Option Pricing in the Presence of Warrants

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Abstract

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The capital structure of the firm is affected by the issuance of warrants due to an eventual dilution effect. Issuing warrants also impacts the generating process of the stock prices. The equity rate of return is not expected to be constant.

This study estimates option prices via a recombining binomial tree incorporating the effect of warrant dilution on the capital structure of the firm (Warrant Dilution Option-Pricing model –WDOP-). The binomial lattice is constructed on the value of the firm under the assumption of constant volatility of the rate of return of the firm value.

The mean percentage prediction error and the mean absolute value of the mean prediction error indicate that the predicted option prices deviate significantly from the observed option prices. The weak predicting performance of the WDOP model doesn't seem to follow a systematic pattern. This is true even when the sample is divided into subcategories based on option moneyness, and option and warrant life.

We tested the WDOP model with one parameter, the volatility of the rate of return of the firm value, estimated out-of-sample. In- and out-of-sample theoretical option prices displayed similar deviations when compared to the observed market prices. The tendency towards increasing differences between in- and out-of-sample option prices as we go from in- to out-of-the-money options seems to be consistent with the volatility smile. These discrepancies, related to the moneyness of the option, may be an indicator of an underlying stochastic process (volatility of the rate of return of the firm value) that differs from the lognormal.

We compare the WDOP model with the Black Schole option-pricing model (hereafter B/S) applied to the options without taking into account the equity dilution effect. The B/S model exhibits the least biases for in- and at-the-money options and the most efficiency in establishing prices for options overall. However the B/S model does not have the best option pricing performance and predictability. The mean percentage predictor error and absolute value of the percentage deviation may not be considered sufficiently small.

Finally, we calculate the warrant prices throughout the sample. Theoretical warrant prices estimated using the B/S model diverge from the sample warrant market prices. This finding may again suggest that the process of the volatility of the rate of return of the firm value follows a non-lognormal stochastic process, which may further explain the limited predicting performance of the WDOP model.

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1. INTRODUCTION

The concept of options, their pricing and related strategies occupies a major place in the field of Corporate Finance, Investment and risk management. The foundations of option trading can be traced to Europe and the United States as far back as the early eighteenth century. Only recently, however has the trading of these instruments been conducted through organized, regulated exchanges known as option markets. The volume of option trades, by both institutional and individual investors, has increased exponentially over the past decade during which period, innovation has brought about a variety of highly complex option-like instruments under an assortment of nomenclatures, from the basic "plain vanilla option" to exotic options.

Options traded on organized exchanges are typically written on underlying financial assets such as common stock, stock indices, foreign currencies and futures contracts. Throughout the present study, we are only going to consider call options written on common stock.

A call option is a contract that gives the right (not the obligation) to the option holder to buy a pre-determined underlying financial asset at a pre-established price by a specific date. Call options can be either European or American. European options can only be exercised on the option's expiry date, whereas American options can be exercised at any time up to and including their expiry date. Call options traded on organized exchanges are usually American options[1] which, due to the possibility of early exercise, makes the

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[1] Hull, J., C. Futures and Options Markets, Chapter 7, 3rd edition, Hull, Pretice Hall

valuation of these instruments more complex when there are dividends prior to expiration.

Another instrument similar to a call option is a warrant. Warrants provide the holder the right to purchase a pre-determined number of underlying assets (such as common stocks) at a pre-established price (exercise price) on or before a specified date. Warrants can also be either Europeans or Americans, where the American warrant can usually be exercised after a minimum holding period. The four major distinctions between call options and warrants are; i) the instrument's life span, ii) their origination, iii) the dilution effect, and iv) the issuer's strategy.

i) A warrant typically has a life span of 2 to 5 years (depending on the firm's underlying strategy), compared to the exchange traded option whose maximum life span is usually limited to 9 months.

ii) While call options can be written and sold by both the underlying security issuer (corporate entity) or the security holder (institutional or individual investor), warrants are only issued by the underlying security issuer (corporate entity) usually accompanying a new debt issue.

iii) Call options, being issued on outstanding common stock, have no dilution effect on the underlying firm's shares when exercised. Warrants require the issuance of new common stock when exercised thereby dividing the corporate "pie" of the firm into more pieces and diluting the claim of the original common stockholders.

iv) Both call options and warrants are commonly offered to a firm's senior executives as a component of their remuneration and incentive package. It is clear that the option is intended to inspire short-term performance and a warrant is more of a loyalty\long-term performance instrument. However, exchange traded call options, due to their shorter life span, are most commonly used as hedging or speculation instruments while warrants, due to their longer term to maturity, as an investment incentive from the security issuer.

The most commonly known warrants are:

- Equity Warrants - warrants written on common stocks;
- Compound Warrants - multiple exercise prices across successive expiry dates;
- Piggy Back Warrants - the right to acquire shares and additional warrants;
- Callable warrants - contain a special provision for early redemption by the issuer.

The Equity Warrant, given its widespread application, will be the central focus of this study.

Purpose

The purpose of this study is to develop a new theoretical pricing model for the call options of firms that have warrants in their capital structure and to test this approach empirically using actual market quotes. The objective is to incorporate the influence of the warrant's dilution effect on the firm's outstanding common shares into the valuation of the call option.

Crouhy and Galai in their (1991a) study show that the issuance of warrants has an impact on the capital structure of the firm as a result of their eventual dilution effect and therefore become an influencing factor in the determination of the underlying stock prices. The authors also argue that once the warrants are issued, the original equity volatility is reduced because it is spread across both the equity and warrant holders as a result of the eventual dilution. The equity volatility is not expected to remain constant and will change with the variation in the firm's assets (equity + warrants).

Our study is the first one to determine the price of call options while considering the dilution effect of the warrants. Section II presents the literature review, which consists of a review of the binomial option pricing model and the B/S option-pricing model using the latter as a basis of comparison for our study. The concept of the "volatility smile" and its effect will be briefly reviewed as well. Studies which use call options, among other parameters related to a firm's equity and its volatility, to establish the value of the warrant, as well as empirical work that studies the importance of the warrant's dilution effect will also be reviewed. And finally, we will review studies which propose alternate warrant pricing models, some of which use the B/S model as a basis of comparison.

Section III systematically presents the methodology of the "Warrant-Dilution Option Pricing Model" (hereafter "WDOP" model). A step-by-step approach is used to demonstrate how a call option can be priced while considering the presence of warrants within the firm's capital structure. We then compare the WDOP model predicted option prices with the actual option prices observed in the market. Next we evaluate the WDOP model's ability to predict option prices using both in and out-of-sample volatility parameters in order to ascertain whether firm variance is constant, consistent with the null hypothesis. The WDOP model is also compared to the B/S model, which does not consider the capital structure of the firm. Both models function under the B/S assumption that the volatility of the value of the firm is constant. We estimate out-of-sample warrant prices using the B/S model, comparing these to actual market prices in order to evaluate whether the variance of the firm is constant. In section III, the sample and the data used for the purposes of our analysis are described in detail. The results of our study are then presented in the fourth section of the study. We examine the effectiveness of the WDOP model's ability to estimate the option market prices. The conclusion and suggestions for further research are presented in the fifth and final section.

2. LITERATURE REVIEW

2.1. Binomial and B/S Option Pricing Model.

Cox, Ross and Rubinstein's paper (1979) (hereafter CRR) and Rendleman and Bartter (1979) pioneered the binomial method of option pricing in which the stock prices follow a two-state process. The valuation method is based on the premise that there are no arbitrage opportunities. The model's theoretical base starts from the construction of a portfolio that it is comprised of a long position in a stock and a short position in an option written on the same stock (known as a hedging portfolio). According to the no-arbitrage theory, this portfolio should have a rate of return equal to the risk-free rate. CRR (1979) (p.249-254) and Rendleman and Bartter's (1979) (p.1099-1109) respective studies prove that the option prices estimated using the binomial model may converge with the prices established using the B/S model in certain cases using specific parameters, specifications and an unlimited number of time periods. As Rendleman and Bartter conclude:" the binomial model opens the door to the understanding of modern option pricing theory without the added complication associated with the solutions to stochastic differential equations" like the B/S model.

The binomial model assumes that the price of the underlying asset can make only two possible movements; up (with probability p) or down (with probability 1-p) at each time period. The "u" (d=1/u) determines the upward (downward) movement. The binomial model is fully specified by the parameter u and d and the probability parameter p. Given these parameters a unique binomial tree can be built using the basic assumption that the variance of the stock's rate-of-return remains constant over time.

CRR (1979), in their evaluation of the B/S model, state that it is the "first completely satisfactory equilibrium option-pricing model." The B/S valuation method is the foundation of the binomial model, by constructing a hedging portfolio and assuming that there are no arbitrage opportunities. Both models assume that the stock prices are lognormal with a variance rate that is stationary over time. Black and Scholes (1973) eventually developed the well-known B/S option pricing formula:

$$C_0 = S_0N(d_1) - Xe^{-rT}N(d_2),$$

Where, $d_1 = \dfrac{\ln(S_0/X) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$;

C_0 is the option price at time 0; S_0 is the current stock price; X is the exercise price of the option, r is the continuously compounded risk-free rate, σ^2 is the variance rate of the stock return; T is the time remaining before the expiration of the option (in years), N(d) is the cumulative normal density function.

The Black and Scholes (1973) study also proposed a possible modification to the B/S model for the pricing of warrants, by suggesting that the variance of the stock return should be replaced by the variance of the firm value's return. The stock in the equity of the firm, is estimated to be "the sum of value of warrants and the value of the common stock, divided by the number of outstanding shares plus the number of shares you can acquire with the exercise of the warrants" thereby accounting for the warrant dilution effect.

Since Black and Scholes' seminal paper in 1973, various extensions of the valuation model have been proposed including discrete or continuous dividend payments. Black's (1975) study examines the valuation of American options thereby relaxing the "no-early exercise" assumption.

According to Ritchken (1987) and Hull (2000), empirical tests suggest that the B/S model is a poor predictor of in- the-money and out-of-the-money options while it is a highly effective predictor of at-the-money options. As Hull notes, the option-pricing models are examined with respect to the volatility smile. "The volatility smile is a plot of the implied volatility of an option as a function of its strike price" (Hull (2000)). Black (1975) found that there is a direct relationship between implied volatilities and the exercise price of an option while Macbeth and Merville (1980) reported the opposite of the Black findings. Rubinstein (1994) found that strike price biases using the B/S model are statistically significant but the direction of the observed biases changes from one period to another. These reported results imply that probability distributions of stock prices may not be lognormal. Some academics expect the B/S model to be outperformed by more recent and advanced option-pricing models due to the volatility smile. However, the majority of empirical studies performed to date, of which only two have been retained for the purposes of our study, suggest that so far the B/S model is a better predictor of option prices than any alternative option-pricing models for out-of-sample results.

The first of these studies is that of Rubinstein (1985) who tests five option-pricing models all of which having different assumptions for the variance process. The models are: a) Pure Jump Model by Cox and Ross (1975); b) Mixed Diffusion-Jump Model by Merton

(1976); c) Constant Elasticity of Variance Diffusion Model by Cox and Ross (1976); f) Compound Option Diffusion Model by Geske (1979); and g) Displaced Diffusion Model by Rubinstein (1983). Rubinstein (1985) uses nonparametric tests (distribution-free) to determine whether there were significant deviations between the theoretical and the market option prices. Comparing all the models for two different periods (Aug. 23, 1976- Oct. 21 1977 and Oct.24, 1977 – Aug. 31, 1978), Rubinstein concludes the following:

1. Out-of–the-money options exhibit an inverse relationship between implied volatilities and time to maturity. Short-term out-of-the-money options are priced significantly higher relative to other calls than the B/S model would predict.
2. There is a time-to-maturity bias for at-the-money options but the direction of the relationship changes from the first period to the second.
3. Strike price biases are statistically significant. The direction of the bias changes from one period to he other.
4. None of the five models seem to be able to explain this oscillation (none of them capture both the strike price and the time to maturity biases) and "provide results consistently superior to the B/S model".

The second study, performed by Dumas, Flemming and Whaley (1998) tests the time series validity of the assumption that the volatility is a deterministic function of the asset price and time. They examine the hedging and predictive abilities of the Deterministic Volatility Function (DVF) in terms of the volatility smile. Dumas et al's (1998) analysis was initiated due to the B/S model deficiencies (strike price and maturity biases). The authors whished to estimate the stock return volatility based on the future expectation and not on past information while considering the future behavior of the option prices. They use out of sample estimation of the volatility function to project the volatility function in the near future

Dumas et al' (1998) use a variety of structural forms to estimate DVF including the B/S model. Their findings are as follows:

- The in-sample parameter estimates of the stock price volatility function shows that they are not constant over time. Using an out-of-sample test Dumas et al' (1998) conclude that their method is inferior to the ad hoc procedure that simply smoothes B/S implied volatilities across strike prices and time to maturity;
- Dumas et al' (1998) show that the hedging portfolio based on the B/S constant volatility model again outperforms the portfolio based on the DVF method;
- Overall the B/S model outperforms the DVF method.

Based on the above arguments, we can conclude that the B/S model is the most effective option-pricing model among the cited studies, and as such, the B/S model has been retained as the appropriate benchmark for the predicting abilities of the WDOP model.

2.2. *Warrant Pricing*

Galai and Schneller (1978) state that many studies value warrants without considering the warrant dilution effect. Galai and Schneller (1978) begin by presenting the pay-off of a warrant at its expiry date and that of a call option written on a pure-equity firm (in the absence of warrants). Comparing the returns of both of these derivatives, Galai and Schneller (1978) show that they are perfectly correlated. As such, the warrant price can be estimated as a proportion of the price of a call option. To be more specific they estimate the warrant value as follows:

$$W_0 = \frac{C_0}{1+q}$$

Where, W_0 is the value of the warrant at time 0; C_0 is the value of the call at time 0; the call is written on a firm identical to the warrant-issuing firm. This identical firm is a pure equity firm, which has not issued warrants; q is the dilution factor and is equal to the ratio of the number of warrants to the number of shares currently outstanding assuming that the warrant holder can buy one share per warrant exercised. This ratio incorporates the dilution effect of the warrant in the capital structure of the firm.

According to Kremer and Roenfeldt (1992), the Galai and Schneller (1978) study implicitly assumes a simultaneous exercise of all-outstanding warrants. This formula is valid in perfect markets where no arbitrage opportunities exist. Kremer and Roenfeldt (1992) assume that the proceeds from the exercise of the warrant are distributed in the form of dividends to the original and new stockowners. Thus, two perfectly correlated assets should have the same value. No formulae for the pricing of call options written on firms in the absence of warrants in their capital structure were presented the by Galai and Schneller (1978). The authors nonetheless suggested to use any option pricing models "that were developed in recent years."

Leornard and Solt (1990) stated that the above adjustment for the dilution effect assumes that the premium paid for each warrant is distributed to the existing common stockholders on a pro-rata basis where the exercise proceeds are given to both old and new stockholders. Thus, in the presence of warrants the firm does not make any other distribution and its capital structure does not affect the value of the firm. These conditions prevent the need to consider dividends and reinvestment issues.

Galai and Scheller (1978) theoretically show that the value of a warrant is equal to the value of a call written on the share of a company that issues warrants assuming that both option and warrant have the same maturity. The Authors also demonstrate that when warrants are issued, the value of the firm cash inflows are greater than before the issuance of warrants due to the expected future proceeds from the exercise of the warrants. Examining the effect of the issuance of warrants on the stock price, Galai and Scheller (1978) show a decrease in the stock price equal to the dilution factor and an inverse relationship between the stock price and the volatility of the firm's value.

In 1981, Noreen and Wolfson empirically tested two forms of the Constant Elasticity Variance option-pricing model (hereafter CEV) adjusted for the pricing of warrants. The first formula is the B/S model (1973) and the second is the Square Root Constant

Elasticity Variance Model (hereafter SRCEV) (Cox and Ross 1976 pp. 161). Both models are adjusted for the dilution effect and incorporate dividend yields. The adjustment for the dilution effect is made according to the Galai and Schneller (1978) approach. The variance used in the two models is calculated using 50 weekly stock rates of return all of which were collected prior to the measurement date. The purpose of the article is to test the pricing performance of the models using warrants that are close to or at-the-money, thereby resembling the executive stock options.

Both models generate similar results and depart from the actual warrant prices within a range of 5 percent. From the analysis of three dilution factors and two different types of default-free interest rates Noreen and Wolfson (1981) found that in most cases the mean error is negative. The authors indicate that focusing on warrants with dilution factors of greater than or equal to .9 or with a 5 percent range of the stock and exercise price or with expirations of more than three years doesn't significantly change the prediction accuracy of either model. Finally, changing the ex-ante to ex-post dividend yield and standard deviation of the stock return gives larger prediction errors. Noreen and Wolfson (1981) conclude that the two models could be considered identical in their ability to price warrants throughout their sample.

Galai (1989) argues that Noreen and Wolfson's (1981) paper contains two errors. The first is that the standard deviation that is estimated using historical time series of the stock rate of return is underestimated. The second error is that there is no need for a dilution effect adjustment because of Galai and Schneller's (1978) theoretical argument that an option, written on the stock of a firm with warrants outstanding is equal to the value of a warrant written on a similar firm. Based on Galai and Schneller's (1978) argument, Galai (1989) states that "the equity price in a firm with warrants should reflect the potential dilution". These two errors explain the downward biases referred to by Norren and Wolfson (1891) with respect to the two models. Galai (1989) also states that if the dilution adjustment is omitted from the warrant valuation formula then the observed Norren and Wolfson (1981) prediction error goes from –0.114 to –0.032. An alternative proxy for the risk-free rate changes this error to a low positive value.

