Percentile Pension Cost Methods with Random Retirement Age

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Abstract

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A new family of methods for pension valuation is studied; Ramsay (1993) who originally proposed them called them percentile cost methods. These are compared to traditional cost methods and their differences are discussed. A numerical illustration is presented.

In addition, this thesis models the retirement age as a random variable. The traditional and percentile cost methods are redefined under this more general random context.

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To my parents

Rosa María and Carlos Luis

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Special Symbols

In percentile methods, the asterisk in some symbols must be replaced by α, the confidence level.

Symbol	Description
$\vee$	$x \vee y := \max\{x, y\}$
α	confidence level, $0 \leq \alpha \leq 1$
β	function that measures the portion of future normal costs to be paid at the beginning of year t (aggregate methods)
$\ddot{\gamma}_x^{(m)}$	index of skewness of $\ddot{Y}_x^{(m)}$
δ	$= \ln(1+i)$ force of interest
$\ddot{\zeta}_x^{(m)}$	kurtosis of $\ddot{Y}_x^{(m)}$
${}_\alpha\ddot{\theta}_x^{(m)}$	balancing item
$\ddot{\nu}_x^{(m)}$	coefficient of variation of $\ddot{Y}_x^{(m)}$
${}_\alpha\ddot{\xi}_x^{(m)}$	$\alpha-$confidence function
${}_{k-x\vert\alpha}\ddot{\xi}_x^{(12)}$	$= {}_{k-x}p_x^{(\tau)} v^{k-x} {}_\alpha\ddot{\xi}_k^{(12)}$, deferred percentile function
$\ddot{\sigma}_x^{(m)}$	standard deviation of $\ddot{Y}_x^{(m)}$
σ_t, γ_t	standard deviation and skewness of X_t
$(\sigma a)_t$, $(\gamma a)_t$	standard deviation and skewness of $(Xa)_t$
$(\sigma a)_t^j$, $(\gamma a)_t^j$	standard deviation and skewness of $(Xa)_t^j$
$(\sigma r)_t$, $(\gamma r)_t$	standard deviation and skewness of $(Xr)_t$
$(\sigma r)_t^j$, $(\gamma r)_t^j$	standard deviation and skewness of $(Xr)_t^j$
${}_\alpha\psi_t$	proportional adjustment coefficient
$\Delta_\alpha\psi_t$	is the change in ${}_\alpha\psi_t$ during year t
ω	first age at which (x) can not survive
${}_*\Pi$	PVFB, subindexes and superscripts are appended to it
${}_*\Pi_t$	PVFB for the entire group at time t
${}_*(\Pi a)_t$	PVFB for $\mathcal{A}_t$ at time t

Special Symbols (*Continuation...*)

Symbol	Description
$_*(\Pi a)^j_t$	PVFB for $j \in \mathcal{A}_t$ at time t
$_*(\widetilde{\Pi a})^j_{t+1}$	expected PVFB for j at t calculated based on the salary–scale
$_*\Pi W_0$	PVFB calculated at age of hire a
Φ	distribution function of the $N(0,1)$ distribution
a	earliest possible age in the active population
$\ddot{a}_x$	$= \mathbb{E}[Y_1]$
$\ddot{a}^{(m)}_x$	$= \mathbb{E}[\ddot{Y}^{(m)}_x]$
$\ddot{a}_{x:\overline{n}\rceil}$	$= \mathbb{E}[Y_2]$, temporary life annuity
$^s\ddot{a}_{x:\overline{n}\rceil}$	salary–based temporary life annuity
$_{y-x\mid}\ddot{a}^{(12)}_x$	deferred annuity $_{y-x}p^{(\tau)}_x v^{y-x}\ddot{a}^{(12)}_y$
$\mathcal{A}_t$	active population at instant t
A_x	$= \mathbb{E}[Z]$, whole life insurance
$A^{(m)}_x$	$= A_x$ payable at the end of the $m-$th of the year of death
$^nA^{(m)}_x$	$= A^{(m)}_x$ calculated at a the force of interest $n\delta$
$_*AAN$	abbreviation for *Attained–Age Normal* cost method
AL	abbreviation of *Actuarial Liability*
$_*AL_t$	AL for the entire group at time t
$_*(ALa)_t$	AL for $\mathcal{A}_t$ at time t
$_*(ALa)^j_t$	AL for $j \in \mathcal{A}_t$ at time t
$_*(ALr)_t$	AL for $\mathcal{P}_t$ at time t
$_*(ALr)^j_t$	AL for $j \in \mathcal{P}_t$ at time t
$_*(\widetilde{ALa})^j_{t+1}$	expected AL for $j \in \mathcal{A}_t$ at time t calculated
	by assuming salaries increase according to the salary–scale function
$_*(\widetilde{ALr})^j_{t+1}$	expected AL for $j \in \mathcal{P}_t$ at time t calculated assuming no benefit changes
b_t, b^j_t	benefit function. If $j \in \mathcal{P}_t$, then it is the amount of pension

Special Symbols (*Continuation...*)

Symbol	Description
B_t	pension payments paid to members of set $\mathcal{R}_t \cup \mathcal{P}_t$ during year t
C_t	collected contributions during year t
d	$= 1 - v$ rate of interest in advance
$d^{(m)}$	nominal annual rate compounded m times a year
$\mathcal{D}_t$	subset of $\mathcal{A}_t$ who die during year t. The same symbol is used for $\mathcal{P}_t$
$\mathcal{E}_t$	subset of $\mathcal{A}_t$ who leave by any cause during year t
$_*EAN$	abbreviation for *Entry–Age Normal* cost method
F_t	observed fund balance of year t for the entire plan, $F_t = (Fa)_t + (Fr)_t$
$(Fa)_t$	observed fund balance of year t for the active group
$(Fr)_t$	observed fund balance of year t for the retired group
$_*FIL$	abbreviation for *Frozen Initial Liability* cost method
$_*FNC_t$	future normal cost at instant t
$_*FNC_t^j$	future normal cost at instant t of employee $j \in \mathcal{A}_t$
FS_t	present value of future salaries at t
FSW_0	FS_t calculated at age of hire a
G	abbreviation for *Gain*
$_*G_t$	gain for the entire plan observed during year t
$_*(Ga)_t$	gain for $\mathcal{A}_t$ observed at t
$_*(Gr)_t$	gain for $\mathcal{R}_t$ observed at t
$_*G_t^{(i)}$	interest gain for $\mathcal{A}_t$ and/or $\mathcal{R}_t$.
$_*G_t^{(\tau)}$	unexpected decrements gain in $\mathcal{A}_t$
	$_*G_t^{(\tau)} = {_*G_t^{(d)}} + {_*G_t^{(w)}} + {_*G_t^{(h)}} + {_*G_t^{(r)}}$
$_*G_t^{(s)}$	unexpected salary changes gain in $\mathcal{A}_t$
$_*G_t^{(b)}$	unexpected pension payment changes gain
$_*G_t^{(p)}$	gain due to unexpected pension payments to members of set $\mathcal{P}_t$

