The Wigner Function and its relation with the wavelet transform for certain Lie groups

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Abstract

"The Wigner Function and its relation with the wavelet transform for certain Lie groups"

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We study Wigner functions on general Lie groups when the group admits square-integrable representations. We develop a relation between Wigner functions and wavelet transforms. In the main part of this thesis, we build Wigner functions and study the connection between Wigner functions and wavelet transforms on the group $G_{ab} = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mid a > 0, b \in \mathbb{R} \; and \; c \; is \; a \; fixed \; constant, \; c \neq 1 \right\}$. Further we check the covariance property, marginality and overlap conditions for the Wigner functions on G_{ab}.

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Contents

Chapter 1

1.1 Introduction

The main aim of this thesis is to build Wigner functions on the group

$$G_{ab} = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mid a > 0, b \in \mathbb{R} \ and \ c \ is \ a \ fixed \ constant, \ c \neq 1 \right\} \quad (1.1)$$

Further we will establish a connection between Wigner functions and wavelet transforms on the same group.

The Wigner function was introduced by E. Wigner in 1932 [12] and it has been studied by many mathematicians and physicists. The idea of building a Wigner function on the group G_{ab} originated from [1] where the theoretical basis of such a construction was laid and an example was given, following a general construction.

We divide this thesis into three chapters. In Chapter-1, we give the prelimenaries. In order to arrange the preliminaries in a decent mathematical way we divide this chapter into three sections. In Section 1.2, we explain how one can construct the Lie algebra of the group G_{ab}. In order to do integration on the group we will determine the invariant measures on it. In Section 1.3 we study the adjoint and coadjoint actions of a group . Finally we explain the covariant coadjoint representations in Section 1.4 In Chapter-2, we will have three sections. In section 2.1, we will study the theoretical nature of Wigner functions and in Section 2.2, we will introduce wavelet transforms. In the final section (Section 2.3), we give the connection between Wigner functions and wavelet transforms.

The first two chapters are completely a survey of Wigner functions. In chapter-3, we study Wigner functions on G_{ab}. Such functions are presented for the first time in this thesis. However the method of construction is an adaptation of earlier work.

1.2 Lie algebra of a group and Haar measures

Here our concentration will be on Lie group. adjoint. coadjoint actions and their orbits.

Definition 1.1 *An abstract group G is said to be a Lie group if*

(i) G is an C^∞ manifold.

(ii) The mapping $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ is C^∞.

Definition 1.2 *Let L be a finite dimensional vector space over the field $\mathbb{K}$ of real or complex numbers. L is called a Lie algebra over $\mathbb{K}$ if there is defined on it a bracket operation*

$$[,] : L \times L \to L \quad by\ (X, Y) \mapsto [X, Y]$$

satisfying the following:

(i) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ *for* $\alpha, \beta \in \mathbb{K}$

(ii) $[X, Y] = -[Y, X]$

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

1.3 Lie algebra generated by a Lie group

Let G be a Lie group and $e \in G$ its identity element. Let

$T(e) = \{f \mid f$ is a differentiable function of class C^1 defined on a neighbourhood of $e\}$

and such that there exists $x : [a, b] \to G$, $t \mapsto x(t)$, a homomorphism of class C^1 such that $x(0) = e$. Now the tangent vector to the curve $x(t)$ at e is the map $A : T(e) \to \mathbb{R}$

$$Af = \frac{df(x(t))}{dt} \Big|_{t=0} \tag{1.2}$$

In a local coordinate system $\{x^1, ..., x^n\}$ at e we have

$$Af = \frac{df(x(t))}{dt} \Big|_{t=0} = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \Big|_{x^j = x^j(e)} \frac{dx^j}{dt} \Big|_{t=0} = \sum_{j=1}^{n} a^j L_j(e) f$$

where

$$L_j(e) f = \frac{\partial f}{\partial x^j} \Big|_{x^j = x^j(e)} \quad \text{and} \quad a^j = \frac{dx^j(t)}{dt} \Big|_{t=0} \quad j = 1....n \tag{1.3}$$

$a^1, ..., a^n$ are the components of the vector A. Indeed, we can represent the tangent vector (1.2) by its components i.e., $A = (a^1, ..., a^n)$. Let $\mathcal{G}$ be the tangent space at e. We convert this vector space into a Lie algebra. Set

$$c^i = [A, B]^i = c^i_{jk} a^j b^k \tag{1.4}$$

where $A = \sum_{j=1}^n a^j L_j(e)$; $B = \sum_{j=1}^n b^j L_j(e)$ and the c^i_{jk} are called structure constants. Now $\mathcal{G}$ together with [,] forms a Lie algebra and is called the Lie algebra of the Lie group G.

Definition 1.3 *A homomorphism $\phi : \mathbb{R} \to G$ is called a 1-parameter subgroup of G.*

Definition 1.4 *Let G be a Lie group and let $\mathcal{G}$ be its Lie algebra. Let $X \in \mathcal{G}$ then $\lambda\frac{d}{dx} \mapsto \lambda X$, $\lambda \in \mathbb{R}$ is a homomorphism of the algebra of $\mathbb{R}$ onto $\mathcal{G}$. i.e., If $\mathcal{R}$ is the Lie algebra of R, then $\phi : \mathcal{R} \to \mathcal{G}$ such that $\lambda\frac{d}{dr} \mapsto \lambda X$ is a homomorphism .*

Theorem 1.1 *Let G and H be Lie groups with Lie algebras $\mathcal{G}$ and $\mathcal{H}$ respectively and with G simply connected. Let $\psi : \mathcal{G} \to \mathcal{H}$ be a homomorphism. Then there exists a unique homomorphism $\phi : G \to H$ such that*

$$d\phi = \psi. \tag{1.5}$$

Proof. : See [7] ■

Note 1.1 Since the real line is simply connected, there exists, by theorem(1.1), a unique 1-parameter subgroup $\exp_X : R \to G$ such that

$$d\exp_X(\lambda\frac{d}{dr}) = \lambda X \tag{1.6}$$

i.e., $\{\exp_X(t), t \in R\}$ is a 1-parameter subgroup of G.

Definition 1.5 *Let G be a Lie group and $\mathcal{G}$ be its Lie algebra then the mapping* $\exp : \mathcal{G} \to G$ *by*

$$\exp(X) = \exp_X(1) \tag{1.7}$$

is called the exponential map from $\mathcal{G}$ to G.

Theorem 1.2 *Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Let $X \in \mathcal{G}$ then*

(i) $\exp(tX) = \exp_X(t) \quad \forall t \in \mathbb{R}$

(ii) $\exp(t_1 + t_2)X = \exp(t_1 X)\exp(t_2 X) \quad \forall t_1, t_2 \in \mathbb{R}$

(iii) $\exp(-tX) = (\exp tX)^{-1} \quad \forall t \in \mathbb{R}$

(iv) $\exp : \mathcal{G} \to G$ *is C^∞ and $d\exp : N_0 \mapsto V_e$ is the identity map where N_0 is a neighbourhood of 0 in $\mathcal{G}$ and V_e is a neighbourhood of e in G. i.e.,* exp *gives a diffeomorphism of N_0 onto V_e.*

Proof. : proof can be found in standard texts of Lie groups. For example see reference [11] ∎

Corollary 1.1 *Let G be a Lie group and $\mathcal{G}$ be its Lie algebra. Then the exponential map* $\exp : \mathcal{G} \to G$ *where*

$$\exp(X) = g \tag{1.8}$$

is a topological homeomorphism between an open neighbourhood N_0 *of* 0 *in* $\mathcal{G}$ *and an open set* V_e *around the identity* e *in* G.

Note 1.2 Now $\exp : N_0 \longrightarrow V_e$ is a homeomorphism. So we can define the inverse map $Log : V_e \longrightarrow N_0$ by

$$Log(g) = X. \tag{1.9}$$

Definition 1.6 *Let* G *be a locally compact group and let* $C_0(G)$ *and* $C_0^+(G)$ *denote the space of continuous and continuous non-negative functions on* G *with compact supports, respectively. A positive Radon measure is a positive linear form* μ *on* $C_0(G)$ *which is non-negative on* $C_0^+(G)$. *i.e.,* $\mu(f) \geq 0$ *for* $f \in C_0^+(G)$.

Definition 1.7 *A positive Radon measure* μ *which is left invariant.*

$$\mu(T_g^L f) = \mu(f) \ \ where \ \ T_g^L f(x) = f(g^{-1}x) \tag{1.10}$$

$\forall\ x, g \in G$ *is called a left-Haar measure and it is denoted by* μ_l. *Similarly, a right Haar measure* μ_r *satisfies*

$$\mu_r(T_g^R f) = \mu_r(f) \tag{1.11}$$

where $(T_g^R f)(x) = f(xg)$. $\forall x, g \in G$.

Definition 1.8 *A measure* μ *which is both left and right invariant is called the invariant Haar measure.*

Definition 1.9 *The left and right invariant Haar measures are equivalent,*

$$d\mu_l(g) = \triangle(g) d\mu_r(g) \tag{1.12}$$

where $\triangle : G \to R^+$ *is a measurable function , called the modular function.*

Theorem 1.3 *The modular function* $\triangle(g)$ *is a group homomorphism*

(i) $\triangle(g_1 g_2) = \triangle(g_1)\triangle(g_2) \quad \forall g_1, g_2 \in G.$

(ii) $\triangle(e) = 1.$

Proof. : See reference [4] ■

1.4 Coadjoint orbits of a Lie algebra

In this section we will concentrate on adjoint and coadjoint actions of a Lie algebra.

Definition 1.10 *A Lie group has a natural action on its Lie algebra, which is called the adjoint action. Let* G *be a group and* $\mathcal{G}$ *be its Lie algebra. Then for* $g \in G$ $\Phi_g : G \to G$ *given by* $g' \mapsto gg'g^{-1}$ *is a differentiable map (inner automorphism) from* G *to itself. Now, for* $X \in \mathcal{G}$, $\exp(X) \in G$

$$\Phi_g(\exp(X)) = g \exp(X) g^{-1} = \exp(Ad_g X)$$

The linear map $Ad_g : \mathcal{G} \to \mathcal{G}$ *is called the adjoint action.*

Note 1.3 (i) If G is a matrix group, so that $\mathcal{G}$ consists of matrices then $Ad_g : \mathcal{G} \longrightarrow \mathcal{G}$ is given by $X \longmapsto Ad_g(X)$

$$Y = Ad_g(X) = gXg^{-1}. \tag{1.13}$$

(ii) Ad_g is an automorphism of $\mathcal{G}$.

(iii) $\mathcal{G}^* = \{X^* : \mathcal{G} \longrightarrow R \mid X^* \ linear\}$ is called the dual of the Lie algebra $\mathcal{G}$. $\mathcal{G}^*$ is a vector space.

(iv) $\langle ; \rangle : \mathcal{G}^* \times \mathcal{G} \longrightarrow \mathbb{R}$ where $(X^*, X) \longmapsto \langle X^*; X\rangle$ is called the dual paring and is defined as follows. Let $\{X_i\}_{i=1}^n$ be a basis in $\mathcal{G}$ and $\{X_i^*\}_{i=1}^n$ be a basis in $\mathcal{G}^*$. Let $X \in \mathcal{G}$ and $X^* \in \mathcal{G}^*$ then

$$X = \sum_{i=1}^{n} x^i X_i, \quad X^* = \sum_{i=1}^{n} \xi_i X_i^*$$

where $x_i, \xi_i \in \mathbb{R}$ $\forall i = 1, 2, ...n$ Now $\langle ; \rangle$ is defined as

$$\langle X^*; X\rangle = \vec{x}.\vec{\xi} \tag{1.14}$$

where $\vec{x} = (x^1, ..., x^n)$ *and* $\vec{\xi} = (\xi^1, ..., \xi^n)$.

(v) With the above notation, we write the Lebesgue measure on $\mathcal{G}$ and $\mathcal{G}^*$ as

$$dX \longrightarrow d\vec{x} = dx^1 \wedge ... \wedge dx^n; \ dX^* \longrightarrow d\vec{\xi} = d\xi_1 \wedge ... \wedge d\xi_n.$$

Definition 1.11 *The coadjoint action Ad_g^* of $g \in G$ is a mapping $Ad_g^* : \mathcal{G}^* \longrightarrow \mathcal{G}^*$ and is defined as*

$$\langle Ad_g^*(X^*); X\rangle = \langle X^*; Ad_g^{-1}(X)\rangle, \ X^* \in \mathcal{G}^*, \ X \in \mathcal{G}. \tag{1.15}$$

Theorem 1.4 *Let $\mathcal{G}$ be a Lie algebra and $\mathcal{G}^*$ be its dual and $\{X^i\}_{i=1}^n$, $\{X_i^*\}_{i=1}^n$ bases for $\mathcal{G}$ and $\mathcal{G}^*$ respectively. If M is the matrix representation of the adjoint action then $(M^{-1})^T$ is the matrix representation of the coadjoint action.*

Proof. : It can be proved using elementary linear algebra techniques. ■

Definition 1.12 *Let $\mathcal{G}^*$ be the dual of a Lie algebra $\mathcal{G}$. Let X_0^* be a fixed element in $\mathcal{G}^*$ then the coadjoint orbit of X_0^* is defined as*

$$\mathcal{O}^* = \{Ad_g^* X_0^* \mid g \in G\}. \tag{1.16}$$

Note 1.4 (i) $\mathcal{O}^*$ comes naturally equipped with a measure $d\Omega$ such that

$$d\Omega(X^*) = d\Omega(Ad_g^* X^*) : \tag{1.17}$$

$X^* \in \mathcal{O}^*$ [8]

(ii) Two coadjoint orbits are either distinct or else they coincide entirely.

(iii) The collection of all coadjoint orbits exhausts $\mathcal{G}^*$. i.e.,

$$\bigcup_{\lambda \in J} \mathcal{O}_\lambda^* = \mathcal{G}^* \tag{1.18}$$

where λ can be a descrete or a continuous parameter.

(iv) We denote the invariant measure on the coadjoint orbit $\mathcal{O}_\lambda^*$ by $d\Omega_\lambda$.

(v) Let $X^* \in \mathcal{G}^*$ then by (1.17) $X^* \in \mathcal{O}_\lambda^*$ for some $\lambda \in J$. So we could denote X^* by X_λ^*. With this notation we assume the following decomposition of the

Lebesgue measure on $\mathcal{G}^*$

$$dX^* = d\kappa(\lambda)\sigma_\lambda(X_\lambda^*)d\Omega_\lambda(X_\lambda^*) \tag{1.19}$$

where $\sigma_\lambda : \mathcal{O}_\lambda^* \to \mathbb{R}$ is a positive non-vanishing function.

1.5 Adjoint and Coadjoint representations

Through out this section $\mathbb{H}$ and $\mathbb{H}^*$ stand for the Hilbert spaces $L^2(\mathcal{G}, dx)$ and $L^2(\mathcal{G}^*, dX^*)$, respectively.

Definition 1.13 *The adjoint representation $V : G \to L(\mathbb{H})$ is defined as*

$$(V(g)\ F)(X) = \parallel Ad_g \parallel^{-\frac{1}{2}}\ F(Ad_g^{-1}X) \tag{1.20}$$

where $X \in \mathcal{G}$, $F \in \mathbb{H}$ and $g \in G$. $\parallel Ad_g \parallel$ is the determinant of the linear transformation Ad_g on G.

Theorem 1.5 *The adjoint representation is unitary.*

Proof. Consider

$$\begin{aligned}
&\parallel V(g)F \parallel^2 \\
= &\int_{\mathcal{G}} \mid (V(g)F)(X) \mid^2 dX \\
= &\int_{\mathcal{G}} \parallel Ad_g \parallel^{-1} \mid F(Ad_g^{-1}X) \mid^2 dX
\end{aligned}$$

by a change of variables, $Y = Ad_{g^{-1}}X$,.

$$\begin{aligned} & \| V(g)F \|^2 \\ = & \int_{\mathcal{G}} | (F(Y) |^2 \, dY \\ = & \| F \|^2 \end{aligned}$$

Thus $V(g)$ is unitary. ■

Definition 1.14 *The coadjoint representation $V^{*}: G \longrightarrow L(\mathbb{H}^{*})$ is defined as*

$$(V^{*}(g)\hat{F})(X^{*}) = \| Ad_g^{*} \|^{-\frac{1}{2}} \hat{F}(Ad_{g^{-1}}^{*}X^{*}) \tag{1.21}$$

where $X^{*} \in \mathcal{G}^{*}$, $g \in G$ and $\hat{F} \in \mathbb{H}^{*}$. $\| Ad_g^{*} \|$ is the determinant of the linear transformation Ad_g^{*} on $\mathcal{G}^{*}$.

