APPLICATION OF STATISTICS TO THE STUDY OF COGNITIVE PROCESSES IN MATHEMATICS EDUCATION - A CASE STUDY OF THEORETICAL THINKING AS A FACTOR OF STUDENTS' SUCCESS IN LINEAR ALGEBRA

by

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ABSTRACT

APPLICATION OF STATISTICS TO THE STUDY OF COGNITIVE PROCESSES IN MATHEMATICS EDUCATION - A CASE STUDY OF THEORETICAL THINKING AS A FACTOR OF STUDENTS' SUCCESS IN LINEAR ALGEBRA

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Factors influencing students' success in mathematics in general and linear algebra in particular are examined, drawing on the literature of mathematics and science education. Based on the findings from a sample of 14 high achieving university undergraduate students, certain features of theoretical thinking are identified as necessary factors of the students' success in two linear algebra courses.

The nature of the resulting data presented a peculiar problem. An extensive study of the possibility of applying different forms of statistical analysis to these data is conducted from different perspectives on statistical methodology namely, descriptive/exploratory statistics and statistical modeling. A number of indices for measuring theoretical behavior tendencies are defined and a new method of analysis is developed.

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This research has been motivated by an ongoing study centered around finding ways of improving the teaching and learning of Linear Algebra. I am grateful to all that have in one way or the other helped to advance research in this area. And to the research leader and supervisor of this thesis, Dr. Anna Sierpinska, I remain highly indebted for the support and encouragement I really needed to carry on this work.

DEDICATION

To

Kelechi, Osita and their parents

&

Helena and her parents

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Chapter 1. Introduction

For a long time, the learning of mathematics was perceived mainly in terms of skill acquisition. However, reforms would always start by stating the students' notorious lack of understanding of mathematical concepts in spite of satisfactory procedural skill acquisition. As such, there have been numerous studies addressing the questions of what students' difficulties in the learning and understanding of different areas of mathematics are, and how to find ways of teaching that can help students overcome these difficulties. However, the current trend is a shift in focus in the way mathematics is perceived from the traditional view in terms of skill acquisition towards a view in terms of cognitive processes. This change in view from acquisition of skills to that of thinking processes is being bolstered by evolution of research interests from the study of students' difficulties towards that of factors of students' understanding that underlie these difficulties. In most cases, such studies do involve the collection of data containing information to be summarized, analyzed and interpreted in order to provide answers to research questions. In order to obtain useful summaries and meaningful interpretations of data that go beyond opinions, ambitious or humble, a scientific framework is needed. Such a framework would enable proper and scientifically accountable ways of collecting, collating and interpreting research data and findings based on them.

Cognizant of the current wave of interest in studying students' cognitive processes, this thesis will seek to examine the possibilities of applying statistical methods to the quantitative analysis of data arising from such studies. An effort being made in a pioneer research to explore cognitive factors of some high achieving undergraduate university students' success in linear algebra will be used as a case study. Before going into details, a brief discussion of traditional applications of statistics to research in this area is in order.

1.1 Traditional Applications of Statistics to Research in Mathematics Education

Classical descriptive statistics methods and the closely related method of exploratory data analysis are often used in analyzing data resulting from studies in mathematics education. Although both methods may serve in the formulation of laws or testing of hypotheses, the scope of their application is limited to data at hand. This is in sharp contrast with the method of inferential statistics, which is concerned with formulation and tests of hypotheses aimed at generalizing from a random sample of data to a larger population of possible data. However, the notion of limiting the usefulness of the descriptive and exploratory methods to the data at hand does not make them any worse or better methods to be disliked or preferred to statistical inference methods. It depends on the nature, aims and objectives of the study, the research question(s) and subsequently the nature of the resulting data. About the resulting data, one has to consider issues such as the sample size, the question of randomness of sample, the method and cost of data collection, the type of subjects (variable or individual), the number of characteristics, attributes or variables of interest and subsequently, the dimension of the data. Consider for example, a pioneer study that is aimed at probing for the first time, a conjecture about a certain characteristic of students' behavior. It is unlikely that the researcher would have much need at the first stage of the study to formulate and test hypotheses aimed at generalizing his or her findings to a population of data other than the one under study. Very often, the idea is that of developing techniques and frameworks that can be applied to similar research studies as opposed to obtaining results that have to be generalized to a wider population of data.

Also, the type of questions that have predominantly been treated statistically in mathematics education include among others, national and international surveys of mathematical skills and the assessment of achievement [see for example Robitaille and Stuart Donn (1993), Kilpatrick (1993)]. More frequently, researchers dealing with issues such as factors of students' success in mathematics have been addressing questions that warrant bilateral forms of analysis aimed at examining the extent to which some less known variable(s) can be positively or negatively associated with some better known variable(s). In this direction, such forms of analysis may be fairly straightforward when dealing for example with certain pedagogical, socio-cultural, affective or demographic factors such as age, sex, civil status etc. But sometimes it may be important to consider unilateral forms of analysis that require dealing with the variable(s) of interest alone without necessarily having to arrive at conclusions dependent on the associations with other variables. Perhaps this is the light in which Gras (1992) suggested, among others, an "implicative analysis" method, which involves statistical modeling that will be considered in chapter three. Unilateral forms of analysis may be necessary when dealing with cognitive factors of students' success, which involve observing, measuring and interpreting human cognitive processes through behavior. However, in this regard a number of problems can arise. Some of the possible problems are considered in the next section.

1.2 Some Problems of using Statistical Analysis in the study of Cognitive Processes

In general, many statistical populations are comprised of naturally distinguishable members, as is the case with tosses of a coin or rolls of a die or, say, number of people in a community, etc. But there exist situations where some populations are not so comprised, as is the case with, for example, a quantity of water or, of particular interest in the sequel, human behavior. An important question that arises then is that of how to apply the concept of group or aggregation to populations of this latter type? Defining the population member arbitrarily as a litre of water or some unit may accomplish this since the whole population may be thought to be composed of an aggregate of such members. In the study of cognitive processes and behavior, much of the data that are analyzed and interpreted arise from the conduct of experiments with human subjects. Thus researchers studying human behavior frequently concern themselves with arbitrarily defined bits of behavior that are modeled or conceived as discrete variables. However, the question remains whether behavior should not be better conceived as a continuous flow or sequence.

Also there is the issue of reliability of data. Questions such as: who are to conduct the interviews? Should they be persons, other than the designers, who do not know which question is which, lest unconsciously they give the interviewees a clue? Or should the designers themselves who know the questions better and perhaps stand a better chance of obtaining a reliable data conduct the interviews? This issue is very important since the validity of any conclusions or research findings are subject to the validity of the data upon which such conclusions are based.

Another important problem is that of sampling. What is the sample size? Does the data constitute a random sample? Considering the question of cost, time, subject availability and other technicalities involved in human sociology, psychology, physiology, political and legal issues, this question may not be a very simple one. Very often, the availability and consent of the subjects can impose an unexpected constraint on the possibility or impossibility of realizing a random sample of a certain desired size. This leads to yet another important issue, namely the question of generalization of results.

Should research findings be generalized to a wider population of data or limited to the sample of data? For statistical inference purposes, a result can be generalized to a wider target population only if the data upon which the result is based constitutes a random sample from the target population. However, in general, this would depend on the spirit, scope, aims and objectives of the research. It turns out that abuse and misjudgment of the application of statistical methods to the analysis of data usually stems from lack of clear vision of research intent on the part of the researcher, or lack of proper understanding of this intent by critics or both. Frequently, needs do arise for case studies, like the one that will be considered in this thesis, the findings of which mainly serve as guidelines or frameworks for future research. It is not very necessary at the very first stage of an inquiry to attempt generalizing results and as such the question at this stage is not that of how to arrive at a valid and generalized statistical result from the first sample data. Instead the main preoccupation of the researcher at this stage is to develop a framework of inquiry along with methods of analysis and to figure out which of these methods would be most appropriate for the type of framework developed and the resulting data to be analyzed. Thus to analyze and report a sample data arising in this context statistically, whether by an inferential, a descriptive or a data analysis method, assumptions are needed. If in the process of a study, inference techniques for generalization are developed, assumption of random sampling, among others, has to be included in the framework. If only descriptive or exploratory data analysis techniques are developed, an assumption of absence of need for generalization would have to be included and explained in the framework, otherwise the usefulness of a research result that cannot be generalized because of lack of randomness of data would be in question.

In chapter two in which a case study of theoretical thinking as a factor of students' success in Linear Algebra is analyzed, more insight into the nature of these types of problems discussed above, will be gained. Before that, it would be opportune to consider, in the next section, some of the factors of students' success in mathematics of which factors of students' success in linear algebra form a subset.

1.3 Factors of Students' Success in Mathematics (FOSSIMs)

In the mathematics education literature, both theoretical results and research findings point to the fact that cognitive as well as non-cognitive factors play very important roles in students' success in mathematics. Boero *et al.*(1997) considered an important pedagogical factor as the problem of bridging the gap between "the expressive forms of students' everyday knowledge and the expressive forms of theoretical knowledge; between the students' spontaneous ways of getting knowledge through facts, and theoretical deduction; and between students' intuitions, and the counterintuitive content of some theories." Another pedagogical factor which has been considered detrimental to students' success is the spiraling curriculum because it is said to encourage repetition and could therefore lead to procrastination of learning; Stigler & Perry (1990), Stevenson & Stigler (1992) and U.S. Department of Education (1988).

Bishop (1983) considered the effect of the socio-cultural context in which mathematics is carried out and argued that context influences the way people approach mathematical problems. Stevenson et al. (1993) considered the effect of cultural attitudes towards learning and found that in countries such as Japan and China where exemplary mathematics achievement of students has been documented, the students are more oriented toward effort than ability in explaining the reasons for success and failure than their American counterparts. Schoenfeld (1989a, 1989b), Lester et al. (1989), Philippou and Christou (1998), all have investigated and found that students' beliefs and concept development are significant factors in the mathematics learning process. In particular, Schoenfeld (1989b) and Kroll (1989) bolster the assumption that affective variables such as beliefs and attitudes towards mathematics play a major role in the development of students' problem solving abilities. Also investigations into the mathematical beliefs of students have been carried out at different school levels. For such studies at the elementary and secondary school levels there are for example; Fennema and Peterson (1985), Dossey et al. (1988), Lester et al. (1989), Garofalo (1989), Mandler (1989), Schoenfeld (1989b), Thompson and Thompson (1989). For investigation of the beliefs of university students, there are, for example, Schoenfeld (1989b) at the undergraduate level and Carlson (1999) at the graduate level.

Cognitive factors as well as influences on students' metacognitive attitudes about learning constitute peculiar potential sources of impediment or contribution to students' success in mathematics. In this connection, Hatano (1988) distinguished "routine" learners from "adaptive learners" whom he considers to have a "deep conceptual understanding" of mathematics and are flexible in their ways of applying mathematical principles. Schwartz (1995), in distinguishing between what he calls "problem solving", which encourages critical thinking skills, and "problem posing", which encourages the development of exploration and conjectures, sees the latter as being left out in mathematics education to the detriment of skills development among students. Gray et al. (1999) considered knowledge construction and diverging thinking in elementary and advanced mathematics. Recently, in a similar direction, Sierpinska (2000, in Dorier) conjectured that the reasons behind students' lack of "good understanding" (good in the sense of not deviating significantly from the theory) is that most students try to grasp the theory with a practical, as opposed to a theoretical mind. This brings into focus the notions of two forms of thinking; practical thinking (PT) and theoretical thinking (TT) upon which a framework for qualitative as well as quantitative assessments of certain factors of students' success in mathematics can be based.

From the foregoing section, it can be observed that some of the influential factors that impede or abet students' success in mathematics can be identified and classified under four major potential sources of influence mainly: socio-cultural, pedagogical, affective and cognitive perspectives. In what follows, further incursion into some of these influential factors is made under these four main categories. Before that, it is opportune to remark that the nature of mathematics and the nature of students are crucial factors that should be highlighted and considered even before any other since every other factor would in some way tend to lean upon them.

The Nature of Mathematics and The Nature of Students

The presence or lack of a sort of balance or moderation between the nature of mathematics and the nature of students can influence students' success in mathematics. As pointed out earlier in the introduction, mathematics has a peculiar theoretical nature that very often requires the processes of abstraction and analysis. And these processes are known to pose some extraordinary challenge to most students. An important question worth posing at this juncture is; what actually is it that makes the processes of mathematical abstraction and analysis less friendly to most students? Many explanations can come to mind but surely formal mathematical abstraction and analysis are known to require some extra degree of seriousness. Also some extra amount of time may be required for development of the mathematical acumen. Added to this is the nature of students - human nature, which allows for use of both theoretical and practical means in attending to life situations. Students, by human nature, would more frequently prefer to exploit their practical dimension in search of a carefree, easy going, less serious and supposedly time saving ways of life. Also, people everywhere are known to frequently attribute some of their characteristic qualities to nature. For example, some would claim that they are 'naturally' patient, lazy, gifted, serious, extrovert, intelligent etc. It might as well be that all or some of these are mere questions of habit that can be acquired or dropped. However, probing the extent to which such claims may be true or due to nature is not my concern in this study. I have brought these types of issues up in order to elucidate their subtlety as factors of students' success in mathematics. They may neither be completely ignored nor completely accounted for by any framework used for assessing *factor(s) of students' success in mathematics*, hereafter referred to as a *FOSSIM* –

singular and *FOSSIMs* - plural.

Also worth mentioning in this context are naturally occurring demographic factors such as sex, age, race etc., which are beyond human control once in effect. It is common to think and talk in terms of activities best suited for males or females, for children or adults, for people of one racial group or the other. Although human nature with its capricious tendency towards bias cannot guarantee the truth of these classifications, sometimes such distinctions can be sincere, realistic and objective. When such is the case, the question becomes that of how to effectively measure the influences of these factors on students' success in mathematics. Do these factors act in isolation or in association with each other within their area of influence? Do they interact with any other factors outside their peculiar areas? What are these other factors and what is the extent of such associations and interactions and how can they be assessed?

Socio-cultural factors

The socio-cultural context in which mathematics is done has been said to influence the way people approach mathematical problems. Consider for example mathematics in schools and mathematics outside schools. Although both are based on the same logico-mathematical principles, mathematical activities done in and out of school have been noted to have different social organizations. Bishop (1983) argues that this difference is rooted in the fact that everyday activities involve people in mathematizing situations while traditional school mathematics involves dwelling on the results of other people's mathematical activities. Unlike mathematics done outside school, mathematics in many of today's classrooms has been viewed as "a set of symbols, procedures, and definitions to be learned for perhaps some later application" while mathematics done outside school has been likened to modeling [D'Ambrosio (1986) cited in Nunes (1993)]. This outside school modeling metaphor is said to be a way of representing a reality so that further knowledge about the reality can be gained from other representations that do not necessarily warrant comparisons of available results with the reality. Thus the differences in social situations and their corresponding mathematics can impact on types of procedures used by students when solving problems and consequently their rates of success.

Cultural attitudes towards learning have been described as critical in the different levels of students' success that researchers have documented (Bempechat, 1998 p.71). This relates to the students' beliefs about learning, beliefs about the link between education and the future, and beliefs about the causes of success and failure in general. Such beliefs may be first transmitted to them through their parents, guardians and immediate society before they come to school. For example, cultural orientation toward effort may be more important than that toward innate ability in determining children's performance in mathematics. As pointed out earlier, students in Japan and China show exemplary mathematics achievement, which has been linked to the fact that the students attribute their success and failure more to effort than ability. Contrary to this type of orientation, their American counterparts attribute more to ability (Stevenson et al. 1993). Thus, while the Asian students' greater focus on effort could lead them to view success in mathematics as a manageable influence, the American students' focus on ability would lead them to view success in mathematics as a fixed and unmanageable influence. Although in each case, the variable is recognized as an internal factor, for the American students, it is a function of intelligence (Weiner, 1994) and therefore beyond their control. It is noteworthy that while a cultural view such as that held by the Asian students could lead to hope and perseverance, the American students' type of view could lead to despair and lethargy.

Pedagogical Factors

Interpersonal relationships and interactions that surround the learning experience are among some of the pedagogical factors that do come into play in determining students' success in mathematics. A teacher with an unfriendly attitude poses a barrier. There is need for good interpersonal communication between teacher and student. In this way the subject matter may be made more accessible. Also peer support and connection can be a powerful boost to doing mathematics, and its absence can be a major barrier. When students feel different and distant from the other students in the room, thoughts of inferiority or superiority complex can easily create an impediment.

The organization and structure of the course itself can also create a sense of distance between the student and the subject matter. Despite the peculiar nature of mathematics mentioned above, mathematics courses are often presented with a discouragingly bitter flavor. Sometimes too much emphasis is laid on correct answers to problems and good grades. Although solutions do of course matter, too much emphasis on them in the syllabus and handouts may overshadow the importance of the processes of abstraction needed to learn how to get to the concepts that help students think about the problems and, ultimately, solve them. A possible implication is that students, as humans, may begin to either think more practically in terms of immediate gratification (e.g. high grade) or begin to shy away by dropping the 'terrible' course when they envision a slim chance of such an immediate reward. In this way, the habit of theoretical thinking would become an unnecessary luxury no student would wish to acquire or even admire.

Another pedagogical factor that serves as a barrier is the use of problems and examples taken out of the context in which they would naturally occur. Such problems may seem unfamiliar and remote from anything students find meaningful. To cite a personal experience, I will never forget a high school physics class in which our teacher used a railway example in a book to explain to us the concept of contraction and expansion of metals. It was in a remote part of Africa where we had neither seen a railway before nor experienced the sharp temperature differences between winter and summer as described in the book. Of course none of that made sense to me until much later when I experienced winter for the first time in Europe. But a well-established principle of learning is that meaningful material is easier to learn than unfamiliar and less meaningful material. Had our teacher applied this principle, he could have found a more familiar and meaningful illustration.

The use of a 'spiraling curriculum', and the quality of both teaching and textbooks, may be potential factors. A spiraling curriculum is a type of curriculum that involves reviewing, each year, previously learned concepts before moving on to new concepts. For example, quite unlike the Japanese, the American mathematics curriculum has been identified as one such curriculum (Stigler & Perry, 1990; Stevenson & Stigler, 1992; U.S. Department of Education, 1988). This type of curriculum encourages repetition and could therefore lead to procrastination of learning since with time both students and teachers get used to the idea of what I may call a 'second chance'. And of course, the authors of textbooks would not be left out since the texts have to be tailored to match the prevailing curriculum.

The traditional methods of teaching and assessment in mathematics have been noted as the likely causes of learners' view of mathematics as a discipline that does not allow for creative thinking, only memorization of formulas and theorems. This has to do with issues such as the emphasis on the product of learning (the one right answer) instead of focusing on the process of learning (questioning and examining, analyzing, exploring alternative means to different solutions). Also assessment methods that emphasize students' abilities to perform routine or rote tasks have been noted as ways that deny them the chance to demonstrate the degree of flexibility of their mathematics understanding (National Mathematics Education Board, 1993).

Schwartz (1995) distinguished between what he calls "problem solving", which encourages critical thinking skills, from "problem posing", which encourages the development of exploration and conjectures. According to him, the latter is being left out in mathematics education to the detriment of exploratory and conjecturing skills development among students. Given the nature of mathematics, it is not unreasonable to expect students' mathematical success to be influenced by such factors.

But not least is the methodical principle upon which the teaching of mathematics is based. For instance, the question of whether to adopt an 'ascent from the concrete to the general' or 'descent from the general to the particular' can be a potential factor of students' success in mathematics. Seeger and Steinbring (1992) offer examples of ways in which the theoretical character of mathematical knowledge could easily be lost when a teaching method based on the former principle is adopted.

Affective factors

It is neither unreasonable nor uncommon to assume that before coming to class, students do have some ideas, attitudes, memories, emotions and beliefs about knowledge in general that can actively influence their individual experiences during and after class [see for example Schommer (1989)]. In particular, there is research evidence [see for example Schoenfeld (1989b), Kroll (1989)] that affective factors such as beliefs and attitudes towards mathematics play an important role in the development of students' problem solving abilities. For instance, Schoenfeld (1989a) observed that the mathematical beliefs of high achieving undergraduate students regarding the usefulness of mathematical knowledge seemed to be the main hindering factor from getting a correct solution to an otherwise easy problem. In the same work, Schoenfeld asserts that beliefs inspire more ineradicable convictions than emotions and attitudes. This perhaps explains why researchers have directed more attention to students' beliefs than to other affective factors [see for example, Schommer (ibid.)]

Cognitive and Metacognitive factors

As pointed out earlier, cognitive factors as well as influences on students' attitudes about learning are potential sources of impediment or contribution to students' success in mathematics. An important factor that has been noted to influence students' metacognitions is their memories of past experiences with mathematics or what Willemsen (1995) calls "metacognitive histories". The idea is that students with negative metacognitive histories may not have formed the kinds of detailed insights into their own learning that their counterparts with positive metacognitive histories have formed. For instance, students who have had histories of successful learning with mathematics will often have formed some ideas about their own learning process. Examples of such metacognitions can be noticed when students make statements such as this one made by a student I am tutoring; "Math tutors help me with my homework assignments and they give me better clues on how to approach mathematical problems". Such metacognitions can help students maintain positive attitudes and serve as important resource to guide them in solving problems.

The notions of "deep conceptual understanding", critical thinking skills, "good understanding" and theoretical thinking have been used to point out possible cognitive factors that influence students' success. Hatano (1988) talks about "routine" learners and "adaptive learners." The former refers to learners who, on demand, can accurately and quickly give back what has been given to them under a given text but would not be able to transfer the knowledge to new contexts. That is, they can solve routine problems and may be very successful in conventional academic terms but they don't have a firm grip of the mathematics concepts. The latter refers to learners with a 'deep conceptual understanding' of mathematics and who are flexible in their ways of applying mathematical principles.

With Sierpinska's (2000) conjecture about the reasons behind students' lack of 'good understanding' and subsequent investigations following it as well as the introduction of the notions of theoretical and practical thinking, a new approach to the study of FOSSIM is underway. For one thing, the notion of good understanding sounds like a fresh alarm alerting researchers about the theoretical nature of mathematics while the notions of theoretical and practical thinking sound like a reminder not to overlook the human nature of students. A careful blending of these notions together gives rise to a framework of analysis which forms the bedrock of the deliberations in the sequel.

1.4 Aim of the thesis

The aim of this thesis is to examine ways in which different perspectives of statistical methodology can be applied to the study of cognitive processes in mathematics education with particular focus on linear algebra. The perspectives are - descriptive statistics, statistical modeling and statistical exploratory data analysis. A careful analysis of statistical methodology in the light of these approaches and their application to the analysis of data resulting from the case study of cognitive factors of students' success in linear algebra, will be carried out. The advantages and drawbacks of these methods, from the theoretical as well as the application points of view, will be exposed, and a new method of analysis that would address the research questions, given the peculiar nature of the data involved, will be developed. Based on the findings, recommendations for future research will be made regarding how the choice of statistical methods can affect research results and their interpretations.

1.5 Plan of the thesis

After this introductory chapter, a case study of theoretical thinking as a cognitive factor of students' success in linear algebra is presented in the second chapter. There, factors of students' success in linear algebra, the notion of theoretical thinking, the research questions, the methodology of data collection and the framework of analysis as well as the nature of data collected, are presented. Chapter three is concerned with the application of statistical methods to the analysis of the data. The chapter is started with a consideration of unilateral forms of analysis under which different statistical methods, their assumptions and implications are presented. Next, bilateral forms of analysis, in which the original data is associated with data on the students' grade in two linear algebra courses, are carried out. Discussions of the advantages and drawbacks associated with the different methods are presented alongside their implementations. In chapter four the thesis is concluded with a summary of the research findings and the effects of the statistical methods used, on these results.

Chapter 2. A case study of Theoretical Thinking as a Factor of Students' Success in Linear Algebra

In this chapter, a case study of theoretical thinking as a cognitive factor of students' success in linear algebra is presented. I will start with a discussion of factors of students' success in linear algebra. After that, the notion of theoretical thinking, the research questions, the methodology of data collection and the framework of analysis as well as the nature of data collected, will be presented.

2.1 Factors of Students' Success in Linear Algebra (FOSSILAs)

It was pointed out in the introduction to chapter one that there have been numerous studies addressing the question of what students' difficulties in the learning and understanding of different areas of mathematics are, and how to find ways of teaching that can help students overcome these difficulties. Linear algebra has not been left out, and its very nature makes it notorious in this regard. It is a full-fledged mathematical theory, and the nature of most university linear algebra courses has traditionally been that of proof-laden courses demanding the often challenging and overwhelming etiquette of justification and explicitness. Assumptions have to be made explicit, statements have to be justified by reference to definitions and already proven facts. Implicit quantifiers, necessary and sufficient conditions, proof techniques, etc., all have to be learnt, studied and understood well, not just memorized, by any student wishing to achieve high success in these courses. These are ideals and great expectations. However, the extent to which these expectations can generally and regularly materialize has not yet been documented. For instance, while it is very reasonable to expect students who study hard and learn linear algebra courses with understanding to pass the examinations, it may not be certain that such students will achieve well in the examinations. Sometimes, it even happens that a student's success in one linear algebra course is not necessarily a guarantee that the student will do as well or better in subsequent linear algebra courses. What factors could be responsible for these? On the other hand, there are students who consistently excel in linear algebra courses. Could this be attributed to mathematical giftedness, or could there be some cognitive factors of their success? If the latter is the case, what are these latent factors and how do they interplay with other known or commonly assumed factors of students' success in mathematics such as demographic, pedagogical, affective, socio-cultural, and metacognitive factors? These types of questions are becoming increasingly pertinent given the present trend of research in mathematics education towards factors of students' understanding underlying their difficulties.

