VECTOR COHERENT STATES WITH MATRICES

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Abstract

Vector Coherent States With Matrices

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In this thesis we develop vector coherent states (VCS) in the form

$$| \mathcal{Z}, j\rangle = \mathcal{N}(|\mathcal{Z}|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m,$$

where $\{\phi_m\}$ is an orthonormal basis of an abstract Hilbert space, $\{\chi^j\}$ is an orthonormal basis of $\mathbb{C}^n$, the n dimensional complex space and $\mathcal{Z}$ is an $n \times n$ matrix. By imposing some conditions on the matrix $\mathcal{Z}$ we develop a general procedure to obtain VCS. We develop VCS on the complex plane and on the unit disc by replacing the matrix $\mathcal{Z}$ by quaternions. We also build VCS with several other classes of matrices. Further, we develop VCS on general Clifford algebras. At last, we build VCS with a class of symmetric matrices with an unbounded frame operator.

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Contents

Introduction

In this introductory chapter we give a brief historical survey of coherent states (CS) and vector coherent states (VCS) suitable to the scope of this thesis. CS can be defined as an over complete family of vectors of a Hilbert space with certain restrictions imposed on it. CS are denoted in Dirac notation by $\mid \nu\rangle$, where ν is a parameter from an appropriate label space. The conventional CS are specific superpositions $\mid z\rangle$, parametrized by a single complex number z, of the eigenstates $\mid n\rangle$ of the harmonic oscillator number operator $N = a^\dagger a$, where $a^\dagger$ and a denote the creation and annihilation operators of the harmonic oscillator respectively. These conventional CS may be constructed in three equivalent ways

(i) by defining them as eigenstates of the annihilation operator a,

(ii) by applying a unitary displacement operator on the vacuum state $\mid 0\rangle$ (where $a \mid 0\rangle = \mid 0\rangle$),

(iii) by considering them as quantum states with a minimum uncertainty relationship.

In recent years, other classes of states have attracted a lot of attention in quantum theories. In every case the set of CS refers to vectors in a Hilbert space. From a mathematical viewpoint, it has been noted [20] that a family of vectors in a Hilbert space can be accepted as a family of CS if it satisfies a minimum set of conditions,

(a) Normalizability;

(b) Continuity in the label z;

(c) Resolution of identity with a positive weight function.

The conditions (a) and (b) are easy to fulfill and condition (b) is not absolutely necessary. But the condition (c) is indeed a difficult task and imposes severe restrictions on the possible set of states. The problem of condition (c) has not been solved for many known CS in the literature. Recently some progress was made in this direction using the inverse Mellin transforms by J-M. Sixdeniers et al. In this thesis, we are mainly interested in a particular class of CS which can be obtained in the following way.

Definition 0.0.1 *Let $\mathbb{H}$ be a Hilbert space with an orthonormal basis $\{\phi_m\}$ and $\mathbb{C}$ be the complex plane with a measure $d\mu$ on it. For $z \in \mathbb{C}$, the states*

$$\mid z\rangle = \mathcal{N}(|z|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} \phi_m \tag{1}$$

are said to form a set of CS if

(a) The states $\mid z\rangle$ are normalized,

(b) The states $\mid z\rangle$ give a resolution of the identity, that is

$$\int_{\Omega} \mid z\rangle W(|z|)\langle z \mid d\mu = I \tag{2}$$

where $\mathcal{N}(|z|)$ is the normalization factor, $\rho(m)$ is a function of m and $W(|z|)$ is a positive function called a weight function.

In the literature condition (b), requiring that the states $\mid z\rangle$ are continuous in their label, is also added. But this condition is always guaranteed by the convergence of the series. The resolution of the identity states that the closed linear span of the set $\{\mid z\rangle \mid z \in \mathbb{C}\}$ is over complete in the Hilbert space. In the following we will discuss some key developments in brief. First of all let us see the bosonic CS.

0.1 Bosonic CS

Let $\{a, a^\dagger, N, I\}$ denote the usual harmonic oscilator algebra, where a is the annihilation operator, $a^\dagger$ is the creation operator and $N = a^\dagger a$ is the number operator. Further, $[a, a^\dagger] = I$. Now introduce the normalized ground state $\mid 0\rangle$ with the property that $a \mid 0\rangle = 0$ and generate a basis for a Hilbert space, $\mathbb{H}_f$ by repeated action of $a^\dagger$ on $\mid 0\rangle$. The Hilbert space so generated is called the Fock space and its basis is denoted by $\{\mid n\rangle \mid n = 0, 1, 2, ...\}$. The action of the operators on the basis is given by

$$a \mid n\rangle = \sqrt{n-1} \mid n-1\rangle, \;\; a^\dagger \mid n\rangle = \sqrt{n+1} \mid n+1\rangle, \;\; \text{and} \;\; N \mid n\rangle = n \mid n\rangle.$$

Let $\mathbb{C}$ denote the set of all complex numbers.

Definition 0.1.1 *For $z \in \mathbb{C}$, the vectors in $\mathbb{H}_f$ form a set of CS*

$$| z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} | m\rangle. \tag{3}$$

A comparison between definitions (0.0.1) and (0.1.1) yields $\mathcal{N} = e^{|z|^2}$ and $\rho(m) = m!$. If we take a measure $d\mu = d(\text{Re}z)d(\text{Im}z) = d^2z$ on $\mathbb{C}$ then in (2) we have to take $W(|z|) = \frac{1}{\pi}$ to get the resolution of the identity. Further it can be easily seen that $a \mid z\rangle = z \mid z\rangle$, that is, CS are the eigenstates of the annihilation operator. Now using the elementary Baker-Campbell-Hausdorff formula,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B \tag{4}$$

when the operators A and B commute with $[A, B]$, we can have the following.

Theorem 0.1.1 *(Exponential form) The CS in the form (3) can be given in the exponential form as*

$$| z\rangle = e^{(za^\dagger - \overline{z}a)} \mid 0\rangle. \tag{5}$$

where $\overline{z}$ stands for the complex conjugate of z.

Now let us take two new operators defined by the relations

$$a^\dagger = \frac{Q - iP}{\sqrt{2}}, \text{ and } a = \frac{Q + iP}{\sqrt{2}} \tag{6}$$

or equivalently

$$Q = \frac{a + a^\dagger}{\sqrt{2}}, \text{ and } P = \frac{a - a^\dagger}{i\sqrt{2}}. \tag{7}$$

The action of these operators and their commutator on CS can easily be verified because the action of the operators a and $a^\dagger$ are known. The Heisenberg uncertainty relation is defined in terms of the operators P and Q as follows.

Definition 0.1.2 *For any state vector $\psi \in \mathbb{H}_f$ with $\|\psi\| = 1$, the Heisenberg uncertainty relation is defined as*

$$\langle \Delta Q \rangle_\psi \langle \Delta P \rangle_\psi \geq \frac{1}{2} \tag{8}$$

where for an arbitrary operator A on $\mathbb{H}_f$,

$$\langle \Delta A \rangle_\psi = [\langle \psi \mid A^2 \psi \rangle - \mid \langle \psi \mid A\psi \rangle \mid^2]^{\frac{1}{2}}.$$

If the equality is achieved for a state, which is said to have the minimum uncertainty.

Theorem 0.1.2 *The states in (3) are minimum uncertainty states.*

Let us take the complex number z as $z = \frac{q-ip}{\sqrt{2}}$. For the states in (3) an exponential relation in terms of Q and P can be obtained as

$$\mid z \rangle = e^{i(pQ-qP)} \mid 0 \rangle = U(q,p) \mid 0 \rangle \tag{9}$$

where $U(q,p)$ is a unitary operator arising from a unitary, irreducible representation of the Weyl-Heisenberg group. Define

$$K(z, z') = \langle \overline{z} \mid z' \rangle. \tag{10}$$

The function $K(z, z')$ satisfies the properties of a reproducing kernel (for the general theory of reproducing kernels see [3] and for the theory related to CS see [1]) . In

CS theory there is a natural way of defining reproducing kernels in terms of CS. Let $\phi \in \mathbb{H}_f$ be an arbitrary vector. Then

$$\begin{aligned} \langle \overline{z} \mid \phi \rangle &= e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{\langle m \mid \phi \rangle}{\sqrt{m!}} z^m \\ &= e^{-\frac{|z|^2}{2}} f(z), \end{aligned}$$

where f is an analytic function of z. Let

$$d\mu = e^{-|z|^2} \frac{dz \wedge d\overline{z}}{2\pi i}$$

then we have the following

Theorem 0.1.3 *The linear function*

$$W : \mathbb{H}_f \longrightarrow L^2(\mathbb{C}, d\mu) \quad \textit{defined by} \ (W\phi)(z) = e^{\frac{|z|^2}{2}} \langle \overline{z} \mid \phi \rangle$$

is an isometry.

0.2 Recent developments

Here we will discuss some developments in generating different classes of CS by changing the parameters and factors in (0.0.1). From the huge literature of CS theory, here we pick only a few recent articles for discussion. They were chosen so that each one gives a new class of CS just by a change in one of the parameters or factors of definition (0.0.1).

0.2.1 Mittag-Leffler CS

Here we discuss a class of CS obtained by replacing $\rho(m)$ by a particular choice. The discussion is based on [36]. For $\alpha, \beta > 0$, the function

$$E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha n + \beta)}$$

was introduced and analyzed by Mittag-Leffler at the begining of this century. In fact, he analyzed it for $\beta = 1$. Thus the series bears his name. A development in generating a new class of CS using this function is as follows:

Theorem 0.2.1 *For $z \in \mathbb{C}$ and $\alpha, \beta > 0$, the states*

$$\mid z; \alpha, \beta\rangle = \left[\mathcal{N}_{\alpha,\beta}(|z|^2)\right]^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m \sqrt{\Gamma(\beta)}}{\sqrt{\Gamma(\alpha m + \beta)}} \mid m\rangle \tag{11}$$

form a set of CS, where $\mid m\rangle$ is as in definition (0.1.1) and the normalization constant is given by

$$\mathcal{N}_{\alpha,\beta}(|z|^2) = \Gamma(\beta) E_{\alpha,\beta}(|z|^2)$$

and the resolution of the identity is obtained as

$$\int\int_{\mathbb{C}} d^2 z \mid z; \alpha, \beta\rangle W_{\alpha,\beta}(x = |z|^2) \langle z; \alpha, \beta \mid = I$$

with

$$W_{\alpha,\beta}(x) = \frac{E_{\alpha,\beta}(x)}{\pi} \frac{x^{\frac{\beta-\alpha}{\alpha}} e^{-x^{\frac{1}{\alpha}}}}{\alpha}.$$

Here the choice for $W_{\alpha,\beta}(x)$ is not unique.

0.2.2 A change in parameter z

Another class of CS is obtained by changing the parameter z. The discussion is based on [28]. In [28] Penson et al. obtained a class of CS by introducing a constant q besides z. Further they have given several calculations for Physical quantities, which are usual for a set of CS, in terms of these CS. Consider the functional equation for a function of the complex variable z

$$\frac{d\epsilon(q,z)}{dz} = \epsilon(q,qz), \quad \epsilon(q,0) = 1, \;\; 0 \leq q \leq 1. \tag{12}$$

When $q = 1$ these are the defining equations for e^z. When $q \neq 1$ we get $\epsilon(q,z) \neq e^z$ but an analytic solution in some neighborhood of $z = 0$ can be assumed and it is given by

$$\epsilon(q,z) = \sum_{m=0}^{\infty} a_m(q) z^m. \tag{13}$$

Equation (12) produces the following recurrence relation,

$$a_{m+1}(q) = a_m(q)\frac{q^m}{(m+1)}, \quad m = 1, 2, ..., \quad a_0 = 1,$$

with solution

$$a_m(q) = \frac{q^{\frac{m(m-1)}{2}}}{m!} \quad \text{and} \quad \epsilon(q,z) = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m-1)}{2}} z^m}{m!}.$$

The series is convergent for all z when $q \leq 1$. We can construct a class of CS in terms of this function.

Theorem 0.2.2 *For $z \in \mathbb{C}$ and $0 \leq q \leq 1$, the states*

$$\mid q, z\rangle = \mathcal{N}(q, |z|^2)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{q^{\frac{m(m-1)}{2}} z^m}{\sqrt{m!}} \mid m\rangle \tag{14}$$

form a set of CS, where $\mid m\rangle$ *is as in definition (0.1.1) and the normalization factor is given by*

$$\mathcal{N}(q,|z|^2) = \epsilon(q^2,|z|^2)$$

and the resolution of identity

$$\int\int d^2z \mid q,z\rangle W(q,|z|^2)\langle q,z \mid = I$$

is obtained with

$$W(q,|z|^2) = \epsilon(q^2,x)[q^2(\pi)^{\frac{3}{2}}]^{-1}\int_{-\infty}^{\infty}\exp\left(-\frac{x}{q^3}e^{2\sqrt{\log(q^{-1})}s} - s^2\right)ds$$

where $x = |z|^2$. *Here the choice for* $W(q,|z|^2)$ *is not unique.*

It should be noted that the CS given in the above theorem are not the eigenstates of the annihilation operator a.

0.2.3 CS generated by binomial distribution

Here we examine a set of CS given by a finite summation formula based on [35] and the references there in. Consider a single spin S interacting with the magnetic field $\overrightarrow{H} = (0,0,h)$, $h > 0$, through $\widehat{H}_S = -h\widehat{S}_\mu$. For complex μ the eigenstates of $\widehat{H}_S$ satisfy

$$\widehat{S}_\mu \mid p\rangle = (S-p)\mid p\rangle, \;\; \langle p \mid p'\rangle = \delta_{p,p'}, \;\; 0 \leq p \leq 2s.$$

Using these states, the spin CS were defined as in the following theorem (due to J.M.Radcliffe, 1971).

Theorem 0.2.3 *For $\mu \in \mathbb{C}$ the states*

$$| \mu\rangle = \mathcal{N}(|\mu|^2)^{-\frac{1}{2}} \sum_{p=0}^{2S} \begin{pmatrix} 2S \\ p \end{pmatrix}^{\frac{1}{2}} \mu^p \mid p\rangle = \mathcal{N}(|\mu|^2)^{-\frac{1}{2}} \exp(\mu \widehat{S}_-) \mid 0\rangle \qquad (15)$$

form a set of CS, where $| 0\rangle$ is the ground state of $\widehat{H}_S$ with $\overrightarrow{\widehat{S}} \| \overrightarrow{H}$ and $\widehat{S}_-$ is the spin lowering operator. The normalization factor is given by

$$\mathcal{N}(|\mu|^2) = (1 + |\mu|^2)^{2S}$$

and the resolution of the identity

$$\int\int_{\mathbb{C}} d^2\mu \mid \mu\rangle V_{2S}(|\mu|^2)\langle \mu \mid = I_{2S}$$

is obtained with

$$V_{2S}(|\mu|^2) = \frac{2S+1}{\pi}(1 + |\mu|^2)^{-2}.$$

The choice for $V_{2S}(|\mu|^2)$ is not unique.

D. Stoler et al. introduced binomial CS in 1985 as

$$| y; M\rangle = \sum_{n=0}^{M} \left[\begin{pmatrix} M \\ n \end{pmatrix} \eta^n (1-\eta)^{M-n} \right]^{\frac{1}{2}} \frac{e^{-i(\theta+\pi)n}}{\sqrt{n!}} (a^\dagger)^n \mid 0\rangle, \quad 1 \leq M < \infty \qquad (16)$$

where $| \ 0\rangle$ and $a^\dagger$ are as in definition (0.1.1) and $y = re^{i\theta}, \eta = \sin^2 r$. Here the normalization and the resolution of the identity were realized exactly as in theorem (0.2.3). Following this work, in 2000 H.C. Fu et al. defined a new class of binomial CS as

$$| z; M\rangle = \mathcal{N}_m^{-\frac{1}{2}}(|z|^2)(a^\dagger + z)^M \mid 0\rangle, \quad 1 \leq M < \infty \qquad (17)$$

where again $\mid 0\rangle$ and $a^\dagger$ are as in definition (0.1.1). In the same year in [35] the completeness of (17) was considered. Further, the states in (15) were generalized using a power of a binomial instead of an exponential. We give these CS in the following theorem followed by the preliminaries. The Mth Laguerre polynomial is denoted by $L_M(x)$ and is given by

$$L_M(x) = \sum_{n=0}^{M} (-1)^n \left(\begin{array}{c} M \\ M-n \end{array} \right) x^n.$$

The Tricomi's integral $\Psi(a;c;z)$ is given by

$$\Psi(a;c;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1} dt, \quad \text{Re}a > 0, \text{Re}z > 0.$$

Theorem 0.2.4 *For $z \in \mathbb{C}$ and $\mid n\rangle$ as in definition (0.1.1), the states*

$$\mid z; M\rangle = \mathcal{N}_M(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{M} \frac{M!}{(M-n)!\sqrt{n!}} z^{M-n} \mid n\rangle \tag{18}$$

form a set of CS, where the normalization factor is given by

$$\mathcal{N}_M(|z|^2) = M! L_M(-|z|^2)$$

and the resolution of the identity

$$\int\int_{\mathbb{C}} d^2z \mid z; M\rangle W_M(x = |z|^2) \langle z; M \mid = I_M$$

is obtained with

$$W_M(x) = L_M(-x) \frac{(M+1)}{\pi} \Gamma(M+2) \Psi(M+2; 1; x).$$

The choice for $W_M(x)$ is not unique.

From the binomial spin CS follows the theorem using the fact that $\widehat{S}_-$ is nilpotent of order $2S+1$, that is $(\widehat{S}_-)^{2S+1} = 0$.

Theorem 0.2.5 *With the notations of theorem (0.2.3), for $\mu \in \mathbb{C}$, the states*

$$| \mu; 2S\rangle = \mathcal{N}_{2S}(|\mu|^2)^{-\frac{1}{2}} \sum_{p=0}^{2S} \begin{pmatrix} 2S \\ p \end{pmatrix} \mu^{2S-p}(\widehat{S}_-)^p \mid 0\rangle \tag{19}$$

form a set of CS, where the normalization factor is given by (with $x = |\mu|^2$)

$$\mathcal{N}_{2S}(x) = [(2S)!]^2 {}_1F_2(-2S; 1, 1; -x)$$

and the resolution of the identity

$$\int\int_{\mathbb{C}} d^2\mu \mid \mu; 2S\rangle W_{2S}(x)\langle\mu; 2S \mid = I_{2S}$$

is obtained with (in terms of Meijer G-function)

$$W_{2S}(x) = \frac{{}_1F_2(-2S; 1, 1; -x)}{\pi(2S)!} G^{3,1}_{1,3}\left(x \mid \begin{array}{ccc} & -2S-1 & \\ 0, & 0, & 0 \end{array} \right)$$

The choice for $W_{2S}(x)$ is not unique.

For the definitions of ${}_1F_2$ and $G^{3,1}_{1,3}$ one could consult [11].

0.2.4 Photon added CS

Here we discuss some recent developments on photon added CS based on [37] and the references cited there. The photon added CS were introduced by G.S.Agarwal and

K.Tara in 1991 as follows.

$$| z; M\rangle = [M!L_M(-|z|^2)]^{-\frac{1}{2}}(a^\dagger)^m \mid z\rangle \longrightarrow \begin{cases} \mid z\rangle & \text{as} \quad M \to 0, \quad z = \text{const} \\ \mid M\rangle & \text{as} \quad z \to 0, \quad M = \text{const} \end{cases}, \tag{20}$$

where $M \in \mathbb{N}$, the set of positive natural numbers and $\mid z\rangle, \mid M\rangle, a^\dagger$ are as in definition (0.1.1). The properties and applications of these states were studied by M.Danka et al. and V.V.Dodonov et al. in 1998. Following this, a development was made in [37]. We summerize the results of [37] in the following theorem.

Theorem 0.2.6 *For $z \in \mathbb{C}, M \in \mathbb{N}$ and $\mid n\rangle$ as in definition (0.1.1), the states*

$$| z; M\rangle = \mathcal{N}_M(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\sqrt{(n+M)!}}{n!} z^n \mid n+M\rangle \tag{21}$$

form a set of CS, where the normalization factor is given by

$$\mathcal{N}_M(|z|^2) = e^{|z|^2} M! L_M(-|z|^2)$$

and the resolution of the identity

$$\int\int_{\mathbb{C}} d^2z \mid z; M\rangle W_M(|z|^2)\langle z; M \mid = I_M = \sum_{n=0}^{\infty} \mid n+M\rangle\langle n+M \mid$$

is obtained with

$$W_M(x) = \frac{M e^x L_M(-x)}{\pi} \sum_{p=0}^{M-1} (-1)^p \begin{pmatrix} M-1 \\ p \end{pmatrix} E_{p+1}(x), \quad M \geq 1, x > 0$$

where $x = |z|^2$ and $E_k(z) = \int_1^\infty \frac{e^{-zt}}{t^k} dt, \ z > 0$ is the generalized exponential of order k. Here the choice for $W_M(x)$ is not unique.

Note that (21) is a linear combination of all the number states starting with $n = M$, that is the first M number states are absent from the wave function $| z; M\rangle$.

0.2.5 Gazeau-Klauder CS

A class of more constrained CS were defined in [15] for semi-bounded Hamiltonian operators $\mathcal{H}$. It was defined by adding two new conditions for a two parameter family of states $\{| J,\gamma\rangle \mid J \geq 0, -\infty < \gamma < \infty\}$. Let us give a formal definition for these states.

Definition 0.2.1 *The set of states* $\{| J,\gamma\rangle \mid J \geq 0, -\infty < \gamma < \infty\}$ *is said to form Gazeau-Klauder CS for a Hamiltonian* $\mathcal{H}$ *if it satisfies*

(i) Continuity: $(J',\gamma') \to (J,\gamma) \Rightarrow | J',\gamma'\rangle \to | J,\gamma\rangle$

(ii) Resolution of unity: $\int | J,\gamma\rangle\langle J,\gamma | d\mu(J,\gamma) = I$

(iii) Temporal stability: $e^{-i\mathcal{H}t} | J,\gamma\rangle = | J,\gamma + \omega t\rangle$, $\omega =$*const*

(iv) Action identity: $\langle J,\gamma | \mathcal{H} | J,\gamma\rangle = \omega J$

The authors have obtained CS obeying conditions for semi-bounded Hamiltonians with discrete spectrum, continuous spectrum and both discrete and continuous parts in the spectrum. Further they have demonstrated the applicability and relevance of these CS. Following this the same class of CS were obtained for periodic, invariant and Morse potentials by others.

As an example, let us discuss CS for discrete dynamics based on [15]. Let $\mathcal{H}$ be a Hamiltonian with discrete spectrum. Suppose the spectrum is bounded below and adjusted so that $\mathcal{H} \geq 0$. Further assume that the eigenstates of $\mathcal{H}$ are non-degenerate. The eigenstates $\mid n\rangle$ are orthonormal vectors that satisfy

$$\mathcal{H} \mid n\rangle = E_n \mid n\rangle, \;\; n \geq 0 \text{ and } 0 = E_0 < E_1 < E_2 < ...$$

Let $E_n = \omega e_n, \;\; \omega > 0$ and fixed, and thereby introduce a sequence of dimensionless real numbers $0 = e_0 < e_1 < e_2 < ....$ Let $\rho_n = e_1 e_2 ... e_n$ along with $\rho_0 = 1$. Take CS for the system as,

$$\mid J, \gamma\rangle = N(J)^{-1} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}} e^{-ie_n\gamma}}{\sqrt{\rho_n}} \mid n\rangle \tag{22}$$

where $0 \leq J$ and $-\infty < \gamma < \infty$, and $N(J)$ is the normalization factor. From $\langle J, \gamma \mid J, \gamma\rangle$ the normalization factor takes the form

$$N(J)^2 = \sum_{n=0}^{\infty} \frac{J^n}{\rho_n}.$$

The domain of allowed J, $0 \leq J < R$ is determined by the radius of convergence $R = \overline{\lim}_{n\to\infty}(\rho_n)^{\frac{1}{n}}$ in the series defining $N(J)^2$. Let $k(J) = N(J)^2\rho(J) \geq 0$ for $0 \leq J < R$ and $k(J) = \rho(J) = 0$ for $J > R$, where $\rho_n = \int_0^R u^n \rho(u) du, \; \rho(u) \geq 0$. The resolution of the identity is obtained as

$$\int \mid J, \gamma\rangle\langle J, \gamma \mid d\mu(J, \gamma) = \mathbb{I}$$

where $d\mu(J, \gamma) = k(J) dJ d\nu(\gamma), 0 \leq J < R$ and $-\infty < \gamma < \infty$. The measure $d\nu(\gamma)$ is defined as

$$\int ... d\nu(\gamma) = \lim_{\Gamma\to\infty} \int_{-\Gamma}^{\Gamma} ... d\gamma.$$

For the states in (22) the temporal stability and the action identity can easily be verified.

0.3 Deformed CS

Here we discuss two types of deformed CS based on [32] and [34] namely q-deformed CS and f-deformed CS (or nonlinear CS). These deformed states can simply be viewed as CS obtained by replacing the $\rho(m)$ in definition (0.0.1) by two different choices. But, they bear this name because they were obtained from deformed oscillator algebras. We will study them in bit more detail in the following subsections.

0.3.1 q-deformed CS

The q-deformed CS were introduced by Arik.M and Coon.D.D in 1976 and further developed by Jannussis et al. and Biedenharn in the 1980s. Here our discussion is mainly based on [32] and the references there in. A q-deformed oscillator algebra, as usual, contains creation, annihilation and an identity operator and they are denoted by $a^\dagger, a, I$ respectively. They act in the Hilbert space $\mathbb{H}$ and satisfy the relations

$$[a, a^\dagger] = aa^\dagger - qa^\dagger a = I, \;\; [a, I] = [a^\dagger, I] = 0.$$

The major differences between the deformed and the ordinary CS are that the commutator contains a parameter q and the orthonormal basis $\mid n\rangle$ in the Hilbert space

$\mathbb{H}$ is defined by

$$| n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}} | 0\rangle$$

where

$$[n]! = [1]\cdot[2]\cdot ....\cdot[n], \;\; [n] = 1+q+...+q^{n-1} = \frac{1-q^n}{1-q},$$

and $| 0\rangle$ is the vacuum vector satisfying $a | 0\rangle =| 0\rangle$. The actions of the operators a and $a^\dagger$ were taken to be

$$a | n\rangle = \sqrt{[n]} | n-1\rangle, \;\text{ and }\; a^\dagger | n\rangle = \sqrt{[n+1]} | n+1\rangle.$$

In order to move further, the usual exponential function was modified as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

The above series converges for $|x| < R_q = (1-q)^{-1}$ for all finite values of x with $|q| > 1$. Further, $[n]! \to n!$ as $q \to 1$. Let $D_q = \{z \mid |z| < R_q\}$. The q-derivative of a function $f(x)$ is defined as

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(x)-f(qx)}{x(1-q)}.$$

Let us see the basic q-deformed CS in the following theorem.

