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Periodicity and Ruin Probabilities for Compound Non-Homogeneous Poisson Processes

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Abstract

Periodicity and Ruin Probabilities for Compound Non-Homogenous Poisson Processes

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Compound non-homogenous Poisson processes with periodic claim intensity rates are studied in this work. A risk process related to a short term periodic environment and the periodicity for its compound claim counting process are discussed. The ruin probabilities of compound non-homogenous Poisson processes with periodic intensity function are also discussed, in which the embedded discrete risk model and the average arrival rate risk model are presented and bounds for the ruin probability of the continuous-time risk model are derived.

We introduce a more general Poisson process model with double periodicity. Here the periodic environment does not repeat the exact same pattern every year but varies the short term peak over a relatively long period, with different levels in each year.

Illustrations of periodicity for short and long term Poisson models and numerical examples for ruin probabilities are also given.

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Contents

List of Figures

List of Tables

Introduction

In classical risk theory, homogeneous Poisson processes are usually used to model some risk related events. But it is far from reality since the claim intensity rate in this process is constant. The more general time dependent case can be modeled by non-homogeneous Poisson processes. This allows the intensity rate to be a function of time t. A special case of this general model is considered here, it assumes a periodic intensity rate function.

There are many natural phenomena involved in a periodic environment or under seasonal conditions. The periodicity of these events may affect the insurance businesses. For example, weather factors affect the insured risk in automobile and fire insurance; factors such as seasonal snowstorms in the North and hurricanes or floods in the South affect property insurance. It is reasonable and tractable using a periodic time-dependent intensity rate to model the claim occurrence process, and therefore the aggregate claim process.

The similarity between the intensity and failure rate functions used in reliability models under a minimal repair policy, gives an advantage to explore the applicability of non-homogeneous Poisson process. Some characterization properties of this process with periodic failure rate have been shown in Chukova et al. (1993) and Dimitrov et al. (1997). Applying these properties to risk models, especially the periodicity in a short term (with period 1) case, are exploited by Garrido et al. (1996). Some ruin problems in a periodic environment are considered by Asmussen and Rolski (1994) and Rolski et al. (1999).

A more practical case is that when the periodic environment does not repeat itself exactly from year to year, but the short term peak changes over a relatively long period, with different levels in each year. This is especially appropriate in catastrophic insurance, such as hurricanes, which have a peak season in the middle of the year, but whose intensity level also depends on the long term climatological effects like La Niña or El Niño. A corresponding Poisson process model with double periodicity is introduced in this work. Its periodicity and related characteristics are also discussed.

A brief review of compound homogeneous Poisson process and its ruin theory are given in Chapter 1. Classical bounds and sharper two-sided bounds for the ultimate ruin probabilities, given by Cai and Garrido (1999), and some asymptotic theory are also summarized.

In Chapter 2, the preliminary results on compound non-homogeneous Poisson processes with periodic intensity rate are presented. The periodicity for short term and related risk characteristics of this model are shown. An embedded discrete risk model and an average arrival rate risk model, which even deals with the time-dependent claim size distribution, are used to give two-sided bounds to the ruin probability. Some special two-sided bounds are derived in this work and numerical illustrations are also given.

Finally, a more general Poisson model with double periodicity is proposed in Chapter 3. Some practical shapes are illustrated for the periodicity of the claim intensity.

Chapter 1

Compound Poisson Process Ruin Theory

1.1 Definitions

First we review the definition of the homogeneous Poisson process and the compound Poisson process (CPP).

Let inter-occurrence times $\{T_n;\ n \geq 1\}$ form a sequence of independent random variables that have a common exponential distribution with parameter $\lambda > 0$. Then the counting process or the number of claims process $\{N_t;\ t \geq 0\}$ is called a homogenous Poisson process with constant rate or intensity λ, a special kind of renewal process, i.e. for all $s, t > 0$, we have

$$Pr\{N_{s+t} - N_s = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

A basic property of the homogenous Poisson process is that it has independent and stationary increments.

Let the individual losses, or the claim severities $\{X_j, j \geq 1\}$ be independent and identically distributed non-negative random variables, independent of N_t, each with cumulative distribution function F_X and mean $\mu = EX_j < \infty$. Thus X_j is the amount of the jth loss. Let S_t be the total loss in $[0, t)$, which is given by $S_t = \sum_{j=1}^{N_t} X_j$ if

3

$N_t > 0$ and $S_t = 0$ if $N_t = 0$. Then for any fixed t, S_t has a compound Poisson distribution. The process $\{S_t; \; t \geq 0\}$ is called compound Poisson process (CPP), denoted as $S_t \sim C.P.[\lambda; \; F_X]$ for $x \geq 0$.

Now consider the risk reserve process in the above compound Poisson risk model, given by

$$R_t = u + \beta t - \sum_{j=0}^{N_t} X_j, \tag{1.2}$$

where $u \geq 0$ is the initial value and $\beta > 0$ is the constant premium rate over time, which satisfies

$$\beta = (1 + \theta)\lambda\mu, \tag{1.3}$$

where $\theta > 0$ is the relative security loading.

Let

$$T(u) = \inf\{t \geq 0; \; R_t < 0\},$$

denote the time to ruin, with initial value u, and let

$$
\begin{aligned}
\Psi(u) &= Pr\{T(u) < \infty\} \\
&= Pr\{R_t < 0, \quad \text{for some } t > 0 \mid R_0 = u\},
\end{aligned} \tag{1.4}
$$

denote the ultimate ruin probability with initial value $u \geq 0$.

We review in the next few sections the classical results in the literature on Ψ.

1.2 Volterra Integro-differential Equation

An integro-differential equation for Ψ is given in this section. The more general probability of ruin in a finite period is considered next.

Theorem 1.1 *The ultimate ruin probability with initial reserve u satisfies the following Volterra integro-differential equation*

$$\Psi'(u) = \frac{\lambda}{\beta}\Psi(u) - \frac{\lambda}{\beta}\int_0^u \Psi(u - x) \, dF_X(x) - \frac{\lambda}{\beta}[1 - F_X(u)], \quad u > 0. \tag{1.5}$$

4

Proof. See Rolski et al. (1999), pp. 162-163. □

The general solution to or a numerical evaluation of $\Psi(u)$ in (1.5) may be obtained subject to the following obvious initial boundary conditions: $\Psi(\infty) = 0$ and $\Psi(u) = 1$ for $u < 0$. The following corollary gives a derivation for the additional boundary condition at $u = 0$. Note there is a discontinuity of Ψ at 0.

Corollary 1.1 *The ruin probability with no initial reserve is given by*

$$\Psi(0) = \frac{1}{1+\theta}, \quad \theta \geq 0. \tag{1.6}$$

Proof. Integrating (1.5) with respect to u from 0 to y gives

$$\Psi(y) - \Psi(0) = \frac{\lambda}{\beta} \int_0^y \Psi(u)\, du - \frac{\lambda}{\beta} \int_0^y \int_0^u \Psi(u - x)\, dF_X(x)\, du$$
$$- \frac{\lambda}{\beta} \int_0^y [1 - F_X(u)]\, du,$$

where

$$\int_0^y \int_0^u \Psi(u - x)\, dF_X(x)\, du = \int_0^y \int_x^y \Psi(u - x)\, du\, dF_X(x)$$
$$= \int_0^y \int_0^{y-x} \Psi(u)\, du\, dF_X(x)$$

This implies that

$$\Psi(y) - \Psi(0) = \frac{\lambda}{\beta} \int_0^y \Psi(u)\, du - \frac{\lambda}{\beta} \int_0^y \Psi(u) \int_0^{y-u} dF_X(x)\, du$$
$$- \frac{\lambda}{\beta} \int_0^y [1 - F_X(u)]\, du,$$
$$= \frac{\lambda}{\beta} \int_0^y \Psi(u)[1 - F_X(y - u)]\, du - \frac{\lambda}{\beta} \int_0^y [1 - F_X(u)]\, du.$$

Letting $y \to \infty$ on both sides of this last equation, and using $\lim_{y\to\infty} \Psi(y) = 0$ yields (1.6). □

Now consider the more general probability of ruin in a finite period, with initial value u. Let

$$\Psi(u, t) = Pr\{R_s < 0 , \quad \text{for some } s, \ 0 < s \leq t\}. \tag{1.7}$$

5

Alternatively, in terms of the time to ruin $T(u)$ as defined by (1.3), we have

$$\Psi(u, t) = Pr\{T(u) \leq t\}, \quad u, \ t \geq 0. \tag{1.8}$$

An equation analogous to (1.5) for $\Psi(u, t)$ is given in the following theorem.

Theorem 1.2 *The probability of ruin before time t, with initial reserve u, satisfies the following partial integro-differential equation for u, t > 0*

$$\frac{\partial}{\partial t}\Psi(u, t) = \beta\frac{\partial}{\partial u}\Psi(u, t) + \lambda[1 - F_X(u)] - \lambda\Psi(u, t) + \lambda \int_0^u \Psi(u - x, t) \ dF_X(x).$$

$$\tag{1.9}$$

Proof. See for example, Panjer and Willmot (1992), p. 389. □

In fact, since the following relation clearly holds

$$\Psi(u) = \lim_{t \to \infty} \Psi(u, t),$$

it is easy to get (1.5) as a special case of (1.9).

1.3 Compoun d Geometric Representation

This section gives an explicit form for $\Psi(u)$ using the technique of Laplace transforms. First note that the ruin probability with no initial reserve in (1.6) is completely specifie d by the security loading,

$$\Psi(0) = \frac{1}{1 + \theta}, \quad \theta \geq 0,$$

independently of the claim frequency rate λ or the single claim amount distribution F_X.

Let

$$G_X(x) = \frac{1}{\mu} \int_0^x \bar{F}_X(y) \ dy, \quad x \geq 0, \tag{1.10}$$

denote the equilibrium distribution function of F_X, and $G_X^{(n)}(x)$ be the n-fold convolution of G_X with itself. It is well-known that $\Psi(u)$ can be expressed as the tail of a compound geometric distribution, a result given below.

Theorem 1.3 *The probability of ultimate ruin* $\Psi(u)$ *is given by the tail of the distribution of a compound geometric distribution, i.e.*

$$\Psi(u) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \bar{G}_X^{(n)}(u), \quad u \geq 0, \tag{1.11}$$

where $\bar{G}_X^{(n)} = 1 - G_X^{(n)}$ *is the tail of* $G_X^{(n)}$, *and* θ *is the relative safety loading factor, satisfying (1.3).*

Proof. See Feller (1971) or Grandell (1991). □

It is difficult to obtain explicit expressions for $\Psi(u)$ in many practical situations. For general claim amount distributions, evaluation via (1.11) is difficult (if not impossible), but we saw that $\Psi(u)$ satisfies a Volterra integro-differential equation. As stated in the following result, it also satisfies a Volterra integral equation, and so $\Psi(u)$ may be obtained numerically using, say, the Laplace transform technique.

Theorem 1.4 *The probability of ultimate ruin* $\Psi(u)$ *satisfies the Volterra type integral equation of the second kind*

$$\Psi(u) = \frac{\bar{G}_X(u)}{1+\theta} + \frac{1}{1+\theta} \int_0^u \Psi(u-y) \, dG_X(y). \tag{1.12}$$

Proof. See Klugman et al. (1998), pp. 543-544. □

Thus, in most situations, $\Psi(u)$ cannot be obtained analytically but can be obtained numerically from equation (1.12). On the other hand, from Theorem 1.3 we know that the ultimate ruin probability $\Psi(u)$ is the survival function of a compound geometric variate. Many results have been derived to find bounds for the ruin probability with initial value u, or to find asymptotic forms of $\Psi(u)$ for large u, based on this compound geometric structure. These are discussed in the next two sections.

7

1.4 Bounds for Ruin Probabilities

We begin our study of bounds for ruin probabilities by introducing the adjustment coefficient, $\kappa > 0$, which is the solution, if any, to the equation

$$\int_0^\infty e^{\kappa x} \bar{F}_X(x) \, dx = \mu(1 + \theta), \tag{1.13}$$

or equivalently

$$\int_0^\infty e^{\kappa x} \, dG_X(x) = 1 + \theta, \tag{1.14}$$

where $G_X(x)$ is given by (1.10).

The adjustment coefficient κ is used in the derivation of useful approximations and bounds for the probability of ruin. A fundamental Cramér-Lundberg's bound is a result in this direction.

Theorem 1.5 *(Cramér-Lundberg) If there exists a $\kappa > 0$ satisfying (1.13), then*

$$\Psi(u) \leq e^{-\kappa x}, \quad u \geq 0. \tag{1.15}$$

Proof. See for instance, Klugman et al. (1998), pp.531-533. □

This is an important result because it gives an upper bound on the probability of ruin on a portfolio of business. However, the Cramér-Lundberg condition does not hold in many practical cases, that is, the related adjustment coefficients do not exist for many distributions. Many results have been derived for bounds on the ruin probability since the work of Lundberg. These are generally based on the fact that the ruin probability $\Psi(u)$ can be expressed as the tail of a compound geometric distribution. Some of these new bounds can be applied to more general claim severity distributions.

There are several approaches to derive general bounds for the ruin probability $\Psi(u)$. One way is to remove the Cramér-Lundberg condition or other similar conditions; for example, De Vylder and Goovaerts (1984) give the following lower bound

for the ruin probability

$$\Psi(u) \geq \frac{\bar{G}_X(u)}{\theta + \bar{G}_X(u)}, \quad u \geq 0. \tag{1.16}$$

Another way is to relax Cramér-Lundberg's condition by replacing the exponential functions in (1.13) by, say new worse than used (NWU) or new better than used (NBU) distribution functions [see Willmot (1997), Cai and Wu (1997)].

Consider a truncating Cramér-Lundberg condition, that is assume that for given $t > 0$, there exists a constant $\kappa(t)$ satisfying

$$\int_0^t e^{\kappa(t)y} \bar{F}_X(y) \, dy = \mu(1 + \theta), \tag{1.17}$$

or, alternatively

$$\int_0^t e^{\kappa(t)y} \, dG_X(y) = 1 + \theta.$$

Under this truncated Cramér-Lundberg condition, Dickson (1994) gives an upper bound for any $0 \leq x \leq t$:

$$\Psi(x) \leq e^{-x\kappa(t)} + \frac{\bar{G}_X(t)}{\theta + \bar{G}_X(x)}. \tag{1.18}$$

In the same spirit Cai and Garrido (1999) derive sharper two-sided bounds for $\Psi(x)$ under the above truncating Cramér-Lundberg condition.

Theorem 1.6 *For any given $t > 0$, if there exists a constant $\kappa(t)$ such that (1.17) holds, then for any $0 \leq x \leq t$,*

$$\frac{\theta\alpha_1(x,t)e^{-x\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)} \leq \Psi(x) \leq \frac{\theta\alpha_2(x,t)e^{-x\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)}, \tag{1.19}$$

where

$$\alpha_1(x,t) = \inf_{0 \leq h \leq x} \alpha(h,t), \quad \alpha_2(x,t) = \sup_{0 \leq h \leq x} \alpha(h,t)$$

and

$$\alpha(h,t) = \frac{e^{h\kappa(t)}[\bar{G}_X(h) - \bar{G}_X(t)]}{\int_h^t e^{y\kappa(t)} \, dG_X(y)}.$$

Proof. See Cai and Garrido (1999). □

Theorem 1.6 implies the following simplified and practical two-sided bounds for $\Psi(u)$.