Theorem 1.6 *The coadjoint representation is unitary.*

Proof. Similar to theorem (1.5) ■

Theorem 1.7 *The adjoint representation V and the coadjoint representation V^{*} are unitarily equivalent.*

$$\mathcal{F}V(g) = V^{*}(g)\mathcal{F} \tag{1.22}$$

for some unitary operator $\mathcal{F}$ from $\mathbb{H}$ to $\mathbb{H}^{*}$.

Proof. Let $D(\mathcal{G})$ be the set of infinitely differential functions with compact support. Let $\mathcal{F}$ be a Fourier transform on $D(\mathcal{G})$

$$F \mapsto \mathcal{F}F, \quad F \in D(\mathcal{G})$$

$$(\mathcal{F}F)(X^{*}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}} \exp^{-i\langle X^{*};X\rangle} F(X)dX \tag{1.23}$$

We have

$$\begin{aligned}
&\| \mathcal{F}F \|^{2} \\
&= \langle \mathcal{F}F \mid \mathcal{F}F \rangle \\
&= \int_{\mathcal{G}^{*}} \overline{\{\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}} \exp^{-i\langle X^{*};X\rangle} F(X)\}dX} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}} \exp^{-i\langle X^{*};Y\rangle} F(Y)dYdX^{*}.
\end{aligned}$$

Since $| \exp^{i\langle X^{*};X-Y\rangle} F(X)F(Y)| \leq |F(X)||F(Y)|$ is integrable, by the Fubini theorem we may interchange the order of integration. [10]

$$\begin{aligned}
\textit{Thus} \quad &\| \mathcal{F}F \|^{2} \\
&= \frac{1}{(2\pi)^{n}} \int_{\mathcal{G}^{*}} \int_{\mathcal{G}} \int_{\mathcal{G}} \exp^{i\langle X^{*};X-Y\rangle} F(X)F(Y)dXdYdX^{*} \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \delta(X-Y)F(X)F(Y)dXdY. [\int_{\mathcal{G}^{*}} \exp^{i\langle X^{*};X-Y\rangle} dX^{*} = (2\pi)^{n}\delta(X-Y)] \\
&= \int_{\mathcal{G}} F(X)F(X)dX \\
&= \int_{\mathcal{G}} | F(X) |^{2} dX \\
&= \| F \|^{2}.
\end{aligned}$$

Thus $\mathcal{F}$ is unitary on $D(\mathcal{G})$. We have $\overline{D(\mathcal{G})} = L^{2}(\mathcal{G})$. Let $F \in L^{2}(\mathcal{G})$, then there exists $\{F_n\} \subseteq D(\mathcal{G})$ such that $F_n \longmapsto F$ in $L^{2}(\mathcal{G})$ -norm. i.e., $||F_n - F|| \to 0$ as $n \to \infty$. So $\{F_n\}$ is a Cauchy sequence and $||\mathcal{F}F_n - \mathcal{F}F_m||_2 = ||F_n - F_m||_2 \to 0$ as $m, n \to \infty$, because the Fourier transform is unitary on $D(\mathcal{G})$. Hence $\{\mathcal{F}F_n\}$ is a cauchy sequence and it is convergent in $L^{2}(\mathcal{G})$. Then there exists $g \in L^{2}(\mathcal{G})$ such that $\lim_{n\to\infty} \mathcal{F}F_n = g = \mathcal{F}F$. Hence $\mathcal{F}$ is unitary in $L^{2}(\mathcal{G})$.

Now we will show $V(g)$ and $V^{\sharp}(g)$ are unitarily equivalent.

$$\mathcal{F}V(g) = V^{\sharp}(g)\mathcal{F} \tag{1.24}$$

Consider

$$\begin{aligned}
&(\mathcal{F}V(g)F)(X^{*}) \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle X^{*};X\rangle}(V(g)F)(X)dX \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle X^{*};X\rangle} \parallel Ad_g \parallel^{-\frac{1}{2}} F(Ad_{g^{-1}}X)dX \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle X^{*};Ad_g Z\rangle} \parallel Ad_g \parallel^{\frac{1}{2}} F(Z)dZ. \;\; [Z = Ad_{g^{-1}}X] \\
&= \frac{\parallel Ad_g \parallel^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle Ad^{\sharp}_{g^{-1}}X^{*};Z\rangle} F(Z)dZ \\
&= \frac{\parallel Ad_g \parallel^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle Ad^{\sharp}_{g^{-1}}X^{*};X\rangle} F(X)dX.
\end{aligned}$$

Other part:

$$\begin{aligned}
(V^{\sharp}\mathcal{F}F)(X^{*}) \quad &= \parallel Ad_{g^{\sharp}} \parallel^{-\frac{1}{2}} \mathcal{F}F(Ad^{\sharp}_{g^{-1}}X^{*}) \\
&= \frac{\parallel Ad_g \parallel^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{\mathcal{G}} \exp^{-i\langle Ad^{\sharp}_{g^{-1}}X^{*};X\rangle} F(X)dX.
\end{aligned}$$

Thus $V(g)$ and $V^{\sharp}(g)$ are unitarily equivalent. ∎

Definition 1.15 (covariant coadjoint representation) *With the earlier notation, for $\lambda \in J$, let $\mathcal{O}^{*}_{\lambda}$ be a coadjoint orbit and $\mathbb{H}_{\lambda} = L^2(\mathcal{O}^{*}_{\lambda}, d\Omega_{\lambda})$. Now define the covariant coadjoint representation $U^{\sharp}_{\lambda} : G \rightarrow L(\mathbb{H}_{\lambda})$ as*

$$(U^{\sharp}_{\lambda}\hat{F}_{\lambda})(X^{*}) = \hat{F}_{\lambda}(Ad^{\sharp}_{g^{-1}}X^{*}) \tag{1.25}$$

where $\hat{F}_\lambda \in \mathbb{H}_\lambda$ *and* $X^* \in \mathcal{O}^*_\lambda$

Theorem 1.8 *The covariant coadjoint representation is unitary.*

Proof. Similar to to theorem(1.5). ■

Remark 1.1 • V^* and U^*_λ are built on the dual of the Lie algebra $\mathcal{G}^*$.

- U^*_λ is more covariant than V^* because U^*_λ is restricted to a single orbit $\mathcal{O}^*_\lambda$ and $d\Omega_\lambda$ is invariant under coadjoint action. But V^* is defined on functions on the entire dual space $\mathcal{G}^*$.

Definition 1.16 *We define a new Hilbert space* $\tilde{\mathbb{H}}$ *as the direct integral over the spaces* $\mathbb{H}_\lambda$.

$$\tilde{\mathbb{H}} = \int_J^\oplus \mathbb{H}_\lambda d\lambda \tag{1.26}$$

which means that $\tilde{\mathbb{H}}$ *is the space of equivalence classes of measurable and square integrable vector fields* $\{\hat{F}_\lambda\}$ *equipped with the scalar product*

$$\langle \hat{F}_\lambda, \hat{P}_\lambda \rangle = \int_J \langle \hat{F}_\lambda(\lambda), \hat{P}_\lambda(\lambda) \rangle_{\Xi_\lambda} d\kappa(\lambda). \tag{1.27}$$

In other words, each $\varphi \in \tilde{\mathbb{H}}$ *is an equivalence class i.e.,* $\varphi = \{\hat{F}_\lambda\}_\lambda \in J;\ \hat{F}_\lambda \in \mathbb{H}_\lambda$ *Further we define the norm of* φ *on* $\tilde{\mathbb{H}}$ *as*

$$\begin{aligned}
& \| \varphi \|^2 \\
&= \langle \varphi, \varphi \rangle \\
&= \int_J \langle \{\hat{F}_\lambda\}, \{\hat{F}_\lambda\} \rangle d\kappa(\lambda)
\end{aligned}$$

$$= \int_J [\int_{\mathcal{O}^*_\lambda} \mid \hat{F}_\lambda(X^*) \mid^2 d\Omega_\lambda(X^*)] d\kappa(\lambda) < \infty \tag{1.28}$$

Now using the following definition we will pack all the covariant coadjoint representations $U^{\ddagger}_\lambda$ defined in Definition(1.24) into one covariant coadjoint representation $U^{\ddagger}$ on $\tilde{\mathbb{H}}$

Definition 1.17 *A new covariant coadjoint representation $U^{\ddagger} : G \longrightarrow L(\tilde{\mathbb{H}})$ is defined as $U^{\ddagger}(g)\varphi = \{U^{\ddagger}_\lambda \hat{F}_\lambda\}_{\lambda \in J}$, $g \in G$.*

$$(U^{\ddagger}(g)\varphi)(X^*_\lambda) = (U^{\ddagger}_\lambda(g)\hat{F}_\lambda)(X^*_\lambda) = \hat{F}_\lambda(Ad^{\ddagger}_{g^{-1}} X^*_\lambda). \tag{1.29}$$

Lemma 1.1 *The operator $\tilde{N} : L^2(\mathcal{G}^*, dX^*) \longrightarrow \tilde{\mathcal{H}}$ given by $\hat{F} \longmapsto \tilde{N}\hat{F} = \varphi = \{\hat{G}'_\lambda\}_{\lambda \in J}$ is unitary . Where*

$$\hat{G}'_\lambda(X^*_\lambda) = [\sigma_\lambda(X^*_\lambda)]^{\frac{1}{2}} \hat{F}(X^*_\lambda) \tag{1.30}$$

$X^*_\lambda \in \mathcal{O}^*_\lambda$ and $\sigma_\lambda : \mathcal{O}^*_\lambda \longrightarrow \mathbb{R}$ is given by $X^*_\lambda \longmapsto \sigma_\lambda(X^*_\lambda)$, and is a positive nonvanishing function.

Proof.

$$\begin{aligned}
& \| \tilde{N}\hat{F} \|^2 \\
= & \int_J \| \tilde{N}\hat{F} \|^2 d\kappa(\lambda) \\
= & \int_J \| [\sigma_\lambda(X^*)]^{\frac{1}{2}} \hat{F}(X^*_\lambda) \|^2 d\kappa(\lambda) \\
= & \int_J \int_{\mathcal{O}^*_\lambda} [\sigma_\lambda(X^*_\lambda) \mid \hat{F}(X^*_\lambda) \mid^2 d\Omega_\lambda(X^*_\lambda)] d\kappa(\lambda)
\end{aligned}$$

$$= \int_J \int_{\mathcal{O}_\lambda^*} |\hat{F}(X_\lambda^*)|^2 \, d\kappa(\lambda)\sigma_\lambda(X_\lambda^*) d\Omega_\lambda(X_\lambda^*)$$

$$= \int_{\mathcal{G}^*} |\hat{F}(X^*)|^2 \, dX^*$$

$$= \| \hat{F} \|^2$$

Thus $\tilde{N}$ is unitary ■

Theorem 1.9 *The coadjoint representation $V^{\ddot{*}}$ given in Definition(1.14) and the co-variant coadjoint representation given in Definition(1.17) are unitarily equivalent.*

$$\tilde{N} V^{\ddot{*}}(g) = U^{\ddot{*}}(g)\tilde{N}, \forall g \in G \tag{1.31}$$

where $\tilde{N}$ is as in Lemma(1.1).

Proof. We have to show that $V^{\ddot{*}}$ and $U^{\ddot{*}}$ are unitarily equivalent. We have the unitary map $\tilde{N} : L^2(\mathcal{G}^*, dX^*) \to \tilde{\mathbb{H}}$. Now we will show

$$\tilde{N} V^{\ddot{*}}(g) = U^{\ddot{*}}(g)\tilde{N} \tag{1.32}$$

We already know

$$dX^* = d\kappa(\lambda)\sigma_\lambda(X_\lambda^*) d\Omega_\lambda(X_\lambda^*) \; ; X_\lambda^* \in \mathcal{O}_\lambda^* \tag{1.33}$$

and

$$d(Ad_g^{\ddot{*}} X^*) = \| Ad_g^{\ddot{*}} \| \, dX^*$$

Now

$$dX^* = d\kappa(\lambda)\sigma_\lambda(X_\lambda^*) d\Omega_\lambda(X_\lambda^*)$$

$$d(Ad_g^{\ddot{*}} X^*) = d\kappa(\lambda)\sigma_\lambda(Ad_g^{\ddot{*}} X_\lambda^*) d\Omega_\lambda(Ad_g^{\ddot{*}} X_\lambda^*)$$

$$\| Ad_g^* \| \, dX^* = d\kappa(\lambda)\sigma_\lambda(Ad_g^* X_\lambda^*)d\Omega_\lambda(X_\lambda^*); \tag{1.34}$$

multiplying (1.33) by $\| Ad_g^* \|$, we get

$$\| Ad_g^* \| \, dX^* = d\kappa(\lambda) \, \| Ad_g^* \| \, \sigma_\lambda(X_\lambda^*)d\Omega_\lambda(X_\lambda^*). \tag{1.35}$$

By (1.34) and (1.35) we get

$$\| Ad_g^* \| \, \sigma_\lambda(X_\lambda^*) = \sigma_\lambda(Ad_g^* X_\lambda^*). \tag{1.36}$$

Consider

$$\begin{aligned}
& (\tilde{N}V^*(g)\hat{F})(X^*) \\
= \;& (\tilde{N}(V^*(g)\hat{F})(X^*) \\
= \;& [\sigma_\lambda(X^*)]^{\frac{1}{2}} V^*(g)\hat{F}(X_\lambda^*) \\
= \;& \| Ad_g^* \|^{-\frac{1}{2}} \, [\sigma_\lambda(X^*)]^{\frac{1}{2}} \hat{F}(Ad_{g^{-1}} X_\lambda^*)
\end{aligned}$$

Now consider

$$\begin{aligned}
& (U^*(g)\tilde{N}\hat{F})(X^*) \\
= \;& (U_\lambda^*(g)N\hat{F}_\lambda)(X_\lambda^*) \\
= \;& \tilde{N}\hat{F}_\lambda(Ad_{g^{-1}}^* X_\lambda^*) \\
= \;& [\sigma_\lambda(Ad_{g^{-1}} X_\lambda^*]^{\frac{1}{2}} \hat{F}(Ad_{g^{-1}} X_\lambda^*) \\
= \;& \| Ad_g^* \|^{-\frac{1}{2}} \, [\sigma_\lambda(X^*)]^{\frac{1}{2}} \hat{F}(Ad_{g^{-1}} X_\lambda^*)
\end{aligned}$$

Thus we get $\quad \tilde{N}V^*(g) = V^*(g)\tilde{N}$ ■

In the following few pages we will concentrate on square integrable representations.

Definition 1.18 *Let U be a unitary irreducible representation of the group G on a Hilbert space $\mathbb{H}$. A vector $\eta \in \mathbb{H}$ is said to be admissible if*

$$I(\eta) = \int_G \mid \langle U(g)\eta \mid \eta\rangle \mid^2 d\mu_l(g) < \infty$$

where $d\mu_l$ is the left Haar measure on G.

Note 1.5 Since $d\mu_r(g) = d\mu_l(g^{-1})$ and $U(g)$ is unitary, we have

$$\begin{aligned} I(\eta) &= \int_G \mid \langle U(g^{-1})\eta \mid \eta\rangle \mid^2 d\mu_l(g) \\ &= \int_G \mid \langle \eta | U(g)\eta\rangle \mid^2 d\mu_r(g) \end{aligned}$$

Thus $I(\eta) = \int_G \mid \langle U(g)\eta \mid \eta\rangle \mid^2 d\mu_r(g)$.

Theorem 1.10 *If $\eta \in \mathbb{H}$ is an admissible vector then $\eta_g = U(g)\eta$ is also an admissible vector for all $g \in G$ whenever $U(g)$ is unitary.*

Proof. It can be found in reference [2] ■

Definition 1.19 (Square integrable representation) *Let $U(g)$ be a strongly continuous unitary representation of G on $\mathbb{H}$. $U(g)$ will be called square integrable if*

(i) U is irreducible.

(ii) There exists at least one non-zero admissible vector in $\mathbb{H}$.

Note 1.6 (i) Any representation, unitarily equivalent to a square integrable representation, is also square-integrable.

(ii) If G is compact, any irreducible representation U of G is square integrable.

(iii) If G is unimodular and if U is a square-integrable representation of G then every vector in $\mathbb{H}$ is admissible. (By Theorem 1.11)

Theorem 1.11 *Let U be a square integrable representation of G, acting on the Hilbert space $\mathbb{H}$. Then there exists a unique self-adjoint operator C in $\mathbb{H}$ such that the following hold :*

(i) The set of admissible vectors coincides with the domain of C.