Theorem 0.3.1 *For $z \in D_q$ the states*

$$| z\rangle_q = \mathcal{N}_q(|z|)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} | n\rangle \tag{23}$$

forms a set of q-deformed CS, where the normalization factor is given by

$$\mathcal{N}_q(|z|) = e_q(|z|^2)$$

and the resolution of the identity (in terms of q-integral)

$$\int\int_{D_q} d_q^2 z \mid z\rangle_q W_q(|z|)_q\langle z \mid = I$$

is obtained with $W_q(|z|) = \frac{1}{\pi}$.

Progress made in obtaining resolution of the identity using ordinary integrals. In 1996 Kar.T.K and Ghosh.G obtained a resolution of identity using inverse Fourier transforms and in 1999 Penson.K.A and Solomon A.I obtained it using Laplace transforms. Further in [32] Quesne.C obtained the following result.

Theorem 0.3.2 *For* $z \in \mathbb{C}$ *and* $q \in (0,1)$, *with* $\mid n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \mid 0\rangle$ *(the usual boson state) and* $[n]! = [1][2]...[n], [0]! = 1$, *where* $[n] = \frac{1-q^{-n}}{q-1} = q^{-n}\{n\}_q$, *the states*

$$\mid z\rangle_q = \mathcal{N}_q(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} \mid n\rangle \tag{24}$$

form a set of q-deformed CS, where the normalization factor is given by (with $x = |z|^2$*)*

$$\mathcal{N}_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{\{n\}_q!}(qx)^n = E_q[(1-q)qx]$$

and the resolution of the identity

$$\int\int_{\mathbb{C}} d^2 z \mid z\rangle_q W_q(|z|^2)_q\langle z \mid = I$$

is obtained with

$$W_q(x) = \frac{1-q}{\ln q^{-1}}\{E_q[(1-q)x]\}^{-1}.$$

In [32], the author also demonstrated the convergence of the function $E_q[(1-q)qx]$ and gave approximations in terms of well known functions.

0.3.2 f-deformed CS

Here we briefly discuss the f-deformed or nonlinear CS based on [34] and the references there in. These CS were obtained by deforming the annihilation and creation operators. Nonlinearity for CS was introduced by Manko et al in 1993. Let $a, a^\dagger$ and N be the usual harmonic oscillator operators. Now introduce two new operators A and $A^\dagger$ by deforming a and $a^\dagger$ as,

$$A = f(N)a \quad A^\dagger = a^\dagger f(N).$$

The operator valued function $f(N)$ is the deforming function. The nonlinearity depends on the form of $f(N)$. The f-oscillators are defined by the Hamiltonian

$$H_f = \frac{1}{2}[AA^\dagger + A^\dagger A].$$

The f-deformed CS are defined as the eigenstates of A, that is

$$A \mid z, f\rangle = z \mid z, f\rangle.$$

With the following notations

$$(f(n))! = f(0)f(1)...f(n), \quad D_f = \{z \mid |z| \leq \lim_{n\to\infty} n[f(n)]^2\}$$

the expansion of these states in the number basis yields the following.

Theorem 0.3.3 *For $z \in D_f$ the states*

$$\mid z, f\rangle = \sum_{n=0}^{\infty} c_n \mid n\rangle \tag{25}$$

form a set of f-deformed CS, where the normalization yields

$$|c_0|^{-2} = \sum_{n=0}^{\infty} \frac{|z|^2}{n![(f(n))!]^2}$$

and the resolution of the identity

$$\int d\mu(z) \mid z, f\rangle\langle z, f \mid = I$$

is obtained if there exist a function $\mu(\rho)$,where $z = \rho e^{i\theta}$, exist to satisfy the moment equation

$$\int_0^{\rho'} \rho^{2n+1}|c_0|^2\mu(\rho)d\rho = n![(f(n))!]^2$$

where ρ' is the maximum value of $|z|$.

0.4 CS of Perelomov type

Let G be an arbitrary Lie group and $U(g)$ its unitary irreducible representation, acting in a Hilbert space $\mathbb{H}$. Fix a vector $\mid \psi_0\rangle$ in the Hilbert space and take $\mid \psi_g\rangle = U(g) \mid \psi_0\rangle$ for $g \in G$. Let

$$H = \{h \mid U(g) \mid \psi_0\rangle = e^{i\alpha(h)} \mid \psi_0\rangle\},$$

where $e^{i\alpha(h)}$ is a character of H. When the subgroup H is maximal, it is called the isotropy subgroup.

Definition 0.4.1 *The system of states $\{\mid \psi_g\rangle = U(g) \mid \psi_0\rangle \mid g \in G\}$ is called the coherent state system and is denoted by $\{U, \mid \psi_0\rangle\}$.*

Let H be the isotropy subgroup for the state $\mid \psi_0\rangle$. Then a CS $\mid \psi_g\rangle$ is determined by a point $x = x(g)$ in the coset space G/H, corresponding to the element g, that is $\mid \psi_g\rangle = e^{i\alpha} \mid x\rangle, \mid \psi_0\rangle = \mid 0\rangle$. Let us consider the action of the operator $U(g)$ upon the state $\mid \psi_0\rangle = \mid 0\rangle$,

$$U(g) \mid 0\rangle = e^{i\alpha(g)} \mid x(g)\rangle, \tag{26}$$

where the function $\alpha(g)$ is defined for any element g of G, while for $h \in H$ it coincides with the function $\alpha(h)$. The CS of the type (26) are said to be in Perelomov's form.

Now let us turn our attention to a particular type of Perelomov's CS, $SU(1,1)$ CS. We discuss the details based on [31] and [13]. Let us consider a unitary irreducible representation $U(g)$ of the group $G = SU(1,1)$, and let $\mid \psi_0\rangle$ be the vector in the representation space, invariant under the action of $K = U(1)$, the maximal compact subgroup. Applying the operator $U(g)$ to this vector gives a CS system parameterized by points of the coset space $X = G/K$. The coset space can be realized as the interior of the unit disc. Selecting an element $g_\zeta \in G$ in the equivalence class $\zeta \in X$, we get the CS system

$$\mid \zeta\rangle = U(g_\zeta) \mid 0\rangle, \quad \mid 0\rangle = \mid \psi_0\rangle. \tag{27}$$

Let us denote the generators of $su(1,1)$ Lie algebra by $\{K_+, K_-, K_3\}$. These generators satisfy the following commutation relations.

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.$$

Further the commutator $[K_+, K_-]$ doesn't commute with K_+ or K_-. Thus the generators K_+ and K_- do not satisfy the Baker-Campbell-Hausdorff identity. But they do satisfy a similar identity in the following form. For $\zeta = \frac{z \tanh |z|}{|z|}$, $z \in \mathbb{C}$ we have

$$e^{zK_+ - \bar{z}K_-} = e^{\zeta K_+} e^{\log(1-|\zeta|^2)K_3} e^{-\bar{\zeta}K_-}. \tag{28}$$

Note that $\zeta \in D = \{z \in \mathbb{C} \mid |z| < 1\}$. Now, using this representation, Perelomov defined the following set of CS,

$$\mid z\rangle = e^{zK_+ - \bar{z}K_-} \mid 0\rangle \tag{29}$$

where $z \in \mathbb{C}$. While replacing g_ζ by

$$g_\zeta = \frac{1}{\sqrt{1-|\zeta|^2}} \begin{pmatrix} 1 & -\zeta \\ -\bar{\zeta} & 1 \end{pmatrix}$$

if we replace the z in (29) by ζ then the CS in (27) and the CS in (29) become equivalent. From now on let $\{K_+, K_-, K_3\}$ stands for a spin K representation of $su(1,1)$. In this case $(K_+)^\dagger = K_-$. Let $\mathbb{H}_K = \{\mid K, n\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ be the Fock space and the action of the generators on this space is

$$\begin{aligned} K_+ \mid K, n\rangle &= \sqrt{(n+1)(2K+n)} \mid K, n+1\rangle \\ K_- \mid K, n\rangle &= \sqrt{n(2K+n-1)} \mid K, n-1\rangle \\ K_3 \mid K, n\rangle &= (K+n) \mid K, n\rangle \end{aligned}$$

where $\mid K, 0\rangle$ is a normalized vacuum with $K_- \mid K, 0\rangle = 0$. Now using the identity (28) together with the above equations the CS in (29) can be decomposed over the orthonormal basis as follows.

$$|\,\zeta\rangle = (1-|\zeta|^2)^K \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+2K)}{n!\Gamma(2K)}\right)^{\frac{1}{2}} \zeta^n \mid K, K+n\rangle. \tag{30}$$

In a little more general set up, let $\{\phi_n\}$ be an orthonormal basis of an abstract Hilbert space $\mathbb{H}$. Let $\{I, A, A^\dagger, N\}$ be a generalized oscillator algebra. Then for $z \in \mathbb{C}$, the CS in the form

$$|\,z\rangle = e^{zA^\dagger - \bar{z}A}\phi_0 \tag{31}$$

are said to be in Perelomov form.

0.5 Vector coherent states

The method of vector coherent state(VCS) theory was independently developed by the research groups of Quesne and Rowe to study representations of Lie groups appearing in physics. Here we briefly discuss the theory of VCS based on [1] and [33]. A sophisticated definition of VCS representation theory was given in [33] by D.J.Rowe and J.Repka in 1991 in terms of induced representations of a group. But an initial definition was given by D.J.Rowe, G.Rosensteel and R.Gilmore in 1985. We begin with the following preliminaries.

Let G be a locally compact group, μ be the left Haar measure on it. H be a closed subgroup of G and $X = G/H$ is the corresponding coset space with a quasi-invariant measure ν. $\sigma : X \to G$ is a global Borel section. Let $\mathbb{H}$ be a Hilbert space.

Definition 0.5.1 *Let U be a unitary representation of G in $\mathbb{H}$ and η^i, $i = 1, ..., n$, be a set of linearly independent vectors in $\mathbb{H}$. Then, if the set of vectors,*

$$\{\eta^i_{\sigma(x)} = U(\sigma(x))\eta^i \mid i = 1, ..., n, x \in X\}$$

is total in $\mathbb{H}$, we call it a family of covariant CS for U.

Consider the closed subgroup H of G that stabilizes a cyclic vector η up to a multiplicative factor; i.e.,

$$U(h)\eta = e^{i\omega(h)}\eta; \quad h \in H$$

where ω is a real valued function on H (in fact $\chi(h) = e^{i\omega(h)}$ is a character of H).

Definition 0.5.2 *Let $X = G/H$, $\sigma : X \longrightarrow G$ be a Borel section and U be a unitary irreducible representation of G. Then the family of covariant CS*

$$\{\eta_{\sigma(x)} = U(\sigma(x))\eta \mid x \in X\}$$

is total in $\mathbb{H}$ and is called Gilmore-Perelomove CS (these are not VCS).

Now let us give a sophisticated definition of VCS based on [1]. For that, let $\eta^1, ..., \eta^n$ be a set of linearly independent vectors in $\mathbb{H}$. $\mathcal{K}$ is an n-dimensional subspace of $\mathbb{H}$ generated by $\eta^1, ..., \eta^n$. Assume that the space $\mathcal{K}$ is stable under the action of the unitary operators $U(h), h \in H$, a closed subgroup of G. Then the functions $\Phi : X = G/H \to \mathbb{C}^n$, with components $\Phi^i(x) = \langle U(\sigma(x))\eta^i \mid \phi\rangle, i = 1, 2, ..., n, \phi \in \mathbb{H}$, give a reproducing kernel Hilbert space, $\mathbb{H}_{\mathcal{K}}$ with matrix kernel $K(x, y)$ having matrix elements $K(x, y)_{ij} = \langle U(\sigma(x))\eta^i \mid U(\sigma(y))\eta^j\rangle, i, j = 1, ..., n$. Now the mapping W_η :

$\mathbb{H} \to \mathbb{H}_{\mathcal{K}}$, by $(W_\eta \phi)^i(x) = \Phi^i(x), i = 1, 2, ..., n, x \in X, \eta \in \mathcal{K}$ is a Hilbert space isometry. Denote the image of $U(g)$ under W_η by $U_\eta(g)$, then it can be written as

$$(U_\eta(g)\Phi)(x) = V(h(g^{-1}, x))^* \Phi(g^{-1}x), \quad \Phi \in \mathbb{H}_{\mathcal{K}}, x \in X$$

where $V(h), h \in H$ is an n-dimensional unitary representation of H on $\mathbb{C}^n$. It is possible to extend U_η to an induced representation $\widehat{U}$ on the Hilbert space $\mathbb{C}^n \otimes L^2(X, d\nu)$, of which $\mathbb{H}_{\mathcal{K}}$ then becomes a subspace.

Definition 0.5.3 *The set of vectors*

$$\mathcal{G}_\sigma = \left\{ \eta^i_{\sigma(x)} = U_\eta(\sigma(x))\eta^i \mid i = 1, 2, ..., n, x \in X \right\}$$

form a set of VCS. The representation U_η is called a VCS representation.

The normalization and resolution of the identity can be studied in terms of the boundedness and square integrability of the involved group representation. Next we give a more general definition of VCS.

Definition 0.5.4 *Let $X = G/H$ be a coset space as in definition (0.5.3) then the set of vectors*

$$\{\eta^i_x \mid i = 1, 2, ..., n, x \in X\}$$

forms a set of VCS if

$$\sum_{i=0}^{n} \int \mid \eta^i_x\rangle\langle\eta^i_x \mid d\nu(x) = \mathbb{I}$$

where $d\nu(x)$ is a Haar measure on X.

The Gilmore-Perelomov CS are a special case of the VCS of definition (0.5.3) when the dimension of the subspace $\mathcal{K}$ is unity. Further the general covariant CS of definition (0.5.1) do not require $\mathcal{K}$ to carry a representation of H. Definition (0.5.1) can be obtained from definition (0.5.3) by taking $\eta = I_{\mathbb{H}}$, the identity of the Hilbert space. Further definitions (0.5.1) and (0.5.3) are special cases of definition (0.5.4).

In this thesis we demonstrate a different class of VCS in the form of definition (0.0.1) by replacing the complex number z by an $n \times n$ matrix.

So far, we have discussed the usual CS and some of the recent developments. From now on, except chapter 2, we present our original contribution to the subject. The chapters are ordered in the following manner. In chapter 1, we present a general procedure to construct VCS by replacing the parameter z of the usual CS by an $n \times n$ matrix $Z = A(r)e^{i\zeta\Theta(k)}$, where $A(r)$ and $\Theta(k)$ are $n \times n$ matrices, and study them under the usual properties. Chapter 2 gives an introduction to quaternions and Clifford algebras which can be considered as a preliminary to the following chapters. In chapter 3, as an example to the general construction, we present explicit VCS based on a matrix realization of the quaternion algebra and study them under the usual properties. In chapter 4, we present a generalization of the quaternionic VCS. In the quaternion case, the matrix $A(r)$ is just a multiple of the identity. Here we construct VCS with a more general matrix $A(r,s)$. Chapter 5 mainly deals with a construction of VCS based on the matrix realization of the Clifford algebras. This construction differs from the general procedure presented in chapter 1. We also present some other examples which are easy consequences of chapter 3 and chapter 4. chapter ends with a partly solved example. In chapter 6, we construct quaternionic VCS on the unit disc by manipulating the results of section 0.4 and prove an interesting disentangling formula similar to the one proved by Perelomov (presented in section 0.4). Chapter 7 gives some possible applications of VCS to physics. We explicitely calculate some physical quantities in terms of the quaternionic VCS of chapter 3. Further, we introduce a connection between spin-orbit-interaction and quaternionic VCS. In the last chapter, VCS are constructed on $n-$copies of the unit disc. This construction is also different from the one presented in chapter 1. The matrix of this chapter cannot be written in the matrix form considered in chapter 1. As an example to this construction and as a preliminary step of the construction of VCS on the classical domains, we present a class of VCS on the unit disc by considering the unit disc as a homogeneous space of the $SU(1,1)$ group.

Chapter 1

Vector Coherent States-General set up

1.1 The set up

In this chapter we generalize the definition (0.0.1) a step further by replacing the complex number z by an $n \times n$ matrix $\mathcal{Z}$. By doing so we generate a new class of VCS. Further, we analyze some of the properties of these VCS.

Let us start with a preliminary set up of the problem. Let $\mathcal{K}$ be a space with probability measure dK, $\mathcal{R}$ be another space with measure $d\mathcal{R}$. Let $\mathbb{R}$ denote the set of all real numbers. For $k \in \mathcal{K}$, $r \in \mathcal{R}$ and $\zeta \in \mathbb{R}$ let

$$\mathcal{Z} = A(r)e^{i\zeta\Theta(k)} \tag{1.1}$$

Where $A(r), \Theta(k)$ are $n \times n$ matrices with the following properties

$$\Theta \text{ is hermitian, that is,} \Theta = \Theta^\dagger \tag{1.2}$$

$$\Theta^2 = \mathbb{I}_n \tag{1.3}$$

$$[A, \Theta] = 0, \text{ that is, } A \text{ and } \Theta \text{ commute} \tag{1.4}$$

$$AA^\dagger = A^\dagger A. \tag{1.5}$$

Lemma 1.1.1 *With the above assumptions on $A(r)$ and $\Theta(k)$, we have*

$$\mathcal{Z} = A(r)e^{i\zeta\Theta(k)} = A(r)[\cos\zeta + i\Theta(k)\sin\zeta]. \tag{1.6}$$

Proof. Let us start with the series representation of the exponential function.

$$\begin{aligned}
\mathcal{Z} &= A(r)e^{i\zeta\Theta(k)} \\
&= A(r)\sum_{m=0}^{\infty}\frac{(i\zeta\Theta(k))^m}{m!} \\
&= A(r)\left[\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{2m}}{(2m)!} + i\Theta(k)\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{2m+1}}{(2m+1)!}\right] \\
&= A(r)(\cos\zeta + i\Theta(k)\sin\zeta).
\end{aligned}$$

Line three in the proof is obtained from line two using the fact that $\Theta^2 = \mathbb{I}_n$. ■

Let $\chi^j, j = 1, 2, ..., n$ be an orthonormal basis in $\mathbb{C}^n$, where $\mathbb{C}$ stands for the set of all complex numbers, and $\{\phi_m\}$ be an orthonormal basis in an abstract Hilbert space $\mathbb{H}$. Now $\{\chi^j \otimes \phi_m\}, j = 1, 2, ..., n$, $m = 1, 2, ..., \infty$ is an orthonormal basis in $\widehat{\mathbb{H}} = \mathbb{C}^n \otimes \mathbb{H}$, where $\otimes$ stands for the tensor product. Now let us define our VCS in the following way

$$| \mathcal{Z}, j\rangle = \mathcal{N}(|\mathcal{Z}|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \tag{1.7}$$

where $\mathcal{N}$ is a normalization factor and $\rho(m)$ is a function of m and both have to be chosen in such a way that $| \mathcal{Z}, j\rangle$ satisfy the following properties of VCS.

$$\text{Normalization:} \quad \sum_{j=0}^{n} \langle \mathcal{Z}, j \mid \mathcal{Z}, j\rangle = 1 \tag{1.8}$$

$$\text{Resolution of unity:} \quad \int_{\mathcal{R}\times\mathcal{K}\times\mathbb{R}} W(|\mathcal{Z}|) \sum_{j=1}^{n} | \mathcal{Z}, j\rangle\langle \mathcal{Z}, j \mid d\mu = \mathbb{I}_n \otimes I, \tag{1.9}$$

where $W(|\mathcal{Z}|)$ is a positive weight function.

1.2 Normalization

Here we are going to define the normalization factor $\mathcal{N}$ in terms of $\rho(m)$. Before going there let us note that

$$\mathcal{Z} = A(r)e^{i\zeta\Theta(k)} = A(r)(\cos\zeta + i\Theta(k)\sin\zeta)$$

implies that

$$\mathcal{Z}^m = A(r)^m e^{im\zeta\Theta(k)} = A(r)^m(\cos m\zeta + i\Theta(k)\sin m\zeta).$$

By polar decomposition every $n \times n$ matrix $A(r)$ can be written as $A(r) = |A(r)|U$, where U is a unitary matrix (if $\det A(r) = 0$ then U is a partial isometry) and $|A(r)|$ is a positive semidefinite hermitian matrix. $|A(r)|$ can also be defined as $|A(r)| = \left[A(r)A(r)^\dagger\right]^{\frac{1}{2}}$. Further, $Tr|A(r)|$ means the trace of the matrix $A(r)$. With these notations we have the following.

With conditions (1.2),(1.4) and (1.5) on (1.1), the states in (1.7) are normalized by the normalization factor

$$\mathcal{N} = \sum_{m=0}^{\infty} \frac{Tr|A(r)|^{2m}}{\rho(m)} \tag{1.10}$$

in the sense of (1.8 Let us consider the sum:

$$\begin{aligned}
\sum_{j=0}^{n} \langle \mathcal{Z}, j \mid \mathcal{Z}, j \rangle &= \sum_{j=0}^{n} \langle \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \mid \mathcal{N}^{-\frac{1}{2}} \sum_{l=0}^{\infty} \frac{\mathcal{Z}^l}{\sqrt{\rho(l)}} \chi^j \otimes \phi_l \rangle_{\hat{\mathbb{H}}} \\
&= \mathcal{N}^{-1} \sum_{j=0}^{n} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\sqrt{\rho(m)\rho(l)}} \langle \mathcal{Z}^{l\dagger} \mathcal{Z}^m \chi^j \mid \chi^j \rangle_{\mathbb{C}^n} \langle \phi_m \mid \phi_l \rangle_{\mathbb{H}}
\end{aligned}$$

The orthonormality of the vectors ϕ_m together with the fact that $\Theta^\dagger = \Theta$ reduces the last line to

$$\mathcal{N}^{-1} \sum_{j=0}^{n} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} \langle e^{-im\zeta\Theta(k)} (A(r)^\dagger)^m A(r)^m e^{im\zeta\Theta(k)} \chi^j \mid \chi^j \rangle_{\mathbb{C}^n}. \tag{1.11}$$

By applying the property $[A, \Theta] = 0$ we can reduce (1.11) to

$$\mathcal{N}^{-1} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} \sum_{j=0}^{n} \langle (A(r)^\dagger)^m A(r)^m \chi^j \mid \chi^j \rangle_{\mathbb{C}^n}. \tag{1.12}$$

Together with the polar decomposition of $A(r)$ and the property $AA^\dagger = A^\dagger A$ we can reduce (1.12) to

$$\mathcal{N}^{-1} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} Tr|A(r)|^{2m} = 1.$$

The last equality is obtained by replacing $\mathcal{N}$ by (1.10). The function $\rho(m)$ must be chosen in a way which guarantees the convergence of (1.7). We will define $\rho(m)$ together with the resolution of identity.

1.3 Resolution of the identity

In this section we will find a condition for the resolution of the identity. We will also give a definition to the function $\rho(m)$ through a moment problem. Further we will discuss the compatibility of the definition of $\rho(m)$ with the convergence of the series in (1.7). Let $d\mu = dRdKd\zeta$ be the measure on the space $\mathcal{R} \times \mathcal{K} \times [0, 2\pi]$, where the spaces $\mathcal{R}$ and $\mathcal{K}$ are as before. With $W(|\mathcal{Z}|)$ denoting a weight function, we prove the following. With the conditions (1.2)-(1.5) on the matrices, the VCS in (1.7) satisfies the resolution of the identity

$$\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]} W(|\mathcal{Z}|) \sum_{j=1}^{n} \mid \mathcal{Z}, j\rangle\langle \mathcal{Z}, j \mid d\mu = \mathbb{I}_n \otimes I \tag{1.13}$$

if the weight function $W(|\mathcal{Z}|)$ can be obtained as a solution to the following moment problem

$$\int_{\mathcal{R}} \frac{2\pi W(|\mathcal{Z}|) \mid A(r) \mid^{2m}}{\mathcal{N}(|\mathcal{Z}|)} d\mathcal{R} = \rho(m)\mathbb{I}_n. \tag{1.14}$$

For, let us start with the required integral.

$$\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]} W(|\mathcal{Z}|) \sum_{j=1}^{n} \mid \mathcal{Z}, j\rangle\langle \mathcal{Z}, j \mid d\mu$$

$$
\begin{aligned}
&= \int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]} W(|\mathcal{Z}|)\sum_{j=1}^{n} \mid \mathcal{N}^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m\rangle\langle\mathcal{N}^{-\frac{1}{2}}\sum_{l=0}^{\infty}\frac{\mathcal{Z}^l}{\sqrt{\rho(l)}}\chi^j\otimes\phi_l \mid d\mu \\
&= \sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]}\frac{W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\sqrt{\rho(m)\rho(l)}} \mid \mathcal{Z}^m\chi^j\otimes\phi_m\rangle\langle\mathcal{Z}^l\chi^j\otimes\phi_l \mid d\mu \\
&= \sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]}\frac{W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\sqrt{\rho(m)\rho(l)}}\mathcal{Z}^m \mid \chi^j\rangle\langle\chi^j \mid \mathcal{Z}^{l\dagger}\otimes \mid \phi_m\rangle\langle\phi_l \mid d\mu \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]}\frac{W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\sqrt{\rho(m)\rho(l)}}A(r)^m e^{im\zeta\Theta(k)}\left(\sum_{j=1}^{n} \mid \chi^j\rangle\langle\chi^j \mid\right) \\
&\qquad \times A(r)^{l\dagger}e^{-il\zeta\Theta(k)^\dagger}\otimes \mid \phi_m\rangle\langle\phi_l \mid d\mu
\end{aligned}
$$

Using the facts

$$
\begin{aligned}
&\sum_{j=1}^{n} \mid \chi^j\rangle\langle\chi^j \mid = \mathbb{I}_n, \\
&\Theta(k)^\dagger = \Theta(k) \quad \text{and} \\
&\int_0^{2\pi} e^{i(m-l)\zeta\Theta(k)}d\zeta = \begin{cases} 0 & \text{if } \ l\neq m \\ 2\pi\mathbb{I}_2 & \text{if } \ l=m \end{cases}
\end{aligned}
$$

where $\mathbb{I}_n$ is the $n\times n$ identity matrix, we can reduce the last line to

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\int_{\mathcal{R}}\int_{\mathcal{K}}\frac{2\pi W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\rho(m)}A(r)^m A(r)^{m\dagger}\otimes \mid \phi_m\rangle\langle\phi_m \mid d\mathcal{R}d\mathcal{K} \\
= &\sum_{m=0}^{\infty}\int_{\mathcal{R}}\int_{\mathcal{K}}\frac{2\pi W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\rho(m)} \mid A(r) \mid^{2m}\otimes \mid \phi_m\rangle\langle\phi_m \mid d\mathcal{R}d\mathcal{K}
\end{aligned}
$$

Since $d\mathcal{K}$ is a probability measure, we have

$$
\begin{aligned}
&\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]}\sum_{j=1}^{n} W(|\mathcal{Z}|) \mid \mathcal{Z},j\rangle\langle\mathcal{Z},j \mid d\mu \\
= &\sum_{m=0}^{\infty}\left(\int_{\mathcal{R}}\frac{2\pi W(|\mathcal{Z}|)}{\mathcal{N}(|\mathcal{Z}|)\rho(m)} \mid A(r) \mid^{2m} d\mathcal{R}\right)\otimes \mid \phi_m\rangle\langle\phi_m \mid
\end{aligned}
$$

Now by choosing

$$\int_{\mathcal{R}} \frac{2\pi W(|\mathcal{Z}|) \mid A(r) \mid^{2m}}{\mathcal{N}(|\mathcal{Z})|} d\mathcal{R} = \rho(m)\mathbb{I}_n \tag{1.15}$$

we can have

$$\int_{\mathcal{R}\times\mathcal{K}\times[0,2\pi]} W(|\mathcal{Z}|) \sum_{j=1}^{n} \mid \mathcal{Z}, j\rangle\langle \mathcal{Z}, j \mid d\mu = \mathbb{I}_n \otimes \sum_{m=0}^{\infty} \mid \phi_m\rangle\langle\phi_m \mid = \mathbb{I}_n \otimes I.$$

Thus we have proved that the states in (1.7) form a set of VCS with conditions (1.10) and (1.14) imposed on them. The compatibility of equations (1.10) and (1.14) will be demonstrated through several interesting examples in the following chapters.

