0, 4, 0, 8}, {4, 8, 0, 4}, {8, 4, 0, 0}},
 {{4, 8, 1, 2}, {0, 4, 2, 2}, {2, 4, 2, 0}, {4, 4, 2, 0}, {0, 4, 2, 4},
  {6, 8, 1, 0}, {0, 0, 3, 0}}, {{4, 8, 1, 2}, {0, 4, 2, 2}, {2, 4, 2, 0},
  {6, 8, 1, 0}, {0, 4, 1, 2}, {2, 4, 1, 0}}, {{4, 8, 1, 2}, {0, 4, 2, 2}, {0, 4, 0, 4},
  {0, 4, 2, 4}, {0, 4, 1, 2}, {0, 4, 0, 8}, {4, 8, 0, 4}}, {{4, 4, 0, 0}, {0, 4, 0, 4},
  {4, 0, 0, 0}, {0, 0, 2, 0}, {0, 4, 1, 2}, {2, 4, 1, 0}, {0, 0, 1, 2}}, {{4, 4, 0, 0},
  {8, 8, 0, 0}, {4, 0, 0, 0}, {2, 4, 2, 0}, {4, 4, 2, 0}, {0, 0, 2, 0}, {4, 0, 1, 0},
  {6, 8, 1, 0}, {2, 4, 1, 0}, {0, 0, 3, 0}, {8, 4, 0, 0}}, {{4, 4, 0, 0}, {4, 8, 1, 2},
  {8, 8, 0, 0}, {0, 4, 0, 4}, {6, 8, 1, 0}, {0, 4, 1, 2}, {2, 4, 1, 0}, {4, 8, 0, 4}},
 {{4, 4, 0, 0}, {8, 8, 0, 0}, {0, 4, 0, 4}, {4, 0, 0, 0}, {0, 4, 0, 8}, {4, 8, 0, 4},
  {8, 4, 0, 0}}}

JJ = Table[Det[{{(t[[i, 2, 1]] - t[[i, 1, 1]]),
      (t[[i, 2, 3]] - t[[i, 1, 3]]), (t[[i, 2, 4]] - t[[i, 1, 4]])},
     {(t[[i, 3, 1]] - t[[i, 1, 1]]), (t[[i, 3, 3]] - t[[i, 1, 3]]),
      (t[[i, 3, 4]] - t[[i, 1, 4]])}, {(t[[i, 4, 1]] - t[[i, 1, 1]]),
      (t[[i, 4, 3]] - t[[i, 1, 3]]), (t[[i, 4, 4]] - t[[i, 1, 4]])}}],
  {i, 1, Length[t]}]
J4 = JJ // Positive
p = Position[J4, True] // Flatten
tp = t[[p]]
tp[[1]]
TT = Table[Flatten[Position[tt, #]&/@tp[[i]]], {i, 1, Length[tp]}]

{16, -16, 16, 16, 0, -4, 0, -4, 0, 16, 0, 0, -16, 0}

{True, False, True, True, False, False, False, False, False, True, False,
 False, False, False}

{1, 3, 4, 10}

{{{0, 4, 0, 8}, {4, 0, 1, 0},
  {0, 0, 1, 4}, {4, 4, 2, 0}, {0, 4, 2, 4}, {0, 0, 3, 0}, {8, 4, 0, 0}},
 {{4, 0, 0, 0}, {4, 0, 1, 0}, {0, 0, 1, 4}, {0, 4, 0, 8}, {8, 4, 0, 0}},
 {{0, 4, 0, 4}, {4, 0, 0, 0}, {0, 0, 1, 4}, {0, 4, 0, 8}, {0, 0, 1, 2}}, {{4, 8, 1, 2},
  {0, 4, 2, 2}, {0, 4, 0, 4}, {0, 4, 2, 4}, {0, 4, 1, 2}, {0, 4, 0, 8}, {4, 8, 0, 4}}}

{{0, 4, 0, 8}, {4, 0, 1, 0}, {0, 0, 1, 4}, {4, 4, 2, 0}, {0, 4, 2, 4}, {0, 0, 3, 0},
 {8, 4, 0, 0}}

{{8, 26, 2, 32, 13, 5, 39}, {25, 26, 2, 8, 39}, {6, 25, 2, 8, 1}, {34, 12, 6, 13, 9, 8, 33}}
```

```
Union[Flatten[tp, 1]]

{{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 3, 0}, {0, 2, 0, 4}, {0, 2, 0, 8},
 {0, 2, 1, 2}, {0, 2, 2, 2}, {0, 2, 2, 4}, {2, 2, 1, 0}, {2, 2, 2, 0}, {4, 0, 0, 0},
 {4, 0, 1, 0}, {4, 2, 0, 0}, {4, 2, 0, 2}, {4, 2, 0, 4}, {4, 2, 1, 0}, {4, 2, 1, 2},
 {4, 2, 2, 0}, {4, 4, 0, 4}, {4, 4, 1, 2}, {6, 2, 0, 2}, {6, 2, 1, 0}, {6, 4, 0, 2},
 {6, 4, 1, 0}, {8, 2, 0, 0}, {8, 4, 0, 0}}

Length[TT]

40
```

APPENDIX 4

Mathematica 3.0 Code

for Construction of Separating Lines

```
<< DiscreteMath`ComputationalGeometry`

Clear[A, as, b, a11, a12, a13, a21, a22, a23, a31, a32, a33, DDD, mu, m];
mu = h * Sqrt[Sqrt[1 - nu^2]];
as[m_, p_, lam_] := -2 * (1 - nu) p^2 + m^2 - (1 - nu^2) * lam * mu^(-4)
   b[m_, p_, lam_] := -(2 - nu) * p^2 * m + m^3
a11[m_, p_, lam_] := -p^2 + (1 - nu) / 2 * m^2 - (1 - nu^2) * lam
a21[m_, p_, lam_] := -(1 + nu) / 2 * p * m
 a12[m_, p_, lam_] := (1 + nu) / 2 * p * m
a31[m_, p_, lam_] := -nu * p
 a13[m_, p_, lam_] := nu * p
a22[m_, p_, lam_] := -(1 - nu) / 2 * p^2 + m^2 + as[m, p, lam] * mu^4
a32[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
 a23[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
a33[m_, p_, lam_] := 1 - (1 - nu^2) * lam + mu^4 * (p^2 - m^2)^2


Clear[alfa];
```

$$\mathtt{DDD[m\_,\ p\_,\ lam\_]\ :=\ Det}\left[\begin{pmatrix} \mathtt{a11[m, p, lam]} & \mathtt{a12[m, p, lam]} & \mathtt{a13[m, p, lam]} \\ \mathtt{a21[m, p, lam]} & \mathtt{a22[m, p, lam]} & \mathtt{a23[m, p, lam]} \\ \mathtt{a31[m, p, lam]} & \mathtt{a32[m, p, lam]} & \mathtt{a33[m, p, lam]} \end{pmatrix}\right]$$

```
DN = Expand[Factor[DDD[m * h^(-alfa), p, lam] * 2 / (1 - nu^2) / (1 - nu)]]
```

$$\begin{aligned}
&2\,\mathrm{lam}^2 - 2\,\mathrm{lam}^3 - 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2 - h^{-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2 + 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2 + 3\,h^{-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2 + h^{4-4\,\mathrm{alfa}}\,m^4 + \\
&3\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4 - h^{-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4 + 2\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^4 - 2\,h^{4-6\,\mathrm{alfa}}\,m^6 - 3\,h^{4-6\,\mathrm{alfa}}\,\mathrm{lam}\,m^6 + h^{4-8\,\mathrm{alfa}}\,m^8 + \\
&2\,\mathrm{lam}^2\,\mathrm{nu} - 2\,\mathrm{lam}^3\,\mathrm{nu} - 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu} + 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + 2\,h^{-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + \\
&4\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4\,\mathrm{nu} + 2\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^4\,\mathrm{nu} - 2\,h^{4-6\,\mathrm{alfa}}\,\mathrm{lam}\,m^6\,\mathrm{nu} + 2\,\mathrm{lam}^3\,\mathrm{nu}^2 - 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2 - \\
&h^{-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2 + h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4\,\mathrm{nu}^2 - 2\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^2 + h^{4-6\,\mathrm{alfa}}\,\mathrm{lam}\,m^6\,\mathrm{nu}^2 + 2\,\mathrm{lam}^3\,\mathrm{nu}^3 - \\
&2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3 - 2\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^3 + 3\,\mathrm{lam}\,p^2 + 4\,h^4\,\mathrm{lam}\,p^2 - 3\,\mathrm{lam}^2\,p^2 - 4\,h^4\,\mathrm{lam}^2\,p^2 - \\
&4\,h^{4-2\,\mathrm{alfa}}\,m^2\,p^2 - 4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,p^2 + 2\,h^{-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,p^2 - 4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,p^2 + 8\,h^{4-4\,\mathrm{alfa}}\,m^4\,p^2 + \\
&9\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4\,p^2 - 4\,h^{4-6\,\mathrm{alfa}}\,m^6\,p^2 + 2\,\mathrm{lam}\,\mathrm{nu}\,p^2 - 2\,\mathrm{lam}^2\,\mathrm{nu}\,p^2 - 4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^2 - \\
&4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}\,p^2 + 6\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4\,\mathrm{nu}\,p^2 - 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^2 + \mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + 8\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + \\
&2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^2 + 4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2\,p^2 - 3\,h^{4-4\,\mathrm{alfa}}\,\mathrm{lam}\,m^4\,\mathrm{nu}^2\,p^2 + 2\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^2 + \\
&4\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3\,p^2 - 4\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^4\,p^2 + p^4 + 4\,h^4\,p^4 - \mathrm{lam}\,p^4 - 4\,h^4\,\mathrm{lam}\,p^4 + 2\,h^4\,\mathrm{lam}^2\,p^4 - \\
&8\,h^{4-2\,\mathrm{alfa}}\,m^2\,p^4 - 9\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,p^4 - 2\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,p^4 + 6\,h^{4-4\,\mathrm{alfa}}\,m^4\,p^4 + h^{8-4\,\mathrm{alfa}}\,m^4\,p^4 + \\
&2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}\,p^4 - 6\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 2\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 4\,h^4\,\mathrm{nu}^2\,p^4 + 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^4 - \\
&2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^4 + 2\,h^{4-2\,\mathrm{alfa}}\,m^2\,\mathrm{nu}^2\,p^4 + 3\,h^{4-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 + 4\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 - \\
&2\,h^{8-4\,\mathrm{alfa}}\,m^4\,\mathrm{nu}^2\,p^4 - 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^3\,p^4 + 4\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^4 - 2\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^4\,p^4 + \\
&h^{8-4\,\mathrm{alfa}}\,m^4\,\mathrm{nu}^4\,p^4 - 2\,h^{8-2\,\mathrm{alfa}}\,\mathrm{lam}\,m^2\,\mathrm{nu}^5\,p^4 + 3\,h^4\,\mathrm{lam}\,p^6 + 4\,h^8\,\mathrm{lam}\,p^6 - 4\,h^{4-2\,\mathrm{alfa}}\,m^2\,p^6 - \\
&4\,h^{8-2\,\mathrm{alfa}}\,m^2\,p^6 + 2\,h^4\,\mathrm{lam}\,\mathrm{nu}\,p^6 - h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 - 8\,h^8\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 + 4\,h^{8-2\,\mathrm{alfa}}\,m^2\,\mathrm{nu}^2\,p^6 + \\
&4\,h^8\,\mathrm{lam}\,\mathrm{nu}^4\,p^6 + h^4\,p^8 + 4\,h^8\,p^8 - 4\,h^8\,\mathrm{nu}^2\,p^8
\end{aligned}$$

```
DN = DN /. lam -> L;
l = DN /. {Minus -> List}
l = Table[l[[i]], {i, 1, Length[l]}]

