Approximate Reinsurance Premiums

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Abstract

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Insurance is a risk transfer mechanism, which allows individuals and firms to reduce the uncertainty about their future cash flows. It provides financial compensation for the effects of misfortune through the establishment of a fund, into which all insured pay premiums and from which benefits are paid when insured events occur.

These uncertainty is usually modeled through two distinct components the claim frequency and the claim severity, since in any given year, neither the number of claims nor their severity is known in advance. The usual stochastic insurance model is thus a random sum called the aggregate claims, where the random number of variables summed represents the claim frequency, while each variable summed represents the claims severities. Each play an important role in the model.

Usually it is difficult to obtain the exact aggregate claims distribution, although it is important to researchers and practitioners in actuarial science. Several approximations have been suggested to this purpose. In particular, Chaubey et. al. [3] proposed a new inverse gaussian-gamma mixture approximation.

The main goal of this thesis is to study approximation methods to calculate stop-loss reinsurance premiums, including a proposal based on the inverse gaussian-gamma mixture approximation.

Various graphical and numerical illustrations are given in support of our conclusions.

Acknowledgments

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Finally, I would also like to thank you for your time and interest. I hope that you enjoy the reading.

Contents

List of Figures

List of Tables

Introduction

The main goal of this thesis is to study approximation methods to calculate stop-loss reinsurance premiums.

The first chapter gives a brief description of basic insurance and reinsurance concepts. It explains in some detail the advantages, disadvantages and other particular characteristics of different reinsurance treaties.

The second chapter is a general survey of approximations to aggregate claim distributions. It confirms the results obtained by Chaubey et. al. [3] about the inverse gaussian-gamma mixture approximation (called IG-mixture here). Numerical examples and graphs are also presented.

Chapter 3 proposes an approximation algorithm. For a given aggregate claims distribution, this algorithm calculates approximate stop-loss premiums at different retention levels, using the gamma, inverse gaussian and IG-mixture approximation. Various numerical illustrations are given.

The last chapter presents a simulation study to validate the different methods to approximate premiums. It also proposes a procedure to approximate stop-loss premiums with the IG-mixture method when the aggregate claims distribution is unknown.

A detailed description of the proposed algorithm and various graphical and numerical results are added to the appendices.

Chapter 1

Preliminaries on Insurance and Reinsurance

This chapter gives an overview of insurance and reinsurance. It starts with a brief description of the main concepts summarized from Hansel [4] and Booth et al. [1]. The last two sections use the notation of Daykin et al. [2] for the definitions of frequency and severity distributions.

A comparison of the different models of reinsurance is also presented. The advantages, disadvantages and the particular characteristics of each model are studied in detail.

1.1 Insurance

Insurance may be defined as a financial compensation program for the effects of misfortune, the payments being made from the accumulated contributions of all parties participating in the scheme. Thus, it may be seen as a kind of fund, into which all who are insured will pay a contribution (called premium). In return those insured will have the right to call on the fund for any appropriate payment, should the insured event occur.

Insurance exists to combat the adverse effects of risk. Risk is inseparable from life and nobody is exempt from it. Insurance is generally used in a pessimistic sense, with the possibility of loss or misfortune in mind. Obviously some people are exposed to greater levels of risk than others, by virtue of their occupation, physical condition, geographical situation or countless other reasons. To a greater or lesser extent, the risks of fire, flood, theft, negligence of others or ourselves and many other perils are constantly with us.

Risk can be classified according to the nature and possible consequences of the hazard involved:

- pure and speculative risks,
- fundamental and particular risks.

A pure risk offers no prospect of gain. It offers only the possibility of loss or at best the preservation of its status (e.g. fire, flood, accident, death). These risks are normally the subject of insurance. Generally, speculative risks are not insurable because these offer the possibility of gain of loss. Trading risk falls within this category.

Most insurable risks are particular risks. These are risk consequences which are comparatively restricted. Fundamental risks tend to affect large sections of society or even the world. Catastrophic risks are fundamental risks. From an actuarial point of view, such events are not insurable. But people living in risky areas need protection against events like earthquakes, hurricanes or flooding and therefore the economy is also protected against it.

Neither all risks are insurable, nor a risk will be insurable only if it is a pure and particular risk. There are another characteristics that a risk should satisfy:

- there must be an insurable interest,
- it must have financial value,
- there must be a large number of similar risk,
- it must be possible to evaluate the risk of loss,
- it must be consistent with public policy,
- it must be random and the probability of occurrence must not be either 0, nor 1.

Insurance can be classified by the main classes of business: Life Insurance and General Insurance. Life insurance provides a vital financial service to individuals and firms who wish to insure themselves against financial losses which might be incurred as a result of any of the following:

- death,
- survival to a particular period,
- sickness or disability.

Life insurance contracts are long term because the policyholder does not need to reapply for renewal of the contract after each coverage year. Renewal is guaranteed over the duration (term) of the contract, as agreed when the policy is issued.

General insurance products offer some protection against the risk that a future event will unexpectedly diminish the insured's normal quality of life. This type of risk is apparent in all aspects of human endeavor. There are different products in general insurance known as short term contracts because the duration is usually 1 year and it is necessary to reapply in order to renew the contract. Among others we find:

- Marine Insurance,
- Fire Insurance,
- Engineering Insurance,
- Aviation Insurance,
- Motor Insurance.

We will only consider general insurance models in this thesis. If you are interested in life insurance refer to Booth et al. [1] or Tiller and Fagerberg [11].

1.2 Reinsurance

Reinsurance is one of the major risk management tools available to insurance companies, providing in with protection against adverse experience fluctuations. Reinsurance is also a powerful financial planning tool. In any given year, reinsurance can be used to increase or decrease the statutory earnings and surplus for either the ceding companies or the reinsurer.

There are two types of reinsurance, proportional and non-proportional. In the first case the primary insurer passes a proportion of the liability from an individual risk, or from a number of risks, and pays to the reinsurer a proportion of the original net insurance premium for those risks. In return, should claims arise, the reinsurer will reimburse the primary insurer for the same proportion of those claims. Non-proportional reinsurance does not contribute proportionally to all losses and does not cost a proportional share of the premium.

The three basic methods of reinsurance are:

- Optional: the primary insurance (ceding) company has the option of submitting claims and the reinsurer has the option of accepting or declining individual risks.
- Automatic Reinsurance: agreement under which the reinsurer must accept or assume risks which meet certain specific criteria based on the ceding company underwriting terms.
- Indemnity Reinsurance: here the risk (but not its administration) is passed to the reinsurer which pays the ceding company for losses covered by the reinsurance agreement (treaty). The ceding company retains its liability and its contractual relationship with the insured.

The principal types of proportional covers are Quota Share and Surplus reinsurance, while for non-proportional covers these are Excess of Loss and Stop Loss reinsurance .

1.3 Frequency, Severity and Aggregate Claims

In insurance it is very important to predict how many claims can occur in a given time of period. The number of claims occured in a time period $[0, t]$, is denoted by N_t and its distribution function given by

$$p_k = \mathbb{P}(N_t \leq k), \qquad k = 0, 1, 2, 3, \ldots$$

and it is known as frequency distribution.

Obviously N_t is a discrete random variable that can be modeled using function such as the Poisson, Binomial or Negative Binomial distributions.

The severity distribution is given by

$$F_X(x) = \mathbb{P}(X \leq x), \quad x \geq 0 \quad ,$$

where X represents the individual claim size.

A natural estimate for the claim size distribution is the empirical distribution, i.e. the distribution function is defined as

$$F_n(x) = \frac{\text{number of claim sizes} \leq x}{\text{total number of claims}} \quad . \tag{1.1}$$

This method is appropriate only where there is a sufficiently large number of claims. This is rarely the case for the tail of the distribution but it is also possible to use continuous distribution functions, such as the gamma, Weibull or lognormal.

The individual claim amount is the quantity that the insurer has to pay on the occurrence of a fire, an accident, death or some other insured event. The sum of the individual claims constitutes the aggregate claim amount, which is one of the key concerns both in the practical management of an insurance company and in theoretical considerations.

Since the number of claims and the size of each claim are stochastic variables we can construct a doubly stochastic (compound) aggregate claim amount model.

Let N be the number of claims for an insurance portfolio over a certain time of period (generally 1 year) and let X_i be the i-th claim size occurring during the time period, then we can write the aggregate claim amount S during that time period as a random sum:

$$S = \sum_{i=1}^{N} X_i \quad .$$

Assume that the claims X_i are independent and identically distributed (i.i.d.) and that they are independent of N.

For a given F_X, the distribution function (d.f.) of X_i and $p_k = \mathbb{P}(N = k)$, the d.f. of S is

$$F_S(x) = \sum_{k=0}^{\infty} p_k F_X^{*k}(x) \quad , \quad x \geq 0 \quad , \tag{1.2}$$

where $F_X^{*k}(x)$ represents the k-th convolution of F_X evaluated at the point x (here $F_X^{*0}(x) = I[x \geq 0]$ and $F_X^{*1} = F_X$).

The Limited Expected Value Function (LEV) $\mathcal{L}$ of a claim size variable X is defined by

$$\mathcal{L}(M) = \mathbb{E}[\min(M, X)] = \int_0^M x dF_X(x) + M[1 - F_X(M)], \quad M \geq 0. \tag{1.3}$$

The value of the function $\mathcal{L}$ at a point M is equal to the expectation of the d.f. F_X truncated at the point M. This function is useful because it represents the claim size distribution in a monetary scale. The d.f. F_X, on the other hand, is expressed on a probability scale. Therefore, it is usually difficult to see, by looking only at F_X, how sensitive the risk premium is to changes in d.f. , while the LEV function shows immediately how different parts of the claim size d.f. contribute to the risk premium.

The function $\mathcal{L}$ has the following important properties:

1. The graph of $\mathcal{L}$ is concave, continuous and increasing,

2. $\mathcal{L}(M) \to m = \mathbb{E}(X)$ as $M \to \infty$,

3. $F_X(M) = 1 - \mathcal{L}'(M)$.

Refer to Daykin [2] for a proof of these properties.

The pure premium (P) is that part of the premium that is sufficient to pay for expected loss and loss adjustment expenses, but not other costs or expenses. It is usually calculated as $P = \mathbb{E}(S) = \mathbb{E}(N)\mathbb{E}(X)$. This expression is obtained from the definition of S and direct properties of the expected value.

1.4 Reinsurance Models

This section presents a brief description of different reinsurance treaties. We start with the following definitions:

- *Cede* : To transfer an insurance risk from the company originally issuing the policy to another insurance company, known as the reinsurer.
- *Ceding Company* : A ceding insurer or a ceding reinsurer. A ceding insurer is an insurer that underwrites and issues an original, primary policy to an insured and contractually transfers (cedes) a portion of the risk to a reinsurer. A ceding reinsurer is a reinsurer that transfers (cedes) a portion of the reinsured risk.
- *Reinsurer:* A reinsurer contractually accepts a portion of the ceding company's risk.
- *Retention:* The dollar amount or percentage of each loss retained by the ceding company under a reinsurer agreement.

1.4.1 Proportional Reinsurance

This class of reinsurance includes the Quota Share and Surplus treaties. These are classified as treaties rather than as optional reinsurance. In proportional reinsurance each claim is shared by the cedant and the reinsurer(s) in a proportion specified in the treaty. An advantage of this type of reinsurance is that the reinsurance premium rating is simpler than in the case of non-proportional reinsurance treaties. Indeed, since the reinsurer pays a certain proportion of every claim, the net reinsurance risk premium is the same proportion of the total net premium.

A drawback of proportional reinsurance is that small claims are shared between the cedant and the reinsurer, as well as the large ones. Thus the cedant's net business volume is further reduced than under non-proportional reinsurance agreements. Surplus reinsurance provides a partial solution to this problem by assigning a different ceded proportion to different risks, according to either size.

Another weakness of proportional reinsurance is that treaties which deal with claims individually do not provide satisfactory cover against fluctuations in the number of claims.

Quota share model

In Quota Share reinsurance any claim, irrespective of its size, is divided between the cedant and reinsurer in a predetermined ratio.

Let the claim size r.v. X have d.f. F_X, density function f_X and mean and variance μ and σ^2, respectively. Assume that the insurer cedes the proportion $(1-\alpha)$, so that the amount retained is the r.v. $Y = \alpha X$. The d.f. of Y is $F_Y(y) = F_X(\frac{y}{\alpha})$

and the density function $f_Y(y) = \frac{1}{\alpha} f_X(\frac{y}{\alpha})$. Thus

$$\mathbb{E}(Y) = \mathbb{E}(\alpha X) = \alpha\mu \quad ,$$
$$\mathrm{Var}(Y) = \mathrm{Var}(\alpha X) = \alpha^2\sigma^2 \quad ,$$

and the coefficient of variation of Y is

$$\mathrm{CV}(Y) = \frac{\sqrt{\mathrm{Var}(Y)}}{\mathbb{E}(Y)} = \frac{\sigma}{\mu} = \mathrm{CV}(X) \quad .$$

Figure 1.1 shows how a portfolio composed by 5 risks of different exposures is shared between the insurer and the reinsurer using a quota share treaty with 75%, 50% and 25% ceding, respectively.

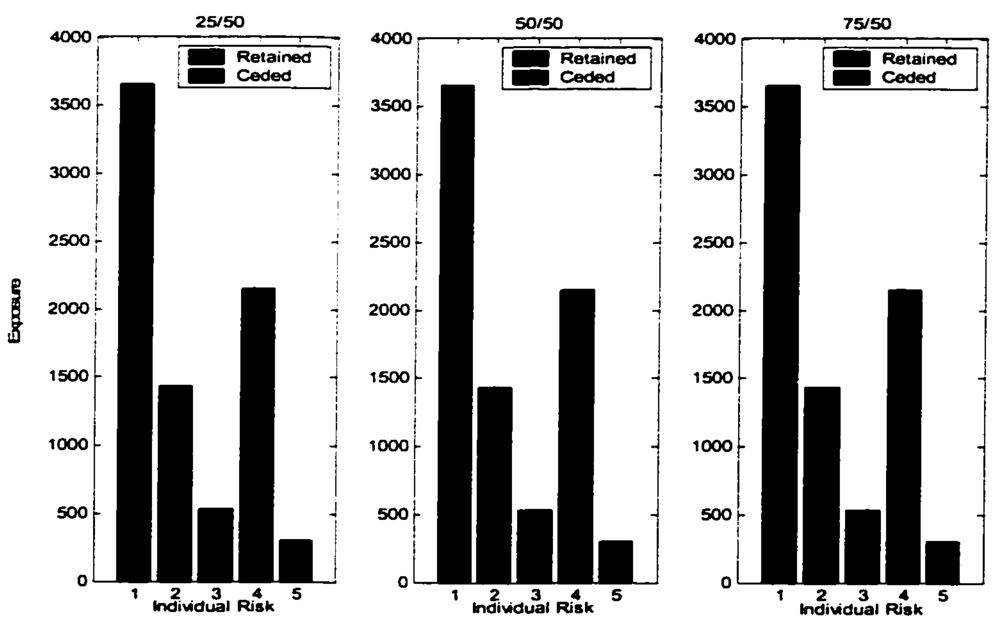

Figure 1.1: Quota Share Reinsurance Treaties.

Surplus model

Whereas quota share reinsurance scales down each risk in the same proportion, a surplus treaty is more flexible: the proportion of each individual risk reinsured may vary and, within the parameters specified in the treaty, is under the control of the ceding office.

The parameters that define the amount ceded are:

1. M, the maximum amount that the insurer may retain,
2. L, the maximum number of lines that may be ceded under the treaty. One line is equal to the amount retained: the maximum amount that can be ceded is $L\,M$.

It follows that the maximum exposure the insurer can underwrite to be covered under the terms of the treaty is the retention, M, plus the amount ceded $L\,M$, that is $(1+L)\,M$.

To calculate the proportional allocations, let the maximum amount $L\,M$ be reinsured, this implies a proportion $\frac{L\,M}{(1+L)\,M} = \frac{L}{1+L}$ of the claim and of the net premium.

On the other hand, for a risk with an exposure E, say, where E is less than $(1+L)\,M$, the insurer has some flexibility in the amount it retains. If the insurer wishes to cede the minimum amount then, as the maximum retention is M, it follows that the amount ceded should be $E - M$. The proportional allocations between the insurer and the reinsurer are $\frac{M}{E}$ retained and $\frac{E-M}{E}$ ceded with $\frac{E-M}{M}$ number of lines used. Conversely, to maximize the amount ceded to the reinsurer the proportion $\frac{L}{1+L}$ as defined in the treaty (using L lines) can be applied to the exposure with $\frac{1}{L}$ retained.

In the case that the initial exposure exceeds $(1+L)\,M$, an insurer must first reduce the exposure so that the residual amount is no more than $(1+L)\,M$.

Suppose that all the risks have exposures less than or equal to $(1+L)\,M$. Let E_i be the exposure for the i-th risk and let α_i be the proportional allocation between the insurer and the reinsurer, then $\alpha_i = \frac{M}{E_i}$ is retained and $1-\alpha_i$ ceded.

Consider a randomly chosen claim X occurring in the portfolio. If X is the size of this claim affecting the j-th risk, the retained amount is then

$$Y = \alpha_j X = \min\left(1\,,\,\frac{M}{E_j}\right)\,X$$

and

$$f_Y(x) = F_X\left(\frac{x}{\alpha_j}\right) \quad ,$$

while the ceded amount is

$$Z = (1-\alpha_j)\,X = \max\left(0\,,\,\frac{E_j - M}{E_j}\right)\,X$$

and

$$f_Z(x) = F_X\left(\frac{x}{1-\alpha_j}\right) \quad .$$

The mean, the variance and the coefficient of variation have been calculated for the retained amount:

$$\mathbb{E}(Y) = \min\left(1\,,\,\frac{M}{E_j}\right)\,\mathbb{E}(X) \quad ,$$
$$\mathrm{Var}(Y) = \min\left(1\,,\,\frac{M}{E_j}\right)^2\,\mathrm{Var}(X)$$

and

$$\mathrm{CV}(Y) = \frac{\sqrt{\mathrm{Var}(Y)}}{\mathbb{E}(Y)} = \frac{\mathrm{Var}(X)}{\mathbb{E}(X)} = \mathrm{CV}(X) \quad .$$

Assuming that our portfolio is composed by k risks, the aggregate loss at the end of the period will be

$$\begin{aligned} S &= \sum_{i=1}^{k} X_{i\bullet} \\ &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} X_{ij} \quad , \end{aligned}$$

where N_i and X_{ij} are the number of claims and the individual claims amount of the i-th risk, respectively. Note that $X_{i\bullet}$ could be zero when no claims have occurred. Then calculating the expectation we obtain

$$\begin{aligned} \mathbb{E}(S) &= \sum_{i=1}^{k} \mathbb{E}(X_{i\bullet}) \\ &= \sum_{i=1}^{k} n_i \mathbb{E}(X_i) \quad , \end{aligned}$$

where n_i is the expected number of claims of the i-th risk.

The expected value of the aggregate net loss and the reinsurer's loss are

$$\mathbb{E}(S_{\text{ret}}) = \sum_{i=1}^{k} n_i \mathbb{E}(X_{i\bullet}) \min\left(1 \, , \, \frac{M}{E_i} \right)$$

and

$$\mathbb{E}(S_{\text{ced}}) = \sum_{i=1}^{k} n_i \mathbb{E}(X_{i\bullet}) \max\left(0 \, , \, \frac{E_i - M}{E_i} \right) \quad ,$$

respectively.

Using these estimates we obtain the following formulas for the cedant's net risk premium and for the reinsurer premium.

$$P_{\text{ret}} = \sum_{i=1}^{k} n_i \mathbb{E}(X_{i\bullet}) \min\left(1 \, , \, \frac{M}{E_i} \right)$$

and

$$P_{\text{ced}} = \sum_{i=1}^{k} n_i \mathbb{E}(X_{i\bullet}) \max\left(0\,,\ \frac{E_i - M}{E_i}\right) .$$

The same portfolio used to illustrate quota share reinsurance is used here for surplus reinsurance treaties, with retentions of \$1,250, \$2,500 and \$3,750, respectively. Figure 1.2 shows how with this treaty the proportion of risk reinsured varies for each individual.