1.4 Reproducing kernel

As we know, reproducing kernels have a strong connection with coherent states. For the VCS in definition (0.5.4), a matrix kernel $K(x, y) = (K_{ij}(x, y))_{n\times n}$ can be defined as [1]

$$K_{ij}(x, y) = \langle \eta_x^i \mid \eta_y^j \rangle, \quad i, j = 1, ..., n.$$

Further, this matrix kernel satisfies the properties of a reproducing kernel. Suppose the matrix kernel is of full rank, that is, $\det K(x, x) \neq 0$ Then for each $x \in X$, the set of vectors $\{\eta_x^1, \eta_x^2, ..., \eta_x^n\}$ is linearly independent and thus the operator

$$T_x = \sum_{i=0}^{n} \mid \eta_x^i\rangle\langle\eta_x^i \mid$$

is of rank n. On the other hand if $\det K(x, x) = 0$ the set $\{\eta_x^1, \eta_x^2, ..., \eta_x^n\}$ is not linearly independent and the operator T_x is not of rank n. When we build a set of VCS it is possible to get one of these cases. For further detail see [1].

Now let us go back to the VCS of this chapter. Even though, in the general set up given in (1.7) we are not going to get a pleasant form for the reproducing kernel, we like to establish it in a form which may give us some idea about it.

The matrix kernel for the set of VCS in (1.7) is given by

$$K(\mathcal{Z}^{\dagger}, \mathcal{Z}') = \left(K_{jk}(\mathcal{Z}^{\dagger}, \mathcal{Z}')\right)_{n\times n}$$

where

$$K_{jk}(\mathcal{Z}^{\dagger}, \mathcal{Z}') = \sum_{m=0}^{\infty} \frac{1}{\rho(m)\sqrt{N(\mathcal{Z})N(\mathcal{Z}')}} \langle e^{-im(\zeta'\Theta(k')-\zeta\Theta(k))} A(r')^{m\dagger} A(r)^m \chi^j \mid \chi^k\rangle. \tag{1.16}$$

For, let us start with the usual idea of getting a reproducing kernel from a set of CS.

$$\begin{aligned}
K_{jk}(\mathcal{Z}^{\dagger}, \mathcal{Z}') &= \langle \mathcal{Z}, j \mid \mathcal{Z}', k\rangle \\
&= \langle \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \mid \mathcal{N}(\mathcal{Z}')^{-\frac{1}{2}} \sum_{l=0}^{\infty} \frac{\mathcal{Z}'^l}{\sqrt{\rho(l)}} \chi^k \otimes \phi_l\rangle \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{\sqrt{N(\mathcal{Z})N(\mathcal{Z}')\rho(m)\rho(l)}} \langle \mathcal{Z}^m \chi^j \mid \mathcal{Z}'^l \chi^k\rangle \langle \phi_m \mid \phi_l\rangle \\
&= \sum_{m=0}^{\infty} \frac{1}{\rho(m)\sqrt{N(\mathcal{Z})N(\mathcal{Z}')}} \langle \mathcal{Z}^m \chi^j \mid \mathcal{Z}'^m \chi^k\rangle \\
&= \sum_{m=0}^{\infty} \frac{1}{\rho(m)\sqrt{N(\mathcal{Z})N(\mathcal{Z}')}} \langle e^{-im\zeta'\Theta(k')} (A(r')^{\dagger})^m A(r)^m e^{im\zeta\Theta(\kappa)^{\dagger}} \chi^j \mid \chi^k\rangle
\end{aligned}$$

Now using the properties (1.2) and (1.4) we can reduce the last line to

$$\sum_{m=0}^{\infty} \frac{1}{\rho(m)\sqrt{N(\mathcal{Z})N(\mathcal{Z}')}} \langle e^{-im(\zeta'\Theta(k')-\zeta\Theta(\kappa))} (A(r')^{\dagger})^m A(r)^m \chi^j \mid \chi^k\rangle.$$

Thus we obtain the reproducing kernel as required. It is interesting to note that:

$$K_{jk}(\mathcal{Z}^\dagger, \mathcal{Z}) = \sum_{m=0}^{\infty} \frac{1}{\rho(m)} \mathcal{N}(\mathcal{Z})^{-1} \langle (A(r')^\dagger)^m A(r)^m \chi^j \mid \chi^k \rangle.$$

Further, the matrix formed by the matrix elements, $K_{jk}(\mathcal{Z}^\dagger, \mathcal{Z})$ has a non-zero determinant.

1.5 Generalized Annihilation, Creation and Number operators

Let $x_m = \frac{\rho(m)}{\rho(m-1)}$ with $x_0! = 1$. Then $\rho(m) = x_m x_{m-1} ... 1 = x_m!$, the formal factorial. Define the annihilation operator on the basis vectors $\{\phi_m\}$ as

$$a\phi_m = \sqrt{x_m}\phi_{m-1} \quad \text{with} \quad a\phi_0 = 0 \tag{1.17}$$

From the annihilation operator we can get the creation operator, $a^\dagger$ and thus the number operator, N' on the same set of vectors.

Lemma 1.5.1 *The action of the creation and number operator on the basis vectors* $\{\phi_m\}$ *is given respectively by*

$$a^\dagger \phi_m = \sqrt{x_{m+1}}\phi_{m+1}, \qquad N'\phi_m = x_m \phi_m. \tag{1.18}$$

Proof. Consider

$$\begin{aligned} \langle a\phi_n \mid \phi_m \rangle &= \langle \phi_n \mid a^\dagger \phi_m \rangle \\ \langle \phi_n \mid a^\dagger \phi_m \rangle &= \langle \sqrt{x_n}\phi_{n-1} \mid \phi_m \rangle \end{aligned}$$

$$= \langle \phi_{n-1} \mid \phi_m \rangle = \begin{cases} 0 & \text{if} \quad m \neq n-1 \\ 1 & \text{if} \quad m = n-1 \end{cases}.$$

Thus we have

$$\frac{1}{\sqrt{x_{m+1}}} \langle \phi_{m+1} \mid a^\dagger \phi_m \rangle = 1.$$

Hence

$$a^\dagger \phi_m = \sqrt{x_{m+1}} \phi_{m+1}.$$

Now we can get the number operator, $N' = a^\dagger a$ in the usual way. Since we have the action of a and $a^\dagger$ the action of N' on the same basis vectors is obtained as

$$N' \phi_m = x_m \phi_m.$$

■

We define the same set of operators for the coherent states, $\{\mid \mathcal{Z}, j\rangle\}$ as follows

$$A = \mathbb{I}_n \otimes a \quad \text{annihilation operator} \tag{1.19}$$

$$A^\dagger = \mathbb{I}_n \otimes a^\dagger \quad \text{creation operator} \tag{1.20}$$

$$N = \mathbb{I}_n \otimes N' \quad \text{number operator.} \tag{1.21}$$

Proposition 1.5.1 *The action of the operators $A, A^\dagger$ and N on $\{\mid \mathcal{Z}, j\rangle\}$ is given respectively by*

$$A\left(\mid \mathcal{Z}, j\rangle\right) = \mathcal{Z} \mid \mathcal{Z}, j\rangle \tag{1.22}$$

$$A^\dagger\left(\mid \mathcal{Z}, j\rangle\right) = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m \sqrt{\rho(m+1)}}{\rho(m)} \chi^j \otimes \phi_{m+1} \tag{1.23}$$

and

$$N\left(\mid \mathcal{Z},j\rangle\right)=\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m-1)}}\chi^j\otimes\phi_m. \tag{1.24}$$

Proof. First, the action of A on $\{\mid \mathcal{Z},j\rangle\}$:

$$\begin{aligned}
A\left(\mid \mathcal{Z},j\rangle\right) &= A\left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m\right)\\
&= A\left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m+\frac{1}{\sqrt{\mathcal{N}(\mathcal{Z})\rho(m)}}\mathbb{I}_n\chi^j\otimes\phi_0\right)\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes a\phi_m+\frac{1}{\sqrt{\mathcal{N}(\mathcal{Z})\rho(m)}}\mathbb{I}_n\chi^j\otimes a\phi_0\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\sqrt{\frac{\rho(m)}{\rho(m-1)}}\chi^j\otimes\phi_{m-1}\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m-1)}}\chi^j\otimes\phi_{m-1}\\
&= \mathcal{Z}\mid \mathcal{Z},j\rangle
\end{aligned}$$

The action of $A^\dagger$ on $\{\mid \mathcal{Z},j\rangle\}$:

$$\begin{aligned}
A^\dagger\left(\mid \mathcal{Z},j\rangle\right) &= A^\dagger\left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m\right)\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes a^\dagger\phi_m\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\sqrt{x_{m+1}}\chi^j\otimes\phi_{m+1}\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m\sqrt{\rho(m+1)}}{\rho(m)}\chi^j\otimes\phi_{m+1}
\end{aligned}$$

Finally the action of N on $\{\mid \mathcal{Z},j\rangle\}$:

$$N\left(\mid \mathcal{Z}\rangle^j\right)=N\left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m\right)$$

$$
\begin{aligned}
&= N\left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes\phi_m+\frac{1}{\sqrt{\mathcal{N}(\mathcal{Z})\rho(m)}}\mathbb{I}_n\chi^j\otimes\phi_0\right)\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}\chi^j\otimes N'\phi_m+\frac{1}{\sqrt{\mathcal{N}(\mathcal{Z})\rho(m)}}\mathbb{I}_n\chi^j\otimes N'\phi_0\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m)}}x_m\chi^j\otimes\phi_m\\
&= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}}\sum_{m=1}^{\infty}\frac{\mathcal{Z}^m}{\sqrt{\rho(m-1)}}\chi^j\otimes\phi_m.
\end{aligned}
$$

Thus we have the proposition. ■

As we know, the conventional bosonic CS can be viewed as the eigenvectors of the annihilation operator, the creation operator is the dual of the annihilation operator and the number operator is the product of both the annihilation and creation operator in a particular order. In this sense, let us compare the operators $A, A^\dagger$ and N with the bosonic ones. The constructed VCS are the eigenvectors of the operator A, the operator $A^\dagger$ is the dual of A and the operator N is taken as a product of A and $A^\dagger$. Thus we can call $A, A^\dagger$ and N as annihilation, creation and number operators respectively. At the same time, if we consider the algebra generated by these operators they differ a lot. In the bosonic case, the algebra $\{I, a, a^\dagger, N\}$ is closed under commutation but the algebra $\{I, A, A^\dagger, N\}$ is not closed under commutation.

1.6 The operators P, Q and commutation relations

Here we will define the operators P and Q in two ways first in terms of a and $a^\dagger$ then in terms of A and $A^\dagger$. We will also investigate their actions on the basis vectors

$\{\phi_m\}$ and on the coherent states $\{| \mathcal{Z}, j\rangle\}$. Finally we will calculate the action of the commutators.

The action of the commutator $[A, A^\dagger]$ on the set of VCS $\{| \mathcal{Z}, j\rangle\}$ is given by

$$[A, A^\dagger] \mid \mathcal{Z}, j\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m (x_{m+1} - x_m)}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m. \tag{1.25}$$

As a first step let us calculate the actions of the commutators $[a, a^\dagger]$.

$$[a, a^\dagger] = (aa^\dagger - a^\dagger a)\phi_m = (x_{m+1} - x_m)\phi_m \tag{1.26}$$

Using equation (1.19)and (1.20) we obtain

$$\begin{aligned} [A, A^\dagger] \mid \mathcal{Z}, j\rangle &= [A, A^\dagger] \left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \right) \\ &= \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes [a, a^\dagger]\phi_m \end{aligned}$$

Using (1.26) we get

$$[A, A^\dagger] \mid \mathcal{Z}, j\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m (x_{m+1} - x_m)}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m.$$

Now from the usual relations

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \quad \text{and} \quad a = \frac{1}{\sqrt{2}}(q + ip),$$

on the Hilbert space $\mathbb{H}$, we get

$$q = \frac{a + a^\dagger}{\sqrt{2}} \quad \text{and} \quad p = \frac{a - a^\dagger}{\sqrt{2}i}. \tag{1.27}$$

On the Hilbert space $\widehat{\mathbb{H}}$ we get similar relations just by replacing the small letters by capital ones as

$$Q = \frac{A + A^\dagger}{\sqrt{2}} \quad \text{and} \quad P = \frac{A - A^\dagger}{\sqrt{2}i}. \tag{1.28}$$

Since we already know the action of the operators A and $A^\dagger$ on the coherent states, we can readily obtain

$$Q \mid \mathcal{Z}, j\rangle = \frac{1}{\sqrt{2}} \left[\mathcal{Z} \mid \mathcal{Z}\rangle^j + \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m \sqrt{\rho(m+1)}}{\rho(m)} \chi^j \otimes \phi_{m+1} \right].$$

In the same way we have

$$P \mid \mathcal{Z}, j\rangle = \frac{1}{\sqrt{2}i} \left[\mathcal{Z} \mid \mathcal{Z}\rangle^j - \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m \sqrt{\rho(m+1)}}{\rho(m)} \chi^j \otimes \phi_{m+1} \right].$$

For the commutators, by direct substitution we get

$$[q,p] = qp - pq = i[a, a^\dagger] \quad \text{and} \quad [Q,P] = QP - PQ = i[A, A^\dagger]. \tag{1.29}$$

Thus, as a consequence, from (1.25) and (1.26) we get

$$[q,p]\phi_m = i(x_{m+1} - x_m)\phi_m \quad \text{and} \tag{1.30}$$

$$[Q,P] \mid \mathcal{Z}, j\rangle = i\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m (x_{m+1} - x_m)}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \tag{1.31}$$

1.7 Uncertainty relation

It is known that, for two hermitian operators X and Y with the commutator $[X, Y] = iC,\ C \neq 0$, the Heisenberg uncertainty relation

$$(\Delta X)^2 (\Delta Y)^2 \geq \frac{\langle C\rangle^2}{4} \tag{1.32}$$

is satisfied [26]. The mean value and dispersion of a given operator X are defined, as usual, by

$$\langle X\rangle = \langle \psi \mid X \mid \psi\rangle, \quad (\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$$

for a normalized state $\mid \psi\rangle$. If we use this fact together with equation (1.29), it is enough to calculate the mean value of the commutators $[a, a^\dagger]$ and $[A, A^\dagger]$. From (1.25) and (1.26) we get

$$\langle \phi_m \mid [a, a^\dagger] \mid \phi_m\rangle = x_{m+1} - x_m \quad \text{and} \tag{1.33}$$

$$\langle \mathcal{Z}, j \mid [A, A^\dagger] \mid \mathcal{Z}, j\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(x_{m+1} - x_m)}{\sqrt{\rho(m)}} \langle \mathcal{Z}^m \chi^j \mid \mathcal{Z}^m \chi^j\rangle \tag{1.34}$$

Now (1.32) together with (1.33) and (1.34) gives

$$(\Delta q)^2 (\Delta p)^2 \geq \frac{(x_{m+1} - x_m)^2}{4} \quad \text{and} \tag{1.35}$$

$$(\Delta Q)^2 (\Delta P)^2 \geq \frac{1}{4} \left(\mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(x_{m+1} - x_m)}{\sqrt{\rho(m)}} \langle \mathcal{Z}^m \chi^j \mid \mathcal{Z}^m \chi^j\rangle \right)^2 \tag{1.36}$$

1.8 Minimum uncertainty states

Here we will illustrate a general approach of obtaining the minimal uncertainty states for a particular choice of $\rho(m)$. In most of our examples the choice of $\rho(m)$ will coincide with this particular choice. In general, we have $\mathcal{Z} = A(r)(\cos\zeta + i\Theta(k)\sin\zeta)$. With an assumption that the eigenvalues of $A(r)$ and $\Theta(k)$ are non-zero, let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $\Theta(k)$ and $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of $A(r)$. Since $[A, \Theta] = 0$, A and Θ can be diagonalized simultaneously. Let $\chi^1_{\mathcal{Z}}, \chi^2_{\mathcal{Z}}, ..., \chi^n_{\mathcal{Z}}$ be the normalized eigenvectors those forms the columns of the matrix which diagonalize A and Θ simultaneously. Then

$$\mathcal{Z}\chi^j_{\mathcal{Z}} = A(r)\chi^j_{\mathcal{Z}}(\cos\zeta + i\lambda_j\sin\zeta) = \mu_j\chi^j_{\mathcal{Z}}(\cos\zeta + i\lambda_j\sin\zeta) = \chi^j_{\mathcal{Z}}(\mu_j\cos\zeta + i\mu_j\lambda_j\sin\zeta)$$

Now by taking

$$r_j = \sqrt{(\mu_j \cos\zeta)^2 + (\mu_j \lambda_j \sin\zeta)^2}, \quad \cos\theta_j = \frac{\mu_j \cos\zeta}{r_j} \quad \text{and} \quad \sin\theta_j = \frac{\mu_j \lambda_j \sin\zeta}{r_j}$$

we can have

$$\mathcal{Z}\chi^j_{\mathcal{Z}} = z_j \chi^j_{\mathcal{Z}} \quad \text{with} \quad z_j = r_j(\cos\theta_j + i \sin\theta_j). \tag{1.37}$$

That is, $z_1, z_2, ..., z_n$ are the eigenvalues of $\mathcal{Z}$. Further $\mathcal{Z}^m \chi^j_{\mathcal{Z}} = z_j^m \chi^j_{\mathcal{Z}}$. With all these we have

$$| \mathcal{Z}, j\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z_j^m}{\sqrt{\rho(m)}} \chi^j_{\mathcal{Z}} \otimes \phi_m. \tag{1.38}$$

The conventional bosonic CS are a typical example of minimum uncertainty states. Let us compare the states in (1.38) with the bosonic CS. The comparison yields that if $\rho(m) = m!$ and $\mathcal{N}(\mathcal{Z}) = e^{|z_j|^2}$ then the states in (1.38) will saturate the uncertainty relation. We will give examples of this nature in the later chapters.

1.9 The natural isometry $\mathcal{W}$

Let $\widetilde{\mathbb{H}} = L^2(\mathcal{K} \times \mathcal{R} \times \mathbb{R}, d\mu)$. Here we intend to look for an isometry, $\mathcal{W}$ between the Hilbert spaces $\mathbb{C}^n \otimes \mathbb{H}$ and $\mathbb{C}^n \otimes \widetilde{\mathbb{H}}$ in the natural way. Define

$$\mathcal{W} : \mathbb{C}^n \otimes \mathbb{H} \longrightarrow \mathbb{C}^n \otimes \widetilde{\mathbb{H}} \quad \text{by} \quad (\mathcal{W}\Psi)^j(\mathcal{Z}) = \langle \mathcal{Z}, j \mid \Psi\rangle. \tag{1.39}$$

From equation (1.37), we know that $z_1, z_2, ..., z_n$ are the eigenvalues of $\mathcal{Z}$ and $\chi^1, \chi^2, ..., \chi^n$ (the subscript $\mathcal{Z}$ is omitted for simplicity) be the corresponding eigenvectors. Denote

$Z = \text{diag}(z_1, z_2, ..., z_n)$, a diagonal matrix with the eigenvalues as diagonal entries. We pick the following expression for the coherent states $\mid \mathcal{Z}, j\rangle$,

$$\mid \mathcal{Z}, j\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m$$

where the vectors χ^j are the eigenvectors of $\mathcal{Z}$. With this choice, let us calculate the image of the isometry.

$$(\mathcal{W}\Psi)^j(\mathcal{Z}) = \langle \mathcal{Z}, j \mid \Psi\rangle = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{1}{\sqrt{\rho(m)}} \langle \mathcal{Z}^m \chi^j \otimes \phi_m \mid \Psi\rangle.$$

Now

$$\langle \mathcal{Z}^m \chi^j \otimes \phi_m \mid \Psi\rangle = \langle U Z^m U^\dagger \chi^j \otimes \phi_m \mid \Psi\rangle \tag{1.40}$$

where U is a unitary matrix (the matrix which diagonalizes $\mathcal{Z}$). Let $U^\dagger \chi^j = c_{j1}\chi^1 + c_{j2}\chi^2 + ... + c_{jn}\chi^n$ then (1.40) becomes

$$\langle U Z^m (c_{j1}\chi^1 + c_{j2}\chi^2 + ... + c_{jn}\chi^n) \otimes \phi_m \mid \Psi\rangle$$

$$= \langle Z^m (c_{j1}\chi^1 + c_{j2}\chi^2 + ... + c_{jn}\chi^n) \otimes \phi_m \mid U^\dagger \Psi\rangle. \tag{1.41}$$

Further, we have $U^\dagger \Psi = \chi^1 \Psi_1 + \chi^2 \Psi_2 + ... + \chi^n \Psi_n$, thus (1.41) becomes

$$\langle Z^m (c_{j1}\chi^1 + c_{j2}\chi^2 + ... + c_{jn}\chi^n) \otimes \phi_m \mid (\chi^1 \Psi_1 + \chi^2 \Psi_2 + ... + \chi^n \Psi_n)\rangle$$

$$= \alpha_{j1} z_1^m + \alpha_{j2} z_2^m + ... + \alpha_{jn} z_n^m = g_j(z_1, z_2, ..., z_n) \tag{1.42}$$

where α_{jl} are constants depending on $r \in \mathcal{R}$, $k \in \mathcal{K}$ and $\langle \phi_m \mid \Psi_l\rangle$. Thus

$$(\mathcal{W}\Psi)^j(\mathcal{Z}) = \mathcal{N}(\mathcal{Z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{g_j(z_1, z_2, ..., z_n)}{\sqrt{\rho(m)}} = f_j(z_1, z_2, ..., z_n). \tag{1.43}$$

We will explain the nature of the coefficients α_{jl} and the nature of the functions $g_j(z_1, z_2, ..., z_n)$ through out some examples. In fact, the nature of these functions depend on the eigenvalues of the matrix $\Theta(k)$. If all the eigenvalues of $\Theta(k)$ are positive then the functions will be holomorphic and if they are all negative then the function will be anti-holomorphic. If some of the eigenvalues are positive and some of them are negative then the function, $(\mathcal{W}\Psi)^j(\mathcal{Z})$ can be written as a linear combinations of holomorphic(the one corresponds to the positive eigenvalues) and anti-holomorphic (the one corresponds to the negative eigenvalues) functions.

Chapter 2

Introduction to Quaternions and Clifford algebras

For the sake of completeness, in this short chapter we discuss quaternions and Clifford algebras to the extent necessary for the purposes of this thesis. The material appearing in this chapter follows from [21], [19], [30], [17] and some of the references cited there.

2.1 Quaternions

Quaternions were discovered by Sir William Rowan Hamilton, who presented the first paper on the subject to the Royal Irish Academy in 1843 [17]. Applications of quaternions were developed in the early 20th century, mainly in physics. In recent years, computer scientists started using them in cartoon animations. The algebra of quaternions **H** is analogous in many ways to the algebra of complex numbers $\mathbb{C}$. In several places quaternions are referred to as hyper-complex numbers.

Definition 2.1.1 *A quaternion is a member of the algebra **H**, where the algebra is generated by* $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. *For* $x_0, x_1, x_2, x_3 \in \mathbb{R}$, *a general quaternion is written as*

$$q(x_0, \overrightarrow{x}) = x_0 + x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k} = x_0 + \underline{i} \cdot \overrightarrow{x}, \tag{2.1}$$

where $\overrightarrow{x} = (x_1, x_2, x_3)$, $\underline{i} = (\hat{\imath}, \hat{\jmath}, \hat{k})$ *and* $\hat{\imath}, \hat{\jmath}, \hat{k}$ *are imaginary units satisfying the following multiplication table*

	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	-1	$\hat{k}$	$-\hat{\jmath}$
$\hat{\jmath}$	$-\hat{k}$	-1	$\hat{\imath}$
$\hat{k}$	$\hat{\jmath}$	$-\hat{\imath}$	-1

It can also be seen that $\hat{\imath}\hat{\jmath}\hat{k} = -1$. *The quaternion in (2.1) is said to be in Cartesian form. Further these are said to be real quaternions and are denoted by* $Q_{\mathbb{R}}$.

The quaternion product is a $\mathbb{R}$-bilinear form obeying the above multiplication table. The scalar part of the quaternion $q(x_0, \overrightarrow{x})$ is x_0 and is denoted by $S(q)$ and the vector part is $x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$ and is denoted by $V(q)$. These notations are in analogy to ordinary complex numbers. The quaternion product enjoys the following properties.

Theorem 2.1.1 *For* $q, r, s \in$ ***H***, *we have*

(a) $q(rs) = (qr)s$, *(associativity)*

(b) $q(r+s) = qr + qs$, *(distributivity)*

(c) For each $q \neq 0$, *there exist* r *such that* $qr = 1$, *(existence of an inverse)*

(d) If $qr = qs$, *then* $r = s$, *(cancelation)*

The quaternion product is not commutative as it can easily be seen through the multiplication table of the generators.

Definition 2.1.2 *For a quaternion* $q(x_0, \overrightarrow{x}) = x_0 + x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$,

(i) If $S(q) = 0$, *it is called a pure quaternion.*

(ii) The conjugate is defined as $q* = S(q) - V(q) = x_0 - x_1\hat{\imath} - x_2\hat{\jmath} - x_3\hat{k}$.

(iii) The modulus is defined as $|q(x_0, \overrightarrow{x})| = |q| = \sqrt{qq*} = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$

(iv) If $|q| = 1$ *it is said to be a unit quaternion.*

(v) $det(q) = |q|^4$

(vi) Two pure quaternions are anticommutative if $qr = -rq$.

Pure quaternions q, r are anti commutative if and only if they are perpendicular. In this sense we have a unit normal $\hat{u}$.

Definition 2.1.3 $q \in Q_{\mathbb{R}}$ *can be written in polar form as*

$$q = |q|(\cos\theta + \hat{u}\sin\theta),$$

where $S(\hat{u}) = 0$ *and* $|\hat{u}| = 1$ *and* $-\pi \leq \theta \leq \pi$.

The complex quaternions are denoted by $Q_{\mathbb{C}}$.

Definition 2.1.4 *A complex quaternion is defined as*

$$q(x_0, \overrightarrow{x}) = x_0 + ix_1\hat{\imath} + ix_2\hat{\jmath} + ix_3\hat{k} = x_0 + i\underline{i} \cdot \overrightarrow{x}, \tag{2.2}$$

where $\overrightarrow{x}$ *and* $\underline{i}$ *has the same meaning as in (2.1.1).*

Definition 2.1.5 *For a complex quaternion* $q(x_0, \overrightarrow{x}) = x_0 + ix_1\hat{\imath} + ix_2\hat{\jmath} + ix_3\hat{k} \in Q_{\mathbb{C}}$,

(i) The scalar part is $S(q) = x_0$.