Galai (1989), as an alternative approach, estimated warrant prices using the value of the firm and its volatility as compared to the Norren and Wolfson (1981) approach which used stock prices, their volatility and the warrant dilution factor. The value of the firm is calculated as a historical time series of the sum of the stock price (S_t) and warrants (W_t) at time t (i.e $0 \leq t \leq T-1$ where T is the expiry date of the warrant) weighted by the outstanding shares (N) and warrants (n) respectively. Then, the estimated standard deviation of the firm value can be used to calculate the warrant price. V_T and W_T can be estimated by simultaneously solving the equation of the value of the firm ($V_T = NS_T + nW_T$) and for warrants ($W_T = C_T$ (N/N+n), Galai (1978)).

In 1986 Ferri, Kremer and Oberhelman test the ability of the B/S model and three CEV models to value warrants. The three CEV models are the SRCEV model (θ=1/2)[2], the "Absolute Model" (θ=0) and CEV model with a θ estimated as the coefficient of the

[2] $\sigma_s = \sigma_1 S^{\theta-1}$, where σ_s is the standard deviation of the stock return when the stock price is S, σ_1 is the standard deviation of the stock return when the stock price is \$1, S is the stock price, and θ is a constant between 0 and 1. Lauterbach and Schultz (1990) pp. 1200.

natural log of the price of the return of a stock when it is regressed on the natural log of the stock price over time. The above option-pricing models are converted into warrant pricing models by adjusting for the warrant dilution effect. The dilution adjustment is consistent with that proposed by Galai and Schneller in their (1978) study.

A comparison of the B/S model with the three CEV models leads to the following conclusions:

- In most cases B/S model is a better predictor of the warrant's market prices than the other models.
- Other models tend to under price the warrants. B/S gives lower prices in specific cases such as out-of money warrants, short term warrants (less than 1 year) and warrants with market prices of less than \$2.00.
- Testing all the models for dividend and non-dividend adjustments, it is shown that dividend adjustments lead to lower warrant prices.

Lauterbach and Schultz (1990) support the use of the B/S model adjusted for the dilution effect for the pricing of warrants. Thus, in their paper the following formula is used to price warrants:

$$W = \left(\frac{N}{N/\gamma + n} \right) \left[\left(S - \sum_i e^{-rt_i} D_i + \frac{n}{N} W \right) N(d_1) - e^{-rT} \chi N(d_2) \right]$$

$$\text{where, } d_1 = \frac{\ln\left(\dfrac{S - \sum_i e^{-rt_i} D_i + (n/N)W}{\chi}\right) + rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \text{ and } d_2 = d_1 - \sigma\sqrt{T}\,;$$

W denotes the warrant price; S is the stock price; S+(n/N)W is the firm value per share; N is the number of the outstanding shares; n is the number of warrants outstanding; X is the warrant exercise price; γ is the number of stocks that can be bought with each exercised warrant: r is the riskless interest rate; T is the time to expiration of the warrant; σ is the standard deviation of equity per share; N(d) is the cumulative normal distribution function evaluated at d; t_i is the period of time until the payment of i^{th} dividend; D_i is the dollar amount (per share) of the i^{th} dividend.

Therefore, three modifications are made to the original B/S model: a) the stock price is replaced by the firm value per share; b) the stock volatility is replaced by the standard deviation of firm value per share; c) the original B/S model is multiplied by N/[N/γ)+n].

The implied standard deviation (ISD) over time is calculated by using the above-adjusted version of the B/S model. ISD on day t is then regressed on the percentage that the warrant is in- or out-of-the-money on t-1 and the t-1 risk-free rate[3]. The regression is run on each warrant for each quarter totaling 451 regressions. The coefficient of the percentage that the warrant is in- or out-of-the money is negative in most cases. According to Lauterbach and Schultz (1990), the negative value of the coefficient

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[3] $ISD_t = \alpha_0 + \alpha_1\left(\dfrac{S - \sum_i D_i e^{-rt_i} - e^{-rT}\chi}{e^{-rT}\chi}\right)_{t-1} + \alpha_2 r_{t-1} + \varepsilon_t$

indicates "a consistent inverse relationship between the ISDs generated by B/S model and underlying stocks prices" contradicting "the transitory biases reported in Rubinstein's (1985) study of options".

One model that allows for such an inverse relationship is the SRCEV model in which the standard deviation is inversely related to the square root of the underlying asset. The model is presented as follows:

$$W=\left(\frac{N}{N/\gamma+n}\right)\left[\left(S-\sum_{t}e^{-rt_t}D_t+\frac{n}{N}W\right)N(q(4))-e^{-rT}\chi N(q(0))\right]$$

where, for υ=0 or 4

$$q(\upsilon)=\frac{1+h(h-1)p-h(h-1)(2-h)(1-3h)\frac{1}{2}p^2-\left(z/(\upsilon+y)\right)^h}{\left\{2h^2p\left(1-(1-h)(1-3h)p\right)\right\}};$$

$$h(\upsilon)=1-\frac{2(\upsilon+y)(\upsilon+3y)}{3(\upsilon+2y)^2};\quad p=\frac{\upsilon+2y}{(\upsilon+y)^2};$$

$$y=\frac{4r\left((S-\sum_{t}e^{-rt_t}D_t+{}^{n}\!/_{N}W\right)}{\sigma^2\left(1-e^{rT}\right)};\quad z=\frac{4rx}{\sigma^2\left(e^{rT}-1\right)};$$

N(q) is the cumulative normal distribution evaluated at q.

In order to compare the two models, Lateurbach and Schultz (1990) obtain an estimate of the ISD each day using both models. The ISD estimates are then used to calculate the warrant prices. A comparison of the dollar difference and percentage difference from the market prices shows that the SRCEV model is a better predictor of actual warrant prices. It is also shown that the accuracy of prediction is greater as time passes from the ISD estimation. The SRCEV model outperforms the B/S model for the fourth month of the quarter as compared to the first. This is also true of the warrant that expires in at least four years compared to the warrant with less than two years of life remaining. The authors conclude that although the SRCEV gives better predictions for the warrant prices this can only be "indirect evidence that the SRCEV model may be more important for pricing warrants than short-lived options".

Leonard and Solt (1990) compare three forms of the B/S model: a) unadjusted B/S model (BS); b) B/S adjusted for dividends (BSDA) and C) B/S model adjusted for dilution effect (GS). The B/S model prices warrants based on the stock price and its volatility, not on the firm value and its standard deviation as did Lateurbach and Schultz (1990). Leonard and Solt (1990) test whether the adjustments for dividend or dilution effect lead to a better-predicted model for warrant market prices. In this article, the two adjustments are made separately and sequentially. They argue that if the two adjustments are made simultaneously, the outcome will be an undervalued warrant price.

According to Leonard and Solt (1990), there are three dominant assumptions to their empirical analysis:

1. The yield curve is flat; thus, the risk-free rate, used for the discounting, is determined by the warrant expiration date;
2. Expected volatility is proxied by the square root of the annualized historical variance returns in order to reflect changing assessments of stock risk;

3. The dividend yield is constant over the life of warrant and is determined by the current dividend yield;

Leonard and Solt's (1990) empirical analysis is based on the percentage prediction error and on their absolute value. The percentage valuation error is estimated by subtracting market price from the model price and then dividing by the market price. The warrant sample is divided into dividend and non-dividend paying stocks. Each of these sub-samples is separated into special categories based simultaneously on warrant life and the ratio of the stock price to the exercise price. Comparing the percentage valuation errors of the three theoretical models over these categories the authors observed the following:

- The BS model underprices warrants with a life of less than two years. Any adjustment exaggerates the under-valuation. The model demonstrates a tendency to overvalue the real warrant price as the sample goes from short to long term warrants and as the warrants move from deep-out to in-the-money.
- Incorporating the dilution effect in the B/S formula usually gives smaller prediction errors for the non-dividend-paying stocks.
- The dividend-adjusted version of the B/S model predicts the market price better for in-the-money warrants. On the other hand, the results for out-of-the money warrants vary and seem to be more sensitive to the dilution factor.

Lim and Phoon's (1991) analysis is based on theoretical findings in Galai and Schneller's (1978) study that an option, written on the stock of a firm with warrants outstanding, is equal to the value of a warrant written of that same firm. Therefore, warrants are undiluted call options on the stock of firms carrying warrants. Lim and Phoon argue that the eventual dilution, brought about by the eventual exercise of the warrant, is already incorporated in the current stock price. Thus, Lim and Phoon (1991) value the warrant using the original B/S model (1973) based on the stock price and its volatility with no adjustment for the dilution effect of the warrant. Galai (1990), on the other hand, based his analysis on the same arguments as Galai and Schneller (1978), but used the value of the firm and its volatility without incorporating the dilution effect of the warrant.

Lim and Phoon's (1991) used the implied volatility of the stock rate of return, which is derived from traded options written on the same stock, instead of using historical standard deviations, in the pricing of the warrants. The one requirement for this procedure is that there be both options and warrants actively traded on the same stock. Lim and Phoon (1991) used seven methods to calculate the standard deviation (σ). The first four methods involved the calculation of an implied volatility using four different approaches for the weighting of the implied volatility of the stock rate of return. The remaining three methods estimate the standard deviation using a historical time series. These standard deviations were used as the input for the original B/S model to price warrants.

Lim and Phoon (1991) found that there was an upward bias for warrants that have a longer time to maturity than those with shorter maturities when comparing the predicted warrant prices with the actual market warrant prices. They conclude that using implied volatilities does not accurately predict the market warrant prices, even after taking transaction costs into account. These results can be attributed to the use of the wrong model. Lim and Phoon's (1991) model is based on stock prices and the volatility of the stock rate of return which is not constant over time (as opposed to the basic assumption of the B/S model) because of the warrant dilution effect on the stock prices. Therefore, if a more precise model were used, we could expect that the upward bias would be substantially reduced knowing that a dilution parameter of less than one can influence the predicted warrant prices. Lim and Phoon (1991) also found that using historical time series to estimate the standard deviation (as in the case of the last three methods) there was an underestimation of the standard deviation thereby supporting Galai's (1989) contentions.

Crouchy and Galai (1991a) claim that in most cases, a warrant's price is estimated as a call option written on the firm's equity, by adjusting the pricing formula for the dilution effect. This approach is appropriate if the call option is written on a firm that is identical to the warrant-issuing firm. This firm must be a pure equity firm, which poses a problem because in the real world most firms attempt to reduce their cost of capital by incorporating at least some leverage in their capital structure. In reality, analysts use the prices of call options written on the common stock of firms with outstanding warrants in their capital structure as an approximation for the warrant's value. According to the authors, the equity rate of return and its estimated volatility are not constant. The authors then suggest an approach for the estimation of the volatility of the stock, which is then used to value the warrants.

Crouchy and Galai (1991a) use the binomial lattice approach to present their model and support their results. A two stage binomial grid is constructed. They denote the initial value of the pure equity firm as $V_0^* = N^* S_0^*$ where N^* is the number of outstanding shares and S_0^* is the stock price at time 0. In addition, the initial value of the firm with warrants outstanding (t=0) is equal to $V_0 = NS_0 + nW_0$, where n is the number of outstanding warrant and W_0 is the initial warrant price. Both values of the firm move by u (upward movement) and d (downward movement) with probability g and 1-g respectively. A basic assumption is that "the proceeds from selling the warrants are assumed to be invested proportionately in the firm, and thus, do not affect the rate of return distribution on total assets". Thus, given the known parameters of S_0^*, N^*, u and d, the equilibrium price of S_0 and W_0 can be determined. Their primary model is presented as follows:

$$S_{2j} = \min\left[\frac{V_{2j}}{N}, \frac{V_{2j} + nX}{N + n}\right]$$

Where, S_{2j} is the value of the stock at time 2 and stage j (j=1-3); V_{2j}/N is the value of firm per original stocks at time 2 and stage j, when the warrant holder does not exercise their right; $(V_{2j} + nX)/(N+n)$ is the value of the diluted shares assuming that all n warrants are exercised; X is the strike price.

Crouchy and Galai (1991a) then constructed a hedging portfolio in order to find the equity price at time 1 and one for each equivalent binomial step (in this case 1 & 2). Each portfolio is comprised of: a) one share long in the firm with warrants outstanding, and b) α number of shares short of the all-equity firm. Under the assumption that these portfolios are default-free, their payoffs at a future stage (t=2) must be equal and the investment in the portfolio must yield the risk-free rate. From the equation of the future payoffs of the portfolio a mathematical expression for α can be found. The approach is identical to that presented by CRR (1979). Crouchy and Galai (1991a) use the equity value of a pure equity firm with warrants outstanding instead of the stock price and t-bills. They also developed an expression for risk neutral probabilities of the firm value given by q= (R-d)/(u-d). R is one plus the default-free rate. The result is the two-period Binomial option-pricing model.

Applying this formula to the situation where the warrant is exercised at time 2 in stage (1,2), Crouchy and Galai (1991a) found that S_0 is a function of the known parameter V_0, N, n, d, X, q and R and does not include the probabilities of the stock price to move up or down.

Following a similar approach and using the stock prices estimated above, Crouchy and Galai (1991a) derive a formula to estimate the warrant price at the initial point of the equivalent binomial tree. Again, assuming that the warrant is exercised at time 2 only in stage (1,2), they estimate the current value of the warrant as an expression of known parameters. In this case, the warrant price is estimated by subtracting the expected future value of the possible proceeds of warrant's exercise from the expected future value of the warrant. They then divide the formula by the post-exercise number of shares to incorporate the dilution effect.

Using a simple numerical example by arbitrarily selecting values for V_0, u, d, R, N and X, Crouchy and Galai (1991a) suggest that the equity value distribution process of the firm with warrants outstanding is non-constant. Crouchy and Galai (1991a) also found that the equity value distribution alters from time to time. The degree of the change becomes greater when the warrant volatility is greater than the volatility of the firm's equity. The authors argue that the equity volatility of a firm with warrants outstanding is expected to be stochastic even if the volatility of the firm value's rate of return distribution is constant. Using the binomial approach, Crouchy and Galai (1991a) argue that "the equity of a firm issuing warrants is less risky than that of a pure-equity firm with identical business risk". This is attributed to the firm being able to replicate the warrants in its capital structure by issuing extra shares and investing the proceeds in government bonds. According to the authors: "An important conclusion is that the time series of the equity of the firm with warrants cannot be used to estimate the volatility." Using the time series of the company's equity will lead to the underestimation of the firm's asset volatility and by extension the use of this volatility in the pricing of warrants will lead to their undervaluation.

At the end of the Crouchy and Galai (1991a) study, a pricing model was developed for warrants under a diffusion process of the stock prices. They substitute the option price estimated by Galai and Schneller (1978) in the B/S option-pricing model for a pure equity firm to arrive at an expression for the warrant value (W_0). By rewriting the formula for the value of the firm as $S_0 = \frac{V_0}{N} - \frac{nW_0}{N}$ and substituting W_0 with the new theoretical expression for warrants, they arrive at the following formula for the calculation of the stock price of a firm with warrants outstanding:

$$S_0 = \frac{V_0}{N}\left[\frac{N + nN(-d_1)}{N+n}\right] + \frac{nKe^{-r\tau}}{(N+n)}N(d_2)$$

Crouchy and Galai (1991a) end up with formula using the value and the variance of the pure equity firm (V_0) and other factors including the dilution effect. The authors suggest that there is nonetheless a way to calculate the volatility of the firm's value which is to be estimated by calculating the value of the firm at each time t (V_t=NS_t+nW_t) and using the time series of ln(V_t-V_{t-1}) to calculate the standard deviation. However, this is possible only if both firm's stocks and outstanding warrants are actively traded.

Crouchy and Galai (1991a) concluded by stating that a simple application of the original B/S model (1973) is inappropriate as adjustments to the basic option-pricing model are required.

The findings of Crouchy and Galai's (1991a) study embody the essence of our approach. We developed the WDOP model based on the premise that the estimate of the volatility of the stock rate of return is not constant. As such, we price warrants using the firm value and its volatility for the firm who's stock and outstanding warrants are actively traded. The variance of the firm value is constant and can be used in Binomial option-pricing and B/S models. We calculate the firm value based on the same formula (V=NS+nW) even though the authors state that "this may be complicated or impossible to estimate the volatility of the rate of returns of the firm's assets" because options and warrants may not be traded simultaneously. The WDOP model goes one step further by pricing options taking the capital structure of the firm into account.

A second article by Crouchy and Galai (1991b) entitled "Common Errors in the Valuation of Warrants and Options on Firms with warrants", notes two commonly made mistakes in the valuation of warrants and options written on the stock of firms with warrants outstanding. They are:

1. There is no need to adjust for the dilution effect of warrants because it is already reflected in the stock price throughout the life of the warrant based on the theoretical argument presented by Galai and Schneller (1978). This then explains Galai and Schneller's (1978) contention that warrants are undiluted call options written on the stock of a firm with warrants outstanding. Lim and Phoon (1991) and Galai (1989) also support the exclusion of the warrant dilution effect from the pricing model based on the same argument.
2. The implicit leverage provided by warrants leads to the non-stationarity of the time series of the stock rate of return. The issuance of warrants incites a reduction of the equity volatility because the warrant holders bear part of the firm risk. Initially, only the original common shareholders boar the firm risk. Thus, volatility estimated by the time series of the stock's rate of return will lead to the undervaluation of the warrant's market price.

These errors are often observed when the original B/S option-pricing formula is used to value warrants with market stock price quotes.