Special Symbols (*Continuation...*)

Symbol	Description
${}_*G_t^{(mr)}$	unexpected mortality gain in $\mathcal{P}_t$
$h_x^{(m)}$	mass function of $\ddot{Y}_x^{(m)}$
$H_x^{(m)}$	distribution function of $\ddot{Y}_x^{(m)}$
$\mathcal{H}_t$	subset of $\mathcal{A}_t$ who get disabled during year t
i	(expected) interest rate
i'	(observed) interest rate
$i^{(m)}$	nominal annual rate compounded m times a year
I_t	observed interest returns during year t for $\mathcal{A}_t \cup \mathcal{P}_t$, $(Ia)_t + (Ir)_t$
$(Ia)_t$	observed interest returns during year t for the active group
$(Ir)_t$	observed interest returns during year t for the retired group
I_t^c	expected interest returns earned on C_t
$I_{t@i'}^c$	observed interest returns earned on C_t, calculated at the rate i'
I_t^{PP}	expected interest earned on PP_t
$I_{t@i'}^{PP}$	observed interest earned on PP_t calculated at the rate i'
I_t^B	interest earned on B_t at the rate i'
I_t^{NB}	interest earned on NB_t at the rate i'
I_t^{OB}	interest earned on OP_t at the rate i'
K	$= \lfloor T \rfloor$, is the discrete random variable of the curtate lifetime of (x)
K_m	$= \lfloor mT \rfloor / m$
m	mass accrual function, is the percentage of PVFB allocated to age x
M	accrual function, M_x is the percentage of PVFB accrued to age x
$\mathcal{N}_t$	set of new entrants during year t
NB_t	pension payments paid to members of set $\mathcal{R}_t$ during year t
${}_*NC_t$	normal cost at instant t

Special Symbols (*Continuation...*)

Symbol	Description
$_*NC_t^j$	normal cost at instant t of employee $j \in \mathcal{A}_t$
$_*NET$	abbreviation for *Net* or *Global* cost method
OB_t	pension payments paid to members of set $\mathcal{P}_t$ during year t
$_tp_x$	$= 1 - {_tq_x}$ probability for (x) to survive to age $x + t$
$_tp_x^{(\tau)}$	probability for (x) to remain in the active population for the next t years
$\mathcal{P}_t$	retired population at instant t
PP_t	amount to "purchase" annuities to members of set R_t
	transferred from $(Fa)_t$ to $(Fr)_t$
$_*PUC$	abbreviation for *Projected Unit Credit* cost method
PVFB	abbreviation for *Present Value of Future Benefits*
$_tq_x$	probability for (x) to die in the next t years
$_tq_x^{(j)}$	probability for (x) to leave the active population in the next t years due
	to cause $j = d$ death, w withdrawal, h disability and r retirement
$_tq_x^{(\tau)}$	probability for (x) to leave the active group within the next t years
$\underline{r}$, $\overline{r}$	earliest and latest possible retirement ages (e.g., $\underline{r} = 60$, $\overline{r} = 70$)
R	discrete random variable representing retirement age
R_2	discrete random variable of age at retirement given that
	the decrement is due to retirement, mass function $p(k) = \Pr[R_2 = k]$
$\mathcal{R}_t$	subset of $\mathcal{A}_t$ who retire during year t
s_x	salary–scale function, usually an increasing function of age
S_t^j	salary of employee $j \in \mathcal{A}_t$
S_t	total of salaries at instant t
T	continuous random variable of the remaining lifetime of (x)
$\mathcal{T}_t$	subset of $\mathcal{A}_t$ who withdraw during year t
$_*U_t$	unit normal cost percentage at t, depends on the aggregate method

Special Symbols (*Continuation...*)

Symbol	Description
UL	abbreviation for *Unfunded Liability*
$_*UL_t$	UL for the entire plan at instant t
$_*(ULa)_t$	UL of $\mathcal{A}_t$ at t
$_*(ULr)_t$	UL of $\mathcal{P}_t \cup \mathcal{R}_t$ at t
v	$= (1+i)^{-1}$ discount factor
(x)	member with of exact age x
X_t	$X_t = (Xa)_t + (Xr)_t$ random variable of the PVFB for the entire plan
$(Xa)_t$	continuous random variable representing the PVFB of $\mathcal{A}_t$
$(Xr)_t$	continuous random variable representing the PVFB of $\mathcal{P}_t$
Y_1	discrete random variable of the present value of a whole life
	annuity of 1 payable at the beginning of each year while (x) survives
Y_2	discrete r. v. of the present value of an $n-$year temporary life
	annuity of 1 payable at the beginning of each year while (x) survives
$\ddot{Y}_x^{(m)}$	discrete random variable of the present value of
	a life annuity due of 1 payable m times per annum issued to x
z_α	100α–th percentile of the $N(0,1)$ distribution
Z	$= v^{K+1}$, discrete r. v. of the present value of a whole life insurance
	issued to (x) of 1 payable at the end of the year of death

Introduction

Traditional cost methods are based on the present value of future benefits (PVFB). This PVFB is the discounted amount of a whole life annuity, times a certain amount defined through a benefit function. This annuity is actually the *expected value* of $\ddot{Y}_x^{(m)}$, the random variable representing a whole life annuity payable m–thly. Such random variables, however, are negatively skewed for most ages, which in turn implies that their mean underestimates the risk associated to not receiving pension benefits in full.

On the other hand, under current methods, it is not possible to provide answers to certain questions. For instance, the actuary may want to know *How much to charge in order to pay retirement benefits with a specified probability* α*?*

These facts motivated Ramsay [6] to introduce the $\alpha-$*confidence function* or the *percentile function* as a substitute for the whole life annuity term that appears in the calculation of PVFB. Thus, the focus is on percentiles of $\ddot{Y}_x^{(m)}$, rather that on mean values.

Besides the introduction of a new family of cost methods, called *percentile cost methods*, an important assumption is taken into account here, namely, that the retirement age is a random variable rather than a fixed value. Traditionally, the possibility of multiple retirement ages is treated as an ancillary benefit. Since formulas become more complicated, it creates communication problems between the plan sponsor and individuals.

Chapter 1 reviews the notation that will be used throughout this work. The random variable $\ddot{Y}_x^{(m)}$ is studied in some detail and the problem mentioned above is stated.

Chapter 2 studies the traditional cost methods. The so–called, *individual* and *aggregate* methods are dealt with separately. For mathematical convenience, the active and the retired populations are assumed to be separate. For individual cost methods, the use of an *accrual function* will play a key role.