(ii) The vector part is $V(q) = ix_1\hat{\imath} + ix_2\hat{\jmath} + ix_3\hat{k}$.

(iii) Conjugate of q *is* $q* = S(q) - V(q)$.

(iv) The Minkowski norm of q *is* $||q|| = |x_0^2 - x_1^2 - x_2^2 - x_3^2|^{\frac{1}{2}}$

(v) If $||q|| \neq 0$ *then the polar form of* q *is*

$$q = ||q||(\cosh\theta + \hat{u}\sinh\theta)$$

, where $S(\hat{u}) = 0, |\hat{u}| = 1$ *and* $\theta \in \mathbb{R}$

Quaternions can also be represented using complex 2×2 matrices. It can be written as a linear combination of the matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where σ_1, σ_2 and σ_3 are the usual Pauli matrices. In this notation the quaternions can be written as

$$q = x_0\sigma_0 + i\underline{x} \cdot \sigma$$

with $\underline{x} = (x_1, x_2, x_3)$ and $\sigma = (\sigma_1, -\sigma_2, \sigma_3)$. That is,

$$q = x_0\sigma_0 + ix_1\sigma_1 - ix_2\sigma_2 + ix_3\sigma_3 = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} \tag{2.3}$$

Theorem 2.1.2 *The quaternions in (2.3) can be written, using polar coordinates, in the form*

$$q = A(r)e^{i\theta\sigma(\hat{n})} \tag{2.4}$$

where

$$A(r) = r\mathbb{I}_2 \quad \textit{and} \quad \sigma(\hat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix} \tag{2.5}$$

and $\mathbb{I}_2$ *stands for the* 2×2 *identity matrix.*

Proof. Now let us replace the rectangular coordinates in (2.3) by the polar coordinates with the following choice

$$x_0 = r\cos\theta, x_1 = r\sin\theta\sin\phi\cos\psi, x_2 = r\sin\theta\sin\phi\sin\psi, x_3 = r\sin\theta\cos\phi.$$

where $r \in [0, \infty), \theta, \phi \in [0, \pi]$ and $\psi \in [0, 2\pi]$. Substitution yields,

$$q = \begin{pmatrix} r\cos\theta + ir\sin\theta\cos\phi & -r\sin\theta\sin\phi\sin\psi + ir\sin\theta\sin\phi\cos\psi \\ r\sin\theta\sin\phi\sin\psi + ir\sin\theta\sin\phi\cos\psi & r\cos\theta - ir\sin\theta\cos\phi \end{pmatrix}$$

$$
\begin{aligned}
&= r\mathbb{I}_2 \left[\cos\theta \mathbb{I}_2 + i \sin\theta \begin{pmatrix} \cos\phi & i\sin\phi\sin\psi + \sin\phi\cos\psi \\ -i\sin\phi\sin\psi + \sin\phi\cos\psi & -\cos\phi \end{pmatrix} \right] \\
&= A(r) \left[\cos\theta \mathbb{I}_2 + i\sigma(\hat{n}) \sin\theta \right] \\
&= A(r) e^{i\theta\sigma(\hat{n})},
\end{aligned}
$$

where

$$
A(r) = r\mathbb{I}_2 \quad \text{and} \quad \sigma(\hat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix}.
$$

We have the theorem. ■

Remark 2.1.1 *For the quaternions we can also have other matrix bases.*

2.2 Clifford algebras

Clifford algebra was discovered by William Kingdon Clifford (1845-1879) and bear his name. Recent authors sometimes refer to these algebras as geometric algebras. Clifford algebras have been playing a popular role in theoretical physics. Here we present these algebras to an extent suitable to this thesis.

Definition 2.2.1 *Let M be an additive group and R be either a commutative ring with unity or a commutative algebra with unity. The map*

$$
R \times M \longrightarrow M, \quad by \quad (\lambda, x) \mapsto \lambda x
$$

is said to be an R-module for X if, for $x, y \in X$ and $\lambda, \mu \in R$, *(i)* $\lambda(x+y) = \lambda x + \lambda y$

(ii) $(\lambda + \mu)x = \lambda x + \mu x$

(iii) $\mu(\lambda x) = (\mu\lambda)x$

(iv) $1x = x$

Note: When R is a field M is said to be an R-linear space. ■

Definition 2.2.2 *A quadratic form is a pair (M, f), where M is an R-module and $f : M \times M \longrightarrow R$ is a symmetric bilinear form, that is, for $a, a', b, b' \in R, x, x', y, y' \in M$*

(i) $f(ax + a'x', y) = af(x, y) + a'f(x', y)$

(ii) $f(x, y) = f(y, x)$

Note:

(1) For an antisymmetric form f condition (ii) in definition (2.2.2) will be replaced by $f(x, y) = -f(y, x)$.

(2) For a sesquilinear form f over $\mathbb{C}$ (ii) is replaced by $f(x, y) = \overline{f(y, x)}$ and (ii) by $f(x, by) = \overline{b}f(x, y)$, where the over line stands for the complex conjugate.

(3) A quadratic form (M, Q) can also be defined as $Q : M \to R$ such that $Q(ax) = a^2Q(x)$ and $f_Q(x, y) = Q(x + y) - Q(x) - Q(y)$.

(4) Let M^+ be the dual of M (for sesquilinear form conjugate dual) and $\langle x, u\rangle = u(x)$ be the canonical pairing. Then the correlation c_f of (M, f) is the unique homomorphism $c_f : M \to M^+$ such that $\langle x, c_f(y)\rangle = f(x, y)$ for all $x, y \in M$.

(5) (M, f) is nondegenerate if c_f is a monomorphism and nonsingular if c_f is an isomorphism.

■

Theorem 2.2.1 *(i) Every quadratic form (M, f) over $\mathbb{C}$ has a basis $e_1, ..., e_n \in M$ such that*

$$f(x, y) = x_1y_1 + ... + x_ny_n$$

whenever $x = x_1e_1 + ... + x_ne_n$ and $y = y_1e_1 + ... + y_ne_n$.

(ii) Every quadratic form (M, f) over $\mathbb{R}$ has a basis $e_1, ..., e_n \in M$ such that

$$f(x, y) = -x_1y_1 - ... - x_ky_k + x_{k+1}y_{k+1} + ... + x_ny_n$$

whenever $x = x_1e_1 + ... + x_ne_n$ and $y = y_1e_1 + ... + y_ne_n$. The integer k is called the index of f.

Definition 2.2.3 *Let (M, f) be a quadratic form over $\mathbb{R}$. The Clifford algebra of (M, f) is a pair $(C(f), \Theta)$, where $C(f)$ is an $\mathbb{R}$-algebra and $\Theta : M \to C(f)$ is a linear function such that*

$$\Theta(x)^2 = f(x, x)1 \tag{2.6}$$

for each $x \in M$. The following universal property will be always assumed. For all linear functions $u : M \to A$, where A is an $\mathbb{R}$-algebra, with $u(x)^2 = f(x,x)1$ there exists an algebra morphism $u' : C(f) \to A$ such that $u'\Theta = u$ and u' is unique with respect to this property.

Note:

(i) If f is a sesquilinear form over $\mathbb{C}$ then (2.6) will be replaced by

$$\Theta(x)\Theta(x)^\dagger = \Theta(x)^\dagger\Theta(x) = \|f(x,x)\|1 \tag{2.7}$$

where $\dagger$ stands for the conjugate transpose.

(ii) A Clifford algebra $(C(f), \Theta)$ exists for each quadratic form (M, f).

■

Theorem 2.2.2 *Let (M, f) be a quadratic form where M has a basis $e_1, ..., e_r$ with $f(e_i, e_j) = 0$ for $i \neq j$ and $a_i = f(e_i, e_i)$. Then $C(f)$ is generated by $e_1, ..., e_r$ with relations*

(i) $e_i^2 = a_i$

(ii) $e_ie_j = e_je_i = 0$ *for* $i \neq j$

The elements $e_{i(1)}, ..., e_{i(s)}$, where $i(i) < ... < i(s)$ and $1 \leq s \leq r$, together with 1 form a basis for $C(f)$. The dimension of $C(f) = 2^r$.

Example 2.2.1 *Let C_k denote $C(f)$ when $f(x,y) = -\langle x \mid y\rangle$ is a form (usual inner product) on $\mathbb{R}^k$. $C_k^c = C(f)$ when $f(z,w) = -\langle z \mid w\rangle$ is a form on $\mathbb{C}^k$. C_k^c is the complexification $C_k \otimes_{\mathbb{R}} \mathbb{C}$. Then,*

(i) C_1 is two dimensional over $\mathbb{R}$ with basis elements 1 and e satisfying $e^2 = -1$. We could identify $e := i$. Thus $C_1 \cong \mathbb{C}$ and $C_1^c \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

(ii) C_2 is four dimensional over $\mathbb{R}$ with basis elements $1, e_1, e_2, e_1e_2$, where $e_1^2 = e_2^2 = -1$ and $e_1e_2 = -e_2e_1$. We can identify $1 := 1, e_1 := \hat{\imath}, e_2 := \hat{\jmath}$ and $e_1e_2 = \hat{k}$. Thus $C_2 \cong \boldsymbol{H}$, the quaternions. $C_2^c \cong \mathbb{C} \otimes_{\mathbb{R}} \boldsymbol{H} \cong C(2)$, 2×2 complex matrices.

Example 2.2.2 *Let $f(x,y) = \langle x \mid y\rangle$ on $\mathbb{R}^k$ and $C_k' = C(f)$. Then C_1' is two dimensional with basis elements 1 and e satisfying $e^2 = 1$. If we identify $1 := (1,1), e := (1,-1)$ then $C_1' \cong \mathbb{R} \oplus \mathbb{R}$. C_2' is four dimensional with basis elements $1, e_1, e_2, e_1e_2$ satisfying $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$. Identify*

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1e_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $C_2' \cong R(2)$, 2×2 real matrices.

Note: Clifford algebras can always be identified with matrix algebras. ■

Chapter 3

Quaternionic Vector Coherent States

In this chapter we build VCS for quaternions and analyze them under the categories of chapter 1. Here we show the quaternionic VCS achieve minimum uncertainty as we proposed in chapter 1. In addition we show that quaternionic VCS can be written in exponential form similar to (5). An interesting application of these VCS will be presented in chapter 8.

3.1 The VCS

For the quaternions, here we take the result of theorem (2.1.2). That is,

$$q = A(r)e^{i\theta\sigma(\hat{n})}, \tag{3.1}$$

where

$$A(r) = r\mathbb{I}_2 \quad \text{and} \quad \sigma(\hat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix}.$$

The matrices $A(r)$ and $\sigma(\hat{n})$ satisfy the conditions (1.2)-(1.5). Thus, with $\{\phi_m\}$ an orthonormal basis of an abstract Hilbert space $\mathbb{H}$ and χ^1, χ^2 an orthonormal basis of $\mathbb{C}^2$, we can have VCS as vectors in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{H}$ in the following form.

$$\mid q\rangle^j = \mathcal{N}(q)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m \quad j = 1, 2 \tag{3.2}$$

where $\mathcal{N}(q)$ and $\rho(m)$ to be chosen appropriately.

3.2 Normalization and the Resolution of the identity

For the quaternionic states in (3.2),we take $r \in [0, \infty), \phi \in [0, \pi]$ and $\theta, \psi \in [0, 2\pi]$. We take the measure $d\mu = rdrd\theta d\Omega$ with $d\Omega = \sin\phi d\phi d\psi$ on $[0, \infty] \times [0, \pi] \times [0, 2\pi] \times [0, 2\pi]$. Note that here we double the range of angle θ for convenience. With these considerations, we have the following.

Theorem 3.2.1 *The states in (3.2) forms a set of VCS, where the normalization factor is given by*

$$\mathcal{N}(q) = 2e^{r^2} \tag{3.3}$$

and the resolution of the identity

$$\int \mid q\rangle^j W(|q|)^j \langle q \mid d\mu = \mathbb{I}_2 \otimes I$$

is obtained with $w(r) = \frac{1}{2\pi^2}$ *and* $\rho(m) = m!$.

Proof. For the states in (3.2), condition (1.10) becomes

$$\mathcal{N}(q) = 2\sum_{m=0}^{\infty} \frac{r^{2m}}{\rho(m)} \tag{3.4}$$

Now with the fact that

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} e^{i(m-l)\theta\sigma(\hat{n})} \sin\phi d\phi d\theta d\psi = \begin{cases} 8\pi^2\mathbb{I}_2 & \text{if} \quad m = l \\ 0 & \text{if} \quad m \neq l \end{cases}$$

(1.14) becomes

$$\int_0^{\infty} \frac{8\pi^2 w(r) r^{2m+1}}{\mathcal{N}(q)} \mathbb{I}_2 dr = \rho(m)\mathbb{I}_2. \tag{3.5}$$

By considering the identity

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt = 2\int_0^{\infty} t^{2z-1}e^{-t^2}dt$$

we have convergence in (3.4) and resolution of the identity with the choices

$$w(r) = \frac{1}{2\pi^2}, \tag{3.6}$$

$$\rho(m) = m!, \tag{3.7}$$

$$\mathcal{N}(q) = 2e^{r^2}. \tag{3.8}$$

The details are easy to verify. ∎

3.3 Reproducing Kernel, Annihilation, Creation and Number operators

From the general case (1.16), without any effort, it can be obtained that the reproducing kernel have the form

$$K_{jk}(q^{\dagger}, q') = \sum_{m=0}^{\infty} \frac{r^m r'^m}{4m!} e^{-(r^2+r'^2)} \langle e^{-im(\theta'\sigma(\hat{n'})-\theta\sigma(\hat{n}))} \chi^j \mid \chi^k \rangle. \tag{3.9}$$

For the annihilation, creation and number operators, in (1.17),(1.18) we replace x_m by m. For (1.19), (1.20) and (1.21), we keep the same form. Thus, the action of the operators on the vectors $\{\phi_m\}$ and $\{\mid q\rangle^j\}$ give us the following,

$$\begin{aligned}
a\phi_m &= \sqrt{m}\phi_{m-1} && (3.10)\\
a^{\dagger}\phi_m &= \sqrt{m+1}\phi_{m+1} && (3.11)\\
N'\phi_m &= m\phi_m && (3.12)\\
A \mid q\rangle^j &= q \mid q\rangle^j && (3.13)\\
A^{\dagger} \mid q\rangle^j &= \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} q^m \sqrt{\frac{m+1}{m!}} \chi^j \otimes \phi_{m+1} && (3.14)\\
N \mid q\rangle^j &= \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \sum_{m=1}^{\infty} \frac{m q^m}{\sqrt{(m)!}} \chi^j \otimes \phi_m. && (3.15)
\end{aligned}$$

3.4 Commutators and the Heisenberg Uncertainty

For the quaternionic states, identities (1.26) and (1.25) take the following form,

$$[a, a^{\dagger}]\phi_m = \phi_m, \quad [A, A^{\dagger}] \mid q\rangle^j = \mid q\rangle^j \tag{3.16}$$

By taking q, p, Q and P as in (1.27) and (1.28), we can have (1.29). Thus from (3.16), identities (1.30) and (1.31) becomes,

$$[q,p]\phi_m = i\phi_m, \quad [Q,P] \mid q\rangle^j = i \mid q\rangle^j. \tag{3.17}$$

For the vectors ϕ_m, by following the same procedure as in section 1.6, from (1.35) and (3.16) we obtain

$$(\Delta q)^2_{\phi_m}(\Delta p)^2_{\phi_m} \geq \frac{1}{4}. \tag{3.18}$$

At the same time, direct calculations show us

$$(\Delta q)_{\phi_m} = \sqrt{\frac{2m+1}{2}} \quad \text{and} \quad (\Delta p)_{\phi_m} = \sqrt{\frac{2m+1}{2}}$$

Thus we can also have

$$(\Delta q)^2_{\phi_m}(\Delta p)^2_{\phi_m} = \frac{(2m+1)^2}{4}.$$

Again, from (1.36) we obtain

$$(\Delta Q)^2_{|q\rangle^j}(\Delta P)^2_{|q\rangle^j} \geq \frac{1}{4}\left(\frac{1}{2}e^{-r^2}\sum_{m=0}^{\infty}\frac{\langle q^m\chi^j \mid q^m\chi^j\rangle}{m!}\right)^2 = \frac{1}{16}. \tag{3.19}$$

3.5 Minimum Uncertainty States

We have $q = r\mathbb{I}_2(\cos\theta\mathbb{I}_2 + i\sigma(\hat{n})\sin\theta)$. The eigenvalues of $\sigma(\hat{n})$ are $\lambda^+ = 1$ and $\lambda^- = -1$. Corresponding normalized eigenvectors are

$$\chi_q^+ = \begin{pmatrix} \cos\frac{\phi}{2}e^{i\psi} \\ \sin\frac{\phi}{2} \end{pmatrix} \quad \text{and} \quad \chi_q^- = \begin{pmatrix} \sin\frac{\phi}{2}e^{i\psi} \\ -\cos\frac{\phi}{2} \end{pmatrix}$$

With these eigenvectors, we have

$$q\chi_q^+ = r\chi_q^+(\cos\theta + i\sin\theta) = re^{i\theta}\chi_q^+ = z\chi_q^+ \tag{3.20}$$

$$q\chi_q^- = r\chi_q^-(\cos\theta - i\sin\theta) = re^{-i\theta}\chi_q^- = \overline{z}\chi_q^- \tag{3.21}$$

Thus, as in section (1.7), we have the minimum uncertainty states as

$$\mid q\rangle^+ = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{q^m}{\sqrt{m!}}\chi_q^+ \otimes \phi_m = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\chi_q^+ \otimes \phi_m \tag{3.22}$$

$$\mid q\rangle^- = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{q^m}{\sqrt{m!}}\chi_q^- \otimes \phi_m = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{\overline{z}^m}{\sqrt{m!}}\chi_q^- \otimes \phi_m. \tag{3.23}$$

Note that from (3.20) and (3.21) we also have the eigenvalues and eigenvectors of q.

3.6 The isometry $\mathcal{W}$

Let $\widetilde{\mathbb{H}} = L^2(\mathbb{R}^2 \times S^2, d\nu d\Omega)$, where S^2 is a two sphere in three dimensions and $d\nu = rdrd\theta$. We are interested in the isometry

$$\mathcal{W} : \mathbb{C}^2 \otimes \mathbb{H} \longrightarrow \mathbb{C}^2 \otimes \widetilde{\mathbb{H}} \quad \text{by} \quad (\mathcal{W}\Psi)^j(q) = \langle \Psi \mid q\rangle^j. \tag{3.24}$$

Theorem 3.6.1 *The image of the isometry (3.24) is given by*

$$\Psi(q) = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\left(F(\overline{z})^\dagger P_1(\widehat{n}) + F(z)^\dagger P_2(\widehat{n})\right). \tag{3.25}$$

where

$$\Psi(q) = \begin{pmatrix} \langle \Psi \mid q\rangle^1 \\ \langle \Psi \mid q\rangle^2 \end{pmatrix}^T, \quad F(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \quad f_j(z) = \sum_{m=0}^{\infty}\frac{\overline{\langle \Psi_j \mid \phi_m\rangle}}{\sqrt{m!}}z^m. \tag{3.26}$$

and $P_1(\widehat{n}), P_2(\widehat{n})$ are projection operators.

Proof. Let

$$U = \begin{pmatrix} \cos\frac{\phi}{2}e^{i\psi} & \sin\frac{\phi}{2}e^{i\psi} \\ \sin\frac{\phi}{2} & -\cos\frac{\phi}{2} \end{pmatrix}.$$

Then by (3.20) and (3.21), we have

$$U^\dagger q U = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}. \tag{3.27}$$

Now consider the projection operators $P_1(\widehat{n})$ and $P_2(\widehat{n})$, given by

$$\begin{aligned} P_1(\widehat{n}) &= U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^\dagger = \begin{pmatrix} \cos^2\frac{\phi}{2} & \frac{1}{2}e^{i\psi}\sin\phi \\ \frac{1}{2}e^{-i\psi}\sin\phi & \sin^2\frac{\phi}{2} \end{pmatrix} \\ &= \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} \quad \text{and} \\ P_2(\widehat{n}) &= U \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U^\dagger = \begin{pmatrix} \sin^2\frac{\phi}{2} & -\frac{1}{2}e^{i\psi}\sin\phi \\ -\frac{1}{2}e^{-i\psi}\sin\phi & \cos^2\frac{\phi}{2} \end{pmatrix} \\ &= \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix}. \end{aligned}$$

Finally let

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us calculate the image of the isometry as follows.

$$\langle \Psi \mid q\rangle^1 = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} \frac{\langle \Psi \mid q^m \chi^1 \otimes \phi_m\rangle}{\sqrt{m!}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} \frac{\langle \chi^1 \Psi_1 + \chi^2 \Psi_2 \mid z^m P_1 \chi^1 \otimes \phi_m + \overline{z}^m P_2 \chi^1 \otimes \phi_m \rangle}{\sqrt{m!}} \\
&= \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \left(\sum_{l=1}^{2} \sum_{m=0}^{\infty} \frac{\langle \Psi_l \mid \phi_m \rangle z^m}{\sqrt{m!}} \langle \chi^l \mid P_1 \chi^1 \rangle + \sum_{l=1}^{2} \sum_{m=0}^{\infty} \frac{\langle \Psi_l \mid \phi_m \rangle \overline{z}^m}{\sqrt{m!}} \langle \chi^l \mid P_2 \chi^1 \rangle \right) \\
&= \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \left(P_{11}^1 \overline{f_1(\overline{z})} + P_{11}^2 \overline{f_1(z)} + P_{21}^1 \overline{f_2(\overline{z})} + P_{21}^2 \overline{f_2(z)} \right),
\end{aligned}
$$

Where

$$
f_j(z) = \sum_{m=0}^{\infty} \frac{\overline{\langle \Psi_j \mid \phi_m \rangle}}{\sqrt{m!}} z^m. \tag{3.28}
$$

Similarly we can have

$$
\langle \Psi \mid q \rangle^2 = \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \left(P_{12}^1 \overline{f_1(\overline{z})} + P_{12}^2 \overline{f_1(z)} + P_{22}^1 \overline{f_2(\overline{z})} + P_{22}^2 \overline{f_2(z)} \right).
$$

With these

$$
\Psi(q) = \begin{pmatrix} \langle \Psi \mid q \rangle^1 \\ \\ \langle \Psi \mid q \rangle^2 \end{pmatrix}^T \quad \text{and} \quad F(z) = \begin{pmatrix} f_1(z) \\ \\ f_2(z) \end{pmatrix},
$$

we can write

$$
\Psi(q) = \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \left(F(\overline{z})^\dagger P_1(\widehat{n}) + F(z)^\dagger P_2(\widehat{n}) \right). \tag{3.29}
$$

Hence the theorem. ∎

Before moving further, let us state the following direct integral decomposition

$$
\mathbb{C}^2 \otimes \widetilde{\mathbb{H}} = \mathbb{C}^2 \otimes L^2(\mathbb{C} \times S^2, d\nu d\Omega) = \int_{S^2}^{\oplus} \mathbb{H}_{\widehat{n}} d\Omega, \tag{3.30}
$$

where $\mathbb{H}_{\widehat{n}} = \mathbb{C}^2 \otimes L^2(\mathbb{C}, d\nu)$. It is interesting to see the following integrals. Recall that on S^2 we have the measure $d\Omega = \sin\phi d\phi d\psi$. With this measure direct integration gives

$$
\frac{1}{2\pi} \int_{S^2} P_j(\widehat{n}) d\Omega = \mathbb{I}_2, \quad j = 1, 2. \tag{3.31}
$$

3.7 Exponential relation

In this section we obtain an exponential form similar to (5) for the quaternionic VCS.

Theorem 3.7.1 *The quaternionic VCS*

$$\mid q\rangle^j = \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \chi^j \otimes \phi_m \tag{3.32}$$

can be written the form

$$\mid q\rangle^j = \frac{1}{\sqrt{2}} e^{q\otimes a^\dagger - q^\dagger \otimes a} \chi^j \otimes \phi_0 \tag{3.33}$$

Proof. We have

$$q = A(r)e^{i\theta\sigma(\hat{n})} = r\mathbb{I}_2 e^{i\theta\sigma(\hat{n})}$$

and

$$qq^\dagger = q^\dagger q = r^2\mathbb{I}_2$$

Now consider

$$\begin{aligned}
[q\otimes a^\dagger, -q^\dagger \otimes a] &= (q\otimes a^\dagger)(-q^\dagger\otimes a) - (-q^\dagger \otimes a)(q\otimes a^\dagger) \\
&= -qq^\dagger \otimes a^\dagger a + q^\dagger q \otimes aa^\dagger \\
&= r^2\mathbb{I}_2 \otimes (aa^\dagger - a^\dagger a) \quad \text{since} \quad qq^\dagger = q^\dagger q = r^2\mathbb{I}_2 \\
&= r^2\mathbb{I}_2 \otimes [a, a^\dagger] \\
&= r^2\mathbb{I}_2 \otimes I \quad \text{since} \quad [a, a^\dagger]\phi_m = \phi_m \Rightarrow [a, a^\dagger] = I
\end{aligned}$$

For two operators A and B, when the commutator $[A, B]$ commutes with both A and B (which is true in our case), by Baker-Campbell-Hausdorff identity we have

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^{A}e^{B}$$

Thus we have

$$e^{q\otimes a^\dagger - q^\dagger \otimes a} = e^{-\frac{1}{2}[q\otimes a^\dagger, -q^\dagger\otimes a]}e^{q\otimes a^\dagger}e^{-q^\dagger\otimes a}$$

Now since $a^m\phi_0 = 0$ for all $m \geq 1$ we have

$$e^{-q\otimes a}\chi^j \otimes \phi_0 = \chi^j \otimes \phi_0.$$

and

$$\begin{aligned}
e^{q\otimes a^\dagger}(\chi^j \otimes \phi_0) &= \sum_{m=0}^{\infty} \frac{(q\otimes a^\dagger)^m}{m!}\chi^j \otimes \phi_0 \\
&= \sum_{m=0}^{\infty} \frac{q^m \chi^j \otimes a^{\dagger m}\phi_0}{m!} \\
&= \sum_{m=0}^{\infty} \frac{q^m \chi^j \otimes \sqrt{m!}\phi_m}{m!} \quad \text{since} \quad (a^\dagger)^m \phi_0 = \sqrt{m!}\phi_m \\
&= \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}}\chi^j \otimes \phi_m
\end{aligned}$$

Thus we have

$$\frac{1}{\sqrt{2}}e^{q\otimes a^\dagger - q^\dagger\otimes a}\chi^j \otimes \phi_0 = \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{q^m}{\sqrt{m!}}\chi^j \otimes \phi_m = \mid q\rangle^j,$$

which completes the proof. ■

3.8 Connection to the Weyl Heisenberg group

Here we establish a connection between quaternionic VCS and Weyl-Heisenberg group. In (9), bosonic CS have written in terms of Weyl-Heisenberg group parameters. The following is, in fact, a generalization of (9) to quaternionic VCS. Let $z = \frac{q-ip}{\sqrt{2}}$. From (3.27) we have

$$U^{\dagger}\mathcal{Z}U = D = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$$

Thus $\mathcal{Z} = UDU^{\dagger}$ and $\mathcal{Z}^{\dagger} = UD^{\dagger}U^{\dagger}$. From which we have

$$\begin{aligned}
& e^{\mathcal{Z}\otimes a^{\dagger} - \mathcal{Z}^{\dagger}\otimes a} \\
= \;& e^{UDU^{\dagger}\otimes a^{\dagger} - UD^{\dagger}U^{\dagger}\otimes a} \\
= \;& Ue^{D\otimes a^{\dagger} - D^{\dagger}\otimes a}U^{\dagger} \\
= \;& U \begin{pmatrix} \eta(q,p) & 0 \\ 0 & \eta'(q,p) \end{pmatrix} U^{\dagger}
\end{aligned}$$

where

$$e^{D\otimes a^{\dagger} - D^{\dagger}\otimes a} = \begin{pmatrix} \eta(q,p) & 0 \\ 0 & \eta'(q,p) \end{pmatrix}.$$

which can be considered as a form similar to (9). Through

$$\mid q\rangle^{j} = e^{\mathcal{Z}\otimes a^{\dagger} - \mathcal{Z}^{\dagger}\otimes a}\chi^{j} \otimes \phi_0,$$

$\mid q\rangle^{j}$ can be intepreted as the VCS of definition 0.5.3 by taking $e^{\mathcal{Z}\otimes a^{\dagger} - \mathcal{Z}^{\dagger}\otimes a}$ as a unitary representation to the Weyl Heisenberg group.

