Performance Analysis of a Multiplexer with Priority Queues and Correlated Arrivals

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ABSTRACT

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Broadband networks are integrating different applications such as data, voice and video in a single network. The priority mechanism allows differentiating among services such that they can meet their QoS requirements. Since a switch output port may be considered as a statistical multiplexer, and the packets coming from an application are correlated, it is very important to obtain a good understanding of the statistical multiplexing with priority queues and correlated arrivals.

In this thesis, we present performance analysis of a discrete-time system with two priority queues and correlated arrivals. A packet is transmitted during a slot if there are packets available in either queue. The packets in the low-priority queue are transmitted only if the high-priority queue is empty. The arrival process to each priority queue consists of the superposition of the traffic generated by a number of independent binary Markov sources and the arrivals to the two queues are independent of each other.

The joint Probability Generating Function (PGF) of the two queue lengths and the number of sources is derived and the unknown boundary function is determined using the busy period distribution of the high-priority queue. From here, we determine closed form expressions for mean and variance of queue lengths as well as mean packet delay. Also we show the correspondence of our results with previous work by reducing our solution to the results of a multiplexer without priority in many special cases. At last we present numerical results, which show the effect of the high-priority traffic on the low-priority traffic and demonstrate the significance of the correlation on the performance of the system.

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Last but not least I would like to thank my wife Tu, Wei, for her love and patience during those struggling times when I was writing my thesis and preparing my defense.

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$a_{i,k}$	Number of On sources for type-i queue during slot k,$i = 0, 1$
$b_{i,k}$	Number of type-i packets that arrive during slot k, $i = 0, 1$
$\tilde{b}$	Duration of the busy period of the high-priority queue in number of slots
c_{ji}	Bernoulli random variable, it assumes the value of $'1'$ if the $j'th$ type-i source remains On in the next slot and $'0'$ otherwise, given that the source is On in the present slot, $i = 0, 1$
$c_i(z_i)$	PGF of c_{ji}, $i = 0, 1$
d_{ji}	Bernoulli random variable, it assumes the value of $'1'$ if the $j'th$ type-i source becomes On in the next slot and $'0'$ otherwise, given that the source is Off in the present slot, $i = 0, 1$
$d_i(z_i)$	PGF of d_{ji}, $i = 0, 1$
$\overline{d}_1$	Mean packet delay for low-priority queue
$f_{ji,k}$	Number of packets generated by the $j'th$ type-i On source during slot k, $i = 0, 1$

$f_i(z_i)$	PGF of the number of packets generated by the $j'th$ type-i On source during slot k, $i = 0, 1$
g_k	It assumes the value of '1' if $i_{0,k} = 0$, and '0' if $i_{0,k} > 0$
$i_{0,k}$	Number of packets in the high-priority queue at the end of slot k
$i_{1,k}$	Number of packets in the low-priority queue at the end of slot k
m_i	Number of sources for type-i traffic, $i = 0, 1$
$p_k(0)$	Probability that the high-priority queue is empty at the end of slot k
$q_k(i_0, y_0, i_1, y_1)$	Joint probability distribution of the number of packets in high and low-priority queues at the end of slot k, the number of On sources for high and low-priority queues during slot k
$z^*(\omega)$	Unique root of equation $z_0 = \omega\lambda_{20}^{m_0}$
$B_i(1)$	$[d_i(y_i \cdot f_i(z_i))]^{m_i}, i = 0, 1$
$B_i^n(k)$	$B_i(k)\Big\vert_{y_i=\phi_i(n)}, i = 0, 1$
$B(1)$	$B_0(1)B_1(1)$
$B(k)$	$B_0(k)B_1(k)$
$C_{1i,2i}$	$\frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i)f_i(z_i) + (\beta_i + \alpha_i f_i(z_i))}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i)f_i(z_i)}}$, $i = 0, 1$
$\overline{N_1}$	Mean high-priority queue length
$P(z_1)$	steady-state PGF of the low-priority queue length
$P_0(\omega)$	Transformation of $p_k(0)$

$Q_k(z_0, y_0, z_1, y_1)$ Joint PGF of $q_k(i_0, y_0, i_1, y_1)$

$Q_k(z_0, y_0, z_1, y_1, \omega)$ Transformation of $Q_k(z_0, y_0, z_1, y_1)$

$Q(\omega)$ Transformation of $Q_k(0, 0, 0, 0)$

$X_i(k)$ $C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k$, $i = 0, 1$

Y_i $\frac{c_i(y_i \cdot f_i(z_i))}{d_i(y_i \cdot f_i(z_i))}$, $i = 0, 1$

$(x)^+$ Operator, $\max(0, x)$

α_i Probability that a type-i source is On in the current slot, given that it was On during previous slot, $i = 0, 1$

β_i Probability that a type-i source is Off in the current slot, given that it was Off during previous slot, $i = 0, 1$

$\lambda_{1i,2i}$ $\frac{\beta_i + \alpha_i f_i(z_i) \mp \sqrt{[\beta_i + \alpha_i f_i(z_i)]^2 + 4(1 - \alpha_i - \beta_i) f_i(z_i)}}{2}$, $i = 0, 1$

$\xi(j)$ Probability of that the busy period of the high-priority queue is j slots

ρ_i Steady-state type-i traffic load, $i = 0, 1$

ρ The total system load, $\rho = \rho_0 + \rho_1$

$\Gamma(\omega)$ PGF of $\xi(j)$

Chapter 1

Introduction

Since the 20th Century, the key technology has been information gathering, processing, and distribution. Among other developments, we have seen the installation of worldwide telephone networks, the birth and unprecedented growth of the computer industry, and the launching of communication satellites [1]. The communications networks is undergoing one of the intense and dramatic changes in its history.

1.1 Communication Networks and Priority Mechanism

Over one hundred years ago, the invention of the telephone launched a revolution in communications that enabled people to communicate efficiently over distance. Nowadays, the communication networks is changing every aspect of our lives-business, entertainment, education, and more. In this section, we give a brief overview of the communication networks, especially ATM and Internet, and also we will introduce some priority mechanisms.

1.1.1 Circuit-Switching and Packet-Switching

Until early 1970's, the long-haul telecommunications network was based on circuit-switching, which was originally designed to handle voice traffic, and the majority of traffic on these networks continues to be voice. A key characteristic of circuit-switching networks is that resources within the network are dedicated to a particular call. However, as circuit-switching networks began to be used as platforms for bursty traffic such as data, two shortcomings became apparent:

- In a circuit-switching based data connection, because the data traffic is bursty, much of the time the line is idle, thus this approach is inefficient.
- A circuit-switching connection transmits data at a constant rate. Thus each of the two devices that are connected to each other must transmit and receive data at the same rate as the other, this limits the utility of the network in interconnecting a variety of host computers and terminals.

The shortcomings of circuit-switching led to the development of packet-switching technology. In packet-switching, data is transmitted in short blocks, called packets. Each packet contains a portion (or all for a short message) of the user's data plus some control information. The control information, at a minimum, includes the information that the network requires to be able to route the packet through the network and deliver it to the intended destination. At each node along the route, the packet is received, stored briefly, and passed on to the next node. This approach has a number of advantages over circuit switching.

- Bandwidth efficiency is greater, because a single node-to-node link can be dynamically shared by many packets over time. The packets are queued up and transmitted as rapidly as possible over the link.

- Two stations of different data rates can exchange packets, because each station connects to its node at its proper data rate.
- On a packet-switching network, when traffic becomes heavy, packets are still accepted with increased delivery delay.
- The traffic may be prioritized, such that the higher-priority packets will experience less delay than lower-priority packets.

However packet-switching also has some disadvantages relative to circuit-switching. For example, it incurs a transmission delay, and this delay may be variable due to processing and queueing in the node, which is called jitter. This may not be desirable for some real-time applications. Another shortcoming is that the control information in each packet reduces the communication capacity available for carrying user information.

Two switching techniques are used in contemporary packet-switching networks: datagram and virtual circuit. In the datagram approach, each packet is treated independently, so the packets with the same destination address do not all follow the same route and they may arrive out of sequence at the exit point, an example of this technique is IP-based network. In the virtual circuit approach, a pre-planned route is established before any packets are sent, all the packets between a pair of communicating parties follow the same route through the network, which is called connection oriented. Asynchronous Transfer Mode (ATM) use this technique. The difference from the datagram approach is that, with virtual circuits, the node needs not make a routing decision for each packet.

1.1.2 ATM and IP-Based network

The objective of integration of the transmission of voice, data, video led to carrying large volumes of traffic with different quality of service (QoS) requirements over networks operating at very high data rates. The types of network facilities that serve as platforms to this are the ATM and IP-based network. Next we give a brief introduction to ATM and Internet networks.

1.1.2.1 The Asynchronous Transfer Mode (ATM)

ATM is a technology that provides a single platform for the transmission of voice, video and data at specified quality of service and at speeds varying from fractional T1 (i.e. nX64 Kbps), to Gbps [2].

ATM was standardized by ITU-T in 1987, it is based on packet-switching and is connection oriented (virtual circuit technique). ATM has been designed to inherit best features of circuit and packet-switching. An ATM packet, known as a cell, is a small fixed-size packet with a payload of 48 bytes and a 5 byte header. ATM does not provide any error detection operations on the user payload inside the cell, and also provides no retransmission services, and only few operations are performed on the small header. The small, fixed size cells allow fast and efficient multiplexing of sources with different QoS constraints. ATM has built-in mechanisms that permit it to provide different quality of service to different types of traffic. Due to its low queueing delay and delay variance, ATM technology networks are well suited for multimedia applications, and can handle any kind of traffic from circuit-switched voice to bursty video streams at any speed.

At ATM switches, capacity is shared by grouped sets of connections to save bandwidth through multiplexing efficiency. If a new connection can be admitted without any adverse effect on the pre-established connections then it is admitted. If not, the request is turned down and the user has to place a new request with easier to meet characteristics. There are five categories of services that characterize connections in ATM networks, which are CBR (constant bit rate), rt-VBR (real-time variable bit-rate), nrt-VBR (no-real-time variable bit-rate), ABR (available bit-rate and UBR (unspecified bit-rate).

1.1.2.2 Internet and Differentiated Services

The growth in the Internet is the dominating factor in the development of new protocols and techniques for data communications and computer networking. Internet is a collection of inter-connected networks running TCP/IP protocols, each machine on the Internet has a IP address. The glue that holds the Internet together is the TCP/IP reference model and TCP/IP protocol stack, which makes universal service possible and can run on the Internet.

Traditionally, the Internet had four main applications: email, news, remote login and file transfer. Up until the early 1990's, the Internet was largely populated by academic, government and industrial researchers. One new application, the WWW (World Wide Web, or short WEB) changed all that and brought millions of new, no-academic users to the Internet. The WWW made it possible for a site to set up a number of pages of information containing text, picture, sound and even video.

Internet was designed to provide best-effort service for delivery of data packets and to run virtually across any network transmission media and system platform. The increasing popularity of IP has shifted the paradigm from "IP over everything," to "everything over IP." In order to manage the multitude of applications such as streaming video, voice over IP, e-commerce, ERP, and others, a network requires quality of service (QoS) in addition to best-effort service. Different applications have varying needs for delay, delay variation (jitter), bandwidth, packet loss, and availability. These parameters form the basis of QoS.

The IP network should be designed to provide the requisite QoS to applications.

For example, VoIP requires very low jitter, and a one-way delay in the order of 100 milliseconds, and guaranteed bandwidth in the range of 8Kbps -> 64Kbps, dependent on the codec used. On the other hand, a file transfer application, based on FTP, doesn't suffer from jitter, while packet loss will be highly detrimental to the throughput.

To facilitate true end-to-end QoS on an IP-network, the Internet Engineering Task Force (IETF) has defined the model of Differentiated Services (DiffServ). DiffServ works on the provisioned-QoS model where network elements are set up to service multiple classes of traffic, with varying QoS requirements. The model can be driven off a policy base, using the CoPS (Common Open Policy Server) protocol.

DiffServ addresses the clear need for relatively simple and coarse methods of categorizing traffic into different classes, also called class of service (CoS), and applying QoS parameters to those classes. To accomplish this, packets are first divided into classes by marking the type of service (ToS) byte in the IP header. A 6-bit bit-pattern (called the Differentiated Services Code Point [DSCP]) in the IPv4 ToS Octet or the IPv6 Traffic Class Octet is used to this end.

Before the IETF defined IP (Layer3) QoS methods, the ITU-T (International Union for Telecommunications, Telecommunications), the Asynchronous Transfer Mode (ATM) Forum, and the Frame-Relay Forum (FRF) had already arrived at standards to do Layer2 QoS in ATM and Frame-Relay networks. The ATM standards define a very rich QoS infrastructure by supporting traffic contracts, many adjustable QoS knobs (such as Peak Cell Rate [PCR], Minimum Cell Rate [MCR], and so on), signaling, and Connection Admission Control (CAC).

1.1.3 Priority Mechanism

In both types of networks, ATM and the IP-based network, dramatic changes are taking place. In the case of Internet, it was designed for bursty data traffic for computer networks and provided best-effort service, now the volume of traffic carried has increased enormously and the character of that traffic has expanded to include multimedia such as voice and video and real-time traffic, DiffServ model has been developed and implemented for providing different QoS to different types of traffic. On the other hand, in the case of ATM, its high data rate has attracted not only voice and video traffic, but also increasing bursty data traffic that used to be based on TCP/IP.

Both in ATM and in IP-based network, different applications need to receive different qualities of services. An important network device for differentiating among services is priority mechanism.

At ATM switches, in order to implement effectively complex scheduling algorithms, so that different connections could be served according to their requested QoS, a number of schedulers have been proposed and implemented. Below, we discuss some of these priority scheduling algorithms. More details will be found in [2] and [4].

Static Priorities: Each output buffer at a ATM switch is organized into four different queues, these queues can be assigned static priorities, which dictate the order in which they are served. These priorities are called static because they do not change over time and they are not effected by the occupancy levels of the queues.

Early Deadline Fist (EDF) Algorithm: In this algorithm, each cell assigned a deadline upon arrival at the buffer. This deadline indicates the time by which the cell should depart from the buffer. The scheduler serves the cells according to their deadlines, so that the one with the earliest deadline gets served first. Using this scheme, cells belonging to delay-sensitive applications, such as voice and video, can be served first by assigning them deadlines closer to their arrival times.

Weighted Round-Robin Scheduler: Each output buffer of an ATM switch is organized into a number of queues, the scheduler serves one cell from each queue in a round robin fashion. The queues are numbered from 1 to M, and they are served sequentially. This sequential servicing of the queues continues until the $M'th$ queue is served. Weighted round-robin scheduling can be used to serve a different number of cells from each queue.

On the Internet, an important component of an implementation is the queueing discipline used at the Internet routers. Traditionally routers used a FIFO (First In First Out) queueing discipline, also known as FCFS (First Come First Serve). One of the drawbacks of FIFO discipline is that no special treatment is given to packets from flows that are of higher priority or are more delay sensitive. Then some other queueing disciplines were proposed and implemented. In the following, some examples are given, more can be found in [4].

Bit-round Fair Queueing (BRFQ): BRFQ is designed to emulate a bit-by-bit round-robin discipline, BRFQ is implemented by computing virtual starting and finishing times on the fly as if PS (processor sharing) were running, whenever a packet finishes transmission, the next packet sent is the one with the smallest value of virtual finishing time. When multiple packets arrive at the same time, according to this discipline, priority is given to short packets.

Generalized Processor Sharing (GPS): BRFQ is not able to provide different amounts of the capacity to different flows. To support QoS, differential allocation capability is needed. GPS is a weighted bit-by-bit round-robin discipline. With GPS, each flow is assigned a weight that determines how many bits are transmitted from that queue during each round. GPS provides a means of responding to different service requests. Equally important, GPS provides a way of guaranteeing that delays for a well-behaved flow which does not exceed some bound.

GPS discipline deals with each flow (or connection) from a traffic source separately, assigning a weight to each connection. But the Internet has to handle huge number of connections at the same time, sometimes it becomes very difficult to implement this algorithm at Internet routers. The simple Priority Mechanism is easier to implement than GPS, which groups connections into limited number of classes, and each class is assigned a priority. All the packets coming from the same class of connections will be put in one queue, and a lower priority queue is served only if higher-priority queues are empty.

1.2 Performance Analysis Issues

Activity in the area of broadband networks has been expanding at a rapid rate, emergence of new network protocols and the integration of packetized voice, video, images and computer-generated data traffic, each with its own multi-objective QoS, requires the development of rather sophisticated models to carry out accurate design and performance evaluation. In this section we will examine the well-used statistical multiplexing model and the Binary Markov On/Off traffic Model.

1.2.1 Statistical Multiplexing

We can view a network as a collection of nodes that are connected by a set of transmission links. Packets are routed from a source node to a destination node, following the store and forward principle. When a packet reaches the nearby node, it is temporarily stored there until the transmission line to the next node becomes available. For this purpose, at each node, switching elements are installed to route the incoming packets to the appropriate output link. For those packets which cannot be transmitted immediately, buffer space has been provided at each switching element.

In a network, hundreds of sources may access a single link, such as a trunk line, then statistical multiplexing is performed on the incoming packets to achieve high bandwidth gain, and buffering is required to absorb traffic fluctuations when the instantaneous rate of the aggregate incoming streams exceeds the limited capacity of the outgoing link. Inside a switching element, packets from different input ports may go to the same output port, again these packets will queue up in buffer and be transmitted according some queueing discipline. Then, an output port can be also considered as a multiplexer.

Therefore, in order to implement efficient admission and flow control strategies to maintain satisfying Quality of Service (QoS), one needs to acquire a very good understanding of the statistical multiplexing of the aggregate traffic generated by multimedia sources (with possibly different characteristics).

From a modeling point of view, the choice of a packet-switching technique leads naturally to the choice of a slotted time axis with synchronized message transmission. In addition, the multiplexing of voice, data and video sources on high capacity links gives rise to a very interesting discrete-time queueing problem, at the multiplexer's level, which involves a deterministic server and a special correlated discrete-time arrival process. Most often, the quantities of interest are the buffer occupancy (number of packets stored in the system, or equivalently, queue length) and the packet delay (or waiting time) experienced by the packets in the buffer.

1.2.2 The Binary Markov On/Off Traffic Model

As mentioned in previous, the current networks must support various communication services, such as data, voice and video, each having different traffic characteristics. In addition, the performance analysis of multiplexers has introduced a significant change in the way uncorrelated traffic (such as Poisson and Bernoulli) dominated the traditional performance evaluation methods. In fact, when dealing with the traffic generated by multimedia sources, the uncorrelated random arrival process assumption becomes inadequate because of the dependency which characterizes the packets stream. For these reasons, traffic characterization has been a major field of research during the past years due to its direct impact on the performance evaluation.

There have been many traffic models proposed in the literature for characterizing individual data traffic sources or a superposition of a multiple sources. For instance, we have Poisson arrival process (continuous time case), geometric inter-arrival process (discrete time case) for data traffic, Interrupted Poisson Process (IPP) for voice traffic and Markov Modulated Poisson Process (MMPP) for data, voice and video traffic. A good review on traffic modeling can be found in [9].

Among those traffic models that have been used for different types of sources, the most versatile one is the binary Markov On/Off model. In this model, each source is characterized by On (corresponding to active bursts) and Off (corresponding to silent duration) periods, which appear in turn. During the silent periods, no packets are generated. Even though, more complicated models, such as the three state model [10], have been proposed for more accurate modeling, analysis of a queueing model with these processes may be very complex and may not be tractable, therefore, these models have rarely been applied in mathematical analysis. On the other hand, the binary Markov On/Off model is very popular and has been often used for the modeling of traffic. For instance a binary Markov model has been successfully applied for modeling the voice source (see [11] and [12]). In addition, in [13], a video source is modeled as a birth-death process, which consists of the superpositions of a number of independent and identical On/Off mini-sources.

Because its versatility and flexibility, the binary Markov On/Off model has been chosen as the basic model for the characterization of input traffic sources. Hence this thesis will be mainly concerned with the analysis of statistical multiplexing with correlated arrivals process which consists of the superposition of many identical independent traffic streams generated by binary Markov sources.

1.3 Previous work on Priority Queues Problem

Typically, networks operate on a best-effort delivery basis, which means that all data traffic has equal priority and an equal chance of being delivered in a prompt manner. However, as more and more different applications are running on the network, priority mechanism is used to meet different QoS requirements. For example, when voice is introduced into a network, it becomes critical that priority be given to the voice packets to insure expected quality of voice calls.

Usually this problem is modeled as priority queues, a simple example is as follows: two $M/G/1$ queues share one transmission line, each queue is fed by a source where source 1 has priority over source 2. When source 1 wishes to transmit, it is given priority irrespective of the backlog from source 2. Then from the viewpoint of queue 2, the server (transmission line) appears to be subject to periodic breakdowns coinciding with the initiation of the transmission of queue 1. In this case, the server is unavailable to the queue 2 for a time interval which is equal to what is called the *busy period* of the queue 1. From the viewpoint of queue 1, the queue 2 doesn't seem to exist, as the transmission line is always available.

The priority queues problem has been considered by several researchers in the literature. Now we give a brief review of the previous work on this problem.

In [14], the PGFs (probability generating functions) of queue lengths have been determined for a model with two-priority queues. The number of arrivals during a slot to each priority queue is determined by one two-state Markov source. The results has also been generalized to a system with $N-$priority queues. This work has the limitation that it assumes that both high and low-priority arrivals depend on the same two-state Markov source and therefore they are dependent on each other. This is not an appropriate assumption, since in real networks, different priority traffics coming from different applications, the sources are independent of each other.

In [15], a model has been analyzed using matrix geometric technique with independent high and low-priority arrivals. The high-priority traffic arrives according to a Markov chain with multiple states. However, low-priority arrivals are assumed to have independence from one slot to the next one. The queue length distributions and the waiting time distributions are provided.

Reference [16] studies a system with high and low-priority traffic arriving according to a different Markov chain with several states. This system has also been studied using matrix geometric technique and it assumes a finite queue for the high-priority traffic and infinite queue for the low priority traffic. A matrix geometric solution for the state probability of the system is provided allowing computation of performance metric of high and low-priority classes. But the presented numerical results have been limited to systems with number of two-state Markov sources no more than eight, because computation will be too difficult to handle larger number of sources or Markov sources states.

Reference [17] studies a system with two priority classes, it is assumed that each class arrives by packet-trains with a fixed size of m packets. If a leading packet of a message arrives in the current slot, the remaining $m - 1$ packets arrive consecutively in the next $m - 1$ slots. The number of leading packets arriving in each slot from each class are i.i.d.(independent and identically distributed) with respective PGFs $A(z)$ and $B(z)$. the joint PGF of the queue lengths and the waiting time distributions have been obtained for each class.

In this thesis, we model a priority-based system with a single server and two priority queues. We assume that priority queue-i is fed by type-i sources with $i = 0, 1$ denoting high and low priority queues and their corresponding sources respectively. The low-priority queue is served only if the high-priority queue is empty. Each type of sources consists of a number of two-state independent Markov sources. Thus the arrival packets to each queue is correlated in time but the arrival process to the two queues are independent of each other. We derive the joint PGF of the queue lengths using the transform techniques and this result may be used to determine closed form expressions for mean queue lengths as well as higher moments. Mean and variance of the queue length may be calculated very easily for any number of two-state sources. These results extend the previous works on this problem, since we have a more general arrival process than those assumed in [14], [15] and [17]. While the arrival process in [16] is more general, however, their results are more numerical because they cannot give closed-form expression and the state-space requirements limit the number of sources that may be handled, as we can see in their numerical results, the number of sources is less than 10.

1.4 Outline of the Rest of Thesis

The objective of this thesis is to offer a simple and efficient approach for the performance analysis of a multiplexer with priority queues and correlated arrivals. The rest of this thesis will be organized as following:

Chapter 2 describes the model of our system under consideration, as well as the definition of the main notations. On the basis of our queueing model, we apply the embedded Markov chain analysis and derive the functional equation that relates the PGF of the system between two consecutive slots.

Chapter 3 presents the performance analysis of the queueing system under consideration. A joint steady-state PGF of the high and low priority queue lengths and the number of On sources will be presented. First, the functional equation will be converted into a mathematically more tractable form. Secondly the busy period distribution of the high priority queue will be used to determine the unknown boundary function $Q_k(0, 0, z_l, y_l)$, then we can perform a transformation on the functional equation which makes it easier to determine the solution. Thirdly, we apply the final value theorem to determine the joint steady-state PGF, then we present some discussions of the solution, the joint PGF, which gives correspondence with the historical results. At last we determine the marginal PGF of the low and high-priority queues as well as the mean queue length, variance and mean packet delay for the low-priority queue.

Chapter 4 studies the characteristics of the priority-based queueing system fed by correlated arrivals by presenting and discussing the numerical results of the queueing behavior. As the behavior of the high-priority queue is not affected by the low-priority queue, in this chapter we present results mainly for the low-priority queue.

Chapter 5 summarizes the research work and concludes the results, in addition, the main contributions of this work are given.

In the appendix, some necessary background knowledge and further proofs of this work are presented. Appendix A determines the mean busy period for the high-priority queue. Appendix B presents the final value theorem and its proof. Appendix C shows that $|\lambda_{1i}\lambda_{2i}| < 1$. Appendix D gives an alternative method to determine $Q(0, 0, z_1, y_1)$. Appendix E shows that the solution, the joint PGF, satisfies the functional equation, which provides further proof for the correctness of the results of this thesis. Appendix F introduces the concept of fixed point, as well as the theorem and its proof.

Chapter 2

System Modeling

In this chapter, we describe the model of a multiplexer with priority queues and correlated arrivals under consideration, as well as define the main notations to be used in the thesis. On the basis of our queueing model, we apply the embedded Markov chain analysis and derive the functional equation that relates PGF of the system between two consecutive slots.

2.1 Queueing Model

We model the system as a discrete-time queueing system with a single deterministic server and two priority queues . Each queue is fed by a number of sources, and has infinite waiting room (see Figure 2.1). The time axis is divided into intervals of equal lengths (slots) and a packet is transmitted at the slot boundaries. It is assumed that a packet which arrives during a slot cannot be transmitted during the slot that it arrives, and that a packet transmission time is equal to one slot.

We assume that priority queue i is fed by type-i sources where $i = 0$ denotes the high priority queue and $i = 1$ denotes the low priority queue. The server serves the low priority queue (type-1) only if the high priority queue is empty. Then from the viewpoint of high priority queue, the server is always available and the low priority queue does not exist.

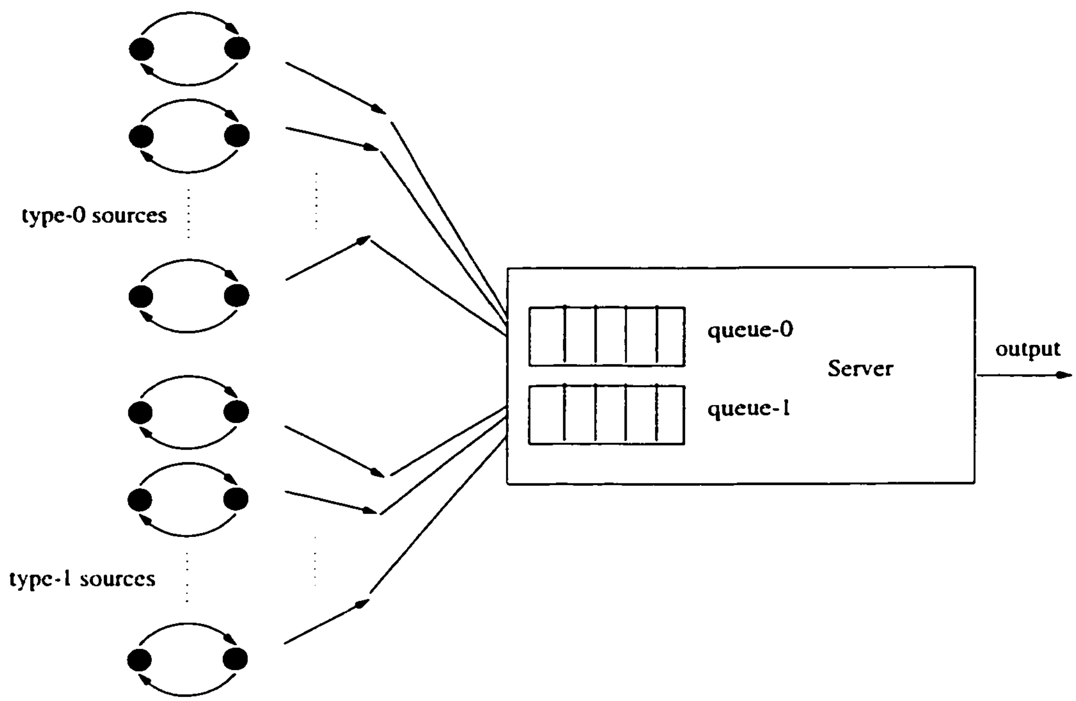

Figure 2.1: Two priority queue model of a multiplexer

Let $i_{0,k}$ be the number of packets in high-priority queue at the end of slot k and $i_{1,k}$ be the number of packets in low-priority queue at the end of slot k. Let $b_{i,k+1}$ be the type-i packets generated during the $(k+1)$'th slot . The evolution of the high priority queue length is given by:

$$i_{0,k+1} = (i_{0,k} - 1)^+ + b_{0,k+1} \tag{2.1}$$

where

$$(i_{0,k} - 1)^+ = \begin{cases} i_{0,k} - 1 & if\ i_{0,k} - 1 \geq 0 \\ 0 & if\ i_{0,k} - 1 < 0 \end{cases}$$

The evolution of the low priority queue length is given by:

$$i_{1,k+1} = (i_{1,k} - g_k)^+ + b_{1,k+1} \tag{2.2}$$

where

$$(i_{1,k} - g_k)^+ = \begin{cases} i_{1,k} - g_k & if\ i_{1,k} - g_k \geq 0 \\ 0 & if\ i_{1,k} - g_k < 0 \end{cases}$$

and

$$g_k = \begin{cases} 1 & if\ i_{0,k} = 0 \\ 0 & if\ i_{0,k} > 0 \end{cases} \tag{2.3}$$

2.2 Source Model Description and Notation

As stated earlier on, a binary Markov model provides a good approximation in modeling arrival traffic to the networks. Because of its simplicity and capability to capture some of the correlation behavior, binary Markov sources have been widely used as basic building blocks to model broadband traffic, including voice and video.

We assume type-i sources consist of m_i mutually independent and identical binary Markov sources, where m_0 is the number of high-priority sources and m_1 is the number of low-priority sources. Each source alternates between On and Off states (see Figure 2.2). State transitions of the sources are synchronized to occur at the slots' boundaries according to a two-state aperiodic and irreducible discrete-time Markov chain. The probability of a transition from an On state to an Off state is $1-\alpha_i$, while a transition from an Off state to an On state occurs with probability $1 - \beta_i$, where $i = 0, 1$ corresponding to the high ($i = 0$) or low priority ($i = 1$) sources respectively(see Figure 2.2). Thus the number of slots that a source spends in On or Off state is geometrically distributed with parameter

α_i or β_i, respectively.

