Bifurcation in Complex Quadratic Polynomial and Some Folk Theorems Involving the Geometry of Bulbs of the Mandelbrot Set

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Abstract

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The Mandelbrot set M is a subset of the parameter plane for iteration of the complex quadratic polynomial $Q_c(z) = z^2 + c$. M consists of those c values for which the orbit of 0 is bounded. This set features a basic cardioid shape from which hang numerous 'bulbs' or 'decorations'. Each of these bulbs is a large disk that is directly attached to the main cardioid together with numerous other smaller bulbs and a prominent 'antenna'. In this thesis we study the geometry of bulbs and some 'folk theorems' about the geometry of bulbs involving spokes of the antenna.

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Chapter 1

Introduction

1.1 General Introduction

In the study of complex dynamical systems the Mandelbrot set is one of the most intricate and beautiful objects in Mathematics. The geometry in the Mandelbrot set is quite complicated. In the last fifteen years many mathematicians such as Paul Blanchard [BL], Pau Atela [A], Bodil Branner [BR], R.L. Devaney [$D2$], E.Lau and D. Schleicher Yoccoz [LS] and recently R.L.Devaney [$D1$] and R.L.Devaney and M.Morenno-Rocha [DM] studied the geometry of the bulbs and antennae in the Mandelbrot set. In this thesis we study the Mandelbrot set and geometry of bulbs. We follow [$D1$], [DM] and [$D2$]. The study of the geometry of the Mandelbrot set needs basic concepts on iteration theory, Fatou set and Julia sets. First we review these basic concepts. Our main work is on 'some folk theorems' about the geometry of bulbs involving spokes of the antenna. Figure 1.1 shows the Mandelbrot set.

Figure 1.1

1.2 Scope of the thesis

In Chapter 2 we review some basic concepts of Discrete Dynamical systems, Complex Analysis, Fatou and Julia sets for a complex polynomial.

In Chapter 3 the definition and construction of the Mandelbrot set and some properties of the Mandelbrot set are given. We also discuss period doubling bifurcation.

We present our main work on the geometry of bulbs in Chapter 4. In this Chapter we discuss the geometry of the bulbs, geometry of the antennas and external rays. The main tool of the discussion is the doubling function and rotation map. We state and prove some folk theorems in this Chapter.

Chapter 2

Preliminaries

In this chapter we review some basic concepts of complex analysis, discrete dynamical systems. Throughout this chapter we follow $[BE], [D3]$ and $[KF]$.

2.1 Fixed points, periodic points and their nature

Definition 2.1 *Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and P be a polynomial of degree $n \geq 1$. A point $z_0 \in \mathbb{C}_\infty$ is said to be a fixed point of P if $P(z_0) = z_0$. Moreover, z_0 is*

(i) attracting if $|P'(z_0)| < 1$;

(ii) super-attracting if $P'(z_0) = 0$;

(iii) repelling if $|P'(z_0)| > 1$;

(iv) neutral if $|P'(z_0)| = 1$;

Definition 2.2 *Let P be a polynomial of degree $n \geq 1$ and $z_0 \in \mathbb{C}_\infty$. The sequence*

$$\{z_0, P(z_0), P^2(z_0), P^3(z_0), \ldots, P^n(z_0), \ldots\}$$

is called the orbit of z_0 under P. The point z_0 is a periodic point of period k if $P^k(z_0) = z_0$ and $P^q(z_0) \neq z_0$ for $1 \leq q < k$. If z_0 is a periodic point of period k, then the sequence $\{z_0, P(z_0), P^2(z_0), P^3(z_0) \ldots P^{k-1}(z_0)\}$ is a k-cycle.

Lemma 2.1 *A point z_0 is an attracting fixed point of a polynomial P if and only if $\exists$ a neighborhood U of z_0 such that*

(i) $P(\overline{U}) \subset U$;

(ii) $\bigcap_{n=0}^{\infty} P^n(U) = \{z_0\}$;

Definition 2.3 *The basin of attraction of z_0 is the set of all points whose orbit tends to z_0 and it is denoted by $B(z_0)$, that is*

$$B(z_0) = \{z : P^n(z) \to z_0, \text{ as } n \to \infty\}.$$

Example 2.1 Let $P : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ be defined by $P(z) = z^2$. Let $U = \{z : |z| < r < 1\}$ be a neighborhood of $z = 0$. Then

$$P(\overline{U}) = \{z : |z| \leq r^2 < 1\} \subset U.$$

Inductively, we get $P^n(U) = \{z : |z| \leq r^{2n}\}$. So $\bigcap_{n=1}^{\infty} P^n(U) = \{0\}$.

Definition 2.4 *Suppose z_0 is a periodic point of period k for the polynomial P. Let $\lambda = (P^k)'(z_0)$. Then the k cycle $\{z_0, P(z_0), P^2(z_0), P^3(z_0), ..., P^{k-1}(z_0)\}$ is* *(i) attracting if* $|\lambda| < 1$;

(ii) super-attracting if $\lambda = 0$;

(iii) repelling if $|\lambda| > 1$;

(iv) neutral if $|\lambda| = 1$;

2.2 Fatou sets and Julia sets

To study the Mandelbrot set we need the concept of Julia set. The Julia set is related to the Fatou set. In this subsection we define Fatou and Julia sets of a polynomial P in terms of equicontinuity of the family $\{P^n\}_{n\geq 0}$. Since equicontinuity is closely related to normal families of analytic functions, using the notion of normality we can get more information about the Fatou and Julia sets. Hence, we discuss normality.

Definition 2.5 *Let* (X, d) *and* (X_1, d_1) *be two metric spaces. A family* $\mathcal{F}$ *of maps of* (X, d) *into* (X_1, d_1) *is equicontinuous at* x_0 *if and only if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x \in X$ *and for all* $f \in \mathcal{F}$

$$d(x_0, x) < \delta \Rightarrow d_1(f(x_0), f(x)) < \epsilon$$

The family $\mathcal{F}$ *is equicontinuous on a subset* X_0 *of* X *if it is equicontinuous at each point* x_0 *of* X_0.

Theorem 2.1 [*BE*] *Let* $\mathcal{F}$ *be any family of maps , each mapping* (X, d) *into* (X_1, d_1). *Then there is a maximal open subset of* X *on which* $\mathcal{F}$ *is equicontinuous. In particular,* *if P maps a metric space (X, d) into itself, then there is a maximal open subset of X on which the family of iterates P^n is equicontinous.*

Definition 2.6 *Let P be a complex polynomial of degree $n \geq 2$. Then the Fatou set $F(P)$ of P is the maximal open subset of $\mathbb{C}_\infty$ on which $\{P^n\}_{n\geq 0}$ is equicontinous and the Julia set $J(P)$ of P is the complement of the Fatou set in $\mathbb{C}_\infty$. In other words, Fatou set is the stable set and Julia set is the unstable set. We see the Fatou and the Julia sets in the following example.*

Definition 2.7 *A family $\mathcal{F}$ of maps from (X_1, d_1) into (X_2, d_2) is said to be a normal or a normal family in X_1 if every infinite sequence of functions from $\mathcal{F}$ contains a subsequence which converges locally uniformly on every compact subset of X_1.*

Example 2.2 Let $P : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ be defined by $P(z) = z^2$. Here fixed points of P are $0, 1$ and ∞. $P'(z) = 2z$ and $|P'(z)|_{z=0} = 0$. Thus 0 is a super-attracting fixed point of P. Also $|P'(z)|_{z=1} = 2 > 1$ and thus 1 is a repelling fixed point of P. If we consider $z \in \mathbb{C}_\infty$ such that $|z| < 1$, i.e., the open unit disc. Then the family $\{P^n(z)\}_{n\geq 0}$ is a normal family (Example 2.1). So the unit disc is a subset of the Fatou set. Similarly if $|z| > 1$, then the family $\{P^n(z)\}_{n\geq 0}$ is also a normal family. We sketch the proof: Let $|z| > R > 1$. Then $|z^2| > R^2$ and $|z^{2n}| > R^{2n}$. Thus $|P^n(z)| \to \infty$. Thus $\{z : |z| > 1\}$ is also a component of Fatou set. If we consider $|z| = 1$ then we have complicated situation and the family $\{P^n(z)\}_{n\geq 0}$ is not a normal family. So from the definition of Fatou and Julia set, the set $\{z : |z| < 1\} \cup \{z : |z| > 1\}$ is the Fatou set for P and the Julia set for P is the unit circle.

Definition 2.8 *Two maps $F,\ G : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ are conjugate to each other if there exists a homeomorphism $h : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ such that $h \circ F = G \circ h$. It is clear that $h \circ F^n = G^n \circ h$. Thus, h maps orbits of F to those of G. Similarly h^{-1} takes orbits of G to those of F. Hence, the conjugacy h gives a one-to-one correspondence between orbits of F and G.*

Theorem 2.2 *(Ascoli-Arzela)[BE] Let D be a subdomain of the complex sphere and let $\mathcal{F}$ be a family of continuous maps of D into the complex sphere. Then $\mathcal{F}$ is equicontinuous in D if and only if it is a normal family in D.*

Theorem 2.3 *(Montel)[KF] Let $\{F_n\}$ be a family of complex analytic functions on an open domain U. If $\{F_n\}$ is not a normal family, then for all $w \in \mathbb{C}$ with at most one exception we have $F_n(z) = w$ for some $z \in U$ and some n.*

Example 2.3 Let $Q_c(z) = z^2 + c$. Let us study the dynamics of Q_c for the following cases:

(i) $Q_c^n(z) \to 0$ if $|z| < \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$;

(ii) $Q_c^n(z) \to \infty$ if $|z| > \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$;

Let $R = \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$ and $|z| < R$. The zeros of $Q_c(z)$ are $\pm\sqrt{c}$ and $|\sqrt{c}| < R$ (shown below). Since $R = \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, we have

$$(R - \frac{1}{2})^2 = \frac{1}{4} - |c| \quad \Rightarrow \quad R^2 - R + \frac{1}{4} = \frac{1}{4} - |c|.$$

$\Rightarrow \quad R - R^2 = |c|$. Since $R > 1/2$ we have $R - R^2 < R^2$ and $|c| < R^2$. Thus $|\sqrt{c}| < R$. So the zeros of $Q_c(z)$ are in $|z| < R$. Thus in $|z| \geq R$, the function

$$\frac{Q_c(z)}{z^2} = \frac{z^2 + c}{z^2}$$

does not have zeros and is holomorphic. Now on the circle $C(0, R)$ we have

$$\left|\frac{Q_c(z)}{z^2}\right| = \left|\frac{z^2 + c}{z^2}\right| = \left|1 + \frac{c}{z^2}\right| < 1 + \left|\frac{c}{R^2}\right| = 1/R.$$

Now, by the maximum principle $\left|\frac{Q_c(z)}{z^2}\right| < \frac{1}{R}$ in the domain $|z| < R$. This implies $|Q_c(z)| < R\left|\frac{z}{R}\right|^2$. Applied inductively

$$|Q_c^n(z)| < R\left|\frac{z}{R}\right|^{2^n} \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \text{ since } \left|\frac{\mathrm{z}}{\mathrm{R}}\right| < 1.$$

Hence,

$$|z| < R \Rightarrow |Q_c^n(z)| \longrightarrow 0 \text{ as n} \longrightarrow \infty.$$

Again, let $R = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ and $|z| > R$. The zeros of $Q_c(z)$ are $\pm\sqrt{c}$ and $|\sqrt{c}| < R$. So the zeros of $Q_c(z)$ are in $|z| < R$ and

$$\left|\frac{Q_c(z)}{z^2}\right| = \left|\frac{z^2 + c}{z^2}\right| = \left|1 + \frac{c}{z^2}\right| > 1 - \left|\frac{c}{z^2}\right| > 1 - \frac{|c|}{R^2} = 1/R.$$

By the minimum principle $|\frac{Q_c(z)}{z^2}| > \frac{1}{R}$ in the domain $|z| > R$. This implies $|Q_c(z)| > R|\frac{z}{R}|^2$. But $|\frac{z}{R}| > 1$. Applied inductively, $|Q_c^n(z)| > R|\frac{z}{R}|^{2^n} \to \infty$.