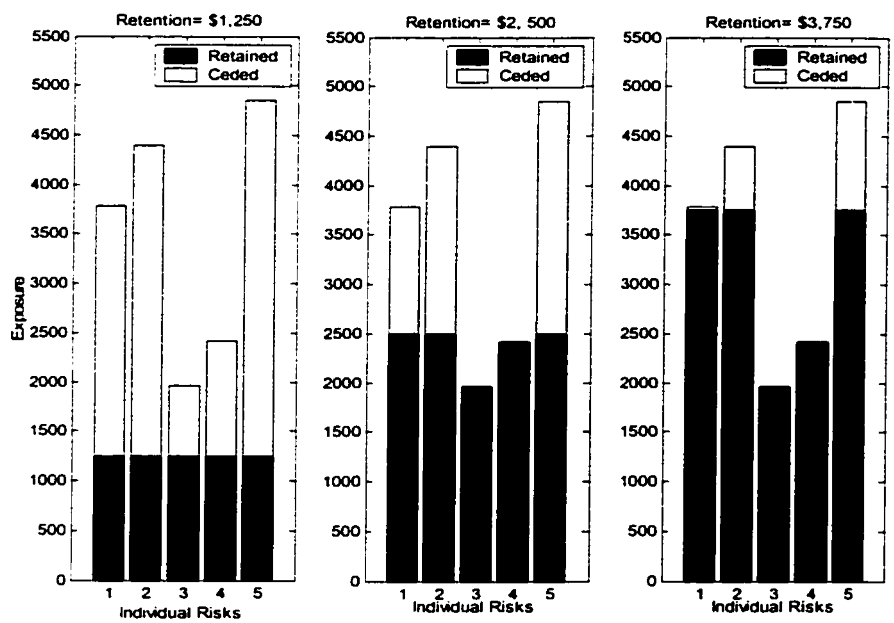

Figure 1.2: Surplus Reinsurance Treaties.

1.4.2 Non-proportional Reinsurance

The available types of non-proportional reinsurance vary widely. Changes are motivated by reinsurance companies need to improve their ability to set a correct price for the risk and to persuade the cedant that the structure and scope of coverage gives an optimal balance between cost and benefit.

The traditional forms of non-proportional reinsurance are known as excess of loss and stop loss. Both have different designs, we study per risk excess of loss reinsurance and stop loss reinsurance based on the aggregate claims, also called aggregate excess of loss cover. The difference is found in the calculation of the retention, in the first contract the retention is based on an individual claim, whereas in the last one the retention is according to the aggregate claims.

Excess of loss model

The excess of loss reinsurance may be written in a variety of ways, for example per risk excess of loss reinsurance and single event excess of loss reinsurance. In both of these, the reinsurer pays the part of each claim amount that exceeds an agreed limit M, the cedant's retention limit. The difference is in how the retention is calculated. In this section we are concerned with per risk excess of loss reinsurance, where the retention M is defined for each claim in a certain group of risks, the reinsurer paying the excess over that amount.

Usually, the reinsurer gives cover for claims which exceed M up to an agreed limit. The region from M to the upper limit of cover is known as a layer of cover. The cedant might arrange for several layers of risk excess cover in succession, each stacking on top of the other, the top layer possibly being unlimited.

It is easy to see that lower layers of cover will be exposed to a higher frequency of loss than in case when the reinsurer is regularly involved on a high proportion of claims.

In general, an excess of loss reinsurance might be described as D in excess of M, it is written as $D \quad \text{xs} \quad M$. This means that the company retains M of each loss and can cover claims that fall in the region M to $M + D$.

If the gross claim is X, the net claim of an excess of loss per risk reinsurance and the reinsurance loss are obtained as shown in Table 1.1.

Gross Claim	Net Claim	Reinsurer Loss
$0 \leq X \leq M$	X	0
$M \leq X \leq D + M$	M	$X - M$
$D + M \leq X$	$M + X - D$	D

Table 1.1: Decomposition of a gross loss X between insurer and reinsurer.

A numerical example illustrating the decomposition of 20 claims under a per risk excess of loss reinsurance of \$150,000 versus \$200,000 is presented in Figure 1.3.

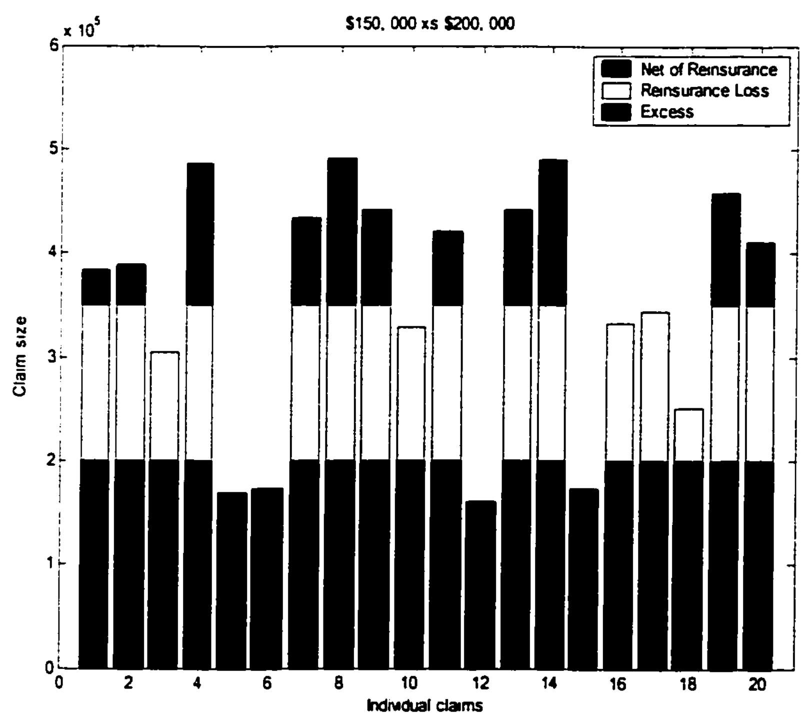

Figure 1.3: Excess of Loss Reinsurance Treaty.

So far we have that in excess of loss reinsurance the reinsurer pays the excess

$$Z_M = (X - M)^+ = X - \min(M, X) \quad .$$

The mean value of the amount to be paid is then,

$$\mathbb{E}(Z_M) = \mathbb{E}(X) - \mathbb{E}[\,\min(M, X)\,] \tag{1.4}$$

$$= \pounds(\infty) - \pounds(M) \quad , \tag{1.5}$$

where $\pounds$ is the limited expected value (LEV) function of the d.f. F_X.

From (1.5) it follows that

$$\mathbb{E}(Z_M) = m\, c(M) \quad ,$$

where

$$c(M) = \left(1 - \frac{\pounds(M)}{m}\right)$$

and

$$m = \pounds(\infty) = \mathbb{E}(X) \quad .$$

c is decreasing and convex function of the retention M, such that $c(0) = 1$ and $c(\infty) = 0$.

Let n be the expected number of claims, the reinsurance premium P_M for the retention M can be calculated as

$$P_M = n\,\mathbb{E}(Z_M) = n\, c(M)\, m$$

$$= P\, c(M) = n[\pounds(\infty) - \pounds(M)] \quad .$$

When the excess of loss cover is limited, as usually, the reinsurer pays the possible excess over the retention limit M, but at most up to an agreed amount A per claim.

If the treaty covers the layer A xs M. Then the reinsurer's share of a claim of size X is

$$Z_M = \min[A, (X - M)^+] = \min[M + A, X] - \min[M, X] \quad .$$

When the reinsurer cover consists of several layers, the risk premium can be calculated for each of them.

The risk premium for these layers is

$$\begin{aligned} P_M &= n\,\mathbb{E}(Z_M) = n\left\{\mathbb{E}[\,\min(M + A\,,\;X)\,] - \mathbb{E}[\,\min(M, X)\,]\right\} \\ &= n\left\{\pounds(\infty) - \pounds(M + A) - \pounds(\infty) + \pounds(M)\right\} \\ &= n\left\{\pounds(M) - \pounds(M + A)\right\} \quad , \end{aligned}$$

where n is the expected number of (all) claims.

The moment about the origin of the cedant and reinsurer's share can also calculated. The cedant's share is given by

$$a_k(M) = \mathbb{E}(Z_{\text{ret}}^k) = \int_0^M X^k dF_X(x) + M^k[1 - F_X(M)] \quad , \quad M \geq 0 \tag{1.6}$$

since

$$Z_{\text{ret}} = \min(M, X) \quad ,$$

and its d.f., F_{ret}, is defined by

$$F_{\text{ret}}(x) = \begin{cases} F_X(x) & , \quad x < M \\ 1 & , \quad x \geq M \end{cases} \quad .$$

By contrast, the moment about the origin of the reinsurer's share takes the following form:

$$a_k^{\text{re}}(Z_M) = \sum_{i=1}^{k} \binom{k}{i} (-M)^{k-1} [a_i^{\text{re}} - a_i^{\text{re}}(M)] \quad , \tag{1.7}$$

where $a_i^{\text{re}} = a_i^{\text{re}}(\infty) = a_i^{\text{re}}(X)$. It follows from the notation $Z_M + Z_{\text{ret}} = X$ and the expression

$$\mathbb{E}(Z_{\text{ret}}) = \mathbb{E}[\,\min(M, X)\,] = \int_0^M X dF_X(x) + M[1 - F_X(M)] \quad . \tag{1.8}$$

When the excess of loss cover is limited, then the first moment in (1.7) is written as

$$a_k^{\text{re}}(Z_{M\text{xs}A}) = \sum_{i=1}^{k} \binom{k}{i} (-M)^{k-1} [a_i^{\text{re}}(M+A) - a_i^{\text{re}}(M)] \quad . \tag{1.9}$$

An example is presented with United Kingdom fire claim data that appeared in Daykin [2]. These claims have been grouped. The total number of claims observed during the period is 16,536 and the mean claim size is 7,009 in the corresponding monetary unit.

From the data it is possible to obtain an expression for the cumulative distribution function of the claim size. Using (1.1) we have:

$$F_X(x) = \begin{cases} F_n(x) & x \leq 10,240 \\ 1 - 7.3208X^{-1.3938} & 10,240 \leq x \leq 100,000 \\ 1 & x \geq 100,000 \end{cases} \quad . \tag{1.10}$$

Assuming that the expected number of claims in a year is $n = 4,134$ then Table 1.2 shows the results.

From left to right Table 1.2 gives the retention value, the cumulative distribution function evaluated at the retention value, the retention value times the probability that the claim size be greater than the retention value, the cedant's and the reinsurer's premium, the second moment about the origin of the cedant's share and the last column is the first risk index.

The risk index is a ratio $r_2 = \frac{a_2(M)}{m_M}$. Usually, it is possible to characterize riskiness as moderate if $r_2 < 30$. Then typically the tail of the distribution is short. On the other hand, if r_2 exceeds, say 200, the distribution is risky.

It can also be seen that the reinsurance premium $P_{\text{re}} = n\ (m_\infty - m_M)$ is relatively small compared with the total risk premium $P = n\ m_\infty$, if the retention limit M is larger than 100,000.

M	$S(M)$	$M*(1-S(M))$	m_M	P_{ced}	P_{re}	$a_2(M)$	$r_2(M)$
0.10	0.261188	0.073881	0.085	349.69	28,624.42	0.008	1.09
0.14	0.309265	0.096703	0.113	467.49	28,506.62	0.015	1.15
0.20	0.364296	0.127141	0.153	631.31	28,342.80	0.028	1.20
0.28	0.422472	0.161708	0.201	831.21	28,142.90	0.051	1.27
0.40	0.488812	0.204475	0.266	1,100.16	27,873.95	0.095	1.34
0.57	0.556604	0.252736	0.347	1,432.79	27,541.32	0.173	1.44
0.80	0.619860	0.304112	0.441	1,821.17	27,152.94	0.301	1.55
1.13	0.678459	0.363341	0.556	2,297.86	26,676.26	0.522	1.69
1.60	0.729439	0.432898	0.694	2,869.92	26,104.20	0.897	1.86
2.26	0.778120	0.501448	0.855	3,535.68	25,438.44	1.513	2.07
3.20	0.820090	0.575714	1.043	4,311.14	24,662.97	2.528	2.32
4.53	0.856495	0.650078	1.256	5,191.97	23,782.14	4.159	2.64
6.40	0.885522	0.732656	1.495	6,180.15	22,793.97	6.746	3.02
9.05	0.908624	0.826956	1.764	7,291.96	21,682.15	10.862	3.49
12.80	0.928520	0.914949	2.066	8,540.74	20,433.38	17.393	4.07
18.10	0.945029	0.994975	2.395	9,899.41	19,074.71	27.435	4.78
25.60	0.957970	1.075955	2.754	11,384.97	17,589.15	42.958	5.66
36.20	0.968372	1.144932	3.140	12,982.05	15,992.07	66.542	6.75
51.20	0.976597	1.198258	3.545	14,655.66	14,318.46	101.470	8.07
72.41	0.983128	1.221722	3.964	16,386.15	12,587.96	152.490	9.71
102.40	0.988449	1.182776	4.381	18,110.99	10,863.12	224.245	11.68
250	0.996671	0.832242	5.271	21,790.79	7,183.32	511.095	18.39
500	0.998733	0.633438	5.776	23,877.78	5,096.33	869.586	26.07
750	0.999280	0.539957	6.013	24,859.12	4,115.00	1160.740	32.10
1000	0.999518	0.482124	6.160	25,466.24	3,507.88	1415.296	37.30
2000	0.999817	0.366955	6.453	26,675.24	2,298.87	2246.000	53.94
3000	0.999896	0.312801	6.590	27,243.74	1,730.37	2920.672	67.25
5000	0.999949	0.255802	6.735	27,842.09	1,132.02	4044.418	89.16
10000	0.999981	0.194697	6.890	28,483.56	490.55	6248.166	131.61
20000	0.999993	0.148188	7.008	28,971.80	2.31	9602.810	195.52
50000	0.999998	0.103301	7.122	29,443.01	0.00	16865.385	332.49
100000	0.999999	0.078625	7.185	29,702.05	0.00	25764.811	499.11

Table 1.2: The Cedant's Premium and Risk index r_2 of the cedant's claim size net of reinsurance. The monetary unit is £1000.

Stop Loss model

In this section stop loss reinsurance is presented as an aggregate type of cover, providing protection not only against large individual claim, but also against fluctuations in the number of claims.

Under this contract, the reinsurer pays the excess $Z = (S - M)^+$ over an agreed limit amount M of the cedant's aggregate claim amount S, accumulated during a certain time period, for example 1 year:

$$S = \sum_{i=1}^{N_t} X_i \quad . \tag{1.11}$$

The cedant's share $Y = S - Z$ of the claim, net of reinsurance, is then

$$Y = \min(S, M) \quad .$$

Note that in stop loss reinsurance the aggregate claim amount is shared between the cedant and the reinsurer, exactly as an individual claim is shared in the case of excess of loss reinsurance. Therefore the formulas obtained in the section above can be applied for stop loss reinsurance.

Hence,

$$\begin{aligned} P &= \mathbb{E}(S) = \mathbb{E}(X)\mathbb{E}(N) \quad , \\ P_{\text{ced}} &= \pounds_{F_S}(M) = \int_0^M x\,dF_S(x) + M\left[1 - F_S(M)\right] \quad , \end{aligned}$$

and

$$P_{\text{re}} = \pounds_{F_S}(\infty) - \pounds_{F_S}(M) = \int_M^\infty [1 - F_S(x)]\,dx \quad , \tag{1.12}$$

where F_S denotes the d.f. of aggregate claims amounts S and L_{F_S} is the corresponding LEV.

If the layer is limited, say A versus M, the reinsurance risk premium is easily obtained by the formula

$$P_{re}(A \,\mathrm{xs}\, M) = \pounds_{F_S}(M + A) - \pounds_{F_S}(M) = P_{re,M} - P_{M+A} \quad . \tag{1.13}$$

The higher moments of Z and Y could be obtained from the results of the section above, keeping in mind to use the aggregate claim amount instead the individual claim.

The distribution function of an individual claim is usually simpler to obtain than the aggregate claim distribution. As in (1.2)

$$\begin{aligned} F_S(x) &= \mathbb{P}(S \le x) \\ &= \mathbb{P}(X_1 + X_2 + \cdots + X_N \le x) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(X_1 + X_2 + \cdots + X_N \le x \,|N = k)\,\mathbb{P}(N = k) \\ &= \sum_{k=1}^{\infty} F_X^{*k}(x)\,\mathbb{P}(N = k) \quad , \quad x \ge 0 \quad , \end{aligned}$$

where F_X^{*k} is the k-th convolution of F_X with itself.

Serious numerical problems have been encountered in the calculation of the exact distribution F_S. Chaubey et al. [3] proposed an inverse Gaussian-gamma mixture approximation, which approximates the true distribution extremely accurately in a large variety of situations. It is described in detail in the next chapter, where we propose to use it to approximate the stop loss reinsurance premium in (1.12).

Chapter 2

Approximations

A general survey of approximations to aggregate claim distributions is given in this chapter (see Chaubey et al. [3]). This will be useful in the following chapters.

Section 2.2 gives expressions for the first moments of the different distributions presented. These form the basis of calculation for the approximate distributions. Using these expressions, algorithms are presented in Section 2.3 to obtain exact and approximated values of the aggregate claims distribution. In turn, these will be used to obtain approximate reinsurance premiums.

Graphical and numerical results for various examples are given at the end of the chapter. These confirm the results obtained by Chaubey et al. [3] on the accuracy of the approximation for probability calculations.

2.1 Introduction

Several approximations are used in actuarial problems. The normal and gamma approximations are simple to use since they are pre-programmed in many mathematical and statistical softwares. However, other distributions are not as simple to compute as the normal or gamma but may produce more accurate approximations.

For instance, the inverse Gaussian and gamma mixture approximation was suggested in Chaubey et al. [3]. In a comparison of different approximations for a large variety of cases, they show that the inverse Gaussian mixture approximates the true distribution extremely accurately.

2.2 Moments and Cumulants of the Aggregate Claims Distribution

A brief summary about the first moments and cumulants for different distribution functions is given in this section. For more details refer to Daykin et al. [2] or Panjer and Willmot [8].

It is known that the j-th moment about 0, μ'_j, of a non-negative random variable is defined as follows

$$\mu'_j = \begin{cases} \mathbb{E}(X^j) = \int_0^\infty x^j \, dF_X(x) & \text{if } X \text{ is continuous} \\ \mathbb{E}(N^j) = \sum_{k=0}^\infty k \, \mathbb{P}(N = k) & \text{if } N \text{ is discrete} \end{cases} .$$

A disadvantage of moments about zero is that they are not additive for independent random variables. It is useful therefore to introduce other characteristics which have this simple additivity feature. A set of characteristics with this property is known as cumulants C_j. They can be expressed in term of the moments

μ'_j and conversely, for example

$$\begin{aligned} C_1 &= \mu'_1 \quad , & (2.1)\\ C_2 &= \mu'_2 - {\mu'_1}^2 \quad , & (2.2)\\ C_3 &= \mu'_3 - 3\,\mu'_1\mu'_2 + 2\,{\mu'_1}^3 \quad , & (2.3)\\ C_4 &= \mu'_4 - 4\,\mu'_1\mu'_3 - 3\,{\mu'_2}^2 + 12\,{\mu'_1}^2\mu'_2 - 6\,{\mu'_1}^4 \quad . & (2.4) \end{aligned}$$

The conventional characteristics of a random variable X can readily be given in terms of the cumulants as

$$\mathbb{E}(X) = C_{X1} \quad , \qquad \text{Mean} \qquad (2.5)$$

$$\mathrm{Var}(X) = C_{X2} = \sigma_X^2 \quad , \qquad \text{Variance} \qquad (2.6)$$

$$\delta_X = \frac{C_{X3}}{C_{X2}^{\frac{3}{2}}} = \frac{C_{X3}}{\sigma_X^3} \quad , \qquad \text{Skewness} \qquad (2.7)$$

$$k_X = \frac{C_{X4}}{C_{X2}^2} = \frac{C_{X4}}{\sigma_X^4} \quad , \qquad \text{Kurtosis} \qquad (2.8)$$

The next few sections will use the first moments of the following distribution functions:

- The Poisson distribution with parameter λ

$$p_n = e^{-\lambda}\,\frac{\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots, \qquad (2.9)$$

  has moments given by:

$$\begin{aligned} \mu'_{N1} &= \lambda \quad , \\ \mu'_{N2} &= \lambda\,(\lambda + 1) \quad , \\ \mu'_{N3} &= \lambda^2\,(\lambda + 2) + \lambda\,(\lambda + 1) \quad , \\ \mu'_{N4} &= \lambda^3\,(\lambda + 3) + 3\lambda^2\,(\lambda + 2) + \lambda\,(\lambda + 1) \quad . \end{aligned}$$

- The binomial distribution with parameters (m, q)

$$p_n = \binom{m}{n} q^n \, (1-q)^{m-n}, \quad n = 0, 1, 2, \ldots, m, \tag{2.10}$$

has moments given by:

$$\begin{aligned}
\mu'_{N1} =& m\, q \quad , \\
\mu'_{N2} =& \mu'^{2}_{N1} + \mu'_{N1}(1-q) \\
\mu'_{N3} =& \mu'^{3}_{N1} - 3\mu'^{2}_{N1}q + 2\mu'_{N1}q^2 + 3\mu'^{2}_{N1} - 3q\mu'_{N1} + \mu'_{N1} \quad , \\
\mu'_{N4} =& \mu'^{4}_{N1} - 6q\mu'^{3}_{N1} + 11q^2\mu'^{2}_{N1} - 6q^3\mu'_{N1} + 6\mu'^{3}_{N1} - 18\mu'^{2}_{N1}q + 12q^2\mu'_{N1} \quad ; \\
& + 7\mu'^{2}_{N1} - 7q\mu'_{N1} + \mu'_{N1} \quad .
\end{aligned}$$