Corollary 1.2 *Under the conditions of Theorem 1.6, for any $0 \leq x \leq t$,*

$$\frac{\theta e^{-(x+t)\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)} \leq \Psi(x) \leq \frac{\theta e^{-x\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)}, \qquad (1.20)$$

and hence, for any $t > 0$ satisfying the above conditions

$$\frac{\theta e^{-2t\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)} \leq \Psi(t) \leq \frac{\theta e^{-t\kappa(t)} + \bar{G}_X(t)}{\theta + \bar{G}_X(t)}.$$

These bounds derived under the truncating Cramér-Lundberg condition, can be applied to any non-negative claim severity distribution F_X with $F_X(0) = 0$ and finite mean.

It is easy to see that the upper bounds in (1.19) and (1.20) are all tighter than that of Dickson (1994) in (1.18), while the lower bounds in (1.19) and (1.20) are all sharper than that of De Vylder & Goovaerts (1984) in (1.16).

1.5   Asymptotic Theory

Let Y be a random variable with distribution function G_X as defined in (1.10) and its corresponding pdf be given by $\frac{1}{\mu}[1 - F_X(y)]$, for $y \geq 0$, where F_X is the distribution function of claim size random variable X.

It was shown in Theorem 1.3 that the ultimate ruin probability $\Psi(u)$ is the survival function of the compound geometric sum of random variables Y. Applying the properties of the compound geometric distribution, we obtain the following approximate formulae for $\Psi(u)$ for large u. These may be used in place of or in addition to the results of the previous section.

Consider first the situation where there exists adjustment coefficient $\kappa > 0$, satisfying

$$1 + (1 + \theta)\kappa\mu = M_X(\kappa), \qquad (1.21)$$

10

or equivalently (1.13), i.e.

$$1 + \theta = M_Y(\kappa). \tag{1.22}$$

If κ exists, then by Theorem 1.5 and the properties of the compound geometric distribution, we have the well-known asymptotic formula, which is referred to as Cramér asymptotic ruin formula, given below. Henceforth we use the notation $\alpha(x) \sim \beta(x)$, $x \to \infty$, to mean $\lim_{x \to \infty} \frac{\alpha(x)}{\beta(x)} = 1$.

Theorem 1.7 *Suppose that $\kappa > 0$ satisfies (1.21), or equivalently (1.22). Then the ruin probability satisfies*

$$\Psi(u) \sim \frac{\theta \mu}{M'_X(\kappa) - \mu(1 + \theta)} e^{-\kappa u} , \quad as\ u \to \infty, \tag{1.23}$$

or equivalently

$$\Psi(u) \sim \frac{\theta}{\kappa M'_Y(\kappa)} e^{-\kappa u} , \quad as\ u \to \infty, \tag{1.24}$$

where $M'_X(\kappa)$ and $M'_Y(\kappa)$ are the derivatives of the moment generating function of X and Y, respectively.

Proof. See Panjer and Willmot (1992), pp. 384-385. $\square$

If an adjustment coefficient exists then we have the asymptotic formula (1.23) and the inequality (1.15) for the probability of ruin. It is easily seen that for light tailed claim severity distributions, such as the exponential or the gamma, there always exists κ satisfying (1.22). However, for the heavy-tailed subexponential distributions, such as Pareto, lognormal or Weibull (for some region of the parameter space), (1.22) is never satisfied.

General asymptotic formulae are obtainable for the probability of ruin with subexponential claim size distributions, for example, Embrechts and Veraverbeke (1982) have shown that

$$\Psi(u) \sim \frac{\int_u^\infty [1 - F_X(y)]\ dy}{\theta \mu} = \frac{\bar{G}_X(u)}{\theta} , \quad as\ u \to \infty, \tag{1.25}$$

11

also see Panjer and Willmot (1992).

For medium tailed distributions, we note that if $M_Y(\gamma) < 1 + \theta$, or alternatively, $M_X(\gamma) < 1 + (1 + \theta)\mu\gamma$, where $\gamma > 0$, and $M_X(t) = \infty$ for any $t > \gamma$, no $\kappa > 0$ exists satisfying (1.22), and so (1.24) does not hold. In this case, however, Embrechts and Veraverbeke (1982) have shown that

$$\Psi(u) \sim \frac{\theta\gamma\mu\bar{F}_X(u)}{[1 + (1 + \theta)\gamma\mu - M_X(\gamma)]^2} , \quad \text{as } u \to \infty. \tag{1.26}$$

Chapter 2

Compound Non-Homogenous Poisson Processes

The previous chapter presented the classical risk model based on compound Poisson process, which has the property that its corresponding claim counting process $\{N_t; \ t \geq 0\}$ has independent and stationary increments. In this chapter, we expand the class of homogeneous Poisson process to non-homogeneous Poisson(NHP) process by introducing the notion of non-homogeneity. This allows for time-dependent arrival rates, for example, the situation where claim occurrence times depend on the time of the year. The corresponding claim counting process $\{N_t; \ t \geq 0\}$ has independent but not necessarily stationary increments. Then a more general point process with a similar structure as compound Poisson process, called compound non-homogeneous Poisson process, is discussed. Further we consider a short term periodic risk model, and discuss its periodicity properties and ruin probabilities.

2.1   Definition and Properties

Let λ be a non-negative, measurable and locally integrable (deterministic) function. Consider the claim arrivals over the time interval $[s, \ t)$, i.e. the number of claims in

$[s, t)$, denoted by $N_{[s, t)}$, for $0 \le s < t$, (denoted N_t when $s = 0$). The definition of a NHP process is given below.

Definition 2.1 *The counting process* $\{N_t; \ t \ge 0\}$ *is said to be a non-homogeneous Poisson (NHP) process with intensity function* λ, *where* $\lambda(t) \ge 0$, *for* $t \ge 0$ *if it satisfies:*

(a) $N_t = 0$, *if* $t = 0$;

(b) $\{N_t; \ t \ge 0\}$ *has independent increments on disjoint intervals;*

(c) $Pr\{N_{t+h} - N_t = 1\} = \lambda(t)h + o(h)$, *for all* t, $h \ge 0$;

(d) $Pr\{N_{t+h} - N_t > 1\} = o(h)$, *for all* t, $h \ge 0$,

where $o(h)$ *means* $\lim_{h \searrow 0} \frac{o(h)}{h} = 0$.

The function Λ defined by

$$\Lambda(t) = \int_0^t \lambda(v)dv, \quad \text{for } t \ge 0, \tag{2.1}$$

is called the hazard function or the cumulative intensity function of the process.

Consider the number, $N_{[\tau, \tau+t)}$, in an interval of the form $[\tau, \ \tau + t)$, where $\tau \ge 0$, $t \ge 0$. The time parameter τ, is called the initial age of the process. It is the beginning of the time interval where claims start to be counted. We refer to Panjer and Willmot (1992), Rolski et al. (1999) and De Vylder (1996) for the following theorems.

Theorem 2.1 *The non-homogeneous Poisson counting process* $\{N_t; \ t \ge 0\}$ *with intensity function* λ *has the following properties:*

(a) The number of claims in the interval $[\tau, \ \tau + t)$ *is Poisson distributed with mean* $\int_\tau^{\tau+t} \lambda(v)dv = \Lambda(\tau + t) - \Lambda(\tau)$. *That is, for all* $\tau, t \ge 0$,

$$Pr\{N_{[\tau, \tau+t)} = n\} = \frac{e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}[\Lambda(\tau + t) - \Lambda(\tau)]^n}{n!}, \quad \text{for } n \in N. \tag{2.2}$$

(b) Let $I, J, \ldots, K$ *be bounded non-overlapping intervals in* $[0, \infty)$. *Then the random variables* $N_I, N_J, \ldots, N_K$, *the number of claims in these intervals, are mutually independent.*

14

Proof. See De Vylder (1996), pp.77-78. □

By the theorem above, it is known that for a NHP process with intensity function λ, $N_{[\tau,\,\tau+t)}$ has a Poisson distribution with mean $\Lambda(\tau + t) - \Lambda(\tau)$, where Λ is defined in (2.1).

In addition, the probability that the first claim occurs in $[t, t + h)$ is given by

$$Pr\{N_{[0,\,t)} = 0, N_{[t,\,t+h)} = 1\} \approx \lambda(t)e^{-\Lambda(t)}h,$$

when h is small enough, or analogously, if claim counting starts at age τ, then the probability that the first claim be counted in interval $[\tau + t,\ \tau + t + h)$ is given by

$$Pr\{N_{[\tau,\,\tau+t)} = 0, N_{[\tau+t,\,\tau+t+h)} = 1\} \approx \lambda(\tau + t)e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}h,$$

where h is also small enough.

Theorem 2.2 *Suppose that $\{N_t^{(1)},\ t \geq 0\}$ and $\{N_t^{(2)},\ t \geq 0\}$ are two independent NHP process with intensity functions λ_1 and λ_2 respectively. The superposition $\{N_t,\ t \geq 0\}$, where $N_t = N_t^{(1)} + N_t^{(2)}$ is also a NHP process with intensity function $\lambda = \lambda_1 + \lambda_2$.*

Proof. From the definition and the property of the Poisson distribution, it is sufficient to prove that $\{N_t,\ t \geq 0\}$ has independent increments. Let $x_k \in \mathbb{N}^+$ and $t_0 = 0 < t_1 < \ldots < t_n = t$. Then,

$$P(\bigcap_{k=1}^{n}\{N_{t_k} - N_{t_{k-1}} = x_k\}) = P(\bigcap_{k=1}^{n}\{[N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)}] + [N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)}] = x_k\})$$

$$= \sum_{\substack{0 \leq y_k \leq x_k \\ k=1,\ldots,n}} P(\bigcap_{k=1}^{n}\{[N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)}] + [N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)}] = x_k\}\ |$$

$$\bigcap_{k=1}^{n}\{N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)} = y_k\})P(\bigcap_{k=1}^{n}\{N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)} = y_k\}).$$

15

By the independence of $N_t^{(1)}$ and $N_t^{(2)}$,

$$P(\bigcap_{k=1}^{n} \{N_{t_k} - N_{t_{k-1}} = x_k\}) = \sum_{\substack{0 \leq y_k \leq x_k \\ k=1,\dots,n}} P(\bigcap_{k=1}^{n} \{N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)} = x_k - y_k\})$$

$$P(\bigcap_{k=1}^{n} \{N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)} = y_k\})$$

$$= \sum_{\substack{0 \leq y_k \leq x_k \\ k=1,\dots,n}} \prod_{k=1}^{n} P\{N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)} = x_k - y_k\}$$

$$P\{N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)} = y_k\}$$

$$= \prod_{k=1}^{n} P\{[N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)}] + [N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)}] = x_k\}$$

$$= \prod_{k=1}^{n} P\{N_{t_k} - N_{t_{k-1}} = x_k\},$$

and therefore $\{N_t, \ t \geq 0\}$ has independent increments. $\square$

It is clear that a NHP process becomes a homogeneous Poisson process when its intensity function λ does not depend on time, i.e. $\lambda(t) = \lambda$, for all $t \geq 0$, and therefore, $\Lambda(t) = \lambda t$.

The intensity or rate function mirrors properly the impact of the environmental conditions at any time t. As proposed by Lawless and Thiagarajah (1994), it allows to include the influence of past events into the model as well. Furthermore, some specific random variables can be constructed using the rate and hazard function, which also consider the impact of the surrounding condition for the model, as suggested by Kotz and Shanbhag (1980) and developed in Chukova et al. (1993) and in Dimitrov et al. (1997). The link between the intensity and failure rate functions, used in reliability models under a minimal repair policy [see for example, Block et al. (1993), Beichelt (1991), or Baxter (1982)], gives an advantage to explore the applicability of NHP processes. Some characterization properties of the NHP process with periodic failure rate have been shown, in Chukova et al. (1993) and Dimitrov et al. (1997), which are also exploited to risk models in Garrido et al. (1996, 2001) and is described in

next section.

If the claim counting process $\{N_t;\ t \geq 0\}$ is a non-homogeneous Poisson process, then its corresponding claim process $\{S_t;\ t \geq 0\}$, given by

$$S_t = \begin{cases} \sum_{j=1}^{N_t} X_j & \text{if } N_t > 0 \\ 0 & \text{if } N_t = 0 \end{cases},$$

is called a compound NHP process, denoted as $S_t \sim C.P.[\Lambda;\ F_X]$ for $x \geq 0$, where $\{X_j\}$ are the claim severities, independent and identically distributed with common c.d.f. F_X and finite mean μ, independent of N_t.

We focus on the compound non-homogeneous Poisson process in the next two sections. There are only a few results based on this claim process in the risk theory literature. A compound non-homogeneous Poisson process with periodic claim intensity rate case, called periodic risk model, was considered by Garrido et al. (1996, 2001). Similar models are also considered by Asmussen and Rolski (1991, 1994), Beard, Pentikäinen and Pesonen (1984) and Dassios and Embrechts (1989) and are discussed later.

2.2 Short Term Periodicity

Now, we consider the case where the risk process evolves in a short term periodic environment, say one year, i.e. the claim arrival rate may depend on the seasons or the claim size distribution may vary with the time of the year. First, assume that only λ, the intensity function of a NHP process $\{N_t;\ t \geq 0\}$, is a periodic function with period 1, so that $t - \lfloor t \rfloor \in [0,\ 1)$ is the time of season, where $\lfloor t \rfloor$ is the integer part of $t \in R$. Also we say that $\{N_t;\ t \geq 0\}$ is a periodic Poisson process.

In fact, there are many seasonal conditions that affect insurance businesses. For example, weather factors affect the insured risk in automobile and fire insurance; factors such as seasonal snowstorms in the North and hurricanes or floods in the South affect property insurance. Therefore, it is reasonable to use a periodic intensity

function to model the claim occurrence process.

Referring to Dimitrov et al. (1997), we have the following properties for the NHP process $\{N_t;\ \geq 0\}$ with periodic intensity function.

Theorem 2.3 *Suppose that the intensity function λ is periodic with period c, then*

(a) The hazard function Λ has the almost linear property

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor \Lambda(c) + \Lambda(t - \lfloor \frac{t}{c} \rfloor c)\ ,\quad for\, t \geq 0.$$

(b) For any integer $n \geq 0$ and $t \geq 0$

$$P\{N_{[nc,\ nc+t)} = k\} = P\{N_t = k\}\ ,\quad for\, k = 0, 1, \ldots\ .$$

Moreover, the random variables N_{nc} and $N_{[nc,nc+t)}$ are mutually independent.

(c) The NHP process has a periodic intensity function λ with period $c > 0$ if and only if the random variables $N_{[0,\ c)}$ and $N_{[c,\ c+t)}$ are mutually independent and distributed as N_c and N_t, respectively.

(d) For any $t \geq 0$ the random variable N_t can be decomposed in the form

$$N_t = \begin{cases} N_{[0,\ t)} & if\, t \leq c \\ M_1 + M_2 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N_{[0,\ t-\lfloor \frac{t}{c} \rfloor c)} & if\, t > c \end{cases},$$

where $\{M_i\}_{i \geq 1}$ are i.i.d. Poisson random variables distributed as $N_{[0,\ c)}$ and independent of $N_{[0,\ t-\lfloor \frac{t}{c} \rfloor c)}$, where $N_{[0,\ t-\lfloor \frac{t}{c} \rfloor c)}$ is a Poisson r.v. distributed as N_y, for $y = t - \lfloor \frac{t}{c} \rfloor c \in [0,\ c)$.