Hence,

$$|z| > R \Rightarrow |Q_c^n(z)| \to \infty \text{ as n} \to \infty.$$

2.3 Some properties of the Julia and the Fatou sets

Definition 2.9 *Let P be a polynomial of degree n and $E \subset \mathbb{C}_\infty$. Then E is*

(i) forward invariant if $P(E) = E$.

(ii) backward invariant if $P^{-1}(E) = E$.

(iii) completely invariant if $P(E) = P^{-1}(E) = E$.

If P is surjective then the concepts of backward invariance and complete invariance coincide.

Proposition 2.1 *Attracting fixed points of a polynomial P lie in the Fatou set of P and repelling fixed points of P lie in the Julia set.*

Proof. Let $P(z_0) = z_0$ and z_0 be an attracting fixed point of P, that is, $|P'(z_0)| = \lambda < 1$. We need to show that $z_0 \in F(P)$. Let

$$U = \{z : |P'(z)| \leq \lambda + \epsilon < 1\} \cap D(z_0, r),$$

where $D(z_0, r)$ is an open disk with radius r. Now $|P^n(z) - z_0| \to 0$ for $z \in U$, because

$$|P^n(z) - P^n(z_0)| \leq sup_{z \in U}|(P^n(z))'||z - z_o| \leq (\lambda + \epsilon)^n r \to 0.$$

It does not depend on z, i.e., $P^n(z) \to z_0$ uniformly on U. Thus U is a subset of the Fatou set for P and consequently $z_0 \in F(P)$.

Now, let z_0 be a repelling fixed point of P. We need to show that $z_0 \in J(P)$. It is clear that $P^n \to \infty$ does not hold uniformly on U, because $P^n(z_0) = z_0,\ n = 1, 2, \cdots$. It is enough to show that $|(P^n)'(z_0)|$ is not bounded. Now

$$|(P^n)'(z_0)| = |(P'(z_0)|^n = \lambda^n \to \infty.$$

Thus $z_0 \in J(P)$. ■

Proposition 2.2 *Let z_0 be an attracting fixed point of P. Then the basin of attraction of z_0, $B(z_0) = \{z : P^n(z) \to z_0,$ as n $\to \infty\}$, is in $F(P)$.*

Proof. In the Proposition [2.1] we have proved that the family $\{P^n\}$ is normal on a neighborhood U, $U \subset B(z_0)$, of z_0. Let $z \in B(z_0)$. Then, for some $k \geq 1$, $P^k(z) \in U$. Let $V \subset U$ be a neighborhood of $P^k(z)$. Then $P^{-k}(V)$ is a neighborhood of z and $P^{-k}(V) \subset B(z_0)$. Then the family $\{P^n\}$ on $P^{-k}(V)$ is normal, because $P^n \to z_0$ uniformly on $P^{-k}(V)$. Thus, $z \in F(P)$. This proves that the basin of attraction of z_0 is in $F(P)$. ■

Proposition 2.3 *Let P be an analytic and suppose that z_0 is a repelling periodic point for P. Then the family of iterates of P at z_0 is not normal at z_0.*

Proof. Let z_0 be a fixed point of P. Assume that $\{P^n\}$ is normal on a neighborhood U of z_0. Since $P^n(z_0) = z_0$ for all n, it follows that $P^n(z)$ does not converge to ∞ on U. Thus assume some sequence of $\{P^n\}$ has a subsequence $\{P^{n_i}\}$ which converges uniformly to G on U. Hence $|(P^{n_i})'(z_0)| \to |G'(z_0)|$. But $|(P^{n_i})'(z_0)| \to \infty$. This contradiction establishes the result for repelling fixed point. ■

Proposition 2.4 *If P is a polynomial, then $J(P)$ is compact.*

Proof. We know by the definition of the Fatou set, $F(P) = \mathbb{C} \setminus J(P)$. The set $z \in \mathbb{C}$ such that there is an open set V with $z \in V$ and $\{P^n\}$ normal on V is an open set. So $J(P)$ has open complement and is closed. Now we need to show that $J(P)$ is bounded. Since P is a polynomial of degree $n \geq 2$, so we may find r such that $|P(z)| \geq 2|z|$ if $|z| \geq r$, which implies that $|P^k(z)| > 2^k r$ if $|z| > r$. Thus $P^k(z) \to \infty$ uniformly on the open set $V = \{z : |z| > r\}$. Now by definition $\{P^k\}$ is normal on V so that

$$V \subset \mathbb{C} \backslash J(P) \Rightarrow V^c \supset (\mathbb{C} \backslash J(P))^c = J(P).$$

$\Rightarrow V^c = \{z : |z| \leq r\} \supset J(P)$. So $J(P)$ is bounded and $J(P)$ is compact. ∎

Proposition 2.5 *$J(P)$ is non-empty.*

Proof. Suppose that $J(P) = \emptyset$. Then for every $r > 0$, the family $\{P^n\}$ is normal on the open disc $B(0, r)$ with center the origin and radius r. Since the closed disc $\overline{B(0, r)}$ is compact and it may be covered by a finite number of open sets on which P^n is normal. Since P is a polynomial, taking r large enough ensures that $B(0, r)$ contains a point z for which $|P^n(z)| \to \infty$ and also contains a fixed point w of P with $P^n(w) = w$ for all n. Thus, we see that it is impossible for any subsequence of $\{P^n\}$ to converge uniformly either to a bounded function or to infinity on any compact subset of $B(0, r)$ which contains both z and w, which is a contradiction for the normality of $\{P^n\}$. ∎

Proposition 2.6 *Let P be a polynomial and $w \in J(P)$ and let U be any neighborhood of w. Then $W = \bigcup_{n=1}^{\infty} P^n(U)$ is the whole of $\mathbb{C}$, except possibly for a single point. Any such exceptional point is not in $J(P)$ and is independent of w and U.*

Proof. We know that for the Julia set $J(P)$, the family $\{P^n\}$ is not normal at w. So by the Montel's Theorem [2.3] we have $\mathbb{C} = W$, except possibly for a single point. Now we need to show that exceptional point is not in $J(P)$. Suppose $v \notin W$. If $P(z) = v$, then since $P(W) \subset W$, it follows that $z \notin W$. As $\mathbb{C} \backslash W$ consists of at most one point, then $z = v$. Hence P is a polynomial of degree n such that the only solution of $P(z) - v = 0$ is v, which implies that $P(z) - v = c(z - v)^n$ for some constant c. If z is sufficiently close to v, then

$$P^k(z) - v \to 0, \quad as \quad k \to \infty,$$

convergence being uniform on $\{z : |z - v| < |2c|^{-\frac{1}{n-1}}\}$. Thus $\{P^n\}$ is normal at v, so the exceptional point $v \notin J(P)$. Clearly v only depends on the polynomial P. ∎

Corollary 2.1 [*KF*] *The following holds for all $z \in \mathbb{C}$ with at most one exception*

(i) If U is an open set intersecting $J(P)$ then $P^{-n}(z)$ intersects U for infinitely many value of n.

(ii) If $z \in J(P)$ then $J(P)$ is the closure of $\bigcup_{n=1}^{\infty} P^{-n}(z)$.

Corollary 2.2 *If P is a polynomial, then $J(P)$ has empty interior.*

Proof. Suppose $J(P)$ contains an open set U. Then $J(P) \supset P^n(U)$ for all n and we know that the Julia set is both forward and backward invariant. So we can write $J(P) \supset \bigcup_{n=1}^{\infty} P^n(U)$. Now by Proposition [2.6] $J(P)$ is all of $\mathbb{C}$ except possibly for one point. Which is a contradiction, since by Proposition [2.4] $J(P)$ is bounded. Hence, $J(P)$ has empty interior. ■

Chapter 3

The Mandelbrot Set and Bifurcation in Complex Quadratic Polynomial

In the study of Complex Dynamical system, the study of the Mandelbrot set is one of the most intricate and beautiful subjects in Mathematics. To study the Mandelbrot set, first we need to study the dynamics of complex quadratic polynomial. In this chapter we study the filled Julia set, the Mandelbrot set and some basic properties of them. At the end of this chapter we also discuss periods of the bulbs and rotation numbers. Throughout this chapter we follow [BE], [$D2$], [$D3$].

3.1 Complex Quadratic polynomial, Filled Julia set and the Mandelbrot set

Definition 3.1 *Consider the Quadratic complex function* $Q_c : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ *defined by* $Q_c(z) = z^2 + c$, *where* c *is a complex parameter.*

In complex dynamics critical points play an important role. Note that 0 *is the only critical point for* $Q_c(z)$.

Proposition 3.1 *Let* $P(z) = az^2 + bz + d$ *with* $a \neq 0$ *and* $b, d \in \mathbb{C}$. *Then* P *is conjugate to* $Q_c(z) = z^2 + c$ *for some* $c \in \mathbb{C}$.

Proof. We need to find a homeomorphism $H : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ such that

$$H \circ P = Q_c \circ H.$$

Let $H(z) = az + b/2$, where $c = ad + b/2 - b^2/4$. Clearly H is a homeomorphism. Now

$$H \circ P(z) = H(P(z)) = H(az^2 + bz + d) = a(az^2 + bz + d) + b/2.$$

$$= a^2z^2 + abz + ad + b/2.$$

Similarly

$$Q_c \circ H(z) = Q_c(H(z)) = Q_c(az + b/2) = (az + b/2)^2 + c.$$

$$= (az + b/2)^2 + ad + b/2 - b^2/4 = a^2z^2 + abz + ad + b/2.$$

Hence P is conjugate to Q_c. ■

Proposition 3.2 *For any complex number $\lambda \neq 0$, there is a $c = c(\lambda)$ for which the quadratic map $P(z) = \lambda z(1-z)$ is analytically conjugate to $Q_c(z)$ with respect to a map of the form $h(z) = az + b$.*

Proof. Let $Q_c(z) = z^2 + c$ and $P(z) = \lambda z(1-z)$. By Proposition [3.1] $a = -\lambda$, $b = \lambda$, and $d = 0$ and $H(z) = az + b/2 = -\lambda z + \lambda/2$, where

$$c(\lambda) = ad + b/2 - b^2/4 = \lambda/2 - \lambda^2/4.$$

Now $Q_c \circ H = H \circ P$. Then,

$Q_c(-\lambda z + \lambda/2) = (-\lambda z + \lambda/2)^2 + c$

$= \lambda^2 z^2 - \lambda^2 z + \lambda^2/4 + \lambda/2 - \lambda^2/4 = \lambda^2 z^2 - \lambda^2 z + \lambda/2.$

Similarly, $H(\lambda z - \lambda z^2) = -\lambda(\lambda z - \lambda z^2) + \lambda/2 = \lambda^2 z^2 - \lambda^2 z + \lambda/2.$

So P is conjugate to Q_c. ∎

Definition 3.2 *(Filled Julia set) The filled Julia set, K_c of the complex quadratic polynomial Q_c is the set of points whose orbits do not tend to ∞.*

That is $K_c = \{z \in \mathbb{C} : Q_c^n(z)$ is bounded $\}$.

Remark 3.1 *In Chapter 2 we saw that if $z \in J(Q_c)$ then the orbit of $Q_c^n(z)$ is bounded. So the Julia set is a subset of the filled Julia set. The filled Julia set also contains any attracting periodic orbit and its basin of attraction, if there is such an orbit. Both K_c and its complement are clearly completely invariant under Q_c.*

Definition 3.3 *(Mandelbrot set) The Mandelbrot set is a subset of the parameter plane which consists of those c values for which the orbit of 0 under the complex* *quadratic function* $Q_c(z) = z^2 + c$ *is bounded (c is complex parameter).*

That is, $M = \{c : Q_c^n(0)$ *is bounded* $\}$.

Example 3.1 $i \in M$ and $2i \notin M$. Since $c = i$ then the critical orbit of Q_i is

$$0, \quad -1+i, \quad -i, \quad -1+i, \quad -i, \cdots$$

which is bounded. Otherwise the critical orbit for Q_{2i} is :

$$0, \quad 2i, \quad -4+2i, \quad 12-14i, \quad -52-334i, \cdots$$

which is unbounded, since $|Q_{2i}^n(0)|$ becomes bigger than max $(2, |c|)$.

