Value at Risk and the Distortion Operator

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Abstract

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VaR is a popular measure for benchmarking market risk based on price or return fluctuations of instruments among institutions. Calculation of VaR depends very much on the model explaining the price changes and volatility of the underlying assets. However, theoretical models can be very unrealistic in comparison to actual historical data. Modifications are required in order for the models to better fit the actual market conditions.

If finding a mathematical tool to bridge to theoretical model with reality were possible, we could expect a better or less expensive estimation of the VaR. Calculation of the actual VaR for an asset can be started by first finding the risk adjusted return under the assumption of a theoretical return model covered by the risk neutral measure, where a new mathematical tool could link the risk adjusted models back to the actual measure.

We propose a distortion operator to serve as such a bridge between the actual and risk neutral distributions.

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Chapter 1

Value at Risk

1.1 Introduction

Value at Risk (VaR) is a firm-wide measure of the market risk of a financial firm's "book", the list of positions in various instruments that expose the firm to the financial risk. Due to its simplicity and relative non-expensiveness for calculation it is gaining popular acceptance among institutions.

Although credit risk is viewed as part of the market risk, it is treated separately in a parallel measure namely, **Credit-VaR**. The methodology of measuring VaR focuses narrowly on the market risk due to price or return fluctuations of the traded instruments over a short time period under a stationary market condition. Other forms of financial risk, such as operational or regulatory risk, are simply ignored or briefly taken into account. With these limitations VaR itself can only be regarded as a benchmark for comparing one instrument against others under a specific time horizon and confidence level. It is misleading to consider VaR as a measure reflecting the true financial risk exposure and capital adequacy of a financial firm.

According to Duffie and Pan (1997) the measure of VaR depends on

(1) A model of random changes in prices of the underlining instruments.

(2) A model for estimating the sensitivity of the prices of derivatives to the prices

of the underlying assets.

The elements of the above models are already available in the markets for the purpose of pricing and hedging derivatives. Measuring market risk by integrating these elements across trading desks however sounds difficult, mathematical modelling comes into the stage under the demand of independent risk management systems.

1.1.1 Basics of VaR

According to Duffie and Pan (1997) Value at Risk is defined as follows:

> For a given time horizon T and confidence level $100p\%$, the Value at Risk is the loss in market value over the time horizon T that is exceeded with probability $1 - p$.

Mathematically speaking the T-period VaR at $100p\%$ confidence level of a portfolio is indeed the distance between the lower bound of the $100(2p - 1)\%$ confidence interval and the current market value of the portfolio (not the expected value) conditioned on the current market information over a T-period time horizon. In order to guarantee a VaR model to be valid a model giving a larger value of VaR is preferred.

The values of T and p vary depending on the standard of a particular institution. For example, the Derivatives Policy Group (DPG) proposed a standard for over-the-counter (OTC) derivatives broker-dealer reports to the Securities and Exchange Commission (SEC) with T of 2 weeks and p of 99%, while J.P. Morgan requires a daily T and p of 95%. Overnight VaR is widely used for internal purposes.

The mathematical challenges in VaR are many including availability of data, simple theoretical and empirical model and computational methods. In Duffie and Pan (1997) the recipe for estimating VaR can be summarized as follows:

(1) Build a model for simulating changes in prices, and volatility across the underlying markets. It can be either parametric or nonparametric.

(2) Build a database of portfolio positions, including derivatives and estimate the corresponding current size.

(3) Develop a model for revaluation of derivatives subjected to changes in the underlying market prices.

(4) Simulate the changes in the market value of portfolios for each scenario of the underlying market returns. Generate independently a sufficient number of scenarios to estimate the critical value of the loss distribution.

We will focus mainly on the price risk model of the underlying markets for VaR calculation.

1.2 Price Risk

The key issues of the underlying price risk model are tail fatness, volatility, skewness and correlations. Volatility is defined as the annualized standard deviation of the price of an underlying asset, and it is usually regarded as merely the standard deviation itself. It is occasionally used as the primary measure of the market risk of a position.

1.2.1 Basic Model of Return Risk

Let R_t be the continuous daily return of some underlying asset (e.g. stock) with price A_t at the end of day t. Then the price of the asset at the end of day $t+1$ will be:

$$A_{t+1} = A_t \, e^{R_{t+1}} \, ,$$

and, conditionally on a sequence of market information $\{P_t\}$ available on day t, the successive return can be modelled as:

$$R_{t+1} = \mu_t + \epsilon_{t+1}\sigma_t \, , \tag{1.1}$$

where $\mu_t = E[R_{t+1}|P_t]$, $\sigma_t = \sqrt{V[R_{t+1}|P_t]} = \sigma[R_{t+1}|P_t]$ and ϵ_{t+1} is the shock to the asset price on day $t+1$ with $E[\epsilon_{t+1}|P_t] = 0$, $\sigma[\epsilon_{t+1}|P_t] = 1$.

If the number of trading days in a year is denoted by n, the volatility of the corresponding annualized asset return on day t will be $\sigma_t\sqrt{n}$. In general, the volatility of an annualized asset return, with price A_t at day t over a period of T days, is $\sigma[R_{t,T}|P_t]\sqrt{\frac{n}{T}}$ where $R_{t,T} = R_{t+1} + R_{t+2} + \ldots + R_{t+T}$. It becomes stochastic if it is allowed to vary randomly over time. For a multiple period horizon the T-period return of an asset at day t can be written as:

$$R_{t,T} = \mu_{t,T} + \epsilon_{t,T}\sigma_{t,T} \ ,$$

where $\mu_{t,T} = E[R_{t,T}|P_t]$ $\sigma_{t,T} = \sigma[R_{t,T}|P_t]$, and $\epsilon_{t,T}$ is the shock over the T-period again with $E[\epsilon_{t,T}|P_t] = 0$ and $\sigma[\epsilon_{t,T}|P_t] = 1$.

For illustration purposes, the analytical $100p\%$-level T-period VaR of the asset with price A_t at day t is:

$$VaR_p[T; A_t] = A_t(1 - e^{\mu_{t,T} - [\epsilon_{t,T}]_p\sigma_{t,T}}) \ ,$$

where $P[\epsilon_{t,T} > [\epsilon_{t,T}]_p] = p$, $[\epsilon_{t,T}]_p$ is the $100(1-p)^{\text{th}}$ percentile of the shock to the T-days annualized return of the asset at day t.

1.2.2 Actual Versus Risk-Neutral VaR

If the current market is fair enough that the price of an asset truly reflects its return, it is reasonable to assume that the asset price at the end of day t is determined by its conditional expected return over day $t+1$ and it must satisfy the martingale condition for any available information on day t :

$$A_t = A_t e^{E[R_{t+1}|P_t]} \ ,$$

or equivalently, $E[R_{t+1}|P_t] = \mu_t = 0$. This is the basic assumption RiskMetrics^{TM} uses for the calculation of VaR and the sequence of the pivotal values $\{\frac{R_t}{\sigma_t}\}$ is assumed to be i.i.d. Normal$(0, 1)$.

However, if the market is assumed to be efficient, the construction of a distorted pseudo-probability Q measure is required to calibrate the expected return with the risk free rate r_f. This measure is used for derivatives pricing, for example Black-Scholes' famous formula. Under this assumption, the price of the asset has to be rewritten as:

$$A_t = A_t e^{E_Q[R_{t+1} - r_{f,t}|P_t]} ,$$

or equivalently, $E_Q[R_{t+1}|P_t] = \mu_t = r_{f,t}$ where $r_{f,t}$ denotes the risk free rate of return at day $t+1$. It is deemed constant with respect to the market information P_t.

Values calculated under this Q measure are regarded as risk-adjusted. The risk-neutral or risk-adjusted VaR can be calculated from the above assumption. According to Harrison and Kreps (1979) the risk neutral distribution has indeed a very different nature from that of the actual in representing the risk of a position. Nevertheless as Duffie and Pan (1997) point out, for most markets there is no significant difference between the risk-adjusted VaR and the actual VaR over a short time period.

1.2.3 Tail Fatness

The tail fatness of the shock ϵ to an asset is also viewed as an indicator of the level of market risk exposure. It is usually measured in terms of kurtosis. Given two assets whose shocks are calibrated to have the same standard deviation, the one with a greater kurtosis value gives a greater overnight VaR. The kurtosis of a normal shock is 3 while the kurtosis of the shocks in most markets is generally greater than this value. Therefore a fat tail model is generally preferred for VaR valuation in order to capture the worse possible situations.

Another measure of the tail fatness, more related to VaR, is the number of standard deviations from the mean associated with the critical $100p\%$-level of downside return. It is indeed the value of ϵ_{VaR_p} mentioned above. For a standard normal shock the corresponding value at a 99% level is 2.33 while it takes a greater value for most markets. If a particular market shows a thinner than normal tail, it is usually negatively skewed, which is not in favour of investors.

Many sources contribute to fat tails. Mixtures of normal shocks can be the major reason, which is the idea behind the assumption of stochastic volatility. A typical example of this mixture of normal shocks is the jump diffusion model.

1.2.4 Volatility

Volatility is one of the key inputs for the VaR calculation. A mixture of normals, leading to fat tails stochastic volatility, is usually assumed for the calculation of VaR. Many models use conditional autoregressive models such as GARCH or EGARCH. GARCH models can also be extended to the cross-market level by taking the covariances between assets in different markets into account. If a market is sophisticated and efficient enough, option-implied stochastic volatility based on the risk neutral assumption is suggested as a better alternative for the estimation of volatility.

1.2.5 Skewness and Correlations

Negative skewness is another major concern in studies of market risk, especially for long positions. It shows a heavier weight on the downside return or loss and it is reflected in the value of VaR. If the negative skewness of the return of an asset is due to the shock and normality is assumed, it will be vanishing over a long time period due to the central limit tendency. Otherwise the skewness is caused by the correlation of the asset return with the market, which is often referred to the systematic risk of the asset in the market. A good assumption about the correlation of a particular asset with the market will be important for the estimation of VaR over a long time horizon.

1.3 Return Risk Models

Consider the simplest form of a return risk model, namely the *plain vanilla model*: the daily return is modelled as

$$R_{t+1} = \mu + \sigma\epsilon_{t+1} ,$$

where $\mu_t = \mu$, $\sigma_t = \sigma$ are constants and $\epsilon_{t+1} \sim$ i.i.d. Normal(0, 1) for all t.

In this setting, the annualized returns R_t are also i.i.d., unconditionally normal with mean μ and volatility $\sigma\sqrt{n}$, where n is the number of trading days in a year. Denoting $R_{t,T} = R_{t+1} + R_{t+2} + \ldots + R_{t+T}$ the T-days period return at day t can be written as:

$$R_{t,T} = \mu T + \epsilon_{t,T}\sigma\sqrt{T} \ ,$$

where $\epsilon_{t,T} \sim$ i.i.d. Normal(0, 1) for all t. The annualized $R_{t,T}$ are i.i.d., unconditionally normal with mean μT and volatility $\sigma\sqrt{\frac{n}{T}}$.

The $100p\%$-level T-day period VaR of the asset with price A_t at day t is:

$$VaR_p[T; A_t] = A_t(1 - e^{\mu T - z_{1-p}\sigma\sqrt{T}}) \ .$$

If two assets with respective prices $A_{1,t}$ and $A_{2,t}$ at the end of day t constitutes a portfolio, it can be verified that:

$$VaR_p[T; A_{1,t} + A_{2,t}] \leq VaR_p[T; A_{1,t}] + VaR_p[T; A_{2,t}] \ ,$$

which shows the benefit of diversification. Equality holds if the two asset returns are perfectly positively correlated. Moreover, perfect negative correlation between returns of identical σ, with both expected values greater or equal to the risk free rate, is not allowed. Otherwise these would imply the existence of pure arbitrage.

In a *jump diffusion* model, where an extra shock ξ_{t+1} is introduced to the plain vanilla model, the daily return can be written as

$$R_{t+1} = \mu + [\epsilon_{t+1} + \xi_{t+1}]\sigma \ ,$$

where $\xi_{t+1} \sim$ i.i.d. compound Poisson, denoted C.P.[Normal$(0, \nu^2)$, $\frac{\lambda}{n}$], and independent of ϵ_{t+1} for all t.

ξ_{t+1} is a compound Poisson variable with an arrival rate of shocks of $\frac{\lambda}{n}$ and a severity distributed as Normal$(0, \nu^2)$, where n is the number of trading days in a year. It has a fatter than normal tail, and it can be verified that the degree of tail-fatness of the jump diffusion model converges to that of normal over an asymptotically long time period, which is consistent with the normal feature of the efficient market assumption. However, both models above can seem unrealistic when compared to actual historical data. Other modifications such as stochastic volatility are used to reconcile theoretical models with reality.

Under the assumption of a specific model based on data for a particular position, a simulation of the price can be performed. The $100p\%$-level VaR can be obtained by finding the $100(1-p)^{\text{th}}$ percentile of the simulated data over a specific time period.

1.4 VaR for Portfolios of Assets and Derivatives

Using the definition of the underlying asset VaR, the estimation of VaR can be extended to a portfolio. By the risk mapping technique, the portfolio risk can be decomposed into a vector of m risk factors, $\{Y_i\}_{i=1}^m$, each with a different degree of exposure, β_i, in the portfolio. Here β_i measures the ratio of change in portfolio return, per change in i^{th} risk factor. These risk factors are usually represented by a list of market information, for example, foreign exchange rates, interest rate, market indices, all assumed to affect the portfolio return. The total market risk of the portfolio can thereby be obtained from the distribution of these risk factors.

In the case of a portfolio whose return is driven by a set of plain vanilla risk factors, denote the i^{th} risk factor over the T-period by $Y_{i,t,T}$ at day t. The i^{th} "surprise" factor $X_{i,t,T}$ is then represented as:

$$X_{i,t,T} = Y_{i,t,T} - E[Y_{i,t,T}|P_t] \ .$$

The return of the T-period portfolio return at day t is written as:

$$R_{p,t,T} = \mu_{p,t,T} + \sum_{i=1}^{m} \beta_i X_{i,t,T} \ ,$$

which is normally distributed with expected value $\mu_{p,t,T} = E[R_{p,t,T}|P_t]$. Denoting by C_{ij} the correlation between the i^{th} and j^{th} risk factors, the total risk of the portfolio,

D, is then given by:

$$\begin{aligned} D &= \sigma[R_{p,t,T}|P_t] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}\beta_i\beta_j C_{ij} \ . \end{aligned}$$

If the market price of the portfolio is $A_{p,t}$ at the end of day t, the $100p\%$-level T-period portfolio VaR will be:

$$VaR_p[T; A_{p,t}] = A_{p,t}(1 - e^{\mu_{p,t,T}+z_{1-p}D}) \ .$$

Moreover, estimation of the VaR for derivatives of a set of underlying assets can be done by the delta and gamma estimations on a pricing formula f over a short time interval. For instance, in the case of derivatives with a single underlying asset of price A_t at day t, the price of the derivatives is $f(A_t)$ at day t. Denoting a "small" change in the underlying asset price over a T period by $\Delta A_{t,T}$, the corresponding delta estimation of the derivatives price will be:

$$f(A_t + \Delta A_{t,T}) \approx f(A_t) + f'(A_t)\Delta A_{t,T} \ .$$

An $100p\%$-level T-period estimate of the derivatives VaR is then:

$$VaR_p[T, f(A_t)] \approx f'(A_t)VaR_p[T, A_t] \ .$$

If the VaR of the underlying asset is large, it will lead to an over-estimation of the derivatives VaR.

The delta-gamma estimation is a simple extention of the delta equation to the second derivative term, which is known as the gamma of the pricing formula. The corresponding estimate of the derivatives price will be:

$$f(A_t + \Delta A_{t,T}) \approx f(A_t) + f'(A_t)\Delta A_{t,T} + 1/2 f''(A_t)\Delta A_{t,T}^2$$

and the estimated risk of the derivatives is:

$$\begin{aligned}\sqrt{V[f(A_t + \Delta A_{t,T})]} &\approx \sqrt{V_f(A_t)} \\ &= f'(A_t)^2 V[\Delta A_{t,T}] + 1/4 f''(A_t)^2 V[\Delta A_{t,T}^2] \\ &\quad + f'(A_t) f''(A_t) CoV[\Delta A_{t,T}, \Delta A_{t,T}^2] \, .\end{aligned}$$

Then one crude estimate of the $100p\%$-level T-period derivatives VaR is $z_{1-p}\sqrt{V_f(A_t)}$. However, the second derivative nature of the gamma term usually results in an under-estimation of the VaR, especially for the Black-Scholes type options. Both delta and delta-gamma estimates of VaR work well over very short time intervals and the accuracy declines as the time horizon increases.

1.5 Conclusion

VaR is a popular measure for benchmarking market risk based on price or return fluctuations of instruments among institutions. Calculation of VaR depends very much on the model explaining the price changes and volatility of the underlying assets. However, theoretical models can be very unrealistic in comparison to actual historical data. Modifications are required in order to better fit the actual market conditions. If finding a mathematical tool to bridge to theoretical model with reality were possible, we could expect a better or less expensive estimation of the VaR. Calculation of the actual VaR for an asset can be started by first finding the risk adjusted return under the assumption of a theoretical return model covered by the Q risk neutral measure, where a new mathematical tool could link the risk adjusted models back to the actual P measure.

Chapter 2

Wang's Distortion Operator

2.1 Introduction

At a time where insurance and financial risks are becoming more integrated, many researchers including Smith (1986), Cummins (1990, 1991) and Embrechts (1996) share the viewpoint that a unified pricing theory is desirable to link both fields.

From the evidence of the influence of expected utility theory in actuarial risk theory Borch (1961), Bühlmann (1980), Goovaerts et al. (1984) and Wang (1995, 1996), the latter proposed an actuarial valuation approach by distorting the survival function of the risk variable. The method is based on Venter's observation for insurance layer prices, coinciding with Yaari's economic theory of risk. Researchers also see a resemblance between Black-Scholes option pricing formula and a stop-loss reinsurance cover, despite differences in the underlying valuation measures and the broader choice of distributions used in reinsurance.

However, difficulties remain in finding a justifiable unified pricing theory. The proportional hazards (PH) transform, as a special member of the general class of transforms defined by Wang (1994), is gaining importance in the field of actuarial science for pricing reinsurance risk layers. It, nevertheless, fails to restore Black-Scholes formula for lognormal risks. Moreover, simultaneous applications of the transform to both assets and liabilities, simply leads to inconsistencies. Applications of the capital asset pricing model (CAPM), from the theory of finance, to insurance pricing also results in serious drawbacks due to the normality assumption on asset returns in the model and estimation errors associated with the underwriting beta [see Cummins and Harrington (1985)].

Many actuaries and financial economists such as d'Arcy and Doherty (1988) or Gerber and Shiu (1994) have tried to bridge the pricing theory in the two fields, a unified theory still is a missing piece in the puzzle.

In this chapter we discuss some basic properties of distortion operators. A new distortion operator proposed by Wang (2000) is discussed in Section 2, which has some interesting properties like reproducing the capital asset pricing model and Black-Scholes formula. The ideas of standard deviation principle and Yaari's dual theory of choice are also being captured under this new distortion operator. Moreover, a new measure of tail fatness for capturing the downside risk can also be derived from it. It seems that this new distortion operator can serve as a good candidate in promoting an unified pricing theory.