In mathematics education, there are many researchers who are inclined to the new viewpoint of perceiving the learning of mathematics more in terms of thinking processes. For example, Arzarello *et al.* (1994) studied the key processes involved in the development of algebraic thinking. In that paper, they focused on "naming" process and pointed out the difference between "sense" and "denotation" of a symbolic expression as a key element of algebraic thinking in problem solving. According to them, the success of the solution to a problem is determined by this naming process insofar as it allows pupils to grasp the algebraic sense of a problem by the "algebraic sense of an expression". The latter is said to reflect the meaning of the problem by means of a suitable "contextualized sense". Other references to the naming process in algebra include Chiappini and Lemut (1991) and Boero (1995). A study of the effect of a unit on proof on subsequent mathematical performance led Bittinger and Rudolph (1974) to conjecture that a supplemental unit in mathematical proof would increase students' chances of success in linear algebra. However, the results of a research referenced by Dorier and Sierpinska (ICMI study 2000) appears to undermine this hypothesis.

Hillel (2000) sees the introduction of the general theory of vector spaces at the undergraduate level as a source of students' difficulties in linear algebra since students end up working mainly with finite dimensional spaces after learning about the isomorphism between an n-dimensional vector space and the R^n spaces. Sierpinska (2000) conducted a series of three teaching experiments, and found out that while it is possible to create situations in which students learn linear algebra with understanding, it may not be easily guaranteed that they learn with 'good understanding' (good in the sense of not deviating significantly from the theory). This finding led them to conjecture that the reason for such circumstances was that most students try to understand the theory with a practical, as opposed to a theoretical mind. This conjecture has given rise to a new perspective from which the factors of students' success in linear algebra can be investigated. And this is the perspective I will be mainly concerned with in this thesis in order to explore latent cognitive factors of students' success in linear algebra. It is hoped that useful information that could help to address the types of questions raised above about students' success in Linear Algebra will be gained. In what follows, a discussion of some of these factors is given.

The nature of students and the nature of Linear Algebra

Historically, the foundation of Linear Algebra dates back to the 1600's and is reckoned as a legacy of René Descartes' 1637 publication, *La Géometrie*. This publication brought an end to the old practice of a dual process of first translating a geometric problem into algebraic equations to be manipulated and solved algebraically (analysis), and then turning back to Geometry for interpretation of the solution (synthesis). From then onwards, the concept of analysis changed from that of a step in a two step process of analyzing a problem to that of an autonomous process whose meaning could stand alone without need for further interpretation. This novelty apart, Descartes' publication brought the idea of a Cartesian plane and analytical geometry that led to the 'algebraization' of Geometry. Thus it became possible to express geometrical curves as algebraic equations in two variables. The two entities, lines and circles, which were previously conceived as basics in Geometry, parted as first degree and second degree equations respectively. With the circle promoted one step higher, the line became a basic entity of algebra and began to enjoy all the attention previously shared with the circle as basics in Geometry. Thus followed many renewed interests, ideas and issues concerning linearity that would culminate in the development of linear algebra as a full-fledged mathematical theory which serves mainly for the unification and generalization of many mathematical concepts and methods. As a result, the ramifications of Linear Algebra are many and varied, and this fact, coupled with its nature as a unifying and generalizing theory, perhaps accounts for some influential factors on students' success in the subject. The types of influences these factors have on students' success can be many and various as the many different interests, issues and ideas that emerged about linearity. In what follows, I will mention but briefly some of these ideas. More detailed studies in this direction are available, see for example, Robinet (1986) and Dorier (1995, 1996, 1997 and 2000, part 1). Among some of these issues and ideas are, for example, the concept of determinants introduced by Cramer (1750) for solving systems of linear equations, the concept of linear dependence introduced by Euler (1750) in a bid to clarify Cramer's paradox, the concept of a matrix, which was developed in a bid to generalize and handle complicated algebraic expressions, and the theory of vector spaces that culminated in the axiomatization of linear algebra. Although the idea of this axiomatization did not solve any new problem, it offered a general and unified approach to the solution of problems in different contexts. The solution of many linear problems can in fact be obtained without using the axiomatic theory of linear algebra. This fact can be disturbing to young students who, upon realizing it, begin to question the need for the extra burden of learning linear algebra.

The foregoing is an idea of the nature of linear algebra that makes it a cognitively and conceptually difficult subject for students who see little or no connections between the concepts or theories they learn and the purpose of these concepts or even the applications of the theories. As a high achieving college linear algebra student lamented in an interview:

> "Linear algebra is abstract, you can't even see it [linear algebra] but you might see Statistics. When ... applying it [statistics], ..., I know what I am getting. If I get a result, I can interpret it, whereas in Linear Algebra you really can't. So Linear Algebra isn't easy."
>
> [Interview with student N1, line 223, p.10]

Later on in the same interview, when the same student was presented with five statements about reasons for taking mathematics courses, he said:

> "[Statement] D is right but it is not why I am personally taking math courses. I am taking math courses to get my CV to work in my field [Actuarial Mathematics] ..."
>
> [ibid, line 370, p.18]

This student's statement is an indication that most students, by virtue of their human nature, may be more likely to engage themselves first in practical thinking than in theoretical thinking. Such students are easily lured by the quest for "good enough understanding" of what they need to know or do to get their CV to work (namely getting a high grade) to the detriment of "good understanding" of the concepts the course was meant to offer them. But in linear algebra, good understanding (understanding that does not deviate significantly from the theory) as Sierpinska (2000) conjectured, is ideally a function of theoretical thinking. On the other hand, "good enough" understanding (I would say, understanding required just to obtain a certain gratification with or without deviating from the theory) does not necessarily entail theoretical thinking. While the acquisition of good understanding may be tantamount to success, the search for good enough understanding can lead students to either success or failure. Success obtained through the quest for good enough understanding can be true or false depending on how much the student knows of what s/he is supposed to know. Failure through the quest for good enough understanding can lead students to all sorts of situations ranging from mere regrets to what I would call, in Willemsen's (1995) terminology, negative "metacognitive histories" or even perpetual hatred for the course. Also, it may be the case that students invest a lot more resources in this direction. This makes one wonder how most students' nature would lead them to that direction. The answer, I think, is connected with the presence or absence of some non-cognitive factors that have been cited in the mathematics education literature as having potential influences on students' success in mathematics in general. In what follows, I will briefly mention some of these factors.

Some non-cognitive factors

Among some potential non-cognitive factors of students' success in linear algebra worth mentioning in connection with the preceding factor are socio-cultural factors, pedagogical factors, and affective factors. In societies where the cultural norm for success and progress in life is essentially in terms of money, and is strictly linked to conventional academic success in courses, students may be more tempted to prefer good enough understanding to good understanding. Simultaneously or with time, pedagogical factors creep in, too, as teachers and textbook authors join the bandwagon preferring quantity to quality. Affective factors are usually present in the form of students' beliefs and attitudes towards linear algebra or mathematics in general [see for example, Schoenfeld (1989b), Kroll (1989), Carlson (1999)]. Students who have developed negative metacognitive histories that are difficult to clear may be more at risk of going for good enough understanding.

Pedagogically, it can be said that students of university linear algebra courses especially at the early stage are disadvantaged. As Carlson (1994) points out, it is often the case that beginning students find it difficult to see how the abstract/axiomatic concepts of linear algebra they are being taught can be related to their previous mathematical knowledge. Given the length of time it took to develop these concepts, to get many mathematicians to abandon their former beliefs and finally have these concepts accepted in the mathematics community, it becomes almost capricious to expect the majority of students to grasp so much in so little time. Also the beauty of the theory of linear algebra lies in its unifying and generalizing power that lends it to a wide area of application to problems - simple and complicated. Usually young students are ideally introduced to its simpler applications. Unfortunately, as goes a popular adage, "beauty is in the eyes of the beholder", the eyes of the young students fail to appreciate the application of a difficult and abstract theory in solving a simple problem that can easily be handled by simpler methods already known to them. This is perhaps the point Harel (2000, p.180) tried to make in his Piaget inspired "principle of concreteness". Also the fact that Linear Algebra has many languages and systems of representation brings another source of influence. According to Hillel (2000), there are the geometric language of lines and planes, the algebraic language of linear equations, *n*-tuples and matrices, and the 'abstract' language of vector spaces and linear transformations. Students need time to make conversions from one form to the other as is often expected of them by teachers, texts and the traditional measures of success - tests. They need to convert from the 'graphical', 'tabular' and 'symbolic' registers of these languages as well as move between the 'Cartesian' and 'parametric' representations of subspaces. Duval (1995) argues that the activity of conversion is ignored in mathematics teaching despite its necessity as a passage for coordinating registers attached to the same concept. This factor has much in connection with the cognitive factors to be discussed in the next section. Other non-cognitive demographic factors such as gender, age, race etc., have been left out in this discussion not because they are irrelevant, but in part because I think that their relevance can easily be studied alone or alongside peculiar factors within an integrated framework as will be shown in the sequel.

Cognitive factors

The different ramifications of Linear Algebra make it necessary for students to be cognitively flexible in order to be able to move between the various languages, registers and modes of representation. For instance, since students' previous understanding would lead them to view, for example, functions as procedures of assigning numbers to other numbers while understanding of linear algebra would require them to view functions as objects in themselves, it becomes imperative that students adapt themselves properly to the latter point of view. These challenges are not very easy ones given the human nature of students that ordinarily would make them prefer the practical to the abstract, the simpler to the more complex, the easier to the harder and the time saving to the time consuming ways of life. As a result, it does appear that students have to understand the need to 'go the extra mile', and to be able to understand this, the students have to be well informed about the changes linear algebra brings, which in turn calls for some form of theoretical mind setting. And it is, perhaps, in this regard that Sierpinska (2000) conjectured that theoretical thinking is necessary for 'good understanding' of linear algebra. This conjecture is the basis of an ongoing research aimed at unveiling the latent features of students' theoretical thinking aptitude that might indeed be necessary for their success in linear algebra.

2.2 The Notion of Good Understanding and Theoretical Thinking (TT)

It is obvious that any task that can be likened to the task of trying to read the mind's construction from the face can be quite onerous. This perhaps explains why research in the area of measuring students' understanding of concepts usually concentrates either on interpretations of behavior that might serve as evidence of the processes of concept acquisition, or on conditions that can lead to understanding. Great caution is needed in designing or determining a framework to be used in assessing students' understanding of concepts. Thus, within the psychological perspective under which theories about students' understandings of concepts are postulated and research studies are carried out, individual students are usually assumed to undergo a transitory process from a level of pre-understanding to that of understanding. In this regard, teachers can be seen as middlemen between the concept and the students. While the concepts can be under the control of the teacher, there is no guarantee that the teacher's effort at effecting and measuring the outcome of the transformation can be completely successful. And this is perhaps what Balacheff (1990, p.262) had in mind when he remarked that:

> "It is not possible to make a direct observation of pupils' conceptions related to a given mathematical concept; one can only infer them from the observation of pupils' behaviors in specific tasks, which is one of the more difficult methodological problems we have to face."

Educators seem to assume that every student will arrive at the same understanding of particular objects in mathematics. Probing the extent to which this assumption can or cannot be justified is not part of the objectives of this study. However, this is not to dismiss the need for a critical analysis of both aspects of the process of understanding as seen from the psychological perspective, namely the notion of the individual and that of knowledge – whether considered absolute or not.

Having cited some documented as well as other considerable sources of factors that do influence students' success in mathematics, it is perhaps pertinent to ask whether there could exist any one source or set of sources that might be a fountain of necessary factors of students' success in mathematics. For example, there is no documented evidence that shows gender, race or age as globally necessary or sufficient factors of students' mathematical success. The same thing applies to all the socio-cultural, pedagogical and affective factors cited earlier in this paper. However, when it comes to the cognitive domain, it can be conjectured that certain factors may indeed be necessary for success in mathematics, irrespective of students' race, sex, age or the pedagogical, socio-cultural, political etc. environment in which they find themselves. Since research provides substantial amount of information and useful knowledge about the non-cognitive sources of influence on students' mathematical success, and since no necessary factors have emerged from such domains of influence, the search for such factors could reasonably be limited first to the cognitive domain and thereafter to some area of intersection of factors from within and outside the cognitive domain.

As pointed out earlier, the highly theoretical nature of mathematics in general, and linear algebra in particular, is commonly acknowledged. Traditionally, the subject of linear algebra as described earlier, involves proofs, which demands the often challenging and overwhelming etiquette of justification and explicitness. Above all, students have to understand the need for all the 'trouble' facing them especially the need for proofs in order for them to appreciate them and be willing to engage in the learning process. To be able to understand and appreciate the need for proofs, the students have to be well informed about the theory and theorems requiring the proofs. This involves the process of abstraction, which in turn calls for some form of theoretical mind setting, and some appreciation and acknowledgement of the nature of mathematics. And it is at this juncture that Sierpinska's (2000) conjecture about theoretical thinking, TT (as opposed to practical thinking, PT) and good understanding comes into play. And a TTPT framework developed on the basis of this conjecture becomes a promising tool for qualitative as well as quantitative analysis that could reveal some necessary factors of (and subsequently conditions for) students' success in linear algebra.

2.3 The Research Questions

On the assumption that theoretical thinking is relevant for a good understanding of Linear Algebra and with interest focused on determining the necessary cognitive conditions of students' success in linear algebra, the following research questions have been set for this thesis:

1. ***What are the features of theoretical thinking that can be identified among the group of high achieving students of linear algebra under study?***

2. ***What quantitative methods, statistical or otherwise could be applied to the study of these features, how can they be effectively applied, and what are the benefits or drawbacks of the application of these methods to the data collected?***

3. ***Based on the statistical methods or otherwise, what are the most and least prominent among these features of theoretical thinking in the high-achieving students? Which features can be considered necessary or unnecessary for the students' success?***

4. ***To what extent a positive association between high achievement in a Linear Algebra course and theoretical thinking tendency, can be validly claimed?***

The answer to the first research question will be given in the next section where the framework of the case study is presented alongside the details of the methodology of data collection. The remaining three questions will be answered in chapter three where the resulting data will be analyzed.

2.4 Framework of Analysis and Research procedures

In this section, a case for the need to have an organized means of analyzing the factors of students' success in linear algebra is made and details of a framework developed on the basis of Sierpinska's (2000) conjecture, are presented. This need is warranted by the difficulties [see, for example, Gras (1992)] that arise in the process of choosing and deciding upon the methods and procedures to take in analyzing data of the type involved in this case study. For instance, data have to be coded, scales of measurement of variables have to be known or decided, statistical procedures have to be chosen and results have to be meaningfully interpreted in a manner consistent with model assumptions. All these tasks can hardly be accomplished harmoniously without a framework of analysis.

There have been efforts geared in directions similar to that of building a framework for qualitative and/or quantitative assessments of FOSSIM. These include the work of Belenky et al. (1986) who, in studying women's ways of knowing, proposed some qualitative categorization of different epistemological positions representing different beliefs about knowledge. Also, Schommer (1989) in investigating students' beliefs about the nature of knowledge and how they affect comprehension, talks about a 'personal epistemology' and the five 'dimensions' of the belief system related to this epistemology. These dimensions include; the structure, certainty and source of knowledge, and the control and speed of knowledge acquisition. Schoenfeld (1989b) presented a theoretical model for the analysis of mathematical behavior. Carlson (1997, 1999) described and used a 'quantitative instrument' designed to assess students' views about knowing and learning mathematics.

Schoenfeld's model has four components: resources, heuristics, control and beliefs. By resources, he meant the mathematical facts and procedures available to the problem solver. Heuristics include the wide range of techniques available for general problem solving. Control includes the global decisions regarding the selection and implementation of the resources and strategies that determine the efficiency with which facts, techniques, and strategies are exploited (e.g., planning, monitoring, decision making etc.). According to him, belief systems shape cognition and determine the perspective with which mathematics and mathematical tasks are approached even when one is not consciously aware of holding these beliefs. Carlson (1999) explored certain non-cognitive factors of students' success in mathematics by investigating their mathematical beliefs, behaviors and backgrounds. In that study, she investigated students' mathematical behavior and methods in the context of completing complex mathematical tasks. For the students' beliefs, she used the *Views About Mathematics Survey* (VAMS), a quantitative survey she developed (Carlson, 1997) for providing a wide classification of students' mathematical views about knowing and learning mathematics. Carlson's study, among other results, reports that for the students under study, when confronted with an unfamiliar task, "their initial problem solving attempts were frequently to classify the problem as one of a familiar type, and they were not always effective in accessing recently taught information or monitoring their solution attempts but were careful to offer only solutions that had a logical foundation."

Inspired by Vygotsky's distinction between scientific concepts and spontaneous concepts, Sierpinska (2000) developed four characteristics of theoretical thinking as opposed to practical thinking. On assuming that theoretical thinking is supported by certain epistemological positions and not others, she used the profiles of 'silence', 'received knowledge', 'subjective knowledge', 'procedural knowledge' and 'constructed knowledge' proposed by Belenky *et al.*, in the design of a questionnaire aimed at revealing students' epistemological positions. Also she took into account the five dimensions of Schommer's 'personal epistemology' and recognized the values therein as closest to what Belenky *et al.* termed 'constructed knowledge'. Based on these characterizations and assumptions, a sample of students from a target population (ideally the high achievers) were interviewed. Their mathematical behavior and declared epistemological beliefs while answering a set of appropriately designed mathematical questions was observed and recorded. The interview involved watching and questioning the individual students as they tried to understand the questions and answer them. Then a transcription of the recorded interview was sifted for symptoms of both theoretical and practical thinking and a coding system (specified a priori and subjected to a posteriori revision) was used to classify the students' reactions (behaviors) according to the observed symptom as either a TT feature or a PT feature.

Thus, a TTPT framework was developed from the initial characterizations, and their subsequent modifications in the light of research evidence obtained by interviewing 14 high achieving specialization and honours university undergraduates who obtained the highest, 'A' grade in the first undergraduate linear algebra course (vector spaces and linear transformations – see course outline in **Appendix LA I**) here in the Department of Mathematics and Statistics, Concordia University, Montreal, Canada, in 1999-2000. Twelve of these students took the second linear algebra course (Jordan canonical forms, quadratic forms and inner product spaces – see course outline in **Appendix LA II**), and 6 of them obtained 'A' grades in this second course. Both courses were taught by the same instructor who was also the course examiner for the courses (i.e. he was responsible for the course outlines and the final examinations questions). From the list, students were telephoned and invited to volunteer in the study. The 14 students who volunteered were then interviewed under conditions made convenient as much as was humanly possible. The interviews, which took, on the average, two hours each, were in order to probe, in a first attempt, what has been put forward as a definition of theoretical thinking and the conjecture that has been made concerning the conditions of success in linear algebra. The interviews were made up of two parts; questions consisting of linear algebra problems and questions designed to ascertain students' epistemological positions. The interview questions are presented in **Appendix A** and the transcription of one of the interviews is given in **Appendix B**.

Careful analysis of the transcription of the interviews yielded a posteriori features of TT and PT and it was found necessary to represent the TT features as formulated in the definition, by features of the students' behavior in responding to the interview questions. That is, each TT feature was assumed to underlie some kind of behavior(s) hereafter called 'Theoretical Behavior' TB as opposed to 'Practical Behavior' PB [see **Appendix C**]. Each student's behavior in a given question with respect to a given feature was coded by the vector [1,0] if the behavior was consistently theoretical, by [0,1] if the behavior was consistently practical, and by [1,1] if the behavior was, at times, theoretical, and at other times, practical (as when the student changed his or her approach as a result of an interviewer's remark or of a conflict between the student's expectations and his or her results in a computation). The resulting data [see **Appendix D**] is a table whose rows correspond to the 14 students and columns correspond to the features of TB-PB in each question. For purposes of quantitative analysis, a number of indices were developed, each defined column-wise for the students as a group (Group indexing) as well as row-wise for each student as an individual (Individual indexing). The conceptual development of these indices and some inherent computational problems associated with different methods of applying them to the data warrants a careful look at the nature of this type of data and the type of questions that can be asked of it. This is pursued in the next section.

2.5 The nature of data collected and questions that can be asked of it

A careful look at the original data in **Appendix D** shows that each one of the 25 TB features can be conceived as a binary variable. Similarly each of the 25 PB features can be conceived as a binary variable. The behavior of each one of the 14 students with respect to each of the (25 × 2) TB-PB features is recorded as any one of four possible sets of events associated with each of the four possible vector outcomes [1,0], [0,1], [1,1] and [0,0] as follows:

$(TT \cap PT^c) \sim [1,0]$ representing an observation of a consistently TB feature
$(TT^c \cap PT) \sim [0,1]$ representing an observation of a consistently PB feature
$(TT \cap PT) \sim [1,1]$ representing an observation of a mixture of TB and PB
$(TT^c \cap PT^c)$ or $\emptyset \sim [0,0]$ representing lack of evidence of either TB or PB

This gives rise to a 14 x 50 data matrix, which can meaningfully be analyzed row-wise, or column-wise depending on the type of research question to be answered. Basically two types of questions can be asked of this type of data layout: questions whose answer would require row-wise analysis and questions whose answers would require column-wise analysis.

If the research question concerns the students as a group, then a column-wise analysis is called for. Also any question concerning a comparison of the features would warrant a column-wise approach to the data. For instance, if one were to answer the third research question as posed in Section 2.4 above, it would be reasonable to consider and compute separately a group index for each of the 25 pairs of TB-PB variables and compare their values. On the other hand, a row-wise approach to the data would address questions about the students' behavior individually. For example, the computation of individual theoretical thinking indices would necessitate a row-wise analysis. Based upon such indices, a bilateral analysis such as correlation or regression analysis can be performed with other variables such as the students' grades in the two linear algebra courses or the students' sex or any other factor from the set of FOSSILAs or FOSSIMs highlighted in earlier sections.

Another kind of question would be the type that requires a simultaneous consideration of rows and columns – for example, when one is interested in interaction patterns such as 'Student-Interviewer' interactions or 'Student-Question' interactions. In other words, one may ask whether the treatments the students received in the course of the interviews are homogeneous? Ideally, one would expect the treatments the students receive in the course of the interviews to be homogeneous. But in interviews of this nature, it is almost impossible to guarantee homogeneity of responses because of the different idiosyncrasies of the interviewees and interviewers. For instance, a student who is an extrovert may want to go one step further in responding to a question while an introvert may not say much unless s/he is asked. Thus while the former could provide extra information, the latter may not even provide enough. Thus there is a dilemma of how to balance or moderate the criteria used for understanding the responses given by the students. The more differently the interviewer/interviewee behavior tended to be, the more the dilemma heightened in each question. However effort was made and double-checked to moderate the situation in the most reasonable way possible. If the data is normally distributed, the analysis of variance (ANOVA) technique may be applied as a statistical means of testing for the presence or absence of any significant interviewer/interviewee interaction effect on the questions.

Chapter 3. Statistical Analysis of the Data

3.1 Unilateral Analysis

Very often, it does happen that certain aspects of the variable(s) of interest in a study are more meaningfully understood by investigating the relationship between them and some other variable(s). This is what I shall later be referring to as bilateral analysis. On the other hand, by unilateral analysis, I shall be referring to a self-contained analysis of the variables of interest that does not involve association with other variables. Thus, in this part of the thesis, effort is made in the direction of trying to understand how data comprising of variables such as the TB and PB features can be organized and have their distributions described. For this purpose, some common statistical descriptive and exploratory data analysis methods will first be employed. After that, classical statistical models will be considered.

3.1.1 Some Common Descriptive and Exploratory Data Analysis Methods

A student's individual total TB score, ***itotalTB*** will be defined as the sum of the total of all observations of TB features whether of the consistent [1,0] type or the mixture [1,1] type. Similarly, a student's individual total PB score, ***itotalPB*** will be defined as the total of all observations of PB features both the consistent [0,1] types and the mixture [1,1] types. In other words, a student's ***itotalTB*** is the number of features on which the student's score was either [1,0] or [1,1] and while ***itotalPB*** represents cases of either [0,1] or [1,1]. The students' individual total scores on the TB and PB features are listed in Table I. The lists indicate that the students' individual total TB scores are strongly inversely correlated ($r = -0.90$) with their individual total PB scores. Noting that the maximum possible score is 25, it can be observed from Table I, that two students, O1 and O2, who scored the lowest 12 and 16 on TB, also scored the highest 19 and 18 respectively on PB. The student O3 who scored the lowest, 5, on PB was also the one who scored the highest 25, on TB. However, it can be observed that some students (S3, S4, N2, V2, V1) who scored very high (21, 19, 21, 23, 23) on TB, had rather high (9, 14, 9, 8, 11) PB scores. The student N1 scored 17 on both TB and PB. Similarly student O4 scored 18 on both TB and PB. This scenario does not give a precisely clear picture of whether the students' behaviors can be judged to be more inclined towards theoretical thinking or to practical thinking. Thus a sharper measure of the students' tendency to theoretical thinking, as a group is needed.