According to Crouchy and Galai (1991b), the value of the firm's equity s composed of the firm's stock and outstanding warrants combined and therefore account for the comprehensive equity risk. A warrant's volatility is not constant over time and is greater than the volatility of the firm value. This is true for an unlevered firm's capital structure.

In the presence of debt, the equity volatility may be higher than the firm value volatility. The grade of non-stationarity is inversely proportional to the remaining life of the warrant and the degree to which they are deep-out-of money.

Crouchy and Galai (1991b) propose an alternative approach for the valuation of warrants. The paper estimates the time series, $V_t = NS_t + nW_t$ (Grouchy and Galia 1991a). Using these time series, a historical volatility estimator of the firm's value can be calculated. Due to the possible lack of observations for W at time t, the authors suggest that the above equation should be solved simultaneously with the formula proposed by Galai and Schneller (1978), $(W=C(N/N+n))$.

A study conducted by Kuwahara and Marsh (1992) states that the B/S model underprices out-of-the-money warrants and overstates the value of in-the-money warrants. Supporting results have been found for the Nikkei-225 index option. The authors assume that the mispricing would be greater for warrants due to their inherently longer maturities and the non-constant stock price volatility. Thus, three models are used to estimate the volatility of the stock prices: GARCH (Generalized Autoregressive Conditional Heteroscedasticity), EGARCH (Exponential Generalized Autoregressive Conditional Heteroscedasticity) and CEV. A brief comparison of the three models shows that the EGARCH model, which accounts for the asymmetry in the change in stock price volatilities attributable to stock price changes, offers a better fit for the stock volatility than the GARCH and CEV models.

Kuwahara and Marsh (1992) then value the warrants using a hopscotch algorithm based on stock prices and their volatility as estimated by the EGARCH model. The main assumption of the paper is that the warrant premium is paid out or invested in a way that has no influence on the firm's investment policy and thereby the stock price and the generating process of the prices are non-stochastic. The authors declare that even after incorporating the stochastic nature of the volatility, they end up with similar results: the theoretical warrant prices are above and below the market prices for deep-in-the money and out-of-the-money warrants respectively.

Kuwahara and Marsh (1992) then compare the "stochastic volatility-hopscotch" warrant values (based on EGARCH) with the estimated warrant values generated by the B/S model which uses 60-day historical stock volatilities. Both models are adjusted for the dilution effect using the Galai and Schneller (1978) process. The "EGARCH-hopscotch" model seems to be a better predictor of the actual warrant prices. They follow the same approach for short-term options and more specifically the near-to-the-money Nikkei call options. In most of the cases, Kuwahara and Marsh (1992) find that the "EGARCH-hopscotch" model generates lower mean-square errors than the B/S model with the only exception being the out-of-the-money option with a ratio of the stock over strike price in the range of [1-1.1]. Kuwahara and Marsh (1992) conclude that while their "EGARCH-hopscotch" model generates smaller prediction errors for observed warrant prices, they are still significant for in-the-money or out-of-the-money warrants.

The Kremer and Roenfeldt's (1992) study examines the pricing ability and accuracy of Jump-Diffusion versus the B/S model in the pricing of warrants. The main reason for the use of a Jump-Diffusion model is that during the life of a warrant (usually 2-5 years), there is an increasing possibility for major news to occur and by extension to provoke stock price jumps. Therefore, the stock price process may differ from the B/S model. The B/S model assumes that the stock prices move randomly, following a geometric Brownian motion process. The Jump-Diffusion model is based on the assumption that the stock prices are governed by two processes: a) Wiener process (continuous price movements) and b) Poisson Process (infrequent price jumps).

The option price C*, estimated by the Jump-Diffusion model, is essentially a weighted average of the B/S model call option price. To be more specific, C* is the sum of option prices C_n on exactly n Poisson jumps weighted by $(\lambda_t)^n/n!$, where λ_t is the mean number of Poisson jumps during time t and n is the number of jumps during the life of the option. The option price C_n is estimated by the B/S formula using a variance, which is a function of a) the variance of stock rate of return when there is no arrival of important news (no Poisson jumps) and b) the variance of Poisson jump magnitude. Both models are adjusted for the dilution effect using the Galai and Schneller (1978) method. The two models are also examined with and without dividend adjustments because of the mixed evidence reported for dividend adjustments on warrant values. The dividend payments are incorporated in both models by substituting the default-free interest rate r with r-D where D is the constant dividend yield paid over the life of the warrant.

Kremer and Roenfeldt (1992) use two methods to estimate the standard deviation of the stock return. The first is based on historical information (181 trading days before the observation date), which is used to calculate stock volatility. The second uses information collected over 90 trading days prior to and 90 trading days after the observation date. Due to the similarity of results only the historical volatility is used. The risk-free rate used in both models for warrants with more than one year to maturity is estimated by the nearest interest payment date of a theoretical bond. The bond value is calculated by an intercept-suppressed regression equation. For warrants with maturities of less than one year, T-bills with maturities closest to warrant maturity date are used.

The mean dollar deviation and mean percentage difference from market warrant prices and in their absolute forms for both models suggests that there is no conclusive evidence that dividend adjustments will help in pricing accuracy. Kremer and Roenfeldt (1992) continue their analysis without dividend adjustment under the argument that there is a "lack of consensus on the proper adjustment".

Kremer and Roenfeldt (1992) conclude that the B/S model is more efficient for pricing warrants as it exhibits smaller mean absolute dollar deviations compared with the Jump-Diffusion model with the exception of warrants with less than one year to maturity. The latter model may be considered less biased as it has less mean dollar and percentage deviation from market prices than those of B/S. This is true for warrants that: a) are written on non-dividend paying stocks, b) are out-of- the-money and c) expire within one year. The Jump-Diffusion model may be preferable for short-term warrants with non-callable and non-senior provisions and for warrants written on stocks with histories of large jump impact.

Schulz and Trautmann (1994) use common stock data to value warrants. Their theoretical analysis is based on two main assumptions: a) the asset volatility of an all-equity firm is stationary and b) the non-existence of dividend payments in order to avoid the premature exercise of warrants. They develop a model based on a hypothetical firm with only stocks and warrants in its capital structure. As a result, the value of the firm is given as V=NS+ nW; where, N is the number of outstanding shares of common stock; S is the stock price; n is the number of outstanding warrants; and W is the warrant price.

Again, Schulz and Trautmann's (1994) warrant valuation model is based on the original B/S option-pricing model adjusted for the dilution effect. In the adjusted B/S model, the stock price is replaced by the V/N and the σ is not the equity volatility but the volatility of the firm's value return. This is the same formula that Lauterbach and Schultz (1990) used.

Crouchy and Galai (1991a) price warrants using the B/S model adjusted for the warrant dilution effect (as mentioned previously) but they used the stock price of a pure-equity firm and its variance instead of the firm value. Crouchy and Galai (1991a) suggest a way to find the volatility of the firm but they state that it is only possible if stock and outstanding warrants are actively traded. "If warrants are not actively traded, the problem becomes more complicated". So Schultz and Trautmann (1994) suggest a way to find the firm value and its volatility when the value of the firm is not readily observable in the market. Schulz and Trautmann (1994) overcome this obstacle (where the V and σ_V may not be observed in the market) by using more easily observable stock prices and their volatility to estimate V and σ_V. They derive the true value of the firm and the asset volatility by solving the two following equations simultaneously with two unknowns:

$$S = \frac{V}{N} - \frac{n}{N} W(V, \sigma_V) \text{ and } \sigma_S = \sigma_V \cdot \varepsilon_{s,v}$$

where, $\varepsilon_{s,v}$ gives the percentage change in the stock's value for the percentage change in the firm's value i.e stock's elasticity[4] allowing "the stock volatility to change over the life of the outstanding warrants even if the firm value volatility is constant".

After deriving the firm value and its volatility by solving the two aforementioned equations simultaneously for each observed stock price and their estimated variances, Schulz and Trautmann (1994) use equation $W(V,\sigma_V)$ to find an estimate of the warrant price. The stock volatility can be found historically from the time series of the stock's returns or, if the stock has listed options, estimated as the implied stock volatility using the B/S partial differential equation using options market quotes. Then, they examine the stock's volatility behavior under a simulation. They set a high dilution factor (l=1), K=100 and σ_v =0.30 and their results can be summarized as follows:

- Out-of-the-money warrants and increasing moneyness ratio: the standard deviation of the stock doesn't change.
- In-the-money warrants and increasing moneyness ratio: the stock volatility changes substantially.

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[4] $$\varepsilon_{s,v} \equiv \frac{\partial S}{\partial V}\frac{V}{S} = \left(\frac{1}{N} - \frac{n}{N}\frac{\partial W(V,\sigma_V)}{\partial V}\right)\frac{V}{S}$$

- σ_S tends to converge to the asset volatility as V moves toward infinity.
- At the expiry of the warrant there will be a jump in the stock's standard deviation. The magnitude of the jump is related to the perceived reinvestment risk of the proceeds from the warrant exercise.
- The two properties of the stock volatility: a) Stock volatility is more sensitive to changes in stock prices when the outstanding warrants approach their maturity date and are at-the-money and b) Stock volatility is below asset volatility as long as $0<\tau<\infty$ and $0<S(V, \tau)< \infty$.

Schulz and Trautmann (1994) compare the warrant prices estimated with their theoretical model ("true-warrant value") and the original B/S model's (1973) (not adjusted for the warrant dilution effect and using the stock price and its volatility instead of the value of the firm) "option-like warrant valuation". They conclude that the bias is not substantial and the B/S formula offers acceptable precision in the pricing of warrants. The B/S formula may need adjustments for the dilution effect only in the case of deep-out-of-the-money warrants.

Schulz and Trautmann (1994) tested the robustness of option-like warrant valuation process in a pure equity firm where the firm value variance is constant with no-dividends payments where premature exercise is not optimal. These conditions may not apply in tandem in realistic valuation situations. Thus, Schulz and Trautmann (1994) use a Constant Variance (hereafter CV) diffusion model that incorporates the premature exercise of warrants. The predicted warrant price estimated using an American CV diffusion model are compared with those observed in the market. They find that the percentage error varies from a minimum –38% to a maximum 90%. Also, after regressing the percentage prediction error on different values of the stock volatility and warrant dilution factor, it is observed that there is a relationship between the percentage error and the extent of the potential dilution. One explanation is that the warrant dilution is not completely anticipated in market stock prices. The American CV model overprices the warrant in the case of out-of-the-money warrants. In the case of in-the-money warrants the percentage prediction error is negatively related to the warrant dilution factor. Finally Schulz and Trautmann (1994) observed that the coefficient of the dilution factor is not constant over time and they conclude that "there is no constant dilution related pricing bias of the American CV model with the stock price as the state variable and this support the empirical robustness of option-like warrant valuation".

The focus of the Bensoussan, Crouhy and Galai (1995) study was the valuation of warrants using the price of the underlying stock. As the authors mention, the warrant pricing model proposed by Galai and Schneller (1978) and Crouchy and Galai (1991) is based on total firm value assuming that it follows a stationary log-normal distribution with constant standard deviation. According to the article this poses a practical problem, as the value of the firm may not be readily observable. Thus, assuming an all-equity firm, Bensoussan, Crouhy and Galai (1995) derive a warrant pricing formula based on the underlying stock and not on the firm value. The authors combine the equation $V=NS+nW$ (same definitions as before) with the pricing model proposed by Galai and Schneller (1978) in which the price of the call option is given by B/S formula $C^{BS}=C(V,t,X,r,\sigma)$, σ is the volatility of the asset and C is a solution to the standard B/S Partial Differential

Equation (PDE)[5]. Stock price can then be expressed as a function of the value of the firm and time. It is important to mention that the authors set the initial outstanding shares (N) equal to 1 in the context of the following formula:

$$S = S(V,t) = V - \frac{nC(V,t)}{n+1}$$

Bensoussan, Crouhy and Galai (1995) show that S is also a solution for the standard B/S PDE. In addition, they show that the warrant, having as an underlying asset the stock price and the instantaneous stock return, can be estimated "by a B/S type of PDE". "The volatility term is the volatility of equity" and is a conditional fraction of the firm's value and the warrant's time to maturity. More specifically:

$$\sigma(S,t) = \sigma \frac{V}{S(V,t)} S_V$$

Then, Bensoussan, Crouhy and Galai (1995) compare a) "the true value" of warrant estimated by Galai and Schneller's (1978) pricing model (the price of the call option is given by $C^{BS}=C(V,t,X,r,\sigma)$) and b) the warrant price given by their approach – "B/S approximation" "based on the stock price and adjusted with an estimate of the instantaneous non-stationary volatility of the stock". The scope of this comparison is to verify the pricing accuracy of their approach when it is compared with the "the true

[5] $-C_t - 0.5\,\sigma^2V^2C_{VV} - rVC_V + rC = 0$ with the boundary condition $C(V,T)=(V-K)^+$ .Subscripts denote the variables used in the partial derivation.

value" of the warrant. If the B/S approximation, solved using a numerical approach, is accurate, then it can be used to value warrants instead of the unadjusted form of the B/S model, which contains a computational limitation for V. Bensoussan, Crouhy and Galai (1995) observe that the B/S approximation gives similar results for in- and at-the-money warrants. The results are consistent for long and short maturities and for high and low volatilities as well. The prediction errors increase rapidly for deep-out-of-the-money warrants but this becomes unimportant as these warrants lose value. Bensoussan, Crouhy and Galai (1995) also conclude that the B/S formula may not need any further adjustments for the warrant dilution effect as it is already reflected in the stock price and its volatility.

Veld and Verboven (1995) empirically test whether options and warrants with similar characteristics have the same value. They compare long-term options and warrants both of which actively trade in the Dutch capital market. Since a direct comparison is not possible, as it is difficult to find options and warrants with the exact same characteristics (maturity date and strike price), the authors compare their implied volatilities. The analysis is based on Merton's (1973) argument that an option's price is directly proportional to the stock rate of return volatility. The implied standard deviation is estimated using three models: the Binomial model (with discrete dividend payments), the Merton model (1973) (with continuous dividend payments) and the SRCEV model (adjusted for continuous dividend payments). Veld and Verboven (1995) explained the exclusion of the warrant dilution effect in their model using the findings of Schulz and Trautmann's (1994) study which stated that the warrant dilution-adjusted and original versions of the B/S model give similar warrant prices.

Veld and Verboven's (1995) null hypothesis is that a warrant and a long-term option have the same value. Initially, they express their concern for the "term structure of volatility". Their reasoning is that as the ISD is attributed to the stock prices, warrants and options should have the same ISD as both are contingent claims on the same underlying stock. Their concern is based on the findings of Becker (1981), which states that series of call options with different maturities appear to have different ISD-estimators. The long-term options appear to have significantly higher ISD estimates than the short-term options. Veld and Verboven (1995) argue that these finding may only affect a study that compares the ISD-values of warrants (long-term to maturity) with those of call options with relatively short life spans. As such, Veld and Verboven (1995) compared warrants and options with similar life spans. A second consideration for their analysis was the "strike price bias" based on the Becker paper (1983). According to Becker (1983), the B/S model seems to exhibit significant prediction errors for in-the-money and out-of-the money options. Veld and Verboven (1995) try to capture the possible effect of the "strike price bias" using the three models for the ISD estimation. They argue that SRCEV captures the "strike price bias".

Veld and Verboven (1995) conclude that the three models overvalued the warrants compared to the long-term options. This would have been corrected if they had used the warrant dilution-effect parameter which by definition is less than zero (outstanding shares over the sum of the outstanding shares and warrants). They proved that their results were free of the time to maturity effect. After comparing long-term versus short-term options and warrants versus short-term options they concluded that time to maturity bias cannot explain their results. In addition, they argue that since most of the time the SRCEV gives similar results as the two other models, "strike price bias" cannot explain the overvaluation of warrants. The authors suggest that other factors (i.e. warrant holders in the Dutch market are small private investors) may explain their findings[6].

Sidenius' (1996) study entitled "Warrant Pricing – Is Dilution a Delusion? (1996), uses a theoretical argument to prove that the warrant pricing could be based on the stock price process as opposed to the firm value process. He describes the theoretical model using the following parameters: T is the warrant exercise date, S_T is the stock price just prior to T, W_T is the warrant price; V_T is the firm value (total equity) just prior to T, N is the original number of shares outstanding; and n is the number of warrants outstanding.

Sidenius (1996) first examines the case of the European warrant. He states that the warrant, just before its expiration date, has no time value and therefore its value rests solely on its intrinsic value. At the warrant exercise date, T, the warrant holder will pay the warrant exercise price (E), which will in turn be distributed to the original and new stockholders (same as Leonard and Solt's (1990) clarification of Galai and Schneller's (1978) study). After the warrant has been exercised, the total equity value of the firm, V_T, will increase to V' and be equal to the sum of firm value just prior the warrant's expiration (V_T) and n*E. After the warrant has been exercised, the stock price becomes S' i.e:

[6] "This makes factors such as transaction costs, which (in case of small amounts) are smaller for warrants than for long term call options, and the fact that warrants generally require a smaller investment than call option, become more important". Veld and Verboven (1995), pp.1142.

$$S' = \frac{V'}{N+n} = \frac{V_T + nE}{N+n}$$ and V' =(V$_T$+ nE)/(N + n)

Thus, the warrant payoff is given by:

$$\max(S' - E{,}0) = \frac{N}{N+n}\max\left(\frac{V_T}{N} - E{,}0\right)$$

Using the formula for the value of the firm, $V_T = NS_T + nW_T$, which leads to a continuous process for the stock price until expiration of the warrant (i.e S_T=S´), the payoff of the warrant can be solved as a function of the stock price:

$$W_T = \frac{N}{N+n}\max\left(S_T - E + \frac{n}{N}W_T{,}0\right)$$

which can further be derived to become Sidenius'(1996) unique solution:

$W_T = \max(S_T - E{,}0)$

Without which, the balance of the study becomes foundless. Unfortunately no effort is made to explain the formula's derivation or to justify the author's contention of the unique solution.