cc = l /. {p -> 1, h -> 1, L -> 1};
A = Table[{cc[[i]], {Exponent[l[[i]], p], Exponent[l[[i]], h], Exponent[l[[i]], L]}},
   {i, 1, Length[cc]}];
ll = Union[Transpose[A][[2]]];
lll = Table[Select[A, #[[2]] == ll[[i]]&], {i, 1, Length[ll]}];
r = Factor[
   Table[{Sum[lll[[i]][[j]][[1]], {j, 1, Length[lll[[i]]]}], lll[[i]][[1]][[2]]},
    {i, 1, Length[lll]}]]
```

$$2 L^2 - 2 L^3 - 2 h^{4-2\,\mathrm{alfa}} L m^2 - h^{-2\,\mathrm{alfa}} L m^2 + 2 h^{4-2\,\mathrm{alfa}} L^2 m^2 + 3 h^{-2\,\mathrm{alfa}} L^2 m^2 + h^{4-4\,\mathrm{alfa}} m^4 + 3 h^{4-4\,\mathrm{alfa}} L m^4 -$$
$$h^{-4\,\mathrm{alfa}} L m^4 + 2 h^{4-4\,\mathrm{alfa}} L^2 m^4 - 2 h^{4-6\,\mathrm{alfa}} m^6 - 3 h^{4-6\,\mathrm{alfa}} L m^6 + h^{4-8\,\mathrm{alfa}} m^8 + 2 L^2 nu - 2 L^3 nu -$$
$$2 h^{4-2\,\mathrm{alfa}} L m^2 nu + 2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu + 2 h^{-2\,\mathrm{alfa}} L^2 m^2 nu + 4 h^{4-4\,\mathrm{alfa}} L m^4 nu + 2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu -$$
$$2 h^{4-6\,\mathrm{alfa}} L m^6 nu + 2 L^3 nu^2 - 2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^2 - h^{-2\,\mathrm{alfa}} L^2 m^2 nu^2 + h^{4-4\,\mathrm{alfa}} L m^4 nu^2 -$$
$$2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu^2 + h^{4-6\,\mathrm{alfa}} L m^6 nu^2 + 2 L^3 nu^3 - 2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^3 - 2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu^3 +$$
$$3 L p^2 + 4 h^4 L p^2 - 3 L^2 p^2 - 4 h^4 L^2 p^2 - 4 h^{4-2\,\mathrm{alfa}} m^2 p^2 - 4 h^{4-2\,\mathrm{alfa}} L m^2 p^2 + 2 h^{-2\,\mathrm{alfa}} L m^2 p^2 -$$
$$4 h^{4-2\,\mathrm{alfa}} L^2 m^2 p^2 + 8 h^{4-4\,\mathrm{alfa}} m^4 p^2 + 9 h^{4-4\,\mathrm{alfa}} L m^4 p^2 - 4 h^{4-6\,\mathrm{alfa}} m^6 p^2 + 2 L nu p^2 - 2 L^2 nu p^2 -$$
$$4 h^{4-2\,\mathrm{alfa}} L m^2 nu p^2 - 4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu p^2 + 6 h^{4-4\,\mathrm{alfa}} L m^4 nu p^2 - 4 h^4 L nu^2 p^2 + L^2 nu^2 p^2 +$$
$$8 h^4 L^2 nu^2 p^2 + 2 h^{4-2\,\mathrm{alfa}} L m^2 nu^2 p^2 + 4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^2 p^2 - 3 h^{4-4\,\mathrm{alfa}} L m^4 nu^2 p^2 +$$
$$2 h^{4-2\,\mathrm{alfa}} L m^2 nu^3 p^2 + 4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^3 p^2 - 4 h^4 L^2 nu^4 p^2 + p^4 + 4 h^4 p^4 - L p^4 - 4 h^4 L p^4 + 2 h^4 L^2 p^4 -$$
$$8 h^{4-2\,\mathrm{alfa}} m^2 p^4 - 9 h^{4-2\,\mathrm{alfa}} L m^2 p^4 - 2 h^{8-2\,\mathrm{alfa}} L m^2 p^4 + 6 h^{4-4\,\mathrm{alfa}} m^4 p^4 + h^{8-4\,\mathrm{alfa}} m^4 p^4 + 2 h^4 L^2 nu p^4 -$$
$$6 h^{4-2\,\mathrm{alfa}} L m^2 nu p^4 - 2 h^{8-2\,\mathrm{alfa}} L m^2 nu p^4 - 4 h^4 nu^2 p^4 + 4 h^4 L nu^2 p^4 - 2 h^4 L^2 nu^2 p^4 +$$
$$2 h^{4-2\,\mathrm{alfa}} m^2 nu^2 p^4 + 3 h^{4-2\,\mathrm{alfa}} L m^2 nu^2 p^4 + 4 h^{8-2\,\mathrm{alfa}} L m^2 nu^2 p^4 - 2 h^{8-4\,\mathrm{alfa}} m^4 nu^2 p^4 - 2 h^4 L^2 nu^3 p^4 +$$
$$4 h^{8-2\,\mathrm{alfa}} L m^2 nu^3 p^4 - 2 h^{8-2\,\mathrm{alfa}} L m^2 nu^4 p^4 + h^{8-4\,\mathrm{alfa}} m^4 nu^4 p^4 - 2 h^{8-2\,\mathrm{alfa}} L m^2 nu^5 p^4 + 3 h^4 L p^6 +$$
$$4 h^8 L p^6 - 4 h^{4-2\,\mathrm{alfa}} m^2 p^6 - 4 h^{8-2\,\mathrm{alfa}} m^2 p^6 + 2 h^4 L nu p^6 - h^4 L nu^2 p^6 - 8 h^8 L nu^2 p^6 +$$
$$4 h^{8-2\,\mathrm{alfa}} m^2 nu^2 p^6 + 4 h^8 L nu^4 p^6 + h^4 p^8 + 4 h^8 p^8 - 4 h^8 nu^2 p^8$$

$$\{2 L^2,\ -2 L^3,\ -2 h^{4-2\,\mathrm{alfa}} L m^2,\ -h^{-2\,\mathrm{alfa}} L m^2,\ 2 h^{4-2\,\mathrm{alfa}} L^2 m^2,\ 3 h^{-2\,\mathrm{alfa}} L^2 m^2,\ h^{4-4\,\mathrm{alfa}} m^4,$$
$$3 h^{4-4\,\mathrm{alfa}} L m^4,\ -h^{-4\,\mathrm{alfa}} L m^4,\ 2 h^{4-4\,\mathrm{alfa}} L^2 m^4,\ -2 h^{4-6\,\mathrm{alfa}} m^6,\ -3 h^{4-6\,\mathrm{alfa}} L m^6,\ h^{4-8\,\mathrm{alfa}} m^8,\ 2 L^2 nu,$$
$$-2 L^3 nu,\ -2 h^{4-2\,\mathrm{alfa}} L m^2 nu,\ 2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu,\ 2 h^{-2\,\mathrm{alfa}} L^2 m^2 nu,\ 4 h^{4-4\,\mathrm{alfa}} L m^4 nu,$$
$$2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu,\ -2 h^{4-6\,\mathrm{alfa}} L m^6 nu,\ 2 L^3 nu^2,\ -2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^2,\ -h^{-2\,\mathrm{alfa}} L^2 m^2 nu^2,$$
$$h^{4-4\,\mathrm{alfa}} L m^4 nu^2,\ -2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu^2,\ h^{4-6\,\mathrm{alfa}} L m^6 nu^2,\ 2 L^3 nu^3,\ -2 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^3,$$
$$-2 h^{4-4\,\mathrm{alfa}} L^2 m^4 nu^3,\ 3 L p^2,\ 4 h^4 L p^2,\ -3 L^2 p^2,\ -4 h^4 L^2 p^2,\ -4 h^{4-2\,\mathrm{alfa}} m^2 p^2,\ -4 h^{4-2\,\mathrm{alfa}} L m^2 p^2,$$
$$2 h^{-2\,\mathrm{alfa}} L m^2 p^2,\ -4 h^{4-2\,\mathrm{alfa}} L^2 m^2 p^2,\ 8 h^{4-4\,\mathrm{alfa}} m^4 p^2,\ 9 h^{4-4\,\mathrm{alfa}} L m^4 p^2,\ -4 h^{4-6\,\mathrm{alfa}} m^6 p^2,\ 2 L nu p^2,$$
$$-2 L^2 nu p^2,\ -4 h^{4-2\,\mathrm{alfa}} L m^2 nu p^2,\ -4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu p^2,\ 6 h^{4-4\,\mathrm{alfa}} L m^4 nu p^2,\ -4 h^4 L nu^2 p^2,$$
$$L^2 nu^2 p^2,\ 8 h^4 L^2 nu^2 p^2,\ 2 h^{4-2\,\mathrm{alfa}} L m^2 nu^2 p^2,\ 4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^2 p^2,\ -3 h^{4-4\,\mathrm{alfa}} L m^4 nu^2 p^2,$$
$$2 h^{4-2\,\mathrm{alfa}} L m^2 nu^3 p^2,\ 4 h^{4-2\,\mathrm{alfa}} L^2 m^2 nu^3 p^2,\ -4 h^4 L^2 nu^4 p^2,\ p^4,\ 4 h^4 p^4,\ -L p^4,\ -4 h^4 L p^4,$$
$$2 h^4 L^2 p^4,\ -8 h^{4-2\,\mathrm{alfa}} m^2 p^4,\ -9 h^{4-2\,\mathrm{alfa}} L m^2 p^4,\ -2 h^{8-2\,\mathrm{alfa}} L m^2 p^4,\ 6 h^{4-4\,\mathrm{alfa}} m^4 p^4,\ h^{8-4\,\mathrm{alfa}} m^4 p^4,$$
$$2 h^4 L^2 nu p^4,\ -6 h^{4-2\,\mathrm{alfa}} L m^2 nu p^4,\ -2 h^{8-2\,\mathrm{alfa}} L m^2 nu p^4,\ -4 h^4 nu^2 p^4,\ 4 h^4 L nu^2 p^4,$$
$$-2 h^4 L^2 nu^2 p^4,\ 2 h^{4-2\,\mathrm{alfa}} m^2 nu^2 p^4,\ 3 h^{4-2\,\mathrm{alfa}} L m^2 nu^2 p^4,\ 4 h^{8-2\,\mathrm{alfa}} L m^2 nu^2 p^4,\ -2 h^{8-4\,\mathrm{alfa}} m^4 nu^2 p^4,$$
$$-2 h^4 L^2 nu^3 p^4,\ 4 h^{8-2\,\mathrm{alfa}} L m^2 nu^3 p^4,\ -2 h^{8-2\,\mathrm{alfa}} L m^2 nu^4 p^4,\ h^{8-4\,\mathrm{alfa}} m^4 nu^4 p^4,$$
$$-2 h^{8-2\,\mathrm{alfa}} L m^2 nu^5 p^4,\ 3 h^4 L p^6,\ 4 h^8 L p^6,\ -4 h^{4-2\,\mathrm{alfa}} m^2 p^6,\ -4 h^{8-2\,\mathrm{alfa}} m^2 p^6,\ 2 h^4 L nu p^6,$$
$$-h^4 L nu^2 p^6,\ -8 h^8 L nu^2 p^6,\ 4 h^{8-2\,\mathrm{alfa}} m^2 nu^2 p^6,\ 4 h^8 L nu^4 p^6,\ h^4 p^8,\ 4 h^8 p^8,\ -4 h^8 nu^2 p^8\}$$

```
{{2 (1 + nu), {0, 0, 2}}, {2 (-1 + nu) (1 + nu)^2, {0, 0, 3}}, {m^8, {0, -4 (-1 + 2 alfa), 0}},
 {-2 m^6, {0, -2 (-2 + 3 alfa), 0}}, {m^6 (-3 + nu) (1 + nu), {0, -2 (-2 + 3 alfa), 1}},
 {m^4, {0, -4 (-1 + alfa), 0}}, {m^4 (1 + nu) (3 + nu), {0, -4 (-1 + alfa), 1}},
 {-2 m^4 (-1 + nu) (1 + nu)^2, {0, -4 (-1 + alfa), 2}},
 {-2 m^2 (1 + nu), {0, -2 (-2 + alfa), 1}}, {-2 m^2 (-1 + nu) (1 + nu)^2, {0, -2 (-2 + alfa), 2}},
 {-m^4, {0, -4 alfa, 1}}, {-m^2, {0, -2 alfa, 1}}, {-m^2 (-3 + nu) (1 + nu), {0, -2 alfa, 2}},
 {3 + 2 nu, {2, 0, 1}}, {(-3 + nu) (1 + nu), {2, 0, 2}}, {-4 (-1 + nu) (1 + nu), {2, 4, 1}},
 {-4 (-1 + nu)^2 (1 + nu)^2, {2, 4, 2}}, {-4 m^6, {2, -2 (-2 + 3 alfa), 0}},
 {8 m^4, {2, -4 (-1 + alfa), 0}}, {-3 m^4 (-3 + nu) (1 + nu), {2, -4 (-1 + alfa), 1}},
 {-4 m^2, {2, -2 (-2 + alfa), 0}}, {2 m^2 (1 + nu) (-2 + nu^2), {2, -2 (-2 + alfa), 1}},
 {4 m^2 (-1 + nu) (1 + nu)^2, {2, -2 (-2 + alfa), 2}}, {2 m^2, {2, -2 alfa, 1}},
 {1, {4, 0, 0}}, {-1, {4, 0, 1}}, {-4 (-1 + nu) (1 + nu), {4, 4, 0}},
 {4 (-1 + nu) (1 + nu), {4, 4, 1}}, {-2 (-1 + nu) (1 + nu)^2, {4, 4, 2}},
 {6 m^4, {4, -4 (-1 + alfa), 0}}, {m^4 (-1 + nu)^2 (1 + nu)^2, {4, -4 (-2 + alfa), 0}},
 {2 m^2 (-2 + nu) (2 + nu), {4, -2 (-2 + alfa), 0}},
 {3 m^2 (-3 + nu) (1 + nu), {4, -2 (-2 + alfa), 1}},
 {-2 m^2 (-1 + nu)^2 (1 + nu)^3, {4, -2 (-4 + alfa), 1}},
 {-(-3 + nu) (1 + nu), {6, 4, 1}}, {4 (-1 + nu)^2 (1 + nu)^2, {6, 8, 1}},
 {-4 m^2, {6, -2 (-2 + alfa), 0}}, {4 m^2 (-1 + nu) (1 + nu), {6, -2 (-4 + alfa), 0}},
 {1, {8, 4, 0}}, {-4 (-1 + nu) (1 + nu), {8, 8, 0}}}
```

```
r[[{2, 15, 26}]]
```

```
{{2 (-1 + nu) (1 + nu)^2, {0, 0, 3}}, {(-3 + nu) (1 + nu), {2, 0, 2}}, {-1, {4, 0, 1}}}
```

```
r1 = Transpose[r] [[1]]
r2 = Transpose[r] [[2]]
```

```
{2 (1 + nu), 2 (-1 + nu) (1 + nu)^2, m^8, -2 m^6, m^6 (-3 + nu) (1 + nu), m^4,
 m^4 (1 + nu) (3 + nu), -2 m^4 (-1 + nu) (1 + nu)^2, -2 m^2 (1 + nu), -2 m^2 (-1 + nu) (1 + nu)^2,
 -m^4, -m^2, -m^2 (-3 + nu) (1 + nu), 3 + 2 nu, (-3 + nu) (1 + nu), -4 (-1 + nu) (1 + nu),
 -4 (-1 + nu)^2 (1 + nu)^2, -4 m^6, 8 m^4, -3 m^4 (-3 + nu) (1 + nu), -4 m^2, 2 m^2 (1 + nu) (-2 + nu^2),
 4 m^2 (-1 + nu) (1 + nu)^2, 2 m^2, 1, -1, -4 (-1 + nu) (1 + nu), 4 (-1 + nu) (1 + nu),
 -2 (-1 + nu) (1 + nu)^2, 6 m^4, m^4 (-1 + nu)^2 (1 + nu)^2, 2 m^2 (-2 + nu) (2 + nu),
 3 m^2 (-3 + nu) (1 + nu), -2 m^2 (-1 + nu)^2 (1 + nu)^3, -(-3 + nu) (1 + nu),
 4 (-1 + nu)^2 (1 + nu)^2, -4 m^2, 4 m^2 (-1 + nu) (1 + nu), 1, -4 (-1 + nu) (1 + nu)}
```

```
{{0, 0, 2}, {0, 0, 3},
 {0, -4 (-1 + 2 alfa), 0}, {0, -2 (-2 + 3 alfa), 0}, {0, -2 (-2 + 3 alfa), 1},
 {0, -4 (-1 + alfa), 0}, {0, -4 (-1 + alfa), 1}, {0, -4 (-1 + alfa), 2},
 {0, -2 (-2 + alfa), 1}, {0, -2 (-2 + alfa), 2}, {0, -4 alfa, 1}, {0, -2 alfa, 1},
 {0, -2 alfa, 2}, {2, 0, 1}, {2, 0, 2}, {2, 4, 1}, {2, 4, 2}, {2, -2 (-2 + 3 alfa), 0},
 {2, -4 (-1 + alfa), 0}, {2, -4 (-1 + alfa), 1}, {2, -2 (-2 + alfa), 0},
 {2, -2 (-2 + alfa), 1}, {2, -2 (-2 + alfa), 2}, {2, -2 alfa, 1}, {4, 0, 0}, {4, 0, 1},
 {4, 4, 0}, {4, 4, 1}, {4, 4, 2}, {4, -4 (-1 + alfa), 0}, {4, -4 (-2 + alfa), 0},
 {4, -2 (-2 + alfa), 0}, {4, -2 (-2 + alfa), 1}, {4, -2 (-4 + alfa), 1}, {6, 4, 1},
 {6, 8, 1}, {6, -2 (-2 + alfa), 0}, {6, -2 (-4 + alfa), 0}, {8, 4, 0}, {8, 8, 0}}
```

```
r2
alfa = 1 / 2
r2
```

```
{{0, 0, 2}, {0, 0, 3},
 {0, -4 (-1 + 2 alfa), 0}, {0, -2 (-2 + 3 alfa), 0}, {0, -2 (-2 + 3 alfa), 1},
 {0, -4 (-1 + alfa), 0}, {0, -4 (-1 + alfa), 1}, {0, -4 (-1 + alfa), 2},
 {0, -2 (-2 + alfa), 1}, {0, -2 (-2 + alfa), 2}, {0, -4 alfa, 1}, {0, -2 alfa, 1},
 {0, -2 alfa, 2}, {2, 0, 1}, {2, 0, 2}, {2, 4, 1}, {2, 4, 2}, {2, -2 (-2 + 3 alfa), 0},
 {2, -4 (-1 + alfa), 0}, {2, -4 (-1 + alfa), 1}, {2, -2 (-2 + alfa), 0},
 {2, -2 (-2 + alfa), 1}, {2, -2 (-2 + alfa), 2}, {2, -2 alfa, 1}, {4, 0, 0}, {4, 0, 1},
 {4, 4, 0}, {4, 4, 1}, {4, 4, 2}, {4, -4 (-1 + alfa), 0}, {4, -4 (-2 + alfa), 0},
 {4, -2 (-2 + alfa), 0}, {4, -2 (-2 + alfa), 1}, {4, -2 (-4 + alfa), 1}, {6, 4, 1},
 {6, 8, 1}, {6, -2 (-2 + alfa), 0}, {6, -2 (-4 + alfa), 0}, {8, 4, 0}, {8, 8, 0}}
```

$\frac{1}{2}$

```
{{0, 0, 2}, {0, 0, 3}, {0, 0, 0}, {0, 1, 0}, {0, 1, 1}, {0, 2, 0}, {0, 2, 1}, {0, 2, 2},
 {0, 3, 1}, {0, 3, 2}, {0, -2, 1}, {0, -1, 1}, {0, -1, 2}, {2, 0, 1}, {2, 0, 2}, {2, 4, 1},
 {2, 4, 2}, {2, 1, 0}, {2, 2, 0}, {2, 2, 1}, {2, 3, 0}, {2, 3, 1}, {2, 3, 2}, {2, -1, 1},
 {4, 0, 0}, {4, 0, 1}, {4, 4, 0}, {4, 4, 1}, {4, 4, 2}, {4, 2, 0}, {4, 6, 0}, {4, 3, 0},
 {4, 3, 1}, {4, 7, 1}, {6, 4, 1}, {6, 8, 1}, {6, 3, 0}, {6, 7, 0}, {8, 4, 0}, {8, 8, 0}}
```

```
Length[r2]
```

```
40
```

```
pt = {{r2[[11]], r2[[3]], r2[[25]]},
   {r2[[11]], r2[[25]], r2[[26]]}, {r2[[11]], r2[[26]], r2[[2]]},
   {r2[[25]], r2[[39]], r2[[26]]}, {r2[[2]], r2[[26]], r2[[39]], r2[[29]]}}
PPP = {r2[[11]], r2[[3]], r2[[25]], r2[[26]], r2[[2]], r2[[39]], r2[[29]]}
ptp = {{11, 3, 25}, {11, 25, 26}, {11, 26, 2}, {25, 39, 26}, {2, 26, 39, 29}}
```

```
{{{0, -4 alfa, 1}, {0, -4 (-1 + 2 alfa), 0}, {4, 0, 0}},
 {{0, -4 alfa, 1}, {4, 0, 0}, {4, 0, 1}}, {{0, -4 alfa, 1}, {4, 0, 1}, {0, 0, 3}},
 {{4, 0, 0}, {8, 4, 0}, {4, 0, 1}}, {{0, 0, 3}, {4, 0, 1}, {8, 4, 0}, {4, 4, 2}}}
```

```
{{0, -4 alfa, 1}, {0, -4 (-1 + 2 alfa), 0}, {4, 0, 0}, {4, 0, 1}, {0, 0, 3},
 {8, 4, 0}, {4, 4, 2}}
```

```
{{11, 3, 25}, {11, 25, 26}, {11, 26, 2}, {25, 39, 26}, {2, 26, 39, 29}}
```

```
PPP (*the list of convex hull points*)
pt(*the list of facets (in points) *)
ptp(*the list of facets (in points numbers)*)
r2(*initial lexicographically sorted list of the points*)
```

```
{{0, -4 alfa, 1}, {0, -4 (-1 + 2 alfa), 0}, {4, 0, 0}, {4, 0, 1}, {0, 0, 3},
 {8, 4, 0}, {4, 4, 2}}
```

