Pricing and Hedging Equity-Linked Products under Stochastic Volatility Models

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ABSTRACT

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Equity-indexed annuities (EIAs) are becoming increasingly interesting for investors as market volatility increases. Simultaneously, they represent a higher risk for insurers, which amplifies the need for hedging strategies that perform well when index returns present unexpected changes in their volatility. In this thesis, we introduce hedging strategies that aim at reducing the risk of the financial guarantees embedded in EIAs.

We first derive closed-form expressions for the price and the Greeks of a point-to-point EIA under the Heston model, which assumes stochastic volatility. To do so, we rely on the similarity between the payoff of a European call option and that of the EIA. We use the Greeks to develop dynamic hedging strategies that aim at reducing equity and volatility risk. Using Monte Carlo simulations to derive the distribution of the resulting hedging errors, we compare the performance of hedging strategies that use the Greeks derived under the Heston model to other strategies based on Greeks developed under Black-Scholes.

We show that, when the market is Hestonian, the performance of hedging strategies developed in a Black-Scholes framework are significantly affected by the calibration of the model and the volatility risk premium. We further show that the performance of a simple delta hedging strategy using Heston Greeks is also reduced by the presence of a volatility risk premium, and that this performance can be improved by incorporating gamma or vega hedging to the strategy. We conclude by recommending the use of a delta-vega hedging strategy to reduce model calibration and volatility risk.

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Introduction

In times of financial turmoil, equity-linked investments offering a minimum return are very attractive to conservative investors who want to benefit from market growth while protecting their initial capital. This explains why equity-indexed annuities (EIAs), which have gained in popularity since their introduction in 1995, have seen their sales increase by 50%, from \$20 to \$30 billion, between 2004 and 2009[1]. By offering a guaranteed minimum payoff, equity-linked products, which include variable annuities (VAs) and EIAs, represent a risk for insurance companies, typical sellers of these products. In this thesis, we explore different strategies to hedge this risk. In particular, we analyze the efficiency of dynamic hedging strategies developed under different market models.

Although EIAs are insurance products, it makes sense to value and hedge them using mathematical finance theory since their payoff, usually dependent on the performance of a stock index, is similar to a financial option. Brennan & Schwartz (1976) and Boyle & Schwartz (1977) were the first to use the Black-Scholes model to value such guarantees. Hardy (2003) discusses product design and pricing techniques, while Tiong (2000) and Lee (2003) presents closed-form expression for the price of the financial guarantees embedded in EIAs. EIAs have also been studied under more general finan-

[1] See www.indexannuity.org

cial models. For example, Lin & Tan (2003) incorporate stochastic interest rates.

All the models cited above assume that the volatility of stock and index prices is constant. However, it is unexpected fluctuations in those prices that represent the biggest financial risk for insurance companies selling EIAs. In that mind set, Lin *et al.* (2009) use regime-switching models to value EIAs. Kurpiel & Roncalli (1998) compare the efficiency of call option hedging strategies that assume stochastic volatility to others that do not. In this thesis, we make use of models that incorporate stochastic volatility to price and hedge point-to-point EIA contracts.

Stochastic volatility models are among the solutions suggested to address some of the flaws of the Black-Scholes model, namely the thick tails of the stock log-return distribution and the varying implied volatilities across option strikes and maturities. The first stochastic volatility model was introduced by Hull & White (1987). Heston (1993) presented an improved version of it by using a square root process to model the dynamics of the volatility, which allowed for a semi-closed form expression for the price of a European call option.

In this thesis, we present hedging strategies and assess their efficiency at reducing the risks inherent to the sale of equity-linked products. In fact, financial derivatives such as equity-linked products can be priced using a replicating portfolio of assets traded on the market. In this thesis, we introduce the pricing of derivatives through such techniques which, under certain models, yield closed and semi-closed form expressions for their prices. These expressions can in turn be differentiated to give the sensitivities of the prices to different parameters. The sensitivities can be used to develop different hedging strategies, which we present and adapt to fit the particularities of equity-linked products. It is those hedging strategies which we shall analyze to identify the ones that are the most efficient at reducing the risks faced by insurers selling EIAs.

In the first chapter, we introduce the Black-Scholes and Heston models, which will be used throughout the thesis. We present a closed-form expression for the price of a European call option under each model. This option will be used to price and hedge point-to-point EIAs, as the payoff of this contract is very similar to the payoff of the call option. For each model, we also describe the hedging strategy underlying the pricing of any derivative. The last part of each section of the first chapter is dedicated to the calibration of the market model to recent market data. The parameters resulting from this calibration will be used in Chapter 4 to perform a numerical example.

In Chapter 2, we present a discrete-time version of the hedging strategies explained in Chapter 1. In doing so, we introduce the Greeks, sensitivities of a financial derivative to different parameters, and use the closed-form expressions for the price of a European call to derive the Greeks for that type of option. In later chapters, those Greeks will be part of the hedging strategy for EIAs. We also introduce the concept of hedging error, which is a consequence of the discretization of the hedging process. Hedging errors play an important role in the analysis of the performance of hedging strategies. In the last section of Chapter 2, we expose discretization schemes for the dynamics of index prices under both the Black-Scholes and Heston models. In the numerical example, those schemes will be necessary to simulate index prices.

Chapter 3 offers an overview of equity-linked products. We describe the different types of contracts, from EIAs to variable annuities, and their particularities. We present a closed-form expression for the price of the point-to-point EIA and show how it can be extended to the pricing and hedging of any equity-linked contract that has a point-to-point design.

In Chapter 4, using previously introduced concepts, we develop hedging strategies specifically adapted to EIAs. Through a numerical example, we assess the performance of each strategy by analyzing the distribution of the resulting hedging errors. More specifically, we apply hedging strategies developed under both Black-Scholes and Heston assumptions to a Hestonian market and assess the performance of each strategy. We also observe the effect of adding call options to a hedge. By doing so, we aim at identifying the hedging strategies that will best protect insurance companies against the risk brought on by the non-constant volatility of stock index returns.

Chapter 1

Financial Models

1.1 The Black-Scholes Model

1.1.1 Motivation

When Black and Scholes first introduced their model, in 1973 (Black & Scholes (1973)), it revolutionized the world of quantitative finance. For many years after, it was extensively used to model stock price fluctuations and to value different types of financial derivatives. Throughout the years and with the evolution of financial markets, the Black-Scholes model has proven to be too simplistic to fit market fluctuations, as will be discussed later in this thesis. As its flaws and limitations were pointed out (see e.g. Rubinstein (1985)), various models were developed to address those problems. Nonetheless, to this day, the Black-Scholes model is still widely used in the industry.

1.1.2 Description

The Black-Scholes model relies on a number of assumptions about financial markets. It assumes that there are no transaction costs or taxes, that securities are infinitely divisible, that anyone can borrow or lend at a constant risk-free rate r and that there are no restriction on short selling. In addition, a very important premise is that the markets are free of arbitrage, which means that any two securities or portfolios of securities that result in the same payoff are assumed to have the same market price.

In this thesis, we also make the assumption that stock indexes pay no dividends. This assumption could however be relaxed without much complication.

The central assumption of the Black-Scholes model is that stock prices follow a geometric Brownian motion with drift. Hence, the dynamics of S_t, the stock price at time t, are given by the following stochastic differential equation:

$$\begin{cases} dS_t = \mu_{BS} S_t dt + \sigma_{BS} S_t dZ_t \quad t > 0, \\ S_0 = s, \end{cases} \tag{1.1}$$

where dZ_t is the derivative of a standard Brownian motion, μ_{BS}, a constant drift term representing the mean rate of return, and σ_{BS}, the constant volatility of the stock prices. The dynamics of the stock price can also be described by the distribution of its returns. Under the Black-Scholes model, over a period of time t, the log of the return of the stock prices follow a normal distribution with mean $(\mu_{BS} - \frac{\sigma_{BS}^2}{2})t$ and variance $\sigma_{BS}^2 t$, such that

$$\log \frac{S_t}{S_0} \sim \mathcal{N}((\mu_{BS} - \frac{\sigma_{BS}^2}{2})t, \sigma_{BS}^2 t), \tag{1.2}$$

where S_t and S_0 are the prices of the stock at time t and 0, respectively, and $\mathcal{N}$ denotes a normal distribution. Expressing the dynamics of the stock prices in terms of the distribution of their log-returns can be useful in many ways, in particular when simulating stock prices.

One of the reasons why the Black-Scholes model is so popular is that its simplicity allows for the derivation of a closed-form expression for the price of many financial derivatives, in particular that of European call and put options. In turn, those can be used to calibrate the model to the market and to price more exotic financial derivatives. In the next section, we show how the formula for the price of a European call option can be obtained.

1.1.3 Price of a European call option

A European call option is a financial derivative which pays, at maturity T, the difference between the price of the stock S_T and the fixed, pre-determined strike price K. Hence, its payoff at maturity, $B^C(K,T)$ is given by

$$B^C(K,T) = \max(S_T - K, 0). \tag{1.3}$$

There are different ways to obtain the price of this derivative. Black & Scholes (1973) construct a risk-free portfolio in which the randomness in the price of the call option is hedged away by short selling the underlying. According to the no-arbitrage principle, this portfolio should yield the risk-free rate. This is used to solve the partial differential equation describing the dynamics of the call. A closed-form expression for the price of the European call is thus obtained.

The same solution can also be obtained through risk-neutral valuation, as presented Harrison & Pliska (1981). The price of the European call option is obtained by taking the expectation of its discounted payoff at maturity. This second technique involves a change of measure. Under the new probability measure, the drift term in the equation of the stock price is r, the risk-free rate, and the discounted price of any asset is a martingale.

Whichever approach is used to obtain the price of the call option, it yields the same result, which is as follows. Under the Black-Scholes framework, the price $C^{BS}(S_t, K, \tau)$ of a European option with time to maturity $\tau = (T - t)$ and payoff as in (1.3) is given by

$$C^{BS}(S_t, K, \tau) = S_t \Phi(d_1) - K e^{-r(\tau)} \Phi(d_2), \tag{1.4}$$

where Φ is the standard normal cumulative distribution function, S_t is the price of the underlying at time t, K is the strike price, r is the risk-free rate and

$$\begin{aligned} d_1 &= \frac{\log \frac{S_t}{K} + \left(r + \frac{\sigma_{BS}^2}{2}\right)(T-t)}{\sigma_{BS}\sqrt{\tau}} \\ d_2 &= d_1 - \sigma_{BS}\sqrt{\tau}, \end{aligned}$$

where σ_{BS} is as described in (1.1).

In this thesis, we will use (1.4) to derive the price of insurance contracts whose value depends on the price of an underlying asset.

1.1.4 Hedging under the Black-Scholes model

In finance, hedging consists in buying or selling financial securities with the objective of reducing the exposure to risks brought on by another position, which we call the initial position. For example, an investor who takes a short position in a call option (sells the call) will have to pay the difference between the price of the underlying and the strike price at maturity of the contract if the former is greater than the latter. Since the price of the underlying fluctuates randomly, the payoff at maturity is random, which represents a risk for the investor. However, the relation between the payoff and the asset price makes it possible for the investor to reduce his or her exposure to that risk by buying or selling the underlying asset.

As explained in Section 1.1.3, the Black-Scholes option pricing formula can be obtained by constructing a risk-free portfolio in which the position in the derivative is hedged by an opposite position in the underlying. This portfolio, as well as the pricing formula, is the basis of the hedging strategy presented here. This strategy aims at eliminating the risk inherent to taking the position in the call option. Here, for simplicity, we hedge a short position in the call. It can easily be extended to the case where the investor has a long position in the call.

The hedging strategy relies on the construction a portfolio replicating the call option payoff. Since the market is assumed to be free of arbitrage, the value of this portfolio at any time t, $0 \leq t \leq T$, must be the same as that of the option. The investor who sells a call option will take a long position in the replicating portfolio, so that at any time between the inception and the expiration of the option, the two positions offset each other. Thus, when the option expires, the value of the replicating portfolio will be exactly the same as the payout of the option, so that by liquidating the portfolio, the investor will have sufficient funds to pay the buyer of the option, if necessary.

It is the option pricing formula that gives us the structure of the replicating portfolio. In fact, by investing $\Phi(d_1)S_t$ in the underlying asset and $Ke^{-r(\tau)}\Phi(d_2)$ in the risk-free asset, with S_t, K, Φ, d_1, d_2 defined as in (1.4), the value of the replicating portfolio will always be equal to that of the call option. The amount invested in the money market represents the difference between the value of the option and the stock part of the hedge. If the latter is greater than the former, the money market part of the hedge becomes negative, representing a loan taken by the investor at the risk-free rate. This ensures that the strategy remains self-financing; the investor does not need to inject additional funds in the replicating portfolio.

Denote H_t the value of the replicating portfolio at t. Then,

$$\begin{aligned} H_t &= S_t\Phi(d_1) - Ke^{-r(\tau)}\Phi(d_2), \\ &= C^{BS}(S_t, K, \tau), \end{aligned} \tag{1.5}$$

Note that both $\Phi(d_1)$ and $e^{-r(\tau)}\Phi(d_2)$ depend on t and S_t, so they vary continuously through time.

This hedging strategy can be used to protect investors against the risk inherent to buying or selling call options. It can also be extended to other types of options. However, the performance of this strategy relies on the assumption that the underlying price follows the Black-Scholes model. In the next section, we present a different model and the resulting strategy.

1.1.5 Calibrating the Black-Scholes model

In order to price non-traded derivatives and to assess the performance of hedging strategies, one needs to calibrate the Black-Scholes model to market data. This section explains the method used to obtain the parameters that will be used to perform the numerical example presented in Chapter 4.

The procedure is performed using the maximum likelihood estimators. The mean and the variance of weekly log-returns of the S&P 500 index is calculated using data[1] ranging from May 20, 1996 to May 16, 2011. We assume the contract is sold on May 20, 2011. We chose to calculate the moments of the log-returns over 15 years.

Using the maximum likelihood estimator for the mean and the variance of the weekly log-returns yields a drift coefficient μ_{BS} of 0.0637 and a volatility term σ_{BS} of 0.19. The value of μ_{BS} that we obtain is somewhat lower than what is usually used in the literature. This is due to the negative returns observed on the markets between 2007 and 2009. We believe that our drift coefficient may be more realistic for the years to come.

To price and hedge the point-to-point EIA, we also need to estimate the risk-free rate r. For short term investments, it is usually approximated by the interest rate on a risk-free investment, typically a 3-month US treasury bill. Here, since we are pricing a long-term investment, we need to look at longer treasury bonds to determine our risk-free rate. In our numerical example, we will focus on an EIA with 10 years to expiration and calculate its price at different times between the inception of the contract and its expiration. Thus, we observe the yield rates for bonds ranging from 1 to 10 years. On May 20, 2011, the yield rates for 1-year, 5-year and 10-year bonds were 0.18%, 1.82% and 3.15%, respectively[2]. To replicate those rates, it would be possible to construct a yield curve and use different rates depending on the horizon of the investment product. However, in this thesis, we use a fixed risk-free rate, which we set at 2.00%. As it falls between the 5-year and the 10-year yield rate, we believe it is a good approximation of

[1] Data from finance.yahoo.com

[2] Data from www.treasury.gov

the average risk-free rate over the next 10 years.

1.2 The Heston Model

1.2.1 Motivation

For many years, the Black-Scholes model has been the academic and industry standard to describe the fluctuations of the price of a stock. As seen in the previous section, it can be used to price various types of financial derivatives. However, it is also known that it presents biases which can significantly impair its performance. One of its main problems is that it assumes a Gaussian distribution for the log-returns while empirical studies show that, in fact, log-returns present higher peaks and heavier tails, which was noted as early as 1963 by Mandelbrot (1963) and Fama (1963). More recently, Haug & Taleb (2008) explain the flaws of the model. Many models that aim at better describing the distribution of the log-returns have been developed over the years, using different methods to reach their goal.

One way to modify the distribution of the log-returns is to relax the assumption that the volatility of the returns is constant. In fact, market data shows that stock returns do not always fluctuate with the same intensity, which has lead to the assumption that their volatility changes in an unpredictable manner through time. Furthermore, under the Black-Scholes model, all options depending on the same underlying stock are priced using the same implied volatility σ_{BS}, regardless of their strikes. Empirical observations prove otherwise: implied volatilities vary with strikes and maturities (see e.g. Rubinstein (1985)), graphing a shape called *volatility smile*. Hence, instead of describing the volatility of the stock price as a constant, it makes more sense to refer to it as a variable evolving randomly through time.

Different stochastic volatility models were suggested, such as those Scott (1987), Hull & White (1987), and Wiggins (1987). Nonetheless, the model presented by Heston (1993) has the advantage of allowing a semi-closed form expression for the price of European options. With an appropriate set of parameters, the model can reproduce the volatility smile relatively well. For those reasons, it has become very popular in the industry.

1.2.2 Description of the model

In this section we present the Heston model. All assumptions about the financial market are as in Section 1.1.2. However, in the Heston model, the price of the stock has two sources of randomness. One is related to the stock price itself, as in the Black-Scholes model, and the other one comes from the fact that the volatility term follows a stochastic process. Hence, to describe the dynamics of the stock price, two stochastic differential equations are used:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_t^{(1)}, \tag{1.6}$$

$$dv_t = \kappa'(\theta' - v_t)dt + \sigma\sqrt{v_t} dZ_t^{(2)}, \tag{1.7}$$

with

$$d\langle Z_\cdot^{(1)} Z_\cdot^{(2)} \rangle_t = \rho \; dt.$$

Here, μ is the mean rate of return, as in the Black-Scholes model, κ' is the speed of reversion to the mean, θ' is the long-term mean of the volatility, σ is the volatility of volatility and ρ is the correlation between the standard Brownian motions $Z_t^{(1)}$ and $Z_t^{(2)}$.

Note that (1.7) describing the dynamics of v_t has the same form as the square-root process described by Cox *et al.* (1985) (CIR). However, in the CIR model, this process is used to model interest rates.

1.2.3 Price of a European call option

This section explains the derivation of the semi-closed form expression for the price of a European call option under the Heston model. The technique presented here uses a valuation equation similar to the one that was presented by Black & Scholes (1973) and Merton (1973), which is based on the no-arbitrage assumption. This equation is also known as the Kolmogorov forward equation (see Kloeden & Platen (1992) for more details). This technique can be used to evaluate the price of any derivative security whose value depends on the price of the underlying. The derivation of the formula presented in this section follows Wilmott (2006) and Gatheral (2006).

We first consider a portfolio which is composed of the derivative to hedge, with value V_t at time t, the underlying, and a third asset whose value at time t, V_t^1, depends on volatility. At any time t between 0 and T, where T is the expiration date of the derivative, the stock and the third asset are held in quantities $\alpha_{2,t}^*$ and $\alpha_{1,t}^*$, respectively. Here, since the stock also depends on stochastic volatility, it is impossible to hedge the derivative perfectly using only the underlying asset. For this reason, the inclusion of the third asset becomes necessary to hedge volatility risk. Hence, the original portfolio is slightly modified since, for Black and Scholes, the stock price had only one source of randomness. Denote by Π_t the value of this modified portfolio, composed of the short position in the derivative and the long position in the stock and the third asset. Then Π_t is given by

$$\Pi_t = V_t - \alpha^*_{1,t} V^1_t - \alpha^*_{2,t} S_t, \tag{1.8}$$

where V_t and V^1_t denote the price of the derivative and the price of the third asset, respectively, at time t, $0 \leq t \leq T$. In order to get the dynamics for this portfolio, we need to differentiate each component using the Itô-Doeblin formula (for more details on the Itô-Doeblin formula, see Shreve (2004)). The dynamics of S_t are given by (1.7). For dV_t and dV^1_t, we have

$$\begin{aligned} dV_t = &\left(\frac{\partial V_t}{\partial t} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \rho \sigma v_t S_t \frac{\partial^2 V_t}{\partial v_t \partial S_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 V_t}{\partial v_t^2} \right) dt \\ &+ \frac{\partial V_t}{\partial S_t} dS_t + \frac{\partial V_t}{\partial v_t} dv_t \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} dV^1_t = &\left(\frac{\partial V^1_t}{\partial t} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V^1_t}{\partial S_t^2} + \rho \sigma v_t S_t \frac{\partial^2 V^1_t}{\partial v_t \partial S_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 V^1_t}{\partial v_t^2} \right) dt \\ &+ \frac{\partial V^1_t}{\partial S_t} dS_t + \frac{\partial V^1_t}{\partial v_t} dv_t. \end{aligned} \tag{1.10}$$

Putting (1.9), (1.10) and the definition of dS_t together, we get

$$\begin{aligned} d\Pi_t = &\left(\frac{\partial V_t}{\partial t} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \rho \sigma v_t S_t \frac{\partial^2 V_t}{\partial v_t \partial S_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 V_t}{\partial v_t^2} \right) dt \\ &- \alpha^*_{1,t} \left(\frac{\partial V^1_t}{\partial t} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V^1_t}{\partial S_t^2} + \rho \sigma v_t S_t \frac{\partial^2 V^1_t}{\partial v_t \partial S_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 V^1_t}{\partial v_t^2} \right) dt \\ &+ \left(\frac{\partial V_t}{\partial S_t} - \alpha^*_{1,t} \frac{\partial V^1_t}{\partial S_t} - \alpha^*_{2,t} \right) dS_t + \left(\frac{\partial V_t}{\partial v_t} - \alpha^*_{1,t} \frac{\partial V^1_t}{\partial v_t} \right) dv_t. \end{aligned} \tag{1.11}$$

Our assumption is that the portfolio is perfectly hedged so that its return is the risk-free rate. Hence, the equation describing its dynamics must reflect this return and contain no random terms, which represent risk. Thus, the coefficients of dS_t and dv_t must be eliminated by finding the quantities $\alpha^*_{1,t}$ and $\alpha^*_{2,t}$ such that

$$\frac{\partial V_t}{\partial S_t} - \alpha^*_{1,t}\frac{\partial V_t^1}{\partial S_t} - \alpha^*_{2,t} = 0 \tag{1.12}$$

and

$$\frac{\partial V_t}{\partial v_t} - \alpha^*_{1,t}\frac{\partial V_t^1}{\partial v_t} = 0. \tag{1.13}$$

From (1.12) and (1.13), we obtain

$$\alpha^*_{1,t} = \frac{\frac{\partial V_t}{\partial v_t}}{\frac{\partial V_t^1}{\partial v_t}} \tag{1.14}$$

and

$$\alpha^*_{2,t} = \frac{\partial V_t}{\partial S_t} + \frac{\frac{\partial V_t}{\partial v_t}}{\frac{\partial V_t^1}{\partial v_t}}\frac{\partial V_t^1}{\partial S_t}. \tag{1.15}$$

The quantities $\alpha^*_{1,t}$ and $\alpha^*_{2,t}$ represent the proportion of the portfolio which must be invested in the stock and the third asset, respectively, to make the portfolio risk-free. This is the main idea behind the hedging strategies presented in Chapter 2. Replacing $\alpha^*_{1,t}$ and $\alpha^*_{2,t}$ in (1.11) by (1.14) and (1.15), the dynamics of Π_t become

$$\begin{aligned} d\Pi_t = (&\frac{\partial V_t}{\partial t} + \frac{1}{2}v_t S_t^2\frac{\partial^2 V_t}{\partial S_t^2} + \rho\sigma v_t S_t\frac{\partial^2 V_t}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 V_t}{\partial v_t^2})dt \\ &- \alpha^*_{1,t}(\frac{\partial V_t^1}{\partial t} + \frac{1}{2}v_t S_t^2\frac{\partial^2 V_t^1}{\partial S_t^2} + \rho\sigma v_t S_t\frac{\partial^2 V_t^1}{\partial v_t \partial St} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 V_t^1}{\partial v_t^2})dt. \end{aligned} \tag{1.16}$$

Since the portfolio is now risk-free, under the no-arbitrage assumption, investing in this portfolio must yield the same return rate as investing in the risk-free asset. One of the model's assumptions is that it is possible to invest in a risk-free asset which returns the risk-free rate r. Hence, since the hedged portfolio is risk-free, it is assumed to yield the risk-free rate r. Thus, we must have

$$d\Pi_t = r\Pi_t \, dt. \tag{1.17}$$

Replacing Π_t by (1.8) in (1.17) leads to

$$d\Pi_t = r(V_t - \alpha^*_{2,t} S_t - \alpha^*_{1,t} V^1_t)dt. \tag{1.18}$$

Equating (1.16) and (1.18) and putting all terms in V_t on one side and all terms in V^1_t on the other, we have

$$\frac{\frac{\partial V_t}{\partial t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 V}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 V_t}{\partial v_t^2} - rV_t + rS_t \frac{\partial V_t}{\partial S_t}}{\frac{\partial V_t}{\partial v_t}} = \frac{\frac{\partial V^1_t}{\partial t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 V^1_t}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 V^1_t}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 V^1_t}{\partial v_t^2} - rV^1_t + rS_t \frac{\partial V^1_t}{\partial S_t}}{\frac{\partial V^1_t}{\partial v_t}}. \tag{1.19}$$

For this equality to be true, both sides have to be equal to another function of the independent variables S_t, t and v_t. It could not be otherwise since V_t and V^1_t are two different assets whose dynamics are determined by their respective contracts, which differ by their maturities, strikes and payoffs. Hence, to satisfy the equality in (1.19), both sides must be equal to a function of S_t, t and v_t independent of V_t and V^1_t. We write (1.19) as:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 V_t}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 V_t}{\partial v_t^2} + rS_t \frac{\partial V_t}{\partial S_t} - rV_t = \left(\kappa'(v_t - \theta') - \lambda v_t\right) \frac{\partial V_t}{\partial v_t}. \tag{1.20}$$

In the above, λ denotes the price of volatility risk. In fact, it has been shown by Lamoureux & Lastrapes (1993) that option prices include a premium for risk caused by stock price volatility. The right-hand side of (1.20) is chosen so that the equation of the dynamics of S_t and v_t can be written in the same form under both the objective and the risk-neutral measure. Following Heston (1993), it is possible to re-write the left-hand side of (1.20) as

$$\kappa(v_t - \theta)\frac{\partial V_t}{\partial v_t},$$

by letting

$$\kappa = \kappa' + \lambda$$
$$\theta = \frac{\kappa'\theta'}{\kappa' + \lambda}. \tag{1.21}$$

Thus, $\kappa'(\theta' - v_t) - \lambda v_t = \kappa(\theta - v_t)$ is the risk-neutral drift rate of volatility. From now on, we will use κ and θ to price the European call option and other derivatives. Hence, (1.20) becomes

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 V_t}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 V_t}{\partial v_t^2} + rS_t \frac{\partial V_t}{\partial S_t} - rV_t = \kappa(v_t - \theta)\frac{\partial V_t}{\partial v_t} \tag{1.22}$$

This equation is similar to the one used by Black and Scholes to retrieve the price of a European call, except that it is applied to the Heston model. The equation can be used to price any financial derivative, since they all share these dynamics. It is the terminal condition, representing the payoff, which specifies the derivative to be price. Here, we use (1.20) to derive the price of a European call option of expiration T, at any time t between 0 and T.