(ii) Let η_1 and η_2 be any two admissible vectors. Let φ_1 and φ_2 be any two vectors in $\mathbb{H}$ then

$$\int_G \overline{\langle U(g)\eta_2 \mid \varphi_2\rangle}\langle U(g)\eta_1 \mid \varphi_1\rangle d\mu(g) = \langle C\eta_1 \mid C\eta_2\rangle\langle\varphi_2 \mid \varphi_1\rangle. \tag{1.37}$$

The above relation is called the orthogonality relation.

(iii) If the group G is unimodular then C is a multiple of the identity.

Proof. It can be found in [2]. ■

Chapter 2

In this chapter, we will introduce the Wigner transform and Wigner functions. In the final chapter, the main part of this thesis, we will construct Wigner functions explicitly. The existence of Wigner functions for a certain class of groups depends on the existence of square integrable representations. The notation introduced in the previous chapter will be used without further explanation. An adequate reference for the material in this chapter is [1].

2.1 Wigner transforms

Definition 2.1 *Let $\eta \in \mathbb{H}$ be an admissible vector of a square integrable unitary representation U and $\psi = C\eta$ where C is as in Theorem (1.11). Let $D(C^{-1})$ be the domain of C^{-1}; then the Wigner transform ϖ is defined as $\varpi : \mathbb{H} \supset D(C^{-1}) \to L^2(G, d\mu)$ by $\rho \mapsto \varpi\rho$ and*

$$(\varpi\rho)(g) = \langle U(g)C^{-1}\psi \mid \varphi \rangle = Tr[U(g)^* \rho C^{-1}] \tag{2.1}$$

where $\rho = | \varphi\rangle\langle\psi | \in \mathbb{B}_2(\mathbb{H})$, *the Hilbert-Schmidt operators on* $\mathbb{H}$, *and* $\| \rho \|_{\mathcal{B}_2} = \{Tr[\rho^*\rho]\}^{\frac{1}{2}}$.

Theorem 2.1 *The Wigner transform* ϖ *satisfies the orthogonality relation.*

$$\int_G (\overline{\varpi\rho_2)(g)}(\varpi\rho_1)(g)d\mu(g) = Tr[\rho_2^*\rho_1] = \langle\rho_2 \mid \rho_1\rangle_{\mathcal{B}_2} \tag{2.2}$$

Proof. Consider

$$\begin{aligned}
&\int_G (\overline{\varpi\rho_2)(g)}(\varpi\rho_1)(g)d\mu(g) \\
= &\int_G (\overline{\langle U(g)C^{-1}\psi_2 \mid \varphi_2\rangle}\langle U(g)C^{-1}\psi_1 \mid \varphi_1\rangle d\mu(g).
\end{aligned}$$

If $\eta_2 = C^{-1}\psi_2$, $\eta_1 = C^{-1}\psi_1$ then

$$\begin{aligned}
&\int_G (\overline{\varpi\rho_2)(g)}(\varpi\rho_1)(g)d\mu(g) \\
&= \langle\psi_1 \mid \psi_2\rangle\langle\varphi_2 \mid \varphi_1\rangle \\
&= \langle\rho_2 \mid \rho_1\rangle \\
&= Tr[\rho_2^*\rho_1].
\end{aligned}$$

Hence the theorem. ■

Theorem 2.2 *The Wigner transform is an isometry.*

Proof. We know from (2.2)

$$\begin{aligned}
&\int_G (\overline{\varpi\rho_2})(g)(\varpi\rho_1)(g)d\mu(g) \\
&= Tr[\rho_2^*\rho_1]
\end{aligned}$$

Let $\rho_m = \sum_{i,j}^n c_{ij}^m |\varphi_i\rangle\langle\psi_j|, \ m = 1, 2.$

$$
\begin{aligned}
\textit{Then.} \qquad & \int_G (\overline{\varpi\rho_2})(g)(\varpi\rho_1)(g)d\mu(g) \\
&= Tr[(\sum_{i,j}^n c_{ij}^2 |\varphi_i\rangle\langle\psi_j|)^*(\sum_{k,l}^n c_{kl}^1 |\varphi_k\rangle\langle\psi_l|)] \\
&= Tr[(\sum_{i,j}^n (c_{ij}^2)^* |\psi_j\rangle\langle\varphi_i|)(\sum_{k,l}^n c_{kl}^1 |\varphi_k\rangle\langle\psi_l|)] \\
&= Tr[\sum_{i,j}^n \sum_{k,l}^n (c_{ij}^2)^* c_{kl}^1 |\psi_j\rangle\langle\varphi_i|\varphi_k\rangle\langle\psi_l|] \\
&= Tr[\sum_{i,j}^n \sum_{k,l}^n (c_{ij}^2)^* c_{kl}^1 |\psi_j\rangle\langle\psi_l|\delta_{ik}] \\
&= Tr[\sum_{i,j}^n \sum_{i,l}^n (c_{ij}^2)^* c_{il}^1 |\psi_j\rangle\langle\psi_l|] \\
&= Tr[\sum_{j,l}^n (\sum_{i,l}^n (c_{ij}^2)^* c_{il}^1) |\psi_j\rangle\langle\psi_l|] \\
&= \sum_i^n \sum_j^n (c_{ij}^2)^* c_{ij}^1
\end{aligned}
$$

Let $\rho_2 = \rho_1 = \rho$

$$
\begin{aligned}
\textit{Then.} \qquad & \int_G (\overline{\varpi\rho_2})(g)(\varpi\rho_1)(g)d\mu(g) \\
&= \sum_i^n \sum_j^n |(c_{ij})|^2 \\
&= \| \rho \|^2 .
\end{aligned}
$$

The Wigner transform is an isometry on the set $\mathbb{H} \otimes D(C^{-1})$. Since $\mathbb{H} \otimes D(C^{-})$ is dense in $\mathcal{B}_2(\mathbb{H})$, as in Theorem 1.7 it can also be extended to an isometry on all of $\mathcal{B}_2(\mathbb{H})$ by continuity of the linear map $\psi \mapsto |\varphi\rangle\langle\psi| = \rho$. Hence the theorem. ∎

Definition 2.2 *The unitary representation $U(g)$ on $\mathbb{H}$ gives another unitary repre-*

sentation $\mathbb{U}_l$ *on* $\mathbb{B}_2(\mathbb{H})$ *and is defined as*

$$\mathbb{U}_l(g)\rho = U(g)\rho. \tag{2.3}$$

Theorem 2.3 *The Wigner transform* ϖ *intertwines* $\mathbb{U}_l(g)$ *on* $\mathbb{B}_2(\mathbb{H})$ *with the left regular representation* U_l *on* $L^2(G, d\mu)$ *i.e.,*

$$\varpi\mathbb{U}_l(g) = U_l(g)\varpi, \qquad \forall g \in G. \tag{2.4}$$

Proof. Consider

$$\begin{aligned} L.H.S = \quad & (\varpi\mathbb{U}_l(g)\rho)(g') \\ = \;& Tr[U(g')^*\mathbb{U}_l(g)\rho C^{-1}] \\ = \;& Tr[U(g')^*U(g)\rho C^1] \end{aligned}$$

If $\mathbb{U}_l(g)\rho = U(g)\rho$ then

$$L.H.S = \quad Tr[U((g')^{-1}g)\rho C^{-1}]$$

Again consider

$$\begin{aligned} R.H.S = \quad & (U_l(g)\varpi\rho)(g') \\ = \;& (\varpi\rho)(g^{-1}g') \\ = \;& Tr[U(g^{-1}g')^*\rho C^{-1}] \\ = \;& Tr[U((g')^{-1}g)\rho C^{-1}] \end{aligned}$$

Thus $\varpi\mathbb{U}_l(g) = U_l(g)\varpi$ ■

Note 2.1 *The operator* C *satisfies the following covariance condition.*

(i) $U(g)^*CU(g) = [\triangle(g)]^{-\frac{1}{2}}C$

2.2 Wigner maps and functions

Before we define the Wigner map and Wigner function let us rewrite the orthogonality relation in a different coordinate system.

Remark 2.1 *(i) The exponential map of the group G is given as $g = e^X$ where $X = \sum_{i=1}^{n} x^i X_i \in \mathcal{G}$ where $\{X_i\}_{i=1}^{n}$ (and $x^i \in \mathbb{R}$) is a basis for the Lie algebra $\mathcal{G}$ and assume that the range of exponential map is dense in G.*

(ii) $d\mu(g) \longrightarrow m(X)dX$ under the coordinate transformation $g = e^X$ where m is a positive Lebesgue measurable function on N_0, where N_0 is a neighbourhood of 0 in $\mathcal{G}$.

(iii) Under the above coordinate transformation the orthogonality relation (2.2) becomes

$$\int_{N_0} \overline{(\varpi\rho_2)(e^X)}(\varpi\rho_1)(e^X)m(X)dX = Tr[\rho_2^*\rho_1] = \langle \rho_2 \mid \rho_1 \rangle_{\mathbb{B}_2}. \tag{2.5}$$

Definition 2.3 *The Wigner map is the Fourier transform of the Wigner transform ϖ. The Wigner map $\mathbb{W} : \mathbb{B}_2(\mathbb{H}) \longrightarrow \tilde{\mathbb{H}}$ is defined as*

$$(\mathbb{W}\rho)(X_\lambda^*) = \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle}(\varpi\rho)(e^X)[m(X)]^{\frac{1}{2}}dX \tag{2.6}$$

where $\sigma_\lambda : \mathcal{G}^* \longrightarrow \mathbb{R}^+$ is as in Lemma(1.1).

Definition 2.4 *The Wigner function corresponding to the Hilbert-Schmidt operator*

ρ *defined on* $\mathcal{G}^*$ *is defined as*

$$W(\rho \mid X^*) = (\mathbb{W}\rho)(X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(e^{-X})\rho C^{-1}][\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX. \tag{2.7}$$

Theorem 2.4 *The Wigner map* $\mathbb{W}$ *is linear.*

Proof. It is easy to verify. So we omit the details. ■

Note 2.2 *The change of co-ordinate* $X \to -X$ *yields*

$$\Delta(e^X)m(-X) = m(X). \tag{2.8}$$

Theorem 2.5 *The Wigner function satisfies sesquilinearity.*

$$W(\rho \mid X^*) = \overline{W(\rho^* \mid X^*)}. \tag{2.9}$$

Proof. We know by eq.(2.7)

$$\begin{aligned} & W(\rho \mid X^*) \\ = & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(e^{-X})\rho C^{-1}][\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX \\ = & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} \langle U(e^X)C^{-1}\psi \mid \varphi\rangle [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX. \end{aligned}$$

Now let $X = -X$ thus $dX = d(-X)$ and $\Delta(e^X)m(-X) = m(X)$. Then

$$\begin{aligned} & W(\rho \mid X^*) \\ = & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;-X\rangle} \langle U(e^{-X})C^{-1}\psi \mid \varphi\rangle [\sigma_\lambda(X^*)m(-X)]^{\frac{1}{2}} d(-X). \end{aligned}$$

Using invariance of N_0 under $X \to -X$, we have

$$
\begin{aligned}
& W(\rho \mid X^*) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{i\langle X^*;X\rangle} \langle U(e^{-X})C^{-1}\psi \mid \varphi\rangle [\frac{\sigma_\lambda(X^*)m(X)}{\Delta(e^X)}]^{\frac{1}{2}} dX \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{i\langle X^*;X\rangle} \overline{\langle \varphi \mid U(e^{-X})C^{-1}\psi\rangle} [\frac{\sigma_\lambda(X^*)m(X)}{\Delta(e^X)}]^{\frac{1}{2}} dX \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{i\langle X^*;X\rangle} \overline{\langle \frac{U(e^X)\varphi}{\Delta(e^X)^{\frac{1}{2}}} \mid C^{-1}\psi\rangle} [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX.
\end{aligned}
$$

let $\Delta(e^X)^{-\frac{1}{2}}U(e^X) = CU(e^X)C^{-1}$ then

$$
\begin{aligned}
& W(\rho \mid X^*) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{i\langle X^*;X\rangle} \overline{\langle CU(e^X)C^{-1}\varphi \mid C^{-1}\psi\rangle} [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX \\
= & \overline{\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} \langle U(e^X)C^{-1}\varphi \mid \psi\rangle [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX} \\
= & \overline{W(\rho^* \mid X^*)}.
\end{aligned}
$$

Thus $W(\rho \mid X^*) = \overline{W(\rho^* \mid X^*)}$ ■

Definition 2.5 *Let U be a square-integrable unitary representation as before. Now we define a new representation $\mathbb{U}_b$ of G on the Hilbert space $\mathbb{B}_2(\mathbb{H})$ of the Hilbert Schmidt operators as follows*

$$
\mathbb{U}_b(g)\rho = U(g)\rho U(g)^* \tag{2.10}
$$

$\forall g \in G$. As can be easily verified, $\mathbb{U}_b$ is unitary.

Lemma 2.1 *With the usual notation we have*

$$
m(Ad_g X) = \frac{m(X)}{\| Ad_g \| \Delta(g)} \quad X \in \mathcal{G},\ g \in G. \tag{2.11}
$$

Proof. We know that

$$\begin{aligned}
e^{(Ad_g X)} &= g(e^X)g^{-1}. \quad where \;\; X \in \mathcal{G} \\
d\mu_l(e^{(Ad_g X)}) &= d\mu_l(g(e^X)g^{-1}) \\
\Rightarrow m(Ad_g X)d(Ad_g X) &= d\mu_l((e^X)g^{-1}) \quad (d\mu_l(e^X) = m(X)dX)
\end{aligned}$$

$$= d\mu_l(ge^{-X})^{-1}$$

$$= d\mu_r(ge^{-X}) \quad (\, d\mu_l(g^{-1}) = d\mu_r(g))$$

$$= \Delta(ge^{-X})^{-1}d\mu_l(ge^{-X}) \quad (d\mu_r(g) = \Delta(g^{-1})d\mu_l(g))$$

$$= \Delta(e^X g^{-1})d\mu_l(e^{-X})$$

$$= \Delta(e^X)\Delta(g^{-1})d\mu_r(e^X)$$

$$= \Delta(e^X)\Delta(g^{-1})\Delta(e^{-X})d\mu_l(e^X). \quad (d\mu_r(e^X) = \Delta(e^{-X})d\mu_l(e^X))$$

$$= \Delta(g^{-1})d\mu_l(e^X).$$

Hence.

$$\begin{aligned}
m(Ad_g X)||Ad_g||dX &= \Delta(g^{-1})m(X)dX \\
m(Ad_g X) &= \frac{\Delta(g^{-1})m(X)}{||Ad_g||} \\
m(Ad_g X) &= \frac{m(X)}{||Ad_g||\Delta(g)}, \quad X \in \mathcal{G}. \;\; g \in G.
\end{aligned}$$

Hence the Lemma. ■

Lemma 2.2 *The Wigner map $\mathbb{W}$ intertwines the representation $\mathbb{U}_b$ with the covariant coadjoint representation $U^{\ddagger}$.*

$$\mathbb{W}\mathbb{U}_b(g_0) = U^{\ddagger}(g_0)\mathbb{W}, \quad \forall g \in G. \tag{2.12}$$

Proof. Consider

$$
\begin{aligned}
&(\mathbb{W}\mathbb{U}_b(g_o)\rho)(X_\lambda^*) \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} (\varpi\mathbb{U}_b(g_0)\rho)(e^X)[m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g)^*\mathbb{U}_b(g_0)\rho C^{-1}](e^X)[m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g)^*U(g_0)\rho U(g_0)^* C^{-1}](e^X)[m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g_0)U(g)^*\rho U(g_0)^* C^{-1}](e^X)[m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g_0)U(e^{-X})\rho C^{-1}[\triangle(g_0)]^{-\frac{1}{2}} U(g_0)^*](e^X) \\
& \times [m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g_0)U(e^{-X})U(g_0)^*\rho C^{-1}[\triangle(g_0)]^{-\frac{1}{2}}](e^X) \\
& \times [m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(g_0 e^{-X} g_0^{-1})\rho C^{-1}[\triangle(g_0)]^{-\frac{1}{2}}](e^X) \\
& \times [m(X)]^{\frac{1}{2}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(e^{Ad_{g_0}(-X)})\rho C^{-1}] \frac{(e^X)[m(X)]^{\frac{1}{2}}}{[\triangle(g_0)]^{\frac{1}{2}}} dX \\
= \ & \frac{[\sigma_\lambda(X_\lambda^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X_\lambda^*;X\rangle} Tr[U(e^{-Ad_{g_0}(X)})\rho C^{-1}](e^X)[\frac{m(X)}{\triangle(g_0)}]^{\frac{1}{2}} dX
\end{aligned}
$$