Based on the above theoretical argument, Sidenius (1996) states that the payoff for a European Warrant as a function of its underlying stock price is the same as the payoff for

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a European option as a function of this same variable without considering the warrant dilution effect.

Sidenius's (1996) statement is based on an error since the payoff of the warrant does not yield "the unique solution" $W_T = \max(S_T - E,0)$. We have:

$$W_T = \max(S' - E,0) = \frac{N}{N+n}\max\left(\frac{V_T}{N} - E,0\right) = \frac{N}{N+n}\max\left(S_T - E + \frac{n}{N}W_T,0\right)$$

$$= \frac{N}{N+n}\left[\frac{n}{N}W_T + \max\left(S_T - E,-\frac{n}{N}W_T\right)\right] = \frac{n}{N+n}W_T + \frac{N}{N+n}\max\left(S_T - E,-\frac{n}{N}W_T\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{N}{N+n}W_T = \frac{N}{N+n}\max\left(S_T - E,-\frac{n}{N}W_T\right) \Leftrightarrow W_T = \max\left(S_T - E,-\frac{n}{N}W_T\right) \neq \max(S_T - E,0)$$

Warrant payoff cannot be both $\max(S_T - E,0)$ and $\max(S' - E,0)$ unless S_T and S' are always equal, which cannot be true.

Sidenius's (1996) study examines the case of American warrants with a one-step binomial tree (CRR, 1979). At the "penultimate time step" T_1 ($T1<T$) the warrant value (W_1) can be considered as equivalent to the maximum of the W_i, the intrinsic value (if the warrant is exercised at T_1) and the "Wait value" W_w (if it is not exercised at T_1). W_i is given by the payoff W_T, presented in the above theoretical argument for the European warrant. However S_T is now S_1 (spot stock price at time 1) and W_T is W_1 i.e. $Wi = \frac{N}{N+n}\max\left(S_1 - E + \frac{n}{N}W_1,0\right) = \max(S_1 - E,0)$, which means that $W_i = W_1$. In the case

of W_w, the warrant can be considered as a European since is not exercised, and thus based again on the European warrant argument, the value of W_w is equal to the corresponding call option value (European option). Thus, W_1 is equal to the max (S_1-E, W_w), which resembles the node value of an option. The same approach can be applied to all the nodes at time T_1. With this approach Sidenius (1996) "establishes the identity between warrant and option values for American-Style options".

Sidenius (1996) concludes that the two pricing approaches (the first using the value of the firm process and the second using the stock price process) are not necessarily contradictory. It is rather a matter of choice. As the two pricing methods give similar results, the criteria of determination for the selection of an approach can be based on the availability of the necessary parameters. On the other hand, he argues that the warrant pricing based on stock prices might be preferable, as this method requires readily available stock price process information and requires less laborious computations.

Unfortunately Sidenius's (1996) conclusions, while providing interesting insights, are based on an error, as shown above.

Hauser and Lauterbach (1997) compare five warrant pricing models; two B/S-type models, two CEV-type models and one warrant-extendible model. The first model is the original B/S model (1973) that ignores the dilution effect and is based on the stock price and not on the firm value. On the other hand, the warrant dilution-adjusted B/S model incorporates the dilution effect and the underlying asset as the value of the firm. Lauterbach and Schultz had proposed this B/S-type model in 1990. The first CEV-type model is the SRCEV model, which is adjusted for the warrant dilution effect. The second CEV-type model is the Free-Theta CEV model, which considers the normal assumption of the SRCEV model (θ is equal to ½) to be a limitation. This model is formulated based on binomial approximations and adjustments for the warrant dilution effect as presented below:

$$W = \frac{n}{N\gamma + n}\left\{\left[\sum_{j=k}^{m}\binom{m}{j}\left(\frac{up}{r}\right)^{j}\left(\frac{d-dp}{r}\right)^{m-j}\right]\left(S_d + \frac{n}{N}W\right) - \frac{X}{r^m}\left[\sum_{j=k}^{m}\binom{m}{j}p^{j}(1-p)^{m-j}\right]\right\}$$

Where, $u = e^{\sigma\left(S_d + \frac{n}{N}W\right)^{\theta-1}\sqrt{\Delta T}}$; d=1/u; p =(r-d)/(u-d);

W is the warrant price; m is number of periods until expiration in the binomial tree; K is minimum number of upward movements in the binomial tree if the warrant is to terminate in the money; θ is the parameter of the CEV model; n is the number of warrants; N is the number of shares; γ is the number of shares of stock that can be bought with each warrant; σ is the instantaneous standard deviation of S+(n/N)W; r is the risk-free rate; T is warrant time to maturity; S_d is $S - \sum_i D_i e^{-rt_i}$ where S is the stock price and D_i is the dividend paid at t_i.

The final model reviewed by Hauser and Lauterbach (1997) is the "writer-extendible call option model" proposed by Longstaff (1990) that incorporates the fact that the warrant life may be extended by the issuer in the case that is to expire out-of-the money. A "writer-extendible call option model" is the dilution adjusted B/S model plus an expression that incorporates the extension value.

The volatility parameter used for all models (Hauser and Lauterbach (1997) study) is the implied volatility based on the previous day's (t-1) market prices. The warrant price is calculated based on the volatility estimator of t-1 day and on the other parameters of day t. Also, they use closing quotes for warrant and stock prices. Based on the mean absolute pricing errors, Hauser and Lauterbach (1997) conclude the following:

1. The models with warrant dilution effect adjustments exhibit smaller prediction pricing errors. Thus, the incorporation of the warrant dilution effect in the warrant pricing process is essential.

   The above conclusion contradicts the findings of Schulz and Trautmann (1994) who found that the B/S model with and without adjustments for the warrant dilution effect yields similar warrant prices. Similarly, Veld and Verboven's (1995) non-warrant dilution adjusted pricing model overvalued the real warrant prices. This overvaluation may be attributed to the fact that they did not adjust their model (dilution parameter<1).

2. Incorporating into the model the possibility of extending the warrant's life may lead to a better prediction of the real prices.
3. CEV models have a better prediction performance than the other models. The effectiveness is more obvious for out-of-the-money warrants. It is known that B/S

models exhibit particularly poor performance in valuing this particular subset of warrants or options.

Hauser and Lauterbach (1997) suggest that although the CEV model is a better price predictor than the dilution-adjusted B/S model, the complexity of estimating the CEV's parameters may give an advantage to the dilution-adjusted B/S model. They consider that this B/S model is "the most cost-effective method" of pricing and that "the CEV appear to be at the mid-point of the line of progress toward better pricing models".

These findings further support our justification of using the B/S model as Benchmark in testing the WDOP model.

Schluz and Trautmann (1994), who compared the original B/S model with and without a warrant dilution effect, found no significant differences between the models. However, more recent studies performed by Hauser and Lauterbach (1996), after comparing five warrant pricing models, found that there was in fact a warrant dilution-related bias. Hauser and Lauterbach's (1997) study further concluded that the model with dilution effect adjustments was a better predictor of the warrant's market value. While the opinions are mixed, the conclusions of the studies performed by Hauser and Lauterbach in 1996 and 1997 substantiate the inclusion of the warrant dilution effect in the WDOP model.

2.3. Implied Volatility

There are two commonly used methods to calculate the volatility of a stock's returns. The first method, the standard deviation, is estimated using historical data of the stock's returns. In the second method, the volatility is estimated as an implicit value by using option market prices and inverting the option-pricing model. Evidence in the literature demonstrates that the actual (ex-post) variance of the stock over the life of the option is better at explaining option prices than the historical data. Norren and Wolfson (1981) and Galai (1989) observed that the standard deviation would be underestimated when a historical time series is used.

Latane and Rendleman (1976) were the first to introduce the implied volatility as a predicting estimator of the future stock price volatility. They support the implied volatility superiority by examining the implied volatilities derived from the B/S formula. According to Chiras and Manaster (1978), historical variance does not predict future volatility (variance) as well as the implied volatility does. Becker (1981) extends the implied volatility concept by incorporating dividend payment. Comparing the predicting ability of historical and implied standard deviations, Becker (1981) found that "the implicit standard deviation is a better predictor of the future stock price variability than past standard deviations" (pp. 372).

Volatility of the stock return is an essential ingredient for the pricing of options. There is a great deal of literature with supporting empirical studies that have dealt with the subject of stock returns variance estimation. According to Scott (1987) the model that is often and traditionally used to estimate the implied volatility is the B/S model. As Geske and Roll (1984) ascertain, the B/S model's biases that are usually observed in its practical application, may be attributed to the non-constant variance.

In addition, Mayhew (1995) states that "traditionally, implied volatility can be calculated using either the B/S option-pricing formula or the CRR binomial model". Under the assumptions of the B/S model, implied volatility is interpreted as the market's estimate of the constant volatility parameter. In cases where the volatility of the underlying asset follows a stochastic process, option-pricing models incorporating a similar stochastic process should be used.

In our analysis we use the B/S model to estimate the implied volatilities of the value of the firm per share, which we consider to be constant.

3. METHODOLOGY

3.1. General Description of a Binomial Tree

Our pricing model is based on the well-known binomial lattice approach. We based our analysis on the binomial model presented by CRR (1979). According to Veld and Verboven (1992): "Overviewing the large body of empirical research, it might appear that option pricing models, especially the B/S and Binomial models, perform quite well for call options" (p.p.294).

Before we the describe WDOP model and pricing approach, we review some important details of the binomial tree. The binomial option-pricing model has some special features. First of all, option pricing is free of investors risk preferences, assuming only that investors prefer more wealth than less. Second, the model is not a function of the probability of the stock price's upward or downward motion, which means that it is free of investor's perception of the probability of the stock price's movement. It depends only on one random variable: the stock price. The interest rate is assumed to be constant and there are no taxes, transaction costs or any margin requirements. Finally, the risk neutral probability, p, is assumed to always generate values between 0 and 1. From this perspective they have the properties of a standard probability. Essentially p may be considered a special case probability when investors are risk-neutral.

CRR (1979) extend their model to incorporate the dividend payment effect in their pricing formula. Assuming a constant dividend yield, the formula is adjusted for dividend payments by subtracting the dividend yield from the stock price. In the WDOP model we use discrete dividend payments. The dividend adjustment is made by subtracting the present value of dividends from the initial firm value per share (V_0/N) calculated as follows:

$$(V_0/N)^* = (V_0/N) - \sum_t D_t e^{-rt_t} \tag{1}$$

where D_i is the dividend paid at time t_i. "This dividend adjustment technique is commonly used in option pricing" (Lauterbach (1990), pp.1183).

In our sample, all the options and warrants are not exercised prior to their expiry date and thus, they are treated as European options. Therefore, early exercise is not a concern requiring the calculation of an intermediate price. There is a backward valuation process that goes directly from the value of the terminal set of nodes to the initial option price.

3.2. Construction of a Binomial Tree on V

We are going to construct a recombining binomial tree on the value of the firm per shares $(V_0/N)^*$ (eq.1). It is assumed that V follows a lognormal diffusion process where the volatility (σ) of V is constant. The binomial tree has a starting time point t_0 and an end point T (the expiration date of the warrant). At t_0 quotes for warrants and options written on the same underlying stock must exist. The expiration date of the option must be prior to the warrant's expiration. The time interval $[t_0,T]$ is selected to cover sufficient time to contain and collect enough observations for the option until the end of its life and sufficient observations for the warrant after the expiry of the option.

As it is known, the binomial tree is defined by three parameters: u, d and the risk neutral probability p. u and d must be chosen to match the volatility of value of the firm. A commonly used approach is u $=e^{\sigma\sqrt{\Delta t}}$ and d= $e^{-\sigma\sqrt{\Delta t}}$, where σ is the implied standard deviation of $(V/N)^{*}$ for the time interval $[t_0, T]$ and Δt is the length of time of one step. As Tian (1999) argued, the risk neutral condition imposes a restriction on probability such that the binomial drift term matches exactly the continuous-time counterpart pu + (1-p)d= $= e^{r\Delta t}$. This means that the hedging probabilities define the risk-neutral distribution of the stock's return. Thus, the expected return of the underlying asset is equal to risk free rate. This leads to the following formula for the risk neutral probability:

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{2}$$

Where, r is the risk-free rate for the life of the binomial tree $[t_0\text{-}T]$.

Once these parameters are estimated, a unique binomial lattice can be constructed. It is important to mention that we can divide each day into any number of steps to capture the continuity concept.

3.3. Value of the Firm

Let's assume that we have a firm whose capital structure is composed only of outstanding warrants and stocks. The WDOP model hypothesis is consistent with the Galai and Schneller's (1978) theory. The proceeds from the sale of warrants are paid on a pro-rata basis to existing common shareholders whereas the exercise proceeds are given to both old and new shareholders.

The initial firm value (V_0) at time t_0 can be estimated as follows:

$$V_0 = NS_0 + nW_0 \qquad (3a)$$

Where, N is the average of the original common shares outstanding for $[t_0\text{-}T]$; S_t is the stock market price at time t_0; n is the number of warrants outstanding; W_0 denotes the price of warrant observed in the market at time t_0.

In the case of the firm that contains Class A and B of common shares, the estimation of the value of the firm changes. The value of the firm in such a case is estimated as follows:

$$V_0 = N_A(S_{A,0} - q) + N_B S_{B,0} + nW_0 \qquad (3b)$$

Where, N_A is the average of the Class A shares outstanding for $[t_0\text{-}T]$; $S_{A,0}$ is the market stock price for the common share A at time t_0; N_B is the average of the Class B shares outstanding for $[t_0\text{-}T]$; $S_{B,0}$ is the stock price for the share B at time t_0; q is the value of voting and is estimated as $S_A - S_B$; n is the number of warrants outstanding; W_0 is warrant price at time t_0.

The warrants in our sample are written on Class B non-voting shares. The price of a voting share is equal to or greater than the non-voting shares. The difference reflects the value of the voting right. Subtracting the value of the vote from the voting share, it becomes equal to the non-voting share. Thus, after these adjustments, the starting value of the firm becomes: $V_0 = (N_a+N_b)S_{B,0} + nW_0$[7].

It is important to mention that preferred shares are subtracted from the capital structure of the firm as they are considered to be a constant. In our sample the amount of preferred shares outstanding remains the same throughout the life of each warrant.

3.4. Estimation of the Implied Volatility

The volatility (σ) in our study is the implied standard deviation of $(V_0/N)^*$. The B/S option-pricing model adjusted for the dilution effect is used. The WDOP model is constructed using the same assumptions as the B/S model which, when used for the estimation of the implied volatility has the following form:

$$W_0 = \left(\frac{N}{N/\gamma + n}\right)\left[\left(\frac{V_0}{N}\right)^{\bullet} N(d_1) - Ke^{-rT} N(d_2)\right] \tag{4}$$

Where, $d_1 = \dfrac{\ln\left(\dfrac{(V_0/N)^*}{K}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$;

W_0 is the warrant price at time t_0, the stating time point of our binomial tree; V_0 is the value of the firm at time t_0 estimated by (3a) or (3b); $(V_0/N)^*$ is the value per share

[7] for more details see Appendix A, part A

adjusted for dividend payments; N is the mean shares outstanding for the period [t_0-.T]; n is the number of warrants outstanding; γ is the number of the shares that can be purchased with the exercise of one warrant. Our sample contains a 1:1 conversion ratio, thus γ is equal to 1; T is the time periods from t_0 until expiry of the warrant; r is the prevailing risk-free rate throughout the period [t_0-T]; K is the warrant strike price; σ is the implied standard deviation of the firm value for the period [t_0-T]; $N(d_j)$ is the cumulative normal density function for variable j=1,2.

3.5. Option Pricing

The option value at time t_0 is estimated from the same binomial tree using the following equation based on the CRR (1979) model:

$$C_0 = e^{-r_{op}t_{op}} \sum_{i=1}^{op} \left[\binom{op}{i} p^i (1-p)^{op-i} C_{i,j} \right] \tag{5}$$

Where, C_0 is the option price at the t_0; r_{op} is the equivalent of the option spot default-free rate (continuously compounded); t_{op} is life span of the option, expressed as a fraction of a year; op is the number of time steps until the expiry of the option; i is the number of upward movements to reach node (i,j) at time step op; j is the number of downward movements to reach node (i,j) at time step op; $C_{i,j}$ is the payoff of the call at the time step op for each node (i,j) - $C_{i,j}$ is the terminal value of the options given by the binomial tree at t_{op}; $\sum_{i=1}^{op}\left[\binom{op}{i}p^{i}(1-p)^{op-i}\right]$[8] which is the probability to reach node (i,j) with i upward movements; p is the risk neutral probability of upward movements until the expiry of the options – p is estimated using equation (2) but with the replacement of r and Δt for those of the option; Here, Δt is t_{op}/op where op is the number of time steps until the expiry of the option.

The payoff of the option at its expiry (t_{op}) is calculated as follows:

$$C_{i,j} = \mathrm{Max}\,(S_{i,j} - X, 0) \tag{6}$$

where, $S_{i,j}$ is the stock price (estimated by equation (7)) at the time step t_{op} for each node (i,j); X is the exercise price of the option.