```
{{{0, -4 alfa, 1}, {0, -4 (-1 + 2 alfa), 0}, {4, 0, 0}},
 {{0, -4 alfa, 1}, {4, 0, 0}, {4, 0, 1}}, {{0, -4 alfa, 1}, {4, 0, 1}, {0, 0, 3}},
 {{4, 0, 0}, {8, 4, 0}, {4, 0, 1}}, {{0, 0, 3}, {4, 0, 1}, {8, 4, 0}, {4, 4, 2}}}
```

```
{{11, 3, 25}, {11, 25, 26}, {11, 26, 2}, {25, 39, 26}, {2, 26, 39, 29}}
```

```
{{0, 0, 2}, {0, 0, 3},
 {0, -4 (-1 + 2 alfa), 0}, {0, -2 (-2 + 3 alfa), 0}, {0, -2 (-2 + 3 alfa), 1},
 {0, -4 (-1 + alfa), 0}, {0, -4 (-1 + alfa), 1}, {0, -4 (-1 + alfa), 2},
 {0, -2 (-2 + alfa), 1}, {0, -2 (-2 + alfa), 2}, {0, -4 alfa, 1}, {0, -2 alfa, 1},
 {0, -2 alfa, 2}, {2, 0, 1}, {2, 0, 2}, {2, 4, 1}, {2, 4, 2}, {2, -2 (-2 + 3 alfa), 0},
 {2, -4 (-1 + alfa), 0}, {2, -4 (-1 + alfa), 1}, {2, -2 (-2 + alfa), 0},
 {2, -2 (-2 + alfa), 1}, {2, -2 (-2 + alfa), 2}, {2, -2 alfa, 1}, {4, 0, 0}, {4, 0, 1},
 {4, 4, 0}, {4, 4, 1}, {4, 4, 2}, {4, -4 (-1 + alfa), 0}, {4, -4 (-2 + alfa), 0},
 {4, -2 (-2 + alfa), 0}, {4, -2 (-2 + alfa), 1}, {4, -2 (-4 + alfa), 1}, {6, 4, 1},
 {6, 8, 1}, {6, -2 (-2 + alfa), 0}, {6, -2 (-4 + alfa), 0}, {8, 4, 0}, {8, 8, 0}}
```

```
s1 = Table[Table[
  p^r2[[ptp[[i]][[j]]]][[1]] * h^r2[[ptp[[i]][[j]]]][[2]] *
    L^r2[[ptp[[i]][[j]]]][[3]], {j, 1, Length[ptp[[i]]]}],
 {i, 1, Length[ptp]}]
```

$\{\{h^{-4\,\text{alfa}}\,L,\ h^{-4\,(-1+2\,\text{alfa})},\ p^4\},\ \{h^{-4\,\text{alfa}}\,L,\ p^4,\ L\,p^4\},\ \{h^{-4\,\text{alfa}}\,L,\ L\,p^4,\ L^3\},\ \{p^4,\ h^4\,p^8,\ L\,p^4\},$
$\{L^3,\ L\,p^4,\ h^4\,p^8,\ h^4\,L^2\,p^4\}\}$

```
l = (# /. {List -> Equal})&/@s1
```

$\{h^{-4\,\text{alfa}}\,L == h^{-4\,(-1+2\,\text{alfa})} == p^4,\ h^{-4\,\text{alfa}}\,L == p^4 == L\,p^4,\ h^{-4\,\text{alfa}}\,L == L\,p^4 == L^3,$
$p^4 == h^4\,p^8 == L\,p^4,\ L^3 == L\,p^4 == h^4\,p^8 == h^4\,L^2\,p^4\}$

```
<< DiscreteMath`ComputationalGeometry`

Clear[A, as, b, a11, a12, a13, a21, a22, a23, a31, a32, a33, DDD, mu, m];
mu = h * Sqrt[Sqrt[1 - nu^2]];
as[m_, p_, lam_] := -2 * (1 - nu) p^2 + m^2 - (1 - nu^2) * lam * mu^(-4)
  b[m_, p_, lam_] := -(2 - nu) * p^2 * m + m^3
a11[m_, p_, lam_] := -p^2 + (1 - nu) / 2 * m^2 - (1 - nu^2) * lam
a21[m_, p_, lam_] := -(1 + nu) / 2 * p * m
 a12[m_, p_, lam_] := (1 + nu) / 2 * p * m
a31[m_, p_, lam_] := -nu * p
 a13[m_, p_, lam_] := nu * p
a22[m_, p_, lam_] := -(1 - nu) / 2 * p^2 + m^2 + as[m, p, lam] * mu^4
a32[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
 a23[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
a33[m_, p_, lam_] := 1 - (1 - nu^2) * lam + mu^4 * (p^2 - m^2)^2


Clear[alfa];
```

$$\mathtt{DDD[m\_,\ p\_,\ lam\_]} := \mathtt{Det}\left[\begin{pmatrix} \mathtt{a11[m,\ p,\ lam]} & \mathtt{a12[m,\ p,\ lam]} & \mathtt{a13[m,\ p,\ lam]} \\ \mathtt{a21[m,\ p,\ lam]} & \mathtt{a22[m,\ p,\ lam]} & \mathtt{a23[m,\ p,\ lam]} \\ \mathtt{a31[m,\ p,\ lam]} & \mathtt{a32[m,\ p,\ lam]} & \mathtt{a33[m,\ p,\ lam]} \end{pmatrix}\right]$$

```
DN = Expand[Factor[DDD[m, p, lam] * 2 / (1 - nu^2) / (1 - nu)]]
```

$$\begin{aligned}
&2\,\mathrm{lam}^2 - 2\,\mathrm{lam}^3 - \mathrm{lam}\,m^2 - 2\,h^4\,\mathrm{lam}\,m^2 + 3\,\mathrm{lam}^2\,m^2 + 2\,h^4\,\mathrm{lam}^2\,m^2 + h^4\,m^4 - \mathrm{lam}\,m^4 + 3\,h^4\,\mathrm{lam}\,m^4 + \\
&2\,h^4\,\mathrm{lam}^2\,m^4 - 2\,h^4\,m^6 - 3\,h^4\,\mathrm{lam}\,m^6 + h^4\,m^8 + 2\,\mathrm{lam}^2\,\mathrm{nu} - 2\,\mathrm{lam}^3\,\mathrm{nu} - 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu} + 2\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + \\
&2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + 4\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu} + 2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu} - 2\,h^4\,\mathrm{lam}\,m^6\,\mathrm{nu} + 2\,\mathrm{lam}^3\,\mathrm{nu}^2 - \mathrm{lam}^2\,m^2\,\mathrm{nu}^2 - \\
&2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2 + h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}^2 - 2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^2 + h^4\,\mathrm{lam}\,m^6\,\mathrm{nu}^2 + 2\,\mathrm{lam}^3\,\mathrm{nu}^3 - 2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3 - \\
&2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^3 + 3\,\mathrm{lam}\,p^2 + 4\,h^4\,\mathrm{lam}\,p^2 - 3\,\mathrm{lam}^2\,p^2 - 4\,h^4\,\mathrm{lam}^2\,p^2 - 4\,h^4\,m^2\,p^2 + 2\,\mathrm{lam}\,m^2\,p^2 - \\
&4\,h^4\,\mathrm{lam}\,m^2\,p^2 - 4\,h^4\,\mathrm{lam}^2\,m^2\,p^2 + 8\,h^4\,m^4\,p^2 + 9\,h^4\,\mathrm{lam}\,m^4\,p^2 - 4\,h^4\,m^6\,p^2 + 2\,\mathrm{lam}\,\mathrm{nu}\,p^2 - 2\,\mathrm{lam}^2\,\mathrm{nu}\,p^2 - \\
&4\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^2 - 4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}\,p^2 + 6\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}\,p^2 - 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^2 + \mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + \\
&8\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^2 + 4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2\,p^2 - 3\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}^2\,p^2 + 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^2 + \\
&4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3\,p^2 - 4\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^4\,p^2 + p^4 + 4\,h^4\,p^4 - \mathrm{lam}\,p^4 - 4\,h^4\,\mathrm{lam}\,p^4 + 2\,h^4\,\mathrm{lam}^2\,p^4 - 8\,h^4\,m^2\,p^4 - \\
&9\,h^4\,\mathrm{lam}\,m^2\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,p^4 + 6\,h^4\,m^4\,p^4 + h^8\,m^4\,p^4 + 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}\,p^4 - 6\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - \\
&2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 4\,h^4\,\mathrm{nu}^2\,p^4 + 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^4 - 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^4 + 2\,h^4\,m^2\,\mathrm{nu}^2\,p^4 + 3\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 + \\
&4\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 - 2\,h^8\,m^4\,\mathrm{nu}^2\,p^4 - 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^3\,p^4 + 4\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^4\,p^4 + \\
&h^8\,m^4\,\mathrm{nu}^4\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^5\,p^4 + 3\,h^4\,\mathrm{lam}\,p^6 + 4\,h^8\,\mathrm{lam}\,p^6 - 4\,h^4\,m^2\,p^6 - 4\,h^8\,m^2\,p^6 + 2\,h^4\,\mathrm{lam}\,\mathrm{nu}\,p^6 - \\
&h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 - 8\,h^8\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 + 4\,h^8\,m^2\,\mathrm{nu}^2\,p^6 + 4\,h^8\,\mathrm{lam}\,\mathrm{nu}^4\,p^6 + h^4\,p^8 + 4\,h^8\,p^8 - 4\,h^8\,\mathrm{nu}^2\,p^8
\end{aligned}$$

```
DN = DN /. lam -> L;
l = DN /. {Minus -> List}
l = Table[l[[i]], {i, 1, Length[l]}]

cc = l /. {p -> 1, h -> 1, m -> 1};
A = Table[{cc[[i]], {Exponent[l[[i]], p],
     Exponent[l[[i]], h] + (betta) * Exponent[l[[i]], L], Exponent[l[[i]], m]}},
   {i, 1, Length[cc]}];
ll = Union[Transpose[A][[2]]];
lll = Table[Select[A, #[[2]] == ll[[i]]&], {i, 1, Length[ll]}];
r = Factor[
  Table[{Sum[lll[[i]][[j]][[1]], {j, 1, Length[lll[[i]]]}], lll[[i]][[1]][[2]]},
   {i, 1, Length[lll]}]]
```

$$2 L^2 - 2 L^3 - L m^2 - 2 h^4 L m^2 + 3 L^2 m^2 + 2 h^4 L^2 m^2 + h^4 m^4 - L m^4 + 3 h^4 L m^4 + 2 h^4 L^2 m^4 - 2 h^4 m^6 -$$
$$3 h^4 L m^6 + h^4 m^8 + 2 L^2 nu - 2 L^3 nu - 2 h^4 L m^2 nu + 2 L^2 m^2 nu + 2 h^4 L^2 m^2 nu + 4 h^4 L m^4 nu +$$
$$2 h^4 L^2 m^4 nu - 2 h^4 L m^6 nu + 2 L^3 nu^2 - L^2 m^2 nu^2 - 2 h^4 L^2 m^2 nu^2 + h^4 L m^4 nu^2 - 2 h^4 L^2 m^4 nu^2 +$$
$$h^4 L m^6 nu^2 + 2 L^3 nu^3 - 2 h^4 L^2 m^2 nu^3 - 2 h^4 L^2 m^4 nu^3 + 3 L p^2 + 4 h^4 L p^2 - 3 L^2 p^2 -$$
$$4 h^4 L^2 p^2 - 4 h^4 m^2 p^2 + 2 L m^2 p^2 - 4 h^4 L m^2 p^2 - 4 h^4 L^2 m^2 p^2 + 8 h^4 m^4 p^2 + 9 h^4 L m^4 p^2 -$$
$$4 h^4 m^6 p^2 + 2 L nu p^2 - 2 L^2 nu p^2 - 4 h^4 L m^2 nu p^2 - 4 h^4 L^2 m^2 nu p^2 + 6 h^4 L m^4 nu p^2 -$$
$$4 h^4 L nu^2 p^2 + L^2 nu^2 p^2 + 8 h^4 L^2 nu^2 p^2 + 2 h^4 L m^2 nu^2 p^2 + 4 h^4 L^2 m^2 nu^2 p^2 - 3 h^4 L m^4 nu^2 p^2 +$$
$$2 h^4 L m^2 nu^3 p^2 + 4 h^4 L^2 m^2 nu^3 p^2 - 4 h^4 L^2 nu^4 p^2 + p^4 + 4 h^4 p^4 - L p^4 - 4 h^4 L p^4 + 2 h^4 L^2 p^4 -$$
$$8 h^4 m^2 p^4 - 9 h^4 L m^2 p^4 - 2 h^8 L m^2 p^4 + 6 h^4 m^4 p^4 + h^8 m^4 p^4 + 2 h^4 L^2 nu p^4 - 6 h^4 L m^2 nu p^4 -$$
$$2 h^8 L m^2 nu p^4 - 4 h^4 nu^2 p^4 + 4 h^4 L nu^2 p^4 - 2 h^4 L^2 nu^2 p^4 + 2 h^4 m^2 nu^2 p^4 + 3 h^4 L m^2 nu^2 p^4 +$$
$$4 h^8 L m^2 nu^2 p^4 - 2 h^8 m^4 nu^2 p^4 - 2 h^4 L^2 nu^3 p^4 + 4 h^8 L m^2 nu^3 p^4 - 2 h^8 L m^2 nu^4 p^4 +$$
$$h^8 m^4 nu^4 p^4 - 2 h^8 L m^2 nu^5 p^4 + 3 h^4 L p^6 + 4 h^8 L p^6 - 4 h^4 m^2 p^6 - 4 h^8 m^2 p^6 + 2 h^4 L nu p^6 -$$
$$h^4 L nu^2 p^6 - 8 h^8 L nu^2 p^6 + 4 h^8 m^2 nu^2 p^6 + 4 h^8 L nu^4 p^6 + h^4 p^8 + 4 h^8 p^8 - 4 h^8 nu^2 p^8$$

$$\{2 L^2, -2 L^3, -L m^2, -2 h^4 L m^2, 3 L^2 m^2, 2 h^4 L^2 m^2, h^4 m^4, -L m^4, 3 h^4 L m^4, 2 h^4 L^2 m^4, -2 h^4 m^6,$$
$$-3 h^4 L m^6, h^4 m^8, 2 L^2 nu, -2 L^3 nu, -2 h^4 L m^2 nu, 2 L^2 m^2 nu, 2 h^4 L^2 m^2 nu, 4 h^4 L m^4 nu,$$
$$2 h^4 L^2 m^4 nu, -2 h^4 L m^6 nu, 2 L^3 nu^2, -L^2 m^2 nu^2, -2 h^4 L^2 m^2 nu^2, h^4 L m^4 nu^2, -2 h^4 L^2 m^4 nu^2,$$
$$h^4 L m^6 nu^2, 2 L^3 nu^3, -2 h^4 L^2 m^2 nu^3, -2 h^4 L^2 m^4 nu^3, 3 L p^2, 4 h^4 L p^2, -3 L^2 p^2, -4 h^4 L^2 p^2,$$
$$-4 h^4 m^2 p^2, 2 L m^2 p^2, -4 h^4 L m^2 p^2, -4 h^4 L^2 m^2 p^2, 8 h^4 m^4 p^2, 9 h^4 L m^4 p^2, -4 h^4 m^6 p^2, 2 L nu p^2,$$
$$-2 L^2 nu p^2, -4 h^4 L m^2 nu p^2, -4 h^4 L^2 m^2 nu p^2, 6 h^4 L m^4 nu p^2, -4 h^4 L nu^2 p^2, L^2 nu^2 p^2,$$
$$8 h^4 L^2 nu^2 p^2, 2 h^4 L m^2 nu^2 p^2, 4 h^4 L^2 m^2 nu^2 p^2, -3 h^4 L m^4 nu^2 p^2, 2 h^4 L m^2 nu^3 p^2, 4 h^4 L^2 m^2 nu^3 p^2,$$
$$-4 h^4 L^2 nu^4 p^2, p^4, 4 h^4 p^4, -L p^4, -4 h^4 L p^4, 2 h^4 L^2 p^4, -8 h^4 m^2 p^4, -9 h^4 L m^2 p^4, -2 h^8 L m^2 p^4,$$
$$6 h^4 m^4 p^4, h^8 m^4 p^4, 2 h^4 L^2 nu p^4, -6 h^4 L m^2 nu p^4, -2 h^8 L m^2 nu p^4, -4 h^4 nu^2 p^4, 4 h^4 L nu^2 p^4,$$
$$-2 h^4 L^2 nu^2 p^4, 2 h^4 m^2 nu^2 p^4, 3 h^4 L m^2 nu^2 p^4, 4 h^8 L m^2 nu^2 p^4, -2 h^8 m^4 nu^2 p^4,$$
$$-2 h^4 L^2 nu^3 p^4, 4 h^8 L m^2 nu^3 p^4, -2 h^8 L m^2 nu^4 p^4, h^8 m^4 nu^4 p^4, -2 h^8 L m^2 nu^5 p^4,$$
$$3 h^4 L p^6, 4 h^8 L p^6, -4 h^4 m^2 p^6, -4 h^8 m^2 p^6, 2 h^4 L nu p^6, -h^4 L nu^2 p^6, -8 h^8 L nu^2 p^6,$$
$$4 h^8 m^2 nu^2 p^6, 4 h^8 L nu^4 p^6, h^4 p^8, 4 h^8 p^8, -4 h^8 nu^2 p^8\}$$