Chapter 3 deals with percentile cost methods. The ideas of Chapter 2 remain the same, except for the change outlined above. For aggregate methods, however, the difference is more obvious, when the analogue to the concept of the PVFB is calculated by approximating the distribution of a certain random variable.

Chapter 4 presents a detailed example to see the effect of the desired confidence level on the valuation methods. The thesis ends with some conclusions, and suggests further developments.

Chapter 1

Preliminaries

1.1 Assumptions and Notation

This section describes some assumptions and the notation that will be used throughout this work (see Bowers et al. [2]).

(a) The symbol (x) denotes a person of exact age x. The first age at which (x) cannot survive is denoted by ω (e.g. $\omega = 110$).

(b) The probability for (x) to die in the next t years is denoted ${}_tq_x$ and the probability for (x) to survive to age $x+t$ is ${}_tp_x := 1 - {}_tq_x$. If $t = 1$ we write q_x and p_x instead. The continuous random variable $T = T(x)$ denotes the remaining lifetime of (x) and its distribution function is

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0\,, \\ {}_tq_x & \text{if } 0 \le t < \omega - x\,\cdot \\ 1 & \text{if } t \ge \omega - x \end{cases} \tag{1.1}$$

Additionally, the discrete random variable $K := \lfloor T \rfloor$, where $\lfloor\cdot\rfloor$ is the integer part function, represents the curtate remaining future lifetime of (x). Its mass function is $\Pr[K = k] = {}_kp_x\, q_{x+k}$, for $k = 0, 1, 2, \ldots, \omega - x - 1$.

(c) The rate of interest is i, while $\delta := \ln(1+i)$ denotes the force of interest, $v := 1/(1+i)$ the discount factor and $d := 1-v$ the rate of interest in advance. When financial operations are made $m-$thly then we need $d^{(m)} := m(1-v^{1/m})$ and $i^{(m)} := m(e^{\delta/m}-1)$, which are nominal annual rates compounded m times a year.

In relation to a pension plan, we have the following assumptions:

(d) Consider an *active population* of employees, whose earliest possible age is denoted by a (e.g. $a = 25$). This group is assumed to be immersed in a *multiple decrement environment*, which is to say that, an active member can leave his/her status due to several (independent) causes. Here we assume the existence of four causes of decrement, namely, death (d), withdrawal (w), disability (h) and retirement (r). If stress on a particular cause of decrement is required, we will adopt the following notation: ${}_tq_x^{(c)}$ is the probability for (x) to leave the active status in the next t years due to cause (c). The sum ${}_tq_x^{(\tau)} := {}_tq_x^{(d)} + {}_tq_x^{(w)} + {}_tq_x^{(h)} + {}_tq_x^{(r)}$ represents the probability for (x) to leave the active status within the next t years due to any cause, while ${}_tp_x^{(\tau)} := 1 - {}_tq_x^{(\tau)}$ is the probability for (x) to reach age $x+t$ within the active group. As usual, if $t = 1$, that subindex is dropped.

(e) We assume the existence of a *retired group*. The retirement age is denoted R; it is a discrete random variable whose domain is the set $\{x \vee \underline{r}, \ldots, \overline{r}, \infty\}$, where $x \vee y := \max\{x, y\}$. Its mass function is

$$\Pr[R = k] = \begin{cases} {}_{k-x}p_x^{(\tau)} q_k^{(r)}, & \text{if } k \in \{x \vee \underline{r}, \ldots, \overline{r}\}; \\ 1 - \sum_{k=x\vee\underline{r}}^{\overline{r}} {}_{k-x}p_x^{(\tau)} q_k^{(r)}, & \text{if } k = \infty. \end{cases} \tag{1.2}$$

When the pension plan allows for retirement at other dates than birthdays, the age last birthday is used for R. The age $\underline{r}$ is known as the *earliest retirement age possible* and $\overline{r}$ as the *latest retirement age*. Unlike an active employee, a retired person faces only one cause of decrement (single decrement environment), namely, death.

(f) Salaries are determined by a *salary–scale* function $\{s_x : x = a, a+1, \ldots\}$. Thus, a person now age x will have a (projected) salary to age $x+k$, equals to his/her current salary times the ratio s_{x+k}/s_x.

Some random variables related with life annuities and life insurance will be used. We summarize those useful for our purposes.

Definition 1.1 Life annuities

(a) A whole life annuity of 1 payable at the beginning of each year while (x) survives has a present value defined as:

$$Y_1 := \ddot{a}_{\overline{K+1}|} = \frac{1 - v^{K+1}}{d} , \qquad k = 0, 1, \ldots, \omega - x - 1 .$$

Its expected value is denoted $\ddot{a}_x$ and it is given by:

$$\ddot{a}_x := \mathbb{E}[Y_1] = \sum_{k=0}^{\omega-x-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x q_{x+k} = \sum_{k=0}^{\omega-x-1} v^k\, {}_kp_x .$$

Notice that $\ddot{a}_x$ is calculated based on the life table, rather than on the service table.

(b) An n–year temporary life annuity of 1 payable at the beginning of each year while (x) survives, calculated by using a service table (multiple decrement environment), has a present value given by:

$$Y_2 := \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{if } K = 0, \ldots, n-1 \\ \ddot{a}_{\overline{n}|} & \text{if } K = n, \ldots, \omega - x - 1 . \end{cases}$$

The expected value (sometimes called the net single premium) of this random variable, denoted $\ddot{a}_{x:\overline{n}|}$, is given by:

$$\ddot{a}_{x:\overline{n}|} := \mathbb{E}[Y_2] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x^{(\tau)} q_{x+k} + \ddot{a}_{\overline{n}|}\, {}_np_x^{(\tau)} = \sum_{k=0}^{n-1} v^k\, {}_kp_x^{(\tau)} .$$

(c) There exists a variation of the $n-$year temporary annuity defined above, that takes into account the progression of salaries as follows:

$$^{s}\ddot{a}_{x:\overline{n}|} := \sum_{k=0}^{n-1} \frac{s_{x+k}}{s_x} v^k\, {}_kp_x^{(\tau)} .$$

Finally, some life insurance functions will be used.

Definition 1.2 Life insurance

(a) The present value of a whole life insurance issued to (x) of 1 payable at the end of the year of death is:

$$Z := v^{K+1} , \qquad K = 0, 1, \ldots, \omega - x - 1 ,$$

while its expected value, denoted A_x, is given by:

$$A_x := \mathbb{E}[Z] = \sum_{k=0}^{\omega-x-1} v^{k+1}\, {}_kp_x\, q_{x+k} .$$

(b) If the above whole life insurance is paid at the end of the m–th of the year of death, rather than at year–ends, then we write $A_x^{(m)}$. Based on a uniform distribution of deaths over integer ages (in short, the UDD assumption), we obtain the approximation (see Bowers et. al.[2, (4.4.6), p. 121],):

$$A_x^{(m)} \approx \frac{i}{i^{(m)}} A_x .$$

(c) An additional variation arises when the values are calculated with a force of interest n times the original one, this is, at $n\delta$ instead of δ. We use the notation ${}^nA_x^{(m)}$ for that purpose.