3.6. Stock Prices

For the valuation of the option at time step t_{op}, we are not using market stock prices, which are only observable at time t_0, but estimated stock prices. The stock prices are calculated for each node (i,j) at the time step t_{op} (the option expiration). From equation (3a) we have the following formula:

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[8] $\binom{op}{i} = \frac{op!}{i!(op-i)!}$

$$S_{i,j} = \left(\frac{V_{i,j}}{N}\right)^{*} - \frac{n}{N} W_{i,j} \tag{7}$$

Where, $S_{i,j}$ is the estimated stock price for each node (i.j) at t_{op}; $V_{i,j}$ is the estimated value of the firm for the node (i.j) at t_{op} and it is estimated by the binomial process as $\left(V_{i,j}/N\right)^{*} = \left(V_0/N\right)^{*} u^i d^j$; N is the average of outstanding shares for the period [t_0-T]; n is the number of warrants outstanding; $W_{i,j}$ is the estimated warrant price for the node (i,j) at time t_{op}.

When the company has both Class A and Class B shares outstanding $S_{i,j}$ is calculated by equation (7) by substituting N with the sum of the number of Class A and Class B shares outstanding (N_A+N_B). For details see Appendix A, part B.

3.7. Warrant Valuation

As it is mentioned above, for the estimation of the stock price at each node (i,j) at time step t_{op}, the warrant price for each node (i,j) at t_{op} should be calculated. The warrant prices are estimated by using the same binomial model:

$$W_{i,j} = \frac{N}{N+n} C_{V^{*}_{i,j}} \left(\left(\frac{V_{i,j}}{N}\right)^{*}, r_f, \sigma, K, T_{W_{i,j}} \right) \tag{8}$$

Where, $W_{i,j}$ is the warrant price at node (i,j) at time step t_{op} ; N/N+n is the dilution factor; N is the mean outstanding number of shares for period [t_0-T]; n is the number of outstanding warrants; $\left(V_{i,j}/N\right)^{*}$ is the value of the firm per share at node (i,j) and time-step t_{op} (same calculation as described in the previous section); K is the strike price of the warrant; $T_{w_{i,j}}$ is the time interval [t_{op}-T] where t_{op} is the option's expiration and T is the time point where the warrant expires (end of the tree); $C_{V^*_{i,j}}\left(\left(\frac{V_{i,j}}{N}\right)^*, r_f, \sigma, K, T_{W_{i,j}}\right)$ is the call option price on $(V_{i,j}/N)^*$ at node (i,j) and time step t_{op};

For the estimation of C_{V^*} we use the same process based on equations (5) and (6) making the following adjustments:

1. C_0 is replaced by $C_{V^*_{i,j}}$;
2. The risk free rate, r_{op} is replaced by the equivalent forward rate (r_f) for the future time interval [t_{op}-T];
3. t_{op} is replaced by $T_{w_{i,j}}$;
4. p is the risk-neutral probability of the price of $(V_{i,j}/N)^*$ to increase by u until the expiry of the warrant;
5. The stock price ($S_{i,j}$) is replaced by $(V_{T_{y,z}}/N)$ - the estimated value of the firm per share for the node (y,z) at T;
6. The option strike price X is replaced by K;

For the estimation of a warrant price at each node (i,j) at time step t_{op}, sub-trees are utilized. These sub-trees are part of our main tree; u, p and σ are the same as the ones used for the construction of our main binomial lattice. All the sub-trees have the same length and number of nodes. Their starting point is the expiry of the option and the final time step is the warrant's expiry. The difference between the sub-trees is that each sub-tree has a different starting value $(V_{i,j}/N)^{*}$.

In the case of the subject company containing both Class A and Class B common shares, the dilution factor in equation (8) changes. The parameter for the dilution effect is equal to $\frac{N_A + N_B}{(N_A + N_B) + n}$ (for details see appendix A, part B).

3.8. Out-of-Sample Tests

Two issues remain to be reviewed in support of the WDOP model: 1) the pricing ability and efficiency of the WDOP model when we go out-of-sample; and 2) the pricing performance of the WDOP model in relation to the benchmark option-pricing model. The model used as a comparison parameter is the original B/S option-pricing model (1973), adjusted for discrete dividend payments and ignoring the capital structure of the firm (hereafter dividend-adjusted B/S). Again, the comparison is made out-of-sample due to the difficulty in estimating the appropriate in-sample implied volatility for the B/S model. Since warrants, through a dilution effect, affect the capital structure of the firm and by extension the stock itself (Galai and Schneller (1978) and Crouhy and Galai (1991a)) our hypothesis is that we expect the WDOP approach to outperform the B/S model which ignores the firm's capital structure. The WDOP model goes one step further by taking into account the capital structure of the firm and the effects of the warrant exercise on the stock itself.

Thus, in this part of our paper two models are used: a) The WDOP binomial-based option pricing model and b) dividend-adjusted B/S option pricing model. For the purposes of this study we are going to use series of options instead of individual options because it facilitates calculations of the equivalent out-of-sample stock rate of return volatility required for the estimation of the out-of-sample option prices using the dividend-adjusted B/S model. Each series contains three or more options written on the same underlying stock, observed on the same day, having the same maturity, but with different strike prices.

3.8.1. Comparison of In-and Out-of-Sample WDOP Option Prices

Assume that we are at time t_0. We estimate the out-of-sample option price using the WDOP approach described in the previous section (eq.1-8). The difference can be found in the parameters used. All the parameters are estimated at time t_0 except for the implied volatility of the rate of return of the firm value (σ). Implied volatility is estimated at time $t_{0\text{-}\alpha}$ (out-of-sample), where α is 1 to 5 trading days before t_0 using parameters observed at $t_{0\text{-}\alpha}$. The out-of-sample implied volatility of the firm's value rate of return ($\sigma_{0\text{-}\alpha}$) is calculated using the B/S model adjusted for the warrant dilution effect and dividend payments (eq.4 and eq.3a or 3b). In- and out-of-sample option prices have been calculated using the WDOP model. The results were then compared to the actual option prices as a basis of comparison to determine the model's ability and efficiency to predict the option prices observed at t_0. This test is in support of our null hypothesis, which states that the firm value volatility is constant.

3.8.2. *Comparison Out-of-Sample WDOP and B/S Option Prices*

In order to find the out-of-sample B/S option price we follow a similar approach as that used to estimate the out-of-sample WDOP option prices. Again, we use the parameters estimated at time t_0 but the implied variance of the stock's rate of return ($\sigma^{s}_{0-\alpha}$) (not of the firm's rate of return as in WDOP) is estimated at $t_{0-\alpha}$. The model used in this portion of the analysis is the original B/S option-pricing model (1973) with no warrant dilution effect considered and using the stock price and its volatility instead of the value of the firm. The original formula is adjusted for discrete dividend payments following the same process as in the WDOP approach (see pp.56). The out-of-sample implied volatility of the stock rate of return $\sigma^{s}_{0-\alpha}$ is estimated 1-5 trading days (α) prior to t_0 with the dividend-adjusted B/S, using parameters derived from time $t_{0-\alpha}$.

The complications imposed in the derivation of the out-of-sample implied volatility using the dividend-adjusted B/S model is that the series of options observed at time t_0 and $t_{0-\alpha}$ may not have the same strike prices. For example, at time t_{0-1} we have three quoted option prices that mature in three months with strike prices of 6,8 and 10 respectively. At t_0 we have three quoted option prices that again mature in three months (this is possible because α is 1 to 5 trading days) with strike prices of 8,12,14 respectively. Obviously, it is difficult to derive the out-of-sample implied volatilities for the options at time t_0[9] using the implied volatilities of the options at time $t_{0-\alpha}$ due to their different strike prices. To

[9] Only options that are traded on each observed day are available in our database and may differ from day to day.

overcome this issue, the out-of-sample stock return volatility of the option (in the series at time $t_{0-\alpha}$) is regressed for each series of options observed at time $t_{0-\alpha}$ on the ratio of the stock price at time $t_{0-\alpha}$ over the strike price of each option at time $t_{0-\alpha}$. Using estimating coefficients a and b for each series of options at time $t_{0-\alpha}$, we can then calculate the equivalent out-of-sample implied volatility for each of options in the series at time t_0. Basically, the implied volatility for each option in the series is assumed to depend on its respective strike price. As such, we consider the shape of the volatility "smile" curve (since options with different exercise prices have different implied volatilities). The formula is presented as follows:

$$\sigma^{s}_{0-\alpha,j} = a + b\, X_{0-\alpha,j}/S_{0-\alpha} \qquad (9)$$

where, $\sigma^{s}_{0-\alpha,j}$ is the estimated implied volatility of the stock's rate of return of option j which is observed at time $t_{0-\alpha}$; $X_{0-\alpha,j}$ is the strike price of option j in the series at time $t_{0-\alpha}$; $S_{0-\alpha}$ is the stock price observed at time $t_{0-\alpha}$.

In order to estimate the equivalent out-of-sample variance of the stock rate of return for each option in each series observed at time t_0, we use the following formula for each series observed at time t_0:

$$\sigma^{s}_{0-\alpha,i} = a + b\, X_{0,i}/S_0 \qquad (10)$$

where, $\sigma^{s}_{0-\alpha,i}$ is out-of-sample implied volatility of the stock's rate of return for option i which is observed at time t_0; $X_{0,i}$ is the strike price of the option i in the series at time t_0; S_0 is the stock price observed at time t_0;

The $\sigma^{i}_{0-\alpha,i}$ can then be used to estimate out-of-sample B/S option prices in each series. The ability of the out-of-sample B/S option prices to predict the real option prices is compared with that of the WDOP model.

3.9. Out-of-Sample Warrant Prices.

In the final step of our tests, an out-of-sample test is conducted based on warrant prices. The purpose of this part of our analysis is to test the validity of our hypothesis for the stability of the variance of the rate of return of the firm value. First of all, we use the B/S warrant-pricing model adjusted for the dilution effect and dividend payments to find the out-of-sample warrant prices. The process is exactly the same as that described in the previous section for the out-of-sample option price estimated with the WDOP model. We use parameters calculated at time t_0 but the volatility is implied from quotes of the day $t_{0-\alpha}$. Equations (3a), (3b) and (4) are used for the purpose of this section. Once the out-of-sample warrant prices have been estimated, we compare them with the warrant prices observed at time t_0;

3.10. Measurement of the Predicting Performance

Our analysis focuses on the deviations between the real and theoretical option prices. Two kinds of measurements are used:

1. The percentage prediction error expressed as: $E_i = (C_i - C_{mkt})/C_{mkt}$, where C_i is the option theoretical price. The subscript i denotes the model used for the estimation of the option price predicted; E_i denotes the average percentage prediction error using C_i; C_{mkt} is the option market price.

The mean of the percentage prediction error is essential in determining if the model has biases in conjunction with the market prices. It is also an indicator of possible systematic mispricing attributed to the model. If E_i is positive (negative), then the theoretical option prices are higher (lower) than the market prices. We are not using the simple dollar difference $C_t - C_{mkt}$ because this can generate misleading conclusions if the sample contains a wide range of option prices. For example, a \$0.5 difference is much more substantial for an option that is worth \$0.8 than for a \$5 option. The pricing range in our sample is relatively large. Thus, we need a more standardized measurement in order to compare options with a wide range of values in order to reach a reliable conclusion.

2. The absolute value of the mean percentage prediction error. This measurement is given by: $ABE_i = |(C_i - C_{mkt})/C_{mkt}|$

This measurement examines the pricing efficiency of the theoretical model. This measurement captures the possibility that a negative percentage error may be cancelled out by a positive percentage error. In the case that underpricing and overpricing observations are almost the same in number and amount, the mean percentage error would appear to be zero. From this perspective the results can be unreliable and misleading. Thus, taking the absolute value gives a more reliable aggregate result free of the "sign effect".

4. SAMPLE AND DATA DESCRIPTION

4.1. Sample

Warrant data is obtained from the Financial Post's annual publication entitled "Preferred Shares and Warrants". The information obtained through this source for the purposes of our study is: a) the name of the company that issued the warrant; b) the warrant basis (the number of warrants to the potentially purchased number of shares); c) the warrant exercise price; d) the warrant expiry date; e) the stock exchange on which the warrant is listed, and f) the ticker symbol. The publication contains information on all the outstanding warrants traded on Canadian stock exchanges within the period from the previous year's publication date. We restrict our sample to warrants that are traded on the Toronto Stock Exchange (TSE) for two reasons: a) to ensure that the necessary information is available and accessible to the public and b) the TSE can be considered as a good proxy for the Canadian market since 87.4% of the country's daily trading activity of publicly listed companies in 2000 took place on the TSE.

Our sample is comprised of equity warrants only. Excluded are: a) compound equity warrants, b) warrants that give the choice to the holder to purchase a number of common shares or preferred shares, c) warrants written exclusively on preferred shares and d) warrants that give the right to buy a number of shares of a company other than the issuing firm and e) callable warrants.

After elimination of the above, a sample of 450 warrants outstanding remains for the period of 1984-2000 inclusively. We have also included two warrants from 2001 taken from " The Toronto Stock Exchange; Monthly Review" in the sample. This monthly publication is also a principal source of warrant issue dates for our study. We eliminate 38 warrants from the sample because their issue dates are before 1982 or because there is a lack of supporting information. Two warrants are also removed from the sample because they are redeemable by the issuing company prior to expiry date. To summarize, we collected 412 equity warrants traded on the TSE during the years 1982-2002.

We then derive option trading information from three main sources: a) "The Toronto Stock Exchange; Official Trading Statistics" the yearly Publication of The Toronto Stock Exchange (1997-1995, 1989, 1987-1983), b)"The Montreal Stock Exchange: Official Trading Statistics" (1992-1985), the yearly publication of the Montreal Exchange and c) " The Vancouver Stock Exchange; "Official Trading Statistics (1982-1995). The first and the last sources contain information for the listing and delisting dates of options. Supplementary information sources are the monthly reviews of the three stock exchanges. We restrict our sample to companies that have traded options during the life of their outstanding warrants. A total of 71 warrants are retained for the purpose of our study. Of these, a further filter for bonds outstanding found in the Financial Post, "FP Bonds: Corporate"[10] annual publication is then conducted. We eliminate all the companies that have outstanding bonds, debentures or notes during the warrant's life, keeping only pure-equity firms in our sample. We examine the special cases where outstanding bonds, debentures or notes expire within the life of the warrant in which case the warrants are retained in our sample if: a) the period from the expiry of the bond and the expiry of warrant is substantially large (more than eight months) and b) there is no other bond,

[10] The old name of this publication is "Corporate Bond Record"; Financial Post, 1982-1997.

debenture or note outstanding. For example, let's assume that a company j issues a warrant with a life span of three years. Also, let's assume that all the firms' bonds expire in a year from the issue date of the warrant. In such a case, we kept the warrant in our sample because there is a substantial period for observations after bond expiration. The starting point of the binomial tree is any date after the expiry of the bond. The number of warrants included in our sample reduces to 30, all of which written on all-equity companies, which makes the estimation of firm value easier.

We include in the final sample only firms whose capital structure contains a single issue of warrants for a significant period of time and in order to eliminate any secondary warrant-dilution effects. In other words, a warrant is deleted from the sample if during its life it is simultaneously traded with any other type of warrant subscribed in any exchange and issued from the same company. For example, the observation period $[t_{0,i}\text{-}T_i]$ for a company i, where $t_{0,i}$ is the issue date or any date after the issue date of the warrant and T is the expiry date of warrant, must contain only one warrant outstanding written from company i.

Following all of our exclusions, 21 warrants issued by 19 companies were found to meet the criteria of our study. Of these 19 companies, a total of 407 warrant and 1520 option observations, collected over time, met the inclusion criteria of our study. For a list of the firms, a summary of the dilution factor and a detailed description of our sample see table 1.

Table 1

Description of the Whole Sample

Id #	Company Name	# of days or warrant observations	# of option observations	Mean % Dilution factor N/(N+n)	Stock Exchange
1	Agnico-Eagle Mines	20	33	0.95	MSE
2	Alcan	23	133	0.95	TSE
3	Asamera Inc	20	108	0.93	TSE
4	Biomira Inc	21	78	0.82	TSE
5	Bow Valley Industries	20	98	0.96	TSE
6	Cambior Inc 86	20	43	0.75	MSE
7	Cambior Inc 93	21	39	0.95	MSE
8	Falconbridge Ltd	20	46	0.96	TSE
9	Infowave	8	11	0.95	TSE
10	Laidlaw Transportation	20	104	0.94	MTL
11	M G I Software Corp.	13	39	0.96	TSE
12	Placer Development	20	153	0.87	TSE
13	Potash Corp of Saskatchewan Inc	20	91	0.92	TSE
14	Quadra Logic Tech.	20	47	0.92	VSE
15	Royal Oak Mines Inc	22	95	0.98	TSE
16	Saskatchewan Oil & Gas	20	28	0.91	MSE
17	Teck Corp. 87	20	52	0.96	VSE
18	Teck Corp. 91	11	12	0.97	VSE
19	Tee-Comm Electrs Inc	20	111	0.91	TSE
20	Total Petroleum (N.A)	26	135	0.91	TSE
21	Westmin Resources Limited	22	64	0.95	TSE
	Whole sample	**407**	**1520**	**0.92**	

4.2. Data

Common shares outstanding, dividends and stock split information are obtained from the Toronto Stock Exchange's "CFMRC-Common Equity" CD-ROM. This is a database of the common equity listed on the Toronto Stock Exchange. Dividends are restricted to cash dividends (regular or extras). Supplementary information on the outstanding number of Common and Preferred shares (if they exist) is derived from the "Toronto Stock Exchange: Monthly Review". The shares outstanding are attributed to the end of the month quotes. To calculate the equivalent number of the outstanding shares for a specific date other than the end of the month, an extrapolation method is used (see Appendix B, part B). The number of warrants outstanding is taken from the "Financial Post; Investors Reports" and from the bulletins of " The Toronto Stock Exchange; Daily Record", a daily publication from the TSE.