First, following Gatheral (2006), we set

$$x_t = log(\frac{F_{t,T}}{K}),$$

where K is the option strike price and $F_{t,T}$ is the T forward price of the underlying observed at time t. It is obtained by accumulating the stock price at time t, S_t, to time T using the risk-free rate r. Since r is assumed to be constant, $F_{t,T}$ is given by

$$F_{t,T} = e^{r(T-t)} S_t$$

where S_t is, as usual, the price of the underlying. To simplify the notation, we let $\tau = T - t$.

We then let $C'(x_t, v_t, \tau)$ denote the future price at expiration of a European call option of moneyness x_t and time to maturity τ, when the instantaneous volatility is v_t, and write $V_t = C^H(x_t, v_t, \tau) = e^{-r\tau}C'(x_t, v_t, \tau)$, with $C^H(x_t, v_t, \tau)$ the price at time t of the European call option. Thus, we have that

$$C^H(x_T, v_T, 0) = C'(x_T, v_T, 0) = (S_T - K)^+. \tag{1.23}$$

Solving (1.22) subject to (1.23) yields a semi-closed form expression for the future price of the European call option.

In order to solve (1.22), we must first calculate the partial derivatives of $V_t = e^{-r\tau}C'(x_t, v_t, \tau)$ with respect to S_t, v_t and t. Below, to simplify the notation, we write C' for $C'(x_t, v_t, \tau)$. Thus, we have

$$\frac{\partial V_t}{\partial t} = e^{-r\tau}\left(rC' - r\frac{\partial C'}{\partial x_t} - \frac{\partial C'}{\partial \tau}\right)$$

$$\frac{\partial V_t}{\partial S_t} = e^{-r\tau}\frac{1}{S_t}\frac{\partial C'}{\partial x_t}$$

$$\frac{\partial^2 V_t}{\partial S_t^2} = e^{-r\tau}\frac{1}{S_t^2}\left(\frac{\partial^2 C'}{\partial x_t^2} - \frac{\partial C'}{\partial x_t}\right)$$

$$\frac{\partial V_t}{\partial v_t} = e^{-r\tau}\frac{\partial C'}{\partial v_t}$$

$$\frac{\partial^2 V_t}{\partial v_t^2} = e^{-r\tau}\frac{\partial^2 C'}{\partial v_t^2}$$

$$\frac{\partial^2 V_t}{\partial v_t \partial S_t} = e^{-r\tau}\frac{1}{S_t}\frac{\partial C'}{\partial v_t \partial x_t}.$$

Using these results, (1.22) becomes

$$-\frac{\partial C'}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 C'}{\partial x_t^2} - \frac{1}{2}v_t\frac{\partial C'}{\partial x_t} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 C'}{\partial v_t^2} + \rho\sigma v_t\frac{\partial C'}{\partial x_t \partial v_t} - \kappa(v_t - \theta)\frac{\partial C'}{\partial v_t} = 0. \tag{1.24}$$

Then, we rely on Heston (1993), who suggests that the equation for the price of the European option under the Heston model has a similar form than the one obtained under Black-Scholes. Gatheral (2006) presents the following version of this equation:

$$C'(x_t, v_t, \tau) = K(e^{x_t} P_1(x_t, v_t, \tau) - P_0(x_t, v_t, \tau)), \tag{1.25}$$

which represents the future value to expiration of the call option. The valuation equation (1.24) also applies to (1.25). As noted by Heston (1993), the first part of his equation, or $Ke^{x_t}P_1(x_t, v_t, \tau)$ in (1.25), is the pseudo expectation of the present value of the stock price give that the option is exercised while the second part, or $KP_0(x_t, v_t, \tau)$ in (1.25), is the present value of the strike payment. Thus, $P_1(x_t, v_t, \tau)$ and $P_0(x_t, v_t, \tau)$ loosely represent probabilities of exercise.

To this end, we take the partial derivatives of (1.25) with respect to τ, x_t and v_t to be able to replace V_t by C' in (1.24). Here again, to simplify the notation, we write P_1 and P_0 for $P_1(x_t, v_t, \tau)$ and $P_0(x_t, v_t, \tau)$, respectively. We get the following partial derivatives:

$$\frac{\partial C'}{\partial \tau} = K(e^{x_t}\frac{\partial P_1}{\partial \tau} - \frac{\partial P_0}{\partial \tau}) \tag{1.26}$$

$$\frac{\partial C'}{\partial x_t} = K(e^{x_t}(P_1 + \frac{\partial P_1}{\partial x_t}) - \frac{\partial P_0}{\partial x_t}) \tag{1.27}$$

$$\frac{\partial^2 C'}{\partial x_t^2} = K(e^{x_t}(P_1 + 2\frac{\partial P_1}{\partial x_t} + \frac{\partial^2 1}{\partial x_t^2}) - \frac{\partial P_0}{\partial x_t^2}) \tag{1.28}$$

$$\frac{\partial C'}{\partial v_t} = K(e^{x_t}\frac{\partial P_1}{\partial v_t} - \frac{\partial P_0}{\partial v_t}) \tag{1.29}$$

$$\frac{\partial^2 C'}{\partial v_t^2} = K(e^{x_t}\frac{\partial^2 P_1}{\partial v_t^2} - \frac{\partial^2 P_0}{\partial v_t^2}) \tag{1.30}$$

$$\frac{\partial^2 C'}{\partial v_t \partial x_t} = K(e^{x_t}\frac{\partial P_1}{\partial v_t} + e^{x_t}\frac{\partial^2 P_1}{\partial x_t \partial v_t} - \frac{\partial^2 P_0}{\partial x_t \partial v_t}). \tag{1.31}$$

The next step is to express (1.24) in terms of P_0 and P_1. First, putting all the terms together, we get

$$\begin{aligned}
&- K\left(e^{x_t}\frac{\partial P_1}{\partial \tau} - \frac{\partial P_0}{\partial \tau}\right) + \frac{1}{2}v_t K\left(e^{x_t}\left(P_1 + 2\frac{\partial P_1}{\partial x_t} + \frac{\partial^2 P_1}{\partial x_t^2}\right) - \frac{\partial^2 P_0}{\partial x_t^2}\right)\\
&\quad - \frac{1}{2}v_t K\left(e^{x_t}\left(P_1 + \frac{\partial P_1}{\partial x_t}\right) - \frac{\partial P_0}{\partial x_t}\right) + \frac{1}{2}\sigma^2 v_t K\left(e^{x_t}\frac{\partial^2 P_1}{\partial v_t^2} - \frac{\partial^2 P_0}{\partial v_t^2}\right) +\\
&\quad \rho\sigma v_t K\left(e^{x_t}\frac{\partial P_1}{\partial v_t} + e^{x_t}\frac{\partial^2 P_1}{\partial x_t \partial v_t} - \frac{\partial^2 P_0}{\partial x_t \partial v_t}\right)\\
&\quad - \kappa(v_t - \theta)K\left(e^{x_t}\frac{\partial P_1}{\partial v_t} - \frac{\partial P_0}{\partial v_t}\right) = 0.
\end{aligned}$$

After dividing this equation by K, we can put all the terms in P_0 and all those in P_1 together. To make the equation equal to 0, we need to have the part in P_0 and the part in P_1 be both equal to zero. That is because the case where they cancel each other would make the equation of the price be equal to 0, which we do not want. Hence, we have

$$\begin{aligned}
-\frac{\partial P_j}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 P_j}{\partial x_t^2} - v(\frac{1}{2} - j)\frac{\partial P_j}{\partial x_t} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 P_j}{\partial v_t^2}&\\
+ \rho\sigma v_t\frac{\partial^2 P_j}{\partial x_t}\partial v_t + (a - b_j v_t)\frac{\partial P_j}{\partial v_t} = 0,&
\end{aligned} \tag{1.32}$$

with $a = \kappa\theta$ and $b_j = \kappa - j\rho\sigma$ for $j = 0, 1$.

Since P_0 and P_1 are probabilities of exercise of the option, (1.32) is subject to the following terminal conditions

$$\lim_{\tau \to 0} P_j(x_t, v_t, \tau) = \begin{cases} 1 & \text{if } x_t > 0, \\ 0 & \text{if } x_t \leq 0, \qquad j = 0, 1. \end{cases} \tag{1.33}$$

In fact, the call option will only be exercised at time T if the price of the stock at that time is above the strike price K, or if $x_T = log(\frac{F_{T,T}}{K}) > 0$.

To solve (1.32) under the conditions given by (1.33), we use a Fourier transform technique. Let $\tilde{P}_j(u, v_t, \tau)$ be the Fourier transform of $P_j(x_t, v_t, \tau)$, defined as

$$\tilde{P}_j(u, v_t, \tau) = \int_{-\infty}^{\infty} e^{-iux} P_j(x_t, v_t, \tau) dx_t. \tag{1.34}$$

Taking the inverse of the Fourier transform gives

$$P_j(x_t, v_t, \tau) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} \tilde{P}_j(u, v_t, \tau) du. \tag{1.35}$$

For further information about Fourier transforms, see Cherubini *et al.* (2010). Since the integral is with respect to u, we can take the partial derivatives of $P_j(x_t, v_t, \tau)$ with respect to τ, x_t and v_t and obtain

$$\frac{\partial P_j}{\partial \tau} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} \frac{\partial \tilde{P}_j}{\partial \tau} du$$

$$\frac{\partial P_j}{\partial x_t} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} iu \tilde{P}_j du$$

$$\frac{\partial^2 P_j}{\partial x_t^2} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} (-u^2) \tilde{P}_j du$$

$$\frac{\partial P_j}{\partial v_t} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} \frac{\partial \tilde{P}_j}{\partial v_t} du$$

$$\frac{\partial^2 P_j}{\partial v_t^2} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} \frac{\partial^2 \tilde{P}_j}{\partial v_t^2} du$$

$$\frac{\partial^2 P_j}{\partial v_t \partial x_t} = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} iu \frac{\partial \tilde{P}_j}{\partial v_t} du, \qquad j = 0, 1.$$

Putting those partial derivatives back in (1.32) and rearranging it so that all the terms are inside the same integral, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t} \left(-\frac{\partial \tilde{P}_j}{\partial \tau} - \frac{1}{2} v_t u^2 \tilde{P}_j - v_t \left(\frac{1}{2} - j \right) iu\tilde{P}_j + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 \tilde{P}_j}{\partial v_t^2} + \right.$$
$$\left. \rho\sigma v_t iu \frac{\partial \tilde{P}_j}{\partial v_t} + (a - b_j v_t) \frac{\partial \tilde{P}_j}{\partial v_t} \right) du = 0, \qquad j = 0, 1. \tag{1.36}$$

The only way to make (1.36) true is to have

$$\left(-\frac{\partial \tilde{P}_j}{\partial \tau} - \frac{1}{2} v_t u^2 \tilde{P}_j - v_t \left(\frac{1}{2} - j \right) iu\tilde{P}_j + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 \tilde{P}_j}{\partial v_t^2} \right.$$
$$\left. + \rho\sigma v_t iu \frac{\partial \tilde{P}_j}{\partial v_t} + (a - b_j v_t) \frac{\partial \tilde{P}_j}{\partial v_t} \right) = 0. \tag{1.37}$$

Then, define

$$\begin{aligned} a_j &= -\frac{u^2}{2} - \frac{iu}{2} + iju \\ \beta_j &= \kappa - \rho\sigma j - \rho\sigma iu \\ \gamma &= \frac{\sigma^2}{2}, \qquad j = 0, 1, \end{aligned}$$

so that (1.37) can be written as

$$v_t \left(a_j \tilde{P}_j - \beta_j \frac{\partial \tilde{P}_j}{\partial v_t} + \gamma \frac{\partial^2 \tilde{P}_j}{\partial v_t^2} \right) + a_j \frac{\partial \tilde{P}_j}{\partial u} - \frac{\partial \tilde{P}_j}{\partial \tau} = 0. \tag{1.38}$$

for $j = 0, 1$. Next, following Heston (1993) and using (1.33), we write $\tilde{P}_j$ as

$$\begin{aligned} \tilde{P}_j(u, v_t, \tau) &= e^{C_j(u,\tau)\theta + D_j(u,\tau)v_t} \tilde{P}_j(u, v_t, 0) \\ &= e^{C_j(u,\tau)\theta + D_j(u,\tau)v_t} \int_{-\infty}^{\infty} e^{-iux_t} P_j(x_t, v_t, 0) dx_t \\ &= e^{C_j(u,\tau)\theta + D_j(u,\tau)v_t} \int_{0}^{\infty} e^{-iux_t} dx_t \\ &= e^{C_j(u,\tau)\theta + D_j(u,\tau)v_t} \frac{1}{iu}. \end{aligned} \tag{1.39}$$

Since the terms $C_j(u,\tau)$ and $D_j(u,\tau)$ are not functions of v_t, it makes $\tilde{P}_j(u,v,0)$ easier to differentiate with respect to v_t:

$$\frac{\partial \tilde{P}_j}{\partial \tau} = \left(\theta \frac{\partial C_j}{\partial \tau} + v_t \frac{\partial D_j}{\partial \tau}\right) \tilde{P}_j$$
$$\frac{\partial \tilde{P}_j}{\partial v_t} = D_j \tilde{P}_j$$
$$\frac{\partial^2 \tilde{P}_j}{\partial v_t^2} = D_j^2 \tilde{P}_j,$$

where we write C_j and D_j for $C_j(u,\tau)$ and $D_j(u,\tau)$, respectively. Thus, (1.38) can be re-written as

$$v_t \left(a\tilde{P}_j - \beta_j D_j \tilde{P}_j + \gamma D_j^2 \tilde{P}_j\right) + a_j D_j \tilde{P}_j - \left(\theta \frac{\partial C_j}{\partial \tau} + v_t \frac{\partial D_j}{\partial \tau}\right) = 0, \qquad j = 0, 1.$$

Dividing by $\tilde{P}_j$ and rearranging the terms, keeping in mind that $a = \kappa\theta$ we get

$$v_t \left(a_j - \beta_j + \gamma D_j^2 - \frac{\partial D_j}{\partial \tau}\right) + \theta \left(\kappa D_j - \frac{\partial C_j}{\partial \tau}\right) = 0, \qquad j = 0, 1. \tag{1.40}$$

In order for (1.40) to be true without having $\tilde{P}_j(u, v_t, \tau) = \tilde{P}_j(u, v_t, 0)$, we need both parts of the equation to be equal to 0. Hence, we obtain

$$\begin{aligned} \frac{\partial D_j}{\partial \tau} &= a_j - \beta_j D_j + \gamma D_j^2 \\ \frac{\partial C_j}{\partial \tau} &= \kappa D_j. \end{aligned} \tag{1.41}$$

For simplification purposes, we write

$$\frac{\partial D_j}{\partial \tau} = \gamma (D_j - r_{j,+})(D_j - r_{j,-})$$

where

$$r_{j,\pm} = \frac{\beta_j \pm \sqrt{\beta_j^2 - 4a_j\gamma}}{2\gamma} = \frac{\beta_j \pm d_j}{\sigma^2}, \qquad j = 0, 1.$$

Next, we need to integrate (1.41) subject to the terminal conditions $C_j(u,0) = 0$ and $D_j(u,0) = 0$, so that

$$\begin{aligned}\tilde{P}_j(u, v_t, 0) &= e^{C_j(u,0)\theta + D_j(u,0)v_t}\tilde{P}_j(u, v_t, 0) \\ &= 1 \times \tilde{P}_j(u, v_t, 0).\end{aligned} \tag{1.42}$$

Integrating (1.41) gives

$$\begin{aligned}D_j(u,\tau) &= r_- \frac{1 - e^{-d_j\tau}}{1 - \omega_j e^{-d_j\tau}} \\ C_j(u,\tau) &= \kappa\left(r_-\tau - \frac{2}{\sigma^2}\log\left[\frac{1 - \omega_j e^{-d_j\tau}}{1 - \omega_j}\right]\right),\end{aligned} \tag{1.43}$$

where

$$\omega_j = \frac{r_{j,+}}{r_{j,-}}, \qquad j = 0, 1.$$

Putting (1.43) back in (1.35), we obtain

$$\begin{aligned}P_j(x, v_t, \tau) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux_t}\left(\frac{e^{C_j(u,\tau)\theta + D_j(u,\tau)v_t}}{iu}\right) du \\ &= \int_{-\infty}^{\infty} \frac{1}{2iu\pi} \exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t) du.\end{aligned} \tag{1.44}$$

According to Gatheral (2006), it is possible to perform the integral in (1.44) to obtain a nicer, real-valued integral of the form

$$P_j(x_t, v_t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{\exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t)}{iu}\right) du, \tag{1.45}$$

for $j = 0, 1$, which is numerically integrable. Hence, the future price to expiration of a European call option under the Heston model can be expressed by

$$C'(x_t, v_t, \tau) = K(e^{x_t} P_1(x_t, v_t, \tau) - P_0(x_t, v_t, \tau)) \tag{1.46}$$

with $P_j(x_t, v_t, \tau)$, $j = 0, 1$, as in (1.45). To obtain $C_t^H(x_t, v_t, \tau)$ the price of the call at time t under the Heston model, $0 < t < T$, it suffices to discount $C'(x_t, v_t, \tau)$ using the risk-free rate r, such that

$$C^H(x_t, v_t, \tau) = Ke^{-r\tau}(e^{x_t} P_1(x_t, v_t, \tau) - P_0(x_t, v_t, \tau)). \tag{1.47}$$

1.2.4 Hedging under the Heston model

As explained in the previous subsection, hedging an option under the Heston model requires the addition of a third asset to the hedging portfolio. This results from the presence of two sources of randomness influencing the price of the option, namely the price of the underlying and the stochastic volatility. As in Section 1.1.4, the hedging strategy relies on a replicating portfolio whose value is the same as that of the call at any time t, $0 \leq t \leq T$. For the hedge to be efficient, holding this portfolio and the call option simultaneously must be risk-free. In other words, the total hedged position must be insensitive to variations of both the price of the underlying and the volatility of its returns.

We show here that the replicating portfolio that hedges a derivative under the Heston model, is based on the hedging ratios $\alpha'_{1,t}$ and $\alpha'_{2,t}$ obtained in Section 1.2.3 when pricing the call option. Thus, H_t, the replicating portfolio at time t, is given by:

$$H_t = \alpha_{2,t} S_t + \alpha_{1,t} V_t^1 + \xi_t, \tag{1.48}$$

where S_t and V_t^1 are the prices of the underlying and the third asset, respectively, $\alpha_{1,t}$ and $\alpha_{2,t}$ are defined as in (1.14) and (1.15) and

$$\xi_t = C^H(x_t, v_t, \tau) - \alpha_{2,t} S_t + \alpha_{1,t} V_t^1. \tag{1.49}$$

That is, at any time t, amounts of $\alpha_{2,t} S_t$ and $\alpha_{1,t} V_t^1$ are invested in the underlying and in the third asset, respectively. Similarly the strategy presented in Section 1.1.4, the strategy presented here is self-financing. The difference between the sum of these amounts and the price of the option is invested in the risk-free asset. This amount may become negative, which signifies that the investor needs to borrow to finance the hedge and ensure that the replicating portfolio and the option keep the same value at any time.

To demonstrate that simultaneously taking a short position in the call and a long one in the replicating portfolio results in a risk-free position, we show that the derivatives of the resulting position with respect to S_t and v_t, the instantaneous volatility, are both equal to 0. In other words, the portfolio is insensitive to both sources of risk.

First, note that ξ_t, the amount invested in the risk-free asset, grows at the risk free rate regardless of the price of the underlying and the volatility of its returns. Hence,

$$\begin{aligned} \frac{\partial \xi_t}{\partial S_t} &= 0 \\ \frac{\partial \xi_t}{\partial v_t} &= 0. \end{aligned} \tag{1.50}$$

Let $\Pi_t^* = H_t - C^H(s_t, v_t, \tau)$ denote the value of the overall position at time t. Thus, using the definition of $\alpha_{1,t}^*$ and $\alpha_{2,t}^*$ given by (1.14) and (1.15), we have that

$$\frac{\partial \Pi_t^*}{\partial S_t} = \alpha_{2,t}^* + \alpha_{1,t}^* \frac{\partial V_t^1}{\partial S_t} - \frac{\partial C_t^H}{\partial S_t} = 0 \tag{1.51}$$

and

$$\frac{\partial \Pi_t^*}{\partial v_t} = \alpha_{1,t}^* \frac{\partial V_t^1}{\partial v_t} - \frac{\partial C_t^H}{\partial v_t} = 0. \tag{1.52}$$

It follows that the expressions $\alpha^*_{1,t}$ and $\alpha^*_{2,t}$, used to eliminate risk from the valuation portfolio given by (1.8) in Section 1.2.3, are also used to derive the hedging strategy implied by the pricing method. In fact, the overall risk-free position Π^*_t is simply the sum of the hedged portfolio defined by (1.8) in Section 1.2.3 and a risk-free investment. Thus, the hedging strategy introduced in this section underlies the pricing approach presented in the previous one.

1.2.5 Calibrating the Heston model

To calibrate the Heston model, one must take many issues into consideration. Stochastic volatility models contain a latent variable, v_t, which is not directly observable in market prices, thus calling for sophisticated filtering techniques. A second issue is that, in addition to estimating the parameters κ', θ', σ, ρ and v_0, which could be obtained from historical prices, one must estimate λ, the volatility risk premium. This last task usually requires the observation of the price of at least one derivative.

When the Heston model is calibrated with the sole purpose of pricing non-traded derivatives, it suffices to estimate the risk-neutral parameter set $\varphi = (\kappa, \theta, \sigma, v_0, \rho)$. This can be done by minimizing the sum of the squares of the difference between observed market prices of call options and model-calculated ones, as explained in Moodley (2005). However, this method fails to estimate the volatility risk premium λ.

To calibrate the full Heston model, including λ, Aït-Sahalia & Kimmel (2007), amongst others, uses maximum likelihood while Garcia *et al.* (2011) proposes a procedure based on the method of moments. Both methods require extensive intraday quotes and historical option prices to estimate volatility and the price of risk. Their implementation is time consuming and their results are not always guaranteed, as they rely on complex system of equations that may not have an solution. The use of such methods, although beyond the scope of this work, might be considered for future research.

In this thesis, the calibration of the model will be done in two steps. We will first estimate the risk-neutral parameter set φ using observed option prices and optimization techniques. The estimation of the volatility risk premium will be performed subsequently.