Table I Correlation between individual total TB and PB scores

Student	Individual total TB score *itotalTB*	Individual total PB score *itotalPB*
O1	12	19
O2	16	18
O3	25	5
O4	18	18
V1	23	11
V2	23	8
V3	19	11
V4	18	16
S1	23	6
S2	17	15
S3	21	9
S4	19	14
N1	17	17
N2	21	9
Correlation coefficient	$r = -0.90772071$	

Researchers do frequently resort to classical descriptive statistics methods as well as new methods of exploratory data analysis that offer simple ways of organizing and presenting data. By these methods, they also aim at statistical procedures of describing the distribution of a variable alone or its relationship or interaction with another variable. Among them are the frequency tables, bar charts, pie charts, histograms, stem and leaf plot, box and whisker plot, and some measures of central tendency such as the mean, median and mode, and measures of dispersion such as the variance, standard deviation, range and quartiles. In this section, some of these techniques are unilaterally considered.

Frequency tables are used for organizing a set of data by grouping them into equal intervals. For purposes of the present study, the raw data consisting of students' TB-PB scores can be totaled and tallied into frequencies of occurrence as shown in Table II. The highest total TB score obtained by a student is 25 and the lowest is 18. Similarly, the highest and lowest total PB scores, obtained by a student, are 23 and 5 respectively. Thus the total scores can be grouped into 3-score intervals starting from the interval 1-3 to 22-24, with 0 and 25 at the ends, as shown in Table II.

Table II Frequency table of total TB and PB scores by number of students

Total individual scores	Number of TB students	Number of PB students
0	0	0
1 – 3	0	0
4 – 6	0	2
7 – 9	0	3
10 – 12	1	2
13 – 15	0	2
16 – 18	5	4
19 – 21	4	1
22 – 24	3	0
25	1	0

It can be observed from Table II that information becomes, in some sense, clearer than it was in the raw data. For example, the interval 16-18 corresponds to the class with the greatest TB frequency (5) while at the same time corresponding to the class with the highest PB frequency (4). However, if, for example, as shown in Table III, we consider the intervals 0-13 as "LOW", 14-19 as "MEDIUM", and 20-25 as "HIGH", it can be difficult to say much about the TT tendency of the students as a group, except that the highest frequency of TB was in the medium range of scores and the highest PB frequency was also in the medium range of PB scores. This might be suggesting that the students could be about as inclined to theoretical thinking as they were to practical thinking.

Table III Grouping total scores into 3 categories

Total scores	Number of TB students	Number of PB students
0 – 13 "LOW"	1	7
14 – 19 "MEDIUM"	7	7
20 – 25 "HIGH"	6	0
Total	14	14

From Table III, we could conclude that the great majority of students were 'capable' of theoretical thinking and a great majority of them also displayed practical behavior Thus we could conjecture that the students would not engage in TB spontaneously when approaching a problem, but would only do so when prompted by the interviewer or by the unexpected outcome of the practical approach. At any rate, the results of the analysis of these scores based on the frequency tables are highly ambiguous. Therefore some more subtle tools for analyzing and interpreting the data are needed.

The frequencies discussed in the preceding paragraphs are concerned mainly with the number of students having scores in the given intervals. Similar tables can be made for frequencies of occurrence of the 25 TB-PB features. However, a much more informative summary table would be Table IV which shows, all in one, the frequencies of occurrence of the different TB-PB features according to the number of students scoring such features. This concatenated table offers the advantage of grouping and highlighting the features according to their degrees of prominence in terms of the number of students exhibiting them. At a glance, it can be understood from Table IV that, in 9 out of the 25 features, all 14 students had TB scores. Also, it can be easily seen that as many as 6 students were associated with the greatest number, 5 of features where PB was observed. In addition, a pie chart can easily and meaningfully be constructed based on this table.

Table IV Categorized Frequency table of total TB and PB scores by number of students and features

# of Students	TB features	PB features	# of TB features	# of PB features	Total TB scores	Total PB scores
14	1.3a, 2.1b, 2.2b, 2.3a, 2.4b, 2.5a, 3.2a, 3.2b, 3.3a	None	9	0	126	0
13	2.5b	None	1	0	13	0
12	1.1b, 3.1a, 5.2a	1.1b, 2.2a, 3.3b	3	3	36	36
11	2.2a, 3.4a, 4.1a	4.1a	3	1	33	11
10	2.4c	1.1a, 2.4b, 2.5a, 2.5b	1	4	10	40
9	**4.3a**	**4.3a**	**1**	**1**	**9**	**9**
8	**4.1b, 5.1a**	**2.1a, 2.4a, 6a**	**2**	**3**	**16**	**24**
7	**3.3b**	**2.2b, 3.2b**	**1**	**2**	**7**	**14**
6	**2.1a, 2.4a, 6a**	**3.1a, 3.4a, 4.1b, 5.1a, 5.2a**	**3**	**5**	**18**	**30**
5	**None**	**2.4c**	**0**	**1**	**0**	**5**
4	1.1a	3.2a	1	1	4	4
3	None	2.3a	0	1	0	3
2	None	None	0	0	0	2
1	None	None	0	0	0	1
0	None	1.3a, 2.1b, 3.3a	0	3	0	0
TOTAL			**25**	**25**	**272**	**179**

One of the drawbacks of the use of frequency tables, as a method of organizing data, is that their usefulness depends on how carefully the choice of intervals is made. When the choice is not well made, a frequency table will not give any more information than the raw data itself. Instead, too large and too few intervals may conceal information, while too small and too many intervals can make it difficult to draw any conclusions from the data. Also, even when the choice of interval is optimal, the frequency table, alone, offers not much more than an organization of data. But it is very useful as a starting point for further methods of analyzing data such as cumulative frequency distribution, bar and pie charts, and histograms. For example, situations do arise when the concern of a researcher is not with the frequencies within the class intervals themselves, but rather with the number or percentage of values "greater than" or "less than" a specified value. In this case, Table V, which shows the cumulative frequencies of total TB scores and total PB scores, can readily furnish such information. The cumulative frequencies are obtained by adding successively the individual frequencies.

Table V Combined Cumulative frequency table of total TB and PB scores by number of students

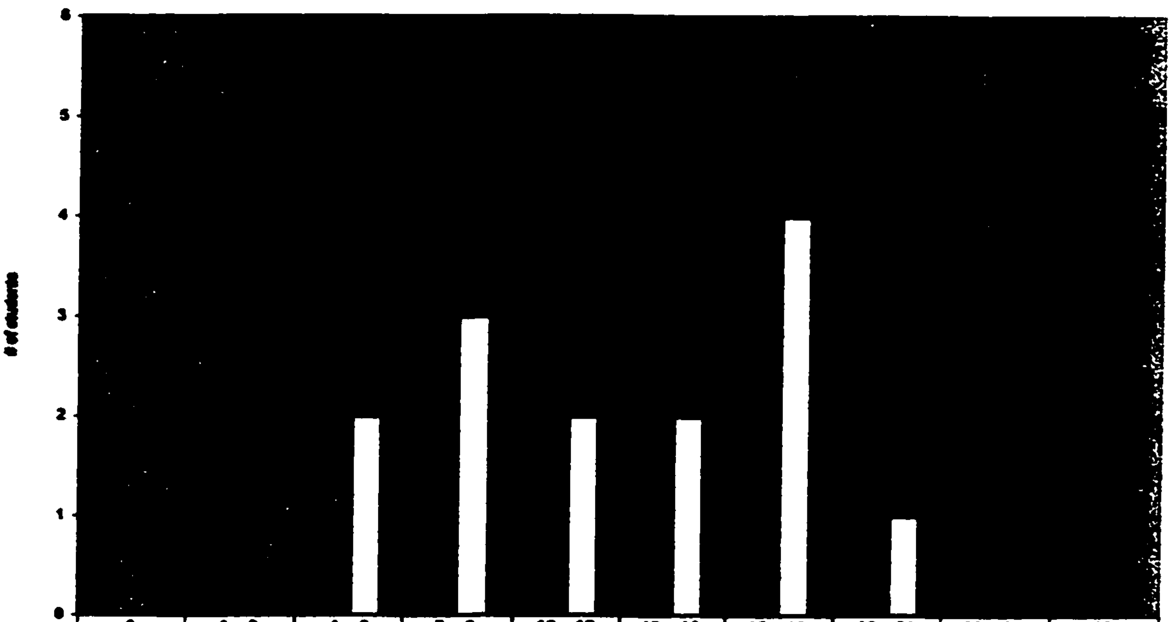
Total individual TB-PB scores (in classes of 3-score intervals)	TB			PB		
	Frequency	*Cumulative*	*Cumulative percentage*	*Frequency*	*Cumulative*	*Cumulative percentage*
0	0	0	0%	0	0	0%
1 – 3	0	0	0%	0	0	0%
4 – 6	0	0	0%	2	2	14.3%
7 – 9	0	0	0%	3	5	35.7%
10 – 12	1	1	7.1%	2	7	50.0%
13 – 15	0	1	7.1%	2	9	64.3%
16 – 18	5	6	43.0%	4	13	92.9%
19 – 21	4	10	71.4%	1	14	100%
22 – 24	3	13	92.9%	0	14	100%
25	1	14	100%	0	14	100%

Table V has been presented as a combined table in which both frequencies can be viewed jointly. In this way, differences or similarities in the students' TB and PB scores can be grasped more quickly. For example, it becomes easier to notice that while none of the students had TB scores less than or equal to 9, about 35.7% of them had PB scores in the same range. Also while the whole group of students (100%) had a maximum PB score less than or equal to 22, only about 71.4% of them had TB scores in this range.

Bar Charts (also known as bar graphs) are commonly used to illustrate a comparison of similar items. In this case, they can be used to visually summarize the data in such a way that a simple description of the number of times the different TB-PB features are scored by the students as a group can quickly be grasped. Because there are two classes of features, TB and PB, **combined bar charts** (also called **compound-bar graphs**) can be used so that the different classes of features can easily be distinguished and compared. From the above chart, a comparative description of the total TB-PB scores of the students, as a group, can be made. The chart in Figure 3.1a, with its combined frequency table attached underneath, suggests more clearly and quickly that the students, as a group, were more inclined to TB than to PB as evidenced by higher TB frequencies for higher scores and higher PB frequencies for lower scores. Although it would have been good to show all the different features on the chart, this type of analysis may be costly to do jointly or even differently for all the 25 TB-PB features. The amount of information to be gained from a multivariate-type bar chart may not be commensurate with the effort, time and resources required to obtain it. Also, even when such can be realized, the information can easily be distorted.

Pie charts (also known as **circle graphs**) are used to show how a quantity can be divided into parts usually expressed as percentages. Although the quantity can be split into many categories, the ideal number of categories is 10 and the notion of 'other' can be used to make up the remaining categories. In this case, to show how the total scores of the students as a group are composed of in terms of the different TB-PB features, two charts can be made; one for the TB totals, the other for the PB totals. The pie chart in Figure 3.1b below shows that 30% of the overall total TB scores are associated with the category under which all the 14 students had TB scores in 9 of the 25 features. The category that makes up the least portion of the pie, 1% is the one in which the smallest number of students, 5 had TB scores in only one feature. The pie chart provides the advantage of emphasizing the relative size or importance of each category of features in relation to the 25 features as a whole set. For the sake of completeness, Figure 3.1c serves to show a similar chart made for the PB scores.

Figure 3.1a Combined bar chart of total TB-PB scores



	0	1 – 3	4 – 6	7 – 9	10 – 12	13 – 15	16 – 18	19 – 21	22 – 24	25
TB	0	0	0	0	1	0	5	4	3	1
PB	0	0	2	3	2	2	4	1	0	0

Figure 3.1b Pie chart of total TB scores by categories

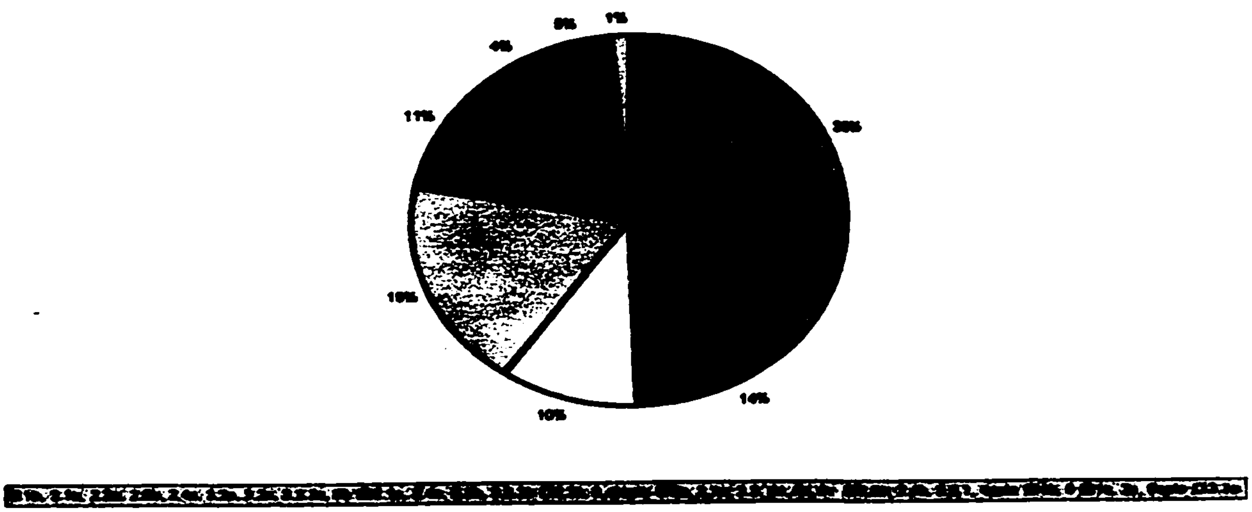
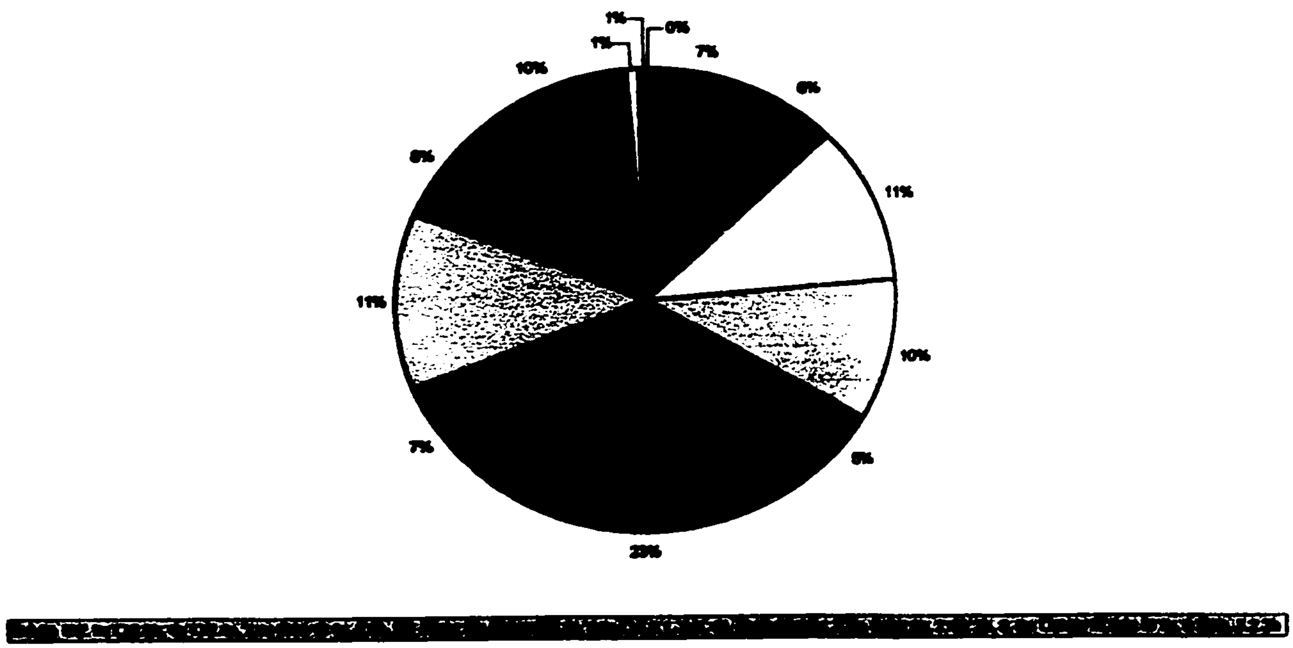
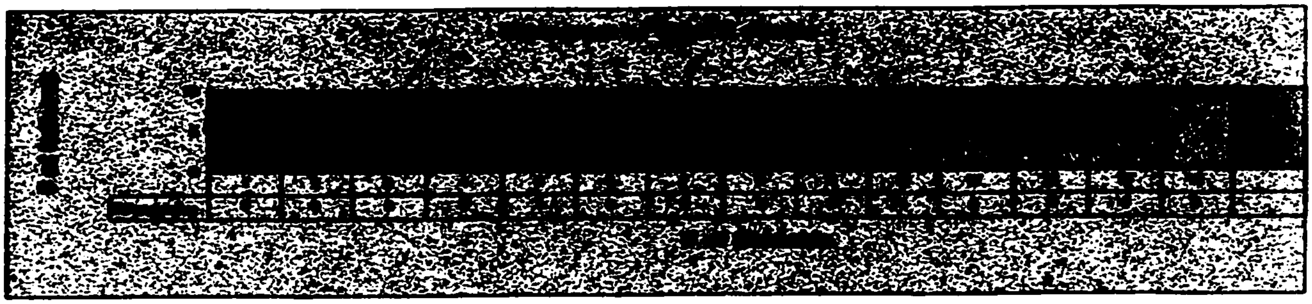


Figure 3.1c Pie chart of total PB scores by categories



Histograms are used to show how often a particular item occurs. They serve as graphical representations of a sample frequency distribution for a variable. Although, they can be constructed in the same way as bar charts, histograms are not used for comparing separate items and, as such, spaces are not included between bars. Instead each bar is placed over the whole interval and the frequency of each category is represented by the height of a bar. For illustration, Figure 3.1d shows a histogram of total TB scores. Here, it provides just about the same information as the bar chart.

Figure 3.1d Histogram of Total TT scores



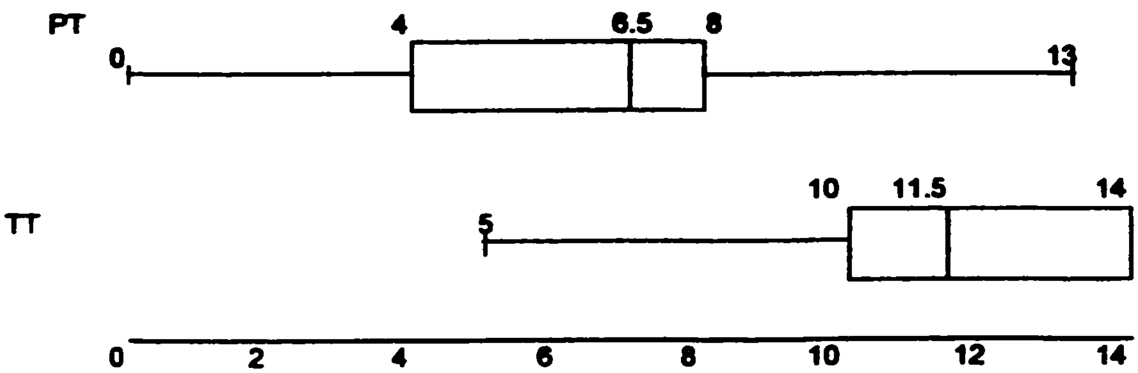
Stem-and-Leaf Plot is an exploratory data analysis technique for organizing data introduced by Tukey (1977) as a simple alternative to a frequency distribution table or graph. Stem-and-leaf plots provide similar presentation of data to histograms in the sense that they give an idea of the shape of the distribution, the degree of dispersion, and indication of outliers if any exist in the data. However, they also provide an advantage over histograms because of the use of actual numeric values in their displays. Thus, any roughness (i.e., when only certain values occur in the range of values) in the data is clearly highlighted and all information in the data are retained. Figures 3.2e and 3.2f show, respectively, stem-and-leaves plots for the total TB and PB scores. They provide similar information like the bar charts except that it is now easier to visualize facts in terms of actual total score values.

Figure 3.2e A Stem-and-Leaves plot of the total TB and PB scores

PB Leaves	Stem	TB Leaves
889465866006754478241748 08	0	665886
1300102	1	41143424040324340431102141

Boxplots, like stem-and-leaf plots, provide information about the shape and dispersion of a distribution. But unlike stem-and-leaf plots, they do not retain information. Their advantage over the stem-and-leaf or histogram lies in their graphical simplicity that allows for quick identification of the median and outliers. The box could be closer to one end meaning that values tend to accumulate at that end or it could be near the middle thereby providing further information about the shape of the distribution. In this case, to construct a boxplot for the total TB scores requires computing the minimum (5) and maximum (14) scores, the first quartile ($Q1 = 10$), the second quartile ($Q2 =$ median $= 11.5$) and the third quartile $Q3 = 14$. Similarly for the PB scores, this set of five values are computed as $\{0, 4, 6.5, 8, 13\}$. Figure 3.2g shows these plots together for the total TB and PB scores. The boxplots suggest that the distributions of the total TB and PB scores are both negatively skewed, although the skewness is much more pronounced in the TB plot as can be corroborated with evidence from Table 3.3 in which the coefficients of skewness are –0.88 and –0.18 respectively. This should be expected since in both cases, the mean and median are each less than the mode. That is, when the distributions are plotted, the longer tails of both the TB and PB plot would be extending toward the lower end of the horizontal axis on the scale when the plots are made separately on different scales. Note however that because of the nature of the data at hand, the longer tails would be in opposite directions when plotted on the same scale. The latter would amount precisely to the same information depicted in the combined bar chart in Figure 3.1a as shown above.

Figure 3.2g Boxplots of total TB and PB scores



Besides the above methods of using frequency tables, graphical charts and plots, data of the type at hand can be summarized and described for easier and quicker understanding by computing some summary statistics as shown in Table VI. The statistical measures in this table are familiar ones and most statistical software packages available today do compute them within seconds even on default. The main task of the researcher, in this regard, is to figure out how to correctly and meaningfully input the data for computation and how to interpret the output from the computer. This process can be simple or complicated depending on the research question as well as the researcher's level of familiarity with statistical methods. I will not dwell on this topic, except to mention that, of particular interest in this research, are the **mean and standard deviation.** Because of the nature of the original data (arrays of the numbers 1 and 0 that remain, for some features, constant for the whole sample), it would not make much sense to think in terms of analyzing the means and standard deviations of the raw data, row-wise across the features separately for each student as an individual, or column-wise, jointly for the students as a group under different features. However, these summary statistics can indeed be of much use (as would become clear in a later section on bilateral analysis) when applied to the totals of scores or to new data derived from the application of certain indices on the original data. In what follows, the need and search for such indices are considered.

Table VI Some common descriptive statistics measures for total TB-PB scores

TB		*PB*	
Mean	10.88	Mean	7.04
Standard Error	0.646322933	Standard Error	0.724522831
Median	12	Median	7
Mode	14	Mode	6
Standard Deviation	3.231614663	Standard Deviation	3.622614157
Sample Variance	10.44333333	Sample Variance	13.12333333
Kurtosis	-0.864520104	Kurtosis	-0.28673691
Skewness	-0.6719589	Skewness	-0.583782645
Range	10	Range	12
Minimum	4	Minimum	0
Maximum	14	Maximum	12
Sum	272	Sum	176
Count	25	Count	25

3.1.2 The Need for an index

In addition to the reasoning in the last paragraph of the foregoing section, the following consideration of an aspect of descriptive analysis will help to further illustrate some other important issues and questions that can arise with ordinary descriptive statistical analysis of data of the type involved in this study. For a more meaningful analysis that would reveal the prominent features of the students' theoretical thinking behavior, the TB scores cannot be considered in isolation to the PB scores. Although effort has been made to incorporate the PB scores into the preceding methods of analysis, information based only on such methods would wind up presenting certain TB-PB features (1.3a, 2.2b, 2.3a, 2.4b, 2.5a, 3.3a, 3.2a, 3.2b) as the most prominent factors of the students' success. However, when the analysis is extended by introducing some 'indices of theoretical thinking', the scenario changes.

In what follows, a number of such indices will be defined. When an index is computed for each feature, column-wise for a group of students, it will be referred to as a group TT behavior index. Similarly, when an index is computed row-wise for an individual student, it will be referred to as individual TT behavior index.