Of special interest is a particular life annuity, that we study in the next section.

1.2 A Problem in the Traditional Cost Methods

Definition 1.3 If $K_m := \lfloor mT \rfloor / m$, then the random variable representing the present value of a life annuity due of 1 payable m times per annum issued to (x) is defined as

$$\ddot{Y}_x^{(m)} := \ddot{a}^{(m)}_{\overline{K_m+1/m}|} = \frac{1 - v^{K_m+1/m}}{d^{(m)}} \ . \tag{1.3}$$

To compute the distribution function of $\ddot{Y}_x^{(m)}$ we apply the distribution function technique to the sequence of random variables $T \rightarrowtail K_m \rightarrowtail \ddot{Y}_x^{(m)}$, as follows. The distribution function of K_m is

$$\begin{aligned}
F_{K_m}(u) &= \Pr[K_m \le u] \\
&= \Pr[\lfloor mT \rfloor \le um] \\
&= \Pr[mT \le \lfloor um \rfloor + 1] \ , \qquad \text{since } \lfloor x \rfloor \le y \Leftrightarrow x \le \lfloor y \rfloor + 1 \ , \\
&= \Pr\left[T \le \frac{\lfloor um \rfloor + 1}{m}\right] \\
&= F_T\left(\frac{\lfloor um \rfloor + 1}{m}\right) \ .
\end{aligned}$$

Then, by (1.1) we have:

$$F_{K_m}(u) = \begin{cases} 0 & \text{if } \frac{\lfloor um \rfloor + 1}{m} < 0 \ , \\ {}_{\frac{\lfloor um \rfloor + 1}{m}}q_x & \text{if } 0 \le \frac{\lfloor um \rfloor + 1}{m} < \omega - x \ , \\ 1 & \text{if } \frac{\lfloor um \rfloor + 1}{m} \ge \omega - x \ . \end{cases} \tag{1.4}$$

This implies that the distribution function of $\ddot{Y}_x^{(m)}$, denoted by $H_x^{(m)}$, can be obtained as follows:

$$\begin{aligned}
H_x^{(m)}(u) &= \Pr[\ddot{Y}_x^{(m)} \le u] \\
&= \Pr\left[\frac{1 - v^{K_m+1/m}}{d^{(m)}} \le u\right] \\
&= \Pr\left[K_m \le -\frac{1}{\delta}\ln\left(1 - ud^{(m)}\right) - \frac{1}{m}\right] \\
&= F_{K_m}\left(-\frac{1}{\delta}\ln\left(1 - ud^{(m)}\right) - \frac{1}{m}\right) \\
&= F_{K_m}(\nu)
\end{aligned}$$

where $\nu := -\frac{1}{\delta}\ln\big(1-ud^{(m)}\big) - \frac{1}{m}$. Note that:

(a) The term $\frac{\lfloor um \rfloor + 1}{m}$ in (1.4), with ν instead of u, turns to be

$$k_m := \frac{1}{m}\Big\lfloor -\frac{m}{\delta}\ln\Big(1-ud^{(m)}\Big)\Big\rfloor .$$

(b) $k_m < 0 \Leftrightarrow u < 0$ and

(c) $0 \le k_m < (\omega - x) \Leftrightarrow 0 \le u < \Big(1 - e^{-\delta/m - \delta/m\lfloor m(\omega-x)\rfloor}\Big)/d^{(m)}$.

Thus, the cdf of $\ddot{Y}_x^{(m)}$ can be written as:

$$H_x^{(m)}(u) = \begin{cases} 0 & \text{if } u < 0\,, \\ {}_{k_m}q_x & \text{if } 0 \le u < \Big(1 - e^{-\delta/m - \delta/m\lfloor m(\omega-x)\rfloor}\Big)/d^{(m)}\,, \\ 1 & \text{if } u \ge \Big(1 - e^{-\delta/m - \delta/m\lfloor m(\omega-x)\rfloor}\Big)/d^{(m)}\,, \end{cases} \tag{1.5}$$

where $k_m = \frac{1}{m}\lfloor -\frac{m}{\delta}\ln(1-ud^{(m)})\rfloor$. Figure 1.1 presents the mass functions $h_{55}^{(12)}$ and $h_{65}^{(12)}$, based on the GAM 1983 male mortality table and $i = 5\%$ as interest rate. Notice the negative skewness of both distributions.

The expected value of $\ddot{Y}_x^{(m)}$ is denoted by $\ddot{a}_x^{(m)}$. Again, under the UDD assumption, (see Bowers et. al. [2, (5.4.11) to (5.4.13), pp. 151–152]) we get the approximation:

$$\ddot{a}_x^{(m)} \approx \left(\frac{id}{i^{(m)}d^{(m)}}\right)\ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}\,. \tag{1.6}$$

It is possible to express the standard deviation and the index of skewness of $\ddot{Y}_x^{(m)}$ in terms of life insurance functions.

Proposition 1.1 The standard deviation $\ddot{\sigma}_x^{(m)}$ and the index of skewness $\ddot{\gamma}_x^{(m)}$ of $\ddot{Y}_x^{(m)}$ are given, respectively, by:

$$\ddot{\sigma}_x^{(m)} = \frac{1}{d^{(m)}}\sqrt{{}^2A_x^{(m)} - \big(A_x^{(m)}\big)^2} \qquad \text{and} \tag{1.7}$$

$$\ddot{\gamma}_x^{(m)} = -\frac{{}^3A_x^{(m)} - 3\big({}^2A_x^{(m)}\big)\big(A_x^{(m)}\big) + 2\big(A_x^{(m)}\big)^3}{\Big[{}^2A_x^{(m)} - \big(A_x^{(m)}\big)^2\Big]^{\frac{3}{2}}}\,. \tag{1.8}$$