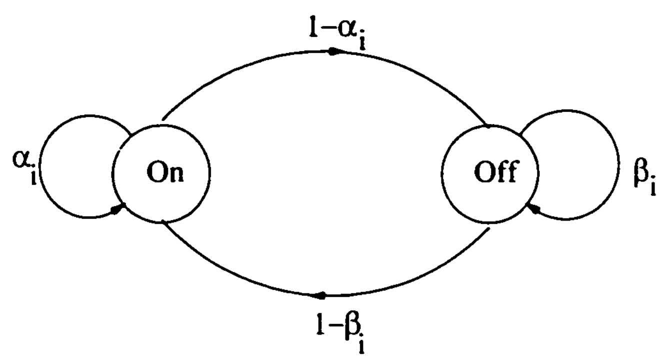

Figure 2.2: State transition diagram of type-i sources

When α_i and β_i are high, the generated packets have tendency to arrive in clusters, alternatively when α_i and β_i are low, the packet arrivals are more dispersed in time. Also, the sum of a source's On and Off probabilities, $\alpha_i + \beta_i$, is an index of the correlation of the arrivals. When $\alpha_i + \beta_i = 1$, we have Bernoulli arrivals and the packets generated by On sources are independent from one slot to the next; while the lower or higher is the sum, $\alpha_i + \beta_i$, there more correlation are between the arrivals in two consecutive slots.

We assume that an On source generates at least one packet during a slot while an Off source will generate no packets during a slot (see Figure 2.3). Let $f_i(z_i)$ denote the PGF of the number of packets generated by an On source of type-i during a slot. Let $f_{ji,k}$ be the number of packets generated by the j 'th type-i On source during slot k ; and assume that $f_{ji,k}$ is i.i.d., then PGF $f_i(z_i) = E[z_i^{f_{ji,k}}]$.

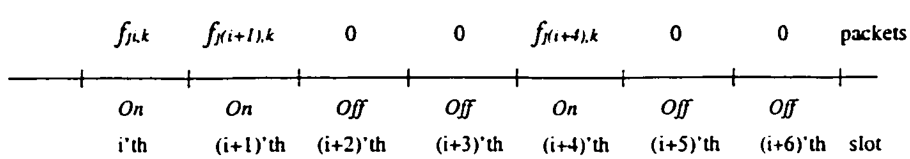

Figure 2.3: The number of packets generated by $j'th$ source of type-i

Let's define the following Bernoulli random variables:

$$c_{ji} = \begin{cases} 1 & a\ type - i\ source\ remains\ On\ in\ the\ next\ slot \\ & given\ it\ s\ On\ in\ the\ present\ slot \\ 0 & otherwise \end{cases}$$

$$d_{ji} = \begin{cases} 1 & a\ type - i\ source\ remains\ On\ in\ the\ next\ slot \\ & given\ it\ is\ Off\ in\ the\ present\ slot \\ 0 & otherwise \end{cases}$$

The PGFs of c_{ji} , d_{ji} are given by

$$c_i(z_i) = 1 - \alpha_i + \alpha_i z_i \tag{2.4}$$

$$d_i(z_i) = \beta_i + (1 - \beta_i) z_i \tag{2.5}$$

Define $a_{i,k}$ as the number of type-i sources On during k'th slot. Since each source generates at least one packet per slot during an On state with PGF $f_i(z_i)$, then $f_i(0) = 0$. This also implies that if the variable $i_{i,k}$ is zero then $a_{i,k}$ must also be zero. Thus if the priority queue-i is empty at the end of slot k, all the type-i sources must be in Off state during slot k since a packet cannot be transmitted during the slot that it has arrived in. Then we have the following relations:

$$b_{i,k} = \sum_{j=1}^{a_{i,k}} f_{ji,k} \tag{2.6}$$

$$a_{i,k=1} = \sum_{j=1}^{a_{i,k}} c_{ji} + \sum_{j=1}^{m_i - a_{i,k}} d_{ji} \tag{2.7}$$

In this thesis, we assume that a summation is empty if its upper limit is smaller than the lower limit. The first term in equation 2.7 represents the number of sources which were in On state during slot k and remain in On state in slot $k+1$, while the second term represents the number of sources which were in Off state during slot k and change to On state during the next slot.

2.3 The Embedded Markov Chain Modeling

In this section, the queueing model under consideration will be formulated as a discrete-time four-dimensional Markov chain, and a joint probability generating function (PGF) $Q_k(z_0, y_0, z_1, y_1)$ with respect to the discrete time k is used to describe the queue model. Following the embedded Markov chain modeling, we will derive the the functional equation that relates the PGF of the system between two consecutive slots.

The state of the system is defined by $(i_{0,k}, a_{0,k}, i_{1,k}, a_{1,k})$, where $i_{i,k}$ denotes the length of priority queue-i at the end of slot k, and $a_{i,k}$ denotes the number of On sources of type-i during slot k. Let us define $Q_k(z_0, y_0, z_1, y_1)$ as the joint PGF of $i_{0,k}, a_{0,k}, i_{1,k}$ and $a_{1,k}$, then,

$$\begin{aligned} Q_k(z_0, y_0, z_1, y_1) &= E[z_0^{i_{0,k}} y_0^{a_{0,k}} z_1^{i_{1,k}} y_1^{a_{1,k}}] \\ &= \sum_{i_0=0}^{\infty} \sum_{j_0=0}^{m_0} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} z_0^{i_0} y_0^{j_0} z_1^{i_1} y_1^{j_1} q_k(i_0, j_0, i_1, j_1) \end{aligned} \tag{2.8}$$

where $q_k(i_0, j_0, i_1, j_1) = Prob(i_{0,k} = i_0, a_{0,k} = j_0, i_{1,k} = i_1, a_{1,k} = j_1)$.

Then $Q_{k+1}(z_0, y_0, z_1, y_1)$ is given by

$$Q_{k+1}(z_0, y_0, z_1, y_1) = E[z_0^{i_{0,k+1}} y_0^{a_{0,k+1}} z_1^{i_{1,k+1}} y_1^{a_{1,k+1}}]$$

In the following we will derive the relation between $Q_{k+1}(z_0, y_0, z_1, y_1)$ and $Q_k(z_0, y_0, z_1, y_1)$.

Substituting for $i_{0,k+1}, i_{1,k+1}$from equation 2.1 and 2.2 in $Q_{k+1}(z_0, y_0, z_1, y_1)$ yields:

$$Q_{k+1}(z_0, y_0, z_1, y_1) = E[z_0^{(i_{0,k}-1)^+ + b_{0,k+1}} y_0^{a_{0,k+1}} z_1^{(i_{1,k}-g_k)^+ + b_{1,k+1}} y_1^{a_{1,k+1}}]$$

First, let us condition on $i_{0,k}$, $i_{1,k}$, $a_{0,k+1}$, $a_{1,k+1}$, and substitute for $b_{0,k+1}$, $b_{1,k+1}$ from equation 2.6, then

$$E\left[z_0^{(i_{0,k}-1)^+ + b_{0,k+1}} y_0^{a_{0,k+1}} z_1^{(i_{1,k}-g_k)^+ + b_{1,k+1}} y_1^{a_{1,k+1}} \middle| i_{0,k}, i_{1,k}, a_{0,k+1}, a_{1,k+1}\right]$$

$$= z_0^{(i_{0,k}-1)^+} y_0^{a_{0,k+1}} z_1^{(i_{1,k}-g_k)^+} y_1^{a_{1,k+1}} E\left[z_0^{\sum_{j=1}^{a_{0,k+1}} f_{j0,k}} z_1^{\sum_{j=1}^{a_{1,k+1}} f_{j1,k}} \middle| i_{0,k}, i_{1,k}, a_{0,k+1}, a_{1,k+1}\right]$$

$$= z_0^{(i_{0,k}-1)^+} y_0^{a_{0,k+1}} z_1^{(i_{1,k}-g_k)^+} y_1^{a_{1,k+1}} [f_0(z_0)]^{a_{0,k+1}} [f_1(z_1)]^{a_{1,k+1}} \quad (2.9)$$

Thus

$$Q_{k+1}(z_0, y_0, z_1, y_1) = E\left[z_0^{(i_{0,k}-1)^+} y_0^{a_{0,k+1}} z_1^{(i_{1,k}-g_k)^+} y_1^{a_{1,k+1}} [f_0(z_0)]^{a_{0,k+1}} [f_1(z_1)]^{a_{1,k+1}}\right]$$

Substituting for $a_{0,k+1}$, $a_{1,k+1}$ from equation 2.7 in the above equation, we have

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= E\left[z_0^{(i_{0,k}-1)^+} (y_0 f_0(z_0))^{\sum_{j=1}^{a_{0,k}} c_{j0} + \sum_{j=1}^{m_0 - a_{0,k}} d_{j0}} z_1^{(i_{1,k}-g_k)^+} (y_1 f_1(z_1))^{\sum_{j=1}^{a_{1,k}} c_{j1} + \sum_{j=1}^{m_1 - a_{1,k}} d_{j1}}\right] \quad (2.10)$$

Let us condition again on $i_{0,k}$, $i_{1,k}$, $a_{0,k}$, $a_{1,k}$, and substituting from equation 2.4 and 2.5, then equation 2.10 becomes

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= z_0^{(i_{0,k}-1)^+}[c_0(y_0 f_0(z_0))]^{a_{0,k}}[d_0(y_0 \cdot f_0(z_0))]^{m_0-a_{0,k}} z_1^{(i_{1,k}-g_k)^+}[c_1(y_1 f_1(z_1))]^{a_{1,k}}[d_1(y_1 f_1(z_1))]^{m_1-a_{1,k}}$$

$$= z_0^{(i_{0,k}-1)^+}\left(\frac{c_0(y_0 f_0(z_0))}{d_0(y_0 f_0(z_0))}\right)^{a_{0,k}}[d_0(y_0 f_0(z_0))]^{m_0} z_1^{(i_{1,k}-g_k)^+}\left(\frac{c_1(y_1 f_1(z_1))}{d_1(y_1 f_1(z_1))}\right)^{a_{1,k}}[d_1(y_1 f_1(z_1))]^{m_1} \tag{2.11}$$

Let us define

$$Y_i = \frac{c_i(y_i f_i(z_i))}{d_i(y_i \cdot f_i(z_i))} \tag{2.12}$$

$$B_i(1) = [d_i(y_i f_i(z_i))]^{m_i} \tag{2.13}$$

$$B(1) = \prod_{i=0}^{1} B_i(1) \tag{2.14}$$

where $i = 0, 1$, then equation 2.11 may be written as

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= E[z_0^{(i_{0,k}-1)^+} Y_0^{a_{0,k}} B_0(1) z_1^{(i_{1,k}-g_k)^+} Y_1^{a_{1,k}} B_1(1)]$$

$$= B(1) E[z_0^{(i_{0,k}-1)^+} Y_0^{a_{0,k}} z_1^{(i_{1,k}-g_k)^+} Y_1^{a_{1,k}}]$$

From equation 2.8 the above equation becomes:

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1) \sum_{i_0=0}^{\infty} \sum_{j_0=0}^{m_0} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} z_0^{(i_0-1)^+} Y_0^{j_0} z_1^{(i_1-g_k)^+} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)$$

removing the $(\ )^+$ operator for $(i_0 - 1)^+$ in the above equation results in:

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\sum_{i_0=1}^{\infty}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{(i_0-1)} Y_0^{j_0} z_1^{i_1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)\right.$$

$$\left.+\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{(i_1-1)^+} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)\right\}$$

In the first multiple summation, letting the initial value of i_0 be 0 instead of 1, and removing the $(\ )^+$ operator for $(i_1 - 1)^+$ in the the second multiple summation, the above equation becomes:

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0}\sum_{i_0=0}^{\infty}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{i_1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)\right.$$

$$-\frac{1}{z_0}\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{i_1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)$$

$$+\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=1}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{i_1-1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)$$

$$\left.+\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{i_1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)\right\}$$

letting the initial value of i_1 be 0 instead of 1 for the third multiple summation in above equation, we have

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0}\sum_{i_0=0}^{\infty}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0} Y_0^{j_0} z_1^{i_1} Y_1^{j_1} q_k(i_0, j_0, i_1, j_1)\right.$$

$$-\frac{1}{z_0}\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$+\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1-1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$-\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1-1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$+\sum_{i_0=0}^{0}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)\Bigg\} \quad (2.15)$$

As stated before, an On source generates at least one packet during a slot, if $j_i > 0$ then $i_i > 0$; therefore if $i_i = 0$ then j_i must be $zero$, where $i = 0, 1$. Thus we have $p_k(i_0 = 0, j_0, i_1, j_1) = 0$ for all the values of $j_0 > 0$, and $p_k(i_0 = 0, j_0, i_1 = 0, j_1) = 0$ for all the values of $j_0 > 0$ and $j_1 > 0$. Therefore equation 2.15 becomes

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\Bigg\{\frac{1}{z_0}\sum_{i_0=0}^{\infty}\sum_{j_0=0}^{m_0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$-\frac{1}{z_0}\sum_{i_0=0}^{0}\sum_{j_0=0}^{0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$+\frac{1}{z_1}\sum_{i_0=0}^{0}\sum_{j_0=0}^{0}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$-\frac{1}{z_1}\sum_{i_0=0}^{0}\sum_{j_0=0}^{0}\sum_{i_1=0}^{0}\sum_{j_1=0}^{0} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)$$

$$+\sum_{i_0=0}^{0}\sum_{j_0=0}^{0}\sum_{i_1=0}^{0}\sum_{j_1=0}^{0} z_0^{i_0}Y_0^{j_0}z_1^{i_1}Y_1^{j_1}q_k(i_0,j_0,i_1,j_1)\Bigg\}$$

Next by recognizing the above summations from 2.8, finally we have

$$Q_{k+1}(z_0, y_0, z_1, y_1) = B(1)\left\{\frac{1}{z_0}Q_k(z_0, Y_0, z_1, Y_1) - \frac{1}{z_0}Q_k(0, 0, z_1, Y_1)\right.$$

$$\left. + \frac{1}{z_1}Q_k(0, 0, z_1, Y_1) - \frac{1}{z_1}Q_k(0, 0, 0, 0) + Q_k(0, 0, 0, 0)\right\}$$

or equivalently

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0}Q_k(z_0, Y_0, z_1, Y_1) + \frac{z_0 - z_1}{z_0 z_1}Q_k(0, 0, z_1, Y_1) + \frac{z_1 - 1}{z_1}Q_k(0, 0, 0, 0)\right\} \quad (2.16)$$

This is the functional equation that relates the PGF of the system between two consecutive slots. In the next chapter, we will present a solution for the above equation.

Chapter 3

Performance Analysis

The objective of this performance analysis is to determine the joint steady-state PGF of the high and low priority queue length and the number of On sources. From this result we will determine the marginal PGF of each priority queue length as well as the mean queue length, variance and mean packet delay.

First, we will convert the functional equation into a mathematically more tractable form. Secondly we will use the busy period distribution of the high priority queue to determine the unknown boundary function $Q_k(0, 0, z_1, y_1)$. Thirdly, we apply the final value theorem to determine the joint steady-state PGF, after this we present several special cases of our solution, which shows the derived result is correct. At last we present the marginal PGFs for the low and high-priority queues as well as the mean queue length, variance and mean packet delay for the low-priority queue.

3.1 Transforming the Functional Equation into a New Form

In chapter 2 we have obtained the functional equation describing the system, but this equation is not mathematically tractable because on the right hand side of the equation 2.16, we have $Q_k(z_0, Y_0, z_1, Y_1)$ instead of $Q_k(z_0, y_0, z_1, y_1)$. In this section we will transform the equation2.16 into another form which lends itself easier to a solution.

3.1.1 Preliminary Results

At first we present some preliminary results that will be used later. Let us define the following:

$$X_i(k+1) = X_i(1)\,[X_i(k)\,|_{y_i=Y_i}] \quad with \quad X_i(0) = 1,\ X_i(1) = \beta_i + (1-\beta_i)y_i f_i(z_i) \quad (3.1)$$

$$U_i(k+1) = X_i(1)\,[U_i(k)\,|_{y_i=Y_i}] \quad with \quad U_i(0) = y_i,\ U_i(1) = 1 - \alpha_i + \alpha_i y_i f_i(z_i) \quad (3.2)$$

where $i = 0, 1$ denoting the high or low priority queues respectively, then $X_i(k)$ and $U_i(k)$ have the recurrence relationships given below (see [22]):

$$X_i(k) = [\beta_i + \alpha_i y_i f_i(z_i)]\,X_i(k-1) + (1 - \alpha_i - \beta_i) f_i(z_i) X_i(k-2), \qquad k \geq 2 \quad (3.3)$$

$$U_i(k) = [\beta_i + \alpha_i y_i f_i(z_i)]\,U_i(k-1) + (1 - \alpha_i - \beta_i) f_i(z_i) U_i(k-2), \qquad k \geq 2 \quad (3.4)$$

Let us define

$$\phi_i(k) = \frac{U_i(k)}{X_i(k)} \qquad with \quad \phi_i(0) = y_i, \quad \phi_i(1) = Y_i \quad (3.5)$$

$$B_i(k) = [X_i(k)]^{m_i} \quad (3.6)$$

$$B(k) = \prod_{i=0}^{1} B_i(k) \tag{3.7}$$

with $i = 0, 1$; we can see that $B_i(k)\,|_{k=1} = [\beta_i + (1-\beta_i)y_i f_i(z_i)]^{m_i}$ which is same as what we have defined in 2.13. It is easy to show that

$$B_i(k+1) = B_i(1)\,[B_i(k)\,|_{y_i=Y_i}] \tag{3.8}$$

if we define $B_i^n(k) = B_i(k)\Big|_{y_i=\phi_i(n)}$ and $B^n(k) = B(k)\Big|_{y_i=\phi_i(n)}$, then 3.8 becomes

$$B_i(k+1) = B_i(1)B_i^1(k), \quad B(k+1) = B(1)B^1(k) \tag{3.9}$$

and from 3.9, it is easy to show that

$$B_i^n(k) = \frac{B_i(k+n)}{B_i(n)} \tag{3.10}$$

Next, we will present the solution of the homogeneous difference equations 3.3 and 3.4, we note that they have the following characteristic equation:

$$\lambda_i^2 - [\beta_i + \alpha_i f_i(z_i)]\,\lambda_i - (1-\alpha_i-\beta_i)f_i(z_i) = 0 \qquad i = 0, 1$$

The roots of the above equation are given by:,

$$\lambda_{1i,2i} = \frac{\beta_i + \alpha_i f_i(z_i) \mp \sqrt{[\beta_i + \alpha_i f_i(z_i)]^2 + 4(1-\alpha_i-\beta_i)f_i(z_i)}}{2} \tag{3.11}$$

The solution of the difference equations 3.3 and 3.4 are given in [22],

$$U_i(k) = D_{1i}\lambda_{1i}^k + D_{2i}\lambda_{2i}^k \tag{3.12}$$

$$X_i(k) = C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k \tag{3.13}$$

where D_{1i}, D_{2i}, C_{1i}, C_{2i}, are constants which satisfy the initial conditions given in 3.1 and 3.2:

$$X_i(0) = C_{1i} + C_{2i} = 1;$$

$$X_i(1) = C_{1i}\lambda_{1i} + C_{2i}\lambda_{1i} = \beta_i + (1-\beta_i)y_i f_i(z_i);$$

$$U_i(0) = D_{1i} + D_{2i} = y_i;$$

$$U_i(1) = D_{1i}\lambda_{1i} + D_{2i}\lambda_{1i} = 1 - \alpha_i + \alpha_i y_i f_i(z_i);$$

From the above equations we can solve for these constants,

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i)f_i(z_i) + (\beta_i + \alpha_i f_i(z_i))}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i)f_i(z_i)}} \tag{3.14}$$

$$D_{1i,2i} = \frac{y_i}{2} \mp \frac{2(1-\alpha_i+\alpha_i y_i f_i(z_i)) - (\beta_i + \alpha_i f_i(z_i))y_i}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i)f_i(z_i)}} \tag{3.15}$$

in the above expressions λ_{1i} , C_{1i} , D_{1i} are taken with the negative sign and λ_{2i}, C_{2i}, D_{2i} with the positive sign.

Also $B_i(k)$ is determined, which is given by

$$B_i(k) = [X_i(k)]^{m_i} = \left[C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{1i}^k\right]^{m_i} \tag{3.16}$$

Substituting 1 for z_i and y_i in 3.11, 3.14, 3.15, 3.16 and 3.5 we have the following results that will be needed later on:

$$\lambda_{1i}\,|_{z_i=1} = \alpha_i + \beta_i - 1, \qquad \lambda_{2i}\,|_{z_i=1} = 1 \tag{3.17}$$

$$C_{1i}\,|_{z_i=1,y_i=1} = 0, \qquad C_{2i}\,|_{z_i=1,y_i=1} = 1 \tag{3.18}$$

$$D_{1i}\,|_{z_i=1,y_i=1} = 0, \qquad D_{2i}\,|_{z_i=1,y_i=1} = 1 \tag{3.19}$$

$$B_i(k)\,|_{z_i=1,y_i=1} = 1 \tag{3.20}$$

$$\phi_i(k)\,|_{z_i=1,y_i=1} = 1 \tag{3.21}$$

3.1.2 New Form of the Functional Equation

Now with the available preliminary results, we are ready to transform the original functional equation 2.16 into a new form which is solvable. Since we have modeled the system as a Markov chain, the steady-state of the system will be independent of the initial conditions. We are only interested in the steady-state behavior of the system, therefore we will choose a zero initial condition to simplify our analysis. We assume that initially the queues are empty and all the sources are in the Off state, then we have:

$$p_0(0,0,0,0) = 1, \quad Q_0(z_0, y_0, z_1, y_1) = 1,$$

$$Q_0(0,0,z_1,y_1) = 1, \quad Q_0(0,0,0,0) = 1$$

Next we will show that the functional equation 2.16 describing the system can be written as follows:

$$Q_k(z_0, y_0, z_1, y_1) = \frac{1}{z_0^{k-1}} B(k) + \frac{z_0 - z_1}{z_1} \sum_{j=1}^{k-1} \frac{B(j)}{z_0^j} Q_{k-j}(0,0,z_1,\phi_1(j))$$

$$+ \frac{z_1 - 1}{z_1} \sum_{j=1}^{k-1} \frac{B(j)}{z_0^{j-1}} Q_{k-j}(0,0,0,0) \qquad k \geq 1 \tag{3.22}$$

as we have stated earlier on, in this thesis, if the upper limit of a summation is less than the lower limit, we assume the summation is empty.

Proof:

The proof of the above result is given through induction. By expanding $Q_{k+1}(z_0, y_0, z_1, y_1)$ in the functional equation 2.16 for the first few values of k , we have the following:

If $k = 0$, the functional equation 2.16 becomes

$$Q_1(z_0, y_0, z_1, y_1) = B(1)$$

If $k = 1$, substituting from 3.5 and 3.9, in equation 2.16, we have

$$Q_2(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0}B^1(1) - \frac{1}{z_0}Q_1(0,0,z_1,\phi_1(1)) + \frac{1}{z_1}Q_1(0,0,z_1,\phi_1(1)) + \frac{z_1 - 1}{z_1}Q_1(0,0,0,0)\right\}$$

$$= \frac{1}{z_0}B(2) + \frac{z_0 - z_1}{z_0 z_1}B(j)Q_1(0,0,z_1,\phi_1(1)) + \frac{z_1 - 1}{z_1}B(1)Q_1(0,0,0,0)$$

If $k = 2$, substituting from 3.5 and 3.9, equation 2.16 becomes

$$Q_3(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0^2}B^1(2) + \frac{z_0 - z_1}{z_0^2 z_1}B^1(1)Q_1(0,0,z_1,\phi_1(2))\right.$$

$$\left. + \frac{z_0 - z_1}{z_0 z_1}B^1(1)Q_1(0,0,0,0) + \frac{z_0 - z_1}{z_0 z_1}Q_2(0,0,z_1,\phi_1(2)) + \frac{z_1 - 1}{z_1}Q_2(0,0,0,0)\right\}$$

$$= \frac{1}{z_0^2}B(3) + \frac{z_0 - z_1}{z_0^2 z_1}B(2)Q_1(0,0,z_1,\phi_1(2)) + \frac{z_0 - z_1}{z_0 z_1}B(2)Q_1(0,0,0,0)$$

$$+ \frac{z_0 - z_1}{z_0 z_1}B(1)Q_2(0,0,z_1,\phi_1(1)) + \frac{z_1 - 1}{z_1}B(1)Q_2(0,0,0,0)$$

$$= \frac{1}{z_0^2}B(3) + \frac{z_0 - z_1}{z_1}\sum_{j=1}^{2}\frac{B(j)}{z_0^j}Q_{3-j}(0,0,z_1,\phi_1(j)) + \frac{z_1 - 1}{z_1}\sum_{j=1}^{2}\frac{B(j)}{z_0^{j-1}}Q_{3-j}(0,0,0,0)$$

We can see that equation 3.22 is true for $k = 1,\ 2$. Now we assume that it is true for order k, then we will show it will also be true for order $k+1$. Substituting 3.22 into 2.16 yields:

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= B(1)\left\{\frac{1}{z_0^k}B^1(k) + \frac{z_0 - z_1}{z_1}\sum_{j=1}^{k-1}\frac{B^1(j)}{z_0^{j+1}}Q_{k-j}(0,0,z_1,\phi_1(j+1))\right.$$

$$\left.\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\frac{B^1(j)}{z_0^j}Q_{k-j}(0,0,0,0) + \frac{z_0 - z_1}{z_0 z_1}Q_k(0,0,z_1,\phi_1(1)) + \frac{z_1 - 1}{z_1}Q_k(0,0,0,0)\right\}$$

substituting for $B(1)B^1(j)$ from equation 3.9,

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= \frac{1}{z_0^k}B(k+1) + \frac{z_0 - z_1}{z_1}\sum_{j=1}^{k-1}\frac{B(j+1)}{z_0^{j+1}}Q_{k-j}(0,0,z_1,\phi_1(j+1))$$

$$+\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\frac{B(j+1)}{z_0^j}Q_{k-j}(0,0,0,0) + \frac{z_0 - z_1}{z_0 z_1}B(1)Q_k(0,0,z_1,\phi_1(1))$$

$$+\frac{z_1 - 1}{z_1}B(1)Q_k(0,0,0,0)$$

or equivalently

$$Q_{k+1}(z_0, y_0, z_1, y_1)$$

$$= \frac{1}{z_0^k}B(k+1) + \frac{z_0 - z_1}{z_1}\sum_{j=2}^{k}\frac{B(j)}{z_0^j}Q_{k+1-j}(0,0,z_1,\phi_1(j))$$

$$+\frac{z_1-1}{z_1}\sum_{j=2}^{k}\frac{B(j)}{z_0^{j-1}}Q_{k+1-j}(0,0,0,0)+\frac{z_0-z_1}{z_0 z_1}B(1)Q_k(0,0,z_1,\phi_1(1))$$

$$+\frac{z_1-1}{z_1}B(1)Q_k(0,0,0,0)$$

Finally, we have

$$Q_{k+1}(z_0,y_0,z_1,y_1)=\frac{1}{z_0^k}B(k+1)+\frac{z_0-z_1}{z_1}\sum_{j=1}^{k}\frac{B(j)}{z_0^{j}}Q_{k+1-j}(0,0,z_1,\phi_1(j))$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\frac{B(j)}{z_0^{j-1}}Q_{k+1-j}(0,0,0,0), \qquad k\geq 0 \tag{3.23}$$

the above shows that 3.22 is also true for $k+1$. and this completes the proof of equation 3.22.

3.2 Determination of the Unknown Boundary Function $Q_k(0,0,z_1,y_1)$

In equations 2.16 and 3.22 we have the unknown boundary function $Q_k(0,0,z_1,y_1)$, which is the PGF of the low priority traffic when high priority queue is empty and high-priority sources are in Off state. In this section, at first we will introduce the busy period distribution of high priority queue and determine the PGF of the busy period, $\Gamma(\omega)$; then we will use the busy period distribution to determine $Q_k(0,0,z_1,y_1)$; at last we will derive the expressions of $\Phi(\omega)$, which will be needed later on.

3.2.1 Busy Period of High-Priority Queue

In this section we will determine the busy period distribution for the high-priority queue, Let us define

$\tilde{b}$ =duration of the busy period of the high-priority queue in number of slots;

$\xi(j)$ =Pr($\tilde{b} = j$ slots) , $j = 0, 1, 2, 3......$

In the above definition, we assume that a zero length busy period corresponds to the high-priority queue being empty for two consecutive slots. We note that according to this definition, the high-priority queue is busy during k consecutive slots; but before and after the k consecutive slots, high-priority queue is idle (see Figure 3.1).

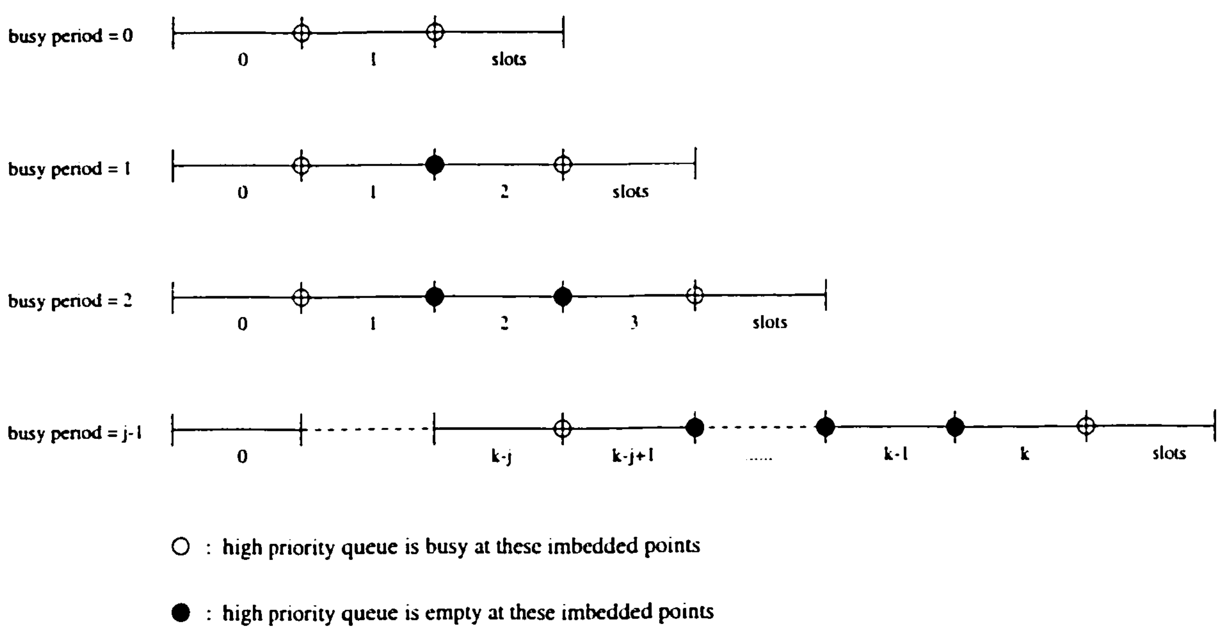

Figure 3.1: Structure of a busy period

Let $p_k(0)$ denote the probability that the high-priority queue is empty at the end of slot k , then it may expressed in terms of mutually exclusive events. Next, we show this for the first few values of $p_k(0)$.