To estimate the risk-neutral parameter set φ, we follow Chapter III of Moodley (2005) closely. The objective is to find the parameter set $\hat{\varphi}$ which minimizes the function

$$M(\varphi) = \Sigma_{i=1}^{N} w_i \left(C_i^{\varphi}(K_i, \tau_i) - C_i^{mkt}(K_i, \tau_i) \right)^2, \tag{1.53}$$

where $C^{mkt}(K_i, \tau_i)$ is the price of a call option of strike K_i and time to maturity τ_i observed on the market, and $C^{\varphi}(K_i, \tau_i)$ is the price of the same option under the Heston model, calculated using (1.47) and parameter set φ. N is the number of options used to calibrate the model and w_i are the weights, which will be discussed later. We choose $C^{mkt}(K_i, \tau_i)$ to be the mid-point between the ask and the bid price of the option.

Since we want to calibrate the model to price 10-year maturity contracts, we use options with the longest maturities possible, from 6 months to over 2 years. However, some of the longer maturity, far-from-the-money options may be less liquid, which will be reflected by a larger bid-ask spread. This may pose a problem, since we assume that market prices of traded securities contain information about the market. The more an option is traded, the more information is contained in its price. Large bid-ask spread signify that the option is less traded and thus, that it contains less information. This is why we let the weights w_i give more importance to the options with smaller bid-ask spreads, defining

$$w_i = \frac{1}{|bid_i - ask_i|},$$

where bid_i and ask_i are the bid and the ask price, respectively, of the i^{th} call option.

The function defined by (1.53) is highly non-linear and contains numeric integrals. It may present numerous local minima, which makes it complicated to minimize. Various optimization methods can be used to solve it, some more efficiently than others. Moodley (2005) points out that the MATLAB function `lsqnonlin` performs relatively well and has the advantage of being fast. This function is based on an *interior-reflective Newton* method, with the disadvantage of returning only a local minimum. This makes the solution highly dependent on the initial parameters. Our solution to this inconvenient was to perform the minimization with different sets of initial parameters and to analyze the results using two criteria. First, any parameter set that does not satisfy

$$\Sigma_{i=1}^{N}(bid_i - ask_i)^2 \ \geq \ \Sigma_{i=1}^{N}\left(C_i^{\varphi}(K_i,\tau_i) - C_i^{mkt}(K_i,\tau_i)\right)^2 \tag{1.54}$$

is rejected. The calibration uses the mid-point of the bid-ask spread, but we accept a parameter set that lets the option prices be, on average, between the bid and ask prices. The optimal parameter set, $\hat{\varphi}$, is the one that returns the lowest value of $M(\varphi)$.

We performed the optimization using the prices of 67 European call options on the

S&P 500 index as of May 20, 2011[1]. We chose options with 0.57 to 2.59 years to maturity, with moneyness ranging from 78.7% to 127.4%. We discarded any option with a trading volume below 10. Using the optimization procedure described above and the a slightly modified version of the code provided by Moodley (2005), we obtained $\hat{\varphi} = (5.1793, 0.0178, 0.1309, 0.0286, -0.7025)$. These results are mostly in line with the parameters found by other authors such as Aït-Sahalia & Kimmel (2007), Gatheral (2006) and Bakshi *et al.* (1997). However, the use of $r = 0.02$, justified by the long horizon of the EIA to price, may have lead to a higher κ and a lower σ than expected. Our choice of options with longer maturity may have also contributed to those results. Nonetheless, this parameter set meets (1.54) and yields $M(\hat{\varphi}) = 171.646$. Furthermore, it satisfies $\frac{4\kappa\theta}{\sigma^2} > 1$, which, as will be seen in Section 2.5.2, facilitates the discretization of the process.

The volatility risk premium, represented by λ, is then chosen so that the variance of simulated weekly log-returns [2] matches the values observed in the historical data. Remember that although λ does not appear explicitly in the volatility process, it is reflected in κ' and θ' through the relation expressed in (1.21).

Denote by $\hat{\lambda}$ the value which, when used in the simulations, yields the smallest difference between the observed and the simulated moments. Testing different values for λ ranging from -3 to 3 by performing 100,000 Monte Carlo simulations for each, we obtain $\hat{\lambda} = 2.62$. The volatility risk premium is generally agreed to be negative and

[1]Data from cboe.com.

[2]The data was simulated using the method described in Section 2.5.2.

Bakshi & Kapadia (2003), amongst others, shows evidence of this. However, in our case, the fact that historical volatility is higher due to the recent financial crisis and the use of $r = 0.02$, which affects our Heston calibration, may explain our positive risk premium. Nonetheless, in our numerical example, we will also test other values of λ to account for cases where the premium is negative.

Chapter 2

Hedging in Discrete Time

2.1 Introduction

In Chapter 1, hedging strategies underlying the valuation of the call price under the Black-Scholes and the Heston model were introduced. Those strategies involve buying the underlying, and a third asset in the Heston case, in certain quantities. Those quantities, being dependent on time, underlying price and volatility, vary continuously in the same manner as the parameters do. In real life, this means that an investor wanting to hedge his position in the derivative would need to adjust his investment in the underlying and the third asset, when necessary, in a continuous manner, which is impossible. In fact, even if he did adjust his position every minute a discretization effect, albeit very slight, would be perceptible in the performance of the strategy. This effect translates into additional costs, and it follows that the strategy applied in a discrete manner is no longer self-financing.

Furthermore, applying such hedging strategies in real life entails certain restrictions.

In fact, while the models assume no transaction costs, they are a reality for investors. In this thesis, we do not account for them explicitly. However, we acknowledge the fact that they limit the frequency with which the investor can rebalance his portfolio. For example, he might decide to adjust his position daily, weekly or even bi-monthly. Such a discretization of a strategy developed under a continuous model gives rise to additional costs, which are described in more details in Section 2.2. To reduce them, investors try to match the sensitivity of the replicating portfolio with those of the call option price, so that they vary in a similar way. Those sensitivities, called the *Greeks* of the option, are introduced in Section 2.3. Hedging strategies using the Greeks are presented in Section 2.4. Section 2.5 explains how to discretize the dynamics of stock prices to simulate them and thus analyze the performance of the different strategies.

2.2 Hedging Errors

Hedging errors are inevitable when a hedging strategy developed in continuous time is applied at discrete time intervals. The proportions of the replicating portfolio invested in each asset depend on parameters that change continuously and thus, those proportions should also be adjusted in a continuous manner. In reality, adjustments occur periodically, causing the value of the replicating portfolio to move away from that of the derivative between adjustments. Thus, hedging errors arise at rebalancing, when the replicating portfolio is adjusted so that its value is the same as that of the position to be hedged. Hedging errors are defined as the difference between the value of the position to be hedged and the actual accumulated value in the replicating portfolio.

The hedging error at time t is defined by

$$HE_t = V_t - H_{t^-}, \tag{2.1}$$

where H_{t^-} is the value of the replicating portfolio at time t before rebalancing occurs. Notice that hedging errors can be negative, forcing the investor to re-invest in the replicating portfolio, or positive, allowing him to withdraw the surplus.

The total discounted hedging error, denoted $PV(HE)$, represents the cost of the strategy at inception of the contract. Assume rebalancing occurs m times a year at equal time intervals. Then, the total discounted hedging error is given by

$$PV(HE) = \sum_{i-1}^{mT} e^{\frac{-ir}{m}} HE_t. \tag{2.2}$$

The distribution of the total discounted hedging errors will later be used to assess the performance of different hedging strategies.

2.3 The Greeks

In finance, the Greeks represent the sensitivities of the price a derivative, or a portfolio of derivatives, to different parameters. The name "Greeks" comes from the fact that most of those sensitivities are denoted by Greek letters. They are central elements of the hedging strategies presented in this chapter.

In this thesis, we will focus on three Greeks in particular: the *delta*, the *gamma* and the *vega*. There exists many other Greeks and they can also be used for hedging, depending on the risk the investor aims at removing (see Wilmott (2006) for more details).

Definition 2.1. *Let V_t and S_t be the price of a derivative of maturity T and the price* *of its underlying,respectively, at time t, $0 \leq t \leq T$. Then the* delta *of the investor's position in the derivative is denoted $\Delta_{V,t}$ and is defined as*

$$\Delta_{V,t} = \frac{\partial V_t}{\partial S_t}. \tag{2.3}$$

The delta measures the sensitivity of the investor's position in the derivative to changes in the price of the underlying. It is a very important element of dynamic hedging strategies, which will be presented in the next section.

Definition 2.2. *The* gamma *of a position V_t is denoted $\Gamma_{V,t}$ and is given by*

$$\Gamma_{V,t} = \frac{\partial^2 V_t}{\partial S_t^2}. \tag{2.4}$$

Thus, the second derivative of the value of the position with respect to the price of the underlying is called the *gamma*. It measures the sensitivity of the delta of a position to variations in the price of the underlying. Since it is the derivative of the delta with respect to the underlying price, it can be used to improve hedging strategies using the delta.

In the Black-Scholes model, the volatility of the return of the underlying is assumed to be constant. However, other models, such as the Heston model presented in Chapter 1, represent the changes in volatility by a stochastic process, so the parameter varies through time.

Definition 2.3. *Let v_t be the instantaneous volatility of the price of the underlying at time t. Let also $\mathcal{V}_{V,t}$ be the* vega *of the position V_t, then*

$$\mathcal{V}_{V,t} = \frac{\partial V_t}{\partial v_t}, \tag{2.5}$$

The vega of a position measures the sensitivity of its price to the variation in the volatility of the underlying. In practice, it is incorporated to hedging strategies in an attempt to eliminate volatility risk.

Closed and semi-closed form expressions for the delta, gamma and vega of a European call option under the Black-Scholes model and the Heston model are presented in the examples that follow. These expressions are very important, not only to hedge positions taken in call options; they can be part of strategies that aim at hedging more exotic derivatives, some of which will be discussed in Chapter 3.

Example 2.1 (Greeks of a European call option under Black-Scholes)**.** *To obtain the Greeks of a European call option in a Black-Scholes setting, it suffices to take the appropriate derivatives of the price of the option, which is given by (1.4). Let* $\Delta_{C,t}^{BS}$*,* $\Gamma_{C,t}^{BS}$ *and* $\mathcal{V}_{C,t}^{BS}$ *be the delta, the gamma and the vega of the call option at time* t*, respectively, under the Black-Scholes model. Then the Greeks of the call option are given by*

$$\begin{aligned}\Delta_{C,t}^{BS} &= \Phi(d_1) \\ \Gamma_{C,t}^{BS} &= \frac{\phi(d_1)}{S_t \sigma_{BS} \sqrt{T-t}} \\ \mathcal{V}_{C,t}^{BS} &= S_t \phi(d_1) \sqrt{(T-t)},\end{aligned} \tag{2.6}$$

where Φ *and* ϕ *are the distribution and the density functions of a standard normal variable, respectively, and*

$$d_1 = \frac{\log \frac{S_t}{K} + \left(r + \frac{\sigma_{BS}^2}{2}\right)(T-t)}{\sigma_{BS}\sqrt{T-t}}.$$

Example 2.2 (Greeks of a European call option under Heston)**.** *One of the advantages of the Heston model is that it allows for a semi-closed expression for the price of the*

European call option. This expression, presented in Chapter 1, contains an integral which can be solved numerically. To obtain the Greeks of the call option in a Heston environment, one must differentiate the expression with respect to the appropriate parameters. Those derivatives are easily obtained since $C_j(u,\tau)$ *and* $D_j(u,\tau)$, $j = 0,1$, *are not functions of* S_t *and* v_t.

Let $P_j(x_t, v_t, \tau)$ *be as defined in (1.45). Let also* $\Delta^H_{C,t}$, $\Gamma^H_{C,t}$ *and* $\mathcal{V}^H_{C,t}$ *be the delta, the gamma and the vega of the call option at time* t, *respectively, under the Heston model. Then the Greeks of the call option are given by*

$$\begin{aligned}
\Delta^H_{C,t} &= P_1 + \frac{\partial P_1}{\partial x_t} - e^{-x_t}\frac{\partial P_0}{\partial x_t} \\
\Gamma^H_{C,t} &= \frac{1}{S_t}\left[\frac{\partial P_1}{\partial x_t} - \frac{\partial^2 P_1}{\partial x_t^2} - e^{-x_t}\left(\frac{\partial^2 P_0}{\partial x_t^2} - \frac{\partial P_0}{\partial x_t}\right)\right] \\
\mathcal{V}^H_{C,t} &= Ke^{-r\tau}\left(e^{x_t}\frac{\partial P_1}{\partial v_t} - \frac{\partial P_0}{\partial v_t}\right),
\end{aligned} \tag{2.7}$$

where $P_j, j = 0,1$ *are short notation for* $P_j(x_t, v_t, \tau)$, *which are defined in (1.45), and*

$$\begin{aligned}
\frac{\partial P_j}{\partial x_t} &= \frac{1}{\pi}\int_0^\infty Re\left(\exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t)\right) du \\
\frac{\partial^2 P_j}{\partial x_t^2} &= \frac{1}{\pi}\int_0^\infty Re\left(iu\exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t)\right) du \\
\frac{\partial P_j}{\partial v_t} &= \frac{1}{\pi}\int_0^\infty Re\left(\frac{D_j(u,\tau)\exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t)}{iu}\right) du,
\end{aligned} \tag{2.8}$$

for $j = 0,1$.

The sensitivities of financial derivatives to the different parameters can be used to hedge risky positions. Some hedging strategies using the Greeks are presented in the next section.

2.4 Dynamic Hedging Strategies

As explained in Chapter 1, the models considered in this thesis assume that the markets are free of arbitrage. This assumption, common to many financial models, leads to the construction of a portfolio replicating the position to be hedged. That portfolio should offset the initial position at any given time and for any underlying price, so that taking the initial position and holding the replicating portfolio simultaneously becomes risk free. Thus, by the no-arbitrage assumption, the return on the overall position is the risk-free rate r.

This replicating portfolio allows us to price and hedge any derivative whose payoff depends on the price of the underlying. This technique was used in Chapter 1 to derive the price of the European call option and to hedge it under both Black-Scholes and Heston models. In this chapter, we use similar replicating portfolios to develop more general hedging strategies. We assume that the position to be hedged is one taken in any financial derivative denoted V_t. For the overall position to be risk-free, or, in other words, to hedge the risks involved in the initial position, the other securities need to be bought or sold in the right quantities. In this section, we assume that the investor takes a short position in the derivative and identify the quantities which make the portfolio risk-free. This can be extended to a long position case by simply inverting the sign of the derivative.

2.4.1 Delta Hedging

The main risk associated with taking a short position in a financial derivative lies in the randomness of the price of the underlying. One way to hedge this risk is to construct a replicating portfolio containing the underlying and the risk-free asset, as presented in Section 1.1.4. To ensure that the replicating portfolio and the price of the derivative vary similarly as the underlying price changes, their first derivatives with respect to the underlying price should be matched. As the strategy is applied in discrete time, this helps to keep the value of the replicating portfolio close to the price of the derivative between portfolio adjustments.

Denote Δ_t^* and ξ_t the position taken in the underlying and the amount invested in the risk-free asset, respectively, at time t, $0 \leq t \leq T$, where T is the maturity of the derivative. Then, the value of the resulting portfolio at time t, denoted H_t^Δ, is given by

$$H_t^\Delta = \Delta_t^* S_t + \xi_t. \tag{2.9}$$

Thus, the overall position is given by

$$\Pi_t^\Delta = H_t^\Delta - V_t. \tag{2.10}$$

For the overall position to be protected against changes in the underlying price, its derivative with respect to S_t must be 0. Hence, we need to choose the hedging ratio Δ_t^* so that

$$\frac{\partial \Pi_t^\Delta}{\partial S_t} = 0. \tag{2.11}$$

As explained in Section 1.2.4, the amount invested in the risk-free asset is insensitive to change in the underlying price. Thus,

$$\frac{\partial \Pi_t^\Delta}{\partial S_t} = \Delta_t^* - \frac{\partial V_t}{\partial S_t}, \tag{2.12}$$

so $\Delta_t^* = \frac{\partial V_t}{\partial S_t}$ satisfies (2.11). Note that $\frac{\partial V_t}{\partial S_t}$ is the delta of the position V_t, denoted $\Delta_{V,t}$. As stated before, this strategy is a generalization of the call option hedging strategy presented in Section 1.1.4, in which the amount invested in the underlying at any time t was given by $\Phi(d_1)S_t$, with $\Phi(d_1)$ defined as in (1.4). This quantity is in fact the delta of the call option, as seen in (2.6).

It is important to realize that $\Delta_{V,t}$ is generally not a constant. It is a function of the underlying price S_t and of the time to expiration of the contract, $T - t$. It can also depend on other parameters according to the model chosen. Hence, the discretization of this strategy will cause additional costs, which arise from the discrepancies between the evolution of the portfolio predicted by the model and the actual changes in the price of the securities. In fact, at time $t = 0$ and after each adjustment, the value of H_t must be equal to V_t. Hence, H_t^{Δ} is adjusted periodically so that the proportion of the replicating portfolio invested in the underlying index remains $\Delta_{V,t}$. Thus, we have that ξ_t, the amount invested in the risk-free account at time t, is given by

$$\xi_t = V_t - \Delta_{V,t} S_t. \tag{2.13}$$

2.4.2 Gamma Hedging

Since the hedging portfolio cannot be rebalanced continuously, hedging errors occur when adjusting the replicating portfolio. A way to reduce those errors without changing the rebalancing frequency is to neutralize both first and second derivatives of the overall position with respect to the underlying price. In other words, we want the first derivative of the delta of the position to be equal to 0. Hence, as the price of the underlying changes slightly, the delta of the position remains constant instead of varying with the fluctuations of the underlying, as it is the case with delta hedging. This contributes to reducing hedging errors.

As seen in the previous section, the second derivative of the position V_t with respect to the price of the underlying is called the gamma of the position and is denoted $\Gamma_{V,t}$. Since the gamma of the underlying is 0, we need to add another security to the replicating portfolio to neutralize the gamma of the initial position. This security must be another derivative of the same underlying and have a gamma different than 0. We assume that this security is always available on the market. In this thesis, without loss of generality, we also assume that this security is an option and we denote V_t^1 its price at time t. Hence, the value of the replicating portfolio at time t is given by:

$$H_t^\Gamma = \alpha_{1,t}^* V_t^1 + \alpha_{2,t}^* S_t + \xi_t, \tag{2.14}$$

where $\alpha_{1,t}^*$ and $\alpha_{2,t}^*$ are the proportions of the portfolio invested in the third asset and in the underlying, respectively, and ξ_t is the amount invested in the risk-free asset. Since the value of the replicating portfolio must be equal to that of the derivative, we have that

$$\xi_t = V_t - \alpha_{1,t}^* V_t^1 + \alpha_{2,t}^* S_t, \tag{2.15}$$

for any t, $0 \leq t \leq T$.

Let Π_t^Γ be the value of the overall position, with value at time t given by

$$\Pi_t^\Gamma = H_t^\Gamma - V_t. \tag{2.16}$$

To neutralize the delta and the gamma of the overall position, we choose $\alpha_{1,t}^*$ and $\alpha_{2,t}^*$ such that

$$\frac{\partial \Pi_t^\Gamma}{\partial S_t} = 0,$$

and

$$\frac{\partial^2 \Pi_t^\Gamma}{\partial S_t^2} = 0.$$

Since

$$\frac{\partial \Pi_t^\Gamma}{\partial S_t} = \alpha_{1,t}^* \frac{\partial V_t^1}{\partial S_t} + \alpha_{2,t}^* - \frac{\partial V_t}{\partial S_t},$$

and

$$\frac{\partial^2 \Pi_t^\Gamma}{\partial S_t^2} = \alpha_{1,t}^* \frac{\partial^2 V_t^1}{\partial S_t^2} - \frac{\partial^2 V_t}{\partial S_t^2},$$

we need to choose $\alpha_{1,t}^* = \alpha_{1,t}^\Gamma$ and $\alpha_{2,t}^* = \alpha_{2,t}^\Gamma$, where

$$\alpha_{1,t}^\Gamma = \frac{\Gamma_{V,t}}{\Gamma_{V^1,t}}, \tag{2.17}$$

$$\alpha_{2,t}^\Gamma = \Delta_{V,t} - \alpha_{1,t}^G \Delta_{V^1,t}, \tag{2.18}$$

and $\Delta_{V,t}$, $\Delta_{V^1,t}$, $\Gamma_{V,t}$ and $\Gamma_{V^1,t}$ are the delta and the gamma of the derivative and the option, respectively.

2.4.3 Hedging Volatility: Vega Hedging

In the Black-Scholes model the volatility of stock returns is assumed to be constant. However, as explained in Section 1.2.1, empirical evidence suggests that it is not the case. Many models, such as the Heston model, assume that the volatility of stock returns follows a stochastic process. Under this assumption, two different sources of randomness influence the value of the initial position in the derivative. Thus, hedging it by neutralizing its delta only is not sufficient, since it does not protect the investor against changes in the volatility of the underlying returns. Incorporating the derivative of the overall position with respect to the instantaneous volatility, the vega, allows for a better performing strategy.

To hedge both sources of randomness, a third asset is necessary. As in gamma hedging, the price of this third asset needs to depend on the price of the underlying and its vega has to be different that 0. Again, we assume that it always exists and, for this thesis, we consider that it is an option and denote its price at time t by V_t^1. Hence, the value of the delta-vega replicating portfolio at time t is given by

$$H_t^{\mathcal{V}} = \alpha_{1,t}^* V_t^1 + \alpha_{2,t}^* S_t + \xi_t, \tag{2.19}$$

where ξ_t, the difference between the value of the derivative and that of $\alpha_{1,t}^* V_t^1 + \alpha_{2,t}^* S_t$, is the amount invested in the risk-free asset. Thus, $\Pi_t^{\mathcal{V}}$, the value of the overall position at time t, $0 \leq t \leq T$, is defined as

$$\Pi_t^{\mathcal{V}} = H_t^{\mathcal{V}} - V_t. \tag{2.20}$$

This setting is very similar to (2.14). However, in this case, we need to choose different $\alpha_{1,t}^*$ and $\alpha_{2,t}^*$ to obtain

$$\frac{\partial \Pi_t^{\mathcal{V}}}{\partial S_t} = 0$$

and

$$\frac{\partial^2 \Pi_t^{\mathcal{V}}}{\partial v_t} = 0,$$

where v_t is the volatility of the price of the underlying at time t. Taking the derivatives of the overall position with respect to S_t and v_t, we find

$$\frac{\partial \Pi_t^{\mathcal{V}}}{\partial S_t} = \alpha_{1,t}^* \frac{\partial V_t^1}{\partial S_t} + \alpha_{2,t}^* \frac{\partial V_t}{\partial S_t}$$

and

$$\frac{\partial \Pi_t^{\mathcal{V}}}{\partial v_t} = \alpha_{1,t}^* \frac{\partial V_t}{\partial v_t} + \alpha_{2,t}^* \frac{\partial S_t}{\partial v_t} - \frac{\partial V_t}{\partial v_t} - .$$

To neutralize the delta and the vega of the portfolio, we need to choose $\alpha_{1,t}^* = \alpha_{1,t}^{\mathcal{V}}$ and $\alpha_{2,t}^* = \alpha_{2,t}^{\mathcal{V}}$, with

$$\alpha_{1,t}^{\mathcal{V}} = \frac{\mathcal{V}_{V,t}}{\mathcal{V}_{V^1,t}} \tag{2.21}$$

and

$$\alpha_{2,t}^{\mathcal{V}} = \Delta_{V,t} - \alpha_{1,t}^{\mathcal{V}} \Delta_{V^1,t}, \tag{2.22}$$

where Δ_{V_t} and $\Delta_{V_t^1}$ are as in (2.18).

Notice that, in the Heston model, when the derivative to hedge is a call option, the hedging proportions $\alpha_{1,t}^{\mathcal{V}} = \alpha_{1,t}$ and $\alpha_{2,t}^{\mathcal{V}} = \alpha_{2,t}$, with $\alpha_{1,t}$ and $\alpha_{2,t}$ defined as in (1.14) and (1.15). Thus, the hedging strategy underlying the valuation of the call option in the Heston model is a delta-vega hedging strategy.

2.5 Discretization

In this thesis, Monte Carlo simulations are used to analyze the distribution of the total discounted hedging error under different hedging strategies. In order to perform those simulations, sample paths of stock prices need to be generated. To do so, we need to discretize the continuous processes that describe stock prices and volatilities of returns. We present here the methods used to discretize those processes and to generate sample paths.

2.5.1 Discretization under Black-Scholes

Stock price dynamics under the Black-Scholes model are given by (1.1). Using the Itô-Doeblin formula, this can be re-written in the integral form:

$$S_t = S_0 \, e^{X_t}, \tag{2.23}$$

where

$$\begin{cases} X_t = X_0 + \int_0^t \left(\mu_{BS} - \frac{\sigma_{BS}^2}{2}\right) dt + \int_0^t \sigma_{BS} dZ_t, \\ X_0 = 1, \end{cases} \tag{2.24}$$

with μ_{BS} and σ_{BS} defined as in (1.1).