- The negative binomial distribution with parameters (r, β)

$$p_n = \binom{r}{n} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^n \quad , \quad n = 0, 1, 2, \ldots \quad , \tag{2.11}$$

has moments given by:

$$\begin{aligned}
\mu'_{N1} =& r\,\beta \quad , \\
\mu'_{N2} =& \mu'^{2}_{N1} + \mu'_{N1}(1-\beta) \quad , \\
\mu'_{N3} =& \mu'^{3}_{N1} + 3\mu'^{2}_{N1}\beta + 2\mu'_{N1}\beta^2 + 3\mu'^{2}_{N1} + 3\beta\mu'_{N1} + \mu'_{N1} \quad , \\
\mu'_{N4} =& \mu'^{4}_{N1} + 6\beta\mu'^{3}_{N1} + 11\beta^2\mu'^{2}_{N1} + 6\beta^3\mu'_{N1} + 6\mu'^{3}_{N1} + 18\mu'^{2}_{N1}\beta + 12\beta^2\mu'_{N1} \\
& + 7\mu'^{2}_{N1} + 7\beta\mu'_{N1} + \mu'_{N1} \quad .
\end{aligned}$$

- The inverse Gaussian distribution with parameters (m, b)

$$F_X(x) = \Phi[(bx)^{-\frac{1}{2}}(x-m)] + \exp\frac{2m}{b}\Phi[-(bx)^{-\frac{1}{2}}(x+m)], \quad x > 0, \tag{2.12}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

The first four raw moments are given by:

$$\begin{aligned}
\mu'_{X1} &= m \quad , \\
\mu'_{X2} &= m^2 + mb \quad , \\
\mu'_{X3} &= m^3 + 3m^2b + 3mb^2 \quad , \\
\mu'_{X4} &= m^4 + 6m^3b + 15m^2b^2 + 15mb^3 \quad ,
\end{aligned}$$

while the r-th cumulant for $r \geq 2$, is given by

$$C_{Xr} = mb^{r-1}(1)(3)\dots(2r-3) \quad . \tag{2.13}$$

- The gamma distribution with parameters (α, λ) has a density function given by

$$f_X(x) = \lambda^\alpha x^{\alpha-1} \frac{e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0 \quad , \tag{2.14}$$

and its k-th raw moments can be obtained by:

$$\mu'_{Xk} = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\lambda^k}, \quad k = 1, 2, \dots \quad . \tag{2.15}$$

- The Weibull distribution with density function

$$f_X(x) = c\lambda x^{c-1} e^{-\lambda x^c}, \quad x > 0 \quad , \tag{2.16}$$

has a k-th raw moment given by:

$$\mu'_{Xk} = \frac{\Gamma(1 + \frac{k}{c})}{\lambda^{\frac{k}{c}}}, \quad k = 1, 2, \dots \quad . \tag{2.17}$$

Now the cumulants of the aggregate claims amount $S = \sum_{i=1}^{N} X_i$ are required. These can be computed directly using the following general formula given in Panjer and Willmot [8, p.167],

$$
\begin{aligned}
C_{S1} &= C_{N1}C_{X1} \\
C_{S2} &= C_{N2}C_{X1}^2 + C_{N1}C_{X2} \\
C_{S3} &= C_{N3}C_{X1}^3 + 3C_{N2}C_{X1}C_{X1} + C_{N1}C_{X3} \\
C_{S4} &= C_{N4}C_{X1}^4 + 6C_{N3}C_{X1}^2C_{X2} + 3C_{N2}C_{X2}^2 + 4C_{N2}C_{X1}C_{X3} + C_{N1}C_{X4} \quad ,
\end{aligned}
$$

where $C_{N\cdot}$ and $C_{X\cdot}$ are the corresponding cumulants of the claim number and claim size distributions, respectively. These are obtained from equations (2.1) to (2.4) and the expressions of the first raw moments of each random variable.

2.3 Exact and Approximate Distribution F_S

2.3.1 Computation of the Exact Distribution F_S

In Chapter 1 we have seen the general form used for the aggregate claims distribution:

$$F_S(x) = \sum_{k=1}^{\infty} F_X^{*k}(x)\, p_k, \quad x \geq 0, \tag{2.18}$$

where p_k is a probability distribution on the integer set $\mathbb{N}$ and F_X^{*k} is the k-th convolution of F_X with itself (see Section 1.3)

The computation of this expression or the corresponding probability (density) function is generally not a simple task, even in the simplest of cases. One approach is to use an approximating distribution to avoid direct calculation, but its disadvantages are significant (see Klugman et al. [5]).

Alternative methods to evaluate the aggregate claims distribution more quickly are discussed in Klugman et al. [5, §4.5-4.7] and Rolski et al. [9, §4.4]. Among them we find *the inversion methods* and *the recursive methods*.

Here we use the Ströter's algorithm, as presented in Panjer and Willmot [8, §D.2]. The main idea of this algorithm is to compute the solution of

$$g(x) = f(x) + \int_0^x K(x,y)\, g(y) dy \quad . \tag{2.19}$$

It is based on the discretization of the integral in (2.19).

For some small interval length $h > 0$, the values of the function g in (2.19) are considered at the points $0, h, 2h, 3h, \ldots$. Using a quadrature rule and solving for $\hat{g}(kh)$ one obtains that,

$$\hat{g}(kh) = \frac{f(kh) + \sum_{j=0}^{k-1} w_{k,j}\, K(kh, jh)\, \hat{g}(jh)}{1 - w_{k,k}\, K(kh, kh)}, \quad k = 1, 2, ..., \tag{2.20}$$

where $\hat{g}(\cdot)$ is an approximation of $g(\cdot)$ and $w_{k,j}$ are fixed weights.

Since the distribution of total claims for continuous severity distributions is the solution of the Volterra integral equation of the second kind (Panjer and Willmot [8, §6]) of the form in (2.19) we can use this algorithm for exact calculation of probabilities.

In our particular problem $\hat{g}(x) = f_S(x)$. Furthermore, when formula (3.2) is used for the $(a, b, 0)$ class of claim frequency distributions (e.g. Poisson, binomial and negative binomial, see [5, §3.5]) then

$$f(x) = p_1\, f_X(x) \quad , \tag{2.21}$$

and

$$K(x,y) = (a + b\,\frac{x-y}{x}) f_X(x-y) \quad , \tag{2.22}$$

where

$$p_1 = (a+b)p_0 \quad . \tag{2.23}$$

Table 2.1 summarizes the values of a, b and p_0 for the Poisson $P(\lambda)$, binomial $B(m,q)$ and negative binomial NB(r,β) distributions.

p_k	a	b	p_0
$P(\lambda)$	0	λ	$e^{-\lambda}$
$B(m, q)$	$\frac{-q}{1-q}$	$\frac{(m+1)q}{1-q}$	$(1-q)^m$
$NB(r, \beta)$	$\frac{\beta}{1+\beta}$	$\frac{(r-1)\beta}{1+\beta}$	$(1+\beta)^{-r}$

Table 2.1: Parameters a, b and p_0 for the Poisson, binomial and negative binomial distributions.

For a fixed set of weights $\{w_{k,j}\ ;\ k = 1,2,\dots\ ;\ j = 0,1,\dots,k-1\}$, equation (3.2) can be used to evaluate $\{\hat{g}(kh)\ ;\ k = 1,2,\dots\}$ recursively, beginning with the initial value $\hat{g}(0) = f(0^+)$. The accuracy of this algorithm depends on the smoothness of the functions $K(x,y)$ and $f(x)$, on the interval size h and on the quadrature rule used to determine the weights $w_{k,j}$.

The values of the weights are determined by applying simple approximate integration formulas repeatedly. We have followed the same idea as in Panjer and Willmot [8, p.417]. When k is even we use a repeated Simpson's rule and when k is odd, a repeated Simpson's rule followed by a single three eighths rule.

The values of the coefficients $w_{k,j}$ are shown in Table 2.2.

k	$j=$	0	1	2	3	4	5	6	7	8
1		$\frac{1}{2}h$	$\frac{1}{2}h$							
2		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{1}{3}h$						
3		$\frac{3}{8}h$	$\frac{9}{8}h$	$\frac{9}{8}h$	$\frac{3}{8}h$					
4		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{1}{3}h$				
5		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{17}{24}h$	$\frac{9}{8}h$	$\frac{9}{8}h$	$\frac{3}{8}h$			
6		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{1}{3}h$		
7		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{17}{24}h$	$\frac{9}{8}h$	$\frac{9}{8}h$	$\frac{3}{8}h$	
8		$\frac{1}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{2}{3}h$	$\frac{4}{3}h$	$\frac{1}{3}h$

Table 2.2: Values of the coefficients $w_{k,j}$.

To obtain the distribution function of total claims the following expression are used

$$\hat{F}_S(kh) = p_0 + \sum_{j=0}^{k} w_{k,j}\, \hat{g}(jh), \quad k = 1, 2, \ldots \tag{2.24}$$

and

$$\hat{F}_S(kh) = F_S(0) = p_0 \quad . \tag{2.25}$$

This algorithm is illustrated using [7, MATLAB version 6], and various numerical examples are shown in the last section of this chapter.

2.3.2 Computation of the Approximation F_S

Chaubey et. al. [3] compare different approximations for $F_S(x)$, such as

- normal,
- normal power,
- Edgeworth,
- Esscher,
- gamma,
- inverse Gaussian and
- The inverse-Gaussian (IG) gamma mixture approximation.

We present the latter in detail in order to apply it to reinsurance premium calculations.

IG-Gamma Mixture Approximation

It has been shown that the IG-gamma mixture approximation produces good results, improving the accuracy over other approximations, especially in the tail of F_S.

Let F_1 be the inverse Gaussian approximation to F_S given by

$$F_S(x) \approx F_1(x) = IG_{m,b}(x - x_0) \quad , \quad x \geq 0 \quad ,$$

where $IG_{m,b}(x)$ is the inverse Gaussian distribution function given in (2.12) and the constants m, b and x_0 are estimated by

$$m = \frac{2C_{S2}^2}{C_{S3}}, \quad b = \frac{C_{S3}}{3C_{S2}} \quad \text{and} \quad x_0 = C_{S1} - m \quad .$$

Now let F_2 be the gamma approximation given by

$$F_S(x) \approx F_2(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha+z\sqrt{\alpha}} e^{-y} y^{\alpha-1} dy \quad , \quad x \geq 0 \quad , \tag{2.26}$$

where Γ is the usual gamma function, $z = \frac{x-C_{S1}}{C_{S2}}$ and $\alpha = \frac{4}{\delta_S^2}$.

The moments $\mu_S = C_{S1}$, $\sigma_S^2 = C_{S2}$ and $\delta_S = \frac{C_{S3}}{\sigma_S^3}$ can be calculated from (2.5)-(2.7).

Matching the first three moments of the exact and approximating mixture distributions

$$F_S(x) \approx wF_1(x) + (1-w)F_2(x) \quad , \quad x \geq 0 \quad ,$$

where the mixture weight is given by the following ratio of kurtosis in (2.8)

$$w = \frac{k_S - k_{F_2}}{k_{F_1} - k_{F_2}} \quad ,$$

defines the IG-gamma approximations. This approximation reproduces the first four moments of F_S.

2.4 Examples

This section presents various examples where different aggregate claims distributions are approximated and compared to exact values. Tables 2.3-2.5 give the distribution parameters used in each example. They also show the mean, variance, skewness and kurtosis of the aggregate claims distributions.

Graphical comparisons of the corresponding density and distributions functions are also shown. The inverse Gaussian, gamma and IG-mixture approximations seem accurate, but the mixture gives uniformly the best results. Appendix A contains tables where the relative approximation error is calculated. For the tail of the distribution F_S, the mixture approximation is the most accurate (see Figures 2.2 and 2.4). These figures zoom on the tail of the density functions for the compound Poisson examples.

A general algorithm was used, valid for any compound distribution with gamma, Pareto, Weibull or inverse Gaussian claim sizes. We have only chosen one example for each distribution.

2.4.1 Compound Poisson

	Parameters of:			Characteristics			
	$F_X(x)$		Poisson	μ_S	σ_S	Skew_S	Kurt_S
Pareto	$\alpha = 24$	$\lambda = 10$	$\lambda = 4$	1.74	1.26	1.14	1.81
Weibull	$\theta = 3.0$	$\lambda = 1.0$	$\lambda = 2$	1.78	1.34	0.82	0.73
I.G.	$\mu = 1.3$	$\beta = 0.1$	$\lambda = 1$	1.30	1.35	1.12	1.34
Gamma	$\alpha = 2.0$	$\lambda = 1.5$	$\lambda = 1$	1.33	1.63	1.63	3.33

Table 2.3: Numerical examples for the Compound Poisson distribution.

Figure 2.1 shows (a) the c.d.f. and (b) the p.d.f. for the compound Poisson distribution with Pareto claims. The same figure gives in (c) the p.d.f. and in (d) the c.d.f. of the compound Poisson distribution with Weibull claims. In both cases the exact distribution and the gamma, IG and mixture approximations are compared.

Figure 2.3 gives a comparison of the same three approximations with the exact values for the compound Poisson with IG claims distribution in (a) and (b), while (c) and (d) are for the compound Poisson with gamma claims distribution.

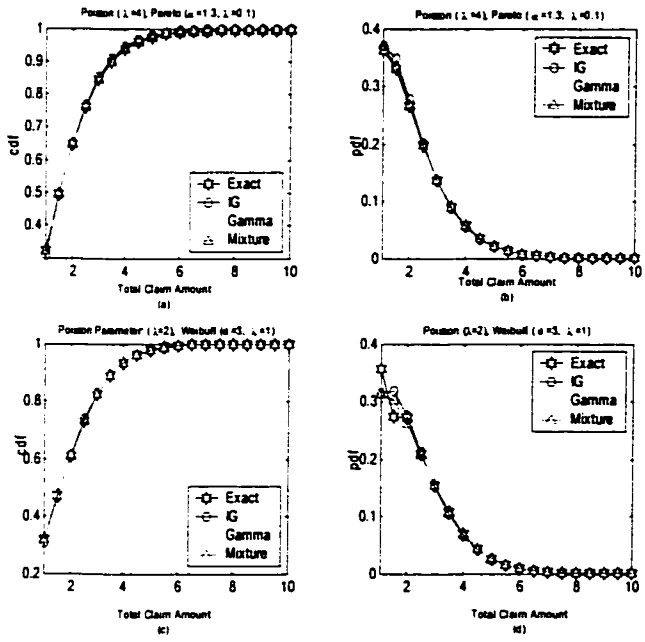

Figure 2.1: Compound Poisson with Pareto and Weibull claims.

2.4.2 Compound Binomial

	Parameters of:				Basic Characteristics			
	$F_X(x)$		Binomial		μ_S	σ_S	Skew$_S$	Kurt$_S$
Pareto	$\alpha = 10$	$\lambda = 3$	$m = 20$	$q = 0.6$	4.00	1.48	0.73	0.93
Weibull	$\theta = 4$	$\lambda = 1$	$m = 15$	$q = 0.2$	2.72	1.47	0.47	0.12
I.G.	$\mu = 1$	$\beta = 1$	$m = 10$	$q = 0.4$	4.00	2.53	1.22	2.39
Gamma	$\alpha = 3$	$\lambda = 3$	$m = 4$	$q = 0.6$	2.40	1.33	0.56	0.29

Table 2.4: Numerical examples for the Compound binomial distribution.

Figure 2.5 (a) shows the c.d.f. and (b) the p.d.f. of the compound Poisson distribution with Pareto claims. The same figure gives in (c) the c.d.f. and in (d) the p.d.f. of the compound Poisson distribution with Weibull claims. In both cases the exact distribution and the gamma, IG and Mixture approximations are compared.

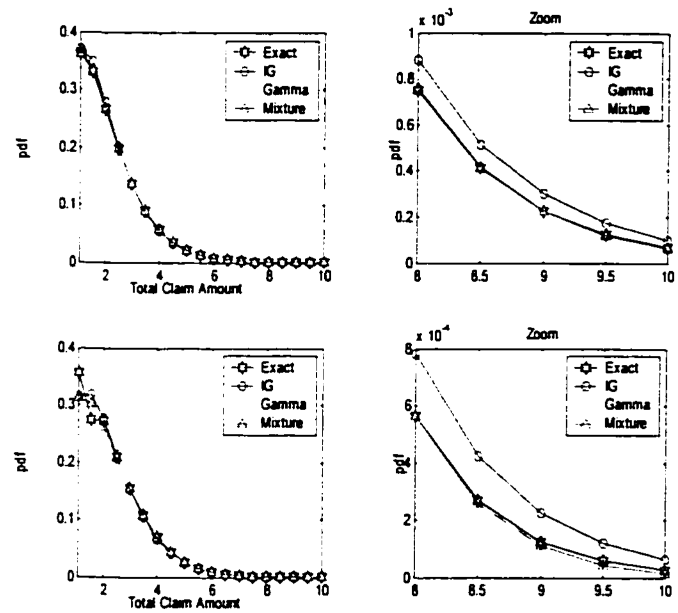

Figure 2.2: Zoom on the p.d.f. tail of the compound Poisson with Pareto and Weibull claims.

Figure 2.6 (a) shows the c.d.f. and (b) the p.d.f. of the compound Binomial with gamma claim distribution. The c.d.f. of the compound binomial with IG claims distribution is shown in (c), while (d) gives its p.d.f..

2.4.3 Compound Negative Binomial

	Parameters of:				Characteristics			
	$F_X(x)$		N.Binomial		μ_S	σ_S	Skew_S	Kurt_S
Pareto	$\alpha = 8$	$\lambda = 3.0$	$r = 15$	$q = 0.6$	3.86	2.20	0.98	1.46
Weibull	$\theta = 7$	$\lambda = 1.0$	$r = 10$	$q = 0.4$	3.74	2.24	0.77	0.80
I.G.	$\mu = 1$	$\beta = 0.7$	$r = 10$	$q = 0.2$	2.00	1.95	1.53	3.32
Gamma	$\alpha = 5$	$\lambda = 3.0$	$r = 4$	$q = 0.2$	1.33	1.76	1.67	3.52

Table 2.5: Numerical examples for the Compound negative binomial distribution.

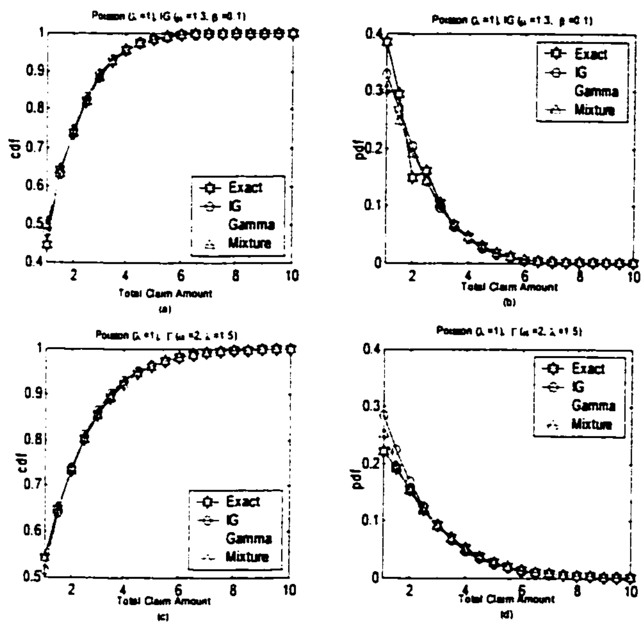

Figure 2.3: Compound Poisson with gamma and IG claims.

Figure 2.7 (a) shows the exact c.d.f. of the compound binomial with Pareto claims distribution and (b) its exact p.d.f. . The approximated p.d.f. and c.d.f. are also included in the same figures.

Figure 2.8 (a) gives the exact and approximated c.d.f. of the compound binomial with gamma claims distribution and (c) shows the exact and approximated c.d.f. of the the compound binomial with IG claims distribution. The same figure in (b) and (d) gives the exact and approximated p.d.f. of these distributions.