Chapter 4

Vector Coherent States with a Non-trivial $A(r)$

4.1 The VCS with a non-trivial $A(r)$

In the case of quaternions we had a more convenient form for the matrix $A(r)$ in which the matrix $A(r)$ is a multiple of the identity. Here we will establish VCS with a non-trivial $A(r)$. The construction follows in more or less in the same way as the case of quaternions. This example can be consireded as a generalization of the quaternion case. Still, it is also an easy example because

$$A(r,s)A(r,s)^{\dagger} = A(r,s)^{\dagger}A(r) = (r^2+s^2)\mathbb{I}_4$$

, but more general than the quaternions. Let

$$A(r,s) = \begin{pmatrix} r\mathbb{I}_2 & -s\mathbb{I}_2 \\ s\mathbb{I}_2 & r\mathbb{I}_2 \end{pmatrix} \quad \text{and}$$

$$\Theta = \Theta(\widehat{n}_1, \widehat{n}_2, \theta) = \begin{pmatrix} \sigma(\widehat{n}_1)\sin\theta & i\sigma(\widehat{n}_2)\cos\theta \\ -i\sigma(\widehat{n}_2)\cos\theta & \sigma(\widehat{n}_1)\sin\theta \end{pmatrix}$$

where $\widehat{n}_1 = (n_{11}, n_{12}, n_{13})$ and $\widehat{n}_2 = (n_{21}, n_{22}, n_{23})$ are unit perpendicular vectors and $\sigma(\widehat{n}_j) = (n_{j1}, n_{j2}, n_{j3}) \cdot (\sigma_1, \sigma_2, \sigma_2)$ with σ_j, the Pauli matrices. Through straight forward calculations we can see that $A(r,s)$ and Θ satisfy conditions (1.2)-(1.5). Let

$$\mathcal{B} = A(r,s)e^{i\zeta\Theta}.$$

With this choice we initiate our VCS with the usual notations as,

$$\mid \mathcal{B}\rangle^j = \mathcal{N}(\mathcal{B})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\mathcal{B}^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m, \quad j = 1,2,3,4. \tag{4.1}$$

Let $r, s \in [0, \infty]$, $\zeta, \theta \in [0, 2\pi]$ and $\widehat{n}_1, \widehat{n}_2 \in \Omega$. For the measure let us take $d\mu = d\nu d\kappa d\Omega$, where $d\nu = rsdrds$, $d\kappa = \frac{1}{2\pi}d\zeta$ and $d\Omega$ is the probability measure on $\Omega \times [0, 2\pi]$.

4.2 Normalization and the Resolution of the identity

Here we choose the factors $\mathcal{N}(\mathcal{B}), \rho(m)$ and a weight function in order to get the normalization and the resolution of the identity.

Theorem 4.2.1 *The VCS in (4.1) acquires normalization and resolution of the identity with*

$$\rho(m) = m! \tag{4.2}$$

$$\mathcal{N}(\mathcal{B}) = 4e^{r^2+s^2} \tag{4.3}$$

$$w(r,s) = \frac{16}{r^2+s^2} \tag{4.4}$$

where $w(r,s)$ is a weight function required to hold the resolution of the identity.

Proof. Since $A(r,s)A(r,s)^\dagger = (r^2+s^2)\mathbb{I}_4$ we have $Tr\,|A(r,s)|^{2m} = 4(r^2+s^2)$. Thus the normalization condition, (1.10) becomes

$$\mathcal{N}(\mathcal{B}) = \sum_{m=0}^{\infty} \frac{4(r^2+s^2)}{\rho(m)}. \tag{4.5}$$

Now for the resolution of the identity, our moment problem, (1.14) becomes

$$\int_0^\infty \int_0^\infty \frac{w(r,s)\,|A(r,s)|^{2m}}{\mathcal{N}(\mathcal{B})} srdsdr = \rho(m)\mathbb{I}_4. \tag{4.6}$$

That is,

$$\int_0^\infty \int_0^\infty \frac{w(r,s)(r^2+s^2)^m}{\mathcal{N}(\mathcal{B})} srdsdr = \rho(m).$$

Now we show the choices

$$\rho(m) = m!, \quad \mathcal{N}(\mathcal{B}) = 4e^{r^2+s^2}, \quad w(r,s) = \frac{16}{r^2+s^2}$$

will satisfy (4.5) and (4.6). For (4.5),

$$\sum_{m=0}^{\infty} \frac{4(r^2+s^2)}{m!} = 4e^{r^2+s^2}$$

and for (4.5), consider

$$\int_0^\infty \int_0^\infty \frac{w(r,s)(r^2+s^2)^m}{\mathcal{N}(\mathcal{B})} srdsdr$$
$$= \int_0^\infty \int_0^\infty \frac{16(r^2+s^2)^m}{4(r^2+s^2)e^{r^2+s^2}} srdsdr$$
$$= \int_0^\infty \int_0^\infty \frac{4(r^2+s^2)^{m-1}}{e^{r^2}e^{s^2}} srdsdr$$

$$= \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \int_0^\infty \int_0^\infty 4rs(r^2)^{m-1-j}(s^2)^j e^{-r^2} e^{-s^2} drds$$
$$= \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \int_0^\infty \left(\int_0^\infty (r^2)^{m-1-j} e^{-r^2} 2rdr \right) (s^2)^j e^{-s^2} 2sds$$
$$= \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \Gamma(m-j)\Gamma(j+1) = m!$$

which completes the proof. ∎

4.3 Reproducing Kernel etc.

Reproducing kernel, annihilation operator, creation operator, number operator, operators P, Q and commutators are of the same form as quaternionic VCS. We just have to replace the matrices A and Θ with the new choice. The minimum uncertainty states and the isometry will differ from the case of quaternions. We will discus them in the following sections.

4.4 Minimum uncertainty states

In order to see the minimum uncertainty states we need to know the eigenvalues of $A(r,s)$ and Θ. We also need to know the unitary matrix which diagonalizes the matrices $A(r,s)$ and Θ simultaneously. To have the eigenvalues and the unitary matrix we have to have a specific form for the matrix Θ. Let us take two unit perpendicular vectors in the unit sphere,

$$\widehat{n}_1 = (\cos\phi\sin\psi, \sin\phi\sin\psi, \cos\psi) \quad \text{and} \quad \widehat{n}_2 = (\cos\phi\cos\psi, \sin\phi\cos\psi, -\sin\psi).$$

With these choices of vectors the matrix Θ becomes

$$\Theta = \begin{pmatrix} \cos\psi\sin\theta & e^{-i\phi}\sin\psi\sin\theta & -i\sin\psi\cos\theta & ie^{-i\phi}\cos\theta\cos\psi \\ e^{i\phi}\sin\psi\sin\theta & -\cos\psi\sin\theta & ie^{i\phi}\cos\psi\cos\theta & i\sin\psi\cos\theta \\ i\cos\theta\sin\psi & -ie^{-i\phi}\cos\theta\cos\psi & \cos\psi\sin\theta & e^{-i\phi}\sin\theta\sin\psi \\ -ie^{i\phi}\cos\psi\cos\theta & -i\sin\psi\cos\theta & e^{i\phi}\sin\psi\sin\theta & -\cos\psi\sin\theta \end{pmatrix}.$$

The eigenvalues of Θ are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 1$ and $\lambda_4 = -1$ and the eigenvalues of $A(r,s)$ are $\mu_1 = r + is, \mu_2 = r + is, \mu_3 = r - is$ and $\mu_4 = r - is$. The unitary matrix which simultaneously diagonalizes $A(r,s)$ and Θ is

$$U=\frac{1}{2}\begin{pmatrix} i\sqrt{(\sin(\theta-\psi)+1)} & i\frac{\sin(\theta-\psi)-1}{\sqrt{(1-\sin(\theta-\psi))}} & \sqrt{(\sin(\theta+\psi)+1)} & \frac{\sin(\theta+\psi)-1}{\sqrt{(1-\sin(\theta+\psi))}} \\ ie^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(\sin(\theta-\psi)+1)}} & ie^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(1-\sin(\theta-\psi))}} & -e^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(\sin(\theta+\psi)+1)}} & -e^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(1-\sin(\theta+\psi))}} \\ \sqrt{(\sin(\theta-\psi)+1)} & \frac{\sin(\theta-\psi)-1}{\sqrt{(1-\sin(\theta-\psi))}} & i\sqrt{(\sin(\theta+\psi)+1)} & i\frac{\sin(\theta+\psi)-1}{\sqrt{(1-\sin(\theta+\psi))}} \\ e^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(\sin(\theta-\psi)+1)}} & e^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(1-\sin(\theta-\psi))}} & -ie^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(\sin(\theta+\psi)+1)}} & -ie^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(1-\sin(\theta+\psi))}} \end{pmatrix}.$$

We have

$$U^{\dagger}A(r,s)U=\begin{pmatrix} r+is & 0 & 0 & 0 \\ 0 & r+is & 0 & 0 \\ 0 & 0 & r-is & 0 \\ 0 & 0 & 0 & r-is \end{pmatrix} \quad \text{and} \quad U^{\dagger}\Theta U=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now let us name the orthonormal eigenvectors (the columns of U in order) as,

$$\chi^1 = \frac{1}{2}\begin{pmatrix} i\sqrt{(\sin(\theta-\psi)+1)} \\ ie^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(\sin(\theta-\psi)+1)}} \\ \sqrt{(\sin(\theta-\psi)+1)} \\ e^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(\sin(\theta-\psi)+1)}} \end{pmatrix}, \quad \chi^2=\frac{1}{2}\begin{pmatrix} i\frac{\sin(\theta-\psi)-1}{\sqrt{(1-\sin(\theta-\psi))}} \\ ie^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(1-\sin(\theta-\psi))}} \\ \frac{\sin(\theta-\psi)-1}{\sqrt{(1-\sin(\theta-\psi))}} \\ e^{i\phi}\frac{\cos(\theta-\psi)}{\sqrt{(1-\sin(\theta-\psi))}} \end{pmatrix},$$

$$\chi^3 = \frac{1}{2}\begin{pmatrix} \sqrt{(\sin(\theta+\psi)+1)} \\ -e^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(\sin(\theta+\psi)+1)}} \\ i\sqrt{(\sin(\theta+\psi)+1)} \\ -ie^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(\sin(\theta+\psi)+1)}} \end{pmatrix} \quad \text{and} \quad \chi^4=\frac{1}{2}\begin{pmatrix} \frac{\sin(\theta+\psi)-1}{\sqrt{(1-\sin(\theta+\psi))}} \\ -e^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(1-\sin(\theta+\psi))}} \\ i\frac{\sin(\theta+\psi)-1}{\sqrt{(1-\sin(\theta+\psi))}} \\ -ie^{i\phi}\frac{\cos(\theta+\psi)}{\sqrt{(1-\sin(\theta+\psi))}} \end{pmatrix}.$$

Now we are in a position to define the minimum uncertainty states. Since $\mathcal{B} = A(r,s)[\cos\zeta + i\Theta\sin\zeta]$, we have

$$\begin{aligned}
\mathcal{B}\chi^1 &= A(r,s)[\cos\zeta + i\Theta\sin\zeta]\chi^1 \\
&= A(r,s)\chi^1[\cos\zeta + i\Theta\sin\zeta] \\
&= (r+is)[\cos\zeta + i\Theta\sin\zeta]\chi^1 \\
&= ([r\cos\zeta - s\sin\zeta] + i[r\sin\zeta + s\cos\zeta])\,\chi^1 \\
&= (a+ib)\chi^1 = z_1\chi^1
\end{aligned}$$

where $a = r\cos\zeta - s\sin\zeta$ and $b = r\sin\zeta + s\cos\zeta$. Similarly,

$$\begin{aligned}
\mathcal{B}\chi^2 &= (b-ia)\chi^2 = \overline{z}_2\chi^2 \\
\mathcal{B}\chi^3 &= (b+ia)\chi^3 = z_2\chi^3 \\
\mathcal{B}\chi^4 &= (a-ib)\chi^4 = \overline{z}_1\chi^4.
\end{aligned}$$

Note also that $z_2 = -iz_1$. Thus, the minimum uncertainty states are

$$\begin{aligned}
\mid\mathcal{B}\rangle^1 &= \frac{1}{2}e^{-\frac{r^2+s^2}{2}}\sum_{m=0}^{\infty}\frac{z_1^m}{\sqrt{m!}}\chi^1\otimes\phi_m, \\
\mid\mathcal{B}\rangle^2 &= \frac{1}{2}e^{-\frac{r^2+s^2}{2}}\sum_{m=0}^{\infty}\frac{\overline{z}_2^m}{\sqrt{m!}}\chi^2\otimes\phi_m, \\
\mid\mathcal{B}\rangle^3 &= \frac{1}{2}e^{-\frac{r^2+s^2}{2}}\sum_{m=0}^{\infty}\frac{z_2^m}{\sqrt{m!}}\chi^3\otimes\phi_m, \qquad \text{and} \\
\mid\mathcal{B}\rangle^4 &= \frac{1}{2}e^{-\frac{r^2+s^2}{2}}\sum_{m=0}^{\infty}\frac{\overline{z}_1^m}{\sqrt{m!}}\chi^4\otimes\phi_m
\end{aligned}$$

4.5 The isometry $\mathcal{W}$

Let $\widetilde{\mathbb{H}} = L^2(r, s, \zeta, \theta, \Omega, d\mu)$. We are interested in the isometry

$$\mathcal{W} : \mathbb{C}^4 \otimes \mathbb{H} \longrightarrow \mathbb{C}^4 \otimes \widetilde{\mathbb{H}} \quad \text{by} \quad (\mathcal{W}\Psi)^j(\mathcal{B}) = \langle \Psi \mid \mathcal{B} \rangle^j, \quad j = 1, 2, 3, 4. \tag{4.7}$$

Let $D_j = \text{diag}(0, .., 1, ..0)_{4\times 4}$ be a 4×4 diagonal matrix with 1 at the jth position and 0 everywhere else. From the previous section, we have

$$U^\dagger \mathcal{B} U = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & \overline{z}_2 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & \overline{z}_1 \end{pmatrix} = \text{diag}(z_1, \overline{z}_2, z_2, \overline{z}_1).$$

Now the projection operators are

$$P_j(\widehat{n}_1, \widehat{n}_2) = U D_j U^\dagger = (P^j_{kl}).$$

Then we have

$$\mathcal{B}^m = U \text{diag}(z_1^m, \overline{z}_2^m, z_2^m, \overline{z}_1^m) U^\dagger = z_1^m P_1 + \overline{z}_2^m P_2 + z_2^m P_3 + \overline{z}_1^m P_4.$$

Let

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now as in the quaternions, we can calculate the image of the isometry as follows,

$$\begin{aligned}
\langle \Psi \mid \mathcal{B}\rangle^1 &= \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\langle \Psi \mid \mathcal{B}^m \chi^1 \otimes \phi_m\rangle}{\sqrt{m!}} \\
&= \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\langle \sum_{j=0}^{4} \chi^j \Psi_j \mid (z_1^m P_1 + \overline{z}_2^m P_2 + z_2^m P_3 + \overline{z}_1^m P_4)\chi^1 \otimes \phi_m\rangle}{\sqrt{m!}} \\
&= \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \left(\sum_{j=0}^{4} \langle \Psi_j \mid \phi_m\rangle \langle \chi^j \mid (z_1^m P_1 + \overline{z}_2^m P_2 + z_2^m P_3 + \overline{z}_1^m P_4)\chi^1\rangle \right) \\
&= \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \left(\sum_{j=0}^{4} \langle \Psi_j \mid \phi_m\rangle \left[P_{j1}^1 z_1^m + P_{j1}^2 \overline{z}_2^m + P_{j1}^3 z_2^m + P_{j1}^4 \overline{z}_1^m \right] \right).
\end{aligned}$$

As before, let

$$f_j(z) = \sum_{m=0}^{\infty} \frac{\overline{\langle \Psi_j \mid \phi_m\rangle}}{\sqrt{m!}} z^m.$$

With this notation we have

$$\langle \Psi \mid \mathcal{B}\rangle^1 = \mathcal{N}^{-\frac{1}{2}} \sum_{j=0}^{4} \left(\overline{f_j(\overline{z}_1)} P_{j1}^1 + \overline{f_j(z_2)} P_{j1}^2 + \overline{f_j(\overline{z}_2)} P_{j1}^3 + \overline{f_j(z_1)} P_{j1}^4 \right)$$

In general,

$$\langle \Psi \mid \mathcal{B}\rangle^k = \mathcal{N}^{-\frac{1}{2}} \sum_{j=0}^{4} \left(\overline{f_j(\overline{z}_1)} P_{jk}^1 + \overline{f_j(z_2)} P_{jk}^2 + \overline{f_j(\overline{z}_2)} P_{jk}^3 + \overline{f_j(z_1)} P_{jk}^4 \right).$$

Let

$$F(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix} \quad \text{and} \quad \Psi(\mathcal{B}) = \begin{pmatrix} \langle \Psi \mid \mathcal{B}\rangle^1 \\ \langle \Psi \mid \mathcal{B}\rangle^2 \\ \langle \Psi \mid \mathcal{B}\rangle^3 \\ \langle \Psi \mid \mathcal{B}\rangle^4 \end{pmatrix}.$$

Then we can write

$$\Psi(\mathcal{B})^T = \frac{1}{2} e^{-\frac{r^2+s^2}{2}} \left(F(\overline{z}_1)^\dagger P_1 + F(z_1)^\dagger P_4 + F(z_2)^\dagger P_2 + F(\overline{z}_2)^\dagger P_3 \right).$$

4.6 Exponential relation

From the proof of theorem (3.7.1) it can easily be noticed that when we have VCS in the form (1.7) if the matrix $\mathcal{Z}$ satisfy the condition

$$\mathcal{Z}\mathcal{Z}^{\dagger} = \mathcal{Z}^{\dagger}\mathcal{Z} = f(|\mathcal{Z}|)\mathbb{I}_n$$

with

$$\mathcal{N}(|\mathcal{Z}|) = be^{-f(|\mathcal{Z}|)}$$

then we can write the VCS in exponential form once the required operators satisfy the Baker-Campbell-Hausdorff identity, where f is some function and b is a constant.

Theorem 4.6.1 *The VCS in (4.1) can be written in exponential form as*

$$|\,\mathcal{B}\rangle^{j} = \frac{1}{2}e^{\mathcal{B}\otimes a^{\dagger}-\mathcal{B}^{\dagger}\otimes a}\chi^{j}\otimes\phi_0$$

for each j.

Proof. Since we have

$$\mathcal{B}\mathcal{B}^{\dagger} = \mathcal{B}^{\dagger}\mathcal{B} = (r^2+s^2)\mathbb{I}_2$$

and

$$\mathcal{N}(|\mathcal{B}|)^{-\frac{1}{2}} = \frac{1}{2}e^{-\frac{r^2+s^2}{2}},$$

as in quaternions, we can have the relation. ∎

Chapter 5

Vector Coherent States on Clifford algebras and some other interesting examples

5.1 Vector Coherent States on Clifford algebras

5.1.1 Introduction

We have introduced Clifford algebras in chapter 2. In this chapter, we construct VCS on a general Clifford algebra. We construct them with an artificial variable introduced for convenience. Here the construction differs from the general construction. We do not consider the matrix in the form $Z = A(r)e^{i\varsigma\Theta(k)}$. Let X be a real linear space and (X, f) is a real quadratic space, by this we mean

$$f = \langle \cdot \mid \cdot \rangle : X \times X \longrightarrow \mathbb{C} \quad (x, y) \mapsto f(x, y)$$

is the usual innerproduct on X. Now the Clifford algebra of (X, f) is a pair $(C(f), \Theta)$, where $C(f)$ is a $\mathbb{R}$-algebra and

$$\Theta : X \longrightarrow C(f) \quad x \mapsto \Theta(x)$$

is a linear function such that

$$\Theta(x)^2 = f(x, x)\mathbb{I}_n = \|x\|^2 \mathbb{I}_n \quad \forall x \in X. \tag{5.1}$$

We take an $n \times n$ matrix representation of $C(f)$ by hermitian matrices and denote these matrices by the same symbol as the elements of the algebra $C(f)$. Now for each $z \in S^1$, the unit circle, we define a new linear function $\Theta_z(x) = z\Theta(x)$. The new function satisfies the relation

$$\Theta_z(x)\Theta_z(x)^\dagger = \Theta_z(x)^\dagger\Theta_z(x) = \Theta(x)^2 = f(x, x)\mathbb{I}_n = \|x\|^2 \mathbb{I}_n \quad \forall x \in X. \tag{5.2}$$

Therefore, for each $z \in S^1$ the pair $(C(f), \Theta_z)$ is again a Clifford algebra. Further notice that for any two different $z \in S^1$ the corresponding Clifford algebras are isomorphic. Let $\chi^1, ..., \chi^n$ be an orthonormal basis of $\mathbb{C}^n$ and $\{\phi_m\}$ an orthonormal basis of an arbitrary Hilbert space $\mathbb{H}$. With these considerations we define VCS on $\mathbb{C}^n \otimes \mathbb{H}$ as,

$$\mid \Theta_z\rangle^j = \mathcal{N}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Theta_z(x)^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m, \tag{5.3}$$

where the normalization constant $\mathcal{N}$ and $\rho(m)$ have to be identified so that (5.3) satisfies the normalization condition and the resolution of the identity.

5.1.2 Normalization and Resolution of the identity

From (5.2) we have $(\Theta_z(x)^\dagger)^m \Theta_z(x)^m = \|x\|^{2m}\, \mathbb{I}_n$. Let us make the following identification.

$$\|\cdot\| : X \longrightarrow \mathbb{R}^+ \ \text{ by } \ \|x\| \mapsto t.$$

Set the measure $d\mu = tdtd\theta$ where $d\theta$ is a measure on S^1 and the elements of S^1 are considered as $e^{i\theta}$. With these we can normalize and get the resolution of the identity for the VCS in (5.3) as follows.

Theorem 5.1.1 *The VCS in (5.3) are normalized in the sense that*

$$\sum_{j=1}^{n} {}^j\langle \Theta_z \mid \Theta_z \rangle^j = 1$$

and the set of VCS achieve its resolution of identity

$$\int_0^\infty \int_0^{2\pi} w \sum_{j=1}^{n} \mid \Theta_z\rangle^{jj}\langle \Theta_z \mid tdtd\theta = \mathbb{I}_n \otimes I$$

with the following choices $\mathcal{N} = ne^{\|x\|^2}$, $\rho(m) = m!$ *and* $w = \frac{n}{\pi}$.

Proof. First let us see the normalization

$$\begin{aligned}
\sum_{j=1}^{n} {}^j\langle \Theta_z \mid \Theta_z \rangle^j &= \mathcal{N}^{-1} \sum_{j=1}^{n} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\sqrt{\rho(m)\rho(l)}} \langle \Theta_z(x)^{l\dagger} \Theta_z(x)^m \chi^j \mid \chi^j \rangle \langle \phi_m \mid \phi_l \rangle \\
&= \mathcal{N}^{-1} \sum_{j=1}^{n} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} \|x\|^{2m} \langle \chi^j \mid \chi^j \rangle \\
&= n\mathcal{N}^{-1} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} \|x\|^{2m} = 1,
\end{aligned}$$

where we take $\rho(m) = m!$ and $\mathcal{N} = ne^{\|x\|^2}$. With a weight function w, consider

$$\int_0^\infty \int_0^{2\pi} w \sum_{j=1}^{n} \mid \Theta_z\rangle^{jj}\langle \Theta_z \mid tdtd\theta$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{w}{\mathcal{N}\sqrt{\rho(m)\rho(l)}} \mid \Theta_z^m \chi^j \otimes \phi_m\rangle\langle \Theta_z^l \chi^j \otimes \phi_l \mid d\mu \\
&= \sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{w}{\mathcal{N}\sqrt{\rho(m)\rho(l)}} \Theta_z^m \mid \chi^j\rangle\langle \chi^j \mid \Theta_z^{l\dagger} \otimes \mid \phi_m\rangle\langle \phi_l \mid d\mu \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{w}{\mathcal{N}\sqrt{\rho(m)\rho(l)}} \Theta_z^m \Theta_z^{l\dagger} \otimes \mid \phi_m\rangle\langle \phi_l \mid d\mu \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{w}{\mathcal{N}\sqrt{\rho(m)\rho(l)}} e^{i(m-l)\theta}\Theta(x)^m \Theta(x)^{l\dagger} \otimes \mid \phi_m\rangle\langle \phi_l \mid d\mu \\
&= \sum_{m=0}^{\infty}\int_0^{\infty} \frac{2\pi w}{\mathcal{N}\rho(m)} \Theta(x)^m \Theta(x)^{m\dagger} \otimes \mid \phi_m\rangle\langle \phi_l \mid tdt \\
&= \sum_{m=0}^{\infty}\int_0^{\infty} \frac{2\pi w}{\mathcal{N}\rho(m)} t^{2m} tdt \otimes \mid \phi_m\rangle\langle \phi_m \mid \\
&= \sum_{m=0}^{\infty}\left(\int_0^{\infty} \frac{2\pi w}{n(m!)} e^{-t^2} t^{2m} tdt\right) \mathbb{I}_n \otimes \mid \phi_m\rangle\langle \phi_m \mid
\end{aligned}
$$

Now the choice $w = \frac{n}{\pi}$ together with a substitution $t^2 = r$ push the last line of the above calculations to

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\int_0^{\infty} e^{-r} r^{(m+1)-1} dr\right) \mathbb{I}_n \otimes \mid \phi_m\rangle\langle \phi_m \mid \\
&= \sum_{m=0}^{\infty}\frac{\Gamma(m+1)}{m!} \mathbb{I}_n \otimes \mid \phi_m\rangle\langle \phi_m \mid \\
&= \sum_{m=0}^{\infty} \mathbb{I}_n \otimes \mid \phi_m\rangle\langle \phi_m \mid \\
&= \mathbb{I}_n \otimes I,
\end{aligned}
$$

where we have used the identity

$$
\int_0^{\infty} e^{-r} r^{(m+1)-1} dr = \Gamma(m+1)
$$

which proves the theorem ■

Theorem (5.1.1) shows that the set of states in (5.3) are VCS. In the above construction we have introduced an additional complex number $e^{i\theta}$ from S^1 to make our calculations easier. It can easily be noted that the new parameter plays only a role in the calculations of resolution of the identity, by bringing the double sum to a single sum through the identity

$$\int_0^{2\pi} e^{i(m-l)\theta} d\theta = \begin{cases} 0 & \text{if} \quad m \neq l \\ 2\pi & \text{if} \quad m = l \end{cases}$$

5.2 Some Interesting Examples

In this short section we give some easy consequences of previous chapters. These are consequences because the construction of VCS follows in more or less in the same way as in the previous chapters. Further to these consequences we also give some partly solved problems.