Group behavior indices

(i) Average Probability index

For a group of n students, and with respect to a given feature, let t, p, and h denote the observed numbers of students who scored [1,0], [0,1] and [1,1] respectively, or the observed numbers of students for whom the probability of behaving in a theoretical way was 1, 0 and ½ respectively. Let T, P, and H be the random variables associated with the observed values such that $\Pr(T = t) = 1$, $\Pr(P = p) = 0$ and $\Pr(H = h) = ½$.

The 'Average probability' Ap of theoretical behavior on a given feature by a randomly chosen student from the group of n students is defined as:

$$Ap = (1*t + 0*p + ½*h)/n = (2t + h)/2n.$$

(ii) Capability index

For a group of n students, the index of 'capability for theoretical thinking', C with respect to a given feature is defined as the ratio of the total number of occurrences of theoretical thinking behavior $(t + h)$ to the total number of students in the group. That is,

$$C = (t + h)/n.$$

(iii) Theoretical Thinking Tendency index

For a group of n individuals, the index of 'group theoretical thinking tendency', Gtt with respect to a given feature is defined as the ratio of the total number of occurrences of theoretical thinking behavior $(t + h)$ to the sum of the total number of occurrences of theoretical and the total number of occurrences of practical thinking behavior, $(t + h) + (p + h)$. That is,

$$Gtt = (t + h)/[(t + p) + (p + h)] = (t + h)/(t + p + 2h).$$

Individual behavior indices expressed as percentages.

By using the similar measures, the individual students' ways of thinking can be analogously characterized. In other to be able to compare the values of these indices with the students' grades in the linear algebra courses, the individual behavior indices are expressed as percentages since the grades are represented as percentages as well.

Here a row-wise analysis is called for. Now, for a set of m features revealed in the whole interview, and with respect to a given student, let it, ip, and ih denote the numbers of features on which the student's score was [1,0], [0,1] and [1,1] respectively, or the numbers of features for which the probability of behaving in a theoretical way was 1, 0 and ½ respectively. The following definitions are in order:

(iv) Individual Average Probability

$$iAp\% = (1*it + 0*ip + ½*ih)/m * 100 = (2*it + ih)/2*m * 100$$

(v) Individual Capability index

$$iC\% = (it + ih)/m * 100$$

(vi) Individual theoretical thinking tendency

$$itt\% = (it + ih)/(it + ip + 2*ih) * 100$$

Application of these indices to the data leads to results different from those obtained under descriptive/exploratory statistical methods. A reason for this difference is that these indices allow for more in-depth consideration of not only the TB scores but also the PB scores. Table VII shows summaries of the values of the indices Ap, C and Gtt obtained from a feature-by-feature consideration of the original data for the whole group of students. The probability that a randomly chosen student would behave theoretically is high ($Ap > 0.80$) only on 5 features. The students in the whole group would be highly capable ($C \geq 0.8$) of behaving theoretically on 13 features while a high tendency of the group to behave theoretically ($Gtt \geq 0.8$) was observed only on 4 features. However, $Gtt = 1$ on three features. The Gtt index appears to give the grimmest assessment of the students' behavior while the Ap index seems to downgrade the role of practical thinking albeit giving the most balanced distribution of high, medium and low scores. It can be observed that the prominence of the features as important elements of the students' theoretical behavior varies with the different indices. This should not be surprising since the indices are by definition, meant to capture information about the same concept but from different points of view. For example, features such as (2.2b, 2.4, 2.5a, 3.2b) that are in the first category when they are ranked according to the index C, lose their position in the hierarchy when ranking is done according to the indices Ap or Gtt.

Table VII Summary of values computed for the Group indices Ap, C and Gtt

	FEATURE	AP	Gtt	C
1.	1.1a Resea	0.29	0.29	0.29
2.	1.1b Intr sig	0.50	0.50	0.86
3.	1.3a Refl	1.00	1.00	1.00
4.	2.1a termino	0.43	0.43	0.43
5.	2.1b meaning	1.00	1.00	1.00
6.	2.2a rep var	0.46	0.48	0.79
7.	2.2b rep gra	0.75	0.67	1.00
8.	2.3a unfam	0.89	0.82	1.00
9.	2.4a rig ex	0.43	0.43	0.43
10.	2.4b quantif	0.64	0.58	1.00
11.	2.4c frm df	0.68	0.67	0.71
12.	2.5a connec	0.64	0.58	1.00
13.	2.5b implic	0.61	0.57	0.93
14	3.1a relat	0.71	0.67	0.86
15	3.2a df alg	0.86	0.78	1.00
16	3.2b df grph	0.75	0.67	1.00
17	3.3a refut	1.00	1.00	1.00
18	3.3b uniq	0.32	0.37	0.50
19	3.4a categ	0.68	0.65	0.79
20	4.1a hy stat	0.50	0.50	0.79
21	4.1b hy gen	0.57	0.57	0.57
22	4.3a rel tru	0.50	0.50	0.64
23	5.1a E prf	0.57	0.57	0.57
24	5.2a D prf	0.71	0.67	0.86
25	6a D cri proc	0.43	0.43	0.43
	Average	0.64	0.62	0.78

Table VIII shows summaries of the values of the individual indices iAp%, iC% and itt% obtained from a general consideration of the original data for the whole group of students. These values are reported alongside their correlation with the students' grades in the first and second linear algebra courses, LAGradeI and LAGradeII respectively. It can be observed that while all three indices are negatively correlated with LAGradeI, they are all positively correlated with LAGradeII. Only two students (O3, S1) had, individually, an average probability of behaving theoretically greater than 0.80 (iap% > 80). Six students (O3, V1, V2, S1, S3, N2) demonstrated high capability (iC% > 80). Here also the theoretical thinking tendency index gives the harshest assessment of the students' behavior; only one student (O3) had itt% > 80. There seems to be something peculiar about the students V1 and N1 that have been highlighted in blue colour. While V1 had exhibited high capability of theoretical thinking (iC% = 92%), the LAGradeII result shows only 72%. On the other hand, N1 who had 90% in LAGradeII could only score iC% = 68. This finding calls for some more elaborate investigation that is considered in the sequel.

Table VIII Summary of values computed for the Individual indices iAp%, iC%, and itt% alongside the correlation with students' grades in the two linear algebra courses, LA1 and LAII

	Student	iAp%	itt%	iC%	LAGrade1	LAGradeII
1	O1	36	39	48	92	70
2	O2	46	47	64	82	50
3	O3	90	83	100	83	90
4	O4	50	50	72	82	63
5	V1	74	68	92	82	70
6	V2	80	74	92	98	85
7	V3	66	63	76	82	80
8	V4	54	53	72	93	60
9	S1	84	79	92	85	
10	S2	54	53	68	88	
11	S3	74	70	84	92	85
12	S4	60	58	76	88	77
13	N1	50	50	68	100	90
14	N2	74	70	84	100	85
	Average	64	61	78	89	75
	Corr LAGradeI	-0.053	-0.051	-0.108		
	Corr LAGradeII	0.619	0.63	0.508		

Table IX shows summaries of the values of weighted individual indices wiAp%, wiC% and witt% obtained from a Question-by-Question consideration of the original data for the whole group of students. Again as in the previous table, these values are reported alongside their correlation with the students' grades in the first and second linear algebra courses, LAGradeI and LAGradeII respectively. The same trend is maintained but extra information is gained. For example, the weighted indices show positive and higher correlation with the students' grades except for the correlation between LAGradeI and wiC% (-0.03). As highlighted in the table, it becomes easier to notice a group of 7 strong theoretical thinkers among the 14 students namely; (O3 - with all three weighted indices > 80%), (V2, S1, N2 – wiAp% and wiC% > 80%) and (V1, V3, S3 – wiC% > 80%).

Table IX Summary of values of weighted averages computed for the individual indices based on a Question-by-Question consideration of the original data

Weighted averages					LAGradeI	LAGradeII
	Student	WiAp%	Witt%	WiC%		
1	O1	40	42	48	92	70
2	O2	42	40	55	82	50
[redacted]						
4	O4	55	55	68	82	63
5	V1	77	72	87	82	70
[redacted]						
7	V3	73	68	81	82	80
8	V4	58	56	68	93	60
[redacted]						
10	S2	60	58	71	88	
11	S3	74	69	81	92	85
12	S4	68	66	77	88	77
13	N1	58	63	68	100	90
[redacted]						
	Average	67	65	76		
	Corr LAGradeI	0.02	0.07	-0.03		
	Corr LAGradeII	0.71	0.78	0.66		

Strong TT-ers, Observed Singularities and possible implications:

The case of the two students V1 and N1 mentioned earlier present singularities. It turns out that the student V1 belongs to the group of 7 strong TT-ers (those with all three weighted indices wiAp%, wiC% and witt% greater than 80% or those with both wiAp% and wiC% > 80% or those with only wiC% > 80%) while N1 is not a strong TT-er. Based on this TT-er versus NonTT-er grouping, we could talk about the relevance of the TT features for success in the two Linear Algebra courses. Two features turn out to be 'necessary' for the students' success namely; [*3.1a - relational thinking*] and [*3.2a - definitional approach to meanings in algebraic contexts*]. Also it becomes easier to identify some features that can be considered 'necessary but not sufficient' as well as those features that are 'not necessary' for the students' success in both courses. These are listed in the summary given in the concluding Chapter 4.

Also the student S2 presents another singularity. This student had low scores (wiAp% = 60%, witt% = 58%) and medium score (wiC% = 71) but had high scores on the 'necessary' features. Further inquiry shows that the student S2 who switched from mathematics to psychology and as such did not take the Linear Algebra II course, could be described thus:

- longed for detailed understanding
- was bothered by her natural logic which contradicted mathematical logic
- was uncomfortable with the lack of control over her reasoning in proofs
- felt insecure in mathematics.

The foregoing highlights the need to minimize the tendency of relying totally or too much on common descriptive methods when faced with data of the type at hand. Even with the method of computing an index, rigorous analysis must be made to ensure that every aspect of the data set is given due consideration. In what follows, I will consider some limitations and consequences of computing Gtt (itt%) simply as the ratio defined above. For this purpose, I will be referring to a new index I will introduce based on further analysis as analytic or adjusted, Agtt (Aitt%).

Some Limitations of Gtt (itt%) index and introduction of Agtt (Aitt%) index

- ***Meaninglessness of 0/0 when the observed score is [0,0]***

  One interesting advantage of the Gtt (itt%) index, among others, is its intuitively appealing and simple definition. However, by using this index the occurrence of the score [0,0] is either assumed impossible or discarded when observed because of the mathematical absurdity of computing 0/(0 + 0) when *t*, *p* and *h* are all equal to zero, especially when the itt% index is needed separately for each feature. Whichever way, some explanation is needed that would not distort the original design of the experiment. When a plausible explanation cannot be found, the [0,0] score must be acknowledged and accounted for. A way of accounting for [0,0] is given in the sequel.

- ***Misleading ranking***

  The [0,0] score on a feature is problematic and crucial not only in the case of the itt%, but also for Gtt. Consider for example a feature on which a group of 14 students had 13 of them scoring [1,0] and one student scoring [0,0]. The Gtt for this group would be 13/(13 + 0) = 1. Similarly if in another feature, 7 students score [1,0] while the rest score [0,0], the Gtt for this feature will also be computed as 7/(7 + 0) = 1. Perhaps to avoid such misleading ranking, features where at least one student scored [0,0] should be dropped from the ranking and treated separately if the design of the experiment permits that. Otherwise further analytic techniques are needed to rectify this setback.

- ***Superfluous use of cardinalities***

As pointed out in Chapter 2, the original data is made up of the 25 TB-PB features, each of which can be conceived as a binary variable. Thus it can be seen that the nature of the original data matrix presents both cases of intersection of type ($TT \cap PT$) associated with the event of observing the vector [1,1] as well as cases of intersection of complements ($TT^c \cap PT^c$) associated with [0,0]. As such, it becomes pertinent to consider the entire sample space of possible events and to view the totality of the elements of the sets TT and PT in terms of union of events given by $(TT \cup PT) = TT + PT - (TT \cap PT)$. Thus the computation of the proportion or percentage of TT in ($TT \cup PT$) ceases to be a matter of simply dividing the cardinalities, card(TT) = $(t + h)$ by [card(TT) + card(PT)] = $[(t + h) + (p + h)]$ otherwise an index computed in that way would be tantamount to equating card($TT \cap PT$) = 0 in every case and therefore counting the elements of TT and PT twice within their sphere of intersection. Also, as pointed out earlier, this would mean ignoring ($TT^c \cap PT^c$) outright and dividing only by the sum of the observed card(TT) and card(PT) instead of dividing by the cardinality of the entire sample space, card(S).

- ***An Analytic Consideration***

Recognizing that the scoring of the TB-PB features are events whose occurrences can take place under different sample space settings or classifications (see Figure 3.3), the Gtt (itt%) indices can be viewed as a piecewise function taking different values for different domains of occurrence of the events. The possibilities are the following:

i) TT and PT may be mutually exclusive but not collectively exhaustive of the sample space, S.

ii) TT and PT may be mutually exclusive but collectively exhaustive.

iii) TT and PT may be intersecting but not collectively exhaustive.

iv) TT and PT may be intersecting and collectively exhaustive.

v) TT may encompass PT but both are not collectively exhaustive.

vi) TT may encompass PT and both are collectively exhaustive.

vii) TT may be encompassed by PT but both are not collectively exhaustive.

viii) TT may be encompassed by PT and both are collectively exhaustive.

ix) TT may occupy the sample space alone but not exhaustively.

x) TT may occupy the sample space alone and exhaustively.

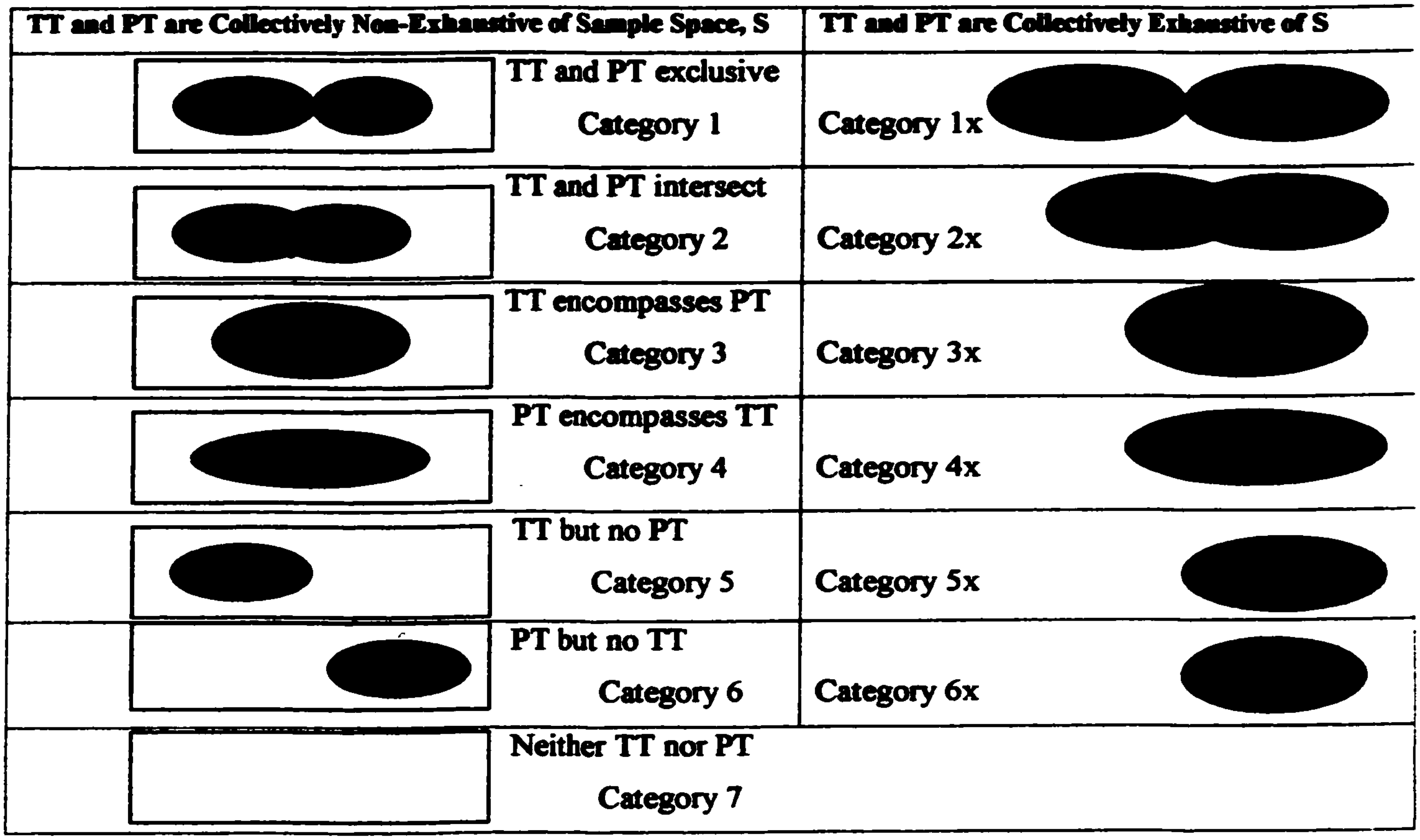
xi) PT may occupy the sample space alone but not exhaustively.

xii) PT may occupy the sample space alone and exhaustively.

xiii) Neither TT nor PT may be present in the sample space.

This type of categorization ensures that the cardinality of the appropriate sample space is used in the denominator when computing an index, individual or group. For example, to compute Gtt for a feature, a small program may be written instructing the computer to first analyze the scenario and figure out which category applies before performing the arithmetic of division. Also, the event [0,0] would not pose any problem since itt% in this case is computable, simply as 0/card(S)*100 = 0%, using category 7 (see figure 3.3) thereby preserving the sample space, S and at the same time assuring an analytically justifiable result.

Figure 3.3 Different Sample Space Settings for Categorization of Features



- ***Implementation of the Analytic Indices***

The implementation of this new method of computing the indices can be achieved by using any of the many available mathematical or statistical software systems such as MAPLE, SPLUS, SAS or SPSS. It can also be done directly on a spreadsheet or database containing the original data provided it allows for some programming. The original data for the present study was stored in Excel and implementation was successfully carried out on this software by writing few lines of code for AGtt (Aitt%) based on the categorization formula;

$$
Att\% = \begin{cases} tt\%1 \text{ if sample space is of type 1} \\ tt\%1x \text{ if sample space is of type 1x} \\ tt\%2 \text{ if sample space is of type 2} \\ tt\%2x \text{ if sample space is of type 2x} \\ tt\%3 \text{ if sample space is of type 3} \\ tt\%3x \text{ if sample space is of type 3x} \\ tt\%4 \text{ if sample space is of type 4} \\ tt\%4x \text{ if sample space is of type 4x} \\ tt\%5 \text{ if sample space is of type 5} \\ tt\%5x \text{ if sample space is of type 5x} \\ tt\%6 \text{ if sample space is of type 6} \\ tt\%6x \text{ if sample space is of type 6x} \\ tt\%7 \text{ if sample space is of type 7} \end{cases}
$$

where

tt%1 = card(TT)/card(S), tt%1x = card(TT)/card(TT u PT)

tt%2 = card(TT ∩ PT^c)/card(S), tt%2x = card(TT ∩ PT^c)/card(TT u PT)

tt%3 = [card(TT)-card(PT)]/card(S) tt%3x = [card(TT)-card(PT)]/card(TT)

tt%4 = card(TT)/card(S), tt%4x = card(TT)/card(PT)

if (TT ∩ PT) = TT.

tt%5 = card(TT)/card(S), tt%5x = card(TT)/card(TT) = 1

if card(PT) = 0.

tt%6 = 0/card(S) = 0, tt%6x = 0/card(PT) = 0

and

tt%7 = 0/card(S) = 0.

Some Implications of using AGtt index and its relationship with previous indices

A general implication of using the analytic indices is a reduction in the magnitude of the values previously computed as, for example, Gtt% (Gtt in percentages). This general trend can be observed from Table X, which shows the Agtt% indices listed alongside the Gtt% indices for the various features.

Table X Comparison of Agtt% and Gtt% indices

Feature	Question	Gtt%		Agtt%	Space Type
[1.3a]	Q5	100		100	5x
[3.3a]	Q4.1	100		100	5x
[3.3b]	Q4.2	100		92.9	5
[2.2c]	Q4.2	92.9		92.9	1x
[2.3b]	Q3.2	85.7		85.7	1x
[2.3a]	Q3.2,Q4.2	77.8		71.4	3x
[3.1a]	Q5	75		71.4	2x
[3.2a]	Q2,Q4.2	73.7		64.3	3x
[2.1b]	Q5	73.3		71.4	2x
[6a]	Q2	73.3		71.4	2x
[2.5b]	Q3.2	71.4		71.4	1x
[2.2b]	Q3.2	66.7		50	3x
[3.2c]	Q3.2	66.7		50	3x
[5b]	Q5	66.7		57.1	2x
[3.4a]	Q1.1	64.7		57.1	2
[3.2d]	Q4.1	62.5		57.1	2x
[5a]	Q4.2	61.5		57.1	1
[6b]	Q4.1,Q4.2	60.9		35.7	3x
[2.4c]	Q4.1	58.8		50	2x
[2.5a]	Q2,Q4.2	58.3		28.6	3x
[6c]	Q3.2	57.9		42.9	2x
[4.1b]	Q3.2	57.1		57.1	1x
[2.5c]	Q4.1	56.5		28.6	2x
[2.4b]	Q2,Q4.2	56		21.4	3x
[4.3a]	Q5	55.6		42.9	2x
[3.2b]	Q2,Q4.1	50		7.1	2x
[4.1a, 4.2a]	Q1.2,Q2	50		21.4	2x
[2.2a]	Q1.1, Q1.2	47.8		14.3	2x
[1.1a, 1.2a]	Q2	42.9		42.9	1x
[2.1a]	Q2	42.9		42.9	1x
[6d]	Q5	42.9		42.9	1x
[2.4a]	Q1.1	38.5		35.7	1

Four rows have been shaded in yellow for purposes of illustration and comparison. Of particular interest is the third shaded row in which the feature 3.2b has Agtt% = 7.1%

and Gtt% = 50%. Earlier, this feature has been noted for the highest PT scores. While Agtt% is able to portray this feature in this position (least in order of TT tendency), Gtt% was unable to capture this information. This is an example of how interpretations of data or results of analysis based on Gtt may lead to conclusions different from those obtained through Agtt. Also the Agtt% index comes with a listing of the category of each feature for which it is computed. In this way, it becomes easier to have two pieces of information in one shot. This would enable an easier and a more objective ranking of features.

Relationship between Agtt index and previous indices

The relation between the Agtt index and the previous indices can be summarized as follows: From Figure 3.3 and the computational formula given above, it can be observed that the Category 2 offers a summary formula from which the rest of the categories are derivable as particular cases. This formula can be expressed in terms of the already defined quantities t, p, h and the quantity $r = \text{card}(TT^c \cap PT^c)$ associated with the event of observing the vector[0,0], as

$$\text{Agtt} = \text{card}(TT \cap PT^c)/\text{card}(S) = t/(t + p + h + r) = t/n.$$

Noting that

$$\text{Ap} = (2t + h)/2n = t/n + ½{*}h/n, \quad C = (t + h)/n = t/n + h/n,$$

$$\text{Gtt} = (t + h)/(t + p + 2{*}h) = (t + h)/(n - r + h) = (t/n + h/n)/(1 - r/n + h/n),$$

it turns out that the following relations hold among the indices. We have that,

$$C = \text{AGtt} + h/n$$

$$\text{Ap} = \text{AGtt} + ½{*}h/n = C - ½{*}h/n$$

$$\text{Gtt} = (\text{AGtt} + h/n)/(1 - r/n + h/n) = C/(1 - r/n + h/n)$$

A limiting result

It is interesting to note that for large sample sizes in which n is much greater than h and r, ($n >> h$, and $n >> r$), all the indices coincide to the same value in the limit as $n \to \infty$ and $h/n \to 0$, namely, Gtt = Ap = C = AGtt = t/n. An implication of this limiting result is that whenever $n >> h$ and/or $n >> r$, it does not really matter which of the indices Gtt or AGtt is used. Problems would arise when n is small, and/or when h and r are fairly large with respect to the size of n. More insight into such and similar problems are being gained from an ongoing and more in-depth statistical study that will be published elsewhere in further research. In the next section, another possible way of analyzing the data from a statistical modeling point of view is presented.