```
{{1, {0, 4, 4}}, {-2, {0, 4, 6}}, {1, {0, 4, 8}},
 {-L, {0, betta, 2}}, {-L, {0, betta, 4}}, {2 L^2 (1 + nu), {0, 2 betta, 0}},
 {-L^2 (-3 + nu) (1 + nu), {0, 2 betta, 2}}, {2 L^3 (-1 + nu) (1 + nu)^2, {0, 3 betta, 0}},
 {-2 L (1 + nu), {0, 4 + betta, 2}}, {L (1 + nu) (3 + nu), {0, 4 + betta, 4}},
 {L (-3 + nu) (1 + nu), {0, 4 + betta, 6}}, {-2 L^2 (-1 + nu) (1 + nu)^2, {0, 2 (2 + betta), 2}},
 {-2 L^2 (-1 + nu) (1 + nu)^2, {0, 2 (2 + betta), 4}}, {-4, {2, 4, 2}},
 {8, {2, 4, 4}}, {-4, {2, 4, 6}}, {L (3 + 2 nu), {2, betta, 0}}, {2 L, {2, betta, 2}},
 {L^2 (-3 + nu) (1 + nu), {2, 2 betta, 0}}, {-4 L (-1 + nu) (1 + nu), {2, 4 + betta, 0}},
 {2 L (1 + nu) (-2 + nu^2), {2, 4 + betta, 2}}, {-3 L (-3 + nu) (1 + nu), {2, 4 + betta, 4}},
 {-4 L^2 (-1 + nu)^2 (1 + nu)^2, {2, 2 (2 + betta), 0}},
 {4 L^2 (-1 + nu) (1 + nu)^2, {2, 2 (2 + betta), 2}},
 {1, {4, 0, 0}}, {-4 (-1 + nu) (1 + nu), {4, 4, 0}},
 {2 (-2 + nu) (2 + nu), {4, 4, 2}}, {6, {4, 4, 4}}, {(-1 + nu)^2 (1 + nu)^2, {4, 8, 4}},
 {-L, {4, betta, 0}}, {4 L (-1 + nu) (1 + nu), {4, 4 + betta, 0}},
 {3 L (-3 + nu) (1 + nu), {4, 4 + betta, 2}}, {-2 L (-1 + nu)^2 (1 + nu)^3, {4, 8 + betta, 2}},
 {-2 L^2 (-1 + nu) (1 + nu)^2, {4, 2 (2 + betta), 0}},
 {-4, {6, 4, 2}}, {4 (-1 + nu) (1 + nu), {6, 8, 2}},
 {-L (-3 + nu) (1 + nu), {6, 4 + betta, 0}}, {4 L (-1 + nu)^2 (1 + nu)^2, {6, 8 + betta, 0}},
 {1, {8, 4, 0}}, {-4 (-1 + nu) (1 + nu), {8, 8, 0}}}
```

```
{{{0, 4, 8}, {0, 2, 4}, {4, 0, 0}}, {{0, 4, 8}, {4, 0, 0}, {8, 4, 0}},
 {{0, 2, 2}, {0, 4, 0}, {4, 0, 0}}, {{0, 2, 2}, {4, 0, 0}, {0, 2, 4}}}

pt = {{r2[[3]], r2[[5]], r2[[25]]}, {r2[[3]], r2[[25]], r2[[39]]},
  {r2[[4]], r2[[6]], r2[[25]]}, {r2[[4]], r2[[25]], r2[[5]]}}
PPP = {r2[[3]], r2[[5]], r2[[25]], r2[[4]], r2[[39]], r2[[6]]}
ptp = {{3, 5, 25}, {3, 25, 39}, {4, 6, 25}, {4, 25, 5}}
```

```
{{{0, 4, 8}, {0, betta, 4}, {4, 0, 0}}, {{0, 4, 8}, {4, 0, 0}, {8, 4, 0}},
 {{0, betta, 2}, {0, 2 betta, 0}, {4, 0, 0}}, {{0, betta, 2}, {4, 0, 0}, {0, betta, 4}}}

{{0, 4, 8}, {0, betta, 4}, {4, 0, 0}, {0, betta, 2}, {8, 4, 0}, {0, 2 betta, 0}}

{{3, 5, 25}, {3, 25, 39}, {4, 6, 25}, {4, 25, 5}}
```

```
PPP (*the list of convex hull points*)
pt(*the list of facets (in points) *)
ptp(*the list of facets (in points numbers)*)
r2(*initial lexicographically sorted list of the points*)
```

```
{{0, 4, 8}, {0, betta, 4}, {4, 0, 0}, {0, betta, 2}, {8, 4, 0}, {0, 2 betta, 0}}
```

```
{{{0, 4, 8}, {0, betta, 4}, {4, 0, 0}}, {{0, 4, 8}, {4, 0, 0}, {8, 4, 0}},
 {{0, betta, 2}, {0, 2 betta, 0}, {4, 0, 0}}, {{0, betta, 2}, {4, 0, 0}, {0, betta, 4}}}
```

```
{{3, 5, 25}, {3, 25, 39}, {4, 6, 25}, {4, 25, 5}}
```

```
{{0, 4, 4}, {0, 4, 6}, {0, 4, 8}, {0, betta, 2}, {0, betta, 4}, {0, 2 betta, 0},
 {0, 2 betta, 2}, {0, 3 betta, 0}, {0, 4 + betta, 2}, {0, 4 + betta, 4}, {0, 4 + betta, 6},
 {0, 2 (2 + betta), 2}, {0, 2 (2 + betta), 4}, {2, 4, 2}, {2, 4, 4}, {2, 4, 6},
 {2, betta, 0}, {2, betta, 2}, {2, 2 betta, 0}, {2, 4 + betta, 0}, {2, 4 + betta, 2},
 {2, 4 + betta, 4}, {2, 2 (2 + betta), 0}, {2, 2 (2 + betta), 2}, {4, 0, 0},
 {4, 4, 0}, {4, 4, 2}, {4, 4, 4}, {4, 8, 4}, {4, betta, 0}, {4, 4 + betta, 0},
 {4, 4 + betta, 2}, {4, 8 + betta, 2}, {4, 2 (2 + betta), 0}, {6, 4, 2}, {6, 8, 2},
 {6, 4 + betta, 0}, {6, 8 + betta, 0}, {8, 4, 0}, {8, 8, 0}}
```

```
s1 = Table[Table[
 p^r2[[ptp[[i]][[j]]]][[1]] *h^r2[[ptp[[i]][[j]]]][[2]] *
   m^r2[[ptp[[i]][[j]]]][[3]], {j, 1, Length[ptp[[i]]]}],
{i, 1, Length[ptp]}]
```

$\{\{h^4 m^8, h^{betta} m^4, p^4\}, \{h^4 m^8, p^4, h^4 p^8\}, \{h^{betta} m^2, h^{2\,betta}, p^4\}, \{h^{betta} m^2, p^4, h^{betta} m^4\}\}$

```
l = (# /. {List -> Equal})&/@s1
```

$\{h^{-4\,alfa} L == h^{-4\,(-1+2\,alfa)} == p^4,\ h^{-4\,alfa} L == p^4 == L\,p^4,\ h^{-4\,alfa} L == L\,p^4 == L^3,$
$p^4 == h^4 p^8 == L\,p^4,\ L^3 == L\,p^4 == h^4 p^8 == h^4 L^2 p^4\}$

APPENDIX 5

Mathematica 3.0 Code for Construction of Short Form of Equation on Separating Line

```
<< DiscreteMath`ComputationalGeometry`

Clear[A, a, b, a11, a12, a13, a21, a22, a23, a31, a32, a33, DDD, mu, m];
mu = h * Sqrt[Sqrt[1 - nu^2]];
a[m_, p_, lam_] := -2 * (1 - nu) p^2 + m^2 - (1 - nu^2) * lam * mu^(-4)
   b[m_, p_, lam_] := -(2 - nu) * p^2 * m + m^3
a11[m_, p_, lam_] := -p^2 + (1 - nu) / 2 * m^2 - (1 - nu^2) * lam
a21[m_, p_, lam_] := -(1 + nu) / 2 * p * m
 a12[m_, p_, lam_] := (1 + nu) / 2 * p * m
a31[m_, p_, lam_] := -nu * p
 a13[m_, p_, lam_] := nu * p
a22[m_, p_, lam_] := -(1 - nu) / 2 * p^2 + m^2 + a[m, p, lam] * mu^4
a32[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
 a23[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
a33[m_, p_, lam_] := 1 - (1 - nu^2) * lam + mu^4 * (p^2 - m^2)^2
```

```
DDD[m_, p_, lam_] := Det[( a11[m, p, lam]  a12[m, p, lam]  a13[m, p, lam]
                           a21[m, p, lam]  a22[m, p, lam]  a23[m, p, lam]
                           a31[m, p, lam]  a32[m, p, lam]  a33[m, p, lam] )]

DN = Expand[Factor[DDD[m, p, lam] * 2 / (1 - nu^2) / (1 - nu)]]
```

$$
\begin{aligned}
&2\,\mathrm{lam}^2 - 2\,\mathrm{lam}^3 - \mathrm{lam}\,m^2 - 2\,h^4\,\mathrm{lam}\,m^2 + 3\,\mathrm{lam}^2\,m^2 + 2\,h^4\,\mathrm{lam}^2\,m^2 + h^4\,m^4 - \mathrm{lam}\,m^4 + 3\,h^4\,\mathrm{lam}\,m^4 +\\
&2\,h^4\,\mathrm{lam}^2\,m^4 - 2\,h^4\,m^6 - 3\,h^4\,\mathrm{lam}\,m^6 + h^4\,m^8 + 2\,\mathrm{lam}^2\,\mathrm{nu} - 2\,\mathrm{lam}^3\,\mathrm{nu} - 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu} + 2\,\mathrm{lam}^2\,m^2\,\mathrm{nu} +\\
&2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + 4\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu} + 2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu} - 2\,h^4\,\mathrm{lam}\,m^6\,\mathrm{nu} + 2\,\mathrm{lam}^3\,\mathrm{nu}^2 - \mathrm{lam}^2\,m^2\,\mathrm{nu}^2 -\\
&2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2 + h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}^2 - 2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^2 + h^4\,\mathrm{lam}\,m^6\,\mathrm{nu}^2 + 2\,\mathrm{lam}^3\,\mathrm{nu}^3 - 2\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3 -\\
&2\,h^4\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^3 + 3\,\mathrm{lam}\,p^2 + 4\,h^4\,\mathrm{lam}\,p^2 - 3\,\mathrm{lam}^2\,p^2 - 4\,h^4\,\mathrm{lam}^2\,p^2 - 4\,h^4\,m^2\,p^2 + 2\,\mathrm{lam}\,m^2\,p^2 -\\
&4\,h^4\,\mathrm{lam}\,m^2\,p^2 - 4\,h^4\,\mathrm{lam}^2\,m^2\,p^2 + 8\,h^4\,m^4\,p^2 + 9\,h^4\,\mathrm{lam}\,m^4\,p^2 - 4\,h^4\,m^6\,p^2 + 2\,\mathrm{lam}\,\mathrm{nu}\,p^2 - 2\,\mathrm{lam}^2\,\mathrm{nu}\,p^2 -\\
&4\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^2 - 4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}\,p^2 + 6\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}\,p^2 - 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^2 + \mathrm{lam}^2\,\mathrm{nu}^2\,p^2 +\\
&8\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^2 + 4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2\,p^2 - 3\,h^4\,\mathrm{lam}\,m^4\,\mathrm{nu}^2\,p^2 + 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^2 +\\
&4\,h^4\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3\,p^2 - 4\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^4\,p^2 + p^4 + 4\,h^4\,p^4 - \mathrm{lam}\,p^4 - 4\,h^4\,\mathrm{lam}\,p^4 + 2\,h^4\,\mathrm{lam}^2\,p^4 - 8\,h^4\,m^2\,p^4 -\\
&9\,h^4\,\mathrm{lam}\,m^2\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,p^4 + 6\,h^4\,m^4\,p^4 + h^8\,m^4\,p^4 + 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}\,p^4 - 6\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 -\\
&2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 4\,h^4\,\mathrm{nu}^2\,p^4 + 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^4 - 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^4 + 2\,h^4\,m^2\,\mathrm{nu}^2\,p^4 + 3\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 +\\
&4\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 - 2\,h^8\,m^4\,\mathrm{nu}^2\,p^4 - 2\,h^4\,\mathrm{lam}^2\,\mathrm{nu}^3\,p^4 + 4\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^4\,p^4 +\\
&h^8\,m^4\,\mathrm{nu}^4\,p^4 - 2\,h^8\,\mathrm{lam}\,m^2\,\mathrm{nu}^5\,p^4 + 3\,h^4\,\mathrm{lam}\,p^6 + 4\,h^8\,\mathrm{lam}\,p^6 - 4\,h^4\,m^2\,p^6 - 4\,h^8\,m^2\,p^6 + 2\,h^4\,\mathrm{lam}\,\mathrm{nu}\,p^6 -\\
&h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 - 8\,h^8\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 + 4\,h^8\,m^2\,\mathrm{nu}^2\,p^6 + 4\,h^8\,\mathrm{lam}\,\mathrm{nu}^4\,p^6 + h^4\,p^8 + 4\,h^8\,p^8 - 4\,h^8\,\mathrm{nu}^2\,p^8
\end{aligned}
$$