2.2 Distortion Operator

Let us focus first on non-negative random loss variables and the distorted transformation of their distribution for the valuation of insurance premiums. Denote the non-negative loss random variable by X and its cumulative distribution F_X. The corresponding survival function is $S_X = 1 - F_X$. From the expected value principle, the pure insurance premium is given by

$$E[X] = \int_0^\infty y dF_X(y) = \int_0^\infty S_X(y) dy \ .$$

An insurance layer $X_{(a,a+m]}$ is defined by its payoff function as:

$$X_{(a,a+m]} = \begin{cases} 0 & 0 \le X < a \\ X - a & a \le X < a + m \\ m & a + m \le X \end{cases}$$

where a is the retention and m is the limit.

Then the survival function of the insurance layer is

$$S_{X_{(a,a+m]}}(y) = \begin{cases} S_X(a+y) & 0 \le y < m \\ 0 & m \le y \end{cases}$$

with a net premium of

$$E[X_{(a,a+m]}] = \int_0^\infty S_{X_{(a,a+m]}}(y)dy = \int_a^{a+m} S_X(x)dx \ .$$

It is important to notice that the net premium based on the expected value principle is not necessarily the market price of the risk layer. Venter (1991) observed that for any given risk, market price by layer always implies a distorted distribution. Wang (1996) therefore suggested a valuation method by distorting the survival function of the risk layer:

$$H_g[X] = \int_0^\infty g[S_X(x)]dx \ ,$$

where $g : [0,1] \longrightarrow [0,1]$ is an increasing function with $g(0) = 0$ and $g(1) = 1$. This function g is known as a distortion operator which transforms the original distribution of X, characterized by S_X, into a new ground-up distribution characterized by $g \circ S_X$. The expected value under this new distorted distribution represents the risk adjusted premium. Similarly, for the case of a risk layer the corresponding risk adjusted premium is

$$H_g[X_{(a,a+m]}] = \int_0^\infty g[S_{X_{(a,a+m]}}]dy = \int_a^{a+m} g[S_X(x)]dx \ .$$

The application of distortion operators can be generalized to assets and losses simultaneously, where an asset A is considered as a negative losses $X = -A$. By making use of the Choquet integral, which is suggested by many authors including Yaari (1987), Wang (1996), Young and Panjer (1997) as a general pricing framework, for any variable X with a survival function S_X the risk premium under the distortion g is:

$$H_g[X] = \int_{-\infty}^{0} \{g[S_X(x)] - 1\}dx + \int_{0}^{\infty} g[S_X(x)]dx \ .$$

We can see that $H_g[X]$ can be further generalized to derivative pricing consistent with the expected value principle, provided that the derivative product is comonotone with respect to its underlying asset. According to Wang (1997):

Definition 2.1 For a risk X and a real-valued function h, $Y = h(X)$ is a derivative of X. If the function h is non decreasing, Y is a comonotone derivative of the underlying risk X.

Theorem 2.1 For a comonotone derivative $Y = h(X)$, the following two methods are equivalent:

(1) Distortion Method: H_g is directly applied to Y

$$H_g[Y] = \int_{-\infty}^{0} \{g[S_Y(y)] - 1\}dx + \int_{0}^{\infty} g[S_Y(y)]dy \ .$$

(2) Transformation Method: The distribution of the underlying risk X is first distorted by g such that $S_{X'}(x) = g \circ S_X(x)$, then the expected value $E[h(X')]$ is taken with respect to X'.

If the derivative is not comonotonic with the underlying risk, an inconsistency with the expected value principle will result with the first distortion method.

By considering an asset as a negative loss $X = -A$, a distortion operator g implies the existence of a dual operator g^*. According to Denneberg (1994) it satisfies the following equality:

$$H_g[-A] = -H_{g^*}[A] \ ,$$

where $g^*(u) = 1 - g(1-u)$, $0 \leq u \leq 1$, is the dual operator of g. Moreover, if g is convex, g^* is concave and vice versa. This indeed coincides with the dual theory of choice of Yaari (1987). However, it is not necessary that g^* belong to the same parametric family or that it have the same desirable properties for a loss than for the corresponding asset.

According to Wang (1996), in order to serve as an distortion operator for pricing insurance risks, the function g has to meet the following criteria for $0 \leq u \leq 1$:

(1) $0 < g(u) < 1$ where $g(0) = 0$ and $g(1) = 1$. This ensures that the distorted distribution remains valid and (non-)zero events remain (non-)zero under the distortion.

(2) g must be an increasing function where $g(u) \geq 0$ so that $g \circ S_X$ defines a probability distribution. It is also consistent with a risk adjusted premium that decreases as the layer increases for fixed limits.

(3) g is concave, where $g''(x) \leq 0$ if it exists. This ensures that the risk load is non-negative and the relative risk loading increases with the level of the layers, for fixed limits.

(4) $g'(0) = +\infty$ for the purpose of attaining an unbounded loading at extremely high layers. It seems reasonable to have unbound relative loadings at high reinsurance layers, based on the observation of market reinsurance premiums by Venter (1991). The loss beta also appears to be unlimited at extremely high levels, as shown by Butsic (1999).

A particular function which satisfies all the above conditions is $g(u) = u^r$, where $0 < r \leq 1$. This is known as the proportional hazards (PH) transform of Wang (1995). Although the PH-transform provides some unique and desirable properties, researchers and practitioners noticed the following drawbacks:

(1) The PH-transform fails to replicate the lognormal distribution under the distortion $g(u) = u^r$. This simply implies that it fails to reproduce the Black-Scholes option pricing formula.

(2) The oversimplified nature of the PH-transform leads to a lack of flexibility in applications. It occasionally generates a relative loading which increases too fast at high layers.

(3) Simultaneous application of the PH-transform on both assets and liabilities is usually not possible. The main reason is due to the difference in shape between the distortion operator $g(u) = u^r$ and its dual $g^*(u) = 1 - (1-u)^r$, which fails to provide a unified approach in pricing insurance and financial risks for a loss and its asset, respectively.

2.3 A New Distortion Operator

Even though the PH-transform fails to provide a unified treatment of assets and liabilities, Wang (2000) suggests a new distortion operator which, apparently, solves the problem. It is defined as:

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha], \quad u \in [0,1] , \tag{2.1}$$

for some real value of α. Φ is the cumulative distribution function of the standard normal distribution with a density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \text{for } x \in \mathbb{R} .$$

It is obvious that this new distortion operator g_α with a positive $\alpha > 0$ satisfies all the four conditions stated by Wang for insurance pricing. Denoting $x = \Phi^{-1}(u)$ we have:

(1) $0 < g_\alpha(u) < 1$, due to its cumulative distribution nature and in the limiting

cases

$$g_\alpha(0) = \lim_{u \searrow 0} g_\alpha(u) = 0 \quad \text{and} \quad g_\alpha(1) = \lim_{u \nearrow 1} g_\alpha(u) = 1 \ .$$

(2) g_α is an increasing function and we have

$$g'_\alpha(u) = \frac{\partial g_\alpha(u)}{\partial u} = \frac{f(x+\alpha)}{f(x)} = e^{-\alpha x - \frac{x^2}{2}} > 0 \ .$$

(3) g_α is concave since

$$\frac{\partial^2 g_\alpha(u)}{\partial u^2} = -\frac{\alpha f(x+\alpha)}{f(x)^2} < 0 \ .$$

(4) For $\alpha > 0$, $g'_\alpha(u)$ is unbounded as u approaches 0.

If we can generalize the application of g to an asset by assuming a negative value $-\alpha < 0$, then the following additional properties hold:

(5) We have a dual operator g^*_α by simply changing the sign of α, due the symmetry of the standard normal property. It is defined as:

$$g^*_\alpha(u) = 1 - g_\alpha(1-u) = g_{-\alpha}(u), \quad \text{for } u \in [0,1] \ .$$

(6) g^*_α is convex by contrast to $g_\alpha(u)$ of the insurance risk in (3).

Under this new distortion operator we have a specific Choquet integral:

$$H[X;\alpha] = \int_{-\infty}^{0} \{g_\alpha[S_X(x)] - 1\}dx + \int_0^\infty g_\alpha[S_X(x)]dx \ .$$

Due to the expected value nature and its dependence on the parameter α it has some very nice properties:

(1) For any constant c, $H[c;\alpha] = c$ and $H[X+c;\alpha] = H[X;\alpha] + c$.

(2) For $b > 0$, $H[bX;\alpha] = bH[X;\alpha]$ and $H[-bX;-\alpha] = -bH[X;-\alpha]$ as a general case of $H[-X;\alpha] = -H[X;-\alpha]$.

(3) For any two variables X_1 and X_2,

$$H[X_1 + X_2; \alpha] \leq H[X_1; \alpha] + H[X_2; \alpha], \quad \text{if } \alpha > 0 ,$$
$$H[X_1 + X_2; \alpha] \geq H[X_1; \alpha] + H[X_2; \alpha], \quad \text{if } \alpha < 0 .$$

This shows the benefit of diversification, except for a pooling of comonotonic risks, which will result in strict equality, or no risk diversification.

(4) For a non constant variable X, $H[X; \alpha]$ is an increasing function for $\alpha \in \mathbb{R}$, such that $\min[X] \leq H[X; \alpha] \leq \max[X]$ and $H[X; 0] = E[X]$. It is important to notice that the distortion can be applied to point at different values of the $100p^{\text{th}}$ percentile of X under an appropriate value of α. If α_p corresponds to the $100p^{\text{th}}$ percentile of X under the distortion g, a cdf-like function is obtained

$$G(\alpha_p) = Pr\{X < H[X; \alpha_p]\} .$$

Then, if α_p is a $100p^{\text{th}}$-percentile parameter of the distortion operator g_α, it must satisfy:

$$\lim_{h \to 0} G(\alpha_p - h) \leq p \leq G(\alpha_p) .$$

One should notice that α_p is unique for any continuous distribution but not for a discrete distribution.

(5) With a positive α for losses or a negative one for assets, $H[X; \alpha]$ preserves the first and second order stochastic dominance (Rothschild and Stiglitz, 1970).

(6) For a positive α, $g'_\alpha(u)$ goes unbounded as u approaches 0. It can be seen under a Bernoulli(θ) risk with $Pr\{X = 1\} = \theta$ that

$$\lim_{\theta \to 0} \frac{H[X; \alpha]}{E[X]} = g'_\alpha(0) = +\infty .$$

This distortion operator g_α is applicable to any probability distribution, including the discrete type. The most obvious discrete case is the calibration of the Bernoulli(θ) risk B with some designated price θ_b:

$$H[B; \alpha_b] = g_{\alpha_b}(\theta) = \theta_b ,$$

which solves for

$$\alpha_b = \Phi^{-1}(\theta_b) - \Phi^{-1}(\theta) \ .$$

In particular, analytical forms of the ground-up distribution exist for normal and lognormal asset returns and prices, respectively, which are the most commonly used distributions in financial mathematics.

If $X \sim \text{Normal}(\mu, \sigma^2)$ with survival function S_X, the ground-up survival function $S_{X'} = g_\alpha \circ S_X$ is given by

$$\begin{aligned} \mathrm{g}_\alpha[\mathrm{S_X(x)}] &= \Phi\{\Phi^{-1}[1 - \Phi(\frac{x-\mu}{\sigma})] + \alpha\} \\ &= \Phi\{\Phi^{-1}[\Phi(\frac{-x+\mu}{\sigma})] + \alpha\} \\ &= \Phi[\frac{-x+\mu+\alpha\sigma}{\sigma}] \\ &= 1 - \Phi[\frac{x-(\mu+\alpha\sigma)}{\sigma}] \\ &= S_{X'}(x) \ , \end{aligned}$$

where $X' \sim \text{Normal}(\mu + \alpha\sigma, \sigma^2)$. Hence,

$$H[X;\alpha] = E[X'] = E[X] + \alpha\sigma[X] \ .$$

This reproduces the traditional standard deviation premium principle with parameter α.

If $Y \sim \text{Lognormal}(\mu, \sigma^2)$ with survival function S_Y, then the ground-up survival function is given by $S_{Y'} = g_\alpha \circ S_Y$ for $Y' \sim \text{Lognormal}(\mu + \alpha\sigma, \sigma^2)$.

Although closed forms do not exist for other distributions, numerical computations are fairly simple. g_α can also be generalized to multivariate distributions by applying to the corresponding joint cumulative distribution function.

2.4 Measure of Downside Risk and Tail-Fatness

One of the major concerns in financial and insurance risk management is the measurement of the down-side risk. It plays an important role in solvency analysis, risk-based-capital requirements, VaR calculations and dynamic financial analysis. The distortion operator g_α can be applied to transform the underlying distribution of a particular risk and obtaining the corresponding risk-adjusted down-side measure. Wang (2000) also suggested that $H[X;\alpha]$ be used to define measures of variability.

As a variation of the expected policyholder deficit (EPD) by Butsic (1994), Artzner et al. (1998) and Artzner (1999) suggested a set of rules for coherent downside risk measure based on the expected deficit in excess of a prescribed $100p^{\text{th}}$ percentile. A risk-adjusted version of the down-side measure can be obtained by applying the distortion operator g_α to the survival function of the excess deficit:

$$g(u) = \begin{cases} g_\alpha(u) & 0 \le u < p \, , \\ g_\alpha(p) & p \le u < 1 \, . \end{cases}$$

If the underlying distribution is lognormal, the expected value of the ground-up distribution is indeed the price of an European call option with a strike price at the $100p^{\text{th}}$ percentile value.

Another application of Wang's distortion operator is the measurement of the tail-fatness, namely right (left) tail deviation, defined respectively for some positive α as:

$$RD_\alpha[X] = \frac{1}{\alpha}\{H[X;\alpha] - E[X]\} \, , \tag{2.2}$$

$$LD_\alpha[X] = \frac{1}{\alpha}\{E[X] - H[X;-\alpha]\} \, . \tag{2.3}$$

As an alternative to kurtosis, which measures the two-side tail-fatness, it is interesting to measure the tail-fatness on only either the right or the left side, especially for the case of financial modelling. Wang suggested the use of a right(left)-tail index which is defined for some positive α as:

$$RTI_\alpha[X] = \frac{RD_{2\alpha}[X]}{RD_\alpha[X]} \, , \tag{2.4}$$

$$LTI_\alpha[X] = \frac{LD_{2\alpha}[X]}{LD_\alpha[X]} \, . \tag{2.5}$$

If $X \sim$ Normal(μ, σ^2), it is easily seen that $RD_\alpha[X] = LD_\alpha[X] = \sigma$ and that $RTI_\alpha[X] = LTI_\alpha[X] = 1$. A tail index greater than 1 indicates a fatter than normal tail.

Based on the expression of the tail deviation it is interesting to see the limiting case when α approaches 0. By (2.2) and (2.3) this limit is seen to be a partial derivative evaluated at $\alpha = 0$:

$$\lim_{\alpha \to 0} RD_\alpha[X] = \lim_{\alpha \to 0} LD_\alpha[X] = \frac{\partial}{\partial \alpha} H[X;\alpha]|_{\alpha=0} .$$

It measures the degree of distortion by g_α from the original distribution of X. Consequently, we define the following index.

Definition 2.2 The distortion index (DI) of a random variable X under the distortion operator g_α, which measures the severity of the distortion from the original distribution of X, is given by

$$DI[X] = \frac{\partial}{\partial \alpha} H[X;\alpha]|_{\alpha=0} . \tag{2.6}$$

For instance, in the case of a Bernoulli(θ) variable X_b,

$$DI[X_b] = \frac{\partial}{\partial \alpha} H[X_b;\alpha]|_{\alpha=0} = f[\Phi^{-1}(\theta)] ,$$

while the DI of the random variable $X \sim$ Normal(μ, σ^2) is always σ for any $\mu \in \mathbb{R}$.

Moreover, the definition can be generalized to any other distortion distribution G.

Definition 2.3 A θ_p-percentile density of a distribution function G with a density function g is defined as:

$$p_d(\theta_p) = g[G^{-1}(\theta_p)] , \tag{2.7}$$

which benchmarks the tail-fatness of the distribution with respective to θ_p.

2.5 Parameter α and Systematic Risk

For the asset price risk in financial markets, the focus is on the asset return, which determines the asset price. Under an efficient market and a plain vanilla return assumption the current price of an asset and its return distribution implies a specific value of α, which has a very special meaning in the context of the capital asset pricing model (CAPM) and the related systematic market risk.

2.5.1 Implied Parameter α

Assuming asset returns are compounded continuously, for an asset with current price A_t at day t, the return over day $t+1$ is:

$$R_{t+1} = \ln(\frac{A_{t+1}}{A_t}), \quad t \geq 0 .$$

As mentioned in Chapter 1, the current asset price is assumed to be determined by the expected return over the future period. If we want to force an efficient market assumption, there exists a risk neutral measure such that:

$$A_t = A_t e^{E_Q[R_{t+1} - r_f]} ,$$

where r_f is the risk free return over the one compounding period. The above equation can be rewritten under the distortion operator $g_{-\alpha}$ as it is suggested by Wang for pricing purposes:

$$A_t = A_t e^{H[R_{t+1} - r_f; -\alpha]} .$$

Assuming a plain vanilla return model where $E[R_t] = \mu$ and $\sigma[R_t] = \sigma$ for all t, under the properties of the ground-up distribution for a normal risk, we have:

$$\begin{aligned} r_f &= H[R_{t+1}; -\alpha] \\ &= \mu - \alpha\sigma , \end{aligned}$$

where

$$\alpha = \frac{\mu - r_f}{\sigma} .$$

Now, if the asset also belongs to a market portfolio with a plain vanilla return $R_{M,t} \sim$ Normal(μ_M, σ_M^2), we have the following market portfolio α_M :

$$\alpha_M = \frac{\mu_M - r_f}{\sigma_M} .$$

It can be identified as the market price of risk (Cummins, 1990) in the expression of the capital market line (CML) in CAPM.

The above plain vanilla return setting for deriving the implied α can be generalized to a multi-period horizon. Denoting by $R_{t,T}$ the net return over a $T-$period starting at day t, and by A_t the current price of the asset at day t, we have:

$$R_{t,T} = \ln \frac{A_{t+T}}{A_t} = \sum_{k=t}^{t+T-1} \ln \frac{A_{k+1}}{A_k} = \sum_{k=t}^{t+T-1} R_k ,$$

and $R_{t,T} \sim$ Normal$(T\mu, T\sigma^2)$ since $R_t \sim$ i.i.d. Normal(μ, σ^2). Assuming that the single period risk free return r_f is constant, the $T-$period risk free return is then Tr_f. Under the same efficient market assumptions there exists an α such that:

$$H(R_{t,T}; -\alpha) = T\mu - \alpha\sqrt{T}\sigma = Tr_f ,$$

and hence,

$$\alpha = \sqrt{T} \left\{ \frac{\mu - r_f}{\sigma} \right\} .$$

For the case of geometric Brownian motion (GBM) returns with a one period rate μ and volatility σ, the asset price A_t satisfies the stochastic differential equation :

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t .$$

The continuous shock, denoted by dW_t, is a white noise. According to Hull (1997) we have a lognormal distribution over the future period $[t, t+T]$:

$$\ln \frac{A_{t+T}}{A_t} \sim \text{Normal } [(\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T] .$$

Under an arbitrage free assumption with a risk free return r_f, a similar condition holds:

$$A_t = e^{-r_f T} H[A_{t+T}; -\alpha] ,$$

which implies again that

$$\alpha = \frac{(\mu - r_f)\sqrt{T}}{\sigma} .$$

Notice how α shares many properties with volatility measures. For a plain vanilla return α is proportional to the square root of the horizon length T, which compares to the volatility. For the case of stochastic volatility, used with a fixed value of μ, the above conditions yield a stochastic α under a conditional normal return.