3.1.3 A New Parametric Statistical Method – 'Inclination Analysis'

Although the data matrix as shown in Appendix D does not contain [0,0] entries, the possible occurrence of such scores in a study like this cannot be generally ruled out. In this section, I will present the details of a new statistical method, 'inclination analysis' which I have developed as a general method for analyzing the type of data at hand, taking into special consideration the third research question. This method of analysis settles the problem of [0,0] scores. In line with the notations introduced in Chapter 2, recall that the behaviors of the students with respect to each of the (25 x 2) TB-PB features can be summarized row-wise into the four possible sets: ($TT \cap PT^c$), ($TT^c \cap PT$), ($TT \cap PT$), and ($TT^c \cap PT^c$), associated with the possible vector outcomes [1,0], [0,1], [1,1] and [0,0] respectively such that the random variables, T = [card($TT \cap PT^c$) + card($TT \cap PT$)] represents the total of all observations of TB and P = [card($TT^c \cap PT$) + card($TT \cap PT$)] represents the total of all observations of PB. As an illustration, consider a feature, i in which a [0,0] score is observed. The data and its analytic set up in this case would look like Table XI. Let the set S denote the total n = card(S) students (considered as a population) and let the sets TT_i and PT_i denote the subsets of students who scored $T = t$ and $P = p$ respectively on the feature i, for $i = 1, .., m$. Note that there is no a priori information about which student would belong to which sub-population. Instead one can imagine a random and independent selection of X_i and Y_i denoting, respectively, any two random parts of S, (the random sets of students who would exhibit the TB_i and PB_i features respectively in either a mixed or consistent manner) such that we would have card(X_i) = card(TT_i), card(Y_i) = card(PT_i), card(X_i^c) = card(TT_i^c), card(Y_i^c) = card(PT_i^c) where X_i^c, TT_i^c, Y_i^c, and PT_i^c denote the complements of X_i, TT_i, Y_i and PT_i respectively.

Table XI An illustration of the 'inclination analysis' setup for the original data

	Feature *i*		
Student	**TB_i**	**PB_i**	
1	0	0	
2	1	0	
3	1	0	
4	0	1	
5	1	0	
6	1	0	
7	0	1	
8	0	1	
9	1	0	
10	0	1	
11	1	0	
12	0	1	
13	1	0	
14	1	0	
Totals	8	5	

PB_i / TB_i	1	0	
1	$card(TT_i \cap PT_i) = 0$	$card(TT_i \cap PT_i^c) = 8$	$t_i = 8$
0	$card(TT_i^c \cap PT_i) = 5$	$card(TT_i^c \cap PT_i^c) = 1$	$n - t_i = 6$
	$p_i = 5$	$n - p_i = 9$	14

Gras (1992) showed that for a data set up in the above way, the cardinal U_i = card($X_i \cap Y_i^c$) can be modeled as a random variable following the Poisson distribution with parameter λ_{i1} = card(TT_i)card(PT_i^c)/n. Similarly, it follows that V_i = card($X_i^c \cap Y_i$) can also be modeled as a random variable following the Poisson distribution with parameter λ_{i2} = card(TT_i^c)card(PT_i)/n. In his work, Gras focused on the question of implication between variables and classes of variables. But in this study, the focus is on a different type of problem: that of the "inclination" between variables. In Gras' work as well as in this work, the original data are set up in similar ways but the research questions are different. Whereas the "inclination analysis" pursued in this study is concerned with determining the extent to which a group of students are inclined towards theoretical thinking, that is, the extent to which evidence from the data would suggest that card(TT_i) > card(PT_i), Gras' "implicative analysis" would be seeking the extent to which one can say that theoretical thinking implies practical thinking, the implication not, a priori, being connoted with causality. Thus, the aim of Gras' work is clearly different from the aim of this study. With this clarification on the similarity and difference between Gras' work and this work out of the way, I will now proceed with details of the inclination analysis. Before going further, I will formalize the following definitions, which will be needed in the sequel.

Definition 3.1.3a: ***Inclination between variables***
Consider data about a group of subjects whose behavior could be recorded according to the presence or absence of a variable feature, A as opposed to another feature B. Let S and R denote the variable sets of subjects exhibiting A and B respectively with S and R not necessarily independent but allowing the possibility of recording some events in their intersection ($S \cap R$), as well as in the intersection of their complements ($S^c \cap R^c$). The data will be said to hold evidence for the subjects' inclination towards A as opposed to B, denoted $A \blacktriangleleft B$, if the cardinality of A, card(A) exceeds that of B, card(B) by at least one unit. Also inclination towards B is similarly defined and denoted by $A \blacktriangleright B$.

Definition 3.1.3b: ***Strength of inclination towards a variable***
The strength of inclination towards A as opposed to B, denoted by $s(A \blacktriangleleft B)$ is the probability of occurrence of the event $A \blacktriangleleft B$. This is given, for A and B discrete variable sets, by

$$s(A \blacktriangleleft B) = \Pr[\text{card}(A) - \text{card}(B) \geq 1].$$

It is conceivable that the random set of students who are more inclined towards TB_i will belong to the set $(X_i \cap Y_i^c)$ while those with more inclination towards PB_i will belong to the set $(X_i^c \cap Y_i)$. Thus for each TB-PB feature i, the extent of a group of students' inclination towards theoretical thinking (Gtt_i tendency) can be determined as follows.

Proposition 3.1.3a

For a given TB-PB feature i, the data would hold evidence for a group of students' inclination towards TB_i as opposed to PB_i (TT_i as opposed to PT_i) if and only if the probability of occurrence of the event $[(U_i - V_i) \geq 1]$ is greater than zero, where U_i and V_i are as defined above.

***Proof*:** Assuming the data holds evidence for a group of students' inclination towards TB_i then the event that the number of students in the set TT_i exceeds those in the set PT_i by at least one unit will occur with a probability greater than zero. Thus,

$$\Pr[(\text{card}(TT_i) - \text{card}(PT_i)) \geq 1] > 0.$$

Recall that,

$$\text{card}(TT_i) = \text{card}(X_i),\ \text{card}(PT_i) = \text{card}(Y_i).$$

Hence,

$$\begin{aligned}\Pr[(\text{card}(TT_i) - \text{card}(PT_i)) \geq 1] &= \Pr[(\text{card}(X_i) - \text{card}(Y_i) \geq 1] \\ &= \Pr[((\text{card}(X_i) - \text{card}(X_i \cap Y_i)) - (\text{card}(Y_i) - \text{card}(X_i \cap Y_i))) \geq 1] \\ &= \Pr[((X_i \cap Y_i^c) - (X_i^c \cap Y_i)) \geq 1] \\ &= \Pr[(U_i - V_i) \geq 1].\end{aligned}$$

Therefore,

$\Pr[(\text{card}(TT_i) - \text{card}(PT_i)) \geq 1] > 0$ is the same as the probability, $\Pr[(U_i - V_i) \geq 1] > 0$.

Assuming the probability of occurrence of the event $[(U_i - V_i) \geq 1]$ is greater than zero, by retracing the steps, the other part of the proposition is proved.

Corollary 3.1.3a

The probability, $Pr[(U_i - V_i) \geq 1]$ gives the strength of inclination of the students towards TB_i as opposed to PB_i. A high value of this probability implies a strong inclination towards TB_i while a low value implies a weak inclination towards PB_i.

Corollary 3.1.3b

The strength of inclination of a group of students towards TB_i as opposed to PB_i on a feature i, is a measure of their group theoretical thinking tendency, Gtt_i. A strong inclination implies a high Gtt_i and a weak inclination implies a low Gtt_i.

Computational Aspects of the Inclination Analysis

Let $W_i = (U_i - V_i)$. Now from previous sections, U_i and V_i are independent and distributed as Poisson random variables. Thus, the probability distribution of the random variable W_i is obtainable as the distribution of the difference of two independent Poisson random variables, which for $W_i > 0$, [see Irwin (1937), Fisz (1953), Johnson and Kotz (1969) or Consul (1986)] is given by

$$f(w_i) = e^{-\lambda_{i1}-\lambda_{i2}}\left(\frac{\lambda_{i1}}{\lambda_{i2}}\right)^{\frac{w_i}{2}} I_{\frac{w_i}{2}}(2\sqrt{\lambda_{i1}\lambda_{i2}}) \qquad (3.1)$$

where $I_r(s)$ is the first kind of modified Bessel function of order r and argument s.

The mean and variance of W_i are given respectively by

$$E(W_i) = (\lambda_{i1} - \lambda_{i2}) \text{ and } Var(W_i) = (\lambda_{i1} + \lambda_{i2})$$

where $\lambda_{i1} = card(TT_i)card(PT_i^c)/n$ and $\lambda_{i2} = card(TT_i^c)card(PT_i)/n$.

Thus for each of the features $i = 1,\ldots,25$, Gtt_i can be computed as the following piecewise function:

$$Gtt_i = \begin{cases} g(-w_i, \lambda_{i1}, \lambda_{i2}), & w_i < 0, \\ 1/2, & w_i = 0, \\ g(w_i, \lambda_{i1}, \lambda_{i2}), & w_i > 0, \end{cases} \qquad (3.2)$$

where for $w > 0$, $g(w,\lambda_{i1},\lambda_{i2})$ is computed as the function given in (3.1) above and for $w < 0$, $g(-w,\lambda_{i1},\lambda_{i2})$ is given by

$$g(-w_i, \lambda_{i1}, \lambda_{i2}) = e^{-\lambda_{i2}-\lambda_{i1}} \left(\frac{\lambda_{i2}}{\lambda_{i1}}\right)^{\frac{w_i}{2}} I_{-\frac{w_i}{2}}(2\sqrt{\lambda_{i1}\lambda_{i2}})$$

For example, corresponding to the first feature [1.1a] for which $i = 1$, we have

λ_{11} = card(TT_1)card(PT^c_1)/n = (4)(4)/14 = 1.14,

λ_{12} = card(TT_1^c)card(PT_1)/n = (10)(10)/14 = 7.14

Since $W_1 = (U_1 - V_1) = (4 - 10) = -6$, $\lambda_{11} = 1.14 < \lambda_{12} = 7.14$, Gtt_1 is therefore computed as g(-6,1.14,7.14) = 0.005613.

Similarly for i = 2, 3, ...25 the values for the other features can be computed accordingly.

Further Aspects of the Inclination Analysis

Now the possibility of generalizing this result for purposes of further statistical inference can be considered as follows. But before that the following proposition that will be needed in the sequel, is in order.

Proposition 3.1.3b

Between TB_i and PB_i, it is admissible at 1-α level of confidence to claim that inclination is towards TB_i, i.e. (TB_i◀PB_i) if and only if

$$Pr(W_i \geq 1) \leq \alpha .$$

This admissible level of confidence is the s(TB_i◀PB_i), that is, the strength of inclination towards TB_i as opposed to PB_i which is the same as the Gtt_i.

The values of *s(TB_i◀PB_i)* for the various features are shown in Table XII alongside those of the previous indices.

Corollary: *For a large sample, the random variable W_i can be standardized to obtain the standard normal random variable*

$$Z_i = \frac{W_i - (\lambda_{i1} - \lambda_{i2})}{\sqrt{(\lambda_{i1} + \lambda_{i2})}} \sim N(0,1).$$

and as such the strength of inclination towards TB_i can be obtained as

$$\Phi(\lambda_{i1}, \lambda_{i2}) = 1 - \Pr\left(Z_i < \frac{1 - (\lambda_{i1} - \lambda_{i2})}{\sqrt{(\lambda_{i1} + \lambda_{i2})}} \right) \cong \Pr(W_i \geq 1)$$

For instance, if the strength of inclination is greater or equal to 0.95, the claim of inclination towards TT_i is admitted with a 5% margin of error.

***Example*:**

For the TB-PB feature [1.1a] as in the illustration above, the Z-score can be obtained by computing the quantities:

$\lambda_1 = \text{card}(TT)\text{card}(PT^c)/n = 1.142$

$\lambda_2 = \text{card}(TT^c)\text{card}(PT)/n = 7.142$

The corresponding strength of inclination can be obtained as

$[1 - \Pr(Z_i < (1\text{-}6)/2.67)] = [1 - \Pr(Zi < -2.88)] = 0.00198844.$

Since this value is less than 0.95, the claim of inclination towards TB_i cannot be admitted with a 5% error margin. Note that this normal approximation albeit the small sample size, $n = 14$, is not very much less than the exact value (0.005613) computed earlier. What would have been gained here is the possibility of being guided by an admissible margin of error that can help in expressing the degree of confidence in any judgement made. Thus when statistical inference is necessary and warranted by the data, one may be able to give not only the value of Gtt, but also the margin of error with which the given value can be considered large enough to permit an admissible claim of inclination towards *TB*.

Table XII. Comparison of *s*(*TT*◄*PT*) with other indices

Feature	Ap	C	Gtt	Agtt	Space Type	*s*(*TT*◄*PT*)
1.1a	0.29	0.29	0.29	1.00	5x	0.005613
1.1b	0.50	0.86	0.50	1.00	5x	0.500000
1.3a	1.00	1.00	1.00	0.93	5	1.000000
2.1a	0.43	0.43	0.43	0.93	1x	0.000000
2.1b	1.00	1.00	1.00	0.86	1x	1.000000
2.2a	0.46	0.79	0.48	0.77	3x	0.000003
2.2b	0.75	1.00	0.67	0.71	2x	1.000000
2.3a	0.89	1.00	0.82	0.64	3x	1.000000
2.4a	0.43	0.43	0.43	0.71	2x	0.000000
2.4b	0.64	1.00	0.58	0.71	2x	1.000000
2.4c	0.68	0.71	0.67	0.71	1x	0.000000
2.5a	0.64	1.00	0.58	0.50	3x	1.000000
2.5b	0.61	0.93	0.57	0.50	3x	0.000061
3.1a	0.71	0.86	0.67	0.57	2x	0.000000
3.2a	0.86	1.00	1.00	0.57	2	1.000000
3.2b	0.75	1.00	0.37	0.57	2x	1.000000
3.3a	1.00	1.00	0.65	0.57	1	1.000000
3.3b	032	0.50	0.50	0.36	3x	0.011095
3.4a	0.68	0.79	0.57	0.50	2x	0.000000
4.1a	0.50	0.79	0.50	0.29	3x	0.500000
4.1b	0.57	0.57	0.57	0.43	2x	0.000000
4.3a	0.50	0.64	0.50	0.57	1x	0.500000
5.1a	0.57	0.79	0.57	0.28	2x	0.000000
5.2a	0.71	0.86	0.67	0.21	3x	0.000000
6a	0.43	0.43	0.43	0.43	2x	0.000000

3.2 Bilateral Analysis

The preceding sections were concerned with unilateral forms of analysis in which the students' theoretical thinking tendency has been analyzed in isolation of other variables. In this section, bilateral forms of analysis will be carried out. The idea is to seek for results that would help in scrutinizing further the results already obtained as well as in probing the underlying assumption of a positive association between theoretical thinking and students' success in linear algebra course.

3.2.1 Investigating relationship between TT tendencies and Students' Grades in Two Linear Algebra Courses

In this section, the fourth research question will be addressed. That is, the relationship between the students' individual theoretical thinking (itt%) tendencies (as computed by the different methods) and their grades in the two linear algebra courses Math 251 and Math 252 respectively denoted by, LAGradeI and LAGradeII, will be investigated. The objective here is to see the extent to which these indices can be associated with the students' grades in the examination. A high positive correlation coefficient with the itt% would be suggesting that the grades could be predicted by a positive linear function of the index or vice versa. A negative correlation would indicate the same except that the function would have a negative sign for the slope of the line. A zero correlation would indicate the absence of a linear relationship between the index and the grade implying that neither the index nor the grade can be predicted from the other by using a linear equation. Tables XIII and XIV show the various correlation values for the first and second linear algebra courses respectively computed under the different indices.

Table XII. Correlation between individual behavior indices and Students' grades in LA I & II

Student	LA Grade1	LA Grade II	IAp%	iC%	Itt%	Aitt%	S(TT?PT)
O1	92	70	36	48	39	28	94
O2	82	50	46	64	47	28	94
O3	83	90	90	100	83	81	100
O4	82	63	50	72	50	38	92
V1	82	70	74	92	68	69	100
V2	98	85	80	92	74	69	100
V3	82	80	66	76	63	53	99
V4	93	60	54	72	53	41	97
S1	85		84	92	79	84	100
S2	88		54	68	53	50	99
S3	92	85	74	84	70	69	100
S4	88	77	60	76	58	50	100
N1	100	90	50	68	50	44	95
N2	100	85	74	84	70	69	100
		Correlation II	**0.62**	**0.51**	**0.63**	**0.71**	**0.59**
	Correlation I		**-0.05**	**-0.05**	**-0.05**	**0.02**	**0.09**

Table XIV. Correlation matrix of individual behavior indices and Students' grades in LA I & II

	LA Grade1	*LA Grade II*	*IAp%*	*IC%*	*itt%*	*Aitt%*	*S(TT?PT)*
LA Grade1	1.00						
LA Grade II	0.47	1.00					
Iap%	-0.05	0.62	1.00				
IC%	-0.11	0.51	0.97	1.00			
Itt%	-0.05	0.63	1.00	0.96	1.00		
Aitt%	0.02	0.71	0.97	0.93	0.97	1.00	
S(TT?PT)	0.09	0.59	0.81	0.74	0.80	0.83	1.00

Correlation analysis of ITT tendencies on the different interview questions and Students' Grades in Linear Algebra courses.

Here, the relationship between the students' individual theoretical thinking (ITT) tendencies on the different interview questions 1-5 and their grades in the two linear algebra courses (Math 251 and Math 252) will be investigated. The objective here is to statistically scrutinize a little further the validity of both the questions and the process of scoring used in the experiment. If the questions were helpful in measuring TT features and the ITT tendencies on the questions were capturing these features, then the itt% scores on the questions would be positively correlated among themselves. A negative correlation would be suggesting something peculiar about the questions involved or the scoring process or the data entry or all of these.

Also, the variable sex will be incorporated into this analysis in order to highlight the role, if any, gender might play in this type of study and also to serve as a 'statistical wedge' that could, in a sense, either expose or confirm any anomalies in the analysis.

The above objectives will be accomplished by a correlation analysis of itt% index on the different questions and the grades as well as a correlation analysis of itt% on different questions and students' gender. Table XV shows that all the scores are positively correlated among themselves. Generally, the students' average itt% score (Aveitt%) on the questions is negatively correlated with LAGradeI (-0.04) and positively correlated (0.72) with LAGradeII.

Table XV. Correlation matrix of itt% by questions

	IttQ1.1%	IttQ1.2%	IttQ2%	IttQ3.2%	Itt4.1%	IttQ4.2%	IttQ5%	AveittQ%	Sex	Grade 1	Grade II
IttQ1.1%	1.00										
IttQ1.2%	0.01	1.00									
IttQ2%	0.01	0.43	1.00								
IttQ3.2%	0.12	0.51	0.34	1.00							
Itt4.1%	0.02	0.14	0.48	0.10	1.00						
IttQ4.2%	0.28	0.25	0.04	0.63	0.19	1.00					
IttQ5%	0.34	0.36	0.41	0.67	0.45	0.82	1.00				
AveittQ%	0.41	0.68	0.65	0.72	0.49	0.64	0.86	1.00			
Sex	-0.10	-0.73	-0.43	-0.34	0.07	0.08	-0.24	-0.46	1.00		
Grade 1	0.02	-0.24	0.25	-0.18	0.02	-0.13	0.11	-0.04	0.05	1.00	
Grade II	0.26	0.47	0.78	0.43	0.38	0.28	0.49	0.72	-0.25	0.47	1.00

This suggests that while, on the average, the students' scores on the questions cannot be used to linearly predict their performance in the first examination, they can indeed be used for predicting the results of the second examination. In particular, the itt% scores on questions 1.2, 3 and 4.2 are all negatively correlated with LAGrade1 while the itt% scores on all the questions are all positively correlated with LAGradeII.

Sex is negatively correlated with the scores on most of the questions suggesting that females (coded 0) were scoring more on most of the questions than males (coded 1). This pattern of correlation is mostly noticeable in Question 1.2, which is most inversely correlated (r = -0.73) with sex. Also, a sex-Ave ittQ% correlation of r = –0.46 indicates that on the average, females were scoring more TT features on the questions than males and that only about 21% (r^2 = 0.2116) of the variance in average itt% tendency can be attributed to gender difference.

Chapter 4 Conclusion

The aim of this thesis was to examine ways in which different perspectives of statistical methodology can be applied to the study of cognitive processes in mathematics education with particular focus on linear algebra. The perspectives are – descriptive/exploratory statistics, and statistical modeling. A careful analysis of statistical methodology in the light of these approaches and their application to the analysis of data resulting from the case study of cognitive factors of students' success in linear algebra, were carried out. The advantages and drawbacks of these methods, from the theoretical as well as the application points of view, were exposed, and a new method of analysis that rectifies a problem concerning the peculiar nature of the data involved, has been developed. Based on the findings, the following reflections on, and answers to the present research questions will be summarized in this concluding chapter.

In summary, as an answer to the first research question, the features of theoretical thinking that can be identified among the group of high achieving students of linear algebra under study are the 25 theoretical behavior features listed in Appendix C [Features of TT]. The detail of the process taken to obtain this list was given in Chapter 2.

Evidently, I have, in this thesis, attempted at exploring some of the traditional applications of statistics to research in mathematics education, focusing especially on the problems of using statistical analysis in the study of cognitive processes. Of particular interest in this sense is the methodological problem involved in analyzing the TB-PB data resulting from the case study presented in Chapter 2. The question of having columns and rows with constant entries and the possibility of observing a [0,0] score in the data matrix warranted much caution in the choice of quantitative methods for their analysis. Starting by scrutinizing the merits and drawbacks of applying, to these data, the most basic forms of descriptive/exploratory statistics methods that are traditionally used, I went on to explore different indices that are defined for measuring variables of interest in the study. In the process, `inclination analysis` has been developed as a new method of analyzing the type of data involved here. In cognizance of the second research question as posed in Chapter 2, it is noteworthy that a typical traditional exploratory/descriptive data analysis alternative to the inclination analysis method would have been to compute the correlation between TT and PT for each feature, and then compare the magnitudes of these correlation values. Since TT and PT are expected to be opposed to each other, a high correlation value between TT and PT on any given feature would be considered an indication of a weak inclination towards TT and a low correlation value would indicate a relatively stronger inclination towards TT. However, looking at the original data matrix, it can be observed that some columns and rows may have constant entries. For such columns or rows, the correlation between TT and PT cannot be computed since they will have zero variances due to the constant values of the entries. This zero variance problem renders this method useless in terms of determining the relative importance of the features. Moreover, it was seen in the first section of Chapter 3, that generally, common descriptive methods although very handy, can be very limited in their application. Also they can be disfiguring in the case of multidimensional data. These and other similar drawbacks of such traditional methods necessitated the development of the methods and indices used in this thesis.

Based on these methods and indices, an answer to the third research question concerning the ranking of the features of theoretical thinking in the high-achieving students is available. It can be recalled that in the case study presented in Chapter 2, all 14 students interviewed achieved an 'A' grade in the first linear algebra course while only 6 of them obtained an 'A' grade also in the second linear algebra course. Since two of the students had not taken LAII by that time, no information was available on their grades. A summary of the results shows that most of the students who obtained an 'A' grade in both courses have high individual theoretical thinking tendencies (itt% ≥ 70%). However, one student N1, in particular who made an A^{*} grade in LAII had an itt% score of only 50%. On looking at the TT features where these students' TT tendencies were most convergent, it turned out that the features, *relational thinking* (*3.1a*) and *definitional approach to meanings in algebraic contexts* (*3.2a*) seem to be necessary factors of success in both linear algebra courses. On the other hand, features that do not seem to be necessary for success in both linear algebra courses include:

- A researchers' attitude to solving mathematical problems (1.1a)
- Appreciation of the intrinsic significance of mathematical concepts (1.1b)
- Sensitivity to mathematical terminology (2.1a)
- Rigorous approach to mathematical expressions (2.4a)
- Logical sensitivity to the structure of statements (2.5a)
- Holding sophisticated epistemological beliefs (4.3a)

In between these necessary and unnecessary features, are some features, which albeit having considerable impact on students' success in both courses, may not be considered sufficient factors. Among them are:

- Analytic approach to unfamiliar representations (2.3a)
- Systemic reasoning in refutation of a general statement (3.3a)

- Reflective approach to the study of mathematics (1.3a)

While the possession of these features can be considered as necessary but not sufficient conditions of success in the linear algebra courses studied, the application of these indices is able to highlight features that cannot be considered as sufficient conditions. When the validity of this information is confirmed, it would undoubtedly have great implication for the teaching and learning of linear algebra and mathematics education in general.

Although the preceding paragraph commands much optimism, a fully exhaustive consideration of factors cannot be claimed at this time because, for instance, in this thesis, emphasis was placed on success as opposed to giftedness. A more exhaustive consideration could be pursued in future research in which the scope of study can be widened and the TTPT framework analysis extended to embrace such a wider scope of research.

In answer to the last research question, the correlation results in Tables XIII and XIV portray the extent to which the underlying assumption of a positive association between high achievement in a linear algebra course and theoretical thinking tendency, can be validly claimed. Although all the various indices show little or no correlation with the students' grades in the first linear algebra course, the second linear algebra grades are positively correlated with all the indices. This suggests that while the students' scores on these indices cannot be used to linearly predict their performance in the first examination, they can indeed be used for predicting the results of the second examination.