Suppose that we divide a year into m time periods and let $\delta = \frac{1}{m}$. Suppose also that we observe the price of the stock at the beginning of each time period. Then the process X_t can be discretized by

$$X_{t+\delta} = X_t + \left(\mu_{BS} - \frac{\sigma_{BS}^2}{2}\right)\delta + \sigma_{BS}\sqrt{\delta}\mathcal{N}(0,1), \tag{2.25}$$

where $\mathcal{N}(0,1)$ is a standard normal random variable. This method performs better than a simple Euler discretization of (1.1) because there are no higher order corrections in the Itô-Taylor expansion of X_t. It is the equivalent of sampling directly from the normal distribution of the log-returns.

2.5.2 Discretization under Heston

Under the Heston model, two processes need to be discretized. The stock price process can be approached the same way as for the Black-Scholes model, such that

$$\begin{cases} X_{t+\delta} = X_t + \left(\mu - \frac{v_t}{2}\right)\delta + \sqrt{v_t\delta}\mathcal{N}(0,1), \qquad 0 \le t \le T, \\ X_0 = 1, \end{cases} \tag{2.26}$$

and

$$S_t = S_0\, e^{X_t}, \qquad 0 \le t \le T. \tag{2.27}$$

The discretization of the volatility process described by (1.7) requires more work. It is possible to apply a simple Euler discretization and get

$$v_{t+h} = v_t + \kappa'\left(\theta' - v_t\right)ht + \sigma\sqrt{v_t ht}\mathcal{N}(0,1). \tag{2.28}$$

However, such a method may simulate negative variance, which is undesirable. To address this problem, we go one order higher in the Itô-Taylor expansion of v_t and use the Milstein scheme. This scheme says that if a stochastic process Y_t is described by

$$dY_t = a(Y_t,t)dt + b(Y_t,t)dZ_t,$$

then Y_t can be discretized as follows:

$$Y_{t+h} = Y_t + \left(a(Y_t,t) - \frac{1}{2} b(Y_t,t) \frac{\partial}{\partial Y_t} b(Y_t,t) \right) h + b(Y_t,t) h \mathcal{N}(0,1) + \frac{1}{2} b(Y_t,t) \frac{\partial}{\partial Y_t} b(Y_t,t) (h\mathcal{N}(0,1))^2. \tag{2.29}$$

For more details on Itô-Taylor expansions and the Milstein scheme, see Kloeden & Platen (1992).

Applying (2.29) to (1.7) gives the following discretization scheme for the volatility process:

$$v_{t+\delta} = v_t + (\kappa'(\theta' - v_t) - \frac{1}{2}\sigma^2)\delta + \sigma\sqrt{v_t\delta}\mathcal{N}(0,1) + \frac{1}{2}\sigma^2\delta(\mathcal{N}(0,1))^2. \tag{2.30}$$

According to Gatheral (2006), the frequency of negative variance is significantly reduced when using the Milstein scheme, compared to a Euler discretization, especially when $\frac{4\kappa\theta}{\sigma^2} > 1$. However, negative variance may still occur. Other schemes have been proposed to solve this problem (see, for example, Glasserman & Kim (2009) and Alfonsi (2010)). Nonetheless, in this thesis, we use the Milstein discretization scheme along with the absorption assumption and let $v_{t+\delta} = \max(0, v'_{t+\delta})$, where $v'_{t+\delta}$ is the value obtained using (2.30). Although this introduces a slight bias, it eliminates the possibility of negative variance.

Chapter 3

Equity-Linked Products

3.1 Introduction

Equity-linked products are investment vehicles offering a participation in financial market growth while protecting the initial capital. Thus, investors can benefit from positive market returns without facing downside risk. To buy an equity-linked product, the investor generally pays a single premium to the insurance company issuing the contract. During the term of the contract, the premium grows as if it were invested in the financial market. Hence, the sum paid out to the investor at maturity of the contract depends on market performance. However, the insurance company guarantees that this sum will not be less than a pre-determined amount, which is based on the initial investment. In addition to this guarantee, the insurer often offers a death benefit should the investor die before the end of the contract. Other guarantees can be incorporated to the equity-linked contract. For example, the investor may have the possibility to convert the accumulated amount at maturity into an immediate annuity at a fixed rate. Throughout the life of the contract, the investor may also be allowed to withdraw from his account, although this often triggers a financial penalty.

Equity-linked products can be divided into two general categories. Equity-indexed annuities and equity-linked insurance, which have very similar designs, make up the first category and will be introduced in Section 3.1.1. The second type of equity-linked products contains variable annuities, products sold in the United States, segregated funds, their Canadian counterparts and unit-linked insurance, which is available in the UK. These contracts will be presented in Section 3.1.2.

3.1.1 Equity-Indexed Annuities

The first equity-indexed annuity (EIA) product was marketed in 1995 by Keyport Life. Since their debut, EIAs have grown increasingly popular and their sales have broken the $20 billion barrier ($23.1 billion) in 2004 and stayed above it since, reaching $26.8 billion in 2008 and $30.2 billion in 2009 (see www.indexannuity.org). EIAs are mainly sold in the United States, although a very similar product called equity-linked insurance is available in Germany.

When buying an EIA, the policy holder typically pays a single premium, an amount which grows for the duration of the contract, usually between 5 and 15 years. The contract guarantees a certain return on the initial investment. That return is typically low, between 0% and 3%. The final payout is also based on the performance of an underlying asset, typically a stock index: if it performs better than the guaranteed rate, part of the index return is added to the guarantee. The way the additional return is capped depends on the contract. While some specify a maximum return, others offer a rate of participation in it, or reduce it by a fixed spread. Limiting the maximum return decreases the risk associated with the random payoff. EIAs also provide a death benefit if the policy holder dies before the end of the contract.

The performance of an EIA depends on the return of a stock index. The way the return of the EIA is calculated and applied to the investor's account depends on the type of contract. In some EIAs, called annual reset, the return on the index is calculated yearly and interest is credited and locked-in every year. The annual index return is generally calculated using its beginning and end of year value, but it can also be obtained by observing its maximum value during the period. Annual reset EIAs reduce the risk of low return caused by a drop in the index value towards the end of the contract. Their payoff, similar to that of a cliquet option, is complex and sensitive to changes in index return volatility. Although they have been gaining in popularity, we do not analyze this type of EIA further in this thesis.

In other cases, such as the one that will be studied here, the payoff of the EIA is based on the value of the underlying index at the beginning and at maturity of the contract. The resulting interest, based on the index's performance during the term, is then applied to the account at maturity. This return is adjusted using different parameters, such as described below.

The focus of this thesis is the point-to-point (PTP) EIA, which offers both a financial guarantee and a death benefit. Assuming that the policyholder survives until maturity of the contract, the financial guarantee pays the highest of the guaranteed rate and the return on the underlying asset, calculated using its price at inception and at maturity. As before, we denote the price of the underlying by S_t, $0 \leq t \leq T$, where T is the maturity of the contract. Hence, $B^{PTP}(S_T, T)$, the PTP payoff at maturity, is given by

$$B^{PTP}(S_T, T) = \max\left(1 + \alpha\left(\frac{S_T}{S_0} - 1\right), \varrho(1+g)^T\right), \tag{3.1}$$

where α, $0 < \alpha \leq 1$, is the participation rate in the index return and ϱ, $0 < \varrho \leq 1$, is the proportion of the premium that earns the guaranteed return g. Here, to simplify the expression, we assume that the premium paid, or the amount invested, is 1. This assumption will remain throughout the thesis, without loss of generality. To apply the results to different premium level, it suffices to multiply them by the premium amount.

For a fixed premium, the value of the PTP financial guarantee depends on the guaranteed return and the participation rates. A higher guaranteed return will increase the value of the financial guarantee, while low participation rates will decrease it. Typically, α is set in a way such that the insurance company makes money by selling those contracts. Thus, participation rates in the index return can vary significantly, depending on market performance.

To price the point-to-point EIA, we assume that mortality risk is diversifiable. Thus, by selling a sufficient number of homogeneous contracts, we can hedge the mortality risk away. Hence, the price of the contract becomes the value of the financial guarantee. To price it, we use the no-arbitrage assumption, which states that two assets or portfolios with the same payoff have the same market price. We first show that the payoff of a

PTP contract is a linear function of a the payoff of a European call option. Since we have closed or semi-closed form expressions for the price of the European call under the Black-Scholes and Heston models, we are able to find similar expressions for the price of the PTP contract under both models.

To simplify the notation, we let $K =: \varrho(1+g)^T$ and $L =: S_0\left(\frac{K-1+\alpha}{\alpha}\right)$. Then, rearranging (3.1), the payoff of the PTP contract can be expressed by

$$B^{PTP}(S_T, T) = K + \frac{\alpha}{S_0} B^C(L, T), \tag{3.2}$$

where $B^C(L,T)$ is the payoff of a European call option of strike L.

To obtain the price of the PTP guarantee at time t, $0 < t < T$, we assume that mortality risk is diversifiable and use risk-neutral valuation, which was briefly explained in section 1.2 (for more details about risk-neutral pricing, see Harrison & Pliska (1981)).

Proposition 3.1. *Let $C(S_t, L, \tau)$ denote the price at time t of a European call option of strike L and maturity T and let $\tau = T - t$. Let L and K be as in (3.2). Then, $P_t(S_t, \tau)$, the price of the PTP contract at time t, is given by:*

$$P_t(S_t, \tau) = Ke^{-r\tau} + \frac{\alpha}{S_0} C(S_t, L, T). \tag{3.3}$$

Proof. Denote the risk-neutral expectation $E^Q[\cdot]$. Using risk-neutral pricing, the price of the PTP contract at time t is given by the expectation of its discounted payoff, such that

$$\begin{aligned} P_t(S_t, \tau) &= E^Q[e^{-r\tau} B_T^{PTP}(S_T, T)] \\ &= E^Q\left[e^{-r\tau}\left(1+\alpha\left(\frac{1}{S_0}\left(L+B^C(L,T)\right)-1\right)\right)\right] \\ &= e^{-r\tau}\left(1+\alpha\left(\frac{1}{S_0}\left(L+E^Q\left[B^C(L,T)\right]\right)-1\right)\right). \end{aligned} \tag{3.4}$$

Since $B^C(L, T)$ is the payoff of a European call option, then

$$E^Q\left[B^C(L,T)\right] = e^{r\tau} C(S_t, L, \tau), \tag{3.5}$$

where $C(S_t, L, \tau)$ is the price of the call option of strike L and time to maturity τ. To obtain (3.3), it suffices to use (3.5) in (3.4) and re-arrange (3.4), remembering that $L = S_0\left(\frac{K-1+\alpha}{\alpha}\right)$. □

One can obtain formulas for the price of the PTP financial guarantee under both the Black-Scholes and Heston models, using (1.4) and (1.46) for the call price, respectively.

3.1.2 Variable Annuities

Variable annuities, segregated funds and unit-linked insurance are American, Canadian and British products, respectively, which have very similar designs. From now on, the term variable annuities will be used to designate the three types of contracts.

Variable annuities are similar to mutual funds, but they offer a guaranteed minimum maturity benefit (GMMB) corresponding to 75 to 100% of the initial premium. The investor pays a single premium, which is invested in a separate account similar to a mutual fund, invested in a mix of stocks, bonds and short-term market securities, chosen by the policyholder. Small investors usually have the choice between portfolios built by the insurer, usually designed to fit different risk profiles. Policyholders investing higher amounts are generally able to compose their own portfolio. The funds are then managed outside of the insurance company. At the beginning of each month, administration fees are paid out of the account to the insurance company. These fees are used to cover the cost of the guarantee. Variable annuities typically offer a guaranteed minimum death benefit (GMDB) of 100% of the initial premium. Some offer additional benefits, such as the possibility to renew the contract at maturity, for 100% of the initial premium or 75% of the accumulated amount.

The payoff of a variable annuity, $B^{VA}(S_T, T)$ is thus the maximum between G and F_T, where G is the guaranteed amount and F_T is the value of the account at maturity T. The value of the account is based on the fund performance in which it is invested. We assume this value is based on an index and denote S_t the value of this index at t, $0 \leq t \leq T$. Let ν, $0 \leq \nu \leq 1$, denote the percentage of the account which is paid out in fees each month. Then the value of the account at maturity, F_T is given by

$$F_T = F_0 \frac{S_T(1-\nu)^{12T}}{S_0}. \tag{3.6}$$

Without loss of generality, we assume that $F_0 = 1$. It then is possible to re-write the payoff B^{VA} in terms of S_T:

$$B^{VA}(S_T, T) = G + \frac{(1-\nu)^{12T}}{S_0}(S_T - L^{VA})^+, \tag{3.7}$$

where $L^{VA} = \frac{G \times S_0}{(1-\nu)^{12T}}$ and $(S_T - L^{VA})^+ = \max(S_T - L^{VA}, 0)$. This expression is used to show that a variable annuity can be valued using (3.3), the pricing formula for a PTP contract.

In fact, variable annuities and point-to-point EIAs have very similar payoffs and results obtained from analyzing PTP contracts can easily be applied to variable annuities. The following proposition illustrates this similarity.

Proposition 3.2. *Let G denote the guaranteed amount and ν, the fee rate. Then, $P_t^{VA}(S_t, \tau)$, the value of a variable annuity with time to expiry τ, is given by (3.3), with $K = G$ and $\alpha = (1-\nu)^{12T}$.*

Proof. This proof is similar to that of Proposition 3.1. □

This similarity will allow us to concentrate on PTP contracts for the rest of the thesis, without dismissing variable annuities.

Proposition 3.2 relies on the assumption, made in Section 3.1.2, that the account's value is based on that of an index. However, in reality, this value depends on the performance of a fund in which the initial premium is invested. While the composition of this fund is chosen by the insurer, its management is done outside the company. In certain cases, approximating the evolution of the fund using an index traded on the market might be appropriate. However, it might not always be possible, which makes it hard to apply Proposition 3.2. In those cases, since the composition of the fund is known, the value of the fund could be estimated by a weighted sum of the N securities composing the portfolio, denoted S_t^*. Thus, we would have

$$S_t^* = \sum_{i=1}^{N} w_t^{(i)} S_t^{(i)}, \tag{3.8}$$

where $S_t^{(i)}$ and $w_t^{(i)}$ are the value of the securities and their weights in the portfolio, respectively, at time t. Hence, pricing the variable annuity using (3.3) would still be possible. Nonetheless, some hedging strategies, such as gamma and vega hedging, involve derivatives of the underlying. Since there are no options on the weighted sum S_t^*, these strategies will not be applicable to variable annuities. Thus, only delta hedging strategies will be available to insurers selling variable annuities, although another alternative to explore would be to hedge the exposure to each component of the fund separately.

Chapter 4

Hedging Point-to-Point EIAs

4.1 Introduction

Insurance companies selling EIAs and variable annuities face a risk caused by the uncertain payoff of these products. In this chapter, we present strategies that aim at reducing that risk. More specifically, we introduce static hedging in Section 4.2 and combine it with the concepts seen in Chapter 2 to build hedges designed for point-to-point (PTP) EIAs. The resulting strategies are then analyzed with a numerical example in Chapter 5.

4.2 Static Hedging

A static hedge is a hedging strategy that does not need rebalancing after it is set up. Since the price of the point-to-point EIA is a deterministic function of the price of a European call, it would be possible to statically hedge the EIA under certain assumptions. In fact, assuming that options of any maturities and strikes are available on the market, the insurance company issuing the EIA could protect itself against the randomness in the payout of the contract by buying the European call option of which the price of the EIA is a function. More specifically, the insurance company would need to build a replicating portfolio that reproduces the payoff of the EIA by using the initial premium to buy $\frac{\alpha}{S_0}$ of the European call option and by investing the rest, $P_0(S_0, \tau) - \frac{\alpha}{S_0}C(S_0, T)$ in the risk-free asset. Let Π_t^s denote the value of the statically hedged portfolio at any time t, $0 \leq t \leq T$, where T is the maturity date of the EIA contract. Then the value of the hedged portfolio at time t is given by

$$\Pi_t^s = \frac{\alpha}{S_0}C(S_t, \tau) + (P_0(S_0, \tau) - \frac{\alpha}{S_0}C(S_0, T))e^{rt} - P_t(S_t, \tau). \tag{4.1}$$

Replacing $P_0(S_0, \tau)$ and $P_t(S_t, \tau)$ by the expression given in (3.3) and re-arranging, we get that the value of the hedged portfolio at any time t is 0. This means that the value of the replicating portfolio is equal to the value of the contract at any time. Specifically, at time T, the payoff of the hedge is equal to that of the EIA contract. Hence, the static hedge allows the insurer to eliminate all the risk embedded in the sale of the EIA.

This strategy would be ideal if it were possible to buy call options of maturities similar to the ones of EIA contracts. However, EIAs usually have 7 to 15 years maturities, while the longest call option sold on a stock index has only $\tilde{\tau}$ years to expiration, with $1 \leq \tilde{\tau} \leq 3$, depending on the index. Thus, the static hedging strategy presented here can only be applied by insurance companies during the last $\tilde{\tau}$ years of the EIA contract.

4.3 Dynamic Hedging

In this section, we develop strategies that take into account the fact that call options on stock indexes with maturities longer than $\tilde{\tau}$ years are not available on the market.

Since it is no longer possible to hedge EIAs statically, except during the last $\tilde{\tau}$ years of contract, we make use of the dynamic hedging strategies described in Section 2.4.

4.3.1 Delta Hedging

As explained in Chapter 2, the aim of the delta hedging strategy is to neutralize the delta of the overall position containing the initial position and the replicating portfolio. This strategy underlies the valuation equation in the Black-Scholes model. Thus, it is the most appropriate when the returns of the underlying has a Gaussian distribution, as in Black-Scholes. However, delta hedging fails to eliminate the risk brought on by varying volatility of the underlying returns. It follow that this hedging strategy is not sufficient under all market models.

Delta hedging is explained in details in Section 2.4.1. To apply this strategy to PTP contracts, it suffices to use the delta of the EIA to determine the proportion of the replicating portfolio to invest in the underlying index. Since this strategy is applied in a discrete manner, it yields hedging errors, which are presented in Chapter 2 and given by (2.1).

Let $P_t(S_t, \tau)$ and $\Delta_{P,t}$ be the value of the EIA and its delta at time t, $0 \leq t \leq T$ as defined in Chapter 3. Let also $H^{\Delta}_{P,t}$ denote the value of the replicating portfolio, or the hedge, at time t. Thus, adapting (2.9) gives us that after each readjustment, $H^{\Delta}_{P,t}$ is given by:

$$H^{\Delta}_{P,t} = \Delta_{P,t}\, S_t + \xi^{\Delta}_{P,t}, \tag{4.2}$$

where $\xi^{\Delta}_{P,t} = P_t(S_t, \tau) - \Delta^H_{P,t}\ S_t$ is the amount invested in the risk-free asset.

Assume the portfolio is rebalanced m times a year. Then, the hedging error occurring at time t is given by (2.1) with $V_t = P_t(S_t, \tau)$ and $H_{t^-} = H^{\Delta}_{P,t^-}$. The total discounted hedging error, whose distribution will be analyzed to assess the performance of this delta hedging strategy, is given by (2.2).

4.3.2 Delta and Static Hedging

Since the value of the point-to-point EIA, given by (3.3), is a linear function of the price of a European call option, it is possible to apply static hedging when the call of the appropriate maturity is available on the market. We use this to improve the delta hedging strategy presented in the previous section. We apply delta hedging until time $t^* = T - \tilde{\tau}$, $\tilde{\tau}$ years before the end of the contract. At that point, we assume it is possible to buy a call option of strike L, with L as defined in Proposition 3.1, and $\tilde{\tau}$ years to expiration. Thus, the dynamic hedge is liquidated and a static hedge based on the value of the EIA contract at time t^*:

$$P_t(S^*_t, \tilde{\tau}) = Ke^{-\tilde{\tau} r} + \frac{\alpha}{S_0} C(S^*_t, T - \tilde{\tau}), \tag{4.3}$$

where K and $C(S^*_t, T - \tilde{\tau})$ are as defined in (3.3). Hence, the static hedge is constructed by purchasing $\frac{\alpha}{S_0}$ European call options of strike L and by investing the rest in the risk-free asset. For this to be possible, we assume that all securities are infinitely divisible and that calls of any strikes are available. It follows that, at time t^*, the value of the static hedging portfolio is equal to that of the contract. At time T, the value of the static hedge, denoted $H^s_{P,T}$, becomes:

$$H^s_{P,T} = K + \frac{\alpha}{S_0} B^C(L, T), \tag{4.4}$$

where $B^C(L,T)$ is the payoff of the call option of strike L. Notice that $H^s_{P,T} = B^{PTP}(S_T,T)$, where $B^{PTP}(S_T,T)$ is the payoff of the EIA contract as defined by (3.2). Thus, the amount obtained by selling the replicating at contract maturity can be used by the insurer to payout the EIA benefit to the policyholder. This way, no hedging errors occur after the set up of the static hedge, which yields a lower total discounted hedging error than a pure delta-hedging strategy.

4.3.3 Gamma and Static Hedging

In order to reduced hedging errors due to discretization, the gamma of the derivative to hedge can be incorporated to a dynamic hedging strategy, as explained in Section 2.4.2. In this section, we attempt to improve the performance of the dynamic hedging strategy applied for the first $T-3$ years by using the gamma to reduce hedging errors. The new strategy calls for the use of a third asset, which depends on the same index as the EIA and has a gamma other than 0. The payoff of the third asset should be similar to the one to hedge, making the call option of strike L, with L defined as in Proposition 3.1, an appropriate choice for the third asset. This selection makes the Greeks of the EIA and those of the hedge more similar, which should improve the efficiency of the strategy.

Since we assume that call options of maturities longer than $\tilde{\tau}$ years are not sold on the market, we use the longest ones available to build our replicating portfolio. The use of longer maturities is justified by the desire to match the payoff of the EIA and those of the option as much as possible, so that their Greeks are similar. Each year, at policy anniversary, the hedge is refreshed by selling the call option held and buying a longer one. This is done at rebalancing times $t^{**} = 1, 2, ..., T$. Since call options of long maturity (1 to 3 years) are only emitted once a year, it is only possible to refresh the hedge annually. Nonetheless, it should still be adjusted regularly during the year. Assuming that the EIA and the call option are issued simultaneously, the hedge thus incorporate call options whose maturities vary from 2 to 3 years, representing the longest maturity available on the market at the moment.

Let S_t, $P_t(S_t, \tau)$ and $\Delta^H_{P,t}$ be as in Section 4.3.1. Let also $\Gamma^H_{P,t}$ be the gamma of the EIA and $H^\Gamma_{P,t}$ denote the value of the replicating portfolio at time t. Define $\bar{\tau} = 3-(\ulcorner\tau\urcorner-\tau)$, where $\ulcorner\cdot\urcorner$ returns the smallest integer greater than τ. Then, following (2.14), the value of the replicating portfolio after each readjustment is given by:

$$H^\Gamma_{P,t} = \alpha^\Gamma_{1,t} C(S_t, L, \bar{\tau}) + \alpha^\Gamma_{2,t} S_t + \xi^\Gamma_{P,t}, \tag{4.5}$$

with

$$\xi^\Gamma_{P,t} = P_t(S_t, \tau) - \alpha^H_{1,t} C(S_t, L, \bar{\tau}) + \alpha^H_{2,t} S_t, \tag{4.6}$$

where $C(S_t, L, \bar{\tau})$ is the call option of strike L and maturity $\bar{\tau}$ and

$$\begin{aligned} \alpha^\Gamma_{1,t} &= \frac{\Gamma^H_{P,t}}{\Gamma^H_{C,t}} \\ \alpha^\Gamma_{2,t} &= \Delta^H_{P,t} - \alpha^H_{1,t}\Delta^H_{C,t}. \end{aligned} \tag{4.7}$$

In the above, $\Delta^H_{C,t}$ and $\Gamma^H_{C,t}$ denote the delta and the gamma, respectively, of the call option $C(S_t, L, \bar{\tau})$.

As for the other strategies, hedging errors arising from discretization are defined by (2.1) and are denoted HE^Γ_t. Note that since a static hedge is applied after time $T - \tilde{\tau}$, we have that

$$HE^\Gamma_t = 0 \qquad \text{for } t = T - \tilde{\tau}, T - \tilde{\tau} + \delta, ..., T. \tag{4.8}$$

A gamma hedging strategy improves the insurer's protection against unpredictable variations in the price of the underlying. However, it does leave the company exposed to changes in index return volatility, which should induce hedging errors and thus lower the performance of the hedge.

4.3.4 Vega and Static Hedging

As seen in section 1.2.4, the vega hedging strategy underlies the pricing of derivative products in the Heston model. In this section, we suggest to enhance the delta and static hedging strategy by neutralizing the delta and vega of the overall position taken by the insurer. Under the same assumptions and considerations as in the last section, we select call options of strike L and longest maturity possible to complete the hedging portfolio. The hedge adjustment and refreshing times are the same as for the gamma and static hedging strategy.