2.5.2 Capital Asset Pricing Model and Systematic Risk

As was already mentioned in the previous section, α_M can be associated with the market price of risk in the CML of CAPM. We review here some basic elements of the CAPM, which links α to the systematic market risk.

A major concern in CAPM is to measure the portfolio risk, under the constraint of a designated expected return in one compounding period, and the assumption of an efficient market. By mixing a risk free return r_f and an efficient market portfolio, which attains a minimal risk for a given level of expected return, we can derive an efficient strategy resulting in a linear relationship between risk and return. Denoting the expected return and risk of this strategy, respectively μ_p and σ_p, we have the following relation, namely the capital market line (CML):

$$\mu_p = r_f + \alpha_M \sigma_p .$$

For the case of a single asset A in a plain vanilla return model $R_A \sim \text{Normal}(\mu_A, \sigma_A^2)$, one can make use of the correlation between A and the market, in terms of market β under a linear model setting, to obtain the security market line (SML). Denoting the market portfolio return as $R_M \sim \text{Normal}(\mu_M, \sigma_M)$ the SML is given by

$$\mu_A = r_f + \beta\{\mu_M - r_f\} ,$$

with

$$\beta = \frac{CoV[R_A, R_M]}{\sigma_M^2} = \rho_{A,M}\frac{\sigma_A}{\sigma_M} .$$

It can be rewritten as

$$\mu_A = r_f + \rho_{A,M}\alpha_M\sigma_A ,$$

and hence the following result is obtained

$$\rho_{A,M}\alpha_M = \frac{\mu_A - r_f}{\sigma_A} = \alpha_A .$$

The SML can be generalized to a multi-period horizon by the same treatment under the plain vanilla assumption. For a GBM return, Merton (1973) proposed a intertemporal version known as ICAPM. Exact results can be obtained from both cases.

Investors in the financial market face two type of risks, namely systematic and non-systematic risks, where only the latter can be reduced by diversification. In other words, the systematic risk is determined by the current market condition and it is reflected by the correlation between the return of their investments and that of the overall market. From the expression above of α_A for a single asset A, we can see that α_A corresponds to the systematic risk in the particular market. For the case of portfolio management a portfolio is priced by its own α based on the SML. If a derivative Y is comonotonic to an underlying asset X, both share the same level of systematic risk as X, where $\rho_{R_Y,R_M} = \rho_{R_X,R_M}$ and hence, $\alpha_Y = \alpha_X$. The derivative should be priced by $H[Y;\alpha_X]$ under a risk-neutral approach.

As the above results are all based on the multivariate normal assumption of the asset returns in the market, one will face serious drawbacks upon insurance pricing where loss distributions are highly skewed. Moreover, in reality asset returns in a market are not all necessarily normally distributed.

Wang (2000) proposes a generalization of CAPM by transforming the components of an aggregate risk portfolio, which can be expressed in terms of a vector of component risks:

$$\{X_1, X_2, \ldots, X_k\} ,$$

for some large k. Denoting the aggregate risk Z we have

$$Z = \sum_{i=1}^{k} X_i .$$

Transforming the vector of the component risks into a vector of multivariate normal variables

$$\{\Phi^{-1}[F_{X_1}(X_1)], \Phi^{-1}[F_{X_2}(X_2)], \ldots, \Phi^{-1}[F_{X_k}(X_k)]\} ,$$

Wang defines a new measure of the systematic risk for the i−th component as

$$\rho^*_{X_i,Z} = \text{Cov}\{\Phi^{-1}[F_{X_i}(X_i)], \Phi^{-1}[F_Z(Z)]\} .$$

As a generalized result from CAPM we have a pricing formula $H[X_i;\alpha_{X_i}]$ where $\alpha_{X_i} = \rho^*_{X_i,Z}\alpha_Z$. However, adjustments to the value α may be required depending on the size of the portfolio and other factors taken into account. A higher value of α is generally preferred in reflecting the market friction and incomplete information.

A similar treatment can be applied to the asset returns in a financial market without assuming plain vanilla returns, except for the market return. We can replace X_i by R_{A_i}, as the return of the i^{th} asset in the market, and Z by R_M, as the market return. However, this is not convincing since the market return is regarded as non-measurable in the theory of financial economics. Instead of generalizing CAPM, the problem is expected to be solved by developing an alternative pricing theory, known as arbitrage pricing theory (APT) which will be discussed in the next chapter.

2.6 Distortion and Black-Scholes Formula

Many researchers see Black-Scholes option pricing formula as a special case of a stop-loss reinsurance under a lognormal risk neutral price. Wang asserts that the distortion operator $g_{-\alpha}$ replicates Black-Scholes formula under the specific α mentioned in the CAPM.

Denoting A_t as the current price of an asset at day t with a strike price of K and a right to exercise after T periods, the payoff of an European call option can be expressed as

$$A_{t+T\ (K,\infty)} = \begin{cases} 0 & if\ A_{t+T} \leq K\ , \\ A_{t+T} - K & if\ A_{t+T} > K\ . \end{cases}$$

The expected payoff is calculated as:

$$E[A_{t+T\ (K,\infty)}] = \int_0^\infty S_{A_{t+T\ (K,\infty)}}(x)dx = \int_K^\infty S_{A_{t+T}}(y)dy \ .$$

Applying the distortion operator $g_{-\alpha}$, the price of the call option is given by

$$e^{-r_f T} H[A_{t+T\ (K,\infty)}; -\alpha] = e^{-r_f T} \int_K^\infty [S_{A'_{t+T}}(y)]dy \ .$$

It can easily be verified that with a GBM return of rate μ and volatility σ under the parameter

$$\alpha = \frac{(\mu - r_f)\sqrt{T}}{\sigma} \ ,$$

the ground-up variables A'_t implies a lognormal distribution

$$\ln \frac{A'_{t+T}}{A'_t} \sim \text{Normal}[(r_f - \frac{1}{2}\sigma^2)T, \sigma^2 T] \ .$$

The distorted setting and resulting pricing formula reproduce exactly Black-Scholes model.

2.7 Comments and Conclusion

As stated by Wang the new distortion operator $g_{-\alpha}$ appears to be one of the missing links in the puzzle of a unifying pricing theory for the four different approaches:

(1) Yaari's economic theory of risk,

(2) traditional standard deviation approach,

(3) capital asset pricing model,

(4) Black-Scholes option pricing formula.

It promotes a unified approach for pricing both financial and insurance risks.

The application of g_α suggested by Wang in measuring the downside risk, and the relation of the parameter α associated with the systematic risk serve as hints in refining methodologies in current market risk measurements, for example VaR modelling and computation. This will be further explored in the next few chapters.

Chapter 3

A New Distortion Operator

3.1 Introduction

Observe the mathematical formulation of Wang's distortion operator in (2.1) and notice that it is a conjugation of the standard normal cumulative distribution on a linear function. Intuitively it can be seen as a special asymmetric form of a more general distortion. The word asymmetric here refers to the fact that the inverse of the distortion can be obtained by simply switching the sign of α. Using different functions over the decumulative values u, different distortion operators can be derived but some of them may not satisfy the conditions for insurance pricing.

In this chapter we focus on the general linear function applied on the decumulative value u and search for a general form of the distortion family which contains Wang's distortion operator. As the latter serves an important role in bridging other financial economic theory, as mentioned in the previous chapter, the attempt is to find a new distortion which retains all of these properties plus an extra distortion in the volatility resulting in a varying risk neutral volatility.

3.2 A Generalized Distortion

3.2.1 General Linear Gaussian Conjugation Class

Recall that Wang's distortion operator with parameter $-\alpha$ takes the form:

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) - \alpha], \quad \text{where } u \in [0, 1].$$

It is indeed a special case of a more general distortion class introduced here.

Definition 3.1 For any real-valued functions F and m defined over all $S \supseteq m(S)$, any real-valued function g that satisfies

$$\begin{aligned} g(y) &= F \circ m \circ F^{-1}(y), \\ &= F[m[F^{-1}(y)]], \quad \text{for } y \in F(S), \end{aligned}$$

is said to be in the same F-conjugation class as m.

Definition 3.2 If $F = \Phi$ and m is a non-decreasing linear real-valued function, any function $h = \Phi \circ m \circ \Phi^{-1}$ belongs to the same general linear Gaussian conjugation (GLGC) class.

We can therefore see that Wang's distortion operator for any real α is in the Gaussian conjugation class of m, where

$$m(x) = x + \alpha, \quad \text{for } x \in \mathbb{R}.$$

Here consider a more general form of m, that is non-decreasing, but with two parameters, namely $\alpha \in \mathbb{R}$ and $\beta \geq 0$, such that

$$m(x) = \beta(x + \alpha), \quad \text{where } x \in \mathbb{R}.$$

Define a general form of the distortion $h_{\alpha,\beta}$, where

$$h_{\alpha,\beta}(u) = \Phi\{\beta[\Phi^{-1}(u) + \alpha]\}, \quad \text{for } u \in [0, 1].$$

3.2.2 A Special Distortion Operator

Let us take one step back and assume that $\alpha = 0$, before studying further the new proposed distortion. This introduces a special distortion operator

$$k_\beta(u) = \Phi[\beta\Phi^{-1}(u)], \quad u \in [0,1] \; , \tag{3.1}$$

for some $\beta \geq 0$.

This special distortion operator k_β satisfies two of the four conditions stated by Wang for insurance pricing. Denoting by $x = \Phi^{-1}(u)$ and f the pdf of the standard normal distribution we have:

(1) $0 < k_\beta(u) < 1$, and the limiting cases

$$k_\beta(0) = \lim_{u \searrow 0} k_\beta(u) = 0 \quad \text{and} \quad k_\beta(1) = \lim_{u \nearrow 1} k_\beta(u) = 1 \; .$$

(2) k_β is an increasing function, and

$$k'_\alpha(u) = \frac{\partial k_\beta(u)}{\partial u} = \frac{\beta f(\beta x)}{f(x)} = \beta e^{\frac{x^2}{2}(1-\beta^2)} > 0 \; .$$

(3) For $\beta > 1$, k_β is concave for $u > \frac{1}{2}$ and convex for $u < \frac{1}{2}$. By contrast, for $\beta < 1$, k_β is convex for $u > \frac{1}{2}$ and concave for $u < \frac{1}{2}$. It can be seen that

$$\frac{\partial^2 k_\beta(u)}{\partial u^2} = \frac{x}{f(x)}(1-\beta^2)\beta e^{\frac{1}{2}(1-\beta^2)} \; .$$

Even though it fails to satisfy the insurance pricing criteria, these properties give an interesting representation of the change in the investor's utility or preference for an asset compared to its expected loss.

(4) For $\beta < 1$, $k'_\beta(u)$ is unbounded as u approaches 0, while for $\beta > 1$, $k'_\beta(u)$ tends to zero as u approaches 0.

$\beta > 1$ implies a risk optimistic perception; an unbounded relative loading is not necessary.

(5) A dual operator $k^*_\beta(u)$ exists and is equal to itself

$$k^*_\beta(u) = 1 - k_\beta(1-u) = k_\beta(u)$$

Hence the distortion that applies to both asset and risk is identical.

(6) An extra property we would like to mention here is the inverse distortion, which will eventually play a role in classifying risks of the same ground-up distribution. The parameter β classifies risks in terms of admissibility to a fixed distorted ground-up distribution. For a fixed ground-up distribution X' there exists a family of distributions of X which can be distorted to X', for some β, satisfying

$$k_\beta[S_X(x)] = S_{X'}(x).$$

This implies an inverse distortion k_β^{-1} such that

$$k_\beta^{-1}[S_{X'}(x)] = S_X(x),$$

and it can be expressed as

$$k_\beta^{-1}(u) = \Phi[\frac{1}{\beta}\Phi^{-1}(u)] = k_{\frac{1}{\beta}}(u).$$

Properties (3) and (4) show that this special distortion operator is not appropriate for insurance pricing. However, these properties have a special interpretation in price return, particularly for the market expectation.

Under the special distortion operator in (3.1), the Choquet integral becomes

$$K[X;\beta] = \int_{-\infty}^{0} \{k_\beta[S_X(x)] - 1\}dx + \int_{0}^{\infty} k_\beta[S_X(x)]dx\ .$$

This means that α, in Wang's distortion, contributes to the mean translation, while β in this special distortion contributes to the median related polarization. In addition, the following properties hold for the distortion in (3.1):

(1) For any constant c, $K[c;\beta] = c$ and $K[X+c;\beta] = K[X;\beta] + c$.

(2) For all $b \in \mathbb{R}$, $K[bX;\beta] = bK[X;\beta]$.

(3) Denoting by $\tilde{\mu}_X$ the median of a non-degenerate variable X,

$$\tilde{\mu}_X = \sup[x : S_X(x) \leq \frac{1}{2}]$$

then $K[X;\beta]$ converges to $\tilde{\mu}_X$ with certainty as β goes unbounded. In other words, the ground-up distribution of X is just a point mass at $\tilde{\mu}_X$. Moreover, as β tends to 0 :

$$K[X;\beta] \longrightarrow \frac{\max[X] + \min[X]}{2}.$$

Notice that for all $\beta \geq 0$ and continuous variable X, $K[X;\beta]$ is equal to $\tilde{\mu}_X$ if $E[X] = \tilde{\mu}_X$.

(4) For any two comonotone variables X_1 and X_2

$$K[X_1 + X_2;\beta] = K[X_1;\beta] + K[X_2;\beta], \quad \text{if} \quad \beta > 0.$$

(5) For any variable X and $\beta_1 > \beta_2 > 1$,

$$\tilde{\mu}_X \leq K[X;\beta_1] \leq K[X;\beta_2] \leq E[X] \quad \text{if} \quad E[X] \geq \tilde{\mu}_X$$
$$\tilde{\mu}_X \geq K[X;\beta_1] \geq K[X;\beta_2] \geq E[X] \quad \text{if} \quad E[X] \leq \tilde{\mu}_X.$$

For $\beta_1 < \beta_2 < 1$,

$$K[X;\beta_1] \geq K[X;\beta_2] \geq E[X] \quad \text{if} \quad E[X] \geq \tilde{\mu}_X$$
$$K[X;\beta_2] \leq K[X;\beta_2] \leq E[X] \quad \text{if} \quad E[X] \leq \tilde{\mu}_X.$$

For $\beta = 1$, $K[X;\beta] = E[X]$ simply implies no distortion.

This distortion operator k_β is applicable to any probability distribution, including discrete ones. For the case of the Bernoulli(θ) risk B, where $\theta \neq \frac{1}{2}$ with some designated price θ_b :

$$K[B;\beta_b] = k_{\beta_b}(\theta) = \theta_b \ ,$$

which solves for

$$\beta_b = \frac{\Phi^{-1}(\theta_b)}{\Phi^{-1}(\theta)} \ .$$

However, according to the inverse distortion property, notice that the solution exists only for an appropriate θ_b, where θ must be θ_b−admissible.

For normal and lognormal asset returns and prices, due to their symmetric nature, we have the following result:

If $X \sim \text{Normal}(\mu, \sigma^2)$ with survival function S_X, the ground-up survival function $S_{X'} = k_\beta \circ S_X$ implies $X' \sim \text{Normal}(\mu, \frac{\sigma}{\beta}^2)$. Hence,

$$H[X;\beta] = E[X'] = E[X],$$

where β has no impact on the expected value of the normal distribution.

If $Y \sim \text{Lognormal}(\mu, \sigma^2)$ with survival function S_Y, then the ground-up survival function is given by $S_{Y'} = k_\beta \circ S_Y$ for $Y' \sim \text{Lognormal}(\mu, \frac{\sigma}{\beta}^2)$. Even though there is no impact on the expected value under normality, the variance is distorted, which makes this special distortion operator interesting for volatility distortion.

3.2.3 A New Combined Distortion Operator

Consider the return risk for the stock market and recall that Wang's distortion operator does not create any volatility distortion for normal and lognormal distributions. This is not consistent with the binomial return model, where the variance cannot be identical for both, risk neutral and actual distributions, if the expected return is not risk neutral . We propose a new distortion by combining Wang's distortion operator with the special distortion operator k_β in (3.1), where β is related to α. We relax his conditions (3) and (4), required for insurance pricing in Section 2.2, as investors may not be as conservative as insurers. On the other hand, one assumption here is based on an observation from CAPM: return is an increasing function of risk, or equivalently, loss is a decreasing function of risk. This translates to an expected loss that is a decreasing function of volatility, and therefore, we assume $\beta = e^\alpha$. This also ensures the absence of distortion at $\alpha = 0$.

Definition 3.3 The combined distortion operator is defined as

$$h_{\alpha,e^\alpha}(u) = g_\alpha \circ k_{e^\alpha}(u) = k_{e^\alpha} \circ g_\alpha(u), \quad u \in [0,1],$$

which is expressed equivalently as

$$h_{\alpha,e^\alpha}(u) = \Phi\{e^\alpha[\Phi^{-1}(u) + \alpha]\}, \quad u \in [0,1]\ , \tag{3.2}$$

for some real value of α.

This new distortion operator h_{α,e^α} satisfies two out of the four conditions stated by Wang for insurance pricing. Denoting $x = \Phi^{-1}(u)$ we have:

(1) $0 < h_{\alpha,e^\alpha}(u) < 1$ and we can again see in the limiting case:

$$h_{\alpha,e^\alpha}(0) = \lim_{u \searrow 0} h_{\alpha,e^\alpha}(u) = 0, \quad \text{and} \quad h_{\alpha,e^\alpha}(1) = \lim_{u \nearrow 1} h_{\alpha,e^\alpha}(u) = 1.$$

(2) h_{α,e^α} is an increasing function and denoting $\beta = e^\alpha$ we have

$$h'_{\alpha,\beta}(u) = \frac{dh_{\alpha,\beta}(u)}{du} = \beta e^{\frac{1}{2}x^2(1-\beta^2) - x\alpha\beta^2 - \frac{1}{2}\alpha^2\beta^2} > 0.$$

(3) The second derivative of $h_{\alpha,\beta}(u)$ with respect to u is

$$\frac{d^2 h_{\alpha,\beta}(u)}{du^2} = \frac{x(1-\beta^2) - \alpha\beta^2}{f(x)} \beta e^{\frac{1}{2}x^2(1-\beta^2) - x\alpha\beta^2 - \frac{1}{2}\alpha^2\beta^2}.$$

For $\alpha > 0$, $h_{\alpha,\beta}$ is convex whenever $u < \Phi(\frac{\alpha\beta^2}{1-\beta^2})$ and h_α is concave whenever $u > \Phi(\frac{\alpha\beta^2}{1-\beta^2})$. For $\alpha < 0$, $h_{\alpha,\beta}$ is concave whenever $u < \Phi(\frac{\alpha\beta^2}{1-\beta^2})$ and $h_{\alpha,\beta}$ is convex whenever $u > \Phi(\frac{\alpha\beta^2}{1-\beta^2})$. This reflects a reasonable preference of the investors towards a return or loss at x such that $S_X(x) = \Phi(\frac{\alpha\beta^2}{1-\beta^2})$.

(4) For $\alpha < 0$, $h'_{\alpha,\beta}(u)$ is unbounded as u approach 0. For $\alpha > 0$, $h'_{\alpha,\beta}(u)$ tends to 0 as u approach 0.