Proposition 3.3 [D3] *Let* $P(z)$ *be an analytic function satisfying* $P(0) = 0$ *and* $P'(0) = \lambda$ *with* $0 < |\lambda| < 1$. *Then there is a neighborhood* U *of* 0 *and an analytic map* $H : U \to \mathbb{C}$ *such that* $P \circ H(z) = H \circ L(z)$, *where* $L(z) = \lambda z$.

Proposition 3.4 [D3] *Let* $|a| < 1$. *Define* $T_a(z) = \frac{z-a}{1-\bar{a}z}$. *Then* T_a *is analytic for* $|z| < |a|^{-1}$. *Moreover* $T_a^{-1} = T_{-a}$ *for* $|z| < 1$ *and* $T_a : D \to D$.

Theorem 3.1 *Let* P *be a polynomial and suppose that* z_0 *is an attracting periodic point of* P. *Then, there is a critical value which lies in the basin of attraction of* z_0.

Proof. We prove this only for an attracting fixed point. Suppose z_0 is an attracting fixed point. Then, by Proposition [3.3] there is a neighborhood U of z_0 and an analytic homeomorphism $H : U \to D$(open unit disc) which linearizes P. Let V be an open set containing U and such that $P : V \to U$ is onto. We need to prove that either P has a critical point in V or else P has an analytic inverse $P^{-1} : U \to V$.

Let us assume that P does not have an analytic inverse on U. Since P is analytic and surjective on V, it follows that P must not be one-to-one. So there exist $z_1, z_2 \in V$ such that $P(z_1) = P(z_2) = q$. Suppose that $H(q) = a$ and let $T_a : D \to D$ be as given in the previous proposition.

Now, consider the circles C_r of radius $r < 1$ centered at 0 in D. T_a^{-1} pulls this family of circles back to nested sequence of simple closed curves surrounding 0. Since $P : V \to U$ is analytic and $P'(z_i) \neq 0$ for each i. It follows that for r small, $P^{-1}(H^{-1} \circ T_a^{-1}(C_r))$ is a pair of families of nested circles, one family centered at z_1 and the other centered at z_2. Here P^{-1} denotes the inverse image, not the inverse map. Now, as r increses there is a smallest r_* for which these two families first intersect. Let p be a point common to both simple closed curves of the form $P^{-1}(H^{-1} \circ T_a^{-1}(C_{r_*}))$. Then, p is easily seen to be a critical point for P.

Thus, we may continue to define P^{-1} on successively larger domains of attraction of z_0 until we either meet a critical point or else exhaust the immediate attractive basin of z_0 by constructing $P^{-k} : U \to \mathbb{C}$ for all positive k. Since for any k, $P^{-k}(U)$ cannot cover all of $\mathbb{C}$, indeed $P^{-k}(U)$ omits the basin of attraction of ∞ which contains $\{z||z| > R\}$ for some sufficiently large R. However, the family of maps P^{-k} is not normal on U, as z_0 is a repelling fixed point for each. So by Montel's Theorem we must have that $\bigcup_{k=0}^{\infty} P^{-k}$ covers $\mathbb{C}$ minus at most one point. This contradiction establishes the result. ■

3.2 Role of the critical orbit

From the definition of the Mandelbrot set M we know that M consists of those c values for which the orbit of the critical point 0 under Q_c is bounded. Theorem [3.1] says that the immediate basin of attraction of an attracting cycle contains a critical point. Since 0 is the only critical point of Q_c, 0 is in the basin of attraction of an attracting cycle. That is, orbit of 0 under Q_c converges to the cycle. So the Mandelbrot set M contains all c values for which Q_c has an attracting cycle. The complement of the Mandelbrot set consists of those c values for which $Q_c^n(0) \longrightarrow \infty$.

3.3 Bifurcation in $Q_c(z)$ and visible bulbs of the Mandelbrot set

We will find values of c for which Q_c has an attracting fixed point. Suppose α and β be two fixed points of $Q_c(z) = z^2 + c$ in $\mathbb{C}$. So $Q_c(\alpha) = \alpha$ and $Q_c(\beta) = \beta$, that is, α and β are the roots of $z^2 - z + c = 0$. Thus

$$\alpha + \beta = 1, \quad \alpha\beta = c.$$

So $Q_c'(\alpha)+Q_c'(\beta) = 2(\alpha+\beta) = 2$. This shows that not both α and β can be attracting. It follows that Q_c can have at most one attracting fixed point, and the condition that one of the fixed points, say α is attracting is $2|\alpha| = |Q_c'(\alpha)| < 1$, and $c = \alpha - \alpha^2$. So the set of c we are seeking is just the image of the disc $\{\alpha : |\alpha| < \frac{1}{2}\}$ under the map

$z \mapsto z - z^2$. This map is fgh, where $h(z) = z - \frac{1}{2}$, $g(z) = z^2$, $f(z) = \frac{1}{4} - z$. The set c is the cardioid illustrated in the following figure.

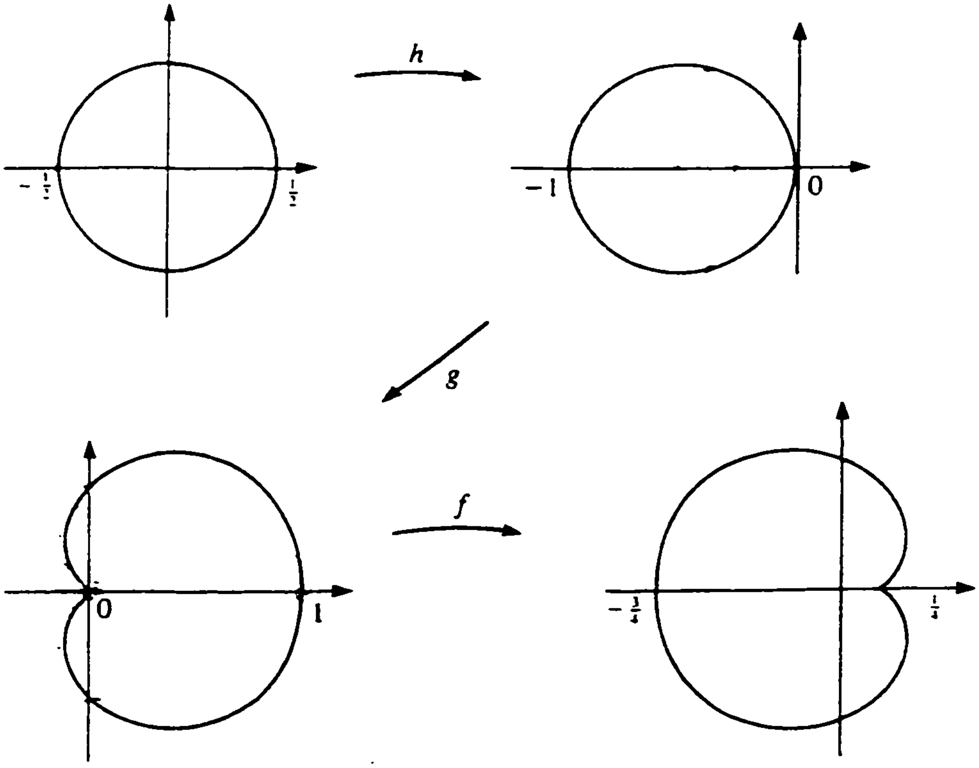


Figure 3.1

Now we want to find all c values for which Q_c has an attracting 2-cycle. The equation for fixed points of Q_c^2 is $Q_c^2(z) = z$. Thus, we have to solve

$$Q_c^{(2)}(z) - z = 0 \Rightarrow (z^2 + c)^2 + c - z = 0.$$

$$\Rightarrow z^4 + 2cz^2 - z + c^2 + c = 0.$$

Since α and β are fixed by Q_c, so they are also fixed by $Q_c^2(z)$. Thus $Q_c(z) - z$ divides $Q_c^2(z) - z$. So we have $z^2 + z + c + 1 = 0$. Let u and v be the roots of $z^2 + z + c + 1 = 0$ and discriminant of $z^2 + z + c + 1 = 0$ is $\Delta = 1 - 4(c+1)$. Roots of Q_c^2 are $\frac{-1 \pm \sqrt{1-4(c+1)}}{2}$.

Let $u = \frac{-1+\sqrt{1-4(c+1)}}{2}$ and $v = \frac{-1-\sqrt{1-4(c+1)}}{2}$. We can write

$$Q_c^2(z) - z = (z^2 - z + c)(z^2 + z + 1 + c) = (z - \alpha)(z - \beta)(z - u)(z - v).$$

For an attracting 2 cycle we have

$$Q_c(u) = v, Q_c(v) = u, u \neq v, \text{ together with } |(Q_c^2)'(u)| < 1, \ |(Q_c^2)'(v)| < 1,$$

so u and v are attracting fixed points of Q_c^2. Now the condition for attracting 2-cycle is $|(Q_c'(u))Q_c'(v))| < 1$. This implies

$$\left|\left(-1 + \sqrt{1 - 4(c+1)}\right)\left((-1 - \sqrt{1 - 4(c+1)}\right)\right| = |1-(1-4(c+1)| < 1 = |4(1+c)| < 1.$$

Hence $|1 + c| < \frac{1}{4}$. So the set c we are looking for is the disc $\{c : |1 + c| < \frac{1}{4}\}$. This

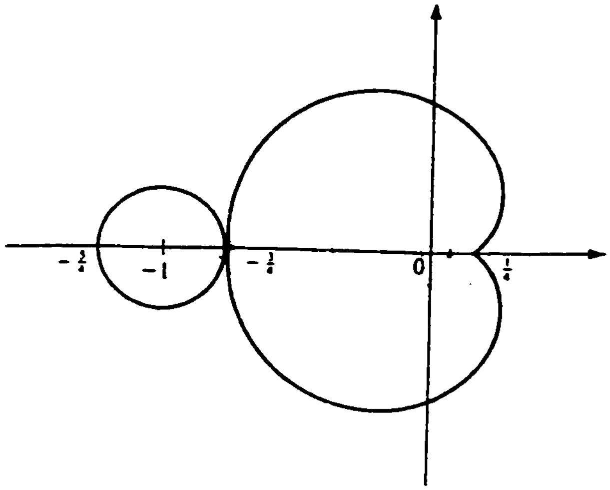


Figure 3.2

disc is illustrated, together with the cardioid obtained previously in the above figure. The point of tangency of the disc and the cardioid is $-\frac{3}{4}$. Similarly we can show the set of c values for which Q_c has an attracting 3- cycle are the bulbs in the top and bottom of the cardioid. We can continuing this process to get the all visible bulbs of the Mandelbrot set. We shall now describe the change in dynamics as c moves from the cardioid to the disc through the value $-\frac{3}{4}$. When c is in the cardioid Q_c has an attracting fixed point α and a repelling 2—cycle $\{u, v\}$. As $c \to -\frac{3}{4}$, the points α. u and v converges to a common value $-\frac{1}{2}$ and when $c = -\frac{3}{4}$, α, u and v coincide at a neutral fixed point of $Q_c(z)$. Since fixed points of $z^2 - z - c = 0$ are $\alpha = \frac{1-\sqrt{1+4c}}{2}$ and $\beta = \frac{1+\sqrt{1+4c}}{2}$. As c now moves into the disc, these points separate again, this time with α being a repelling fixed point and $\{u, v\}$ being an attracting $2-cycle$: thus during this process the attracting nature of α has been transferred to pair $\{u, v\}$. This process is repeated: as parameter c moves further along negative part of x- axis, $\{u, v\}$ losing its attracting nature to a 4- cycle, then to an 8- cycle, and so on. This is known as period doubling and one can easily verify it on a computer by examining the limit of $Q_c^n(0)$ for various values of c.