Let $\mathcal{V}^H_{P,t}$ be the vega of the EIA and let S_t, $P_t(S_t, \tau)$, $\Delta^H_{P,t}$, $C(S_t, L, \bar{\tau})$ and $\bar{\tau}$ be defined as in the previous section. Then, following (2.19), $H^{\mathcal{V}}_{P,t}$ the value of the replicating portfolio after each readjustment is obtained by

$$H^{\mathcal{V}}_{P,t} = \alpha^{\mathcal{V}}_{1,t} C(S_t, L, \bar{\tau}) + \alpha^{\mathcal{V}}_{2,t} S_t + \xi^{\mathcal{V}}_{P,t}, \tag{4.9}$$

with

$$\xi^{\mathcal{V}}_{P,t} = P_t(S_t, \tau) - \alpha^{\mathcal{V}}_{1,t} C(S_t, L, \bar{\tau}) + \alpha^{\mathcal{V}}_{2,t} S_t. \tag{4.10}$$

In the above, $\alpha^{\mathcal{V}}_{1,t}$ and $\alpha^{\mathcal{V}}_{2,t}$ are given by

$$\begin{aligned} \alpha^{\mathcal{V}}_{1,t} &= \frac{\mathcal{V}^H_{P,t}}{\mathcal{V}^H_{C,t}} \\ \alpha^{\mathcal{V}}_{2,t} &= \Delta^H_{P,t} - \alpha^H_{1,t} \Delta^H_{C,t}. \end{aligned} \tag{4.11}$$

By neutralizing the delta and the vega of its overall position, the insurer reduces the risks caused by unexpected changes in the underlying price and the volatility of its returns. The distribution of total hedging errors, obtained by (2.1) and (2.2), will be used to analyze the performance of this strategy.

4.4 Numerical Example

In this section, we consider a 10-year point-to-point EIA on the S&P 500. The contract guarantees a non-negative return ($g = 0\%$) and full participation in that return ($\varrho = 1$). The rate of participation in the index return, α, is set such that the price of the contract calculated using (3.3) is 1. To assess the performance of the hedging strategies presented in the previous sections, we perform Monte Carlo simulations and analyze the distribution of the total discounted hedging error resulting from the application of each strategy. It is important to keep in mind that a positive hedging error results in a loss for the insurer. Thus, we want to avoid hedging strategies that yield a hedging error distribution presenting a heavy right tail. The 95% value-at-risk ($VaR_{95\%}$) will be one of the risk measures used to assess the heaviness of the tail. The best strategies are those that yield hedging errors that are close to the mean. This signifies that the costs associated with the hedge do not vary too much and so, are more predictable. This translates in a low standard deviation and a small difference between the expected value and the 95% conditional tail expectation ($CTE_{95\%}$).

Throughout the numerical example, the Black-Scholes and Heston models are calibrated using the methods introduced in Chapter 1. Section 4.4.1 presents the results obtained when assuming that the market follows the Black-Scholes model. Section 4.4.2 focuses on the distributions obtained under the assumption of the Heston market model. A sample of the code used to produce the graphs displayed in this section can be found in the Appendix.

4.4.1 Black-Scholes hedging strategies in a Black-Scholes market

Here, we consider that the dynamics of the price of the index are given by (1.1). We assume that $r = 0.02$ and let $\mu_{BS} = 0.0637$ and $\sigma_{BS} = 0.19$, such as obtained in Section 1.1.5. Under these assumptions, we set $\alpha = 0.572255$, so that the price of the EIA is 1. We simulate 50,000 sample paths of index prices using the discretization scheme introduced in Section 2.5.1 and we apply the hedging strategies presented in Section 4.3, with the Greeks derived under the Black-Scholes framework. We assume that the insurance company rebalances the hedge weekly, so that $\delta = \frac{1}{52}$. Since the Black-Scholes framework assumes constant volatility, it does not make sense to hedge the portfolio volatility through vega hedging. However, we do assess the performance of delta, delta and static and gamma and static hedging.

Figures 4.1 to 4.3 below illustrate the distributions of the discounted total hedging errors, calculated using (2.2). This hedging error is reported as a percentage of the premium. Figure 4.1 shows the distribution of the hedging errors resulting from the discretization of a pure delta hedging strategy. One can note that the mean of the distribution is 0.005%. In the Black-Scholes model, the total discounted hedging error resulting from the delta hedging strategy is a martingale under the risk-neutral measure. Thus, the empirical distribution should be centered at 0% so that, on average, the hedge is self-financing. The standard deviation, value-at-risk ($VaR_{95\%}$) and conditional tail expectation ($CTE_{95\%}$) at 0.4008%, 0.6502% and 0.9244%, respectively, are relatively low. Thus, in a Black-Scholes market, the delta hedging strategy performs relatively well, which makes sense since it is the strategy that underlies the pricing of the European call option.

It is however possible to improve the performance of the delta hedging strategy by incorporating static hedging during the last 3 years, as presented in Section 4.3.2. The distribution of the hedging errors resulting from the application of this strategy to 50,000 sample paths of index prices is illustrated by Figure 4.2. One can notice an improvement when comparing the performance of this strategy to that of the pure delta hedging. The standard deviation, $VaR_{95\%}$ and $CTE_{95\%}$ of the hedging errors decrease, to 0.2938%, 0.4921% and 0.6416%, respectively, which signifies that the errors resulting from the hedging strategy will tend to be closer to the mean, thus reducing the risk inherent to the sale of the contract.

The most significant way to improve our hedging strategy is to incorporate a third asset, a European call option in our case, and to use it to neutralize the gamma of the portfolio, as seen in Section 4.3.3. The distribution of the hedging errors obtained by using this strategy in 50,000 Monte Carlo simulations is presented in Figure 4.3. With a mean centered at 0% and a standard deviation, $VaR_{95\%}$ and $CTE_{95\%}$ all under to 0.015%, this strategy ensures that the cost of the hedge will almost always be very close to 0%. Applying this strategy in a Black-Scholes world reduces the risk born by the insurance company selling point-to-point EIAs.

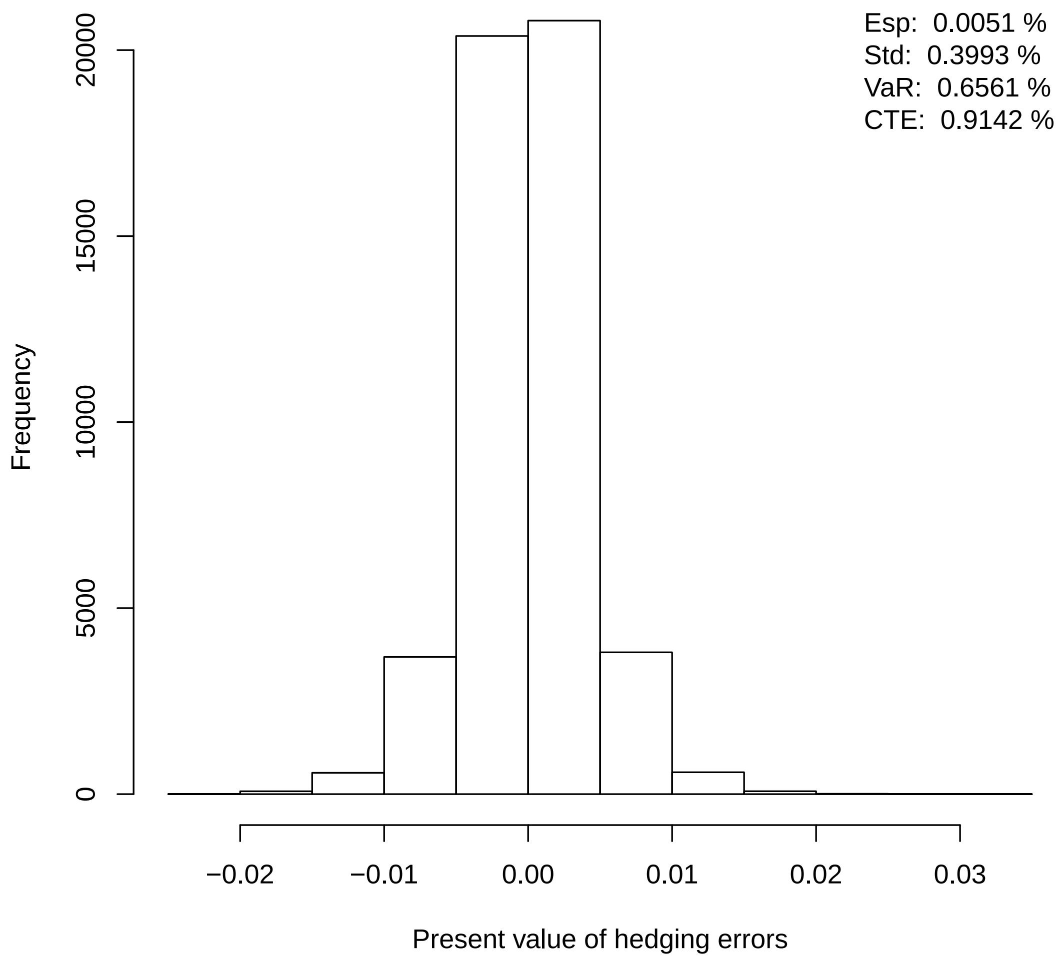

Figure 4.1: Present values of hedging errors resulting from a delta hedging strategy in a Black-Scholes framework

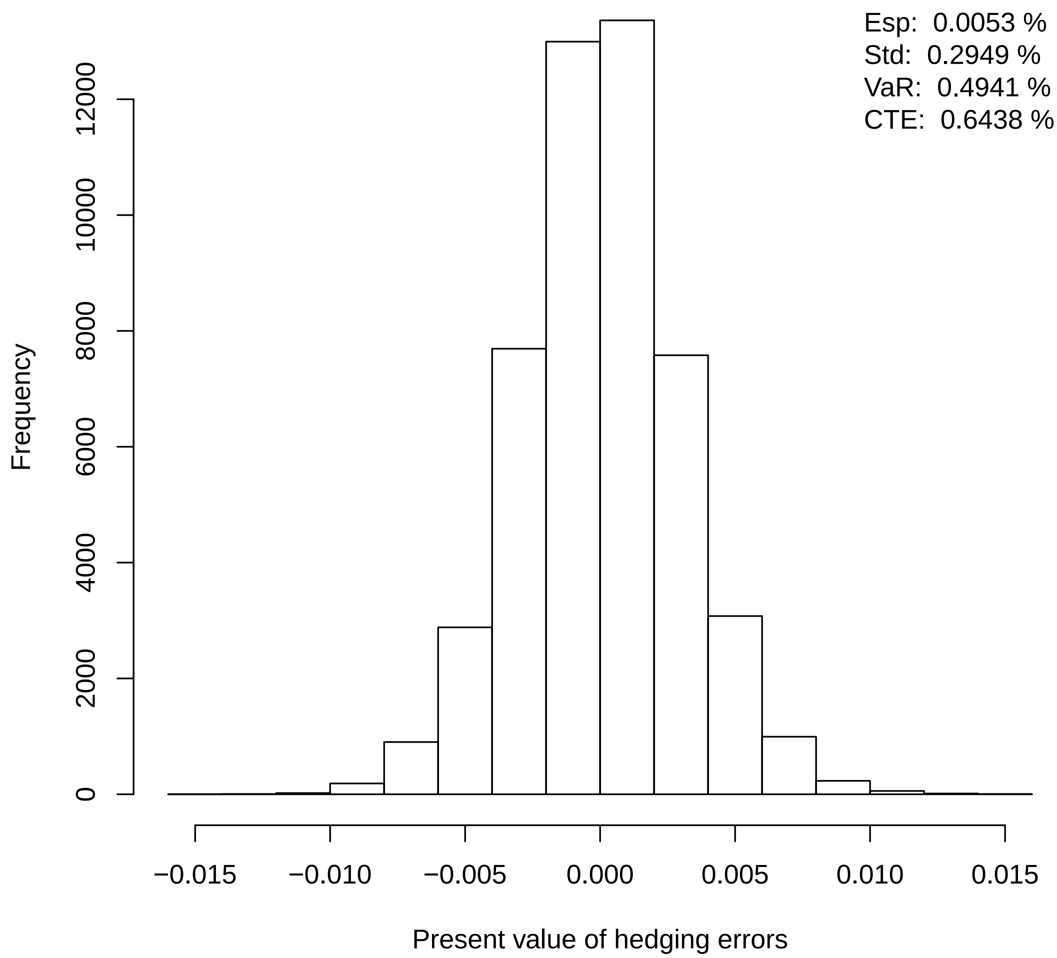

Figure 4.2: Present values of hedging errors resulting from a delta and static hedging strategy in a Black-Scholes framework

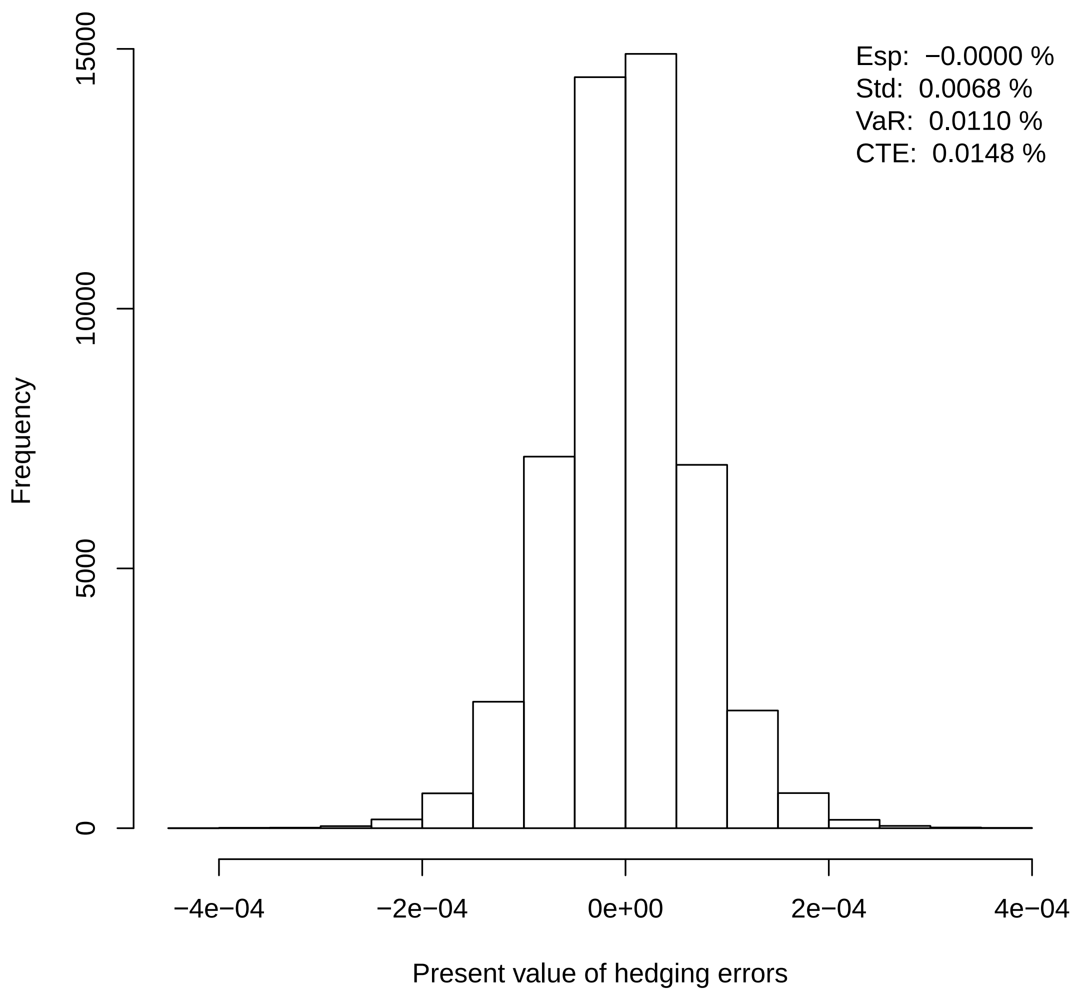

Figure 4.3: Present values of hedging errors resulting from a gamma and static hedging strategy in a Black-Scholes framework

4.4.2 Black-Scholes hedging strategies in a Heston market

The previous section shows that the hedging strategies developed in the Black-Scholes framework perform well when the market is assumed to follow that same model. These results can hardly be applied on the markets since it is well known that the Black-Scholes model fails to capture some very important characteristics of the distribution of index log-returns. In this section, we will show that Black-Scholes hedging strategies do not yield such good results when they are applied to index prices simulated by the Heston model. It is important to remember that the Heston model replicates index returns better than the Black-Scholes model, mainly because it allows for stochastic volatility. Thus, strategies that fail in a Hestonian market are also likely to fail when applied by insurers. Similarly, strategies that perform well in a Heston framework have a better chance at reducing the risks faced by EIA-selling insurers in real life.

To generate the results in this section, we again assume that $r = 0.02$ and we use the parameters obtained in Section 1.2.5, such that $\kappa = 5.1793$, $\theta = 0.0178$, $\sigma = 0.1309$, $v_0 = 0.0286$, $\rho = -0.7025$ and $\lambda = 2.62$, unless we indicate otherwise. The 50,000 sample paths of index prices are generated using the method described in Section 2.5.2 and the real-world parameters κ' and θ' defined in (1.21). In order to assess the performance of the strategies developed assuming a Black-Scholes framework, we let $\alpha = 0.572255$, as in the previous section. This value was calculated so that the price of the contract under the Black-Scholes assumptions is 1. Note that, using the Heston parameters listed above, we need to let $\alpha = 0.696091$ for the price of the EIA to be 1. This indicates that historical returns are more volatile than what the market predicts, through call option prices, for the future. Thus, letting $\alpha = 0.572255$ in our Hestonian market should introduce a first bias. Also, as in the previous section, we use the Greeks calculated under Black-Scholes assumptions, which should increase the hedging errors since they are not designed for the dynamics of the Heston market model.

Again, we analyze the performance of the delta hedging strategy, with and without static hedging, and of the gamma hedging strategy. This time, since the volatility is not constant, we also observe the distribution of the total discounted hedging error resulting from the application of the vega hedging strategy presented in Section 2.4.3.

Since we assume that the index price follows the Heston model, the price of the call option (in the case of gamma and vega hedging) is calculated using (1.47). The numerical integral in (1.45) and its derivatives is performed in R using the function `integrate()`, with 50 as the upper bound of the interval of integration. This is justified by the fact that the integrand converges to zero very quickly, even with the most extreme parameters. In his work, Moodley (2005) uses 100 as the upper bound, but we found that this bound still allows for some numerical instability. Thus, we use 50 as the upper bound for the integral.

4.4.2.1 Delta Hedging

We first test the performance of the delta hedging strategy, with and without static hedging during the last 3 years. The resulting distributions of the total discounted hedging errors are illustrated in Figure 4.4 and Figure 4.5. The application of the Black-Scholes delta hedging strategy to a Hestonian market increases the mean of the hedging errors from 0.005% to 0.0524% for the pure delta hedging strategy. Thus, the average loss per contract sold is 10 times bigger. An increase can also be observed in the standard deviation, the $VaR_{95\%}$ and the $CTE_{95\%}$, making the Black-Scholes delta hedging strategy less efficient in a Hestonian market.

The application of a static hedging strategy during the last three years of the contract gives rise to another type of hedging error, caused by the difference between the funds available at maturity and the actual sum to be paid out. If the assumptions made by the insurance company matched the market, then this error would be 0. However, in this case, we hedge assuming Black-Scholes, thus mispricing the future payoff at time $T-3$. This error causes the mean of the total discounted error to shift left to -0.6472%. This negative shift is due to the fact that our Black-Scholes calibration overprices the EIA by overestimating future volatility, when compared to our Heston calibration. In this case, the resulting shift represents a reduction in hedging costs. However, should the Heston calibration predict a more volatile market, the distribution of the total hedging error would be shifted positively, thus increasing hedging costs. This highlights the sensitivity of the delta and static hedging strategy to the calibration of the Black-Scholes model. One can also note that the difference between the mean and the $CTE_{95\%}$ of the distribution is decreased from 1.5685% to 1.2186% by adding static hedging to the strategy, which indicates a lighter right tail for the second strategy.

Our calibration sets λ, the volatility risk premium, so that the mean and the variance of the log-returns simulated by the Heston model reproduce historical data. These moments were also the ones used to calibrate the Black-Scholes model. Hence, it makes sense that a delta hedging strategy using those Black-Scholes parameters perform relatively well. However, changing the risk premium leads to different moments of the log-return distribution, which affects the performance of the strategy. Figure 4.6 and 4.7 show the results assuming $\lambda = 0$ and $\lambda = -1$.

Changing the risk-premium leads to errors that are no longer centered around 0%, since $\lambda = 2.62$ was chosen so that the moments moments of the log-return distribution resulting from the Heston calibration matched those of the Black-Scholes one. Thus, as λ decreases from 2.62, the mean of the discounted hedging error moves away from 0% and its standard deviation increases. This indicates a worse hedging performance. Although the VaR and CTE decrease as the risk premium becomes negative, representing lower costs, this decrease also signifies that the distribution of the discounted hedging error is no longer centered around 0. In cases where historical data is less volatile than what the Heston calibration predicts, systematic losses could occur. This analysis demonstrates that the Black-Scholes delta hedging strategy is significantly affected by variations of the volatility risk premium.