The relationship established between the students' itt% scores and their performance in the second linear algebra examination is an indication that the TTPT scheme can offer an alternative method to the traditional routine method of assessing students' knowledge of the course material. This new method might prove more competitive since it would in essence be measuring the extent to which students' understanding of the material does or does not depart from the theory. That is to say, in Sierpinska's terminology, it would be measuring directly, students' 'good understanding' of the material. I would suggest that further research into this area be pursued. This suggestion is in order since, as cited in Chapter 1, under the factors of students' success in mathematics, method of assessment has been identified as an important element.

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APPENDIX A

The interview questions - version 2, after the first interview

Question 1

Question 1.1

Here is a pile of expressions, each written on a separate card. You are asked to separate these expressions into at least two groups, according to your own criteria.

$2x + 3 = 7$ $\quad$ $y = x^2 - 4x + 1$ $\quad$ $A = ah/2$ $\quad$ $2x - x^3 + 78$

$x - 13 = x$

Question 1.2

At a final exam in linear algebra, there was a question about the definition of linear independence of vectors. One student wrote:

A set of vectors $\{v_1, \dots, v_n\}$ is called linearly independent if

$a_1 v_1 + \dots + a_n v_n = \mathbf{0}$.

(i) What do you think about this definition?

(ii) The same student then wrote that the set of vectors $\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\}$ is linearly independent.

Would you say that what the student said follows from his definition?

Question 2.

Here is a problem that was given in 1997 on a final exam in the *Vectors and Matrices* course (MATH 204). Without solving it, could you tell me how you would approach the problem?

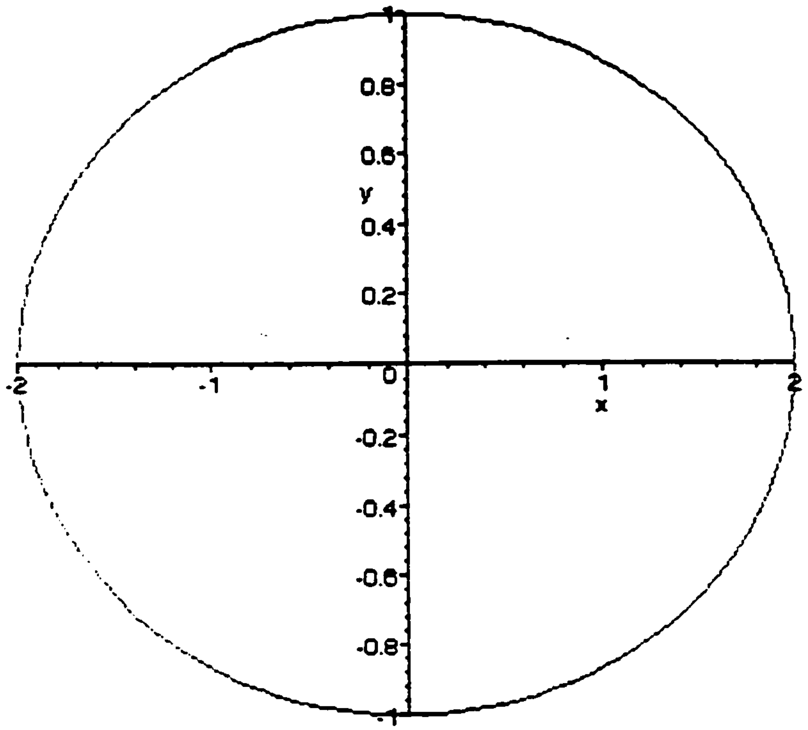
> **Let u, v, and w be vectors in a vector space V over R. Show that the vectors u - v, u - w, and v + w are linearly dependent.**

Question 3.

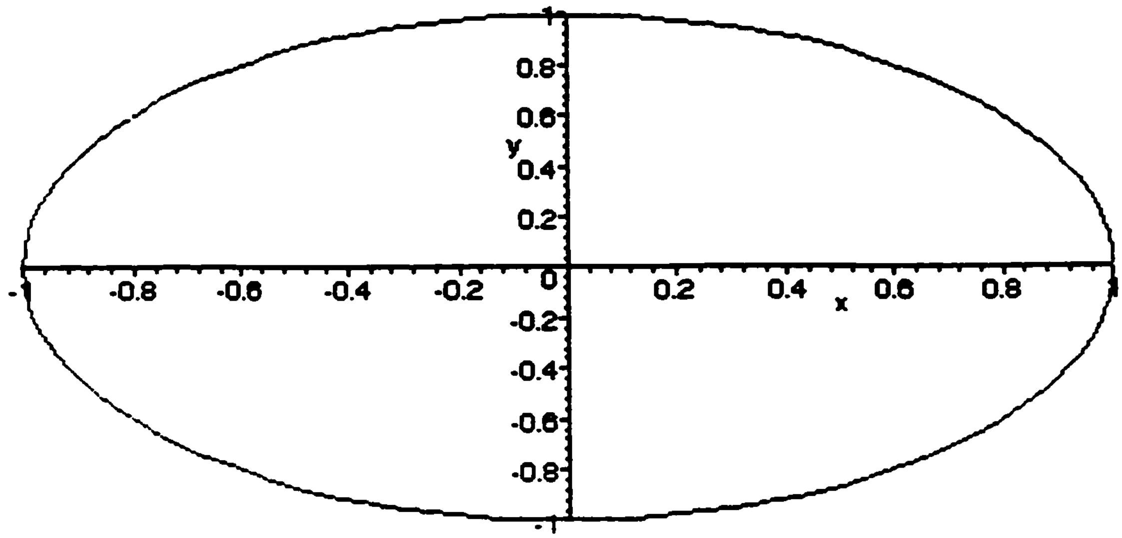
Question 3.1

Does the following diagram represent a circle?

(i)

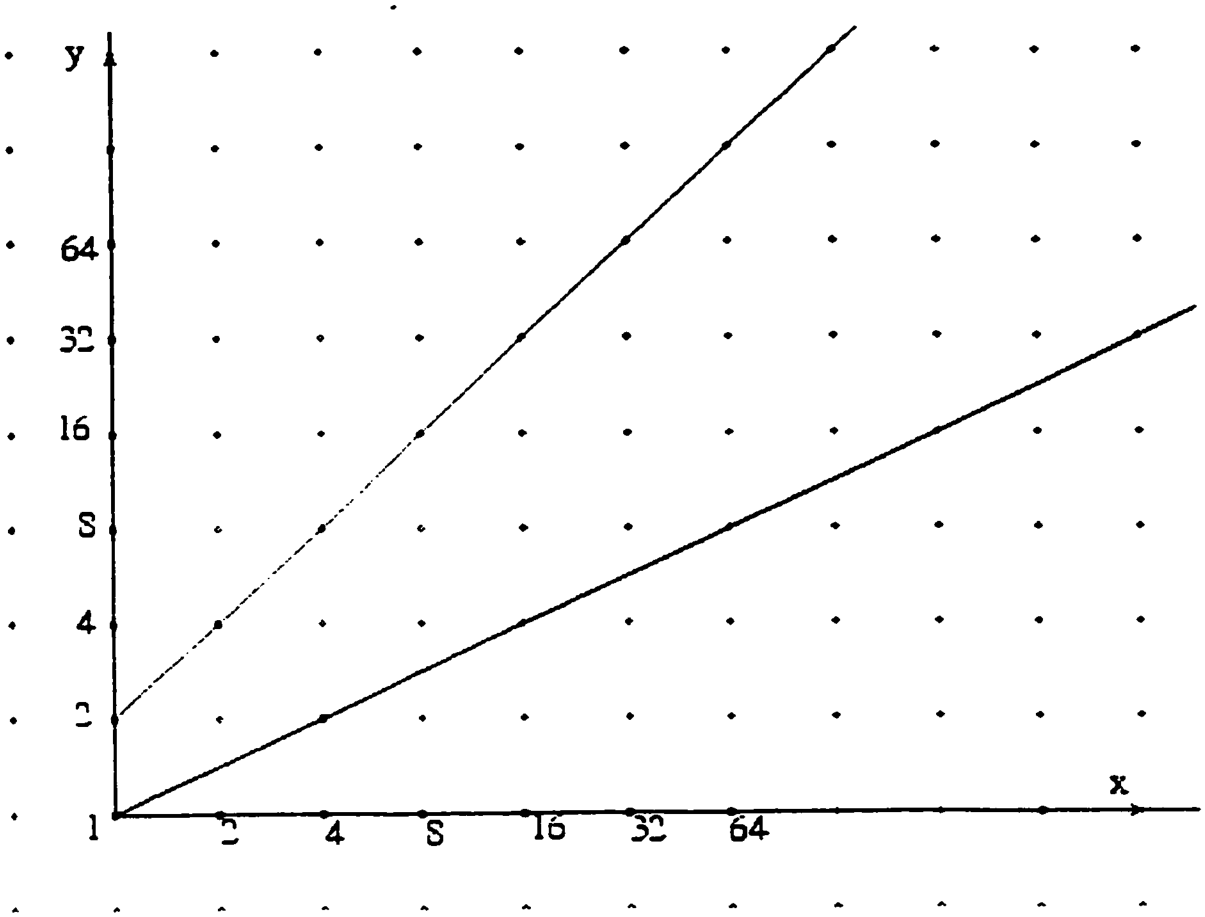


(ii)



Question 3.2

Do the following graphs represent linear functions?



Question 4

Question 4.1

Here are some true statements about brillig numbers. Read them carefully.

Statement 1: Brillig numbers are, for example, 5, 7, 13.

Statement 2: Any brillig number can be written as $p + 2$ where p is a prime number greater than 2.

Statement 3: A brillig number is an integer number b for which there exists an odd prime p such that $b = p + 2$.

Statement 4: All brillig numbers are odd.

Statement 5: To obtain a brillig number, just take any odd prime number and add 2 to it.

(a) Which of the above statements would you pick as a definition of the concept 'brillig number'?

(b) Which one explains best what brillig numbers are?

(c) Is the sum of two brillig numbers a brillig number?

(d) Do you think that 'brillig numbers' is an interesting concept?

Question 4.2

In the set T = {1, 2, 3, 4} we define the operation 'vorpal' (∇) in the following way:

∇	1	2	3	4
1	1	1	4	1
2	2	2	2	4
3	3	3	1	1
4	4	4	4	3

This table reads as follows: vorpal of 1 and 1 is 1, vorpal of 1 and 2 is 1, vorpal of 1 and 3 is 4, vorpal of 3 and 4 is 1, etc.

We write:

$1 \nabla 1 = 1$, $1 \nabla 2 = 1$, $1 \nabla 3 = 4$, $3 \nabla 4 = 1$

(a) Is vorpal a commutative operation?

(b) Does vorpal have a right-hand zero element? (i.e. is there an element z in T such that for any z in T, $x \nabla z = x$?).

(c) Does vorpal have a left-hand zero element?

(d) Try to define a different operation in the set T such that it has both a left-hand zero and a right-hand zero. Can you make these zero elements be distinct, or is this not possible?

Question 5

Question 5.1

<u>Aim</u>: attitude towards scientific truth

Which of the following statements describes best your attitude towards truth?

A. **I am not in a position of knowing the truth or what is truth. I rely on teachers to tell me the truth or on moral authorities to tell me or, even better, show me, what's right and what's wrong.**

B. **Teachers are always more or less right because they have the books to look at and they have passed all these courses before me.**

C. **I know the truth when I feel something's true, because it sort of agrees with my experience. Of course, someone else may feel otherwise; a truth for me need not be a truth for you. I'm having hard time with the premise that truth is scientific knowledge because for me it isn't that at all. For me it's internal knowledge. I don't think we need to ascertain what's right at all. I think we need internal exploration and knowledge of self to know what's right and what is true.**

D. Experts have definite procedures to find the truth. You just apply the prescribed techniques, or methodologies, for example, statistical analysis to a set of data, and you draw conclusions. What you obtain is truth. There are expert procedures even for solving everyday life problems, like child rearing or managing family disputes.

E. Truth is something that is constructed in one's mind and shared among people, not just found in things external to the mind, although objective factors cannot be ignored. In science you don't really want to say that something's true. You realize that you are dealing with a model. Our models are always simpler than the real world. The real world is more complex than anything we can create. We're simplifying everything so that we can work with it, but the thing is really more complex. When you try to describe things, you're leaving the truth because you're simplifying.

Question 5.2

Aim: Probing the student's attitude towards the nature of mathematical knowledge

Which statement, according to you, describes best mathematical knowledge:

A. Mathematical knowledge is a set of rules and formulas for the computation of exact values of some unknown quantities, given the values of some other quantities.

B. Mathematical knowledge is a system of tools for the modeling of the functioning of real life activities such as financial operations (banking, insurance), and for the design of engineering projects and technological devices.

C. Mathematical knowledge is a developing system of abstract theories about the most general aspects of relations between phenomena. Mathematical theories attempt to make their assumptions as clear as possible. Mathematical theories are made of conclusions drawn from these assumptions, rather than from some empirical evidence.

Question 5.3

Aim: Attitude towards the study of mathematics

Which statement best describes the reasons why you are taking mathematics courses?

A. I am taking mathematics courses because I really had no choice. I wanted to enter the profession of .... (specify) and mathematics courses are required for it.

B. I prefer math and sciences to humanities and social sciences. At least in math there are clear criteria whether you are right or wrong. There are absolutes in math and science... [while] work in other courses seems to accomplish nothing. It doesn't really matter whether you are right or wrong because there isn't right or wrong . In math you know what you are going to be evaluated on and how the grade will be computed.

C. I take math because I can do math, I've succeeded in it so far, so it allows me to get a university degree, but I don't think it has anything to do with my life or with truth, for that matter. I don't need math to make decisions in my life. I go by my gut feeling and not by my computations. This is why I think I haven't really learned anything in math. I have just crammed for the exams. The teachers always tell us how math is useful but then they dump on you all this theory and show some 'applications' at the end, but it's all so far fetched that you would never think of using the theory if you were not told that it can be applied. I think the way I've always learned things was through experience and practice. I like to know what's going on, so it's hard for me to explore something on the theory aspect and then go out and get the practical. I like to have the practical first, so I know what's going on, and what it's really like and then look at the theories that way.

D. I am taking math and science because I trust their methods of arriving at truth. Look - the proof of math is in all those things that we, physically weak creatures, can do: fly without having wings, talk to people miles away, walk on the moon. The scientific procedures work, you just have to follow them precisely.

E. I am taking math because it gives you such a lot of space for imagination and creativity. You can actually do math while you are learning it. You start with a little theory and you can immediately go on and solve problems. Indeed, you can invent problems of your own and try to solve them. In social sciences or literature, you have to read pages upon pages of other people's writings and compose long essays analyzing other people's creations before (if ever) you are allowed to create something yourself.

Question 5.4

<u>Aim</u>: Attitude towards mathematical proof

Which sentence describes best your attitude towards proofs in mathematics

A. **Proof is just a type of exercise in mathematics assignments and tests**

B. **Without a proof we wouldn't know if a mathematical statement is true or false.**

C. **Proofs are part of mathematical discourse and style. That's how you write mathematics. Proofs are not necessary to establish the truth of a statement; very often you are convinced just by a few examples.**

D. **Proofs are a way to establish mathematical theorems. A statement without a proof is just a conjecture.**

APPENDIX B

PROTOCOL OF THE INTERVIEW WITH STUDENT N1

Interviewer/Transcriber: ALF = ALFRED NNADOZIE
Observers: AO = ASUMAN OKTAC
AS = ANNA SIERPINSKA

Question 1.1

1. ALF: [presents the first question] Here we have a pile of expressions each written on a separate card. We would expect you to separate them, sort them into groups.
2. [Silence 17 seconds]
3. ALF at least 2 groups
4. [Silence 48 seconds]
5. N1 : Ok, these four (excluding x-13 = x) seem to be like normal equations and this one has no solution. These are the same, there seem to be no solution to this. It's saying that x-13 = x , 0 is −13 ...is no solution.
6. ALF : Interesting. eeehm how about this one ($2x - x^3 + 78$), would you call this one an equation?
7. N1 : [interferes ...well] Yes I will call all of them. It makes sense to me. But this one doesn't.
8. ALF : So, basically you would call eeh, you would believe eeehm, what other reason would you give for grouping these, all these together?
9. N1 : This one you can easily solve, this one is a formula you can use for area, this one can be solved too with a quadratic formula, this one although doesn't have an equality sign is just an ordinary equation, but this one eeem I don't know, not true. It does seem to me to .....

Question 1.2

10. ALF: [presents the second question] What do you think about this definition?
11. [Silence 22 seconds]
12. N1 : Eeehm ... there is something missing ... For the set to be linearly dependent this has to be equal to zero, only if each a_i, a_l, ..., a_n is equal to zero. So something is missing, it is not completely true.
13. ALF : It sounds interesting. And you said only if the coefficients...
14. N1 : (interferes) Each a has to be zero.
15. ALF : Each a_i has to be zero?
16. N1 : That's zero only if each a_i is zero, then the set is linearly dependent. Because all of them are not linked. no linear combination, no one be a linear combination of the other ones.
17. ALF: In the same exam, the same student wrote this [presents question (ii)]

18. N1 : These two [pointing at the vectors] eeem, they are dependent. Ok b_2 is $2b_1$ so then I think they are dependent. So if you give the definition then, let say you give .... these ones right here. They are linearly dependent.

19. ALF: You are saying the set, this set of vectors, is not linearly dependent?

20. N1 : No, it is linearly dependent because 2 fours + 2 fives is 1 two. This one replaces that one so it is .....dependent.

21. ALF: So would you say that this statement [statement of question 1.2 (ii)] follows from the definition, the definition the student gave?

22. N1 : Uuum [silence 4 seconds] – kind of because I mean he didn't add that a_i is zero but if you use his definition you see that 2 times b1 minus 1 times 2 is equal to zero. I mean from what he thinks, is right but he didn't add that ....

Question 2

23. ALF : This is also another problem in the math exam 1997. Here you don't need to solve the problem. But I would like you to tell how you would approach this problem.

24. N1 : [silence 25 seconds – coughs ehee] Ok you want me, I wanna set each individual expression equal to zero. And I will put the coefficients in the matrix, the first matrix, I want the matrix to be eeeem u, v, w equal to 0. And in the first row we have a-b = 0, in the second one we have a + b – c = 0 and so on. And I would solve that and it should show that a, if they were linearly independent, it would show that a = 0, b = 0, c = 0, but it would not so they are linearly dependent.

25. ALF : So in effect, you would go solving and setting as usual, setting coefficients equal to zero?

26. N1 : Interferes – Yes I would put these coefficients, these cofficients from u, v, w in the matrix, am gonna put them (and equate them) to zero ...(column!) and then be sure that the coefficients depend on each other.

27. ALF : And if you get a solution ...

28. N1 : [interferes] – if you get a solution that doesn't say that each coefficient is zero, then you can, if they are all zero, then you say that the are linearly dependent. ....!

29. ALF : But, silence 2 seconds, so you wouldn't have any problem with the vectors u, v, w, you wouldn't have any problem with them being eehm ...

30. AO : [interferes] – let him do it

31. ALF : Oh you would ...

32. ALF : You want to do it [gives N1 paper and pen]

33. AO : Yeah go on and do it. Because you can ....

34. N1 : [silence 25 seconds] – From this you get that a = b, a = c, b = ...c, [another silence 19 seconds]

35. AS : [perhaps opens bottle of fruit juice and prepares some light refreshment]

36. N1 : continues – No this does not make sense. Because I get that b = - c. ...

37. ALF : You are sensing that it wouldn't make sense?

38. N1 : Well it would make sense if c = 0. But here I get that a = b, a = c, so b = c.

39. AL : Uuhm

40. N1 : But in the last one I get, b = -c, so this is not a true ...

41. ALF : So ...

42. N1 : So it sounds like if somethig is wrong.

43. ALF : Eeeh, can you look at the problem a little bit more to see if, may be why you think you are not getting the right answer is because of, may be, anything has anything to do with the question.

44. N1 : silence 6 seconds

45. AO : What is your conclusion from this!

46. N1 : Linear independence – it .... So this .... Silence 32 seconds Ok what if they said show that they are linearly dependent that they must be right. Then it would be

47. AO: But that is enough torture Alfred. - laughs – that's enough torture or no? – laughs

48. ALF : Excuse me

49. N1 : Then I would do something like this, if they are linearly dependent, then the coefficient of this plus the coefficient of this, something like that

50. ALF: Ok, in this problem, there is a typo error in the exam and that is why something is not working out fine. Would it be easy for you, based on this information I am giving you , to figure out what you can do to have things work out fine?

51. N1: So there is an error in the question?

52. ALF: Yes

53. N1: Waaoh !

54. AO: And it really happened in an exam. They gave this question. It was written wrong, So the students had to struggle with it.

55. N1: Linearly independent? From what I got they are independent, so the error would be that the are linearly independent.

56. ALF: No

57. N1: That's not the error?

58. AS: [hums - No it is not like that]

59. ALF: No it is not

60. ALF: If you don't worry about that?

61. N1: You don't worry about that!

62. ALF: Yes

63. N1: [silence 20 seconds] – if that was a minus, actually it would be v-w

64. ALF: v-w ?

65. N1: Yes, you have this, then b = c. If this was minus, then we have, 13 seconds silence - yeah it is ok we have say, minus 1 times v-u plus -w is equal to 2v-w and v would be linearly dependent . Like if you can ... this one from this, you get ... of this. Because you are trying to be linearly dependent. That one of them is linearly dependent on two of them means that they are linearly dependent. „„„

Question 3.1

66. ALF: This one is on graphics. So what do you think about this, does this represent a circle?

67. N1: No, no from this radius this one is .., this one .., it looks like a circle. But not from the graph.

68. ALF: What about this one?

69. N1: It looks like an ellipse but it is a circle. Looking at the graph, the radius is one everywhere. Technically, it is a circle.

70. AS: Technically? –laughs

71. N1: Well, by definition.

72. AS: Yeah yeah I like it.

Question 3.2

73. ALF: Ok. still on graphics. Do you think the following graphs represent linear functions – the green and the red lines?

74. N1: Yeah eehm eeem

75. AO: hums - ....

76. N1: I am looking at the slope.

77. AO: The slope?

78. ALF: Ok, you are looking at the slope?

79. N1: Yeah

80. ALF: Why are you looking at the slope?

81. N1: Well, I am trying to see if the slope keeps changing. It seems to be a straight line so they are linear, but if the slope keeps changing between two points, it is not a linear function.

82. ALF: In which one of them do you think the slope is constant?

83. N1: [silence 17 seconds]

84. AS: Could you speak out loud?

85. N1: Sorry
86. AS: ah aaaaa -laughs
87. N1: If I look here, this is one over two, this is two over four that's a half, here is 4 over sixteen which is a quarter, so the slope changed. So I am saying it is not a linear function.
88. ALF: So the green one is not a linear function?
89. N1: Yeah
90. ALF: The red one?
91. N1: The red one has two over ...
92. AS: Is it six over four?
93. N1: Yeah, here the change in y is six and the change in x is, oh am sorry am is three sorry.
94. ALF: three
95. N1: I just assumed that was zero, fourteen over seven, yeah ok, so the red one seem to have a constant slope of 2 so I would just say that it is linear.
96. ALF: And the green?
97. N1: The green is not.
98. ALF: And the reason is because?
99. N1: interferes – the slope is changing.
100. ALF: Ok.
101. AO: And what is causing that?
102. N1: Eeeh – laughs- the equation, the equation of the eeeeh of this one, ok I don't understand what you are asking. Why is the slope changing?
103. AO: Yeah you said that the look linear.
104. N1: The look linear because it is a straight line but by the way the units are given on the (previous) graphs you think it is a circle but it is not a circle. So the green is not linear. The other one is.

Question 4.1

105. ALF: Now there is this eeh some eeeh, these are true statements about numbers called brillig numbers. You can read these carefully and answer some questions that follow.
106. AO: They are all true
107. N1: Ok. [silence 32 seconds] – Ok
108. ALF: Ok, so which of the above statements would you pick as a definition of the concept brillig numbers?

109.N1: You just told me that all of them are true eh?

110.AO: Yeah

111.ALF: Yes they are all true. All five statements are true. Which one would choose as the definition?

112.N1: The best one of these five?

113.ALF Yes, as a definition.

114.N1: [silence 15 seconds] – ok it seems that 2 and 3 are saying the same thing. It does not look to me like the definition

115.ALF: But which one is a definition?

116.N1: Well am eliminating them.

117.AS: Eliminating the two?

118.N1: May be one. I would just say that from these five it seems to me that brillig numbers are just odd numbers. So let me see number 4 – all brilig numbers are odd numbers.

119.ALF: So you are choosing statement 4 that all brillig numbers are odd as the definition.

120.N1: Yes

121.ALF: What is the reason?

122.N1: Why? Well I can say that am not choosing 1 because it is an example not a definition. Here it says 2 is ``any billig number can be written as p+2 where p is a prime number.`` Eeeh although I know that prime numbers are all odd. these ones are ..., so eem

123.AO: 2 too?

124.N1: Well greater than 2.

125.AO: Ok.