Daily closing stock price quotations are taken from "CFMRC-Common Equity". Daily closing warrant prices are obtained from the "Toronto Stock Exchange; Daily Record". From the "Globe and Mail" newspaper we obtain daily closing option prices, exercise prices, and the expiry month of the options. For the purposes of our study, we use an average of bid and ask prices to capture the possibility of thin trading. According to Becker (1981), a problem occurs if closing price data is used, because the closing price reflects the price at which the last trade took place, while it is unknown whether this trade took place at the bid price, the ask price, or some price in between.

Closing spot foreign exchange rates (CD$/US$) are obtained from the "CFMRC-Common-Equity" CD-ROM. We use the exchange rate in order to convert any dividends expressed in American currency to Canadian dollars. We make the conversion by multiplying the dividend amount by the prevailing spot exchange rate on the ex-dividend date. Four of the 19 companies in our sample pay dividends in American dollars. The "Total Petroleum" company has issued warrants written on Canadian common shares with an exercise price expressed in American dollars. For this case we estimate for each observation an average of the daily spot exchange rates during the time interval used in the valuation of the warrant.

4.3. Interest Rates

Information on long-term zero-coupon bonds or "strips"[11] is not easy to obtain. Therefore, yields of Canada Bonds are used as a proxy for long-term risk-free rates (i.e for maturities in excess of 1 year). The Canada bonds used in our study are traded close to par value and mature near the warrant expiration dates. It is sometimes difficult to find Canada Bonds that are trading close to par having the same maturity as the warrant. Therefore, for each observation, we select two Canada Bonds trading close to par: one with a shorter and one with a longer maturity than the warrant. Then, through interpolation, we obtain the appropriate interest rate (see Appendix B, part A). For maturities of less than one year we use T-bills with a maturity of 1,3 and 6 months and 1

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[11] Several Investment banking firms buy Government of Canada or Provincial coupon-paying bonds and sell the rights to single payments backed by the bonds. These bonds are often called strips. Bodie, Kane, Markus, Perrakis and Ryan (1999), "Investments",3rd Canadian Edition, Toronto:Irwin, McGrae-Hill of Canada, pp.403.

year taken from the "DataStream Database". Again interpolation is used to find the appropriate risk-free rates for option pricing and for estimating the PV of dividends.

Yields, derived from publicly available databases, are usually expressed on an annual basis. Our analysis is based on continuously compounded risk-free rates. Therefore, the annualized risk-free rate is converted into a continuous rate. The formula used for the conversion is the following:

$$(1+r)=e^{r_c t}$$

where, r is the annualized risk-free rate; r_c is the continuously compounded annual risk-free rate; t is equal to 1 since the risk-free rate is annualized.

Spot risk-free rates are used for the valuation of options and the discounting of the dividends. Spot risk-free rates are known only at t_0, the beginning of the constructed binomial tree ("the present"). Risk-free rates at any other time-step of the binomial lattice (in "the future") are unknown. Warrant valuation is achieved at time t_{op} where the option expires In other words, we need risk-free rates that are going to be effective in the future. For these we use the forward rates observed at t_0. Having the appropriate spot risk-free rate before and after the t_{op}, the forward risk-free rate is estimated from the following:

$$e^{-r_f (T-t_{op})}=\frac{e^{-r_T T}}{e^{-r_{op} t_{op}}}$$

Where, r_f is the forward rate; r_T is the spot risk-free rate for the whole life of the binomial tree; T is the life of the binomial grid expressed in years; r_{op} is the prevailing spot risk-free rate at the period from t_0 until the expiry of the; t_{op} is the length of time t_0 until expiry of the option. The use of the forward rated as an estimate of the future riskless rate is consistent with the absence of interest rate uncertainty. In practice the choice of the riskless rate makes little difference, since option values are relatively insensitive to interest rates changes.

Note: all the rates are continuously compounded rates[12].

4.4. Test for Early Exercise

The most commonly traded call options are American options due to the flexibility offered by this derivative instrument's premature exercise feature. As such, the central focus of this study is the American warrant. Merton (1973) showed that rational investors would only exercise a call option early just before the ex-dividend date. Smith (1976) proves that there is an incentive for the holder of an American option to exercise just prior to an ex-dividend date. According to Breen (1991): "The binomial is an efficient means of valuing American calls on dividend paying stock because, as it is well known, the early exercise value of the option need only be computed at each ex-dividend date."(pp. 158). Thus, we test all of the options and warrants for the early exercise using the following condition:

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[12] for the original forward rate formula see appendix C.

It is not optimal to exercise at time t_n, the final ex-dividend date, when:

$$D_n \le K\left[1 - e^{-r(T-t_n)}\right]$$

where, T is the expiry date; r is the risk-free interest rate; K is the strike price of warrant; (for more information see Hull (2000) pp.259-261).

All the warrants and options in our sample are not exercised before their respective expiry dates. Since there is no positive probability of early exercise, option prices in the intermediate nodes of the binomial tree are not taken into account. The pricing process "jumps backwards" from the expiry of options or warrants to the point where the options' or warrants' value must be estimated. In other words, we can treat the options and the warrants in our sample as European derivatives. The valuation process of a European option involves less complicated computations.

In summary, trading information for options, warrants, stocks and bonds must be available at the starting point of each tree. The life of the option is equal to or longer than one month because of lack of appropriate risk-free rates for periods of less than one month. If the option of a company is listed after the issue date of the warrant, we select observations during the period of the listing and expiry of the warrant.

5. RESULTS

5.1. In-Sample Results

In this section, the deviations between the theoretical option prices, derived using the WDOP approach, and the actual market prices are examined. The 1520 option observations (total sample) are divided into categories based on strike price ratios (X/S_0). At-the-money options are considered to be the warrants with strike price ratios in the range of [0.95-1.05]. In-and out-of-the-money options are those with X/S_0 less than 0.95 and greater than 1.05 respectively. This definition for in-, at- and out-of-the-money options is going to be used for the remainder of our study. Table 2 reports the mean average percentage errors and their absolute values for each category separately and for the entire sample. Option observations for the entire sample and each category are also presented in this table. T-tests are given in brackets and are used to test the null hypothesis which states that the mean percentage prediction errors are equal to zero.

Table 2.

Average Percentage Errors Estimated by WDOP Model

		Mean Percentage Prediction Error	Mean Percentage Prediction Error in Absolute Value
moneyness of option – X/S_0	# of option observations	$E_b = \frac{C_b - C_{mkt}}{C_{mkt}}$	$ABE_b = \left\lvert \frac{C_b - C_{mkt}}{C_{mkt}} \right\rvert$
in-the-money	549	-8.77% (-12.20)*	14.99% (30.11)*
at-the-money	461	-0.75% (-0.53)	24.31% (28.90)*
out-of-the-money	510	12.82% (5.13)*	44.48% (27.16)*
Whole Sample	1520	0.90% (0.90)	27.71% (39.12)*

* Significance at level 0.01

The values of the average prediction error, shows that the WDOP model exhibits significant biases in predicting the market prices with the exception of the at-the-money options. This is consistent with the biases reported by Ritchken (1987) and Hull (2000) based on the use of the B/S model.

The average prediction errors vary from –8.77%, for in-the-money options, to 12.82 % for out-of-the-money options and are found to be statistically significant. This is consistent with Rubinstein's findings that strike-price biases exist and are statistically significant. Our model appears to have less biases for at-the-money options having a prediction error equal to –0.75% which is not considered to be significantly different from zero (t-test:–0.53). In addition, the WDOP model tends to underprice the in- and at-the-money options and overprice the out-of-the money options, which is consistent with the Black (1975) and MacBeth and Merville (1979), findings for option prices. Another interesting point is that the WDOP model has more biases for the out-of-the-money options. This result was expected based on the fact that the out-of-the money options are worth less than the balance of the observed options. Thus, a small dollar deviation of theoretical prices from market prices seems to be even greater for options with low prices. The mean prediction error for the entire sample is low (0.90%) and indicates that the WDOP has a tendency of overpricing the actual option price. The overall mean for the entire sample is found to not be significantly different from zero (0.90).

Testing for the efficiency of the WDOP, we use the absolute values of the prediction errors in relation to market prices. The statistically significant values for in-, at- and the out-of-the money options (14.99%, 24.31% and 44.48% respectively) suggest that the

WDOP model demonstrates imprecision in the predicting of option prices. This is also supported by the absolute percentage deviation of the entire sample.

The option-pricing literature has shown that model deviations are linked to time to maturity. The above-reported biases may also be connected to the time to maturity of warrants and options. Under the assumption that the volatility of the V is constant, we expect that the longer the maturity of an option, the lower the resulting biases using the WDOP model. This is based on the fact that short-term options have lower prices compared to options with longer life spans. Based on this same reasoning, we expect this to be even more prevalent for options that are priced using longer maturity warrants. Thus, we divide the three categories of options (in-, at- and out-of-the money) into sub-categories according to warrant and option life spans. Three ranges of option time to maturity (option-maturity constraint) were selected:

- Short-term options: life span of 1 to 2 months;
- Medium-term options: life span of 3 to 4 months;
- Long-term option: life span of 5 months and over;

Similarly, three ranges of warrant life span (warrant-maturity constraint) were selected:

- Short-term warrants: less than 1 year;
- Medium-term warrants: time to maturity equal to or greater than 1 year but less than 2 years;
- Long-term warrants: equal to or greater than 2 years to maturity. Our sample does not contain warrants with maturities of more than 3 years;

Percentage predicting errors and their absolute values are presented in the three-dimensional table 3. The three dimensions are: 1) moneyness (in-, at- and out-of-the-money) of the option, 2) the option time to maturity and 3) warrant life. Panel A presents how the option price observations are distributed across these subcategories. There are

Table 3

Average Percentage Errors Estimated by WDOP model-
Sample Distribution Based on the Option Moneyness and on the Life Span of Option and Warrant

Panel A - Data disribution

	option Life=OP	0.083<=OP<0.25 (1-2 months)			0.25<=OP<0.416 (3-4 months)			0.146<=OP<0.5 (5-more monts)		
	Warrant life =w (in years)	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3
X/S_0	In-the-money	71	62	53	76	73	45	57	65	47
	at-the-money	63	49	36	70	46	35	60	63	39
	out-of-the-money	55	61	30	73	94	34	70	60	33
	Whole Sample	189	172	119	219	213	114	187	188	119

Panel B - Average Percentage Error (E_b)

	option Life=OP	0.083<=OP<0.25 (1-2 months)			0.25<=OP<0.416 (3-4 months)			0.146<=OP<0.5 (5-more monts)		
	Warrant life =w (in years)	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3
X/S_0	In-the-money	-1.21% (-0 89)	-14.43% (-8.58)*	-28.21% (-16 22)*	4.28% (2.95)*	-8.30% (-5.48)*	-24.14% (-14.49)*	8.13% (4 24)*	-8 67% (-5 34)*	-18 58% (-12 75)*
	at-the-money	0 75% (0 27)	-11 09% (-2.60)*	-31 84% (-7 38)*	17 62% (5 69)*	-5 56% (-1 53)	-24 34% (-8 45)*	25 49% (8.72)*	-6 28% (-2.19)*	-13.66% (-4 09)
	out-of-the-money	22 85% (2.39)**	-24 25% (-3.9)*	3.18% (0.31)	45.22% (7 82)*	-2.81% (-0 54)	11 11% (1.50)	49 08% (8 03)*	0 40% (0 07)	8.78% (1 33)
	Whole Sample	9.42% (3.00)*	-17 00% (-6.51)*	-21.39% (-6.53)*	22.19% (8.88)*	-7.62% (-3.14)*	-13.69% (-4.75)*	29.03% (10.30)*	-4.97% (-2.34)**	-9.60% (-3.94)*

Panel C - Average Percentage Error in absolute value (|Eb|)

	option Life=OP	0.083<=OP<0.25 (1-2 months)			0.25<=OP<0.416 (3-4 months)			0.146<=OP<0.5 (5-more monts)		
	Warrant life =w (in years)	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3	w<1	1<=w<2	2<=w<3
X/S_0	In-the-money	8.13% (8.46)*	17.20% (14.54)*	28.21% (16.22)*	8.81% (7.71)*	12.87% (13.18)*	24.15% (14.51)*	11.76% (7.60)*	13.06% (12.20)*	18.58% (12.75)*
	at-the-money	22.32% (12.58)*	27.03% (11.81)*	36.44% (11.74)*	23.59% (9.61)*	20.94% (10.08)*	25.05% (9.29)*	27.11% (10.17)*	18.78% (10.53)*	21.09% (10.11)*
	out-of-the-money	59.25% (9.91)*	45.34% (12.07)*	44.03% (6.78)*	52.42% (10.78)*	38.92% (12.78)*	32.28% (6.20)*	53.75% (9.75)*	31.02% (7.87)*	27.91% (6.00)*
	Whole Sample	27.99% (11.25)*	30.00% (16.71)*	34.69% (16.49)*	28.07% (12.69)*	27.07% (16.45)*	26.85% (14.21)*	32.40% (12.40*)	20.71% (13.51)*	22.04% (13.73)*

* Significant at level 0.01
** Significant at level 0 05

more than 30 observations in each subcategory. Panel B presents the percentage prediction errors E_b (of the WDOP model) and Panel C, the absolute value of E_b. t-ratios are given in brackets.

From panel B it seems that our model does not effectively predict the market prices of options except for the in- and at-of-the-money short-term options priced with warrants having less than one year to maturity (error mean: -1.21% and 0.75% respectively) and out-of-the-money long-term options estimated using warrants of medium life span (error mean 0.40%). The corresponding t-tests for each of these cases show that the three means are not statistically significant. In all cases, our model produce large absolute percentage errors, with the results varying from 8.73% to 31.02%.

Panel B shows that in-the-money options using the option-maturity constraint (short-, medium- and long-term options), tends to have a mean prediction error range that changes from a positive value to negative values as we move from short-term to medium-term warrants with increasingly negative values for longer-term warrants. For example, the average percentage error for options that a) are in-the-money and b) have a life span of 3 to 4 months, moves from 4.28% (for less than one year warrants) to –8.30% (for warrants within 1 and 2 years) to a greater negative value of -24.14% for warrants with more than 2 years to maturity. These mean prediction errors are statistically significant at a level 0.01 with the exception of short term-options priced with short-term warrants Thus, in-the-money options priced with short-term warrants are less biased relative to the options priced with warrants having a longer term to maturity. Of the in-the-money options, priced with short-term warrants, the WDOP model offers fewer biases for the short-term options. Again, at-the-money options priced with short-term warrants tend to be overpriced and options with medium- and long-term warrants are underpriced. No specific price prediction bias pattern for the WDOP model can be concluded for out-of-the-money options and for the sample of options as a whole.

Panel C indicates that our method is not efficient enough to predict the options' market prices having a $|E_b|$ in excess of 8.13%. All the means within the sub-categories are significantly different from zero. For in-the-money options, we can say that $|E_b|$ tends to increase as we move from short to long-term warrant maturities within the same category based on the option life span, thereby supporting the biases reported in the panel A for in-the-money options. The opposite tendency exists for out-of-the-money options. For at-the-money options the results are mixed.

The WDOP model's efficiency improves for in- and at-the-money options priced with short-term warrants, as we go from long to short-term options. Moving from long to short-term options priced with warrants with less than one year to maturity, the absolute value of the prediction error ($|E_b|$) tends to decrease (11.76%, 8.81% and 8.13% for in-the-money options and 27.11%, 23.59% and 22.32% for at-the-money options). The opposite can be said for out-of-the-money options. Examining the two sub-categories based on the medium- and long-term warrants, it can be said for in-, at- and out-of-the-money options that as we move from long- to short-terms options $|E_b|$ tends to increase which means that the pricing efficiency of the WDOP model deteriorates. For example, at-the-money options priced with medium-term warrants have a mean prediction error that moves from 18.78%, 20.94% to 27.03 and from 21.09%, 25.05% to 36.44% for options priced with long-term warrants. Following the same analysis for the entire sample of options, $|E_b|$ tends to exhibit the same pattern for the pricing efficiency as in- and at-the-money options.

Overall, table 2 shows that our model exhibits statistically significant deviations from the observed market prices. There is no clear predictability pattern when we divide our sample into sub-categories based on the option time to maturity, warrant life and the ratio of strike price to the stock price at time t_0. However, we can conclude that the pricing performance of the WDOP model works better for short-term options and warrants, which may indicate that the volatility of the rate of return of the firm value is not constant over time.

5.2. Out-of-Sample Option-Pricing Tests

For this part of our analysis, a sub-sample of the entire sample is used for which there are observations on series of three or more options with the same expiration date but different degrees of moneyness. Dividing the options into series of three or more options, we end up with options issued by 13 companies. Table 4 gives a detailed description of this sub-sample. A total of 536 option observations are used creating a total of 167 option series observed over 130 days.