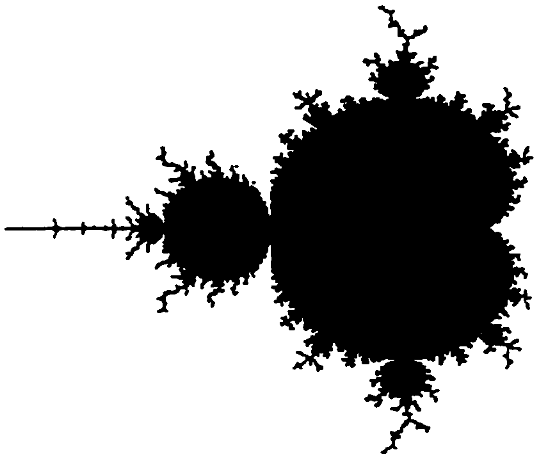


Figure 3.3

Proposition 3.5 *Let $Q_c(z) = z^2 + c$, $c_n = Q_c^n(0)$ and $\alpha = \frac{1-\sqrt{1-4c}}{2}$, $\beta = \frac{1+\sqrt{1-4c}}{2}$ Then*

(i) if $c < -2$, then $c_n \geq 2 + n|c+2|$ when $n \geq 2$;

(ii) if $-2 \leq c < 0$, then Q_c maps $[-\beta, \beta]$ into itself;

(iii) if $0 \leq c \leq \frac{1}{4}$, then Q_c maps $[0, \alpha]$ into itself;

(iv) if $c > \frac{1}{4}$, then $c_n \geq n(c - \frac{1}{4})$;

Moreover, the intersection of the Mandelbrot set with the real line is $[-2, \frac{1}{4}]$.

Proof. (i) Suppose $c < -2$ and $n \geq 2$. Let $n = 2$ then

$$c_2 = Q_c^2(0) = Q_c(Q_c(0)) = Q_c(c) = c^2 + c \geq 2 + 2|c+2|.$$

So it is true for $n = 2$. Assume that it is true for k. Then,

$$c_k \geq 2 + k|c+2|.$$

Now we need to show that it is true for $k+1$.

We have

$$c_{k+1} = c_k^2 + c \geq 4 + 4k|c+2| + k^2|c+2|^2 + c$$

$$= 2 + (k+1)|c+2| + (4k - (k+1) - 1)|c+2| + c + 2 + |c+2| + k^2|c+2|^2$$

$$> 2 + (k+1)|c+2|,$$

since $(4k - (k+1) - 1)|c+2| + c + 2 + |c+2| + k^2|c+2|^2 > 0$. The last inequality follows from $k \geq 2$ and $t + |t| \geq 0$. So by mathematical induction it is true for any $n \geq 2$, i.e., $c_n \geq 2 + n|c+2|, \forall n \geq 2$.

(ii) Suppose $-2 \leq c < 0$ and $Q_c : [-\beta, \beta] \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$, where $\beta = \frac{1+\sqrt{1-4c}}{2}$ which is equivalent to $\beta^2 + c = \beta$. Let $z \in [-\beta, \beta] \Rightarrow |z| \leq \beta \Rightarrow z^2 \leq \beta^2$. Thus $Q_c(z) = z^2 + c \leq \beta^2 + c = \beta$. Hence $Q_c(z) \leq \beta$.

Now $\beta = \frac{1+\sqrt{1-4c}}{2}$ and $-2 \leq c < 0$. This implies $1 < \beta \leq 2$. Thus, $z \in [-\beta, \beta] \Rightarrow 1 < z^2 \leq 4$. Now $-2 \leq c < 0$, $1 < z^2 \leq 4$, $-2 \leq -\beta < -1$, which implies that $z^2 + c \geq -\beta$, $\forall z \in [-\beta, \beta]$. So $-\beta \leq Q_c(z)$. Combining this two result we get

$$Q_c(z) \in [-\beta, \beta], \quad \forall z \in [-\beta, \beta].$$

(iii) Suppose $0 \leq c \leq \frac{1}{4}$ and $Q_c : [0, \alpha] \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$, where $\alpha = \frac{1-\sqrt{1-4c}}{2}$. Thus, $0 \leq c \leq \frac{1}{4} \Rightarrow 0 \leq \alpha \leq \frac{1}{2}$ and hence,

$$z \in [0, \alpha] \Rightarrow 0 \leq z^2 \leq \frac{1}{4} \text{ and } 0 \leq z^2 \leq \frac{1}{4},\ 0 \leq c \leq \frac{1}{4}.$$

Thus, $0 \leq z^2 + c \leq \frac{1}{2}$. Hence, $Q_c(z) \in [0, \alpha]$.

(iv) Suppose $c > \frac{1}{4}$. Let $S(n) : c_n \geq n(c - \frac{1}{4})$. So

$$S(1) : c_1 = Q_c(0) = c_n \geq (c - \frac{1}{4}).$$

Thus $S(n)$ is true for $n = 1$. Suppose $S(k)$ is true then $c_k \geq k(c - \frac{1}{4})$. We have

$$c_{k+1} = c_k^2 + c \geq k^2(c - 1/4)^2 + c$$

$$= k^2(c-1/4)^2 + (c - 1/4) + 1/4 \geq (k+1)(c - 1/4),$$

since $k^2t^2 - kt + 1/4 \geq 0$ for all t. So $S(n)$ is true for $k+1$. By mathematical induction $Q_c(n)$ is true $\forall n \geq 1$. Hence $c_n \geq n(c - \frac{1}{4})$. From the definition of the Mandelbrot set we conclude that $M \bigcap \mathbb{R} = [-2, \frac{1}{4}]$. ■

Theorem 3.2 *The Mandelbrot set M is contained in the closed disk of radius 2. i.e.,*

$$M \subset \{c : |c| \leq 2\}.$$

Proof. Suppose that $|c| > 2$. Then, if $|z| \geq |c|$ we have

$$|Q_c(z)| = |z^2 + c| = |z|\left|z + \frac{c}{z}\right| \geq |z|\left(|z| - \frac{|c|}{|z|}\right) \geq |z|(|c| - 1) = |z|r,$$

where $r = |c| - 1 > 1$. Now $Q_c(0) = c$, $|Q_c^2(0)| = |Q_c(c)| = |c^2 + c| \geq |c|r$. So it is true for $k = 2$. Suppose that it is true for k, i.e, $Q_c^k(0) \geq |c|r^{k-1}$. We need to show that $Q_c^{k+1}(0) \geq |c|r^k$. Now

$$|Q_c^{k+1}(0)| = |Q_c Q_c^k(0)| \geq |Q_c^k(0)|r \geq |c|r^k.$$

So by mathematical induction $|Q_c^k(0)| \geq |c|r^k$ for all k. Since $r > 1$ so $|Q_c^k(0)| \geq |c|r^k \to \infty$ as $n \to \infty$. Consequently $c \notin M$. ■

3.4 Algorithm for the Mandelbrot set

The above discussion in Section 3.3 gives us an algorithm for computing the Mandelbrot set. We consider a rectangular grid in the c-plane. As we know, the Mandelbrot set M is contained in the circle of radius 2 centered at the origin, so we will always assume that our grid contained inside the square centered at the origin whose sides have length 4. For each point c in this grid, we compute the corresponding orbit of 0 under Q_c and ask whether or not this orbit tends to infinity. If the orbit does not escape, then our original point is in M, so we will color the original point black. If the orbit escapes then we leave the original point white. More precisely we have the following algorithm: Choose a maximum number of iterations N. For each point c in a grid, compute the first N points on the orbit of 0 under Q_c. If $|Q_c^i(0)| > 2$ for some $i \leq N$, then stop iterating and color c white. If $|Q_c^i(0)| \leq 2$ for some $i \leq N$, then color c black.

3.5 Periods of the bulb

The bulbs which are directly attached to the main cardioid are called the primary bulbs. In this section we will only discuss the primary bulbs. From the discussion in Section 3.3 we see that for each c values such that Q_c has an attracting k-cycle and we have a bulb which is a part of the Mandelbrot set M. So we can assign a number for each bulb of the Mandelbrot set. This number is called the period of the corresponding bulb. For example, the biggest bulb in the left of the cardioid has period 2, because this bulb corresponds to all c values for which Q_c has an attracting 2 cycle. Another example, we see is the topmost bulb on the cardioid has period 3, because this bulb corresponding to all c values for which Q_c has an attracting 3 cycle. Again consider the period 5 bulbs. Because these bulbs correspond to all c values for which

Q_c has an attracting 5 cycle. Similarly we can find periods 7, 12 and other primary bulbs. The following figure (figure 3.4) shows various type of periods of bulbs.

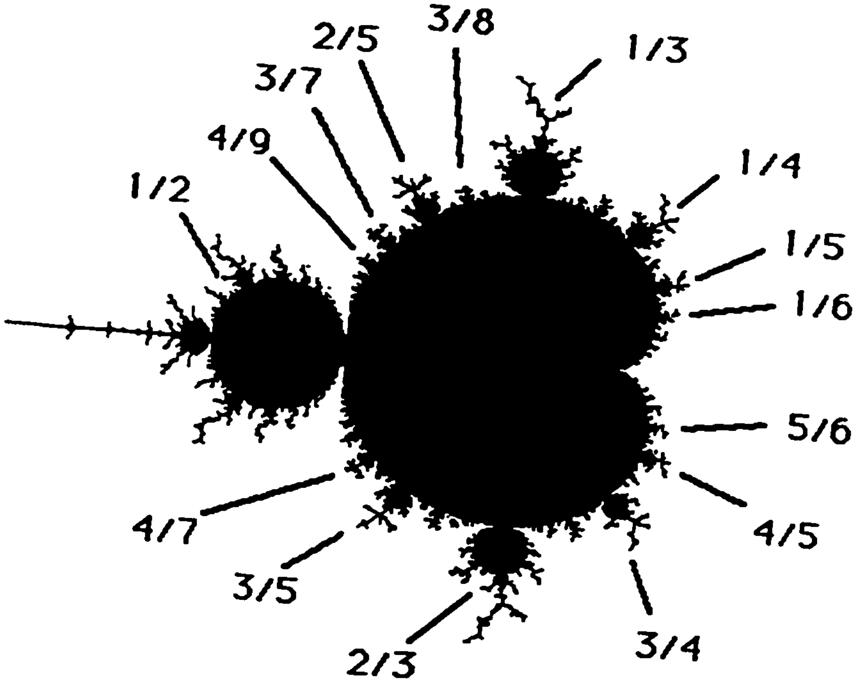


Figure 3.4

Now we can also read off the period of the primary bulbs by simply counting the spokes of the antenna attached to this decoration [$D2$]. For a rough proof consider the following example. Note that each bulb features a large antenna that contains a junction point from which a number of spokes emanate. It is a fact that the number of those spokes is exactly the period of the period of the bulb. When counting these spokes, it is important to count the main spoke (the spoke that is attached directly to the bulb), this spoke is called the principal spoke. Consider a bulb which has attracting 3 cycle. Then we see that this bulb has 3 spokes of antenna attached to this bulb and one spoke is directly attached to this bulb. Now counting clockwise direction from the main spoke we have 3 spokes of antenna. So this bulb has period 3. Consider a bulb which has attracting 4 cycle. We see that this bulb has 4 spokes of antenna attached to this bulb and one spoke is directly attached to this bulb. Now counting clockwise direction from the main spoke we have 4 spokes of antenna. So this bulb has period 4. The following figure (figure 3.5) shows the antena and spokes of period 3 and period 4 bulbs.

Period 3 bulb

Period 4 bulb

Figure 3.5

Another example, consider a bulb which has attracting 5 cycle. Then, we see that this bulb has 5 spokes of antenna attached to this bulb and one spoke is directly attached to this bulb. Now counting clockwise direction form the main spoke we have 5 spokes of antenna. So this bulb has period 5. Similarly we can find periods of other bulbs. In the following figure (figure 3.6) we have displayed several of the primary decorations and their periods.

Period 7 bulb **Period 5 bulb**

Figure 3.6

3.6 Rotation numbers

In Section 3.5, we saw that for each primary bulb we have a number q which is the period of the bulb. In this section we will associate a number p/q with each bulb where q is the period of the bulb. The rational number p/q is called a rotation number for the bulb. We have to determine the number p.