Now consider the other part.

$$
\begin{aligned}
& (U^{\sharp}(g_0)\mathbb{W}\rho)(X_\lambda^*) \\
= \ & (\mathbb{W})(Ad^{\sharp}_{g_0^{-1}} X_\lambda^*)
\end{aligned}
$$

$$= \frac{[\sigma_\lambda(Ad^{*}_{g_0^{-1}}X^{*}_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle Ad^{*}_{g_0^{-1}}X^{*}_\lambda;X\rangle} Tr[U(g)^{*}\rho C^{-1}][m(X)]^{\frac{1}{2}}dX$$

$$= \frac{\| Ad^{*}_{g_0^{-1}} \|^{\frac{1}{2}} [\sigma_\lambda(X^{*}_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*}_\lambda;Ad_{g_0}X\rangle} Tr[U(g)^{*}\rho C^{-1}][m(X)]^{\frac{1}{2}}dX.$$

Put $X = Ad_{g_0}X$ then

$$\begin{aligned}
&(U^{*}(g_0)\mathbb{W}\rho)(X^{*}_\lambda)\\
= \;& \frac{\| Ad^{*}_{g_0^{-1}} \|^{\frac{1}{2}} [\sigma_\lambda(X^{*}_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*}_\lambda;X\rangle} Tr[U(e^{-X})\rho C^{-1}]\\
&\times [m(Ad_{g_0}X)]^{\frac{1}{2}} d(Ad_{g_0}X)\\
= \;& \frac{[\sigma_\lambda(X^{*}_\lambda)]^{\frac{1}{2}}}{\| Ad_{g_0} \|^{\frac{1}{2}} (2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*}_\lambda;X\rangle} Tr[U(e^{-(Ad_{g_0})(X)})\rho C^{-1}]\\
&\times \frac{\| Ad_{g_0} \| [m(X)]^{\frac{1}{2}}}{\| Ad_{g_0} \|^{\frac{1}{2}} \triangle(g_0)^{\frac{1}{2}}} dX\\
= \;& \frac{[\sigma_\lambda(X^{*}_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*}_\lambda;X\rangle} Tr[U(e^{-Ad_{g_0}(X)})\rho C^{-1}](e^{X})\\
&\times [\frac{m(X)}{\triangle(g_0)}]^{\frac{1}{2}} dX.
\end{aligned}$$

Thus $\mathbb{W}\mathbb{U}_b(g_0) = U^{*}(g_0)\mathbb{W}$. ■

In the following theorem we prove the covariance property of the Wigner function.

Theorem 2.6 *With the usual notation we have*

$$W(U(g_0)\rho U(g_0)^{*} \mid X^{*}) = W(\rho \mid Ad^{*}_{g_0^{-1}}X^{*}), \quad g_0 \in G,\ X^{*} \in \mathcal{G}^{*}. \tag{2.13}$$

Proof. Using Definition(2.4) of the Wigner function

$$
\begin{aligned}
& W(U(g_0)\rho U(g_0)^*) \mid X^*) \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(e^{-X})U(g_0)\rho U(g_0)^* C^{-1}][\sigma(X^*)m(X)]^{\frac{1}{2}} dX \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(g_0)U(e^{-X})\rho U(g_0)^* C^{-1}][\sigma(X^*)m(X)]^{\frac{1}{2}} dX \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(g_0)U(e^{-X})\rho C^{-1}[\triangle(g_0)]^{-\frac{1}{2}} U(g_0)^*] \\
& \times [\sigma(X^*)m(X)]^{\frac{1}{2}} dX.
\end{aligned}
$$

Using Note(2.3)

$$
\begin{aligned}
& W(U(g_0)\rho U(g_0)^*) \mid X^*) \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(g_0)U(e^{-X})U(g_0)^* \rho C^{-1}[\triangle(g_0)]^{-\frac{1}{2}}] \\
& \times [\sigma(X^*)m(X)]^{\frac{1}{2}} dX \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(g_0 e^{-X} g_0^{-1})\rho C^{-1}][\frac{\sigma(X^*)m(X)}{\triangle(g_0)}]^{\frac{1}{2}} dX \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X\rangle} Tr[U(e^{-Ad_{g_0}X})\rho C^{-1}][\frac{\sigma(X^*)m(X)}{\triangle(g_0)}]^{\frac{1}{2}} dX
\end{aligned}
$$

Consider the other part

$$
\begin{aligned}
& W(\rho \mid Ad^*_{g_0^{-1}} X^*) \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle Ad^*_{g_0^{-1}} X^*;X\rangle} Tr[U(e^{-X})\rho C^{-1}][\sigma(Ad^*_{g_0^{-1}} X^*)m(X)]^{\frac{1}{2}} dX \\
= \;& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;Ad_{g_0}X\rangle} Tr[U(e^{-X})\rho C^{-1}][\| Ad^*_{g_0^{-1}} \| \sigma(X^*)m(X)]^{\frac{1}{2}} dX.
\end{aligned}
$$

Let $X = Ad_{g_0}X$ then

$$\begin{aligned}
& W(\rho \mid Ad^{*}_{g_0^{-1}}X^{*}) \\
= & \frac{\| Ad^{*}_{g_0^{-1}} \|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*};X\rangle} Tr[U(e^{-Ad_{g_0}X})\rho C^{-1}] \\
& \times [\sigma(X^{*})m(Ad_{g_0}X)]^{\frac{1}{2}} d(Ad_{g_0}X).
\end{aligned}$$

Using lemma(2.2)

$$\begin{aligned}
& W(\rho \mid Ad^{*}_{g_0^{-1}}X^{*}) \\
= & \frac{\| Ad^{*}_{g_0^{-1}} \|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*};X\rangle} Tr[U(e^{-Ad_{g_0}X})\rho C^{-1}] \frac{[\sigma(X^{*})m(X)]^{\frac{1}{2}}}{\| Ad_{g_0} \|^{\frac{1}{2}} [\triangle(g_0)]^{\frac{1}{2}}} \\
& \times \| Ad_{g_0} \| \, d(X) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^{*};X\rangle} Tr[U(e^{-Ad_{g_0}X})\rho C^{-1}][\frac{\sigma(X^{*})m(X)}{\triangle(g_0)}]^{\frac{1}{2}} dX.
\end{aligned}$$

Thus $W(U(g_0)\rho U(g_0)^{*} \mid X^{*}) = W(\rho \mid Ad^{*}_{g_0^{-1}}X^{*})$. ■

Note 2.3 *The Wigner map is an isometry because it preserves scalar products (Theorem 2.2). Thus for any two* $\rho_1, \rho_2 \in \mathbb{B}_2(\mathbb{H})$, *we have*

$$\langle \mathbb{W}\rho_1 \mid \mathbb{W}\rho_2 \rangle_{\hat{\mathbb{H}}} = \langle \rho_1 \mid \rho_2 \rangle_{\mathbb{B}_2}. \tag{2.14}$$

Theorem 2.7 *With the usual notation the following overlap condition holds.*

$$\int_{\mathcal{G}_{*}} \overline{W(\rho_1 \mid X^{*})} W(\rho_2 \mid X^{*}) [\sigma(X^{*})]^{-1} dX^{*} = Tr[\rho_1^{*}\rho_2]. \tag{2.15}$$

Proof. Using the Definition (2.4)

$$
\begin{aligned}
&\int_{\mathcal{G}_*} \overline{W(\rho_1 \mid X^*)} W(\rho_2 \mid X^*)[\sigma(X^*)]^{-1} dX^* \\
= & \int_{\mathcal{G}_*} \overline{\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X_1\rangle} \langle U(g)C^{-1}\psi_1 \mid \varphi_1\rangle [\sigma(X^*)m(X_1)]^{\frac{1}{2}} dX_1} \\
& \times \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*;X_2\rangle} \langle U(g)C^{-1}\psi_2 \mid \varphi_2\rangle [\sigma(X^*)m(X_2)]^{\frac{1}{2}} dX_2 [\sigma(X^*)]^{-1} dX^* \\
= & \frac{1}{(2\pi)^n} \int_{\mathcal{G}_*} \int_{N_0} \int_{N_0} e^{i\langle X^*;X_1\rangle} e^{-i\langle X^*;X_2\rangle} \overline{\langle U(g)C^{-1}\psi_1 \mid \varphi_1\rangle} \\
& \times \langle U(g)C^{-1}\psi_2 \mid \varphi_2\rangle [m(X_1)m(X_2)]^{\frac{1}{2}} dX_1 dX_2 dX^* \\
= & \frac{1}{(2\pi)^n} \int_{\mathcal{G}_*} \int_{N_0} \int_{N_0} e^{i\langle X^*;(X_1-X_2)\rangle} \overline{\langle U(g)C^{-1}\psi_1 \mid \varphi_1\rangle} \\
& \times \langle U(g)C^{-1}\psi_2 \mid \varphi_2\rangle [m(X_1)m(X_2)]^{\frac{1}{2}} dX_1 dX_2 dX^*.
\end{aligned}
$$

Using the delta function, $\int_{\mathcal{G}_*} e^{i\langle X^*;(X_1-X_2)\rangle} dX^* = (2\pi)^n \delta(X_1 - X_2)$ we have

$$
\begin{aligned}
&\int_{\mathcal{G}_*} \overline{W(\rho_1 \mid X^*)} W(\rho_2 \mid X^*)[\sigma(X^*)]^{-1} dX^* \\
= & \frac{1}{(2\pi)^n} \int_{N_0} \int_{N_0} (2\pi)^n \delta(X_1 - X_2) \overline{\langle U(g)C^{-1}\psi_1 \mid \varphi_1\rangle} \langle U(g)C^{-1}\psi_2 \mid \varphi_2\rangle \\
& \times [m(X_1)m(X_2)]^{\frac{1}{2}} dX_1 dX_2.
\end{aligned}
$$

Using $\int_{N_0} f(X_1)\delta(X_1 - X_2) dX_1 = f(X_2)$, we have

$$
\begin{aligned}
&\int_{\mathcal{G}_*} \overline{W(\rho_1 \mid X^*)} W(\rho_2 \mid X^*)[\sigma(X^*)]^{-1} dX^* \\
= & \int_{N_0} \overline{\langle U(e^{X_2})C^{-1}\psi_1 \mid \varphi_1\rangle} \langle U(e^{X_2})C^{-1}\psi_2 \mid \varphi_2\rangle m(X_2) dX_2 \\
= & \int_{N_0} \overline{\langle U(e^{X})C^{-1}\psi_1 \mid \varphi_1\rangle} \langle U(e^{X})C^{-1}\psi_2 \mid \varphi_2\rangle m(X) dX.
\end{aligned}
$$

Using the orthogonality relation(1.37), we have

$$\begin{aligned}
&\int_{\mathcal{G}_*} \overline{W(\rho_1 \mid X^*)} W(\rho_2 \mid X^*)[\sigma(X^*)]^{-1} dX^* \\
&= \langle CC^{-1}\psi_2 \mid \psi_1\rangle \langle CC^{-1}\varphi_1 \mid \varphi_2\rangle \\
&= \langle \psi_2 \mid \psi_1\rangle \langle \varphi_1 \mid \varphi_2\rangle \\
&= \langle \rho_1 \mid \rho_2\rangle \\
&= Tr[\rho_1^* \rho_2].
\end{aligned}$$

Hence the theorem. ∎

Note 2.4 *The Wigner function satisfies the following*

$$\int_{\mathcal{G}_*} \overline{W(\mid \varphi\rangle\langle\psi \mid\mid X^*)} W(\rho \mid X^*)[\sigma(X^*)]^{-1} dX^* = \langle \varphi \mid \rho\psi\rangle \tag{2.16}$$

where $\rho_2 = \rho$ *and* $\rho_1 = \mid \varphi\rangle\langle\psi \mid$.

Theorem 2.8 *Using the Wigner function, we can have the following reconstruction formula*

$$\rho = \frac{1}{(2\pi)^{\frac{1}{n}}} \int_{\mathcal{G}_*} [\int_{\mathcal{N}_0} e^{i\langle X^*; X_2\rangle} W(\rho \mid X^*) U(e^X) C^{-1} [\frac{m(X)}{\sigma(X^*)}]^{\frac{1}{2}} dX] dX^*. \tag{2.17}$$

Proof. Consider

$$
\begin{aligned}
&\overline{\int_{\mathcal{G}_\bullet} \overline{W(|\varphi\rangle\langle\psi \,||\, X^\bullet)} W(\rho \mid X^\bullet)[\sigma(X^\bullet)]^{-1} dX^\bullet} \\
=& \int_{\mathcal{G}_\bullet} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^\bullet ; X\rangle} \langle U(e^X)C^- \psi | \varphi\rangle [\sigma(X^\bullet) m(X)]^{\frac{1}{2}} dX W(\rho \mid X^\bullet) \\
& \times [\sigma(X^\bullet)]^{-1} dX^\bullet \\
=& \int_{\mathcal{G}_\bullet} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{N}_0} e^{i\langle X^\bullet ; X\rangle} \overline{\langle U(e^X)C^- \psi | \varphi\rangle} [\sigma(X^\bullet) m(X)]^{\frac{1}{2}} dX W(\rho \mid X^\bullet) \\
& \times [\sigma(X^\bullet)]^{-1} dX^\bullet \\
=& \int_{\mathcal{G}_\bullet} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{N}_0} e^{i\langle X^\bullet ; X\rangle} \langle \varphi | U(e^X)C^- \psi\rangle W(\rho \mid X^\bullet) [\frac{m(X)}{\sigma(X^\bullet)}]^{\frac{1}{2}} dX dX^\bullet \\
=& \langle \varphi | \int_{\mathcal{G}_\bullet} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{N}_0} e^{i\langle X^\bullet ; X\rangle} U(e^X)C^- W(\rho \mid X^\bullet) [\frac{m(X)}{\sigma(X^\bullet)}]^{\frac{1}{2}} dX dX^\bullet \psi\rangle \\
=& \langle \varphi | \rho \psi\rangle \;\; by \; Note(2.4)
\end{aligned}
$$

$$
\Rightarrow \rho = \frac{1}{(2\pi)^{\frac{1}{n}}} \int_{\mathcal{G}_\bullet} [\int_{\mathcal{N}_0} e^{i\langle X^\bullet ; X_2\rangle} W(\rho \mid X^\bullet) U(e^X) C^{-1} [\frac{m(X)}{\sigma(X^\bullet)}]^{\frac{1}{2}} dX] dX^\bullet . \tag{2.18}
$$

Hence the theorem. ■

2.3 The Wigner function and wavelet transform

Definition 2.6 *Let U be a square integrable representation and $\eta \in \mathbb{A}$ an admissible vector with $I(\eta)$ as in definition(1.18). The wavelet transform $f_{\eta,\varphi}$ of an arbitrary $\varphi \in \mathbb{H}$ is defined as*

$$
f_{\eta,\varphi}(g) = \frac{1}{[I(\eta)]^{\frac{1}{2}}} \langle U(g)\eta \mid \varphi\rangle, \quad g \in G \tag{2.19}
$$

Note 2.5 *The wavelet transform is a square-integrable function on G and is an element of the Hilbert-space $L^2(G, d\mu)$.*

Theorem 2.9 *The map $\varphi \to f_{\eta,\varphi}$ is an isometry.*

$$\int_G \mid f_{\eta,\varphi}(g) \mid^2 d\mu(g) = \parallel \varphi \parallel^2 \tag{2.20}$$

Proof. We know from the orthogonality relation (1.37)

$$\int_G \overline{\langle U(g)\eta_2 \mid \varphi_2\rangle}\langle U(g)\eta_1 \mid \varphi_1\rangle d\mu(g) = \langle C\eta_1 \mid C\eta_2\rangle\langle \varphi_2 \mid \varphi_1\rangle \tag{2.21}$$

If $\eta_1 = \eta_2 = \varphi_1 = \varphi_2 = \eta$ then

$$\langle C\eta \mid C\eta\rangle\langle \eta \mid \eta\rangle = \int_G \overline{\langle U(g)\eta \mid \eta\rangle}\langle U(g)\eta \mid \eta\rangle d\mu(g)$$

$$\langle C\eta \mid C\eta\rangle = \frac{1}{\parallel \eta \parallel^2}\int_G \mid \langle U(g)\eta \mid \eta\rangle \mid^2 d\mu(g)$$

again if $\eta_1 = \eta_2 = \eta$. (2.21)$\Rightarrow$

$$\int_G \overline{\langle U(g)\eta \mid \varphi_2\rangle}\langle U(g)\eta \mid \varphi_1\rangle d\mu(g) = \langle C\eta \mid C\eta\rangle\langle \varphi_2 \mid \varphi_1\rangle$$

$$= \frac{1}{\parallel \eta \parallel^2}\int_G \mid \langle U(g)\eta \mid \eta\rangle \mid^2 d\mu(g)\langle \varphi_2 \mid \varphi_1\rangle \tag{2.22}$$

Let U be a representation of the group G on a Hilbert space $\mathbb{H}$ and η be an admissible vector such that

$$c(\eta) = \frac{1}{\parallel \eta \parallel^2}\int_G \mid \langle U(g)\eta \mid \eta\rangle \mid^2 d\mu(g) < \infty$$

(2.22)$\Rightarrow$

$$\int_G \overline{\langle U(g)\eta \mid \varphi_2\rangle}\langle U(g)\eta \mid \varphi_1\rangle d\mu(g) = c(\eta)\langle \varphi_2 \mid \varphi_1\rangle$$

let $\varphi_2 = \varphi_1 = \varphi$ then

$$\int_G \overline{\langle U(g)\eta \mid \varphi\rangle}\langle U(g)\eta \mid \varphi\rangle d\mu(g) = \parallel \varphi \parallel^2$$

$$\int_G | f_{\eta,\varphi}(g) |^2 \, d\mu(g) = \| \varphi \|^2 .$$

Hence the theorem. ■

In the following definition we define the term Coherent state. We will not establish any theoretical results about coherent states. For details on coherent states, one may refer to [2].