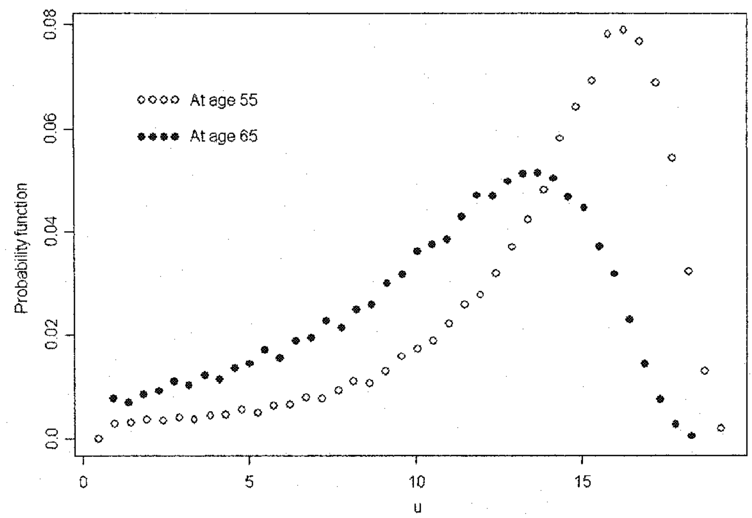

Figure 1.1: Mass functions $h_{55}^{(12)}$ and $h_{65}^{(12)}$ at $i = 5\%$

Proof. Expression (1.7) can be found in Bowers et al. [2, (5.4.4), p. 149]. The proof of (1.8) is straightforward. □

Tables 1.1 and 1.2 display values for the coefficient of variation $\ddot{v}_x^{(12)} := \ddot{\sigma}_x^{(12)}/\ddot{a}_x^{(12)}$ and the skewness, respectively. The calculations use the GAM 1983 male mortality table and the UDD assumption.

Note that if X is a random variable whose skewness is small, then, by using the normal approximation, $\Pr[X \leq \mathbb{E}[X]] \approx 50\%$, with equality if the skewness of X is zero. On the other hand, if the distribution of X is negatively skewed, then the above probability will be less than 50%. This tells us something about the random variable $\ddot{Y}_x^{(m)}$: if its index of skewness is negative, then the chance of paying pension benefits in full is less than 50%. As a matter of fact, Table 1.3 presents $\Pr[\ddot{Y}_x^{(12)} \leq \ddot{a}_x^{(12)}]$ for ages between 50 and 75, for distinct interest rates, based on the GAM 1983 mortality table for males. Note that all the values are less than 50%, as is expected, since the distribution of $\ddot{Y}_x^{(12)}$ has a negative skewness over this range of ages.

Table 1.1: Coefficient of variation $\ddot{v}_x^{(12)}$ at different ages and interest rates

Age x	$i = 5\%$	$i = 6\%$	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$
50	0.2315	0.2145	0.1998	0.1871	0.1761	0.1665
55	0.2692	0.2515	0.2360	0.2224	0.2105	0.1999
60	0.3154	0.2971	0.2809	0.2664	0.2534	0.2418
65	0.3745	0.3561	0.3395	0.3245	0.3109	0.2986
70	0.4384	0.4202	0.4035	0.3883	0.3744	0.3617
75	0.5064	0.4883	0.4717	0.4565	0.4423	0.4292

Table 1.2: Skewness $\ddot{\gamma}_x^{(12)}$ at different ages and interest rates

Age x	$i = 5\%$	$i = 6\%$	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$
50	−1.7227	−1.9909	−2.2566	−2.5183	−2.7748	−3.0249
55	−1.3489	−1.5647	−1.7774	−1.9865	−2.1915	−2.3918
60	−0.9882	−1.1585	−1.3248	−1.4870	−1.6452	−1.7995
65	−0.6648	−0.8011	−0.9321	−1.0584	−1.1803	−1.2981
70	−0.3633	−0.4768	−0.5850	−0.6885	−0.7876	−0.8827
75	−0.0609	−0.1570	−0.2482	−0.3348	−0.4175	−0.4965

Table 1.3: $H_x(\ddot{a}_x^{(12)}) = \Pr[\ddot{Y}_x^{(12)} \leq \ddot{a}_x^{(12)}]$ at different ages and interest rates

x	$i = 5\%$	$i = 6\%$	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$
50	0.3580	0.3401	0.3225	0.3035	0.2872	0.2716
55	0.3784	0.3616	0.3452	0.3319	0.3164	0.3012
60	0.4011	0.3886	0.3734	0.3585	0.3466	0.3324
65	0.4271	0.4133	0.4030	0.3893	0.3759	0.3660
70	0.4569	0.4450	0.4330	0.4211	0.4131	0.4014
75	0.4881	0.4788	0.4694	0.4550	0.4505	0.4410

At younger ages, the mean value $\ddot{a}_x^{(12)}$ underestimates the risk associated with the payment of the retirees' lifetime annuities. This fact pushed Ramsay [6] to ask the two following questions:

1. Should the degree of assurance be included in the actuarial cost methods?

2. Is the traditional actuarial liability a good measure of the plan's cost?

It is worth mentioning that, in practice, pension actuaries have compensated the inadequacies in estimated actuarial liabilities (based on a mean value), by using "conservative" assumptions. These implicitly build safety margins (thus, loading the mean). Amortizing any resulting losses (or gains), serves as a sponge that absorbs the deviations of the expected from the actual experience.

The next section addresses a possible answer to the above questions, by defining a new cost function that incorporates a level of confidence.

1.3 Percentile Cost Methods: A Possible Solution

The main idea behind the percentile cost method is to fix a desired level of confidence α that the random variable $\ddot{Y}_x^{(m)}$ will ultimately cover pension benefits. For this, we define the *α-confidence function*, denoted ${}_\alpha\ddot{\xi}_x^{(m)}$.

Definition 1.4 Given a confidence level α, age x and frequency of payments m, ${}_\alpha\ddot{\xi}_x^{(m)}$ is defined as the amount needed to ensure that a whole life annuity due of 1 per annum (payable m times per annum) to (x) is paid in its entirety with probability α, that is:

$$\Pr\left[\ddot{Y}_x^{(m)} \leq {}_\alpha\ddot{\xi}_x^{(m)}\right] = \alpha \,.$$

From this definition and the distribution function (1.5) we have:

Proposition 1.2 The $\alpha-$confidence function satisfies

$${}_\alpha\ddot{\xi}_x^{(m)} = \begin{cases} \ddot{a}_{\overline{k}|}^{(m)} & \text{if } \alpha = {}_kq_x \text{ and } k = 1/m, 2/m, \dots\,, \\ \text{undefined} & \text{otherwise}\,. \end{cases} \tag{1.9}$$