Proof. See Dimitrov et al. (1997) pp.508-509. □

Consider $\{N_{[\tau,\ \tau+t)},\ t \geq 0\}$, the claim counting process of initial age τ, with periodic intensity function λ and period $c = 1$. Its corresponding hazard function Λ has a special structure, given below.

Theorem 2.4 *Suppose that the intensity function λ is periodic of period 1, then the hazard function during the time period $[\tau,\ \tau+t)$, i.e. $\Lambda(\tau+t) - \Lambda(\tau)$, has the following*

18

structure.

(i) If t is an integer, then

$$\Lambda(\tau + t) - \Lambda(\tau) = \Lambda(1)t.$$

(ii) If τ is an integer but t is not, then

$$\Lambda(\tau + t) - \Lambda(\tau) = \Lambda(1)\lfloor t \rfloor + \Lambda(t - \lfloor t \rfloor).$$

(iii) If neither τ nor t are integers, then

$$\Lambda(\tau + t) - \Lambda(\tau) = \begin{cases} \Lambda(1)\lfloor t \rfloor + \Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) - \Lambda(\tau - \lfloor \tau \rfloor) \\ \qquad\qquad if \ \lfloor \tau \rfloor + 1 - \tau \geq t - \lfloor t \rfloor \\ \Lambda(1)(\lfloor t \rfloor + 1) + \Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor - 1) - \Lambda(\tau - \lfloor \tau \rfloor) \\ \qquad\qquad otherwise \end{cases}.$$

Proof. According to the definition of $\Lambda(\tau + t) - \Lambda(\tau)$, that is,

$$\Lambda(\tau + t) - \Lambda(\tau) = \int_{\tau}^{\tau+t} \lambda(v)dv,$$

and the periodicity of function λ, results follow. $\qquad\qquad\qquad\qquad\qquad\square$

Now we consider a compound NHP claim process with periodic intensity function λ and again period $c = 1$. Assume that claim severities, $\{X_j\}_{j\geq 1}$, are i.i.d. random variables with c.d.f. F_X, independent of time. Then aggregate claims over the time interval $[\tau, \ \tau + t)$ are given by

$$S_{[\tau, \ \tau+t)} = \sum_{n=1}^{N_{[\tau, \ \tau+t)}} X_n, \qquad\qquad\qquad (2.3)$$

where $N_{[\tau, \ \tau+t)}$ is supposed to be a NHP process and $S_{[\tau, \ \tau+t)} = 0$ if $N_{[\tau, \ \tau+t)} = 0$. For this process with initial age τ, the following results for $S_{[\tau, \ \tau+t)}$ hold:

(i) If t is an integer, independent of the initial age value τ, then by Theorem 2.3-(d) $S_{[\tau, \ \tau+t)}$ can be decomposed as

$$S_{[\tau, \ \tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor}, \qquad\qquad\qquad (2.4)$$

19

where all S_i's are i.i.d. random variables distributed as

$$S_1 = \sum_{n=1}^{N_1} X_n,$$

and N_1 is a Poisson r.v. with parameter $\Lambda(1) = \int_0^1 \lambda(v)dv$.

(*ii*) If τ is an integer but t is not, then the claim counting process $\{N_{[\tau, \ \tau+t)}, \ t \geq 0\}$ is equivalent to the process which has the same time period but starts from $\tau = 0$, i.e. $\{N_{[0, \ t)}, \ t \geq 0\}$. Thus $S_{[\tau, \ \tau+t)}$ can be decomposed as

$$S_{[\tau, \ \tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S_{t-\lfloor t \rfloor}, \tag{2.5}$$

where the last term is also a compound Poisson random variable, with parameter $\Lambda(t - \lfloor t \rfloor)$ for $t - \lfloor t \rfloor \in [0, \ 1)$, independent of other aggregate claims S_i.

(*iii*) If neither τ nor t are integer, then $S_{[\tau \ \tau+t)}$ may contain one or two incomplete terms. When $\lfloor \tau \rfloor + 1 - \tau \geq t - \lfloor t \rfloor$, only one incomplete term appears as if aggregating claims on the time interval $[\tau - \lfloor \tau \rfloor, \ \tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) \subset [0, \ 1)$. In this case, $S_{[\tau, \ \tau+t)}$ can be decomposed as

$$S_{[\tau, \ \tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S_{[\tau-\lfloor \tau \rfloor, \ \tau-\lfloor \tau \rfloor+t-\lfloor t \rfloor)}, \tag{2.6}$$

where the last term is a Poisson random variable with parameter $\Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) - \Lambda(\tau - \lfloor \tau \rfloor)$, independent of other Poisson random variables with parameter $\Lambda(1)$.

Otherwise, two incomplete terms appear in the decomposition form of $S_{[\tau, \ \tau+t)}$. One is equivalent to the accumulated claims from time $\tau - \lfloor \tau \rfloor$ to the end of the year, while the other is equal to accumulate claims from the beginning of the year up to $t - \lfloor t \rfloor - (\lfloor \tau \rfloor + 1 - \tau)$. Consequently, $S_{[\tau, \ \tau+t)}$ can be decomposed as

$$S_{[\tau, \ \tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S_{t-\lfloor t \rfloor-(\lfloor \tau \rfloor+1-\tau)} + S_{[\tau-\lfloor \tau \rfloor, \ 1)}, \tag{2.7}$$

where both incomplete terms are mutually independent Poisson random variables, with parameters $\Lambda(t - \lfloor t \rfloor - (\lfloor \tau \rfloor + 1 - \tau))$ and $\Lambda(1) - \Lambda(\tau - \lfloor \tau \rfloor)$ respectively, independent of other aggregate claims as well.

20

Moreover, the moment generating function of $S_{[\tau, \tau+t)}$ can be gotten as

$$E(e^{rS_{[\tau, \tau+t)}}) = e^{[\Lambda(\tau+t)-\Lambda(\tau)][M_X(r)-1]}, \tag{2.8}$$

and moments of $S_{[\tau, \tau+t)}$ also can be obtained from (2.8). For example, the total initial premium is

$$E(S_{[\tau, \tau+t)}) = [\Lambda(\tau+t) - \Lambda(\tau)]E(X_1).$$

2.3 Ruin Probabilities

This section, discusses the ruin problem for a general compound NHP process, with known intensity function λ and premium rate β. First, a Volterra integral equation similar to (1.9) is derived. Then the embedded discrete risk model, in short term periodic case, is presented and bounds for the ruin probability of the continuous-time risk model are derived. Finally, we introduce a more general periodic Poisson process and give corresponding two-sided bounds for the probability of ruin.

As in the classical risk model, assume that claim severities are i.i.d. with common cumulative distribution function F_X, independent of time t, and finite mean μ.

Consider the risk reserve process, over the time interval $[\tau, \tau+t)$, for a compound NHP model with initial value of u at time τ. We denote it by $R_{[\tau, \tau+t)}(u)$ and it is given as

$$R_{[\tau, \tau+t)}(u) = u + \beta t - \sum_{j=1}^{N_{[\tau, \tau+t)}} X_j, \quad \text{for } \tau, \ t \geq 0, \tag{2.9}$$

where $N_{[\tau, \tau+t)}$ is the corresponding NHP claim counting process, with intensity function λ.

Let

$$T_\tau(u) = \inf\{t \geq 0 ; \quad R_{[\tau, \tau+t)}(u) < 0\}$$

denote the time to ruin for the above reserve process, with initial value u at initial

age τ, and define the probability that ruin occurs before time $\tau + t$ as

$$
\begin{aligned}
\Psi_{[\tau,\,\tau+t)}(u) &= \Pr\{T_\tau(u) \le t\} \\
&= \Pr\{R_{[\tau,\,\tau+s)}(u) < 0 \,, \quad \text{for some } 0 < s \le t \,\}.
\end{aligned}
\tag{2.10}
$$

The ultimate ruin probability with initial value u and initial age τ is then

$$
\Psi(\tau,\,u) = \lim_{t\to\infty} \Psi_{[\tau,\,\tau+t)}(u).
\tag{2.11}
$$

Like the parallel definition for the classical risk model, shown in (1.7), this is a finite time ruin problem and it depends not only on the initial value u and finite time period t, but also on the initial time age τ.

2.3.1   Volterra Integral Equation

Referring to Garrido et al. (1996, 2001), an analogous Volterra integral equation for $\Psi_{[\tau,\,\tau+t)}$ is given by the following theorem.

Theorem 2.5 *The probability of ruin to time t beginning with initial reserve u and initial age τ, satisfies the integral equation:*

$$
\begin{aligned}
\Psi_{[\tau,\,\tau+t)}(u) &= \int_0^t \lambda(\tau + v)e^{-[\Lambda(\tau+v)-\Lambda(\tau)]}[1 - F_X(u + \beta v)]dv \\
&\quad + \int_0^t \lambda(\tau + v)e^{-[\Lambda(\tau+v)-\Lambda(\tau)]} \int_0^{u+\beta v} \Psi_{[\tau+v,\,\tau+t)}(u + \beta v - y)dF_X(y)dv,
\end{aligned}
\tag{2.12}
$$

where λ, Λ are the intensity and hazard function respectively, and F_X is the common distribution function of claim severity $\{X_j\}_{j\ge 1}$.

Proof.   Consider what will happen on the first claim. The time until the first claim occurs is exponentially distributed with known intensity function λ, then the probability density function is $\lambda(\tau + v)e^{-\int_0^v \lambda(\tau+s)ds} = \lambda(\tau + v)e^{-[\Lambda(\tau+v)-\Lambda(\tau)]}$. If the claim occurs at time $v > 0$, the surplus available to pay the claim at time v is $u + \beta v$.

Thus, ruin occurs on the first claim if the amount of the claim exceeds $u + \beta v$. The probability that this happens is $1 - F_X(u + \beta v)$.

On the other hand, if the amount of the claim is y, where $0 \le y \le u + \beta v$, ruin does not occur on the first claim. After payment of the claim, there is still a surplus of $u + \beta v - y$ remaining. Ruin can still occur on the rest of the time interval $[\tau + v, \ \tau + t)$ with initial reserve $u + \beta v - y$ at time $\tau + v$ with probability $\Psi_{[\tau+v, \ \tau+t)}(u + \beta v - y)$. Therefore, by the law of total probability, we have the recursive equation.

Let R_v denote the reserve value at time v, and T_τ be the ruin time with initial age τ.

$$
\begin{aligned}
\Psi_{[\tau, \ \tau+t)}(u) &= Pr\{T_\tau(u) \le t\} = Pr\{T_\tau \le t \mid R_\tau = u\} \\
&= \int_0^t \int_0^\infty Pr\{T_\tau \le t \mid R_\tau = u, \ T_1 = v, \ X_1 = y\} dF_X(y) \\
&\quad \lambda(\tau + v)e^{-[\Lambda(\tau+v) - \Lambda(\tau)]} dv \\
&= \int_0^t \int_0^\infty Pr\{T_\tau \le t - v \mid R_{\tau+v} = u + \beta v - y\} dF_X(y) \\
&= \int_0^t \lambda(\tau + v)e^{-[\Lambda(\tau+v) - \Lambda(\tau)]} \\
&\quad \int_0^\infty \Psi_{[\tau+v, \ \tau+t)}(u + \beta v - y) dF_X(y) dv \\
&= \int_0^t \lambda(\tau + v)e^{-[\Lambda(\tau+v) - \Lambda(\tau)]} \\
&\quad \{\int_0^{u+\beta v} \Psi_{[\tau+v, \ \tau+t)}(u + \beta v - y) dF_X(y) + [1 - F_X(u + \beta v)]\} dv.
\end{aligned}
$$

$\square$

For some special cases of (2.12), we get the following corollaries.

Corollary 2.1 *In (2.12), let $\tau = 0$, then the probability of ruin to time t, beginning with initial reserve u and initial age 0, satisfies the integral equation*

$$
\begin{aligned}
\Psi_{[0, \ t)}(u) &= \int_0^t \lambda(v)e^{-\Lambda(v)}[1 - F_X(u + \beta v)] dv \\
&\quad + \int_0^t \lambda(v)e^{-\Lambda(v)} \int_0^{u+\beta v} \Psi_{[v, \ t)}(u + \beta v - y) dF_X(y) dv.
\end{aligned}
$$

Corollary 2.2 *In (2.12), let $t \uparrow \infty$, then the ultimate probability of ruin, beginning with initial reserve u and initial age τ, satisfies the integral equation*

$$\Psi(\tau, u) = \int_0^\infty \lambda(\tau + v) e^{-[\Lambda(\tau+v)-\Lambda(\tau)]}[1 - F_X(u + \beta v)]dv$$

$$+ \int_0^\infty \lambda(\tau + v) e^{-[\Lambda(\tau+v)-\Lambda(\tau)]}$$

$$\int_0^{u+\beta v} \Psi(\tau + v, u + \beta v - y)dF_X(y)dv$$

Particularly, when $\tau = 0$ and $\beta = 1$, the corresponding Volterra integral equation is given as

$$\Psi(0, u) = \int_u^\infty \lambda(s - u) e^{-\Lambda(s-u)}[1 - F_X(s)]ds$$

$$+ \int_u^\infty \lambda(s - u) e^{-\Lambda(s-u)} \int_0^s \Psi(s - u, s - y)dF_X(y)ds$$

Remark 2.1 *In (2.12), if $\tau = 0$ and $\lambda(t) = \lambda$, after taking a derivative with respect to time t, we get (1.9), the partial integro-differential equation about $\Psi(u, t)$ in the classical case with constant initial value u.*

2.3.2 Embedded Discrete Risk Model

It is difficult to solve (2.12) analytically, even for the exponential claim size case. Consider the special case where the intensity function λ is periodic with period $c = 1$. As in the classical case, an embedded discrete risk model can be used to give a two-sided bound for the ruin probability $\Psi_{[0,t)}(u)$ [see Garrido et al. (1996, 2001)].

By Theorem 2.3-(d), the surplus process $R_{[0,t)}(u)$, with zero initial age, can be represented as follows (all equalities are in distribution):

$$R_{[0,t)}(u) = u + \beta t - \sum_{n=1}^{N_t} X_n$$

$$= u + \beta \lfloor t \rfloor + \beta(t - \lfloor t \rfloor) - \sum_{k=1}^{\lfloor t \rfloor} \sum_{n=1}^{M_k} X_n^{(k)} - \sum_{n=1}^{N_{t-\lfloor t \rfloor}} X_n^{(\lfloor t \rfloor+1)}$$

$$= u + \sum_{k=1}^{\lfloor t \rfloor} [\beta - \sum_{n=1}^{M_k} X_n^{(k)}] + [\beta(t - \lfloor t \rfloor) - \sum_{n=1}^{N_{t-\lfloor t \rfloor}} X_n^{(\lfloor t \rfloor+1)}],$$

where $t - \lfloor t \rfloor \in [0, 1)$, $\{X_n^{(k)}\}_{n \geq 0}$ are independent sequences of i.i.d. random variables in period k, for $k = 1, 2, \ldots$, with common claim size distribution function as X_1, and M_k, as well as $N_{t-\lfloor t \rfloor}$, are i.i.d. Poisson variables with parameter $\Lambda(1)$ and $\Lambda(t - \lfloor t \rfloor)$ respectively, independent of $X_n^{(k)}$.