There are some methods involving Julia set J_c to determine the number p. In this thesis we want to determine the number p without computing the Julia set. The method involves only looking at the antenna attached to the bulb. To find p, note that there are q spokes emanating from the junction point in the main antenna attached to the bulb. Locate the shortest of these. for most bulbs this shortest spoke is located p/q revolutions in the counterclockwise direction from the main spoke. Now this is not always true if we measure the length using the usual Euclidean distance. Rather we should use a distance that assigns a shorter length to the spoke closer to the main spoke.

Consider the period 3 bulb which has 3 spokes of antenna emanating from the junction point and one spoke (principal spoke) directly attached to this bulb. We see that the shortest spoke is located 1/3 of a turn in the counterclockwise direction from the principal spoke. So this bulb is called 1/3 bulb.

1/3 bulb **3/7 bulb**

Figure 3.7

Consider the period 7 bulb which has 7 spokes of antenna emanating from the junction point and one spoke (principal spoke) directly attached to this bulb.We see that the shortest spoke located 3/7 of a turn in the counter clockwise direction from the principal spoke. So this is called 3/7 bulb.

11/12 bulb **5/12 bulb**

Figure 3.8

Similarly we can find 11/12 bulb and 5/12 bulb and so on. The following figure shows that various type of bulbs:

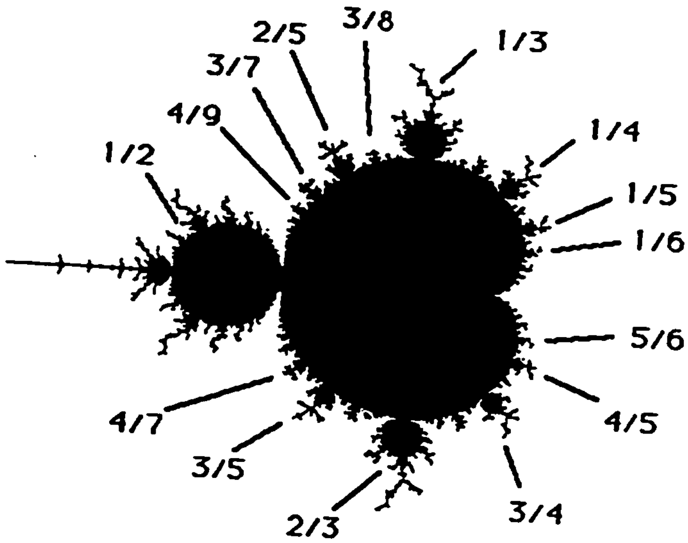


Figure 3.9

Chapter 4

Geometry of bulbs in the Mandelbrot set, Farey tree and Fibonacci sequence

4.1 Introduction

So far we have discussed the Mandelbrot set in Chapter 3. It turns out that the Mandelbrot set M consists of c values for which the orbit of 0 under the quadratic map $Q_c(z) = z^2 + c$ is bounded. In the Mandelbrot set, the bulbs attached directly to the main cardioid are called the primary bulbs. Each of these bulbs has a rotation number p/q. Recently R.L. Devaney and M.Moreno Rocha $[DM]$, R.L.Devaney $[D2]$ studied the geometry of primary bulbs in the Mandelbrot set. In this chapter we discuss the method of finding period of bulbs and rotation number of bulbs, size of limbs using the notion of external rays.

4.2 Farey addition, Farey tree, Fibonacci sequence

The Mandelbrot set features a basic cardioid shape from which hang numerous "bulbs" or "decorations" (Figure 4.1). Each of these bulbs is a large disk that is directly attached to the main cardioid, together with numerous other smaller decorations and a prominent "antenna".

Figure 4.1

In Chapter 3, we showed that each of these large disks turns contains c values for which Q_c admits an attracting cycle with period q and rotation number p/q. That is, the attracting cycle of Q_c tends to rotate about a central fixed point, turning on average p/q bulb revolutions at each iteration. For this reason this bulb is called the p/q bulb. Each of the c values in this bulb has essentially the same dynamical behavior.

In Chapter 3 we showed that we can recognize the p/q bulb from the geometry of the bulb itself. That is we can read off dynamical information from the geometric information contained in the Mandelbrot set. In fact we can recognize the p/q bulb by locating the "smallest" spoke in the antenna and determining its location relative to the main spoke. This is our first folk theorem that we want to discuss.

Now consider 1/2 bulb and topmost 1/3 bulb. In between these two bulbs there are infinitely many bulbs directly attached to the main cardioid. Now we want to find the largest bulb of all other bulbs between 1/2 and 1/3 bulb. To find this we need the following definition.

Definition 4.1 *(Farey addition) Let $p_1/q_1, p_2/q_2 \in [0,1]$. If $p_1q_2 - p_2q_1 = \pm 1$ then p_1/q_1 and p_2/q_2 are called Farey neighbors. The Farey addition is defined by*

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}.$$

i.e., we simply add the corresponding fractions just the way we always wanted to add them, namely by adding the numerators and denominators.

Thus,

$$\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}.$$

Our second folk theorem that we want to discuss will state that 2/5 bulb is the largest bulb of the bulbs between 1/2 bulb and 1/3 bulb. That is the largest bulb between two bulbs is the bulb which we get from the Farey addition. We will show that the largest bulb of bulbs between 1/2 and 1/3 is 2/5 bulb.

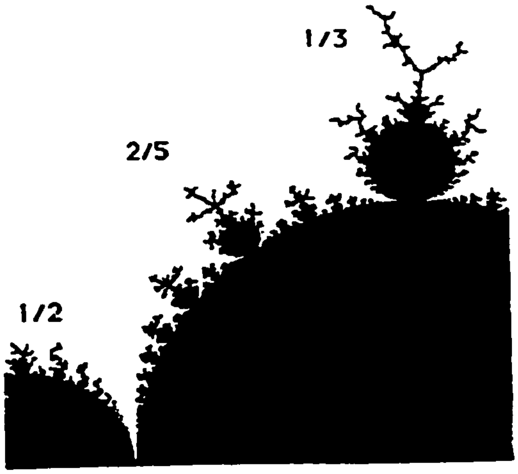


Figure 4.2

Similarly the largest bulb of bulbs between 1/3 and 2/5 is 3/8 bulb. Similarly the largest bulb of bulbs between 3/8 and 2/5 is 5/13 bulb. Continuing in this way we get a special sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \dots$$

whose numerators and denominators correspond to the Fibonacci sequence. We will also discuss this connection.

Lemma 4.1 *Every rational number* $p/q \in (0,1)$, $(p,q) = 1$, *can be expressed uniquely as a Farey sum*

$$\frac{p}{q} = \frac{p_1}{q_1} \ominus \frac{p_2}{q_2}.$$

Proof. Arrange q points around the circle and mark a distinguised first point.

Move counterclockwise around the circle, each time marking the point which is p points beyond the one previously marked. Continue as long as no two adjacent points are marked. Let q_1 be the number of marked points. By rotational symmetry, the next marked point would be adjacent to the first one, so that

$$q_1 p = p_1 q \pm 1$$

for some integer p_1, which satisfies $p_1 < p$ since $q > q_1$. If we set Euclidean algorithm.

$$p_2 = p - p_1, \quad \textit{and} \quad q_2 = q - q_1,$$

then

$$q_1 p_2 - p_1 q_2 = q_1 p - p_1 q = \pm 1.$$

$$\frac{p}{q} = \frac{p_1}{q_1} \oplus \frac{p_2}{q_2}.$$

It remains to prove uniqueness. Suppose p_1'/q_1' and p_2'/q_2' are a second pair with p/q as the Farey sum; without loss of generality we assume their order is the same as that of the non-primed pair of fractions. Then

$$pq_1 - p_1 q = pq_1' - p_1' q = \pm 1$$

and $p|q_1 - q_1'| = q|p_1 - p_1'|$. Since p and q coprime, and $|q_1 - q_1'| < q$, the last equality is impossible unless $q_1 - q_1' = p_1 - p_1' = 0$. ■

Definition 4.2 *(The Farey Tree) The Farey tree is a tree which contains all of the rationals between* 0 *and* 1.

Let us understand this notion with an example: let 0/1 and 1/1 be a pair of neighbors (rationals). Using the Farey addition we get a Farey child

$$\frac{0}{1} \oplus \frac{1}{1} = \frac{0+1}{1+1} = \frac{1}{2}$$

which is the rational between 0/1 and 1/1 and whose denominators is the smallest [JF]. We say that in this case 0/1 and 1/1 are Farey parent of the Farey child 1/2. Thus we get

$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}.$$

Now if we consider 0/1, and 1/2 as Farey parent we get a new Farey child 1/3 and if we consider 1/2 and 1/1 as Farey parent we get another new Farey child 2/3. In this stage we get

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}.$$

In this next stage we get

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1},$$

and so on. From lemma 4.1 we know that the Farey tree contains all rationals between 0 and 1.

Recall from Chapter 3 that the visible bulbs in M correspond to c values for which Q_c has an attracting cycle of some given period. For example, the main central cardioid in M consists of c values for which Q_c has an attracting fixed point. This can be seen by solving for the fixed points of $Q_c(z) = z^2 + c$ that are attracting:

$|Q_c'(z)| = |2z| < 1$. Solving these equations simultaneously we see that the boundary of this region is given by $c = z - z^2$, where $z = \frac{1}{2}e^{2\pi i\theta}$. That is,

$$c(\theta) = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$$

parametrizes the boundary of the cardioid with $0 \leq \theta \leq 1$. At $c(\theta)$, $Q_c(\theta)$ has a fixed point that is neutral and the derivative of $Q_c(\theta)$ at this point is $e^{2\pi i\theta}$. For each rational value of θ, there is a bulb tangent to the main cardioid at $c(\theta)$. For c values in the bulb attached to the cardioid at $c(p/q)$, Q_c has an attracting cycle of period q. We call this bulb the p/q bulb attached to the main cardioid and denoted it by $B_{p/q}$.

It is known that as c passes from the main cardioid through $c(p/q)$, and into $B_{p/q}$, Q_c undergoes a p/q bifurcation. By this we mean when c lies in the main cardioid near $c(p/q)$, Q_c has an attracting fixed point with a nearby repelling cycle of period q. At $c(p/q)$ the attracting fixed point and repelling cycle merge to produce a neutral fixed point with derivative $e^{2\pi i\frac{p}{q}}$. When c lies in $B_{p/q}$, Q_c now has an attracting cycle of period q and a repelling fixed point. When $c = c(p/q)$, the local(linearized) dynamics are given by rotation through angle $2\pi(p/q)$. As a consequence, for nearby $c \in B_{p/q}$, the attracting cycle rotates about the repelling fixed point by jumping approximately $2\pi(p/q)$ radians at each iteration.

The doubling map and the rotation map is the main tool of the discussion of the geometry of bulbs. We discuss the doubling map in the next section and the rotation map in Section 4.7.

4.3 Angle doubling mod 1

Definition 4.3 *The doubling function, which is defined on the circle considered as the reals modulo one, is given by* $D(\theta) = 2\theta$ *mod* 1, *i.e.,*

$$D(\theta) = \begin{cases} 2\theta & \text{for } \theta \in [0, 1/2), \\ \\ 2\theta - 1 & \text{for } \theta \in [1/2, 1). \end{cases}$$

Lemma 4.2 *The angle* θ *is periodic under doubling function* D *if and only if* θ *is a rational of the form* p/q *with* q *odd.*

Proof. Let $\theta = p/q$, then $D(p/q) = 2p/q$, or $D(p/q) = (2p - q)/q$. Similarly, we have $D^n(\theta) = 2^n\theta - integer$. Assume that θ is eventually periodic then we have

$$D^m(\theta) = D^k(\theta) \qquad m \neq k, \quad or$$

$$2^m\theta - integer_1 = 2^k\theta - integer_2.$$

Since there is only finite number of possible numerators θ is eventually periodic. Thus

$$\theta = \frac{integer_1 - integer_2}{2^m - 2^k},$$

which means that θ is a rational number. Thus every eventually periodic point is rational.