```
DN = DN /. lam -> L
l = DN /. {Plus -> List}
cc = l /. {p -> 1, h -> 1, L -> 1, m -> 1}
```

$$2 L^2 - 2 L^3 - L m^2 - 2 h^4 L m^2 + 3 L^2 m^2 + 2 h^4 L^2 m^2 + h^4 m^4 - L m^4 + 3 h^4 L m^4 + 2 h^4 L^2 m^4 - 2 h^4 m^6 - 3 h^4 L m^6 + h^4 m^8 + 2 L^2 nu - 2 L^3 nu - 2 h^4 L m^2 nu + 2 L^2 m^2 nu + 2 h^4 L^2 m^2 nu + 4 h^4 L m^4 nu + 2 h^4 L^2 m^4 nu - 2 h^4 L m^6 nu + 2 L^3 nu^2 - L^2 m^2 nu^2 - 2 h^4 L^2 m^2 nu^2 + h^4 L m^4 nu^2 - 2 h^4 L^2 m^4 nu^2 + h^4 L m^6 nu^2 + 2 L^3 nu^3 - 2 h^4 L^2 m^2 nu^3 - 2 h^4 L^2 m^4 nu^3 + 3 L p^2 + 4 h^4 L p^2 - 3 L^2 p^2 - 4 h^4 L^2 p^2 - 4 h^4 m^2 p^2 + 2 L m^2 p^2 - 4 h^4 L m^2 p^2 - 4 h^4 L^2 m^2 p^2 + 8 h^4 m^4 p^2 + 9 h^4 L m^4 p^2 - 4 h^4 m^6 p^2 + 2 L nu p^2 - 2 L^2 nu p^2 - 4 h^4 L m^2 nu p^2 - 4 h^4 L^2 m^2 nu p^2 + 6 h^4 L m^4 nu p^2 - 4 h^4 L nu^2 p^2 + L^2 nu^2 p^2 + 8 h^4 L^2 nu^2 p^2 + 2 h^4 L m^2 nu^2 p^2 + 4 h^4 L^2 m^2 nu^2 p^2 - 3 h^4 L m^4 nu^2 p^2 + 2 h^4 L m^2 nu^3 p^2 + 4 h^4 L^2 m^2 nu^3 p^2 - 4 h^4 L^2 nu^4 p^2 + p^4 + 4 h^4 p^4 - L p^4 - 4 h^4 L p^4 + 2 h^4 L^2 p^4 - 8 h^4 m^2 p^4 - 9 h^4 L m^2 p^4 - 2 h^8 L m^2 p^4 + 6 h^4 m^4 p^4 + h^8 m^4 p^4 + 2 h^4 L^2 nu p^4 - 6 h^4 L m^2 nu p^4 - 2 h^8 L m^2 nu p^4 - 4 h^4 nu^2 p^4 + 4 h^4 L nu^2 p^4 - 2 h^4 L^2 nu^2 p^4 + 2 h^4 m^2 nu^2 p^4 + 3 h^4 L m^2 nu^2 p^4 + 4 h^8 L m^2 nu^2 p^4 - 2 h^8 m^4 nu^2 p^4 - 2 h^4 L^2 nu^3 p^4 + 4 h^8 L m^2 nu^3 p^4 - 2 h^8 L m^2 nu^4 p^4 + h^8 m^4 nu^4 p^4 - 2 h^8 L m^2 nu^5 p^4 + 3 h^4 L p^6 + 4 h^8 L p^6 - 4 h^4 m^2 p^6 - 4 h^8 m^2 p^6 + 2 h^4 L nu p^6 - h^4 L nu^2 p^6 - 8 h^8 L nu^2 p^6 + 4 h^8 m^2 nu^2 p^6 + 4 h^8 L nu^4 p^6 + h^4 p^8 + 4 h^8 p^8 - 4 h^8 nu^2 p^8$$

$$\{2 L^2, -2 L^3, -L m^2, -2 h^4 L m^2, 3 L^2 m^2, 2 h^4 L^2 m^2, h^4 m^4, -L m^4, 3 h^4 L m^4, 2 h^4 L^2 m^4, -2 h^4 m^6, -3 h^4 L m^6, h^4 m^8, 2 L^2 nu, -2 L^3 nu, -2 h^4 L m^2 nu, 2 L^2 m^2 nu, 2 h^4 L^2 m^2 nu, 4 h^4 L m^4 nu, 2 h^4 L^2 m^4 nu, -2 h^4 L m^6 nu, 2 L^3 nu^2, -L^2 m^2 nu^2, -2 h^4 L^2 m^2 nu^2, h^4 L m^4 nu^2, -2 h^4 L^2 m^4 nu^2, h^4 L m^6 nu^2, 2 L^3 nu^3, -2 h^4 L^2 m^2 nu^3, -2 h^4 L^2 m^4 nu^3, 3 L p^2, 4 h^4 L p^2, -3 L^2 p^2, -4 h^4 L^2 p^2, -4 h^4 m^2 p^2, 2 L m^2 p^2, -4 h^4 L m^2 p^2, -4 h^4 L^2 m^2 p^2, 8 h^4 m^4 p^2, 9 h^4 L m^4 p^2, -4 h^4 m^6 p^2, 2 L nu p^2, -2 L^2 nu p^2, -4 h^4 L m^2 nu p^2, -4 h^4 L^2 m^2 nu p^2, 6 h^4 L m^4 nu p^2, -4 h^4 L nu^2 p^2, L^2 nu^2 p^2, 8 h^4 L^2 nu^2 p^2, 2 h^4 L m^2 nu^2 p^2, 4 h^4 L^2 m^2 nu^2 p^2, -3 h^4 L m^4 nu^2 p^2, 2 h^4 L m^2 nu^3 p^2, 4 h^4 L^2 m^2 nu^3 p^2, -4 h^4 L^2 nu^4 p^2, p^4, 4 h^4 p^4, -L p^4, -4 h^4 L p^4, 2 h^4 L^2 p^4, -8 h^4 m^2 p^4, -9 h^4 L m^2 p^4, -2 h^8 L m^2 p^4, 6 h^4 m^4 p^4, h^8 m^4 p^4, 2 h^4 L^2 nu p^4, -6 h^4 L m^2 nu p^4, -2 h^8 L m^2 nu p^4, -4 h^4 nu^2 p^4, 4 h^4 L nu^2 p^4, -2 h^4 L^2 nu^2 p^4, 2 h^4 m^2 nu^2 p^4, 3 h^4 L m^2 nu^2 p^4, 4 h^8 L m^2 nu^2 p^4, -2 h^8 m^4 nu^2 p^4, -2 h^4 L^2 nu^3 p^4, 4 h^8 L m^2 nu^3 p^4, -2 h^8 L m^2 nu^4 p^4, h^8 m^4 nu^4 p^4, -2 h^8 L m^2 nu^5 p^4, 3 h^4 L p^6, 4 h^8 L p^6, -4 h^4 m^2 p^6, -4 h^8 m^2 p^6, 2 h^4 L nu p^6, -h^4 L nu^2 p^6, -8 h^8 L nu^2 p^6, 4 h^8 m^2 nu^2 p^6, 4 h^8 L nu^4 p^6, h^4 p^8, 4 h^8 p^8, -4 h^8 nu^2 p^8\}$$

$$\{2, -2, -1, -2, 3, 2, 1, -1, 3, 2, -2, -3, 1, 2 nu, -2 nu, -2 nu, 2 nu, 2 nu, 4 nu, 2 nu, -2 nu, 2 nu^2, -nu^2, -2 nu^2, nu^2, -2 nu^2, nu^2, 2 nu^3, -2 nu^3, -2 nu^3, 3, 4, -3, -4, -4, 2, -4, -4, 8, 9, -4, 2 nu, -2 nu, -4 nu, -4 nu, 6 nu, -4 nu^2, nu^2, 8 nu^2, 2 nu^2, 4 nu^2, -3 nu^2, 2 nu^3, 4 nu^3, -4 nu^4, 1, 4, -1, -4, 2, -8, -9, -2, 6, 1, 2 nu, -6 nu, -2 nu, -4 nu^2, 4 nu^2, -2 nu^2, 2 nu^2, 3 nu^2, 4 nu^2, -2 nu^2, -2 nu^3, 4 nu^3, -2 nu^4, nu^4, -2 nu^5, 3, 4, -4, -4, 2 nu, -nu^2, -8 nu^2, 4 nu^2, 4 nu^4, 1, 4, -4 nu^2\}$$

```
A = Table[{cc[[i]], {Exponent[l[[i]], p],
    Exponent[l[[i]], h], Exponent[l[[i]], L], Exponent[l[[i]], m]}},
   {i, 1, Length[cc]}]
```

```
{{2, {0, 0, 2, 0}}, {-2, {0, 0, 3, 0}}, {-1, {0, 0, 1, 2}}, {-2, {0, 4, 1, 2}},
 {3, {0, 0, 2, 2}}, {2, {0, 4, 2, 2}}, {1, {0, 4, 0, 4}}, {-1, {0, 0, 1, 4}},
 {3, {0, 4, 1, 4}}, {2, {0, 4, 2, 4}}, {-2, {0, 4, 0, 6}}, {-3, {0, 4, 1, 6}},
 {1, {0, 4, 0, 8}}, {2 nu, {0, 0, 2, 0}}, {-2 nu, {0, 0, 3, 0}}, {-2 nu, {0, 4, 1, 2}},
 {2 nu, {0, 0, 2, 2}}, {2 nu, {0, 4, 2, 2}}, {4 nu, {0, 4, 1, 4}}, {2 nu, {0, 4, 2, 4}},
 {-2 nu, {0, 4, 1, 6}}, {2 nu^2, {0, 0, 3, 0}}, {-nu^2, {0, 0, 2, 2}}, {-2 nu^2, {0, 4, 2, 2}},
 {nu^2, {0, 4, 1, 4}}, {-2 nu^2, {0, 4, 2, 4}}, {nu^2, {0, 4, 1, 6}}, {2 nu^3, {0, 0, 3, 0}},
 {-2 nu^3, {0, 4, 2, 2}}, {-2 nu^3, {0, 4, 2, 4}}, {3, {2, 0, 1, 0}}, {4, {2, 4, 1, 0}},
 {-3, {2, 0, 2, 0}}, {-4, {2, 4, 2, 0}}, {-4, {2, 4, 0, 2}}, {2, {2, 0, 1, 2}},
 {-4, {2, 4, 1, 2}}, {-4, {2, 4, 2, 2}}, {8, {2, 4, 0, 4}}, {9, {2, 4, 1, 4}},
 {-4, {2, 4, 0, 6}}, {2 nu, {2, 0, 1, 0}}, {-2 nu, {2, 0, 2, 0}}, {-4 nu, {2, 4, 1, 2}},
 {-4 nu, {2, 4, 2, 2}}, {6 nu, {2, 4, 1, 4}}, {-4 nu^2, {2, 4, 1, 0}}, {nu^2, {2, 0, 2, 0}},
 {8 nu^2, {2, 4, 2, 0}}, {2 nu^2, {2, 4, 1, 2}}, {4 nu^2, {2, 4, 2, 2}}, {-3 nu^2, {2, 4, 1, 4}},
 {2 nu^3, {2, 4, 1, 2}}, {4 nu^3, {2, 4, 2, 2}}, {-4 nu^4, {2, 4, 2, 0}}, {1, {4, 0, 0, 0}},
 {4, {4, 4, 0, 0}}, {-1, {4, 0, 1, 0}}, {-4, {4, 4, 1, 0}}, {2, {4, 4, 2, 0}},
 {-8, {4, 4, 0, 2}}, {-9, {4, 4, 1, 2}}, {-2, {4, 8, 1, 2}}, {6, {4, 4, 0, 4}},
 {1, {4, 8, 0, 4}}, {2 nu, {4, 4, 2, 0}}, {-6 nu, {4, 4, 1, 2}}, {-2 nu, {4, 8, 1, 2}},
 {-4 nu^2, {4, 4, 0, 0}}, {4 nu^2, {4, 4, 1, 0}}, {-2 nu^2, {4, 4, 2, 0}}, {2 nu^2, {4, 4, 0, 2}},
 {3 nu^2, {4, 4, 1, 2}}, {4 nu^2, {4, 8, 1, 2}}, {-2 nu^2, {4, 8, 0, 4}}, {-2 nu^3, {4, 4, 2, 0}},
 {4 nu^3, {4, 8, 1, 2}}, {-2 nu^4, {4, 8, 1, 2}}, {nu^4, {4, 8, 0, 4}}, {-2 nu^5, {4, 8, 1, 2}},
 {3, {6, 4, 1, 0}}, {4, {6, 8, 1, 0}}, {-4, {6, 4, 0, 2}}, {-4, {6, 8, 0, 2}},
 {2 nu, {6, 4, 1, 0}}, {-nu^2, {6, 4, 1, 0}}, {-8 nu^2, {6, 8, 1, 0}}, {4 nu^2, {6, 8, 0, 2}},
 {4 nu^4, {6, 8, 1, 0}}, {1, {8, 4, 0, 0}}, {4, {8, 8, 0, 0}}, {-4 nu^2, {8, 8, 0, 0}}}
```

```
l1 = Union[Transpose[A][[2]]]
```

```
{{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0}, {0, 4, 0, 4},
 {0, 4, 0, 6}, {0, 4, 0, 8}, {0, 4, 1, 2}, {0, 4, 1, 4}, {0, 4, 1, 6}, {0, 4, 2, 2},
 {0, 4, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 4, 0, 2}, {2, 4, 0, 4},
 {2, 4, 0, 6}, {2, 4, 1, 0}, {2, 4, 1, 2}, {2, 4, 1, 4}, {2, 4, 2, 0}, {2, 4, 2, 2},
 {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 4, 0, 0}, {4, 4, 0, 2}, {4, 4, 0, 4}, {4, 4, 1, 0},
 {4, 4, 1, 2}, {4, 4, 2, 0}, {4, 8, 0, 4}, {4, 8, 1, 2}, {6, 4, 0, 2}, {6, 4, 1, 0},
 {6, 8, 0, 2}, {6, 8, 1, 0}, {8, 4, 0, 0}, {8, 8, 0, 0}}
```

```
lll = Table[Select[A, #[[2]] == ll[[i]]&], {i, 1, Length[ll]}]
```

```
{{{-1, {0, 0, 1, 2}}}, {{-1, {0, 0, 1, 4}}}, {{2, {0, 0, 2, 0}}, {2 nu, {0, 0, 2, 0}}},
 {{3, {0, 0, 2, 2}}, {2 nu, {0, 0, 2, 2}}, {-nu^2, {0, 0, 2, 2}}}, {{-2, {0, 0, 3, 0}},
  {-2 nu, {0, 0, 3, 0}}, {2 nu^2, {0, 0, 3, 0}}, {2 nu^3, {0, 0, 3, 0}}}, {{1, {0, 4, 0, 4}}},
 {{-2, {0, 4, 0, 6}}}, {{1, {0, 4, 0, 8}}}, {{-2, {0, 4, 1, 2}}, {-2 nu, {0, 4, 1, 2}}},
 {{3, {0, 4, 1, 4}}, {4 nu, {0, 4, 1, 4}}, {nu^2, {0, 4, 1, 4}}},
 {{-3, {0, 4, 1, 6}}, {-2 nu, {0, 4, 1, 6}}, {nu^2, {0, 4, 1, 6}}},
 {{2, {0, 4, 2, 2}}, {2 nu, {0, 4, 2, 2}}, {-2 nu^2, {0, 4, 2, 2}}, {-2 nu^3, {0, 4, 2, 2}}},
 {{2, {0, 4, 2, 4}}, {2 nu, {0, 4, 2, 4}}, {-2 nu^2, {0, 4, 2, 4}}, {-2 nu^3, {0, 4, 2, 4}}},
 {{3, {2, 0, 1, 0}}, {2 nu, {2, 0, 1, 0}}}, {{2, {2, 0, 1, 2}}},
 {{-3, {2, 0, 2, 0}}, {-2 nu, {2, 0, 2, 0}}, {nu^2, {2, 0, 2, 0}}}, {{-4, {2, 4, 0, 2}}},
 {{8, {2, 4, 0, 4}}}, {{-4, {2, 4, 0, 6}}}, {{4, {2, 4, 1, 0}}, {-4 nu^2, {2, 4, 1, 0}}},
 {{-4, {2, 4, 1, 2}}, {-4 nu, {2, 4, 1, 2}}, {2 nu^2, {2, 4, 1, 2}}, {2 nu^3, {2, 4, 1, 2}}},
 {{9, {2, 4, 1, 4}}, {6 nu, {2, 4, 1, 4}}, {-3 nu^2, {2, 4, 1, 4}}},
 {{-4, {2, 4, 2, 0}}, {8 nu^2, {2, 4, 2, 0}}, {-4 nu^4, {2, 4, 2, 0}}},
 {{-4, {2, 4, 2, 2}}, {-4 nu, {2, 4, 2, 2}}, {4 nu^2, {2, 4, 2, 2}}, {4 nu^3, {2, 4, 2, 2}}},
 {{1, {4, 0, 0, 0}}}, {{-1, {4, 0, 1, 0}}},
 {{4, {4, 4, 0, 0}}, {-4 nu^2, {4, 4, 0, 0}}}, {{-8, {4, 4, 0, 2}}, {2 nu^2, {4, 4, 0, 2}}},
 {{6, {4, 4, 0, 4}}}, {{-4, {4, 4, 1, 0}}, {4 nu^2, {4, 4, 1, 0}}},
 {{-9, {4, 4, 1, 2}}, {-6 nu, {4, 4, 1, 2}}, {3 nu^2, {4, 4, 1, 2}}},
 {{2, {4, 4, 2, 0}}, {2 nu, {4, 4, 2, 0}}, {-2 nu^2, {4, 4, 2, 0}}, {-2 nu^3, {4, 4, 2, 0}}},
 {{1, {4, 8, 0, 4}}, {-2 nu^2, {4, 8, 0, 4}}, {nu^4, {4, 8, 0, 4}}},
 {{-2, {4, 8, 1, 2}}, {-2 nu, {4, 8, 1, 2}}, {4 nu^2, {4, 8, 1, 2}},
  {4 nu^3, {4, 8, 1, 2}}, {-2 nu^4, {4, 8, 1, 2}}, {-2 nu^5, {4, 8, 1, 2}}},
 {{-4, {6, 4, 0, 2}}}, {{3, {6, 4, 1, 0}}, {2 nu, {6, 4, 1, 0}}, {-nu^2, {6, 4, 1, 0}}},
 {{-4, {6, 8, 0, 2}}, {4 nu^2, {6, 8, 0, 2}}},
 {{4, {6, 8, 1, 0}}, {-8 nu^2, {6, 8, 1, 0}}, {4 nu^4, {6, 8, 1, 0}}}, {{1, {8, 4, 0, 0}}},
 {{4, {8, 8, 0, 0}}, {-4 nu^2, {8, 8, 0, 0}}}}
```

```
r = Factor[
  Table[{Sum[lll[[i]][[j]][[1]], {j, 1, Length[lll[[i]]]}], lll[[i]][[1]][[2]]},
   {i, 1, Length[lll]}]]
```

```
{{-1, {0, 0, 1, 2}}, {-1, {0, 0, 1, 4}},
 {2 (1 + nu), {0, 0, 2, 0}}, {-(-3 + nu) (1 + nu), {0, 0, 2, 2}},
 {2 (-1 + nu) (1 + nu)^2, {0, 0, 3, 0}}, {1, {0, 4, 0, 4}}, {-2, {0, 4, 0, 6}},
 {1, {0, 4, 0, 8}}, {-2 (1 + nu), {0, 4, 1, 2}}, {(1 + nu) (3 + nu), {0, 4, 1, 4}},
 {(-3 + nu) (1 + nu), {0, 4, 1, 6}}, {-2 (-1 + nu) (1 + nu)^2, {0, 4, 2, 2}},
 {-2 (-1 + nu) (1 + nu)^2, {0, 4, 2, 4}}, {3 + 2 nu, {2, 0, 1, 0}},
 {2, {2, 0, 1, 2}}, {(-3 + nu) (1 + nu), {2, 0, 2, 0}}, {-4, {2, 4, 0, 2}},
 {8, {2, 4, 0, 4}}, {-4, {2, 4, 0, 6}}, {-4 (-1 + nu) (1 + nu), {2, 4, 1, 0}},
 {2 (1 + nu) (-2 + nu^2), {2, 4, 1, 2}}, {-3 (-3 + nu) (1 + nu), {2, 4, 1, 4}},
 {-4 (-1 + nu)^2 (1 + nu)^2, {2, 4, 2, 0}}, {4 (-1 + nu) (1 + nu)^2, {2, 4, 2, 2}},
 {1, {4, 0, 0, 0}}, {-1, {4, 0, 1, 0}}, {-4 (-1 + nu) (1 + nu), {4, 4, 0, 0}},
 {2 (-2 + nu) (2 + nu), {4, 4, 0, 2}}, {6, {4, 4, 0, 4}}, {4 (-1 + nu) (1 + nu), {4, 4, 1, 0}},
 {3 (-3 + nu) (1 + nu), {4, 4, 1, 2}}, {-2 (-1 + nu) (1 + nu)^2, {4, 4, 2, 0}},
 {(-1 + nu)^2 (1 + nu)^2, {4, 8, 0, 4}}, {-2 (-1 + nu)^2 (1 + nu)^3, {4, 8, 1, 2}},
 {-4, {6, 4, 0, 2}}, {-(-3 + nu) (1 + nu), {6, 4, 1, 0}},
 {4 (-1 + nu) (1 + nu), {6, 8, 0, 2}}, {4 (-1 + nu)^2 (1 + nu)^2, {6, 8, 1, 0}},
 {1, {8, 4, 0, 0}}, {-4 (-1 + nu) (1 + nu), {8, 8, 0, 0}}}
```