* $p_0(0)$

Under the assumption that the queue is initially empty, we have:

$$p_0(0) = 1;$$

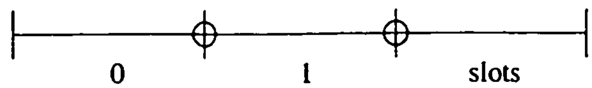

Figure 3.2: Queue is empty at the end of 1'st slot

* $p_1(0)$

If at the end of 1'st slot, the queue is empty, that is $p_1(0)$, then from figure 3.2, we see the preceding busy period duration must be 0. Thus we have:

$$p_1(0) = p_0(0)\xi(0);$$

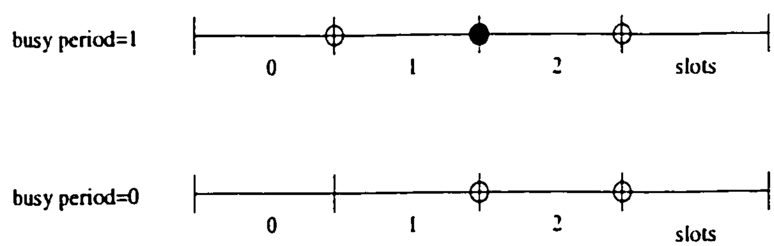

Figure 3.3: Queue is empty at the end of 2'nd slot

* $p_2(0)$

If at the end of 2'nd slot, the queue is empty, which is $p_2(0)$, then from figure3.3, we see $p_2(0)$ may be expressed in terms of two mutually exclusive events. Thus we have:

$$p_2(0) = p_0(0)\xi(1) + p_1(0)\xi(0);$$

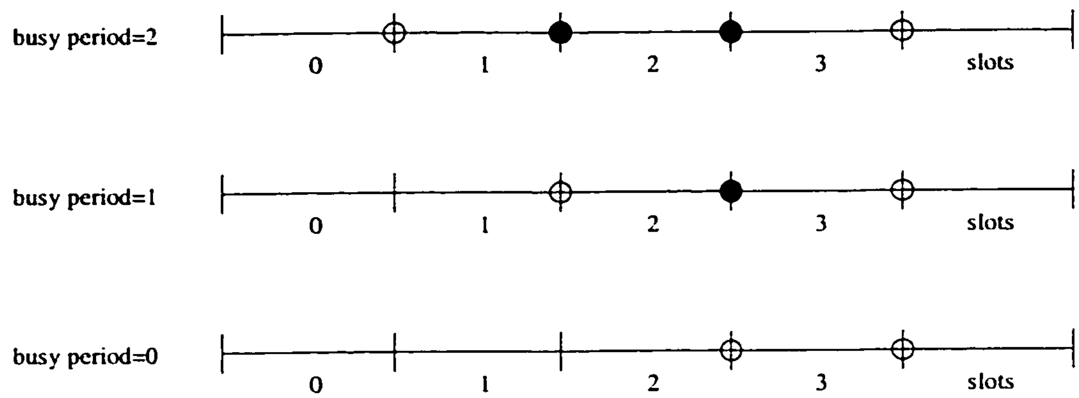

Figure 3.4: Queue is empty at the end of 3'rd slot

* $p_3(0)$

If at the end of 3'rd slot, the queue is empty, which is $p_3(0)$, then from figure 3.4, we see $p_3(0)$ may be expressed in terms of three mutually exclusive events. Thus we have,

$$p_3(0) = p_0(0)\xi(2) + p_1(0)\xi(1) + p_2(0)\xi(0);$$

Following a similar analysis, we have,

$$p_4(0) = p_0(0)\xi(3) + p_1(0)\xi(2) + p_2(0)\xi(1) + p_3(0)\xi(0);$$

$$\vdots$$

Finally, from above we can see that, $p_k(0) = \sum_{j=1}^{k}$Pr(high-priority queue is empty at the end of $(k-j)'th$ slot which is followed by a busy period of $(j-1)$ slots) . Then we have the expression:

$$p_k(0) = \sum_{j=1}^{k} \xi(j-1)p_{k-j}(0) \tag{3.24}$$

3.2.2 Determination of the PGF of Busy Period, $\Gamma(\omega)$

In subsection 3.2.1, we have defined busy period probability distribution $\xi(j)$ for the high-priority queue. Now, we will derive the PGF of $\xi(j)$.

Let us define the PGF of $\xi(j)$ as:

$$\Gamma(\omega) = \sum_{j=0}^{\infty} \xi(j)\omega^j \tag{3.25}$$

and define the transformation of $p_k(0)$ as

$$P_0(\omega) = \sum_{k=0}^{\infty} p_k(0)\omega^k \tag{3.26}$$

Substituting equation 3.24 into 3.26 yields:

$$P_0(\omega) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \xi(j-1)p_{k-j}(0)\omega^k + p_0(0)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{k} \xi(j-1)p_{k-j}(0)\omega^k + 1$$

Interchanging the order of summations, we have

$$P_0(\omega) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \xi(j-1)p_{k-j}(0)\omega^k + 1$$

Let $m = k - j$, then $k = m + j$ and

$$P_0(\omega) = \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \xi(j-1)p_m(0)\omega^{m+j} + 1$$

from the definition 3.26, above becomes:

$$P_0(\omega) = P_0(\omega)\sum_{j=1}^{\infty}\xi(j-1)\omega^j + 1,$$

letting $n = j - 1$, then $j = n + 1$ and

$$P_0(\omega) = P_0(\omega)\sum_{n=0}^{\infty}\xi(n)\omega^{n+1} + 1,$$

from definition 3.25 we have

$$P_0(\omega) = P_0(\omega)\Gamma(\omega)\omega + 1,$$

thus

$$\Gamma(\omega) = \frac{P_0(\omega) - 1}{P_0(\omega)\omega} = \frac{1}{\omega}\left(1 - \frac{1}{P_0(\omega)}\right) \quad (3.27)$$

But $P_0(\omega)$ has already been determined and from [18], the transform $P_0(\omega)$ for the high-priority queue is given as follows:

$$P_0(\omega) = \frac{1}{1 - z^*(\omega)} \quad (3.28)$$

where $z^*(\omega)$ is the unique root of equation $z_0 = \omega\lambda_{20}^{m_0}$ within the unit circle. Therefore

$$\Gamma(\omega) = \frac{z^*(\omega)}{\omega} \quad (3.29)$$

In the following, we will show that $\Gamma(\omega)\,|_{\omega=1} = 1$. First we know $z^*(\omega)$ is the unique root of equation $z_0 = \omega\lambda_{20}^{m_0}$, if $\omega \to 1$, the equation becomes $z_0 = \lambda_{20}^{m_0}$. Because $\lambda_{20}\,|_{z_0=1} = 1$ (see 3.17), $z_0 = 1$ is the root of equation $z_0 = \lambda_{20}^{m_0}$, but we know this root is unique, then $z_0 = 1$ is the unique root of equation $z_0 = \omega\lambda_{20}^{m_0}$ at $\omega = 1$, therefore $z^*(\omega)\mid_{\omega=1} = 1$. Finally we have

$$\Gamma(\omega)\mid_{\omega=1} = \frac{z^*(\omega)}{\omega}\mid_{\omega=1} = 1 \tag{3.30}$$

this completes the proof. The result $\Gamma(\omega)\mid_{\omega=1} = 1$ further shows that $\Gamma(\omega)$ is a PGF.

Let $\bar{b}$ be the mean busy period of the high-priority queue, then $\bar{b} = \Gamma'(\omega)\mid_{\omega=1}$. In the following, we will determine $\bar{b}$. We may show that (see Appendix A equation A.6),

$$\frac{\partial z^*(\omega)}{\partial\omega}\mid_{\omega=1} = \frac{1}{1-\rho_0}$$

From 3.29, we have

$$z^*(\omega) = \omega\Gamma(\omega)$$

Taking derivative of bother sides in the above equation with respect to ω results in

$$\frac{\partial z^*(\omega)}{\partial\omega} = \Gamma(\omega) + \omega\Gamma'(\omega)$$

substituting 1 for ω, we have

$$\frac{\partial z^*(\omega)}{\partial\omega}\mid_{\omega=1} = \Gamma(\omega)\mid_{\omega=1} + \Gamma'(\omega)\mid_{\omega=1}$$

substituting 1 for $\Gamma(\omega)\mid_{\omega=1}$ and $\frac{1}{1-\rho_0}$ for $\frac{dz^*(\omega)}{d\omega}\mid_{\omega=1}$ we have

$$\Gamma'(\omega)\mid_{\omega=1} = \frac{\rho_0}{1-\rho_0}$$

thus the mean busy period of the high-priority queue is

$$\bar{b} = \frac{\rho_0}{1 - \rho_0}$$

3.2.3 Expression of $Q_k(0, 0, z_1, y_1)$

After having introduced the busy period of the high-priority queue, we are able to determine the expression for $Q_k(0, 0, z_1, y_1)$. The analysis will be similar with that of the busy period.

Clearly, during the busy period of the high-priority queue, service cannot be given to the low-priority queue and the arrivals to the low-priority queue will accumulate. The busy period includes only the slots during which the high-priority packets are served, thus the arrival slot of the packets initiating a busy period is not a part of the busy period. Assuming that the last time when the high-priority queue is empty is at the end of $(k - j)'th$ slot, then a low-priority packet may be transmitted during $(k - j + 1)'th$ slot if the low priority queue is not empty at the end of $(k - j)'th$ slot. Therefore it is possible that we express $Q_k(0, 0, z_1, y_1)$ in terms of mutually exclusive events as the following.

* $Q_0(0, 0, z_1, y_1)$

At the beginning, under the assumption that the queue is empty initially, we have

$$Q_0(0, 0, z_1, y_1) = 1$$

* $Q_1(0, 0, z_1, y_1)$

If at the end of the $1'st$ slot, there are no high priority packets and high priority On sources in the system, which corresponds to $Q_1(0, 0, z_1, y_1)$, then during the $1'st$ slot there must be no high priority packets in the system which means the busy period of high-priority queue is 0, that is $\xi(0)$, (see Figure 3.2). Thus we have the following

$$Q_1(0,0,z_1,y_1) = \xi(0)B_1(1)Q_0(0,0,z_1,\phi_1(1)) = \xi(0)B_1(1)$$

where $B_1(1)$ accounts for low-priority packets accumulation during a slot.

* $Q_2(0,0,z_1,y_1)$

If at the end of the $2'nd$ slot, there are no high-priority packets in the system and all high-priority sources in are in Off state, which corresponds to $Q_2(0,0,z_1,y_1)$ as well as $p_2(0)$, then there are two mutually exclusive events (see Figure 3.3), i): During the $1'st$ slot the system is busy for the high-priority queue; ii): during $2'nd$ slot there are no high priority packets generated in the system and at the end of the $1'st$ slot, there are no high-priority packets in the system and all high-priority sources are in Off state. Thus we have

$$Q_2(0,0,z_1,y_1)$$

$$= \xi(1)B_1(2)Q_0(0,0,z_l,\phi_1(2))$$
$$+\xi(0)B_1(1)\left\{\frac{1}{z_1}\left[Q_1(0,0,z_1,\phi_1(1)) - Q_1(0,0,0,0)\right] + Q_1(0,0,0,0)\right\}$$

where $B_1(2)$ accounts for low-priority packets accumulation during two slots, and $B_1(1)$ accounts for low-priority packets accumulation during one slot; z_1 in the denominator accounts for a low-priority packet transmission during slot 1 if the corresponding queue is non-empty.

* $Q_3(0,0,z_1,y_1)$

If at the end of the $3'rd$ slot, there are no high-priority packets and all high-priority sources are in Off state, which corresponds to $Q_3(0,0,z_1,y_1)$, then there are three mutually exclusive events (see Figure 3.4). i): both during the $1'st$, and $2'nd$ slots, the system is busy for the high-priority queue; ii): during $2'nd$ slot the system is busy for the high-priority queue, and, at the end of the $1'st$ slot there are no high priority-packets in the system and all high-priority sources are in Off state; iii): during $3'rd$ slot there are no high-priority packets generated in the system and at the end of the $2'nd$ slot, there are no high-priority packets and all high-priority sources are in Off state. Thus we have

$$
\begin{aligned}
&Q_3(0,0,z_1,y_1) \\
&= \xi(2)B_1(3)Q_0(0,0,z_1,\phi_1(3)) \\
&+\xi(1)B_1(2)\left\{\frac{1}{z_1}[Q_1(0,0,z_1,\phi_1(2)) - Q_1(0,0,0,0)] + Q_1(0,0,0,0)\right\} \\
&+\xi(0)B_1(1)\left\{\frac{1}{z_1}[Q_2(0,0,z_1,\phi_1(1)) - Q_2(0,0,0,0)] + Q_2(0,0,0,0)\right\}
\end{aligned}
$$

where the meaning of $B_1(3)$, $B_1(2)$, $B_1(1)$, and z_1 is the same as explained before.

Following the same logic it is easy to find that

$$
\begin{aligned}
&Q_k(0,0,z_1,y_1) \\
&= \xi(k-1)B_1(k)Q_0(0,0,z_1,\phi_1(k)) \\
&\quad +\frac{1}{z_1}\sum_{j=1}^{k-1}\xi(j-1)B_1(j)\left[Q_{k-j}(0,0,z_1,\phi_1(j)) - Q_{k-j}(0,0,0,0)\right] \\
&\quad +\sum_{j=1}^{k-1}\xi(j-1)B_1(j)Q_{k-j}(0,0,0,0)
\end{aligned}
$$

combining the similar terms in the two summations above, we have

$$Q_k(0,0,z_1,y_1)$$

$$= \xi(k-1)B_1(k) + \frac{1}{z_1}\sum_{j=1}^{k-1}\xi(j-1)B_1(j)Q_{k-j}(0,0,z_1,\phi_1(j))$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k-1}\xi(j-1)B_1(j)Q_{k-j}(0,0,0,0) \quad (3.31)$$

after change of the subscriptions, equation 3.31 can be written as

$$Q_k(0,0,z_1,y_1)$$

$$= \xi(k-1)B_1(k) + \frac{1}{z_1}\sum_{j=1}^{k-1}\xi(k-j-1)B_1(k-j)Q_j(0,0,z_1,\phi_1(k-j))$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k-1}\xi(k-j-1)B_1(k-j)Q_j(0,0,0,0) \quad (3.32)$$

Next we will transform $Q_k(0,0,z_1,y_1)$ into a new form. At first let us define the followings:

$$C^n(k) = \xi(k-1)B_1^n(k) \quad (3.33)$$

$$I_{k-1}^n = \frac{z_1-1}{z_1}\sum_{j=1}^{k-j}C^n(k-j)Q_j(0,0,0,0) \quad (3.34)$$

Substituting 3.33, 3.34 into 3.32 gives us

$$Q_k(0,0,z_1,y_1) = C^0(k) + \frac{1}{z_1}\sum_{j=1}^{k-1}C^0(k-j)Q_j(0,0,z_1,\phi_1(k-j)) + I_{k-1}^0 \quad (3.35)$$

Next, we will present $Q_k(0,0,z_1,y_1)$ for the first few values of k.

* $Q_1(0,0,z_1,y_1)$

From 3.35 for $k = 1$, we have

$$Q_1(0,0,z_1,y_1) = C^0(1)$$

* $Q_2(0,0,z_1,y_1)$

From 3.35 for $k = 2$, we have

$$Q_2(0,0,z_1,y_1) = C^0(2) + \frac{1}{z_1}C^0(1)Q_1(0,0,z_1,\phi_1(1)) + I_1^0$$

$$= C^0(2) + \frac{1}{z_1}C^0(1)C^1(1) + I_1^0$$

* $Q_3(0,0,z_1,y_1)$

From 3.35 for $k = 3$, we have

$$Q_3(0,0,z_1,y_1) = C^0(3) + \frac{1}{z_1}\sum_{j=1}^{2} C^0(3-j)Q_j(0,0,z_1,\phi_1(3-j)) + I_2^0$$

$$= C^0(3) + \frac{1}{z_1}C^0(2)\left[C^2(1)\right] + \frac{1}{z_1}C^0(1)\left[C^1(2) + \frac{1}{z_1}C^1(1)C^2(1) + I_1^1\right] + I_2^0$$

$$= C^0(3) + \frac{1}{z_1}C^0(2)C^2(1) + \frac{1}{z_1}C^0(1)C^1(2) + \frac{1}{z_1^2}C^0(1)C^1(1)C^2(1) + \frac{1}{z_1}C(1)I_1^1 + I_2^0$$

$$= C^0(3) + \left[\frac{1}{z_1}C^0(1)\right]C^1(2) + \left[\frac{1}{z_1}C^0(2) + \frac{1}{z_1^2}C^0(1)C^1(1)\right]C^2(1) + \frac{1}{z_1}C^0(1)I_1^1 + I_2^0$$

* $Q_4(0,0,z_1,y_1)$

From 3.35 for $k = 4$, we have

$$Q_4(0,0,z_1,y_1) = C^0(4)+\frac{1}{z_1}\sum_{j=1}^{3} C^0(4-j)Q_j(0,0,z_1,\phi_1(4-j))+I_3^0$$

$$= C^0(4)+\frac{1}{z_1}C^0(3)Q_1(0,0,z_1,\phi_1(3))+\frac{1}{z_1}C^0(2)Q_2(0,0,z_1,\phi_1(2))+\frac{1}{z_1}C^0(1)Q_1(0,0,z_1,\phi_1(1))+I_2^0$$

$$= C^0(4)+\frac{1}{z_1}C^0(3)C^3(1)+\frac{1}{z_1}C^0(2)\left[C^2(2)+\frac{1}{z_1}C^2(1)C^3(1)+I_1^2\right]$$
$$+\frac{1}{z_1}C^0(1)\left\{C^1(3)+\left[\frac{1}{z_1}C^1(1)\right]C^2(2)+\left[\frac{1}{z_1}C^1(2)+\frac{1}{z_1^2}C^1(1)C^2(1)\right]\right.$$
$$\left.+\frac{1}{z_1}C^1(1)I_1^2+I_2^1\right\}+I_2^0$$

$$= C^0(4)+\left[\frac{1}{z_1}C^0(1)\right]C^1(3)+\left[\frac{1}{z_1}C^0(2)+\frac{1}{z_1^2}C^0(1)C^1(1)\right]C^2(2)$$
$$+\left\{\frac{1}{z_1}C^0(3)+\frac{1}{z_1^2}C^0(2)C^2(1)+\frac{1}{z_1^2}C^0(1)C^1(2)+\frac{1}{z_1^3}C^0(1)C^1(1)C^2(1)\right\}C^3(1)$$
$$+I_3^0+\left[\frac{1}{z_1}C^0(1)\right]I_2^1+\left[\frac{1}{z_1}C^0(2)+\frac{1}{z_1^2}C^0(1)C^1(1)\right]I_1^2$$

From the above, we can find that $Q_k(0,0,z_1,y_1)$ can be expressed as:

$$Q_k(0,0,z_1,y_1) = \sum_{i=1}^{k} a_{k-i}C^{k-i}(i) + \sum_{i=1}^{k-1} a_{k-i-1}I_i^{k-i-1}, \qquad k \geq 1 \tag{3.36}$$

where a_i are the following:

$$a_0 = 1;$$

$$a_1 = \frac{1}{z_1}C^0(1)$$
$$= \frac{1}{z_1}C^0(1)a_0;$$

$$a_2 = \frac{1}{z_1}C^0(2)+\frac{1}{z_1^2}C^0(1)C^1(1)$$
$$= \frac{1}{z_1}C^0(2)a_0+\frac{1}{z_1}C^1(1)a_1;$$

$$\begin{aligned}
a_3 &= \frac{1}{z_1}C^0(3) + \frac{1}{z_1^2}C^0(2)C^2(1) + \frac{1}{z_1^2}C^0(1)C^1(2) + \frac{1}{z_1^3}C^0(1)C^1(1)C^2(1) \\
&= \frac{1}{z_1}C^0(3) + \frac{1}{z_1^2}C^0(1)C^1(2) + \frac{1}{z_1}C^2(1)\left[\frac{1}{z_1}C^0(2) + \frac{1}{z_1^2}C^0(1)C^1(1)\right] \\
&= \frac{1}{z_1}C^0(3)a_0 + \frac{1}{z_1}C^1(2)a_1 + \frac{1}{z_1}C^2(1)a_2
\end{aligned}$$

From the above, we can see a_i may be written as:

$$a_i = \frac{1}{z_1}\sum_{j=0}^{i-1} a_j C^j(i-j), \qquad i \geq 1 \tag{3.37}$$

Finally, a_i may be expressed as :

$$a_i = \sum_{\bar{r}=0}^{\bar{r'}(r_0)} \prod_{n=1}^{r_0} \frac{1}{z_1^{\varepsilon(\bar{r})}} C^{r_n}(r_{n-1} - r_n)\,|_{r_0=i}\,. \qquad i \geq 1 \tag{3.38}$$

where

$$\sum_{\bar{r}=0}^{\bar{r'}(r_0)} = \sum_{r_1=0}^{r_0-1}\sum_{r_2=0}^{r_1-1}\sum_{r_3=0}^{r_2-1}\cdots\sum_{r_{r_0}=0}^{r_{r_0-1}-1}$$

and

$$\varepsilon(\bar{r}) = 1 + \sum_{j=1}^{r_0-1} u_j \qquad with \quad u_j = \begin{cases} 0 & if \quad r_j = 0 \\ 1 & if \quad i_j > 0 \end{cases} \qquad 1 \leq j \leq r_0 \tag{3.39}$$

we note that if $u_{j_1} = 0$, then $u_{j_2} = 0$ for $\forall j_2 > j_1$.

Next, substituting 3.34 and 3.38 into 3.36 yields:

$$Q_k(0,0,z_1,y_1)$$

$$= \sum_{i=1}^{k} \sum_{\bar{r}=0}^{\bar{r}\cdot(k-i)} \prod_{n=1}^{k-i} \frac{1}{z_1^{\varepsilon(\bar{r})}} C^{k-i}(i) C^{r_n}(r_{n-1} - r_n)$$

$$+\frac{z_1-1}{z_1}\sum_{i=1}^{k-1}\sum_{h=1}^{i}\sum_{\bar{r}=0}^{\bar{r}\cdot(k-i-1)}\prod_{n=1}^{k-i-1}\frac{1}{z_1^{\varepsilon(\bar{r})}}C^{k-1-i}(i+1-h)C^{r_n}(r_{n-1}-r_n)Q_h(0,0,0,0) \quad (3.40)$$

Substituting equation 3.33 into equation 3.40 yields:

$$Q_k(0,0,z_1,y_1)$$

$$=\sum_{i=1}^{k}\sum_{\bar{r}=0}^{\bar{r}\cdot(k-i)}\prod_{n=1}^{k-i}\frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(i-1)B_1^{(k-i)}(i)\xi(r_{n-1}-r_n-1)B_1^{r_n}(r_{n-1}-r_n)$$

$$+\frac{z_1-1}{z_1}\sum_{i=1}^{k-1}\sum_{h=1}^{i}\sum_{\bar{r}=0}^{\bar{r}\cdot(k-i-1)}\prod_{n=1}^{k-i-1}\frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(r_{n-1}-r_n-1)B_1^{r_n}(r_{n-1}-r_n)$$

$$\cdot\xi(i-h)B_1^{(k-i-1)}(i+1-h)Q_h(0,0,0,0)$$

substituting equation 3.10 into the above equation, we have

$$Q_k(0,0,z_1,y_1)$$

$$=\sum_{i=1}^{k}\sum_{\bar{r}=0}^{\bar{r}\cdot(k-i)}\prod_{n=1}^{k-i}\frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(i-1)\frac{B_1(k)}{B_1(k-i)}\xi(r_{n-1}-r_n-1)\frac{B_1(r_{n-1})}{B_1(r_n)}$$

$$+\frac{z_1-1}{z_1}\sum_{i=1}^{k-1}\sum_{h=1}^{i}\sum_{\bar{r}=0}^{\bar{r}\cdot(k-i-1)}\prod_{n=1}^{k-i-1}\frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(r_{n-1}-r_n-1)\frac{B_1(r_{n-1})}{B_1(r_n)}$$

$$\cdot\xi(i-h)\frac{B_1(k-h)}{B_1(k-i-1)}Q_h(0,0,0,0)$$

In the first multiple summation, $B_1(i_{k-i}) = B_1(0) = 1$; in the second multiple summation, $B_1(i_{k-i-1}) = B_1(0) = 1$. Thus all the factors in the denominators in the above equation will be cancelled out by the factors in the numerators, and we have:

$$Q_k(0,0,z_1,y_1)$$

$$= \sum_{i=1}^{k} \sum_{\bar{r}=0}^{\bar{r}'(k-i)} \prod_{n=1}^{k-i} \frac{1}{z_1^{\varepsilon(\bar{r})}} \xi(i-1) B_1(k) \xi(r_{n-1} - r_n - 1)$$

$$+ \frac{z_1 - 1}{z_1} \sum_{i=1}^{k-1} \sum_{h=1}^{i} \sum_{\bar{r}=0}^{\bar{r}'(k-i-1)} \prod_{n=1}^{k-i-1} \frac{1}{z_1^{\varepsilon(\bar{r})}} \xi(r_{n-1} - r_n - 1)$$

$$\cdot \xi(i-h) B_1(k-h) Q_h(0,0,0,0) \tag{3.41}$$

Finally, the equation 3.41 may be expressed as:

$$Q_k(0,0,z_1,y_1) = \sum_{i=1}^{k} \varphi(k-i) \xi(i-1) B_1(k)$$

$$+ \frac{z_1 - 1}{z_1} \sum_{i=1}^{k-1} \sum_{h=1}^{i} \varphi(k-i-1) \xi(i-h) B_1(k-h) Q_h(0,0,0,0), \qquad k \geq 1 \tag{3.42}$$

where

$$\varphi(k) = \sum_{\bar{r}=0}^{\bar{r}'(k)} \prod_{n=1}^{k} \frac{1}{z_1^{\varepsilon(\bar{r})}} \xi(r_{n-1} - r_n - 1), \qquad k \geq 1 \qquad and \quad \varphi(0) = 1 \tag{3.43}$$

In the Appendix F, the equivalence of equations 3.42 and 3.32 is shown through induction.

3.2.4 Determination of the Expression of $\Phi(\omega)$

In subsection 3.2.3 we defined a function of $\varphi(k)$ (see 3.43), in the next, we will determine the transformation of $\varphi(k)$, which is needed later on. Let us define the transformation of $\varphi(r)$ as

$$\Phi(\omega) = \sum_{r=0}^{\infty} \varphi(r) \omega^r \tag{3.44}$$

In the following we will use the PGF of busy period, $\Gamma(\omega)$, to determine $\Phi(\omega)$. At first, since $\varphi(k)$ is a multiple level summation, we will find a way to reduce $\varphi(k)$ to a single level summation. As may be seen from 3.39 the power of z_1, $\varepsilon(\bar{r})$, is equal to the number of nonempty summations. Grouping by the power of z_1 in equation , we rewrite 3.43 for the first values of k.

* $\varphi(1)$

$$\varphi(1) = \sum_{r_1=0}^{0} \frac{1}{z_1}\xi(0);$$

* $\varphi(2)$

$$\varphi(2) = \sum_{r_1=0}^{1} \sum_{r_2=0}^{r_1-1} \frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(2 - r_1 - 1)\xi(r_1 - r_2 - 1)$$

$$= \sum_{r_1=0}^{0} \frac{1}{z_1}\xi(1)$$

$$+ \sum_{r_1=0}^{1} \sum_{r_2=0}^{0} \frac{1}{z_1^2}\xi(2 - r_1 - 1)\xi(r_2 - r_3 - 1);$$

* $\varphi(3)$

$$\varphi(3) = \sum_{r_1=0}^{2} \sum_{r_2=0}^{r_1-1} \sum_{r_3=0}^{r_2-1} \frac{1}{z_1^{\varepsilon(\bar{r})}}\xi(3 - r_1 - 1)\xi(r_1 - r_2 - 1)\xi(r_2 - r_3 - 1)$$

$$= \sum_{i_1=0}^{0} \frac{1}{z_1}\xi(2)$$

$$+ \sum_{r_1=1}^{2} \sum_{r_2=0}^{0} \frac{1}{z_1^2}\xi(3 - r_1 - 1)\xi(r_1 - r_2 - 1)$$

$$+\sum_{r_1=2}^{2}\sum_{r_2=1}^{r_1-1}\sum_{r_3=0}^{0}\frac{1}{z_1^3}\xi(3-r_1-1)\xi(r_1-r_2-1)\xi(r_2-r_3-1);$$

* $\varphi(4)$

$$\varphi(4)=\sum_{r_1=0}^{3}\sum_{r_2=0}^{r_1-1}\sum_{r_3=0}^{r_2-1}\sum_{r_4=0}^{r_3-1}\frac{1}{z_1^{\epsilon(\bar{r})}}\xi(4-r_1-1)\xi(r_1-r_2-1)\xi(r_2-r_3-1)\xi(r_3-r_4-1)$$

$$=\sum_{r_1=0}^{0}\frac{1}{z_1}\xi(3)$$

$$+\sum_{r_1=1}^{3}\sum_{r_2=0}^{0}\frac{1}{z_1^2}\xi(4-r_1-1)\xi(r_1-r_2-1)$$

$$+\sum_{r_1=2}^{3}\sum_{r_2=1}^{r_1-1}\sum_{r_3=0}^{0}\frac{1}{z_1^3}\xi(3-r_1-1)\xi(r_1-r_2-1)\xi(r_2-r_3-1)$$

$$+\sum_{r_1=3}^{3}\sum_{r_2=2}^{r_1-1}\sum_{r_3=1}^{r_2-1}\sum_{r_4=0}^{0}\frac{1}{z_1^3}\xi(3-r_1-1)\xi(r_1-r_2-1)\xi(r_2-r_3-1)\xi(r_3-r_4-1)$$

Finally, $\varphi(k)$ can be expressed as the sum of $\psi_j(k)$:

$$\varphi(k)=\sum_{j=1}^{k}\psi_j(k), \qquad k\geq 1 \tag{3.45}$$

where $\psi_j(k)$ denotes sum of the terms in 3.39 which have the same factor z_1^j in their denominators, $\psi_j(k)$ is given by the following subset of the sample space of $\varphi(k)$

$$\psi_j(k)=\frac{1}{z_1^j}\sum_{j_1=j-1}^{r_0-1}\sum_{j_2=j-2}^{r_1-1}\sum_{j_3=j-3}^{r_2-1}\cdots\sum_{j_{j-1}=1}^{r_{j-2}-1}\sum_{j_j=0}^{0}\prod_{n=1}^{k}\xi(r_{n-1}-r_n-1)\,|_{r_0=k}\,, \quad j\geq 2 \tag{3.46}$$

and

$$if\ j=1, \quad \psi_1(k)=\frac{1}{z_1}\xi(k-1), \tag{3.47}$$

From the definition of $\Phi(w)$ (see 3.44), and noting that if $k = 0$, $\varphi(0) = 1$, we have

$$\Phi(\omega) = \sum_{k=0}^{\infty} \varphi(k)\omega^k = \sum_{k=1}^{\infty}\sum_{j=1}^{k} \psi_j(k)\omega^k + 1$$

Interchanging the order of summations, above becomes

$$\Phi(\omega) = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \psi_j(k)\omega^k + 1$$

Let $m = k - j$, then $k = m + j$ and we have

$$\Phi(\omega) = \sum_{j=1}^{\infty}\sum_{m=0}^{\infty} \psi_j(m+j)\omega^{m+j} + 1$$

If we define

$$\Psi_j(\omega) = \sum_{m=0}^{\infty} \psi_j(m+j)\omega^m,$$

then

$$\Phi(\omega) = \sum_{j=1}^{\infty} \Psi_j(\omega)\omega^j + 1 \tag{3.48}$$

Next, we will find the expression for $\Psi_j(\omega)$. Let us determine $\Psi_j(\omega)$ for the first few values of j,

* $\Psi_1(\omega)$

$$\Psi_1(\omega) = \sum_{m=0}^{\infty} \psi_1(m+1)\omega^m$$

Substituting from 3.47, and using the definition of $\Gamma(\omega)$ from 3.25, $\Psi_1(\omega)$ becomes

$$\Psi_1(\omega) = \sum_{m=0}^{\infty} \frac{1}{z_1}\xi(m)\omega^m = \frac{1}{z_1}\Gamma(\omega) \tag{3.49}$$

* $\Psi_2(\omega)$

$$\Psi_2(\omega) = \sum_{m=0}^{\infty} \psi_2(m+2)\omega^m$$

Substituting from 3.46, $\Psi_2(\omega)$ becomes

$$\Psi_2(\omega) = \sum_{m=0}^{\infty} \sum_{r_1=1}^{m+1} \sum_{r_2=0}^{0} \frac{1}{z_1^2}\xi(m+1-r_1)\xi(r_1-r_2-1)\omega^m$$

$$= \sum_{m=0}^{\infty} \sum_{r_1=1}^{m+1} \frac{1}{z_1^2}\xi(m+1-r_1)\xi(r_1-1)\omega^m$$

letting $r = r_1 - 1$,

$$\Psi_2(\omega) = \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{1}{z_1^2}\xi(m-r)\xi(r)\omega^m,$$

interchanging the order of summations

$$\Psi_2(\omega) = \frac{1}{z_1^2} \sum_{r=0}^{\infty} \sum_{m=r}^{\infty} \xi(m-r)\xi(r)\omega^m.$$

letting $n = m - r$,

$$\Psi_2(\omega) = \frac{1}{z_1^2} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \xi(n)\xi(r)\omega^{n+r}$$

$$= \frac{1}{z_1^2}\Gamma(\omega)\Gamma(\omega),$$

thus

$$\Psi_2(\omega) = \left(\frac{\Gamma(\omega)}{z_1}\right)^2 \tag{3.50}$$

* $\Psi_3(\omega)$

$$\Psi_3(\omega) = \sum_{m=0}^{\infty} \psi_3(m+3)\omega^m$$

Substituting from 3.46, $\Psi_3(\omega)$ becomes

$$\Psi_3(\omega) = \sum_{m=0}^{\infty}\sum_{r_1=2}^{m+2}\sum_{r_2=1}^{r_1-1}\sum_{r_3=0}^{0}\frac{1}{z_1^3}\xi(m+2-r_1)\xi(r_1-r_2-1)\xi(r_2-r_3-1)\omega^m$$

$$= \sum_{m=0}^{\infty}\sum_{r_1=2}^{m+2}\sum_{r_2=1}^{r_1-1}\frac{1}{z_1^3}\xi(m+2-r_1)\xi(r_1-r_2-1)\xi(r_2-1)\omega^m,$$

letting $r = r_1 - 2$,

$$\Psi_3(\omega) = \frac{1}{z_1^3}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\sum_{r_2=1}^{r+1}\xi(m-r)\xi(r-r_2+1)\xi(r_2-1)\omega^m,$$

letting $h = r_2 - 1$,

$$\Psi_3(\omega) = \frac{1}{z_1^3}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\sum_{h=0}^{r}\xi(m-r)\xi(r-h)\xi(h)\omega^m,$$

interchanging the order of summations,

$$\Psi_3(\omega) = \frac{1}{z_1^3}\sum_{r=0}^{\infty}\sum_{m=r}^{\infty}\sum_{h=0}^{r}\xi(m-r)\xi(r-h)\xi(h)\omega^m.$$

letting $s = m - r$,

$$\Psi_3(\omega) = \frac{1}{z_1^3}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{h=0}^{r}\xi(s)\xi(r-h)\xi(h)\omega^{s+r}$$

$$= \frac{1}{z_1^3}\Gamma(\omega)\sum_{r=0}^{\infty}\sum_{h=0}^{r}\xi(r-h)\xi(h)\omega^r,$$

interchanging the order of summation, and letting $k = r - h$ gives

$$\Psi_3(\omega) = \frac{1}{z_1^3}\Gamma(\omega)\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\xi(k)\xi(h)\omega^{k+h}$$

$$= \frac{1}{z_1^3}\Gamma(\omega)\Gamma(\omega)\Gamma(\omega),$$

thus

$$\Psi_3(\omega) = \left(\frac{\Gamma(\omega)}{z_1}\right)^3; \tag{3.51}$$

From equations 3.49, 3.50, 3.51 we can conclude that:

$$\Psi_j(\omega) = \left(\frac{\Gamma(\omega)}{z_1}\right)^j \tag{3.52}$$

Substituting 3.52 into 3.48 gives us:

$$\Phi(\omega) = \sum_{j=1}^{\infty}\Psi_j(\omega)\omega^j + 1 = \sum_{j=1}^{\infty}\left(\frac{\Gamma(\omega)}{z_1}\right)^j\omega^j + 1 = \sum_{j=1}^{\infty}\left(\frac{\Gamma(\omega)}{z_1}\omega\right)^j + 1,$$

finally we have the result,

$$\Phi(\omega) = \frac{z_1}{z_1 - \omega\Gamma(\omega)} \tag{3.53}$$

3.3 Steady-State Solution of the Functional Equation

In this section, we will complete the solution of the functional equation 2.16, given by 3.22. The objective is to determine the four-dimensional joint steady-state PGF of the queue lengths and the number of On sources of both high and low priority traffic. From this solution, we will be able to determine the marginal PGFs of the queue length for both high and low-priority queues, and consequently the mean queue length, mean delay and variance.