Now we have to prove that periodic points are k/odd. Assume that θ = odd/even then θ is not periodic. Because

$$D(\theta) = 2(odd)/even \quad or \quad 2(odd)/even - 1.$$

So

$$\frac{odd}{even/2} \quad or \quad \frac{odd - even/2}{even/2}$$

and all further images have denominator $even/2$. Thus θ cannot be periodic.

Now if $\theta = k/q$ with q odd then θ is periodic. Now we want to prove that every image $D^n(k/q)$ is of the form p/q , where $0 \leq p \leq q$ and we also proved that θ is periodic. Let

$$D^n(k/q) = D^j(k/q) = y, \qquad n > j$$

and let (n, j) be the smallest such pair. Then

(i) if $j = 0$ then θ is periodic.

(ii) if $j > 0$ then θ is not periodic.

If $j > 0$ we can consider $D^{n-1}(k/q)$ and $D^{j-1}(k/q)$. Since the pair (n, j) is minimal so $D^{n-1}(\theta)$ and $D^{j-1}(\theta)$ must be different and be on different sides of $1/2$. Let $D^{n-1}(\theta) = p_1/q$ and $D^{j-1}(\theta) = p_2/q$. Then

$$2p_1/q = 2p_2/q - 1 \quad or \quad 2p_1 = 2p_2 - q.$$

This implies q is even. Which is a contradiction. Thus the only possibilities is (i) and θ is periodic with q odd. ■

Remark 4.1 *The rationals with odd denominator are periodic and the rational with even denominator are all strictly preperiodic.*

The following are some examples of periodic, pre periodic and eventually fixed points.

Example 4.1 The D-orbit of 2/3 is

$$\frac{2}{3} \to \frac{1}{3} \to \frac{2}{3} \to \dots,$$

which has period 2. Similarly the D- orbit of 1/3 is

$$\frac{1}{3} \to \frac{2}{3} \to \frac{1}{3} \to \dots,$$

which has period 2.

Example 4.2 The rational 3/7 has period 3 under doubling :

$$\frac{3}{7} \to \frac{6}{7} \to \frac{5}{7} \to \frac{3}{7} \to \dots,$$

Similarly 1/7 and 2/7 have period 3 under doubling:

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \to \dots,$$

$$\frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \to \frac{2}{7} \to \dots,$$

Example 4.3 The rational 2/5 has period 4:

$$\frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5} \to \frac{2}{5} \to \dots,$$

Similarly 1/15, 3/15 and 7/15 have period 4:

$$\frac{1}{15} \to \frac{2}{15} \to \frac{4}{15} \to \frac{8}{15} \to \frac{1}{15} \to \dots,$$

and

$$\frac{3}{15} \to \frac{6}{15} \to \frac{12}{15} \to \frac{9}{15} \to \frac{3}{15} \to \dots,$$

and

$$\frac{7}{15} \to \frac{14}{15} \to \frac{13}{15} \to \frac{11}{15} \to \frac{7}{15} \to \dots,$$

Example 4.4 The rational 9/31 has period 5 :

$$\frac{9}{31} \to \frac{18}{31} \to \frac{5}{31} \to \frac{10}{31} \to \frac{20}{31} \to \frac{9}{31} \ldots,$$

Similarly the rational 1/31 has period 5:

$$\frac{1}{31} \to \frac{2}{31} \to \frac{4}{31} \to \frac{8}{31} \to \frac{16}{31} \to \frac{1}{31} \ldots,$$

and 21/31 has period 5:

$$\frac{21}{31} \to \frac{11}{31} \to \frac{22}{31} \to \frac{13}{31} \to \frac{26}{31} \to \frac{21}{31} \ldots,$$

Example 4.5 Consider 1/10 ,where q is even.The rationals with even denominator are eventually periodic but not periodic. The D-orbit of 1/10 is

$$\frac{1}{10} \to \frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5} \to \ldots,$$

So 1/10 lies on an eventual 3-cycle.

Example 4.6 The rational with even denominator 1/4 is eventually fixed :

$$\frac{1}{4} \to \frac{1}{2} \to 0 \to 0 \to \ldots,$$

4.4 Itinerary and Binary Expansion:

Using the doubling function on the circle we can get the binary expansion of θ where $0 \leq \theta < 1$ by noting the itinerary of θ in the circle relative to D.

To define the itinerary, we denote the upper semicircle $0 \leq \theta < \frac{1}{2}$ by I_0 and the lower semicircle $\frac{1}{2} \leq \theta < 1$ by I_1. Given θ, we attach an infinite string of $B(\theta)$ as follows:

$$B(\theta) = (s_0 s_1 s_2 \ldots),$$

where

$$s_j = \begin{cases} 0 & \text{when } D^j(\theta) \in I_0, \\ 1 & \text{when } D^j(\theta) \in I_1 \end{cases} \tag{4.1}$$

That is ,we simply watch the orbit of θ in the circle under doubling and assign 0 or 1 to the itinerary whenever $D^j(\theta)$ lands in the arc I_0 or I_1.

Lemma 4.3 *The itinerary $B(\theta)$ is the binary expansion of θ.*

Proof. Let

$$\theta = \sum \frac{\theta_i}{2^i} = \frac{\theta_1}{2} + \frac{\theta_2}{2^2} + \frac{\theta_3}{2^3} + \cdots,$$

where θ_i are either 0 or 1, be the binary expansion of $\theta \in (0, 1)$.

Apply the doubling function D then

$$D(\theta) = \theta_1 + \frac{\theta_2}{2} + \frac{\theta_3}{2^2} + \cdots$$

Here θ_1 is the integer part which is either 0 or 1. Now

$$\theta^* = \frac{\theta_2}{2} + \frac{\theta_3}{2^2} + \cdots$$

and

$$D(\theta^*) = \theta_2 + \frac{\theta_3}{2} + \frac{\theta_4}{2^2} + \frac{\theta_5}{2^3} + \cdots.$$

Here θ_2 is the integer part which is again 0 or 1. We proceed in this way and we get

$$B(\theta) = \theta_1\theta_2\theta_3\cdots$$

is the itinerary of θ. ■

Example 4.7 If $\theta = 2/3$, then $\theta \in I_1$, while $D(\theta) \in I_\circ$ and $D^2(\theta) = \theta$. Hence $B(2/3)$ is the repeating sequence $\overline{10}$, which of course is the binary expansion of 2/3. Similarly, $B(3/7) = \overline{011}$, while $B(2/5) = \overline{0110}$,

$B(9/31) = \overline{01001}$

and $B(10/31) = \overline{01010}$

4.5 External Rays

In order to make precise the folk theorems mentioned in the introduction, we recall some beautiful results of Douady and Hubbard [DH2] concerning the external rays of the Mandelbrot set.

Let $E = \{z : |z| > 1\}$ denote the exterior of the unit circle in the plane. According to Douady and Hubbard, there is a unique analytic isomorphism Φ that maps E to the exterior of the Mandelbrot set. The mapping Φ takes ∞ to ∞ and positive reals to positive reals. The mapping is the uniformization of the exterior of the Mandelbrot, or the exterior Riemann map.

The importance of Φ stems from the fact that the image under Φ of the straight rays $\theta =$ constant in E have dynamical significance. In the Mandelbrot set, we define the external ray with external angle θ_0 to be the Φ- image of $\theta = \theta_0$. It is known that an external ray whose angle θ_0 is rational "lands" on M. That is $\lim_{r \to 1} \Phi(re^{2\pi i \theta_0})$ exists and is a unique point on the boundary of M. This c value is called the landing point of the ray with angle θ_0.

The ray with angle 0 lies on the real axis and lands on M at the cusp of the main cardioid, namely $c = 1/4$. Also the ray with angle $1/2$ lies on the negative real axis and lands on M at the tip of the "tail" of M, which can be shown to be $c = -2$. Consider now the interior of M. The interior consists of infinitely many simple connected regions. A bulb of M is a component of the interior of M in which each c-value corresponds to a quadratic function that admits an attracting cycle. The period of this cycle is constant over each bulb. In many cases, a bulb is attached to a component of lower period at a unique point called the root point of the component.

Theorem 4.1 [$DH2$] *Suppose a bulb B consists of c-values for which the quadratic map has an attracting q- cycle. Then the root point of this bulb is the landing point of exactly 2 rays, and the angles of each of these rays have period q under doubling.*

Example 4.8 The angles of the external rays of M determine the ordering of the bulbs in M. The large bulb directly attached to the left of the main cardioid is the $1/2$ bulb, so two rays with period 2 under doubling must land there. Now the only angles with period 2 under doubling are $1/3$ and $2/3$, so these are the angles of rays that land at the root point of $B_{1/2}$. In this case root point of $B_{1/2} = -3/4$.

Example 4.9 Consider the 1/3 bulb atop the main cardioid. This bulb lies "between" the rays 0 and 1/3. There are only two angles between 0 and 1/3 that have period 3 under doubling, namely 1/7 and 2/7, so these are the rays that land at the root point of $B_{1/3}$.

Example 4.10 Consider the 2/5 bulb.The 2/5 bulb lies between the 1/3 and 1/2 bulbs. Hence the rays that land at $c(2/5)$ must have period 5 under doubling and lie between 2/7 and 1/3.The only angles that have this property are 9/31 and 10/31, so these rays must land at $c(2/5)$. we have seen in the above figure.

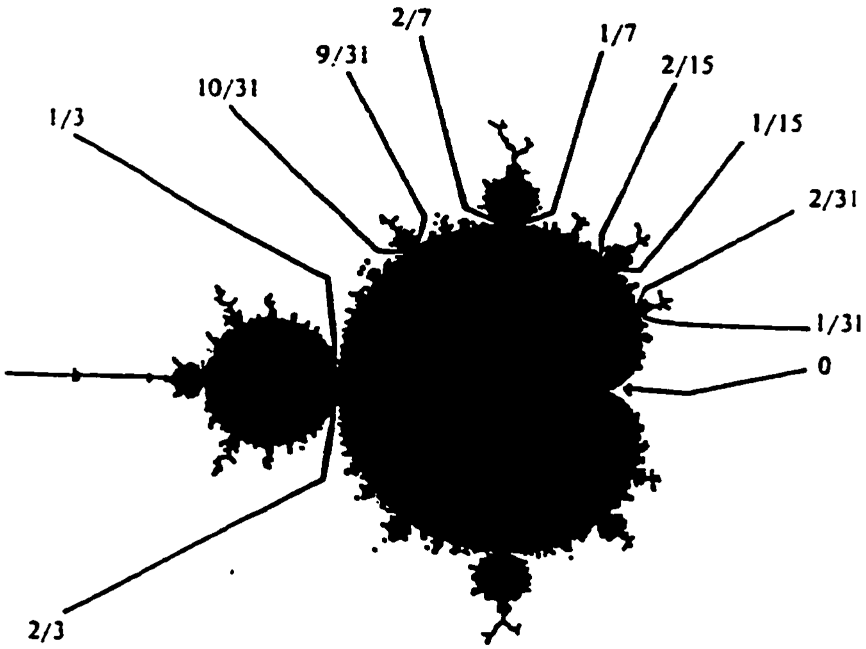


Figure 4.3

4.6 Idea of Limbs

The idea of external rays allow us to measure the "largeness" or "smallness" of portions of the Mandelbrot set. Suppose we have two rays with angles θ_- and θ_+ that both land at a point c_* in the boundary of M. Let M^* be the component of M which touches the main cardiod at c_*. Then, by the isomorphism Φ, all rays with angles between θ_- and θ_+ must approach the component of $M^* \setminus \{c_*\}$ cut off by θ_- and θ_+. Thus, it is natural to measure the size of this portion M^* of M by the length of the interval $[\theta_-, \theta_+]$. The root point of the p/q bulb of M divides M into two sets. The component M^* containing the p/q bulb is called the p/q limb. We can then measure the size of the p/q limb if we know the external rays that land on the root point of the p/q bulb.

4.7 Rays landing on the p/q bulb

In order to make the notion of 'large' or 'small' precise in the statement of the folk theorems, we need a way to determine the angles of the rays landing at the root point of $B_{p/q}$. We denote the angles of these two rays in binary by $\overline{l_\pm(p/q)}$, where $\overline{l_-(p/q)} < \overline{l_+(p/q)}$. We call $\overline{l_-(p/q)}$ the lower angle of $B_{p/q}$ and $\overline{l_+(p/q)}$ the upper angle.