```
r1 = Transpose[r] [[1]]
r2 = Transpose[r] [[2]]
```

```
{-1, -1, 2 (1 + nu), -(-3 + nu) (1 + nu), 2 (-1 + nu) (1 + nu)^2, 1,
 -2, 1, -2 (1 + nu), (1 + nu) (3 + nu), (-3 + nu) (1 + nu), -2 (-1 + nu) (1 + nu)^2,
 -2 (-1 + nu) (1 + nu)^2, 3 + 2 nu, 2, (-3 + nu) (1 + nu), -4, 8, -4, -4 (-1 + nu) (1 + nu),
 2 (1 + nu) (-2 + nu^2), -3 (-3 + nu) (1 + nu), -4 (-1 + nu)^2 (1 + nu)^2, 4 (-1 + nu) (1 + nu)^2,
 1, -1, -4 (-1 + nu) (1 + nu), 2 (-2 + nu) (2 + nu), 6, 4 (-1 + nu) (1 + nu),
 3 (-3 + nu) (1 + nu), -2 (-1 + nu) (1 + nu)^2, (-1 + nu)^2 (1 + nu)^2, -2 (-1 + nu)^2 (1 + nu)^3,
 -4, -(-3 + nu) (1 + nu), 4 (-1 + nu) (1 + nu), 4 (-1 + nu)^2 (1 + nu)^2, 1, -4 (-1 + nu) (1 + nu)}
```

```
{{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0}, {0, 4, 0, 4},
 {0, 4, 0, 6}, {0, 4, 0, 8}, {0, 4, 1, 2}, {0, 4, 1, 4}, {0, 4, 1, 6}, {0, 4, 2, 2},
 {0, 4, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 4, 0, 2}, {2, 4, 0, 4},
 {2, 4, 0, 6}, {2, 4, 1, 0}, {2, 4, 1, 2}, {2, 4, 1, 4}, {2, 4, 2, 0}, {2, 4, 2, 2},
 {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 4, 0, 0}, {4, 4, 0, 2}, {4, 4, 0, 4}, {4, 4, 1, 0},
 {4, 4, 1, 2}, {4, 4, 2, 0}, {4, 8, 0, 4}, {4, 8, 1, 2}, {6, 4, 0, 2}, {6, 4, 1, 0},
 {6, 8, 0, 2}, {6, 8, 1, 0}, {8, 4, 0, 0}, {8, 8, 0, 0}}
```

```
BB = A[[ {35, 1, 14, 31, 42, 56}]]
(*{2,15,22,28,33,43,48,58,60,66,71,76,81,85,86,90}
  {8,5,17,23,2,15,22,28,36,33,43,48,58}
  {3,1,14,31,42,56}
  {13,12,21,27,36,57,69,58,81,85,86,90}
*)
```

```
{{-4, {2, 4, 0, 2}}, {2, {0, 0, 2, 0}}, {2 nu, {0, 0, 2, 0}}, {3, {2, 0, 1, 0}},
 {2 nu, {2, 0, 1, 0}}, {1, {4, 0, 0, 0}}}
```

```
coef = r1 = Transpose[BB] [[1]]
PPP = r2 = Transpose[BB] [[2]]
```

```
{-4, 2, 2 nu, 3, 2 nu, 1}
```

```
{{2, 4, 0, 2}, {0, 0, 2, 0}, {0, 0, 2, 0}, {2, 0, 1, 0}, {2, 0, 1, 0}, {4, 0, 0, 0}}
```

```
s1 = Table[
 p^PPP[[i]] [[1]] *h^PPP[[i]] [[2]] *
   L^PPP[[i]] [[3]] *m^PPP[[i]] [[4]], {i, 1, Length[PPP]}]
eq = s1 . coef
```

$\{h^4 m^2 p^2, L^2, L^2, L p^2, L p^2, p^4\}$

$2 L^2 + 2 L^2 nu + 3 L p^2 - 4 h^4 m^2 p^2 + 2 L nu p^2 + p^4$

```
eee = Expand[(p^2 + m^2)^4]
Expand[(p^2 - 2 (1 + nu) L) * (p^2 - (1 - nu^2) L)]
```

$m^8 + 4\, m^6\, p^2 + 6\, m^4\, p^4 + 4\, m^2\, p^6 + p^8$

$2\, L^2 + 2\, L^2\, nu - 2\, L^2\, nu^2 - 2\, L^2\, nu^3 - 3\, L\, p^2 - 2\, L\, nu\, p^2 + L\, nu^2\, p^2 + p^4$

```
Solve[eee == 0, {p}]
```

{{p → -I m}, {p → -I m}, {p → -I m}, {p → -I m}, {p → I m}, {p → I m}, {p → I m}, {p → I m}}

APPENDIX 6

Mathematica 3.0 Code

for Construction of Short Form of Equation

In[1]:=

```
<< DiscreteMath`ComputationalGeometry`
```

In[2]:=

```
Clear[A, a, b, a11, a12, a13, a21, a22, a23, a31, a32, a33, DDD, mu, m];
mu = Sqrt[h * Sqrt[1 - nu^2]];
a[m_, p_, lam_] := -2 * (1 - nu) p^2 + m^2 - (1 - nu^2) * lam * mu^(-4)
   b[m_, p_, lam_] := -(2 - nu) * p^2 * m + m^3
a11[m_, p_, lam_] := -p^2 + (1 - nu) / 2 * m^2 - (1 - nu^2) * lam
a21[m_, p_, lam_] := -(1 + nu) / 2 * p * m
 a12[m_, p_, lam_] := (1 + nu) / 2 * p * m
a31[m_, p_, lam_] := -nu * p
 a13[m_, p_, lam_] := nu * p
a22[m_, p_, lam_] := -(1 - nu) / 2 * p^2 + m^2 + a[m, p, lam] * mu^4
a32[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
 a23[m_, p_, lam_] := (m + b[m, p, lam] * mu^4)
a33[m_, p_, lam_] := 1 - (1 - nu^2) * lam + mu^4 * (p^2 - m^2)^2
```

In[13]:=

$$\mathtt{DDD[m\_,\ p\_,\ lam\_]\ :=\ Det}\left[\begin{pmatrix} \mathtt{a11[m,\ p,\ lam]} & \mathtt{a12[m,\ p,\ lam]} & \mathtt{a13[m,\ p,\ lam]} \\ \mathtt{a21[m,\ p,\ lam]} & \mathtt{a22[m,\ p,\ lam]} & \mathtt{a23[m,\ p,\ lam]} \\ \mathtt{a31[m,\ p,\ lam]} & \mathtt{a32[m,\ p,\ lam]} & \mathtt{a33[m,\ p,\ lam]} \end{pmatrix}\right]$$

```
DN = Expand[Factor[DDD[m, p, lam] * 2 / (1 - nu^2) / (1 - nu)]]
```

Out[14]=

$$\begin{aligned}
&2\,\mathrm{lam}^2 - 2\,\mathrm{lam}^3 - \mathrm{lam}\,m^2 - 2\,h^2\,\mathrm{lam}\,m^2 + 3\,\mathrm{lam}^2\,m^2 + 2\,h^2\,\mathrm{lam}^2\,m^2 + h^2\,m^4 - \mathrm{lam}\,m^4 + 3\,h^2\,\mathrm{lam}\,m^4 + \\
&2\,h^2\,\mathrm{lam}^2\,m^4 - 2\,h^2\,m^6 - 3\,h^2\,\mathrm{lam}\,m^6 + h^2\,m^8 + 2\,\mathrm{lam}^2\,\mathrm{nu} - 2\,\mathrm{lam}^3\,\mathrm{nu} - 2\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu} + 2\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + \\
&2\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu} + 4\,h^2\,\mathrm{lam}\,m^4\,\mathrm{nu} + 2\,h^2\,\mathrm{lam}^2\,m^4\,\mathrm{nu} - 2\,h^2\,\mathrm{lam}\,m^6\,\mathrm{nu} + 2\,\mathrm{lam}^3\,\mathrm{nu}^2 - \mathrm{lam}^2\,m^2\,\mathrm{nu}^2 - \\
&2\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2 + h^2\,\mathrm{lam}\,m^4\,\mathrm{nu}^2 - 2\,h^2\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^2 + h^2\,\mathrm{lam}\,m^6\,\mathrm{nu}^2 + 2\,\mathrm{lam}^3\,\mathrm{nu}^3 - 2\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3 - \\
&2\,h^2\,\mathrm{lam}^2\,m^4\,\mathrm{nu}^3 + 3\,\mathrm{lam}\,p^2 + 4\,h^2\,\mathrm{lam}\,p^2 - 3\,\mathrm{lam}^2\,p^2 - 4\,h^2\,\mathrm{lam}^2\,p^2 - 4\,h^2\,m^2\,p^2 + \\
&2\,\mathrm{lam}\,m^2\,p^2 - 4\,h^2\,\mathrm{lam}\,m^2\,p^2 - 4\,h^2\,\mathrm{lam}^2\,m^2\,p^2 + 8\,h^2\,m^4\,p^2 + 9\,h^2\,\mathrm{lam}\,m^4\,p^2 - 4\,h^2\,m^6\,p^2 + \\
&2\,\mathrm{lam}\,\mathrm{nu}\,p^2 - 2\,\mathrm{lam}^2\,\mathrm{nu}\,p^2 - 4\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^2 - 4\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu}\,p^2 + 6\,h^2\,\mathrm{lam}\,m^4\,\mathrm{nu}\,p^2 - \\
&4\,h^2\,\mathrm{lam}\,\mathrm{nu}^2\,p^2 + \mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + 8\,h^2\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^2 + 2\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^2 + 4\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^2\,p^2 - \\
&3\,h^2\,\mathrm{lam}\,m^4\,\mathrm{nu}^2\,p^2 + 2\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^2 + 4\,h^2\,\mathrm{lam}^2\,m^2\,\mathrm{nu}^3\,p^2 - 4\,h^2\,\mathrm{lam}^2\,\mathrm{nu}^4\,p^2 + p^4 + \\
&4\,h^2\,p^4 - \mathrm{lam}\,p^4 - 4\,h^2\,\mathrm{lam}\,p^4 + 2\,h^2\,\mathrm{lam}^2\,p^4 - 8\,h^2\,m^2\,p^4 - 9\,h^2\,\mathrm{lam}\,m^2\,p^4 - 2\,h^4\,\mathrm{lam}\,m^2\,p^4 + \\
&6\,h^2\,m^4\,p^4 + h^4\,m^4\,p^4 + 2\,h^2\,\mathrm{lam}^2\,\mathrm{nu}\,p^4 - 6\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}\,p^4 - 4\,h^2\,\mathrm{nu}^2\,p^4 + \\
&4\,h^2\,\mathrm{lam}\,\mathrm{nu}^2\,p^4 - 2\,h^2\,\mathrm{lam}^2\,\mathrm{nu}^2\,p^4 + 2\,h^2\,m^2\,\mathrm{nu}^2\,p^4 + 3\,h^2\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 + 4\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^2\,p^4 - \\
&2\,h^4\,m^4\,\mathrm{nu}^2\,p^4 - 2\,h^2\,\mathrm{lam}^2\,\mathrm{nu}^3\,p^4 + 4\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^3\,p^4 - 2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^4\,p^4 + h^4\,m^4\,\mathrm{nu}^4\,p^4 - \\
&2\,h^4\,\mathrm{lam}\,m^2\,\mathrm{nu}^5\,p^4 + 3\,h^2\,\mathrm{lam}\,p^6 + 4\,h^4\,\mathrm{lam}\,p^6 - 4\,h^2\,m^2\,p^6 - 4\,h^4\,m^2\,p^6 + 2\,h^2\,\mathrm{lam}\,\mathrm{nu}\,p^6 - \\
&h^2\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 - 8\,h^4\,\mathrm{lam}\,\mathrm{nu}^2\,p^6 + 4\,h^4\,m^2\,\mathrm{nu}^2\,p^6 + 4\,h^4\,\mathrm{lam}\,\mathrm{nu}^4\,p^6 + h^2\,p^8 + 4\,h^4\,p^8 - 4\,h^4\,\mathrm{nu}^2\,p^8
\end{aligned}$$

In[15]:=