This property represents that investors are perceiving a more certain loss at a more conservative loss premium. On the other hand they will see a more fluctuating loss or gain over a less conservative loss premium which leads to an unbounded relative loading.

(5) A dual operator $h^*_{\alpha,e^\alpha}(u)$ defined as:

$$\begin{aligned} h^*_{\alpha,e^\alpha}(u) &= 1 - h_{\alpha,e^\alpha}(1-u) \\ &= \Phi\{e^\alpha[\Phi^{-1}(u) - \alpha]\} \\ &= h_{-\alpha,e^\alpha}(u) \end{aligned}$$

Here assets and risks are distorted under the same general family in a broader sense.

(6) An inverse distortion defined as

$$h^{-1}_{\alpha,e^\alpha}(u) = \Phi\{e^{-\alpha}[\Phi^{-1}(u) - \alpha e^\alpha]\} = h_{-\alpha e^\alpha, e^{-\alpha}}(u)$$

Under this new distortion operator Choquet's integral takes the form

$$H[X;\alpha,e^\alpha] = \int_{-\infty}^{0} \{h_{\alpha,e^\alpha}[S_X(x)] - 1\}dx + \int_0^\infty h_{\alpha,e^\alpha}[S_X(x)]dx.$$

It has the following properties:

(1) For any constant c, $H[c;\alpha,e^\alpha] = c$ and $H[X+c;\alpha,e^\alpha] = H[X;\alpha,e^\alpha] + c$.

(2) For $b > 0$, $H[bX;\alpha,e^\alpha] = bH[X;\alpha,e^\alpha]$ and $H[-bX;\alpha,e^\alpha] = -bH[X;-\alpha,e^\alpha]$. In particular, $H[-X;\alpha,e^\alpha] = -H[X;-\alpha,e^\alpha]$.

(3) For any two comonotone variables X_1 and X_2,

$$H[X_1 + X_2;\alpha,e^\alpha] = H[X_1;\alpha,e^\alpha] + H[X_2;\alpha,e^\alpha], \quad \text{if} \quad \alpha > 0,$$

(4) For a non-degenerate variable X, $h_{\alpha,e^\alpha}[S_x(x)]$ is an increasing function in $\alpha \in [\alpha_0, \alpha_1] \subseteq \mathbb{R}$, whenever $S_x(x) \geq \gamma$ for

$$\gamma = 1 - \Phi(1 + \alpha_0)$$

and

$$\min[X] \leq H[X;\alpha,e^\alpha] \leq \max[X],$$

If $\alpha = 0$ then $H[X;0,1] = E[X]$. Due to its composite nature, $H[X;\alpha,e^\alpha]$ inherits the increasing nature of Wang's distortion operator with respect to α. Therefore for any symmetric distributed variable X, $H[X;\alpha,e^\alpha]$ increases from $\min[X]$ to $\max[X]$ as α increases.

(5) For a negative α, $h'_{\alpha,e^\alpha}(u)$ goes unbounded as u approaches 0. It can be seen that for a Bernoulli(θ) risk with $Pr\{X=1\}=\theta$

$$\lim_{\theta\to 0}\frac{H[X;\alpha,e^\alpha]}{E[X]} = h'_{\alpha,e^\alpha}(0) = +\infty.$$

The distortion operator h_{α,e^α} can be applied to any probability distribution, including discrete ones. To calibrate a Bernoulli(θ) risk B with an appropriate designated price θ_b, the equation to be solved is

$$\begin{aligned} H[B;\alpha_b,e^{\alpha_b}] &= h_{\alpha_b,e^{\alpha_b}}(\theta), \\ &= \theta_b, \end{aligned}$$

then α is the solution of

$$\Phi[e^{\alpha_b}\Phi^{-1}(\theta)+\alpha_b] = \theta_b.$$

Again analytical forms of the ground-up distribution exist in the normal and lognormal cases.

If $X \sim$ Normal(μ,σ^2) with survival function S_X, we can see that the ground-up survival function $S_{X'} = h_{\alpha,e^\alpha}\circ S_X$:

$$\begin{aligned} h_{\alpha,e^\alpha}[S_X] &= \Phi[e^\alpha\{\Phi^{-1}[1-\Phi(\tfrac{X-\mu}{\sigma})]+\alpha\}], \\ &= \Phi[e^\alpha\{\Phi^{-1}[\Phi(\tfrac{-X+\mu}{\sigma})]+\alpha\}], \\ &= \Phi[\tfrac{-X+\mu+\alpha\sigma}{e^{-\alpha}\sigma}], \\ &= 1-\Phi[\tfrac{X-(\mu+\alpha\sigma)}{e^{-\alpha}\sigma}], \\ &= S_{X'}. \end{aligned}$$

It defines $X' \sim$ Normal$(\mu+\alpha\sigma, e^{-2\alpha}\sigma^2)$. Hence

$$H[X;\alpha,e^\alpha] = E[X'] = E[X]+\alpha\sigma[X].$$

As in Wang's case, this distortion operator restores the traditional standard deviation principle.

If $Y \sim \text{Lognormal}(\mu, \sigma^2)$ with survival function S_Y, then the ground-up survival function $S_{Y'} = h_{\alpha,e^\alpha} \circ S_Y$ defines $Y' \sim \text{Lognormal}(\mu + \alpha\sigma, e^{-2\alpha}\sigma^2)$.

For other distributions, numerical computations can easily be performed in EXCEL. h_{α,e^α} can also be extended to multivariate variables by an application to the corresponding joint cumulative distribution function.

3.3 CAPM Revisited

Due to the invariant nature of $H[X;\alpha,e^\alpha]$ with respect to $H[X;\alpha]$ under normality, this new combined distortion operator preserves all the properties of Wang's operator in reproducing the CAPM.

3.3.1 Capital Market Line

Assuming asset returns are compounded continuously, for an asset with current price A_t at day t, the return over day $t+1$ is:

$$R_{t+1} = \ln(\frac{A_{t+1}}{A_t}), \quad t \geq 0 .$$

We want to replicate a risk neutral measure such that:

$$A_t = A_t e^{E_Q[R_{t+1} - r_f]} ,$$

where r_f is the risk free return over the one compounding period. The above equation can be rewritten under the distortion operator $h_{-\alpha,e^\alpha}$, in a way similar to Wang's pricing equation:

$$A_t = A_t e^{H[R_{t+1} - r_f; -\alpha, e^\alpha]} .$$

Under a plain vanilla return model where $E[R_t] = \mu$ and $\sigma[R_t] = \sigma$ for all t we have

$$\begin{aligned} r_f &= H[R_{t+1}; -\alpha, e^\alpha] \\ &= \mu - \alpha\sigma. \end{aligned}$$

Therefore it provides identical results to Wang's distortion,

$$\alpha = \frac{\mu - r_f}{\sigma}.$$

If the asset belongs to a market portfolio with a plain vanilla return, that is $R_{M,t} \sim$ Normal(μ_M, σ_M^2), we replicate the market portfolio α_M :

$$\alpha_M = \frac{\mu_M - r_f}{\sigma_M} .$$

It is exactly the market price of risk (Cummins, 1990) in the expression of the capital market line (CML) in CAPM.

For a multi-period horizon, denote $R_{t,T}$ as the net return over a $T-$period starting at day t, and by A_t the current price of the asset at day t, then

$$R_{t,T} = \ln \frac{A_{t+T}}{A_t} = \sum_{k=t}^{t+T-1} \ln \frac{A_{k+1}}{A_k} = \sum_{k=t}^{t+T-1} R_k ,$$

and $R_{t,T} \sim$ Normal$(T\mu, T\sigma^2)$ since $R_t \sim$ i.i.d. Normal(μ, σ^2). Assuming the single period risk free return as r_f, the $T-$period risk free return is then Tr_f. Under the same efficient market assumptions there exists an α and a time adjusted distortion such that:

$$H[R_{t,T}; -\alpha, e^{\frac{\alpha}{\sqrt{T}}}] = T\mu - \alpha\sqrt{T}\sigma = Tr_f ,$$

and hence,

$$\alpha = \sqrt{T} \left\{ \frac{\mu - r_f}{\sigma} \right\} .$$

This differs from Wang's result in the case of geometric Brownian motion (GBM) returns with a one period rate μ and volatility σ. The asset price A_t then satisfies the stochastic differential equation :

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t .$$

Again we have a lognormal distribution over the future period $[t, t+T]$:

$$\ln \frac{A_{t+T}}{A_t} \sim \text{Normal } [(\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T] .$$

Under an arbitrage free assumption with a risk free return r_f, for an GBM level α^* a similar condition holds:

$$A_t = e^{-r_f T} H[A_{t+T}; -\alpha^*, e^{\frac{\alpha^*}{\sqrt{T}}}] \, ,$$

which implicitly defines α^* satisfying

$$\alpha^* = \sqrt{T}\left[\frac{(\mu - r_f)}{\sigma} + \frac{1}{2}\sigma(1 - e^{-2\alpha^*})\right] . \tag{3.3}$$

Here α^* is similar to the case of the expected return. For a GBM return the above expression indicates that α is adjusted by an additional term related to the volatility. Comparing with the $T-$period expected return we have

$$E\left[\ln\frac{A_{t+T}}{A_t}\right] = (\mu - \frac{1}{2}\sigma^2)T.$$

3.3.2 Security Market Line

Under our new distortion operator, α_M is identical with the market price of risk in the CML of CAPM. For the case of a single asset A we can again replicate the security market line (SML) identical to that of Wang's distortion operator produced. Denoting the market portfolio return as $R_M \sim$Normal(μ_M, σ_M) the SML is given by

$$\begin{aligned} \mu_A &= r_f + \rho_{A,M}\frac{\sigma_A}{\sigma_M}(r_f - \mu_M) \\ &= r_f + \rho_{A,M}\alpha_M\sigma_A, \end{aligned}$$

and

$$\rho_{A,M} = \frac{\frac{\mu_A - r_f}{\sigma_A}}{\alpha_M} = \frac{\alpha_A}{\alpha_M} \, .$$

Identical results can be obtained over a multi-period horizon by the same treatment under the plain vanilla assumption. For a GBM return, we can expect a minor deviation resulting from the discussion on α^* in (3.3). By rearranging the expression of the SML, one obtains

$$
\begin{aligned}
\rho_{A,M} &= \frac{\frac{\mu_A - r_f}{\sigma_A}}{\frac{\mu_M - r_f}{\sigma_M}} = \frac{\left[\frac{\mu_A - r_f}{\sigma_A} + \frac{1}{2}\sigma_A(1 - e^{-2\alpha_A^*})\right] - \frac{1}{2}\sigma_A(1 - e^{-2\alpha_A^*})}{\left[\frac{\mu_M - r_f}{\sigma_M} + \frac{1}{2}\sigma_M(1 - e^{-2\alpha_M^*})\right] - \frac{1}{2}\sigma_M(1 - e^{-2\alpha_M^*})} \\
&= \frac{\alpha_A^* - \frac{1}{2}\sigma_A(1 - e^{-2\alpha_A^*})}{\alpha_M^* - \frac{1}{2}\sigma_M(1 - e^{-2\alpha_M^*})}
\end{aligned}
$$

This expression is important to infer the α_A^*, for a particular asset A from the whole market α_M^*, under an GBM assumption.

Again if a derivative Y is comonotonic to an underlying asset X, both share the same level of systematic risk as X, where $\rho_{R_Y,R_M} = \rho_{R_X,R_M}$ and hence, $\alpha_Y = \alpha_X$. The derivative should be priced by $H[Y; -\alpha_X, e^{\frac{\alpha_X}{\sqrt{T}}}]$ under a risk-neutral approach.

Without assuming plain vanilla returns the above results from CAPM can still be applied in a similar manner but on the assumption of conditional normality characterized by different values of α. A mixture of normal distributions results. This is reasonable as many researchers observed that there is no one loss and profit (L&P) distribution that can fit all the market information for a particular asset.

3.4 Arbitrage Pricing Theory

Let us revisit Wang's result on the α implied by CAPM, as promised in Chapter 2. Recall the relation between the α_A of a particular asset and the whole market α_M

$$\alpha_A = \rho_{A,M}\alpha_M.$$

Simply assume that the α_A can be obtained directly from the correlation between the asset and market returns under a known α_M. As seen in the previous chapter this assumption is problematic since the market return μ_M is considered non-measurable in the context of financial economics. This implies that α_M is also non-measurable as

$$\alpha_M = \frac{\mu_M - r_f}{\sigma_M}.$$

Another approach, based on arbitrage pricing theory (APT), is an alternative to CAPM that can resolve this problem. It uses the underlying ideas behind the portfolio VaR method mentioned in Chapter 1.

The basic assumption of APT is that the asset return R_A is driven by a finite number m of normal independent economic factors (or returns) R_i. These factors have to be identified by the risk mapping technique. For a designated level of expected return μ_A, it asserts that

$$\mu_A = r_f + \sum_{i=1}^{m} \beta_i(\mu_i - r_f),$$

where

$$\begin{aligned} \mu_i &= E[R_i] \\ \beta_i &= \frac{CoV[R_A, R_i]}{\sigma[R_i]^2}. \end{aligned}$$

In terms of α's it can be rewritten as

$$\mu_A = r_f + \sum_{i=1}^{m} \rho_{R_A,R_i} \alpha_i \sigma[R_A],$$

where

$$\alpha_i = \frac{\mu_i - r_f}{\sigma_i}.$$

Now all the components α_i can be obtained since these component returns are measurable and the particular asset α_A can be obtained as follow

$$\alpha_A = \frac{\mu_A - r_f}{\sigma[R_A]} = \sum_{i=1}^{m} \rho_{R_A,R_i} \alpha_i.$$

3.5 Black-Scholes Formula Revisited

As an alternative to Wang's distortion operator, our time adjusted combined distortion operator $h_{-\alpha, e^{\frac{\alpha}{\sqrt{T}}}}$ replicates Black-Scholes formula under the α^* in (3.3) for the CAPM at GBM return level.

Denote by A_t the current price of an asset at day t and assume a strike price of K for the right to exercise an option after T periods. Recall the payoff of an European call option

$$A_{t+T\ (K,\infty)} = \begin{cases} 0 & if\ A_{t+T} \leq K \\ A_{t+T} - K & if\ A_{t+T} > K \end{cases}$$

and the expected payoff, calculated as

$$E[A_{t+T\ (K,\infty)}] = \int_0^\infty S_{A_{t+T\ (K,\infty)}}(x)dx = \int_K^\infty S_{A_{t+T}}(y)dy\ .$$

Applying the new distortion operator $h_{-\alpha^*,e^{\frac{\alpha^*}{\sqrt{T}}}}$, the price of the call option is given by

$$e^{-r_f T} H[A_{t+T\ (K,\infty)}; -\alpha^*, e^{\frac{\alpha^*}{\sqrt{T}}}] = e^{-r_f T} \int_K^\infty [S_{A'_{t+T}}(y)]dy\ .$$

It can easily be verified that with a GBM return of rate μ, volatility σ and a parameter α^* satisfying

$$\alpha^* = \sqrt{T}[\frac{(\mu - r_f)}{\sigma} + \frac{1}{2}\sigma(1 - e^{-2\alpha^*})]\ ,$$

the ground-up variable A'_t implies a lognormal distribution

$$\ln \frac{A'_{t+T}}{A'_t} \sim \text{Normal}[(r_f - \frac{1}{2}e^{-2\alpha^*}\sigma^2)T, e^{-2\alpha^*}\sigma^2 T]\ .$$

If we accept the new ground-up volatility $e^{-2\alpha^*}\sigma^2$ as the risk neutral volatility σ_Q, the distorted setting and resulting pricing formula reproduces Black-Scholes model over a completely risk neutral GBM return (r_f, σ_Q). It is more reasonable and preferable to have an implied volatility different from the actual volatility σ; as mentioned by Duffie and Pan (1997), the risk neutral and actual distributions have complete different natures.

3.6 Comments and Conclusion

Our new combined distortion operator $h_{-\alpha,e^\alpha}$ is an alternative to Wang's distortion operator $g_{-\alpha}$ for asset pricing and preserves the bridges between the four different pricing approaches of Chapter 2:

(1) Yaari's economic theory of risk,

(2) Traditional standard deviation approach,

(3) Capital asset pricing model,

(4) Black-Scholes option pricing formula.

Even though it may not promote a unified approach for pricing financial and insurance risks, as it fails to satisfy all the insurance pricing criteria, it captures other interesting features like the market preference or utility and a possible risk return parity in the exchange market. One should however notice that by comparing the return to the severity of insurance risk, the time adjusted distortion operator converges to Wang's original distortion operator as the time factor tends to ultimate under a fixed α, which still lays a hint to the unified pricing theory.

The operator $h_{-\alpha,e^{\alpha}}$ can be applied under the assumption of conditional normality, with some α, to generate a mixture of normal distributions. This can result in a two layer L&P distribution, further discussed in the next chapter. We will see how under an appropriate setting, our distortion in volatility reproduces the implied volatility smile observed in the last decade by market modellers for post-crash markets.

Chapter 4

A State Return Model

4.1 Introduction

The previous chapter shows how our new combined distortion operator $h_{-\alpha,e^\alpha}$ can serve as an alternative to Wang's distortion operator $g_{-\alpha}$, linking the four different pricing approaches in finance. However it does not help to unify finance and insurance pricing.

This chapter focuses on generalizations and applications of the four basic price return models generally used in the VaR calculation:

(1) The plain vanilla model,

(2) The jump diffusion model,

(3) The Gaussian kernel density estimation model,

(4) The stochastic volatility equivalent model.

A new two layer model, we call "State Return Model", is proposed as a way to unify the above basic price return models.

We will see that the four different models originate from the same underlying setting that lead to the new distortion operator $h_{-\alpha,e^\alpha}$, but with differences in the α

assumption. To motivate the definition of this new two layer state return model we start with a simple example. Its interpretation will help understand the behaviour of investors in an efficient market. We then relate the four basic pricing models above to our new proposed model. Mathematical and economical aspects are then discussed in the sections that follow.

The value of α is a key feature of this new return state model; it can be obtain directly or indirectly, depending on the model setting.

4.2 Motivation

As a risk example, consider the 6/49 lottery. It is well known that for each dollar of ticket purchase (price) the expected payoff P is far less than a dollar. Now assume that the one dollar price per draw has a market of its own. In terms of distortion operator $h_{-\alpha,e^{\alpha}}$, we see that the distortion is severe, due to the extremely large value of α in the public perception. It must satisfy

$$H[P; -\alpha, e^{\alpha}] = 1, \quad \text{where } E[P] \ll 1.$$

Now assume that you have purchased a limited number of hypothetical win-or-nothing lottery tickets and that you are allowed to re-sell them to the public, or to another dealer, in a secondary market. The initial conditions are:

(1) Each ticket has a payoff $P_0 \in \{0, 1\}$ such that

$$P_0 \sim \text{Bernoulli}(r_0), \quad \text{where } E[P_0] = r_0$$

and with the return expressed as $R_0 = P_0 - r_0$, with $E[R_0] = 0$.

(2) You purchase each ticket at r_0 and you can be sell them at r_i dollars, the price of public bids from the i-th preferences (or utility) class. Clearly nobody will pay you more than 1 dollar for a ticket, but you can choose not to participate in the draw by selling all your tickets. On the other hand, you are forced to participate on the unsold tickets.