As we will see , $l_\pm(p/q)$ is a string of q digits (0 *or* 1) and so $\overline{l_\pm(p/q)}$ denotes the infinite repeating sequence whose basic block is $l_\pm(p/q)$. Douady and Hubbard

[$DH1$] have a geometric method involving Julia sets to determine these angles. Our method is more combinatorial and resembles algorithms due to Aleta [A], La Vaurs [LV], and Lau and Schleicher[LS].

To describe this algorithm, let $R_{p/q}(\theta)$ denote rotation of the unit circle through p/q turns, i.e.,

$$R_{p/q}(\theta) = e^{2\pi i(\theta+p/q)}.$$

We consider the itineraries of points in the circle under R using two different partitions of the circle. The lower partition of the circle is defined as follows. Let

$$I_0^- = \{\theta | 0 < \theta \leq 1 - \frac{p}{q}\} \quad and \quad I_1^- = \{\theta | 1 - \frac{p}{q} < \theta \leq 1\}.$$

The boundary point 0 belongs to I_1^- and $-p/q = 1 - p/q$ belongs to I_0^-. We then define $\overline{s_-(p/q)}$ to be the itinerary of p/q under $R_{p/q}$ relative to this partition. We call the basic repeating block of this itinerary, $s_-(p/q)$, the lower itinerary of p/q. That is

$$s_-(p/q) = s_1 s_2 \cdots s_q$$

where s_j is either 0 or 1 and the digits

$$s_j = \begin{cases} 0 & \text{when } R_{p/q}^{j-1}(p/q) \in I_0^-, \\ 1 & \text{when } R_{p/q}^{j-1}(p/q) \in I_1^- \end{cases} \tag{4.2}$$

We also define the upper partition I_0^+ and I_1^+ as follows:

$$I_0^+ = [0, 1 - p/q), \quad I_1^+ = [1 - p/q, 1).$$

The upper itinerary of p/q, $s_+(p/q)$is then the repeating block of the itinerary of p/q relative to this partition. Note that I_0^+ and I_1^+ differ from I_0^- and I_1^- only at the end points.

Example 4.11 Consider $s_-(1/3) = \overline{001}$, since

$$I_0^- = (0, 2/3], \quad I_1^- = (2/3, 1],$$

and the orbit $1/3 \to 2/3 \to 1 \to 1/3 \cdots$ lies in I_0^-, I_0^-, I_1^-, respectively. Similarly $s_+(1/3) = \overline{010}$, since

$$I_0^+ = [0, 2/3), \quad I_1^+ = [2/3, 1),$$

and the orbit is

$$1/3 \to 2/3 \to 0 \to 1/3 \cdots .$$

This orbit starts in I_0^+, hops to I_1^+, and then returns to I_0^+ before cycling and the orbit lies in I_0^+, I_1^+, I_0^+ respectively.

Similarly $s_-(2/3) = \overline{101}$, since $I_0^- = (0, 1/3]$, $I_1^- = (1/3, 1]$ and the orbit $2/3 \to 1/3 \to 1 \to 2/3 \cdots$ lies in I_1^-, I_0^-, I_1^-, respectively. Similarly $s_+(2/3) = \overline{110}$, since

$$I_0^+ = [0, 1/3], \quad I_1^+ = [1/3, 1),$$

and the orbit $2/3 \to 1/3 \to 0 \to 2/3 \cdots$ lies in I_1^+, I_1^+, I_0^+ respectively.

Example 4.12 Consider 3/5 then the lower itinerary of 3/5 is $s_-(3/5) = \overline{10101}$, since

$$I_0^- = (0, 2/5], \quad I_1^- = (2/5, 1],$$

and the orbit is $3/5 \to 1/5 \to 4/5 \to 2/5 \to$ lies in $I_1^-, I_0^-, I_1^-, I_0^-, I_1^-$, respectively. Similarly the upper itinerary of 3/5 is $s_+(3/5) = \overline{10110}$, since

$$I_0^+ = [0, 2/5), \quad I_1^+ = [2/5, 1),$$

and the orbit is $3/5 \to 1/5 \to 4/5 \to 2/5 \to$ lies in $I_1^+, I_0^+, I_1^+, I_1^+, I_0^+$, respectively.

Example 4.13 Consider $s_-(2/5) = \overline{01001}$ since the orbit is $2/5 \to 4/5 \to 1/5 \to 3/5 \to 2/5 \to \cdots$ and

$$I_0^- = (0, 3/5], \quad I_1^- = (3/5, 1].$$

and the orbit lies in $I_0^-, I_1^-, I_0^-, I_0^+$ and I_1^- Similarly $s_+(2/5) = \overline{01010}$, since the orbit is $2/5 \to 4/5 \to 1/5 \to 3/5 \to 2/5 \to \cdots$ and

$$I_0^+ = [0, 3/5), \quad I_1^+ = [3/5, 1).$$

This orbit starts in I_0^+, hops to I_1^+, and then returns to I_0^+ before cycling and the orbit lies in $I_0^+, I_1^+, I_0^+, I_1^+, I_0^+$, respectively.

4.8 An algorithm for computing the angles of rays landing at $c(p/q)$.

Theorem 4.2 $[DH1, DM]$ *The rays $l_\pm(p/q)$ landing at the root point $c(p/q)$ of the p/q bulb are given by $s_-(p/q)$ and $s_+(p/q)$.*

Example 4.14 Let $p/q = 1/4$. Then the lower external angle $l_-(1/4) = \overline{0001}$ and the upper external angle $l_+(1/4) = \overline{0010}$, so that $s_-(1/4) = 1/15 = \overline{0001}$ in binary and $s_+(1/4) = 2/15 = \overline{0010}$ in binary. Since from the binary expansion we have

$$\overline{0001} = \frac{0}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{0}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \cdots$$
$$= \frac{1}{2^4}[1 + \frac{1}{2^4} + \frac{1}{2^8} + \ldots] = \frac{1}{2^4}.\frac{1}{1-\frac{1}{2^4}}$$
$$= \frac{1}{2^4}.\frac{2^4}{2^4-1} = \frac{1}{15}.$$

Similarly we have

$$\overline{0010} = \frac{0}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \ldots$$
$$= \frac{1}{2^3}[1 + \frac{1}{2^4} + \frac{1}{2^8} + \ldots = \frac{1}{2^3}.\frac{1}{1-\frac{1}{2^4}}.$$
$$= \frac{1}{2^3}.\frac{2^4}{2^4-1} = \frac{2}{15}.$$

Thus

$$l_-(1/4) = s_-(1/4) = \overline{0001} = \frac{1}{15} \quad \textit{and} \quad l_+(1/4) = s_+(1/4) = \overline{0010} = \frac{2}{15}.$$

Example 4.15 Consider $p/q = 3/5$, then the lower angle $l_-(3/5) = \overline{10101}$ and the upper angle $l_+(3/5) = \overline{10101}$, so that $s_-(3/5) = 21/31 = \overline{10101}$ in binary and $s_+(3/5) = 22/31 = \overline{10110}$ in binary. From binary expansion we have

$$\overline{10101} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^8}\frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{13}} + \frac{1}{2^{15}} + \cdots$$
$$= \frac{1}{2} + [1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \cdots] + \frac{1}{2^3}[1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \ldots] + \frac{1}{2^6}[1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \ldots]$$
$$= \frac{1}{2} + \frac{1}{2^5}.\frac{1}{1-\frac{1}{2^5}} + \frac{1}{2^3}.\frac{1}{1-\frac{1}{2^5}} + \frac{1}{2^6}.\frac{1}{1-\frac{1}{2^5}}$$
$$= \frac{1}{2} + \frac{1}{2^5}.\frac{2^5}{31} + \frac{1}{2^3}.\frac{2^5}{31} + \frac{1}{2^6}.\frac{2^5}{31}$$

$$= \frac{1}{31}.\frac{1}{2}[31+2+8+1] = \frac{21}{31}.$$

Similarly

$$\overline{10110} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{1}{2^{11}} + \frac{1}{2^{13}} + \frac{1}{2^{14}} + \dots$$

$$= \frac{1}{2} + \frac{1}{2^3}[1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \dots] + \frac{1}{2^4}[1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \dots] + \frac{1}{2^6}[1 + \frac{1}{2^5} + \frac{1}{2^{10}} + \dots]$$

$$= \frac{1}{2} + \frac{1}{2^3}.\frac{1}{1-\frac{1}{2^5}} + \frac{1}{2^4}.\frac{1}{1-\frac{1}{2^5}} + \frac{1}{2^6}.\frac{1}{1-\frac{1}{2^5}}$$

$$= \frac{1}{2} + \frac{1}{2^3}.\frac{2^5}{31} + \frac{1}{2^4}.\frac{2^5}{31} + \frac{1}{2^6}.\frac{2^5}{31}$$

$$= \frac{1}{31}.\frac{1}{2}[31+4+8+1] = \frac{22}{31}.$$

Thus

$$l_-(3/5) = s_-(3/5) = \overline{10101} = \frac{21}{31}$$

and

$$l_+(3/5) = s_+(3/5) = \overline{10110} = \frac{22}{31}.$$

Remark 4.2 *Note that $s_\pm(p/q)$ differ only in their last two digits provided $q \geq 2$. Indeed we may write*

$$s_-(p/q) = s_1 s_2 \cdots s_{q-2}01,$$

$$s_+(p/q) = s_1 s_2 \cdots s_{q-2}10$$

The reason for this is that the upper and lower itineraries are the same except at $R^{q-2}_{p/q}(p/q) = -p/q$ and $R^{q-1}_{p/q}(p/q) = 0$, which form the endpoints of the two partitions of the circle.

We now define the size of the p/q limb to be the length of interval $[s_-(p/q), s_+(p/q)]$. That is, the size of the p/q limb is given by the number of external rays that approach this limb. We may compute the size of these bulbs explicitly by using the fact that $s_\pm(p/q)$ differ only in the last two digits.

Theorem 4.3 *The size of the p/q limb is $1/(2^q - 1)$. That is,*

$$\overline{s_+(p/q)} - \overline{s_-(p/q)} = \frac{1}{2^q - 1}$$

Proof. We write the binary expansion of the difference in the form

$$\begin{aligned}\overline{s_+(p/q)} - \overline{s_-(p/q)} &= \left(\frac{1}{2^{q-1}} + \frac{1}{2^{2q-1}} + \frac{1}{2^{3q-1}} + \cdots\right) \\ &\quad - \left(\frac{1}{2^q} + \frac{1}{2^{2q}} + \frac{1}{2^{3q}} + \cdots\right) \\ &= 2\left(\frac{1}{2^q} + \frac{1}{2^{2q}} + \frac{1}{2^{3q}} + \cdots\right) - \left(\frac{1}{2^q} + \frac{1}{2^{2q}} + \frac{1}{2^{3q}} + \cdots\right). \\ &= \frac{1}{2^{q-1}}.\frac{2^q}{2^q - 1} - \frac{1}{2^q}.\frac{2^q}{2^q - 1}.\end{aligned}$$

■

Example 4.16 Consider 1/3 limb then,

$$\overline{s_+(1/3)} - \overline{s_-(1/3)} = 2/7 - 1/7 = 1/7 = \frac{1}{2^3 - 1}.$$

Similarly the size of the 2/5 limb is 1/31, since

$$\overline{s_+(2/5)} - \overline{s_-(2/5)} = 10/31 - 9/31 = 1/31 = \frac{1}{2^5 - 1}.$$

As we see in the following Figure , the visual size of the bulbs does indeed correspond to the size as defined above.