Table 4

Description of the Sub-Sample for Out-of-Sample Tests

Id #	Company	# of days	# of series	# of option observations	Stock Exchange
1	Alcan	10	16	55	TSE
2	Asamera Inc	10	14	43	TSE
3	Biomira Inc	10	10	30	TSE
4	Bow Valley Industries	10	12	37	TSE
5	Falconbridge Ltd	10	10	30	TSE
6	Laidlaw Transportation	10	17	54	MTL
7	M G I Software Corp.	10	10	30	TSE
8	Placer Development	10	20	67	TSE
9	Potash Corp of Saskatchewan Inc.	10	13	45	TSE
10	Royal Oak Mines Inc	10	11	33	TSE
11	Tee-Comm Electrs Inc	10	11	41	TSE
12	Total Petroleum (N.A)	10	13	40	TSE
13	Westmin Resources Limited	10	10	31	TSE
	Sum	**130**	**167**	**536**	

5.2.1. In- and Out-of-Sample Comparison for WDOP Model.

In this section a comparison between the in- and out-of-sample predicted option prices is conducted using the WDOP model. Table 5 presents the corresponding results for the whole sub-sample and for in-, at- and out-of-the money options separately. Table 5 reports the in- and out-of-sample average percentage prediction errors - $\left(E_b^{in}\right)$ and $\left(E_b^{out}\right)$ (Panel A). Average percentage deviations from the options' market prices are given in the absolute value as well (Panel B). Table 5 summarizes the differences between the in- and out-of-sample mean percentage prediction errors. The difference is estimated as

Table 5

Comparison of Prediction Errors Estimated by WDOP Model - In- and Out-of-Sample Estimated Implied Volatility

	# of Options Observation	**Panel A: Average Precentage Prediction Error for WDOP Model**		
		in-sample $E_b^{in} = \frac{C_b^{in} - C_{mkt,0}}{C_{mkt,0}}$	out-of-sample $E_b^{out} = \frac{C_b^{out} - C_{mkt,0}}{C_{mkt,0}}$	$D = E_b^{in} - E_b^{out}$
in-the-money	175	-8.67% (-7.09)*	-7.91% (-5.97)*	-0.76% (-2.33)**
at-the-money	153	-5.43% (-2.43)**	-4.60% (-1.91)	-0.82% (-1.08)
out-of-the-money	208	18.92% (4.15)*	20.42% (4.12)*	-1.50% (-1.21)
Whole Sample	536	1.27% (-0.72)	2.29% (1.20)	-1.02% (-2.15)**

	# of Options Observation	**Panel B: Average Precentage Prediction Error for WDOP Model in Absolute Value**		
		in-sample $ABE_b^{in} = \left\lvert\frac{C_b^{in} - C_{mkt,0}}{C_{mkt,0}}\right\rvert$	out-of-sample $ABE_b^{out} = \left\lvert\frac{C_b^{out} - C_{mkt,0}}{C_{mkt,0}}\right\rvert$	$ABD = \lvert E_b^{in} - E_b^{out}\rvert$
in-the-money	175	15.21% (17.69)*	15.93% (17.50)*	-0.72% (-2.32)**
at-the-money	153	23.35% (18.40)*	24.82% (18.10)*	-1.48% (-2.36)**
out-of-the-money	208	49.71% (16.89)*	53.72% (16.66)*	-4.01% (-3.37)*
Whole Sample	536	28.80% (22.92)*	30.81% (22.46)*	-2.01% (-4.49)*

* Significant at level 0.01
** Significant at level 0.05

$D = E_b^{in} - E_b^{out}$ where the absolute value is, $ABD = \lvert E_b^{in} - E_b^{out}\rvert$. Again, t-tests results are presented in the brackets. The t-tests for D and ABD determine whether the two mean prediction errors are significantly different from one another.

As was discussed for the out-of-sample methodology, the parameters used to calculate the theoretical option prices are estimated at time t_0, while the implied volatility of the rate of return of the firm value is estimated at $t_{0-\alpha}$. The following measurements of the pricing performance are used:

$$E_b^{in} = \frac{C_b^{in} - C_{mkt,0}}{C_{mkt,0}} \text{ and } E_b^{out} = \frac{C_b^{out} - C_{mkt,0}}{C_{mkt,0}}$$

Where, E_b^{in} and E_b^{out} are the average percentage prediction errors for in- and out-of-sample theoretical option prices respectively; C_b^{in} denotes the in-sample predicted option prices estimated using the WDOP model. The estimation of the parameter, used to calculate C_b^{in}, is made using actual option related quotes at time t_0; C_b^{out} is the predicted option price estimated by the WDOP model using the out-of-sample implied volatility ($\sigma_{0-\alpha}$); $C_{mkt,0}$ is the option market price at time t_0;

Comparing the in- and out-of-sample average percentage errors for the in-the-money options, the results do not differ substantially (-8.67% v.s –7.61%) and are both statistically significant. The difference between the two means is small (equal to –0.76%). T-test shows that the two means differ at a 0.05 level. We have similar results for at-the-money options (-5.43% v.s-4.60) having a difference of –0.82%. For out-of-the-money options the difference between the in- and out-of-sample average percentage errors increases (-1.50%). These results were expected for the out-of-the-money options, as their price is low (usually less than $1). A dollar difference represents a substantial deviation as a proportion of a price less than 1. The t-tests for at- and out-of-the-money options (-1.08 and -1.21 respectively) indicate that the difference between in- and out-of-sample mean prediction errors (E_b^{in} and E_b^{out} respectively) is not significantly different from zero.

When we are out-of-sample, the in- and at-the-money options tend to be less underpriced using the WDOP theoretical model while the out-of-the-money options tend to be more overpriced compared to in-sample option prices. For example in the case of in-the-money options, E_b^{in} is equal to –8.67% whereas E_b^{out} is equal to –7.61%. This indicates that the in-sample theoretical option prices are smaller than the out-of-sample theoretical option prices. According to Merton (1973), the option prices have a direct relationship with the volatility, and therefore we can say that the out-of-sample implied volatility is larger than the in-sample volatility. The difference between the two implied volatilities cannot be as large as the deviations between E_b^{in} and E_b^{out} which are either small or are not statistically significantly different from each other.

In addition, D increases as we move from in- to out-of-the-money options. This may happen for two reasons. First, because of the decrease in the option prices as the strike price increases. Secondly, because of the implied volatility difference that becomes more acute moving from in- to out-of-the-money options. We must keep in mind that in all cases, D is quite small and predominantly not different from zero. Overall, in-sample mean percentage deviation is smaller than the out-of-sample mean percentage deviation. The difference between the two means is –1.02 with a significance level of 0.05.

Comparing the absolute values of E_b^{in} and E_b^{out}, we obtain similar results. ABD is quite small with the exception of the out-of-the-money options, where it is equal to –4.01%. For in- and at-the-money options, ABDs are equal to –0.72 and –1.48% respectively and are different from zero at a 0.05 level. Out-of-the-money ABD can be considered large and statistically different from zero. We observe that in-sample predicted option prices are smaller than the out-of-sample option prices, thereby reinforcing our conclusion regarding the findings in panel A. ABD increases as we move from in- to out-of-the-money options. In this case, the differences between in-, at- and out-of-the-money options are greater and statistically significant, thus making the hypothesis of a constant volatility of the value of the firm even more questionable. The small values of D and ABD make the conclusions from the previous section (5.1) more robust and reliable.

5.2.2. *Out-of-Sample Comparison for WDOP versus B/S Model.*

In this part of our study the pricing ability and performance of our model is compared with the one of B/S model. The comparison is made going out-of-sample for both models. We use the following measurements for this part of our analysis:

- E_b^{out} as previously defined;
- $E_{B/S}^{out} = \dfrac{C_{B/S}^{out} - C_{mkt,0}}{C_{mkt,0}}$, where $C_{B/S}^{out}$ is the out-of-sample predicted option price estimated using the dividend-adjusted B/S model (see section 3.8.2); $C_{mkt,0}$ denotes the real option price observed at time 0.

The average absolute value of the percentage deviations of both models is also presented. T-tests are again presented in brackets. D expresses the difference between the average mean prediction error percentage deviations of the two models. D is estimated as $D = E_b^{out} - E_{B/S}^{out}$. ABD represents the absolute value of the estimated mean prediction errors, $ABD = \left| E_b^{out} - E_{B/S}^{out} \right|$. The purpose of this analysis is to test the ability of the WDOP model to predict the market prices compared to the selected benchmark model (B/S model).

Table 6 summarizes the results of this section. Panel A gives the out-of-sample mean prediction errors for the WDOP and dividend-adjusted B/S models. Panel B gives their

Table 6

Comparison of Prediction Errors Estimated by WDOP and B/S Model - Out-of-Sample Estimated Implied Volatility

		Panel A: Average Percentage Difference between Predicted & Market Prices		
	# of Options Observations	out-of-sample $E_b^{out} = \frac{C_b^{out} - C_{mkt,0}}{C_{mkt,0}}$	out-of-sample $E_{B/S}^{out} = \frac{C_{B/S}^{out} - C_{mkt,0}}{C_{mkt,0}}$	$D = E_b^{out} - E_{B/S}^{out}$
in-the-money	175	-7.91% (-5.97)*	2.48% (2.97)*	-10.39% (-8.79)*
at-the-money	153	-4.60% (-1.91)	3.47% (2.30)**	-8.07% (-3.28)*
out-of-the-money	208	20.42% (4.12)*	6.92% (1.43)	13.50% (2.09)**
Whole Sample	536	2.29% (1.20)	4.21% (2.52)*	-1.93% (-0.83)

		Panel B: Average Absolute value of Percentage Difference between Predicted & Market Prices		
	# of Options Observations	out-of-sample $ABE_b^{out} = \left\| \frac{C_b^{out} - C_{mkt,0}}{C_{mkt,0}} \right\|$	out-of-sample $ABE_{B/S}^{out} = \left\| \frac{C_{B/S}^{out} - C_{mkt,0}}{C_{mkt,0}} \right\|$	$ABD = \left\| E_b^{out} - E_{B/S}^{out} \right\|$
in-the-money	175	15.93% (17.50)*	5.59% (7.38)*	10.34% (11.14)*
at-the-money	153	24.82% (18.10)*	10.13% (7.84)*	14.69% (8.44)*
out-of-the-money	208	53.72% (16.66)*	27.25% (6.16)*	26.47% (4.76)*
Whole Sample	536	30.81% (22.46)*	13.96% (8.89)*	16.85% (8.71)*

* Significant at level 0.01
** Significant at level 0.05

absolute values. Our sample is again divided into categories based on strike-price ratios (X/S_0). The B/S model appears to outperform WDOP model in most, but not in all cases. In-, at- and out-of-the-money theoretical option prices, estimated with the B/S model, are less biased compared to the option prices estimated using the WDOP model (2.48%, 3.47% and 6.92% for B/S model v.s –7.91%, -4.60% and 20.42% for WDOP model respectively). $E^{out}_{B/S}$ is not statistically significant for out-of-the-money options and E^{out}_{b} is not significant for at-the-money options. Ds are –10.36%, -8.07% and 13.50% for in-, at- and out-of-the-money options respectively. The differences between the mean prediction errors for both models are statistically different than zero.

Overall the B/S model is more biased than WDOP model in the estimation of out-of-sample option prices but their mean prediction errors are not statistically different from each other (t-ratio –0.83). There is a tendency for at-the-money D to become smaller in magnitude (-8.07% v.s –10.39% and 13.50%) but this tendency is not confirmed when we use the absolute values. The difference in the mean prediction errors (ABD) increases as we move from in- to out-of-the-money options (10.36% to 26.47%). The B/S model not only seems to be a less biased model for in-, and at-the-money options when compared to the WDOP model, but also exhibits a systematic pattern for predicting option prices. In all cases, the B/S model consistently overestimates the option market prices, whereas no systematic pattern appears for the WDOP model.

The mean prediction errors in their absolute form favor the B/S model supporting its superiority as an option market price predictor compared to the WDOP model. In all cases (in-, at- and out-of-the-money options), ABD is positive, indicating that the WDOP model exhibits larger deviations from the real option prices than the B/S model. Thus, it can be said that the B/S model appears to be a more efficient model for the prediction of real option prices. T-ratios seem to be substantial enough to support the statistical significance of all mean prediction errors (ABE_{b}^{out} and $ABE_{B/S}^{out}$). The overall ABD is positive (16.85%) and statistically significant, confirming the pricing efficiency of the B/S model compared to the WDOP model.

The WDOP model, in its attempt to incorporate the warrant-dilution effect, does not demonstrate the expected improvement over a model that does not consider the capital structure of the firm. It is important to mention that while the B/S model may have better results than the WDOP model (in terms of $E_{B/S}^{out}$ and $ABE_{B/S}^{out}$) it may still not be a good model for the pricing of options as $E_{B/S}^{out}$ and $ABE_{B/S}^{out}$ can be considered large, since they have values in excess of 2.48%, while its biases are systematic and statistically significantly different than zero.

5.2.3. In- and Out-of-Sample Warrant Prices.

In- and out-of-sample warrant prices are compared in terms of their average percentage deviation from the option market prices. The sample of warrants is divided into sub-categories based on the ratio $RT=K/W_{mkt,0}$, where K is the warrant exercise price and $W_{mkt,0}$ is the warrant market price observed at time t_0; Again, we consider at-the-money warrants to have a ratio within the range [0.95-10.5]. In-the-money warrants have an RT less than 0.95, whereas out-of-the-money one have a ratio greater than 1.05. Table 7 presents the number of warrants in each of these categories for the entire sample. A total of 130 warrant prices are used for this analysis.

Two types of measurements are used in for the purpose of this analysis:

- $E_W^{out} = \dfrac{W_b^{out} - W_{mkt.0}}{W_{mkt.0}}$, where E_W^{out} is the warrant average percentage deviation. We compare the out-of-sample predicted prices with the market prices observed at time 0; W_b^{out} is the out-the-sample predicted warrant value calculated by equations (4) (3α) and (3b). The parameters used in (4) are derived from t_0 except for the implied standard deviations that is estimated at time $t_{0-\alpha}$; α varies from 1 to 5 trading days; $W_{mkt.0}$ is the warrant market price observed at time t_0;

- The absolute value of E_W^{out} is expressed as $ABE_W^{out} = \left|\dfrac{W_b^{out} - W_{mkt.0}}{W_{mkt.0}}\right|$;

Table 7 presents the means of the percentage prediction errors and their absolute values. T-ratios are presented in brackets testing the null hypothesis that the errors are zero. The mean prediction errors suggest that the predicted warrant prices do not deviate significantly from the market warrant prices. This is even more obvious for the out-of-the-money warrants, which have a mean percentage error equal to 0.08%. The t-test (0.04) shows that the mean prediction error is not significantly different from zero. The equivalent mean prediction errors for in- and at-the-money warrants are 0.40% and –0.33% respectively and are not significantly different from zero. E_W^{out} suggests an

Table 7

**Percentage Difference between B/S Theoretical and Market Warrant Prices-
Implied Volatility Estimated Out-of-Sample**

	Number of Options Observations	Average Precentage Prediction Error $E_W^{out} = \frac{W_b^{out} - W_{mkt,0}}{W_{mkt,0}}$	Average Percentage Prediction Error in Absolute Value $ABE_W^{out} = \left\lvert \frac{W_b^{out} - W_{mkt,0}}{W_{mkt,0}} \right\rvert$
in-the-money	32	0.40% (0.58)	3.06% (7.62)*
at-the-money	17	-0.33% (-0.42)	2.65% (5.88)*
out-of-the-money	81	0.08% (0.04)	10.14% (7.16)*
Whole Sample	130	0.10% (0.09)	7.42% (7.95)*

* Significant at level 0.01

overestimation of the real warrant prices for in- and out-of-the-money warrants but an underestimation for at-the-money warrants. Overall, predicted prices are greater than market prices having an average percentage error of 0.10%, which is not significantly different from zero. Based on E_W^{out}, we can say that the volatility of the value of the firm tends to remain constant over the period $t_{0-\alpha}$, thereby appearing to support the null hypothesis.

Absolute values of the mean prediction error percentage suggest that the predicted prices deviate from market prices in excess of 2.65%. The discrepancies become larger as we go from in- and at- to out-of-the-money warrants (from 3.06% and 2.65% to 10.14%). This may be explained by the fact that the out-of-the-money warrants have smaller market prices than in- and at-the-money warrants.

At-the-money warrants are better predicted than in- and out-of-the-money warrants (2.65% v.s 3.06 and 10.14% respectively). The t-ratios are sufficiently large to show that the difference of the mean prediction errors is significantly different from zero. Thus, the previous conclusion is driven by large positive and negative biases, canceling each other. The large values for ABE_W^{out} for all the categories and the sample as a whole (7.42%) may indicate that the volatility of the value of the firm may not be consistent over time. This may explain the large deviation of our predicted option prices from those observed in the market. Furthermore, this would provide ample justification to reject the null hypothesis.

6. CONCLUSION AND RECOMMENDATION FOR FURTHER RESEARCH

Galai and Schneller (1978) and Crouhy and Galai (1991a) argued that when warrants are issued the capital structure of the issuing firm is affected because of the eventual dilution effect and therefore the "process of generating the stock prices"[13] is not expected to be constant over time. The stochastic nature of the equity's rate of return is supported by the possible existence of a "volatility smile". In this paper, we developed a new option-pricing model that incorporates the capital structure of the firm but ignores the volatility smile issue.

We based ourselves on the CRR (1979) binomial option-pricing model. We constructed a binomial lattice on the value of the firm using the assumption of the B/S model that the volatility of the rate of return of the firm value remains constant. Our model has been tested empirically using actual warrant and call option quotes. All the firms in our sample are pure-equity firms containing only shares and warrants outstanding in their capital structure. After testing for the early exercise of both options and warrants, we selected warrants and options that will not be exercised prematurely thereby allowing us to treat them like European options.