The question is:

How would you justify the price r_0 of each ticket in the current market?

The problem can be solved by a two layer model. There exists a set of preference indexes reflected by a set of parameters α_i in the distortion operator $h_{-\alpha_i, e^{\alpha_i}}$, together with weights assigned to each α_i, representing the preference distribution in the public. Recalling the observation from Venter (1991) we would suggest a price obtained from the expected value of the random variable under distortion with respect to each preference class. Each participant with common preference index α_i will follow a pre-distorted return in the secondary market $R_i \in \{1 - r_i, -r_i\}$ with winning probability r_0. R_i has a ground-up return distribution with expectation $H[R_i; -\alpha_i, e^{\alpha_i}] = 0$. For all preference classes participants perceive the same ground-up distorted expected return.

Notice that any participant follows a pre-distorted return R_j with the index $\alpha_j = 0$, where $H[R_j; 0, 1] = E[R_0] = 0$ implies a risk neutral preference. On the other hand under the inverse distortion of the original payoff we have that $0 \leq H[P_0; -\alpha_i e^{-\alpha_i}, e^{-\alpha_i}] \leq 1$ implies that all possible expected prices with respect to each α_i are within the range $[0, 1]$. Denoting by $P_i = R_i + r_0$ for a given $r_0 > \frac{1}{2}$, $\min_{-\alpha_i}[P_i] > 0$ indicates a supporting point of the ticket, from the technical analyst point of view. According to this idea each ticket must be priced as:

$$P = E_{\alpha_i}\{H[P_i; -\alpha_i, e^{\alpha_i}]\} = r_o,$$

which justifies the price of the ticket in the lottery market. We easily see that the primary layer is the pre-distorted payoff distribution agreed by the participant of a particular preference index α_i. The second layer is the distribution of the preference α_i within the public. However, you would be more interested in the actual payoff of the tickets in the secondary market of this hypothetical setting. From a speculation point of view, the return on the ticket generated from the secondary market is driven by the market preferences.

Obviously this is just an hypothetical example but it mimics the trading behaviour in the stock market. If we can see the initial purchase of the tickets as the initial public offering (IPO) of a stock, then the ticket re-selling is indeed the stock tradings and it is driven by the market preference instead of the original ticket payoff.

4.3 Plain Vanilla Model

Recall the basic assumption of the CAPM for the 1-period return of an underlying asset i in the plain vanilla model. Let R_{t+1} be the next period return of an asset A with price A_t at day t. This model assumes that $R_{t+1} \sim$ i.i.d. Normal(μ, σ^2). From our result in Section 3.3.1, under the distortion operator $h_{-\alpha, e^{\alpha}}$ for the 1-period α and risk free return r_f we have

$$\alpha = \frac{\mu - r_f}{\sigma} \tag{4.1}$$

and hence,

$$\mu = r_f + \alpha\sigma.$$

Therefore $R_{t+1} \sim$ i.i.d. Normal$(r_f + \alpha\sigma, \sigma^2)$

$$R_{i,t+1} = r_f + (\alpha_i + \epsilon_{t+1})\sigma,$$

where $\epsilon_{t+1} \sim$ i.i.d. Normal$(0, 1)$ for all t. Then the $100p$%-level overnight VaR defined in Section 1.1.1 is here

$$VaR_p[1, A_t] = A_t[1 - e^{r_f + (\alpha - z_{1-p})\sigma}].$$

For a multiple T-period plain vanilla return model we have the T-period $\alpha_{t,T} = \sqrt{T}\alpha$ and return,

$$R_{t,T} = r_f T + \sqrt{T}(\alpha + \epsilon_{t,T})\sigma,$$

where $\epsilon_{t,T} \sim$ i.i.d. Normal$(0, 1)$ is the white noise for all t. The $100p$%-level T-period VaR is then

$$VaR_p[T, A_t] = A_t[1 - e^{r_f T + \sqrt{T}(\alpha - z_{1-p})\sigma}].$$

Notice that VaR depends solely on the value of α and σ. The expected value of the return $E[R_{t,T}] = \mu T$ disappears from the expression. However, with this derivation it is not clear that the plain vanilla can be considered a two layers model. If it is, there does not seem to be variability in the parameter α.

An alternative formulation is to impose synthetic conditional normal returns with a fixed volatility:

$$R_{t+1} \quad \text{given} \quad \alpha_{t+1} \sim \text{Normal}(r_f + \alpha_{t+1}\eta, \eta^2).$$

Then the returns can be modelled as

$$\begin{aligned} R_{t+1} &= r_f + \alpha_{t+1}\eta + \epsilon_{t+1}\eta \\ &= r_f + (\alpha_{t+1} + \epsilon_{t+1})\eta. \end{aligned}$$

where α_{t+1} is independent of the white noise $\epsilon_{t+1} \sim \text{Normal}(0, 1)$.

If $\alpha_t \sim \text{Normal}(\alpha, \nu^2)$, then

$$\begin{aligned} E[R_{t+1}] &= E_{\alpha_{t+1}}\{E[R_{t+1}|\alpha_{t+1}]\} \\ &= E[r_f + \alpha_{t+1}\eta] \\ &= r_f + \alpha\eta. \end{aligned}$$

The volatility $\sigma[R_{t+1}]$ of the asset return is then

$$\begin{aligned} V[R_{t+1}] &= E_{\alpha_{t+1}}\{V[R_{t+1}|\alpha_{t+1}]\} + V_{\alpha_{t+1}}\{E[R_{t+1}|\alpha_{t+1}]\} \\ &= E[\eta^2] + V[r_f + \alpha_{t+1}\eta] \\ &= \eta^2 + \nu^2\eta^2 = (1 + \nu^2)\eta^2, \\ \sigma[R_{t+1}] &= \sqrt{1 + \nu^2}\eta. \end{aligned}$$

Notice that $\alpha_{t+1} + \epsilon_{t+1}$ is normally distributed, therefore

$$R_{t+1} \sim \text{Normal}(\mu, \sigma^2),$$

where

$$\begin{aligned} \mu &= r_f + \alpha\eta \\ \sigma^2 &= (1 + \nu^2)\eta^2. \end{aligned}$$

It replicates exactly a plain vanilla return. Moreover, by assuming different distributions for the varying α, different return models can be generated.

4.4 Jump Diffusion Model

Recall our result on CAPM in Section 3.4 where the α of an asset A corresponds to its systematic risk in the market. Denoting by $\alpha_M = \frac{\mu_M - r_f}{\sigma_M}$ the market price of risk for the market portfolio which has a Normal(μ_M, σ_M^2) plain vanilla 1-period

$$\alpha_t = \rho_{A,M}\alpha_M$$

for all t provided the market is frictionless with complete information.

In reality the market itself is seldom frictionless and information complete, therefore the price of an asset does not necessarily follow the SML in CAPM (see Section 3.3.2). We can suspect the existence a small independent deviations from the theoretical α, say ξ_{t+1}, for each compounding period. A new α_{t+1} can then be used by assuming that the asset return is conditionally normal such that R_{t+1} given α_{t+1} is Normal(μ_{t+1}, η^2) and

$$\mu_{t+1} = r_f + \alpha_{t+1}\eta,$$

where

$$\alpha_{t+1} = \alpha + \xi_{t+1}.$$

ξ_{t+1} is an i.i.d. compound Poisson jump shock $\sim$ C.P.[Normal$(0, \nu^2)$, $\frac{\lambda}{n}$] where n denotes the number of trading periods in a year. As a result we can model the 1-period return as

$$\begin{aligned} R_{t+1} &= r_f + (\alpha_{t+1} + \epsilon_{t+1})\eta \\ &= r_f + (\alpha + \xi_{t+1} + \epsilon_{t+1})\eta. \end{aligned}$$

This is exactly the jump diffusion model mentioned in Duffie (1997) if the ξ_{t+1} are assumed to be independent of the ϵ_{t+1}. Under the assumption of a constant volatility σ for asset A, we can interpret the jump shock ξ_{t+1} in the jump diffusion model as the result of the fluctuation of α_{t+1} in the conditional return, with expectation $E[R_{t+1}|\alpha_{t+1}] = r_f + \alpha_{t+1}\eta$, due to the friction or information incompleteness of the market.

Given an α_M for the whole market at time t, under the assumption of a plain vanilla market return $R_{M,t} \sim \text{Normal}(\mu_M, \sigma_M^2)$ for all t, assume that our result on α holds for the conditional distribution of an individual asset. Then

$$\begin{aligned}\alpha_{A,t+1} &= \alpha_{t+1} \\ &= \rho_{A,M,t+1}\alpha_M \\ &= \rho_{A,M,t+1}\frac{\mu_M - r_f}{\sigma_M}.\end{aligned}$$

We can conclude that the change in expected return implied by α_{t+1} is the result of the fluctuation on the conditional correlation $\rho_{A,M,t+1}$ between the asset return $R_{A,t+1}$ and the market return $R_{M,t+1}$, provided that the market conditions remained unchanged.

Notice that if the expected return is a fixed constant a stochastic volatility model will result.

The 100p%-level overnight VaR for this jump diffusion model with the asset price A_t at day t is given by

$$VaR_p[1, A_t] = A_t(1 - e^{r_f + \alpha\eta + [\xi_{1+t} + \epsilon_{1+t}]_p\eta}),$$

where $[\xi_{1+t} + \epsilon_{1+t}]_p$ is the $100(1-p)^{\text{th}}$ percentile of the sum of the jump shock and the white noise.

For a multiple T-period model, the T-period $\alpha_{t,T}$ is given by

$$\alpha_{t,T} = \sqrt{T}\alpha + \xi_{t,T}.$$

The T-period return at day t can be written as:

$$\begin{aligned}R_{t,T} &= r_f T + (\sqrt{T}\alpha + \xi_{t,T} + \sqrt{T}\epsilon_{t,T})\eta \\ &= (r_f + \alpha)T + (\xi_{t,T} + \sqrt{T}\epsilon_{t,T})\eta,\end{aligned}$$

where $\xi_{t,T} \sim$ i.i.d. C.P.[Normal$(0, \nu^2)$, $\frac{\lambda T}{n}$] are the jump shocks and $\epsilon_{t,T} \sim$ i.i.d. Normal$(0,1)$ are the white noise for all t, all being mutually independent. Therefore the 100p%-level T-period VaR is

$$VaR_p[T, A_t] = A_t(1 - e^{(r_f + \alpha\eta)T + [\xi_{t,T} + \sqrt{T}\epsilon_{t,T}]_p\eta}),$$

where $[\xi_{t,T} + \sqrt{T}\epsilon_{t,T}]_p$ is the $100(1-p)^{\text{th}}$ percentile of the sum of the jump shock and $\sqrt{T}$ times the white noise. However, no closed form formula is available for the percentile in this type of model, numerical methods are required.

4.5 Gaussian Kernel Equivalent Model

Recall the following identity in (4.1) for a 1-period plain vanilla model

$$\alpha = \frac{\mu - r_f}{\sigma}.$$

In the previous section we have seen that the above result can be applied even though the asset return is not plain vanilla, as long as the return is conditionally normal with respect to α_t with a CAPM assumption at the conditional level. We will see that the kernel estimation model and stochastic volatility model indeed come from the same conditional setting, based on $h_{-\alpha,e^{\alpha}}$, by fixing either the rate of return or the volatility and varying the other over the available market information.

If we assume that the asset return R_t is not plain vanilla but conditionally normal with a constant volatility σ with respect to a particular α_t, for each compounding period we have the following condition at all t

$$R_t \quad \text{given} \quad \alpha_t \sim \text{Normal}(\mu_t, \eta^2).$$

Under an efficient market assumption and if all possible returns are reflected by conditional expectations, given the available market information, we have a sequence of α_t values. $h_{-\alpha_t,e^{\alpha_t}}$ can be applied with a risk free return r_f to obtain the following result

$$\alpha_{t+1} = \frac{\mu_{t+1} - r_f}{\eta}.$$

Hence,

$$\mu_{t+1} = r_f + \alpha_{t+1}\eta.$$

Notice again that α_{t+1} corresponds to μ_{t+1} and we have the following return model

$$R_{t+1} = r_f + \alpha_{t+1}\eta + \epsilon_{t+1}\eta,$$

where ϵ_{t+1} is the white noise and a certain distribution is assumed for α_t. For simplicity, if we assume that all α_t are independent and from a discrete distribution, we get a discrete mixture of normal returns. This type of model serves the purpose of generating fatter than normal tails in order to capture the worse market scenarios.

If we choose each empirical return to correspond to the conditional expected return, where $\mu_{t+1} = r_f + \alpha_{t+1}\eta$ takes on the values of $\hat{R}_i$, for $i = 1, 2, \ldots, N$, with weight $\frac{1}{N}$, it replicates the standard Gaussian kernel with component mean $\mu_{t+1,i} = \hat{R}_i$ and bandwidth $h = \sigma$. The kernel density estimator is then

$$f(R_{t+1}) = \sum_{i=1}^{N} \frac{1}{N\eta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{R_{t+1}-\mu_{t+1,i}}{\eta})^2}.$$

The $100p\%$-level overnight VaR is the $100p^{\text{th}}$ percentile of the above density.

For a multiple T-period, the same approach gives

$$R_{t,T} = Tr_f + \alpha_{t,T}\eta\sqrt{T} + \epsilon_{t,T}\eta\sqrt{T},$$

where

$$\alpha_{t,T} = \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \alpha_{t+i-1}.$$

A similar estimation density can be obtained by matching the conditional expected return $\mu_{t,T} = Tr_f + \alpha_{t,T}\eta\sqrt{T}$ with the T-period empirical return $\hat{R}_{t,T,j} = \sum_{i=1}^{T} \hat{R}_{i+j-1}$. The $100p\%$-level VaR is obtained accordingly.

Finally, notice that the α of the jump diffusion and in the Gaussian kernel models are related to a stochastic expected return. The value of η is quite synthetic which is not necessarily equal to the spot volatility of the asset due to its conditional nature. This explains why η can serve as the bandwidth in the kernel density for smoothing purposes. Under a different setting, η can also be the risk neutral volatility; this is further discussed in the next section.

4.6 Stochastic Volatility

In a way similar to the previous model we can construct an alternative return model by fixing the expected return μ and varying the volatility over time, to correspond to the information reflected by α. Assuming that, given α_t for each compounding period, the asset return R_t is conditionally normal with a constant expected return μ, we have at all t

$$R_t \quad \text{given} \quad \alpha_t \sim \text{Normal}(\mu, \eta_t).$$

By the same argument as in the previous section we obtain a sequence of α_t values. Under the CAPM, the conditional normal return gives

$$\alpha_{t+1} = \frac{\mu - r_f}{\eta_{t+1}}.$$

Notice that a stochastic α_{t+1} with a fixed μ implies a stochastic volatility η_{t+1}. Hence the following stochastic volatility α return model is obtained

$$R_{t+1} = \mu + \epsilon_{t+1}\frac{\mu - r_f}{\alpha_{t+1}},$$

where ϵ_{t+1} is the white noise and a certain distribution is assumed for α_{t+1}. As mentioned in Section 1.2.4, the stochastic volatility model is generated from a mixture of normal distributions, the same result can be expected here for the stochastic α model. Again here the parameter representing the conditional expected return μ is synthetic. It can be inferred from the estimated α. Finally, notice that under this assumption an increase in the conditional risk of an asset reduces the market price of risk α_t.

The estimation of VaR for this type of model can be achieved by simulating the stochastic α, a good model assumption for α will be essential in the estimation process. The $100p$%-level VaR of the stochastic α model will be the $100p^{\text{th}}$ percentile of the simulated returns for a sufficiently large simulation size.

An auto-regressive model for α_t, compared to that of the volatility, can also be considered. We can therefore develop a Markovian stochastic α model of the form:

$$\alpha_{t+1} = F(\alpha_t, z_{t+1}, t),$$

where $\{z_{t+1}\}$ is a sequence of white noises. A standard stochastic volatility model is obtained under a log-autoregressive assumption

$$\log \sigma_{t+1}^2 = \beta + \gamma \log \sigma_t^2 + \kappa z_{t+1},$$

where β, γ and κ are some constants. γ represents the persistence factor and $|\gamma| < 1$ if the model is non-explosive. Similarly, we have an α version of this model by simply substituting the σ_t in terms of the above expression:

$$\log \alpha_{t+1}^2 = \beta' + \gamma' \log \alpha_t^2 + \kappa' z_{t+1},$$

where

$$\begin{aligned} \beta' &= 2(1-\gamma)\log(\mu - r_f) - \beta, \\ \gamma' &= \gamma, \\ \kappa' &= -\kappa. \end{aligned}$$

Here

$$\tilde{\alpha} = \lim_t E(\alpha_t) = \exp\{\frac{\beta'}{1-\gamma'} + \frac{1}{2}\frac{\kappa'^2}{1-\gamma'^2}\}$$

is the steady-state α.

Moreover, referring to GARCH and EGARCH models for the stochastic volatility we can again expect an α version of these models. We will not explore further these models as we try to propose a new state return approach. Further research is expected in this direction.

4.7 Return State Model

4.7.1 Parameter α

For all models with a stochastic α, such as the jump diffusion, the Gaussian kernel density estimation and the stochastic volatility models, the estimation of α plays an important role setting up the conditional normal distribution of the return R_t given α_t .

For the jump diffusion and Gaussian kernel density models, the conditional expected return

$$R_t \quad \text{given} \quad \alpha_t \sim \text{Normal}(r_f + \alpha_t \eta, \eta^2),$$

such that

$$E[R_t] = E_{\alpha_t}\{E[R_t|\alpha_t]\} = r_f + E[\alpha_t]\eta = r_f + \alpha\eta,$$

where $E[\alpha_t] = \alpha$. It implies that $\eta = \frac{E[R_t]-r_f}{\alpha}$. We can infer the value of η from the estimated α and the empirical average return $\bar{R}_t$.

For the stochastic volatility α model, the conditional expected return

$$R_t \quad \text{given} \quad \alpha_t \sim \text{Normal}(\mu, \frac{\mu - r_f}{\alpha_t}),$$

where

$$\mu = r_f + \alpha_t \eta_t = r_f + \tilde{\alpha}\tilde{\eta} \approx r_f + \alpha\sigma[R_t].$$

Again, the value of the conditional expected return μ can be inferred from the estimated α and the empirical standard deviation S_{R_t}.

Once all the parameters are set, an appropriate assumption for the behaviour of the stochastic α is needed. It can be done either by assuming a prior distribution and using Bayesian inference or by establishing a filtration model. Here we take the Bayesian approach and assume a discrete marginal α_i distribution, which leads to a set of conditionally normal return states over a particular time horizon.

4.7.2 α State Return Model

Wang (2000) suggests that the price of a portfolio be determined by its systematic risk, reflected by α. If CAPM holds at the conditional normal return level, then the conditional return state is determined by its conditional price of risk α_i. Under the new proposed model we use a simple diffusion over α_i and fix the value of the conditional volatility η through time, as in the Gaussian kernel density model.