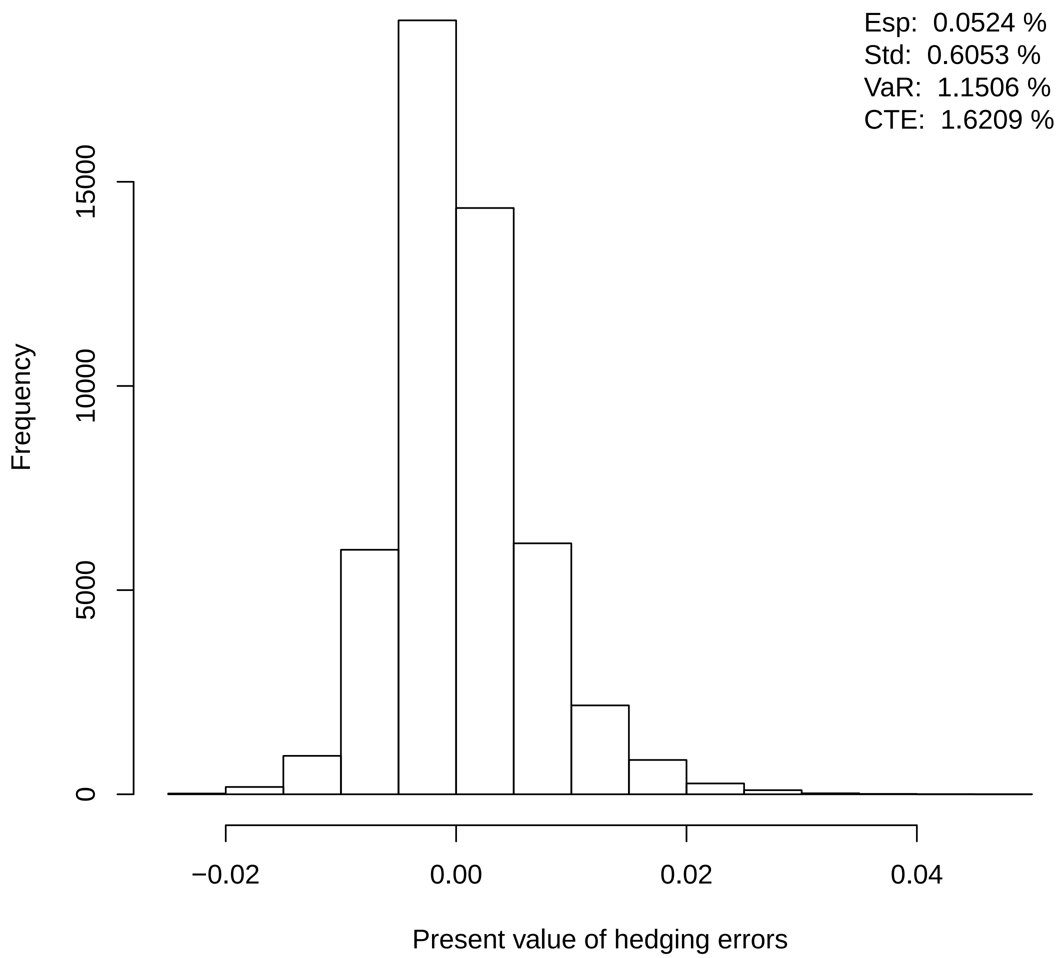

Figure 4.4: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy in a Heston framework, $\lambda = 2.62$

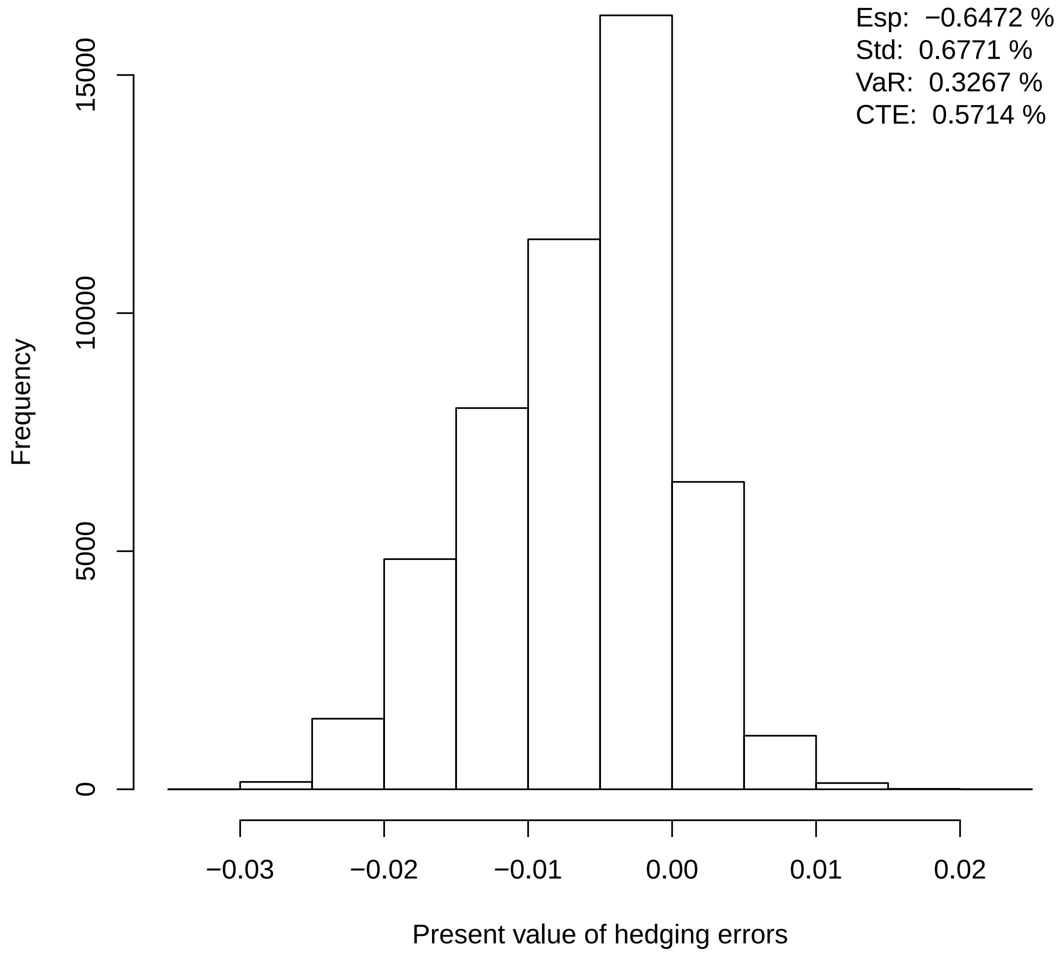

Figure 4.5: Present values of hedging errors resulting from a Black-Scholes delta and static hedging strategy in a Heston framework, $\lambda = 2.62$

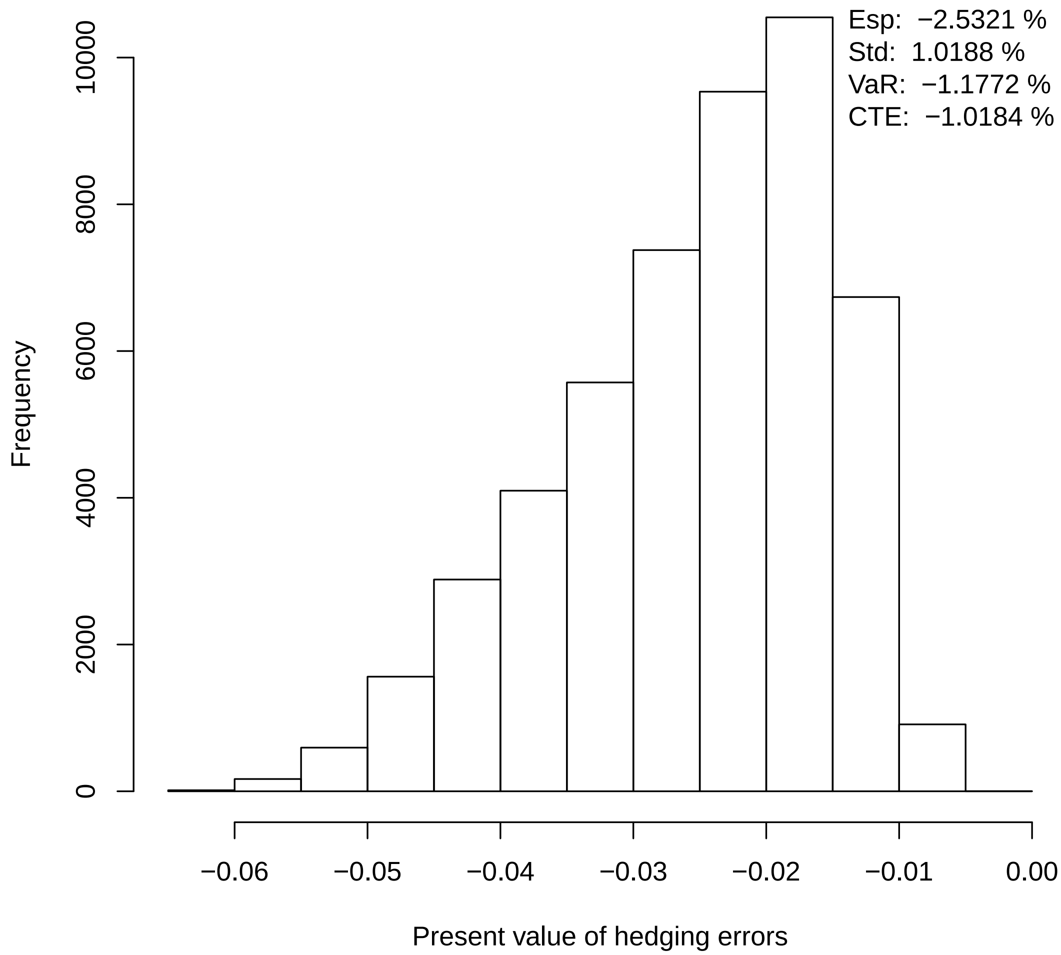

Figure 4.6: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy in a Heston framework, $\lambda = 0$

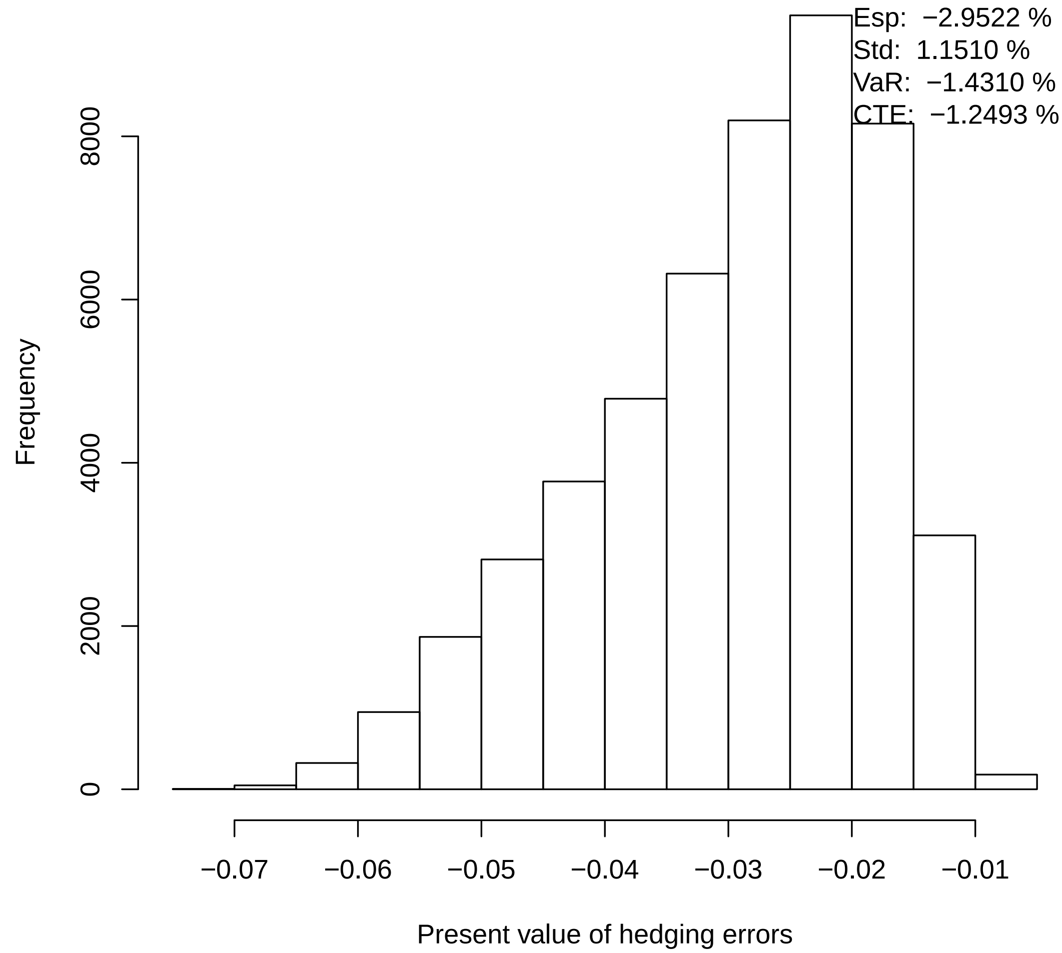

Figure 4.7: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy in a Heston framework, $\lambda = -1$

4.4.2.2 Gamma and Vega Hedging

The usual way to improve a delta hedging strategy is to neutralize the gamma or the vega of the portfolio. In a Black-Scholes market, as seen in Section 4.4.1, neutralizing the gamma tends to correct the errors caused by the discretization of a continuous process. In a Heston market, such strategies are affected by the calibration of the models and by the presence of a volatility risk premium. The distribution of the total discounted hedging error resulting from the application of the strategies introduced in Sections 4.3.3 and 4.3.4 are presented in Figures 4.8 and 4.9, respectively.

While the delta-gamma and delta-vega hedging strategies should lead to better results, it is not the case here. The volatility risk premium causes an important difference between the prices of the call options calculated using our Black-Scholes calibration, reflected in the Greeks used to implement the strategy, and the actual market prices of the options, which are based on the Heston model. For the delta-gamma strategy, this results in a mean of the discounted hedging error of -2.8256%, a $VaR_{95\%}$ of -2.1826% and a $CTE_{95\%}$ of -2.0656%. These results are worse than those obtained using a simple delta-hedging strategy, since buying calls sold on the market introduces a systematic bias. This bias stems from the gap between the call option prices used to calculated the Greeks of the hedging strategy and the actual market prices. However, the addition of the gamma of the portfolio to the hedge reduces the standard deviation to 0.3523%. Thus, the distribution of the total hedging error is no longer centered, but it is less dispersed. In our case, since our Heston calibration predicts a less volatile market than the one reflected by historical data, the use of the Black-Scholes Greeks in the hedging strategy creates systematic profits. Nonetheless, if the predicted market was more volatile, a systematic loss would occur. This reflects an additional risk caused by the use of a hedging strategy based on the Black-Scholes model. Similar results can be observed in Figure 4.9, which illustrates the distribution of the total discounted error under a Black-Scholes delta-vega strategy. The mean of the distribution, at -6.5080%, is even further away from 0%, which implies a significant amount of risk in the case where the market predicted by the Heston model is more volatile than the one implied by the Black-Scholes calibration. Since, in this section, we assume that the insurer calibrates the Black-Scholes model to historical data without taking Heston predictions into account, applying a delta-gamma or delta-vega strategy adds risk. In fact, because of the volatility risk premium, the distribution of the hedging errors may be negative or positive, which, in the worst case, will result in a systematic loss. Thus, the presence of a volatility risk premium significantly affects the performance of the Black-Scholes delta-gamma and delta-vega hedging strategy.

To further illustrate this effect, the same analysis was conducted using different volatility risk premiums, namely setting $\lambda = 0$ and $\lambda = -1$. The results are presented in Figure (4.10) and Figure (4.11). Although Figure (4.10) assumes no risk premium, the resulting distribution of hedging errors is not improved, with a mean of -2.9470%, a standard deviation of 0.286%, a $VaR_{95\%}$ of -2.4514% and a $CTE_{95\%}$ of -2.3474%. In fact, without the risk premium, the Black-Scholes and Heston calibrations yield different moments of the stock returns. This mismatch leads to a bias which cancels the effect of the absence of risk premium. Similar results are observed assuming $\lambda = -1$.

Furthermore, since the Black-Scholes vega is simply the Black-Scholes gamma multiplied by the coefficient $S_t^2 \sigma_{BS} \tau$, changing the value of $\hat{\lambda}$ should affect the performance of the Black-Scholes vega hedging strategy in a similar manner.

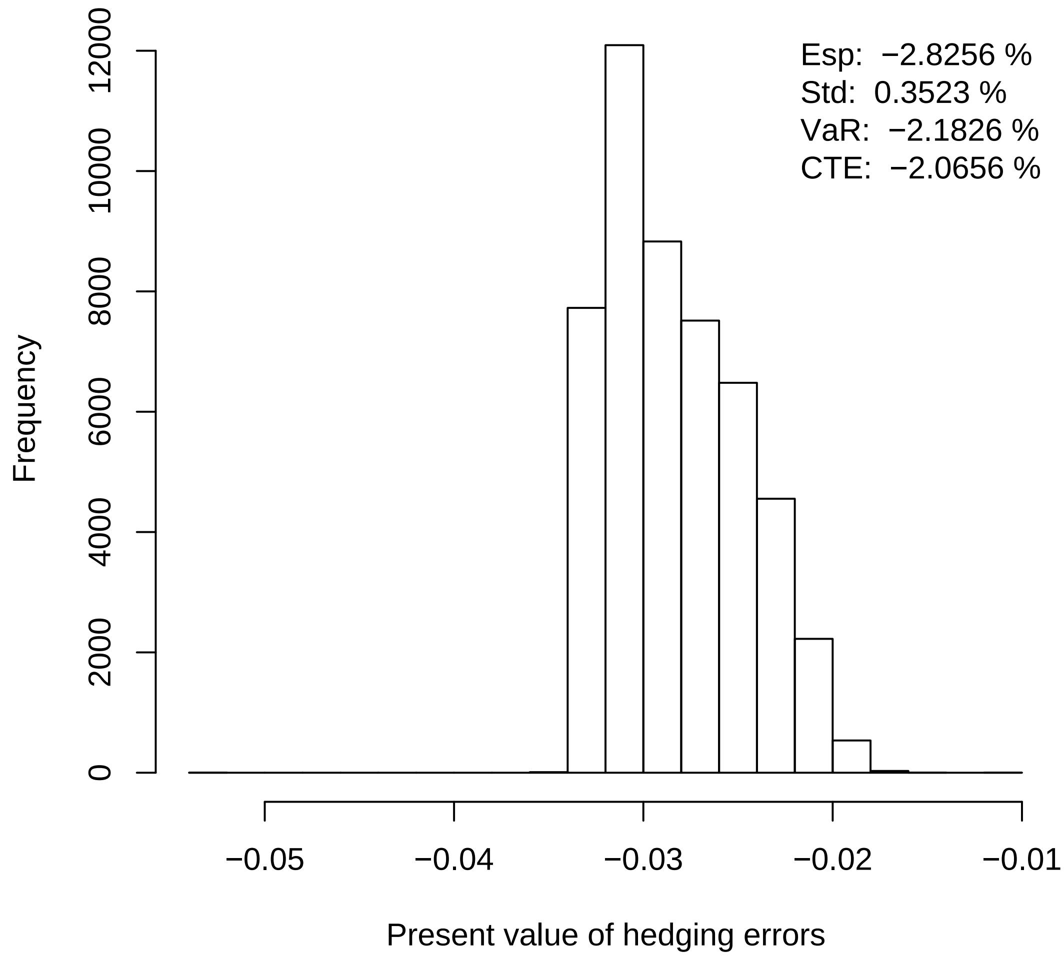

Figure 4.8: Present values of hedging errors resulting from a Black-Scholes delta-gamma hedging strategy in a Heston framework, $\lambda = 2.62$

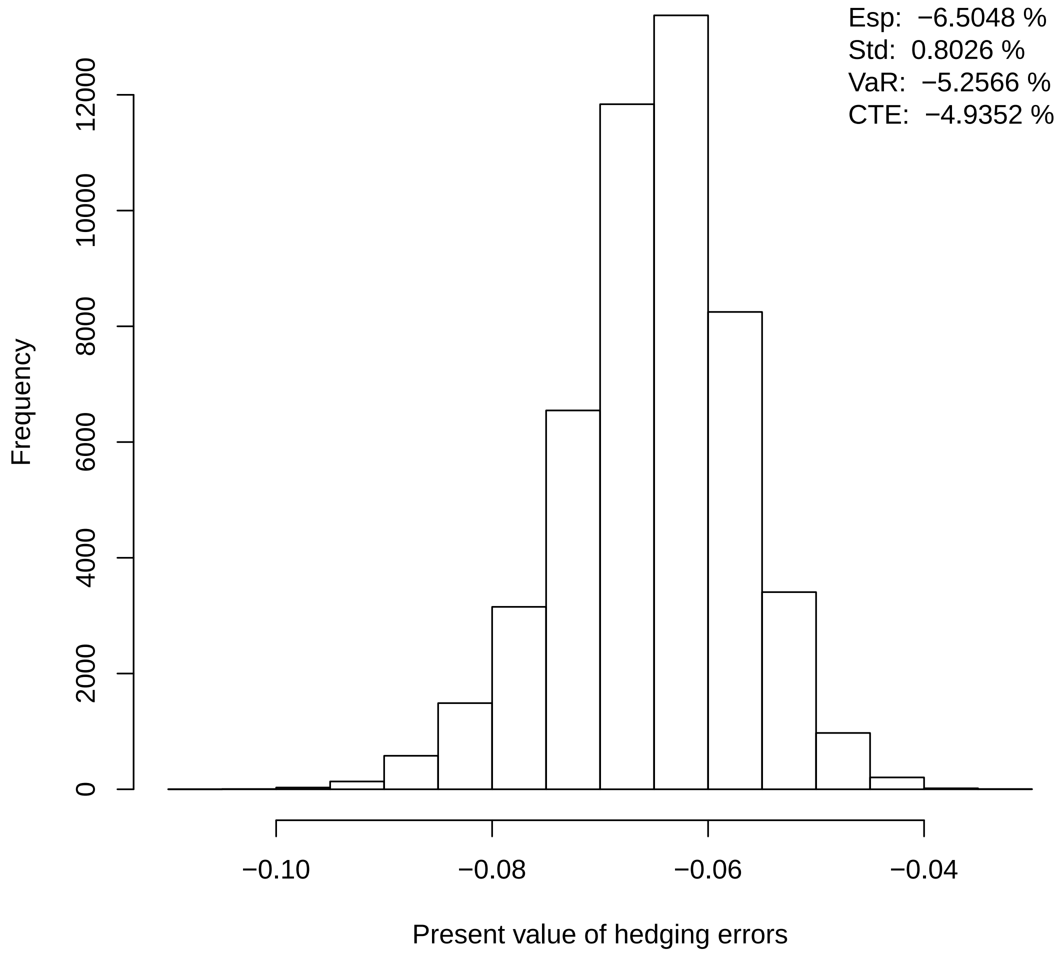

Figure 4.9: Present values of hedging errors resulting from a Black-Scholes delta-vega hedging strategy in a Heston framework, $\lambda = 2.62$

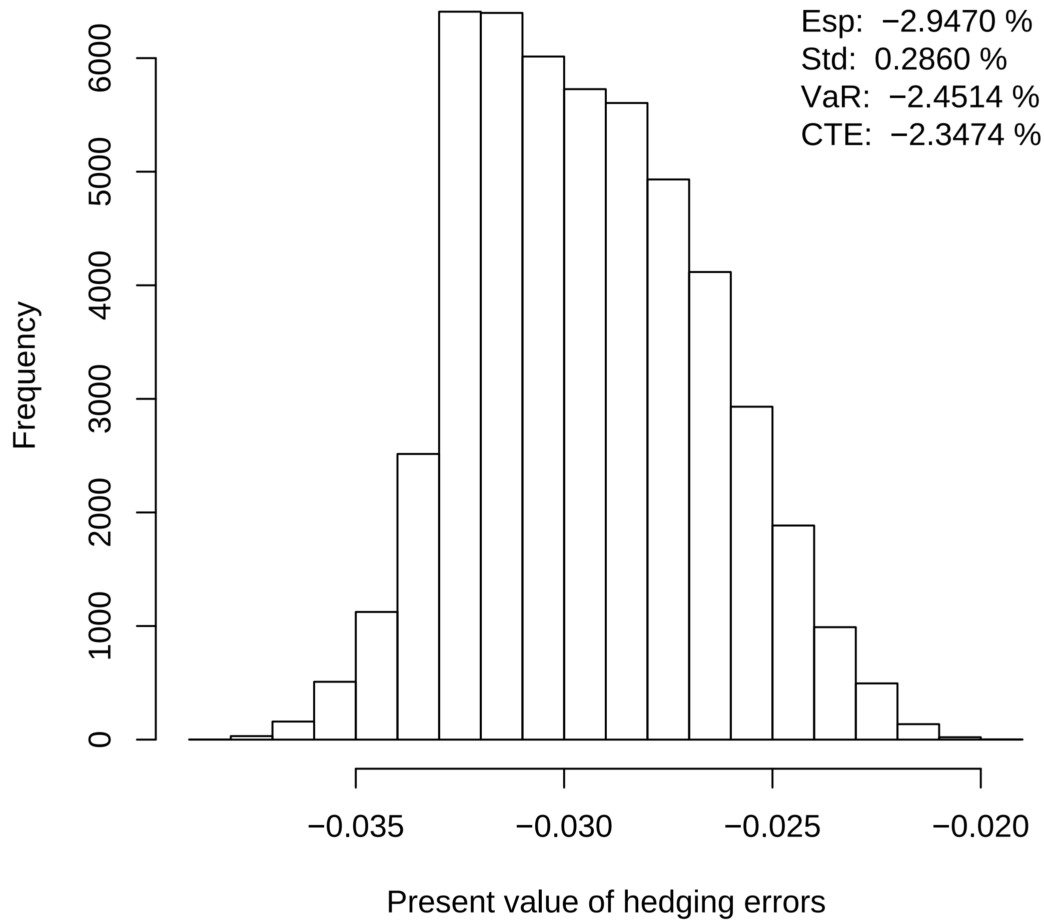

Figure 4.10: Present values of hedging errors resulting from a Black-Scholes delta-gamma hedging strategy in a Heston framework, $\lambda = 0$

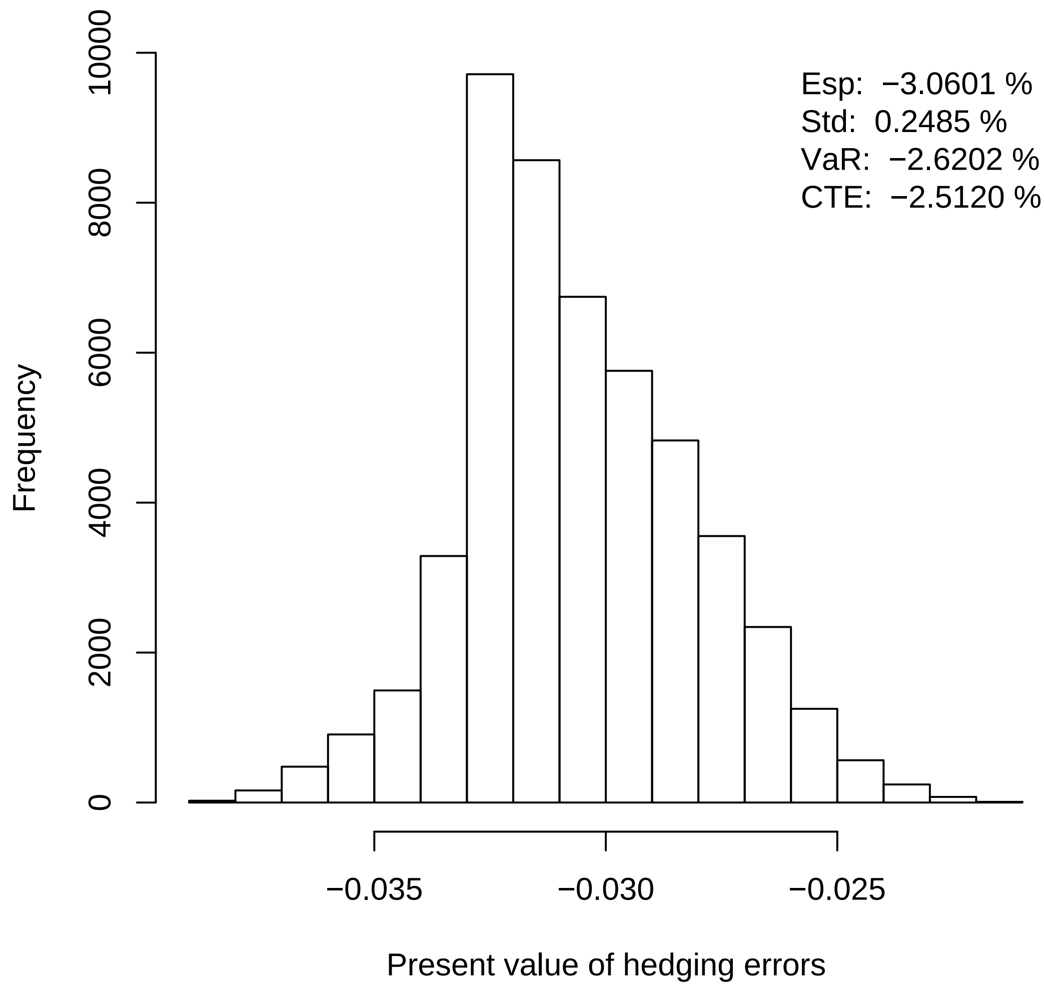

Figure 4.11: Present values of hedging errors resulting from a Black-Scholes delta-gamma hedging strategy in a Heston framework, $\lambda = -1$

4.4.2.3 Comments on the results

Most hedging error distributions presented in this section have a negative expected value. As explained earlier, this translates into a financial gain for the insurer. This is mostly due to the fact that the calibration of the Heston model, based on the market prices of options, predicts a less volatile market that the one reflected in historical data, from which the Black-Scholes parameters are obtained. In our case, this difference results in financial gains for the insurer. However, the opposite situation could occur, in which case the insurance company would suffer systematic losses. In this section, we assume that the company only considers historical data and the Black-Scholes model when developing it hedging strategy. Thus, it may result in systematic gains or losses. This uncertainty represents an additional risk.