Definition 2.7 *As before let U be a square integrable representation and $\eta \in \mathbb{H}$ be an admissible vector. Let $\eta_g = U(g)\eta$ then the following resolution of the identity holds*

$$\frac{1}{I(\eta)} \int_{\mathcal{G}} | \eta_g \rangle \langle \eta_g | \, d\mu(g) = I. \tag{2.23}$$

Further, the vectors $[c(\eta)]^{-\frac{1}{2}}\eta$ are called the coherent states of the group G .

2.4 A relation between the Wigner function and wavelet transform

In this section we establish a relation between the Wigner function $W(\rho \mid X^*)$ and the wavelet transform $f_{\eta,\varphi}$ of $\varphi \in \mathbb{H}$. In Theorem(2.11) we write a Wigner function in terms of the wavelet tranform and in Theorem(2.12) we do the reverse.

Theorem 2.10 *For a fixed admissible vector $\eta \in \mathbb{A}$ and arbitrary $\varphi \in \mathbb{H}$ we have*

$$W(\rho_{\eta,\varphi} \mid X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} f_{\eta,\varphi}(e^X)[\sigma(X^*)m(X)]^{\frac{1}{2}} dX \tag{2.24}$$

where, $\rho_{\eta,\varphi} = \frac{1}{[c(\eta)]^{\frac{1}{2}}} \mid \varphi \rangle \langle \eta \mid C.$

Proof. We have by eqn.(2.7)

$$\begin{aligned}
& W(\rho \mid X^*) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} Tr[U(e^{-X})\rho C^{-1}][\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} \langle U(e^{X})C^{-1}\psi \mid \varphi\rangle [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX.
\end{aligned}$$

Let $\rho = \rho_{\eta,\varphi} = \frac{1}{[C(\eta)]^{\frac{1}{2}}} \mid \varphi\rangle\langle\eta \mid C$, then

$$\begin{aligned}
& W(\rho_{\eta,\varphi} \mid X^*) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} \langle \frac{1}{[c(\eta)]^{\frac{1}{2}}} U(e^{X})C^{-1}\eta \mid C\varphi\rangle [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} \frac{1}{[c(\eta)]^{\frac{1}{2}}} \langle U(e^{X})C^{-1}\eta \mid C\varphi\rangle [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} f_{\eta,\varphi}(e^{X}) [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX.
\end{aligned}$$

Hence the theorem. ■

Theorem 2.11 *With the same notation as in Theorem(2.10) we have*

$$f_{\eta,\varphi}(e^{X}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}^*} e^{i\langle X^*;X\rangle} W(\rho_{\eta,\varphi} \mid X^*) [\sigma_\lambda(X^*)m(X)]^{-\frac{1}{2}} dX^*. \tag{2.25}$$

Proof. We know from the relation between the Wigner function and the wavelet transform

$$\begin{aligned}
& W(\rho_{\eta,\varphi} \mid X^*) \\
= & \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{N}_0} e^{-i\langle X^*;X\rangle} f_{\eta,\varphi}(e^{X}) [\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} dX
\end{aligned}$$

This relation is easily inverted.

$$
\begin{aligned}
& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}^*} e^{i\langle X^*;Y\rangle} W(\rho_{\eta,\varphi} \mid X^*)[\sigma_\lambda(X^*)m(X)]^{-\frac{1}{2}} dX^* \\
= & \frac{1}{(2\pi)^n} \int_{\mathcal{G}^*} \int_{\mathcal{N}_0} e^{i\langle X^*;Y\rangle} e^{-i\langle X^*;X\rangle} f_{\eta,\varphi}(e^X)[\sigma_\lambda(X^*)m(X)]^{\frac{1}{2}} \\
& \times [\sigma_\lambda(X^*)m(Y)]^{-\frac{1}{2}} dX^* dX \\
= & \frac{1}{(2\pi)^n} \int_{\mathcal{G}^*} \int_{\mathcal{N}_0} e^{i\langle X^*;Y-X\rangle} f_{\eta,\varphi}(e^X)[m(X)]^{\frac{1}{2}}[m(Y)]^{-\frac{1}{2}} dX^* dX.
\end{aligned}
$$

Now using the delta function

$$
\frac{1}{(2\pi)^n} \int_{\mathcal{G}^*} e^{i\langle X^*;Y-X\rangle} dX^* = \delta(Y - X) \quad and \quad \int_{\mathcal{N}_0} f(X)\delta(Y - X) dX = f(Y).
$$

we have

$$
\begin{aligned}
& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}^*} e^{i\langle X^*;Y\rangle} W(\rho_{\eta,\varphi} \mid X^*)[\sigma_\lambda(X^*)m(X)]^{-\frac{1}{2}} dX^* \\
= & \int_{\mathcal{N}_0} \delta(Y - X) f_{\eta,\varphi}(e^X)[m(X)]^{\frac{1}{2}}[m(Y)]^{-\frac{1}{2}} dX \\
= & f_{\eta,\varphi}(e^Y).
\end{aligned}
$$

Thus

$$
f_{\eta,\varphi}(e^X) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{G}^*} e^{i\langle X^*;X\rangle} W(\rho_{\eta,\varphi} \mid X^*)[\sigma_\lambda(X^*)m(X)]^{-\frac{1}{2}} dX^*.
$$

Hence the theorem. ■

Chapter 3

In this final chapter we build Wigner functions on the group G_{ab}. Then we establish a connection between the Wigner function and the wavelet transform on the same group. Basically, we will go through all the concepts discussed in the first two chapters on this particular group. Similar work can be found in [1] for the affine group and in [3] for the SIM(2) group.

3.1 Lie algebra of the group G_{ab}.

Consider the group

$$G_{ab} = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mid a > 0, b \in \mathbb{R} \ and \ c \ is \ a \ fixed \ constant, c \neq 1 \right\}.$$

Let X_1 and X_2 be two elements of the Lie algebra $\mathcal{G}_{ab}$ of the group G_{ab}. Take $\begin{pmatrix} a & 0 \\ 0 & a^c \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ as a one parameter subgroups of G_{ab}. Let $a = \exp(\lambda)$.

So $\lambda = \log a$. Now

$$\exp(\lambda X_1) = \begin{pmatrix} \exp(\lambda) & 0 \\ 0 & \exp(\lambda c) \end{pmatrix}, \ \exp(bX_2) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore the basis for the Lie algebra $\mathcal{G}_{ab}$ is

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

Now the general element in the Lie algebra can be written as

$$X = x^1 X_1 + x^2 X_2 = \begin{pmatrix} x^1 & 0 \\ 0 & cx^1 \end{pmatrix} + \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix} \tag{3.2}$$

The group element obtained from the exponential map is

$$\begin{aligned} g &= \exp(X) \\ &= \exp\begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix} \\ &= I + \begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix}^3 + \ldots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} (x^1)^2 & (1+c)x^1x^2 \\ 0 & c(x^1) \end{pmatrix}^2 \\ &+ \frac{1}{3!}\begin{pmatrix} (x^1)^3 & (1+c)(x^1)^2x^2 + c^2(x^1)^2x^2 \\ 0 & c(x^1) \end{pmatrix}^3 + \ldots \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 + x^1 + \frac{1}{2!}(x^1)^2 + \frac{1}{3!}(x^1)^3 + \ldots & 0 + x^2 + \frac{(1+c)x^1x^2}{2!} + \frac{(1+c+c^2)(x^1)^2x^2}{3!} + \ldots \\ 0 & 1 + cx^1 + \frac{(cx^1)^2}{2!} + \frac{(cx^1)^3}{3!} + \ldots \end{pmatrix} \\
&= \begin{pmatrix} \exp(x^1) & \frac{x^2}{x^1(c-1)}(\exp(cx^1) - \exp(x^1)) \\ 0 & \exp(cx^1) \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & a^c \end{pmatrix}
\end{aligned}
$$

Thus $a = \exp(x^1)$, $b = \frac{x^2}{x^1(c-1)}(\exp(cx^1) - \exp(x^1))$, which gives us

$$x^1 = \log a, \ \ x^2 = \frac{(c-1)b\log a}{a^c - a}. \tag{3.3}$$

Every $X \in \mathcal{G}_{ab}$ is mapped to $g \in G_{ab}$ by the exponential map and we can use $\vec{x} = (x^1, x^2) \in \mathbb{R}^2$ as the coordinates for the elements of the Lie algebra $\mathcal{G}_{ab}$.

3.2 Haar Measure

Here we build left and right invariant Haar measures on the group G_{ab}. Let $g = \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}$. Fix an element $g_0 = \begin{pmatrix} a_0 & b_0 \\ 0 & a_0^c \end{pmatrix}$.

Then the left action of g_0 on g is

$$
\begin{aligned}
g_0 g &= \begin{pmatrix} a_0 & b_0 \\ 0 & a_0^c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \\
&= \begin{pmatrix} a_0 a & a_0 b + b_0 a^c \\ 0 & a_0^c a^c \end{pmatrix}.
\end{aligned}
$$

Let $a' = a_0 a$, then $da' = a_0 da$ and $b' = a_0 b + b_0 a^c$. Thus $db' = a_0 db$. Now $\frac{da'db'}{a'^2} = \frac{dadb}{a^2}$. Thus we have obtained our left Haar measure as $d\mu(g_0 g) = \frac{dadb}{a^2}$. The coordinate transformation $a = \exp(x^1)$ and $b = \frac{x^2}{x^1(c-1)}(\exp(cx^1) - \exp(x^1))$ yeilds $da = \exp(x^1)dx^1$ and $db = \frac{\exp(cx^1)-\exp(x^1))dx^2}{(c-1)x^1}$. Let us calculate $\frac{dadb}{a^2}$ under this coordinate transformation.

$$
\begin{aligned}
&\frac{dadb}{a^2} \\
&= \frac{\exp(x^1)(\exp(cx^1) - \exp(x^1))}{(c-1)x^1\exp(2x^1)}dx^1dx^2 \\
&= \frac{\exp(cx^1 + x^1 - 2x^1) - 1}{(c-1)x^1}dx^1dx^2 \\
&= \frac{\exp(cx^1 - x^1) - 1}{(c-1)x^1}dx^1dx^2 \\
&= \frac{1 - \exp(-(1-c)x^1)}{(1-c)x^1}dx^1dx^2.
\end{aligned}
$$

Right action of g_0 on g gives us
$\begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ 0 & a_0^c \end{pmatrix} = \begin{pmatrix} aa_0 & ab_0 + ba_0^c \\ 0 & (aa_0)^c \end{pmatrix}$. Now let $a' = aa_0$ then $da' = a_0 da$ and $b' = ab_0 + ba_0^c$ then $db' = a_0 db$. $\frac{da'db'}{a'} = \frac{dadb}{aa^c}$. Thus we have obtained the right Haar measure as $d\mu_r(gg_0) = \frac{dadb}{a^{1+c}}$. The coordinate transformation $a = \exp(x^1)$ and $b = \frac{x^2(\exp(cx^1)-\exp(x^1))}{x^1(c-1)}$ yeilds $da = \exp(x^1)dx^1$ and $db = \frac{\exp(cx^1)-\exp(x^1)}{(c-1)x^1}dx^2$. Let us calculate $\frac{dadb}{a^{1+c}}$ under this coordinate transformation.

$$
\begin{aligned}
&\frac{dadb}{a^{1+c}} \\
&= \frac{\exp(x^1)(\exp(cx^1) - \exp(x^1))}{(c-1)x^1\exp((1+c)x^1)}dx^1dx^2 \\
&= \frac{\exp((1+c)x^1) - \exp(2x^1)}{(c-1)x^1\exp((1+c)x^1)}dx^1dx^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-\exp(2x^1 - x^1 - cx^1))}{(c-1)x^1} dx^1 dx^2 \\
&= \frac{(1-\exp((1-c)x^1))}{(c-1)x^1} dx^1 dx^2 \\
&= \frac{\exp((1-c)x^1) - 1}{(1-c)x^1} dx^1 dx^2.
\end{aligned}
$$

The modular function is

$$\triangle(g) = \frac{1}{a^{(1-c)}}. \tag{3.4}$$

We know from remark (2.1) $d\mu(g) \to m(\vec{x})d\vec{x}$ under the coordinate transformation $g = \exp(X)$. Now $\frac{dadb}{a^2} = m(\vec{x})d\vec{x} = \frac{1-\exp(-(1-c)x^1)}{(1-c)x^1} dx^1 dx^2$. Thus

$$m(x^1, x^2) = \frac{1-\exp(-(1-c)x^1)}{(1-c)x^1}. \tag{3.5}$$

3.3 Adjoint and Coadjoint action

In this section we define adjoint and coadjoint actions on the group G_{ab}. Let $g = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}, a > 0, b \in \mathbb{R} \; and \;\; c \; is \; a \; fixed \; constant, c \neq 1 \right\}$. Then the adjoint action of the group G_{ab} on an element of Lie algebra $\mathcal{G}_{ab}$ is

$$
\begin{aligned}
& Ad_g X \\
&= gXg^{-1} \\
&= \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \begin{pmatrix} x^1 & x^2 \\ 0 & cx^1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}^{-1} \\
&= \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \begin{pmatrix} a^{-1}x^1 & -ba^{-1}a^{-c}x^1 + a^{-c}x^1 \\ 0 & cx^1 a^{-c} \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} x^1 & -ba^{-c}x^1 + a^{1-c}x^2 + cx^1ba^{-c} \\ 0 & cx^1 \end{pmatrix}$$

$$= \begin{pmatrix} x^1 & (c-1)ba^{-c}x^1 + a^{1-c}x^2 \\ 0 & cx^1 \end{pmatrix}$$

Thus the matrix of this transformation which acts on vectors $\vec{x} = (x_1, x_2)^T \in \mathbb{R}^2$ is

$$M(g) = \begin{pmatrix} 1 & 0 \\ (c-1)ba^{-c} & a^{1-c} \end{pmatrix}. \tag{3.6}$$

On the dual of the Lie algebra $\mathcal{G}_{ab}$, $X^* \in \mathcal{G}^*_{ab}$ has coordinates $\vec{\xi} = (\xi_1, \xi_2)^T$. The coadjoint action is represented by the inverse transpose matrix.

$$M^*(g) = (M(g^{-1}))^T = \begin{pmatrix} 1 & (1-c)ba^{-1} \\ 0 & a^{c-1} \end{pmatrix}. \tag{3.7}$$

Now the determinants of these matrices are

$$\| Ad_g \| = a^{1-c} = \| Ad^*_g \|^{-1} \tag{3.8}$$

Th coadjoint representation of the group is carried by the Hilbert space $L^2(\mathbb{R}^2, d\vec{\xi})$.

$$\begin{aligned} & (V^*(g)\hat{F})(\vec{\xi}) \\ = \; & \| Ad^*_g \|^{-\frac{1}{2}} \hat{F}(Ad^*_{g^{-1}} X) \\ = \; & a^{\frac{(1-c)}{2}} \hat{F} \begin{pmatrix} 1 & (c-1)ba^{-c} \\ 0 & a^{1-c} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \end{aligned}$$

$$\Rightarrow (V^*(g)\hat{F})(\vec{\xi}) = a^{\frac{(1-c)}{2}} \hat{F}(\xi_1 + (c-1)ba^{-c}\xi_2, \xi_2 a^{1-c}) \tag{3.9}$$