3.3.1 Transformation of $Q_k(z_0, y_0, z_1, y_1)$

First, we will determine the transformation of $Q_k(z_0, y_0, z_1, y_1)$ with respect to discrete-time k.

As may be seen from equation 3.22, $Q_k(z_0, y_0, z_1, y_1)$ contains the unknown boundary function $Q_{k-j}(0, 0, z_1, \phi(j))$ which we will determine next,

From 3.42 we have

$$
\begin{aligned}
&Q_{k-j}(0,0,z_1,y_1)\\
&=\sum_{i=1}^{k-j}\varphi(k-j-i)\xi(i-1)B_1(k-j)\\
&+\sum_{i=1}^{k-1-j}\sum_{h=1}^{i}\frac{z_1-1}{z_1}\varphi(k-1-i-j)\xi(i-h)B_1(k-j-h)Q_h(0,0,0,0),
\end{aligned}
$$

then,

$$
\begin{aligned}
&Q_{k-j}(0,0,z_1,\phi_1(j))\\
&=\sum_{i=1}^{k-j}\varphi(k-j-i)\xi(i-1)B_1^j(k-j)\\
&+\sum_{i=1}^{k-1-j}\sum_{h=1}^{i}\frac{z_1-1}{z_1}\varphi(k-1-i-j)\xi(i-h)B_1^j(k-j-h)Q_h(0,0,0,0)
\end{aligned}
$$

Substituting above equation into 3.22 yields:

$$Q_k(z_0, y_0, z_1, y_1)$$

$$= \frac{1}{z_0^{k-1}}B(k) + \frac{z_0 - z_1}{z_1}\sum_{j=1}^{k-1}\sum_{i=1}^{k-j}\frac{B(j)}{z_0^j}\varphi(k-j-i)\xi(i-1)B_1^j(k-j)$$

$$+\frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\sum_{i=1}^{k-1-j}\sum_{h=1}^{i}\frac{B(j)}{z_0^j}\varphi(k-1-i-j)\xi(i-h)B_1^j(k-j-h)Q_h(0,0,0,0)$$

$$+\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\frac{B(j)}{z_0^{j-1}}Q_{k-j}(0,0,0,0), \qquad k \geq 1$$

from 3.7 and 3.10 we have $B(k) = B_0(k)B_1(k)$, $B_1^j(k-j) = \frac{B_1(k)}{B_1(j)}$ and $B_1^j(k-j-h) = \frac{B_1(k-h)}{B_1(j)}$, substituting these results in the above equation gives us

$$Q_k(z_0, y_0, z_1, y_1)$$

$$= \frac{1}{z_0^{k-1}}B(k) + \frac{z_0 - z_1}{z_1}\sum_{j=1}^{k-1}\sum_{i=1}^{k-j}\frac{B_0(j)}{z_0^j}\varphi(k-j-i)\xi(i-1)B_1(k)$$

$$+\frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\sum_{i=1}^{k-1-j}\sum_{h=1}^{i}\frac{B_0(j)}{z_0^j}\varphi(k-1-i-j)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)$$

$$+\frac{z_1 - 1}{z_1}\sum_{j=1}^{k-1}\frac{B(j)}{z_0^{j-1}}Q_{k-j}(0,0,0,0), \qquad k \geq 1 \tag{3.54}$$

Let us define the transformation of $Q_k(z_0, y_0, z_1, y_1)$ as

$$Q(z_0, y_0, z_1, y_1, \omega) = \sum_{k=0}^{\infty} Q_k(z_0, y_0, z_1, y_1)\omega^k, \tag{3.55}$$

and define the transformation of $Q_0(0,0,0,0)$ with respect to discrete-time as:

$$Q(\omega) = \sum_{h=0}^{\infty} Q_h(0,0,0,0)\omega^h \tag{3.56}$$

then

$$Q(z_0, y_0, z_1, y_1, \omega) = Q_0(z_0, y_0, z_1, y_1)\omega^0 + \sum_{k=1}^{\infty} Q_k(z_0, y_0, z_1, y_1)\omega^k$$

substituting equation 3.54 into the above equation results in:

$$Q(z_0, y_0, z_1, y_1, \omega)$$

$$= 1+\sum_{k=1}^{\infty} \frac{1}{z_0^{k-1}} B(k)\omega^k + \frac{z_0 - z_1}{z_1} \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} \frac{B_0(j)}{z_0^j} \varphi(k-j-i)\xi(i-1)B_1(k)\omega^k$$

$$+\frac{z_0 - z_1}{z_1} \frac{z_1 - 1}{z_1} \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \sum_{i=1}^{k-1-j} \sum_{h=1}^{i} \frac{B_0(j)}{z_0^j} \varphi(k-1-i-j)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k$$

$$+\frac{z_1 - 1}{z_1} \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{B(j)}{z_0^{j-1}} Q_{k-j}(0,0,0,0)\omega^k \tag{3.57}$$

Let us define A_1, A_2 and A_3 as:

$$A_1 = 1+\sum_{k=1}^{\infty} \frac{1}{z_0^{k-1}} B(k)\omega^k + \frac{z_0 - z_1}{z_1} \sum_{k=2}^{\infty} \sum_{j=1}^{k} \sum_{i=1}^{k-j} \frac{B_0(j)}{z_0^j} \varphi(k-j-i)\xi(i-1)B_1(k)\omega^k \tag{3.58}$$

$$A_2 = \frac{z_0 - z_1}{z_1} \frac{z_1 - 1}{z_1} \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \sum_{i=1}^{k-1-j} \sum_{h=1}^{i} \frac{B_0(j)}{z_0^j} \varphi(k - 1 - i - j)$$

$$\cdot\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k \tag{3.59}$$

$$A_3 = \frac{z_1 - 1}{z_1} \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \frac{B(j)}{z_0^{j-1}} Q_{k-j}(0,0,0,0)\omega^k \tag{3.60}$$

In the following, we will determine A_1, A_2 and A_3 one by one.

*** Determining the expression for A_1:**

Interchanging the order of outer summations in the second term of A_1,

$$A_1 = 1+\sum_{k=1}^{\infty} \frac{1}{z_0^{k-1}} B_0(k) B_1(k)\omega^k + \frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \sum_{i=1}^{k-j} \frac{B_0(j)}{z_0^j} \varphi(k-j-i)\xi(i-1) B_1(k)\omega^k.$$

next, interchanging the order of inner summation,

$$A_1 = 1+\sum_{k=1}^{\infty} \frac{1}{z_0^{k-1}} B_0(k) B_1(k)\omega^k + \frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{k=i+j}^{\infty} \frac{B_0(j)}{z_0^j} \varphi(k-j-i)\xi(i-1) B_1(k)\omega^k,$$

letting $m = k - i - j$,

$$A_1 = 1 + \sum_{k=1}^{\infty} \frac{1}{z_0^{k-1}} B_0(k) B_1(k)\omega^k$$

$$+ \frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \frac{B_0(j)}{z_0^j} \varphi(m)\xi(i-1) B_1(m+i+j)\omega^{m+i+j}$$

Substituting for $B_1(k)$ and $B_1(m + i + j)$ from 3.16 in above equation and then expanding it using binomial theorem yields:

$$A_1 = 1+\sum_{k=1}^{\infty} \sum_{n=0}^{m_1} \frac{1}{z_0^{k-1}} B_0(k) \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^k\right)^n \left(C_{21}\lambda_{21}^k\right)^{m_1-n} \omega^k$$

$$+ \frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{m_1} \frac{B_0(j)}{z_0^j} \varphi(m)\xi(i-1)$$

$$\cdot \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^{m+i+j}\right)^n \left(C_{21}\lambda_{21}^{m+i+j}\right)^{m_1-n} \omega^{m+i+j}$$

replacing $(i-1)$ with i in the second multiple summation,

$$A_1 = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{m_1} \frac{1}{z_0^{k-1}} B_0(k) \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^{k}\right)^{n} \left(C_{21}\lambda_{21}^{k}\right)^{m_1-n} \omega^{k}$$

$$+\frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m_1} \frac{B_0(j)}{z_0^{j}} \varphi(m)\xi(i)$$

$$\cdot \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^{m+i+1+j}\right)^{n} \left(C_{21}\lambda_{21}^{m+i+1+j}\right)^{m_1-n} \omega^{m+i+1+j}$$

putting together those terms with power m or i, we have ,

$$A_1 = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{m_1} \frac{1}{z_0^{k-1}} B_0(k) \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^{k}\right)^{n} \left(C_{21}\lambda_{21}^{k}\right)^{m_1-n} \omega^{k}$$

$$+\frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m_1} \begin{pmatrix} m_1 \\ n \end{pmatrix} \frac{B_0(j)}{z_0^{j}} \varphi(m) \left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)^{m} \xi(i) \left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)^{i}$$

$$\cdot \left(C_{11}\lambda_{11}^{1+j}\right)^{n} \left(C_{21}\lambda_{21}^{1+j}\right)^{m_1-n} \omega^{1+j}$$

substituting from 3.44 and 3.25,

$$A_1 = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{m_1} \frac{1}{z_0^{k-1}} B_0(k) \begin{pmatrix} m_1 \\ n \end{pmatrix} \left(C_{11}\lambda_{11}^{k}\right)^{n} \left(C_{21}\lambda_{21}^{k}\right)^{m_1-n} \omega^{k}$$

$$+\frac{z_0 - z_1}{z_1} \sum_{j=1}^{\infty}\sum_{n=0}^{m_1} \begin{pmatrix} m_1 \\ n \end{pmatrix} \frac{B_0(j)}{z_0^{j}} \Phi\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right) \Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)$$

$$\cdot \left(C_{11}\lambda_{11}^{1+j}\right)^{n} \left(C_{21}\lambda_{21}^{1+j}\right)^{m_1-n} \omega^{1+j}$$

Substituting for $B_0(k)$ and $B_0(j)$ from 3.16 in above equation and then expanding it using binomial theorem yields:

$$A_1 = 1+\frac{1}{z_1}\sum_{k=1}^{\infty}\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\binom{m_0}{l}\binom{m_1}{n}\frac{1}{z_0^k}\left(C_{10}\lambda_{10}^k\right)^l\left(C_{20}\lambda_{20}^k\right)^{m_0-l}\left(C_{11}\lambda_{11}^k\right)^n\left(C_{21}\lambda_{21}^k\right)^{m_1-n}\omega^k$$

$$+\frac{z_0-z_1}{z_1}\sum_{j=1}^{\infty}\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\binom{m_0}{l}\binom{m_1}{n}\frac{1}{z_0^j}\left(C_{10}\lambda_{10}^j\right)^l\left(C_{20}\lambda_{20}^j\right)^{m_0-l}$$

$$\cdot\left(C_{11}\lambda_{11}^{j+1}\right)^n\left(C_{21}\lambda_{21}^{j+1}\right)^{m_1-n}\Phi\left(\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)\Gamma\left(\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)\omega^{j+1},$$

interchanging the order of summations and rearranging the terms in above equation results in:

$$A_1 = 1+\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\sum_{k=1}^{\infty}\left(\frac{1}{z_0}\lambda_{10}^l\lambda_{20}^{m_0-l}\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)^k\binom{m_0}{l}\binom{m_1}{n}C_{10}^lC_{20}^{m_0-l}C_{11}^nC_{21}^{m_1-n}$$

$$+\frac{z_0-z_1}{z_1}\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\sum_{j=1}^{\infty}\left(\frac{1}{z_0}\lambda_{10}^l\lambda_{20}^{m_0-l}\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)^j\binom{m_0}{l}\binom{m_1}{n}C_{10}^lC_{20}^{m_0-l}$$

$$\cdot\left(C_{11}\lambda_{11}\right)^n\left(C_{21}\lambda_{21}\right)^{m_1-n}\Phi\left(\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)\Gamma\left(\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)\omega.$$

Because

$$\sum_{j=1}^{\infty}\left(\frac{1}{z_0}\lambda_{10}^l\lambda_{20}^{m_0-l}\lambda_{11}^n\lambda_{21}^{m_1-n}\omega\right)^j = \frac{\lambda_{10}^l\lambda_{20}^{m_0-l}\lambda_{11}^n\lambda_{21}^{m_1-n}\omega}{z_0-\lambda_{10}^l\lambda_{20}^{m_0-l}\lambda_{11}^n\lambda_{21}^{m_1-n}\omega},$$

and from 3.53, $\Phi\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right) = \frac{z_1}{z_1-\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)}$, A_1 becomes

$$A_1 = 1+\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\begin{pmatrix} m_0 \\ l \end{pmatrix}\begin{pmatrix} m_1 \\ n \end{pmatrix}\frac{(C_{10}\lambda_{10})^{l}(C_{20}\lambda_{20})^{m_0-l}(C_{11}\lambda_{11})^{n}(C_{21}\lambda_{21})^{m_1-n}\omega}{z_0-\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega}$$

$$+\frac{z_0-z_1}{z_1}\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\begin{pmatrix} m_0 \\ l \end{pmatrix}\begin{pmatrix} m_1 \\ n \end{pmatrix}C_{10}^{l}C_{20}^{m_0-l}(C_{11}\lambda_{11})^{n}(C_{21}\lambda_{21})^{m_1-n}$$

$$\cdot\frac{\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega}{z_0-\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega}\frac{z_1\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)\omega}{z_1-\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)},$$

$$A_1 = 1+\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\begin{pmatrix} m_0 \\ l \end{pmatrix}\begin{pmatrix} m_1 \\ n \end{pmatrix}\frac{(C_{10}\lambda_{10})^{l}(C_{20}\lambda_{20})^{m_0-l}(C_{11}\lambda_{11})^{n}(C_{21}\lambda_{21})^{m_1-n}\omega}{z_0-\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega}$$

$$+(z_0-z_1)\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\begin{pmatrix} m_0 \\ l \end{pmatrix}\begin{pmatrix} m_1 \\ n \end{pmatrix}(C_{10}\lambda_{10})^{l}(C_{20}\lambda_{20})^{m_0-l}(C_{11}\lambda_{11})^{n}(C_{21}\lambda_{21})^{m_1-n}$$

$$\cdot\frac{\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)\omega}{\left(z_0-\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)\left[z_1-\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)\right]},$$

from 3.27 we have $\omega\Gamma(\omega) = 1-\frac{1}{P_0(\omega)}$, thus

$$\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\Gamma\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right) = 1-\frac{1}{P_0\left(\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega\right)}, \tag{3.61}$$

substituting 3.61 into A_1 gives:

$$A_1 = 1+\sum_{l=0}^{m_0}\sum_{n=0}^{m_1}\begin{pmatrix} m_0 \\ l \end{pmatrix}\begin{pmatrix} m_1 \\ n \end{pmatrix}\frac{(C_{10}\lambda_{10})^{l}(C_{20}\lambda_{20})^{m_0-l}(C_{11}\lambda_{11})^{n}(C_{21}\lambda_{21})^{m_1-n}\omega}{z_0-\lambda_{10}^{l}\lambda_{20}^{m_0-l}\lambda_{11}^{n}\lambda_{21}^{m_1-n}\omega}$$

$$+ (z_0 - z_1) \sum_{l=0}^{m_0} \sum_{n=0}^{m_1} \begin{pmatrix} m_0 \\ l \end{pmatrix} \begin{pmatrix} m_1 \\ n \end{pmatrix} (C_{10}\lambda_{10})^l (C_{20}\lambda_{20})^{m_0-l} (C_{11}\lambda_{11})^n (C_{21}\lambda_{21})^{m_1-n}$$

$$\cdot \frac{\left[P_0\left(\lambda_{11}^n \lambda_{21}^{m_1-n}\omega\right) - 1\right]\omega}{\left(z_0 - \lambda_{10}^l \lambda_{20}^{m_0-l} \lambda_{11}^n \lambda_{21}^{m_1-n}\omega\right)\left[(z_1 - 1) P_0\left(\lambda_{11}^n \lambda_{21}^{m_1-n}\omega\right) + 1\right]} \quad (3.62)$$

*** Determining the expression for A_2 which is defined in 3.59**

Interchanging the order of summations of A_2,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1} \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \sum_{i=1}^{k-1-j} \sum_{h=1}^{i} \frac{B_0(j)}{z_0^j} \varphi(k-1-i-j)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k,$$

interchanging the order of summations again,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{k=i+j+1}^{\infty} \sum_{h=1}^{i} \frac{B_0(j)}{z_0^j} \varphi(k-1-i-j)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k.$$

and again

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{h=1}^{i} \sum_{k=i+j+1}^{\infty} \frac{B_0(j)}{z_0^j} \varphi(k-1-i-j)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k,$$

letting $m = k - i - j$,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{h=1}^{i} \sum_{m=1}^{\infty} \frac{B_0(j)}{z_0^j} \varphi(m-1)\xi(i-h)B_1(m+i+j-h)Q_h(0,0,0,0)\omega^{m+i+j},$$

interchanging the order of inner summations

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1} \sum_{j=1}^{\infty} \sum_{h=1}^{\infty} \sum_{i=h}^{\infty} \sum_{m=1}^{\infty} \frac{B_0(j)}{z_0^j} \varphi(m-1)\xi(i-h)B_1(m+i+j-h)Q_h(0,0,0,0)\omega^{m+i+j},$$

letting $n = i - h$,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{B_0(j)}{z_0^j}\varphi(m-1)\xi(n)B_1(m+j+n)Q_h(0,0,0,0)\omega^{m+n+h+j},$$

replacing $(m-1)$ with m,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_0(j)}{z_0^j}\varphi(m)\xi(n)B_1(m+1+j+n)Q_h(0,0,0,0)\omega^{m+1+n+h+j},$$

substituting from 3.56,

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}[Q(\omega) - 1]\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_0(j)}{z_0^j}\varphi(m)\xi(n)B_1(m+1+j+n)\omega^{m+1+n+j}$$

Substituting for $B_1(m+1+j+n)$ from 3.16 in above equation and then expanding it using binomial theorem yields:

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}[Q(\omega) - 1]\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{m_1}\begin{pmatrix} m_1 \\ i \end{pmatrix}\left(C_{11}\lambda_{11}^{m+1+j+n}\right)^i\left(C_{21}\lambda_{21}^{m+1+j+n}\right)^{m_1-i}$$

$$\cdot\frac{B_0(j)}{z_0^j}\varphi(m)\xi(n)\omega^{m+1+n+j}$$

putting together those terms with the power of m or n, we have:

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}[Q(\omega) - 1]\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{m_1}\begin{pmatrix} m_1 \\ i \end{pmatrix}\frac{B_0(j)}{z_0^j}\left(C_{11}\lambda_{11}^{j+1}\right)^i\left(C_{21}\lambda_{21}^{j+1}\right)^{m_1-i}$$

$$\cdot\omega^{j+1}\varphi(m)\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)^m\xi(n)\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)^n$$

substituting from 3.44 and 3.25, above equation becomes:

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\left[Q(\omega) - 1\right]\sum_{j=1}^{\infty}\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{B_0(j)}{z_0^j}\left(C_{11}\lambda_{11}^{j+1}\right)^i\left(C_{21}\lambda_{21}^{j+1}\right)^{m_1-i}$$

$$\cdot\omega^{j+1}\Phi\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)$$

substituting for $\Phi\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)$ from 3.53 , we have:

$$A_2 = \frac{z_0 - z_1}{z_1}\frac{z_1 - 1}{z_1}\left[Q(\omega) - 1\right]\sum_{j=1}^{\infty}\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{B_0(j)}{z_0^j}\left(C_{11}\lambda_{11}^{j+1}\right)^i\left(C_{21}\lambda_{21}^{j+1}\right)^{m_1-i}\omega^{j+1}$$

$$\cdot\frac{z_1\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)}{z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)}$$

Substituting for $B_0(j)$ from 3.16 in above equation and then expanding it using binomial theorem yields:

$$A_2 = \frac{(z_0 - z_1)(z_1 - 1)}{z_1}\left[Q(\omega) - 1\right]\sum_{j=1}^{\infty}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\left(C_{10}\lambda_{10}^j\right)^k\left(C_{20}\lambda_{20}^j\right)^{m_0-k}$$

$$\cdot\left(C_{11}\lambda_{11}^{j+1}\right)^i\left(C_{21}\lambda_{21}^{j+1}\right)^{m_1-i}\cdot\frac{\omega^{j+1}}{z_0^j}\cdot\frac{\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)}{z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)},$$

interchanging the order of summations,

$$A_2 = \frac{(z_0 - z_1)(z_1 - 1)}{z_1}\left[Q(\omega) - 1\right]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}C_{10}^kC_{20}^{m_0-k}$$

$$\cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \omega\Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)}{z_1 - \lambda_{11}^i \lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)} \sum_{j=1}^{\infty}\left(\frac{1}{z_0}\lambda_{10}^k \lambda_{20}^{m_0-k}\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)^j$$

$$A_2 = \frac{(z_0 - z_1)(z_1 - 1)}{z_1}[Q(\omega) - 1]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix} C_{10}^k C_{20}^{m_0-k}$$

$$\cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \omega\Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)}{z_1 - \lambda_{11}^i \lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)} \cdot \frac{\lambda_{10}^k \lambda_{20}^{m_0-k}\lambda_{11}^i \lambda_{21}^{m_1-i}\omega}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k}\lambda_{11}^i \lambda_{21}^{m_1-i}\omega}.$$

substituting for $\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)$ from 3.61 in above equation gives:

$$A_2 = \frac{(z_0 - z_1)(z_1 - 1)}{z_1}[Q(\omega) - 1]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\omega\left[P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right) - 1\right]}{\left(z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k}\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right)\left[(z_1 - 1)P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i}\omega\right) + 1\right]} \tag{3.63}$$

*** Determining the expression for A_3, which is defined in 3.60**

Interchanging the order of summations, A_3 becomes:

$$A_3 = \frac{z_1 - 1}{z_1}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\frac{B_0(j)B_1(j)}{z_0^{j-1}}Q_{k-j}(0,0,0,0)\omega^k,$$

letting $l = k - j$,

$$A_3 = \frac{z_1 - 1}{z_1}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{B_0(j)B_1(j)}{z_0^{j-1}}Q_l(0,0,0,0)\omega^{l+j},$$

substituting from 3.56,

$$A_3 = \frac{z_1 - 1}{z_1}[Q(\omega) - 1]\sum_{j=1}^{\infty} \frac{B_0(j)B_1(j)}{z_0^{j-1}}\omega^j,$$

Substituting for $B_0(j)$ and $B_1(j)$ from 3.16 in the above equation, then expanding them using binomial theorem yields:

$$A_3 = \frac{z_1 - 1}{z_1}[Q(\omega) - 1]\sum_{j=1}^{\infty}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\left(C_{10}\lambda_{10}^j\right)^k\left(C_{20}\lambda_{20}^j\right)^{m_0-k}$$

$$\cdot\left(C_{11}\lambda_{11}^j\right)^i\left(C_{21}\lambda_{21}^j\right)^{m_1-i}\frac{\omega^j}{z_0^{j-1}}$$

putting together those terms with the power of j,

$$A_3 = \frac{z_1 - 1}{z_1}z_0[Q(\omega) - 1]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}C_{10}^kC_{20}^{m_0-k}C_{11}^iC_{21}^{m_1-i}$$

$$\cdot\sum_{j=1}^{\infty}\left(\frac{1}{z_0}\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)^j$$

$$A_3 = \frac{z_1 - 1}{z_1}z_0[Q(\omega) - 1]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}C_{10}^kC_{20}^{m_0-k}C_{11}^iC_{21}^{m_1-i}$$

$$\cdot\frac{\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}$$

finally we have:

$$A_3 = \frac{z_1 - 1}{z_1} z_0 \left[Q(\omega) - 1\right] \sum_{k=0}^{m_0} \sum_{i=0}^{m_1} \binom{m_0}{k} \binom{m_1}{i} \cdot \frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k} (C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \omega}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k} \lambda_{11}^i \lambda_{21}^{m_1-i} \omega} \tag{3.64}$$

Now by putting 3.62, 3.63 and 3.64 together, we are able to determine $Q(z_0, y_0, z_1, y_1, \omega)$ as follows:

$$\begin{aligned}
&Q(z_0, y_0, z_1, y_1, \omega) \\
&= 1 + \sum_{k=0}^{m_0} \sum_{i=0}^{m_1} \binom{m_0}{k} \binom{m_1}{i} \frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k} (C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \omega}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k} \lambda_{11}^i \lambda_{21}^{m_1-i} \omega} \\
&+ (z_0 - z_1) \sum_{k=0}^{m_0} \sum_{i=0}^{m_1} \binom{m_0}{k} \binom{m_1}{i} (C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k} \\
&\qquad \cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \left[P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) - 1\right] \omega}{\left(z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k} \lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) \left[(z_1 - 1) P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) + 1\right]} \\
&+ \frac{(z_0 - z_1)(z_1 - 1)}{z_1} \left[Q(\omega) - 1\right] \sum_{k=0}^{m_0} \sum_{i=0}^{m_1} \binom{m_0}{k} \binom{m_1}{i} (C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k} \\
&\qquad \cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} \omega \left[P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) - 1\right]}{\left(z_0 - \lambda_{10}^k \lambda_{20}^{m_0-k} \lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) \left[(z_1 - 1) P_0\left(\lambda_{11}^i \lambda_{21}^{m_1-i} \omega\right) + 1\right]} \\
&+ \frac{z_1 - 1}{z_1} z_0 \left[Q(\omega) - 1\right] \sum_{k=0}^{m_0} \sum_{i=0}^{m_1} \binom{m_0}{k} \binom{m_1}{i} (C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}
\end{aligned}$$

$$\cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\omega}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}$$

we note that the second term and the third term of the RHS of the above equation have some common factors, after combining these terms, we have the transformation as the following result:

$$Q(z_0, y_0, z_1, y_1, \omega)$$

$$= 1+\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\omega}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}$$

$$+\frac{z_0 - z_1}{z_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}$$

$$\cdot\frac{\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right) - 1\right]\omega}{\left(z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)\left[(z_1 - 1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right) + 1\right]}$$

$$+\frac{(z_0 - z_1)(z_1 - 1)}{z_1}Q(\omega)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\omega\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right) - 1\right]}{\left(z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)\left[(z_1 - 1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right) + 1\right]}$$

$$+\frac{z_1 - 1}{z_1}z_0\left[Q(\omega) - 1\right]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\omega}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega} \quad (3.65)$$

3.3.2 The Steady-State Solution of the Functional Equation, $Q(z_0, y_0, z_1, y_1)$

In the previous subsection, we have obtained the transformation of $Q_k(z_0, y_0, z_1, y_1)$ with respect to discrete time k (see equation 3.65), now we are ready to determine the steady-state joint PGF $Q(z_0, y_0, z_1, y_1)$, which may be determined from $Q_k(z_0, y_0, z_1, y_1, \omega)$ through the application of final value theorem.