It is important to remember that, in this section, we assume that the contract is priced under Black-Scholes assumptions, which yields a participation rate in the guaranteed return of $\alpha = 0.572255$. However, under the Heston parameters used to simulate the sample paths of index prices, this rate goes up to 0.696091, which indicates that the contract analyzed here is overpriced. This is the main reason why hedging errors tend to be negative. Nonetheless, in real life, an insurer selling a contract with $\alpha = 0.572255$ would most likely not be competitive when call option market prices indicate that this rate could be significantly higher. Thus, the high probability of gain would probably be offset by a decrease in sales.

4.4.3 Heston hedging strategies in a Heston market

In this section, we assess the performance of hedging strategies based on Greeks derived using Heston assumptions, such as those defined in Example 2.2. Since our market is assumed to follow the Heston model, those strategies are expected to perform better than the Black-Scholes ones used in the previous section. We will show that, while a simple delta hedging strategy can be greatly affected by the volatility risk premium, neutralizing the gamma or the vega of the portfolio reduces the risk faced by the insurer, even in the presence of such a premium. Since the Heston model reproduces more characteristics observed from empirical data than the Black-Scholes model does, we believe that a strategy that performs well for index prices simulated using the Heston model is more likely to yield satisfactory results when applied to real data.

In this section, we consider $r = 0.02$, as always, and the same Heston parameters as in the previous one. However, since we are now pricing the EIA under Heston assumptions, we set the participation rate α to 0.696091, which yields of a price of 1 for the contract described at the beginning of this numerical example. As in the previous section, we simulate $50,000$ sample paths of index prices to estimate the distribution of the total discounted hedging error under each strategy. We use the same method as before, which is described in Section 2.5.2.

We first analyze the delta hedging strategy, with and without static hedging during the last 3 years. Although we are using Heston Greeks, this type of strategy does not suffice to hedge volatility risk since we now allow for randomness in the volatility of index returns. This type of risk can be reduced by using call options to neutralize the gamma or the vega of the hedging portfolio. We assess the performance of gamma hedging and vega hedging in the second part of this section.

4.4.3.1 Delta Hedging

We analyze the distribution of the total hedging error resulting from the application of a delta hedging strategy with and without static hedging during the last 3 years. These distributions are illustrated in Figure 4.12 and Figure 4.13, respectively. One can easily notice than the performance of the hedging strategy is pretty poor. The expected value of the total hedging errors, at 3.1782%, indicates that an insurer would lose over 3%, on average, on each contract. In the worst cases, he could face losses of almost 8% of the value of the contract. Switching to a static hedge for the last 3 years would only reduce this loss to 5.2968%. As we will see later, by repeating the simulations with a different value for λ, the volatility risk premium greatly affects the performance of the delta hedge. One can also observe a positive skew in the distributions illustrated in Figure 4.12 and Figure 4.13. This skew is caused by the fact that the volatility is not hedged in a simple delta strategy. This tends to create very high absolute values of hedging errors, especially when no static hedging is applied.

In order to assess the effect of the risk premium on the distribution of the total discounted hedging error, we have repeated the same analysis for the delta hedging strategy, this time assuming $\lambda = 0$ and $\lambda = -1$. The resulting distributions are displayed in Figure 4.14 and Figure 4.15. Two main observations can be made from the graphs. First, when there is no volatility risk premium ($\lambda = 0$), the distribution is centered around 0, with an expected value of 0.0079%. However, when the premium is negative ($\lambda = 1$), the expected value also becomes negative and reaches -0.5627%. Thus, a smaller volatility risk premium ensures an expected value that is closer to 0.

The second observation concerns the variance of the results. In fact, as the volatility risk premium decreases, the variance of the distribution does so, too. The same trend can be observed in the difference between the $CTE_{95\%}$ and the expected value, which goes from 4.7761% when $\lambda = 2.62$ to 1.1158% for $\lambda = 0$ and 0.9324% when $\lambda = -1$. Although those results may seem surprising at first, the can be explained by the fact that a decrease in the volatility risk premium, when the risk-neutral parameters κ and θ remain the same translates into a decrease in the variance of the log-returns. This decrease comes from the change in the objective parameters κ' and θ', which are linked to the risk-neutral ones by the risk premium. One can however observe extreme values in the distribution of the total hedging error, even when the variance of the log-returns is smaller. That is an effect of the stochastic volatility.

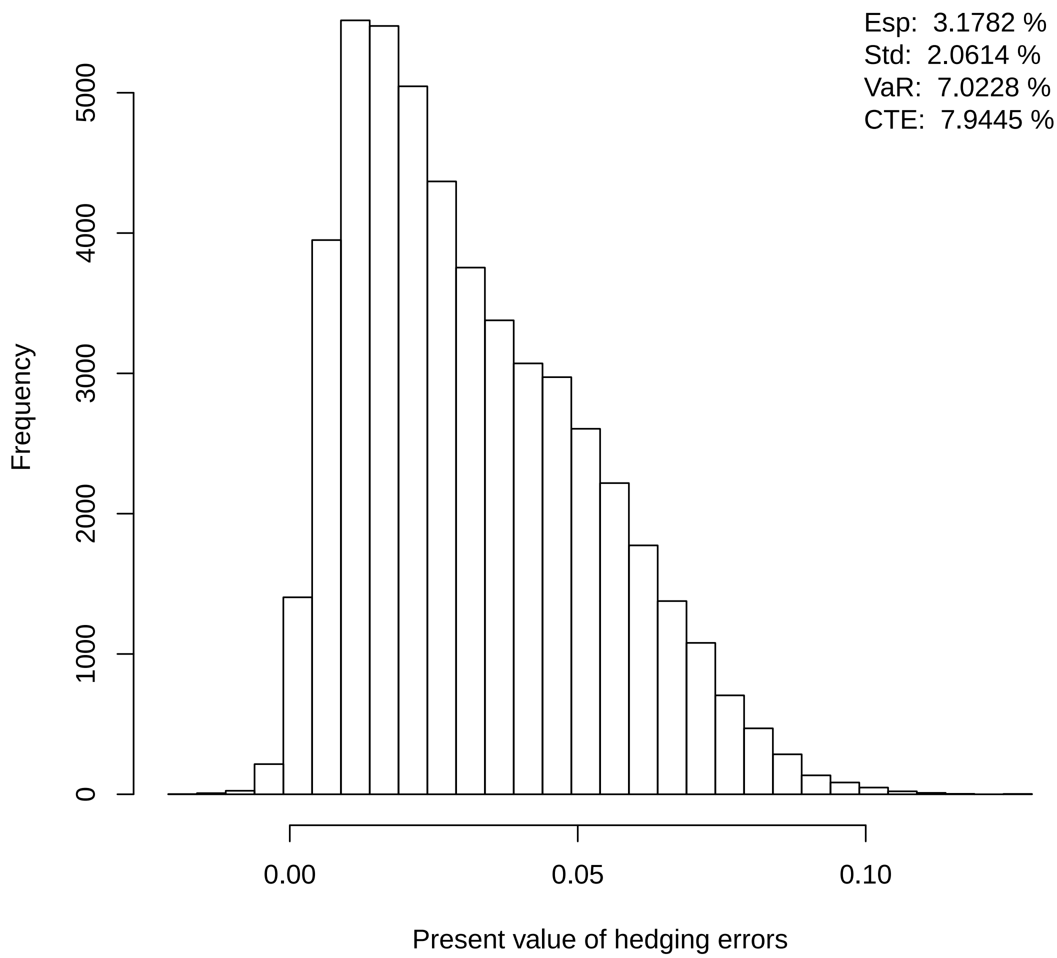

Figure 4.12: Present values of hedging errors resulting from a Heston delta hedging strategy in a Heston framework, $\lambda = 2.62$

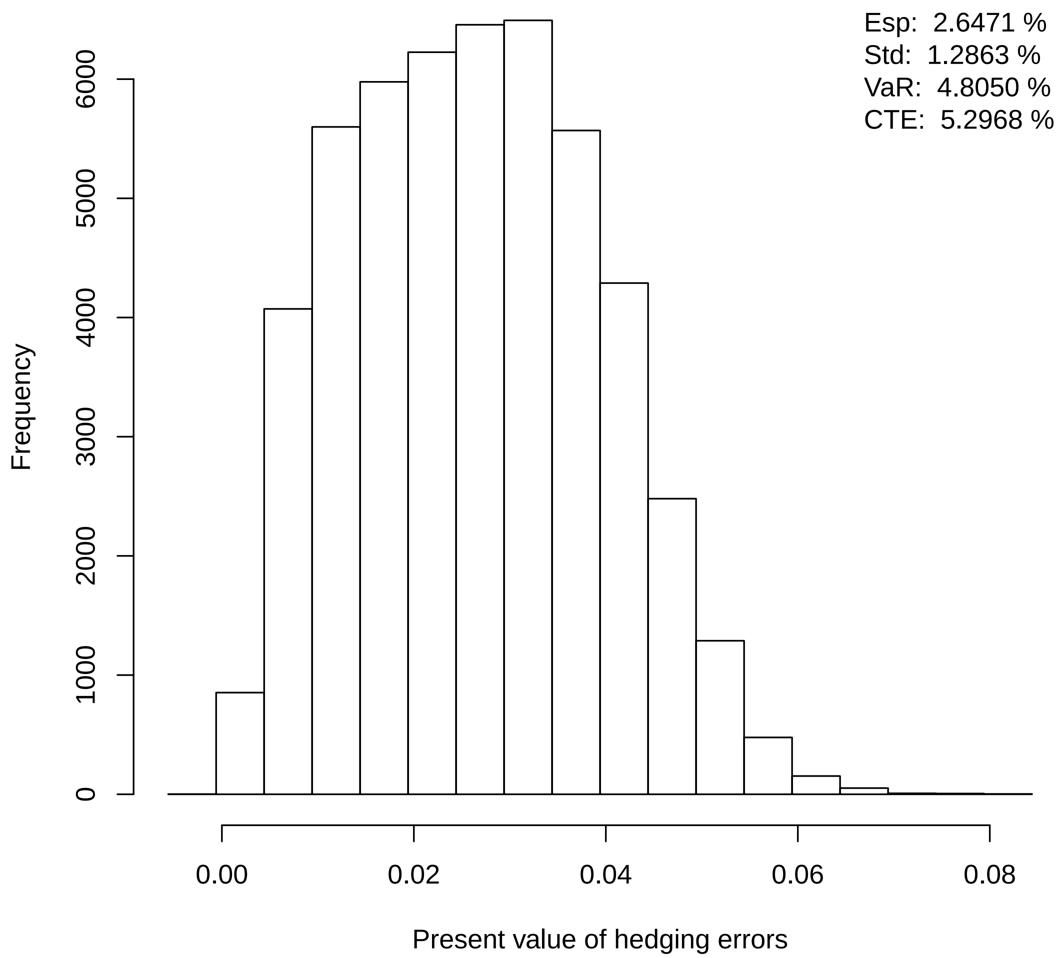

Figure 4.13: Present values of hedging errors resulting from a Heston delta and static hedging strategy in a Heston framework, $\lambda = 2.62$

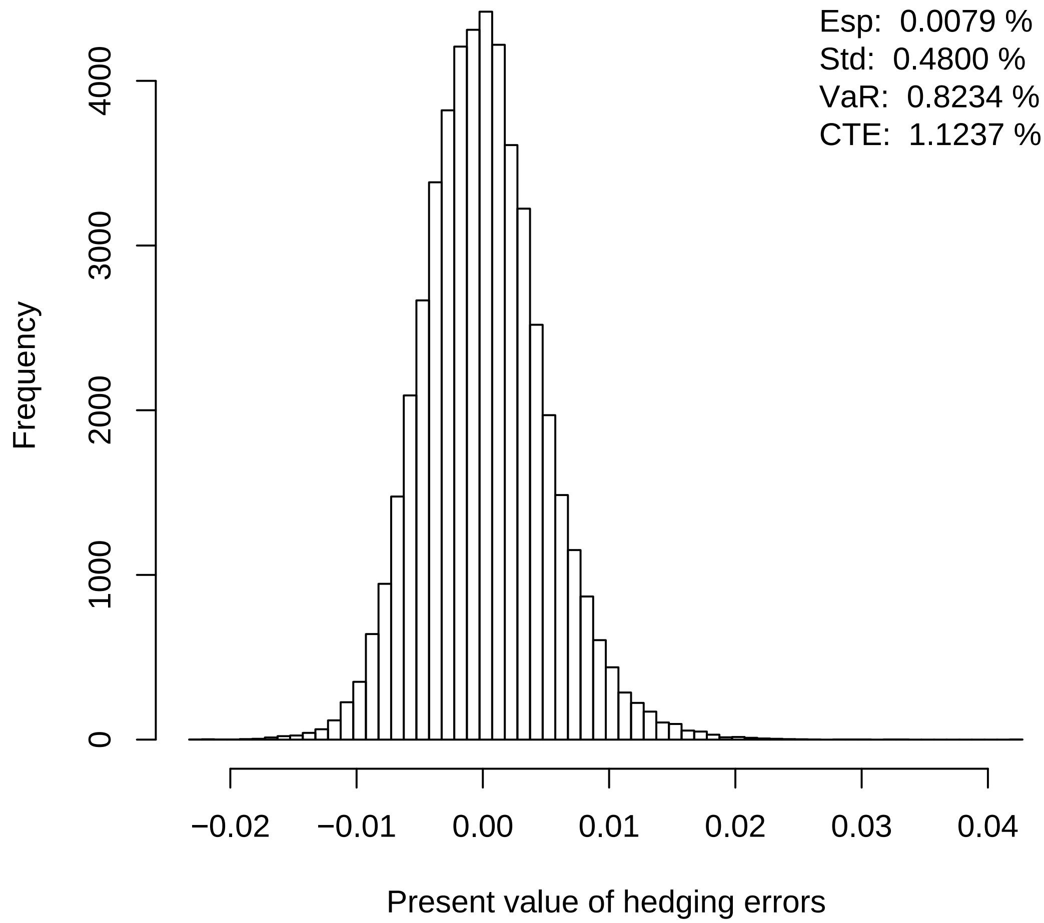

Figure 4.14: Present values of hedging errors resulting from a Heston delta hedging strategy in a Heston framework, $\lambda = 0$

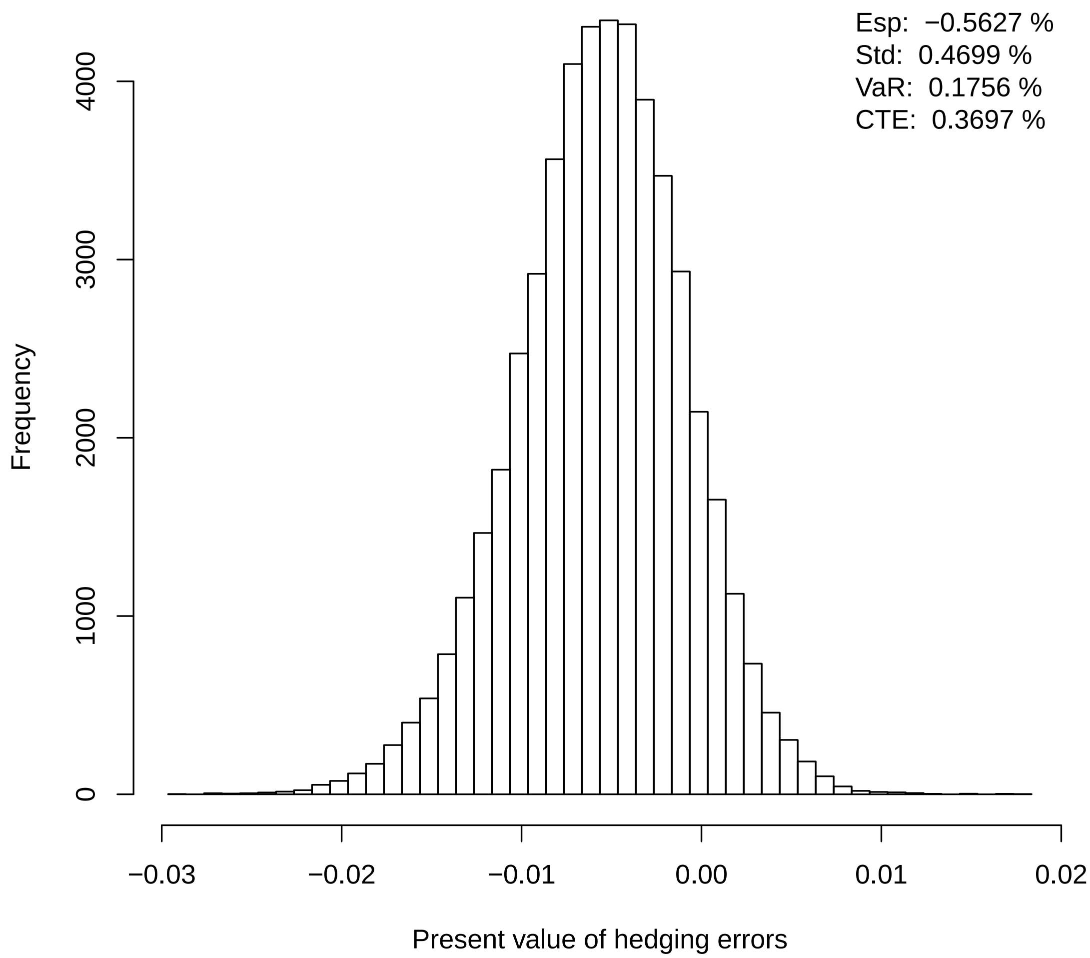

Figure 4.15: Present values of hedging errors resulting from a Heston delta and static hedging strategy in a Heston framework, $\lambda = -1$

4.4.3.2 Gamma Hedging

In the presence of stochastic volatility, neutralizing the Heston gamma of the portfolio will not directly protect the insurer against volatility risk. However, it should reduce the errors caused by the unexpected variations of the stock returns resulting from stochastic volatility. In this section, we assess the performance of the gamma and static hedging presented in section 4.3.3. The Greeks are derived using the Heston model, as in Example 2.2.

The distribution of the total discounted hedging error resulting from the gamma hedging strategy is presented in Figure 4.16. Incorporating call options to the replicating portfolio greatly improves the performance of the hedging strategy. In fact, the distribution of the total error, even in the presence of a volatility risk premium, is much more centered than in Figure 4.12. Both the standard deviation of the distribution and the distance between the mean and the $CTE_{95\%}$ are significantly reduced, from 2.0562% to 0.0851% in the first case and from 4.7761% to 0.2162% in the second. Although the gamma hedging strategy greatly reduces the risk involved in the sale of the point-to-point EIA, it is still possible to observe positive skewness in the distribution of the total hedging error. This may be caused by the volatility premium and by the fact that the volatility is not directly hedged.

To assess the effect of the volatility risk premium on the performance of the gamma hedging strategy, we performed a similar analysis as in Figure 4.16, but this time using $\lambda = 0$. The resulting distribution is illustrated in Figure 4.17 and shows that the risk premium does affect the performance of the gamma hedging strategy. In its absence, the mean of the hedging error is very close to 0 and the standard deviation of the distribution is reduced to 0.0179%. The presence of a volatility risk premium also thickens the right tail of the distribution. In fact, with $\lambda = 2.62$, the $CTE_{95\%}$ is 0.2202% while it drops to 0.0369% for $\lambda = 0$. Thus, although the gamma hedging strategy significantly reduces the risk involved in the sale of the EIA, the presence of a volatility risk premium moderates its efficiency.