Let

$$U_k = \beta - \sum_{n=1}^{M_k} X_n^{(k)}, \quad \text{for } k \geq 1, \tag{2.13}$$

and $U_0 = u$, denote the gain random variables, where $\sum_{n=0}^{M_k} X_n^{(k)}$, for $k = 1, 2, \ldots$, the aggregate claims during the kth period, are i.i.d. compound Poisson distributed with parameter $\Lambda(1)$. Thus, these random variables $\{U_k\}_{k \geq 1}$ are i.i.d., with common moment generating function given by

$$M_U(r) = e^{r\beta - \Lambda(1)[M_X(r) - 1]}.$$

Now, consider the random walk $\{(k, \, S_k(u)); \; k \geq 0\}$, or discrete time surplus process, defined by

$$S_k(u) = U_0 + U_1 + \cdots + U_k, \tag{2.14}$$

and let

$$\widetilde{T}(u) = \inf\{k \geq 0 \; ; \quad S_k(u) < 0\} \tag{2.15}$$

denote its first passage time to a negative value, that is, the time of ruin in a discrete-time risk model, with initial value u at time 0. Let

$$\widetilde{\Psi}(u, \, k) = Pr\{\widetilde{T}(u) \leq k\} \tag{2.16}$$

denote the finite time ruin probability that the random walk hits a negative value within k steps, with initial value u.

Since $R_{[0, \, t)}$ and $S_t(u)$ are related at integer values of t, that is,

$$R_{[0, \, \lfloor t \rfloor)}(u) = S_{\lfloor t \rfloor}(u), \quad \text{for any } t \geq 0, \tag{2.17}$$

25

it is possible to use the time to ruin for the embedded discrete risk model, $\widetilde{T}(u)$ in (2.15), to approach the time to ruin for the original continuous-time risk model with periodic NHP claim counting process [see Garrido et al. (1996, 2001)]. A basic result related to this is given below.

Theorem 2.6 *The time to ruin for the continuous-time risk model is stochastically equivalent to the corresponding time to ruin for the embedded discrete model, in the sense that the following inequalities hold:*

$$P\{\widetilde{T}(u) > \lfloor t \rfloor + 1\} \leq P\{T_0(u) > t\} \leq P\{\widetilde{T}(u) > \lfloor t \rfloor\}.$$

In other words, for the ruin probability $\Psi_{[0,\,t)}(t)$, the following holds:

$$\widetilde{\Psi}(u,\ \lfloor t \rfloor) \leq \Psi_{[0,\,t)}(u) \leq \widetilde{\Psi}(u,\ \lfloor t \rfloor + 1). \tag{2.18}$$

Proof. See Garrido et al. (1996, 2001) . $\qquad\qquad\qquad\qquad\qquad\qquad$ □

As shown in (2.18), a two-sided bound for the finite time ruin probability $\Psi_{[0,\,t)}$, is given by the corresponding consecutive finite time ruin probability for the embedded discrete risk model.

In addition, the finite time ruin probabilities $\widetilde{\Psi}(u,\ k)$ satisfy the following recursive equation:

$$\widetilde{\Psi}(u,\ k+1) = 1 - F_U(u + \beta) + \int_0^{u+\beta} \widetilde{\Psi}(u + \beta - y,\ k)dF_U(y), \tag{2.19}$$

where F_U is the common c.d.f. of the U_k's, under the boundary condition that $\widetilde{\Psi}(u,\ 0) = 0$ for all $u \geq 0$.

The inequalities in (2.18) imply that the probability of ultimate ruin in the embedded discrete model is the same as that in continuous-time surplus process.

A lower bound for the ultimate ruin probability $\Psi_0(u) = \lim_{t \to \infty} \Psi_{[0,\,t)}(u)$ can also be found [see, for instance, Bowers et al. (1997)]. It satisfies

$$\Psi_0(u) \geq \frac{e^{-\tilde{\gamma} u}}{E\{e^{-\tilde{\gamma} S_{\widetilde{T}(u)}}\ \mid\ \widetilde{T}(u) < \infty\}},$$

where $\tilde{\gamma}$ is the positive solution of the equation

$$\Lambda(1)[M_X(r) - 1] - \beta r = 0.$$

2.3.3 Average Arrival Rate Risk Model

In practical situations, apart from the claim arrival rate that may vary with the time of the year, the claim size distribution may also depend on the seasons. We focus on the time-dependent claim size case in this section.

Let $\{F_t(x), \ t \geq 0\}$ be a family of distribution functions such that the mapping $t \to \int_0^\infty g(x) dF_t(x)$ is measurable and periodic with period 1 for all integrable functions g.

Assume that claims arrive according to the NHP process $\{N_t\}$, with periodic intensity function λ of period 1, and if a claim arrives at time t, then the claim size distribution is F_t, independent of everything else. Denote the moment generating function of F_t by

$$M_t(s) = \int_0^\infty e^{sx} dF_t(x).$$

Denote the average arrival rate by

$$\bar{\lambda} = \int_0^1 \lambda(v) dv = \Lambda(1), \tag{2.20}$$

while

$$F_X^0(x) = \frac{\int_0^1 \lambda(v) F_v(x) dv}{\bar{\lambda}} \tag{2.21}$$

is the distribution function of a typical claim size; a weighted average of distribution F_t for different time values t. It has mean

$$\mu_X^0 = \int_0^\infty x \, dF_X^0(x),$$

and moment generating function

$$M_{F_X^0}(s) = \frac{\int_0^1 \lambda(v) M_v(s) dv}{\bar{\lambda}}.$$

Thus, the claim counting process $\{N_t, \ t \geq 0\}$ is a special homogeneous Poisson process with average arrival rate $\bar{\lambda}$. Its corresponding aggregate claims process $\{S_t, \ t \geq 0\}$, where $S_t = \sum_{j=0}^{N_t} X_j$ with $S_t = 0$ if $N_t = 0$, is also a special compound

Poisson process with common cumulative distribution function F_X^0. Furthermore, its related risk reserve process $\{R_t, \ t \geq 0\}$, shown in (1.2), is called average arrival rate risk model.

Now consider the income process $\{R_t - u, \ t \geq 0\}$ instead of risk reserve process $\{R_t, \ t \geq 0\}$. First, the Laplace-Stieltjes transform of $R_t - u$, denoted as $L(t; \ s) = Ee^{-s(R_t-u)}$, is given in the following lemma.

Lemma 2.1 *For $s, \ t \geq 0$,*

$$L(t; \ s) = e^{-\beta st + \int_0^t \lambda(v)[M_v(s)-1]dv}, \tag{2.22}$$

where β is the constant premium rate.

Proof. See Rolski et al. (1999), p.526. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We refer to Asmussen & Rolski (1994) and Rolski et al. (1999) for the following average arrival rate risk model and its related two-sided bounds and asymptotic formula for the ruin probability $\Psi(u)$.

Let $s_0 = \sup\{s \geq 0 : \sup_{v \in [0, 1)} M_v(s) < \infty\}$ and define

$$\theta^*(s) = \bar{\lambda}[M_{F_X^0}(s) - 1] - \beta s. \tag{2.23}$$

Let $\gamma > 0$ be the solution of (2.23), that is, $\int_0^1 \lambda(v)[M_v(\gamma) - 1]dv = \beta\gamma$, called the adjustment coefficient for the average arrival rate risk model. Then $\gamma > 0$ fulfills $\theta^*(\gamma) = 0$. We assume that such a γ exists in $[0, \ s_0)$ in the sequel of this section since $\theta^*(0) = 0$ and the derivative of $\theta^*(s)$ at zero is $\bar{\lambda}\mu_X^0 - \beta < 0$. The convexity of $\theta^*(s)$ ensures that

$$(\theta^*)'(\gamma) = \bar{\lambda} \int_0^\infty x e^{\gamma x} dF_X^0(x) - \beta > 0.$$

Let $x_v = \sup\{y : F_v(y) < 1\}$. We have following two-sided bound for $\Psi(u)$ in the periodic Poisson model, which has time-dependent claim size distribution.

Theorem 2.7 *For the ruin probability $\Psi(u)$ in the periodic Poisson model, where the claim size distribution is time-dependent as F_t, the following inequalities hold:*

$$a_- e^{-\gamma u} \leq \Psi(u) \leq a_+ e^{-\gamma u}, \tag{2.24}$$

28

where

$$a_- = \inf_{0 \le v \le 1} L(v; \gamma) \alpha_-(v), \quad a_+ = \sup_{0 \le v \le 1} L(v; \gamma) \alpha_+(v),$$

$$\alpha_-(v) = \inf_{0 \le x \le x_v} \alpha(x, v), \quad \alpha_+(v) = \sup_{0 \le x \le x_v} \alpha(x, v),$$

and

$$\alpha(x, v) = \frac{\bar{F}_v(x)}{\int_x^\infty e^{\gamma(y-x)} dF_v(y)}.$$

Proof. See Rolski et al. (1999), p.529. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now consider the special case in the above periodic Poisson model, where the claim size distribution is no longer time-dependent, that is, $F_t = F_X$. Thus, its moment generating function is $M(s) = \int_0^\infty e^{sx} dF_X(x)$, and the Laplace-Stieltjes transform (2.22) has the form $L(t; s) = e^{-\beta st + [M_X(s)-1]\Lambda(t)}$, where $\Lambda(t) = \int_0^t \lambda(v) dv$. Moreover, in this case, the average arrival rate risk model can be described as having the following features

$$\bar{\lambda} = \int_0^1 \lambda(v) dv, \quad F_X^0(x) = F_X(x);$$

$$\mu_X^0 = \int_0^\infty x dF_X(x) = \mu, \quad M_{F_X^0}(s) = M_X(s).$$

Thus, we have $s_0 = \sup\{s \ge 0 : M_X(s) < \infty\}$ and (2.23) has the form

$$\theta^*(s) = \bar{\lambda}[M_X(s) - 1] - \beta s, \qquad\qquad\qquad (2.25)$$

which is exactly the same as in the classical case. Still assume that there exists a $\gamma > 0$ such that $\theta^*(\gamma) = 0$. Then we obtain two-sided bounds for ruin probabilities $\Psi(u)$ in this special periodic Poisson model, below. Let $x_0 = \sup\{y : F_X(y) < 1\}$.

Corollary 2.3 *For the ruin probability $\Psi(u)$ in the periodic Poisson model, where the claim size distribution does not depend on the time, the following inequalities hold:*

$$a_-^* e^{-\gamma u} \le \Psi(u) \le a_+^* e^{-\gamma u}, \qquad\qquad\qquad (2.26)$$

where

$$a_-^* = \alpha_-^* \inf_{0 \le v \le 1} e^{-\beta \gamma [v - \frac{\Lambda(v)}{\bar{\lambda}}]}, \quad a_+^* = \alpha_+^* \sup_{0 \le v \le 1} e^{-\beta \gamma [v - \frac{\Lambda(v)}{\bar{\lambda}}]},$$

$$\alpha_-^* = \inf_{0 \le x \le x_0} \alpha(x), \quad \alpha_+^* = \sup_{0 \le x \le x_0} \alpha(x), \tag{2.27}$$

and

$$\alpha(x) = \frac{\bar{F}_X(x)}{\int_x^\infty e^{\gamma(y-x)} dF_X(y)}.$$

Proof. Under assumption of the model, γ is the solution of (2.25), that is,

$$\theta^*(\gamma) = \bar{\lambda}[M_X(\gamma) - 1] - \beta\gamma = 0.$$

Hence, the Laplace-Stieltjes transform $L(v; \gamma)$ is obtained as

$$\begin{aligned}
L(v; \gamma) &= e^{-\beta\gamma v + [M_X(\gamma) - 1]\Lambda(v)} \\
&= e^{-\beta\gamma(v - \frac{\Lambda(v)}{\lambda})}.
\end{aligned}$$

Then (2.26) holds by Theorem 2.6. □

Corollary 2.4 *Assume there is λ_{max}, such that*

$$\lambda(t) \le \lambda_{max}, \quad for\ 0 \le t \le 1. \tag{2.28}$$

Then under the same condition of Corollary 2.3, the following inequalities hold:

$$a_-^{**} e^{-\gamma u} \le \Psi(u) \le a_+^{**} e^{-\gamma u}, \tag{2.29}$$

where

$$a_-^{**} = \alpha_-^* e^{-\beta\gamma}, \quad a_+^{**} = \alpha_+^* e^{\beta\gamma \frac{\lambda_{max}}{\lambda}},$$

and α_-^*, α_+^* are shown in (2.27).

Proof. The relation (2.28) implies that

$$\Lambda(v) = \int_0^v \lambda(s) ds \le \lambda_{max} v \le \lambda_{max}, \quad for\ 0 \le v \le 1,$$

then

$$L(v; \gamma) \le e^{\beta\gamma \frac{\Lambda(v)}{\lambda}} \le e^{\beta\gamma \frac{\lambda_{max}}{\lambda}},$$

and

$$L(v; \gamma) \ge e^{-\beta\gamma(1 - \frac{\Lambda(v)}{\lambda})} \ge e^{-\beta\gamma}.$$

Hence, (2.29) holds. □

Corollary 2.5 *Let $\lambda(t)$ take a specific parametric form from the beta family:*

$$\lambda(t) = \lambda_0 t^{p-1}(1-t)^{q-1}, \quad \text{for } 0 \le t \le 1, \ p, \ q > 1$$

and $g(v) = v - \frac{\Lambda(v)}{\bar\lambda} = v - \frac{B(p,\,q,\,v)}{B(p,\,q)}$, for $0 \le v \le 1$, where $B(p,\ q,\ v)$ is the incomplete beta function. Assume that v_{max} exists such that

$$\sup_{0 \le v \le 1} g(v) = g(v_{max}),$$

then

$$a_-^* e^{-\gamma u} \le \Psi(u) \le a_+^* e^{-\gamma u}, \tag{2.30}$$

where

$$a_-^* = \alpha_-^* e^{-\beta\gamma\left[v_{max} - \frac{B(p,\,q,\,v_{max})}{B(p,\,q)}\right]},$$

$$a_+^* = \alpha_+^* e^{\frac{\beta\gamma}{B(p,\,q)} \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}},$$

and $\alpha_-^*, \ \alpha_+^*$ are shown in (2.27).

Proof. Since in this case,

$$a_-^* = \alpha_-^* \inf_{0 \le v \le 1} e^{-\beta\gamma g(v)} \ \ge \ \alpha_-^* e^{-\beta\gamma g(v_{max})}$$

$$= \ \alpha_-^* e^{-\beta\gamma\left[v_{max} - \frac{B(p,\,q,\,v_{max})}{B(p,\,q)}\right]}$$

and $\bar\lambda = \lambda_0 B(p,\ q)$, $\lambda_{max} = \lambda_0 \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}$, the result holds. $\square$

Asmussen and Rolski (1994) also mentions that under some additional conditions, there is a Cramér-Lundberg type approximation for $\Psi(u)$ in the form of

$$\Psi(u) \sim c e^{-\gamma u}, \quad u \to \infty.$$

2.4 Illustrations

2.4.1 Short-term Periodicity for the Counting Process

In this section, we consider some specific periodic intensity functions and illustrate their corresponding short-term periodic Poisson counting processes.

Example 2.1 *Properties of the short-term periodic Poisson counting process with beta-type claim intensity rate.*

As in Corollary 2.5, consider a beta-type intensity rate

$$\lambda(t) = \lambda(t - \lfloor t \rfloor)^{p-1}\{1 - (t - \lfloor t \rfloor)\}^{q-1} , \quad t > 0, \tag{2.31}$$

where the parameters λ, p, $q > 0$. It is a periodic intensity function which reflects the cyclic environmental condition in period $[0, 1)$, say one year. The particular shape depends on the values of the shape parameters p and q, and covers many possible forms of annual claim intensities. For instance, when both p and q are less then 1, a concave shape is obtained, while a convex shape is obtained when both p and q are greater than 1. By selecting the values of parameters p, q appropriately, many other intermediate shapes are obtained.