For simplicity, consider a one-stage binomial model, with risk neutral initial state P_0, but where the return R eventually differentiates into two term structure conditional return states, respectively the upside and downside states, denoted by P_u and P_d. For a given risk free return r_f we have

$$\begin{aligned} P_u &\sim \text{Normal}(r_f + \alpha_u \eta, \eta^2) \\ P_d &\sim \text{Normal}(r_f + \alpha_d \eta, \eta^2), \end{aligned}$$

where the weights of each state are determined by an underlying prior distribution of α. Here we adopt a binomial type distribution where,

$$\Pr[\alpha = A] = \begin{cases} p & \text{if} \quad A = \alpha_u \\ 1 - p & \text{if} \quad A = \alpha_d. \end{cases}$$

Using our new distortion operator $h_{-\alpha, e^{\alpha}}$ on each state we obtain a pair of risk neutral states

$$\begin{aligned} P_u^Q &\sim \text{Normal}(r_f, e^{-2\alpha_u}\eta^2) \\ P_d^Q &\sim \text{Normal}(r_f, e^{-2\alpha_d}\eta^2). \end{aligned}$$

Under this setting we have

$$\begin{aligned} E[R] &= r_f + [\alpha_d + p(\alpha_u - \alpha_d)]\eta \\ \sigma[R] &= \eta\sqrt{1 + p(1-p)\alpha_u - \alpha_d)^2}, \end{aligned}$$

and under the risk neutral measures

$$\begin{aligned} E^Q[R] &= r_f \\ \sigma^Q[R] &= \eta\sqrt{1 + p(e^{-2\alpha_u} - e^{-2\alpha_d})}. \end{aligned}$$

A refinement of this binomial tree lets α evolve over N states. We choose a Bernstein binomial-like α with a probability function defined as

$$f(A = \alpha; N, p, \alpha_u, \alpha_d) = \begin{pmatrix} N \\ N\frac{\alpha - \alpha_d}{\alpha_u - \alpha_d} \end{pmatrix} p^{N\frac{\alpha - \alpha_d}{\alpha_u - \alpha_d}} (1-p)^{N\frac{\alpha_u - \alpha}{\alpha_u - \alpha_d}},$$

where

$$\alpha = \alpha_d + i\frac{\alpha_u - \alpha_d}{N} \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

We recognize here a discrete version of a bounded diffusion process for α. We can see that the actual return and the risk neutral volatility are stochastic, down the α binomial tree. Volatilities are not identical, as desired for both actual and risk neutral returns. Finally the key input of this model is the set of spot European put-call prices and its implied volatility, from which the value of the state α can be inferred.

For a sufficiently large number of states, α tends to be normally distributed. As a result we see that the risk actual expected return tends to a normal distribution and the risk neutral volatility approaches lognormality. This just coincides with the standard assumption for the expected return and stochastic volatility, justifying our choice of $\beta = e^{\alpha}$.

4.7.3 Geometric Brownian Motion α Implied Tree

We can generalize our new return model under a geometric Brownian motion (GBM) in order to recover Black Scholes formula.

For simplicity, again consider an one stage binomial model with a risk neutral initial state P_0. The return R eventually differentiates into two GBM term structure conditional return states, where the upside and downside states are respectively denoted by P_u and P_d. For α^* defined in (3.3), Section 3.3.1 and a given risk free return r_f we have

$$\begin{aligned} P_u &\sim \quad \mathrm{GBM}(r_f + \alpha_u^* \eta + \tfrac{1}{2}\eta^2(1 - e^{-2\alpha_u^*}), \eta^2) \\ P_d &\sim \quad \mathrm{GBM}(r_f + \alpha_d^* \eta + \tfrac{1}{2}\eta^2(1 - e^{-2\alpha_d^*}), \eta^2), \end{aligned}$$

where the weights of each state are determined by an underlying prior distribution of α^* defined in the same way as the continuous return model of the previous section. Here we obtain a pair of risk neutral states

$$\begin{aligned} P_u^Q &\sim \quad \mathrm{GBM}(r_f, e^{-2\alpha_u^*}\eta^2) \\ P_d^Q &\sim \quad \mathrm{GBM}(r_f, e^{-2\alpha_d^*}\eta^2). \end{aligned}$$

Under this setting we have

$$\begin{aligned} E[R] &= r_f + [\alpha_d^* + p(\alpha_u^* - \alpha_d^*)]\eta - \tfrac{1}{2}[e^{-2\alpha_d^*} + p(e^{-2\alpha_u^*} - e^{-2\alpha_d^*})]\eta^2 \\ \sigma[R] &= \eta\sqrt{1 + p(1-p)[(\alpha_u^* - \alpha_d^*) + \tfrac{1}{2}(e^{-2\alpha_u^*} - e^{-2\alpha_d^*})\eta]^2}, \end{aligned}$$

and under the risk neutral measures

$$\begin{aligned} E^Q[R] &= r_c - \tfrac{1}{2}[e^{-2\alpha_d^*} + p(e^{-2\alpha_u^*} - e^{-2\alpha_d^*})]\eta^2 \\ \sigma^Q[R] &= \eta\sqrt{e^{-2\alpha_d^*} + p(e^{-2\alpha_u^*} - e^{-2\alpha_d^*})}. \end{aligned}$$

The same prior distribution of implied tree can be generated for a GBM α. By calibrating the set of spot calls with the weighted average of the state call price at the end-node of the implied tree, both the actual and risk neutral distributions can be obtained.

4.8 Comments and Conclusion

The application of the new distortion operator $h_{-\alpha,e^\alpha}$ can be used to replicate the basic return risk models used for the computation of VaR under the assumption of CAPM at the conditional level. We see that the return state model based on a conditionally normal return implied by the stochastic α hints to a possible bridge between the actual and the risk neutral distribution, as in financial economics. This model shares some properties with the actual expected return and the implied risk neutral volatility, which provides a possibility in the search for a link between the actual P-measure and the risk neutral Q-measure.

We can see that all four models, plain vanilla, jump diffusion, Gaussian estimation kernel and volatility related stochastic α are two layers stochastic α models. They have a primary conditional normal return state and a secondary return state distribution, determined by the behaviour of the changing α with a binomial type distribution assumption. For convenience, an implied α binomial tree is suggested. A detailed fit of this model to real data is discussed in the following chapter.

By fixing one parameter of the conditional normal return and varying the other, we can use the expected α to infer the fixed parameter. Hopefully, our new distortion operator $h_{-\alpha,e^{\alpha}}$ allows to vary both parameters and have them implied from a risk neutral assumption, like in Black-Scholes formula. We hope that this can explain the distortion of the option implied volatility, namely, the "smile" shape of volatilities over different striking prices of a call option at a single spot moment.

Chapter 5

Two Estimation Methods

5.1 Introduction

In this final chapter we will make use of the results developed in Chapters 3 and 4 for the new distortion operator $h_{-\alpha,e^{\alpha}}$ and the implied α tree model, to implement a practical return estimation model. A discussion of the concept of risk neutrality is presented in the first section. A brief discussion of the implied volatility smile follows. The estimation models are then discussed using a set of spot put-call prices or implied volatilities.

We propose two basic estimation methods. Generalizations to option non-available positions can be achieved provided a market in index options is available. We will first borrow the idea of the Gaussian kernel density estimation and then go back to the implementation of the α implied binomial model.

Details of the model are discussed. An analysis of the strengths and limitations of the model are given in the conclusion.

5.2 Risk Neutrality

Reconsider the risk-neutral assumption in Black-Scholes' model. Given an actual return with mean μ and volatility σ, Black-Scholes' formula corresponds to a risk adjusted price for an European option, under the assumption of a risk free rate that follows a GBM with a risk neutral volatility. However, under the assumption of the new distortion, for a particular risk neutral state, the pre-distorted risk actual state is not unique and depends on the value of α_t.

Consider the two parameters of a normal distribution; the expected return μ and volatility σ constitute an ordered pair of risk-return parity. For our new distortion operator, each possible risk neutral state actually contains a class of ordered pairs of risk-return parity. Given a risk free return r_f under a one-stage continuous return, look at the inverse distortion from a given risk neutral state P^Q, where

$$P^Q \quad \text{given} \quad \alpha \quad \sim \quad \text{Normal}(r_f, \eta_Q^2).$$

The following relation exists between the class of risk actual state P_α, for some α,

$$P_\alpha \quad \text{given} \quad \alpha \quad \sim \quad \text{Normal}(\mu_\alpha, \eta^2).$$

Under the distortion $h_{-\alpha,e^\alpha}$ the following condition holds,

$$\begin{aligned} r_f &= H[P_\alpha; -\alpha, e^\alpha] \\ &= \mu_\alpha - \alpha\eta, \end{aligned}$$

and

$$\eta_Q = e^{-\alpha}\eta.$$

It can be rewritten as

$$\begin{aligned} \mu_\alpha &= H[P_Q; \alpha e^{-\alpha}, e^{-\alpha}] \\ &= r_f + \alpha e^\alpha \eta_Q, \end{aligned}$$

and

$$\eta = e^\alpha \eta_Q.$$

Then the set of α satisfying the following equation

$$\mu_\alpha = r_f + \alpha e^\alpha \eta_Q,$$

indeed defines a family of normal return state P_α of the form

$$P_\alpha \quad \text{given } \alpha \quad \sim \quad \text{Normal}(r_f + \alpha e^\alpha \eta_Q, e^\alpha \eta_Q).$$

We can easily see that

$$\text{Normal}(\mu_\alpha, \eta^2) \to \text{Normal}(r_f, \eta_Q^2) \quad \text{as} \quad \alpha \to 0,$$

which represents the distortion to risk neutrality. However, we have another limiting case

$$\text{Normal}(\mu_\alpha, \eta^2) \to r_f \quad \text{as} \quad \alpha \to -\infty,$$

which represents the distortion to the risk free return. This justified our choice of $\beta = e^\alpha$ in the GLGC distortion of Chapter 3, distinguishing the risk free from the risk neutral distribution. Notice also that the above expression for α is convex and a minimum of μ_α is attainable, which features a supporting point, in technical analyst terms, for a particular risk neutral state. On the other hand, if we are able to determine the risk neutral return, we then know the lowest admissible risk actual return state. In addition notice the existence of a positive expected state return, beyond r_f, of extreme low volatility, which may represent some sort of low risk investment at the given interest rate (for example convertible bonds). Further investigation of this idea is needed.

One last thing we have to mention is that the families of all risk neutral states will cover the whole risk-return parity plane in $\mathbb{R}^2$. Therefore extreme negative returns are possible in the market.

5.3 Implied Volatility Smile

Theoretically, under Black-Scholes' formula an European option with different striking prices should have a constant volatility. For a given set of option prices for different striking prices we can invert Black Scholes' formula to solve for the underlying volatility, which is known as the implied volatility. However, researchers notice that this volatility exhibits some sort of variability over different striking prices which cannot be attributed to the white noise. The variation of the implied volatility versus the striking price produces a graphical curve known as the volatility smile.

Fix the time to expiration for a underlying spot price and take a close look at Black-Scholes' formula over different volatility levels, we discover that any average of two price curves crosses some other price curves. Inverting the values on the average price, we can actually see a variation of implied volatility along the striking prices. Under our state return model together with the distortion operator $h_{-\alpha,e^{\alpha}}$ it is possible to capture this type of variability.

A set of M possible return states $\{P_1, P_2, \dots, P_M\}$, gives a set of M risk neutral return states $\{P_1^Q, P_2^Q, \dots, P_M^Q\}$, under the distortion induced by a set $\{\alpha_1, \alpha_2, \dots, \alpha_M\}$. We therefore define a modified option pricing formula which captures the set of actual spot option prices $\{\hat{o}_{K_1}, \hat{o}_{K_2}, \dots, \hat{o}_{K_W}\}$ off different striking prices $\{K_1, K_2, \dots, K_W\}$, expressed as

$$\hat{o}(K_j) = \sum_{i=1}^{M} \Pr[A = \alpha_i] o_{\alpha_i}(K_j) \quad \text{for all} \quad j = 1, 2, \dots, W,$$

where $o_{\alpha_i}(K_j)$ is the option price of the i^{th} risk neutral state evaluated by Black-Scholes' formula replicated by our distortion at striking price K_j. This is the key component of the estimation method proposed in the next section.

5.4 α Gaussian Kernel Density

5.4.1 Model Objective

Consider the estimation problem for our Gaussian kernel model and assume the following return density of R

$$f(R) = \sum_{i=1}^{N} \frac{1}{N\eta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{R_{t+1}-\mu_{t+1,i}}{\eta})^2}.$$

Our objective is to estimate the bandwidth parameter η under a set of N slack parameters $\{\alpha_i\}$, by calibrating the set of spot puts $\{\hat{c}(K_j)\}$ of W different striking prices with that of the modified option prices stated in the previous section under the distortion $h_{-\alpha_i, e^{\alpha_i}}$. Least squares is applied to minimize the following expression

$$\text{Min:} \quad \sum_{j=1}^{W} [\{\frac{1}{N} \sum_{i=1}^{N} c_{\alpha_i}(K_j)\} - \hat{c}_{K_j}]^2,$$

where $c_{\alpha_i}(K_j)$ is the call price evaluated under the new distortion with state parameter α_i, at striking price K_j.

5.4.2 Source Data

The required source date are:

(1) A set of N daily continuous historical prices $\{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_N\}$ such that

$$\hat{R}_i - \frac{1}{2}\eta^2 = \ln(\frac{\hat{A}_j}{\hat{A}_{j-1}}), \quad \text{for} \quad i = 1, 2, \ldots, N,$$

where $\hat{R}_i$ is the historical return under the GBM assumption at time for $j = 1, 2, \ldots, N+1$.

(2) The risk free rate r_f is assumed constant and it can be inferred from interest instruments such as bonds.

(3) A set of call prices $\{\hat{p}_{K_1}, \ldots, \hat{p}_{K_W}\}$ with different striking prices $\{K_1, \ldots, K_W\}$, respectively. Note that the set of spot put prices would work as well if the implied volatility of the puts does not deviate too much from that of the calls.

5.4.3 Model Constraints

Several constraints are needed to infer all the parameters in this model.

(1) As we suggested in Chapter 4, each historical return $\hat{R}_i$, can be considered as the expected return of a return state $\{P_1, P_2, \ldots, P_N\}$ with a fixed bandwidth η for the set of assumed α values $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$

$$P_i \sim \quad \text{GBM}(\hat{R}_i, \eta^2) \quad \text{for} \quad i = 1, 2, \ldots, N.$$

Under the distortion to risk neutrality of GBM returns and the risk free rate r_f, $\hat{R}_i$ can be expressed as

$$\hat{R}_i = r_f + \alpha_i \eta - \frac{1}{2} e^{-2\alpha_i} \eta^2 + \eta^2$$

and the expected state price is $e^{\hat{R}_i}$.

(2) For a good fit in the optimization we have to make sure that the average implied volatility call price falls between the maximum and minimum call curves evaluated on the set of the distorted risk neutral states. Therefore, for all K_j

$$\min_i [c_{\alpha_i}(K_j)] \leq \bar{c}(K_j) \leq \max_i [c_{\alpha_i}(K_j)],$$

where $\bar{c}(K_j)$ is the call curve evaluated at the average implied volatility of the smile, which can be obtained by inverting Black-Scholes' formula.

(3) The bandwidth obviously must be positive such that $\eta > 0$.

5.4.4 Comments

Note that it is possible to solve for all α_i values; as we have already mentioned, they play the role of slack parameters. The constraint in (1) can be expressed as

$$e^{r_f - \hat{R}_i} = e^{-\eta^2} e^{-\alpha_i \eta + \frac{1}{2} e^{-2\alpha_i} \eta^2}, \quad \text{for} \quad i = 1, 2, \ldots, N,$$

where the second term on the right hand side is indeed a moment generating function of a normal variable Y_i, evaluated at η, where

$$Y_i \sim \text{Normal}(-\alpha_i, e^{-2\alpha_i}), \quad \text{for} \quad i = 1, 2, \ldots, N.$$

Therefore the set of α_i's is just equivalent to a single bandwidth parameter η.

Puts are preferred over calls, as this input is more likely to show the convexity of the smile.

Under this estimation approach we are indeed looking for a L&P density such that parameters are inferred not only from the historical data as in the classical way. We also try to extract information from option prices, which may reflect the market preference and anticipation, in order to capture the current momentum of the market.

However, in the same way as most VaR models, this density does not give a better expected return, which is just the average of the historical return. For an improvement in estimating both return and risk we introduce an alternative model in the next section.

5.5 α Binomial Model

5.5.1 Model Objective

For simplicity we consider only a one period GBM binomial model mentioned in the previous chapter. Our objective is to estimate the state parameters α_u and α_d and the upstate weight p. By calibrating the set of spot calls $\{\hat{c}(K_j)\}$ of W different striking prices with that of the modifies calls

$$c(K_j) = pc_{\alpha_u}(K_j) + (1-p)c_{\alpha_d}(K_j), \quad \text{for} \quad j = 1, 2, \ldots, W,$$

under the distortion $h_{-\alpha_i, e^{\alpha_i}}$, and assuming that the bandwidth parameter η equals the average of the implied volatility, we invert Black-Scholes' formula on the input calls. Again least squares is applied to minimize the following expression

$$\text{Min:} \quad \sum_{j=1}^{W}[c(K_j) - \hat{c}_{K_j}]^2.$$

5.5.2 Source Data

Historical returns are not used in this model. The key input data are:

(1) The risk free rate r_f is again assumed constant; it can be inferred from interest instruments such as bonds.

(2) A set of spot call prices $\{\hat{c}_{K_1}, \hat{c}_{K_2}, \ldots, \hat{c}_{K_W}\}$ with different striking prices $\{K_1, K_2, \ldots, K_W\}$, respectively. Again the set of spot puts works well.

(3) By inverting Black-Scholes' formula on the calls for each striking price, we can obtain a set of implied volatilities. We are assuming that η is the average of the volatilities. One should notice that the value of η is essential in reflecting tail-fatness of the return density.

5.5.3 Model Constraints

Fewer constraints are needed for inferring all the parameters in this model.

(1) For a good fit in the optimization we have to make sure the actual call curve falls between the call curves evaluated on the upside and downside of the distorted risk neutral states. Under the volatility distortion induced by $\beta = e^{\alpha}$ and the fixed bandwidth η assumption, the distorted volatility of the upside state is less than that of the downside state. Therefore

$$c_{\alpha_u}(K_j) \leq \hat{c}(K_j) \leq p_{\alpha_d}(K_j).$$

(2) $\alpha_u > 0$ and $\alpha_d < 0$ in order to ensure that the model captures both possible upside and downside price movements.

Refinements and modifications are needed for a multiple stage model.

5.5.4 Comments

This estimation approach is actually prospective, as pure information is drawn from the option prices. However, numerical procedures may be complicated for solving three variables in a multiple stages model. One possible way would be to first find the optimal set of feasible risk neutral distorted volatilities and state weights, then solve for the α_i values. Moreover as mentioned above, puts are preferred over calls.

Unlike other VaR estimation methods, this one appears to be more practical for individual investors or speculators as more information will be reflected by having the estimated return heavily depend on the option prices. Moreover, the accuracy of this model depends on the time to expiration. As an option is close to its expiration, the estimated density tends to be more accurate under normal market condition as the market expectation becomes increasingly clear. We notice that estimation is done within a certain time to expiration; the best estimation range would be up to the expiration date. Interpolation up to expiration is possible but extrapolation after the expiration is possible but with reduced accuracy.

For historical data reconciliation, we can choose an additional constraint in calibrating the expected return of each return states with a set of historical prices, in a similar fashion as the kernel estimation method in the previous section. The size of the historical data have to be sufficiently large in order to obtain a smooth density. In this case the estimation becomes a prospective-retrospective mixture.