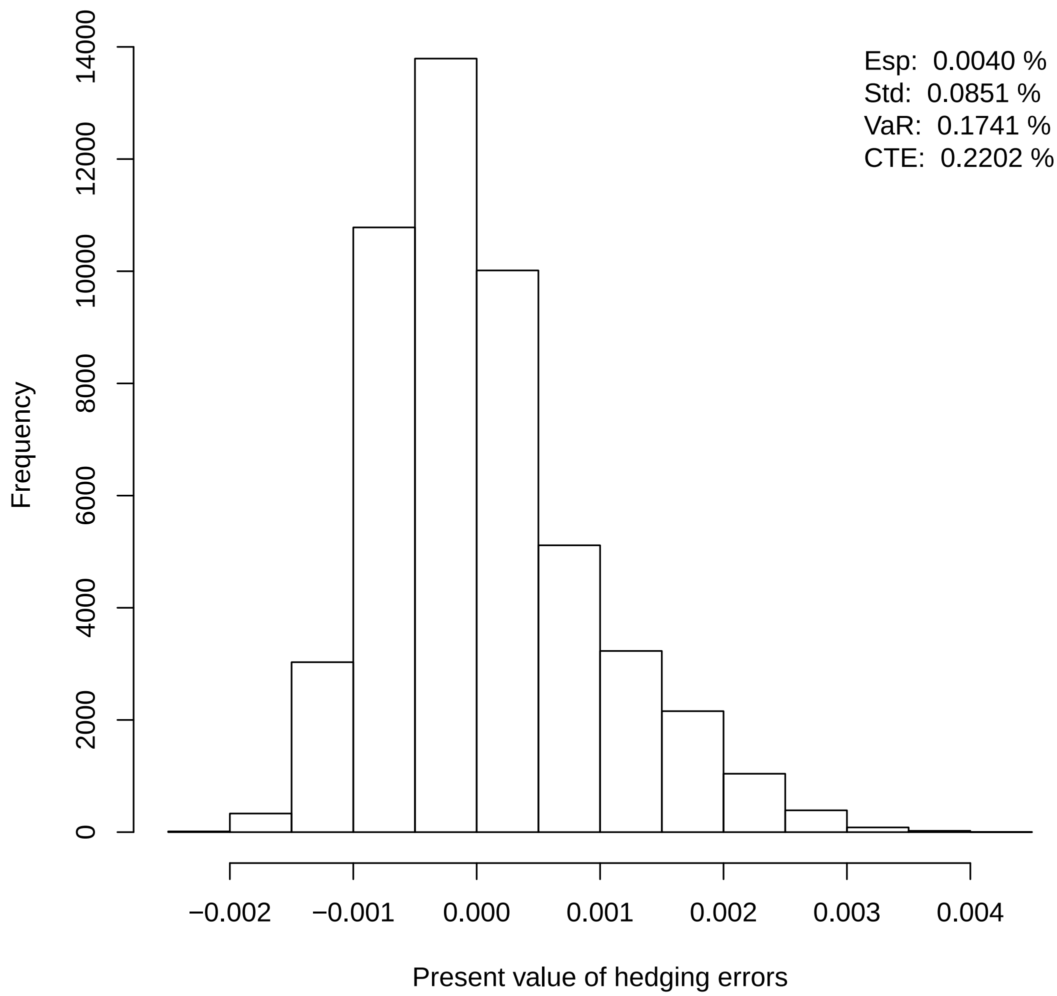

Figure 4.16: Present values of hedging errors resulting from a Heston gamma and static hedging strategy in a Heston framework, $\lambda = 2.62$

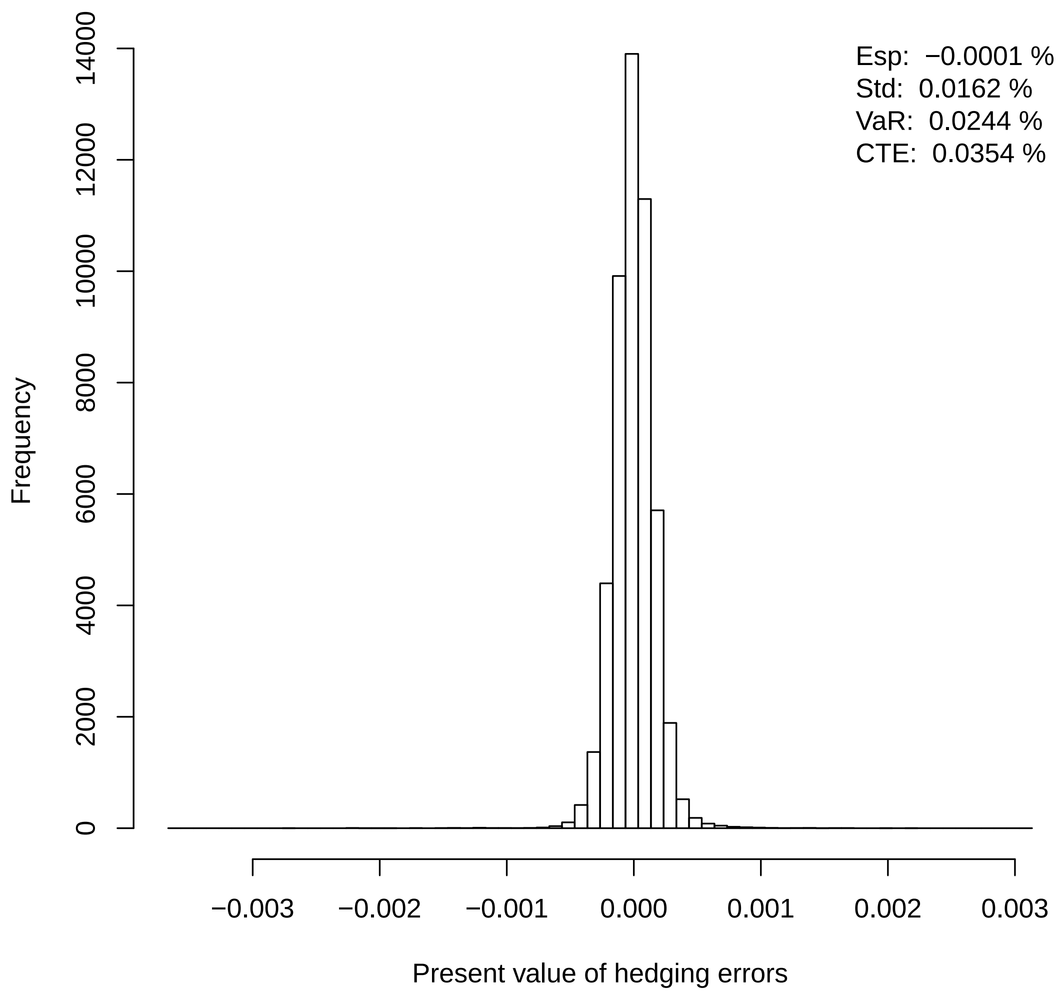

Figure 4.17: Present values of hedging errors resulting from a Heston gamma and static hedging strategy in a Heston framework, $\lambda = 0$

4.4.3.3 Vega Hedging

As it is the strategy that underlies the pricing of the European call option, and thus the point-to-point EIA, the vega hedging strategy should be the most efficient at reducing the total discounted hedging error. In this section, we assess the performance of the vega and static hedging strategy, presented in Section 4.3.4. We use the Heston Greeks such as in Example 2.2.

In order to smooth the values of $\alpha_{1,t}^{\mathcal{V}}$ and $\alpha_{2,t}^{\mathcal{V}}$, which may explode when S_t and v_t take extreme values, we make some adjustments in their calculation. First of all, we replace $\mathcal{V}_{C,t}^{H}$ and $\Delta_{C,t}^{H}$ by 0.001 if their actual value is lower. In the determination of $\mathcal{V}_{C,t}^{H}$, we let $S_t = 40$ if its actual value is lower. The bias introduced by these modifications is insignificant in comparison to the important errors caused by the extreme values.

Figure 4.18 presents the distribution of the total hedging error resulting from the application of the vega and static hedging strategy to $50,000$ sample paths of index prices, simulated as described in at the beginning of this section. As expected, this strategy performs better than the gamma hedging one. It is more centered around 0, with a mean of -0.0014%, compared to 0.004% for gamma hedging. However, the greatest improvement resides in the significant decrease in the standard deviation of the distribution and the difference between its $CTE_{95\%}$ and its mean. In fact, for the gamma hedging strategy, these figures were 0.0851% and 0.2162%, respectively. Applying vega hedging instead reduces them to 0.0246% and 0.053%. This signifies that the distribution is less dispersed around its mean and results in more predictable costs for the insurer. It reduces the probability of incurring important costs on the sale of an EIA contract.

The same analysis was performed, this time assuming $\lambda = 0$, to observe the effect of the volatility risk premium on the performance of the vega hedging strategy. The resulting distribution of the total discounted hedging error is presented in Figure 4.19.

One can notice that the performance of the vega hedging strategy in the absence of a volatility risk premium is improved. The mean of the distribution is brought back to 0% and the standard deviation is reduced to 0.0163%. Similarly, the $CTE_{95\%}$ is reduced by half, to 0.0234%. Thus, the vega hedging strategy is also affected by the volatility risk premium. However, even in the presence of such a risk premium, it performs fairly well and represents an important improvement over the other hedging strategy analyzed in this thesis. This leads us to believe that, under the assumptions made in this thesis, insurers would greatly benefit from the implementation of the Heston vega hedging strategy.

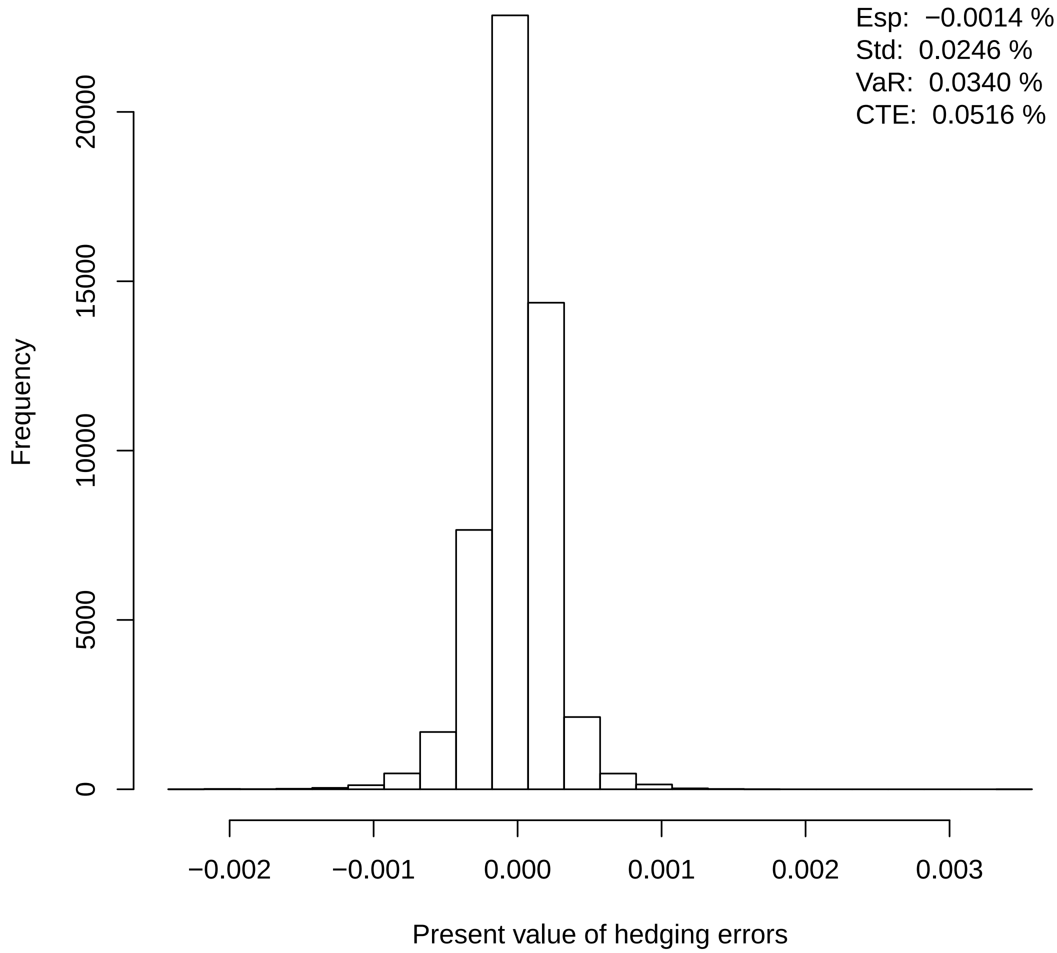

Figure 4.18: Present values of hedging errors resulting from a Heston vega and static hedging strategy in a Heston framework, $\lambda = 2.62$

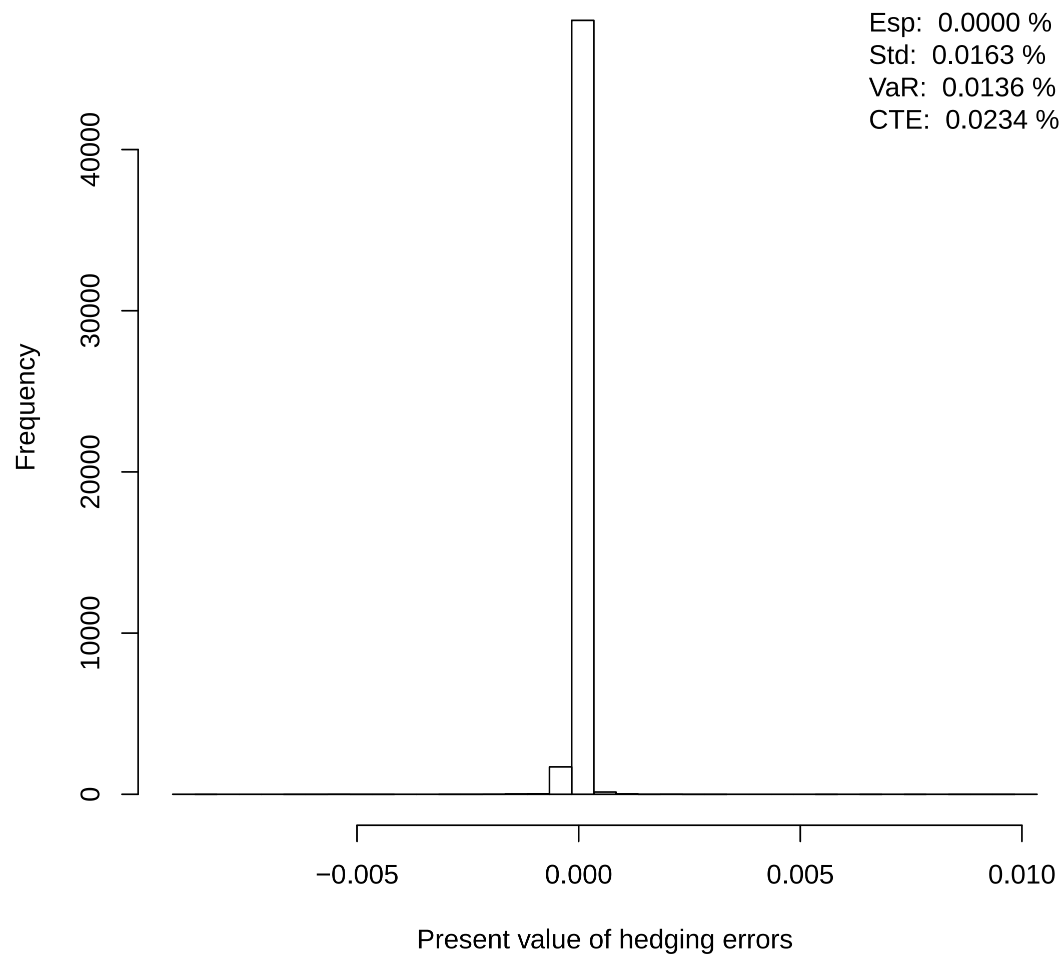

Figure 4.19: Present values of hedging errors resulting from a Heston vega and static hedging strategy in a Heston framework, $\lambda = 0$

Even when we relax some of the assumptions underlying our results, we believe that the use of Heston gamma and vega hedging strategies should be explored by insurers. In fact, we show here that these strategies perform very well in a Heston framework. Although this model may fail to capture some characteristics of the distribution of index returns, it does replicate it better than the Black-Scholes model does. For this reason, and since Heston gamma and vega hedging strategies perform better in a Heston framework, it makes sense to use Heston Greeks to hedge the sale of point-to-point EIAs.

Conclusion

Equity-linked products have grown in popularity in the past 15 years and they are still very attractive, especially for conservative investors who want to take advantage of market growth without facing downside risk. The sale of investment products with a payoff depending on the equity market represents a risk for insurance companies. This is particularly true in a market where the volatility of stock returns are not constant. Varying stock return volatility causing important changes in stock prices is likely to cause significant financial losses if the insurer is not protected against them. In order for the company to have sufficient funds to pay the policyholder at maturity of the contract, hedging strategies need to be implemented.

In this thesis, we analyzed the performance of strategies based on the Black-Scholes model and we presented alternatives using the Heston model. The choice of the Heston model is motivated by the fact that it is able to better reproduce the higher peaks and heavier tails observed in the empirical distribution of stock returns. Moreover, it allows for a semi-closed form expression for the price of a European call option. In turn, this allows for a similar expression for the price of a point-to-point EIA and its Greeks.

We showed that the hedging strategies based on Black-Scholes assumptions are efficient at reducing the risk inherent to the sale of a point-to-point EIA in a Black-Scholes framework. However, these results cannot be used to recommend hedging strategies for the real markets, as the Black-Scholes model fails to replicate the characteristics that make those markets risky. To demonstrate this, we applied the hedging strategies to index prices simulated by the Heston model. Since this model is better able to capture some of the risky characteristics of the market, strategies that fail when used with Heston data are likely to fail when applied to empirical data. Our results show that the differences between the Black-Scholes and Heston calibration methods reduce the efficiency of the hedge. In addition, it may lead the insurer who prices the EIA using the Black-Scholes model to over- or underprice the contract. This will affect the hedging strategy by creating systematic losses or gains. It can also decrease the competitiveness of the company. We have also demonstrated that Black-Scholes hedging strategies are affected by the volatility risk premium. Since it has been shown that such a premium exists and that it varies through time (see Bollerslev *et al.* (2011)), strategies that are very sensitive to it are less efficient at reducing risks.

We also demonstrated that the performance of hedging strategies can be improved by using Heston Greeks. Since the Heston model assumes stochastic volatility, a simple delta hedging strategy does not suffice to hedge all the risk. This strategy is significantly affected by the volatility risk premium and fails to hedge the volatility risk. Our results show that incorporating call options to the replicating portfolio, either through gamma or vega hedging, improves the performance of the strategy. In fact, even in the presence of an important volatility risk premium, the distributions of the total hedging error resulting from the gamma and the vega hedging strategy remain centered around

0. In both cases, low variances, $VaR_{95\%}$ and $CTE_{95\%}$ signify that the cost of the hedge will remain low. Thus, the use of Heston Greeks and the addition of call options to the replicating portfolio significantly reduces the risk inherent to the sale of the point-to-point EIA in a Heston market.

It is interesting to see the extent to which the addition of stochastic volatility to the market model affects the hedging of the point-to-point EIA. In fact, this design should be less sensitive to stochastic volatility than other types of designs, such as the annual reset. Further research could focus on this very popular design. The inclusion of jumps, transaction costs and a varying volatility risk premium should also be explored to extend this work.

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Appendix

This appendix contains a sample of the code used to produce the graphs in Section 4.4. The code displayed below contains the functions to calculate the price and the Greeks of a point-to-point EIA and a European call option. The function `PTP.dv.HE()` use the Heston model to simulate a sample path of index prices and calculates the discounted hedging error resulting from the application of the Heston vega hedging strategy. The histogram representing the distribution of that hedging error is obtained by running this function 50,000 times and producing a graph with the output.

```
PTP.price.H<-function(S0,St,t.rem,n,r,kap,theta,sigma,v,rho,g,alpha,rhoptp){
K<-rhoptp*(1+g)^n;L<-S0*((K-1+alpha)/alpha)
K*exp(-r*t.rem)+(alpha/S0)*hes.call.price(St,L,r,t.rem,kap,theta,sigma,v,rho)}

PTP.del.H<-function(S0,St,t.rem,n,r,kap,theta,sigma,v,rho,g,alpha,rhoptp){
K<-rhoptp*(1+g)^n;L<-S0*((K-1+alpha)/alpha)
x<-log(St*exp(r*t.rem)/L);i<-complex(real=0,imaginary=1)
(alpha/S0)*(Pj(1,x,kap,theta,sigma,v,rho,t.rem)+
        Pj.dx(1,x,kap,theta,sigma,v,rho,t.rem)-
```

```
        (exp(-x))*Pj.dx(0,x,kap,theta,sigma,v,rho,t.rem))}


PTP.gamma.H<-function(S0,St,t.rem,n,r,kap,theta,sigma,v,rho,g,alpha,rhoptp){
K<-rhoptp*(1+g)^n;L<-S0*((K-1+alpha)/alpha)
x<-log(St*exp(r*t.rem)/L);i<-complex(real=0,imaginary=1)
(alpha/(S0*St))*(Pj.dx(1,x,kap,theta,sigma,v,rho,t.rem)+
        Pj.dx2(1,x,kap,theta,sigma,v,rho,t.rem)+
        (L*exp(-r*t.rem)/St)*(Pj.dx(0,x,kap,theta,sigma,v,rho,t.rem)-
        Pj.dx2(0,x,kap,theta,sigma,v,rho,t.rem)))}


PTP.vega.H<-function(S0,St,t.rem,n,r,kap,theta,sigma,v,rho,g,alpha,rhoptp){
K<-rhoptp*(1+g)^n;L<-S0*((K-1+alpha)/alpha)
x<-log(St*exp(r*t.rem)/L);i<-complex(real=0,imaginary=1)
   (alpha/S0)*(St*Pj.dv(1,x,kap,theta,sigma,v,rho,t.rem)-
        L*exp(-r*t.rem)*Pj.dv(0,x,kap,theta,sigma,v,rho,t.rem))}


C.price<-function(S,K,r,n,kappa,theta,sigma,v,rho){
x<-log(S*exp(r*n)/K)
S*Pj(1,x,kappa,theta,sigma,v,rho,n)-
        K*exp(-r*n)*Pj(0,x,kappa,theta,sigma,v,rho,n)}


C.delta<-function(S,K,r,n,kappa,theta,sigma,v,rho){
x<-log(S*exp(r*n)/K)
(Pj(1,x,kappa,theta,sigma,v,rho,n)+
```

```
        Pj.dx(1,x,kappa,theta,sigma,v,rho,n)-
        (exp(-x))*Pj.dx(0,x,kappa,theta,sigma,v,rho,n))}


C.gamma<-function(S,K,r,n,kappa,theta,sigma,v,rho){
x<-log(S*exp(r*n)/K)
(1/S)*(Pj.dx(1,x,kappa,theta,sigma,v,rho,n)+
        Pj.dx2(1,x,kappa,theta,sigma,v,rho,n)-
        exp(-x)*(Pj.dx2(0,x,kappa,theta,sigma,v,rho,n)-
        Pj.dx(0,x,kappa,theta,sigma,v,rho,n)))}


C.vega<-function(S,K,r,n,kap,theta,sigma,v,rho){
    x<-log(S*exp(r*n)/K)
    K*exp(-r*n)*(exp(x)*Pj.dv(1,x,kap,theta,sigma,v,rho,n)-
        Pj.dv(0,x,kap,theta,sigma,v,rho,n))}


PTP.dv.HE<-function(S0,kap,theta,sigma,v0,rho,mu,lambda,nstep,nrebal,
nyears,g,r,alpha,rhoptp,prem){
library(MASS)
kap.s<-kap-lambda;theta.s<-(theta*kap)/(kap-lambda)
nstep<-ceiling(nstep/nrebal)*nrebal;simx<-0;simS<-S0
    ssize<-nstep/nrebal;v<-v0;;v.vec<-v
L<-S0*(rhoptp*(1+g)^nyears-1+alpha)/alpha
del.PTP<-PTP.del.H(S0,S0,nyears,nyears,r,kap,theta,sigma,v,rho,
        g,alpha,rhoptp)
```

```
del.C<-C.delta(S0,L,r,3,kap,theta,sigma,v,rho)
vega.PTP<-PTP.vega.H(S0,S0,nyears,nyears,r,kap,theta,sigma,v,rho,
        g,alpha,rhoptp)
vega.C<-C.vega(S0,L,r,3,kap,theta,sigma,v,rho)
PTP.V<-PTP.price.H(S0,S0,nyears,nyears,r,kap,theta,sigma,v0,rho,
        g,alpha,rhoptp)
call.price.old<-C.price(S0,L,r,3,kap,theta,sigma,v,rho)
call.price.new<-C.price(S0,L,r,3,kap,theta,sigma,v,rho)
HE<-PTP.V-prem
for (i in (1:(nstep*nyears))){
z<-t(mvrnorm(1,c(0,0),cbind(c(1,rho),c(rho,1))))
simx<-simx+(mu-max(0,v)*0.5)/nstep+sqrt(max(0,v)/nstep)*z[2]
v<-(sqrt(max(0,v))+(sigma/2)*sqrt(1/nstep)*z[1])^2-
            kap.s*(max(0,v)-theta.s)*(1/nstep)-(sigma^2)/(4*nstep)
v.vec<-c(v.vec,max(10^-8,v))
simS<-c(simS,S0*exp(simx))}
year.vec<-c(1,(1:nyears)*nstep+1)
    rebal.vec<-c(1,(1:(nrebal*(nyears-3)))*ssize+1)
simS.year<-simS[year.vec];simS.rebal<-simS[rebal.vec]
v.rebal<-v.vec[rebal.vec]
t.rem.new<-c(2,rep(seq(3-1/nrebal,2,by=-1/nrebal),nyears-3))
t.rem.old<-c(rep(seq(3,2+1/nrebal,by=-1/nrebal),nyears-3),3)
for (i in (2:length(simS.rebal))){
del.PTP<-c(del.PTP,PTP.del.H(S0,simS.rebal[i],nyears-(i-1)/nrebal,
```

```
            nyears,r,kap,theta,sigma,v.rebal[i],rho,g,alpha,rhoptp))
del.C<-c(del.C,C.delta(simS.rebal[i],L,r,t.rem.old[i],kap,theta,
            sigma,v.rebal[i],rho))
vega.PTP<-c(vega.PTP,PTP.vega.H(S0,simS.rebal[i],nyears-(i-1)/nrebal,
            nyears,r,kap,theta,sigma,v.rebal[i],rho,g,alpha,rhoptp))
vega.C<-c(vega.C,max(C.vega(max(simS.rebal[i],40),L,r,t.rem.old[i],
            kap,theta,sigma,v.rebal[i],rho),10^-3))
PTP.V<-c(PTP.V,PTP.price.H(S0,simS.rebal[i],nyears-(i-1)/nrebal,
            nyears,r,kap,theta,sigma,v.rebal[i],rho,g,alpha,rhoptp))
call.price.old<-c(call.price.old,C.price(simS.rebal[i],L,r,
            t.rem.old[i],kap,theta,sigma,v.rebal[i],rho))
call.price.new<-c(call.price.new,C.price(simS.rebal[i],L,r,
            t.rem.new[i],kap,theta,sigma,v.rebal[i],rho))}
alp.1<-vega.PTP[-length(vega.PTP)]/vega.C[-length(vega.C)]
alp.2<-del.PTP[-length(del.PTP)]-alp.1*del.C[-length(del.C)]
St<-simS.rebal[-length(simS.rebal)];St1<-simS.rebal[-1]
    Vt<-PTP.V[-length(PTP.V)];Vt1<-PTP.V[-1]
Ct1<-call.price.new[-1];Ct<-call.price.old[-length(call.price.old)]
disc<-exp(-r*(1:(nrebal*(nyears-3)))/nrebal)
HE<-c(HE,disc*(Vt1-Vt*exp(r/nrebal)-
        alp.1*(Ct1-Ct*exp(r/nrebal))-alp.2*(St1-St*exp(r/nrebal))))
sum(HE)}
```