The corresponding hazard function $\Lambda(t)$ is given by

$$\begin{aligned}
\Lambda(t) &= \int_0^t \lambda(v)dv = \int_0^t \lambda v^{p-1}(1 - v)^{q-1}dv \\
&= \lambda\{\lfloor t \rfloor B(p, q) + B(p, q; t - \lfloor t \rfloor)\} , \quad \text{for } t > 0,
\end{aligned} \tag{2.32}$$

where

$$B(p, q) = \int_0^1 v^{p-1}(1 - v)^{q-1}dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}$$

is the beta function at p, $q > 0$, and

$$B(p, q; t) = \int_0^t v^{p-1}(1 - v)^{q-1}dv , \quad \text{at } t \in [0, 1)$$

is the usual incomplete beta function. Figure 2.1 shows the shape of $\lambda(t)$ and $\Lambda(t)$ when $\lambda = 50, p = 5$, and $q = 2$.

Consider one year period insured events. $N_{[\tau, \tau+t)}$ denotes the number of claims, with similar policy in a portfolio, recorded in the time interval $[\tau, \tau + t)$, and is assumed to follow the non-homogeneous Poisson distribution with parameter $\lambda(t)$ as in (2.31). By Theorems 2.1, 2.3 and 2.4, we get the following results.

$\lambda(t)$–solid black curve, $\Lambda(t)$–dotted blue curve.
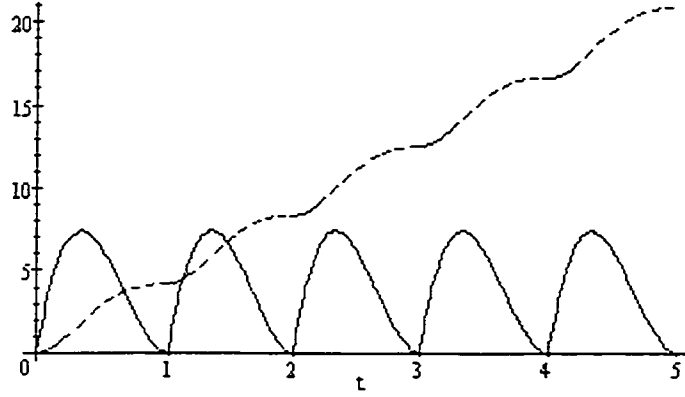
$\lambda = 50,\ p = 2$ and $q = 3$.

Figure 2.1: Functions $\lambda(t)$ and $\Lambda(t)$.

(i) The probability of $n \in \mathbb{N}$ claims in the time interval $[\tau,\ \tau + t)$, is given by

$$Pr\{N_{[\tau,\ \tau+t)} = n\} = \frac{\left[\int_\tau^{\tau+t} \lambda(v)dv\right]^n}{n!} e^{-\int_\tau^{\tau+t} \lambda(v)dv}$$

$$= \frac{[\Lambda(\tau + t) - \Lambda(\tau)]^n}{n!} e^{-[\Lambda(\tau+t)-\Lambda(\tau)]},$$

where $\Lambda(\tau + t) - \Lambda(\tau)$ is derived from Theorem 2.4 and (2.32) as

$$\Lambda(\tau + t) - \Lambda(\tau) = \begin{cases} \lambda B(p,q)\lfloor t \rfloor & \text{if } t \in \mathbb{N} \\ \lambda\{B(p,q)\lfloor t \rfloor + B(p,q,t - \lfloor t \rfloor)\} & \text{if } \tau \in \mathbb{N},\ t \notin \mathbb{N} \\ \lambda\{B(p,q)\lfloor t \rfloor + B(p,q,\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) \\ \quad - B(p,q,\tau - \lfloor \tau \rfloor)\} & \text{if } \tau, t \notin \mathbb{N} \text{ and } \lfloor \tau \rfloor + 1 - \tau \geq t - \lfloor t \rfloor \\ \lambda\{B(p,q)(\lfloor t \rfloor + 1) + B(p,q,\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor - 1) \\ \quad - B(p,q,\tau - \lfloor \tau \rfloor)\} & \text{otherwise} \end{cases}$$

$$(2.33)$$

The shape of $\Lambda(\tau + t) - \Lambda(\tau)$ as a function of initial age $\tau \in [0,1)$, when $\lambda = 50$, $p = 2$, $q = 3$ and $t = 3.8$ is shown in Figure 2.2. This is also the expected number of claims in the interval $[\tau,\ \tau + t)$. The shape of $\Lambda(\tau + t) - \Lambda(\tau)$ depends on the shape

33

of the intensity rate $\lambda(t)$ and time t. Since the intensity rate function is skewed to the left and $t = 3.8$, it is clear that when τ changes over one period, the expected number of claims over $[\tau, \ \tau + t)$ first decreases to a minimum and then increases over the rest of the interval.

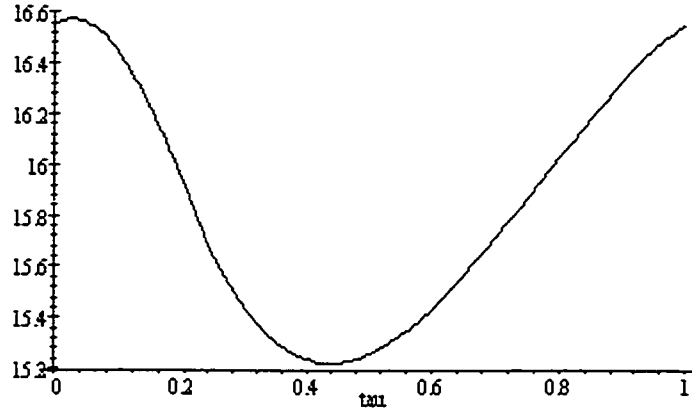$$\lambda = 50, \ p = 2, \ q = 3 \text{ and } t = 3.8.$$

Figure 2.2: $\Lambda(\tau + t) - \Lambda(\tau)$ as a function of initial age $\tau \in [0, 1)$.

Particularly, the number of claims over one year, N_1, is distributed as

$$
\begin{aligned}
Pr\{N_1 = n\} &= \frac{\left[\int_0^1 \lambda(v)dv\right]^n}{n!} e^{-\int_0^1 \lambda(v)dv} \\
&= \frac{[\lambda B(p, \ q)]^n}{n!} e^{-\lambda B(p, \ q)}, \quad \text{for } n = 0, \ 1 \ \ldots \ .
\end{aligned}
$$

(ii) The moment generating functions of $N_{[\tau, \ \tau+t)}$ and N_1 are obtained as

$$E\left(e^{rN_{[\tau, \ \tau+t)}}\right) = e^{[\Lambda(\tau+t) - \Lambda(\tau)](e^r - 1)}$$

and

$$E\left(e^{rN_1}\right) = e^{\Lambda(1)(e^r - 1)} = e^{\lambda B(p, \ q)(e^r - 1)},$$

respectively, where $\Lambda(\tau + t) - \Lambda(\tau)$ is given by (2.33).

(iii) The expected number of claims during any time interval $[\tau, \ \tau + t)$, equals its variance and is given by

$$E(N_{[\tau, \ \tau+t)}) = V(N_{[\tau, \ \tau+t)}) = \Lambda(\tau + t) - \Lambda(\tau),$$

34

and similarly

$$E(N_1) = V(N_1) = \Lambda(1) = \lambda B(p, q).$$

(iv) The probability to survive the time interval $[\tau, \ \tau + t)$ is

$$e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}$$

and for a one year period is

$$\alpha = e^{-\Lambda(1)} = e^{-\lambda B(p, \ q)}.$$

(v) The waiting time T_1 for the first claim in time interval $[0, \ t)$ has an almost-lack-of-memory distribution [see Dimitrov et al. (1997)], and is obtained by

$$\begin{aligned} Pr\{T_1 \leq t\} &= 1 - Pr\{N_t = 0\} = 1 - e^{-\Lambda(t)} \\ &= 1 - \alpha^{\lfloor t \rfloor} e^{-\lambda B(p, \ q, \ t - \lfloor t \rfloor)}. \end{aligned}$$

Thus, the p.d.f. of T_1 is

$$f_{T_1}(t) = \alpha^{\lfloor t \rfloor} e^{-\lambda B(p,q,t-\lfloor t \rfloor)} \lambda(t - \lfloor t \rfloor),$$

and the expectation of T_1 is derived as

$$\begin{aligned} E(T_1) &= \frac{\alpha + \lambda \int_0^1 t^p (1-t)^{q-1} e^{-\lambda B(p,q,t)} dt}{1 - \alpha} \\ &= \frac{\int_0^1 e^{-\lambda B(p, \ q, \ t)} dt}{1 - \alpha}. \end{aligned}$$

Figure 2.3 shows the behaviour of the survival probability $e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}$ on the time interval $[\tau, \ \tau + t)$ as a function of initial age $\tau \in [0, 1)$, when $\lambda = 50$, $p = 2$, $q = 3$, and $t = 3.8$.

Generally, if the NHP process has a periodic intensity function of period $c > 0$, then (2.31) is alternatively defined as

$$\lambda(t) = \lambda \tau^{p-1} (1 - \tau)^{q-1}, \quad \text{for } \tau = t - \lfloor \frac{t}{c} \rfloor c, \ t > 0.$$

The related results, as mentioned in Example 2.1, could be acquired similarly.

35

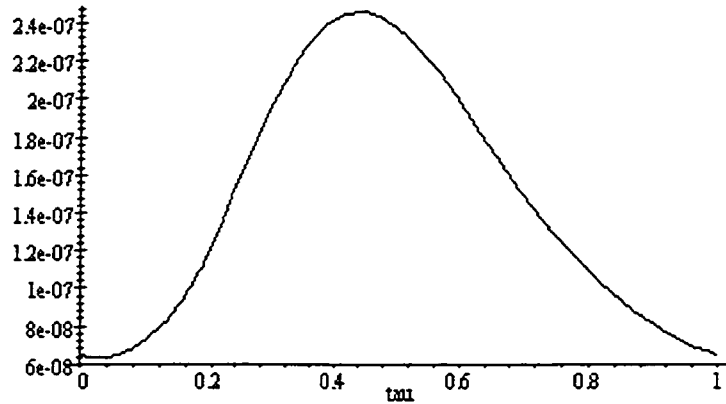$$\lambda = 1, \quad p = 2, \quad q = 3 \text{ and } t = 3.8.$$

Figure 2.3: $e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}$ as a function of initial age $\tau \in [0, 1)$.

Now consider two periodic intensity functions λ_1 and λ_2, as well as their corresponding NHP processes $\{N_t^{(1)}, \ t \geq 0\}$ and $\{N_t^{(2)}, \ t \geq 0\}$. By Theorem 2.2, the composition of these two processes is also a NHP process with intensity function $\lambda = \lambda_1 + \lambda_2$.

A possible interpretation with real insurance events could be when the number of claims in the process is counted under the effect of two independent random environments, each with its own repeated periodic behavior.

Example 2.2 *Properties of the short-term periodic Poisson counting process, with the superposition of two exponential-type claim intensity rates.*

In the following example, we suppose there are two intensity functions. One has monotonic increasing claim tendency and the other has monotonic decreasing claim tendency, with period 1 in its claim counting process.

Let

$$\lambda_1(t) = e^\tau, \quad \text{for } \tau = t - \lfloor t \rfloor \in [0, \ 1), \ t \geq 0 \tag{2.34}$$

and

$$\lambda_2(t) = e^{1-\tau}, \quad \text{for } \tau = t - \lfloor t \rfloor \in [0, \ 1), \ t \geq 0. \tag{2.35}$$

36

Obviously, this produces a periodic behaviour if

$$\lambda(t + n) = \lambda(t) , \quad \text{for any } n \in N, \ t \geq 0,$$

where $\lambda(t) = \lambda_1(t) + \lambda_2(t)$.

Let Λ_1, Λ_2 denote the hazard functions of λ_1 and λ_2 respectively. Thus, the corresponding hazard function Λ is of the form $\Lambda = \Lambda_1 + \Lambda_2$ with

$$\Lambda_1(t) = \lfloor t \rfloor (e - 1) + (e^{t - \lfloor t \rfloor} - 1)$$

and

$$\Lambda_2(t) = \lfloor t \rfloor (e - 1) - [e^{1 - (t - \lfloor t \rfloor)} - e].$$

Therefore,

$$\Lambda(t) = 2(e - 1)\lfloor t \rfloor + [e^{t - \lfloor t \rfloor} - e^{1 - (t - \lfloor t \rfloor)} + e - 1] , \quad \text{for } t \geq 0. \tag{2.36}$$

Figure 2.4 shows the shape of $\lambda_1(t)$, $\lambda_2(t)$ and $\Lambda(t)$ over two periods.

$\lambda_1(t)$–solid black curve, $\lambda_2(t)$–dotted blue curve, $\Lambda(t)$–dashed red curve.
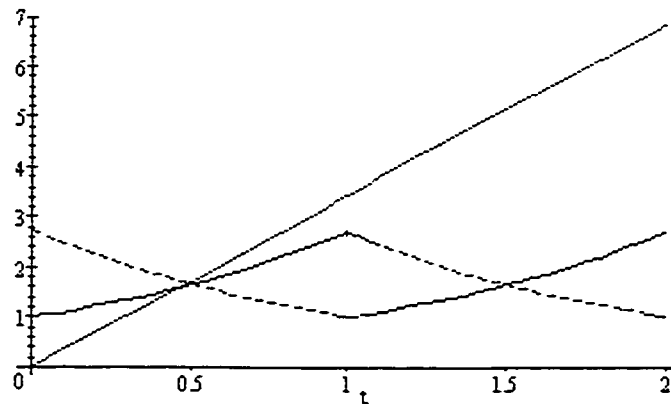

Figure 2.4: Functions $\lambda_1(t)$, $\lambda_2(t)$ and $\Lambda(t)$ over two periods.

Analogously, consider $N_{[\tau, \tau+t)}$, the number of claims in the time interval $[\tau, \tau+t)$, and assumed to follow a NHP distribution with parameter $\Lambda(t) = \Lambda_1(t) + \Lambda_2(t)$, as shown in (2.36). The resulting properties are summarized in the following.

(i) The number of claims over time interval $[\tau, \ \tau + t)$ and over one year are distributed respectively as

$$N_{[\tau, \ \tau+t)} \sim \text{Poisson}\{[\Lambda_1(\tau+t) - \Lambda_1(\tau)] + [\Lambda_2(\tau+t) - \Lambda_2(\tau)]\}, \quad \text{for } t > 0, \ \tau \geq 0,$$

and

$$\Pr\{N_1 = n\} = \frac{[2(e-1)]^n}{n!} e^{-2(e-1)}, \quad \text{for } n = 0, 1, \ldots,$$

where $\Lambda(\tau + t) - \Lambda(\tau)$, can be derived from (2.36).