Example 5.2.1 *It is known that the $su(1,1)$ algebra is generated by the basis*

$$X_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Now, by adding the 2×2 identity matrix to the algebra we extend the algebra to a larger class and we call it extended $su(1,1)$ Lie algebra. In order to obtain a class of coherent states on this extended algebra we take a special linear combination (with *complex coefficients) of the basis elements as follows.*

$$g = x_0\mathbb{I}_2 + ix_1X_- + x_2X_0 + ix_3X_+ = \begin{pmatrix} x_0 + ix_2 & ix_1 - x_3 \\ ix_1 + x_3 & x_0 - ix_2 \end{pmatrix}$$

Now as before let us make the following polar substitution,

$$x_0 = r\cos\theta, x_1 = r\sin\theta\sin\phi\cos\psi, x_2 = r\sin\theta\sin\phi\sin\psi, x_3 = r\sin\theta\cos\phi$$

With this substitution the matrix g becomes

$$g = r\mathbb{I}_2(\cos\theta\mathbb{I}_2 + i\Theta\sin\theta)$$

where

$$\Theta = \begin{pmatrix} \sin\phi\sin\psi & \sin\phi\cos\psi + i\cos\phi \\ \sin\phi\cos\psi - i\cos\phi & -\sin\phi\sin\psi \end{pmatrix}.$$

Now a straight forward calculation shows that $A(r) = rI_2$ and Θ satisfy conditions (1.2)-(1.5). Further the eigenvalues of Θ are 1 and -1 thus everything follows as in the discussion of quaternions.

As we know, the basis presented above forms a real $su(1,1)$ algebra. By adding the identity we can only enlarge the algebra as a real algebra. But we have used complex coefficients, so the name extended $su(1,1)$ Lie algebra doesn't make much sense here. It should be understood as presented above, while the name is just a formal one.

Example 5.2.2 *The construction in quaternionic VCS can also be carried out by replacing the Pauli matrices by Dirac matrices (or γ-matrices), with one exception,* *the matrix γ^0 replaced by the identity matrix*

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3.$$

In fact, if we take $\mathcal{D} = x_0\gamma^0 + (x_1, x_2, x_3) \cdot (\gamma^1, -\gamma^2, \gamma^3) = x_0\gamma^0 + x_1\gamma^1 - x_2\gamma^2 + x_3\gamma^3$ then by the same polar substitution (as in quaternions) $\mathcal{D}$ can be written as

$$\mathcal{D} = A(r)(\cos\theta \mathbb{I}_2 + i\gamma(\widehat{n})\sin\theta)$$

where $A(r) = r\mathbb{I}_4$ and

$$\gamma(\widehat{n}) = \begin{pmatrix} 0 & 0 & -i\cos\phi & -ie^{i\psi}\sin\phi \\ 0 & 0 & -ie^{-i\psi}\sin\phi & i\cos\phi \\ i\cos\phi & ie^{i\psi}\sin\phi & 0 & 0 \\ ie^{-i\psi}\sin\phi & -i\cos\phi & 0 & 0 \end{pmatrix}.$$

It can be easily seen that $A(r)$ and $\gamma(\widehat{n})$ satisfy conditions (1.2)-(1.5). The remaining details follows exactly as in quaternions.

Example 5.2.3 *The construction in chapter 4 can also be carried out by replacing the matrix Θ by*

$$\frac{1}{\sqrt{2\cosh^2\theta - 1}} \begin{pmatrix} \sigma(\widehat{n}_1)\sinh\theta & i\sigma(\widehat{n}_2)\cosh\theta \\ -i\sigma(\widehat{n}_2)\cosh\theta & \sigma(\widehat{n}_1)\sinh\theta \end{pmatrix}.$$

The details follows as in chapter 4 with some complicated calculations.

Example 5.2.4 *The construction in chapter 4 can also be carried out by replacing the matrix Θ by the matrix $\gamma(\widehat{n})$ of example (5.2.2).*

Example 5.2.5 *Let*

$$A(r,\theta) = \begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & 1 \end{pmatrix} \quad \text{and} \quad \Theta = \mathbb{I}_2,$$

with $r \in [0,\infty), \theta \in [0, 2\pi]$. Then $A(r,\theta)$ and Θ satisfy conditions (1.2)-(1.5). Thus it is possible to have VCS in the form (1.7), if there exist $\rho(m)$ and $W(r)$ satisfying conditions (1.10) and (1.14). In order to see these conditions more closely, let us calculate the following. The eigenvalues of $|A(r,\theta|$ are

$$\lambda_1 = 1 - 2r\sin\theta + r^2 \quad \text{and} \quad \lambda_2 = 1 + 2r\sin\theta + r^2$$

, and the corresponding eigenvectors are

$$\begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

respectively. Let

$$P = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Thus

$$P^{-1}|A(r,\theta)|P = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

Therefore

$$|A(r,\theta)|^{2m} = P \begin{pmatrix} \lambda_2^m & 0 \\ 0 & \lambda_1^m \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{1}{2}(\lambda_1^m + \lambda_2^m) & \frac{i}{2}(\lambda_1^m - \lambda_2^m) \\ \frac{i}{2}(\lambda_1^m - \lambda_2^m) & \frac{1}{2}(\lambda_1^m + \lambda_2^m) \end{pmatrix} = B(say).$$

Thus we have $Tr|A(r,\theta)|^{2m} = \lambda_1^m + \lambda_2^m$. *The conditions (1.10) and (1.14) become*

$$\mathcal{N} = 2\sum_{m=0}^{\infty} \frac{\lambda_1^m + \lambda_2^m}{\rho(m)} \tag{5.4}$$

$$\int_0^{\infty}\int_0^{2\pi} \frac{W(r)}{\mathcal{N}\rho(m)} Br d\theta dr = \mathbb{I}_2 \tag{5.5}$$

If there exist $\rho(m)$ *and* $W(r)$ *to satisfy (5.4) and (5.5), we can have VCS.*

Chapter 6

Quaternionic States on the Unit disc

In this chapter, we introduce quaternionic VCS on the unit disc. It can be seen as a generalization of Perelomov CS. For the quaternionic VCS we prove a disentangling formula similar to (28). Using this formula we also get VCS in the form (29).

6.1 The VCS

We have the quaternions as $q = A(r)e^{i\zeta\Theta}$ with

$$A(r) = r\mathbb{I}_2 \quad \text{and} \quad \Theta = \begin{pmatrix} \cos\phi & e^{i\psi}\sin\phi \\ e^{-i\psi}\sin\phi & -\cos\phi \end{pmatrix},$$

where $r \in [0,\infty), \phi, \psi \in [0,\pi]$ and $\zeta \in [0, 2\pi]$. Now we define the coherent states as (with j a positive integer)

$$| \,\omega\rangle^n = \frac{1}{\sqrt{2}}(1-s^2)^j \sum_{k=0}^{\infty} \sqrt{\frac{(2j)_k}{k!}} \omega^k \chi^n \otimes \phi_k \quad n = 1, 2 \tag{6.1}$$

where

$$\omega = \frac{q}{r}\tanh r, \quad s = \tanh r \qquad \text{and} \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

Thus $s \in [0, 1)$ and $\omega = A(s)e^{i\zeta\Theta}$. The parameters s and ζ of the matrix ω form a unit disc. The VCS constructed with the matrix ω can be considered as VCS on the unit disc. We set the measure as $d\mu = d\nu d\Omega$ with $d\nu = sdsd\zeta$ and $d\Omega = \sin\phi d\phi d\psi$.

6.2 Normalization

Here we prove that the states in (6.1) are in fact normalized VCS.

Theorem 6.2.1 *The VCS in (6.1) are normalized in the sense that*

$$\sum_{n=1}^{2} {}^n\langle \omega \mid \omega\rangle^n = 1$$

Proof. Consider

$$\begin{aligned}
\sum_{n=1}^{2} {}^n\langle \omega \mid \omega\rangle^n &= \frac{1}{2}(1-s^2)^{2j} \sum_{n=1}^{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \sqrt{\frac{(2j)_k (2j)_l}{k!l!}} \langle \omega^{l\dagger}\omega^k \chi^n \mid \chi^n\rangle\langle \phi_k \mid \phi_l\rangle \\
&= \frac{1}{2}(1-s^2)^{2j} \sum_{n=1}^{2}\sum_{k=0}^{\infty} \frac{(2j)_k}{k!} \langle \omega^{l\dagger}\omega^k \chi^n \mid \chi^n\rangle \\
&= \frac{1}{2}(1-s^2)^{2j} \sum_{k=0}^{\infty} \frac{(2j)_k}{k!} Tr|A(s)|^{2k} \\
&= \frac{1}{2}(1-s^2)^{2j} \sum_{k=0}^{\infty} \frac{(2j)_k}{k!} s^{2m} = 1.
\end{aligned}$$

We have the theorem. ■

6.3 Resolution of Unity

He we prove that the states in (6.1) has a resolution of the identity.

Theorem 6.3.1 *The states in (6.1) satisfy the resolution of the identity*

$$\int_{D\times\Omega} W(s)\sum_{n=1}^{2} \mid \omega\rangle^{nn}\langle\omega \mid d\mu = \mathbb{I}_2 \otimes I \tag{6.2}$$

where $W(s)$ is a weight function and is given by

$$W(s) = \frac{2j-1}{2\pi^2(1-s^2)^2}$$

Proof. Consider the following integral with the weight function $W(s)$

$$\begin{aligned}
&\int_{D\times\Omega} W(s)\sum_{n=1}^{2} \mid \omega\rangle^{nn}\langle\omega \mid d\mu \\
= &\sum_{n=1}^{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{2}(1-s^2)^{2j}W(s)\sqrt{\frac{(2j)_k(2j)_l}{k!l!}}\omega^k \mid \chi^n\rangle\langle\chi^n \mid \omega^{l\dagger}\otimes \mid \phi_k\rangle\langle\phi_l \mid d\mu \\
= &\frac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\int_{D\times\Omega}(1-s^2)^{2j}W(s)\sqrt{\frac{(2j)_k(2j)_l}{k!l!}}A(s)^k e^{i(k-l)\zeta\Theta}A(s)^{l\dagger}\otimes \mid \phi_k\rangle\langle\phi_l \mid d\mu
\end{aligned}$$

Since

$$\int_0^{2\pi} e^{i(k-l)\zeta\Theta}d\zeta = \begin{cases} 0 & \text{if} \quad k \neq l \\ 2\pi\mathbb{I}_2 & \text{if} \quad k = l \end{cases} \qquad \text{and} \qquad \int_0^{\pi}\int_0^{\pi} \sin\phi d\phi d\psi = 2\pi$$

we have

$$\begin{aligned}
&\int_{D\times\Omega} W(s)\sum_{n=1}^{2} \mid \omega\rangle^{nn}\langle\omega \mid d\mu \\
=\ & 4\pi^2\sum_{k=0}^{\infty}\frac{(2j)_k}{k!}\int_0^1 (1-s^2)^{2j}W(s)|A(s)|^{2k}sds\otimes \mid \phi_k\rangle\langle\phi_k \mid \\
=\ & 4\pi^2\sum_{k=0}^{\infty}\frac{(2j)_k}{k!}\int_0^1 (1-s^2)^{2j}W(s)s^{2k}sds\mathbb{I}_2\otimes \mid \phi_k\rangle\langle\phi_k \mid .
\end{aligned}$$

With the choice

$$W(s)=\frac{2j-1}{2\pi^2(1-s^2)^2}$$

the last line of the above equation becomes

$$\sum_{k=0}^{\infty}\frac{(2j)_k(2j-1)}{k!}\int_0^1 2(1-s^2)^{2j-2}s^{2k}sds\mathbb{I}_2\otimes \mid \phi_k\rangle\langle\phi_k \mid . \qquad (6.3)$$

The substitution $t=r^2$ changes (6.3) to

$$\sum_{k=0}^{\infty}\frac{\Gamma(2j+k)}{\Gamma(2j-1)\Gamma(k+1)}\int_0^1 (1-t)^{2j-2}t^k dt\mathbb{I}_2\otimes \mid \phi_k\rangle\langle\phi_k \mid . \qquad (6.4)$$

Now by the identities

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{and} \quad B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

we get

$$\int_{D\times\Omega} W(s)\sum_{n=1}^{2} \mid \omega\rangle^{nn}\langle\omega \mid d\mu = \mathbb{I}_2\otimes\sum_{k=0}^{\infty} \mid \phi_k\rangle\langle\phi_k \mid = \mathbb{I}_2\otimes I. \qquad (6.5)$$

We have the theorem. ■

6.4 su(1,1) algebra and its connection

Let $\{K_+, K_-, K_3\}$ be a Weyl basis of the Lie algebra $su(1,1)$, where

$$K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, K_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, K_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and they satisfy the commutation relations

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_-, K_+] = 2K_3.$$

Now let us define the action of these operators on the basis vectors $\{\phi_k\}$, as usual, by

$$\begin{aligned} K_+\phi_k &= \sqrt{(j+1)(2k+j)}\phi_{k+1} \\ K_-\phi_k &= \sqrt{j(2k+j-1)}\phi_{k-1} \\ K_3\phi_k &= (k+j)\phi_k. \end{aligned}$$

For the coherent states, we define the corresponding operators as

$$A_+ = \mathbb{I}_2 \otimes K_+, A_- = \mathbb{I}_2 \otimes K_-, A_3 = \mathbb{I}_2 \otimes K_3.$$

Here our aim is to establish the following connection

$$|\,\omega\rangle^n = \frac{1}{\sqrt{2}} e^{qA_+ - q^\dagger A_-} \chi^n \otimes \phi_0 = \frac{1}{\sqrt{2}} e^{q\otimes K_+ - q^\dagger \otimes K_-} \chi^n \otimes \phi_0. \tag{6.6}$$

In order to establish the relation (6.6), let us state the following disentangling formulas. Formula(6.7) is already known[31] and (6.8) can be obtained in the same way. For the sake of completeness we give a short proof.

Theorem 6.4.1 *When* $z = r(\cos\zeta + i\sin\zeta)$ *and* $\rho = s(\cos\zeta + i\sin\zeta)$ *with* $s = \tanh r$, *we have*

$$e^{zK_+ - \overline{z}K_-} = e^{\rho K_+} e^{\log(1-|\rho|^2)K_3} e^{-\overline{\rho}K_-} \tag{6.7}$$

$$e^{\overline{z}K_+ - zK_-} = e^{\overline{\rho}K_+} e^{\log(1-|\rho|^2)K_3} e^{-\rho K_-} \tag{6.8}$$

Proof. The proof depends on the Gaussian decomposition of matrices. We have

$$e^{zK_+ - \overline{z}K_-} = \exp\begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix} = \begin{pmatrix} \cosh r & e^{i\zeta}\sinh r \\ e^{-i\zeta}\sinh r & \cosh r \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{i\zeta}\tanh r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-\tanh^2 r} & 0 \\ 0 & \frac{1}{\sqrt{1-\tanh^2 r}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-i\zeta}\tanh r & 1 \end{pmatrix}$$

$$= e^{\rho K_+} e^{\log(1-|\rho|^2)K_3} e^{-\overline{\rho}K_-}.$$

Similarly

$$e^{\overline{z}K_+ - zK_-} = \exp\begin{pmatrix} 0 & \overline{z} \\ z & 0 \end{pmatrix} = \begin{pmatrix} \cosh r & e^{-i\zeta}\sinh r \\ e^{i\zeta}\sinh r & \cosh r \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{-i\zeta}\tanh r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-\tanh^2 r} & 0 \\ 0 & \frac{1}{\sqrt{1-\tanh^2 r}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{i\zeta}\tanh r & 1 \end{pmatrix}$$

$$= e^{\overline{\rho}K_+} e^{\log(1-|\rho|^2)K_3} e^{-\rho K_-}.$$

We have the theorem. ■

Using the results in the theorem, let us derive a disentangling formula for the VCS in (6.1). The eigenvalues of q are z and $\overline{z}$. The eigenvalues of ω are ρ and $\overline{\rho}$. Since

$qq^\dagger = q^\dagger q$, the same unitary matrix, U diagonalizes both q and $q^\dagger$. That is

$$U^\dagger q U = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}, U^\dagger q^\dagger U = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}. \tag{6.9}$$

Further, it is also true that

$$U^\dagger \omega U = \begin{pmatrix} \rho & 0 \\ 0 & \overline{\rho} \end{pmatrix}, U^\dagger w^\dagger U = \begin{pmatrix} \overline{\rho} & 0 \\ 0 & \rho \end{pmatrix}. \tag{6.10}$$

Theorem 6.4.2 *Let $q \in \boldsymbol{H}$, w be as in (6.1), K_+, K_-, K_3 be a $su(1,1)$ basis. Then we have*

$$e^{q\otimes K_+ - q^\dagger \otimes K_-} = e^{\omega \otimes K_+} e^{\log(1-|\rho|^2)\mathbb{I}_2 \otimes K_3} e^{-\omega^\dagger \otimes K_-}. \tag{6.11}$$

(6.11) is a disentangling formula for quaternions.

Proof. With the U of (6.9), consider

$$\begin{aligned}
& U^\dagger \otimes \mathbb{I}_2 \left(e^{q\otimes K_+ - q^\dagger\otimes K_-}\right) U \otimes \mathbb{I}_2 \\
= \; & e^{U^\dagger q U \otimes K_+ - U^\dagger q^\dagger U \otimes K_-} \\
= \; & \begin{pmatrix} e^{zK_+ - \overline{z}K_-} & 0 \\ 0 & e^{\overline{z}K_+ - zK_-} \end{pmatrix} \\
= \; & \begin{pmatrix} e^{\rho K_+} e^{\log(1-|\rho|^2)K_3} e^{-\overline{\rho}K_-} & 0 \\ 0 & e^{\overline{\rho}K_+} e^{\log(1-|\rho|^2)K_3} e^{-\rho K_-} \end{pmatrix}
\end{aligned}$$

Thus

$$e^{q\otimes K_+ - q^\dagger\otimes K_-} = U\otimes \mathbb{I}_2 \begin{pmatrix} e^{\rho K_+}e^{\log(1-|\rho|^2)K_3}e^{-\overline{\rho}K_-} & 0 \\ 0 & e^{\overline{\rho}K_+}e^{\log(1-|\rho|^2)K_3}e^{-\rho K_-} \end{pmatrix} U^\dagger \otimes \mathbb{I}_2$$

$$= \exp\left[U\begin{pmatrix} \rho & 0 \\ 0 & \overline{\rho}\end{pmatrix}U^\dagger\otimes K_+\right]\exp\left[U^\dagger \log(1-|\rho|^2)U\otimes K_3\right]$$

$$\times \exp\left[-U\begin{pmatrix} \overline{\rho} & 0 \\ 0 & \rho\end{pmatrix}U^\dagger\otimes K_-\right]$$

$$= e^{\omega\otimes K_+}e^{\log(1-|\rho|^2)\mathbb{I}_2\otimes K_3}e^{-\omega^\dagger\otimes K_-},$$

which completes the proof. ■

The following theorem gives an exponential form for the VCS defined in (6.1)

Theorem 6.4.3 *The VCS in (6.1) can be written in the following form*

$$\mid \omega\rangle^n = \frac{1}{\sqrt{2}}e^{q\otimes K_+ - q^\dagger\otimes K_-}\chi^n\otimes\phi_0$$

for each n.

Proof. With the disentangling formula

$$e^{q\otimes K_+ - q^\dagger\otimes K_-} = e^{\omega\otimes K_+}e^{\log(1-|\rho|^2)\mathbb{I}_2\otimes K_3}e^{-\omega^\dagger\otimes K_-}.$$

and the facts

$$(K_-)^k\phi_0 = 0, \quad \forall k\geq 1, \qquad K_3\phi_0 = j\phi_0, \qquad (K_+)^k\phi_0 = \sqrt{k!(2j)_k}\phi_k$$

we have

$$\begin{aligned}
\frac{1}{\sqrt{2}} e^{q\otimes K_+ - q^\dagger \otimes K_-} \chi^n \otimes \phi_0 &= \frac{1}{\sqrt{2}} e^{\omega\otimes K_+} e^{\log(1-|\rho|^2)\mathbb{I}_2\otimes K_3} e^{-\omega^\dagger\otimes K_-} \chi^n \otimes \phi_0 \\
&= \frac{1}{\sqrt{2}} e^{\omega\otimes K_+} e^{\log(1-|\rho|^2)\mathbb{I}_2\otimes K_3} \chi^n \otimes \phi_0 \\
&= \frac{1}{\sqrt{2}} (1-s^2)^j e^{\omega\otimes K_+} \chi^n \otimes \phi_0 \\
&= \frac{1}{\sqrt{2}} (1-s^2)^j \sum_{k=0}^{\infty} \frac{\omega^k \otimes (K_+)^k}{k!} \chi^n \otimes \phi_0 \\
&= \frac{1}{\sqrt{2}} (1-s^2)^j \sum_{k=0}^{\infty} \sqrt{\frac{(2j)_k}{k!}} \omega^k \chi^n \otimes \phi_k = | \omega\rangle^n .
\end{aligned}$$

We have the theorem. ■

Chapter 7

Applications

CS have applications in many branches of quantum physics. It was even said that CS are the natural language of quantum physics [1]. The early articles related to several applications are well documented in [20]. Once we have a set of CS it is natural to compute certain associated physical quantities, namely photon number distribution, signal-to-quantum noise ratio etc. It is also of interest to analyze the squeezing properties and time evolution of the CS. There is also a great deal of interest in applying CS theory to quantum computation, quantum cryptography and quantum teleportation. In this chapter we will discuss these aspects based on [20], [13] and [28] and the references cited there.

The above mentioned physical quantities are related to a physical system. The quantities can be calculated using CS when a set of CS describe a physical system. As far as we know, a physical system, which can be described by quaternionic VCS, is not known. Still, it may be interesting to calculate quantities analogue to the quantities of the bosonic CS for quaternionic VCS. In this regard, we calculate some of these physical quantities for our newly formed VCS. Finally we give an interesting application of our quaternionic VCS.

7.1 Photon number distribution

In quantum mechanics, it is natural to discuss the statistics of the measured quantities rather than the quantities itself, for example, in light beams the light itself is not measured, but the photocurrent from a detector is measured (on which the light beam penetrates), that is, the statistics of the photocurrent is studied. In quantum mechanical treatment the light beam and the detector are considered quantum mechanical. In this treatment, for a single mode of the electric field, the field is not a classical random variable, but it is a quantum mechanical operator $\widehat{E}$ (which can be written in terms of annihilation and creation operators together with some other factors). The annihilation and creation operators $a, a^\dagger$ for the photons in the field satisfy $[a, a^\dagger] = 1$ (in the basic bosonic case). The operator which yields the number of photons in the field is $N = aa^\dagger$, the number operator. For the states that the field can occupy, the Fock states $|n\rangle$, the eigenstates of N form an orthonormal basis. The effect of a creation (annihilation) operator on a number state, $|n\rangle$ is to increase (decrease) by one the number of photons in the state. If an electric field is in a state

$|\psi\rangle$, then the probability that the field will contain n photons is given by

$$P(n) = |\langle n \mid \psi\rangle|^2 \tag{7.1}$$

For example if the field is in the number state, $|m\rangle$ then $P(n) = \delta_{nm}$, the field contains a definite number of photons.

Some of the most interesting statistical quantities are the moments of the photon number distribution. For a field in state $|\psi\rangle$ the mean and the variance of the photon number are

$$\langle N\rangle = \langle\psi \mid N \mid \psi\rangle \tag{7.2}$$

$$\langle(\Delta N)^2\rangle = \langle\psi \mid N^2 \mid \psi\rangle - \langle\psi \mid N \mid \psi\rangle^2 \tag{7.3}$$

Again for the number state, $|m\rangle$,$\langle N\rangle = m$ and $\langle(\Delta N)^2\rangle$. That is, a state containing a definite number of photons has no fluctuations, meaning that this state would show zero noise in its photo count.

Let us move to the CS theory. CS are of interest here because they are the closest equivalent to classical electromagnetic waves. If the field is in a CS (bosonic)$|z\rangle$ then the probability that the state will contain n photons is given by

$$P(n) = \frac{|z|^{2n}}{n!}e^{-|z|^2}. \tag{7.4}$$

That is, CS contain a Poisson distribution of photons (in the case of bosonic CS) with a mean and variance,

$$\langle N\rangle = |z|^2 \tag{7.5}$$

$$\langle(\Delta N)^2\rangle = |z|^2 \tag{7.6}$$

The probability distribution is not necessarily Poisson for all classes of CS. For example a sub-Poisson CS was given in [28]. Further they differ a lot from Poisson in the case of deformed CS.