Let us apply the final value theorem to both sides of equation 3.65,

$$Q(z_0, y_0, z_1, y_1) = \lim_{\omega\to 1}(1-\omega)Q(z_0, y_0, z_1, y_1, \omega)$$

$$= \lim_{\omega\to 1}(1-\omega)\left[1+\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\frac{(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\omega}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}\right]$$

$$+\lim_{\omega\to 1}(1-\omega)\left\{\frac{z_0-z_1}{z_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}\right.$$

$$\left.\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)-1\right]\omega}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)\left[(z_1-1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)+1\right]}\right\}$$

$$+\lim_{\omega\to 1}(1-\omega)\left\{\frac{(z_0-z_1)(z_1-1)}{z_1}Q(\omega)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}\right.$$

$$\left.\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\omega\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)-1\right]}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)\left[(z_1-1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)+1\right]}\right\}$$

$$+\lim_{\omega\to 1}(1-\omega)\left\{\frac{z_0(z_1-1)}{z_1}[Q(\omega)-1]\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}\right.$$

$$\left.\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\omega}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}\right\}$$

Because

$$\lim_{\omega\to 1}(1-\omega)\left[1+\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\frac{(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\omega}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\omega}\right]$$

$= 0,$

$$\lim_{\omega\to 1}(1-\omega)Q(\omega)=Q(0,0,0,0)=1-\rho,$$

$$\lim_{\omega\to 1}(1-\omega)\left[Q(\omega)-1\right]=Q(0,0,0,0)=1-\rho,$$

where ρ is the total system load (see [22])

$$\rho=\sum_{i=0}^{1}\rho_i \qquad with\ \rho_i=\frac{m_i(1-\beta_i)\overline{f_i}}{2-\alpha_i-\beta_i}=\frac{d\lambda_{2i}^{m_i}}{dz_i},\ \overline{f_i}=f_i'(z_i)|_{z_i=1}$$

the joint PGF becomes:

$$Q(z_0,y_0,z_1,y_1)$$

$$=\frac{z_0-z_1}{z_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}$$

$$\cdot\frac{\lim_{\omega\to 1}(1-\omega)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\omega\right)}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[(z_1-1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)+1\right]}$$

$$+\frac{(z_0-z_1)(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)-1\right]}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[(z_1-1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)+1\right]}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}};$$

Since $|\lambda_{1i}\lambda_{2i}|<1$ (see Appendix C), thus $|\lambda_{11}|<1$, $|\lambda_{12}|\le 1$, we have

$$\begin{array}{ll}\lambda_{11}^j\lambda_{21}^{m_1-j}=1, & if\ j=0\ and\ z_1=1\\ -1<\lambda_{11}^j\lambda_{21}^{m_1-j}<1, & otherwise\end{array};$$

therefore

$$\lim_{\omega\to 1}(1-\omega)P_0\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)=\lim_{k\to\infty}p_k(0)\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\right)^k=\begin{cases}1-\rho_0, & if\ j=0\ and\ z_1=1\\ 0, & otherwise\end{cases}\tag{3.66}$$

then the first term of the RHS of above equation is equal to 0 except at $z_1=1$. Since the limit operation includes continuous values of a variable, we ignore the point $z_1=1$ and therefore the first term will be zero and it is not part of the steady-state PGF. Thus we have

$$Q(z_0,y_0,z_1,y_1)$$

$$=\frac{(z_0-z_1)(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\left[P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)-1\right]}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[(z_1-1)P_0\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)+1\right]}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot \frac{(C_{11}\lambda_{11})^{i}(C_{21}\lambda_{21})^{m_1-i}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}} \tag{3.67}$$

From 3.61 we have

$$\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)=1-\frac{1}{P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)},$$

then

$$P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)=\frac{1}{1-\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)},$$

substituting this result in equation 3.67, we have the final solution of the joint four-dimensional PGF:

$$Q(z_0,y_0,z_1,y_1)$$

$$=\frac{(z_0-z_1)(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^{k}(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^{i}(C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)}{\left(z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^{k}(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^{i}(C_{21}\lambda_{21})^{m_1-i}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}} \tag{3.68}$$

This is the main result of this work.

3.4 Some Discussions of the Solution

In this section, we will determine the marginal PGF for the high-priority queue as well as the low-priority queue, and also we will discuss the results of the solution $Q(z_0, y_0, z_1, y_1)$ in some special cases.

3.4.1 The Marginal PGF for the High-Priority Queue, $Q(z_0, y_0, 1, 1)$

In order to obtain the marginal PGF of the high-priority queue, we substitute 1 for z_1, y_1 in equation 3.68. Let us define A_1, A_2 as follows:

$$A_1 = \frac{(z_0 - z_1)(z_1 - 1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

$$A_2 = \frac{z_0(z_1 - 1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}}$$

First we consider A_1. Let us substituting 1 for z_1 y_1, then the numerator $(z_1 - 1)$ is zero, and $(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i} = 0$ except at $i = 0$, also noting that $\Gamma(1) = 1$, we have $\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right) = 1$ at $i = 0$. Thus the first term A_1 is zero except at $i = 0$. therefore,

$$A_1\,|_{z_1=y_1=1}$$

$$= \frac{(z_0 - z_1)(z_1 - 1)(C_{21}\lambda_{21})^{m_1} \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})}{z_1^2 - z_1 \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})} \Big|_{z_1 = y_1 = 1}$$

$$\cdot (1 - \rho) \sum_{k=0}^{m_0} \begin{pmatrix} m_0 \\ k \end{pmatrix} \frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0 - k}}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0 - k}}$$

we can see that both the numerator and denominator are zero at $z_1 = y_1 = 1$, then it is $\frac{0}{0}$type. After we apply the L'Hospital's rule with respect to z_1, we have

$$\frac{(z_0 - z_1)(z_1 - 1)(C_{21}\lambda_{21})^{m_1} \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})}{z_1^2 - z_1 \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})} \Big|_{z_1 = y_1 = 1} = \frac{z_0 - 1}{1 - \frac{\partial(\lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1}))}{\partial z_1}}$$

Since (see Appendix A, equation A.8),

$$\left. \frac{\partial(\lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1}))}{\partial z_1} \right|_{z_1 = 1} = \frac{\rho_1}{1 - \rho_0}$$

the above equation becomes

$$\frac{(z_0 - z_1)(z_1 - 1)(C_{21}\lambda_{21})^{m_1} \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})}{z_1^2 - z_1 \lambda_{21}^{m_1} \Gamma(\lambda_{21}^{m_1})} \Big|_{z_1 = y_1 = 1} = \frac{(1 - \rho_0)(z_0 - 1)}{1 - \rho_0 - \rho_1} = \frac{(1 - \rho_0)(z_0 - 1)}{1 - \rho}$$

Substituting the above result in A_1 gives us

$$A_1 \big|_{z_1 = y_1 = 1} = (1 - \rho_0)(z_0 - 1) \sum_{k=0}^{m_0} \begin{pmatrix} m_0 \\ k \end{pmatrix} \frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0 - k}}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0 - k}}$$

After substituting 1 for z_1 y_1, A_2 becomes:

$$A_2 \big|_{z_1 = y_1 = 1} = 0$$

At last, by adding up A_1 and A_2, the marginal PGF of the high-priority queue is:

$$Q(z_0, y_0, 1, 1) = (1-\rho_0)(z_0-1)\sum_{k=0}^{m_0}\binom{m_0}{k}\frac{(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}}.$$

We find that as expected $Q(z_0, y_0, 1, 1)$ is the same as the result of a multiplexer with single type of traffic given in [18] . This result conforms that the high-priority queue "does not see" the low priority-queue and the behavior of the high-priority queue is not affected by the low-priority queue.

3.4.2 The Marginal PGF for the Low-Priority Queue $Q(1, 1, z_1, y_1)$

In order to obtain the marginal PGF of the high-priority queue which is $Q(1, 1, z_1, y_1)$, we substitute 1 for z_0, y_0 in equation 3.68.

Let us define A_1, A_2 as before in subsection 3.4.1.

Substituting 1 for z_0 and y_0 from 3.17 and 3.18, we have

$$C_{10}\lambda_{10}\,|_{z_0=y_0=1} = 0, \quad C_{20}\lambda_{20}\,|_{z_0=y_0=1} = 1;$$

$$\lambda_{10}\,|_{z_0=1} = \alpha_0+\beta_0-1, \quad \lambda_{20}\,|_{z_0=1} = 1$$

then the first term A_1 becomes:

$$A_1 = -\frac{(z_1-1)^2}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(1-\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

And the second term A_2 becomes:

$$A_2 = \frac{(z_1 - 1)}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{1-\lambda_{11}^i\lambda_{21}^{m_1-i}}$$

Therefore,

$$\begin{aligned}
&Q(1,1,z_1,y_1)\\
&= -\frac{(z_1-1)^2}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(1-\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}\\
&+\frac{(z_1-1)}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{1-\lambda_{11}^i\lambda_{21}^{m_1-i}} \qquad (3.69)
\end{aligned}$$

Since $Q(1,1,z_1,y_1)$ is the PGF of the low priority queue, it should satisfy the normalization condition, that is $Q(1,1,z_1,y_1)\,|_{z_1=y_1=1} = 1$. In the next, we will show this is true.

We note that if we substitute 1 for z_1, y_1, the numerators of $Q(1,1,z_1,y_1)$ is zero, and the denominators of $Q(1,1,z_1,y_1)$ is non-zero except at $i=0$; then we have:

$$\begin{aligned}
&Q(1,1,z_1,y_1)\,|_{z_1=y_1=1}\\
&= -\frac{(z_1-1)^2}{z_1}(1-\rho)\left.\frac{(C_{21}\lambda_{21})^{m_1}\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)}{\left(1-\lambda_{21}^{m_1}\right)\left[z_1-\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}\right|_{z_1=y_1=1}\\
&+\frac{(z_1-1)}{z_1}(1-\rho)\left.\frac{(C_{21}\lambda_{21})^{m_1}}{1-\lambda_{21}^{m_1}}\right|_{z_1=y_1=1}
\end{aligned}$$

combining the above two terms, we have:

$$Q(1,1,z_1,y_1)\,|_{z_1=y_1=1}$$

$$= \left. \frac{(1-\rho)(z_1-1)(C_{21}\lambda_{21})^{m_1}\left[1-\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{\left(1-\lambda_{21}^{m_1}\right)\left[z_1-\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]} \right|_{z_1=y_1=1}$$

We can see that both the numerator and the denominator are zero, let us apply L'Hospital's rule with respect to z_1. As the first order derivatives of the numerator and denominator are still zero, we have to take the second order derivative of the numerator and denominator, and noting that

$$C_{21}\lambda_{21}\,|_{z_1=y_1=1} = 1, \qquad \lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)|_{z_1=y_1} = 1, \qquad \lambda_{21}^{m_1}\,|_{z_1=y_1} = 1$$

we have

$$Q(1,1,z_1,y_1)\,|_{z_1=y_1=1} = \left. \frac{-2(1-\rho)\frac{\partial\left[\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{\partial z_1}}{-2\frac{\partial\lambda_{21}^{m_1}}{\partial z_1}\left(1-\frac{\partial\left[\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{\partial z_1}\right)} \right|_{z_1=y_1=1}$$

because

$$\frac{\partial\lambda_{21}^{m_1}}{\partial z_1} = \rho_1, \qquad \frac{\partial\left[\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{\partial z_1} = \frac{\rho_1}{1-\rho_0}$$

(see Appendix A, equations A.7 and A.8); finally we have the result:

$$Q(1,1,z_1,y_1)\,|_{z_1=y_1=1} = \frac{(1-\rho)\frac{\rho_1}{1-\rho_0}}{\rho_1\left(1-\frac{\rho_1}{1-\rho_0}\right)} = \frac{1-\rho}{1-\rho_0-\rho_1} = 1$$

this shows that the PGF $Q(1,1,z_1,y_1)$ satisfies the normalization condition.

Further, we can see that the high-priority traffic affects the low-priority queue through the function $\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)$. If the high-priority traffic load $\rho_0 = 0$, then its busy period will also approach to zero, thus

$$\lim_{\rho_0 \to 0} \Gamma(w) = 1;$$

also we note that if $\rho_0 = 0$, $\rho = \rho_0 + \rho_1 = \rho_1$; therefore

$$\begin{aligned}
&Q(1, 1, z_1, y_1) \mid_{\rho_0=0} \\
&= -\frac{(z_1 - 1)^2}{z_1}(1 - \rho_1)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}}{\left(1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\right]} \\
&\quad + \frac{(z_1 - 1)}{z_1}(1 - \rho_1)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{1 - \lambda_{11}^i\lambda_{21}^{m_1-i}} \\
&= (1 - \rho_1)(z_1 - 1)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}}
\end{aligned}$$

We can see that the above result is the same as the PGF for the high-priority queue. This is expected, since when there is no high-priority queue traffic, the low-priority queue should behave like the high-priority queue.

3.4.3 $Q(z_0, y_0, z_1, y_1)$ at $m_0 = 0$

If $m_0 = 0$, then there is no high-priority traffic in the system, thus the high-priority traffic load $\rho_0 = 0$, as stated in the previous subsection, $\rho = \rho_0 + \rho_1 = \rho_1$ and $\lim_{\rho_0 \to 0} \Gamma(\omega) = 1$, therefore, 3.68 becomes:

$$Q(z_0, y_0, z_1, y_1) \mid_{m_0=0}$$

$$= \frac{(z_0 - z_1)(z_1 - 1)}{z_1}(1-\rho_1)\sum_{i=0}^{m_1}\begin{pmatrix} m_1 \\ i \end{pmatrix}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}}{\left(z_0 - \lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\right]}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho_1)\sum_{i=0}^{m_1}\begin{pmatrix} m_1 \\ i \end{pmatrix}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{z_0 - \lambda_{11}^i\lambda_{21}^{m_1-i}}$$

$$= (1-\rho_1)(z_1-1)\sum_{i=0}^{m_1}\begin{pmatrix} m_1 \\ i \end{pmatrix}\frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1-i}}{z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}}$$

As we expect, the result is again reduced to the same PGF of a multiplexer with single queue.

3.4.4 $Q(z_0, y_0, z_1, y_1)$ at $m_1 = 0$

If $m_1 = 0$, then there is no low-priority traffic in the system, and the low-priority queue doesn't exist, we have $\rho_1 = 0$, thus $\rho = \rho_0$. In this case, the system should also be reduced to a multiplexer with single queue. Now let us show this is true.

Let us substitute 0 for m_0 in 3.68, we have:

$$Q(z_0, y_0, z_1, y_1)$$

$$= \frac{(z_0 - z_1)(z_1 - 1)}{z_1}(1-\rho_0)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}}{\left(z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}\right)(z_1 - 1)}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho_0)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{(C_{10}\lambda_{10})^k (C_{20}\lambda_{20})^{m_0-k}}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}}$$

$$= (1-\rho_0)(z_0-1)\sum_{k=0}^{m_0}\binom{m_0}{k}\frac{(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}}$$

We can see that as expected the system has been reduced to a multiplexer with single queue.

3.4.5 $Q(z, y_0, z, y_1)$

If we drop the priority then the two queues reduce to a single queue. In this case, again the system should reduce to a multiplexer with two types of traffic but without priority. In $Q(z, y_0, z, y_1)$, we should not distinguish between z_1 and z_0.

Let us define A_1, A_2 as before in subsection 3.4.1

Letting $z_0 = z_1 = z$, A_1, A_2 becomes :

$$A_1 = \frac{(z-z)(z-1)}{z}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(z-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

$$A_2 = \frac{z(z-1)}{z}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}}{z-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}}$$

It is easy to see that the first term $A_1 = 0$. Only the second term A_2 remains there. Thus

$Q(z, y_0, z, y_1)$

$$= (z-1)(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}}{z-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}}$$

we can see that as expected this result is same as that of a multiplexer with two types of traffic without priority given in [22].

3.4.6 $Q(0, 0, z_1, y_1)$

$Q(0,0,z_1,y_1)$ is the steady-state PGF of the boundary function, in order to get this result, we should substitute 0 for z_0 and y_0 in our solution 3.68. Recall $f_i(0) = 0$, from equation 3.11 we have $\lambda_{10}\,|_{z_0=0} = 0$ and $\lambda_{20}\,|_{z_0=0} = \beta_0$; from equation 3.14 we have $C_{10}\,|_{z_0=0} = 0$ and $C_{20}\,|_{z_0=0} = 1$. Therefore, we have

$$Q(0,0,z_1,y_1)$$

$$= \frac{(0-z_1)(z_1-1)}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\beta_0^{m_0}\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(0-\beta_0^{m_0}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

$$= (1-\rho)(z_1-1)\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\frac{\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\right)}{z_1-\lambda_{11}^j\lambda_{21}^{m_1-j}\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\right)} \quad (3.70)$$

Actually, we have an alternative way to determine $Q(0,0,z_1,y_1)$. In section 3.2, we have obtained the unknown boundary function $Q_k(0,0,z_1,y_1)$ with respect to the discrete time k, then we may perform transformation on it and finally apply the final value theorem to determine the steady-state PGF of the boundary function $Q(0,0,z_1,y_1)$. We show this method and the corresponding result in Appendix D, from there we can see that the result through the transformation technique is the same as the result in 3.70.

3.4.7 $Q(0,0,1,1)$

Next, let us consider the specific value of $Q(0,0,1,1)$. According to the definition of $Q(0,0,z_1,y_1)$, $Q(0,0,1,1)$ should be equal to $1-\rho_0$. In the following, we will show it is true for our solution. Substituting 1 for z_1 and y_1 in 3.70, we have $(C_{11}\lambda_{11})^j\,(C_{21}\lambda_{21})^{m_1-j} = 0$ except at $j=0$, thus

$$Q(0,0,1,1) = (1-\rho)(z_1-1)\left.\frac{\Gamma\left(\lambda_{21}^{m_1}\right)}{z_1-\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)}\right|_{z_1=1}$$

Since $\Gamma\left(\lambda_{21}^{m_1}\right)|_{z_1=1} = \Gamma(1) = 1$, $Q(0,0,1,1)$ is $\frac{0}{0}$ type, after we apply L'Hospital's rule with respect to z_1, and noting (see Appendix A equation A.8),

$$\left.\frac{d\left[\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{dz_1}\right|_{z_1=1} = \frac{\rho_1}{1-\rho_0}$$

we have

$$Q(0,0,1,1) = \frac{1-\rho}{1-\left.\frac{d\left[\lambda_{21}^{m_1}\Gamma\left(\lambda_{21}^{m_1}\right)\right]}{dz_1}\right|_{z_1=1}} = \frac{(1-\rho)(1-\rho_0)}{1-\rho_0-\rho_1} = \frac{(1-\rho)(1-\rho_0)}{1-\rho} = 1-\rho_0$$

This is what we expect.

3.5 Performance Analysis of the Low-Priority Queue

Since we have shown that the marginal PGF of the high-priority queue $Q(z_0,y_0,1,1)$ is the same as the multiplexer result with single type of traffic given in [18], and, the behavior of the high-priority queue is not affected by the low-priority queue, then the performance analysis of the high-priority queue will be the same as in [18]. Therefore, in the following analysis, we will focus on the performance analysis of the low-priority queue.

3.5.1 The PGF of the Length Distribution for the Low-Priority Queue

Let $P(z_1)$ be the steady-state PGF of the low-priority queue length distribution, then $P(z_1) = Q(1, 1. z_1, 1)$. Let us substitute 1 for y_1 in the marginal PGF 3.69 for the low-priority queue, we have $P(z_1)$ as the following:

$$P(z_1) = -\frac{(z_1 - 1)^2}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i\left(\tilde{C}_{21}\lambda_{21}\right)^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(1-\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{(z_1 - 1)}{z_1}(1-\rho)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i\left(\tilde{C}_{21}\lambda_{21}\right)^{m_1-i}}{1-\lambda_{11}^i\lambda_{21}^{m_1-i}} \tag{3.71}$$

where $\tilde{C}_{11} = C_{11}\,|_{y_1=1}$, $\tilde{C}_{21} = C_{21}\,|_{y_1=1}$.

Following the similar steps as we have derived in subsection 3.5.2, if the high-priority traffic load $\rho_0 = 0$, then its busy period will also approach to zero, thus $\lim_{\rho_0\to 0}\Gamma(w) = 1$; once again, if $\rho_0 = 0$, $\rho = \rho_0 + \rho_1 = \rho_1$; therefore,

$$P(z_1) = -\frac{(z_1 - 1)^2}{z_1}(1-\rho_1)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i\left(\tilde{C}_{21}\lambda_{21}\right)^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}}{\left(1-\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^i\lambda_{21}^{m_1-i}\right]}$$

$$+\frac{(z_1 - 1)}{z_1}(1-\rho_1)\sum_{i=0}^{m_1}\binom{m_1}{i}\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i\left(\tilde{C}_{21}\lambda_{21}\right)^{m_1-i}}{1-\lambda_{11}^i\lambda_{21}^{m_1-i}}$$

After we combine these two term, we have

$$P(z_1) = (z_1 - 1)(1 - \rho_1) \sum_{i=0}^{m_1} \binom{m_1}{i} \frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1 - i}}{z_1 - \lambda_{11}^i \lambda_{21}^{m_1 - i}},$$

as expected, $P(z_1)$ approaches to the single multiplexer result given in [22].

3.5.2 Mean Queue Length of the Low-priority Queue

With the steady-state PGF of the low-queue length $P(z_1)$, it is easy to calculate the mean queue length of the low priority queue, which is $\overline{N}_1 = P'(z_1)\,|_{z_1=1}$.

We note that in the equation 3.71, if $z_1 \to 1$, except at $i = 0$,

$$\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1 - i} \lambda_{11}^i \lambda_{21}^{m_1 - i} \Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1 - i}\right)}{\left(1 - \lambda_{11}^i \lambda_{21}^{m_1 - i}\right)\left[z_1 - \lambda_{11}^i \lambda_{21}^{m_1 - i} \Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1 - i}\right)\right]} = 0$$

and

$$\frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1 - i}}{1 - \lambda_{11}^i \lambda_{21}^{m_1 - i}} = 0$$

thus let us define:

$$F(z_1) = \sum_{i=1}^{m_1} \binom{m_1}{i} \frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1 - i} \lambda_{11}^i \lambda_{21}^{m_1 - i} \Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1 - i}\right)}{\left(1 - \lambda_{11}^i \lambda_{21}^{m_1 - i}\right)\left[z_1 - \lambda_{11}^i \lambda_{21}^{m_1 - i} \Gamma\left(\lambda_{11}^i \lambda_{21}^{m_1 - i}\right)\right]};$$

$$E(z_1) = \sum_{i=0}^{m_1} \binom{m_1}{i} \frac{\left(\tilde{C}_{11}\lambda_{11}\right)^i \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1 - i}}{1 - \lambda_{11}^i \lambda_{21}^{m_1 - i}};$$

then we have, $F(z_1)\,|_{z_1=1} = 0$, $E(z_1)\,|_{z_1=1} = 0$.

Let us define again that $G(z_1) = \left(\tilde{C}_{21}\lambda_{21}\right)^{m_1}$, $H(z_1) = \lambda_{21}^{m_0}$; then $G(z_1)\,|_{z_1=1} = 1$,

$H(z_1)\,|_{z_1=1} = 1$. With these definitions, $P(z_1)$ simplifies to:

$$P(z_1) = -\frac{(z_1-1)^2}{z_1}(1-\rho)\left[E(z_1) + \frac{G(z_1)H(z_1)\Gamma\left(H(z_1)\right)}{\left[1-H(z_1)\right]\left[z_1 - H(z_1)\Gamma\left(H(z_1)\right)\right]}\right]$$

$$+\frac{(z_1-1)}{z_1}(1-\rho)\left[F(z_1) + \frac{G(z_1)}{1-H(z_1)}\right]$$

We note that $H(z_1)$ always comes up with $\Gamma\left(H(z_1)\right)$, then let us define

$$\Theta(z_1) = H(z_1)\Gamma\left(H(z_1)\right) \tag{3.72}$$

then $P(z_1)$ becomes:

$$P(z_1) = -\frac{(z_1-1)^2}{z_1}(1-\rho)\left[E(z_1) + \frac{G(z_1)\Theta(z_1)}{\left[1-H(z_1)\right]\left[z_1 - \Theta(z_1)\right]}\right]$$

$$+\frac{(z_1-1)}{z_1}(1-\rho)\left[F(z_1) + \frac{G(z_1)}{1-H(z_1)}\right],$$

When we take first order derivative of $P(z_1)$ with respect to z_1 and then substitute 1 for z_1 in above equation, we will have 0 in both the numerator and denominator; therefore first let us move the denominator to the LHS in above equation, then we have:

$$z_1\left[1-H(z_1)\right]\left[z_1 - \Theta(z_1)\right]P(z_1)$$

$$= -(z_1-1)^2(1-\rho)\left\{E(z_1)\left[1-H(z_1)\right]\left[z_1-\Theta(z_1)\right] + G(z_1)\Theta(z_1)\right\}$$

$$+z_1(z_1-1)(1-\rho)\left\{F(z_1)\left[1-H(z_1)\right]\left[z_1-\Theta(z_1)\right] + G(z_1)\left[z_1-\Theta(z_1)\right]\right\} \tag{3.73}$$

Let us take third order derivative of both sides of equation 3.73 with respect to z_1, then let $z_1 \to 1$, and we note that $F(z_1)\,|_{z_1=1} = 0$, $E(z_1)\,|_{z_1=1} = 0$, $G(z_1)\,|_{z_1=1} = 1$,

$H(z_1)\,|_{z_1=1} = 1$, then we have:

$$\overline{N}_1 = P'(z_1)\,|_{z_1=1}$$

$$= \frac{-2H'(1) - H''(1) + [2 - 2\rho + 2H'(1) + H''(1) + 2G'(1)(1-\rho)]\,\Theta'(1) + [1 - \rho + H'(1)]\,\Theta''(1)}{2H'(1)\,[1 - \Theta'(1)]}$$

where

$$G'(1) = \frac{m_1(1-\beta_1)(3 - 2\alpha_1 - 2\beta_1)}{(2 - \alpha_1 - \beta_1)^2}\overline{f}_1.$$

$$H'(1) = \frac{m_1(1-\beta_1)\overline{f}_1}{2 - \alpha_1 - \beta_1} = \rho_1,$$

$$H''(1) = \frac{m_1(m_1 - 1)(1-\beta_1)^2}{(2 - \alpha_1 - \beta_1)^2}(\overline{f}_1)^2 + \frac{2m_1(1-\alpha_1)(1-\beta_1)(\alpha_1 + \beta_1 - 1)}{(2 - \alpha_1 - \beta_1)^3}(\overline{f}_1)^2 + \frac{m_1(1-\beta_1)}{2 - \alpha_1 - \beta_1}f_1''(1)$$

the closed-form expressions for $\Theta'(1)$ and $\Theta''(1)$ are given in Appendix A. Now every term in the expression of mean queue length for the low-priority queue, $\overline{N}_1$, is determined, so we can calculate the mean queue length easily.

3.5.3 Mean Delay of the Low-Priority Packets

From the well known Little's result, the mean packet delay of the low-priority queue $\overline{d}_1$ is given by,

$$\overline{d}_1 = \frac{\overline{N}_1}{\rho_1} = \frac{\overline{N}_1(2 - \alpha_1 - \beta_1)}{m_1(1-\beta_1)\overline{f}_1}$$

3.5.4 Variance of the Low-Priority Queue Length

According to the queueing theory, the variance of the low-priority queue length, $\sigma^2_{N_1}$ is given by :

$$\sigma^2_{N_1} = P''(1) + P'(1)\left[1 - P'(1)\right],$$

where $P''(1) = P''(z_1)\left|_{z_1=1}\right.$, $P'(1) = P'(z_1)\left|_{z_1=1}\right.$.

$P'(1)$ is the mean queue length of the low-priority queue which has been determined. In the next, we will determine the second order moment of the low-priority queue length, $P''(1)$.

Let us take fourth order derivative of both sides of equation 3.73 with respect to z_1, then let $z_1 \to 1$, and we note that $F(z_1)\left|_{z_1=1}\right. = 0$, $E(z_1)\left|_{z_1=1}\right. = 0$, $G(z_1)\left|_{z_1=1}\right. = 1$, $H(z_1)\left|_{z_1=1}\right. = 1$, then we have:

$$P''(1) = \frac{F_1 + F_2 + F_3 + F_4 + F_5}{6H'(1)\left[\Theta'(1) - 1\right]}$$

where

$$F_1 = -6H''(1)\Theta'(1) + 6\rho\Theta''(1) + 12\rho G'(1)\Theta'(1) + 2H'''(1) + 6H''(1) - 6H'(1)\Theta''(1)$$

$$F_2 = -6\Theta''(1) - 2\Theta'''(1) - 12G'(1)\Theta'(1) - 6G''(1)\Theta'(1) - 6G'(1)\Theta''(1) - 6H'(1)\Theta''(1)P'(1)$$

$$F_3 = -12\rho F'(1)H'(1)\Theta'(1) - 3H''(1)\Theta''(1) - 2H'(1)\Theta'''(1) + 6H''(1)P'(1) + 12H'(1)P'(1)$$

$$F_4 = 12F'(1)H'(1)\Theta'(1) - 6H''(1)P'(1)\Theta'(1) - 12H'(1)P'(1)\Theta'(1) + 12\rho F'(1)H'(1)$$

$$F_5 = 6\rho G''(1)\Theta'(1) + 6\rho G'(1)\Theta''(1) - 2H'''(1)\Theta'(1) + 2\rho\Theta'''(1) - 12F'(1)H'(1))$$

In the expression of $P''(1)$, $F'(1) = F'(z_1)\left|_{z_1=1}\right.$, which is easy to compute, and $G'(1)$,

$H'(1)$, $H''(1)$ are already determined in the previous section, $G''(1)$, $H'''(1)$ are easy to compute by taking derivatives and then substituting 1 for z_1. In Appendix A, we present the closed-form expressions for $\Theta'(1)$, $\Theta''(1)$ and $\Theta'''(1)$. Therefore we can determine $P''(1)$ so that finally we can compute the the variance of the low-priority queue length, $\sigma^2_{N_1}$.

Chapter 4

Numerical Results

In this chapter we present some numerical examples regarding the results of our performance analysis. As has been shown that the behavior of the high-priority queue is not affected by the low-priority queue, and it is determined by a multiplexer with no priority mechanism (see [18]). Thus in this chapter we present results mainly for the low-priority queue. In our simulation, for all the numerical results, we assume that an On source generates a single packet during a slot, that is we assume $f_i(z_i) = z_i$.

Figure 4.1 presents the mean length of low-priority queue $\overline{N}_1$ versus the low-priority traffic load ρ_1 with high-priority traffic load as a parameter. We set $\alpha_0 = \alpha_1 = 0.75$, $m_0 = 100$, $m_1 = 120$, and β_0, β_1 vary corresponding to different values of ρ_0 and ρ_1 respectively. From left to right, the curves are for $\rho_0 = 0.3$, $\rho_0 = 0.2$, $\rho_0 = 0.1$, $\rho_0 = 0.01$, $\rho_0 = 0$ respectively. We may see how the system load ρ_0 for the high-priority queue affects the low-priority queue: as ρ_0 decreases, low-priority queue length decreases, because the server can spend more time on serving the low priority packets. Also we can see as ρ_0 decreases, low-priority mean queue length approaches to that of the mean queue length without priority mechanism which is $\rho_0 = 0$. For the curve of that $\rho_0 = 0.3$, the mean low-priority queue length approaches to infinity when $\rho_1$approaches 0.7, because at this point, the total system load $\rho = \rho_0 + \rho_1 = 1$. For the curve of that $\rho_0 = 0.2$, the mean low-priority queue length approaches to infinity when $\rho_1$approaches 0.8, and so on and so forth. This shows that the mean queue length of low-priority queue is determined by the total system traffic load, not only by its own traffic load.