126.N1: Prime numbers greater than 2 are odd ....

127.ALF: But remember these are all true statements. All these statements are true about brillig numbers and

128.N1: [interferes] – Yes Yeeees I know

129.ALF: Eh?

130.N1: Yes I know.

131.ALF: Ok.

132.N1: But here like you are saying, p is a prime number.

133.ALF: And we are interested in which one is a definition, which one will you choose as a definition?

134.N1: Right so here says that - any brillig number can be written as p+2 where p is a prime number greater than 2. and here it says - a brillig number is an integer number b for which there exists an odd prime p such that b = p+2. So for me these are saying the same thing. So I am eliminating one of them right away. Anyway for numbers greater than 2, if you take a prime number, it has to be odd because any even number greater than 2 is not prime. So if you take any odd number and you add 2 to it, it is still an odd number. [Statement] 5 says so basically a brillig number, actually [statement] 4.

135.ALF: So you are choosing 4?

136.N1: - all brillig numbers are odd.

137.ALF: And what problem do you have with #5. Why wouldn't you choose #5?

138.N1: silence 10 seconds - Well if you take eehm here it states that - any brillig number can be written as p+2 where p is a prime number greater than 2. Statement 5 doesn't specify that. So if you take 2 and add 2 to it, you get 4 which is not a prime number, which is even and all these reasons seem to prove the fact that all brillig numbers are odd. So I don't think statement 5 is correct.

139.ALF: So you are choosing statement #4 that – all brillig numbers are odd?

140.N1: Yes

141.ALF: Would you, how about the fact, are all odd numbers brillig numbers?

142.N1: Eeehm – [silence 7 seconds] – just from 4, if you just take 4 it doesn't ,,,, if you look at all true statements, I will say that yes – all brillig numbers are odd. But 4 itself doesn't ....converse is true. But if you take any other one, if you look at all of them together, you ... Statement 3, you just add 2 to any brillig number, it is still odd.

143.AS: So, is 3 odd?

144.N1: Eeehm

145.AS: Is 3 an ordinary, is it a brillig number?

146.N1: [silence – 59 seconds] - uuhm

147.ALF: Did you get the question right?

148.N1: Yeah, I got the question, am just eehm

149.ALF: Is 3 odd, and if it is a brillig number.

150.N1: From what I get, ok from what I get, that one would be ...

151.AS: Laughs

152.ALF: But may I ask you, is 3 an odd number?

153.N1: 3 is odd, I don't think it is brillig.

154.AS: Yeah, it is a brillig.

155.N1: Why is it brillig?

156.AS: Why is not brillig?

157.ALF: Why is not brillig?

158.N1: Well because a brillig number is – [reads out statement #3] So there is no odd prime such that , I was stuck at that one, isn't it? That's right. So the only way that a brillig number could be 3 is 1+2 but 1 is not prime so it is mistaken. [Statement] 3 is false.

159.ALF: Ok, why would you prefer to look at statement #3 to answer this question?

160.N1: Because they are all true, so ...

161.ALF: So you just chose at random or for some particular reason?

162.N1: I am going through others

163.ALF: Why did you stop at #3 to answer that question?

164.N1: well because 3 is 1+2 and 1 isn't a prime so

165.ALF: I mean you found statement #3 more convenient for purposes of answering that question?

166.N1: Yes, what you are getting at is why did I choose 4.

167.ALL laugh

168.ALF: Why?

169.N1: Ok, the thing about, ok, I'm saying all brillig numbers are odd. That's what I'm saying. All odd numbers are brillig, I would like to change my mind about that one.

170.ALF: Ok, I have another question for you. Which one of the five statements explains best what brillig numbers are?

171.N1: [silence – 25 seconds] – ok, definitely not 1.

172.ALF: So you are eliminating #1?

173.N1: Yes, now by looking at [#]5, 2, 3 and 5 to me are saying exactly the same thing.

174.ALF: 2, 3, and 5 are saying exactly the same thing?

175.N1: Yes, so – [silence 5 seconds]

176.ALF: You have eliminated statement #1?

177.N1: Let me still leave #1. Statement #4, am not sure if I want that any more. I will take 4 that – all brillig numbers are odd - because it seems to me that brillig numbers are just odd numbers greater than 3.

178.ALF: So if statement #4 were to be modified,

179.AS: How about eleven? Is eleven a brillig number?

180.N1: Eeeehm you got me here – silence 4 seconds

181.ALF: She got you there?

182.N1: Yeah, statement #4 is too general. Am not, it is too general for me. They are all odd. But 3 and 11 are not brillig numbers

183.ALF: They are not exhaustive of all information you need?

184.N1: laughs – so eeehm from what I have seen of definitions, I would say that 3 seems to be the most formal one ....

185.ALF: So statement #3 explains best what brillig numbers are?

186.N1: Yes.

187.ALF: Ok. Is the sum of two brillig numbers a brillig number?

188.N1: ok, eehm, oh no.

189.ALF: Why?

190.N1: [reasons aloud] 5 + 7 is 12. 12 is obviously not a brillig number. Whatever, from statement #3, a brillig number is a prime that you add 2 to it so 12 will be 10 + 2 that's not fine, no.

191.ALF: So statement #3 is playing a more important role out of this five.

192.N1: Well, no am just using one of them to show

193.ALF: Randomly or,

194.N1: yeah just randomly, well I mean, if you disprove one of them, then

195.AS: In that case you could use #4 which you like so much

196.N1: eehm

197.ALF: In two questions, you have always gone to #3. There must be something particular about that number.

198.[all laugh]

199.N1: Yeah I was just using that formula to make sure that

200.AS: may I ask you a question? What are definitions for you. How do you know that a statement is a definition. Not these ones, in general?

201.N1: In general, my rule is that you don't have to prove a definition. But you have to prove theorems or whatever. But definitions you don't prove them, you just give them.

202.ALF: Do you think that the concept of brillig numbers is an interesting concept?

203.N1: I don't really see it useful. They could be interesting to think about it, to figure out exactly, basically it is not difficult to see what a brillig number is, but if you put all these together ….. It is interesting, it is an interesting concept yes. I don't see the usefulness but I think it is interesting.

204.ALF: Why would you consider it an interesting concept?

205.N1: Ah just because it makes me think about it.

206.AO: It is challenging.

207.N1: Well not that I know what it is.

208.ALF: What other concepts do you consider interesting in mathematics?

209.N1: What other concepts?

210.ALF: Yeah.

211.N1: knocks on the table – 4 seconds silence

212.ALF: Like in the courses you have taken this semester

213.N1 : Well this semester, the first thing that attracts me to mathematics is eeeh just arithmetic calculations, when I was small, tricks to add, multiply and minus, figure out multiples of numbers ... but I don't know, the concepts are like today that interest me most in mathematics are mostly statistical applications, distributions,

214.ALF: probability

215.N1: Yeah probability applications testing hypothesis, this is what I like most.

216.ALF: Why?

217.N1: It is more straight forward. Linear algebra is abstract, you can't even see it but you might see statistics. It seems like when you are applying it you know, I know what I am getting. If I get a result, I can interpret it, where as in Liner Algebra you really can't. What I can calculate is the volume ..... So linear algebra isn't easy.

218.AS: I would like to ask you a question. Which of the following two concepts are you more interested in – congruence of matrices or similarity of matrices?

219.N1: For congruence, the formulas are almost the same, but I guess similarity because I use it more. I saw it more in CEGEP and in the two linear algebra courses whereas congruence I only saw part of it a little while but I guess if I were to say which one I find more interesting, it has to be similarity because I studied it more and I know more about it.

220.AS: Because it is used in more occasions?

221.N1: Yes. I used it in all three classes.

222.AS: Thanks.

223.ALF: Let's go to some other interesting question.

Question 4.2

224.ALF: In the set T = {1,2,3,4} we define the operation Vorpal in the following way – [shows the table to N1]

225.N1: Ok

226.AS: You have to explain how you read the table.

227.ALF: This how it reads. We say vorpal of 1 and 1 is 1, vorpal of 1 and 2 is again 1, and the vorpal of 1and 3 is 4 and so on. And we write,

228.N1: So

229.AO: Just to make sure, what is vorpal of 3and 3?

230.N1: One.

231.ALF: Ok. Now would you consider vorpal a commutative operation?

232.N1: Eeehm [silence – 6seconds] eeeh no. You see vorpal of 1 and 3 is 4 and vorpal of 3 and 1 is 3, so it is not commutative.

233.ALF: You are saying vorpal of 1 and 1 is?

234.N1: No, no. I am saying that the vorpal operator or whatever am saying is not commutative. Because vorpal of 1 and 3 is 4 and vorpal of 3 and 1 is 3 so they are not the same so it is commutative.

235.ALF: Now does vorpal have a right-hand zero element? You need to understand what we mean by right-hand zero element. It means that if there is an element z in the set such that for any x in the set T, x vorpal of z is equal to x.

236.N1: [silence- 10 seconds] Yeah I suppose that 1 vorpal 1 is one,

237.AO: It has to work for every element.

238.N1: Oh, for every element?

239.AO: Yeah.

240.N1: [silence - 8 seconds] No it doesn't seem so. ....take eeh, ok yes, yes, yes, 1 and 2. Let's start with this one: 1 vorpal 1 is 1, 2 vorpal 1 is 2, 3 vorpal 1 is 3, 4 vorpal 1 is 4. Same thing for z = 2. So for z equals to 1 and 2 then, it does have right-hand zero element.

241.ALF: Ah ok. Does vorpal have a left-hand zero element?

242.N1: [silence – 16 seconds] Eeehm no.

243.ALF: Why?

244.N1: What if you did I guess may be rows 1,2,3,4 ......??

245.ALF: Can you try to define a different operation in the set T such that it has both a left-hand zero and a right-hand zero element?

246.AS: That's one question there. And the second is? There are two questions in one.

247.ALF: Yeah the second question is if you can make these zero elements distinct, i.e. whether it is possible or not possible.

248.N1: Looking at them all, the first .....

249.AS: [serves some refreshment] We get some drink, cookies here.

250.AO: We have got something to eat.

251.N1: Uh thanks. Um... I have a question... like, okay, this vorpal thing, does it have something like an arithmetic operation in it that gives the statement?

252.AO & AS: No, You don't have to, you don't need that.

253.N1: Yeah but am saying it has that right?

254.AS: No

255.N1: No? Like the table is just written down?

256.AS & AO: Yeah correct.

257.N1: So then I can just write down my own table?

258.AS: And you can give it your own symbol. There was a girl here who wrote a little heart for the symbol of love.

259.N1: Hmmm yeah

260.AS: You can write a question mark – [laughs.]

261.AO: Dollar sign.

262.N1: Oh yeah, what I will use is Batman.

263.[Silence - 27 seconds] – Can I use your pencil? [silence – 3minutes 27 seconds]

264.ALF: You have some idea? You want something?

265.N1: I have to use these same numbers right?

266.ALF: Uuuhm.

267.N1: Here I just have to have one right. It doesn't have to be all right-handed all the time eehm?

268.ALF: Yeah you have to have...

269.N1: And I just want one of the elements here

270.ALF: what do you mean by one of the elements here?

271.AO: But there are two parts to the question. In the first one,

272.N1: Ok, here it says try to find an operation in the set such that it has both left-hand zero and right-hand zero.

273.AS: Because this one had a right-hand zero .....

274.N1: Ok so you don't have to change any row, if I put 2,3,4 here, then this is

275.AS: And what is this, zero?

276.N1: x eeh, ok, x vorpal zero has to be x, so 1 vorpal 1 is 1, 2 vorpal 1 is 2, 3 vorpal 1 is 1, 4 vorpal 1 is 4 so that eeh, right-hand?

277.ALF: Uhmm

278.N1: And then z vorpal x : 1 vorpal 1 is 1, 2 vorpal 1 is 2, 3 vorpal 1 is 3, 4 vorpal 1 is 4 so this is left-hand. So

279.AO: What is the right-hand zero and what is the left-hand zero?

280.N1: This would be the left.

281.AO: So you are showing the column?

282.N1: Yes, the column is the left and the row is the right hand.

283.ALF: But here, when you had it this way, you had it as right-hand or left-hand?

284.N1: Here the left-hand was a column, the column ..., that was a left-hand zero.

285.AS: And what was the z?

286.N1: The z or the z is fixed at 1.

287.AS: Yeah and in your situation?

288.N1: It is the same thing. It is fixed at one. And then x vorpal z for 1, 2,3,4 you get 1,2,3,4 again. And the other way round. z is fixed at one here, so z vorpal x , you get x.

289.ALF: And inside here?

290.N1: Well it doesn't really matter because you just wanna ...

291.AS: The second question

292.ALF: Can you make these zero elements distinct?

293.AS: Here you have left-hand and right-hand are both 1. The number 1 is both left and right hand to it.

294.N1: Can I make this bigger please?

295.AO: What is this, Batman?

296.N1: Yes.

297.AS: Ok, Batman, you have to do it, 1 Batman 1 is 1.

298.N1: Eeehm ok - [silence 15 seconds] – I don't think it is possible because to have right-hand zero you got your row has to start at 1, yeah your row has to start at 1 and then has to go 1,2,3,4 and the only way that this could happen is in the first row but if you put in the first row, then your set would be the same. Even if you shift this to the second row, then you have to have a 2 in the second element of the, I am sorry, if you shift this to the second column, you need a 2 in second element of this row so you can only have them here. Basically the elements on all of them have to be the same. So I don't think it is possible to have distinct zero elements.

Question 5.1

299.ALF: [presents question] I would like you to read out this loud.

300.AS: And the question is, read the question.

301.N1: Which of the following statements describes best your attitude towards truth? Ok. I read this aloud too.

302.ALF: Yeah, so the question is as you read it out. ....

303.N1: Ok.

304.ALF: Now there are about A,B,C,D,E of them and I would like you to read out A first and then comment.

305.AS: Comment, comment and then you go to the second and then at the end you tell us which one you like best.

306.N1: Ok, [reads out statement A] Eeeee, well I guess I can't really rely on the teacher to tell me - are we speaking like moral truth or mathematics truth or any truth?

307.AO: So you are making a distinct between those truths?

308.N1: Oh yes because I feel in class a teacher could give you a theorem or whatever. Like he can give you a theorem completely wrong and you won't know. You just kind of think that he is teaching you something right. So,

309.ALF: But if in the church the priest gives you a theorem?

310.N1: Yes I believe it because it is my faith. But eeehm – I guess it can really make sense but I would require a form of explanation, like in class, if you tell me something, then I would expect the teacher to prove it to show me that it is the truth. The same thing in a church, I would expect the priest to explain and demonstrate what could be the truth and the wrong. But I don't have any problem with A. Because I do rely on my teachers or priest to tell me what is true or not.

311.ALF: And B?

312.N1: [reads out B] Yeah I have thought that once or twice.

313.ALF: You have thought that this statement is true?

314.N1: Yeah before - Teachers always seem to know a lot. But I guess I know ahead before class they have to practice it a billion times so anything that you see for the first time is confusing when they are teaching you, you just know it but eehm yeah B is, I agree with B.

315.ALF: What about C?

316.[all laugh]

317.AO: That's a long one.

318.ALF: Yeah it is a little longer.

319.N1: [reads out C] Eeehm ok, kind of, the way I think is funny a bit. If you're talking about science and religion, I practice both. So for me, in science, what's true for me has to be true for you but it is proven like I don't know.

320.AS: [interferes] Objective.

321.N1: Speed of light, 3 times 10 to the eight, ... Ok, but eeehm religiously, I mean for me as a catholic, I believe in God, blablabla so I mean there other types of religion. What is true for me doesn't have to be true for you because you can believe whatever you want. Am not saying it is wrong. It may be wrong but you can think whatever you want and I will think whatever I want. So scientifically we should all arrive at the same truth. Religiously, if we were in a perfect world, we would have the same truth but we don't but I don't think it is wrong either.

322.ALF: Let's move on to D.

323.N1: [reads D] – what's child rearing?

324.ALF, AS & AO: bringing up children.

325.N1: Ok, I agree with first part with the statistical analysis, methods and procedures to a set of data and you obtain your conclusions which is right assuming you use the correct procedures. But they are experts so they must know. For everyday life problems I don't think there is one expert procedure on how to raise your child or manage your family disputes. I think it is like the religion thing some people do one way, some people do the other. It doesn't mean that one is right or one is wrong. But there should be distinct between science and morals. So I agree with the first part but I don't agree that there are expert procedures for solving problems like child raising and family disputes.

326.ALF: So you do agree to the fact that experts have definite procedures to find the truth?

327.N1: Yes, scientifically I think so, follow the steps, apply that and you will find it work. But for every day problems in society or whatever, I don't think so.

328.ALF: Let's go to E.

329.N1: [reads E] Ok, I think it is kind of right. It just depends on what you are talking about in science. I think eehm, I think if you look at science like Physics or calculus, there is more of that. There is a lot of estimating, but I think if you study physics, you pretty well, you're pretty precise to what should be happening in physical world. From all that I have studied but if you go like in statistics, you are making lots of assumptions, so you are not really representing the real world but like I agree that the world is very complex that you can't simulate it perfectly. But in some fields of science, you find the reason it works much better described in reasoning.

330.ALF: So, actually you are saying that, in general, truth is what is constructed in one's mind and shared among people?

331.N1: Am sorry what am I saying? Could you repeat what you said?

332.ALF: Am actually reading out the statement.

333.N1: [reads statement again] Aaah yeah I read that, although objective truth cannot be ignored. Yeah you can make your own truth and share it with people. But I mean some things are true like you know they say the speed of light that's true you can't make out something different.

334.AS: How about Newton's mechanics and relativity theory? Which one is true?

335.N1: Eeehm Newton's mechanics and what ? relativity? I think they are looking at it from different perspectives. .....a particle .....

336.AS: Can you really say that Newton's theory is true or false? Can you say that?

337.N1: Well, they are not sure. One works better in some situations, the other works better in other situations. So scientists use the one that works better lets say I have forgotten I think Newton's mechanics doesn't work at light speed or something like that and relativity is just, what I mean is one is true for something so in a sense you can use that and once you get past that you have to use a different theory. I think they both say the truth.

338.AS: Yeah

339.ALF: Would think that there is a contradiction between D and E?

340.N1: Kind of. D says that experts have definite procedures to find the truth and E is saying that in science you don't really wanna say that something is true. Eeeehm me personally, I think that scientists – if I were a scientist I would try to push to find the exact truth and (justice). Personally for me there is a big distinction between scientific truth and moral truth but eeehm

341.ALF: I hope you did get the question right.

342.N1: Yes I understand.

343.ALF: That is, if there is some contradiction in what statement D is saying compared to what E is saying.

344.N1: D is saying that experts, I see it D is saying that experts or scientists try to find the truth and they apply it, the procedures, and they find the truth and E is saying that they don't really want to say that something is true. So they are just modeling. So they are estimating what is true. So there is distinction for that. I don't really agree that they are modeling.

345.ALF: You don't agree that they are modeling, scientists are modeling?

346.N1: Well not really, I don't know. I do a lot of physics and it's kind of what you are saying ....something. Like if physics wasn't like, Physics is modeled, I guess it is modeled to a certain extent. I see it is really precise. Because if physics wasn't that really precise, then a lot of things wouldn't work. So I don't really think it is really all modeled. There is some contradiction between Newton's theory and Einstein's theory but they both describe pretty much exactly what is true in one situation and in another.

347.[all laugh]

Question 5.2

348.ALF: [presents statements] Which of these statements according to you describes best mathematical knowledge.

349.N1: Ok, [reads statements] From what I have studied, the best definition would be A. Because here it says you have this and this by that you should be able to learn that eeeh .... I would say is more like eeeh, I think it is more something like that. Something like C, but you assume something and then try proving it and get a conclusion out of it because really it has to be since it is mathematics you just want to obtain this. For me, mathematics is a language so like you have, - C you can't explain things like how much ... it works. You have here inside the definitions eehm how it works and how you present and you cannot say it is a language. The way to learn mathematics is A, and the application of mathematics is B. I mean they are, I mean my best description would beee , would be C.

Question 5.3

350.ALF: Now, we go to another A,B,C,D,E question.

351.AS: Here the sentences are long. You have to read one by one. So you get to know them well.

352.ALF: This is about your reason for taking mathematics courses. Which statement best describes the reasons why you are taking mathematics courses.

353.N1: [reads statement A] .... actuarial mathematics ...No, no that's not me. I wanted to take, I had no choice? No, that's not why, I wanted to take mathematics courses that's why I took it but eeeh that is not for me. I don't, A kind of says that eeeh I want stay away from math but my program required so I took it. So personally, me I don't think so. I am not A.

354.N1: [reads B] Here pretty much. It's like me. Eeeh not that eeh that eeh, from the humanities courses like philosophy or history I have taken, I mean I don't mind listening, I enjoy learning it, I just read without applying it but in maths and science, you know you do problem, you know you are going to the same problem, you know you are right or wrong. So now I guess that this one is the one that pretty much best describes it.

355.ALF: Ok. Let's see C.

356.[all laugh]

357.ALF: C is always longer than others.

358.N1: [reads C] Now I think you need to know how it works before you can apply it. Figure out how it works and then apply it. You can't apply it and then figure out how it works. Like if you wanna tow a car. You don't tow a car unless you know how it works. If you don't know how a car works, you can't put it together. So eeehm I mean from the beginning, I guess I do math because I know how to do it but eeeh eventually I think eventually you can use it, I mean if you gonna teach math which I don't mind it's done with (reasoning?) mind because eeeh it is only teaching ....as a historian. We entering the .... area. [All laugh] But eeeh like the math I am really in Actuarial science is the math I wanna use every day of my life so I need to know how it works before I use it. Am not gonna go into an insurance company you know, make them loose millions of dollars, so C is wrong.

359.ALF: ok.

360.N1: [reads D] D is right but it is not why I am, personally, taking math courses. I am taking math courses like get my CV to work in my field but eeehm. It makes sense but that's not me.

361.AS: We should have a video camera because your face tells a lot more than what you are saying. ... The way you raise your left brow and ...

362.[all laugh]

363.N1: Ok E. [reads E] Yeah. well it's kind of true. Eeeehm ok but taking math does expand your forms of .... I mean if you read a lot you tend to find a way of, how to go about solving something. .... You have a problem you can figure out how to solve it , but eeeh, I mean I don't think in social science like in literature, you are restricted from expressing yourself, I think they actually allow you to express yourself but personally I think your reward comes only when you are dead. [All laugh]. E agrees with what I think . It allows me to expand myself in thinking but eehm my best description would be B, which is in science you know either you are right or wrong. You are not in the middle, you are not abstract.

364.ALF: So that is what describes best the reason why you are taking mathematics.

365.N1: Yeah

Question 5.4

366.ALF: Lastly, we have these [presents the statements of question 5.4 to N1].

367.N1: [reads the question – which statement best describes your attitude towards proofs - and laughs]

368.AS: Why are you laughing?

369.N1: I hate proofs. Everybody hates proofs. [reads statements A, B, and C] Eeeehm , ok eee, my attitude towards proofs would be A. The reason I take the time to learn and understand a proof is because I know it's gonna be in the assignement or test but I know that when you learn a theorem, you have to realize why it is true, you have to convince yourself it is true. But when you use it, you just use the result anyway in problem solving. You just use the result, so why I learn the proof is A because I know it's gonna be in the test. So I learn it for that reason. But eeeh, B, C, and D are better reasons but there are two stories to proofs. But like I said first I just learn them for the tests.

370.ALF: Generally you would consider B, subscribe to D and C, but personally, that is, objectively you would subscribe to B and C but subjectively you think A describes best your attitude?

371.N1: Yeah I just learn it because I 'm gonna be tested on it. I wanna get a good grade, so I take time to read it. But I know that proofs are to convince ourselves, to show that it is true and I mean once you see other proofs a lot of them tend to come out the same way so you get more ideas showing a proof or whatever, but personally I just learn them because I have to.

372.AO: Do you think that once you finish school, for example, working you would need to look at proofs or results?

373.N1: Eeeehm I don't think so. Well, it all depends, if you gonna teach, you get more proofs. It all depends on what you gonna do in life, I mean, I don't think an actuary sits down and proves something. I think he just knows what to use and uses it.

374.AS: Let me know something. It could be that you have a situation in the consulting company for which nothing of what you already know actually fits. So you have to figure out a modification of a model or a completely new model. You have a very specific very strange situation. How would you make sure that this makes sense, that there is no contradiction in your model.

375.N1: Well am assuming that the mathematical model without ... I think I am not able to do it.

376.But to show that the model makes sense would just be eeeh testing the model and see if it is, verifying that

377.AO: See if it does like that

378.N1: Yeah, OH NO NO, just first discussing it like try to see if it makes sense. Because mathematics works like aaah the theorems are .... But eeeh, if I have to come up with a new model, I mean, I would have to face the fact.

379.AS: How do you face the fact?