3.4 Coadjoint orbits of the group

The coadjoint orbits $\mathcal{O}^*_+$, $\mathcal{O}^*_-$ and $\mathcal{O}^*_\lambda$ are defined as follows:

1. The orbit obtained by acting with the matrices $M(g^{-1})^T$ on the column vector $(0,1)^T$,

$$
\begin{aligned}
&\mathcal{O}^*_+ \\
&= \{M(g^{-1})^T(0,1)^T\} \\
&= \left\{\begin{pmatrix} 1 & (1-c)ba^{-1} \\ 0 & a^{c-1} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \\
&= \left\{\begin{pmatrix} (1-c)ba^{-1} \\ a^{c-1} \end{pmatrix}\right\} \\
&= \{\vec{\xi_+} = (\xi_1,\xi_2) \in \mathbb{R}^2, \xi_2 > 0\} = \mathbb{R} \times \mathbb{R}^+_*.
\end{aligned}
$$

2. The orbit obtained by acting with the matrices $M(g^{-1})^T$ on the column vector $(0,-1)^T$,

$$
\begin{aligned}
&\mathcal{O}^*_- \\
&= \{M(g^{-1})^T(0,-1)^T\} \\
&= \left\{\begin{pmatrix} 1 & (1-c)ba^{-1} \\ 0 & a^{c-1} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} \\
&= \left\{\begin{pmatrix} -(1-c)ba^{-1} \\ -a^{c-1} \end{pmatrix}\right\} \\
&= \{\vec{\xi_-} = (\xi_1,\xi_2) \in \mathbb{R}^2, \xi_2 < 0\} = \mathbb{R} \times \mathbb{R}^-_*.
\end{aligned}
$$

3. Applying the matrices to the column vector $(\alpha, 0)^T$, for each $\alpha \in \mathbb{R}$, we obtain an orbit that consists of the single point $(\alpha, 0)^T$.

$$
\begin{aligned}
&\mathcal{O}^*_\alpha \\
&= \{M(g^{-1})^T(\alpha, 0)^T\} \\
&= \left\{ \begin{pmatrix} 1 & (1-c)ba^{-1} \\ 0 & a^{c-1} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}.
\end{aligned}
$$

From the construction $\mathcal{O}^*_+$, $\mathcal{O}^*_-$, $\mathcal{O}^*_\alpha$ are disjoint. Further

$$\mathbb{R}^2 = \bigcup_{\lambda \in J} \mathcal{O}^*_\lambda \tag{3.10}$$

where $J = \{+, -, \mathbb{R}\}$.

Note 3.1 *(i) The Lebesgue measure of the set of orbits* $\mathcal{O}^*_\alpha$, $\alpha \in \mathbb{R}$ *is zero in* $\mathbb{R}^2$.

(ii) $dX^* = d\kappa(\lambda)\sigma_\lambda(X^*_\lambda)d\Omega_\lambda(X^*_\lambda)$ *Thus* $d\Omega_\pm(\vec{\xi}) = \frac{d\xi_1 d\xi_2}{|\xi_2|}$, $\vec{\xi} \in \mathcal{O}^*_\pm$

(iii) The direct integral Hilbert space $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$, *where* $\mathbb{H}_\pm = L^2(\mathcal{O}^*_\pm, d\Omega_\pm)$

3.5 Covariant coadjoint representaion

The covariant coadjoint representation on our group can be written as

$$(U^*(g)\hat{F})(\vec{\xi}) = \hat{F}(M(g)^T\vec{\xi}) = \hat{F}(\xi_1 + (c-1)ba^{-c}\xi_2, a^{1-c}\xi_2) \tag{3.11}$$

where ξ_1 is the translation in the group and ξ_2 is the scale parameter. In order to construct the Wigner function for the group, we need its unitary irreducible representations. Consider the representation $U(g)$ on the Hilbert-space $L^2(\mathbb{R}, dt)$,

$$(U(g)\varphi)(t) = a^{\frac{c-1}{2}}\varphi(\frac{t-(1-c)ba^{-c}}{a^{1-c}}), \ \varphi \in L^2(\mathbb{R}, dt), \ g = (a, b) \in \ G_{ab}. \tag{3.12}$$

Now let us show $U(g)$ is unitary representation.

$$\begin{aligned}
& (U(g_1)U(g_2)\varphi)(t) \\
= \ & a_1^{\frac{c-1}{2}} U(g_2)\varphi(\frac{t-(1-c)b_1 a_1^{-c}}{a_1^{1-c}}) \\
= \ & a_1^{\frac{c-1}{2}} a_2^{\frac{c-1}{2}} \varphi(\frac{t-(1-c)b_1 a_1^{-c} - (1-c)b_2 a_2^{-c} a_1^{1-c}}{(a_1 a_2)^{1-c}}) \\
= \ & (a_1 a_2)^{\frac{c-1}{2}} \varphi(\frac{t-(1-c)(b_1 a_1^{-c} + b_2 a_2^{-c} a_1^{1-c})}{(a_1 a_2)^{1-c}}) \\
= \ & (a_1 a_2)^{\frac{c-1}{2}} \varphi(\frac{t+(c-1)(a_1 b_2 + b_1 a_2^{c})(a_1 a_2)^{-c}}{(a_1 a_2)^{1-c}}) \\
= \ & (U(g_1 g_2)\varphi)(t).
\end{aligned}$$

i,e) $(U(g_1)U(g_2)\varphi)(t) = (U(g_1 g_2)\varphi)(t)$.

$$\begin{aligned}
& (U(e)\varphi)(t) \\
= \ & 1 \times \varphi(\frac{t-(1-c) \times 0 \times 1}{1^{1-c}}) \\
= \ & \varphi(t).
\end{aligned}$$

i,e) $(U(e)\varphi)(t) = \varphi(t)$.

Thus it is a representation.

Unitarity:

$$
\begin{aligned}
& \| U(g)\varphi \|^2 \\
= & \int_{G_{ab}} | U(g)\varphi |^2 \, dt \\
= & \int_{G_{ab}} | a^{\frac{c-1}{2}} \varphi(\frac{t-(1-c)ba^{-c}}{a^{1-c}}) |^2 \, dt \\
= & \int_{G_{ab}} | \varphi(\omega) |^2 \, d\omega, \quad where \ \omega = \frac{t-(1-c)ba^{-c}}{a^{1-c}} \\
= & \| \varphi \|^2 .
\end{aligned}
$$

This representation is unitary but not irreducible. To find its irreducible components, we take the Fourier transformation of (3.12).

$$
\begin{aligned}
& \int_{-\infty}^{\infty} (U(g)\varphi)(t) e^{-i\omega t} dt \\
= & \int_{-\infty}^{\infty} a^{\frac{c-1}{2}} \varphi(\frac{t-(1-c)ba^{-c}}{a^{1-c}}) e^{-i\omega t} dt, \ \ let \ z = \frac{t-(1-c)ba^{-c}}{a^{1-c}} \\
= & \int_{-\infty}^{\infty} a^{\frac{c-1}{2}} a^{1-c} \varphi(z) e^{-i\omega((1-c)ba^{-c}+a^{1-c}z)} dz \\
= & \int_{-\infty}^{\infty} a^{\frac{1-c}{2}} \varphi(z) e^{-i\omega(1-c)ba^{-c}} e^{-i\omega a^{1-c}z} dz \\
= & \int_{-\infty}^{\infty} a^{\frac{1-c}{2}} \varphi(z) e^{-i\omega a^{1-c}z} e^{-i\omega(1-c)ba^{-c}} dz \\
= & a^{-\frac{c-1}{2}} \varphi(a^{1-c}\omega) e^{-i(1-c)ba^{-c}\omega}.
\end{aligned}
$$

Thus the unitary irreducible representation of G_{ab} on the Fourier-transformed Hilbert space $L^2(\mathbb{R}, d\omega)$ is

$$
(\hat{U}(g)\hat{\varphi})(\omega) = a^{\frac{1-c}{2}} \hat{\varphi}(a^{1-c}\omega) e^{-i(1-c)ba^{-c}\omega}, \ \ \hat{\varphi} \in L^2(\mathbb{R}, d\omega), \ g \in G_{ab}. \tag{3.13}
$$

Note 3.2 *(i) Each of the two subspaces of the functions defined on the intervals* $(0, \infty)$, *and* $(-\infty, 0)$ *are denoted as* $\mathbb{H}^{\pm} = L^2(\mathbb{R}^{\pm}, d\omega)$. *Further* $\mathbb{H}^{\pm}$ *are* *stable under the action of $\hat{U}(g)$. In fact $\mathbb{H}^{\pm}$ are irreducible subspaces under this action.*

(ii) $U^{\pm}(g)$ are square integrable representations of G_{ab} on $\mathbb{H}^{\pm}$ respectively in the sense of Definitions(1.18) and (1.19).

3.6 Wigner functions for $\hat{U}(g)^{+}$

In this section we build the Wigner function for $\hat{U}(g)^{+}$ on the group G_{ab}. In a similar way we do it for $\hat{U}(g)^{-}$ too. For this purpose first we will find the operator C. We know the orthogonality relations by (1.37)

$$\int_{G} \overline{\langle \hat{U}(g)\eta_2 \mid \varphi_2\rangle} \langle \hat{U}(g)\eta_1 \mid \varphi_1\rangle d\mu(g) = \langle C\eta_1 \mid C\eta_2\rangle \langle \varphi_2 \mid \varphi_1\rangle .$$

let $\eta_1 = \eta_2 = \hat{\eta}$ and $\varphi_1 = \varphi_2 = \varphi$ then

$$\langle C\hat{\eta} \mid C\hat{\eta}\rangle \langle \varphi \mid \varphi\rangle = \int_{G_{ab}} \overline{\langle U(g)\hat{\eta} \mid \varphi\rangle} \langle U(g)\hat{\eta} \mid \varphi\rangle d\mu(g).$$

Thus,

$$\begin{aligned}
&\| C\hat{\eta} \|^2 \\
&= \frac{1}{\| \varphi \|^2} \int_{G_{ab}} \mid \langle \hat{U}(g)\hat{\eta} \mid \varphi\rangle \mid^2 d\mu(g) \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \overline{\langle \hat{U}(g)\hat{\eta} \mid \varphi\rangle} \langle \hat{U}(g)\hat{\eta} \mid \varphi\rangle \frac{dadb}{a^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\varphi(\omega)} \hat{U}(g) \hat{\eta}(\omega) \ \overline{\hat{U}(g) \hat{\eta}(\omega') \varphi(\omega')} d\omega d\omega' \frac{dadb}{a^2} \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\varphi(\omega)} a^{\frac{1-c}{2}} \hat{\eta}(a^{1-c}\omega) e^{-i(1-c)ba^{-c}\omega} a^{\frac{1-c}{2}} \varphi(\omega') \\
&\times \ \overline{\hat{\eta}(a^{1-c}\omega')} e^{i(1-c)ba^{-c}\omega'} \frac{dadbd\omega d\omega'}{a^2} \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^{1+c}} \overline{\varphi(\omega)} \varphi(\omega') \hat{\eta}(a^{1-c}\omega) \overline{\hat{\eta}(a^{1-c}\omega')} \\
&\times \ e^{i(1-c)ba^{-c}(\omega'-\omega)} dadbd\omega d\omega'
\end{aligned}
$$

Let $b' = (1-c)ba^{-c}$ then $db' = (1-c)a^{-c}db$ then

$$
\begin{aligned}
& \| C\hat{\eta} \|^2 \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^{1+c}} \overline{\varphi(\omega)} \varphi(\omega') \hat{\eta}(a^{1-c}\omega) \overline{\hat{\eta}(a^{1-c}\omega')} \\
&\times \ e^{ib'(\omega'-\omega)} \frac{db'}{(1-c)a^{-c}a^{1-c}} dad\omega d\omega' \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\varphi(\omega)} \varphi(\omega') \hat{\eta}(a^{1-c}\omega) \overline{\hat{\eta}(a^{1-c}\omega')} e^{ib'(\omega'-\omega)} \\
&\times \frac{db'}{(1-c)a} dad\omega d\omega'.
\end{aligned}
$$

Using $\int_{-\infty}^{\infty} e^{ib'(\omega'-\omega)} db' = 2\pi\delta(\omega'-\omega)$ and $\int_0^{\infty} f(\omega')\delta(\omega'-\omega)d\omega = f(\omega)$.we have

$$
\begin{aligned}
& \| C\hat{\eta} \|^2 \\
&= \frac{1}{\| \varphi \|^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi\delta(\omega'-\omega) \overline{\varphi(\omega)} \varphi(\omega') \hat{\eta}(a^{1-c}\omega) \overline{\hat{\eta}(a^{1-c}\omega')} \\
&\times \frac{1}{(1-c)a} dad\omega d\omega' \\
&= \frac{1}{\| \varphi \|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\pi \overline{\varphi(\omega)} \varphi(\omega) \hat{\eta}(a^{1-c}\omega) \overline{\hat{\eta}(a^{1-c}\omega)} \frac{1}{(1-c)a} dad\omega
\end{aligned}
$$

Again let $a' = a^{1-c}\omega$ then $da' = \omega(1-c)a^{-c}da$.

$$\begin{aligned}
& \| C\hat{\eta} \|^2 \\
= & \frac{1}{\| \varphi \|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\pi a^c}{a\omega(1-c)^2} \overline{\varphi(\omega)} \varphi(\omega) \hat{\eta}(a') \overline{\hat{\eta}(a')} da' d\omega \\
= & \frac{1}{\| \varphi \|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\pi}{(1-c)^2} \overline{\varphi(\omega)} \varphi(\omega) \frac{\hat{\eta}(a') \overline{\hat{\eta}(a')}}{a'} da' d\omega \\
= & \frac{1}{\| \varphi \|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\pi}{(1-c)^2} \mid \varphi(\omega) \mid^2 \frac{\hat{\eta}(a') \overline{\hat{\eta}(a')}}{a'} da' d\omega \\
= & \frac{1}{\| \varphi \|^2} \| \varphi \|^2 \| \sqrt{\frac{2\pi}{a'}} \frac{\hat{\eta}(a')}{1-c} \|^2 \\
= & \sqrt{\frac{2\pi}{a'}} \frac{\hat{\eta}(a')}{1-c}.
\end{aligned}$$

Thus the operator C is obtained. i.e.,

$$C(\hat{\eta})(\omega) = \sqrt{\frac{2\pi}{\omega}} \frac{\hat{\eta}(\omega)}{1-c}, \quad \omega \geq 0. \tag{3.14}$$

Now we build the Wigner function. Consider the Wigner function

$$\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= & \frac{1}{2\pi} \int_{\hat{N}_0} e^{-i\vec{\xi}.\vec{x}} \langle U(g) C^{-1} \hat{\psi} \mid \hat{\varphi} \rangle [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} \\
= & \frac{1}{2\pi} \int_{\hat{N}_0} e^{-i\vec{\xi}.\vec{x}} \langle C^{-1} \hat{\psi} \mid U(g^{-1}) \hat{\varphi} \rangle [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} \\
= & \frac{1}{2\pi} \int_{\hat{N}_0} \int_0^{\infty} e^{-i\vec{\xi}.\vec{x}} \overline{C^{-1} \hat{\psi}(\omega)} (U(g^{-1}) \hat{\varphi})(\omega) [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} d\omega.
\end{aligned}$$

Using(3.13), we have

$$\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= & \frac{1}{2\pi} \int_{\hat{N}_0} \int_0^{\infty} e^{-i\vec{\xi}.\vec{x}} (1-c) \sqrt{\frac{\omega}{2\pi}} \overline{\hat{\psi}(\omega)} a^{-\frac{(1-c)}{2}} \hat{\varphi}(a^{c-1}\omega) \\
& \times e^{-i(1-c)(-ba^{(-1-c)}a^c\omega)} [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} d\omega.
\end{aligned}$$

Let $a = e^{x^1}$ $\quad b = \frac{x^2}{x^1(1-c)}(e^{x^1} - e^{cx^1})$. Then

$$\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= \; & \frac{1-c}{(2\pi)^{\frac{3}{2}}} \int_{\tilde{N}_0} \int_0^\infty e^{-i(\xi_1 x^1 + \xi_2 x^2)} \sqrt{\omega} \overline{\hat{\psi}(\omega)} \varphi(e^{(c-1)x^1}\omega) \\
& \times e^{\frac{(c-1)x^1}{2}} e^{x i_2 (\frac{1-e^{(c-1)x^1}}{x^1})\omega} [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} d\omega \\
= \; & \frac{1-c}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \sqrt{\omega} \overline{\hat{\psi}(\omega)} \hat{\varphi}(e^{(c-1)x^1}\omega) \\
& \times e^{ix^2(\frac{1-e^{(c-1)x^1}}{x^1}\omega - \xi^2)} [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} dx^1 dx^2 d\omega.
\end{aligned}$$