Figure 4.2 presents the mean queue length versus the number of low-priority sources in the system with high and low-priority traffic load as parameters. We set $\alpha_0 = \alpha_1 = 0.75$, and fix $m_0 = 20$. The traffic load ρ_0 and ρ_1 are also fixed for each curve, for the upper two curves, the total load $\rho = 0.9$, while for the lower two curves, the total load is $\rho = 0.7$. Here, it may be seen that our analysis method can handle large number of sources. As the traffic generated by a source approaches to zero or one, the burstiness of a source increases (see [20]). Since the low-priority load is held constant in this figure, as the number of low-priority sources, m_1 increases, the traffic generated by each source approaches to zero, therefore its burstiness increases. On the other hand, as a result of the statistical multiplexing, increasing the number of the sources smoothes out the superposed traffic. From this figure, it can be seen that for constant values of traffic loads, low-priority queue length increases slightly with the number of sources which means the burstiness over-weighs traffic smoothing. Another conclusion we can draw from this figure is that if the number of sources is not small, $m_1 > 20$, the mean queue length is not affected much by the number of sources, as we may see the four curves are almost flat when $m_1 > 20$.

Figure 4.3 presents the mean packet delay $\overline{d}_1$ for the low-priority traffic as a function of the high-priority traffic load ρ_0 for constant values of low-priority traffic load. It can be seen that mean packet delay increases as the low-priority traffic load increases, it also can be seen that mean packet increases as the high-priority traffic load increases for any given value of low-priority queue load.

Figure 4.4 demonstrates the significance of the correlation in the arrival process. This figure presents the mean packet delay $\bar{d}_1$ for low-priority traffic as a function of the sum of the *On* and *Off* probabilities of a low-priority traffic source, $\alpha_1 + \beta_1$, for constant values of high and low-priority traffic loads ρ_0 and ρ_1. As well known, an uncorrelated source corresponds to $\alpha_1 + \beta_1 = 1$ and it generates Bernoulli arrivals which are independent from one slot to the next. It may be seen that an uncorrelated source is at the low end of the delay curve. As the correlation increases, the mean packet delay increases although the high and low-priority traffic loads ρ_0 and ρ_1 are kept constant.

Figure 4.5 presents the standard deviation of the low-priority queue length versus the low-priority traffic load ρ_1 with the number of low-priority sources m_1 as a parameter. It shows that the queue length variation increases as the traffic load ρ_1 increases. We can see the two curves, $m_1 = 20$ and $m_1 = 100$, are very close to each other, this is in agreement with the result of figure 4.2, showing that the standard deviation, as well as the mean queue length, is not affected much by the number of sources when $m_1 > 20$.

Figure 4.6 also presents the standard deviation of the low-priority queue length versus the low-priority traffic load ρ_1 but with the high-priority traffic load ρ_0 as a parameter. From the figure, we can see that the deviation of the low priority queue length increases with the high-priority traffic load ρ_0 , which means larger variations from the low-priority queue length will occur as ρ_0 increases.

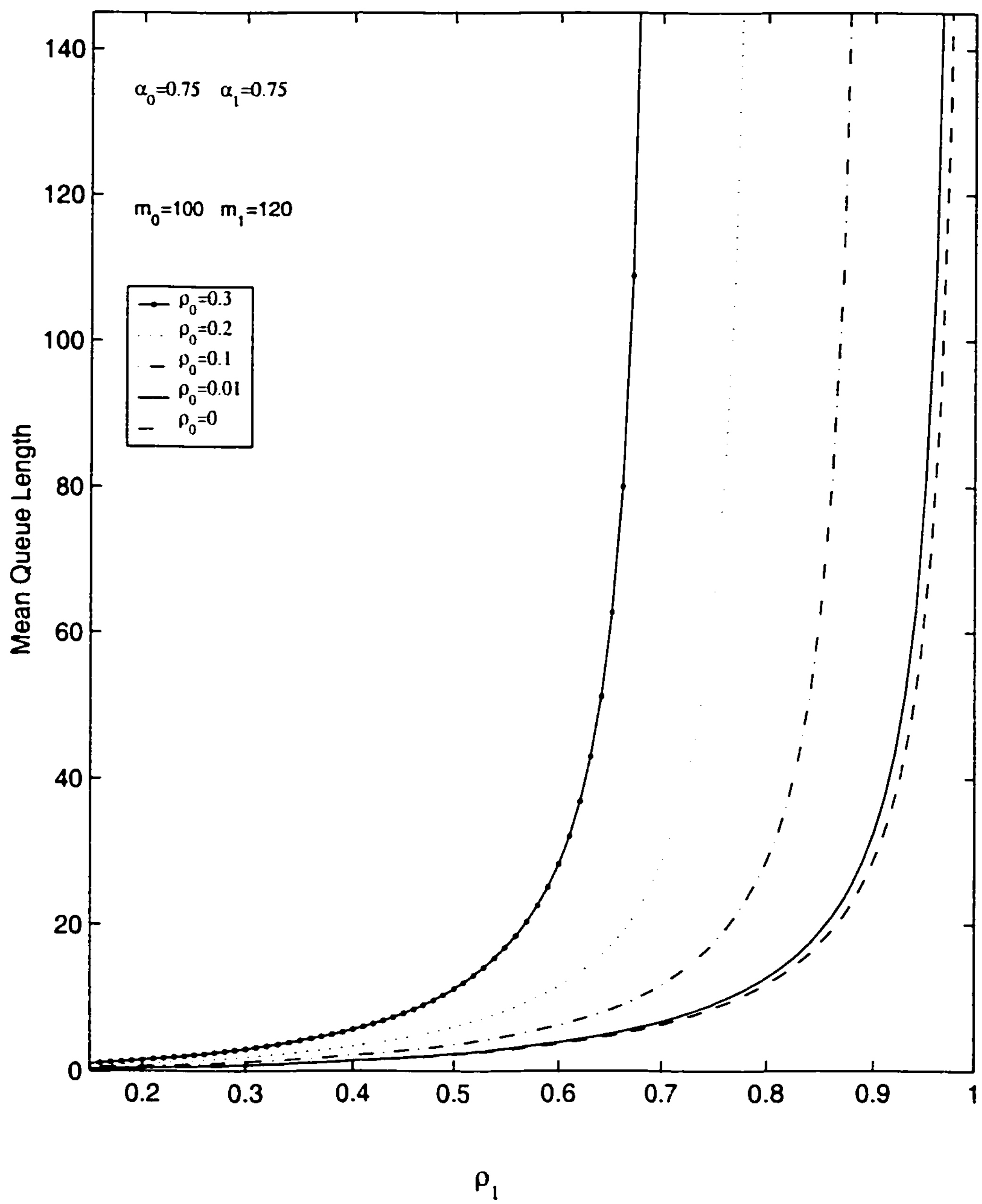

Figure 4.1: Mean queue length versus its load for the low-priority queue

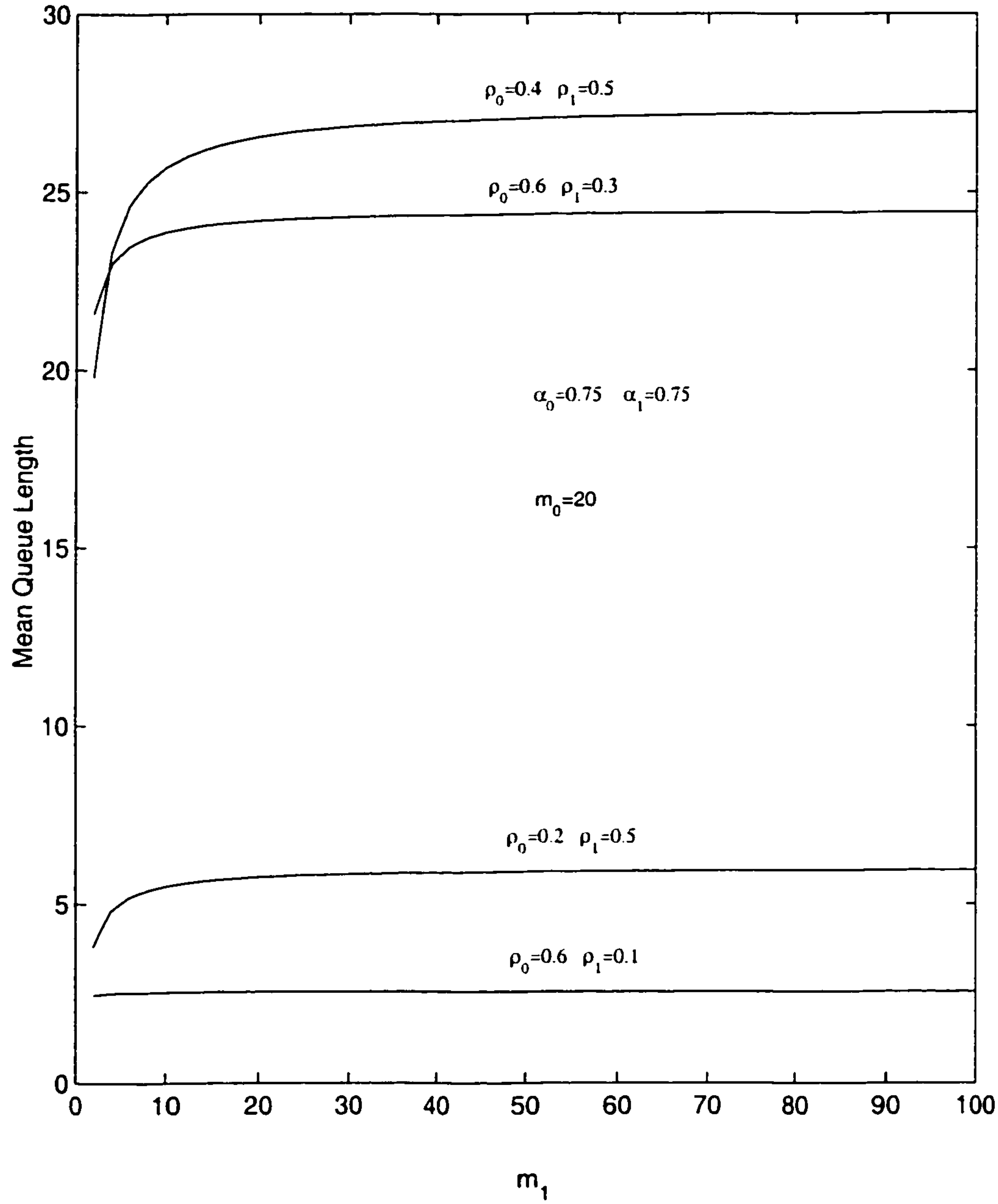

Figure 4.2: Mean queue length versus the number of sources for the low-priority traffic

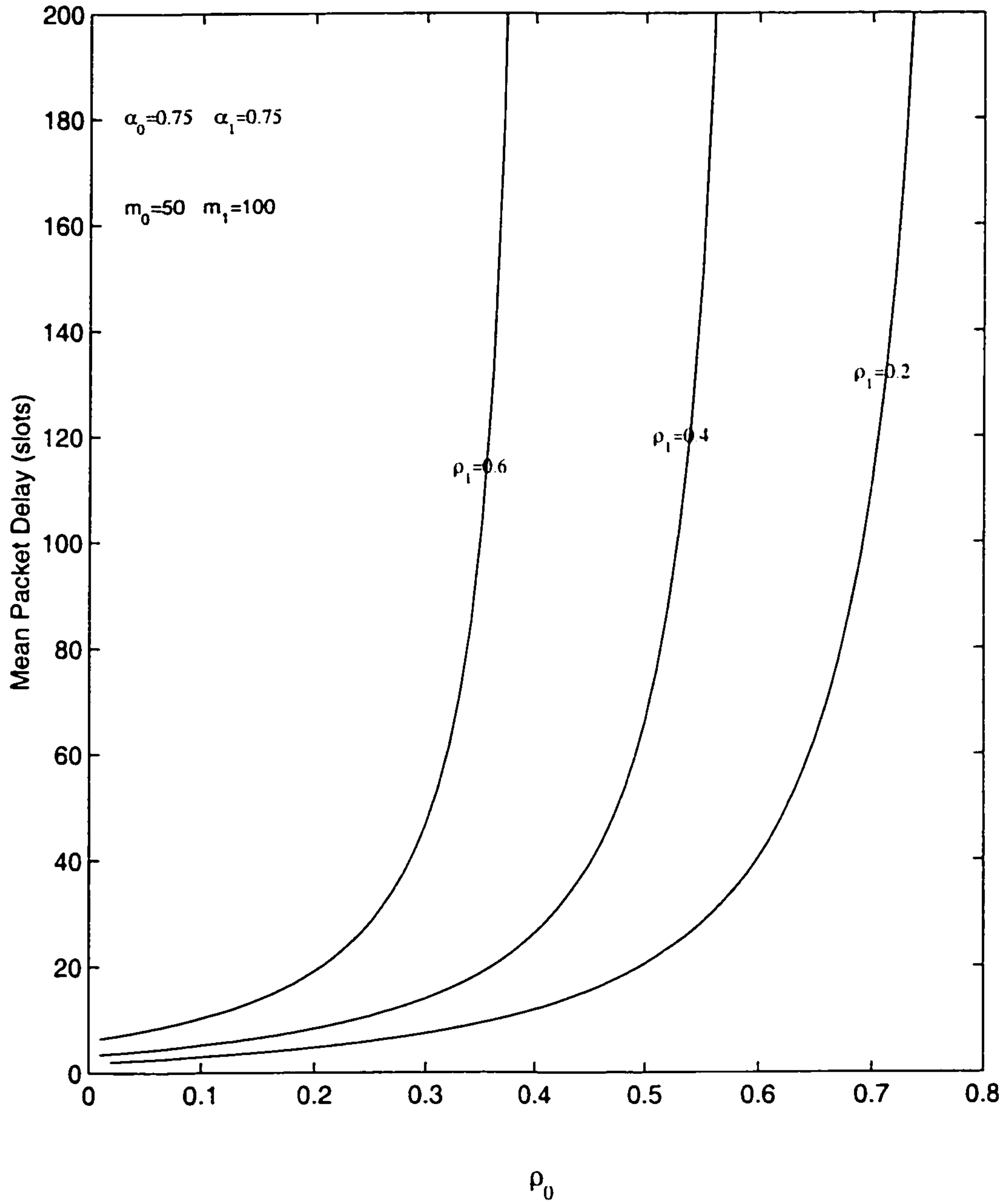

Figure 4.3: Mean packet delay for low-priority traffic versus high-priority traffic load

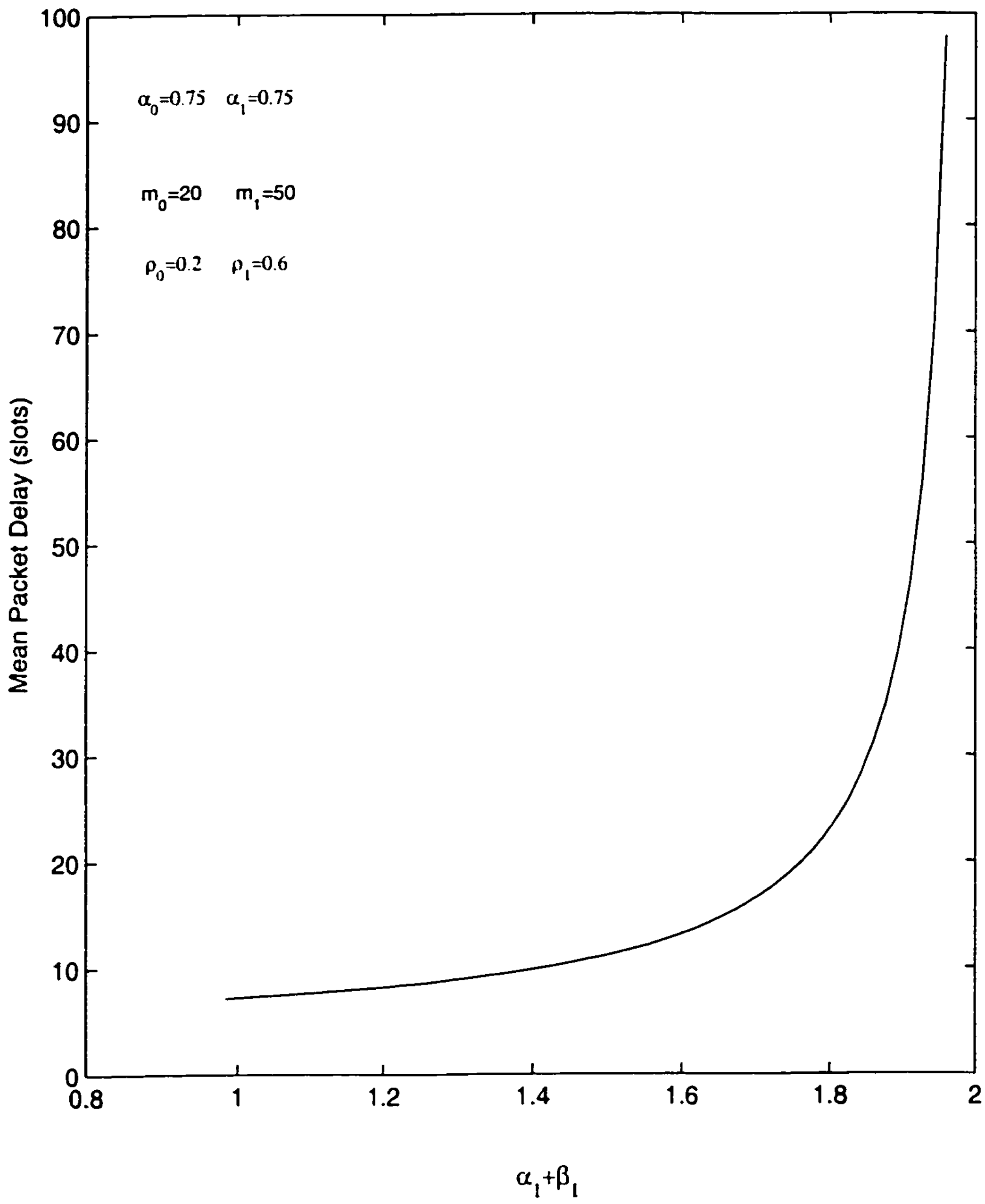

Figure 4.4: Mean packet delay versus sum of a source's *On* and *Off* probabilities for low-priority queue

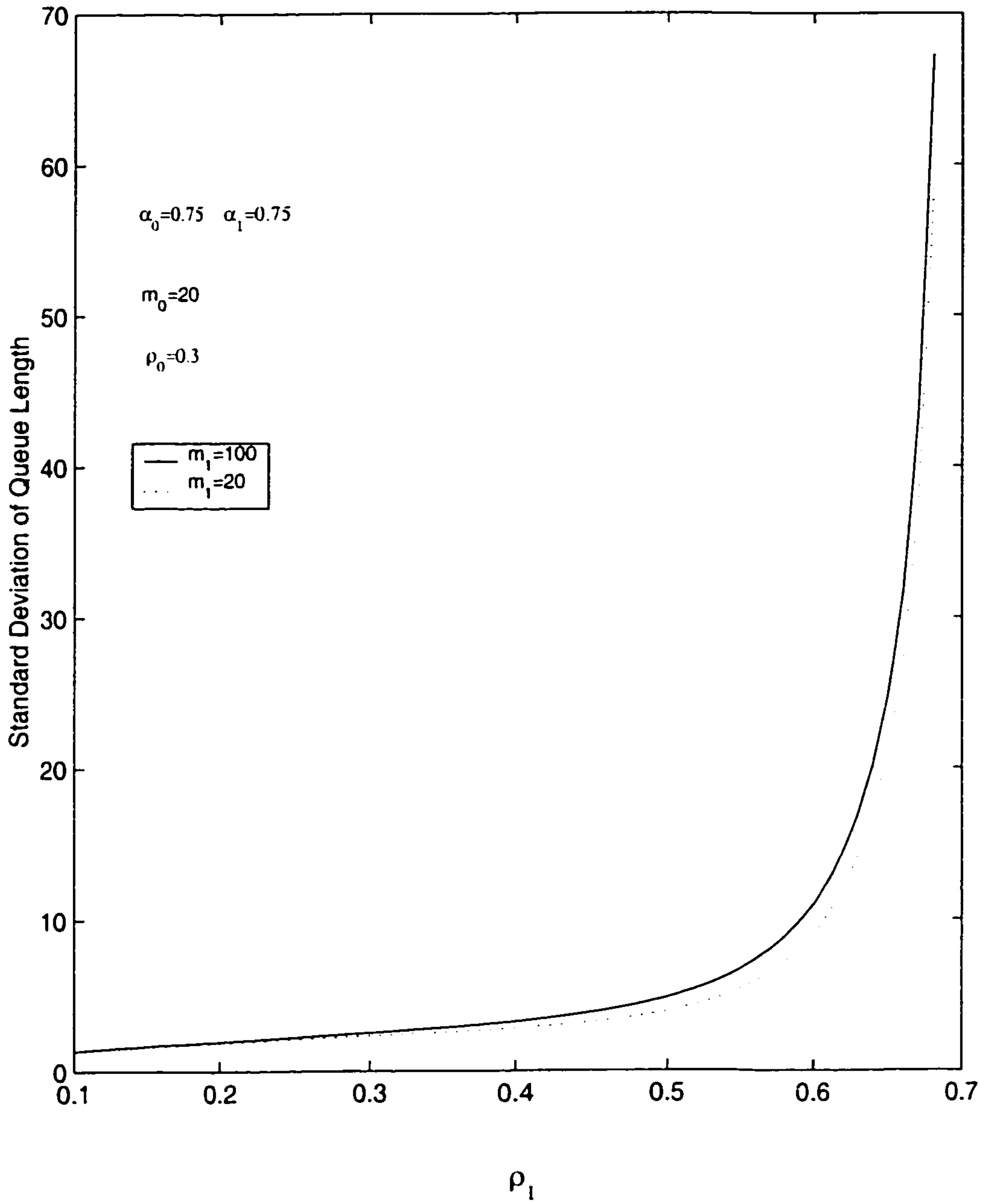

Figure 4.5: Standard deviation of the low-priority queue length versus its traffic load

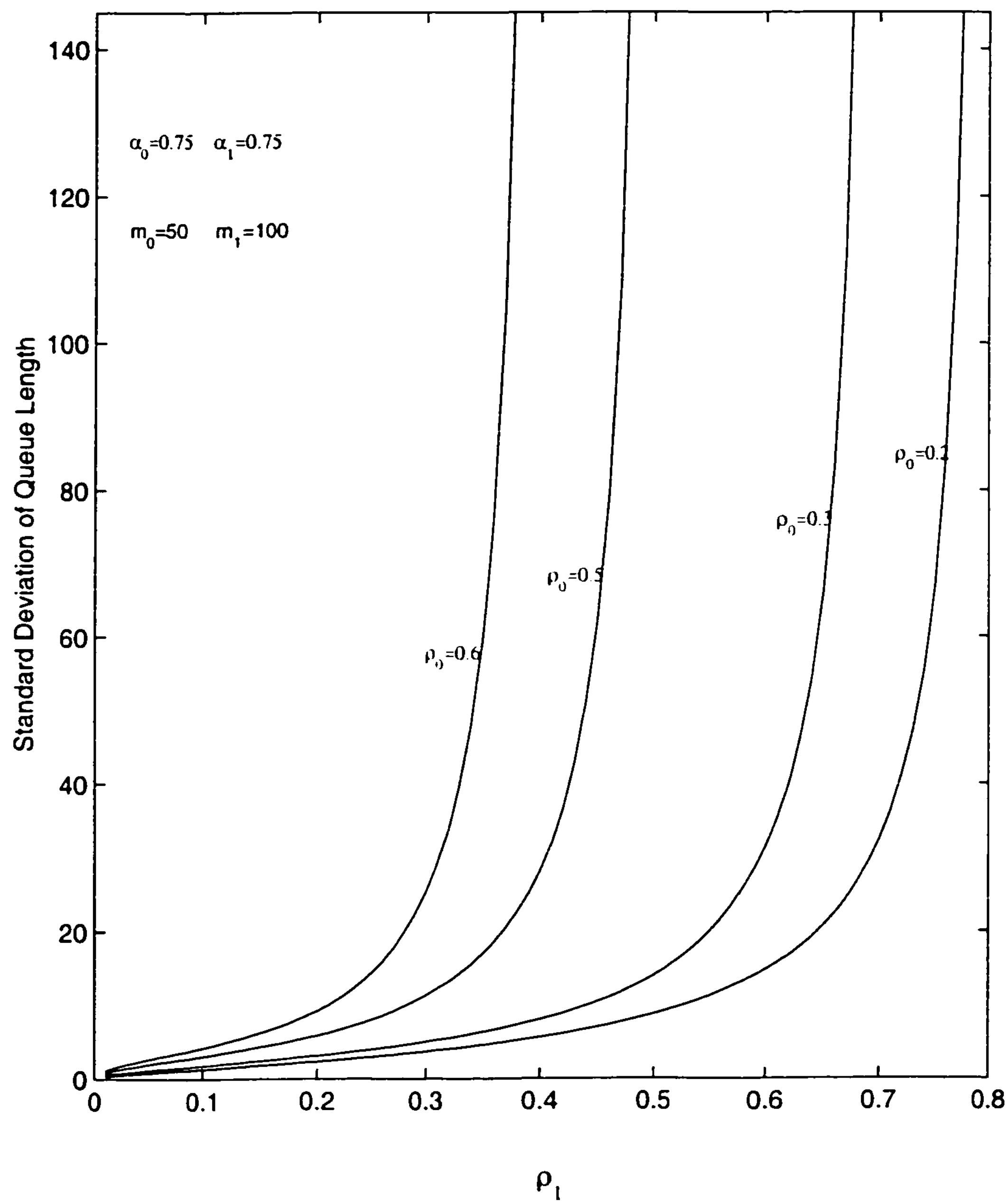

Figure 4.6: Standard deviation of the low-priority queue length versus its traffic load

Chapter 5

Contributions and Conclusions

5.1 Main Contributions

The main contributions of this thesis can be briefly summarized as follows:

First, we derive explicit closed-form expressions for the joint Probability Generating Function (PGF) of the system, as well as for the high and low priority queues. Comparing with the results using matrix geometric technique given in [15] and [16], our results are easier to follow.

Second, we obtain the closed-form expression for the mean queue length, mean packet delay and variance of queue length for the low-priority queue. These results are easy for people to follow and calculate specific values, and we can handle very large number of sources such as over 10,000, since we don't have the limitation of state-space requirements for matrix computation.

Third, we have derived our results for more general arrival process compared with those assumed in [14], [15] and [17]. We have two types of independent sources, within each type, the sources are independent of each others. This assumption is closer to the real networking environment, as the networking traffic is generated by different applications, and these applications are independent of each other. But in our model, the number of packets generated by the same source in different slots are correlated, which again reflects the reality better since the packets from the same message source such as a big file or image have strong correlation.

Forth, since the server is unavailable to the low-priority queue for a interval which is equal to what is called the *busy period* of the high-priority queue, the busy period of the high-priority queue has been used to determine the unknown boundary function $Q_k(0, 0, z_1, y_1)$. This boundary function has been a difficulty in many performance analysis problems. Also we have determined the transform of the busy period distribution, $\Gamma(\omega)$. As from the viewpoint of low-priority queue, the server is available only intermittently, the technique we have developed is general and it can be applied to solve problems that the sever (or transmitting line, output port...) is available intermittently.

5.2 Conclusions

This thesis presents a performance analysis for a discrete-time multiplexer with two priority queues and correlated arrivals. In the queue model the low-priority queue under consideration is served only if the high-priority queue is empty and the correlated arrival process consists of two types of binary Markov On/Off sources. We have obtained the joint steady-state PGF of the high and low priority queue lengths and the number of On sources, from which, the performance metrics such as mean queue length, mean delay and queue length variance are expressed in closed-forms that are easy to calculate for any number of sources.

At first, through an embedded Markov chain analysis, a functional equation relating the PGFs of the system between two consecutive slots is obtained. Then we transform the functional equation into a suitable new form which is mathematically tractable. Since the low-priority queue could be served only when the high-priority queue is not in its busy period, we introduced the busy period of the high-priority queue to determine the unknown boundary function in the new form. Finally we can determine the joint steady-state PGF of the high and low priority queue lengths and the number of On sources. From this result, we derive the marginal PGFs of the high and low-priority queues, as well as the PGF of the low-priority queue length. It has been shown that the results can reduce to that of a multiplexer without priority when priority is dropped given in [18] and [22]. In the end, we present some numerical examples regarding the results of this thesis.

The analysis shows that the high-priority queue *does not see* the low-priority queue, its behavior is not affected by the low-priority traffic. But the the low-priority queue is affected by the high-priority queue dramatically. When the high-priority traffic load increases, the mean queue length and variance of the low-priority queue increase correspondingly. In the closed-form of the PGF of the low-priority queue, it can be seen that the high-priority queue affects the low-priority queue through the busy period distribution, which is determined in our analysis.

The results also show that the low-priority queue length increases slightly with the number of sources. The source burstiness increases with the number of sources, but this is over-weighed by the smoothing effect of the statistical multiplexing due to increased number of sources.

The numerical results demonstrate the significance of the correlation in the arrival process. As the correlation increases, the mean queue length and mean packet delay for the low priority queue increase correspondingly, although the traffic for both queues are kept constant.

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Appendix A

The Derivatives of Function $\Theta(z_1)$

In chapter 3, we have defined $\Theta(z_1) = H(z_1)\Gamma(H(z_1))$, where $H(z_1) = \lambda_{21}^{m_1}$ (see equation 3.72). As we need the first order, second order and third order derivatives of $\Theta(z_1)$ at $z_1 = 1$ to determine the mean and variance of queue length of low-priority queue, in this appendix, we will determine the closed-form expressions for $\Theta'(1)$, $\Theta''(1)$ and $\Theta'''(1)$.

*** Determining** $\Theta'(1)$

From equation 3.29 we have

$$\omega\Gamma(\omega) = z^*(\omega),$$

thus

$$\Theta(z_1) = z^*(H(z_1)) \tag{A.1}$$

where $z^*(\omega)$ is the unique root of equation $z_0 = \omega\lambda_{20}^{m_0}$. Thus the first order derivative of $\Theta(z_1)$ is

$$\Theta'(z_1) = \frac{\partial z^*(H(z_1))}{\partial z_1} = \left[\frac{\partial z^*(\omega)}{\partial \omega}\Big|_{\omega=H(z_1)}\right]\frac{\partial H(z_1)}{\partial z_1}$$

we note that at $z_1 = 1$, we have $\omega = H(z_1)\,|_{z_1=1} = 1$, thus

$$\Theta'(z_1)|_{z_1=1} = \left[\frac{\partial z^*(\omega)}{\partial \omega}\,|_{\omega=1}\right] \frac{\partial H(z_1)}{\partial z_1}\bigg|_{z_1=1} \tag{A.2}$$

In the next we shall determine $\frac{\partial z^*(\omega)}{\partial \omega}\Big|_{\omega=1}$ and $\frac{\partial H(z_1)}{\partial z_1}\Big|_{z_1=1}$.