7.1.1 An analogue of photon number distribution for the quaternionic VCS

Here we calculate the quantities of photon number distribution for the quaternionic VCS of chapter 3. We have VCS

$$\mid q\rangle^{j}=\frac{1}{\sqrt{2}}e^{-\frac{r^{2}}{2}}\sum_{m=0}^{\infty}\frac{q^{m}}{\sqrt{m!}}\chi^{j}\otimes\phi_{m}$$

where q is of the form (3.1) satisfying (1.2)-(1.5). We have the action of the number operator, N on the VCS given by (3.15). With these first we calculate the mean,

$$\begin{aligned}
\langle N\rangle={}^{j}\langle q\mid N\mid q\rangle^{j} &= \frac{1}{2}e^{-r^{2}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\langle\frac{q^{m}}{\sqrt{m!}}\chi^{j}\otimes\phi_{m}\mid\frac{kq^{k}}{\sqrt{k!}}\chi^{j}\otimes\phi_{k}\rangle\\
&= \frac{1}{2}e^{-r^{2}}\sum_{m=0}^{\infty}\frac{1}{(m-1)!}\langle q^{m}\chi^{j}\mid q^{m}\chi^{j}\rangle\\
&= \frac{1}{2}e^{-r^{2}}\sum_{m=0}^{\infty}\frac{r^{2m}}{(m-1)!}\\
&= \frac{r^{2}}{2}.
\end{aligned}$$

In order to see the variance, as in (3.15) we get

$$N^{2}\mid q\rangle^{j}=\frac{1}{\sqrt{2}}e^{-\frac{r^{2}}{2}}\sum_{m=1}^{\infty}\frac{m^{2}q^{m}}{\sqrt{m!}}\chi^{j}\otimes\phi_{m}.$$

Thus

$$\begin{aligned}
\langle N^2 \rangle = {}^j\langle q \mid N \mid q \rangle^j &= \frac{1}{2} e^{-r^2} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \langle \frac{q^m}{\sqrt{m!}} \chi^j \otimes \phi_m \mid \frac{k^2 q^k}{\sqrt{k!}} \chi^j \otimes \phi_k \rangle \\
&= \frac{1}{2} e^{-r^2} \sum_{m=0}^{\infty} \frac{m^2}{(m)!} \langle q^m \chi^j \mid q^m \chi^j \rangle \\
&= \frac{1}{2} e^{-r^2} \sum_{m=0}^{\infty} \frac{m r^{2m}}{(m-1)!} \\
&= \frac{r^2(r^2+1)}{2}.
\end{aligned}$$

Thus the variance is

$$\langle (\Delta N)^2 \rangle = \langle \psi \mid N^2 \mid \psi \rangle - \langle \psi \mid N \mid \psi \rangle^2 = \frac{r^2(r^2+1)}{2} - \left(\frac{r^2}{2} \right)^2 = \frac{r^2(r^2+2)}{4}.$$

Now the probability distribution can be computed as follows.

$$\begin{aligned}
P(m) &= \mid \langle \chi^j \otimes \phi_m \mid q \rangle^j \mid^2 = \mid \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \frac{q^m}{\sqrt{m!}} \mid^2 \\
&= \frac{1}{2} \frac{|q|^{2m}}{m!} e^{-r^2} = \frac{1}{2} \frac{r^2}{m!} e^{-r^2} \mathbb{I}_2.
\end{aligned}$$

Thus we get a Poisson distribution as in the bosonic case.

7.2 Signal-to-quantum noise ratio

Noise is, loosely, any disturbance tending to interfere with the normal operation of a device or system. But noise is often defined as the uncertainty in a signal due to random fluctuations in that signal. For example, an X-ray beam emerging from an X-ray tube is statistical in nature, that is, the number of photons emitted from the source per unit time varies according to a Poisson distribution. Further, photons incident on a charge-coupled detector (an electron bombardment tube) convert to photoelectrons, these photoelectrons contain the signal but also carry a statistical fluctuation in the photon arrival rate at a given point. This phenomenon is known as photon noise and follows Poisson statistics. Signal-to-noise ratio (SNR) is a ratio between the desired signal and noise signal. The calculation of SNR in experiments represents an important tool for experimental data analysis. It provides quantitative information on the measured signal and the noise inherent to the experiment. As an example, suppose that we take one picture of a scene, then another, and another, and another. Because of random variations in light, the random distribution of molecules of air in the path of the light etc, we will never quite get exactly the same picture twice. We can consider the picture $f(x)$ as a random variable from which we sample an ensemble of images from the space of all possibilities. This ensemble has a mean (average) image, $\overline{f(x)}$. Thus if we sample enough images, the ensemble mean $\overline{f(x)}$ approaches the noise-free original signal. If we compare the strength of a signal or image (the mean of the ensemble) to the variance between individual acquired images we get a signal-to-noise ratio (SNR). A high SNR indicates a relatively clean signal or image, a low SNR indicates that the noise dominate the signal.

Early in the development of quantum mechanics, Dirac constructed operators corresponding to photon number and phase of a CS, these operators do not commute, result an uncertainty principle.

$$\Delta N \Delta \phi \geq \frac{1}{2}. \tag{7.7}$$

This was used as the basis for the treatment of noise in coherent detection, that is, the act of coherent detection (small $\Delta\phi$) adds noise (uncertainty in N is inversely proportional to $\Delta\phi$). This extra noise in N, which appears in addition to any background fluctuations, is called quantum noise.

Unfortunately, (7.7) is not quite correct. It was realized that Dirac's phase operator is not hermitian and thus did not correspond to an observable quantity. In order to correct this the self adjoint quadrature operators,

$$Q = \frac{a + a^\dagger}{\sqrt{2}} \quad \text{and} \quad P = \frac{a - a^\dagger}{i\sqrt{2}} \tag{7.8}$$

were introduced, where the operators a and $a^\dagger$ bear the usual meaning. They satisfy $[Q, P] = iI$ and the variance of Q and P in any state $\mid z\rangle$ satisfies the Heisenberg uncertainty principle. In terms of the operator Q the signal-to-quantum noise ratio is defined as

$$\sigma = \frac{\langle Q\rangle^2}{(\Delta Q)^2}. \tag{7.9}$$

Let us calculate the SNR for the bosonic CS. The following quantities can be easily calculated.

$$\begin{aligned}
\langle z \mid a \mid z\rangle &= \overline{z} \\
\langle z \mid a^\dagger \mid z\rangle &= z \\
\langle z \mid a^2 \mid z\rangle &= \overline{z}^2 \\
\langle z \mid (a^\dagger)^2 \mid z\rangle &= z^2
\end{aligned}$$

$$\begin{aligned}\langle z \mid aa^\dagger \mid z\rangle &= (|z|^2+1)\\ \langle z \mid a^\dagger a \mid z\rangle &= |z|^2.\end{aligned}$$

Thus

$$\langle z \mid Q \mid z\rangle = \frac{1}{\sqrt{2}}(\overline{z}+z)$$

and

$$(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2 = \langle z \mid Q^2 \mid z\rangle - \langle z \mid Q \mid z\rangle^2 = \frac{1}{2}.$$

From the above calculations, SNR can be obtained as,

$$\sigma = (\overline{z}+z)^2.$$

7.2.1 SNR for the quaternionic VCS

Here we calculate an analogue of SNR for the quaternionic VCS. The calculations are bit different from the bosonic case. Let us take the quaternions as

$$q = \begin{pmatrix} r\cos\theta + ir\sin\theta\cos\phi & -r\sin\theta\sin\phi\sin\psi + ir\sin\theta\sin\phi\cos\psi \\ r\sin\theta\sin\phi\sin\psi + ir\sin\theta\sin\phi\cos\psi & r\cos\theta - ir\sin\theta\cos\phi \end{pmatrix}.$$

The quaternionic VCS are of the form

$$\mid q\rangle^j = \frac{1}{\sqrt{2}} e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \chi^j \otimes \phi_m.$$

Let us calculate

$$
\begin{aligned}
{}^1\langle q \mid A \mid q\rangle^1 &= {}^1\langle q \mid A \mid q\rangle^1 \\
&= \frac{1}{2}e^{-r^2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\langle \frac{q^m}{\sqrt{m!}}\chi^1 \mid \frac{q^{k+1}}{\sqrt{k!}}\chi^1\rangle\langle \phi_m \mid \phi_k\rangle \\
&= \frac{1}{2}e^{-r^2}\sum_{m=0}^{\infty}\frac{r^{2m}}{m!}\langle \chi^1 \mid q\chi^1\rangle \\
&= \frac{1}{2}\langle \chi^1 \mid q\chi^1\rangle \\
&= \frac{r}{2}[\cos\theta + i\sin\theta\cos\phi].
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
{}^1\langle q \mid A^\dagger \mid q\rangle^1 &= \frac{1}{2}\langle q\chi^1 \mid \chi^1\rangle = \frac{r}{2}[\cos\theta - i\sin\theta\cos\phi] \\
{}^1\langle q \mid A^2 \mid q\rangle^1 &= \frac{1}{2}\langle \chi^1 \mid q^2\chi^1\rangle = \frac{1}{2}[2r^2\cos^2\theta + 2ir^2\cos\theta\sin\theta\cos\phi - r^2] \\
{}^1\langle q \mid (A^\dagger)^2 \mid q\rangle^1 &= \frac{1}{2}\langle q^2\chi^1 \mid \chi^1\rangle = \frac{1}{2}[2r^2\cos^2\theta - 2ir^2\cos\theta\sin\theta\cos\phi - r^2] \\
{}^1\langle q \mid AA^\dagger \mid q\rangle^1 &= \frac{1}{2}e^{-r^2}\sum_{m=0}^{\infty}\frac{|q|^{2m}}{m!}(m+1) = \frac{1}{2}(r^2+1) \\
{}^1\langle q \mid A^\dagger A \mid q\rangle^1 &= \frac{1}{2}\langle q\chi^1 \mid q\chi^1\rangle = \frac{r^2}{2}.
\end{aligned}
$$

Thus

$$
{}^1\langle q \mid Q \mid q\rangle^1 = \frac{r\cos\theta}{\sqrt{2}}
$$

and

$$
(\Delta Q)_1^2 = \langle Q^2\rangle - \langle Q\rangle^2 = {}^1\langle q \mid Q^2 \mid q\rangle^1 - {}^1\langle q \mid Q \mid q\rangle^{12} = \frac{1}{4}[2r^2\cos^2\theta + 1]. \quad (7.10)
$$

From the above calculations, for the state $\mid q\rangle^1$, SNR can be obtained as,

$$\sigma_1 = \frac{2r^2\cos^2\theta}{2r^2\cos^2\theta + 1}.$$

For $\mid q\rangle^2$, a similar calculation shows that

$$\sigma_2 = \frac{2r^2\cos^2\theta}{2r^2\cos^2\theta + 1}.$$

7.3 Squeezing properties

In recent years, the problem of generating squeezed quantum states received a lot of attention both from the mathematical and application points of view. Most often squeezed states play an important role in electromagnetic fields. Squeezed states are non classical states for which the fluctuations in one of the two quadratures (P,Q) of the phase amplitudes of the electromagnetic field drop below the level of the field. Squeezed states therefore provide a field which is in some sense quieter than the vacuum state and hence can be employed to improve measurements beyond the standard quantum limits, a quantity set by the quantum noise,$(\Delta Q)^2$ and $(\Delta P)^2$ through the uncertainty principle.

Definition 7.3.1 *The variances of the quadrature operators Q, P in any state $\mid \psi\rangle$ satisfy the Heisenberg uncertainty relation*

$$(\Delta Q)^2(\Delta P)^2 \geq \frac{1}{4}.$$

A state $\mid \psi\rangle$ is called squeezed for the quadrature Q if $(\Delta Q)^2 < \frac{1}{2}$.

Although the Heisenberg uncertainty principle cannot be circumvented, it is possible to redistribute the fluctuations in Q and P. In this way we can have the variance of one quadrature below the standard quantum limit at the expense of excess noise in the other.

In the simplest case, take a single mode quantum field which is described by the creation and annihilation operators, $a^\dagger, a$. Quantum fluctuations, determined by the Heisenberg uncertainty principle, are represented diagrammatically in the QP plane of the quadrature components by a circle for a CS and by an ellipse (a squeezed circle) for a squeezed state.

7.3.1 Squeezing of the quaternionic VCS

As we mentioned before, we do not have a physical system for the quaternionic VCS. Thus a physical interpretation of squeezing is not possible. But mathematically we can check an analogue of the squeezing condition. From (7.10) we have

$$(\Delta Q)_1^2 = \frac{1}{4}[2r^2\cos^2\theta + 1].$$

Thus the squeezing condition, $(\Delta Q)^2 < \frac{1}{2}$ yields

$$\frac{1}{4}[2r^2\cos^2\theta + 1] < \frac{1}{2}.$$

That is

$$r^2\cos^2\theta < \frac{1}{2}.$$

We have taken $r \in [0, \infty)$ and $\theta \in [0, \pi]$ in our quaternionic VCS, therefore the squeezing condition is valid.

7.4 Time evolution

The outcomes of quantum processes are usually analyzed in terms of time independent techniques, which do not tell anything about what happens during the process. To know how the state of the system evolves in time, one has to solve the quantum mechanical equation of motion, the time-dependent Schrödinger equation. Solving a time-dependent Schrödinger equation is not easy. In fact, there are only a few oversimplified examples for which this can be done analytically. There are time consuming numerical methods available for the rest. For CS, it can be analyzed in the following manner.

For a quantum state $\mid \psi(0)\rangle$, the time evolution of the state is denoted by $\mid \psi(t)\rangle$ (the state at time t). The time-dependent Schrödinger equation (with $\hbar = 1$) is given by

$$i\frac{\partial}{\partial t} \mid \psi(t)\rangle = H \mid \psi(t)\rangle \tag{7.11}$$

where $H = \omega a^\dagger a$ (ω constant) is the free field Hamiltonian. The time-dependent state $\mid \psi(t)\rangle$ is obtained by the action of the time-evolution operator $U(t) = e^{-iHt}$ on $\mid \psi(0)\rangle$. Note that if we are given the time evolution operator, $U(t)$ and, in addition, know how $U(t)$ acts on the state $\mid \psi(0)\rangle$, it is not necessary to bother with the Schrödinger equation for the state at time t. All we have to do is to apply $U(t)$ to $\mid \psi(0)\rangle$ to get a state at time t. If the state is an eigenstate of the Hamiltonian, it remains so at all times. The most that can happen is a phase modulation. In this sense it is a constant of the motion. But if the state in question is not an eigenstate of the Hamiltonian, it is natural to analyze its time evolution. It is also of interest to answer the question: is the time evolution of a CS is again a CS that is, answering the question of temporal stability.

7.4.1 Time-evolution of quaternionic VCS

Here we study the time evolution of the quaternionic VCS of chapter 3. We use the following notations. $\mid q\rangle^j = \mid q, 0\rangle^j$ denotes the state at time $t=0$ and $\mid q,t\rangle^j$ denotes the same state at time t that is, $\mid q,t\rangle^j = U(t) \mid q, 0\rangle^j$. For these VCS the free field Hamiltonian takes the form $H = \omega A^\dagger A$, where $A^\dagger$ and A are as in chapter 3. With the details of chapter 3, it can be easily seen that

$$U(t)\chi^j \otimes \phi_m = e^{-iHt}\chi^j \otimes \phi_m = e^{-im\omega t}\chi^j \otimes \phi_m. \tag{7.12}$$

With (7.12), let us see the action of $U(t)$ on the VCS.

$$\begin{aligned}
\mid q,t\rangle^j &= U(t) \mid q, 0\rangle^j \\
&= e^{-iHt}\frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{q^m}{\sqrt{m!}}\chi^j \otimes \phi_m \\
&= \frac{1}{\sqrt{2}}e^{-\frac{r^2}{2}}\sum_{m=0}^{\infty}\frac{q^m e^{-im\omega t}}{\sqrt{m!}}\chi^j \otimes \phi_m \\
&= \frac{1}{\sqrt{2}}e^{-\frac{|q(t)|^2}{2}}\sum_{m=0}^{\infty}\frac{q(t)^m}{\sqrt{m!}}\chi^j \otimes \phi_m
\end{aligned}$$

where $q(t) = qe^{-i\omega t}$. Thus the quaternionic VCS stay as VCS after the time evolution, in other words the quaternionic VCS are temporally stable for the free field Hamiltonian.

7.5 Spin-orbit interaction and the quaternionic VCS

Here we explain the spin-orbit interaction between a spinning electron and an external magnetic field, $\overrightarrow{B}$ in terms of quaternionic VCS. Due to the spin the electron displays an intrinsic angular momentum S, thus one might expect a magnetic moment. With e the electron charge, m the electron's mass and c the velocity of light, the intrinsic magnetic moment of the electron is given by

$$\mu = -g_s \mu_b \frac{S}{\hbar} \tag{7.13}$$

where

$$\mu_b = \frac{e\hbar}{2mc} \tag{7.14}$$

is a fundamental unit of magnetic moment called the Bohr magneton. The number g_s is called the spin gyromagnetic ratio of the electron, expected from Dirac theory to be exactly 2 but known experimentally to be 2.00233....

The energy levels of the electron are affected by the interaction between the electron-spin magnetic moment and the external magnetic field. The interaction energy takes the form

$$V_{int} = \overrightarrow{\mu} \cdot \overrightarrow{B} = k \overrightarrow{S} \cdot \overrightarrow{B},$$

where k is a constant determined by (7.13) and (7.14). Thus the interaction part of the Hamiltonian for the system is $\mathcal{H} = V_{int}$ (the total Hamiltonian takes the form $\mathcal{H} = \mathcal{H}_0 + V_{int}$, where $\mathcal{H}_0$ is the free Hamiltonian). Now the time evolution operator for the electron is given by $U(t) = e^{-it\mathcal{H}}$. From equation (3.1), the quaternions can be written as $q = r\mathbb{I}_2 e^{i\theta\sigma(\hat{n})}$. $\sigma(\hat{n})$ can be written as

$$\sigma(\hat{n}) = (-\sin\phi\cos\psi, \sin\phi\sin\psi, -\cos\phi)\cdot(\sigma_1, \sigma_2, \sigma_3) = \hat{n}\cdot\hat{\sigma}.$$

It is well known that the spin, $\overrightarrow{S}$ and $\hat{\sigma}$ are proportional, that is $\overrightarrow{S} = \frac{\hbar}{2}\hat{\sigma}$. Suppose the magnetic field is $\overrightarrow{B} = \hat{n}$. Thus the time evolution operator of the electron can be written as

$$U(t) = e^{-it\mathcal{H}} = e^{ic\theta\hat{n}\cdot\hat{\sigma}}$$

with an interpretation that $-it = ic\theta$. In this interpretation the quaternion q takes the form $q = r\mathbb{I}_2 e^{-\frac{it}{c}\mathcal{H}}$ and the quaternionic states take the form

$$|\,q, j\rangle = \frac{1}{\sqrt{2}} e^{\frac{-r^2}{2}} \sum_{m=0}^{\infty} \frac{r^m e^{-\frac{imt}{c}\mathcal{H}}}{\sqrt{m!}} \chi^j \otimes \phi_m. \tag{7.15}$$

The states in (7.15) indicate a connection between the spin-orbit interaction energy and the quaternionic VCS. A proper explanation has yet to be worked out.

Chapter 8

Vector Coherent States with an Unbounded frame operator

In this chapter we present VCS in the domain $D \times D \times .... \times D$ (n-copies), where D is the complex unit disc, using a highly symmetric hermitian matrix. Further, as an example, we build vector coherent states in the unit disc by considering the unit disc as a homogeneous space of the group $SU(1,1)$. This example is much different from the example 5.2.5 because in example 5.2.5 the range of the parameter r is $[0,\infty)$ and the matrix was taken in a form similar to (1.1). Further, the matrix in example 5.2.5 cannot be considered as a matrix from the homogeneous space of the group $SU(1,1)$ and the parameters do not take range in the unit disc. The VCS construction of this chapter is completely different from the construction of chapter 1 because here we do not consider the matrix in the form (1.1) and the matrix of this chapter cannot be written in the form (1.1) with the required conditions. The example of this chapter can be considered as a preliminary step of building VCS on the classical domains. For the VCS we obtain the resolution of the identity on a dense subset of the homogeneous space. For the remaining part of the homogeneous space we give an approximate resolution of the identity through a limit process. The unboundedness arises as a consequence of the resolution of the identity. At the end of this chapter we discuss whether the difficulty faced in obtaining a perfect resolution of identity can be avoided or not.

8.1 Vector coherent states

Let D be the complex unit disc, $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For $z_1, z_2, ..., z_{n-1} \in D$, let $Z = (z_1, z_2, ..., z_{n-1})$. Define the $n \times n$ hermitian matrix, $\mathcal{Z}$ as

$$\mathcal{Z} = \mathbb{I}_n + \begin{pmatrix} 0 & Z \\ \overline{Z}^T & 0 \end{pmatrix}_{n\times n} \tag{8.1}$$

Let $\mathbb{H}$ be an abstract Hilbert space and $\{\phi_n\}$ be an orthonormal basis of it. Further,

$$\begin{aligned} x^{(1)} &= \begin{pmatrix} 1 & 0 & 0 & . & . & . & . & 0 \end{pmatrix}^T_{n\times 1}, \\ x^{(2)} &= \begin{pmatrix} 0 & 1 & 0 & . & . & . & . & 0 \end{pmatrix}^T_{n\times 1}, \\ \cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^{(n)} &= \begin{pmatrix} 0 & 0 & 0 & . & . & . & . & 1 \end{pmatrix}^T_{n\times 1} \end{aligned}$$

is the natural basis of $\mathbb{C}^{2n}$. We form a new Hilbert space $\widetilde{\mathbb{H}} = \mathbb{C}^{2n} \otimes \mathbb{H}$, where $\otimes$

is the tensor product. set

$$\begin{aligned}
\Phi_{1m} &= \begin{pmatrix} \psi_m & 0 & 0 & . & . & . & . & 0 \end{pmatrix}^T_{n\times 1}, \\
\Phi_{1m} &= \begin{pmatrix} 0 & \psi_m & 0 & . & . & . & . & 0 \end{pmatrix}^T_{n\times 1}, \\
\ldots\ldots & \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Phi_{1m} &= \begin{pmatrix} 0 & 0 & 0 & . & . & . & . & \psi_m \end{pmatrix}^T_{n\times 1}
\end{aligned}$$

where $\psi_m = \frac{\phi_m}{m+1}$. The set, $\{\Phi_{jm}\}, j = 1...n, m = 0...\infty$, is a basis of $\widetilde{\mathbb{H}}$. With the above set up we form the set of coherent states in $L^2(D \times D \times ... \times D, d\mu)$, where we take

$$d\mu = r_1 r_2 ... r_{n-1} dr_1 d\theta_1 dr_2 d\theta_2 ... dr_{n-1} d\theta_{n-1} \tag{8.2}$$

by taking $z_j = r_j e^{i\theta_j}, j = 1, 2, ..n-1$,as

$$|\mathcal{Z}\rangle^j = \mathcal{N}(|\mathcal{Z}|) \sum_{m=0}^{\infty} R(m) \mathcal{Z}^m \Phi_{jm}, \qquad j = 1, 2, ..., n \tag{8.3}$$

The number $\mathcal{N}$ and the $n\times n$ matrix $R(m)$ have to be chosen suitably. First of all let us calculate $\mathcal{Z}^m$ by diagonalizing the matrix $\mathcal{Z}$. The eigenvalues of $\mathcal{Z}$ are $1, 1+a$ and $1-a$ with multiplicities $n-2, 1$ and 1 respectively, where $a = \|Z\| = \sqrt{r_1^2 + r_2^2 + ... + r_{n-1}^2}$. A set of orthogonal normalized eigenvectors corresponding to the eigen value, 1 are

$$\begin{aligned}
V_1 &= \frac{1}{\sqrt{b_1}} \begin{pmatrix} 0 & -z_2 & z_1 & 0 & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix} \\
V_2 &= \frac{\sqrt{b_1}}{r_1\sqrt{b_2}} \begin{pmatrix} 0 & -\frac{r_1^2}{b_1} z_3 & -\frac{1}{b_1} z_1 \overline{z_2} z_3 & z_1 & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix}
\end{aligned}$$

$$V_3 = \frac{\sqrt{b_2}}{r_1\sqrt{b_3}}\begin{pmatrix} 0 & -\frac{r_1^2}{b_2}z_4 & -\frac{1}{b_2}z_1\overline{z_2}z_4 & -\frac{1}{b_2}z_1\overline{z_3}z_4 & z_1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$V_4 = \frac{\sqrt{b_3}}{r_1\sqrt{b_4}}\begin{pmatrix} 0 & \frac{-r_1^2z_5}{b_3} & \frac{-z_1\overline{z_2}z_5}{b_3} & \frac{-z_1\overline{z_3}z_5}{b_3} & \frac{-z_1\overline{z_4}z_5}{b_3} & z_1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$V_5 = \frac{\sqrt{b_4}}{r_1\sqrt{b_5}}\begin{pmatrix} 0 & \frac{-r_1^2z_6}{b_4} & \frac{-z_1\overline{z_2}z_6}{b_4} & \frac{-z_1\overline{z_3}z_6}{b_4} & \frac{-z_1\overline{z_4}z_6}{b_4} & \frac{-z_1\overline{z_5}z_6}{b_4} & z_1 & 0 & \dots & 0 \end{pmatrix}$$

....

$$V_{n-2} = \frac{\sqrt{b_{n-4}}}{r_1\sqrt{b_{n-3}}}\begin{pmatrix} 0 & \frac{-r_1^2z_{n-1}}{b_{n-3}} & \frac{-z_1\overline{z_2}z_{n-1}}{b_{n-3}} & \frac{-z_1\overline{z_3}z_{n-1}}{b_{n-3}} & \cdot\ \cdot\ \cdot & \frac{-z_1\overline{z_{n-2}}z_{n-1}}{b_{n-3}} & z_1 \end{pmatrix}$$

where $b_j = r_1^2 + r_2^2 + \dots r_{j+1}^2$, $j = 1, 2 \dots n-3$. Eigenvectors corresponding to the eigenvalues $1+a$ and $1-a$ are

$$V_{n-1} = \frac{1}{\sqrt{2}a}\begin{pmatrix} -a & \overline{z_1} & \overline{z_2} & \dots & \overline{z_{n-1}} \end{pmatrix}$$

$$V_n = \frac{1}{\sqrt{2}a}\begin{pmatrix} a & \overline{z_1} & \overline{z_2} & \dots & \overline{z_{n-1}} \end{pmatrix}$$

respectively. The set $\{V_1, V_2, \dots, V_n\}$ is an orthonormal set with unit vectors. We form a matrix P by placing V_js as columns, i.e,

$$P = \begin{pmatrix} V_1^T & V_2^T & \dots & V_n^T \end{pmatrix}.$$

Then $P^T\mathcal{Z}P = D$, a diagonal matrix with the eigenvalues over the diagonal. Thus we obtain $\mathcal{Z}^m = PD^mP^\dagger = (c_{lj})_{n\times n}$, where

$$c_{11} = E_m$$

$$c_{l1} = \frac{O_m\overline{z_{l-1}}}{a}, \quad l = 2, 3, \dots, n$$

$$
\begin{aligned}
c_{1j} &= \frac{O_m z_{j-1}}{a}, \; j = 2,3,...,n \\
c_{ll} &= \frac{r_{l-1}^2 E_m + a^2 - r_{l-1}^2}{a^2}, \; l = 2,3,..,n \\
c_{lj} &= \frac{z_{j-1}\overline{z_{l-1}}}{a^2}(E_m - 1), \; l \neq j, l \neq 1, j \neq 1
\end{aligned}
$$

with

$$
\begin{aligned}
E_m &= \frac{1}{2}[(1+a)^m + (1-a)^m] && (8.4) \\
O_m &= \frac{1}{2}[(1+a)^m - (1-a)^m]. && (8.5)
\end{aligned}
$$