Proof. As $\alpha = \Pr\left[\ddot{Y}_x^{(m)} \leq {}_\alpha\ddot{\xi}_x^{(m)}\right] = H_x^{(m)}\left({}_\alpha\ddot{\xi}_x^{(m)}\right)$ then, from (1.5):

$$\alpha = \begin{cases} 0 & \text{if } {}_\alpha\ddot{\xi}_x^{(m)} < 0, \\ {}_{k_m}q_x & \text{if } 0 \leq {}_\alpha\ddot{\xi}_x^{(m)} < \left(1 - e^{-\delta/m - \delta/m\lfloor m(\omega - x)\rfloor}\right)/d^{(m)}\,, \\ 1 & \text{if } {}_\alpha\ddot{\xi}_x^{(m)} \geq \left(1 - e^{-\delta/m - \delta/m\lfloor m(\omega - x)\rfloor}\right)/d^{(m)}\,, \end{cases}$$

where $k_m = \frac{1}{m}\lfloor -\frac{m}{\delta}\ln(1 - {}_\alpha\ddot{\xi}_x^{(m)} d^{(m)})\rfloor = \frac{j}{m} = k$ for some integer $j \in \mathbb{N}$. In particular, if $\alpha = {}_kq_x$ then, ${}_{k_m}q_x = {}_kq_x$ and $k_m = k$, which in turn implies that

$$\begin{aligned} & \left\lfloor -\frac{m}{\delta}\ln\left(1 - {}_\alpha\ddot{\xi}_x^{(m)} d^{(m)}\right)\right\rfloor = j \\ \Rightarrow \quad & j \leq -\frac{m}{\delta}\ln\left(1 - {}_\alpha\ddot{\xi}_x^{(m)} d^{(m)}\right) < j + 1 \\ \Rightarrow \quad & \frac{1 - v^k}{d^{(m)}} \leq {}_\alpha\ddot{\xi}_x^{(m)} < \frac{1 - v^{k+1/m}}{d^{(m)}} \\ \Rightarrow \quad & \ddot{a}_{\overline{k}|}^{(m)} \leq {}_\alpha\ddot{\xi}_x^{(m)} < \ddot{a}_{\overline{k+1/m}|}^{(m)} \\ \Rightarrow \quad & {}_\alpha\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{k}|}^{(m)}\,, \end{aligned}$$

with the convention that the percentile is the smallest possible value. □

Since the random variable $\ddot{Y}_x^{(m)}$ is discrete, the percentile ${}_\alpha\ddot{\xi}_x^{(m)}$ is not uniquely defined in some parts of its domain, where a conventional definition can be given, as long as it is consistent. Note that

$$ {}_\alpha\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{k}|}^{(m)} = \frac{1-v^k}{d^{(m)}} = \frac{\delta}{d^{(m)}}\left(\frac{1-v^k}{\delta}\right) = \frac{\delta}{d^{(m)}}\bar{a}_{\overline{k}|} \,. $$

This fact suggests to redefine ${}_\alpha\ddot{\xi}_x^{(m)}$ as follows:

Definition 1.5 Given the confidence level $0 \leq \alpha \leq 1$. Then

$$ {}_\alpha\ddot{\xi}_x^{(m)} := \frac{\delta}{d^{(m)}}\bar{a}_{\overline{t}|}\,, \qquad \text{for } t > 0 \text{ such that } {}_tq_x = \alpha. \tag{1.10} $$

Note that Definition 1.5 coincides with (1.9) when $\alpha = {}_kq_x$, for some $k = 1/m, 2/m, \ldots$. To compare the $\alpha-$confidence function and the whole life annuity, Table 1.4 shows their ratios at $i = 5\%$, for distinct levels of confidence. For instance, that with $i = 5\%$, ${}_{0.5}\ddot{\xi}_x^{(12)}$ is "more expensive" than $\ddot{a}_x^{(12)}$ for $x \leq 75$. At age 65 in particular, the percentile function ${}_{0.5}\ddot{\xi}_{65}^{(12)}$ is about 7% more expensive than the mean $\ddot{a}_{65}^{(12)}$.

Table 1.4: $\ddot{a}_x^{(12)}$ and ${}_\alpha\ddot{\xi}_x^{(12)} = (\delta/d^{(12)})\bar{a}_{\overline{t}|}$ with $i = 5\%$

Age	$\ddot{a}_x^{(12)}$	${}_\alpha\ddot{\xi}_x^{(12)}/\ddot{a}_x^{(12)}$				
		$\alpha = 50\%$	$\alpha = 60\%$	$\alpha = 70\%$	$\alpha = 80\%$	$\alpha = 90\%$
50	14.82592	1.06872	1.10654	1.14023	1.17302	1.20934
55	13.62833	1.07415	1.12460	1.16999	1.21450	1.26413
60	12.24298	1.07618	1.14371	1.20533	1.26652	1.33536
65	10.67885	1.07196	1.16253	1.24719	1.33211	1.42884
70	9.06222	1.05333	1.17170	1.28576	1.40309	1.53950
75	7.46558	1.01729	1.16678	1.31634	1.47500	1.66456

We close this chapter with a recursive relation satisfied by ${}_{\alpha}\ddot{\xi}_x^{(m)}$, which will be helpful in the analysis of *gains* (a mechanism to measure the deviation of our prior actuarial assumptions) for the retired population under the percentile approach.

Recall that the equation (see Bowers et. al.[2, (5.3.4), p. 144])

$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1} \ , \tag{1.11}$$

forms the basis for analysis of gains for the retired population in the classical valuation methods. Let

$$I_x := \begin{cases} 1 & \text{if } (x) \text{ survives to age } x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \alpha &= \Pr[\ddot{Y}_x^{(m)} \le {}_{\alpha}\ddot{\xi}_x^{(m)}] \\ &= q_x \Pr[\ddot{Y}_x^{(m)} \le {}_{\alpha}\ddot{\xi}_x^{(m)} | I_x = 0] + p_x \Pr[\ddot{Y}_x^{(m)} \le {}_{\alpha}\ddot{\xi}_x^{(m)} | I_x = 1] \\ &= q_x + p_x \Pr[\ddot{a}_{\overline{1}|}^{(m)} + v\ddot{Y}_{x+1}^{(m)} \le {}_{\alpha}\ddot{\xi}_x^{(m)}] \ , \end{aligned}$$

since $\Pr[\ddot{Y}_x^{(m)} \le {}_{\alpha}\ddot{\xi}_x^{(m)} | I_x = 0] = 1$ and $[\ddot{Y}_x^{(m)} | I_x = 1] = \ddot{a}_{\overline{1}|}^{(m)} + v\ddot{Y}_{x+1}^{(m)}$. Then

$$\frac{\alpha - q_x}{p_x} = \Pr[\ddot{Y}_{x+1}^{(m)} \le (1+i)({}_{\alpha}\ddot{\xi}_x^{(m)} - \ddot{a}_{\overline{1}|}^{(m)})] \ ,$$

which yields the recursion,

$${}_{\alpha}\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} + v\,{}_{\beta}\ddot{\xi}_{x+1}^{(m)} \ , \tag{1.12}$$

where

$$\beta := \frac{\alpha - q_x}{p_x} \ .$$

However, the above recursion does not involve ${}_{\alpha}\ddot{\xi}_x^{(m)}$ and ${}_{\alpha}\ddot{\xi}_{x+1}^{(m)}$. Based on (1.11), a recursive relation for ${}_{\alpha}\ddot{\xi}_x^{(m)}$ will be of the form:

$${}_{\alpha}\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} + vp_x\,{}_{\alpha}\ddot{\xi}_{x+1}^{(m)} + {}_{\alpha}\ddot{\theta}_x^{(m)} \ , \tag{1.13}$$

where ${}_{\alpha}\ddot{\theta}_x^{(m)}$ is a balancing item required to ensure that the right–hand sides of (1.12) and (1.13) are equal. Therefore,

$${}_{\alpha}\ddot{\theta}_x^{(m)} = v\,{}_{\beta}\ddot{\xi}_{x+1}^{(m)} - vp_x\,{}_{\alpha}\ddot{\xi}_{x+1}^{(m)} \ .$$

Table 1.5 give values of ${}_{\alpha}\ddot{\theta}_x^{(12)}$ for $i = 5\%$, and for various values of x and α.