Let us take first order derivative of both sides of equation $z^*(\omega) = \omega\lambda_{20}^{m_0}\Big|_{z_0=z^*(\omega)}$ with respect to ω, then we have the following,

$$\frac{\partial z^*(\omega)}{\partial \omega} = \lambda_{20}^{m_0}\Big|_{z_0=z^*(\omega)} + m_0\omega\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial z_0}{\partial \omega}\Big|_{z_0=z^*(\omega)}$$

$$\frac{\partial z^*(\omega)}{\partial \omega} = \lambda_{20}^{m_0}\Big|_{z_0=z^*(\omega)} + m_0\omega\left[\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}\bigg|_{z_0=z^*(\omega)}\right]\frac{\partial z^*(\omega)}{\partial \omega}$$

Solving for $\frac{\partial z^*(\omega)}{\partial \omega}$ in the above equation, we have

$$\frac{\partial z^*(\omega)}{\partial \omega} = \frac{\lambda_{20}^{m_0}}{1 - m_0\omega\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}}\Bigg|_{z_0=z^*(\omega)}$$

Since we need to determine the specific value $\frac{\partial z^*(\omega)}{\partial \omega}\Big|_{\omega=1}$, we need the root of equation $z^*(\omega) = \omega\lambda_{20}^{m_0}\Big|_{z_0=z^*(\omega)}$ at $\omega = 1$. This root is $z^*(1) = 1$, and because $\lambda_{20}^{m_0}|_{z_0=z^*(1)=1} = 1$, therefore

$$\frac{\partial z^*(\omega)}{\partial \omega}\bigg|_{\omega=1} = \frac{1}{1 - m_0\left(\frac{\partial \lambda_{20}}{\partial z_0}\Big|_{z_0=z^*(1)=1}\right)} \tag{A.3}$$

from 3.11, we have:

$$\lambda_{20} = \beta_0 + \alpha_0 f_0(z_0) + \sqrt{[\beta_0 + \alpha_0 f_0(z_0)]^2 + 4(1-\alpha_0-\beta_0)f_0(z_0)} \tag{A.4}$$

From the above we have

$$\frac{\partial \lambda_{20}}{\partial z_0} = \frac{1}{2}\left[\alpha_0 f_0'(z_0) + \frac{1}{2}\frac{2\left(\beta_0 + \alpha_0 f_0(z_0)\right)\alpha_0 f_0'(z_0) + 4(1-\alpha_0-\beta_0)f_0'(z_0)}{\sqrt{[\beta_0 + \alpha_0 f_0(z_0)]^2 + 4(1-\alpha_0-\beta_0)f_0(z_0)}}\right]$$

next we substitute 1 for z_0, then

$$\left.\frac{\partial \lambda_{20}}{\partial z_0}\right|_{z_0=1} = \frac{1}{2}\left[\alpha_0 f_0'(1) + \frac{1}{2}\frac{2\left(\beta_0 + \alpha_0 f_0(1)\right)\alpha_0 f_0'(1) + 4(1-\alpha_0-\beta_0)f_0'(1)}{\sqrt{[\beta_0 + \alpha_0 f_0(1)]^2 + 4(1-\alpha_0-\beta_0)f_0(1)}}\right]$$

noting that $f_0(1) = 1$, $f_0'(1) = \overline{f_0}$

$$\left.\frac{\partial \lambda_{20}}{\partial z_0}\right|_{z_0=1} = \frac{1}{2}\left[\alpha_0 \overline{f_0} + \frac{1}{2}\frac{2\left(\beta_0 + \alpha_0\right)\alpha_0 \overline{f_0} + 4(1-\alpha_0-\beta_0)\overline{f_0}}{2-\alpha_0-\beta_0}\right]$$

$$= \frac{1-\beta_0}{2-\alpha_0-\beta_0}\overline{f_0} \tag{A.5}$$

Substituting the above result into A.3 we have

$$\left.\frac{\partial z^*(\omega)}{\partial \omega}\right|_{\omega=1} = \frac{1}{1 - m_0\left(\left.\frac{\partial \lambda_{20}}{\partial z_0}\right|_{z_0=1}\right)} = \frac{1}{1 - m_0\frac{1-\beta_0}{2-\alpha_0-\beta_0}\overline{f_0}}$$

but $m_0\frac{1-\beta_0}{2-\alpha_0-\beta_0}\overline{f_0} = \rho_0$, then we have

$$\left.\frac{\partial z^*(\omega)}{\partial \omega}\right|_{\omega=1} = \frac{1}{1-\rho_0} \tag{A.6}$$

Now let us determine $\left.\frac{\partial H(z_1)}{\partial z_1}\right|_{z_1=1}$, from A.5, we have

$$\left.\frac{\partial H(z_1)}{\partial z_1}\right|_{z_1=1} = \left.\frac{\partial(\lambda_{21}^{m_1})}{\partial z_1}\right|_{z_1=1} = m_1\lambda_{21}^{m_1-1}\left.\frac{\partial(\lambda_{21})}{\partial z_1}\right|_{z_1=1} = \rho_1 \tag{A.7}$$

finally from equation A.2, we have

$$\Theta'(1) = \Theta'(z_1)|_{z_1=1} = \left[\frac{\partial z^*(\omega)}{\partial \omega}\bigg|_{\omega=1}\right]\frac{\partial H(z_1)}{\partial z_1}\bigg|_{z_1=1} = \frac{\rho_1}{1-\rho_0} \tag{A.8}$$

*** Determining** $\Theta''(1)$

In the next we will determine $\Theta''(1)$. From equation A.1, we have the second order derivative of $\Theta(z_1)$ as follows:

$$\Theta''(z_1) = \frac{\partial^2 z^*(H(z_1))}{\partial z_1^2} = \left[\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\bigg|_{\omega=H(z_1)}\right]\left(\frac{\partial H(z_1)}{\partial z_1}\right)^2 + \left[\frac{\partial z^*(\omega)}{\partial \omega}|_{\omega=H(z_1)}\right]\frac{\partial^2 H(z_1)}{\partial z_1^2}$$

when we substitute 1 for z_1, we have $\omega = H(z_1)|_{z_1=1} = 1$, thus

$$\Theta''(z_1)|_{z_1=1} = \left[\frac{\partial^2 z^*(\omega)}{\partial \omega^2}|_{\omega=1}\right]\left(\frac{\partial H(z_1)}{\partial z_1}\bigg|_{z_1=1}\right)^2 + \left[\frac{\partial z^*(\omega)}{\partial \omega}|_{\omega=1}\right]\frac{\partial^2 H(z_1)}{\partial z_1^2}\bigg|_{z_1=1} \tag{A.9}$$

In the above equation, we can see that $\frac{\partial H(z_1)}{\partial z_1}\big|_{z_1=1}$ and $\frac{\partial z^*(\omega)}{\partial \omega}|_{\omega=1}$ are determined, but $\frac{\partial^2 z^*(\omega)}{\partial \omega^2}|_{\omega=1}$ is unknown. In the next we will determine it.

After taking second order derivative of both sides of equation $z^*(\omega) = \omega\lambda_{20}^{m_0}\big|_{z_0=z^*(\omega)}$ with respect to ω, we have the following,

$$\frac{\partial^2 z^*(\omega)}{\partial \omega^2} = 2m_0\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial z_0}{\partial \omega}|_{z_0=z^*(\omega)} + m_0(m_0-1)\omega\lambda_{20}^{m_0-2}\left(\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial z_0}{\partial \omega}\right)^2|_{z_0=z^*(\omega)}$$

$$+m_0\omega\lambda_{20}^{m_0-1}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\left(\frac{\partial z_0}{\partial \omega}\right)^2|_{z_0=z^*(\omega)} + m_0\omega\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 z_0}{\partial \omega^2}|_{z_0=z^*(\omega)}$$

or equivalently

$$\frac{\partial^2 z^*(\omega)}{\partial \omega^2}$$

$$= 2m_0 \left[\lambda_{20}^{m_0-1} \frac{\partial \lambda_{20}}{\partial z_0}\Big|_{z_0=z^*(\omega)} \right] \frac{\partial z^*(\omega)}{\partial \omega} + m_0(m_0-1)\omega \left[\lambda_{20}^{m_0-2} \left(\frac{\partial \lambda_{20}}{\partial z_0} \right)^2 \Big|_{z_0=z^*(\omega)} \right] \left(\frac{\partial z^*(\omega)}{\partial \omega} \right)^2$$

$$+ m_0\omega \left[\lambda_{20}^{m_0-1} \frac{\partial^2 \lambda_{20}}{\partial z_0^2}\Big|_{z_0=z^*(\omega)} \right] \left(\frac{\partial z^*(\omega)}{\partial \omega} \right)^2 + m_0\omega \left[\lambda_{20}^{m_0-1} \frac{\partial \lambda_{20}}{\partial z_0}\Big|_{z_0=z^*(\omega)} \right] \frac{\partial^2 z^*(\omega)}{\partial \omega^2}$$

Solving for $\frac{\partial^2 z^*(\omega)}{\partial \omega^2}$ in the above equation, we have

$$\frac{\partial^2 z^*(\omega)}{\partial \omega^2}$$

$$= \frac{2m_0\lambda_{20}^{m_0-1} \frac{\partial \lambda_{20}}{\partial z_0} \frac{\partial z^*(\omega)}{\partial \omega} + m_0(m_0-1)\omega\lambda_{20}^{m_0-2} \left(\frac{\partial \lambda_{20}}{\partial z_0} \frac{\partial z^*(\omega)}{\partial \omega} \right)^2 + m_0\omega\lambda_{20}^{m_0-1} \frac{\partial^2 \lambda_{20}}{\partial z_0^2} \left(\frac{\partial z^*(\omega)}{\partial \omega} \right)^2}{1 - m_0\omega\lambda_{20}^{m_0-1} \frac{\partial \lambda_{20}}{\partial z_0}} \Bigg|_{z_0=z^*(\omega)}$$

Because if $\omega = 1$, $z_0 = z^*(\omega) = z^*(1) = 1$, and $\lambda_{20}|_{z_0=1} = 1$, we have

$$\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\Bigg|_{\omega=1}$$

$$= \frac{2m_0 \frac{\partial \lambda_{20}}{\partial z_0} \frac{\partial z^*(\omega)}{\partial \omega} + m_0(m_0-1) \left(\frac{\partial \lambda_{20}}{\partial z_0} \frac{\partial z^*(\omega)}{\partial \omega} \right)^2 + m_0 \frac{\partial^2 \lambda_{20}}{\partial z_0^2} \left(\frac{\partial z^*(\omega)}{\partial \omega} \right)^2}{1 - m_0 \frac{\partial \lambda_{20}}{\partial z_0}} \Bigg|_{z_0=1,\omega=1} \qquad \text{(A.10)}$$

in the above equation A.10, $\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\Big|_{z_0=1}$ is unknown, but it is easy to determine by taking second order derivative of λ_{20} in A.4 with respect to z_0 and then substituting 1 for z_0, which gives us

$$\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\Bigg|_{z_0=1} = (m_0 - 1) \left(\frac{1-\beta_0}{2-\alpha_0-\beta_0} \right)^2 f_0'(1)$$

$$+ \left[\frac{2(1-\alpha_0)(1-\beta_0)(\alpha_0+\beta_0-1)^2}{(2-\alpha_0-\beta_0)^3} [f_0'(1)]^2 + \frac{1-\beta_0}{2-\alpha_0-\beta_0} f_0''(1) \right] \qquad \text{(A.11)}$$

substituting this result into A.10, we have

$$\left.\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\right|_{\omega=1}$$

$$= \frac{2\rho_0}{(1-\rho_0)^2} + \frac{m_0 - 1}{m_0}\frac{2\rho_0^2}{(1-\rho_0)^3} + \frac{m_0}{(1-\rho_0)^3}(m_0 - 1)\left(\frac{1-\beta_0}{2-\alpha_0-\beta_0}\right)^2 f_0'(1)$$
$$+\frac{m_0}{(1-\rho_0)^3}\left[\frac{2(1-\alpha_0)(1-\beta_0)(\alpha_0+\beta_0-1)^2}{(2-\alpha_0-\beta_0)^3}[f_0'(1)]^2 + \frac{1-\beta_0}{2-\alpha_0-\beta_0}f_0''(1)\right] \tag{A.12}$$

Now we can determine the second order derivative of $\Theta(z_1)$ at $z_1 = 1$. Substituting A.5, A.6, A.11 and A.12 into A.9, we have

$$\Theta''(1) = \Theta''(z_1)\,|_{z_1=1}$$

$$= \frac{2\rho_0^3}{(1-\rho_0)^2} + \frac{m_0 - 1}{m_0}\frac{2\rho_0^4}{(1-\rho_0)^3} + \frac{(m_0-1)(1-\rho_0)^2 + (m_0-1)m_0\rho_0^2}{(1-\rho_0)^3}\left(\frac{1-\beta_0}{2-\alpha_0-\beta_0}\right)^2 f_0'(1)$$
$$+\frac{m_0\rho_0^2 + (1-\rho_0)^2}{(1-\rho_0)^3}\left[\frac{2(1-\alpha_0)(1-\beta_0)(\alpha_0+\beta_0-1)^2}{(2-\alpha_0-\beta_0)^3}[f_0'(1)]^2 + \frac{(1-\beta_0)\rho_0^2}{2-\alpha_0-\beta_0}f_0''(1)\right] \tag{A.13}$$

*** Determining** $\Theta'''(1)$

At last we show how to determine $\Theta'''(1)$. From equation A.1, we have the third order derivative of $\Theta(z_1)$ as follows:

$$\Theta'''(z_1) = \frac{\partial^3 z^*(H(z_1))}{\partial z_1^3}$$

$$\Theta'''(z_1) = \left[\left.\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\right|_{\omega=H(z_1)}\right]\left(\frac{\partial H(z_1)}{\partial z_1}\right)^3 + 3\left[\left.\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\right|_{\omega=H(z_1)}\right]\frac{\partial^2 H(z_1)}{\partial z_1^2}\frac{\partial H(z_1)}{\partial z_1}$$

$$+\left[\frac{\partial z^*(\omega)}{\partial \omega}\Big|_{\omega=H(z_1)}\right]\frac{\partial^3 H(z_1)}{\partial z_1^3}$$

Once again, because we have $\omega = H(z_1)|_{z_1=1} = 1$, thus

$$\Theta'''(z_1)|_{z_1=1} = \left[\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\Big|_{\omega=1}\right]\left(\frac{\partial H(z_1)}{\partial z_1}\Big|_{z_1=1}\right)^3 + 3\left[\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\Big|_{\omega=1}\right]\frac{\partial^2 H(z_1)}{\partial z_1^2}\frac{\partial H(z_1)}{\partial z_1}\Big|_{z_1=1}$$

$$+\left[\frac{\partial z^*(\omega)}{\partial \omega}\Big|_{\omega=1}\right]\frac{\partial^3 H(z_1)}{\partial z_1^3}\Big|_{z_1=1} \tag{A.14}$$

In the above equation, we can see that $\frac{\partial^3 H(z_1)}{\partial z_1^3}\Big|_{z_1=1}$ and $\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\Big|_{\omega=1}$ are unknown. The former expression is easy to determine, in the next we will show how to determine $\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\Big|_{\omega=1}$.

Let us take third order derivative of both sides of equation $z^*(\omega) = \omega\lambda_{20}^{m_0}\big|_{z_0=z^*(\omega)}$ with respect to ω, then we have,

$$\frac{\partial^3 z^*(\omega)}{\partial \omega^3} = \Bigg\{3m_0(m_0-1)\lambda_{20}^{m_0-2}\left(\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial \lambda_{20}}{\partial z_0}\right)^2 + 3m_0\lambda_{20}^{m_0-1}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^2$$

$$+m_0(m_0-1)(m_0-2)\omega\lambda_{20}^{m_0-3}\left(\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial z^*(\omega)}{\partial \omega}\right)^3$$

$$+2m_0(m_0-1)\omega\lambda_{20}^{m_0-2}\left(\frac{\partial \lambda_{20}}{\partial z_0}\right)^2\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0\omega\lambda_{20}^{m_0-1}\frac{\partial^3 \lambda_{20}}{\partial z_0^3}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^3$$

$$+2m_0\omega\lambda_{20}^{m_0-1}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0(m_0-1)\lambda_{20}^{m_0-2}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 z^*(\omega)}{\partial \omega^2}$$

$$+m_0(m_0-1)\omega\lambda_{20}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0(m_0-1)\omega\lambda_{20}^{m_0-2}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^3 z^*(\omega)}{\partial \omega^3}$$

$$+3m_0(m_0-1)\omega\lambda_{20}^{m_0-2}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^3 + 2m_0\lambda_{20}^{m_0-1}\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\Bigg\}\Bigg|_{z_0=z^*(\omega)}$$

Once again, because if $\omega = 1$, $z_0 = z^*(\omega) = z^*(1) = 1$, and $\lambda_{20}|_{z_0=1} = 1$, then we have

$$\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\bigg|_{\omega=1} = \Bigg\{ 3m_0(m_0-1)\left(\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial \lambda_{20}}{\partial z_0}\right)^2 + 3m_0\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^2$$

$$+m_0(m_0-1)(m_0-2)\left(\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial z^*(\omega)}{\partial \omega}\right)^3$$

$$+2m_0(m_0-1)\left(\frac{\partial \lambda_{20}}{\partial z_0}\right)^2\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0\frac{\partial^3 \lambda_{20}}{\partial z_0^3}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^3$$

$$+2m_0\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0(m_0-1)\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 z^*(\omega)}{\partial \omega^2}$$

$$+m_0(m_0-1)\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\frac{\partial z^*(\omega)}{\partial \omega}\frac{\partial^2 z^*(\omega)}{\partial \omega^2} + m_0(m_0-1)\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^3 z^*(\omega)}{\partial \omega^3}$$

$$+3m_0(m_0-1)\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 \lambda_{20}}{\partial z_0^2}\left(\frac{\partial z^*(\omega)}{\partial \omega}\right)^3 + 2m_0\frac{\partial \lambda_{20}}{\partial z_0}\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\Bigg\}\bigg|_{z_0=1,\omega=1}$$

After we solve the above equation with respect to $\frac{\partial^3 z^*(\omega)}{\partial \omega^3}$, and follow the similar steps as we did for $\frac{\partial^2 z^*(\omega)}{\partial \omega^2}\Big|_{\omega=1}$, we can determine $\frac{\partial^3 z^*(\omega)}{\partial \omega^3}\Big|_{\omega=1}$. But the expression is too complicated and will not be presented here. It may be best determined using symbolic software such as Maple.

Now every term in equation A.14 is determined, so we are able to determine $\Theta'''(1)$. Again, because the expression of $\Theta'''(1)$ is too complicated, it will not be presented here.

Appendix B

Final Value Theorem

Assume $x(n)$ is causal sequence,

if

$$X(z) = Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^n$$

and $\sum_{n=0}^{\infty} x(n) z^n$ converges for $|z| < 1$

then

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} [(z-1) X(z)]$$

this theorem is also called Abel's theorem

Proof of the theorem:

Since

$$Z[x(n+1) - x(n)] = \frac{1}{z} X(z) - \frac{1}{z} x(0) - X(z) = \left(\frac{1}{z} - 1\right) X(z) - \frac{1}{z} x(0)$$

then

$$(1-z) X(z) = x(0) + z \cdot Z[x(n+1) - x(n)]$$

taking the limit of both side

$$\lim_{z\to 1}(1-z)\,X(z) = x(0) + \lim_{z\to 1}\sum_{n=0}^{\infty} z\cdot[x(n+1)-x(n)]\,z^{n}$$

$$= x(0) + [x(1)-x(0)] + [x(2)-x(1)] + [x(3)-x(2)] + \cdots$$

$$= x(0) - x(0) + x(\infty)$$

therefore

$$\lim_{z\to 1}(z-1)X(z) = x(\infty)$$

Appendix C

Proof of $|\lambda_{1i}\lambda_{2i}| < 1$

From the characteristic equation:

$$\lambda_i^2 - [\beta_i + \alpha_i f_i(z_i)]\,\lambda_i - (1 - \alpha_i - \beta_i) f_i(z_i) = 0 \qquad i = 0, 1$$

we have

$$\lambda_{1i}\lambda_{2i} = -(1 - \alpha_i - \beta_i) f_i(z_i),$$

thus

$$|\lambda_{1i}\lambda_{2i}| = |(1 - \alpha_i - \beta_i) f_i(z_i)| = |1 - \alpha_i - \beta_i|\,|f_i(z_i)|\,;$$

since $f_i(z_i)$ is a PGF, $|f_i(z_i)| \leq 1$; and $0 < \alpha_i < 1$, $0 < \beta_i < 1$, thus $|1 - \alpha_i - \beta_i| < 1$;

Therefore, $|1 - \alpha_i - \beta_i|\,|f_i(z_i)| < 1$, this shows that $|\lambda_{1i}\lambda_{2i}| < 1$.

Appendix D

Determination of Function $Q(0, 0, z_1, y_1)$

In subsection 3.4.5, we have determined $Q(0, 0, z_1, y_1)$ by just substituting 0 for z_0, y_0 in 3.68. Now we will use another method to determine $Q(0, 0, z_1, y_1)$. We perform transformation on $Q_k(0, 0, z_1, y_1)$, and then we apply the final value theorem to obtain $Q(0, 0, z_1, y_1)$. We will see the result of the alternative method is the same as in subsection 3.4.5.

D.1 Transformation of $Q_k(0, 0, z_1, y_1)$

In this subsection, we will determine the transformation of $Q_k(0, 0, z_l, y_l)$, so that we can apply the final value theorem to find out $Q(0, 0, z_1, y_1)$, which is the steady state PGF of the low priority traffic when there are no high-priority packets and high-priority On sources in the system.

Define the transformation of $Q_k(0, 0, z_1, y_1)$ as

$$Q(0,0,z_1,y_1,w) = \sum_{k=0}^{\infty} Q_k(0,0,z_1,y_1)\omega^k \tag{D.1}$$

from equation 3.42 we have

$$\sum_{k=1}^{\infty} Q_k(0,0,z_1,y_1)\omega^k$$

$$= \sum_{k=1}^{\infty}\sum_{i=1}^{k} \varphi(k-i)\xi(i-1)B_1(k)\omega^k + \sum_{k=0}^{\infty}\sum_{i=1}^{k-1}\sum_{h=1}^{i} \frac{z_1-1}{z_1}\varphi(k-i-1)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k$$

Let A_1, A_2 be the following:

$$A_1 = \sum_{k=1}^{\infty}\sum_{i=1}^{k} \varphi(k-i)\xi(i-1)B_1(k)\omega^k$$

$$A_2 = \sum_{k=1}^{\infty}\sum_{i=1}^{k-1}\sum_{h=1}^{i} \frac{z_1-1}{z_1}\varphi(k-i-1)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k$$

Interchanging the order of summations, A_1 becomes

$$A_1 = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \varphi(k-i)\xi(i-1)B_1(k)\omega^k$$

Let $r = k - i$, then $k = i + r$

$$A_1 = \sum_{i=1}^{\infty}\sum_{r=0}^{\infty} \varphi(r)\xi(i-1)B_1(i+r)\omega^{i+r}$$

Substituting from 3.16 in above equation and then expanding it by binomial theorem yields:

$$A_1 = \sum_{i=1}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{m_1} \binom{m_1}{j} \left(C_{11}\lambda_{11}^{r+i}\right)^j \left(C_{21}\lambda_{21}^{r+i}\right)^{m_1-j} \varphi(r)\xi(i-1)\omega^{r+i}$$

rearranging the terms,

$$A_1 = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \varphi(r)\xi(i-1) \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right)^r \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right)^{i-1} \omega$$

Let us define the transformation of $\varphi(r)$ as

$$\Phi(\omega) = \sum_{r=0}^{\infty} \varphi(r)\omega^r \tag{D.2}$$

then above becomes

$$A_1 = \sum_{j=0}^{m_1} \sum_{i=1}^{\infty} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \Phi\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \xi(i-1) \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\right)^{i-1} \omega$$

let $h = i - 1$ then

$$A_1 = \sum_{j=0}^{m_1} \sum_{h=0}^{\infty} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \Phi\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \xi(h) \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\right)^{h} \omega$$

Let us define the transformation of $\xi(h)$ as:

$$\Gamma(\omega) = \sum_{h=0}^{\infty} \xi(h)\omega^h \tag{D.3}$$

finally A_1 becomes

$$A_1 = \sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \Phi\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \Gamma\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \omega \tag{D.4}$$

Interchanging the order of summations, A_2 becomes:

$$A_2 = \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \sum_{h=1}^{i} \frac{z_1 - 1}{z_1} \varphi(k-i-1)\xi(i-h)B_1(k-h)Q_h(0,0,0,0)\omega^k$$

let $n = k - i - 1$, then $k = n + i + 1$

$$A_2 = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \sum_{h=1}^{i} \frac{z_1 - 1}{z_1} \varphi(n)\xi(i-h)B_1(n+i+1-h)Q_h(0,0,0,0)\omega^{n+i+1}$$

Interchanging the order of summations again, A_2 becomes

$$A_2 = \sum_{n=0}^{\infty} \sum_{h=1}^{\infty} \sum_{i=h}^{\infty} \frac{z_1 - 1}{z_1} \varphi(n)\xi(i-h)B_1(n+i+1-h)Q_h(0,0,0,0)\omega^{n+i+1}$$

let $r = i - h$,

$$A_2 = \sum_{n=0}^{\infty} \sum_{h=1}^{\infty} \sum_{r=0}^{\infty} \frac{z_1 - 1}{z_1} \varphi(n)\xi(r)B_1(n+r+1)Q_h(0,0,0,0)\omega^{n+r+h+1}$$

Let us define the transformation of $Q_0(0,0,0,0)$ as:

$$Q(\omega) = \sum_{h=0}^{\infty} Q_h(0,0,0,0)\omega^h \tag{D.5}$$

$$A_2 = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{z_1 - 1}{z_1} \varphi(n)\xi(r)B_1(n+r+1)\omega^{n+r+1}\left[Q(\omega) - 1\right]$$

Substituting from 3.16 in above equation and then expanding it by binomial theorem yields:

$$A_2 = [Q(\omega) - 1] \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{m_1} \begin{pmatrix} m_1 \\ j \end{pmatrix} \left(C_{11}\lambda_{11}^{n+r+1}\right)^j \left(C_{21}\lambda_{21}^{n+r+1}\right)^{m_1-j} \frac{z_1 - 1}{z_1} \varphi(n)\xi(r)\omega^{n+r+1}$$

Rearranging the right hand side of A_2 , we have:

$$A_2 = [Q(\omega) - 1] \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{m_1} \frac{z_1 - 1}{z_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1 - j}$$

$$\cdot \varphi(n)\xi(r) \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right)^n \left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right)^r \omega$$

Substituting from the definitions of 3.44 and 3.25, finally A_2 becomes:

$$A_2 = [Q(\omega) - 1] \frac{z_1 - 1}{z_1} \sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \Phi\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \Gamma\left(\lambda_{11}^j \lambda_{21}^{m_1-j}\omega\right) \omega \tag{D.6}$$

Now we have

$$\sum_{k=1}^{\infty} Q_k(0,0,z_1,y_1)\omega^k = A_1 + A_2$$

which is equivalent to

$$\sum_{k=0}^{\infty} Q_k(0,0,z_1,y_1)\omega^k - 1 = A_1 + A_2$$

substituting for the summation from definition D.1 we have,

$$Q(0,0,z_1,y_1,\omega) = 1 + A_1 + A_2$$

Substituting for A_1, A_2 from D.4 and D.6 yields

$Q(0,0,z_1,y_1,\omega)$

$$= 1+\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\,\Phi\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega$$

$$+[Q(\omega)-1]\frac{z_1-1}{z_1}\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\,\Phi\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega$$

In the above equation, we note that the first summation and the second summation have some common factors, combining some term, we have

$$Q(0,0,z_1,y_1,\omega)$$

$$= 1+\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\,\Phi\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega$$

$$+Q(\omega)\frac{z_1-1}{z_1}\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\,\Phi\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega \tag{D.7}$$

D.2 Determination of the Steady-state Boundary Function $Q(0,0,z_1,y_1)$

With the results we have obtained in section 3.2.3 and 3.2.4, we are able to determine $Q(0,0,z_1,y_1)$ by applying final value theorem to $Q(0,0,z_1,y_1,\omega)$. $Q(0,0,z_1,y_1)$ is the marginal stead-state joint PGF when there are no high priority packets and no high priority On sources in the system.

Substituting 3.53 into D.7, we have

$$Q(0,0,z_1,y_1,\omega)$$

$$= 1+\sum_{j=0}^{m_1}\begin{pmatrix} m_1 \\ j \end{pmatrix}(C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j}\frac{z_1\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1-\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)}$$

$$+\left[Q(w)-1\right]\frac{z_1-1}{z_1}\sum_{j=0}^{m_1}\begin{pmatrix} m_1 \\ j \end{pmatrix}(C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j}\frac{z_1\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1-\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)}$$

Combining the common terms in the first and second summations, we have

$$Q(0,0,z_1,y_1,\omega)$$

$$= 1+\frac{1}{z_1}\sum_{j=0}^{m_1}\begin{pmatrix} m_1 \\ j \end{pmatrix}(C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j}\frac{\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1-\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)}$$

$$+Q(\omega)\frac{z_1-1}{z_1}\sum_{j=0}^{m_1}\begin{pmatrix} m_1 \\ j \end{pmatrix}(C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j}\frac{z_1\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1-\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)} \tag{D.8}$$

From 3.27 we have

$$\lambda_{11}^j\lambda_{21}^{m_1-i}\omega\Gamma\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right) = 1-\frac{1}{P_0\left(\lambda_{11}^j\lambda_{21}^{m_1-j}\omega\right)}$$

then the above equation becomes

$$Q(0,0,z_1,y_1,\omega)$$

$$= 1 + \sum_{j=0}^{m_1} \binom{m_1}{j} C_{11}^{j} C_{21}^{m_1-j} \frac{P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right) - 1}{(z_1 - 1)\, P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right) + 1}$$

$$+Q(\omega)\frac{z_1 - 1}{z_1} \sum_{j=0}^{m_1} \binom{m_1}{j} \left(C_{11}\lambda_{11}\right)^{j} \left(C_{21}\lambda_{21}\right)^{m_1-j} \frac{z_1 \Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1 - \lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\right)}$$

Applying the final value theorem gives us:

$$Q(0,0,z_1,y_1) = \lim_{k\to\infty} Q_k(0,0,z_1,y_1) = \lim_{\omega\to 1}(1-\omega)Q(0,0,z_1,y_1,\omega)$$

$$= \lim_{\omega\to 1}(1-\omega)$$

$$+ \lim_{\omega\to 1}(1-\omega)\sum_{j=0}^{m_1} \binom{m_1}{j} C_{11}^{j} C_{21}^{m_1-j} \frac{P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right) - 1}{(z_1 - 1)\, P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right) + 1}$$

$$+ \lim_{\omega\to 1}(1-\omega)\left[Q(\omega) - 1\right](z_1 - 1)\sum_{j=0}^{m_1} \binom{m_1}{j} \left(C_{11}\lambda_{11}\right)^{j} \left(C_{21}\lambda_{21}\right)^{m_1-j}$$

$$\cdot \frac{\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\right)\omega}{z_1 - \lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\right)}$$

because

$$\lim_{\omega\to 1}(1-\omega) = 0;$$

$$\lim_{\omega\to 1}(1-\omega)Q(\omega) = \lim_{k\to\infty}(1-\omega)Q_k(0,0,0,0) = 1-\rho$$

Finally we have

$$Q(0,0,z_1,y_1)$$

$$= \sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \left[\frac{\lim_{\omega\to 1}(1-\omega) P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right)}{(z_1-1)\, P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)+1} \right]$$

$$+(1-\rho)(z_1-1)\sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \frac{\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)}{z_1-\lambda_{11}^{j}\lambda_{21}^{m_1-j}\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)} \quad \text{(D.9)}$$

Let A_1 , A_2 be the follows:

$$A_1 = \sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \left[\frac{\lim_{\omega\to 1}(1-\omega) P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right)}{(z_1-1)\, P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)+1} \right]$$

$$A_2 = (1-\rho)(z_1-1)\sum_{j=0}^{m_1} \binom{m_1}{j} (C_{11}\lambda_{11})^j (C_{21}\lambda_{21})^{m_1-j} \frac{\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)}{z_1-\lambda_{11}^{j}\lambda_{21}^{m_1-j}\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)}$$

then we have $Q(0,0,z_1,y_1) = A_1 + A_2$.