380.N1: Well I mean in actuary science, basically everything is in averages. I mean a life table says probably that you guys die in a year or whatever it means doesn't mean you gonna be exactly sure of the number of people that are gonna die when they are supposed to die. It's just an average. There's no exact formula that says some many people gonna die, it's all just an average. So if you have to create your own model, you would test it to see if it fits the population. I don't think you use it right away. You compare it to your population for a couple of years and then if you apply it fits. If it is similar to the way you know the population behaves then you can use it. Like the proofs, I don't think I need to know. There are some basic proofs you need to know but I mean

381.AS: I think you have a very specific, special notion of proof. Like if it is something a proof that is written in a text is written in the book which follows eeeh a theorem. But how would you call the activity, your own activity, in testing your model, by looking at the assumptions, suppose at one point in your model you assume that the person is going to die in one year. And in the second place, you use the formula which assumes that people are going to be dying not earlier than five years. If you notice the contradiction, isn't that something that could be called part of an activity of proofing? You don't consider that as part of proving, finding logical connections between things, deducing what follows from this assumption. That is not proving for you?

382.N1: Eeehm no, it is more like verifying. A proof to me is, am saying this and I write something down and it gives me something and that's why it is that. But if you are modeling, you can't exactly model the world. So I don't really think that any model wants a proof of that's the way people, population works the way the evolve, when they die or when they live. But to me a proof exactly explains what a sentence says.

383.AS: It is an explanation.

384.N1: Yeah, it is exactly.

385.ALF: Your were objectively subscribing to D and C and subjectively preferring A because it best describes your attitude sincerely. What about D?

386.N1: I agree, I agree, B, C, and D they are kind of like inside I know I should learn proofs because of B, C, and D. But I learn them because of A because I know am gonna be tested on that.

387.ALF: Does there seem to be any contradiction between C and D?

388.N1: C and D [silence – 6seconds] There is a contradiction because eeeh D is saying if you don't have a proof, then it is just like you said a contradiction but C [reads C again]. There is a little bit of difference between C and D. But C also says that proofs are required for mathematical discourse and style. That's the way you write mathematics. I think that's right but people are convinced by just a few examples I mean that's most of the population, but if you really like doing math, then you shouldn't think like that. You should get the proof done, and understand. So there is a distinction between C and D but I think both of them are, I don't think so, people should have proofs.

389.ALF: Ok. Interesting. That's the end. Thanks so much. mille grazie. Grazie mille!

390.N1: Prego

391.[all laugh]

392.AS: What did you get in the first Linear Algebra course?

393.N1: An A.

394.AS: Did you find it a hard course? Which one was harder?

395.N1: Well I guess the second one. Like linear algebra, understanding it, to me it is almost impossible, but applying it is not a problem. But eeem I think I found the theorems in the second one harder. Because the theorem in the first one was kind of just finding ....??? So like in Cegep there wasn't much .....????

396.AS: You had to learn a little bit and spend some time on it.

397.N1: Yeah, linear functions eeem

398.AS: linear transformations?

399.N1: Yeah, am still confused by linear transformations but it is very abstract, I find it is not difficult proving things, I try to visualize things and it gets confusing.

400.AS: You like to visualize things? Like in calculus, see volumes

401.N1: yeah, tangents , in algebra, I don't know it is really ..

402.AS: Thanks so much.

$u - v \qquad v - w \qquad v + w$

$$\begin{matrix} a & -b & 0 & 0 \\ a & & -c & 0 \\ & b & -c & 0 \end{matrix}$$

$a = b = c$
$a = c$
$b = -c$

} if $b = 0 = c$
$a = 0$

$u - v \qquad u - w \qquad v - w$

$$-1(u - v) + (u - w) = v - w$$

[N1: Notes in Question 2]

[illegible]	1	2	3	4
1	1			
2	2			
3	3			
4	4			

$2 \bigtriangledown x = x$

$\checkmark \bigtriangledown 2 = \checkmark$

[N1: Notes in Question 4.2]

APPENDIX C

Postulated defining features of theoretical thinking

1. *TT is reflective*, i.e.,

    TT1.1 self-serving,

    TT1.2 self-referential, and

    TT1.3 voluntary

2. *TT is analytic*, i.e., it is

    TT2.1 mediated through language, which is an object of both reflection and invention;

    it is aware of

    TT2.2 the conventional/symbolic character of language in general, and mathematical notations and graphical representations in particular, and of

    TT2.3 the possibility of inventing/designing an artificial language.

    Theoretical thinking is also sensitive to

    TT2.4 syntax and mathematical syntax in particular, and especially to the quantification of variables, and to

    TT2.5 the logical rules of drawing conclusions and negating statements.

3. *TT is systemic*, i.e., it is

    TT3.1 relational, and it features

    TT3.2 a definitional approach to meanings, as well as

    TT3.3 a systemic approach to validation.

    Moreover, it

    TT3.4 uses systemic categorization (or 'formal categorization' in the sense of Bruner, Goodnow & Austin, 1960, p. 5-6).

4. *TT is hypothetical*, i.e. it is

    TT4.1 aware of the conditional character of mathematical statements;

    TT4.2 concerned not only with the plausible or the realistic but also with the hypothetically possible, and it

    TT4.3 believes in the relativity of truth.

5. *TT is concerned with validation*, i.e., it is

    TT5.1 fuelled by doubt and uncertainty and hence considers validation as an important problem. and it

    TT5.2 considers proofs in mathematics as necessary for the establishment of knowledge.

6. *TT has a critical attitude towards standard procedures,* i.e., it considers procedures and their underlying concepts problematic, it does not take them for granted and does not accept them just because they have a stamp of authority.

Postulated features of theoretical behavior corresponding to the above model of theoretical thinking

1. Reflective thinking

1.1 The self-serving character of theoretical thinking

1.1a. A researcher's attitude towards mathematics

Is the student carried away by a problem and does more than just solve it? Does he or she formulate interesting conjectures or questions that go beyond what is required in the problem?

1.1b An appreciation of the intrinsic significance of mathematical concepts

Does the student appreciate the intrinsic significance of mathematical concepts: the judgment of their significance on the basis of the mathematical questions that can be asked about them and not only their extra-mathematical applications or value (e.g. for 'the development of logical thinking').

1.2 The self-referential character of theoretical thinking

There are two aspects of this feature. A first aspect is the concern for the global internal consistency of a system of concepts. Another one is the tendency to derive meaning of concepts from definitions rather than from analogies with some extraneous experience. This tendency to definitional meaning can be considered as part of systemic thinking and treated under this heading.

1.3 The voluntary character of theoretical thinking

We assessed this feature under the aspect of conscious reflection on knowledge and knowing.

1.3a *Reflective approach to the study of mathematics*

Has the student developed opinions about knowledge and knowing?

2. Analytic thinking

2.1 Theoretical thinking is mediated through language which is an object of reflection and invention

We assumed that the following behaviors 2.1a and 2.1b represented this feature.

2.1a *Analytical sensitivity to mathematical terminology*

Is the student articulate and does he or she use appropriate terminology?

2.1b *Analytical sensitivity to the meaning of words*

Is the student picky in using words and does he or she make careful distinctions concerning their meanings?

2.2 Theoretical thinking is aware of the conventional/symbolic character of language in general, and of the mathematical notations and graphical representations, in particular

2.2a *Analytic-representational approach to variables in algebra*

Does the student interpret letters in formulas as representing variables whose domains are conventionally defined (TT) or does he or she interpret letters as names for fixed domains of objects (PT); for example, in the context of the Pythagorean theorem, 'a' and 'b' would be interpreted as the sides of the right angle, and 'c' as the hypotenuse, even if the equation is written as '$b^2 + c^2 = a^2$'.

2.2b *Analytic-representational approach to graphs*

We assumed that the theoretically thinking student analyzes the points on a graph of a function as representing a relationship between variables, unlike the practically thinking student who recognizes a graph through its shape. We also assumed that a theoretically thinking student distinguishes between the 'background' of a graph (the coordinate systems, the scales) and the graph itself, unlike the practically thinking student who would view straight lines synthetically as shapes (with the axes being 'part of the picture' without any systematic meaning for the graph).

2.3 Theoretical thinking is aware of the possibility of inventing an artificial language

2.3a *Analytic approach to unfamiliar representations*

We assumed that if a familiar symbol is used in a new meaning defined by a formal definition, the theoretically thinking student disregards the familiar meaning and reasons from the definition.

2.4 Theoretical thinking is sensitive to the syntax of language and to quantifiers, in particular

2.4a *Analytic-rigorous approach to mathematical expressions*

Does the student attend to the form of a mathematical expression, or does he read into it what is not really there (but might be, in a school exercise situation)? We assumed that a theoretically thinking student takes an algebraic expression literally for what it is and not for what it might be in a particular context; for example - does not claim that $x^2 - 1$ is an equation because it might appear in an equation on a test.

2.4b *Analytic-logical sensitivity to quantifiers*

Does the student have a sense of the quantification of variables in mathematical statements?

We assumed that a theoretically thinking student notices the universal and existential quantification of variables in mathematical statements and its order and applies them correctly.

2.4c *Analytic-logical sensitivity to the form of definitions*

Does the student distinguish between a definition and an explanation?

We assumed that a theoretically thinking student has a sense that a definition gives a name to an object, and distinguishes between a definition and an explanation.

2.5 Theoretical thinking is sensitive to the logical rules of drawing conclusions and negating statements

2.5a *Analytic-logical sensitivity to logical connectives*

We assumed that a theoretically thinking student has a sense that 'and', 'or', and 'if...then' have a special meaning in mathematics and negates them correctly.

2.5b Analytic-logical sensitivity to implications

Does the student distinguish between implication and equivalence?

We assumed that a theoretically thinking student distinguishes between implication and equivalence.

3. Systemic thinking

3.1 Theoretical thinking is relational

We asked the following question about the students' behavior with respect to this feature:

In discussing an issue, is the student talking about relations between matters or always about his personal feelings about matters?

3.1a Relational thinking

We assumed that a theoretically thinking student talks mostly about relations between meanings and matters rather than about his or her own emotional or factual relations to meanings and matters.

3.2 Theoretical thinking has a definitional approach to meanings

We asked the following questions about the students' behavior with respect to this feature:

Does the student refer to a definition to decide whether a given object belongs to a given category... in algebraic contexts (3.2a), ... in graphical contexts (3.2b)?

3.2a Systemic-definitional approach to meanings in algebraic contexts

We assumed that a theoretically thinking student refers to a definition in deciding upon the meaning of a term in an algebraic context not involving graphical representations.

3.2b Systemic-definitional approach in graphical contexts

We assumed that in categorizing a graph as representing a function, a theoretically thinking student refers to definitions and other equivalent characterizations of functions and not only to the shape of the graph (e.g. 'the function is linear because it looks linear').

3.3 Theoretical thinking features a systemic approach to validation

We looked at the students' arguments when they were justifying their statements. The question of whether they referred to definitions in their argument was already addressed in the 'systemic-definitional approach' features above. Here, we only asked the following questions:

Were the students trying to find a logical contradiction in refuting a general statement? (3.3a)

Were they proposing systematic-analytic (see below) arguments in justifying the uniqueness of an object? (3.3b)

3.3a Systemic reasoning in refutation of a general statement

We assumed that a theoretically thinking student uses some form of contradiction to disprove a general statement.

3.3b Systemic reasoning in proving the uniqueness of an object

We assumed that a theoretically thinking student uses a systematic-analytic argument to prove that two objects satisfying certain properties must be the same object: using variables to name the objects, stating the properties in the form of equations and concluding with an equality of these variables.

3.4 Theoretical thinking uses systemic categorization

We asked:

Does the student categorize given mathematical expressions according to a single feature and is this feature mathematical? (3.4a)

3.4a Formal categorization

We assumed that a theoretically thinking student categorizes a set of algebraic expressions according to a single feature, which appears relevant from the point of view of a system of concepts. A practically thinking student puts the expressions into groups changing the reasons from one group to another.

4. Hypothetical thinking

We asked,

4.1a Hypothetical approach to mathematical statements

Is the student aware that mathematical reasoning does not establish 'facts' but implications?
We assumed that a theoretically thinking student interprets mathematical statements as conditional and tries to identify their assumptions and conclusions; spontaneously engages in hypothetical reasoning, formulating his or her claims in a conditional form.

4.1b Hypothetical approach in generalization

Is the student aware that generalization from a few particular cases produces a hypothesis but not a true mathematical statement? We assumed that a theoretically thinking student is cautious in extrapolating from an observation of a few particular cases.

4.2a Concern with the hypothetically possible cases

Is the students concerned not only with the plausible or realistic but with also with the hypothetically possible?

4.3a Belief in the relativity of truth

Is the student convinced that mathematical and scientific knowledge produces conditional statements rather than facts?
We assumed that a theoretically thinking student expresses belief in the relativity of scientific truth, i.e., its dependence on assumptions. A practically thinking student may believe that truth is in empirical facts, in (universal) logic, in what everybody agrees on, intuitive revelations.

5. Concern with validation

We postulated that theoretical thinking is fuelled by doubt and uncertainty and hence considers validation as an important problem.
We asked,
Is the student concerned with the validation of his claims? Does he or she aim at obtaining a satisfactory proof? (5.1a)
Does the student believe that proofs are necessary in mathematics? (5.2a)

5.1a Enacted belief in the necessity of proof in mathematics

We assumed that a theoretically thinking student aims at obtaining a valid mathematical proof of a claim he or she is not sure of.

5.2a *Declared belief in the necessity of proofs in mathematics*

We assumed that a theoretically thinking student declares the belief in the necessity of proofs in mathematics, both for the experts and the students.

6. Critical attitude towards procedures

Has the student a critical attitude towards procedures (a) in action (6a), (b) in declaration (6b)?

6a *Declared problematizing approach to the validity of scientific procedures*

We assumed that a theoretically thinking student declares his or her belief in the limited applicability and validity of scientific procedures.

6b. Enacted problematizing attitude to the validity of scientific procedures

We assumed that a theoretically thinking student would not take the standard ways of solving problems proposed in a course for granted but would look for their justification and limitations of applicability.

APPENDIX D

Question	Classification						Lin. indep. definition			
Feature	2.2a repr var		2.4a rigor exp		3.4a categorization		2.2a repr var		4.1a hypo statements	
TT-PT	TT	PT	TT	PT	TT	PT	TT	PT	TT	PT
O1	0	1	0	1	0	1	0	1	0	1
O2	0	1	1	0	0	1	0	1	0	1
O3	1	0	1	0	1	0	1	0	1	0
O4	1	0	0	1	1	0	1	0	0	1
V1	1	0	0	1	1	1	1	0	1	0
V2	1	0	1	0	0	1	1	0	0	1
V3	1	0	0	1	1	0	1	0	1	0
V4	1	0	1	0	1	0	1	0	0	1
S1	0	1	0	1	1	1	0	1	1	0
S2	1	0	0	1	1	0	1	0	0	1
S3	0	1	1	0	1	0	0	1	0	1
S4	1	0	1	0	1	0	1	0	1	0
H1	1	0	0	1	1	1	1	0	1	0
H2	1	0	0	1	1	0	1	0	1	0

Lin. dep. typo

2.1 termino		2.4b quantif		2.5a connect		3.2 def mean		4.1a hypo statements	
TT	PT	TT	PT	TT	PT	TT	PT	TT	PT
1	0	1	0	1	0	1	0	0	1
0	1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1	0
0	1	0	1	0	1	1	1	1	1
1	0	0	1	0	1	1	0	1	1
1	0	1	0	1	0	1	0	1	1
0	1	1	0	1	0	1	0	1	1
0	1	0	1	0	1	1	1	0	1
1	0	1	0	1	0	1	0	1	0
0	1	1	0	1	0	1	0	1	1
0	1	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	0
0	1	1	0	1	0	1	0	0	1
0	1	1	0	1	0	1	0	1	1

Log-log scales									
1.1a researcher's		2.2b repre gr		2.3a unfam relx		3.2b def graph		4.1b hypo generaliz	
TT	PT	TT	PT	TT	PT	TT	PT	TT	PT
0	1	1	1	1	0	1	1	0	1
1	0	1	1	1	0	1	1	0	1
1	0	1	1	1	0	1	1	1	0
0	1	1	1	1	0	1	1	1	0
0	1	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	0	1
1	0	1	0	1	0	1	0	1	0
0	1	1	1	1	0	1	1	0	1
0	1	1	0	1	0	1	0	1	0
0	1	1	0	1	0	1	1	1	0
0	1	1	1	1	0	1	0	1	0
0	1	1	1	0	1	1	0	0	1
0	1	1	0	0	1	1	1	1	0
0	1	1	0	1	0	1	0	0	1

Brillig

1.1b Intr sig		2.4c form def		2.5b implic		3.3a refutation	
TT	PT	TT	PT	TT	PT	TT	PT
0	1	1	0	1	0	1	0
1	1	0	1	1	1	1	0
1	1	1	0	1	0	1	0
1	1	0	1	1	1	1	0
1	1	1	0	1	1	1	0
1	0	1	0	1	0	1	0
1	0	1	0	1	1	1	0
1	1	1	0	1	1	1	0
1	1	1	0	1	0	1	0
1	1	0	1	1	1	1	0
1	1	1	0	1	1	1	0
1	1	1	1	1	1	1	0
0	1	0	1	0	1	1	0
1	1	1	0	1	1	1	0

Vorpal											
2.3a unfamiliar repr		2.4b quantifiers		2.5a connectives		3.2a def algebra		3.3b uniqueness		5.1a enacted nec proof	
TT	PT	TT	PT	TT	PT	TT	PT	TT	PT	TT	PT
1	1	1	1	1	1	0	1	0	1	0	1
1	0	1	1	1	1	1	0	1	1	1	0
1	0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	0	1	0	1
1	0	1	0	1	0	1	0	1	1	1	0
1	0	1	1	1	1	1	0	1	0	1	0
1	0	1	1	1	1	1	0	0	1	0	1
1	0	1	0	1	0	1	0	0	1	0	1
1	0	1	0	1	0	1	0	1	1	1	0
1	0	1	1	1	1	1	0	0	1	0	1
1	0	1	0	1	0	1	0	0	1	1	0
1	0	1	1	1	1	1	0	0	1	0	1
1	0	1	1	1	1	1	0	1	1	1	0
1	0	1	0	1	0	1	0	1	1	1	0

Beliefs											
1.3a reflective		2.1b meanings		3.1a Relational		4.3a relativity truth		5.2a decl neces proofs		6a Decl crit proc	
TT	PT	TT	PT	TT	PT	TT	PT	TT	PT	TT	PT
1	0	1	0	0	1	0	1	0	1	0	1
1	0	1	0	0	1	0	1	1	1	0	1
1	0	1	0	1	0	1	0	1	1	1	0
1	0	1	0	1	1	1	1	1	0	0	1
1	0	1	0	1	0	1	0	1	1	1	0
1	0	1	0	1	1	1	1	1	0	1	0
1	0	1	0	1	1	0	1	1	0	0	1
1	0	1	0	1	0	1	1	1	0	0	1
1	0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	0	1	1	1	0	1
1	0	1	0	1	1	1	0	1	0	1	0
1	0	1	0	1	0	0	1	1	0	0	1
1	0	1	0	1	0	1	1	0	1	0	1
1	0	1	0	1	0	1	0	1	0	1	0

APPENDIX LA I

Concordia University

Fall 1999

Department of Mathematics & Statistics

MATH 251
Linear Algebra I

Instructor: ______________________

Office/Tel No.: ______________________

Office Hours: ______________________

Course Examiner: Dr. R. Raphael, Loyola-HB 221, 848-3253, e-mail: raphael@alcor.concordia.ca

Text: *Linear Algebra and Matrix Theory* by Gilbert & Gilbert (Academic Press)

Assignments: Assignments are given on a weekly basis. The questions are fairly typical of those which will be asked on the final examination. Students should solve the problems, consult the Math Help Centre for the difficult ones and ask their teachers any questions they may have. Solutions will be available in the copy centre on a weekly basis. Assignments are to be handed in every week. No late assignments will be accepted.

* **Only the underlined Assignments should be handed in.**

Test: There will be one class test in the seventh week and it will cover the first five weeks of the term. There will be no make-up test.

Final Exam: The final examination will be three hours long. It covers material from the entire course.

Final Mark: The final grade will be based on the higher of (a) or (b) below:

a) 10% for the assignments, 30% for the test, and 60% for the final.

or

b)100% for the final examination.

Week	Section	Topics	Assignments
1	1.1-1.3	Vector Spaces R^n and subspaces of R^n	p. 10: 1c, 3b, 14, 17, 18 p. 18: 6b, 7b, 9, 10
2	1.4-1.5	Geometric Interpretation of R^2, R^3 Bases and Dimension	p. 27: 4b, 7b, 12 p. 37: 1b, 2b, 12b
3	2.1-2.4	Elementary Operations and their Inverses Elementary Operations and Linear Independence Standard Bases for Subspaces	p. 45: 2, 6, 10 2, 4, 5b p. 56: 2, 5b, 6d, 9b
4	3.1-3.4	Matrices of Transition Properties of Matrix Multiplication Invertible Matrices	p. 64: 4, 5b, 6b, 7b, 8b p. 71: 4 p. 82: 14
5	3.5-3.6	Column and Row Operations Column-Echelon and Row-Echelon Forms	p. 90: 3b, 6d p. 98: 10, 14, 18
6	3.7-3.8	Row and Column Equivalence Rank and Equivalence	p. 105: 10, 11 p. 111: 4, 8b, 10
7	CLASS TEST		
8	4.1-4.4	Vector Spaces, Subspaces Isomorphisms	p. 121: 10, 14 p. 126: 4b, 5b, 6b p. 129: 3b
9	4.5-4.7	Bases for Subspaces Systems of Equations	p. 133: 2b, 3b, 5b p. 144: 18, 20
10	5.1-5.3	Linear Transformations and Matrices	p. 155: 8, 9d, 10b, 15 p. 166: 8, 10, 14
11	5.3-5.5	(Continued) Change of Basis Composition of Linear Transformations	p. 168: 16 p. 175: 8, 12, 18 p. 183: 4, 8, 12
12	7.1-7.4	Eigenvalues and Eigenvectors Eigen Spaces and Similarity Representation by Diagonal Matrices	p. 222: 20, 24 p. 229: 10, 12, 14 p. 236: 8, 12
13	REVIEW		

APPENDIX LA II

Winter 1999

Concordia University

Department of Mathematics & Statistics

MATH 252
Linear Algebra II

Instructor: ______________________

Office/Tel No.: ______________________

Office Hours: ______________________

Course Examiner: Prof. R. Raphael - Loyola - HB 221 - 848-3253

Text: *Linear Algebra and Matrix Theory* by Gilbert & Gilbert (Academic Press)

Reference Text: *Linear Algebra,* Lipschutz, Schaim's outline (used copies abound)

Assignments: Assignments are given on a weekly basis. The questions are fairly typical of those which will be asked on the final examination. Students should solve the problems, consult the Math Help Centre for the difficult ones and ask their teachers any questions they may have. Solutions will be available in the copy centre on a weekly basis. Assignments are to be handed in every week. No late assignments will be accepted.
* **Only the underlined Assignments should be handed in.**

Test: There will be one class test in the seventh week and it will cover the first five weeks of the term. There will be no make-up test.

Final Exam: The final examination will be three hours long. It covers material from the entire course.

Final Mark: The final grade will be based on the higher of (a) or (b) below:

a) 10% for the assignments, 30% for the test, and 60% for the final.

b) 100% for the final examination.

Week	Section	Topics	Assignments
1	8.2	Linear Functionals	p. 244: 6, 8, 10b, 11b, 12b, 14b, 16, 17b
2	8.3	Real Quadratic Forms	p. 252: 2b, 3b, 8, 9
	8.4	Orthogonal Matrices	p. 257: 2, 4b, 9, 10, 11
3	8.5	Reduction of Real Quadratic Forms	p. 263: 1b, 9, 10
	8.6	Classification of Real Quadratic Forms	p. 269: 1b, 2b, 3b, 4b
4	8.7	Bilinear Forms	p. 274: 1b, 2b, 3c, 4b, 8

Week	Section	Topics	Assignments
5	8.8	Symmetric Bilinar Forms	p. 281: 1b, 2b, 7
	8.9	Hermitian Forms	p. 291: 1d, 2d, 17
6	9.2	Inner Products	p. 297: 4b, 7, 11, 12
	9.3	Norms and Distances	p. 301: 4, 13, 14
	9.4	Orthonormal Bases	p. 305: 6, 9, 11
7		CLASS TEST	
8	9.5	Orthogonal Complements	p. 308: 4, 6
	9.6	Isometries	p. 312 4, 8, 9
9	9.7	Normal Matrices	p. 317: 5, 7, 8
	9.8	Normal Linear Operators	p. 322: 3b, 5, 8
10	10.2	Projections and Direct Sums	p. 329: 1, 2, 3, 7, 9
11	10.3	Spectral Decompositions	p. 334: 1, 3, 4
	10.4	Minimal Polynomials	p. 345: 1b, 2b, 4b, 5b, 7, 8
12	10.5	Nilpotent Transformations	p. 355; 2b, 3, 4, 5, 6
	10.6	Jordan Canonical Forms	p. 362: 2b, 2d, 6
13	REVIEW		