5.6 Conclusion

We have presented two estimation methods based on the idea developed from Wang's distortion operator. A new distortion operator is proposed which leads to the return state model proposed of the previous chapter. By calibrating the spot European option prices we notice that our model is able to generate the implied volatility smile. On the other hand the $\beta = e^{\alpha}$ parameter in the GLGC class features some properties which may be significant in the exchange market in terms of supporting point. A change in utility over the return value is also featured in the properties of the new distortion operator.

The main problem of this model is the option non-availability of a position. However, this situation can indeed be overcome if the market itself is index European option available and the position is correlated with the index. We can first impose our model over the index together with its option. Then we will make use an assumption that the return states correlation between the position and the index is a constant ρ across α_i :

$$\rho = \rho_{A_i,M_i} = \frac{\alpha^*_{Ai} - \frac{1}{2}\eta_{Ai}(1 - e^{-2\alpha^*_{Ai}})}{\alpha^*_{Mi} - \frac{1}{2}\eta_{Mi}(1 - e^{-2\alpha^*_{Mi}})}, \quad \text{for all} \quad i.$$

Least squares can also be applied and the solution is rather easy to obtain in EXCEL. The worse case is when even the index option is not available; our estimation must then resort to the traditional Gaussian kernel density or simply be inapplicable.

Further research can be done in several directions. Some suggestions are:

(1) A better β parameter may exist in relating the risk and return.

(2) The volatility related stochastic α model may lead to better estimation of the stochastic volatility as the market price of risk is concerned.

(3) Generalization to the American option or intermediate type option, for example, the mid-Atlantic option can also be developed.

(4) Further investigation of $\beta = e^\alpha$ in the new distortion may be important as it already features some interesting properties in the market preference in the exchange market.

Bibliography

[1] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., 1998, "Coherent Risk Measures", *Mathematical Finance*, **9**, 3, 203-228.

[2] Artzner, P., 1999, "Application of coherent risk measures to capital requirements in insurance", *North American Actuarial Journal*, **3**, 2, 11-25.

[3] Borch, K.H., 1961. "The utility concept applied to the theory of insurance", *ASTIN Bulletin*, **11**, 52-60.

[4] Bühlmann, H., 1980, "An economic premium principle", *ASTIN Bulletin*, **11**, 52-60.

[5] Butsic, R.P., 1994, "Solvency measurement for risk-based capital application", *Journal of Risk and Insurance*, **61**, 656-690.

[6] Butsic, R.P., 1999, "Capital alloction for property-liablity insurers: catastrophe reinsurance application", Casualty Actuarial Society Forum, Spring 1999 *Reinsurance Call for Papers*, 1-70.

[7] Cummins, J.D. and Harrington, S., 1985, "Property-liability insurance rate regulation: estimation of underwriting beta using quarterly profit data", *Journal of Risk and Insurance*, **52**, 16-43.

[8] Cummins, J.D., 1990, " Asset pricing models and insurance ratemaking", *ASTIN Bulletin*, **20**, 2, 125-166.

[9] Cummins, J.D., 1991, "Startistical and financial model insurance pricing", *Journal of Risk and Insurance*, **58**, 261-166.

[10] d'Arcy, S.P., and Doherty, N.A., 1988, *The Financial Theory of Pricing Property-Liability Insurance Contracts*, The S.S. Huebner Foundation for Insurance Education, University of Pennsylvania, Philadelphia, PA.

[11] Denneberg, D., 1994, *Non-Additive Measure and Integral*,Kluwer Academic Publishers, Boston.

[12] Duffie,D., and Pan, J., 1997, " An overview of value at risk", *The Journal of Derivatives*, **Spring 1997**, 1-49.

[13] Embrechts, P., 1996, " Actuarial versus financial pricing of insurance", *Proc. of the Conference on Risk Management of Insurance Firms*, The Wharton School of the University of Pennsylvania, May 15 -17, 1996.

[14] Gerber, H.U., and Shiu, E.S.W., 1994, "Martingale approach to pricing perpetual American option", *ASTIN Bulletin*, **24**, 195-220.

[15] Goovaerts, M.J., de Vylder, F. and Haezendonck, J., 1984, *Insurance Premiums: Theory and Applications*, North-Holland, Amsterdam.

[16] Harrison, M., and Kreps, D., 1979, "Martingales and arbitrage in multiperiod securities markets", *Journal of Economic Theory*, **20**, 381-408.

[17] Hull, J., 1997, *Options, Futures and Other derivatives*, 3rd edition, Prentice Hall, Upper Saddle River, New Jersey.

[18] Merton, R.C., 1973, "An intertemporal capital asset pricing model", *Econometrica*, **41**, 867-880.

[19] Rothschild, M., and Stiglitz, J.E., 1970, "Increasing risk I: a definition", *Journal of Risk and Insurance*, **53**, 693-717.

[20] Smith, C.W., Jr., 1986, "On convergence of insurance and finance research", *Journal of Risk and Insurance*, **53**, 692-717.

[21] Venter, G.G., 1991, "Premium implication of reinsurance without arbitrage", *ASTIN Bulletin*, **21**, 2, 223-230.

[22] Wang, S.S., 1995, "Insurance pricing and increased limits ratemarking by proportional hazards transform", *Insurance: Mathematics and Economics*, **17**, 43-54.

[23] Wang, S.S., 1996, "Premium calculation by transforming the layer premium density", *ASTIN Bulletin*, **26**, 1, 71-92.

[24] Wang, S.S., Young, V.R., and Panjer, H.H., 1997 "Axiomatic characterization of insurance press", *Insurance: Mathematics and Economics*, **21**, 173-183

[25] Wang, S.S., 2000, "A class of distortion operator for pricing Financial and Insurance Risk", *Journal of Risk and Insurance*, **67**, 1, 15-36.

[26] Yaari, M.E., 1987, "The dual theory of choice under risk", *Econometrica*, **55**, 95-115.

[27] Young, V.R., 1999, "Discussion of Christofides' Conjecture Regarding Wang's Premium Principle." *ASTIN Bulletin*, to appear.

Appendix A

Numerical Results

We derive the estimation based on a 4-stage α binomial GBM model on the sets of implied volatility of puts over different striking prices of the following underlying assets:

(1) Nortel (pre-crash, 2000) on October, 12 with option expiring on December,16.

(2) Nortel (up-move, 2000) on October, 18 with option expiring on October, 21.

(3) US index (post-crash, 1990) with option expiring over 164 days.

Denoting as S_0 initial price of the underlying asset, as S_τ the price at the expiration date of the option, as r_f the risk free rate, as τ the time to expiration of the option in unit year and as η the bandwidth, the input data and the estimated expected price on the option expiration date under both actual and risk neutral measures are shown in the Table A.1. The subscripts next to the expected price indicates the corresponding continuous rate of return.

Table A.2 shows the values of α_u and α_d with their corresponding state weights, and also the VaRs of actual, risk neutral and plain vanilla models, where their subscripts indicate the lower bound of the price at the expiration day of the call.

The recalibration errors on calls $\hat{c}$ (which can be obtained by put-call parity) and that of implied volatilities $\hat{IV}$ for each striking price K are shown in the Table A.3.

Graphs of the return density, call price and implied volatility smile are shown subsequently.

Notice that the high VaR of the Nortel pre-crash prices together with a fat downside tail on the return density indicate a possible down slide of the stock price, even thought the expected price at the date of expiration does not given much indication or warning signal. In the Nortel up-move case, the expected price reflects a possible up-move of the stock price together with a very positively skewed return density. One should also notice that the actual VaR is indeed less than that of the plain vanilla model. For the US index post-crash case, the high VaR together with the fat lower tail of the return density indicates the "Crash-o-phobia" anxiety among investors.

The above examples indicate that our proposed α binomial model can capture the market preferences and reflect the market movements up to the expiration date of the options, on both pre-crash and post-crash prices.

Table A.1: 4-Stage Nortel Pre-Crash, 2000

S_0	90.90		$E(S_\tau)$	91.99 $_{7.24\%}$
r_f	4.88%		$E_Q(S_\tau)$	91.63 $_{4.88\%}$
τ	0.16		S_τ	55.6 $_{-299.04\%}$
η	89.58%			

Table A.2: 4-Stage Nortel Pre-Crash, 2000

α	p		$\text{VaR}_{0.99}$	$\text{VaR}^{Q}_{0.99}$	Plain Vanilla $\text{VaR}_{0.99}$
α_u = 0.057630	0.992724		51.14 $_{39.76}$	45.51 $_{45.39}$	30.00 $_{60.90}$
α_d =-3.554319	0.007276				

Table A.3: 4-Stage Nortel Pre-Crash, 2000

K	c	$\hat{c}$	$(c-\hat{c})^2$	IV	$\hat{\text{IV}}$	$(\text{IV}-\hat{\text{IV}})^2$
65	29.009124	29.151050	0.020143	90.20%	92.00%	0.000323
70	25.260991	25.304538	0.001896	89.55%	90.00%	0.000021
75	21.848182	22.513372	0.442478	89.12%	95.00%	0.003454
80	18.782233	19.178564	0.157078	88.86%	92.00%	0.000987
85	16.061586	16.101262	0.001574	88.71%	89.00%	0.000008
90	13.673994	13.581119	0.008626	88.65%	88.00%	0.000042
95	11.599207	11.649102	0.002489	88.66%	89.00%	0.000012
100	9.811830	9.557770	0.064547	88.73%	87.00%	0.000300
110	6.985776	6.563740	0.178115	89.04%	86.00%	0.000923
120	4.968056	4.650186	0.101041	89.55%	87.00%	0.000650
130	3.552821	3.523219	0.000876	90.27%	90.00%	0.000007
			0.088988			0.000612

Table A.4: 4-Stage Nortel Up-Move, 2000

S_0	94.65		$E(S_\tau)$	95.78 $_{144.56\%}$
r_f	4.88%		$E_Q(S_\tau)$	94.69 $_{4.88\%}$
τ	0.01		S_τ	103.75 $_{1166.88\%}$
η	144.92%			

Table A.5: 4-Stage Nortel Up-Move, 2000

α	p		$\text{VaR}_{0.99}$	$\text{VaR}^{Q}_{0.99}$	Plain Vanilla $\text{VaR}_{0.99}$
α_u = 1.341547	0.880342		2.86 $_{91.79}$	12.29 $_{82.36}$	3.34 $_{91.31}$
α_d = -2.558773	0.119658				

Table A.6: 4-Stage Nortel Up-Move, 2000

K	c	$\hat{c}$	$(c-\hat{c})^2$	IV	$\hat{\text{IV}}$	$(\text{IV}-\hat{\text{IV}})^2$
85	10.565441	10.411485	0.023702	122.85%	115.00%	0.006160
90	6.139630	6.190174	0.002555	98.22%	100.00%	0.000316
95	2.814179	3.000214	0.034609	86.57%	92.00%	0.002953
100	1.474899	1.336393	0.019184	99.77%	95.00%	0.002276
105	1.012685	0.771419	0.058209	120.92%	110.00%	0.011929
110	0.752278	0.580335	0.029564	139.89%	130.00%	0.009790
115	0.591513	0.631432	0.001594	157.36%	160.00%	0.000698
120	0.490474	0.708574	0.047568	174.08%	190.00%	0.025348
130	0.374923	0.776744	0.161461	205.44%	240.00%	0.119462
			0.042049			0.019881

Table A.7: 4-Stage US Index, 1990

S_0	349.19		$E(S_\tau)$	372.66 14.28%
r_f	8.62%		$E_Q(S_\tau)$	363.17 8.62%
τ	0.46			
η	18.89%			

Table A.8: 4-Stage US Index, 1990

α	p		$\text{VaR}_{0.99}$	$\text{VaR}^{Q}_{0.99}$	Plain Vanilla $\text{VaR}_{0.99}$
α_u = 1.059332	0.884987		194.48 $_{154.71}$	137.13 $_{212.06}$	54.62 $_{294.57}$
α_d = -3.010166	0.115013				

Table A.9: 4-Stage US Index, 1990

K	c	$\hat{c}$	$(c-\hat{c})^2$	IV	$\hat{\text{IV}}$	$(\text{IV}-\hat{\text{IV}})^2$
250	110.362426	109.876033	0.236578	34.50%	32.00%	0.000627
300	64.064408	63.613065	0.203711	24.75%	23.70%	0.000110
325	41.953520	42.422089	0.219557	20.17%	20.90%	0.000053
330	37.680155	38.342361	0.438517	19.24%	20.20%	0.000092
335	33.497023	34.287025	0.624104	18.33%	19.40%	0.000115
340	29.447771	30.415433	0.936371	17.47%	18.70%	0.000152
345	25.594649	26.842553	1.557262	16.70%	18.20%	0.000225
350	22.013960	23.261056	1.555248	16.07%	17.50%	0.000204
355	18.784158	19.669575	0.783964	15.62%	16.60%	0.000095
360	15.969257	16.544963	0.331437	15.38%	16.00%	0.000038
365	13.603476	13.647590	0.001946	15.35%	15.40%	0.000000
370	11.683071	11.187345	0.245744	15.53%	15.00%	0.000028
375	10.168570	8.989467	1.390284	15.90%	14.60%	0.000168
380	8.995772	7.319476	2.809968	16.42%	14.50%	0.000368
385	8.090603	5.484004	6.794360	17.06%	13.90%	0.000996
			1.208603			0.000218