Let $\omega' = \frac{1-e^{(c-1)x^1}}{x^1}\omega$ thus $d\omega = \frac{x^1}{1-e^{(c-1)x^1}} d\omega'$. Then

$$\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= \; & \frac{1-c}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \sqrt{\frac{x^1\omega'}{1-e^{(c-1)x^1}}} \; \overline{\hat{\psi}(\frac{x^1\omega'}{1-e^{(c-1)x^1}})} \\
& \times \hat{\varphi}(e^{(c-1)x^1} \frac{\omega' x^1}{1-e^{(c-1)x^1}}) e^{x i_2(\omega' - \xi^2)} [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} \frac{x^1}{1-e^{(c-1)x^1}} dx^1 dx^2 d\omega'.
\end{aligned}$$

Put $\int_{-\infty}^\infty e^{x^2(\omega' - \xi_2)} dx^2 = 2\pi\delta(\omega' - \xi_2)$ *and* $\int_0^\infty f(\omega')\delta(\omega' - \xi_2) d\omega' = f(\xi_2)$. Then

$$\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= \; & \frac{1-c}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \sqrt{\frac{x^1\omega'}{1-e^{(c-1)x^1}}} \; \overline{\hat{\psi}(\frac{x^1\omega'}{1-e^{(c-1)x^1}})} \\
& \times \hat{\varphi}(e^{(c-1)x^1} \frac{\omega' x^1}{1-e^{(c-1)x^1}}) \delta(\omega' - \xi_2) [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} \frac{x^1}{1-e^{(c-1)x^1}} dx^1 d\omega' \\
= \; & \frac{1-c}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \sqrt{\frac{x^1\xi_2}{1-e^{(c-1)x^1}}} \; \overline{\hat{\psi}(\frac{x^1\xi_2}{1-e^{(c-1)x^1}})} \\
& \times \hat{\varphi}(e^{(c-1)x^1} \frac{\xi_2 x^1}{1-e^{(c-1)x^1}}) [\sigma(\vec{\xi}) \frac{1-e^{(c-1)x^1}}{(1-c)x^1}]^{\frac{1}{2}} \frac{x^1}{1-e^{(c-1)x^1}} dx^1 \\
= \; & \frac{1-c}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^\infty \overline{\hat{\psi}(\frac{x^1\xi_2}{1-e^{(c-1)x^1}})} \; \hat{\varphi}(e^{(c-1)x^1} \frac{\xi_2 x^1}{1-e^{(c-1)x^1}}) \xi_2 e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \\
& \times \sqrt{\frac{x^1\xi_2}{1-e^{(c-1)x^1}} \frac{1-e^{(c-1)x^1}}{(1-c)x^1} \frac{x^1}{1-e^{(c-1)x^1}}} dx^1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-c}{(2(1-c)\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \overline{\hat{\psi}(\frac{x^1\xi_2}{1-e^{(c-1)x^1}})}\ \hat{\varphi}(e^{(c-1)x^1}\frac{\xi_2 x^1}{1-e^{(c-1)x^1}})\xi_2 e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \\
&\quad \times \frac{x^1}{1-e^{(c-1)x^1}} dx^1 \\
&= \sqrt{\frac{(1-c)}{2\pi}} \int_{-\infty}^{\infty} \overline{\hat{\psi}(\frac{x^1\xi_2}{1-e^{(c-1)x^1}})}\ \hat{\varphi}(e^{(c-1)x^1}\frac{\xi_2 x^1}{1-e^{(c-1)x^1}})\xi_2 e^{-i(\xi_1 x^1)} e^{\frac{(c-1)x^1}{2}} \\
&\quad \times \frac{x^1}{1-e^{(c-1)x^1}} dx^1.
\end{aligned}
$$

Now we know $\operatorname{sinch}(u) = \frac{\sinh u}{u}$.

$$
\begin{aligned}
&\operatorname{sinch}(\frac{(c-1)x^1}{2}) \\
&= \frac{e^{\frac{(c-1)x^1}{2}} - e^{\frac{-(c-1)x^1}{2}}}{(c-1)x^1} \\
&= \frac{e^{-\frac{(c-1)x^1}{2}}(e^{(c-1)x^1}-1)}{(c-1)x^1} \\
&= \frac{e^{\frac{(1-c)x^1}{2}}(1-e^{(c-1)x^1})}{(1-c)x^1}
\end{aligned}
$$

So $\operatorname{sinch}(\frac{(1-c)x^1}{2}) = \frac{e^{-\frac{(1-c)x^1}{2}}(e^{(1-c)x^1}-1)}{(1-c)x^1}$

Again.

$$
\begin{aligned}
\frac{x^1}{1-e^{(c-1)x^1}} &= \frac{1}{\frac{e^{(c-1)x^1}-1}{-x^1}} \\
&= \frac{1}{\frac{e^{-\frac{(1-c)x^1}{2}}(e^{-\frac{(1-c)x^1}{2}} - e^{\frac{(1-c)x^1}{2}}}{-x^1}} \\
&= \frac{x^1}{e^{-\frac{(1-c)x^1}{2}}(e^{\frac{(1-c)x^1}{2}} - e^{-\frac{(1-c)x^1}{2}})}.
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{\frac{(1-c)x^1}{2}}}{(1-c)\frac{e^{-\frac{(1-c)x^1}{2}}(e^{(1-c)x^1}-1)}{(1-c)x^1}} \\
&= \frac{e^{\frac{(1-c)x^1}{2}}}{(1-c)\operatorname{sinch}(\frac{(1-c)x^1}{2})}
\end{aligned}
$$

Similarly $\frac{x^1 e^{(c-1)x^1}}{1-e^{(c-1)x^1}} = \frac{e^{-\frac{(1-c)x^1}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})}$ and $\frac{x^1 e^{\frac{(c-1)x^1}{2}}}{1-e^{(c-1)x^1}} = \frac{1}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})}$.

$$
\begin{aligned}
& W(\rho \mid \vec{\xi}) \\
= & \sqrt{\frac{(1-c)}{2\pi}} \int_{-\infty}^{\infty} \overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x^1}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})})} \\
& \times \hat{\varphi}(\frac{\xi_2 e^{-\frac{(1-c)x^1}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})}) \frac{\xi_2 e^{-i(\xi_1 x^1)}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})} dx^1.
\end{aligned}
$$

Thus the Wigner function is

$$
\begin{aligned}
W(\hat{\psi}, \hat{\varphi} \mid \xi_1, \xi_2) = & \sqrt{\frac{1}{(1-c)2\pi}} \int_{-\infty}^{\infty} \overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x^1}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})})} \\
& \times \hat{\varphi}(\frac{\xi_2 e^{-\frac{(1-c)x^1}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x^1}{2})}) \frac{\xi_2 e^{-i(\xi_1 x^1)}}{\text{sinch}(\frac{(1-c)x^1}{2})} dx^1.
\end{aligned}
\tag{3.15}
$$

which is the Wigner function for the irreducible representation $\hat{U}^+$ supported on the orbit $\mathcal{O}_+^*$ where $\xi_2 > 0$. Similarly we can find an analogous function for the irreducible representation $\hat{U}^-$ supported on $\mathcal{O}_-^*$. Thus the Wigner function for the reducible representation $\hat{U} = \hat{U}^+ + \hat{U}^-$ for arbitrary $\hat{\varphi} \in L^2(\mathbb{R}, d\omega)$ and $\hat{\psi} \in L^2(\mathbb{R}, d\omega)$ is

$$
\begin{aligned}
W(\hat{\psi}, \hat{\varphi} \mid \xi_1, \xi_2) = & \frac{1}{\sqrt{2\pi(1-c)}} \int_{-\infty}^{\infty} \overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})})} \\
& \times \hat{\varphi}(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}) \frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})} dx.
\end{aligned}
\tag{3.16}
$$

which is valid for all $\vec{\xi} \in \mathbb{R}^2$.

3.7 Covariance

Here we check the covariance relation for the Wigner function(3.16).

$W(\hat{U}(g)\hat{\psi}, \hat{U}(g)\hat{\varphi} \mid \vec{\xi}) = W(\hat{\psi}, \hat{\varphi} \mid M^T(g)\vec{\xi}), g \in G_{ab}, \;\; and \;\; \hat{\psi}, \hat{\varphi} \in \mathbb{H}$

$$W(\hat{U}(g)\hat{\psi}, \hat{U}(g)\hat{\varphi} \mid \vec{\xi}) = W(\hat{\psi}, \hat{\varphi} \mid \xi_1 + (c-1)ba^{-c}\xi_2, \xi_2 a^{1-c}) \tag{3.17}$$

Since $\;(M(g^{-1}))^T = \begin{pmatrix} 1 & (1-c)ba^{-1} \\ 0 & a^{c-1} \end{pmatrix}.$

$$\begin{aligned}
L.H.S &= W(\hat{U}(g)\hat{\psi}, \hat{U}(g)\hat{\varphi} \mid \vec{\xi}) \\
&= \frac{1}{\sqrt{2\pi(1-c)}} \int_{-\infty}^{\infty} \overline{\hat{U}(g)\hat{\psi}\Big(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)sinch(\frac{(1-c)x}{2})}\Big)} \\
&\quad \times \hat{U}(g)\hat{\varphi}\Big(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\Big) \frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})} dx \\
&= \frac{1}{\sqrt{2\pi(1-c)}} \int_{-\infty}^{\infty} \overline{a^{\frac{1-c}{2}} \hat{\psi}\Big(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\Big) e^{i(1-c)ba^{-c}\big(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\big)}} \\
&\quad \times a^{\frac{1-c}{2}} \hat{\varphi}\Big(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\Big) e^{-i(1-c)ba^{-c}\big(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\big)} \frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})} dx \\
&= \frac{1}{\sqrt{2\pi(1-c)}} \int_{-\infty}^{\infty} \overline{\hat{\psi}\Big(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\Big)} \\
&\quad \times e^{i(1-c)ba^{-c}\big(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})} - \frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\big)} \hat{\varphi}\Big(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\Big) \\
&\quad \times \frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})} dx.
\end{aligned}$$

Now

$$e^{i(1-c)ba^{-c}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}-\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)}$$

$$= e^{i(1-c)ba^{-c}\xi_2 x}$$

$$\begin{aligned} L.H.S &= \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\overline{\hat{\psi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)e^{i(1-c)ba^{-c}\xi_2 x}} \\ &\quad \times \hat{\varphi}\left(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)\frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})}dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\overline{\hat{\psi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)e^{i(1-c)ba^{-c}\xi_2 x}} \\ &\quad \times \hat{\varphi}\left(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)\frac{\mid \xi_2 \mid e^{-i(\xi_1 x)}}{\text{sinch}(\frac{(1-c)x}{2})}dx \\ &= \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\overline{\hat{\psi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)}\ \hat{\varphi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right) \\ &\quad \times \frac{e^{\frac{(1-c)x}{2}} \mid \xi_2 \mid e^{-i(\xi_1+(c-1)ba^{-c}\xi_2)x}}{\text{sinch}(\frac{(1-c)x}{2})}dx. \end{aligned}$$

$$\begin{aligned} R.H.S &= W(\hat{\psi},\hat{\varphi} \mid \xi_1+(c-1)ba^{-c}\xi_2, \xi_2 a^{1-c}) \\ &= \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\overline{\hat{\psi}\left(\frac{a^{(1-c)}\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)} \\ &\quad \times\hat{\varphi}\left(\frac{a^{(1-c)}\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)\frac{a^{(1-c)} \mid \xi_2 \mid e^{-i(\xi_1+(c-1)ba^{-c}\xi_2)x}}{\text{sinch}(\frac{(1-c)x}{2})}dx. \end{aligned}$$

put $a^{1-c} = e^{(1-c)X}$.

$$\begin{aligned} R.H.S &= \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\overline{\hat{\psi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)} \\ &\quad \times\hat{\varphi}\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}\right)\frac{e^{\frac{(1-c)x}{2}} \mid \xi_2 \mid e^{-i(\xi_1+(c-1)ba^{-c}\xi_2)x}}{\text{sinch}(\frac{(1-c)x}{2})}dx. \end{aligned}$$

Thus

$$W(\hat{U}(g)\hat{\psi},\hat{U}(g)\hat{\varphi}\mid\vec{\xi})=W(\hat{\psi},\hat{\varphi}\mid\xi_1+(c-1)ba^{-c}\xi_2,\xi_2a^{1-c}). \qquad (3.18)$$

3.8 Overlap condition

Here we check the overlap condition.

$$\int_{\mathcal{G}^*}\overline{W(\hat{\psi_1},\hat{\varphi_1}\mid\vec{\xi})}W(\hat{\psi_2},\hat{\varphi_2}\mid\vec{\xi})[\sigma(\vec{\xi})]^{-1}d\vec{\xi}=\langle\hat{\varphi_1}\mid\hat{\varphi_2}\rangle\langle\hat{\psi_2}\mid\hat{\psi_1}\rangle. \qquad (3.19)$$

Proof. Consider

$$\begin{aligned}
&\int_{\mathcal{G}^*}\overline{W(\psi_1,\varphi_1\mid\vec{\xi})}W(\psi_2,\varphi_2\mid\vec{\xi})[\sigma(\vec{\xi})]^{-1}d\vec{\xi}\\
=\;&\frac{1}{2\pi(1-c)}\int_{\mathcal{G}^*}\overline{\int_{-\infty}^{\infty}\overline{\hat{\psi}_1(\frac{\xi_2e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})}\,\hat{\varphi}_1(\frac{\xi_2e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\frac{\mid\xi_2\mid e^{-i(\xi_1x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx}\\
&\times\int_{-\infty}^{\infty}\overline{\hat{\psi}_2(\frac{\xi_2e^{\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})}\,\hat{\varphi}_2(\frac{\xi_2e^{-\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})\frac{\mid\xi_2\mid e^{-i(\xi_1y)}}{\mathrm{sinch}(\frac{(1-c)y}{2})}dy[\xi_2]^{-1}d\vec{\xi}\\
=\;&\frac{1}{2\pi(1-c)}\int_{\mathcal{G}^*}\int_{-\infty}^{\infty}\hat{\psi}_1(\frac{\xi_2e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\,\overline{\hat{\varphi}_1(\frac{\xi_2e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})}\frac{\mid\xi_2\mid e^{i(\xi_1x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx\\
&\times\int_{-\infty}^{\infty}\overline{\hat{\psi}_2(\frac{\xi_2e^{\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})}\,\hat{\varphi}_2(\frac{\xi_2e^{-\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})\frac{e^{-i(\xi_1y)}}{\mathrm{sinch}(\frac{(1-c)y}{2})}dyd\vec{\xi}\\
=\;&\frac{1}{2\pi(1-c)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\psi}_1(\frac{\xi_2e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\overline{\hat{\varphi}_1(\frac{\xi_2e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})}\\
&\times\overline{\hat{\psi}_2(\frac{\xi_2e^{\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})}\hat{\varphi}_2(\frac{\xi_2e^{-\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})})\\
&\times\frac{\mid\xi_2\mid e^{i\xi_1(x-y)}}{\mathrm{sinch}(\frac{(1-c)x}{2})\mathrm{sinch}(\frac{(1-c)y}{2})}dydxd\xi_1d\xi_2
\end{aligned}$$

$$= \frac{1}{(1-c)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}_1\left(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})}\right) \overline{\hat{\varphi}_1\left(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})}\right)}$$

$$\times \overline{\hat{\psi}_2\left(\frac{\xi_2 e^{\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})}\right)} \hat{\varphi}_2\left(\frac{\xi_2 e^{-\frac{(1-c)y}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)y}{2})}\right)$$

$$\times \frac{|\xi_2| \delta(x-y)}{\mathrm{sinch}(\frac{(1-c)x}{2})\mathrm{sinch}(\frac{(1-c)y}{2})} dydxd\xi_2.$$