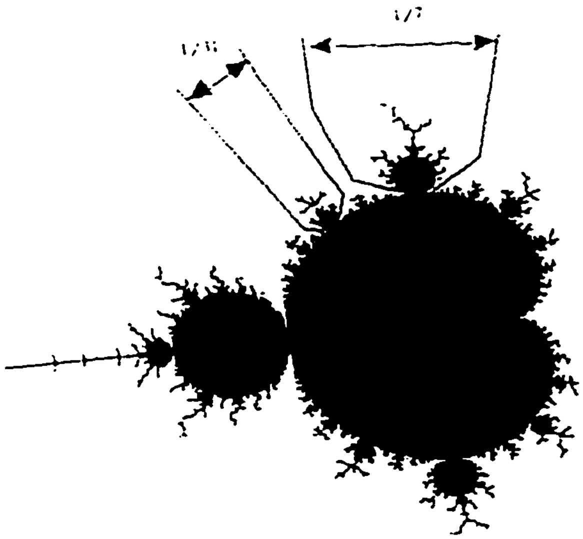


Figure 4.4

4.9 The size of Limbs and the Farey Tree

In this section we relate the size of a p/q limb to the size of the limbs corresponding to the Farey parents of p/q. The following proposition relates the upper and lower itineraries of p/q and its Farey parents.

Proposition 4.1 *Suppose α/β and γ/δ are the Farey parents of p/q and that $0 < \alpha/\beta < \gamma/\delta < 1$. Then the lower itinerary $s_-(p/q)$ consists of the first q digits of the upper angle $\overline{s_+(\alpha/\beta)}$ of the smaller parent, and the upper itinerary $s_+(p/q)$ consists of the first q digits of the lower angle $\overline{s_-(\gamma/\delta)}$ of the larger parent.*

Proof. We consider only $s_+(p/q)$; the proof for $s_-(p/q)$ is similar. From Definition of Farey addition we have $p\delta - q\gamma = \pm 1$, so we have

$$\frac{\gamma}{\delta} - \frac{p}{q} = \frac{p\delta - q\gamma}{q\delta} = \frac{1}{q\delta}.$$

Consider the orbits of p/q and γ/δ relative to the respective rotations $R_{p/q}$ and $R_{\gamma/\delta}$. Since γ/δ rotates faster than p/q, the distance between these orbits advances by $1/q\delta$ at each iteration. We thus have

$$R^j_{\gamma/\delta}(\gamma/\delta) - R^j_{p/q}(p/q) = \frac{j+1}{q\delta}.$$

It follows that $R^j_{p/q}(p/q)$ lies within $1/\delta$ units of $R^j_{\gamma/\delta}(\gamma/\delta)$ provided $j < q-1$. Since points on the orbit of γ/δ under $R^j_{\gamma/\delta}(\gamma/\delta)$ under $R_{\gamma/delta}$ lie exactly $1/\delta$ units apart on the circle, it follows that the first $q-1$ entries in the itineraries of p/q and γ/δ are the same, provided we choose the lower itinerary for γ/δ and the upper itinerary for p/q. The reason for this is that the orbit of γ/δ lies ahead of that of p/q in the counterclockwise direction, but by no more than $1/\delta$ units. Choosing the upper itinerary for p/q and the lower for γ/δ forces the corresponding digits to be the same. When $j = q-1$, we have $R^{q-1}_{p/q}(p/q) = 0$ and

$$R^{q-1}_{\gamma\delta}(\gamma/\delta) - R^{q-1}_{p/q}(p/q) = \frac{1}{\delta}.$$

Hence

$$R^{q-1}_{\frac{\gamma}{\delta}}(\gamma/\delta) = \frac{1}{\delta}.$$

Therefore the qth digit in $s_+(p/q)$ is 0 and so is the qth digit of γ/δ, as long as $\gamma/\delta \neq 1$. ■

Example 4.17 Consider 1/3 and 1/2 are the Farey parents of 2/5 and $0 < \alpha < \beta < 1$ and the upper itinery of 1/3 is

$$\overline{s_+(1/3)} = \overline{010}.$$

So by Proposition 4.1 we have $s_-(2/5) = \overline{01001}$, since $q = 5$.

Similarly the lower angle of 1/2 is

$$\overline{s_-(1/2)} = \overline{01}.$$

So by Proposition 4.1 we have

$$s_+(2/5) = \overline{01010},$$

since $q = 5$

If one of the Farey parents is 0 or 1, we must modify Proposition 4.1

Proposition 4.2 *Suppose that* 0 *is a Farey parent of* p/q*. Then the* q *digits in the lower itinerary of* p/q *are* $s_-(p/q) = 0\cdots01$*. If* 1 *is a Farey parent of* p/q *then* $s_+(p/q) = 1\cdots10$.

Proof. For $s_-(p/q)$, we first consider 0/1 is a Farey parent of p/q then from the definition of Farey addition we have $p/q = \frac{\alpha+\gamma}{\beta+\delta}$ and $p\beta - q\alpha = 1$ and $p\delta - q\gamma = -1$, where $\alpha = 0$ and $\beta = 1$, so we must have $p = 1$. Thus, $s_-(p/q)$ is given by the itinerary of $1/q$ unit under counterclockwise rotation by $1/q$ units. We therefore have

$$I_0^- = (0, (q-1)/q], \quad I_1^- = ((q-1)/q, 1].$$

It follows that the first $q-1$ digits of $s_-(1/q)$ are 0, and the last digit is 1.

If a Farey parent is 1/1, the proof is similar. For $s_+(p/q)$, we note that, since 1/1 is a Farey parent, we must have $p = q - 1$, since $\gamma = 1$ and $\delta = 1$. Thus, $s_+(p/q)$ is given by the itinerary of $(q-1)/q$ unit under counterclockwise rotation by $(q-1)/q$ units. We therefore have

$$I_0^+ = [0, 1/q), \quad I_1^+ = [1/q, 1).$$

It follows that the first $q-1$ digits of $s_+((q-1)/q)$ are 1, and the last digit is 0. Thus $s_+((q-1)/q) = \overline{1\cdots 10}$. ■

We now complete the proof of one of the folk theorems mentioned in the introduction.

Theorem 4.4 *Suppose α/β and γ/δ are the Farey parents of p/q and that $0 \leq \alpha/\beta < \gamma/\delta \leq 1$. Then the size of the p/q limb is larger than the size of any other limb between the α/β and γ/δ limbs.*

Proof. Assume first that neither parent is 0 or 1. Proposition 4.1 and 4.2 ensure that $\overline{s_-(p/q)}$ and $\overline{s_+(\alpha/\beta)}$ agree in their first q digits. Using these binary representations, we have

$$\overline{s_-(p/q)} - \overline{s_+(\alpha/\beta)} \leq \frac{1}{2^q}.$$

Similarly

$$\overline{s_-(\gamma/\delta)} - \overline{s_+(p/q)} \leq \frac{1}{2^q}.$$

This implies that the arc of rays between the p/q limb and either of its parents' limbs has length no longer than $1/2^q$. From Theorem 4.3, we know that

$$\overline{s_+(p/q)} - \overline{s_-(p/q)} = \frac{1}{2^q - 1}.$$

As this quantity is larger than $1/2^q$, it follows that the p/q limb attracts the largest number of rays between its two parents.

If one of the parents of $1/q$ is $0/1$, then the size of the $1/q$ bulb is again $1/(2^q - 1)$, while the gap between $0/1$ and $\overline{s_-(p/q)} = \overline{0\cdots01}$ is also $1/(2^q - 1)$. But then any limb between the $1/q$ limb and the cusp of the cardioid must have size strictly smaller than $1/(2^q - 1)$, again showing that the $1/q$ limb is the largest.

Now if one of the parents of $1/q$ is $1/1$,then the size of the $1/q$ bulb is again $1/(2^q - 1)$, while the gap between $1/1$ and $\overline{s_+(p/q)} = \overline{1\cdots10}$ is also $1/(2^q - 1)$. But then any limb between the $1/q$ limb and the cusp of the cardioid must have size strictly smaller than $1/(2^q - 1)$, again showing that the $1/q$ limb is the largest. ■

Example 4.18 Consider one of the $1/3$ is $0/1$, then the size of the $1/3$ bulb is

$$\overline{s_+(1/3)} - \overline{s_-(1/3)} = 2/7 - 1/7 = \frac{1}{2^3 - 1} = 1/7.$$

Again if we consider $1/4$ limb between $1/3$ and the cusp of the cardioid must have strictly smaller than $\frac{1}{2^3-1}$, since

$$\overline{s_+(1/4)} - \overline{s_-(1/4)} = 2/15 - 1/15 = 1/15 = 1/(2^4 - 1) < 1/(2^3 - 1) = 1/7.$$

Thus $1/3$ is the largest limb.

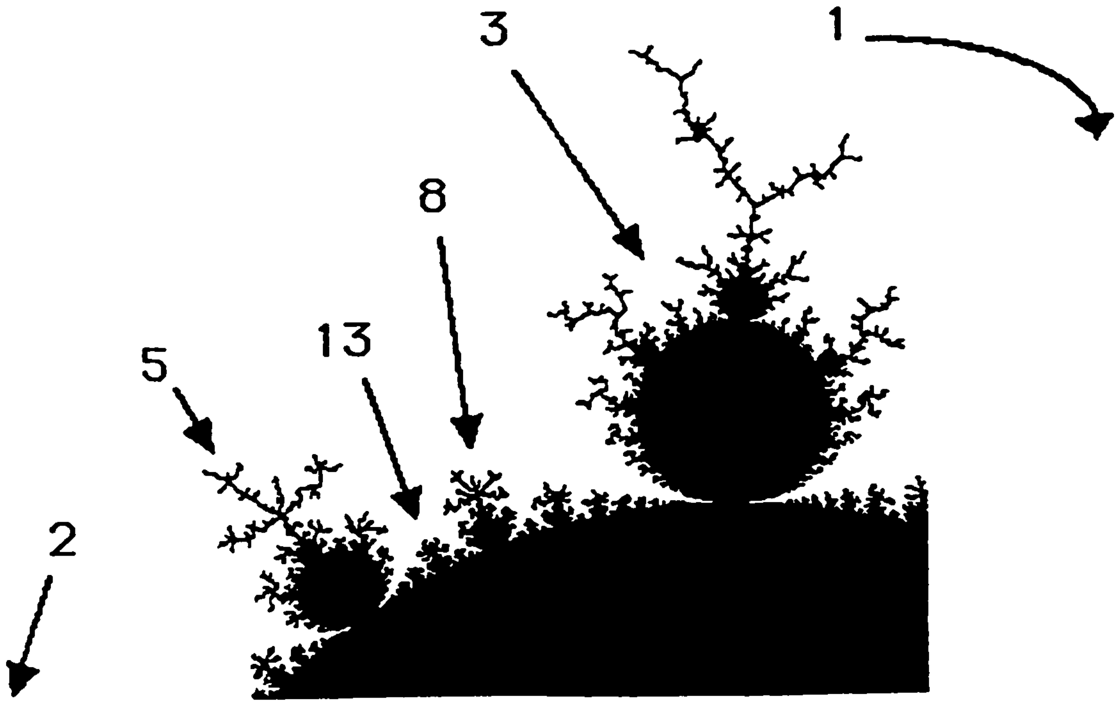
4.10 The Fibonacci sequence

Theorem 4.4 shows that the Fibonacci sequence appears in the Mandelbrot set. As we have seen in Figure 4.2 the largest bulb between the $1/2$ and $1/3$ bulb is the $2/5$

bulb, and the largest between the 1/3 and 2/5 bulbs is the 3/8 bulb. This progression continues, with the numerators and denominators forming the Fibonacci sequence.

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \ldots$$

This sequence of bulbs actually converges to a single point on the boundary of the main cardiod. The following figure shows that the corresponding bulbs forming the Fibonacci sequence.



The Fibonacci sequence.

Figure 4.5

4.11 Conclusion

The technique of measuring the size of certain portions of the Mandelbrot set by the length of interval of rays that land on the portion provides justification for other folk theorems involving the size of the Mandelbrot set M. For example this technique is used to identify the p/q bulb using the length of the spokes in its antenna. Once we know these rays we can easily compute the lengths of the various spokes.

We can also use the two rays separating the principal spoke from the rest of the antenna to determine a list of the q rays that land on the junction point. Then we can determine that the shortest is located p/q turns in the counterclockwise direction from the principal spoke.

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