(ii) The moment generating function of $N_{[\tau, \ \tau+t)}$ and N_1 are

$$
\begin{aligned}
E(e^{rN_{[\tau, \ \tau+t)}}) &= E(e^{rN^{(1)}_{[\tau, \ \tau+t)}})E(e^{rN^{(2)}_{[\tau, \ \tau+t)}}) \\
&= e^{[\Lambda_1(\tau+t) - \Lambda_1(\tau) + \Lambda_2(\tau+t) - \Lambda_2(\tau)](e^r - 1)},
\end{aligned}
$$

and

$$E(e^{rN_1}) = e^{2(e-1)(e^r - 1)}.$$

(iii) The expected claims over time interval $[\tau, \ \tau + t)$ and over one year are

$$
\begin{aligned}
E(N_{[\tau, \ \tau+t)}) &= E(N^{(1)}_{[\tau, \ \tau+t)}) + E(N^{(2)}_{[\tau, \ \tau+t)}) \\
&= [\Lambda_1(\tau+t) - \Lambda_1(\tau)] + [\Lambda_2(\tau + t) - \Lambda_2(\tau)],
\end{aligned}
$$

and

$$E(N_1) = \Lambda(1) = 2(e - 1).$$

(iv) The probability to survive for one year is

$$\alpha = e^{-\Lambda(1)} = e^{-2(e-1)}.$$

(v) By analogy with Example 2.1, the waiting time T_1 for the first claim in $[0, \ t)$ also has an almost-lack-of-memory distribution,

$$\Pr\{T_1 \leq t\} = 1 - \alpha^{\lfloor t \rfloor} e^{-[e^{t - \lfloor t \rfloor} - e^{1 - (t - \lfloor t \rfloor)} + e - 1]}.$$

Thus, the p.d.f. of T_1 is

$$f_{T_1}(t) = \alpha^{\lfloor t \rfloor} e^{-[e^{t - \lfloor t \rfloor} - e^{1 - (t - \lfloor t \rfloor)} + e - 1]}[e^{t - \lfloor t \rfloor} - e^{1 - (t - \lfloor t \rfloor)}],$$

and its expectation is given by

$$E(T_1) = \frac{e^{-(e-1)} \int_0^1 e^{-(e^u - e^{1-u})} du}{1 - e^{-2(e-1)}}.$$

2.4.2 Numerical Examples for Ruin Probabilities

This section first explores the one-year ruin probability for the embedded discrete risk model, as a function of the initial reserve u, or the initial age at onset τ, with a beta-type claim intensity rate and exponentially distributed claim severities X_j of mean μ. Then under the same conditions, we compute two-sided bounds for ultimate ruin probabilities $\Psi(u)$, for the compound NHP process, with a periodic claim intensity function and zero initial age.

Example 2.3 *The ruin probability within the first year period, as a function of initial reserve u or the age at onset τ.*

In the first year period, ruin occurs if $S_1(u) < 0$, i.e. $\sum_{i=0}^{M_1} X_i > u + \beta$, where M_1 is a Poisson random variable of parameter $\Lambda(1) = \lambda_0 B(p, q)$ and $F_X(x) = 1 - e^{-\frac{x}{\mu}}$, for $x \geq 0$. By the law of total probability and the fact that the sum of independent exponentials has an Erlang distribution, $\tilde{\Psi}_{[0, 1)}(u)$ can be represented as

$$\begin{aligned}
\tilde{\Psi}_{[0, 1)}(u) &= \Pr\{\sum_{i=1}^{M_1} X_i > u + \beta\} \\
&= \sum_{n=1}^{\infty} \Pr\{\sum_{i=1}^{M_1} X_i > u + \beta \mid M_1 = n\} \Pr\{M_1 = n\} \\
&= \sum_{n=1}^{\infty} \Pr\{X_1 + X_2 + \cdots + X_n > u + \beta\} \frac{[\Lambda(1)]^n}{n!} e^{-\Lambda(1)}.
\end{aligned}$$

Since $\sum_{i=1}^{i=n} X_i \sim \text{Erlang}(n, \mu^{-1})$, we have

$$
\begin{aligned}
\Pr\{X_1 + X_2 + \cdots + X_n > u + \beta\} &= \int_{u+\beta}^{\infty} \frac{(\frac{1}{\mu})^n x^{n-1} e^{-\frac{x}{\mu}}}{\Gamma(n)} dx \\
&= \frac{[\frac{1}{\mu}(u+\beta)]^{n-1} e^{\frac{1}{\mu}(u+\beta)}}{\Gamma(n)} + \int_{u+\beta}^{\infty} \frac{(\frac{1}{\mu})^{n-1} x^{n-2} e^{\frac{x}{\mu}}}{\Gamma(n-1)} dx \\
&\vdots \\
&= \sum_{k=0}^{n-1} \frac{[\frac{1}{mu}(u+\beta)]^k}{k!} e^{-\frac{1}{\mu}(u+\beta)},
\end{aligned}
$$

and hence

$$
\tilde{\Psi}_{[0, 1)}(u) = \sum_{n=1}^{\infty} \frac{[\Lambda(1)]^n}{n!} e^{-\Lambda(1)} \sum_{k=0}^{n-1} \frac{[\frac{1}{\mu}(u+\beta)]^k}{k!} e^{-\frac{1}{\mu}(u+\beta)}. \tag{2.37}
$$

Table 2.1 and Figure 2.5 show the dependence in (2.37) on u, for the case when $\lambda_0 = 50$, $p = q = 2$, $E(X_1) = \mu = 1$ and $\beta = 10$, for values of u between 0 and 50. The value in graph is of $\ln[\tilde{\Psi}_{[0, 1)}(u)]$ versus u. Clearly, the larger the initial reserve, the smaller the ruin probability. We see that the decline is at a (negative) exponential rate.
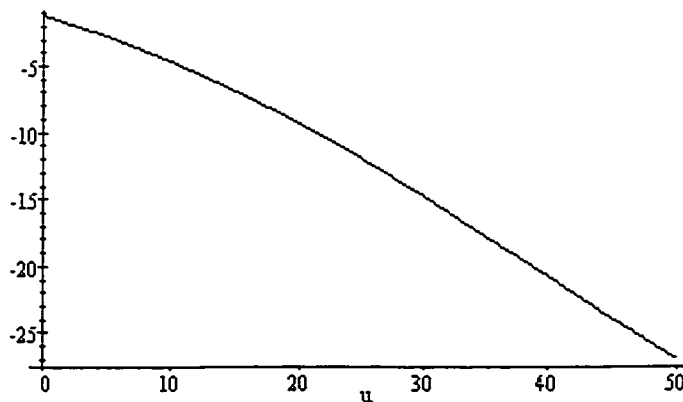

Figure 2.5: The log ruin probability as a function of the initial reserve u.

Similarly, under the same parameter values, we present the probability of ruin at the end of the first year, $\tilde{\Psi}_{[\tau, 1)}(u)$, as a function of the age at onset, τ, with an initial reserve u. Note that the aggregate premium over $[\tau, 1)$ is $\beta(1 - \tau)$ and that $N_{[\tau, 1)}$ is

Table 2.1: Ruin probability as a function of the initial reserve u.

u	$\tilde{\Psi}_{[0,\,1)}(u)$	$\ln[\tilde{\Psi}_{[0,\,1)}(u)]$
0	0.305816752	-1.184769205
5	0.066686493	-2.707752858
10	0.009632438	-4.642618871
15	0.001031333	-6.876903329
20	8.79265E-05	-9.339009105
25	6.26068E-06	-11.98122140
30	3.84941E-07	-14.77017607
35	2.09408E-09	-17.68156856
40	1.02652E-11	-20.69708713
45	4.59923E-11	-23.80254707
50	1.90469E-12	-26.98670431

Poisson distributed with parameter $\Lambda(1) - \Lambda(\tau)$. The probability of ruin at the end of the first year is then

$$
\begin{aligned}
\tilde{\Psi}_{[\tau,\,1)}(u) &= P\{\sum_{i=1}^{N_{[\tau,\,1)}} X_i > u + \beta(1-\tau)\} \\
&= \sum_{n=1}^{\infty} P\{\sum_{i=1}^{N_{[\tau,\,1)}} X_i > u + \beta(1-\tau) \mid N_{[\tau,\,1)} = n\} P\{N_{[\tau,\,1)} = n\} \\
&= \sum_{n=1}^{\infty} P\{X_1 + \cdots + X_n > u + \beta(1-\tau)\} \frac{[\Lambda(1) - \Lambda(\tau)]^n}{n!} e^{-[\Lambda(1)-\Lambda(\tau)]} \\
&= \sum_{n=1}^{\infty} \frac{[\Lambda(1) - \Lambda(\tau)]^n}{n!} e^{-[\Lambda(1)-\Lambda(\tau)]} \\
&\quad \times \sum_{k=0}^{n-1} \frac{\{\frac{1}{\mu}[u + \beta(1-\tau)]\}^k}{k!} e^{-\frac{1}{\mu}[u+\beta(1-\tau)]}.
\end{aligned}
\tag{2.38}
$$

Table 2.2: Ruin probability as a function of the initial age τ.

τ	$\Lambda(1) - \Lambda(\tau)$	$\tilde{\Psi}_{[\tau,\ 1)}(u)$
0	8.333333	0.176412708
0.1	8.100000	0.217247741
0.2	7.466667	0.232953691
0.3	6.533333	0.224567384
0.4	5.400000	0.196778149
0.5	4.166667	0.156507354
0.6	2.933333	0.112096044
0.7	1.800000	0.072325408
0.8	0.866667	0.044245890
0.9	0.233333	0.023468509

Table 2.2 and Figure 2.6 show the dependence in (2.38) on τ, for the same parameter values of $\lambda_0 = 50$, $p = q = 2$, $\mu = 1$, $\beta = 10$ and $u = 2$, for τ between 0 and 0.9. The ruin probability over period $[\tau,\ 1)$ is affected by the intensity rate and the length of the interval. When τ is small, the claim intensity rate is also small and so does the ruin probability. After reaching a maximum value, the ruin probability decreases together with the intensity rate and as the length of the interval increases.
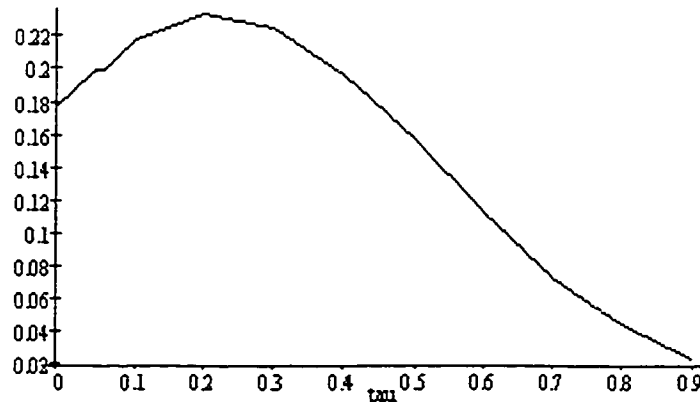

Figure 2.6: Ruin probabilities as a function of the initial age τ.

Example 2.4 *Two-sided bounds for the ultimate ruin probability* $\Psi(u)$, *with zero initial age.*

Assume that the compound NHP process has a beta-type periodic claim intensity rate,

$$\lambda(t) = \lambda_0 t^{p-1}(1-t)^{q-1}, \quad \text{for } 0 \le t < 1,$$

and the claim size random variables X_j are exponentially distributed with mean μ.

In this case, the moment generating function of X_j is $M_X(s) = \frac{1}{1-\mu s}$, and then (2.25) takes the form

$$\theta^*(s) = \bar{\lambda}\left(\frac{1}{1-\mu s} - 1\right) - \beta s,$$

where $\bar{\lambda} = \lambda_0 B(p, q)$ is the average arrival rate. From $\theta^*(\gamma) = 0$, we get

$$\gamma = \frac{\beta - \mu\bar{\lambda}}{\beta\mu} = \frac{\beta - \mu\lambda_0 B(p, q)}{\beta\mu} > 0, \tag{2.39}$$

and therefore,

$$
\begin{aligned}
\alpha(x) &= \frac{\bar{F}_X(x)}{\int_x^\infty e^{\gamma(y-x)} dF_X(y)} \\
&= \frac{e^{-\frac{x}{\mu}}}{\int_x^\infty e^{\gamma(y-x)} \frac{1}{\mu} e^{-\frac{y}{\mu}} dy} \\
&= 1 - \mu\gamma = \frac{\mu}{\beta}\lambda_0 B(p, q), \tag{2.40}
\end{aligned}
$$

is a constant, denoted as α^*.

Hence, the two-sided bounds for $\Psi(u)$ of Corollary 2.5, are obtained from (2.30)

$$a_-^* = \alpha^* e^{-\beta\gamma[v_{max} - \frac{B(p, q, v_{max})}{B(p, q)}]},$$

$$a_+^* = \alpha^* e^{\frac{\beta\gamma}{B(p, q)} \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}},$$

and v_{max} can be found numerically, as the root of the equation

$$g(v) = v - \frac{B(p, q, v)}{B(p, q)} = 0,$$

where γ and α^* are given in (2.39) and (2.40), respectively.

Table 2.3: Two-sided Bounds for $\Psi(u)$.

u	Lower Bound	Upper Bound
15	0.058268	0.833333
20	0.025323	0.362165
25	0.011005	0.157396
30	0.004783	0.068404
35	0.002079	0.029728
40	0.000903	0.012920
45	0.000393	0.005615
50	0.000171	0.002440

Table 2.3 and Figure 2.7 show the lower and upper bounds for $\Psi(u)$, with zero initial age, for the case when $\lambda_0 = 50, p = q = 2, \mu = 1$, $\beta = 10$ and values of u between 15 and 50, with increments of 5.

Clearly these bounds produce larger intervals for small u values, but estimate the ruin probability quite accurately for large u.

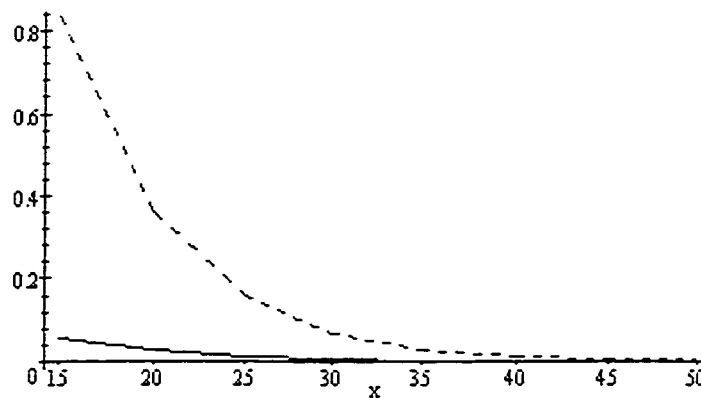Lower bound–solid curve, upper bound–dotted curve.

Figure 2.7: The log ultimate ruin probability as a function of the initial reserve u.

Chapter 3

Long Term Periodicity for the Poisson Model

In the previous chapter we studied the short term periodic Poisson model, that is, a NHP process with periodic intensity function of period 1. Its interpretation is that of a risk process that evolves in a short-term periodic environment, for instance the seasons in a year. However, it is more realistic to consider the case where the periodic environment does not exactly repeat itself every year but also varies over a relatively long period, for instance, four-years, with different levels in each year. This is especially appropriate in catastrophic insurance, such as earthquakes, tidal waves and hurricanes, which have peak seasons in a year but whose intensity also depends on long term climatological effects like El Niño.

Consider a NHP claim counting process $\{N_t, \ t \geq 0\}$, with periodic intensity function λ of period c, where $c > 1 \in \mathbb{N}^+$. The periodicity for a case $c = 2$ is discussed first. In this case, λ is a periodic function, with different shape or levels of claim intensity in the first and second year period. The periodicity for a general case where the period $c > 1 \in \mathbb{N}^+$ is then discussed; another periodic function will then describe the periodic levels when the intensity function is beta-type. Illustrations are given in the last section of the chapter.