Further notice that $(\mathcal{Z}^m)^\dagger = \mathcal{Z}^m$. Before we choose $\mathcal{N}$ and $R(m)$ suitably, let us write $R(m)$ tentatively as $R(m) = (\alpha_{lj})$ where

$$
\alpha_{11} = R_m e^{im\theta}, \;\; \alpha_{ll} = S_m e^{im\theta}, l \neq 1, \;\; \alpha_{lj} = 0, l \neq j
$$

with

$$
\theta = \theta_1 + \theta_2 + ... + \theta_{n-1}.
$$

Now we are in a position to calculate

$$
\begin{aligned}
|\mathcal{Z}\rangle^{11}\langle\mathcal{Z}| &= \mathcal{N}^2 \mid \sum_{m=0}^{\infty} R(m)\mathcal{Z}^m\Phi_{1m}\rangle\langle\sum_{k=0}^{\infty} R(k)\mathcal{Z}^k\Phi_{1k} \mid \\
&= \mathcal{N}^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \mid R(m)\mathcal{Z}^m\Phi_{1m}\rangle\langle R(k)\mathcal{Z}^k\Phi_{1k} \mid \\
&= \mathcal{N}^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} R(m)\mathcal{Z}^m\Omega_1 \left(\mathcal{Z}^k\right)^\dagger (R(k))^\dagger \mid \psi_m\rangle\langle\psi_k \mid
\end{aligned}
$$

where $\Omega_1 = \left(s_{lj}^{(1)}\right)_{n\times n}$ with $s_{11}^{(1)} = 1$ and $s_{lj}^{(1)} = 0$ for $l \neq 1, j \neq 1$. The matrix multiplication yeilds,

$$
R(m)\mathcal{Z}^m\Omega_1 \left(\mathcal{Z}^k\right)^\dagger (R(k))^\dagger = \left(f_{lj}^{(1)}\right)_{n\times n}
$$

where

$$
\begin{aligned}
f_{11}^{(1)} &= R_m R_k E_m E_k e^{i\theta(m-k)} \\
f_{ll}^{(1)} &= S_m S_k O_m O_k \frac{r_{l-1}^2}{a^2} e^{i\theta(m-k)}, \quad l = 2, 3, ..., n \\
f_{1l}^{(1)} &= R_k E_k S_m O_m \frac{r_{l-1}}{a} e^{i(m\theta - k\theta + \theta_{l-1})}, \quad l = 2, 3, ..., n \\
f_{l1}^{(1)} &= R_k E_k S_m O_m \frac{r_{l-1}}{a} e^{i(m\theta - k\theta - \theta_{l-1})}, \quad l = 2, 3, ..., n \\
f_{lj}^{(1)} &= S_m S_k O_m O_k r_{j-1} r_{l-1} e^{i(m\theta - k\theta + \theta_{j-1} - \theta_{l-1})}, \quad j, l = 2, 3, ..., n; j \neq l.
\end{aligned}
$$

Thus we obtain

$$
|\mathcal{Z}\rangle^{11}\langle\mathcal{Z}| = \mathcal{N}^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mid \psi_m\rangle\langle\psi_k \mid \left(f_{lj}^{(1)}\right)_{n\times n}. \tag{8.6}
$$

Similarly, in order to obtain $|\mathcal{Z}\rangle^{22}\langle\mathcal{Z}|$ we calculate

$$
R(m)\mathcal{Z}^m \Omega_2 \left(\mathcal{Z}^k\right)^\dagger (R(k))^\dagger = \left(f_{lj}^{(2)}\right)_{n\times n}
$$

where

$$
\begin{aligned}
f_{11}^{(2)} &= R_m R_k O_m O_k \frac{r_1^2}{a^2} e^{i\theta(m-k)} \\
f_{22}^{(2)} &= S_m S_k \frac{(r_1^2 E_m + a^2 - r_1^2)}{a^4} (r_1^2 E_k + a^2 - r_1^2) e^{i\theta(m-k)} \\
f_{ll}^{(2)} &= S_k S_m \frac{(E_m - 1)(E_k - 1)}{a^4} r_1^2 r_{l-1}^2 e^{i\theta(m-k)}, \quad l = 3, 4, ..., n \\
f_{lj}^{(2)} &= A_{ljmk}^{(2)} e^{i(m\theta - k\theta - \theta_{l-1})}, \quad j = 1, l = 2, 3, ..., n \\
f_{lj}^{(2)} &= B_{ljmk}^{(2)} e^{i(m\theta - k\theta + \theta_{j-1})}, \quad l = 1, j = 2, 3, ..., n; j \neq l. \\
f_{lj}^{(2)} &= C_{ljmk}^{(2)} e^{i(m\theta - k\theta + \theta_{j-1} - \theta_{l-1})}, \quad \text{otherwise}
\end{aligned}
$$

Where $A^{(2)}_{ljmk}, B^{(2)}_{ljmk}$ and $C^{(2)}_{ljmk}$ are some functions of m, k, r_l, and r_j (exact expressions are irrelevant for our purpose). Thus we obtain

$$|\mathcal{Z}\rangle^{22}\langle\mathcal{Z}| = \mathcal{N}^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \mid \psi_m\rangle\langle\psi_k \mid \left(f^{(2)}_{lj}\right)_{n\times n}. \tag{8.7}$$

In general, in order to get $|\mathcal{Z}\rangle^{qq}\langle\mathcal{Z}|$, we calculate

$$R(m)\mathcal{Z}^m\Omega_q\left(\mathcal{Z}^k\right)^{\dagger}(R(k))^{\dagger} = \left(f^{(q)}_{lj}\right)_{n\times n}$$

where

$$\begin{aligned}
f^{(q)}_{11} &= R_mR_kO_mO_k\frac{r^2_{q-1}}{a^2}e^{i\theta(m-k)}\\
f^{(q)}_{22} &= S_mS_k\frac{(r^2_{q-1}E_m + a^2 - r^2_{q-1})}{a^4}(r^2_{q-1}E_k + a^2 - r^2_{q-1})e^{i\theta(m-k)}\\
f^{(q)}_{ll} &= S_kS_m\frac{(E_m-1)(E_k-1)}{a^4}r^2_{q-1}r^2_{l-1}e^{i\theta(m-k)}, \quad l\neq q, l = 2,4,...,n\\
f^{(q)}_{lj} &= A^{(q)}_{ljmk}e^{i(m\theta-k\theta-\theta_{l-1})}, \quad j=1, l=2,3,...,n\\
f^{(q)}_{lj} &= B^{(q)}_{ljmk}e^{i(m\theta-k\theta+\theta_{j-1})}, \quad l=1, j=2,3,...,n; j\neq l.\\
f^{(q)}_{lj} &= C^{(q)}_{ljmk}e^{i(m\theta-k\theta+\theta_{j-1}-\theta_{l-1})}, \quad \text{otherwise}
\end{aligned}$$

Thus we obtain

$$|\mathcal{Z}\rangle^{qq}\langle\mathcal{Z}| = \mathcal{N}^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \mid \psi_m\rangle\langle\psi_k \mid \left(f^{(q)}_{lj}\right)_{n\times n}. \tag{8.8}$$

Now, when we compute

$$\sum_{q=1}^{n}\int_{D\times...\times D} \mid \mathcal{Z}\rangle^{qq}\langle\mathcal{Z} \mid d\mu.$$

All the off-diagonal terms vanish with one of the $\theta_1, \theta_2, ..., \theta_{n-1}$ integrals and only in the diagonals terms with $m = k$ survive. Thus we obtain

$$\sum_{q=1}^{n}\int_{D\times...\times D} \mid \mathcal{Z}\rangle^{qq}\langle\mathcal{Z} \mid d\mu = (D_{lj})_{n\times n}$$

where

$$\begin{aligned}
D_{11} &= \sum_{q=1}^{n} f_{11}^q = \sum_{m=0}^{\infty} \int_{D\times\ldots\times D} \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) d\mu \mid \psi_m\rangle\langle\psi_m \mid \\
D_{ll} &= \sum_{q=1}^{n} f_{ll}^q \\
&= \sum_{m=0}^{\infty} \int_{D\times\ldots\times D} \mathcal{N}^2 S_m^2 \frac{r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2}{a^2} d\mu \mid \psi_m\rangle\langle\psi_m \mid, \quad l = 1, 2, \ldots, n \\
D_{lj} &= 0 \quad \text{otherwise.}
\end{aligned}$$

Choose

$$\mathcal{N} = \frac{a}{\sqrt{(2\pi)^{n-1}}}. \tag{8.9}$$

Then

$$D_{11} = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \int_0^1 \ldots \int_0^1 a^2 R_m^2 (E_m^2 + O_m^2) dr_1 \ldots dr_{n-1}$$

and

$$D_{ll} = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \int_0^1 \ldots \int_0^1 S_m^2 (r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2) dr_1 \ldots dr_{n-1}, \quad l = 2, 3, \ldots n.$$

The following integrals are finite and positive

$$\begin{aligned}
N_{1m} &= \int_0^1 \ldots \int_0^1 a^2 R_m^2 (E_m^2 + O_m^2) dr_1 \ldots dr_{n-1} \\
N_{2m} &= \int_0^1 \ldots \int_0^1 S_m^2 (r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2) dr_1 \ldots dr_{n-1}.
\end{aligned}$$

Further $N_{1m} = (n-1)N_{2m} - \frac{1}{4}(n-2)$. Now choose

$$R_m = \frac{1}{\sqrt{N_{1m}}} \text{ and } S_m = \frac{1}{\sqrt{N_{2m}}}. \tag{8.10}$$

Then we obtain

$$\sum_{q=1}^{n} \int_{D\times\ldots\times D} \mid \mathcal{Z}\rangle^{qq}\langle \mathcal{Z} \mid d\mu = \sum_{m=1}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \mathbb{I}_n.$$

Now let

$$T = \sum_{m=1}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \mathbb{I}_n, \tag{8.11}$$

which is a bounded invertible operator with $KerT = \{0\}$, for

$$KerT = \{\Phi \in \tilde{\mathbb{H}} \mid T\Phi = 0\}.$$

Every vector Φ in $\tilde{\mathbb{H}}$ can be written as

$$\Phi = \left(\sum_{k=1}^{\infty} \alpha_{k1}\phi_k, \sum_{k=1}^{\infty} \alpha_{k2}\phi_k, \ldots. \sum_{k=1}^{\infty} \alpha_{kn}\phi_k\right)^T$$

Thus $T\Phi = 0$ gives

$$diag\left(\sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \mid \phi_m\rangle\langle\phi_m \mid, \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \mid \phi_m\rangle\langle\phi_m \mid, \ldots. \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \mid \phi_m\rangle\langle\phi_m \mid\right) \cdot$$

$$\left(\sum_{k=1}^{\infty} \alpha_{k1}\phi_k, \sum_{k=1}^{\infty} \alpha_{k2}\phi_k, \ldots. \sum_{k=1}^{\infty} \alpha_{kn}\phi_k\right)^T = (0, 0, \ldots.0)$$

Hence, for each i, we get

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \alpha_{mi}\phi_m = 0.$$

With the fact that $\{\phi_m\}$ is an orthonormal basis, for each m and for each i, we have $\alpha_{mi} = 0$. Thus we have $KerT = \{0\}$. The inverse of T exists and the domain of T^{-1}, $\mathcal{D}(T^{-1})$ is dense in $\tilde{\mathbb{H}}$. For $\phi \in \mathcal{D}(T^{-1})$, consider

$$T(T^{-1}\phi) = \left[\sum_{q=1}^{n} \int_D \mid \mathcal{Z}\rangle^{qq}\langle \mathcal{Z} \mid d\mu\right] T^{-1}\phi.$$

That is

$$\phi = \sum_{q=1}^{n} \int_D \mid \mathcal{Z}\rangle^{qq} \langle \mathcal{Z} \mid T^{-1}\phi\rangle d\mu.$$

Thus, for the vectors in $\mathcal{D}(T^{-1})$ we have a proper decomposition in the above sense. Now let $\phi \in \tilde{\mathbb{H}}$, then there exists a sequence $\{\phi_m\} \subset \mathcal{D}(T^{-1})$ such that $\phi_m \to \phi$ as $m \to \infty$ in $\tilde{\mathbb{H}}$. Further for each ϕ_m we have,

$$\phi_m = \sum_{q=1}^{n} \int_D \mid \mathcal{Z}\rangle^{qq} \langle \mathcal{Z} \mid T^{-1}\phi_m\rangle d\mu.$$

Now by taking limit both sides as $m \to \infty$, we can have

$$\phi = \lim_{m\to\infty} \left(\sum_{q=1}^{n} \int_D \mid \mathcal{Z}\rangle^{qq} \langle \mathcal{Z} \mid T^{-1}\phi_m\rangle d\mu \right).$$

In this regard, we have a week decomposition for the vectors in $\tilde{\mathbb{H}} - \mathcal{D}(T^{-1})$. The continuity in the label follows from the convergence of the defining series.

8.2 Example: Vector coherent states with SU(1,1)

In this section we build vector coherent states arising from the $SU(1,1)$ group. In order to introduce the concept we need the following preliminaries.

The non-compact group $SU(1,1)$ is defined as,

$$SU(1,1) = \left\{ g | g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} ; \alpha, \beta \in \mathbb{C}, \det g = |\alpha|^2 - |\beta|^2 = 1 \right\}$$

and its maximal compact subgroup K is given by

$$K = \left\{ k | k = \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{-i\phi}{2}} \end{pmatrix} ; 0 \leq \phi \leq 2\pi \right\}.$$

The Cartan decomposition of an arbitrary $g \in SU(1,1)$ is,

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = |\alpha| \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix}$$

where $z = \beta\alpha^{-1}$ and $|\alpha| = (1 - |z|^2)^{-\frac{1}{2}}$.

Thus the coset space $SU(1,1)/K$ can be identified with the unit disc $D = \{z \in \mathbb{C} | |z| < 1\}$. Further it is known that the measure

$$d\nu(z, \overline{z}) = \frac{1}{2\pi i} \frac{dz \wedge d\overline{z}}{(1 - |z|^2)^2}$$

on D is invariant under the action of $SU(1,1)$. In polar coordinates

$$\mathcal{Z} = \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} = \begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & 1 \end{pmatrix}$$

$$d\nu(z, \overline{z}) = d\nu(r, \theta) = \frac{rdrd\theta}{\pi(1 - r^2)^2}$$

where $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$. Let $\mathbb{H}$ be an abstract Hilbert space and $\{\phi_n\}$ be an orthonormal basis of it and

$$x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; x^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the natural basis of $\mathbb{C}^2$. Now we form a new Hilbert space $\widetilde{\mathbb{H}} = \mathbb{C}^2 \otimes \mathbb{H}$, where $\otimes$ is the tensor product. Set

$$\Phi_{1m} = \begin{pmatrix} \psi_m \\ 0 \end{pmatrix}; \Phi_{2m} = \begin{pmatrix} 0 \\ \psi_m \end{pmatrix}$$

where $\psi_m = \frac{\phi_m}{m+1}$. The set $\{\Phi_{jm}\}$ is a basis of $\widetilde{\mathbb{H}}$. With the above set up we form the set of coherent states in $L^2(D, d\mu)$ as,

$$|\mathcal{Z}\rangle^j = \mathcal{N}(|\mathcal{Z}|)^{\frac{1}{2}} \sum_{m=0}^{\infty} R_m \mathcal{Z}^m \Phi_{jm}, \qquad j = 1, 2$$

by suitably choosing the number $\mathcal{N}(|\mathcal{Z}|)$ and a 2×2 matrix $R(m)$.

As we did before through diagonalization we obtain

$$\mathcal{Z}^m = \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{i\theta} & E_m \end{pmatrix}$$

with

$$\begin{aligned} E_m &= \frac{1}{2}[(1+r)^m + (1-r)^m] \\ O_m &= \frac{1}{2}[(1+r)^m - (1-r)^m]. \end{aligned}$$

Again as we did before let us take

$$R(m) = \begin{pmatrix} R_m e^{im\theta} & 0 \\ 0 & R_m e^{im\theta} \end{pmatrix}.$$

Now with the above choices, matrix multiplication yields

$$| \mathcal{Z}\rangle^1 = \mathcal{N} \sum_{m=0}^{\infty} \begin{pmatrix} R_m e^{im\theta} E_m \psi_m \\ R_m e^{im\theta} O_m \psi_m \end{pmatrix} \tag{8.12}$$

$$| \mathcal{Z}\rangle^2 = \mathcal{N} \sum_{m=0}^{\infty} \begin{pmatrix} R_m e^{im\theta} O_m \psi_m \\ R_m e^{im\theta} E_m \psi_m \end{pmatrix} \tag{8.13}$$

First let us calculate the following finite integrals

$$
\begin{aligned}
\int_0^1 E_m^2 r dr &= \int_0^1 \frac{1}{4}(1+r)^{2m} + \frac{1}{2}(1-r^2)^m + \frac{1}{4}(1-r)^{2m} r dr \\
&= \frac{1}{4}\left(\sum_{j=0}^{2m}\begin{pmatrix}2m\\ j\end{pmatrix}\frac{1}{j+2} + \sum_{j=0}^{2m}\begin{pmatrix}2m\\ j\end{pmatrix}\frac{(-1)^j}{j+2}\right) + \frac{1}{4(m+1)} \\
&= \frac{1}{2}\frac{4^m+1+m}{(2m+1)(m+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 O_m^2 r dr &= \int_0^1 \frac{1}{4}(1+r)^{2m} - \frac{1}{2}(1-r^2)^m + \frac{1}{4}(1-r)^{2m} r dr \\
&= \frac{1}{4}\left(\sum_{j=0}^{2m}\begin{pmatrix}2m\\ j\end{pmatrix}\frac{1}{j+2} + \sum_{j=0}^{2m}\begin{pmatrix}2m\\ j\end{pmatrix}\frac{(-1)^j}{j+2}\right) - \frac{1}{4(m+1)} \\
&= \frac{m}{2}\frac{4^m-1}{(2m+1)(m+1)}.
\end{aligned}
$$

With the above two integrals we can also have

$$
\int_0^1 (E_m^2 + O_m^2) r dr = \frac{2\times 4^m m + 1}{2(2m+1)(m+1)}.
$$

Further notice that $\langle \psi_m \mid \psi_m \rangle = \frac{1}{(1+m)^2}$. Next simple calculations yields

$$
{}^1\langle \mathcal{Z} \mid \mathcal{Z} \rangle^1 = \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) \frac{r d\theta dr}{\pi(1+r^2)^2} \tag{8.14}
$$

$$
{}^2\langle \mathcal{Z} \mid \mathcal{Z} \rangle^2 = \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) \frac{r d\theta dr}{\pi(1+r^2)^2}. \tag{8.15}
$$

Now let us choose

$$
\mathcal{N} = \frac{\sqrt{6}}{\pi}(1+r^2) \tag{8.16}
$$

$$
R_m = \sqrt{\frac{(2m+1)(m+1)}{2\times 4^m m + 1}}. \tag{8.17}
$$

With these choices we have

$$
\begin{aligned}
{}^1\langle \mathcal{Z} \mid \mathcal{Z}\rangle^1 =^2 \langle \mathcal{Z} \mid \mathcal{Z}\rangle^2 &= \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 \frac{6}{\pi^2} \frac{(2m+1)(m+1)}{2 \times 4^m m + 1} (E_m^2 + O_m^2) r d\theta dr \\
&= \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \frac{6}{\pi^2} \\
&= 1
\end{aligned}
$$

Let us turn our attention to the resolution of identity by calculating

$$
\begin{aligned}
\mid \mathcal{Z}\rangle^{11}\langle \mathcal{Z} \mid &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{N}^2 \mid R_m \mathcal{Z}^m \Phi_{1m}\rangle\langle R_k \mathcal{Z}^k \Phi_{1k} \mid \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{N}^2 R_m \mathcal{Z}^m \mid \Phi_{1m}\rangle\langle \Phi_{1k} \mid \overline{R_k \mathcal{Z}^k}^T \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{N}^2 \begin{pmatrix} R_m e^{im\theta} & 0 \\ 0 & R_m e^{im\theta} \end{pmatrix} \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{-i\theta} & E_m \end{pmatrix} \times \\
&\qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{-i\theta} & E_m \end{pmatrix} \begin{pmatrix} R_m e^{-im\theta} & 0 \\ 0 & R_m e^{-im\theta} \end{pmatrix} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{N}^2 \begin{pmatrix} R_m R_k E_m E_k e^{i\theta(m-k)} & R_m R_k E_m O_k e^{i\theta(m+k+1)} \\ R_m R_k O_m E_k e^{-i\theta(m+k+1)} & R_m R_k O_m O_k e^{i\theta(m-k)} \end{pmatrix}
\end{aligned}
$$

Thus we get

$$
\int_D W \mid \mathcal{Z}\rangle^{11}\langle \mathcal{Z} \mid d\mu = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
$$

where

$$
\begin{aligned}
A &= \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle \psi_m \mid \int_0^{2\pi} \int_0^1 \mathcal{N}^2 R_m^2 E_m^2 W \frac{r d\theta dr}{\pi(1+r^2)^2} && (8.18) \\
B &= \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle \psi_m \mid \int_0^{2\pi} \int_0^1 \mathcal{N}^2 R_m^2 O_m^2 W \frac{r d\theta dr}{\pi(1+r^2)^2}. && (8.19)
\end{aligned}
$$

Similarly we get

$$\int_D W \mid \mathcal{Z}\rangle^{22}\langle \mathcal{Z} \mid d\mu = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}.$$

With all these we have

$$\int_D W \mid \mathcal{Z}\rangle^{11}\langle \mathcal{Z} \mid d\mu + \int_D W \mid \mathcal{Z}\rangle^{22}\langle \mathcal{Z} \mid d\mu = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix}. \tag{8.20}$$

Now let us calculate $A+B$

$$A+B = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) W \frac{r d\theta dr}{\pi(1+r^2)^2}$$

In order to get

$$\int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) W \frac{r d\theta dr}{\pi(1+r^2)^2} = 1$$

let us choose

$$W = \frac{\pi^2}{6}.$$

With this choice we have

$$\begin{aligned} & \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) W \frac{r d\theta dr}{\pi(1+r^2)^2} \\ = & \int_0^{2\pi}\int_0^1 \frac{6(1+r^2)^2}{\pi^2} \frac{(2m+1)(m+1)}{2\times 4^m m+1} (E_m^2 + O_m^2) \frac{\pi^2}{6} \frac{r d\theta dr}{\pi(1+r^2)^2} \\ = & \frac{2\times 4^m m+1}{2(2m+1)(m+1)} \frac{6}{\pi^2} \frac{(2m+1)(m+1)}{2\times 4^m m+1} \frac{\pi^2}{6} 2\pi \\ = & \ 1 \end{aligned}$$

Now we have

$$A+B = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \tag{8.21}$$

which yields

$$\int_D W \mid \mathcal{Z}\rangle^{11}\langle \mathcal{Z} \mid d\mu + \int_D W \mid \mathcal{Z}\rangle^{22}\langle \mathcal{Z} \mid d\mu = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \mathbb{I}_2.$$

Let

$$T = \sum_{m=0}^{\infty} \mid \psi_m\rangle\langle\psi_m \mid \mathbb{I}_2.$$

Thus we have an operator T as in equation (8.11) with $n = 2$. The decomposition of any vector ϕ in $\widetilde{\mathbb{H}}$ follows by replacing $n = 2$ in the discussion which we have right after equation (8.11).

8.3 Remarks and Discussions

In this section we will discuss some other possibilities of our choices over the construction and resulting difficulties.

- We have chosen a basis of the Hilbert space in an unusual way as $\psi_m = \frac{\phi_m}{m+1}$. We explain the unavoidability of this choice using the $SU(1,1)$ example. Instead of this choice, if we just take ϕ_m, in equations (8.12) and (8.13) we will have ϕ_m instead ψ_m, which will change equations (8.14) and (8.15) as

$$^1\langle \mathcal{Z} \mid \mathcal{Z}\rangle^1 = \sum_{m=0}^{\infty} \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2)\frac{rd\theta dr}{\pi(1+r^2)^2} \qquad (8.22)$$

$$^2\langle \mathcal{Z} \mid \mathcal{Z}\rangle^2 = \sum_{m=0}^{\infty} \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2)\frac{rd\theta dr}{\pi(1+r^2)^2}. \qquad (8.23)$$

Further, in equation (8.20) ψ_m will be replaced by ϕ_m, i.e,

$$A + B = \sum_{m=0}^{\infty} \mid \phi_m\rangle\langle\phi_m \mid \int_0^{2\pi}\int_0^1 \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) W \frac{rd\theta dr}{\pi(1+r^2)^2}. \qquad (8.24)$$

So in (8.24) by choosing

$$\begin{aligned} \mathcal{N} &= \frac{(1+r^2)}{\sqrt{2}} \\ R_m &= \frac{1}{\sqrt{N_m}} \qquad \text{and} \\ W &= 1 \end{aligned}$$

with

$$N_m = \int_0^1 r(E_m^2 + O_m^2)dr = \frac{2 \times 4^m m + 1}{2(2m+1)(m+1)}$$

we can have the resolution of identity. But these choices will make the series in (8.22) and (8.23) diverge. In order to have the convergence in (8.22) and (8.23), at least we have to have

$$R_m^2 \cdot \frac{2 \times 4^m m + 1}{2(2m+1)(m+1)} \sim \frac{1}{m^p}$$

with $p > 1$. If we make such a choice for R_m, our moment problem will be

$$\int_0^1 R_m^2(E_m^2 + O_m^2)Wrdr = 1$$

or

$$\int_0^1 (E_m^2 + O_m^2)Wrdr = \frac{2 \times 4^m m + 1}{2(2m+1)(m+1)} m^p.$$

We have experienced difficulty in solving this moment problem by taking W as a function of r only, but it can be solved if we take $W = f(r, m)$. Generally, as a weight function, dependence of W on m is not allowed. This stems from the choice $\psi_m = \frac{\phi_m}{m+1}$. Notice also that ψ_m can be chosen in many ways in a similar fashion.

- We have obtained proper decomposition for the vectors in our intended space up to a dense subset. The effort of getting the decomposition for the remaining nowhere dense set is limited by the unboundedness of the inverse operator T^{-1}. In this case, we have an approximation for the decomposition.

- In equation (8.2), we take the measure $d\mu$ on $D \times D \times ... \times D$ as

$$r_1 r_2 ... r_{n-1} dr_1 d\theta_1 ... dr_{n-1} d\theta_{n-1}$$

(the usual Lebesgue measure). In general, the measure $d\mu$ comes with a weight $\rho(r_1, r_2 ... r_{n-1}, \theta_1, ... \theta_{n-1})$, in such a case we can always replace $\mathcal{N}$ by

$$\frac{\mathcal{N}}{\rho(r_1, r_2 ... r_{n-1}, \theta_1, ... \theta_{n-1})}$$

to reduce the problem to our case. In this sense, even for a different choice of measure on the domain, our argument remains the same as long as the measure $d\mu$ keeps the form $d\mu = \rho d\lambda$, where $d\lambda$ is the Lebesgue measure.

Further, as long as N_{1m} and N_{2m} remain finite, a different choice of domain (here we mean a larger or smaller disc) will not change our construction unless we make a different choice for R_m and S_m.

- Our $SU(1,1)$ example is a preliminary step of constructing vector coherent states on classical domains in this way. The construction directly depends on the explicit form of the elements of the classical groups. As we know, many of the matrix groups consist of elements with an unpleasant explicit form in calculations. Further, in our case, we have real eigenvalues for the matrix, which made our construction easier. In this regard, construction of coherent states on other classical domains needs little more effort. For example, matrices of $SU(2,1)$ or $SU(2,2)$ cannot be considered in an exact similar way as we did. One can also consider a similar approach to other matrix groups.

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