Table 1.5: Balancing item, ${}_{\alpha}\ddot{\theta}_x^{(12)}$, with $i = 5\%$

	Level of confidence α				
Age	50%	60%	70%	80%	90%
50	0.0463	0.0522	0.0566	0.0603	0.0638
55	0.0611	0.0723	0.0808	0.0880	0.0949
60	0.0719	0.0910	0.1058	0.1194	0.1321
65	0.0848	0.1212	0.1513	0.1770	0.2053
70	0.0875	0.1534	0.2096	0.2654	0.3176
75	0.0561	0.1561	0.2488	0.3438	0.4343

Chapter 2

Traditional Cost Methods

In this chapter we review the concepts related with the valuation of pension plans. The active and retired populations are dealt with separately. When studying the active population, a small dilemma is faced: there are several ways to classify actuarial cost methods. A very comprehensive presentation on this subject can be found in McGill [4]. We will divide the cost methods in two groups. By *individual cost methods*, we mean those in which the calculations of the normal cost and the actuarial liability (concepts defined in Section 2.1) are made on an individual basis, the aggregate cost being just the summation of the individual components. We call *aggregate cost methods*, on the other hand, those that look at the group as a whole.

For mathematical convenience, we keep separately the active and the retired populations. As we will notice, a central concept in pension valuation is the *present value of future benefits*, which will appear repeatedly throughout this work. To ease the presentation, hereon we abbreviate to PVFB, and denoted with the greek letter Π. In the sequel, time t is assumed discrete.

2.1 Individual Cost Methods

Bowers et. al [2] suggest a unified approach, where the concept of *accrual function* plays a key role. Within this approach, the philosophy behind a particular cost method is hidden. A second methodology can be found in Anderson [1], where a cost method is defined in terms of some premises. Our presentation adopts a combination of both approaches.

The difference between two cost methods lies in the way the pension liabilities are recognized by the plan sponsor, during the employees' service time. In other words, to each individual cost method, we can associate an *accrual function* $M : [a, \infty) \to [0, 1]$. $M_x := M(x)$ represents that fraction of the actuarial present value of future benefits accrued at age x. The reader may want to take a glance at Figure 2.1, page 32. We will assume that:

(a) $M_a = 0$, that is, a newly hired (age a) employee has no accrued liability;

(b) M is nondecreasing and right-continuous; and

(c) $M_x = 1$ for $x \geq r$, that is, full benefits are given to those who reach retirement age.

The resemblance between the concepts of *accrual function* and *cumulative distribution function* suggests to consider also the analogue to the *probability mass function*. As a matter of fact, a cost method can also be defined by means of its *accrued mass function*, denoted m and linked to the accrued function by $M_x = \sum_{y=a}^{x-1} m_y$. In particular, $m_x = 0$ for $x > r$.

Assume a benefit function b_t, updated to year $t = 0, 1, 2, \ldots$, with b_t^j being the benefit function at t of member j. Consider the following division of active pension plan members $\mathcal{A}_t$ at time $t = 0, 1, 2, \ldots$:

$\mathcal{T}_t :=$ the subset of $\mathcal{A}_t$ who withdraw in $[t, t+1)$,

$\mathcal{H}_t :=$ the subset of $\mathcal{A}_t$ who become disable in $[t, t+1)$,

$\mathcal{D}_t :=$ the subset of $\mathcal{A}_t$ who die in $[t, t+1)$,

$\mathcal{R}_t :=$ the subset of members of $\mathcal{A}_t$ who retire in $[t, t+1)$,

$\mathcal{E}_t := \mathcal{T}_t \cup \mathcal{H}_t \cup \mathcal{D}_t \cup \mathcal{R}_t$ is the subset of members of $\mathcal{A}_t$ who leave by any cause in $[t, t+1)$,

$\mathcal{N}_t :=$ the set of new entrants into the plan during the year t .

Symbolically the set of active members at $t+1$ is given by:

$$\mathcal{A}_{t+1} = \mathcal{A}_t - \mathcal{E}_t + \mathcal{N}_t \ , \qquad t = 0, 1, 2, \ldots \ . \tag{2.1}$$

In what follows, some functions for the active population will be introduced.

Present Value of Future Benefits $(\Pi a)_t$

If the plan is such that it is mandatory to retire when the person is exact age r (in other words, the random variable R takes only two values, r and ∞), as in Ramsay [6], then the present value of future benefits for employee now age $x = x_j$ is given by

$$b_t^j \ v^{r-x} \ {}_{r-x}p_x^{(\tau)} \ \ddot{a}_r^{(12)} \ , \qquad t = 0, 1, 2, \ldots \ ,$$

which is interpreted as:

$$\begin{pmatrix} \text{benefit} \\ \text{amount} \end{pmatrix} \times \begin{pmatrix} \text{discount} \\ \text{factor} \end{pmatrix} \times \begin{pmatrix} \text{probability for } (x) \\ \text{to reach age } r \end{pmatrix} \times \begin{pmatrix} \text{whole life annuity} \\ \text{payable monthly} \\ \text{issued to } (r) \end{pmatrix} .$$

The above expression can be written shorter, by introducing the $(k-x)$–year deferred annuity:

$$_{k-x|}\ddot{a}_x^{(12)} := v^{k-x} \, {}_{k-x}p_x^{(\tau)} \, \ddot{a}_k^{(12)} \ , \qquad k \geq x \ .$$

The deferred annuity is based on the service table for active members, while the annuity on the right–hand side is based on the life table for retired members. When the sum over all the active participants is taken, the PVFB is obtained. A more general approach is obtained when considering various possible retirement ages. Recall that the retirement age random variable R has mass function

$$\Pr[R = k] = \begin{cases} {}_{k-x}p_x^{(\tau)} q_k^{(r)}, & \text{if } k \in \{x \vee \underline{r}, \dots, \overline{r}\} , \\ 1 - \sum_{k=x\vee\underline{r}}^{\overline{r}} {}_{k-x}p_x^{(\tau)} q_k^{(r)}, & \text{if } k = \infty. \end{cases}$$

Definition 2.1 The *present value of future benefits for the active group*, denoted $(\Pi a)_t$, is given by

$$(\Pi a)_t := \sum_{j \in \mathcal{A}_t} (\Pi a)_t^j = \sum_{j \in \mathcal{A}_t} \sum_{k=x\vee\underline{r}}^{\overline{r}} b_t^j \; {}_{k-x|}\ddot{a}_x^{(12)} q_k^{(r)} , \qquad t \in \mathbb{N}. \tag{2.2}$$

The quantity

$$(\Pi a)_t^j = \sum_{k=x\vee\underline{r}}^{\overline{r}} b_t^j \; {}_{k-x|}\ddot{a}_x^{(12)} q_k^{(r)}$$

is interpreted as the PVFB for employee $j \in \mathcal{A}_t$.