3.1 Periodicity for Period $c = 2$

Let

$$\lambda(t) = \begin{cases} \lambda_L(t) & \text{if } 0 \le t < 1 \\ \lambda_U(t) & \text{if } 1 \le t < 2 \end{cases} \tag{3.1}$$

be the intensity function in one period, where λ_L and λ_U are two different functions. For $0 \le t \le 1$, denote two half period hazard functions related to λ_L and λ_U by

$$\Lambda_L(t) = \int_0^t \lambda_L(v)dv$$

and

$$\Lambda_U^*(t) = \Lambda_U(1+t) - \Lambda_U(1) = \int_1^{1+t} \lambda_U(v)dv = \int_0^t \lambda_U(v+1)dv,$$

where $\Lambda_U(t) = \int_0^t \lambda_U(v)dv$.

By a analogy to Theorem 2.3, we have the following properties for the corresponding hazard function Λ and the NHP process $\{N_t, \ t \ge 0\}$.

Theorem 3.1 *Suppose that the intensity function λ in (3.1) is with period 2. Then*

(a) The hazard function Λ has the almost linear property, given as

$$\Lambda(t) = \lfloor \tfrac{t}{2} \rfloor \Lambda^*(1) + \Lambda(t - 2\lfloor \tfrac{t}{2} \rfloor), \quad \text{for } t \ge 0, \tag{3.2}$$

where

$$\Lambda^*(1) = \int_0^1 [\lambda_L(v) + \lambda_U(v+1)]dv = \Lambda_L(1) + \Lambda_U^*(1), \tag{3.3}$$

and

$$\Lambda(t - 2\lfloor \tfrac{t}{2} \rfloor) = \begin{cases} \Lambda_L(t - \lfloor t \rfloor) & \text{if } 0 \le t - 2\lfloor \tfrac{t}{2} \rfloor < 1 \\ \Lambda_U^*(t - \lfloor t \rfloor) + \Lambda_L(1) & \text{if } 1 \le t - 2\lfloor \tfrac{t}{2} \rfloor < 2 \end{cases}.$$

(b) For any $t \ge 0$, the random variable N_t is decomposed in the form

$$N_t = M_1^* + \cdots + M_{\lfloor \frac{t}{2} \rfloor}^* + N_{t-\lfloor t \rfloor}^*, \tag{3.4}$$

where

$$N^*_{t-\lfloor t \rfloor} = \begin{cases} N_{t-\lfloor t \rfloor} & \text{if } 0 \leq t - 2\lfloor \frac{t}{2} \rfloor < 1 \\ N'_{t-\lfloor t \rfloor} + N_1 & \text{if } 1 \leq t - 2\lfloor \frac{t}{2} \rfloor < 2 \end{cases}$$

and $\{M^*_i\}_{i \geq 1}$ are i.i.d. Poisson random variables of parameter $\Lambda^*(1)$ in (3.3), independent of N_1, $N_{t-\lfloor t \rfloor}$ and $N'_{t-\lfloor t \rfloor}$, which are Poisson distributed with parameters $\Lambda_L(1)$, $\Lambda_L(t - \lfloor t \rfloor)$ and $\Lambda^*_U(t - \lfloor t \rfloor)$, respectively.

Proof. (a) By setting $c = 2$ in Theorem 2.3-(a), it follows that

$$\Lambda(t) = \lfloor \frac{t}{2} \rfloor \Lambda(2) + \Lambda(t - 2\lfloor \frac{t}{2} \rfloor),$$

where

$$\begin{aligned} \Lambda(2) &= \int_0^1 \lambda_L(v)dv + \int_1^2 \lambda_U(v)dv \\ &= \int_0^1 \lambda_L(v)dv + \int_0^1 \lambda_U(v+1)dv \\ &= \Lambda^*(1) \end{aligned}$$

and

$$\begin{aligned} \Lambda(t - 2\lfloor \frac{t}{2} \rfloor) &= \int_0^{t-2\lfloor \frac{t}{2} \rfloor} \lambda(v)dv \\ &= \begin{cases} \int_0^{t-\lfloor t \rfloor} \lambda_L(v)dv & \text{if } 0 \leq t - 2\lfloor \frac{t}{2} \rfloor < 1 \\ \int_0^{t-\lfloor t \rfloor} \lambda_U(v+1)dv + \int_0^1 \lambda_L(v)dv & \text{if } 1 \leq t - 2\lfloor \frac{t}{2} \rfloor < 2 \end{cases} \\ &= \begin{cases} \Lambda_L(t - \lfloor t \rfloor) & \text{if } 0 \leq t - 2\lfloor \frac{t}{2} \rfloor < 1 \\ \Lambda^*_U(t - \lfloor t \rfloor) + \Lambda_L(1) & \text{if } 1 \leq t - 2\lfloor \frac{t}{2} \rfloor < 2 \end{cases}. \end{aligned}$$

Hence, (3.2) holds.

(b) By Theorem 2.3-(d), (3.4) is a decomposition of N_t, where

$$M^*_i = N_{[2(i-1),\, 2i)} + N'_{[2(i-1),\, 2i)}, \quad \text{for } i = 1, 2, \ldots,$$

are i.i.d. Poisson r.v.'s with parameter $\Lambda^*(1) = \Lambda_L(1) + \Lambda^*_U(1)$, and $N_{[2(i-1),\, 2i)}$, $N'_{[2(i-1),\, 2i)}$ are Poisson r.v.'s with parameters $\Lambda_L(1)$ and $\Lambda^*_U(1)$ respectively. The term $N^*_{t-\lfloor t \rfloor}$ is derived naturally. Hence (b) holds. $\quad\square$

Consider that (3.1) is a beta-type function but with different peak levels, given as

$$\lambda(t) = \begin{cases} \lambda_L t^{p-1}(1-t)^{q-1} & \text{if } 0 \le t < 1 \\ \lambda_U t^{p-1}(1-t)^{q-1} & \text{if } 1 \le t < 2 \end{cases} , \tag{3.5}$$

then we have the following corollary.

Corollary 3.1 *Suppose that the intensity function λ is given as (3.5) in Theorem 3.1. Then*

(a) The hazard function Λ has the form

$$\Lambda(t) = \begin{cases} \lfloor \frac{t}{2} \rfloor(\lambda_L + \lambda_U)B(p,q) + \lambda_L B(p,q,t-\lfloor t \rfloor) & \text{if } 0 \le t - 2\lfloor \frac{t}{2} \rfloor < 1 \\ \lfloor \frac{t}{2} \rfloor(\lambda_L + \lambda_U)B(p,q) + \lambda_U B(p,q,t-\lfloor t \rfloor) + \lambda_L B(p,q) & \text{if } 1 \le t - 2\lfloor \frac{t}{2} \rfloor < 2 \end{cases} . \tag{3.6}$$

(b) For any $t \ge 0$, N_t is decomposed as (3.4), where M_i^*'s, $N_{t-\lfloor t \rfloor}$ and $N'_{t-\lfloor t \rfloor}$ are mutually independent Poisson r.v.'s with parameters $(\lambda_L + \lambda_U)B(p,q)$, $\lambda_L B(p,q,t-\lfloor t \rfloor)$ and $\lambda_U B(p,q,t-\lfloor t \rfloor)$ respectively.

Proof. Since $\lambda(t)$ has the form in (3.5), it is easy to check that

$$\begin{aligned} \Lambda^*(1) &= \int_0^1 \lambda_L v^{p-1}(1-v)^{q-1}dv + \int_1^2 \lambda_U(v-1)^{p-1}(2-v)^{q-1}dv \\ &= \int_0^1 (\lambda_L + \lambda_U)v^{p-1}(1-v)^{q-1}dv \\ &= (\lambda_L + \lambda_U)B(p,q), \\ \Lambda_L(t) &= \int_c^t \lambda_L v^{p-1}(1-v)^{q-1}dv = \lambda_L B(p,q,t) \end{aligned}$$

and

$$\begin{aligned} \Lambda_U^*(t) &= \int_1^{1+t} \lambda_U(1-v)^{p-1}(2-v)^{q-1}dv \\ &= \int_0^t \lambda_U v^{p-1}(1-v)^{q-1}dv \\ &= \lambda_U B(p,q,t). \end{aligned}$$

Therefore, the result follows by Theorem 3.1. $\qquad\qquad\square$

Figure 3.1 shows the shape of the intensity function λ and its hazard function Λ in Corollary 3.1, when $p = 2, q = 4, \lambda_L = 25$ and $\lambda_U = 40$.

48

$\lambda(t)$-solid curve, $\Lambda(t)$-dashed curve.

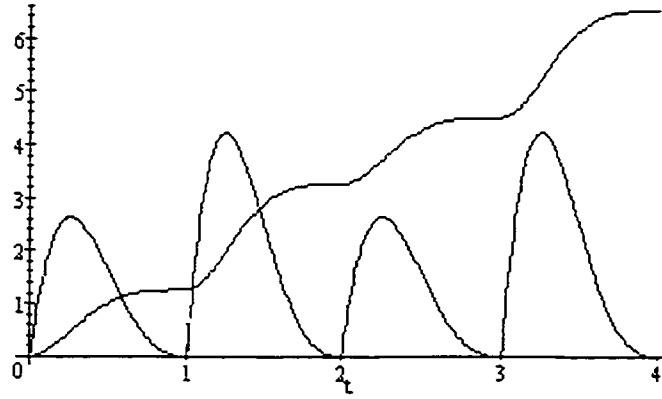$p = 2$, $q = 4$, $\lambda_L = 25$ and $\lambda_U = 40$.

Figure 3.1: $\lambda(t)$ and $\Lambda(t)$ as functions of time t.

Remark 3.1 *For a periodic Poisson model with period 2, N_t is decomposed in different Poisson r.v.'s, distributed as the corresponding Poisson r.v.'s in time interval* [0, 1].

3.2 Periodicity for Period $c \geq 2$

Consider a period $c \in \mathbb{N}^+$, $c \geq 2$ and let

$$\lambda(t) = \begin{cases} \lambda_0(t) & \text{if } 0 \leq t < 1 \\ \lambda_1(t) & \text{if } 1 \leq t < 2 \\ \vdots & \vdots \\ \lambda_{c-1}(t) & \text{if } c - 1 \leq t < c \end{cases} \tag{3.7}$$

be the intensity function in one period. Using the same argument as in Section 3.1, we get a parallel result to Theorem 3.1. Here consider that $\lambda(t)$ is beta-type but with different levels in each year, for $t \geq 0$ given as

$$\lambda(t) = g(\lfloor t \rfloor)(t - \lfloor \tfrac{t}{c} \rfloor c - \lfloor t - \lfloor \tfrac{t}{c} \rfloor c \rfloor)^{p-1}[1 - (t - \lfloor \tfrac{t}{c} \rfloor c - \lfloor t - \lfloor \tfrac{t}{c} \rfloor c \rfloor)]^{q-1}, \tag{3.8}$$

where g is a periodic function with the same period c. Figure 3.2 shows the shape of $\lambda(t)$, when $g(t) = 5 + 4\sin\frac{\pi}{2}(t-1)$, and its periodicity when $g_1(t) = \frac{2\sqrt{3}}{9}[5 + 4\sin\frac{\pi}{2}(t - $

$\frac{5}{3}$)]. Its hazard function Λ and corresponding claim counting process $\{N_t,\ t \geq 0\}$ are given in the following theorem.

$\lambda(t)$–solid black curve, $g(t)$–dotted red curve, $g_1(t)$–dashed blue curve.
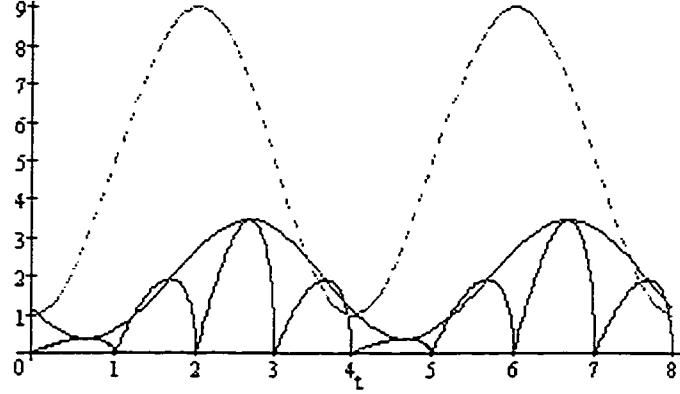
$$p = 2 \text{ and } q = \tfrac{3}{2}.$$

Figure 3.2: Functions $\lambda(t)$ and $g(t)$.

Theorem 3.2 *Assume the intensity function λ is given in (3.8), then*

(a) The hazard function Λ has the almost linear property, given by

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor \Lambda^*(c) + B(p,q) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} g(j) + g(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor) B(p,q,t - \lfloor t \rfloor), \qquad (3.9)$$

where

$$\Lambda^*(c) = B(p,q) \sum_{j=0}^{c-1} g(j). \qquad (3.10)$$

(b) For any $t \geq 0$, the random variable N_t is decomposed in the form

$$N_t = M_1^* + \cdots + M_{\lfloor \frac{t}{c} \rfloor}^* + N_{t-\lfloor t \rfloor}^*. \qquad (3.11)$$

where

$$N_{t-\lfloor t \rfloor}^* = \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N_c^{(j)} + N_{t-\lfloor t \rfloor}^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)} \qquad (3.12)$$

50

and $\{M_i^*\}_{i\geq 1}$ are i.i.d. Poisson r.v.'s of parameter $\Lambda^*(c)$ in (3.10), independent of $N_c^{(j)}$ and $N_{t-\lfloor t\rfloor}^{(j)}$, distributed with Poisson parameters $g(j)B(p,q)$ and $g(j)B(p,q,t-\lfloor t\rfloor)$, for $j = 0,1,2,\ldots,\lfloor t - \lfloor \frac{t}{c}\rfloor c\rfloor$, respectively.