Put $\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})} = \frac{x\xi_2}{1-e^{-(1-c)x}}$, $\quad \frac{\xi_2 e^{\frac{-(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})} = \frac{x\xi_2}{e^{(1-c)x}-1}$ and $\int_{-\infty}^{\infty} f(x)\delta(x-y)dx = f(y)$. Then

$$\int_{\mathcal{G}^*} \overline{W(\psi_1,\varphi_1 \mid \vec{\xi})} W(\psi_2,\varphi_2 \mid \vec{\xi})[\sigma(\vec{\xi})]^{-1} d\vec{\xi}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}_1\left(\frac{y\xi_2}{1-e^{-(1-c)y}}\right)\overline{\hat{\varphi}_1\left(\frac{y\xi_2}{e^{(1-c)y}-1}\right)}$$

$$\times \overline{\hat{\psi}_2\left(\frac{y\xi_2}{1-e^{-(1-c)y}}\right)}\hat{\varphi}_2\left(\frac{y\xi_2}{e^{(1-c)y}-1}\right)\frac{\xi_2 y^2(1-c)}{e^{-\frac{(1-c)2y}{2}}(e^{(1-c)y}-1)^2} dyd\xi_2.$$

Let $\omega = \frac{y\xi_2}{1-e^{-(1-c)y}}$ and $\omega' = \frac{x\xi_2}{e^{(1-c)x}-1}$ thus $d\omega d\omega' = \frac{\xi_2 y^2(1-c)}{e^{-\frac{(1-c)2y}{2}}(e^{(1-c)y}-1)^2} dyd\xi_2$.

$$\int_{\mathcal{G}^*} \overline{W(\psi_1,\varphi_1 \mid \vec{\xi})} W(\psi_2,\varphi_2 \mid \vec{\xi})[\sigma(\vec{\xi})]^{-1} d\vec{\xi}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}_1(\omega)\overline{\hat{\varphi}_1(\omega')}\overline{\hat{\psi}_2(\omega)}\hat{\varphi}_2(\omega')d\omega d\omega'$$

$$= \int_{-\infty}^{\infty} \overline{\hat{\varphi}_1(\omega')}\hat{\varphi}_2(\omega')d\omega' \int_{-\infty}^{\infty} \overline{\hat{\psi}_2(\omega)}\hat{\psi}_1(\omega)d\omega$$

$$= \langle\hat{\varphi}_1 \mid \hat{\varphi}_2\rangle\langle\hat{\psi}_2 \mid \hat{\psi}_1\rangle.$$

Hence the theorem. ■

3.9 Marginality relations of the Wigner function

Here we check the marginality relations of the Wigner function $W(\hat{\psi},\hat{\varphi} \mid \xi_1,\xi_2)$. The integration with respect to the variable ξ_1 yields a nice form but the integration with respect to ξ_2 does not yeild a nice form.

$$W(\hat{\psi},\hat{\varphi}\mid\xi_1,\xi_2)$$
$$= \sqrt{\frac{1}{(1-c)2\pi}}\int_{-\infty}^{\infty}\overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})}})\hat{\varphi}(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\frac{\xi_2 e^{-i(\xi_1 x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx.$$

Now integration with respect to ξ_1:

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}W(\hat{\psi},\hat{\psi}\mid\xi_1,\xi_2)\frac{d\xi_1}{\mid\xi_2\mid}\\
=\;&\frac{1}{(2\pi)^{\frac{1}{2}}}\sqrt{\frac{1}{(1-c)2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})}\\
&\times\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\xi_2\frac{e^{-i(\xi_1 x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx\frac{d\xi_1}{\mid\xi_2\mid}\\
=\;&\frac{1}{(2\pi)\sqrt{1-c}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mid\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\mid^2\frac{e^{-i(\xi_1 x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx\,d\xi_1
\end{aligned}$$

$$\begin{aligned}
=\;&\frac{1}{\sqrt{1-c}}\int_{-\infty}^{\infty}\mid\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\mid^2\frac{\delta(x-0)}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx\\
=\;&\frac{1}{\sqrt{1-c}}\mid\hat{\psi}(\frac{\xi_2}{(1-c)})\mid^2
\end{aligned}$$

Integration with respect to ξ_2:

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}W(\hat{\psi},\hat{\psi}\mid\xi_1,\xi_2)\frac{d\xi_2}{\mid\xi_2\mid}\\
=\;&\frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\hat{\psi}(\frac{\xi_2 e^{\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})}\\
&\times\hat{\psi}(\frac{\xi_2 e^{-\frac{(1-c)x}{2}}}{(1-c)\mathrm{sinch}(\frac{(1-c)x}{2})})\xi_2\frac{e^{-i(\xi_1 x)}}{\mathrm{sinch}(\frac{(1-c)x}{2})}dx\frac{d\xi_2}{\mid\xi_2\mid}.
\end{aligned}$$

let $\omega = \frac{\xi_2}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}$ thus $d\omega = \frac{d\xi_2}{(1-c)\text{sinch}(\frac{(1-c)x}{2})}$. Then

$$
\begin{aligned}
& \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} W(\hat{\psi},\hat{\psi} \mid \xi_1,\xi_2)\frac{d\xi_2}{\mid \xi_2 \mid} \\
= & \frac{1}{\sqrt{2\pi(1-c)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\hat{\psi}(\omega e^{\frac{(1-c)x}{2}})}\ \hat{\psi}(\omega e^{\frac{-(1-c)x}{2}})e^{-i(\xi_1 x)}(1-c)dxd\omega \\
= & \sqrt{\frac{(1-c)}{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x)}\overline{\hat{\psi}(\omega e^{\frac{(1-c)x}{2}})}\ \hat{\psi}(\omega e^{\frac{-(1-c)x}{2}})dxd\omega.
\end{aligned}
$$

3.10 Coherent states of the group

Consider the representation of the group G_{ab} given in (3.12). A mother wavelet is any vector $\hat{\eta}$ in Hilbert space $L^2(\mathbb{R}^+, d\omega)$ of the representation which satisfies the admissibility condition

$$\int_0^{\infty} \frac{2\pi}{\mid \omega \mid} \mid \frac{\hat{\eta}(\omega)}{1-c} \mid^2 d\omega < \infty \tag{3.20}$$

Now use the irreducible representation on $\mathbb{R}^+$.

$$
\begin{aligned}
& c(\hat{\eta}) \\
= & \int_0^{\infty}\int_{-\infty}^{\infty} \mid \langle \hat{U}(a,b)\hat{\eta} \mid \hat{\eta}\rangle \mid^2 \frac{dadb}{a^2} \\
= & \int_0^{\infty}\int_{-\infty}^{\infty} \overline{\langle \hat{U}(a,b)\hat{\eta} \mid \hat{\eta}\rangle}\langle \hat{U}(a,b)\hat{\eta} \mid \hat{\eta}\rangle \frac{dadb}{a^2} \\
= & \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \overline{\hat{\eta}(\omega)}\hat{U}(a,b)\hat{\eta}(\omega)\overline{\hat{U}(a,b)\hat{\eta}\omega')}\hat{\eta}(\omega')\frac{dadb}{a^2}d\omega d\omega' \\
= & \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \overline{\hat{\eta}(\omega)}a^{\frac{1-c}{2}}\hat{\eta}(a^{1-c}\omega)e^{-i(1-c)ba^{-c}\omega} \\
& \qquad \times \overline{(a^{\frac{1-c}{2}}\hat{\eta}(a^{1-c}\omega')e^{-i(1-c)ba^{-c}\omega'}}\hat{\eta}(\omega')\frac{dadb}{a^2}d\omega d\omega' \\
= & \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \overline{\hat{\eta}(\omega)}\hat{\eta}(\omega')\overline{\hat{\eta}(a^{1-c}\omega')}a^{1-c}\hat{\eta}(a^{1-c}\omega)e^{i(1-c)ba^{-c}(\omega'-\omega)}\frac{dadb}{a^2}d\omega d\omega'
\end{aligned}
$$

Let $b' = (1-c)ba^{-c}$ which gives $db' = (1-c)a^{-c}db$.

Thus,

$$\begin{aligned} & c(\hat{\eta}) \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \overline{\hat{\eta}(\omega)}\hat{\eta}(\omega')\overline{\hat{\eta}(a^{1-c}\omega')}\hat{\eta}(a^{1-c}\omega)e^{ib'(\omega'-\omega)}\frac{dadb'd\omega d\omega'}{a(1-c)} \end{aligned}$$

since $\int_{-\infty}^{\infty} e^{ib'(\omega'-\omega)}db' = 2\pi\delta(\omega'-\omega)$

$$\begin{aligned} & c(\hat{\eta}) \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \overline{\hat{\eta}(\omega)}\hat{\eta}(\omega')\overline{\hat{\eta}(a^{1-c}\omega')}\hat{\eta}(a^{1-c}\omega)2\pi\delta(\omega'-\omega)\frac{dad\omega d\omega'}{a(1-c)} \\ = & 2\pi\int_0^\infty \int_0^\infty \overline{\hat{\eta}(\omega)}\hat{\eta}(\omega)\overline{\hat{\eta}(a^{1-c}\omega)}\hat{\eta}(a^{1-c}\omega)\frac{dad\omega}{a(1-c)} \end{aligned}$$

Let $a = (a^{1-c}\omega)$ then $da' = \omega(1-c)a^{-c}da$. Thus

$$\begin{aligned} & c(\hat{\eta}) \\ = & 2\pi\int_0^\infty \int_0^\infty \mid \hat{\eta}(\omega) \mid^2 \overline{\hat{\eta}(a')}\hat{\eta}(a')\frac{da'd\omega}{\omega(1-c)^2a^{1-c}} \\ = & \int_0^\infty \int_0^\infty \mid \hat{\eta}(\omega) \mid^2 \frac{2\pi}{a'} \mid \frac{\hat{\eta}(a')}{1-c} \mid^2 da'd\omega \\ = & \parallel C\hat{\eta} \parallel^2 \parallel \hat{\eta} \parallel^2 c(\hat{\eta}) = \parallel C\hat{\eta} \parallel^2 \parallel \hat{\eta} \parallel^2 . \end{aligned}$$

$$c(\hat{\eta}) = \parallel C\hat{\eta} \parallel^2 \parallel \hat{\eta} \parallel^2 \tag{3.21}$$

Now using the mother wavelet $\hat{\eta}$. We define a family of wavelets or equivalently coherent states of the group as

$$\hat{\eta}_{a,b} = \hat{U}(a,b)\hat{\eta}, \ \ (a,b) \in G_{ab}. \tag{3.22}$$

3.11 Wigner-wavelet relations

Consider an arbitrary signal $\hat{\varphi} \in L^2(\mathbb{R}, dw)$ and its wavelet transform in the translation and scale parameter a,b of the wavelet family

$$f_{\hat{\eta},\hat{\varphi}}(a,b) = \int |a|^{\frac{1-c}{2}} \hat{\eta}(a^{1-c}\omega) e^{-i(1-c)ba^{-c}\omega} \hat{\varphi}(\omega) d\omega = \langle \hat{\eta}_{a,b} | \hat{\varphi} \rangle \tag{3.23}$$

we build a connection between the wavelet transform and Wigner function. This connection points the way to for practical applications of the Wigner function. We know that

$$\begin{aligned} & W(\hat{\eta}, \hat{\varphi} \mid \vec{\xi}) \\ = \; & \frac{1}{2\pi} \int_{\mathfrak{X}^2} e^{-i\vec{\xi}.\vec{x}} \langle U(e^X) C^{-1} \hat{\eta} \mid \hat{\varphi} \rangle [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} \end{aligned}$$

Put $\hat{\eta} = C\hat{\eta}$. Then

$$\begin{aligned} & W(C\hat{\eta}, \hat{\varphi} \mid \vec{\xi}) \\ = \; & \frac{1}{2\pi} \int_{\mathfrak{X}^2} e^{-i\vec{\xi}.\vec{x}} \langle U(e^X) C^{-1} C \hat{\eta} \mid \hat{\varphi} \rangle [\sigma(\vec{\xi}) m(\vec{x})]^{\frac{1}{2}} d\vec{x} \\ = \; & \frac{1}{2\pi} \int_{\mathfrak{X}^2} e^{-i\vec{\xi}.\vec{x}} \langle U(e^X) \hat{\eta} \mid \hat{\varphi} \rangle [\frac{\mid \xi_2 \mid 1 - e^{-(1-c)x^1}}{(1-c)x^1}]^{\frac{1}{2}} d\vec{x} \\ = \; & \frac{1}{2\pi} \int_{\mathfrak{X}^2} e^{-i\vec{\xi}.\vec{x}} \langle U(e^X) \hat{\eta} \mid \hat{\varphi} \rangle [\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x}. \end{aligned}$$

Put $f_{\hat{\eta},\hat{\varphi}}(g) = \langle \hat{U}(g)\hat{\eta} \mid \hat{\varphi}\rangle$,where $g = e^X \in G$

$$
\begin{aligned}
& W(C\hat{\eta},\hat{\varphi} \mid \vec{\xi}) \\
= \;& \frac{1}{2\pi}\int_{\Re^2} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, \frac{x^2(e^{x^1}-e^{cx^1})}{x^1(1-c)})[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x} \\
= \;& \frac{1}{2\pi}\int_{\Re^2} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, \frac{x^2(e^{\frac{(1-c)x^1}{2}}-e^{\frac{-(1-c)x^1}{2}})(e^{\frac{x^1}{2}}+e^{\frac{cx^1}{2}})}{x^1(1-c)}) \\
& \times[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x} \\
= \;& \frac{1}{2\pi}\int_{\Re^2} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \operatorname{sinch}(\frac{(1-c)x^1}{2})) \\
& \times[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x}.
\end{aligned}
$$

Thus the connection between Wavelet transform and Wigner function is

$$
W(C\hat{\eta},\hat{\varphi} \mid \vec{\xi}) = \frac{1}{2\pi}\int_{\Re^2} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \operatorname{sinch}(\frac{(1-c)x^1}{2}))[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x}. \tag{3.24}
$$

We know

$$
W(C\hat{\eta},\hat{\varphi} \mid \vec{\xi}) = \frac{1}{2\pi}\int_{\Re^2} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \operatorname{sinch}(\frac{(1-c)x^1}{2}))[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}} d\vec{x}
$$

Integrating both sides with respect to $d\vec{\xi}$ gives,

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{\Re^2} e^{i\vec{\xi}.\vec{y}} W(C\hat{\eta},\hat{\varphi} \mid \vec{\xi})[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{y^1}{2}}{e^{(1-c)\frac{y^1}{2}}}]^{-\frac{1}{2}} d\vec{\xi} \\
= \;& \frac{1}{(2\pi)^2}\int_{\Re^2}\int_{\Re^2} e^{i\vec{\xi}.\vec{y}} e^{-i\vec{\xi}.\vec{x}} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \operatorname{sinch}(\frac{(1-c)x^1}{2})) \\
& \qquad \times[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}}[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{y^1}{2}}{e^{(1-c)\frac{y^1}{2}}}]^{-\frac{1}{2}} d\vec{x}d\vec{\xi} \\
= \;& \frac{1}{(2\pi)^2}\int_{\Re^2}\int_{\Re^2} e^{i\vec{\xi}.(\vec{y}-\vec{x})} f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \operatorname{sinch}(\frac{(1-c)x^1}{2})) \\
& \qquad \times[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}}[\frac{\mid \xi_2 \mid \operatorname{sinch}(1-c)\frac{y^1}{2}}{e^{(1-c)\frac{y^1}{2}}}]^{-\frac{1}{2}} d\vec{x}d\vec{\xi}
\end{aligned}
$$

$$= \int_{\Re^2} \delta(\vec{y}-\vec{x}) f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \mathrm{sinch}(\frac{(1-c)x^1}{2}))[\frac{\mid \xi_2 \mid \mathrm{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{\frac{1}{2}}$$

$$\times[\frac{\mid \xi_2 \mid \mathrm{sinch}(1-c)\frac{y^1}{2}}{e^{(1-c)\frac{y^1}{2}}}]^{-\frac{1}{2}} d\vec{x}$$

$$= f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \mathrm{sinch}(\frac{(1-c)x^1}{2}))$$

Thus the wavelet transform in terms of the Wigner function is

$$f_{\hat{\eta},\hat{\varphi}}(e^{x^1}, x^2 e^{\frac{(1+c)x^1}{2}} \mathrm{sinch}(\frac{(1-c)x^1}{2})) = \frac{1}{2\pi} \int_{\Re^2} e^{i\vec{\xi}.\vec{x}} W(C\hat{\eta}, \hat{\varphi} \mid \vec{\xi})[\frac{\mid \xi_2 \mid \mathrm{sinch}(1-c)\frac{x^1}{2}}{e^{(1-c)\frac{x^1}{2}}}]^{-\frac{1}{2}} d\vec{\xi} \tag{3.25}$$

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