Since $|\lambda_{1i}\lambda_{2i}| < 1$ (see Appendix C), thus $|\lambda_{11}| < 1$, $|\lambda_{12}| \le 1$, we have

$$\begin{array}{ll} \lambda_{11}^{j}\lambda_{21}^{m_1-j} = 1, & if\ j=0\ and\ z_1 = 1 \\ -1 < \lambda_{11}^{j}\lambda_{21}^{m_1-j} < 1, & otherwise \end{array} ;$$

therefore

$$\lim_{\omega\to 1}(1-\omega)P_0\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\omega\right) = \lim_{k\to\infty} p_k(0)\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)^k = \begin{cases} 1-\rho_0, & if\ j=0\ and\ z_1=1 \\ 0, & otherwise \end{cases} \quad \text{(D.10)}$$

$$\lim_{k\to\infty} p_k(0) = 1-\rho_0$$

where ρ_0 is the system load for the high-priority queue.

Since $|\lambda_{1i}\lambda_{2i}| < 1$(see Appendix B), thus $|\lambda_{11}| < 1$, $|\lambda_{12}| \leq 1$, we have

$$\begin{array}{ll} \lambda_{11}^{j}\lambda_{21}^{m_1-j} = 1, & if\ j = 0\ and\ z_1 = 1 \\ -1 < \lambda_{11}^{j}\lambda_{21}^{m_1-j} < 1, & otherwise \end{array} ;$$

therefore from D.10 we have:

$$\lim_{\omega\to 1}(1-\omega)P_0\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\omega\right) = \begin{cases} 1-\rho_0, & if\ i = 0\ and\ z_1 = 1 \\ 0, & otherwise \end{cases},$$

thus A_1 is equal to 0 except at $z_1 = 1$. Since the limit operation includes continuous values of a variable, we ignore the point $z_1 = 1$ and therefore A_1 will be zero and it is not part of the steady-state PGF, and only A_2 remains there. Therefore

$$Q(0,0,z_1,y_1)$$

$$= (1-\rho)(z_1-1)\sum_{j=0}^{m_1}\binom{m_1}{j}(C_{11}\lambda_{11})^j(C_{21}\lambda_{21})^{m_1-j}\frac{\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)}{z_1-\lambda_{11}^{j}\lambda_{21}^{m_1-j}\Gamma\left(\lambda_{11}^{j}\lambda_{21}^{m_1-j}\right)}$$

Comparing this result with that in section 3.4.5, we can see that as expected they are equal to each other.

Appendix E

The Solution Satisfies the Functional Equation

Since we have derived everything on the basis of the original functional equation 2.16, the solution 3.68 together with the boundary function 3.70 that we have obtained should satisfy 2.16. In this section, we show that the solution of $Q(z_0, y_0, z_1, y_1)$ and $Q(0, 0, z_1, y_1)$ do satisfy the original functional equation, which provides further proof for the correctness of the results that we have obtained.

E.1 Preliminary Results

Before we present the proof, let us show some preliminary results that will be used in the proof. At first let us show the following result:

$$C_{1i} = \frac{\lambda_{2i} - X_i(1)}{\lambda_{2i} - \lambda_{1i}}, \qquad C_{2i} = \frac{X_i(1) - \lambda_{1i}}{\lambda_{2i} - \lambda_{1i}} \tag{E.1}$$

Proof: From 3.11, we have

$$\lambda_{1i,2i} = \frac{\beta_i + \alpha_i f_i(z_i) \mp \sqrt{[\beta_i + \alpha_i f_i(z_i)]^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}}{2},$$

thus

$$\lambda_{1i} + \lambda_{2i} = \beta_i + \alpha_i f_i(z_i), \qquad \lambda_{2i} - \lambda_{1i} = \sqrt{[\beta_i + \alpha_i f_i(z_i)]^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}$$

and

$$\lambda_{1i}\lambda_{2i} = \frac{[\beta_i + \alpha_i f_i(z_i)]^2 - \left\{[\beta_i + \alpha_i f_i(z_i)]^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)\right\}}{4} = -(1-\alpha_i-\beta_i) f_i(z_i)$$

From 3.1 , we have $X_i(1) = \beta_i + (1-\beta_i) y_i f_i(z_i)$,

thus $(1-\beta_i) y_i f_i(z_i) = X_i(1) - \beta_i$.

From 3.14, we have

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i) f_i(z_i) + (\beta_i + \alpha_i f_i(z_i))}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}}$$

thus

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(1-\beta_i) y_i f_i(z_i) - 2\alpha_i f_i(z_i) + \lambda_{1i} + \lambda_{2i}}{2\sqrt{(\lambda_{1i} + \lambda_{2i})^2 - 4\lambda_{1i}\lambda_{2i}}},$$

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2[X_i(1) - \beta_i] - 2\alpha_i f_i(z_i) + \lambda_{1i} + \lambda_{2i}}{2\sqrt{(\lambda_{2i} - \lambda_{1i})^2}},$$

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2X_i(1) - 2[\beta_i + \alpha_i f_i(z_i)] + \lambda_{1i} + \lambda_{2i}}{2(\lambda_{2i} - \lambda_{1i})},$$

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2X_i(1) - 2(\lambda_{1i} + \lambda_{2i}) + \lambda_{1i} + \lambda_{2i}}{2(\lambda_{2i} - \lambda_{1i})} = \frac{1}{2} \mp \frac{2X_i(1) - (\lambda_{1i} + \lambda_{2i})}{2(\lambda_{2i} - \lambda_{1i})};$$

Finally,

$$C_{1i} = \frac{1}{2} - \frac{2X_i(1) - (\lambda_{1i} + \lambda_{2i})}{2(\lambda_{2i} - \lambda_{1i})} = \frac{\lambda_{2i} - X_i(1)}{\lambda_{2i} - \lambda_{1i}}$$

$$C_{2i} = \frac{1}{2} + \frac{2X_i(1) - (\lambda_{1i} + \lambda_{2i})}{2(\lambda_{2i} - \lambda_{1i})} = \frac{X_i(1) - \lambda_{1i}}{\lambda_{2i} - \lambda_{1i}};$$

this completes the proof.

Let us define $C^*_{1i} = C_{1i}\,|_{y_i=Y_i}$, $C^*_{2i} = C_{2i}\,|_{y_i=Y_i}$, $X^*_{1i}(1) = X_i(1)\,|_{y_i=Y_i}$, then from E.1, we have

$$C^*_{1i} = \frac{\lambda_{2i} - X^*_i(1)}{\lambda_{2i} - \lambda_{1i}}, \qquad C^*_{2i} = \frac{X^*_i(1) - \lambda_{1i}}{\lambda_{2i} - \lambda_{1i}} \tag{E.2}$$

From 3.3,

$$X_i(2) = [\beta_i + \alpha_i y_i f_i(z_i)]\, X_i(1) + (1 - \alpha_i - \beta_i) f_i(z_i) X_i(0),$$

$$X_i(2) = [\beta_i + \alpha_i y_i f_i(z_i)]\, X_i(1) + (1 - \alpha_i - \beta_i) f_i(z_i)$$

then substitute for $\beta_i + \alpha_i y_i f_i(z_i)$ and $(1 - \alpha_i - \beta_i) f_i(z_i)$, we have

$$X_i(2) = (\lambda_{1i} + \lambda_{2i}) X_1(1) - \lambda_{1i}\lambda_{2i} \tag{E.3}$$

E.2 Proof of that the Solution Satisfies the Functional Equation

Now with the available preliminary results, we are ready to show the solution satisfies the original functional equation through substitution.

Letting $k \to \infty$, the original functional equation 2.16 becomes:

$$Q(z_0, y_0, z_1, y_1) = B(1)\left\{\frac{1}{z_0}Q(z_0, Y_0, z_1, Y_1) + \frac{z_0 - z_1}{z_0 z_1}Q(0, 0, z_1, Y_1) + \frac{z_1 - 1}{z_1}Q(0, 0, 0, 0)\right\} \tag{E.4}$$

Let us define

$$A_1 = \frac{1}{z_0}B(1)Q(z_0, Y_0, z_1, Y_1),$$

$$A_2 = \frac{z_0 - z_1}{z_0 z_1}B(1)Q(0, 0, z_1, Y_1)$$

$$A_3 = B(1)\frac{z_1 - 1}{z_1}Q(0, 0, 0, 0)$$

then E.4 becomes $Q(z_0, y_0, z_1, y_1) = A_1 + A_2 + A_3$, in the following, we will show that this is true.

First let us consider the part A_1 , substituting our solution 3.68 in A_1 we have

$$A_1 = \frac{1}{z_0}B(1)Q(z_0, Y_0, z_1, Y_1) = \frac{1}{z_0}[X_0(1)]^{m_0}[X_1(1)]^{m_1}Q(z_0, Y_0, z_1, Y_1)$$

$$= (1 - \rho_0)\frac{z_0 - 1}{z_0}[X_0(1)]^{m_0}[X_1(1)]^{m_1}C_{21}^{*m_1}\delta(z_1 - 1)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{(C_{10}^*\lambda_{10})^k(C_{20}^*\lambda_{20})^{m_0-k}}{z_0 - \lambda_{10}^k\lambda_{20}^{m_0-k}}$$

$$+\frac{(z_0 - z_1)(z_1 - 1)}{z_0 z_1}(1 - \rho)[X_0(1)]^{m_0}[X_1(1)]^{m_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}$$

$$\cdot\frac{\left(C_{10}^{*}\lambda_{10}\right)^{k}\left(C_{20}^{*}\lambda_{20}\right)^{m_0-k}\left(C_{11}^{*}\lambda_{11}\right)^{i}\left(C_{21}^{*}\lambda_{21}\right)^{m_1-i}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)}{\left(z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_0z_1}(1-\rho)\left[X_0(1)\right]^{m_0}\left[X_1(1)\right]^{m_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}$$

$$\cdot\frac{\left(C_{10}^{*}\lambda_{10}\right)^{k}\left(C_{20}^{*}\lambda_{20}\right)^{m_0-k}\left(C_{11}^{*}\lambda_{11}\right)^{i}\left(C_{21}^{*}\lambda_{21}\right)^{m_1-i}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}$$

Substituting for C_{10}^{*}, C_{20}^{*}, C_{11}^{*}, C_{21}^{*}, from E.2, above equation becomes

$$A_1=(1-\rho_0)\frac{z_0-1}{z_0}\left[X_1(1)\frac{X_1^{*}(1)-\lambda_{11}}{\lambda_{21}-\lambda_{11}}\right]^{m_1}\delta(z_1-1)\sum_{k=0}^{m_0}\binom{m_0}{k}\frac{\lambda_{10}^{k}\lambda_{20}^{m_0-k}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}}$$

$$\cdot\left(X_0(1)\frac{\lambda_{20}-X_0^{*}(1)}{\lambda_{20}-\lambda_{10}}\right)^{k}\left(X_0(1)\frac{X_0^{*}(1)-\lambda_{10}}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}$$

$$+\frac{(z_0-z_1)(z_1-1)}{z_0z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{2i}\lambda_{21}^{2(m_1-i)}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)$$

$$\cdot\frac{\left(X_0(1)\frac{\lambda_{20}-X_0^{*}(1)}{\lambda_{20}-\lambda_{10}}\right)^{k}\left(X_0(1)\frac{X_0^{*}(1)-\lambda_{10}}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}\left(X_1(1)\frac{\lambda_{21}-X_1^{*}(1)}{\lambda_{21}-\lambda_{11}}\right)^{i}\left(X_1(1)\frac{X_1^{*}(1)-\lambda_{11}}{\lambda_{21}-\lambda_{11}}\right)^{m_1-i}}{\left(z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_0z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\frac{\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}\left(X_0(1)\frac{\lambda_{20}-X_0^{*}(1)}{\lambda_{20}-\lambda_{10}}\right)^{k}$$

$$\cdot\left(X_0(1)\frac{X_0^{*}(1)-\lambda_{10}}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}\left(X_1(1)\frac{\lambda_{21}-X_1^{*}(1)}{\lambda_{21}-\lambda_{11}}\right)^{i}\left(X_1(1)\frac{X_1^{*}(1)-\lambda_{11}}{\lambda_{21}-\lambda_{11}}\right)^{m_1-i}.$$

noting that $X_i(2)=X_i(1)X_i^{*}(1)$ (see 3.1), we can find that A_1 becomes:

$$A_1 = (1-\rho_0)\,\frac{z_0-1}{z_0}\left[\frac{X_1(2)-\lambda_{11}X_1(1)}{\lambda_{21}-\lambda_{11}}\right]^{m_1}\delta(z_1-1)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{\lambda_{10}^{k}\lambda_{20}^{m_0-k}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}}$$

$$\cdot\left(\frac{\lambda_{20}X_0(1)-X_0(2)}{\lambda_{20}-\lambda_{10}}\right)^{k}\left(\frac{X_0(2)-\lambda_{10}X_0(1)}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}$$

$$+\frac{(z_0-z_1)(z_1-1)}{z_0 z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{2i}\lambda_{21}^{2(m_1-i)}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)$$

$$\cdot\frac{\left(\frac{\lambda_{20}X_0(1)-X_0(2)}{\lambda_{20}-\lambda_{10}}\right)^{k}\left(\frac{X_0(2)-\lambda_{10}X_0(1)}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}\left(\frac{\lambda_{21}X_1(1)-X_1(2)}{\lambda_{21}-\lambda_{11}}\right)^{i}\left(\frac{X_1(2)-\lambda_{11}X_1(1)}{\lambda_{21}-\lambda_{11}}\right)^{m_1-i}}{\left(z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_0 z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}\frac{\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}\left(\frac{\lambda_{20}X_0(1)-X_0(2)}{\lambda_{20}-\lambda_{10}}\right)^{k}$$

$$\cdot\left(\frac{X_0(2)-\lambda_{10}X_0(1)}{\lambda_{20}-\lambda_{10}}\right)^{m_0-k}\left(\frac{\lambda_{21}X_1(1)-X_1(2)}{\lambda_{21}-\lambda_{11}}\right)^{i}\left(\frac{X_1(2)-\lambda_{11}X_1(1)}{\lambda_{21}-\lambda_{11}}\right)^{m_1-i};$$

Substituting $(\lambda_{1i}+\lambda_{2i})X_1(1)-\lambda_{1i}\lambda_{2i}$ for $X_i(2)$ from E.3 and noting that $C_{1i}=\frac{\lambda_{2i}-X_i(1)}{\lambda_{2i}-\lambda_{1i}}$, $C_{2i}=\frac{X_i(1)-\lambda_{1i}}{\lambda_{2i}-\lambda_{1i}}$ results in:

$$A_1 = (1-\rho_0)\,\frac{z_0-1}{z_0}C_{21}^{m_1}\delta(z_1-1)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{C_{10}^{k}C_{20}^{m_0-k}\lambda_{10}^{2k}\lambda_{20}^{2(m_0-k)}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}}$$

$$+\frac{(z_0-z_1)(z_1-1)}{z_0 z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}\lambda_{10}^{2k}\lambda_{20}^{2(m_0-k)}\lambda_{11}^{3i}\lambda_{21}^{3(m_1-i)}$$

$$\cdot \frac{C_{10}^{k} C_{20}^{m_0-k} C_{11}^{i} C_{21}^{m_1-i} \Gamma\left(\lambda_{11}^{i} \lambda_{21}^{m_1-i}\right)}{\left(z_0 - \lambda_{10}^{k} \lambda_{20}^{m_0-k} \lambda_{11}^{i} \lambda_{21}^{m_1-i}\right)\left[z_1 - \lambda_{11}^{i} \lambda_{21}^{m_1-i} \Gamma\left(\lambda_{11}^{i} \lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_0 z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\binom{m_0}{k}\binom{m_1}{i}\frac{\lambda_{10}^{2k}\lambda_{20}^{2(m_0-k)}\lambda_{11}^{2i}\lambda_{21}^{2(m_1-i)}}{z_0-\lambda_{10}^{k}\lambda_{20}^{m_0-k}\lambda_{11}^{i}\lambda_{21}^{m_1-i}}$$

$$\cdot C_{10}^{k} C_{20}^{m_0-k} C_{11}^{i} C_{21}^{m_1-i}; \qquad \text{(E.5)}$$

Next let us consider the second part A_2 . Substituting 3.70 and following the similar steps as we did for A_1, we have

$$A_2 = \frac{z_0 - z_1}{z_0 z_1} B(1) Q(0, 0, z_1, Y_1)$$

$$A_2 = (1-\rho_0)\frac{z_0 - z_1}{z_0 z_1} B_0(1) C_{21}^{m_1} \delta(z_1 - 1)$$

$$+(1-\rho)\frac{(z_0 - z_1)(z_1 - 1)}{z_0 z_1} B_0(1) \sum_{i=0}^{m_1}\binom{m_1}{i} C_{11}^{i} C_{21}^{m_1-i} \frac{\lambda_{11}^{2i}\lambda_{21}^{2(m_1-i)}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)}{z_1 - \lambda_{11}^{i}\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^{i}\lambda_{21}^{m_1-i}\right)};$$

From 3.16 we have

$$B_0(1) = [X_0(1)]^{m_0} = \left[C_{10}\lambda_{10}^{k} + C_{20}\lambda_{10}^{k}\right]^{m_0} = \sum_{k=0}^{m_0}\binom{m_0}{k}(C_{10}\lambda_{10})^{k}(C_{20}\lambda_{20})^{m_0-k}$$

thus

$$A_2 = (1-\rho_0)\frac{z_0 - z_1}{z_0 z_1} C_{21}^{m_1}\delta(z_1 - 1)\sum_{k=0}^{m_0}\binom{m_0}{k}(C_{10}\lambda_{10})^{k}(C_{20}\lambda_{20})^{m_0-k}$$

$$+(1-\rho)\frac{(z_0-z_1)(z_1-1)}{z_0 z_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}\lambda_{10}^{2k}\lambda_{20}^{2(m_0-k)}\lambda_{11}^{3i}\lambda_{21}^{3(m_1-i)}$$

$$\frac{C_{10}^k C_{20}^{m_0-k} C_{11}^i C_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{z_1-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}; \tag{E.6}$$

The third part is:

$$A_3 = B(1)\frac{z_1-1}{z_1}Q(0,0,0,0) = [X_0(1)]^{m_0}[X_1(1)]^{m_1}\frac{z_1-1}{z_1}(1-\rho)$$

$$= (1-\rho)\frac{z_1-1}{z_1}\left[C_{10}\lambda_{10}^k + C_{20}\lambda_{10}^k\right]^{m_0}\left[C_{11}\lambda_{11}^i + C_{21}\lambda_{11}^i\right]^{m_1}$$

$$= (1-\rho)\frac{z_1-1}{z_1}\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k} \tag{E.7}$$

Now, adding E.5, E.6 and E.7 up, and then combining some terms, we have

$$A_1 + A_2 + A_3$$

$$= (1-\rho_0)(z_0-1)C_{21}^{m_1}\delta(z_1-1)\sum_{k=0}^{m_0}\begin{pmatrix} m_0 \\ k \end{pmatrix}\frac{(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}}{z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}}$$

$$+\frac{(z_0-z_1)(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot\frac{(C_{11}\lambda_{11})^i(C_{21}\lambda_{21})^{m_1-i}\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)}{\left(z_0-\lambda_{10}^k\lambda_{20}^{m_0-k}\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\left[z_1-\lambda_{11}^i\lambda_{21}^{m_1-i}\Gamma\left(\lambda_{11}^i\lambda_{21}^{m_1-i}\right)\right]}$$

$$+\frac{z_0(z_1-1)}{z_1}(1-\rho)\sum_{k=0}^{m_0}\sum_{i=0}^{m_1}\begin{pmatrix} m_0 \\ k \end{pmatrix}\begin{pmatrix} m_1 \\ i \end{pmatrix}(C_{10}\lambda_{10})^k(C_{20}\lambda_{20})^{m_0-k}$$

$$\cdot \frac{(C_{11}\lambda_{11})^i (C_{21}\lambda_{21})^{m_1 - i}}{z_0 - \lambda_{10}^k \lambda_{20}^{m_0 - k} \lambda_{11}^i \lambda_{21}^{m_1 - i}};$$

Comparing the above equation with 3.68, we find that $Q(z_0, y_0, z_1, y_1) = A_1 + A_2 + A_3$. This completes the proof that the solution satisfies the original functional equation, which provides further proof for the correctness of our performance analysis.

Appendix F

Proof of Equivalence of Equations 3.32 and 3.42

In section 3.2.4 we have derived equation 3.42 from 3.32, but we didn't give strict proof of that derivation. In this appendix, we will show that these two equations are equivalent through induction.

The equivalence of the two equations for the first few values of k is straight forward and will not be given here. In the next, we assume that these equations are equivalent for order k, and we will show they will also be equivalent for oder $(k+1)$.

From 3.32 we have

$$
\begin{aligned}
&Q_{k+1}(0,0,z_1,y_1) \\
&= \xi(k)B_1(k+1) + \frac{1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,z_1,\phi_1(k+1-j)) \\
&+ \frac{z_1-1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,0,0) \qquad \text{(F.1)}
\end{aligned}
$$

From 3.42 we have

$$Q_{k+1}(0,0,z_1,y_1)$$

$$= \sum_{i=1}^{k+1} \varphi(k+1-i)\xi(i-1)B_1(k+1) + \frac{z_1-1}{z_1}\sum_{i=1}^{k}\sum_{h=1}^{i}\varphi(k-i)\xi(i-h)B_1(k+1-h)Q_h(0,0,0,0) \tag{F.2}$$

Next we will show F.1 and F.2 are equivalent.

Substituting j for k and substituting $\phi_1(k+1-j)$ for y_1 in 3.42 we have:

$$Q_j(0,0,z_1,\phi_1(k+1-j))$$

$$= \sum_{i=1}^{j} \varphi(j-i)\xi(i-1)B_1^{k+1-j}(j) + \frac{z_1-1}{z_1}\sum_{i=1}^{j-1}\sum_{h=1}^{i}\varphi(j-i-1)\xi(i-h)B_1^{k+1-j}(j-h)Q_h(0,0,0,0) \tag{F.3}$$

Substituting F.3 in F.1 gives us:

$$Q_{k+1}(0,0,z_1,y_1)$$

$$= \xi(k)B_1(k+1) + \frac{1}{z_1}\sum_{j=1}^{k}\sum_{i=1}^{j}\varphi(j-i)\xi(i-1)\xi(k-j)B_1(k+1-j)B_1^{k+1-j}(j)$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\sum_{i=1}^{j-1}\sum_{h=1}^{i}\varphi(j-i-1)\xi(k-j)\xi(i-h)B_1(k+1-j)B_1^{k+1-j}(j-h)Q_h(0,0,0,0)$$

noting that

$$B_i^n(k) = \frac{B_i(k+n)}{B_i(n)}$$

the above equation becomes:

$$Q_{k+1}(0,0,z_1,y_1)$$

$$= \xi(k)B_1(k+1) + \frac{1}{z_1}\sum_{j=1}^{k}\sum_{i=1}^{j}\varphi(j-i)\xi(i-1)\xi(k-j)B_1(k+1)$$

$$+\frac{z_1-1}{z_1^2}\sum_{j=1}^{k}\sum_{i=1}^{j-1}\sum_{h=1}^{i}\varphi(j-i-1)\xi(k-j)\xi(i-h)B_1(k+1-h)Q_h(0,0,0,0)$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,0,0)$$

Exchanging the order of summations, we have

$$Q_{k+1}(0,0,z_1,y_1)$$

$$= \xi(k)B_1(k+1) + \frac{1}{z_1}\sum_{i=1}^{k}\sum_{j=i}^{k}\varphi(j-i)\xi(i-1)\xi(k-j)B_1(k+1)$$

$$+\frac{z_1-1}{z_1^2}\sum_{j=1}^{k}\sum_{h=1}^{j-1}\sum_{i=h}^{j-1}\varphi(j-i-1)\xi(k-j)\xi(i-h)B_1(k+1-h)Q_h(0,0,0,0)$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,0,0)$$

letting $n = j - i, \quad m = i - h$, the above equation becomes:

$$Q_{k+1}(0,0,z_1,y_1)$$

$$= \xi(k)B_1(k+1) + \frac{1}{z_1}\sum_{i=1}^{k}\sum_{n=0}^{k-i}\varphi(n)\xi(i-1)\xi(k-n-i)B_1(k+1)$$

$$+\frac{z_1-1}{z_1^2}\sum_{j=1}^{k}\sum_{h=1}^{j-1}\sum_{m=0}^{j-1-h}\varphi(j-m-h-1)\xi(k-j)\xi(m)B_1(k+1-h)Q_h(0,0,0,0)$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,0,0) \tag{F.4}$$

From the definition of $\varphi(k)$ in 3.43 we have

$$\varphi(k) = \sum_{r_1=0}^{k-1} \sum_{r_2=0}^{r_1-1} \cdots \sum_{r_k=0}^{r_{k-1}-1} \prod_{n=1}^{k} \frac{1}{z_1^{\varepsilon(\bar{r})}} \xi(r_{n-1} - r_n - 1)$$

expanding the product

$$\varphi(k) = \frac{1}{z_1} \sum_{r_1=0}^{k-1} \sum_{r_2=0}^{r_1-1} \cdots \sum_{r_k=0}^{r_{k-1}-1} \frac{1}{z_1^{\varepsilon(\bar{r})-1}} \xi(r_0 - r_1 - 1)\xi(r_1 - r_2 - 1) \cdots \xi(r_{k-1} - r_k - 1)$$

here, we assume that if the upper limit is less than zero then the summation is equal to 1 instead of zero.

$$\varphi(k) = \frac{1}{z_1} \sum_{r_1=0}^{k-1} \xi(k - 1 - r_1) \sum_{r_2=0}^{r_1-1} \cdots \sum_{r_k=0}^{r_{k-1}-1} \frac{1}{z_1^{\varepsilon(\bar{r})-1}} \xi(r_1 - r_2 - 1) \cdots \xi(r_{k-1} - r_k - 1)$$

therefore

$$\varphi(k) = \frac{1}{z_1} \sum_{r_1=0}^{k-1} \xi(k - 1 - r_1)\varphi(r_1) \tag{F.5}$$

From this result, we have

$$\varphi(k + 1 - i) = \frac{1}{z_1} \sum_{n=0}^{k-i} \xi(k - i - n)\varphi(n)$$

$$\varphi(j - h) = \frac{1}{z_1} \sum_{m=0}^{j-1-h} \xi(j - h - 1 - m)\varphi(m)$$

substituting the above results into F.4

$$Q_{k+1}(0, 0, z_1, y_1)$$

$$= \xi(k)B_1(k + 1) + \frac{1}{z_1} \sum_{i=1}^{k} \varphi(k + 1 - i)\xi(i - 1)$$

$$+\frac{z_1-1}{z_1^2}\sum_{j=1}^{k}\sum_{h=1}^{j-1}\varphi(j-h)\xi(k-j)\xi(m)B_1(k+1-h)Q_h(0,0,0,0)$$

$$+\frac{z_1-1}{z_1}\sum_{j=1}^{k}\xi(k-j)B_1(k+1-j)Q_j(0,0,0,0)$$

combining the terms, we finally have that

$$Q_{k+1}(0,0,z_1,y_1)$$

$$=\sum_{i=1}^{k+1}\varphi(k+1-i)\xi(i-1)B_1(k+1)+\frac{z_1-1}{z_1}\sum_{i=1}^{k}\sum_{h=1}^{i}\varphi(k-i)\xi(i-h)B_1(k+1-h)Q_h(0,0,0,0)$$

Thus that equations 3.42 and 3.32 are equivalent for oder $(k+1)$. This completes the introduction.

Appendix G

Determining Busy Period Distribution from First Principles

In section 3.2.1, we have determined the expression for busy period of the high-priority queue (see equation 3.24) as

$$p_k(0) = \sum_{j=1}^{k} \xi(j-1) p_{k-j}(0) \tag{G.1}$$

where $\xi(j-1)$ is the probability of that the busy period of high-priority queue is $j-1$ slots, and $p_k(0)$ is the probability of that the high-priority queue is empty at the end of slot k. In this appendix, we will show this equation is correct by some examples.

In the next, we will compute the first few values of $\xi(j)$. From equation G.1, we have

$$p_1(0) = \sum_{j=1}^{1} \xi(j-1) p_{k-j}(0) = p_k(0) = \xi(0) p_0(0) = \xi(0)$$

$$p_2(0) = \sum_{j=1}^{2} \xi(j-1) p_{2-j}(0) = \xi(0) p_1(0) + \xi(1) p_0(0) = \xi(0) p_1(0) + \xi(1)$$

$$p_3(0) = \sum_{j=1}^{3} \xi(j-1) p_{k-j}(0)$$
$$= \xi(0)p_2(0) + \xi(1)p_1(0) + \xi(2)p_0(0) = \xi(0)p_2(0) + \xi(1)p_1(0) + \xi(2)$$

Therefore

$$\xi(0) = p_1(0)$$

$$\xi(1) = p_2(0) - \xi(0)p_1(0)$$

$$\xi(2) = p_3(0) - \xi(0)p_2(0) - \xi(1)p_1(0)$$

The expression for $p_k(0)$ has been determined in [18], which is given by:

$$p_k(0) = \beta^{m_0 k} + \sum_{t=1}^{k} \sum_{j=0}^{t} \frac{1}{t} \binom{t}{j} \binom{m_0 k - j - 1}{t-1} m_0(k-t)\alpha_0^{t-j}(1-\alpha_0-\beta_0)^j \beta_0^{m_0 k - j - t}, \ k \geq 1$$

where it assumes that the high-priority queue is initially empty and all sources are in the Off state; also it assumes that a On source generates one packet during a slot. With these results, it is easy for us to compute the probability distribution of busy period for the high-priority queue.

Assuming that $m_0 = 4$, $\alpha_0 = 0.75$, $\beta_0 = 0.97$, we have first few values of $\xi(j)$ as:

$$\xi(0) = 0.88529281, \ \xi(1) = 0.0249891596, \ \xi(2) = 0.0178987364 \qquad \text{(G.2)}$$

Also, we can use an alternative method to compute the probability distribution of busy period by first principles. Let us define $P_i(k)$= Probability that the high-priority sources generate i packets during $k'th$ slot, and assuming that an On source generates only one packet during a slot, then we have:

$$\xi(0) = P_0(1) = \beta_0^{m_0}$$

$$\begin{aligned}\xi(1) &= P_1(1)P_0(2)\\ &= \left[\begin{pmatrix} m_0 \\ 1 \end{pmatrix} (1-\beta_0)\beta_0^{m_0-1}\right]\left[(1-\alpha_0)\beta_0^{m_0-1}\right]\end{aligned}$$

$$\begin{aligned}\xi(2) &= P_1(1)P_1(2)P_0(3) + P_2(1)P_0(2)P_0(3)\\ &= \left[\begin{pmatrix} m_0 \\ 1 \end{pmatrix} (1-\beta_0)\beta_0^{m_0-1}\right]\left(\alpha_0\beta_0^{m_0-1}\right)\left[(1-\alpha_0)\beta_0^{m_0-1}\right]\\ &+ \left[\begin{pmatrix} m_0 \\ 1 \end{pmatrix} (1-\beta_0)\beta_0^{m_0-1}\right]\left[(1-\alpha_0)\begin{pmatrix} m_0 \\ 1 \end{pmatrix} (1-\beta_0)\beta_0^{m_0-2}\right]\left[(1-\alpha_0)\beta_0^{m_0-1}\right]\\ &+ \left[\begin{pmatrix} m_0 \\ 2 \end{pmatrix} (1-\beta_0)^2\beta_0^{m_0-2}\right]\left[(1-\alpha_0)^2\beta_0^{m_0-2}\right]\beta_0^{m_0}\end{aligned}$$

Again assuming that $m_0 = 4$, $\alpha_0 = 0.75$, $\beta_0 = 0.97$, we have first few values of $\xi(j)$ as:

$$\xi(0) = 0.8853,\ \xi(1) = 0.025,\ \xi(2) = 0.018 \qquad \text{(G.3)}$$

Comparing the result of G.3 with G.2, we find that they are equal to each other. This provides further proof that equation 3.24 is correct.