Proof. (a) By definition, the hazard function Λ is derived as following:

$$
\begin{aligned}
\Lambda(t) &= \int_0^v \lambda(v)dv \\
&= \int_0^t g(\lfloor v\rfloor)(v - \lfloor\tfrac{v}{c}\rfloor c - \lfloor v - \lfloor\tfrac{v}{c}\rfloor c\rfloor)^{p-1}\{1 - (v - \lfloor\tfrac{v}{c}\rfloor c - \lfloor v - \lfloor\tfrac{v}{c}\rfloor c\rfloor)\}^{q-1}dv \\
&= \lfloor\tfrac{t}{c}\rfloor \int_0^c g(\lfloor v\rfloor)(v - \lfloor v\rfloor)^{p-1}\{1 - (v - \lfloor v\rfloor)\}^{q-1}dv \\
&\quad + \int_0^{t-\lfloor\frac{t}{c}\rfloor c} g(\lfloor v\rfloor)(v - \lfloor v\rfloor)^{p-1}\{1 - (v - \lfloor v\rfloor)\}^{q-1}dv \\
&= \lfloor\tfrac{t}{c}\rfloor \sum_{j=0}^{c-1}\int_j^{j+1} g(j)(v-j)^{p-1}(j+1-v)^{q-1}dv \\
&\quad + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\int_j^{j+1} g(j)(v-j)^{p-1}(j+1-v)^{q-1}dv \\
&\quad + \int_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}^{t-\lfloor\frac{t}{c}\rfloor c} g(\lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor)(v - \lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor)^{p-1}(\lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor + 1 - v)^{q-1}dv \\
&= \lfloor\tfrac{t}{c}\rfloor \sum_{j=0}^{c-1}g(j)\int_0^1 v^{p-1}(1-v)^{q-1}dv + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1} g(j)\int_0^1 v^{p-1}(1-v)^{q-1}dv \\
&\quad + g(\lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor)\int_0^{t-\lfloor t\rfloor} v^{p-1}(1-v)^{q-1}dv \\
&= \lfloor\tfrac{t}{c}\rfloor B(p,q)\sum_{j=0}^{c-1}g(j) + B(p,q)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1} g(j) + g(\lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor)B(p,q,t-\lfloor t\rfloor).
\end{aligned}
$$

(b) By Theorem 2.3-(d), (3.12) is a decomposition of N_t, where

$$
M_i^* = \sum_{j=0}^{c-1} N_{[2(i-1),\,2i)}^{(j)}, \quad \text{for } i = 1,2,\ldots,
$$

are i.i.d. Poisson r.v.'s with parameter $\Lambda^*(c) = B(p,q)\sum_{j=0}^{c-1}g(j)$. The term $N_{t-\lfloor t\rfloor}^*$ is derived by the value of $\lfloor t - \lfloor\tfrac{t}{c}\rfloor c\rfloor$. Hence (b) holds. $\square$

Theorem 3.2 presents the case where the intensity function has the same beta-type in every year but with different levels in each period, given by $g(\lfloor t\rfloor)$. It may be

possible to use another periodic function h, with peak values over periods when the intensity function has a convex shape, instead of $g(\lfloor t \rfloor)$.

For example, in the illustrations that follow we use

$$g(t) = 5 + 4\sin(\frac{\pi}{2}t - \frac{\pi}{2}),$$

with parameters $c = 4$ and $p = q$ for the beta-type functions. In particular function g takes the following four values

$$g(0) = 1, \ g(1) = 5, \ g(2) = 9, \ g(3) = 5.$$

In this case, we can define

$$h(t) = \frac{5}{4} + \sin(\frac{\pi}{2}t - \frac{3}{4}\pi) = \frac{1}{4}h_1(t),$$

where $h_1(t) = 5 + 4\sin(\frac{\pi}{2}t - \frac{3}{4}\pi)$ is a periodic function that satisfies

$$h_1(\frac{1}{2}) = 1, \ h_1(\frac{3}{2}) = 5, \ h_1(\frac{5}{2}) = 9, \ h_1(\frac{7}{2}) = 5$$

and the functions h and λ have the relation

$$h(p^* + i) = \lambda(p^* + i), \quad \text{for } p^* = \frac{1}{2} \text{ and } i = 0, \ 1, \ 2, \ \dots \ ,$$

where p^* is the peak reached at

$$p^* = \frac{p-1}{p+q-2} \tag{3.13}$$

when beta-type function λ is convex over $0 \le t < 1$, i.e. for $p, \ q > 1$. The relation of $g(t)$ to $h(t)$ is shown in Figure 3.3, where $c = 4$ and $p = q = 2$.

Now assume that intensity function λ has the same beta shape in each year, but with different peak levels, given by function h. That is

$$\lambda(t) = h(\lfloor s \rfloor + p^*)(s - \lfloor s \rfloor)^{p-1}\{1 - (s - \lfloor s \rfloor)\}^{q-1}, \quad \text{for } s = t - \lfloor \frac{t}{c} \rfloor c, \ t \ge 0, \tag{3.14}$$

where h is a periodic function with the same period c.

Analogously, we have the following properties for its hazard function Λ and claim counting process $\{N_t, \ t \ge 0\}$.

$\lambda(t)$–solid black curve, $h(t)$–dot dashed blue curve.

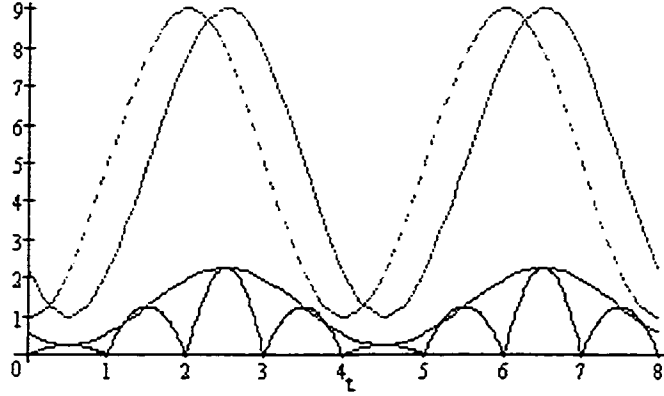$g(t)$–dotted brown curve, $h_1(t)$–dashed red curve.

$$p = q = 2.$$

Figure 3.3: The relation between functions $g(t)$ and $h(t)$.

Corollary 3.2 *Assume that the intensity function λ is given by (3.14), then*

(a) The hazard function Λ has the almost linear property, given as

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor \Lambda^*(c) + B(p,q) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} h(j + p^*) + h(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + p^*) B(p, q, t - \lfloor t \rfloor),$$

$$(3.15)$$

where

$$\Lambda^*(c) = B(p, q) \sum_{j=0}^{c-1} h(j + p^*) \qquad (3.16)$$

and p^* is given by (3.13).

(b) For any $t \geq 0$, the random variable N_t is decomposed in the form

$$N_t = M_1^* + \cdots + M_{\lfloor \frac{t}{c} \rfloor}^* + N_{t-\lfloor t \rfloor}^*, \qquad (3.17)$$

where

$$N_{t-\lfloor t \rfloor}^* = \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N_c^{(j)} + N_{t-\lfloor t \rfloor}^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}, \qquad (3.18)$$

53

and $\{M_i^*\}_{i \geq 1}$ are i.i.d. Poisson r.v.'s of parameter $\Lambda^*(c)$ in (3.10), independent of $N_c^{(j)}$ and $N_{t-\lfloor t \rfloor}^{(j)}$, distributed with Poisson parameters $h(j+p^*)B(p,q)$ and $h(j+p^*)B(p,q,t-\lfloor t \rfloor)$, for $j = 0, 1, 2, \ldots, \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor$, respectively.

Proof. Follows by the same argument as in Theorem 3.2 with $g(t) = h(t + p^*)$. $\square$

3.3 Illustrations

In this section we consider some possible periodic intensity functions with period $c \geq 2$ and their corresponding properties on long-term periodic Poisson counting processes.

Example 3.1 *The long-term periodic Poisson counting process with beta-type and symmetry in every period claim intensity rate.*

Let λ be a periodic function with period 5, given by

$$\lambda(t) = g(\lfloor t \rfloor)(s - \lfloor s \rfloor)^{p-1}[1 - (s - \lfloor s \rfloor)]^{q-1}, \quad \text{for } s = t - 5\lfloor \frac{t}{5} \rfloor, t \geq 0 \qquad (3.19)$$

and parameters $p, q > 1$, where

$$g(t) = \mid \sin \frac{\pi t}{4} \mid + 1 \qquad (3.20)$$

is a periodic function also with period 5. Figure 3.4 shows the periodicity and symmetry of the function λ when the peaks of λ, as a function of t, are given by a polynomial

$$p(t) = \frac{3 - 2\sqrt{2}}{48}(t - \frac{5}{2})^4 + \frac{-15 + 8\sqrt{2}}{48}(t - \frac{5}{2})^2 + \frac{1}{2}.$$

By Theorem 3.2, its hazard function Λ is derived as follows:

$$\Lambda(t) = \lfloor \frac{t}{5} \rfloor \Lambda^*(5) + B(p,q) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor - 1} g(j) + g(\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor) B(p,q,t-\lfloor t \rfloor)$$

$$= \lfloor \frac{t}{5} \rfloor B(p,q) \sum_{j=0}^{4} g(j) + B(p,q) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor - 1} [| \sin \frac{\pi j}{4} | +1]$$

$$+ [| \sin \frac{\pi(\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor)}{4} | +1] B(p,q,t-\lfloor t \rfloor)$$

$$= \lfloor \frac{t}{5} \rfloor B(p,q)(7+\sqrt{2}) + B(p,q) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor - 1} [| \sin \frac{\pi j}{4} | +1]$$

$$+ [| \sin \frac{\pi(\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor)}{4} | +1] B(p,q,t-\lfloor t \rfloor). \tag{3.21}$$

Denote

$$\Lambda^* = (7+\sqrt{2})B(p,q), \tag{3.22}$$

then the number of claims N_t is decomposed in the form

$$N_t = M_1^* + \cdots + M_{\lfloor \frac{t}{5} \rfloor}^* + N_{t-\lfloor t \rfloor}^*,$$

where

$$N_{t-\lfloor t \rfloor}^* = \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor - 1} N_5^{(j)} + N_{t-\lfloor t \rfloor}^{(\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor)},$$

and $\{M_i^*\}_{i \geq 1}$ are i.i.d. Poisson r.v.'s of parameter Λ^* in (3.22), independent of $N_5^{(j)}$ and $N_{t-\lfloor t \rfloor}^{(j)}$, distributed with Poisson parameters $(| \sin \frac{\pi j}{4} | +1)B(p,q)$ and $(| \sin \frac{\pi j}{4} | +1)B(p,q,t-\lfloor t \rfloor)$, for $j = 0,1,2,\ldots,\lfloor t - \lfloor \frac{t}{5} \rfloor 5 \rfloor$, respectively.

Moreover, the moment generating function of N_5 is given by

$$E(e^{rN_c}) = e^{\Lambda^*(e^r-1)},$$

therefore the expected number of claims during one period, coincides with the variance:

$$E(N_c) = V(N_c) = \Lambda^*.$$

The probability to survive for one period is

$$\alpha = e^{-\Lambda^*},$$

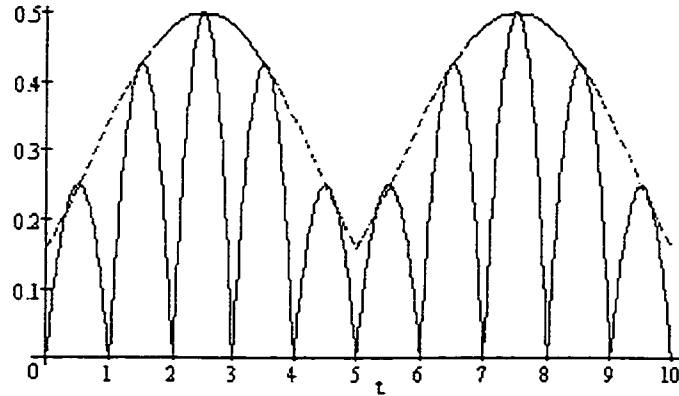$\lambda(t)$–solid black curve, $p(t)$–dotted blue curve.

$c = 5$ and $p = q = 2$.

Figure 3.4: The periodicity and symmetry of $\lambda(t)$.

where Λ^* is shown in (3.22).

Example 3.2 *The long-term periodic Poisson counting process with beta-type and peak values given by function h.*

Let λ be a periodic function with period 4, given by

$$\lambda(t) = h(\lfloor s \rfloor + p^*)(s - \lfloor s \rfloor)^{p-1}[1 - (s - \lfloor s \rfloor)]^{q-1}, \quad \text{for } s = t - 4\lfloor \frac{t}{4} \rfloor, t \geq 0 \quad (3.23)$$

and parameters $p, q > 1$, where

$$h(t) = \frac{5}{4} + \sin(\frac{\pi}{2}t - \frac{3}{4}\pi) \quad (3.24)$$

is a periodic function also with period 4. Figure 3.5 shows the periodicity of the function λ. By Theorem 3.3, its hazard function Λ is obtained when $p = q$ as follows

(note that in this case $p^* = \frac{1}{2}$):

$$
\begin{aligned}
\Lambda(t) &= \lfloor\frac{t}{4}\rfloor\Lambda^*(4) + B(p,q)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor-1} h(j+p^*) + h(\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor + p^*)B(p,q,t-\lfloor t\rfloor)\\
&= \lfloor\frac{t}{4}\rfloor B(p,q)\sum_{j=0}^{3} h(j+\frac{1}{2}) + B(p,q)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor-1}[\frac{5}{4}+\sin(\frac{\pi}{2}(j+\frac{1}{2})-\frac{3}{4}\pi)]\\
&\quad +\{\frac{5}{4}+\sin[\frac{\pi}{2}(\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor + \frac{1}{2})-\frac{3}{4}\pi]\}B(p,q,t-\lfloor t\rfloor)\\
&= 5\lfloor\frac{t}{4}\rfloor B(p,q) + B(p,q)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor-1}[\frac{5}{4}+\sin(\frac{\pi}{2}(j+\frac{1}{2})-\frac{3}{4}\pi)]\\
&\quad +\{\frac{5}{4}+\sin[\frac{\pi}{2}(\lfloor t-\lfloor\frac{t}{4}\rfloor 4\rfloor + \frac{1}{2})-\frac{3}{4}\pi]\}B(p,q,t-\lfloor t\rfloor).
\end{aligned}
\tag{3.25}
$$

$\lambda(t)$–solid black curve, $h_1(t)$–dotted blue curve.
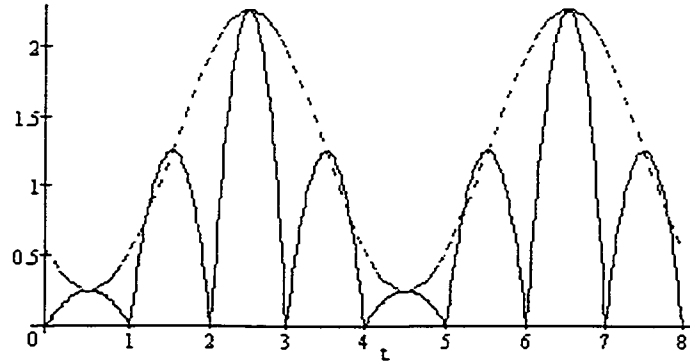
$c = 4$ and $p = q = 2$.

Figure 3.5: The periodicity of $\lambda(t)$.

In a similar way to Example 3.1, the related risk characteristics can be obtained.

Conclusion

Compound non-homogenous Poisson processes with periodic claim intensity rate are appropriate for modeling risk processes under seasonal conditions. Based on the short-term Poisson model and its periodicity, we introduce a more general Poisson process model with double periodicity. The interpretation is that the risk process evolves in a relatively long period where the periodic environment does not repeat itself exactly. The short periodicity peak varies over a relatively long period, with different levels in each short period. This makes the model more practical than the short-term one for some risks, such as hurricanes. Some analytical and more general shapes (e.g. beta, sine) are illustrated for the periodicity of the claim intensity.

An embedded discrete risk model, which is related to the time-continuous periodic claim process and an average arrival rate risk model are presented. The latter even deals with the time-dependent claim size distribution. These models can be used to derive some explicit analytical and useful results for ruin probabilities. In some case, ruin probabilities and bounds can also be evaluated numerically, in which the related risk characteristics can be recognized.

More work on general compound non-homogenous Poisson risk models is still needed. The Volterra integral equation for the ruin probability $\Psi_{[\tau, \tau+t)}(u)$ is already derived. Methods are needed to analytically or numerically evaluate it, under regularity conditions, as in the classical case. More realistically, the ruin problem for a long term Poisson model can be considered, based on the ruin model in the short term case.

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