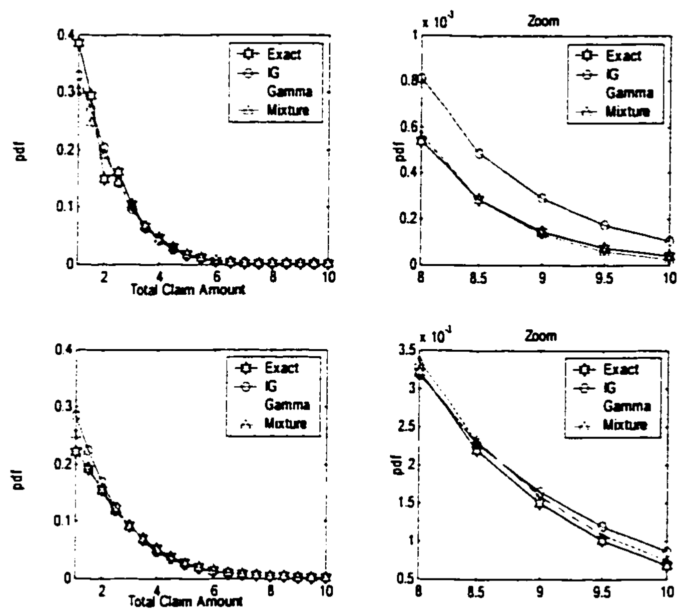

Figure 2.4: Zoom on the p.d.f. tail of the compound Poisson with gamma and IG claims

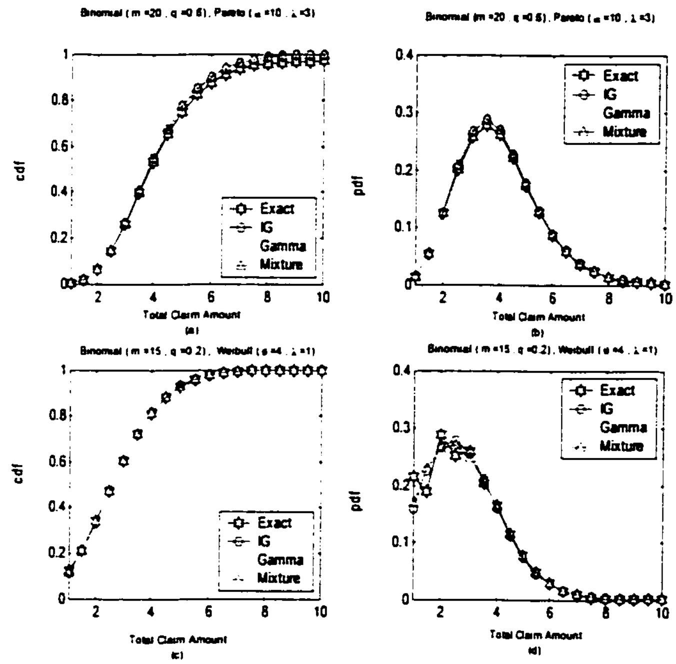

Figure 2.5: Compound Binomial with Pareto and Weibull claims.

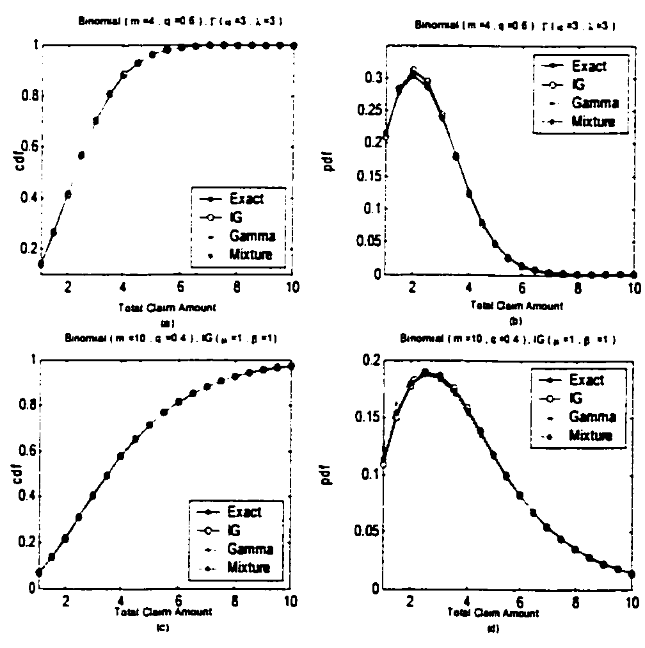

Figure 2.6: Compound Binomial with gamma and IG claims.

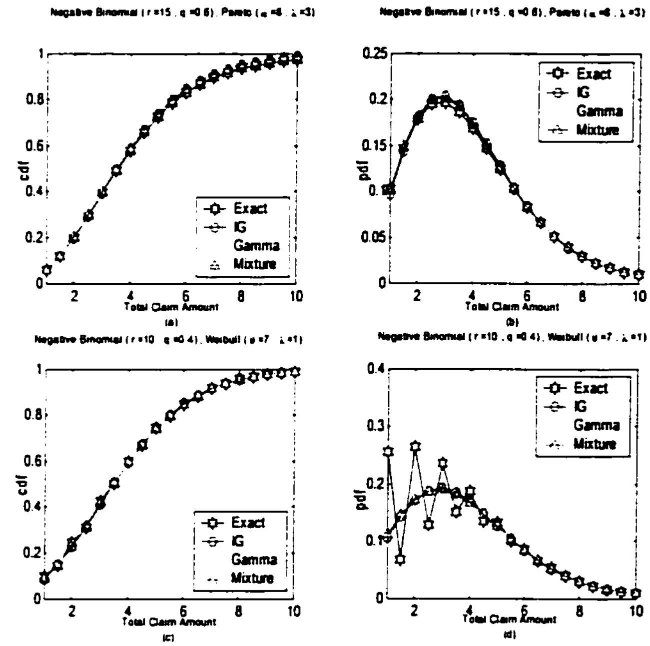

Figure 2.7: Compound Negative Binomial with Pareto and Weibull claims.

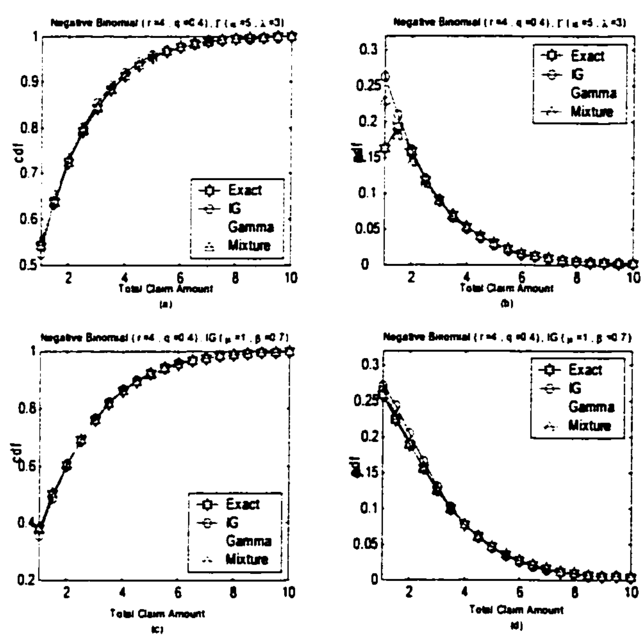

Figure 2.8: Compound Negative Binomial with gamma and IG claims.

Chapter 3

Computation of Stop Loss Reinsurance Premiums

From Section 1.4 we know the definition of reinsurance risk premiums and how to calculate them for each type of reinsurance.

Assuming a stop loss reinsurance treaty, the reinsurance premium given by

$$\pi(M) = \mathbb{E}[\,(S - M)^+\,] = \int_M^\infty [1 - F_S(x)]\, dx\ , \quad M \geq 0\ , \tag{3.1}$$

where F_S is the aggregate claim distribution function.

Since the true distribution F_S is usually unknown, equation (3.1) is also difficult to evaluate. In this chapter we propose an algorithm to obtain approximate values for (3.1). The idea is to use the inverse gaussian-gamma mixture, in Section 2.3.2, as an approximation to F_S.

We have seen that the IG-gamma approximation is appropriate for F_S. Hopefully, this choice will also yield good approximate values for the corresponding stop loss reinsurance premiums.

3.1 Exact Methods

We need to compare the premiums obtained with our proposed algorithm to those obtained with exact methods. One route is to use in (3.1) the exact distribution F_S, as presented in Section 2.3.1. In this case $F_S(x)$ values are available in tabular form, where the number of points depends on h [see (2.24)]. Smaller h values will produce more accurate $F_S(x)$ values, but will slow down the algorithm. Once values $\bar{F}_S(x) = 1 - F_S(x)$ are known, we calculate the integral in (3.1) numerically, using a classical method like Simpson's rule.

When $\bar{F}_S$ is given in closed form, we will be able to evaluate the accuracy of this numerical "exact method". This case is described in the next section.

3.1.1 An Analytical Expression for $\bar{F}_S$

When claim severities are exponentially distributed with mean θ, an expression for the aggregate claims distribution is given in Klugman et al. [5, §4.4] for the whole $(a, b, 0)$ class of claim frequency distributions:

$$F_S(x) = 1 - e^{-\frac{x}{\theta}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{x}{\theta}\right)^j \sum_{n=j+1}^{\infty} p_n \quad , \tag{3.2}$$

where $p_n = \mathbb{P}(N = n)$ is the probability that the number of claims, N, be equal to n.

For the different distributions in the $(a, b, 0)$ class, we have:

1. The Poisson distribution with parameter λ

$$\bar{F}_S(x) = e^{-\frac{x}{\theta}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{\theta}\right)^j \sum_{n=j+1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} . \tag{3.3}$$

2. The binomial distribution with parameters (m, q)

$$\bar{F}_S(x) = e^{-\frac{x}{\theta}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{\theta}\right)^j \sum_{n=j+1}^{m} \binom{m}{n} q^n (1-q)^{m-n} . \tag{3.4}$$

3. The geometric distribution with parameter β

$$\bar{F}_S(x) = \frac{\beta}{1+\beta} \exp\left\{\frac{-x}{\theta(1+\beta)}\right\} . \tag{3.5}$$

4. The negative binomial distribution with parameters (r, β)

$$\bar{F}_S(x) = e^{-\frac{x}{\theta}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x}{\theta}\right)^j \left(\frac{1}{1+\beta}\right)^r \sum_{n=j+1}^{\infty} \binom{r}{n} \left(\frac{\beta}{1+\beta}\right)^n , \tag{3.6}$$

   or equivalently,

$$\bar{F}_S(x) = e^{-\frac{x}{\theta}} \sum_{n=1}^{r} \binom{r}{n} \left(\frac{1}{1+\beta}\right)^{r-n} \left(\frac{\beta}{1+\beta}\right)^n \sum_{j=0}^{n-1} \frac{\left[\frac{x}{\theta(1+\beta)}\right]^j \exp\left\{\frac{-x}{\theta(1+\beta)}\right\}}{j!} .$$

When the claim severity distribution is not necessarily exponential, but closed under convolutions, similar expressions $\bar{F}_S(x)$ are available.

For instance, using (2.18) for gamma claim severities we have

$$F_S(x) = \sum_{n=1}^{\infty} p_n \Gamma(n\alpha, \beta x) ,$$

while for inverse gaussian claim severities

$$F_S(x) = \sum_{n=1}^{\infty} p_n IG(n\mu, \beta x) .$$

Other exact methods can also be used to calculate stop loss reinsurance premiums. The following section describes the Laplace transform technique.

3.1.2 Laplace Transforms for Stop Loss Premiums

Consider the exact expression for the stop-loss premium $\pi(M)$ in (3.1), where $S = \sum_{i=1}^{N} X_i$ is a compound sum and F_S its c.d.f. .

By definition, the Laplace transform of π, say $\hat{\pi}(z)$ is given by

$$\begin{aligned}
\hat{\pi}(z) &= \int_0^\infty e^{-zx}\pi(x)dx , \quad z > 0 , \\
&= \int_0^\infty \int_x^\infty e^{-zx}\bar{F}_S(y)dydx \\
&= \int_0^\infty \int_0^y e^{-zx}\bar{F}_S(y)dxdy \\
&= \int_0^\infty \left(\frac{1 - e^{-zy}}{z}\right) \bar{F}_S(y)dx = \frac{E(S)}{z} - \frac{\hat{\bar{F}}_S(z)}{z} .
\end{aligned}$$

To obtain $\hat{\pi}(z)$ values we need $E(S) = E(N)E(X)$ and $\hat{\bar{F}}_S(z)$.

Under the usual conditions, the latter is simply given by

$$\begin{aligned}\hat{\bar{F}}_S(z) &= \int_0^\infty e^{-zx}\bar{F}_S(x)dx\,, \quad z>0\,,\\ &= \int_0^\infty e^{-zx}\sum_{n=0}^\infty p_n\bar{F}_X^{*n}(x)dx\\ &= \sum_{n=0}^\infty p_n\int_0^\infty e^{-zx}\bar{F}_X^{*n}(x)dx = \sum_{n=0}^\infty p_n\hat{\bar{F}}_S^{*n}(x)\,.\end{aligned}$$

Once the transformed values $\hat{\pi}(z)$ are know, a (usually numerical) inversion of the transform gives the stop-loss premiums $\pi(x)$.

When S is compound geometric(β) with exponential(θ) claims, an analytical expression is obtained for stop loss premiums. Using MAPLE 7 (see [6]) we obtained

$$\pi(x) = \theta\beta\exp\left\{\frac{-x}{\theta(1+\beta)}\right\}, \quad x\geq 0\,. \tag{3.7}$$

3.2 Approximation Algorithm

We now turn to the problem of approximating the integral given in (3.1). The following algorithm was programmed in MATLAB 6 (see [7]) for its vast array of predefined elementary functions.

Step

1. Choose the claims frequency distribution and its parameters.

2. Choose the claims severity distribution and its parameters.

3. Calculate the cumulants of $F_S(x)$.

4. Determine step-size, h, for Simpson's rule.

5. Discretize the interval from 0 to $\max\{10, 5\,\sigma_S\}$ in sub-intervals of length h.

6. Calculate the exact value of $F_S(x_i)$, for $i = 1, 2, \ldots, k$, where k is the number of sub-intervals obtained at step 5.

7. Calculate the approximate values for $F_S(x)$ using the incomplete gamma, inverse gaussian and IG-Gamma mixture approximations.

8. Define the retention values.

9. Evaluate (3.1) at each retention value of step 8 using the approximations in step 7 and the exact values obtained in step 6.

10. Calculate the resulting approximation errors.

11. Check the accuracy of the exact values in step 6 against analytical formulas, whenever possible.

12. Analyze the program output.

13. End

Appendix B gives a detailed description of each of the above steps.

3.3 Numerical Examples

All the examples illustrated here are for compound Poisson or binomial distributions F_S, with gamma or inverse gaussian claims severity. We use neither Pareto nor Weibull claims in this section, since these do not yield analytical expressions for F_S, making it difficult to verify the accuracy of our approximations. Results for the latter are presented in Chapter 4.

Each compound distribution is illustrated with four examples. Tables 3.1 to 3.4 give the parameter values and the first four moments for each case, while Tables 3.5 to 3.8 report the results.

Exact premiums obtained from the analytical expressions for F_S appear in the second column, $\pi(M)$. These are followed by the absolute errors of the approximate premium obtained from the mixture, gamma and IG approximations, respectively. Exact premiums $\pi(M)$, from analytical expressions, are also compared in the last column to the exact premiums obtained from direct numerical integration in (3.1).

Absolute errors are used instead of the relative ones. This is due to the fact that when reinsurance premiums get close to zero (≤ 0.5), relative errors become unstable and it is difficult to assess the accuracy of the approximations. However, the evaluation of absolute errors is reliable for all values of the retention level M.

	Parameter			Basic Characteristics			
	IG		Poisson				
Example	μ	β	λ	Mean	Std.Dev.	Kurt.	Skew.
(i)	0.70	0.50	1.00	0.70	0.92	6.26	2.08
(ii)	1.00	0.50	1.00	1.00	1.22	4.28	1.77
(iii)	1.30	0.10	1.00	1.30	1.35	1.34	1.12
(iv)	1.30	0.50	1.00	1.30	1.53	3.33	1.59

Table 3.1: Compound Poisson-IG examples

	Parameter				Basic Characteristics			
	IG		Binomial					
Example	μ	β	m	β	Mean	Std.Dev.	Kurt.	Skew.
(i)	0.7	1.00	10.00	0.67	4.20	2.32	2.75	1.28
(ii)	0.7	0.70	10.00	0.67	4.20	2.03	1.73	1.02
(iii)	1	1.00	10.00	1.50	1.22	5.39	4.00	2.53
(iv)	1	0.70	10.00	4.00	2.00	1.73	2.91	1.40

Table 3.2: Compound binomial-IG examples

	Parameter			Basic Characteristics			
	Gamma		Poisson				
Example	α	λ	λ	Mean	Std.Dev.	Kurt.	Skew.
(i)	2.00	1.50	1.00	1.33	1.63	3.33	1.63
(ii)	2.00	1.50	2.00	2.67	2.31	1.66	1.15
(iii)	3.20	1.50	1.00	2.13	2.44	2.40	1.42
(iv)	2.60	3.30	2.00	1.57	1.31	1.38	1.06

Table 3.3: Compound Poisson-gamma examples

	Parameter				Basic Characteristics			
	Gamma		Binomial					
Example	α	λ	m	β	Mean	Std.Dev.	Kurt.	Skew.
(i)	5.00	3.00	4.00	4.00	1.33	1.49	1.26	1.16
(ii)	3.00	3.00	4.00	0.67	2.40	1.33	0.29	0.56
(iii)	5.00	3.00	6.00	4.00	2.00	1.82	0.84	0.95
(iv)	3.00	3.00	6.00	1.50	2.40	1.50	0.44	0.67

Table 3.4: Compound binomial-gamma examples

Some simple conclusions clearly stand-out from the analysis of these examples. First, that the gamma approximation produces the worst results of all. Then that the inverse gaussian approximation seems to be better, in general, than the gamma. But what is clear is that, as expected, the IG-gamma mixture approximation always produces the best results for large retention limits M, which is the case of interest in practice.

In some cases, the IG-gamma mixture absolute errors are even smaller than those of the (numerical) exact values. This is usually the case for compound binomial and compound negative binomial distributions. It is explained by the fact that we used a discretization step of $h = 0.025$, which is not sufficiently small in all cases. Instead an $h = 0.001$ would have produced better exact (numerical) results, but would have also made the algorithm excessively slow.

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	0.418990	0.288	0.271	0.284	0.552	0.695447	0.461	0.033	0.271	0.673
1.0	0.245515	0.155	0.826	0.323	0.324	0.465557	0.391	1.146	0.726	0.466
1.5	0.143288	0.021	0.633	0.174	0.187	0.307438	0.182	1.263	0.662	0.313
2.0	0.083439	0.035	0.330	0.056	0.107	0.200761	0.030	0.953	0.440	0.206
2.5	0.048533	0.044	0.106	0.007	0.060	0.129894	0.047	0.564	0.225	0.132
3.0	0.028217	0.035	0.020	0.031	0.032	0.083397	0.073	0.245	0.069	0.082
3.5	0.016405	0.023	0.076	0.036	0.017	0.053196	0.071	0.030	0.026	0.049
4.0	0.009540	0.012	0.091	0.032	0.008	0.033743	0.057	0.094	0.073	0.027
4.5	0.005551	0.005	0.086	0.025	0.003	0.021301	0.040	0.151	0.089	0.013
5.0	0.003231	0.000	0.072	0.018	0.000	0.013390	0.025	0.165	0.088	0.004
5.5	0.001883	0.002	0.057	0.013	0.002	0.008386	0.014	0.157	0.077	0.001
6.0	0.001097	0.003	0.043	0.009	0.003	0.005235	0.005	0.137	0.064	0.005
6.5	0.000640	0.003	0.031	0.006	0.003	0.003259	0.000	0.113	0.050	0.007
7.0	0.000374	0.002	0.023	0.004	0.003	0.002023	0.003	0.090	0.038	0.009
7.5	0.000218	0.002	0.016	0.002	0.003	0.001253	0.004	0.070	0.029	0.010
8.0	0.000128	0.002	0.011	0.002	0.003	0.000775	0.005	0.053	0.021	0.010
8.5	0.000075	0.001	0.008	0.001	0.002	0.000478	0.005	0.040	0.015	0.010
9.0	0.000044	0.001	0.005	0.001	0.002	0.000295	0.004	0.030	0.011	0.011
9.5	0.000026	0.001	0.004	0.000	0.002	0.000181	0.004	0.022	0.007	0.011
10.0	0.000015	0.000	0.002	0.000	0.001	0.000111	0.003	0.016	0.005	0.011