Exhibit 1: Nortel Pre-Crash, 2000

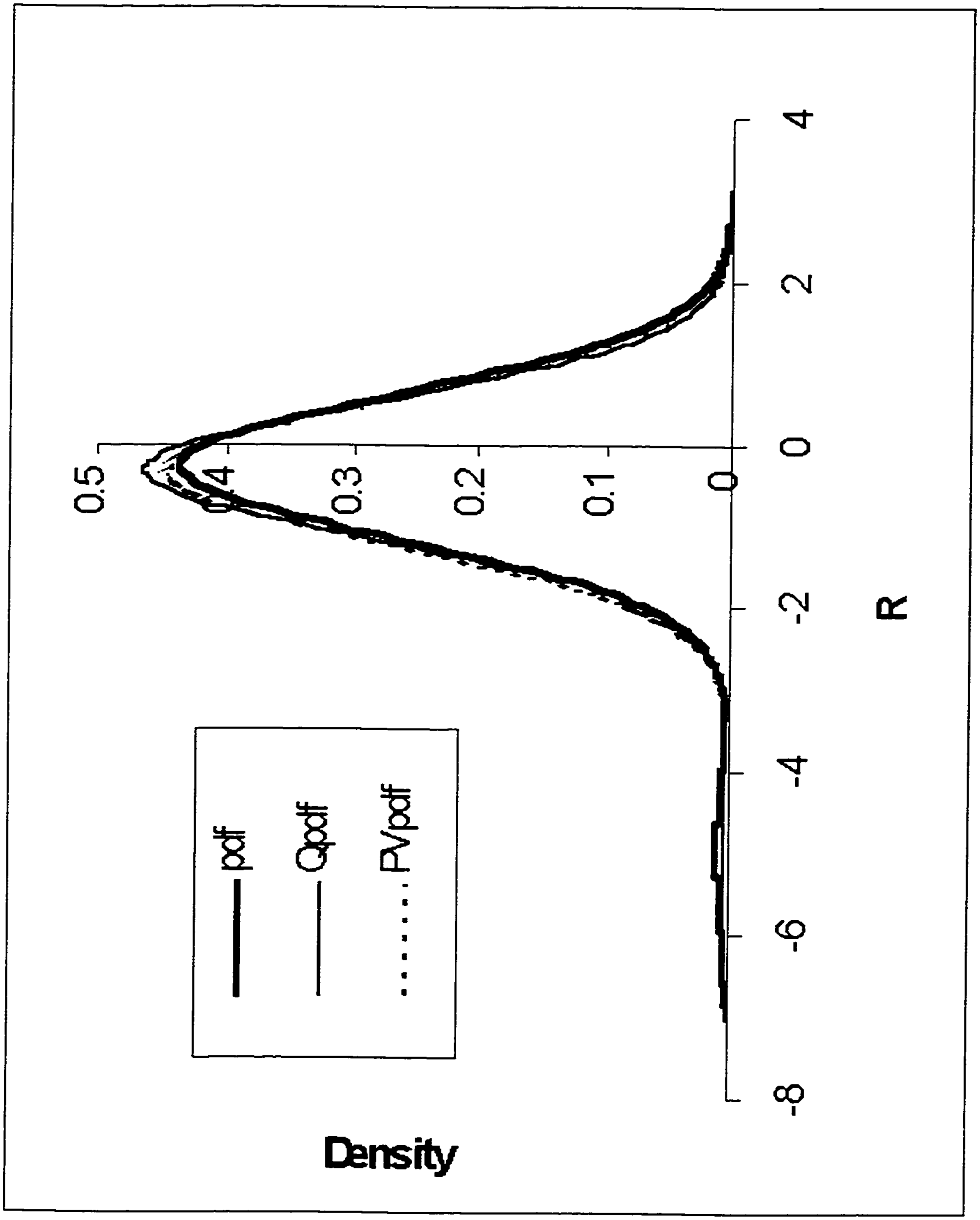

Figure A.1: Nortel Return PDF

Exhibit 1: Nortel Pre-Crash, 2000

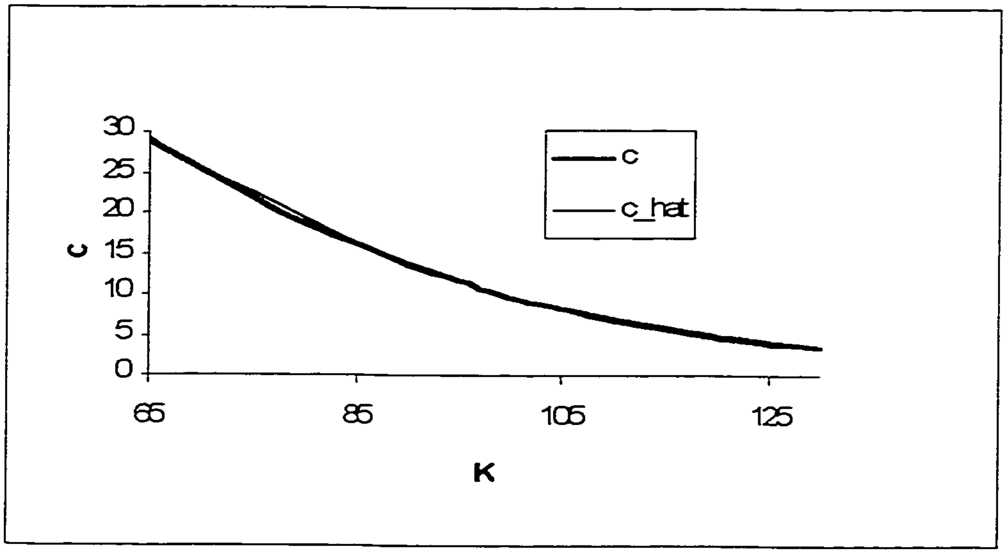

Figure A.2: Nortel Call Prices

Exhibit 1: Nortel Pre-Crash, 2000

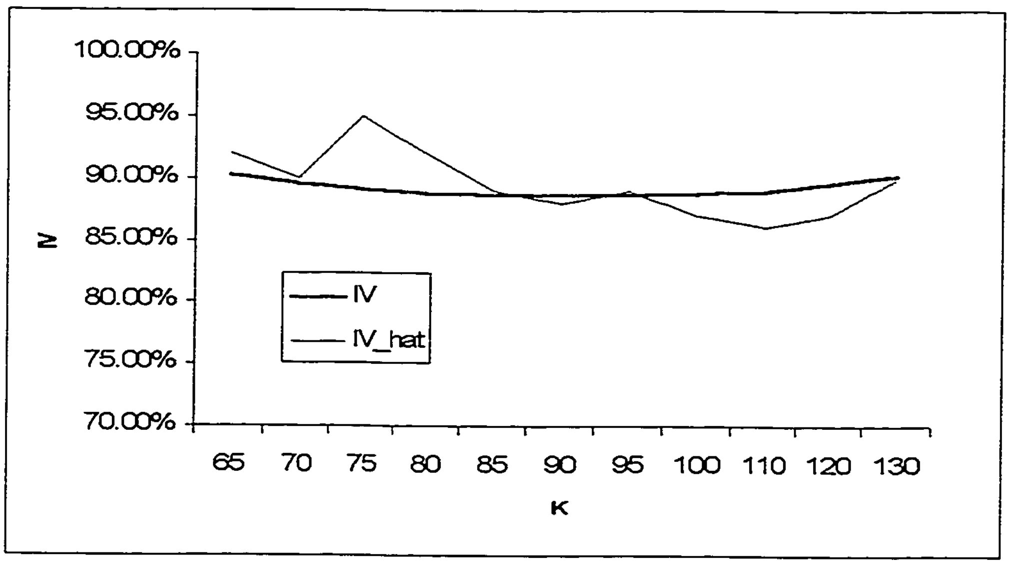

Figure A.3: Nortel Implied Volatility Smile

Exhibit 2: Nortel Up-Move, 2000

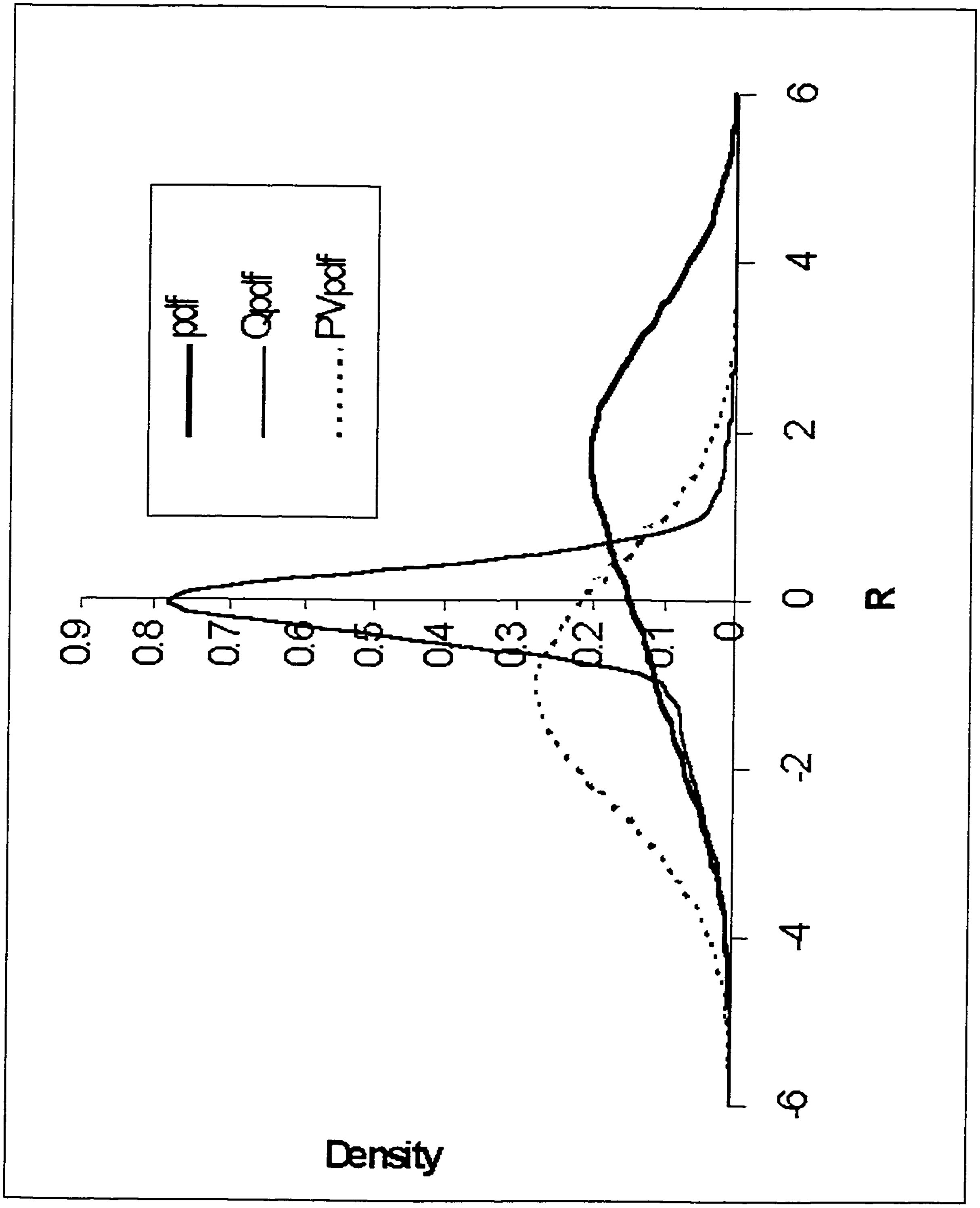

Figure A.4: Nortel Return PDF

Exhibit 2: Nortel Up-Move, 2000

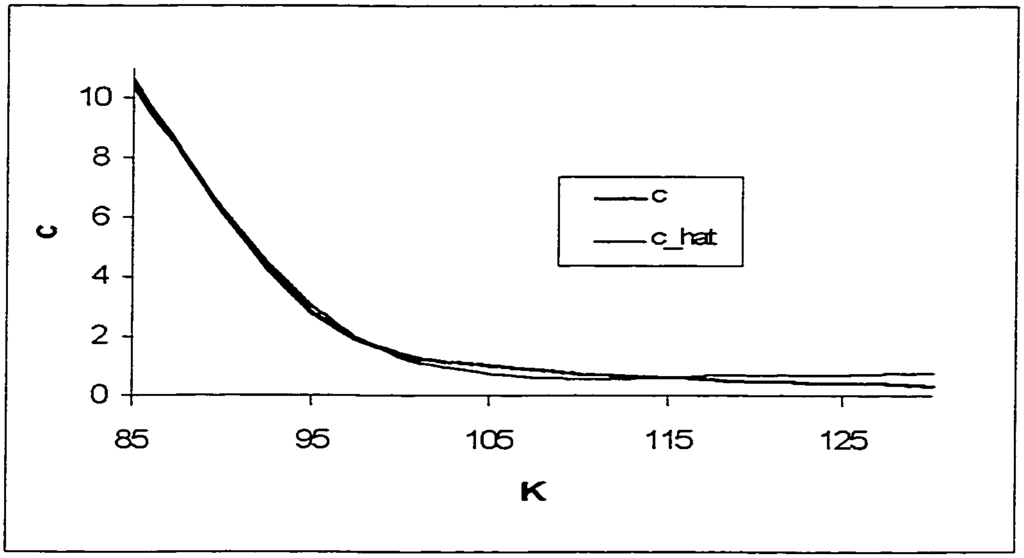

Figure A.5: Nortel Call Prices

Exhibit 2: Nortel Up-Move, 2000

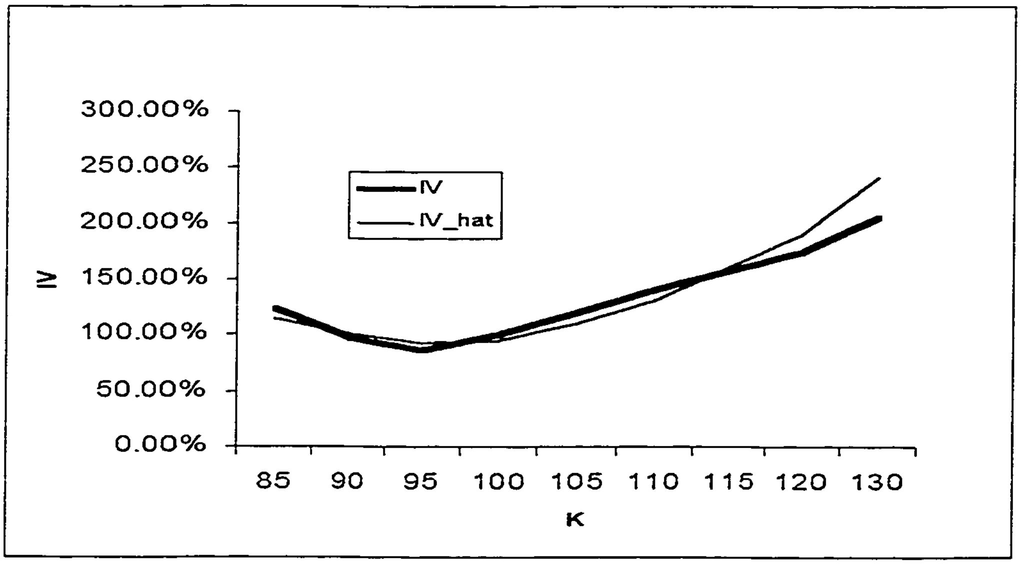

Figure A.6: Nortel Implied Volatility Smile

Exhibit 3: US Index, 1990

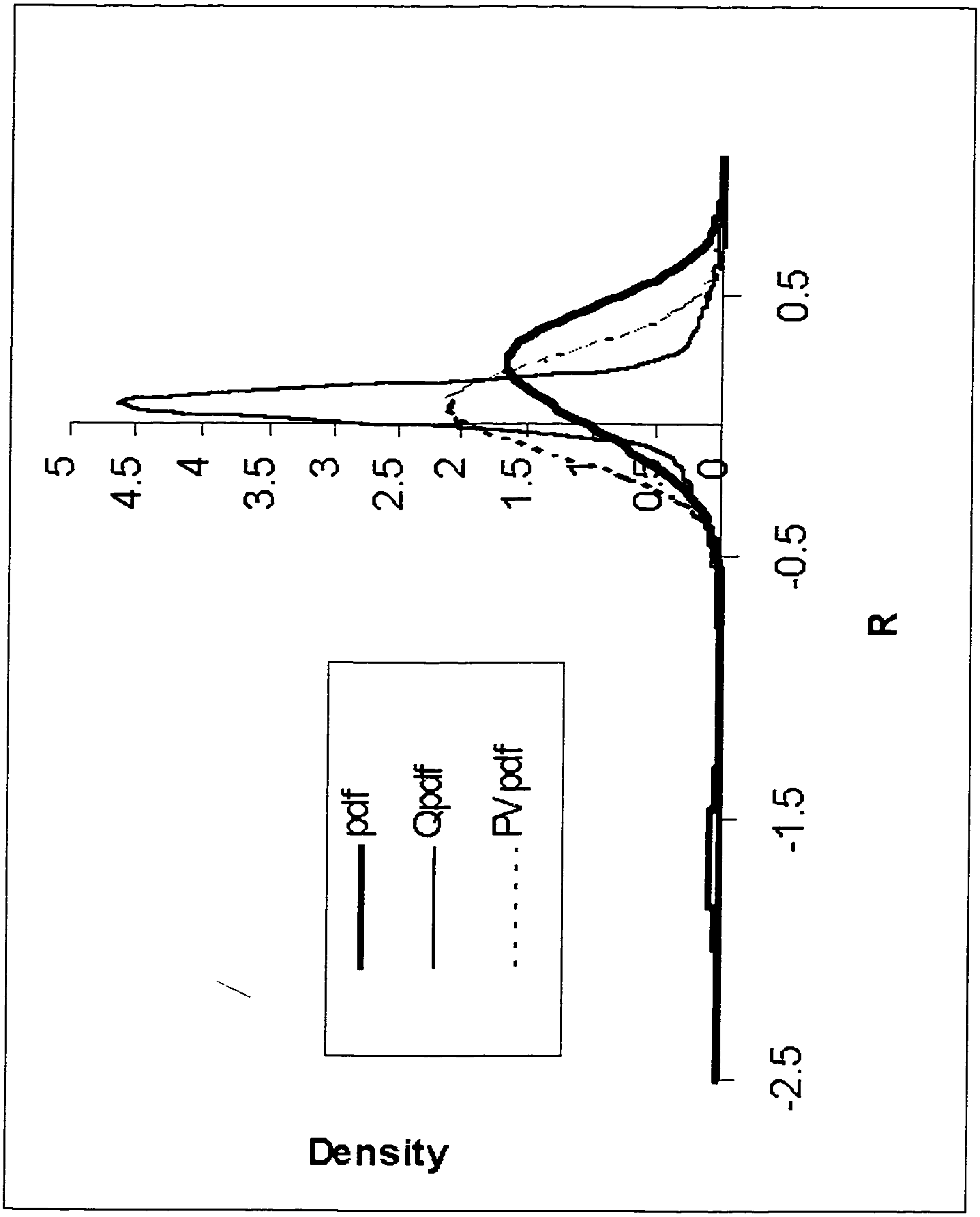

Figure A.7: US Index Return PDF

Exhibit 3: US Index, 1990

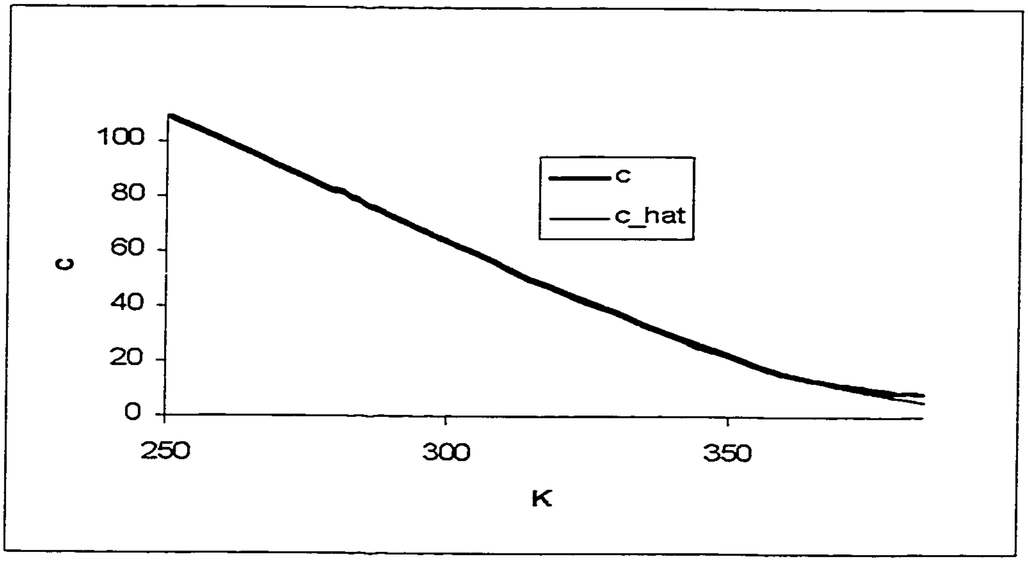

Figure A.8: US Index Call Prices

Exhibit 3: US Index, 1990

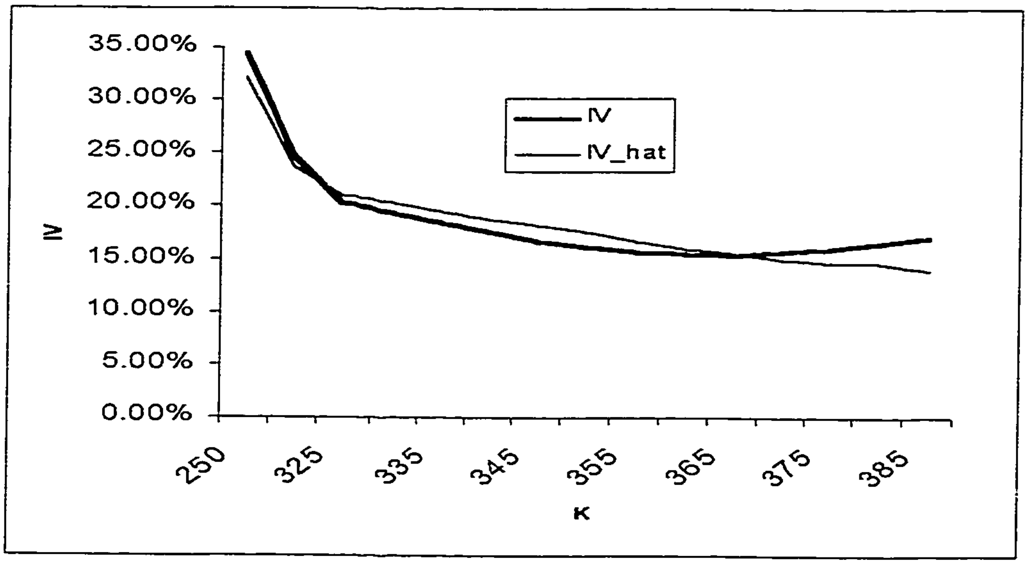

Figure A.9: US Index Implied Volatility Smile

Appendix B

Glossary

1 **APT** - Arbitrage pricing theory. Alternative to the CAPM in which the market portfolio plays no special role. Pp.26, 40.

2 **BS-Model** - Black-Schole Model. The option pricing model for European options of a non-dividend paying underlying stock under the assumption of a risk-neutral GBM return at the risk-free rate r_f and volatility σ. Denoting S_0 as the current spot price of a stock, S_τ as the stock price at the date of expiration, τ as time to expiration and K as the striking price, the standard expression of the call c and put p options are:

$$\begin{aligned} c &= e^{-r_f\tau}E_Q[(S_\tau - K)_+] &&= S_0\Phi(d_1) - Ke^{-r_f\tau}\Phi(d_2)\ , \\ p &= e^{-r_f\tau}E_Q[(K - S_\tau)_+] &&= Ke^{-r_f\tau}\Phi(-d_2) - S_0\Phi(-d_1)\ , \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{Ln(\frac{S_0}{K}) + (r_f + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\ , \\ d_2 &= d_1 - \sigma\sqrt{\tau}\ . \end{aligned}$$

An alternative expression in terms of survival functions is used here. Pp.27

3 **CAPM** - Capital asset pricing model. A model based on the proposition that any stock required rate of return is the risk-free rate plus its risk premium. Pp.22.

4 **CML** - Capital market line. A graphical representation of the relationship between risk and the required rate of return. Pp.23-24.

5 **DI** - Distortion index. A measure of the severity of distortion upon the expected value induced by a distortion operator. Pp.21.

6 **EPD** - Expected policyholder deficit. A downside risk measure of an insurance policy. With an asset denoted by A and a random loss denoted by L the EPD of the policy is expressed as $E[(L - A)_+]$. Pp.20.

7 **GBM** - Geometric Brownian motion. Details can be found in Chapter 2 and 3. Pp.23, 25.

8 **GLGC Class** - General linear Gaussian conjugation class. A generalized class of distortion operators based on distortion operator of Wang (1999). Pp.29.

9 **ICAPM** - Intertemporal capital asset pricing model. An GBM version of CAPM. Pp.25.

10 **PH-Transform** - Proportional hazard transform. A transformation of random variables induced by the following distortion operator over the survival value u

$$g(u) = u^r$$

where $0 < r \leq 1$. Pp.11.

11 **SML** - Security market line. The line that shows the relationship between risk and rate of return of individual securities.