```
DN = DN /. lam -> L
l = DN /. {Plus -> List}
cc = l /. {p -> 1, h -> 1, L -> 1, m -> 1}
```

Out[15]=

$$2 L^2 - 2 L^3 - L m^2 - 2 h^2 L m^2 + 3 L^2 m^2 + 2 h^2 L^2 m^2 + h^2 m^4 - L m^4 + 3 h^2 L m^4 + 2 h^2 L^2 m^4 - 2 h^2 m^6 -$$
$$3 h^2 L m^6 + h^2 m^8 + 2 L^2 nu - 2 L^3 nu - 2 h^2 L m^2 nu + 2 L^2 m^2 nu + 2 h^2 L^2 m^2 nu + 4 h^2 L m^4 nu +$$
$$2 h^2 L^2 m^4 nu - 2 h^2 L m^6 nu + 2 L^3 nu^2 - L^2 m^2 nu^2 - 2 h^2 L^2 m^2 nu^2 + h^2 L m^4 nu^2 - 2 h^2 L^2 m^4 nu^2 +$$
$$h^2 L m^6 nu^2 + 2 L^3 nu^3 - 2 h^2 L^2 m^2 nu^3 - 2 h^2 L^2 m^4 nu^3 + 3 L p^2 + 4 h^2 L p^2 - 3 L^2 p^2 - 4 h^2 L^2 p^2 -$$
$$4 h^2 m^2 p^2 + 2 L m^2 p^2 - 4 h^2 L m^2 p^2 - 4 h^2 L^2 m^2 p^2 + 8 h^2 m^4 p^2 + 9 h^2 L m^4 p^2 - 4 h^2 m^6 p^2 + 2 L nu p^2 -$$
$$2 L^2 nu p^2 - 4 h^2 L m^2 nu p^2 - 4 h^2 L^2 m^2 nu p^2 + 6 h^2 L m^4 nu p^2 - 4 h^2 L nu^2 p^2 + L^2 nu^2 p^2 +$$
$$8 h^2 L^2 nu^2 p^2 + 2 h^2 L m^2 nu^2 p^2 + 4 h^2 L^2 m^2 nu^2 p^2 - 3 h^2 L m^4 nu^2 p^2 + 2 h^2 L m^2 nu^3 p^2 +$$
$$4 h^2 L^2 m^2 nu^3 p^2 - 4 h^2 L^2 nu^4 p^2 + p^4 + 4 h^2 p^4 - L p^4 - 4 h^2 L p^4 + 2 h^2 L^2 p^4 - 8 h^2 m^2 p^4 - 9 h^2 L m^2 p^4 -$$
$$2 h^4 L m^2 p^4 + 6 h^2 m^4 p^4 + h^4 m^4 p^4 + 2 h^2 L^2 nu p^4 - 6 h^2 L m^2 nu p^4 - 2 h^4 L m^2 nu p^4 - 4 h^2 nu^2 p^4 +$$
$$4 h^2 L nu^2 p^4 - 2 h^2 L^2 nu^2 p^4 + 2 h^2 m^2 nu^2 p^4 + 3 h^2 L m^2 nu^2 p^4 + 4 h^4 L m^2 nu^2 p^4 - 2 h^4 m^4 nu^2 p^4 -$$
$$2 h^2 L^2 nu^3 p^4 + 4 h^4 L m^2 nu^3 p^4 - 2 h^4 L m^2 nu^4 p^4 + h^4 m^4 nu^4 p^4 - 2 h^4 L m^2 nu^5 p^4 +$$
$$3 h^2 L p^6 + 4 h^4 L p^6 - 4 h^2 m^2 p^6 - 4 h^4 m^2 p^6 + 2 h^2 L nu p^6 - h^2 L nu^2 p^6 - 8 h^4 L nu^2 p^6 +$$
$$4 h^4 m^2 nu^2 p^6 + 4 h^4 L nu^4 p^6 + h^2 p^8 + 4 h^4 p^8 - 4 h^4 nu^2 p^8$$

Out[16]=

$$\{2 L^2, -2 L^3, -L m^2, -2 h^2 L m^2, 3 L^2 m^2, 2 h^2 L^2 m^2, h^2 m^4, -L m^4, 3 h^2 L m^4, 2 h^2 L^2 m^4, -2 h^2 m^6,$$
$$-3 h^2 L m^6, h^2 m^8, 2 L^2 nu, -2 L^3 nu, -2 h^2 L m^2 nu, 2 L^2 m^2 nu, 2 h^2 L^2 m^2 nu, 4 h^2 L m^4 nu,$$
$$2 h^2 L^2 m^4 nu, -2 h^2 L m^6 nu, 2 L^3 nu^2, -L^2 m^2 nu^2, -2 h^2 L^2 m^2 nu^2, h^2 L m^4 nu^2, -2 h^2 L^2 m^4 nu^2,$$
$$h^2 L m^6 nu^2, 2 L^3 nu^3, -2 h^2 L^2 m^2 nu^3, -2 h^2 L^2 m^4 nu^3, 3 L p^2, 4 h^2 L p^2, -3 L^2 p^2, -4 h^2 L^2 p^2,$$
$$-4 h^2 m^2 p^2, 2 L m^2 p^2, -4 h^2 L m^2 p^2, -4 h^2 L^2 m^2 p^2, 8 h^2 m^4 p^2, 9 h^2 L m^4 p^2, -4 h^2 m^6 p^2, 2 L nu p^2,$$
$$-2 L^2 nu p^2, -4 h^2 L m^2 nu p^2, -4 h^2 L^2 m^2 nu p^2, 6 h^2 L m^4 nu p^2, -4 h^2 L nu^2 p^2, L^2 nu^2 p^2,$$
$$8 h^2 L^2 nu^2 p^2, 2 h^2 L m^2 nu^2 p^2, 4 h^2 L^2 m^2 nu^2 p^2, -3 h^2 L m^4 nu^2 p^2, 2 h^2 L m^2 nu^3 p^2,$$
$$4 h^2 L^2 m^2 nu^3 p^2, -4 h^2 L^2 nu^4 p^2, p^4, 4 h^2 p^4, -L p^4, -4 h^2 L p^4, 2 h^2 L^2 p^4, -8 h^2 m^2 p^4,$$
$$-9 h^2 L m^2 p^4, -2 h^4 L m^2 p^4, 6 h^2 m^4 p^4, h^4 m^4 p^4, 2 h^2 L^2 nu p^4, -6 h^2 L m^2 nu p^4, -2 h^4 L m^2 nu p^4,$$
$$-4 h^2 nu^2 p^4, 4 h^2 L nu^2 p^4, -2 h^2 L^2 nu^2 p^4, 2 h^2 m^2 nu^2 p^4, 3 h^2 L m^2 nu^2 p^4, 4 h^4 L m^2 nu^2 p^4,$$
$$-2 h^4 m^4 nu^2 p^4, -2 h^2 L^2 nu^3 p^4, 4 h^4 L m^2 nu^3 p^4, -2 h^4 L m^2 nu^4 p^4, h^4 m^4 nu^4 p^4, -2 h^4 L m^2 nu^5 p^4,$$
$$3 h^2 L p^6, 4 h^4 L p^6, -4 h^2 m^2 p^6, -4 h^4 m^2 p^6, 2 h^2 L nu p^6, -h^2 L nu^2 p^6, -8 h^4 L nu^2 p^6,$$
$$4 h^4 m^2 nu^2 p^6, 4 h^4 L nu^4 p^6, h^2 p^8, 4 h^4 p^8, -4 h^4 nu^2 p^8\}$$

Out[17]=

$$\{2, -2, -1, -2, 3, 2, 1, -1, 3, 2, -2, -3, 1, 2 nu, -2 nu, -2 nu, 2 nu, 2 nu, 4 nu,$$
$$2 nu, -2 nu, 2 nu^2, -nu^2, -2 nu^2, nu^2, -2 nu^2, nu^2, 2 nu^3, -2 nu^3, -2 nu^3, 3, 4,$$
$$-3, -4, -4, 2, -4, -4, 8, 9, -4, 2 nu, -2 nu, -4 nu, -4 nu, 6 nu, -4 nu^2, nu^2,$$
$$8 nu^2, 2 nu^2, 4 nu^2, -3 nu^2, 2 nu^3, 4 nu^3, -4 nu^4, 1, 4, -1, -4, 2, -8, -9, -2, 6,$$
$$1, 2 nu, -6 nu, -2 nu, -4 nu^2, 4 nu^2, -2 nu^2, 2 nu^2, 3 nu^2, 4 nu^2, -2 nu^2, -2 nu^3,$$
$$4 nu^3, -2 nu^4, nu^4, -2 nu^5, 3, 4, -4, -4, 2 nu, -nu^2, -8 nu^2, 4 nu^2, 4 nu^4, 1, 4, -4 nu^2\}$$

```
In[18]:= A = Table[{cc[[i]], {Exponent[l[[i]], p],
            Exponent[l[[i]], h], Exponent[l[[i]], L], Exponent[l[[i]], m]}},
          {i, 1, Length[cc]}]

Out[18]= {{2, {0, 0, 2, 0}}, {-2, {0, 0, 3, 0}}, {-1, {0, 0, 1, 2}}, {-2, {0, 2, 1, 2}},
          {3, {0, 0, 2, 2}}, {2, {0, 2, 2, 2}}, {1, {0, 2, 0, 4}}, {-1, {0, 0, 1, 4}},
          {3, {0, 2, 1, 4}}, {2, {0, 2, 2, 4}}, {-2, {0, 2, 0, 6}}, {-3, {0, 2, 1, 6}},
          {1, {0, 2, 0, 8}}, {2 nu, {0, 0, 2, 0}}, {-2 nu, {0, 0, 3, 0}}, {-2 nu, {0, 2, 1, 2}},
          {2 nu, {0, 0, 2, 2}}, {2 nu, {0, 2, 2, 2}}, {4 nu, {0, 2, 1, 4}}, {2 nu, {0, 2, 2, 4}},
          {-2 nu, {0, 2, 1, 6}}, {2 nu^2, {0, 0, 3, 0}}, {-nu^2, {0, 0, 2, 2}}, {-2 nu^2, {0, 2, 2, 2}},
          {nu^2, {0, 2, 1, 4}}, {-2 nu^2, {0, 2, 2, 4}}, {nu^2, {0, 2, 1, 6}}, {2 nu^3, {0, 0, 3, 0}},
          {-2 nu^3, {0, 2, 2, 2}}, {-2 nu^3, {0, 2, 2, 4}}, {3, {2, 0, 1, 0}}, {4, {2, 2, 1, 0}},
          {-3, {2, 0, 2, 0}}, {-4, {2, 2, 2, 0}}, {-4, {2, 2, 0, 2}}, {2, {2, 0, 1, 2}},
          {-4, {2, 2, 1, 2}}, {-4, {2, 2, 2, 2}}, {8, {2, 2, 0, 4}}, {9, {2, 2, 1, 4}},
          {-4, {2, 2, 0, 6}}, {2 nu, {2, 0, 1, 0}}, {-2 nu, {2, 0, 2, 0}}, {-4 nu, {2, 2, 1, 2}},
          {-4 nu, {2, 2, 2, 2}}, {6 nu, {2, 2, 1, 4}}, {-4 nu^2, {2, 2, 1, 0}}, {nu^2, {2, 0, 2, 0}},
          {8 nu^2, {2, 2, 2, 0}}, {2 nu^2, {2, 2, 1, 2}}, {4 nu^2, {2, 2, 2, 2}}, {-3 nu^2, {2, 2, 1, 4}},
          {2 nu^3, {2, 2, 1, 2}}, {4 nu^3, {2, 2, 2, 2}}, {-4 nu^4, {2, 2, 2, 0}}, {1, {4, 0, 0, 0}},
          {4, {4, 2, 0, 0}}, {-1, {4, 0, 1, 0}}, {-4, {4, 2, 1, 0}}, {2, {4, 2, 2, 0}},
          {-8, {4, 2, 0, 2}}, {-9, {4, 2, 1, 2}}, {-2, {4, 4, 1, 2}}, {6, {4, 2, 0, 4}},
          {1, {4, 4, 0, 4}}, {2 nu, {4, 2, 2, 0}}, {-6 nu, {4, 2, 1, 2}}, {-2 nu, {4, 4, 1, 2}},
          {-4 nu^2, {4, 2, 0, 0}}, {4 nu^2, {4, 2, 1, 0}}, {-2 nu^2, {4, 2, 2, 0}},
          {2 nu^2, {4, 2, 0, 2}}, {3 nu^2, {4, 2, 1, 2}}, {4 nu^2, {4, 4, 1, 2}}, {-2 nu^2, {4, 4, 0, 4}},
          {-2 nu^3, {4, 2, 2, 0}}, {4 nu^3, {4, 4, 1, 2}}, {-2 nu^4, {4, 4, 1, 2}}, {nu^4, {4, 4, 0, 4}},
          {-2 nu^5, {4, 4, 1, 2}}, {3, {6, 2, 1, 0}}, {4, {6, 4, 1, 0}}, {-4, {6, 2, 0, 2}},
          {-4, {6, 4, 0, 2}}, {2 nu, {6, 2, 1, 0}}, {-nu^2, {6, 2, 1, 0}}, {-8 nu^2, {6, 4, 1, 0}},
          {4 nu^2, {6, 4, 0, 2}}, {4 nu^4, {6, 4, 1, 0}}, {1, {8, 2, 0, 0}}, {4, {8, 4, 0, 0}},
          {-4 nu^2, {8, 4, 0, 0}}}

In[19]:= l1 = Union[Transpose[A][[2]]]

Out[19]= {{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0}, {0, 2, 0, 4},
          {0, 2, 0, 6}, {0, 2, 0, 8}, {0, 2, 1, 2}, {0, 2, 1, 4}, {0, 2, 1, 6}, {0, 2, 2, 2},
          {0, 2, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 2, 0, 2}, {2, 2, 0, 4},
          {2, 2, 0, 6}, {2, 2, 1, 0}, {2, 2, 1, 2}, {2, 2, 1, 4}, {2, 2, 2, 0}, {2, 2, 2, 2},
          {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 2, 0, 0}, {4, 2, 0, 2}, {4, 2, 0, 4}, {4, 2, 1, 0},
          {4, 2, 1, 2}, {4, 2, 2, 0}, {4, 4, 0, 4}, {4, 4, 1, 2}, {6, 2, 0, 2}, {6, 2, 1, 0},
          {6, 4, 0, 2}, {6, 4, 1, 0}, {8, 2, 0, 0}, {8, 4, 0, 0}}
```

```
In[20]:= lll = Table[Select[A, #[[2]] == ll[[i]]&], {i, 1, Length[ll]}]

Out[20]= {{{-1, {0, 0, 1, 2}}}, {{-1, {0, 0, 1, 4}}}, {{2, {0, 0, 2, 0}}, {2 nu, {0, 0, 2, 0}}},
  {{3, {0, 0, 2, 2}}, {2 nu, {0, 0, 2, 2}}, {-nu^2, {0, 0, 2, 2}}}, {{-2, {0, 0, 3, 0}},
   {-2 nu, {0, 0, 3, 0}}, {2 nu^2, {0, 0, 3, 0}}, {2 nu^3, {0, 0, 3, 0}}}, {{1, {0, 2, 0, 4}}},
  {{-2, {0, 2, 0, 6}}}, {{1, {0, 2, 0, 8}}}, {{-2, {0, 2, 1, 2}}, {-2 nu, {0, 2, 1, 2}}},
  {{3, {0, 2, 1, 4}}, {4 nu, {0, 2, 1, 4}}, {nu^2, {0, 2, 1, 4}}},
  {{-3, {0, 2, 1, 6}}, {-2 nu, {0, 2, 1, 6}}, {nu^2, {0, 2, 1, 6}}},
  {{2, {0, 2, 2, 2}}, {2 nu, {0, 2, 2, 2}}, {-2 nu^2, {0, 2, 2, 2}}, {-2 nu^3, {0, 2, 2, 2}}},
  {{2, {0, 2, 2, 4}}, {2 nu, {0, 2, 2, 4}}, {-2 nu^2, {0, 2, 2, 4}}, {-2 nu^3, {0, 2, 2, 4}}},
  {{3, {2, 0, 1, 0}}, {2 nu, {2, 0, 1, 0}}}, {{2, {2, 0, 1, 2}}},
  {{-3, {2, 0, 2, 0}}, {-2 nu, {2, 0, 2, 0}}, {nu^2, {2, 0, 2, 0}}}, {{-4, {2, 2, 0, 2}}},
  {{8, {2, 2, 0, 4}}}, {{-4, {2, 2, 0, 6}}}, {{4, {2, 2, 1, 0}}, {-4 nu^2, {2, 2, 1, 0}}},
  {{-4, {2, 2, 1, 2}}, {-4 nu, {2, 2, 1, 2}}, {2 nu^2, {2, 2, 1, 2}}, {2 nu^3, {2, 2, 1, 2}}},
  {{9, {2, 2, 1, 4}}, {6 nu, {2, 2, 1, 4}}, {-3 nu^2, {2, 2, 1, 4}}},
  {{-4, {2, 2, 2, 0}}, {8 nu^2, {2, 2, 2, 0}}, {-4 nu^4, {2, 2, 2, 0}}},
  {{-4, {2, 2, 2, 2}}, {-4 nu, {2, 2, 2, 2}}, {4 nu^2, {2, 2, 2, 2}}, {4 nu^3, {2, 2, 2, 2}}},
  {{1, {4, 0, 0, 0}}}, {{-1, {4, 0, 1, 0}}},
  {{4, {4, 2, 0, 0}}, {-4 nu^2, {4, 2, 0, 0}}}, {{-8, {4, 2, 0, 2}}, {2 nu^2, {4, 2, 0, 2}}},
  {{6, {4, 2, 0, 4}}}, {{-4, {4, 2, 1, 0}}, {4 nu^2, {4, 2, 1, 0}}},
  {{-9, {4, 2, 1, 2}}, {-6 nu, {4, 2, 1, 2}}, {3 nu^2, {4, 2, 1, 2}}},
  {{2, {4, 2, 2, 0}}, {2 nu, {4, 2, 2, 0}}, {-2 nu^2, {4, 2, 2, 0}}, {-2 nu^3, {4, 2, 2, 0}}},
  {{1, {4, 4, 0, 4}}, {-2 nu^2, {4, 4, 0, 4}}, {nu^4, {4, 4, 0, 4}}},
  {{-2, {4, 4, 1, 2}}, {-2 nu, {4, 4, 1, 2}}, {4 nu^2, {4, 4, 1, 2}},
   {4 nu^3, {4, 4, 1, 2}}, {-2 nu^4, {4, 4, 1, 2}}, {-2 nu^5, {4, 4, 1, 2}}},
  {{-4, {6, 2, 0, 2}}}, {{3, {6, 2, 1, 0}}, {2 nu, {6, 2, 1, 0}}, {-nu^2, {6, 2, 1, 0}}},
  {{-4, {6, 4, 0, 2}}, {4 nu^2, {6, 4, 0, 2}}},
  {{4, {6, 4, 1, 0}}, {-8 nu^2, {6, 4, 1, 0}}, {4 nu^4, {6, 4, 1, 0}}},
  {{1, {8, 2, 0, 0}}}, {{4, {8, 4, 0, 0}}, {-4 nu^2, {8, 4, 0, 0}}}}

In[21]:= r = Factor[
   Table[{Sum[lll[[i]][[j]][[1]], {j, 1, Length[lll[[i]]]}], lll[[i]][[1]][[2]]},
    {i, 1, Length[lll]}]]

Out[21]= {{-1, {0, 0, 1, 2}}, {-1, {0, 0, 1, 4}},
  {2 (1 + nu), {0, 0, 2, 0}}, {-(-3 + nu) (1 + nu), {0, 0, 2, 2}},
  {2 (-1 + nu) (1 + nu)^2, {0, 0, 3, 0}}, {1, {0, 2, 0, 4}}, {-2, {0, 2, 0, 6}},
  {1, {0, 2, 0, 8}}, {-2 (1 + nu), {0, 2, 1, 2}}, {(1 + nu) (3 + nu), {0, 2, 1, 4}},
  {(-3 + nu) (1 + nu), {0, 2, 1, 6}}, {-2 (-1 + nu) (1 + nu)^2, {0, 2, 2, 2}},
  {-2 (-1 + nu) (1 + nu)^2, {0, 2, 2, 4}}, {3 + 2 nu, {2, 0, 1, 0}},
  {2, {2, 0, 1, 2}}, {(-3 + nu) (1 + nu), {2, 0, 2, 0}}, {-4, {2, 2, 0, 2}},
  {8, {2, 2, 0, 4}}, {-4, {2, 2, 0, 6}}, {-4 (-1 + nu) (1 + nu), {2, 2, 1, 0}},
  {2 (1 + nu) (-2 + nu^2), {2, 2, 1, 2}}, {-3 (-3 + nu) (1 + nu), {2, 2, 1, 4}},
  {-4 (-1 + nu)^2 (1 + nu)^2, {2, 2, 2, 0}}, {4 (-1 + nu) (1 + nu)^2, {2, 2, 2, 2}},
  {1, {4, 0, 0, 0}}, {-1, {4, 0, 1, 0}}, {-4 (-1 + nu) (1 + nu), {4, 2, 0, 0}},
  {2 (-2 + nu) (2 + nu), {4, 2, 0, 2}}, {6, {4, 2, 0, 4}}, {4 (-1 + nu) (1 + nu), {4, 2, 1, 0}},
  {3 (-3 + nu) (1 + nu), {4, 2, 1, 2}}, {-2 (-1 + nu) (1 + nu)^2, {4, 2, 2, 0}},
  {(-1 + nu)^2 (1 + nu)^2, {4, 4, 0, 4}}, {-2 (-1 + nu)^2 (1 + nu)^3, {4, 4, 1, 2}},
  {-4, {6, 2, 0, 2}}, {-(-3 + nu) (1 + nu), {6, 2, 1, 0}},
  {4 (-1 + nu) (1 + nu), {6, 4, 0, 2}}, {4 (-1 + nu)^2 (1 + nu)^2, {6, 4, 1, 0}},
  {1, {8, 2, 0, 0}}, {-4 (-1 + nu) (1 + nu), {8, 4, 0, 0}}}
```