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	0.983942	2.054	1.732	1.824	0.788	0.986911	0.656	0.201	0.199	0.693
1.0	0.678026	0.688	1.457	1.241	0.696	0.718893	0.675	1.238	0.976	0.526
1.5	0.452004	0.196	1.066	0.711	0.447	0.516259	0.422	1.720	1.115	0.377
2.0	0.300390	0.198	1.384	1.051	0.325	0.366388	0.191	1.615	0.952	0.260
2.5	0.189497	0.132	0.691	0.460	0.226	0.257256	0.029	1.234	0.672	0.172
3.0	0.116720	0.210	0.204	0.088	0.142	0.178927	0.063	0.796	0.396	0.105
3.5	0.070704	0.143	0.051	0.076	0.090	0.123404	0.104	0.414	0.173	0.056
4.0	0.041557	0.116	0.228	0.196	0.055	0.084472	0.111	0.126	0.016	0.021
4.5	0.023903	0.078	0.289	0.229	0.032	0.057432	0.100	0.065	0.081	0.004
5.0	0.013503	0.040	0.279	0.210	0.018	0.038808	0.081	0.176	0.132	0.021
5.5	0.007471	0.015	0.239	0.175	0.010	0.026077	0.060	0.229	0.150	0.034
6.0	0.004057	0.001	0.189	0.135	0.005	0.017434	0.041	0.242	0.148	0.042
6.5	0.002167	0.009	0.141	0.098	0.002	0.011601	0.025	0.232	0.135	0.048
7.0	0.001139	0.013	0.101	0.068	0.000	0.007686	0.013	0.208	0.117	0.052
7.5	0.000589	0.012	0.070	0.046	0.001	0.005072	0.005	0.179	0.097	0.054
8.0	0.000300	0.011	0.047	0.030	0.001	0.003335	0.001	0.149	0.078	0.056
8.5	0.000151	0.008	0.031	0.019	0.001	0.002185	0.005	0.121	0.062	0.057
9.0	0.000075	0.006	0.020	0.012	0.002	0.001427	0.007	0.096	0.048	0.058
9.5	0.000037	0.004	0.013	0.007	0.002	0.000929	0.008	0.076	0.037	0.059
10.0	0.000018	0.002	0.008	0.004	0.002	0.000604	0.007	0.059	0.028	0.059

Table 3.5: Compound Poisson-IG premiums: absolute errors ×100

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	3.70057	0.008	0.005	0.057	2.653	3.70036	0.010	0.007	0.034	1.615
1.0	3.20636	0.018	0.006	0.480	2.679	3.20396	0.022	0.007	0.215	1.630
1.5	2.72869	0.014	0.031	0.940	2.743	2.71891	0.019	0.010	0.445	1.670
2.0	2.28203	0.002	0.049	0.994	2.848	2.25810	0.001	0.033	0.520	1.742
2.5	1.87868	0.015	0.047	0.675	2.983	1.83542	0.021	0.044	0.379	1.840
3.0	1.52579	0.020	0.029	0.195	3.136	1.46175	0.031	0.035	0.104	1.950
3.5	1.22518	0.018	0.005	0.258	3.291	1.14273	0.028	0.015	0.181	2.058
4.0	0.97460	0.012	0.016	0.584	3.442	0.87872	0.018	0.008	0.389	2.152
4.5	0.76934	0.006	0.031	0.758	3.581	0.66603	0.007	0.025	0.488	2.223
5.0	0.60350	0.001	0.038	0.797	3.706	0.49857	0.003	0.034	0.488	2.269
5.5	0.47099	0.003	0.038	0.741	3.818	0.36923	0.009	0.035	0.419	2.290
6.0	0.36604	0.005	0.035	0.626	3.917	0.27093	0.011	0.031	0.315	2.289
6.5	0.28351	0.006	0.028	0.486	4.005	0.19724	0.011	0.023	0.205	2.267
7.0	0.21897	0.005	0.022	0.343	4.083	0.14262	0.010	0.016	0.104	2.230
7.5	0.16874	0.005	0.015	0.214	4.152	0.10253	0.008	0.009	0.023	2.180
8.0	0.12978	0.004	0.009	0.105	4.215	0.07333	0.005	0.003	0.035	2.120
8.5	0.09967	0.003	0.004	0.018	4.272	0.05223	0.003	0.001	0.072	2.052
9.0	0.07644	0.002	0.000	0.046	4.325	0.03705	0.002	0.004	0.093	1.979
9.5	0.05857	0.002	0.003	0.090	4.375	0.02620	0.001	0.006	0.100	1.901
10.0	0.04484	0.001	0.004	0.118	4.422	0.01848	0.000	0.006	0.099	1.820

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	3.50605	0.116	0.279	0.264	4.169	1.56652	0.001	0.884	0.059	0.774
1.0	3.02841	0.070	0.336	0.549	4.225	1.19749	0.221	0.142	0.197	0.594
1.5	2.57909	0.008	0.270	0.653	4.309	0.89933	0.249	0.453	0.262	0.422
2.0	2.16782	0.076	0.127	0.547	4.413	0.66577	0.174	0.737	0.212	0.272
2.5	1.80078	0.113	0.029	0.307	4.527	0.48713	0.079	0.764	0.124	0.147
3.0	1.48040	0.117	0.156	0.027	4.644	0.35303	0.002	0.641	0.044	0.048
3.5	1.20601	0.098	0.236	0.222	4.756	0.25383	0.045	0.460	0.012	0.029
4.0	0.97479	0.066	0.268	0.401	4.858	0.18131	0.066	0.280	0.043	0.087
4.5	0.78256	0.033	0.262	0.500	4.949	0.12881	0.068	0.131	0.055	0.130
5.0	0.62457	0.003	0.232	0.527	5.028	0.09109	0.059	0.021	0.054	0.162
5.5	0.49596	0.019	0.188	0.500	5.096	0.06417	0.046	0.053	0.047	0.186
6.0	0.39213	0.033	0.141	0.437	5.152	0.04506	0.033	0.095	0.037	0.203
6.5	0.30887	0.040	0.095	0.354	5.198	0.03155	0.021	0.115	0.027	0.215
7.0	0.24250	0.041	0.055	0.266	5.235	0.02204	0.011	0.119	0.018	0.223
7.5	0.18986	0.039	0.023	0.181	5.266	0.01537	0.004	0.114	0.011	0.230
8.0	0.14828	0.034	0.002	0.106	5.290	0.01070	0.001	0.103	0.006	0.234
8.5	0.11556	0.028	0.021	0.044	5.309	0.00743	0.004	0.090	0.003	0.238
9.0	0.08989	0.021	0.033	0.006	5.324	0.00516	0.005	0.076	0.000	0.240
9.5	0.06982	0.016	0.040	0.042	5.336	0.00358	0.006	0.063	0.001	0.242
10.0	0.05415	0.010	0.044	0.068	5.346	0.00248	0.006	0.051	0.002	0.243

Table 3.6: Compound binomial-IG premiums: absolute errors ×100

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	1.02944	0.116	0.987	0.545	0.612	2.24340	0.330	2.068	1.372	0.308
1.0	0.77313	0.843	1.485	1.229	0.475	1.85720	0.251	0.774	0.363	0.419
1.5	0.57111	0.693	2.292	1.654	0.345	1.51808	0.461	0.362	0.402	0.540
2.0	0.41669	0.407	2.243	1.509	0.237	1.22676	0.475	1.210	0.916	0.657
2.5	0.30084	0.170	1.806	1.152	0.151	0.98081	0.387	1.699	1.175	0.766
3.0	0.21516	0.012	1.259	0.761	0.085	0.77635	0.263	1.854	1.218	0.864
3.5	0.15256	0.079	0.749	0.419	0.035	0.60875	0.140	1.754	1.109	0.950
4.0	0.10732	0.122	0.340	0.156	0.002	0.47312	0.039	1.494	0.912	1.024
4.5	0.07493	0.132	0.044	0.026	0.030	0.36465	0.035	1.157	0.681	1.087
5.0	0.05196	0.124	0.148	0.138	0.050	0.27884	0.082	0.806	0.451	1.138
5.5	0.03580	0.106	0.258	0.196	0.064	0.21165	0.107	0.483	0.248	1.180
6.0	0.02452	0.084	0.308	0.218	0.075	0.15952	0.115	0.212	0.081	1.214
6.5	0.01670	0.063	0.317	0.214	0.082	0.11943	0.111	0.001	0.045	1.241
7.0	0.01131	0.043	0.301	0.197	0.087	0.08885	0.101	0.154	0.132	1.262
7.5	0.00762	0.027	0.271	0.173	0.090	0.06570	0.086	0.255	0.187	1.278
8.0	0.00511	0.015	0.235	0.146	0.093	0.04830	0.069	0.313	0.215	1.291
8.5	0.00341	0.006	0.198	0.120	0.095	0.03532	0.053	0.338	0.224	1.301
9.0	0.00227	0.000	0.163	0.097	0.096	0.02569	0.039	0.338	0.218	1.308
9.5	0.00150	0.005	0.133	0.077	0.097	0.01860	0.026	0.322	0.203	1.313
10.0	0.00099	0.007	0.106	0.060	0.097	0.01340	0.015	0.295	0.183	1.318

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	1.81871	0.498	2.688	1.917	1.001	1.15953	0.136	0.513	0.272	0.941
1.0	1.51844	1.542	1.013	1.200	1.060	0.81439	0.270	0.429	0.371	0.734
1.5	1.24879	1.712	2.647	2.320	1.149	0.55094	0.196	0.940	0.664	0.539
2.0	1.01666	1.314	3.268	2.583	1.244	0.36013	0.070	0.943	0.620	0.374
2.5	0.82152	0.863	3.357	2.482	1.333	0.22817	0.021	0.665	0.411	0.246
3.0	0.65965	0.496	3.126	2.203	1.410	0.14051	0.063	0.329	0.184	0.154
3.5	0.52654	0.223	2.702	1.833	1.477	0.08430	0.070	0.060	0.012	0.090
4.0	0.41787	0.030	2.181	1.427	1.533	0.04939	0.058	0.105	0.087	0.048
4.5	0.32977	0.102	1.637	1.028	1.580	0.02831	0.040	0.180	0.128	0.022
5.0	0.25884	0.185	1.124	0.665	1.619	0.01591	0.023	0.193	0.130	0.007
5.5	0.20212	0.231	0.674	0.358	1.651	0.00877	0.010	0.173	0.112	0.002
6.0	0.15705	0.248	0.303	0.111	1.678	0.00475	0.001	0.140	0.088	0.006
6.5	0.12146	0.245	0.012	0.076	1.699	0.00253	0.004	0.105	0.065	0.008
7.0	0.09352	0.228	0.202	0.210	1.716	0.00133	0.006	0.076	0.045	0.009
7.5	0.07169	0.203	0.350	0.297	1.730	0.00069	0.006	0.052	0.030	0.008
8.0	0.05474	0.174	0.442	0.347	1.740	0.00035	0.006	0.035	0.020	0.007
8.5	0.04163	0.144	0.491	0.368	1.749	0.00018	0.005	0.023	0.012	0.006
9.0	0.03154	0.115	0.507	0.368	1.756	0.00009	0.003	0.014	0.008	0.005
9.5	0.02380	0.089	0.499	0.353	1.761	0.00004	0.002	0.009	0.005	0.004
10.0	0.01791	0.065	0.474	0.329	1.765	0.00002	0.002	0.006	0.003	0.002

Table 3.7: Compound binomial-gamma premiums: absolute errors ×100

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	1.03890	1.254	0.869	0.959	0.725	1.91292	0.045	0.383	0.305	1.181
1.0	0.76133	1.874	2.957	2.714	0.640	1.46180	0.056	0.194	0.137	1.071
1.5	0.53293	0.757	2.567	2.159	0.492	1.06457	0.142	0.180	0.171	0.915
2.0	0.36376	0.060	1.875	1.465	0.353	0.73631	0.123	0.459	0.383	0.731
2.5	0.24365	0.160	1.241	0.926	0.247	0.48292	0.044	0.513	0.406	0.545
3.0	0.15966	0.217	0.645	0.452	0.170	0.30027	0.032	0.375	0.282	0.379
3.5	0.10204	0.234	0.144	0.061	0.114	0.17714	0.069	0.164	0.111	0.248
4.0	0.06360	0.222	0.199	0.202	0.073	0.09930	0.069	0.015	0.027	0.152
4.5	0.03871	0.186	0.382	0.335	0.045	0.05300	0.048	0.117	0.101	0.088
5.0	0.02305	0.137	0.440	0.369	0.027	0.02701	0.023	0.147	0.119	0.048
5.5	0.01343	0.089	0.423	0.345	0.015	0.01317	0.004	0.133	0.104	0.025
6.0	0.00766	0.050	0.366	0.292	0.007	0.00616	0.007	0.101	0.077	0.013
6.5	0.00428	0.022	0.297	0.232	0.002	0.00278	0.011	0.069	0.050	0.006
7.0	0.00234	0.003	0.229	0.175	0.000	0.00121	0.010	0.042	0.030	0.003
7.5	0.00125	0.007	0.170	0.127	0.002	0.00051	0.008	0.025	0.017	0.001
8.0	0.00066	0.012	0.123	0.090	0.003	0.00021	0.005	0.013	0.009	0.001
8.5	0.00034	0.012	0.086	0.061	0.003	0.00008	0.003	0.007	0.005	0.000
9.0	0.00017	0.010	0.059	0.041	0.004	0.00003	0.002	0.004	0.002	0.000
9.5	0.00008	0.007	0.040	0.027	0.004	0.00001	0.001	0.002	0.001	0.000
10.0	0.00004	0.004	0.027	0.017	0.004	0.00000	0.000	0.001	0.000	0.000

	(i)					(ii)				
		Absolute Errors ×100					Absolute Errors ×100			
M	$\pi(M)$	Mix.	Γ	IG	Exact	$\pi(M)$	Mix.	Γ	IG	Exact
0.5	1.63180	0.841	0.155	0.072	0.855	1.92903	0.035	0.596	0.460	1.142
1.0	1.27982	1.333	1.372	1.364	0.774	1.49386	0.086	0.261	0.177	1.021
1.5	0.97534	0.661	1.644	1.423	0.630	1.11466	0.174	0.266	0.244	0.867
2.0	0.72962	0.284	1.827	1.478	0.487	0.80049	0.152	0.655	0.533	0.699
2.5	0.53618	0.128	1.796	1.419	0.368	0.55327	0.073	0.771	0.602	0.535
3.0	0.38621	0.004	1.459	1.129	0.269	0.36823	0.005	0.646	0.488	0.389
3.5	0.27252	0.120	0.964	0.719	0.186	0.23620	0.054	0.398	0.289	0.270
4.0	0.18856	0.186	0.477	0.328	0.120	0.14618	0.071	0.143	0.091	0.179
4.5	0.12807	0.199	0.088	0.024	0.068	0.08740	0.065	0.050	0.054	0.113
5.0	0.08545	0.181	0.181	0.180	0.030	0.05054	0.048	0.161	0.133	0.068
5.5	0.05601	0.147	0.340	0.295	0.002	0.02831	0.029	0.199	0.158	0.038
6.0	0.03608	0.110	0.410	0.340	0.018	0.01537	0.012	0.191	0.148	0.020
6.5	0.02285	0.075	0.417	0.338	0.032	0.00811	0.001	0.159	0.121	0.009
7.0	0.01423	0.045	0.384	0.306	0.041	0.00416	0.006	0.121	0.090	0.003
7.5	0.00872	0.021	0.332	0.260	0.048	0.00207	0.008	0.086	0.063	0.000
8.0	0.00526	0.004	0.273	0.211	0.052	0.00101	0.009	0.057	0.041	0.002
8.5	0.00312	0.006	0.216	0.164	0.055	0.00048	0.007	0.037	0.026	0.003
9.0	0.00183	0.012	0.166	0.124	0.056	0.00022	0.006	0.023	0.016	0.004
9.5	0.00105	0.015	0.124	0.091	0.057	0.00010	0.004	0.013	0.009	0.004
10.0	0.00060	0.014	0.091	0.066	0.058	0.00004	0.003	0.008	0.005	0.004

Table 3.8: Compound Poisson-gamma premiums: absolute errors ×100

Chapter 4

Simulation

This chapter presents a model validation using simulations. Once the approximate stop loss reinsurance premiums are known, using the proposed algorithm, new losses are generated following the same distribution. Presumably, simulated losses are shared between the insurer and reinsurer, according to the same retention levels. The reinsurer's losses are analyzed. The difference between the approximate premiums and the reinsurer's losses are studied; if this difference is close to zero this validates our proposed approximate premiums as acceptable.

4.1 Simulation Process

Here aggregate losses were generated using MATLAB version 6.0 (see [7]). The following steps summarize the algorithm that was used:

1. Choose the claim frequency and severity distributions.

2. Given a retention level, say M, approximate the corresponding stop-loss premium $\pi(M)$ using the algorithm described in Section 3.2.

3. Accordingly, generate the number of claims, say N, following a Poisson, binomial or negative binomial distribution.

4. For $i = 1$ to N, generate the claim amounts X_i.

5. Calculate the aggregate loss $S = \sum_{1=1}^{N} X_i$.

6. Determine the reinsurer's loss $Z = (S - M)^+$.

7. Evaluate the difference between the stop-loss premium and the reinsurer's loss, $U = \pi(M) - Z$.

This algorithm is repeated 10,000 times. An empirical confidence interval for U, obtained in step 7, is estimated using the mean and standard deviation of U over the 10,000 generated observations. This interval is given by

$$\left[\text{mean} - 2\,\frac{\text{Standard deviation}}{\sqrt{\text{No. iterations}}}, \text{mean} + 2\,\frac{\text{Standard deviation}}{\sqrt{\text{No. iterations}}}\right]$$

The predefined M-function of MATLAB was used to generate the random variables N and X_i, except in the inverse gaussian distribution. In that case there is no predefined MAT LAB function and a rejection sampling method was used instead (see Ross [10, §5.2] for details).

4.2 Examples

Various simulations are presented here. The relative errors of the approximate premiums, compared to the exact ones, are calculated for the mixture approximation. The exact method used in these examples is based on Simpson's rule, since in the Pareto and Weibull cases F_S does not have an analytical expression.

4.2.1 Compound Poisson with Pareto Claims Distribution

Table 4.1 illustrates two examples of compound Poisson distributions with Pareto claims, reporting the relative errors for different retention values.

	Poisson $\lambda = 4$	Pareto $\alpha = 24$, $\lambda = 10$		Poisson $\lambda = 2$	Pareto $\alpha = 10$, $\lambda = 6$	
M	Mixture	Exact	% Error	Mixture	Exact	% Error
0.0	1.7407	1.7589	1.03	1.3477	1.3436	0.31
0.5	1.2783	1.2957	1.34	0.9559	0.9629	0.73
1.0	0.8938	0.9106	1.84	0.6693	0.6798	1.54
1.5	0.6003	0.6148	2.36	0.4656	0.4747	1.92
2.0	0.3898	0.4017	2.96	0.3229	0.3289	1.82
2.5	0.2459	0.2555	3.76	0.2235	0.2266	1.37
3.0	0.1514	0.1592	4.90	0.1545	0.1555	0.64
3.5	0.0912	0.0978	6.75	0.1068	0.1064	0.38
4.0	0.0539	0.0597	9.72	0.0738	0.0727	1.51
4.5	0.0313	0.0365	14.25	0.0510	0.0496	2.82
5.0	0.0179	0.0226	20.80	0.0353	0.0338	4.44
5.5	0.0101	0.0143	29.37	0.0244	0.0230	6.09
6.0	0.0056	0.0093	39.78	0.0169	0.0156	8.33
6.5	0.0031	0.0063	50.79	0.0117	0.0105	11.43
7.0	0.0017	0.0045	62.22	0.0081	0.0070	15.71
7.5	0.0009	0.0032	71.88	0.0056	0.0046	21.74
8.0	0.0005	0.0023	78.26	0.0039	0.0030	30.00
8.5	0.0003	0.0016	81.25	0.0027	0.0018	50.00
9.0	0.0001	0.0010	90.00	0.0019	0.0010	90.00
9.5	0.0001	0.0005	80.00	0.0013	0.0004	225.00
10.0	0.0000	0.0000	-	0.0009	0.0000	-

Table 4.1: Premiums for the compound Poisson with Pareto claims

See how the relative error increase with the retention level. However, in simulated cases the approximate premiums are closer to exact values, even at high retention levels. Table 4.2 shows the confidence intervals estimated for the difference of the stop-loss premiums minus the reinsurer's losses. This difference decreases to zero as the retention level increases. A graphical representation of these results is also given in Figure C.1 (see Table 4.3 for the summary statistics of the distributions used).

Poisson ($\lambda = 4$) and Pareto ($\alpha = 24$, $\lambda = 10$)

		Premium - Reinsurer's Loss			
Retention	Premium	Mean	Std.Dev.	Lower Limit	Upper Limit
4.25	0.0410	-0.001526	0.2656	-0.0038	0.0068
7.00	0.0017	-0.0001709	0.04627	-0.00075446	0.0011
9.28	0.0001	-0.00006152	0.003848	-0.00001544	0.00013848

Poisson ($\lambda = 2$) and Pareto ($\alpha = 10$, $\lambda = 6$)

		Premium - Reinsurer's Loss			
Retention	Premium	Mean	Std.Dev.	Lower Limit	Upper Limit
4.16	0.0655	-0.003352	0.4144	-0.0116	0.0049
6.99	0.0081	-0.00011816	0.1502	-0.0031	0.0029
9.82	0.0010	-0.00040196	0.0421	-0.00044053	0.0012

Table 4.2: Simulation results: compound Poisson with Pareto claims

In both cases the aggregate claims distribution is skewed to the right with a heavy tail. Reinsured claims are usually those losses that we find in the tail of the distribution. Therefore retention limits should be large, for instance M could be equal to the mean aggregate claims plus five standard deviations.