The large values of the mean percentage prediction errors and the mean absolute value of the mean prediction error based on the WDOP model suggest that theoretical option prices deviate substantially from observed option market prices. Even after dividing our sample into subcategories based on the moneyness of the option, option and warrant life,

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[13] Crouhy, M. and Galai D. (1991a) pp. 215.

the imprecision in pricing remained and did not exhibit a systematic pattern based on the selected descriptive parameters.

We tested our model out-of-sample using only an out-of-sample rate of return of the firm value volatility estimator. Mean prediction errors show that in- and out-of-sample predicted option prices are similar in most cases. In- and out-of-sample errors differences are relatively small for in- and at-the-money options but become larger for out-of-sample options. The discrepancies increase when the mean absolute prediction errors are used. Theses discrepancies, related to the moneyness of the option, are consistent with the volatility smile reported for equities and may be an indication that the process governing the volatility of the value of the firm may not be constant. The volatility may decrease as the stock price increases, thus supporting the contentions of the CEV type models. Volatility can also be governed by other stochastic processes which could impose the use of other estimation models such as the GARCH (1,1) where the today's volatility depends on past volatilities with weights decreasing geometrically and placing more emphasis on recent variance changes.

We also compare the results of the WDOP option-pricing model with the B/S model applied to the firm's equity, which we selected as a benchmark. In all cases, the B/S model outperformed the WDOP model which despite its consideration of the warrant-dilution effect, did not exhibit the expected improvement over a model in which the capital structure of the firm is not taken into account. The B/S model seems to be the least biased for in- and out-the-money options and a more efficient model for the pricing of call options compared with the WDOP model. However, the B/S model may not be the best at pricing options given that the errors that it exhibits are highly significant in most cases. A comparison of the out-of-sample warrant prices, estimated using the dilution adjusted B/S model, with the observed market warrant prices, indicates that there is a substantial inconsistency with the market warrant prices and the B/S model seems to be a poor predictor of warrant prices.

The mean percentage prediction errors and their absolute values, in the case of the out-of-sample warrant prices, are small and they are not statistically significant. Overall, the predicted warrant prices deviate from real prices by 7.42%. This is another indication that the volatility of the rate of return of the firm value may not be constant and may follow an unspecified stochastic process. This may explain the imprecision of the WDOP theoretical model.

A new model based on the WDOP option-pricing approach but using a stochastic process of the firm value may improve the test results. A number of authors have constructed option-pricing models incorporating the concept of the stochastic volatility. Few of them have developed binomial option-pricing models under a non-constant volatility. We refer to two of them here.

Hilliard and Schwartz (1996) use a two-step transformation process to create a path-independent and recombining tree. The binomial tree is based on two processes: one for the underlying asset and one for the volatility. First of all, they transform the stochastic process of the volatility to a constant process with volatility equal to one, an essential transformation for a tree to be recombining. Then, they transform the underlying asset to a new variable with a unit constant volatility. After these transformations, the instantaneous volatility of the constructed binomial grid becomes constant. According to Hilliard and Schwartz (1996) this path–independent lattice can be used to value American options whose "stock price and volatility diffusions may have non-zero correlations", which makes the model "consistent with the volatility smile".

Chang and Fu (2001) develop a binomial model that is a hybrid of the continuous-time binomial option-pricing model of Hilliard and Schwartz (1996) and of Amin's (1993) approach. Amin (1993) developed a discrete time option-pricing model to price American options when a jump-diffusion process governs the underlying asset. The new combined model can value American options when there are both a stochastic volatility and a jump process in the distribution of the underlying asset. Three are the main findings in their study, summarized as follows: a) "the jump parameters affect the option values significantly and especially for the near at-the-money options", b) "the stochastic volatility and jump have significant impact on hedge ratios" and c) "there is a positive correlation between the option value and the correlation coefficient of the stock price and its volatility". In addition, they show that their theoretical option-pricing model captures the volatility smile observed in the market.

All of these articles try to solve the problem of the stochastic nature of the volatility of the stock rate of return but it can also be applied to a stochastic firm value. The Hilliard and Schwartz (1996) and Chang and Fu's (2001) models could price American options under the assumption that the volatility of the firm's value is non-constant or follows a jump-diffusion process. The application of these models for the WDOP approach could be possible if someone were to overcome the numerical difficulty of estimating the model parameters. Said estimation is beyond the scope of our study.

Another possible extension of the WDOP model is the analysis of non-pure equity firms, specifically those with outstanding bond issues in their capital structure. As such, the value of the firm would be different due to the inclusion of bonds in the firm's capital structure. While the WDOP model can be extended to perform the above-suggested analysis, it should be noted that the required data is missing due to the infrequent trading of bonds.

7. References

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Web Sites:

Montreal Stock Exchange: www.me.org

Toronto Stock Exchange: www.TSE.com

Social Science Research Network (SSRN): www.ssrn.com

Hoadley Option Software: www.hoadley.net/options

8. APPENDICES

Appendix A.

Change in the value of the firm and stock prices when there are Class A and B shares in the capital structure

We are going to use the following notations:

N_A is the number of outstanding class A common shares;
N_B is the number of outstanding Class B common shares;
S_A is the price of the Class A common share;
S_B is the price of Class B common share;
n is the number of outstanding warrants;
W_B is the warrant price written on the non-voting Class B share;

A. Value of Firm

Let's denote the value of the vote as $q = S_A - S_B$.

$$V = N_A(S_A - q) + N_B S_B + nW_B = N_A(S_A - (S_A - S_B)) + N_B S_B + nW_B = N_A S_B + N_B S_B + nW_B =$$

$$= (N_A + N_B)S_B + nW_B$$

B. Stock Price

$$V = (N_A + N_B)S_B + nW_B \Leftrightarrow S_B = \frac{V}{N_A + N_B} - \frac{n}{N_A + N_B} W_B S_B$$

Appendix B – Extrapolation method

A. Interest Rates

Suppose we want to estimate the risk-free rate r_k that prevails at time t_k The formula used for the interpolation is the following:

$$r_k = r_{k-\alpha} + \frac{t_k - t_{k-\alpha}}{t_{k+\beta} - t_{k-\alpha}}\left(r_{k+\beta} - r_{k-\alpha}\right)$$

where,

$r_{k-\alpha}$ is the yield of a bond that matures prior to t_k; $t_{k-\alpha}$ is the life-span of the bond (expressed in years) that matures prior to t_k; $r_{k+\beta}$ is the yield of the bond that matures after t_k; and $t_{k+\beta}$ is the life span of the bond (expressed in years) that matures after t_k.

B. Outstanding shares

Suppose we have the number of the outstanding shares for the end of January (N_J) and the end of February (N_F). The corresponding outstanding shares $\left(N^{\bullet}\right)$ for the β^{th} day of February is estimated as followed:

$$N^{\bullet} = N_J + \frac{\beta}{d}(N_F - N_J)$$

where,

d denotes the number of days from the end of the first month till the end of the second month. In the above example d will be equal to 28.

Appendix C – Forward Rates

Forward rate is the rate that is going to prevail in a future period under interest rates certainty and is estimated using today's data.

Assume that the forward interest rate for the period k is f_k. The relationship of the spot rates and the forward rate is:

$$f_k = \frac{r_k T_k - r_{k-1} T_{k-1}}{T_k - T_{k-1}}$$

where,
T_k is length of period K expressed in years;
T_{k-1} is length of period K-1 expressed in years;
r_k is the spot rate of the period K;
r_{k-1} is the spot rate of the period K-1;

Note that all the rates are continuously compound rates;

Appendix D – Main program for option pricing

Our kind gratitude to Michael for supplying the test programming

```
program sweet_georgia_1
implicit none
integer, parameter :: long=selected_real_kind(15,200)
real, parameter :: e_long = 2.71828182845905_long
real, parameter :: tolerance = 0.00001
real (kind=long), dimension (:), allocatable :: ARR_pq, ARR_comb, ARR_ipq,
ARR_co_s, ARR_stock
real (kind=long), dimension (:), allocatable :: ARR_inter, ARR_zero, ARR_zeros,
ARR_temp
integer :: A, B, I, J, K, L, M, NN, P, Q, R, S, T, E, X, Y, Z, DD, ios
real (kind=long) :: st_price, time_l, time_s, strike_1, strike, RR, RF, disc, f_disc, ret_vol,
m_n
real (kind=long) :: up_A, down_A, R_l, p_ul, p_dl, R_s, p_us, p_ds, temp, temp_2,
temp_4, check
real (kind=long) :: AA, BB, DDD, ca_ll
character(len=22), parameter :: NBIG = "Total steps"
character(len=22), parameter :: NSMA = "Short steps"
character(len=22), parameter :: WARX = "Warrant X"
character(len=22), parameter :: CALX = "Call X"
character(len=22), parameter :: VALS = "Value per share"
character(len=22), parameter :: TLON = "Total time"
character(len=22), parameter :: TSHO = "Short time"
character(len=22), parameter :: IRAT = "Rate"
character(len=22), parameter :: FRAT = "Forward rate"
character(len=22), parameter :: VOLA = "Volatility"
character(len=22), parameter :: COEF = "Coefficient"
character(len=22), parameter :: cal_l = "CALL PRICE"
character(len=2), parameter :: star = "**"
character(len=12) :: outfile
!Reading data
print *,"Total number of steps"
read *,A
print *,"Smaller number of steps"
read *,B
print *,"Warrant strike"
read *,strike
print *,"Call strike"
read *,strike_1
print *,"Value per share"
read *,st_price
print *,"Total time to maturity (in years)"
read *,time_l
```

```
print *,"Shorter time to maturity (in years)"
read *,time_s

print *,"Interest rate"
read *,RR
print *,"Forward rate"
read *,RF
print *,"Volatility (per annum)"
read *,ret_vol
print *,"M/N"
read *,m_n
print *, "Enter output file name (up to 10 characters)."
read *, outfile
open(unit=1, action="write", status="new", file=outfile)
!!!Preliminary calculations
DD = A - B
AA = real(A)
BB = real(B)
DDD = real(DD)
disc = e_long ** (-1.0 * RR * time_l)
f_disc = e_long ** (-1.0 * RF * time_s)
up_A = e_long ** (ret_vol * sqrt(time_l / A))
down_A = 1.0 / up_A
R_l = e_long ** ((RR * time_s) / B)
p_ul = (R_l - down_A) / (up_A - down_A)
p_dl = 1.0 - p_ul
R_s = e_long ** ((RF * (time_l - time_s)) / DD)
p_us = (R_s - down_A) / (up_A - down_A)
p_ds = 1.0 - p_us
!!! Calculations of binomial distr. coefficients for big number
allocate(ARR_pq(0:B))
ARR_pq(0) = p_dl ** B
!print*,"0",ARR_pq(0)
 do I = 1, B
   ARR_pq(I) = ((ARR_pq(I - 1) * p_ul) * ((BB + 1 - I) / I)) / p_dl
!print*,I,ARR_pq(I)
 end do
!!! Calculations of binomial distr. coefficients for smaller number
allocate(ARR_ipq(0:DD))
ARR_ipq(0) = p_ds ** DD
!print*, "0",ARR_ipq(0)
 do I = 1, DD
   ARR_ipq(I) = ((ARR_ipq(I - 1) * p_us) * ((DDD + 1 - I) / I)) / p_ds
```

```
!print*, I,ARR_ipq(I)
 end do
!!Calculations of ending stock prices
allocate(ARR_stock(0:A))
do I = 0, A
  ARR_stock(I) = st_price * (up_A ** I) * (down_A ** (A - I))
 end do
!!Calculations of intermediate stock prices
allocate(ARR_inter(0:B))
 do I = 0, B
  ARR_inter(I) = st_price * (up_A ** I) * (down_A ** (B - I))
 end do
!!Calculations of intermediate call payoffs & stock prices
allocate(ARR_zero(0:A))
allocate(ARR_temp(0:B))
ARR_zero = 0.0
ARR_stock = ARR_stock - strike
ARR_stock = max(ARR_stock, ARR_zero)
 do J = B, 0, -1
   temp = 0.0
   do K = DD, 0, -1
    check = ARR_stock(J + K)
!      if(check <= tolerance) then
!        exit
!      end if
    check = ARR_ipq(K) * check
    temp = temp + check
   end do
   ARR_temp(J) = temp
 end do
allocate(ARR_zeros(0:B))
ARR_temp = (f_disc * m_n) * ARR_temp
ARR_inter = ARR_inter - ARR_temp
ARR_inter = ARR_inter - strike_I
ARR_inter = max(ARR_inter, ARR_zeros)
ARR_pq = ARR_pq
ARR_inter = ARR_pq * ARR_inter
ca_ll = 0.0
do J = B, 0, -1
ca_ll = ca_ll + ARR_inter(J)
!print*,ARR_inter(J)
!  if(ca_ll <= tolerance) then
!    exit
!  end if
```

```
end do
!ca_ll = sum(ARR_inter)
ca_ll = (ca_ll * disc)
write(unit=1, fmt="(A, F10.5)") cal_l, ca_ll
write(unit=1, fmt="(A, F10.5)") NBIG,AA
write(unit=1, fmt="(A, F10.5)") NSMA,BB
write(unit=1, fmt="(A, F10.5)") WARX,strike
write(unit=1, fmt="(A, F10.5)") CALX,strike_1
write(unit=1, fmt="(A, F10.5)") VALS,st_price
write(unit=1, fmt="(A, F10.5)") TLON,time_l
write(unit=1, fmt="(A, F10.5)") TSHO,time_s
write(unit=1, fmt="(A, F10.5)") IRAT,RR
write(unit=1, fmt="(A, F10.5)") FRAT,RF
write(unit=1, fmt="(A, F10.5)") VOLA,ret_vol
write(unit=1, fmt="(A, F10.5)") COEF,m_n
close(unit=1)
print *,"The output file:", outfile
end program sweet_georgia_1
```

Appendix E - B/S software

We test Hoadley software with Derivagem software. Both of software gave the same results in all cases.

A. Option price

HoadleyOptions1

Returns the option price or a hedge parameter. Designed for use from a worksheet although can also be called from a VBA module

Note: You need to install the addin from the Tools/Add-ins menu

Syntax

HoadleyOptions1(calculation_type,pricing_model,option_type,strike,spot,deal_date,expiry_date,volatility,risk_free_rate,dividend_type,dividend_details,binomial_steps)

Explanation & example

Argument	Example		Remarks
calculation_type	P		P=Price,D=Delta,G=Gamma,V=Vega,T=Theta,R-Rho
pricing_model	1		1=Black-Scholes European, 2 = Binomial American, 3 = Binomial European
option_type	C		C=Call, P=Put
strike	13.00		Strike price
spot	12.50		Spot price of the underlying asset
deal_date	1/14/00		
expiry_date	3/25/00		
volatility	30%		
risk_free_rate	5.75%		Risk free interest rate
dividend_type	D		N=None, D=Discrete payments, C=Continuous yield
dividend_details	0.20	2/1/00	For discrete payments specify a range consisting of an unlimited
	0.50	3/1/00	number of amount & ex-dividend date pairs. The example on the left
			shows two dividends. The dividends can be in columns (like th example)
			or all on one row. Dividends must be in date order with no gaps.
			Dividends with an ex-date prior to the deal date or after the expiration date are ignored
			For a continuous yield specify one cell containing the dividend rate
			If there are no dividends, the dividend details parameter should equal zero
binomial_steps	50		Number of calculation steps for binomial pricing model, zero for Black-Scholes

Results:	Calc type	Result
Price	P	0.25517
Delta	D	0.28146
Gamma	G	0.21607
Vega	V	0.01757
Theta	T	-0.00419
Rho	R	0.00617

www.hoadley.net/options

Appendix E - B/S software (continues)

B. Implied Volatility

HoadleyImpliedVolatility1
Returns the implied volatility of an option

Note: You need to install the addin from the Tools/Add-ins menu.

The function calculates implied volatility using the Newton-Raphson method which is fast and accurate. If no solution can be found the function returns a value of 9999

Syntax
HoadleyImpliedVolatility1(pricing_model,option_type,strike,spot,deal_date,expiry_date,option_price,risk_free_rate,dividend_type,dividend_details,binomial_steps)

Explanation & example

Argument	Example		Remarks
pricing_model	1		1=Black-Scholes European, 2 = Binomial American, 3 = Binomial European
option_type	C		C=Call, P=Put
strike	13.00		Strike price
spot	12.50		Spot price of the underlying asset
deal_date	1/14/2000		
expiry_date	3/25/2000		
option_price	0.2552		The price of the option for which implied volatility is to be calculated
risk_free_rate	5.75%		Risk free interest rate
dividend_type	D		N=None, D=Discrete payments, C=Continuous yield
dividend_details	0.20	2/1/2000	For discrete payments specify a range consisting of an unlimited
	0.50	3/1/2000	number of amount & ex-dividend date pairs. The example on the left
			shows two dividends. The dividends can be in columns (like th example)
			or all on one row. Dividends must be in date order with no gaps
			Dividends with an ex-date prior to the deal date or after the expiration date are ignored
			For a continuous yield specify one cell containing the dividend rate
			If there are no dividends, the dividend details parameter should equal zero
binomial_steps	50		Number of calculation steps for binomial pricing model, zero for Black-Scholes
			Note: As the implied volatility calculation involves iteration, setting binomial_steps
			to a high value (eg 500) will cause calculation time to increase exponentially
			Normally a value of 50 will produce an accurate-enough result with a fast calculation time

Result: **30.0%**

www.hoadley.net/options