```
In[32]:=
        r1 = Transpose[r][[1]]
        r2 = Transpose[r][[2]]

Out[32]= {-1, -1, 2 (1 + nu), -(-3 + nu) (1 + nu), 2 (-1 + nu) (1 + nu)^2, 1, -2, 1, -2 (1 + nu),
          (1 + nu) (3 + nu), (-3 + nu) (1 + nu), -2 (-1 + nu) (1 + nu)^2, -2 (-1 + nu) (1 + nu)^2, 3 + 2 nu,
          2, (-3 + nu) (1 + nu), -4, 8, -4, -4 (-1 + nu) (1 + nu), 2 (1 + nu) (-2 + nu^2),
          -3 (-3 + nu) (1 + nu), -4 (-1 + nu)^2 (1 + nu)^2, 4 (-1 + nu) (1 + nu)^2, 1, -1,
          -4 (-1 + nu) (1 + nu), 2 (-2 + nu) (2 + nu), 6, 4 (-1 + nu) (1 + nu), 3 (-3 + nu) (1 + nu),
          -2 (-1 + nu) (1 + nu)^2, (-1 + nu)^2 (1 + nu)^2, -2 (-1 + nu)^2 (1 + nu)^3, -4,
          -(-3 + nu) (1 + nu), 4 (-1 + nu) (1 + nu), 4 (-1 + nu)^2 (1 + nu)^2, 1, -4 (-1 + nu) (1 + nu)}

Out[33]= {{0, 0, 1, 2}, {0, 0, 1, 4}, {0, 0, 2, 0}, {0, 0, 2, 2}, {0, 0, 3, 0}, {0, 2, 0, 4},
          {0, 2, 0, 6}, {0, 2, 0, 8}, {0, 2, 1, 2}, {0, 2, 1, 4}, {0, 2, 1, 6}, {0, 2, 2, 2},
          {0, 2, 2, 4}, {2, 0, 1, 0}, {2, 0, 1, 2}, {2, 0, 2, 0}, {2, 2, 0, 2}, {2, 2, 0, 4},
          {2, 2, 0, 6}, {2, 2, 1, 0}, {2, 2, 1, 2}, {2, 2, 1, 4}, {2, 2, 2, 0}, {2, 2, 2, 2},
          {4, 0, 0, 0}, {4, 0, 1, 0}, {4, 2, 0, 0}, {4, 2, 0, 2}, {4, 2, 0, 4}, {4, 2, 1, 0},
          {4, 2, 1, 2}, {4, 2, 2, 0}, {4, 4, 0, 4}, {4, 4, 1, 2}, {6, 2, 0, 2}, {6, 2, 1, 0},
          {6, 4, 0, 2}, {6, 4, 1, 0}, {8, 2, 0, 0}, {8, 4, 0, 0}}

       (*
       h1=r2[[{11,24,25,26}]]//Simplify
       h2=r1[[{11,24,25,26}]]
       DE=Table[{h2[[i]],h1[[i]]},{i,1,Length[h1]}]
       AA=Transpose[A][[2]]
       Position[AA,#]&/@h1//Flatten
       *)

       PPP = Transpose[A[[{8, 36, 56, 58}]]][[2]]
       coef = Transpose[A[[{8, 36, 56, 58}]]][[1]]

Out[44]= {{0, 0, 1, 4}, {2, 0, 1, 2}, {4, 0, 0, 0}, {4, 0, 1, 0}}

Out[45]= {-1, 2, 1, -1}

In[46]:= s1 = Table[
          p^PPP[[i]][[1]] *h^PPP[[i]][[2]] *
            L^PPP[[i]][[3]] *m^PPP[[i]][[4]], {i, 1, Length[PPP]}]
         eq = s1 . coef
```

Out[46]= $\{L\,m^4,\ L\,m^2\,p^2,\ p^4,\ L\,p^4\}$

Out[47]= $-L\,m^4 + 2\,L\,m^2\,p^2 + p^4 - L\,p^4$

APPENDIX 7

Mathematica 3.0 Code
for Analysis of Characteristic Equation for Different
Boundary Conditions and Frequency Domains

Low frequency vibrations, bbb- simply supported, aaa-clamped edges

```
bbb := {{p1*u1, p2*u2, 0, 0, p5*u5, p6*u6}, {w1, w2, 0, 0, w5, w6},
   {w1*p1^2, w2*p2^2, 0, 0, w5*p5^2, w6*p6^2}, {p1*u1*Exp[p1*L],
    p2*u2*Exp[p2*L], p3*u3*Exp[p3*L], p4*u4*Exp[p4*L], 0, 0}, {w1*Exp[p1*L],
    w2*Exp[p2*L], w3*Exp[p3*L], w4*Exp[p4*L], 0, 0}, {w1*p1^2*Exp[p1*L],
    w2*p2^2*Exp[p2*L], w3*p3^2*Exp[p3*L], w4*p4^2*Exp[p4*L], 0, 0}};

aaa :=
 {{u1, u2, 0, 0, u5, u6}, {w1, w2, 0, 0, w5, w6}, {w1*p1, w2*p2, 0, 0, w5*p5, w6*p6},
  {u1*Exp[p1*L], u2*Exp[p2*L], u3*Exp[p3*L], u4*Exp[p4*L], 0, 0},
  {w1*Exp[p1*L], w2*Exp[p2*L], w3*Exp[p3*L], w4*Exp[p4*L], 0, 0},
  {w1*p1*Exp[p1*L], w2*p2*Exp[p2*L], w3*p3*Exp[p3*L], w4*p4*Exp[p4*L], 0, 0}};


zz = Sqrt[2] / 2;
p1 = Sqrt[lamda] * I; p2 = -Sqrt[lamda] * I; p3 = (zz + I * zz) / h;
p4 = (zz - I * zz) / h; p5 = (-zz + I * zz) / h; p6 = (-zz - I * zz) / h;
u1 = p1; u2 = p2; u3 = u4 = u5 = u6 = nu; Exp[p4 * L] == 0;
w1 = w2 = -nu * lamda; w3 = p3; w4 = p4; w5 = p5; w6 = p6;

Simplify[Det[bbb]]
```

$$-\frac{4\, E^{\frac{L\left(\sqrt{2}-I\, h\sqrt{\text{lamda}}\right)}{h}} \left(-1+E^{2\, I\, L\sqrt{\text{lamda}}}\right) \text{lamda}^2 \left(-1+\text{nu}^2\right)^2}{h^8}$$

```
Simplify[Det[aaa]]
```

$$\frac{1}{h^6}\left(2\, E^{\frac{L\left(\sqrt{2}-I\, h\sqrt{\text{lamda}}\right)}{h}}\, \text{lamda}\, \left(-1+E^{2\, I\, L\sqrt{\text{lamda}}} - 2\, I\sqrt{2}\, \left(1+E^{2\, I\, L\sqrt{\text{lamda}}}\right) h\sqrt{\text{lamda}}\; \text{nu}^2 + 2\, I\sqrt{2}\, \left(1+E^{2\, I\, L\sqrt{\text{lamda}}}\right) h^3\, \text{lamda}^{3/2}\, \text{nu}^4 + \left(-1+E^{2\, I\, L\sqrt{\text{lamda}}}\right) h^4\, \text{lamda}^2\, \text{nu}^4 - 2\left(-1+E^{2\, I\, L\sqrt{\text{lamda}}}\right) h^2\, \text{lamda}\; \text{nu}^2\, \left(1+\text{nu}^2\right)\right)\right)$$

Low frequency vibrations, bbb- simply supported, aaa-clamped edges

```
bbb := {{p1*u1, p2*u2, 0, p4*u4, 0, 0, p7*u7, p8*u8}, {w1, w2, 0, w4, 0, 0, w7, w8},
   {w1*p1^2, w2*p2^2, 0, w4*p4^2, 0, 0, w7*p7^2, w8*p8^2},
   {v1, v2, 0, v4, 0, 0, v7, v8}, {0, p2*u2*Exp[p2*L], p3*u3*Exp[p3*L],
    p4*u4*Exp[p4*L], p5*u5*Exp[p5*L], p6*u6*Exp[p6*L], 0, 0}, {0, w2*Exp[p2*L],
    w3*Exp[p3*L], w4*Exp[p4*L], w5*Exp[p5*L], w6*Exp[p6*L], 0, 0},
   {0, w2*p2^2*Exp[p2*L], w3*p3^2*Exp[p3*L], w4*p4^2*Exp[p4*L],
    w5*p5^2*Exp[p5*L], w6*p6^2*Exp[p6*L], 0, 0}, {0, v2*Exp[p2*L],
    v3*Exp[p3*L], v4*Exp[p4*L], v5*Exp[p5*L], v6*Exp[p6*L], 0, 0}};

zz = Sqrt[2] / 2;
p1 = -m*Sqrt[Sqrt[lamda - h^4*m^4]]; p2 = I*m*Sqrt[Sqrt[lamda - h^4*m^4]];
p3 = m*Sqrt[Sqrt[lamda - h^4*m^4]]; p4 = -I*m*Sqrt[Sqrt[lamda - h^4*m^4]];
p5 = (zz + I*zz) / h; p6 = (zz - I*zz) / h; p7 = (-zz + I*zz) / h; p8 = (-zz - I*zz) / h;
u1 = p1 / (m^2); u2 = p2 / (m^2); u3 = p3 / (m^2);
u4 = p4 / (m^2); u5 = nu / p5; u6 = nu / p6; u7 = nu / p7; u8 = nu / p8;
w1 = w2 = w3 = w4 = w5 = w6 = w7 = w8 = 1;
v1 = v2 = v3 = v4 = -1 / m; v5 = m / (p5^2); v6 = m / (p6^2); v7 = m / (p7^2); v8 = m / (p8^2);

Simplify[Det[bbb]]
```

$$\frac{16\,E^{\frac{L\left(\sqrt{2}-(1-I)\,h\,m\,(lamda-h^4\,m^4)^{1/4}\right)}{h}}\left(-1+E^{2\,I\,L\,m\,(lamda-h^4\,m^4)^{1/4}}\right)\,(-lamda+h^4\,m^4)\,(-1+h^4\,m^4\,nu)^2}{h^4\,m^2}$$

```
aaa := {{u1, u2, 0, u4, 0, 0, u7, u8},
  {v1, v2, 0, v4, 0, 0, v7, v8}, {w1, w2, 0, w4, 0, 0, w7, w8},
  {w1*p1, w2*p2, 0, w4*p4, 0, 0, w7*p7, w8*p8}, {0, u2*Exp[p2*L], u3*Exp[p3*L],
   u4*Exp[p4*L], u5*Exp[p5*L], u6*Exp[p6*L], 0, 0}, {0, v2*Exp[p2*L],
   v3*Exp[p3*L], v4*Exp[p4*L], v5*Exp[p5*L], v6*Exp[p6*L], 0, 0},
  {0,
   w2*Exp[p2*L], w3*Exp[p3*L], w4*Exp[p4*L], w5*Exp[p5*L], w6*Exp[p6*L], 0, 0},
  {0, w2*p2*Exp[p2*L], w3*p3*Exp[p3*L], w4*p4*Exp[p4*L], w5*p5*Exp[p5*L],
   w6*p6*Exp[p6*L], 0, 0}};

Simplify[Det[aaa]]
```

$$\frac{4\,I\,E^{\frac{L\left(\sqrt{2}-(1-I)\,h\,m\,(lamda-h^4\,m^4)^{1/4}\right)}{h}}\left(1+E^{2\,I\,L\,m\,(lamda-h^4\,m^4)^{1/4}}\right)\sqrt{lamda-h^4\,m^4}\,\left(1-h^4\,m^4\,nu+h^2\,m^2\,(1+nu)\right)^2}{h^2\,m^4}$$

High frequency vibrations (bbb - simply supported edges, aaa -clamped edges).

```
bbb := {{p1 * u1, p2 * u2, 0, p4 * u4, p5 * u5, p6 * u6}, {w1, w2, 0, w4, w5, w6},
  {w1 * p1 ^ 2, w2 * p2 ^ 2, 0, w4 * p4 ^ 2, w5 * p5 ^ 2, w6 * p6 ^ 2}, {p1 * u1 * Exp[p1 * L],
   p2 * u2 * Exp[p2 * L], p3 * u3 * Exp[p3 * L], 0, p5 * u5 * Exp[p5 * L], p6 * u6 * Exp[p6 * L]},
  {w1 * Exp[p1 * L], w2 * Exp[p2 * L], w3 * Exp[p3 * L], 0, w5 * Exp[p5 * L], w6 * Exp[p6 * L]},
  {w1 * p1 ^ 2 * Exp[p1 * L], w2 * p2 ^ 2 * Exp[p2 * L], w3 * p3 ^ 2 * Exp[p3 * L], 0,
   w5 * p5 ^ 2 * Exp[p5 * L], w6 * p6 ^ 2 * Exp[p6 * L]}};

aaa := {{u1, u2, 0, u4, u5, u6}, {w1, w2, 0, w4, w5, w6},
  {w1 * p1, w2 * p2, 0, w4 * p4, w5 * p5, w6 * p6}, {u1 * Exp[p1 * L], u2 * Exp[p2 * L],
   u3 * Exp[p3 * L], 0, u5 * Exp[p5 * L], u6 * Exp[p6 * L]}, {w1 * Exp[p1 * L], w2 * Exp[p2 * L],
   w3 * Exp[p3 * L], 0, w5 * Exp[p5 * L], w6 * Exp[p6 * L]}, {w1 * p1 * Exp[p1 * L],
   w2 * p2 * Exp[p2 * L], w3 * p3 * Exp[p3 * L], 0, w5 * p5 * Exp[p5 * L], w6 * p6 * Exp[p6 * L]}};


p1 = Sqrt[lamda] * I * Sqrt[1 - nu ^ 2];
p2 = -Sqrt[lamda] * I * Sqrt[1 - nu ^ 2]; p3 = lamda ^ (1 / 4) / h;
p4 = -lamda ^ (1 / 4) / h; p5 = I * lamda ^ (1 / 4) / h; p6 = -I * lamda ^ (1 / 4) / h;
u1 = p1; u2 = p2; u3 = u4 = u5 = u6 = nu;
w1 = nu; w2 = nu; w3 = p3; w4 = p4; w5 = p5; w6 = p6;
bbb;
Simplify[Det[bbb]]
```

$$-\frac{1}{h^8}\left(4\,E^{-\frac{I\,L\,lamda^{1/4}\left((1+I)+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}\left(-1+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}\right)\left(-1+E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}\right)lamda^2\right.$$
$$\left.(lamda+nu^2-lamda\,nu^2)^2\right)$$

```
Simplify[Det[aaa]]
```

$$\frac{1}{h^6}\left(2\,E^{-\frac{I\,L\,lamda^{1/4}\left((1+I)+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}\,lamda\right.$$
$$\left(2\left(-1+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}\right)\left(-1+E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}\right)h^2\,nu^4+(2+2\,I)\left(1-I\,E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}+\right.\right.$$
$$\left.E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}-(2-2\,I)\,E^{\frac{I\,L\,lamda^{1/4}\left(1+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}-I\,E^{\frac{2\,I\,L\,lamda^{1/4}\left(1+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}\right)$$
$$h\,lamda^{3/4}\,nu^2\sqrt{1-nu^2}-$$
$$(2+2\,I)\left(-I+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}-I\,E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}-\right.$$
$$\left.(2-2\,I)\,E^{\frac{I\,L\,lamda^{1/4}\left(1+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}+E^{\frac{2\,I\,L\,lamda^{1/4}\left(1+h\,lamda^{1/4}\sqrt{1-nu^2}\right)}{h}}\right)h^3\,lamda^{1/4}\,nu^4\sqrt{1-nu^2}+$$
$$I\left(1+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}\right)\left(-1+E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}\right)lamda^{3/2}\,(-1+nu^2)+$$
$$2\left(-1+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}\right)\left(-1+E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}\right)h^2\,lamda\,nu^2\,(-1+nu^2)-$$
$$\left.\left.I\left(1+E^{\frac{2\,I\,L\,lamda^{1/4}}{h}}\right)\left(-1+E^{2\,I\,L\sqrt{lamda}\,\sqrt{1-nu^2}}\right)h^4\sqrt{lamda}\,nu^4\,(-1+nu^2)\right)\right)$$