In the first case (I) we used retention values of 4.25, 7 and 9.28, which correspond to two, four and six standard deviations around the mean, respectively.

	Poisson	Pareto		Mean	Std.Dev.	Skewness	Kurtosis
I	$\lambda = 4$	$\alpha = 24$	$\lambda = 10$	1.74	1.26	1.14	1.81
II	$\lambda = 2$	$\alpha = 10$	$\lambda = 6$	1.33	1.41	1.82	8.14

Table 4.3: Summary statistics: compound Poisson with Pareto claims

Appendix C presents graphical representations of these results in Figures C.1 to C.3.

In case (II), F_S shows greater skewness, so its tail is heavier. See from Table 4.1 how when $M \geq 9.0$, the relative error takes much larger values than in the first example. However, in Table 4.2 the difference between approximate premiums and reinsurer's losses remains close to zero. Figures C.4, C.5 and C.6 illustrate graphically these results for the simulated examples.

4.2.2 Compound Poisson with Weibull claim distribution

The examples in this section correspond to compound Poisson distributions with Weibull claim severities. The chosen parameter values and the resulting first four moments are reported in Table 4.4.

	Poisson	Weibull		Mean	Std.Dev.	Skewness	Kurtosis
I	$\lambda = 3$	$\theta = 1$	$\lambda = 1$	3.00	2.45	1.22	2.00
II	$\lambda = 2$	$\theta = 1$	$\lambda = 2$	1.00	1.00	1.50	3.00

Table 4.4: Summary statistics: compound Poisson with Weibull claims

Here we used absolute instead of relative errors, as in the previous section, since the latter was unstable in this example. Table 4.5 reports the results.

	Poisson $\lambda = 3$	Weibull $\theta = 1$, $\lambda = 1$		Poisson $\lambda = 2$	Weibull $\theta = 1$, $\lambda = 2$	
M	Exact	Mixture	Error	Exact	Mixture	Error
0.0	3.01130	3.00770	0.36	1.0116	1.0080	0.36
0.5	2.55520	2.54960	0.56	0.6421	0.6329	0.92
1.0	2.14310	2.13500	0.81	0.3914	0.3841	0.73
1.5	1.77830	1.76890	0.94	0.231	0.2265	0.45
2.0	1.46120	1.45190	0.93	0.1328	0.1303	0.25
2.5	1.18980	1.18140	0.84	0.0747	0.0733	0.14
3.0	0.96070	0.95380	0.69	0.0413	0.0405	0.08
3.5	0.76970	0.76440	0.53	0.0226	0.0220	0.06
4.0	0.61220	0.60840	0.38	0.0122	0.0117	0.05
4.5	0.48370	0.48110	0.26	0.0066	0.0062	0.04
5.0	0.37970	0.37810	0.16	0.0036	0.0032	0.04
5.5	0.29630	0.29550	0.08	0.002	0.0016	0.04
6.0	0.22990	0.22970	0.02	0.0012	0.0008	0.04
6.5	0.17740	0.17770	0.03	0.0007	0.0004	0.03
7.0	0.13620	0.13670	0.05	0.0005	0.0002	0.03
7.5	0.10400	0.10480	0.08	0.0003	0.0001	0.02
8.0	0.07910	0.07990	0.08	0.0002	0	0.02
8.5	0.05980	0.06070	0.09	0.0002	0	0.02
9.0	0.04500	0.04590	0.09	0.0001	0	0.01
9.5	0.03370	0.03460	0.09	0	0	0
10.0	0.02500	0.02600	0.10	0	0	0

Table 4.5: Premiums for the compound Poisson with Weibull claims

Absolute errors decrease as the retention level increases. As observed with the Pareto claims of the previous section, we can see here in Table 4.6 that approximate premiums are close to simulated reinsurer's losses, even at high retention levels. Table 4.6 also reports the estimated confidence intervals for the difference between stop-loss premiums and reinsurer's losses. Again, this difference decreases to zero as the retention level increases.

A graphical representation of these results is given in Appendix C. Figure C.7 to C.9 correspond to case I, while Figures C.10 and C.11 correspond to case II. In both cases, the aggregate claims distribution is skewed to the right with a heavy tail. The chosen retention limits were two, four and six standard deviations around the mean, respectively.

Poisson ($\lambda = 3$) and Pareto ($\theta = 1$, $\lambda = 1$)

		Premium - Reinsurer's Loss			
Retention	Premium	Mean	Std.Dev.	Lower Limit	Upper Limit
7.90	0.08443	-0.001956	0.559170	-0.0131399	0.0092268
12.80	0.00485	-0.001175	0.151093	-0.0041972	0.0018465
17.70	0.00018	0.000017	0.000949	0.0001515	0.0001895

Poisson ($\lambda = 2$) and Pareto ($\theta = 1$, $\lambda = 2$)

		Premium - Reinsurer's Loss			
Retention	Premium	Mean	Std.Dev.	Lower Limit	Upper Limit
3	0.04050	-0.00049607	0.2662102	-0.0058203	0.0048281
5	0.00318	-0.00063668	0.0546720	-0.0004567	0.0017301
7	0.00018	-0.00002638	0.0242193	-0.0007482	0.0007482

Table 4.6: Simulation results: compound Poisson with Weibull claims

So far, our model seems validated by these simulation results. For the illustrated aggregate claims distributions, the IG-mixture approximation provides accurate stop-loss reinsurance premium approximations. To complete the study we also consider a case more common in practice, when the aggregate claims distributions is not known but its moments can be estimated from claims data. The following section illustrates various examples in this setting.

4.3 Unknown Aggregate Claims Distribution

Every year, natural catastrophies produce large materials losses. In actuarial work, it is common that the frequency and severity distributions be unknown, while proper reinsurance coverages are still needed. Given a retention level for these reinsurance treaties, the corresponding stop-loss premium is then required.

The algorithm described in Section 3.2 cannot be used when the distribution of aggregate claims is unknown. However, it is possible to use the IG-mixture approximation directly, since this function only needs the first four cumulants that can be estimated from claims data.

As an illustration of the method, this section features four examples. Claims data for each example is simulated using Weibull, gamma, Pareto and inverse gaussian severity distributions. Then, assuming that these distributions are not known, we estimate their first four moments. An histogram for each example is given in Figure 4.1.

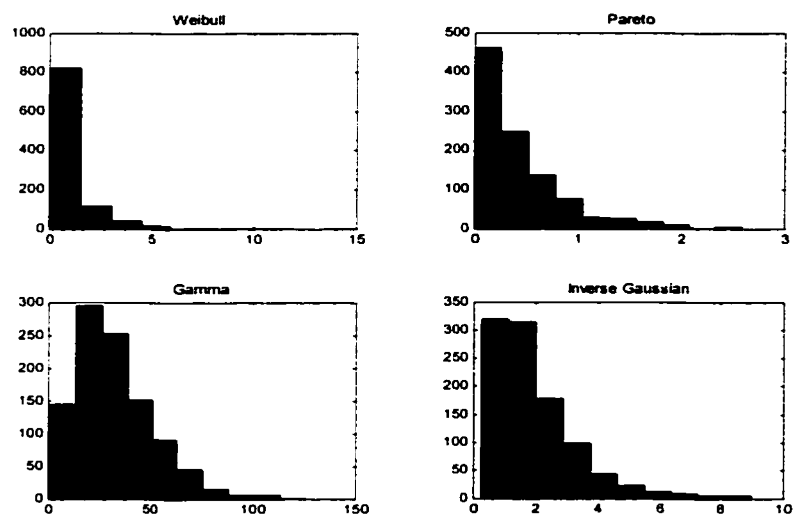

Figure 4.1: Histogram of simulated losses

With these moments the first four cumulants are calculated, the approximate stop-loss reinsurance premiums were calculated using the IG-mixture approximation. Table 4.7 reports the results for each example.

The $\pi(M)$ column gives the approximate premiums for the different retention levels $M = \mathbb{E}(X) + i\sqrt{\mathrm{Var}(X)}$, where $i = 1, 2, \ldots, 10$ are the corresponding rows and X the simulated claim severity random variable.

To validate these approximate premiums we simulate additional claim severities with the same distributions. Then using the algorithm in Section 3.2, steps 6 and 7 are performed with the generated claim amounts. This procedure is repeated 10,000 times and the results are reported in Table 4.7.

Weibull				
M	$\pi(M)$	U	Lower	Upper
2.50	0.2141	0.0391	0.0218	0.0563
3.99	0.1157	0.0242	0.0107	0.0378
5.48	0.0644	0.0203	0.0111	0.0294
6.96	0.0365	0.0152	0.0093	0.0210
8.45	0.0209	0.0083	0.0027	0.0138
9.94	0.0120	0.0019	-0.0035	0.0072
11.43	0.0070	0.0009	-0.0034	0.0051
12.92	0.0040	-0.0009	-0.0042	0.0025
14.41	0.0024	-0.0007	-0.0029	0.0014
15.90	0.0014	-0.0014	-0.0053	0.0025

Pareto				
M	$\pi(M)$	U	Lower	Upper
0.84	0.0594	-0.0080	-0.0130	-0.0030
1.27	0.0207	-0.0069	-0.0102	-0.0036
1.69	0.0068	-0.0046	-0.0065	-0.0026
2.11	0.0021	-0.0045	-0.0062	-0.0027
2.54	0.0006	-0.0023	-0.0037	-0.0010
2.96	0.0001	-0.0011	-0.0020	-0.0003
3.38	0.0000	-0.0005	-0.0010	0.0001
3.81	0.0000	-0.0004	-0.0007	0.0000
4.23	0.0000	-0.0004	-0.0010	0.0002
4.66	0.0000	0.0000	0.0000	0.0000

Gamma				
M	$\pi(M)$	U	Lower	Upper
51.42	2.2708	0.4239	0.2826	0.5651
70.34	0.5972	0.2451	0.1921	0.2981
89.25	0.1393	0.0223	-0.0022	0.0468
108.17	0.0293	0.0003	-0.0180	0.0187
127.09	0.0055	0.0055	0.0055	0.0055
146.01	0.0009	0.0009	0.0009	0.0009
164.93	0.0001	0.0001	0.0001	0.0001
183.85	0.0000	0.0000	0.0000	0.0000
202.77	0.0000	0.0000	0.0000	0.0000
221.69	0.0000	0.0000	0.0000	0.0000

IG				
M	$\pi(M)$	U	Lower	Upper
3.36	0.1875	-0.0117	-0.0265	0.0031
4.73	0.0640	-0.0150	-0.0256	-0.0045
6.10	0.0207	-0.0118	-0.0181	-0.0056
7.47	0.0064	-0.0072	-0.0115	-0.0029
8.83	0.0018	-0.0037	-0.0062	-0.0012
10.20	0.0005	-0.0014	-0.0029	0.0000
11.57	0.0001	-0.0002	-0.0005	0.0001
12.94	0.0000	-0.0003	-0.0007	0.0002
14.31	0.0000	-0.0002	-0.0006	0.0002
15.68	0.0000	0.0000	0.0000	0.0000

Table 4.7: Premium comparison and simulation results

As we can see, the difference U between approximate premiums and the reinsurer's losses goes to zero as the retention limit increases. Empirical confidence intervals for U are also given in the last two columns of Table 4.7.

Again we see that the approximation is more accurate for large retention values M, which is the case of interest in practice. In this semi-parametric case the upper and lower bounds on the difference U are not around 0 when M is small. This shows systematic under or over estimation of the premium for small M. But the problem disappears as M takes values greater than the mean plus two or three standard deviations.

This analysis provides further evidence that the IG-mixture approximation is valid for stop-loss premiums, even in semi-parametric cases, when only a few claims sample moments are known.

Conclusion

Chaubey et al. [3] showed that the IG-mixture approximation provides a good fit to aggregate claims distributions. We can now conclude from our analysis that it also provides a good approximation to stop-loss reinsurance premiums.

We compared the IG-mixture approximation to other approximations commonly used in the actuarial literature, like the gamma and inverse gaussian approximations. For the computation of stop-loss reinsurance premiums, we found that the best results were obtained using an IG-mixture approximation. The popular gamma approximation produced the worst stop-loss premiums, with relative errors uniformly larger than those produced by the inverse gaussian and IG-mixture approximations.

An important feature of the IG-mixture approximation is its non-parametric character. Like the gamma approximation, it can be used to calculate reinsurance premiums without specifying the aggregate claims distribution F_S. It is sufficient to estimate the first four cumulants of F_S from the claims data.

The algorithm presented in Section 3.2 for stop-loss premiums can be adapted to approximate premiums for more general reinsurance treaties, like reinsurance layers. The approximate premium formula would then depend on the retention level at each layer. This could be a subject for further research.

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Appendix A

Approximation of Aggregate Claim Distributions

This appendix summarizes the numerical results of Section 2.4, page 35.

	Gamma claim size distribution				IG claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.43300	0.17280	0.13600	0.08070	0.36800	0.16340	0.14440	0.09600
1.0	0.54290	0.05390	0.03350	0.00300	0.44360	0.07800	0.08860	0.11550
1.5	0.64650	0.00870	0.00180	0.00850	0.64240	0.02150	0.02000	0.01620
2.0	0.73270	0.00830	0.00790	0.00750	0.73930	0.00990	0.00790	0.00270
2.5	0.80110	0.01310	0.00980	0.00490	0.81840	0.01730	0.01450	0.00710
3.0	0.85390	0.01280	0.00880	0.00290	0.88570	0.00700	0.00460	0.00150
3.5	0.89380	0.01040	0.00680	0.00140	0.92700	0.00480	0.00320	0.00090
4.0	0.92370	0.00760	0.00480	0.00050	0.95480	0.00250	0.00160	0.00060
4.5	0.94560	0.00510	0.00310	0.00000	0.97320	0.00030	0.00000	0.00090
5.0	0.96160	0.00310	0.00170	0.00030	0.98430	0.00060	0.00060	0.00060
5.5	0.97310	0.00160	0.00080	0.00040	0.99100	0.00100	0.00080	0.00040
6.0	0.98130	0.00050	0.00010	0.00040	0.99500	0.00100	0.00080	0.00020
6.5	0.98710	0.00010	0.00020	0.00040	0.99720	0.00090	0.00070	0.00010
7.0	0.99110	0.00050	0.00040	0.00040	0.99850	0.00070	0.00050	0.00000
7.5	0.99390	0.00070	0.00050	0.00030	0.99920	0.00050	0.00040	0.00000
8.0	0.99590	0.00070	0.00050	0.00020	0.99960	0.00040	0.00030	0.00000
8.5	0.99720	0.00070	0.00050	0.00020	0.99980	0.00030	0.00020	0.00010
9.0	0.99810	0.00070	0.00040	0.00010	0.99990	0.00020	0.00010	0.00000
9.5	0.99880	0.00060	0.00040	0.00010	0.99990	0.00010	0.00010	0.00000
10.0	0.99920	0.00050	0.00030	0.00000	1.00000	0.00010	0.00000	0.00000

	Pareto claim size distribution				Weibull claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.14710	0.05220	0.03320	0.02180	0.16735	0.03752	0.02318	0.01338
1.0	0.32210	0.03590	0.01340	0.00010	0.31727	0.03765	0.02648	0.00200
1.5	0.49740	0.00800	0.00110	0.00660	0.47371	0.01685	0.01219	0.00029
2.0	0.64690	0.00600	0.00670	0.00720	0.61150	0.00592	0.00619	0.00688
2.5	0.76210	0.01040	0.00800	0.00650	0.73272	0.00677	0.00513	0.00094
3.0	0.84470	0.01040	0.00750	0.00580	0.82286	0.00776	0.00583	0.00090
3.5	0.90090	0.00890	0.00670	0.00540	0.88857	0.00496	0.00348	0.00031
4.0	0.93750	0.00730	0.00600	0.00520	0.93258	0.00259	0.00171	0.00051
4.5	0.96050	0.00620	0.00560	0.00520	0.96076	0.00079	0.00043	0.00050
5.0	0.97460	0.00540	0.00530	0.00520	0.97797	0.00026	0.00030	0.00039
5.5	0.98310	0.00510	0.00520	0.00530	0.98802	0.00071	0.00058	0.00024
6.0	0.98810	0.00500	0.00520	0.00530	0.99369	0.00080	0.00061	0.00012
6.5	0.99090	0.00500	0.00520	0.00540	0.99677	0.00070	0.00051	0.00004
7.0	0.99260	0.00510	0.00530	0.00540	0.99839	0.00054	0.00038	0.00001
7.5	0.99350	0.00510	0.00530	0.00540	0.99922	0.00038	0.00026	0.00003
8.0	0.99400	0.00520	0.00530	0.00540	0.99963	0.00025	0.00017	0.00004
8.5	0.99430	0.00530	0.00540	0.00540	0.99983	0.00016	0.00011	0.00004
9.0	0.99450	0.00530	0.00540	0.00540	0.99992	0.00010	0.00006	0.00003
9.5	0.99450	0.00530	0.00540	0.00540	0.99996	0.00006	0.00004	0.00002
10.0	0.99460	0.00540	0.00540	0.00540	0.99998	0.00003	0.00002	0.00002

Table A.1: Relative errors for the compound Poisson distribution

	Gamma claim size distribution				IG claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.0565	0.0043	0.0072	0.0168	0.0242	0.0983	0.2637	0.0106
1.0	0.1445	0.0459	0.0394	0.0175	0.0693	0.0025	0.0616	0.0203
1.5	0.2694	0.0268	0.0213	0.0024	0.1367	0.0169	0.0012	0.0114
2.0	0.4167	0.0084	0.0058	0.0029	0.2203	0.0144	0.0172	0.0049
2.5	0.5653	0.0020	0.0022	0.0030	0.3123	0.0094	0.0176	0.0013
3.0	0.6975	0.0056	0.0047	0.0017	0.4059	0.0051	0.0135	0.0005
3.5	0.8030	0.0052	0.0041	0.0004	0.4956	0.0022	0.0088	0.0011
4.0	0.8795	0.0032	0.0024	0.0003	0.5781	0.0004	0.0048	0.0012
4.5	0.9305	0.0013	0.0009	0.0005	0.6514	0.0006	0.0019	0.0010
5.0	0.9621	0.0001	0.0001	0.0005	0.7150	0.0011	0.0001	0.0007
5.5	0.9804	0.0005	0.0005	0.0003	0.7690	0.0012	0.0013	0.0005
6.0	0.9903	0.0007	0.0006	0.0001	0.8142	0.0012	0.0019	0.0003
6.5	0.9954	0.0006	0.0005	0.0000	0.8515	0.0010	0.0020	0.0001
7.0	0.9979	0.0004	0.0003	0.0000	0.8820	0.0008	0.0020	0.0000
7.5	0.9991	0.0003	0.0002	0.0001	0.9066	0.0006	0.0018	0.0001
8.0	0.9996	0.0002	0.0001	0.0000	0.9264	0.0005	0.0015	0.0001
8.5	0.9998	0.0001	0.0001	0.0000	0.9422	0.0003	0.0012	0.0001
9.0	0.9999	0.0000	0.0000	0.0000	0.9547	0.0002	0.0009	0.0001
9.5	1.0000	0.0000	0.0000	0.0000	0.9646	0.0001	0.0006	0.0001
10.0	1.0000	0.0000	0.0000	0.0000	0.9724	0.0000	0.0004	0.0001

	Pareto claim size distribution				Weibull claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.0001	0.6531	0.6944	1.4264	0.0432	0.1408	0.1377	0.1191
1.0	0.0029	0.0829	0.2839	0.2934	0.1215	0.0814	0.0774	0.0541
1.5	0.0180	0.0156	0.0742	0.0671	0.2119	0.0180	0.0135	0.0131
2.0	0.0612	0.0158	0.0093	0.0195	0.3405	0.0229	0.0202	0.0040
2.5	0.1427	0.0243	0.0360	0.0176	0.4720	0.0029	0.0020	0.0038
3.0	0.2590	0.0300	0.0404	0.0240	0.6029	0.0034	0.0032	0.0018
3.5	0.3947	0.0321	0.0375	0.0290	0.7179	0.0057	0.0050	0.0006
4.0	0.5307	0.0321	0.0334	0.0313	0.8103	0.0059	0.0051	0.0004
4.5	0.6520	0.0312	0.0304	0.0317	0.8800	0.0036	0.0030	0.0004
5.0	0.7504	0.0303	0.0287	0.0312	0.9278	0.0019	0.0016	0.0003
5.5	0.8246	0.0296	0.0281	0.0305	0.9591	0.0003	0.0002	0.0004
6.0	0.8772	0.0293	0.0282	0.0300	0.9779	0.0005	0.0004	0.0002
6.5	0.9126	0.0293	0.0286	0.0296	0.9887	0.0008	0.0007	0.0001
7.0	0.9356	0.0294	0.0291	0.0295	0.9945	0.0008	0.0007	0.0001
7.5	0.9499	0.0295	0.0295	0.0295	0.9975	0.0006	0.0005	0.0000
8.0	0.9587	0.0297	0.0298	0.0296	0.9989	0.0004	0.0004	0.0000
8.5	0.9639	0.0298	0.0299	0.0297	0.9995	0.0003	0.0002	0.0000
9.0	0.9669	0.0298	0.0300	0.0297	0.9998	0.0002	0.0001	0.0000
9.5	0.9687	0.0299	0.0300	0.0298	0.9999	0.0001	0.0001	0.0000
10.0	0.9697	0.0299	0.0300	0.0298	1.0000	0.0000	0.0000	0.0000

Table A.2: Relative errors for the compound binomial distribution

	Gamma claim size distribution				IG claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.48823	0.22263	0.18864	0.13898	0.03534	0.20886	0.17317	0.13896
1.0	0.54179	0.03516	0.01423	0.01634	0.09705	0.11721	0.11447	0.11184
1.5	0.63498	0.00846	0.01618	0.02746	0.14215	0.03828	0.05150	0.06418
2.0	0.72341	0.01207	0.01241	0.01290	0.24347	0.07104	0.05915	0.04775
2.5	0.79171	0.01282	0.00987	0.00556	0.31212	0.01364	0.02334	0.03265
3.0	0.84322	0.01275	0.00877	0.00297	0.42453	0.02981	0.02413	0.01869
3.5	0.88282	0.01105	0.00720	0.00157	0.50145	0.01022	0.01297	0.01560
4.0	0.91326	0.00850	0.00526	0.00052	0.60045	0.00822	0.00773	0.00727
4.5	0.93634	0.00598	0.00348	0.00016	0.66968	0.00853	0.00763	0.00677
5.0	0.95361	0.00387	0.00208	0.00053	0.74440	0.00007	0.00151	0.00302
5.5	0.96639	0.00225	0.00107	0.00066	0.79746	0.00604	0.00426	0.00255
6.0	0.97578	0.00108	0.00037	0.00067	0.84782	0.00184	0.00018	0.00142
6.5	0.98263	0.00028	0.00008	0.00061	0.88380	0.00346	0.00207	0.00074
7.0	0.98761	0.00023	0.00035	0.00052	0.91488	0.00131	0.00026	0.00074
7.5	0.99120	0.00053	0.00049	0.00042	0.93705	0.00150	0.00078	0.00010
8.0	0.99377	0.00068	0.00053	0.00032	0.95490	0.00043	0.00001	0.00040
8.5	0.99561	0.00073	0.00053	0.00024	0.96755	0.00032	0.00012	0.00006
9.0	0.99691	0.00071	0.00049	0.00017	0.97721	0.00014	0.00017	0.00020
9.5	0.99784	0.00066	0.00044	0.00011	0.98398	0.00022	0.00014	0.00006
10.0	0.99849	0.00059	0.00038	0.00006	0.98895	0.00037	0.00022	0.00008

	Pareto claim size distribution				Weibull claim size distribution			
		Relative Errors				Relative Errors		
x	$F_S(x)$	IG	Γ	Mixture	$F_S(x)$	IG	Γ	Mixture
0.5	0.01691	0.29047	0.04735	0.09661	0.03534	0.20886	0.17317	0.13896
1.0	0.05625	0.03336	0.00279	0.00454	0.09705	0.11721	0.11447	0.11184
1.5	0.11948	0.01141	0.00102	0.00150	0.14215	0.03828	0.05150	0.06418
2.0	0.20186	0.01236	0.00862	0.00437	0.24347	0.07104	0.05915	0.04775
2.5	0.29604	0.00365	0.01389	0.01033	0.31212	0.01364	0.02334	0.03265
3.0	0.39438	0.00527	0.01673	0.01441	0.42453	0.02981	0.02413	0.01869
3.5	0.49041	0.01194	0.01792	0.01671	0.50145	0.01022	0.01297	0.01560
4.0	0.57943	0.01617	0.01813	0.01774	0.60045	0.00822	0.00773	0.00727
4.5	0.65860	0.01842	0.01786	0.01797	0.66968	0.00853	0.00763	0.00677
5.0	0.72665	0.01931	0.01740	0.01779	0.74440	0.00007	0.00151	0.00302
5.5	0.78351	0.01935	0.01692	0.01742	0.79746	0.00604	0.00426	0.00255
6.0	0.82989	0.01893	0.01652	0.01701	0.84782	0.00184	0.00018	0.00142
6.5	0.86696	0.01832	0.01622	0.01664	0.88380	0.00346	0.00207	0.00074
7.0	0.89607	0.01770	0.01602	0.01636	0.91488	0.00131	0.00026	0.00074
7.5	0.91857	0.01714	0.01591	0.01616	0.93705	0.00150	0.00078	0.00010
8.0	0.93573	0.01669	0.01587	0.01604	0.95490	0.00043	0.00001	0.00040
8.5	0.94867	0.01636	0.01588	0.01597	0.96755	0.00032	0.00012	0.00006
9.0	0.95832	0.01613	0.01591	0.01596	0.97721	0.00014	0.00017	0.00020
9.5	0.96545	0.01599	0.01596	0.01597	0.98398	0.00022	0.00014	0.00006
10.0	0.97067	0.01592	0.01602	0.01600	0.98895	0.00037	0.00022	0.00008

Table A.3: Relative errors for the compound negative binomial distribution

Appendix B

Algorithms

This appendix gives a detailed description of each step in the algorithm defined in Section 3.2.

B.1 Detailed Description of the Algorithm

First, the claim frequency and severity distributions are selected in step 1. Figure B.1 shows the claim frequency distribution menu.

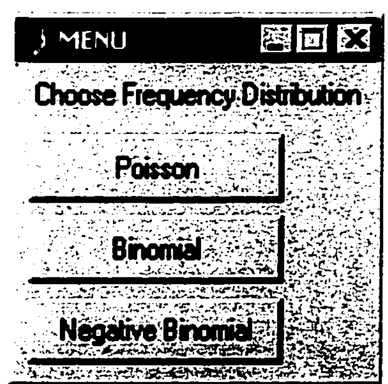


Figure B.1: Claim frequency distribution menu

Then the corresponding parameters are entered. For example assume that the number of claims is Poisson distributed. Figure B.2 illustrates the dialog-box of our MATLAB program, where *Lambda* would be the corresponding Poisson parameter.

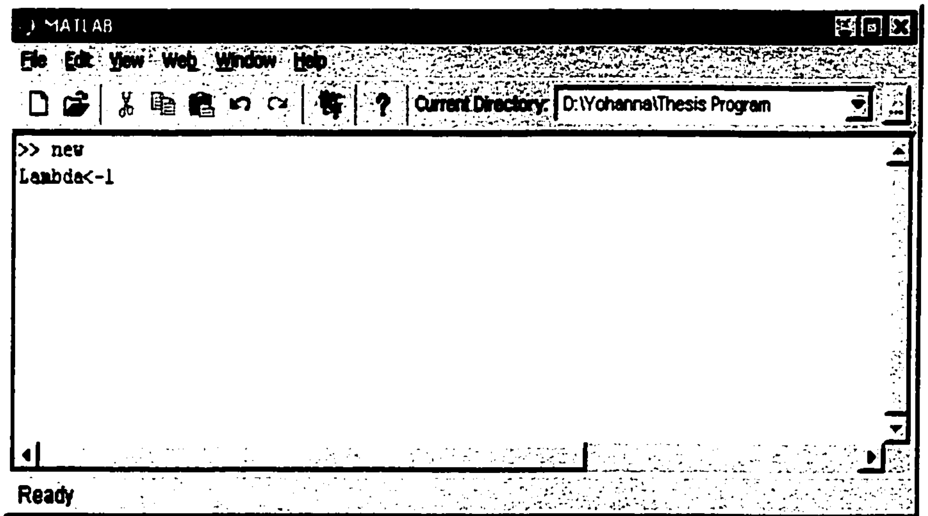


Figure B.2: Defining the parameter value for a Poisson distribution

Given these parameter values, the first four cumulants of the frequency distribution p_n are calculated. After step 2 the claim severity distribution is chosen and the corresponding parameter are entered. The selection menu appears in Figure B.3.

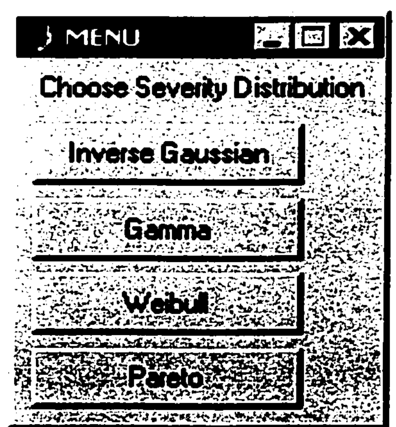


Figure B.3: Claim severity distribution menu

This example assumes a $\Gamma(\alpha = 1, \beta = 2)$ severity distribution, where the parameters are entered as shown in Figure B.4.

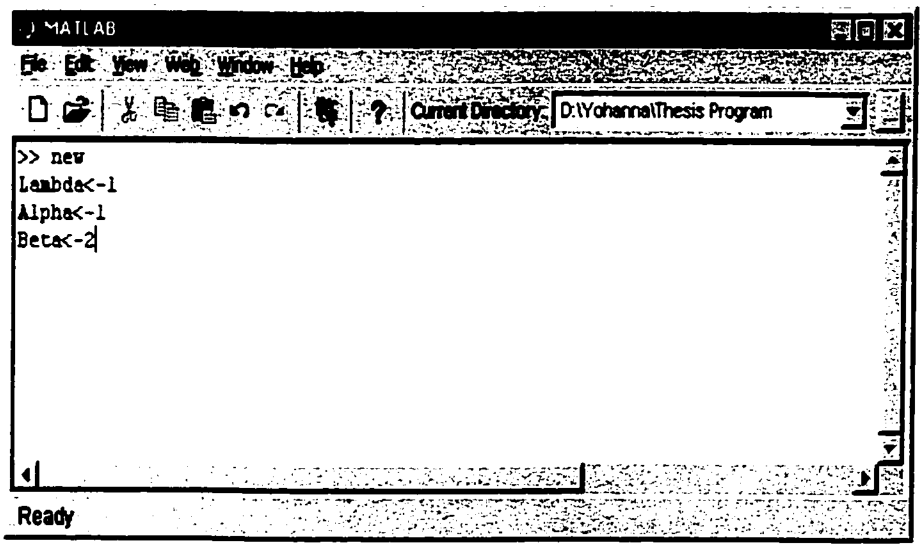


Figure B.4: Defining the parameter values for a $\Gamma(\alpha, \beta)$ distribution

Then the first four moments, and therefore the first four cumulants of the severity distribution F_X are obtained.

Once the cumulants of p_n have been obtained in step 1 and the cumulants of F_X in step 2, the program calculates the cumulants of F_S, as defined in Section 2.2. Figure B.5 summarizes the results for this example.

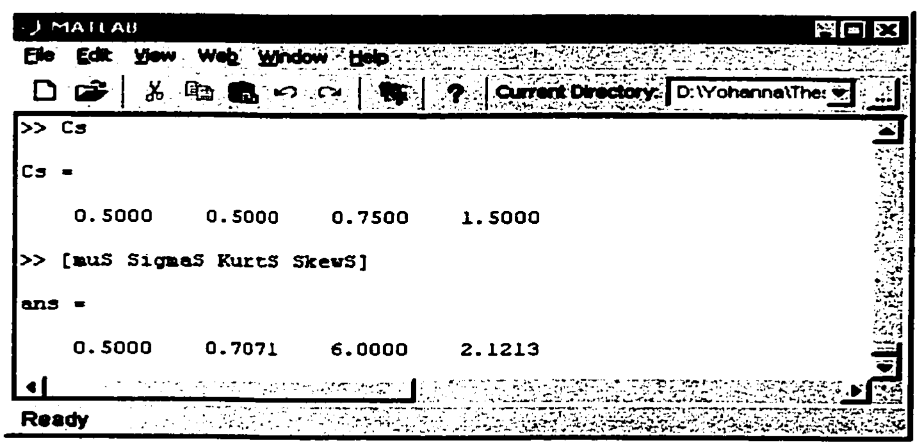


Figure B.5: Cumulants (C_S) and moments of F_S

In step 4, h is entered for Simpson's rule, from a menu similar to that in Figure B.6. Three possibilities are offered: 0.025 0.016 and 0.01.

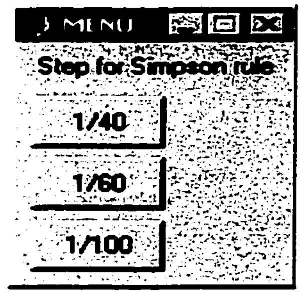


Figure B.6: Simpson's rule h menu

The next step discretizes the interval $[0, m]$ into k sub-intervals of length h, where $m = \max\{10, 5\ \sigma_S\}$. For $h = \frac{1}{40}$ the first and the last 26 sub-intervals of this example are shown in Figure B.7.

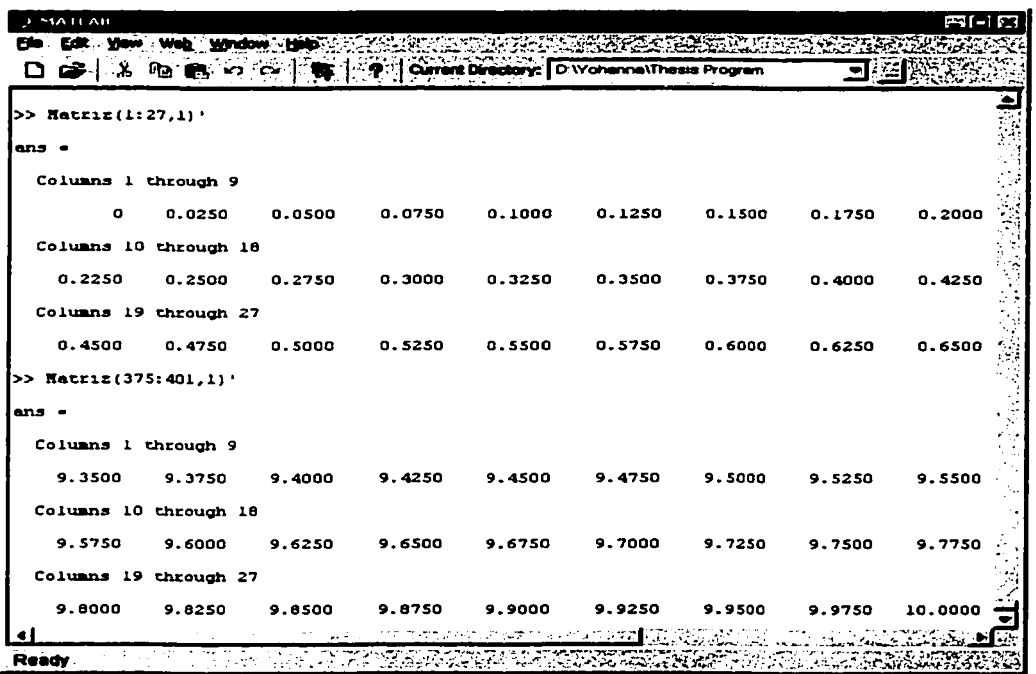


Figure B.7: Discretization of $[0, m]$

Now another MATLAB function calculates a matrix with the p.d.f. and c.d.f. exact values for the aggregate claims distribution with the chosen parameters. The evaluation is at each end point of the partition. The exact values at the required points x_i are extracted from this matrix. The corresponding approximate values are also calculated at each x_i with the different approximations.

Let the retention M take values from 0 to five standard deviations, or from 0 to ten, whichever is greater. As explained in step 6, each approximation method is used to evaluate (3.1) in MATLAB.

The function syntax is

$$q = quad\,(\,\mathbf{fun}\,, a\,, b\,)\,,$$

which approximates the integral of function **fun** from a to b, to an error of within 10^6, using recursive adaptive Simpson's quadrature. **fun** accepts a vector x and returns a vector y. The function **fun** being evaluated at each element of x. A simple change of variable is needed in (3.1) in order to produce a definite integral: let $y = \frac{1}{1+x}$ be the new variable, with $dy = -\frac{1}{(1+x)^2}$. Then (3.1) becomes

$$\pi(M) = \int_M^\infty [1 - F_S(x)]\,dx = \int_0^{\frac{1}{1+M}} \left[1 - F_S\left(\frac{1}{y} - 1\right)\right] y^2\,dy\,, \tag{B.1}$$

where F_S is replaced by one of the approximations to the aggregate claims distribution, as well as its exact values.

Then, the exact value for the reinsurance premium is obtained numerically. Using a tabular form of F_S at the points $x_0, x_1, \ldots, x_k$, where $x_k - x_{k-1} = h$, as in step 6, we evaluate (B.1) as follows:

$$\int_M^\infty [1 - F_S(x)]\,dx \approx \sum_{j=n_1}^{n_2} h\,[1 - F_S(x_j)]\,, \tag{B.2}$$

where n_1, n_2 are such that x_{n_1} is the closest value to M and $x_{n_2} = m$ is the upper limit of our interval.

The relative errors, used to assess the accuracy of the approximations, is defined as

$$\frac{\text{Exact value} - \text{Approximated value}}{\text{Exact value}}\,.$$

This exact method is not always as precise as needed, since it was obtained from a numerical method, itself containing sources of error. To verify the precision of the method, we compare it with the values obtained from an analytical expression for F_S, whenever possible (see Section 3.1.1).

The illustrated choices of frequency and severity distributions for which an analytical expression is available are summarized in the menu of Figure B.8.

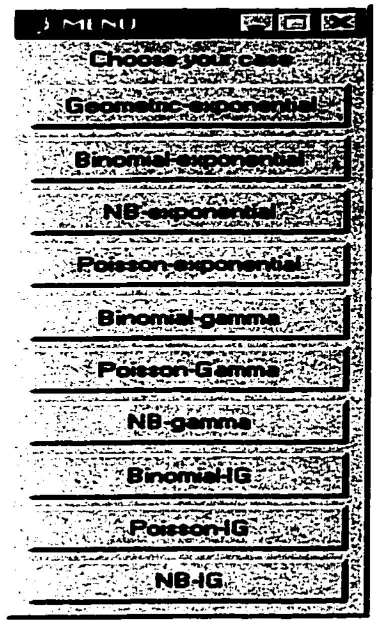


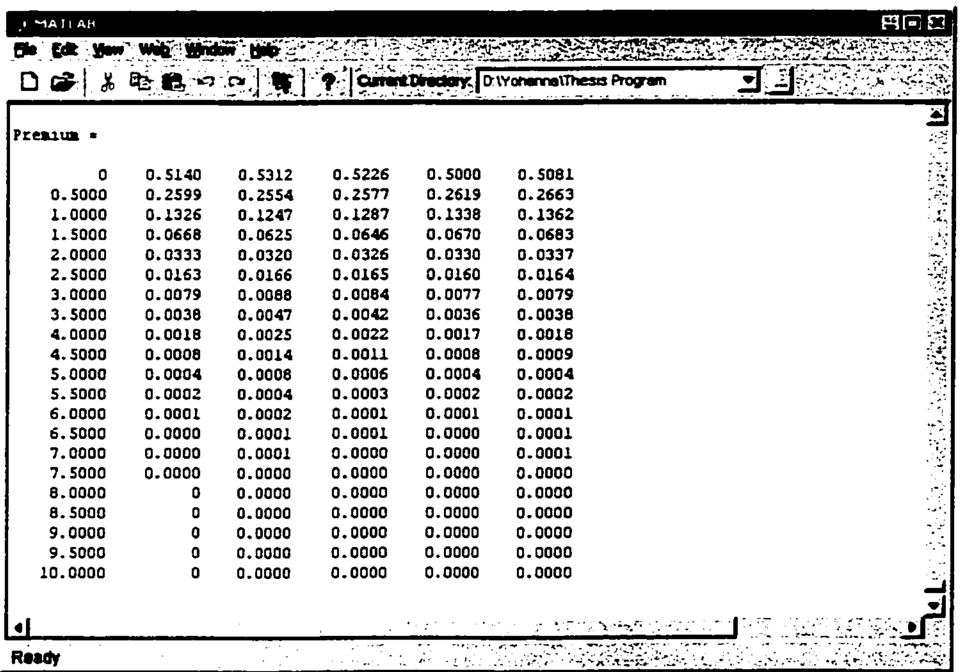
Figure B.8: Cases with Analytical Expression for F_S

In each case, the results are obtained using the analytical expression of F_S in (B.1). Here the Poisson-gamma case is illustrated with the values obtained for each retention level. The relative errors in the values obtained in step 6 and step 11 are compared in Figure B.9.

The first column gives the different retention levels used. The three following columns report approximated stop loss reinsurance premiums obtained through the mixture, IG and gamma approximations, respectively. The last column gives the premiums obtained in step 6, while the before-last column reports the premiums calculated using the analytical expression in step 11.

As we can see, the last two columns are very similar in this specific example. The accuracy of F_S in step 6 depends on h. When F_S is heavy tailed, the numerical errors increase and the relative errors, comparing the exact and approximate values, also get larger.

Various numerical illustrations of this algorithm are presented in Section 3.3 and Appendix A.



Premium =

0	0.5140	0.5312	0.5226	0.5000	0.5081
0.5000	0.2599	0.2554	0.2577	0.2619	0.2663
1.0000	0.1326	0.1247	0.1287	0.1338	0.1362
1.5000	0.0668	0.0625	0.0646	0.0670	0.0683
2.0000	0.0333	0.0320	0.0326	0.0330	0.0337
2.5000	0.0163	0.0166	0.0165	0.0160	0.0164
3.0000	0.0079	0.0088	0.0084	0.0077	0.0079
3.5000	0.0038	0.0047	0.0042	0.0036	0.0038
4.0000	0.0018	0.0025	0.0022	0.0017	0.0018
4.5000	0.0008	0.0014	0.0011	0.0008	0.0009
5.0000	0.0004	0.0008	0.0006	0.0004	0.0004
5.5000	0.0002	0.0004	0.0003	0.0002	0.0002
6.0000	0.0001	0.0002	0.0001	0.0001	0.0001
6.5000	0.0000	0.0001	0.0001	0.0000	0.0001
7.0000	0.0000	0.0001	0.0000	0.0000	0.0001
7.5000	0.0000	0.0000	0.0000	0.0000	0.0000
8.0000	0	0.0000	0.0000	0.0000	0.0000
8.5000	0	0.0000	0.0000	0.0000	0.0000
9.0000	0	0.0000	0.0000	0.0000	0.0000
9.5000	0	0.0000	0.0000	0.0000	0.0000
10.0000	0	0.0000	0.0000	0.0000	0.0000

Figure B.9: Program Output

Appendix C

Graphs for the Compound Poisson Examples

This appendix contains the graphs used with the examples of Section 4.2. They correspond to compound Poisson distributions with Pareto and Weibull claim severities.

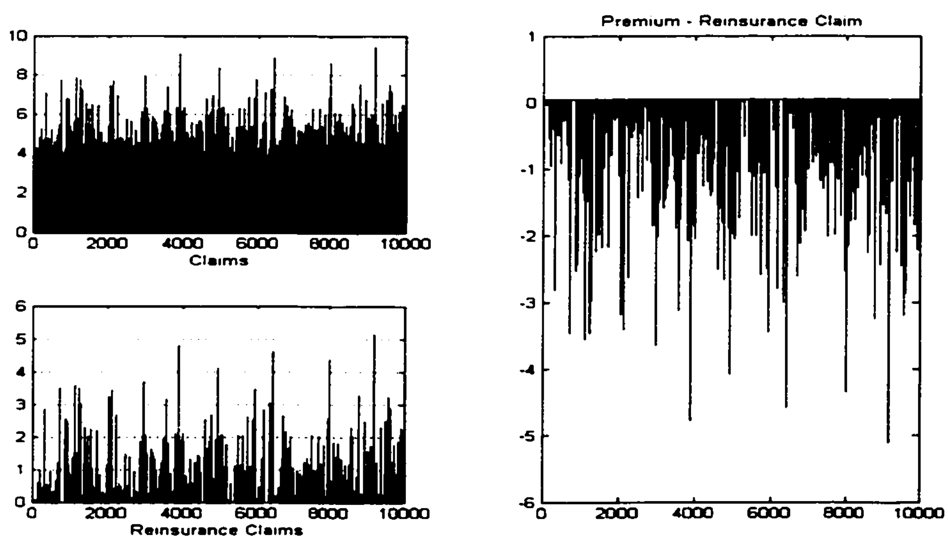


Figure C.1: Compound Poisson-Pareto (I): retention level = mean + 2 st.dev.

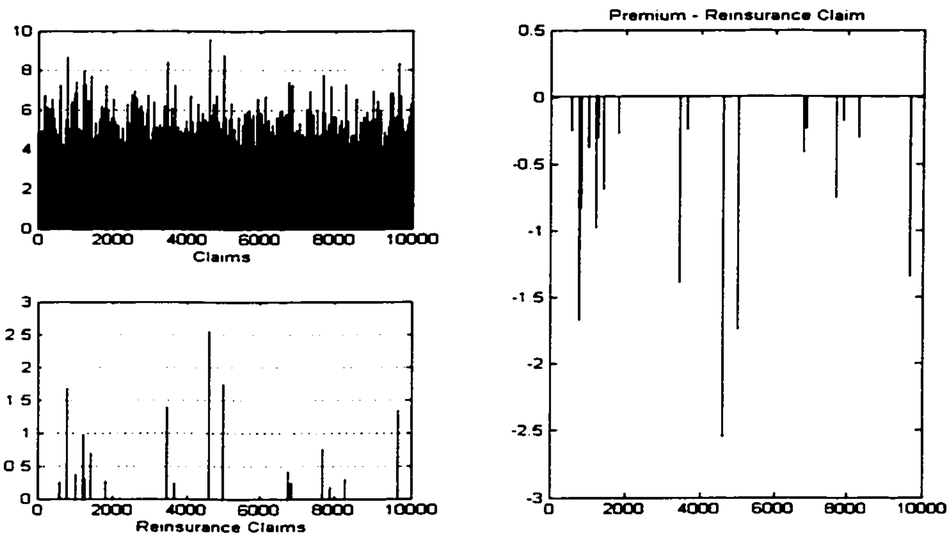


Figure C.2: Compound Poisson-Pareto (I): retention level = mean + 4 st.dev.

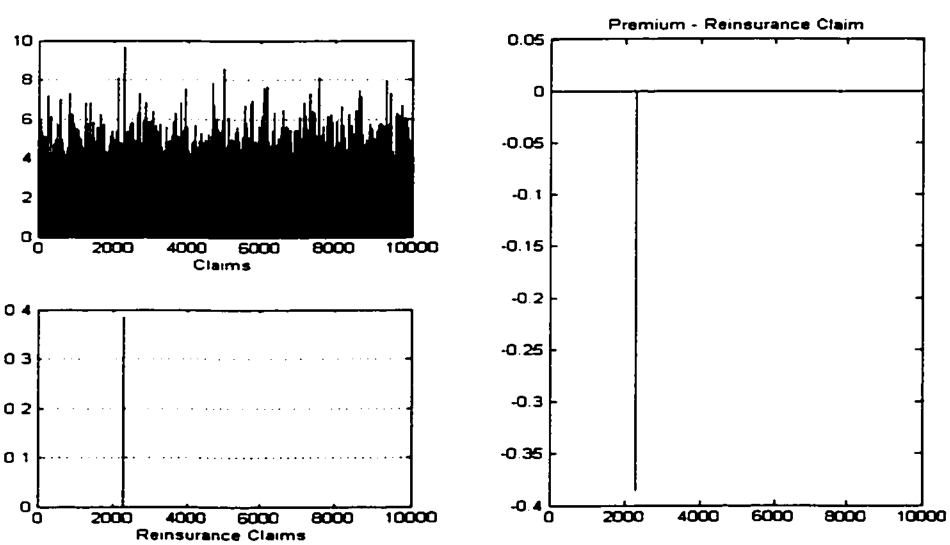


Figure C.3: Compound Poisson-Pareto (I): retention level = mean + 6 st.dev.

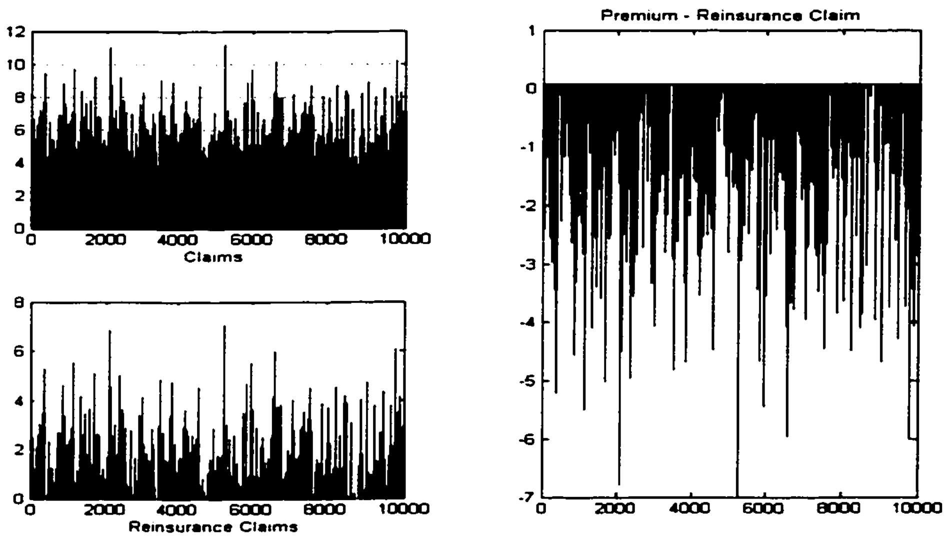


Figure C.4: Compound Poisson-Pareto (II): retention level = mean + 2 st.dev.

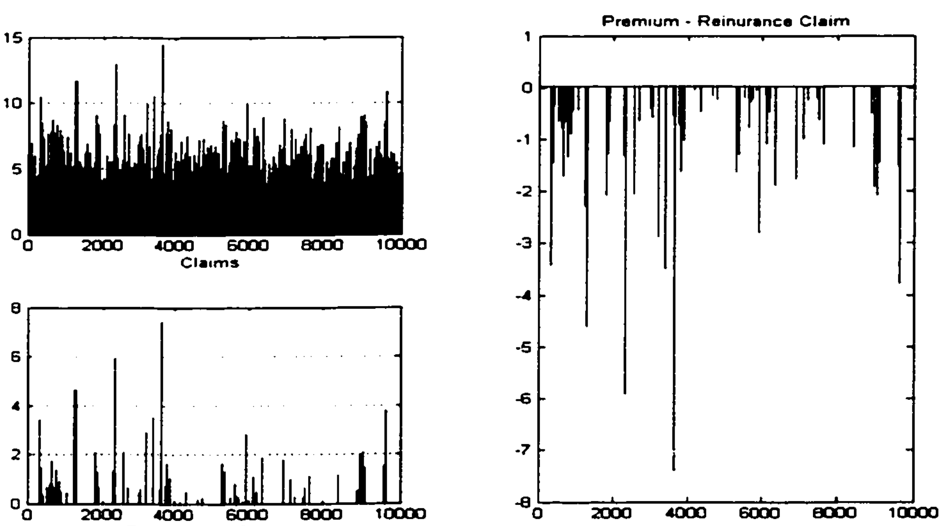


Figure C.5: Compound Poisson-Pareto (II): retention level = mean + 4 st.dev.

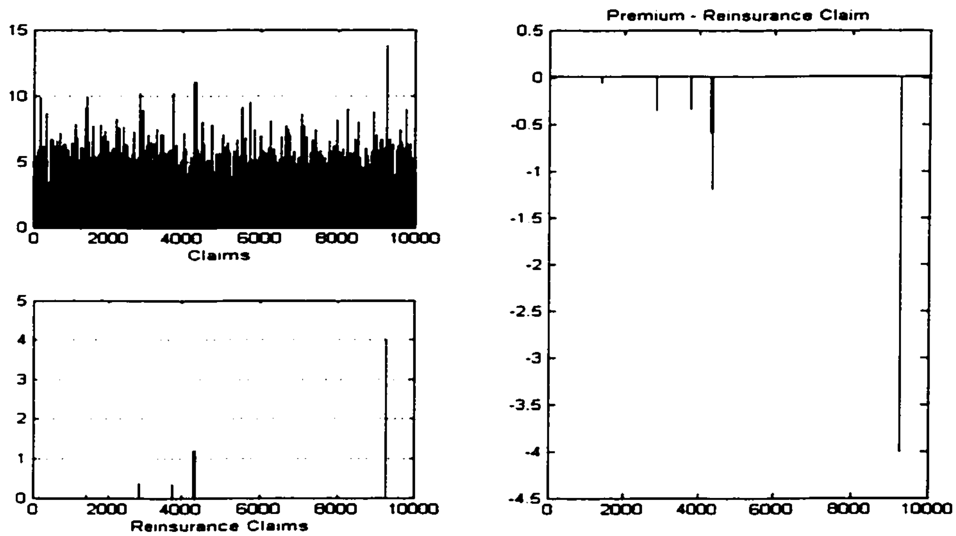


Figure C.6: Compound Poisson-Pareto (II): retention level = mean + 6 st.dev.

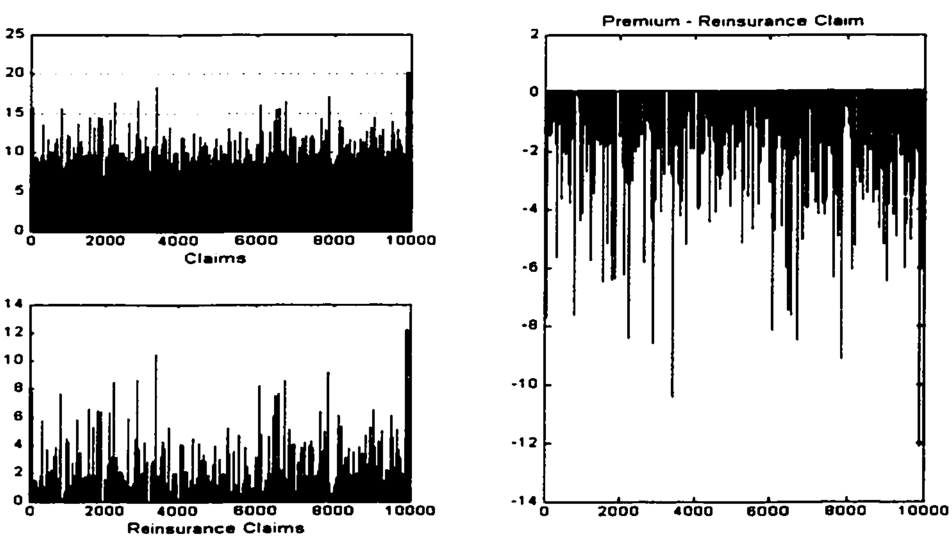


Figure C.7: Compound Poisson-Weibull (I): retention level = mean + 2 st.dev.

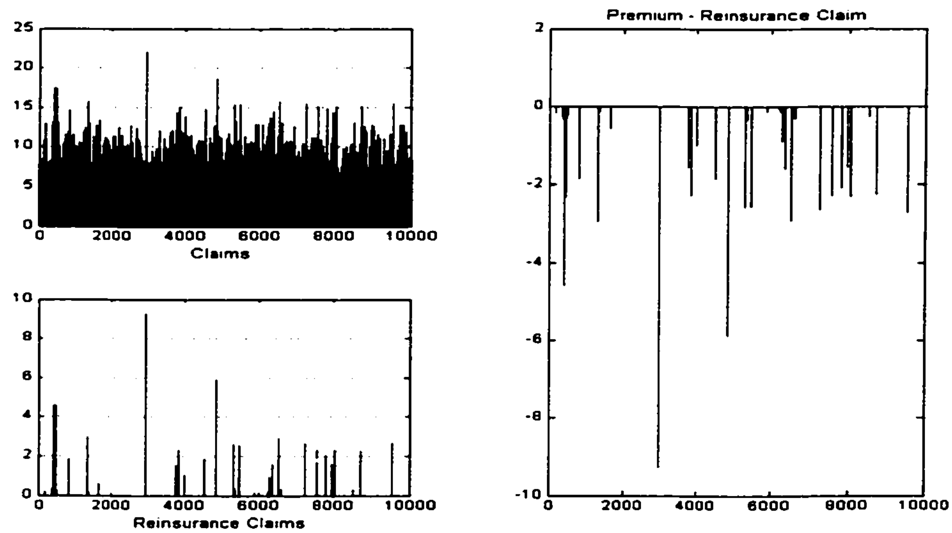


Figure C.8: Compound Poisson-Weibull (I): retention level = mean + 4 st.dev.

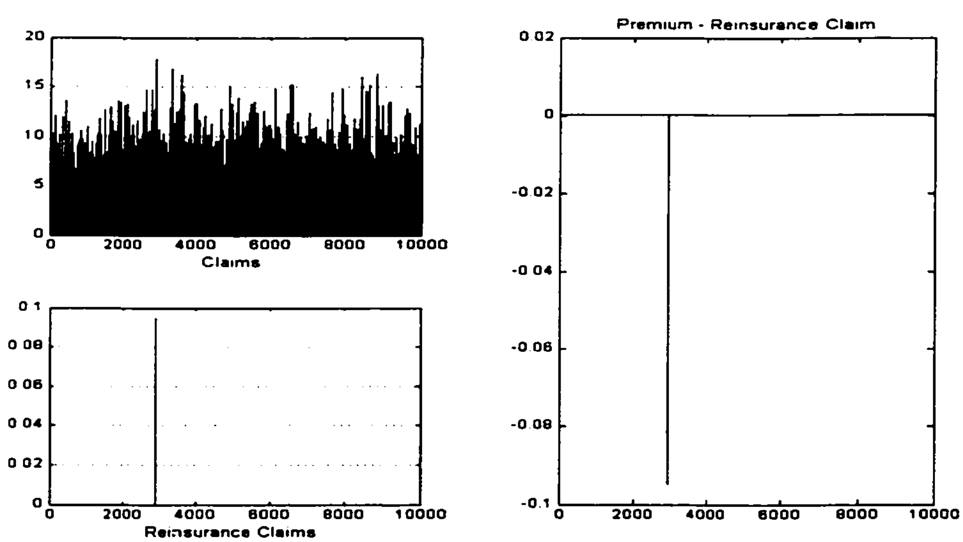


Figure C.9: Compound Poisson-Weibull (I): retention level = mean + 6 st.dev.

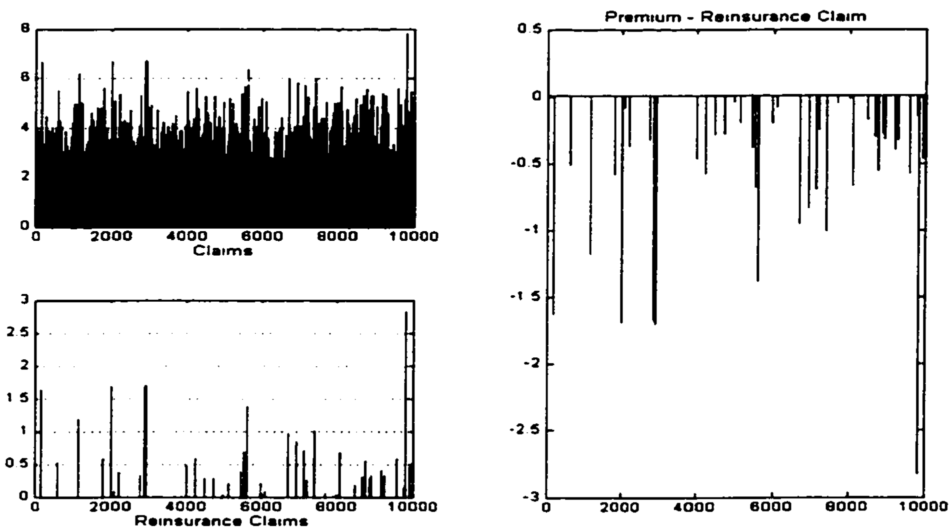


Figure C.10: Compound Poisson-Weibull (II): retention level = mean + 4 st.dev.

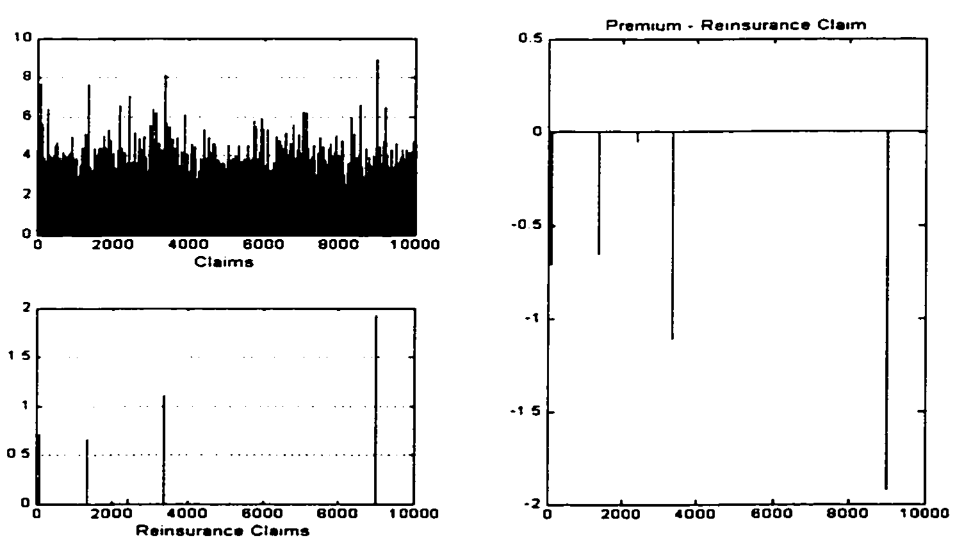


Figure C.11: Compound Poisson-Weibull (II): retention level = mean + 6 st.dev.