Actuarial Liability $(ALa)_t$

The actuarial liability for an individual $j \in \mathcal{A}_t$ is defined as the portion of the PVFB allocated to date, in other words, as the product

$$\begin{pmatrix} \text{cumulative} \\ \text{percentage} \end{pmatrix} \times \begin{pmatrix} \text{present value of} \\ \text{future benefits} \end{pmatrix} .$$

Definition 2.2 The *actuarial liability of the active population* at instant t, denoted $(ALa)_t$, is given by:

$$\begin{aligned} (ALa)_t := \sum_{j \in \mathcal{A}_t} (ALa)_t^j &= \sum_{j \in \mathcal{A}_t} M_x \, (\Pi a)_t^j \\ &= \sum_{j \in \mathcal{A}_t} \sum_{k=x\vee\underline{r}}^{\overline{r}} M_x \, b_t^j \; {}_{k-x|}\ddot{a}_x^{(12)} q_k^{(r)} , \qquad t \in \mathbb{N} . \end{aligned} \tag{2.3}$$

The amount $(ALa)_t^j$ can be interpreted as the actuarial liability of employee $j \in \mathcal{A}_t$, currently age x. Overall, $(ALa)_t$ is interpreted as the *ideal fund balance*, or desired amount of assets to have at hand to face future liabilities.

The following identity, which is easily proved, (see Anderson [1, Exercise 2.2.1, p. 14]) will be useful:

$$ {}_{k-x-1|}\ddot{a}_{x+1}^{(12)} = {}_{k-x|}\ddot{a}_x^{(12)}(1+i) + q_x^{(\tau)}\,{}_{k-x-1|}\ddot{a}_{x+1}^{(12)} \,. \tag{2.4} $$

It is interesting to see the evolution of $(ALa)_t$ between instants t and $t+1$:

$$ \begin{aligned} (ALa)_{t+1} &= \sum_{j\in\mathcal{A}_{t+1}} (ALa)_{t+1}^j \\ &= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (ALa)_{t+1}^j + \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j \,, \qquad \text{since } \mathcal{A}_{t+1} - \mathcal{N}_t = \mathcal{A}_{t+1}\cap\mathcal{A}_t \,, \\ &= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (\widetilde{ALa})_{t+1}^j + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] + \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j \\ &= \sum_{j\in\mathcal{A}_t} \sum_{k=(x+1)\vee\underline{r}}^{\overline{r}} M_{x+1}\, b_t^j \underbrace{{}_{k-x-1|}\ddot{a}_{x+1}^{(12)}}_{\text{Use (2.4)}} - \sum_{j\in\mathcal{E}_t} (\widetilde{ALa})_{t+1}^j \\ &\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] + \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j \\ &= \Big[(ALa)_t + \sum_{j\in\mathcal{A}_t} m_x\,(\Pi a)_t^j\Big](1+i) - \Big[\sum_{j\in\mathcal{E}_t} (\widetilde{ALa})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)} (\widetilde{ALa})_{t+1}^j\Big] \\ &\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] + \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j \,, \end{aligned} \tag{2.5} $$

where

$$ (\widetilde{ALa})_{t+1}^j = \sum_{k=x\vee\underline{r}}^{\overline{r}} M_{x+1}\, b_t^j\, {}_{k-x-1|}\ddot{a}_{x+1}^{(12)}\, q_k^{(r)} $$

is the *expected* actuarial liability, calculated by assuming that salaries increase according to the salary–scale function. Notice that $(ALa)_{t+1}$ has the form

$$ (ALa)_{t+1} = \Big[(ALa)_t + \Big(\textstyle\sum_{j\in\mathcal{A}_t} m_x(\Pi a)_t^j\Big)\Big](1+i) + \begin{pmatrix} \text{other terms that are zero} \\ \text{in normal circumstances} \end{pmatrix} . $$

This analysis motivates the introduction of the next function.

Normal Cost NC_t

If all the actuarial assumptions hold exactly in practice, the second term in (2.5) will be zero, assuming the service table is correct. Also, the third term is zero, if salaries increase according to the salary–scale, in other words, if $b^j_{t+1} = b^j_t$.

Definition 2.3 The *normal cost of the plan*, denoted NC_t, is given by:

$$NC_t := \sum_{j \in \mathcal{A}_t} NC^j_t = \sum_{j \in \mathcal{A}_t} m_x \, (\Pi a)^j_t \,, \qquad t \in \mathbb{N} \,. \tag{2.6}$$

Thus, in *normal circumstances*, the quantity NC_t is required to keep the fund at its proper level. The term NC^j_t is interpreted as the normal cost for employee $j \in \mathcal{A}_t$, aged x at instant t. This normal cost is obtained once the actuarial liability in (2.3) has been established. However, it is also possible to first define the normal cost and then to obtain the resulting actuarial liability. Historically the choice of a particular cost method has usually been the result of premises, either on the accrued liability $(ALa)_t$ or on the normal cost NC_t.

Future Normal Cost FNC_t

Definition 2.4 The *present value of future normal costs for the active population*, denoted FNC_t, is given by:

$$FNC_t := (\Pi a)_t - (ALa)_t \,, \qquad t \in \mathbb{N} \,. \tag{2.7}$$

From the definitions of $(\Pi a)_t$ and $(ALa)_t$ we have that

$$FNC_t = \sum_{j \in \mathcal{A}_t} FNC^j_t = \sum_{j \in \mathcal{A}_t} (1 - M_x)(\Pi a)^j_t \,.$$

The quantity FNC^j_t can be interpreted as the amount to be allocated from the current age x to retirement age for employee $j \in \mathcal{A}_t$.

From (2.7) we can also write

$$(ALa)_t = (\Pi a)_t - FNC_t\,, \qquad t \in \mathbb{N}\,, \tag{2.8}$$

which is sometimes taken as the definition of the actuarial liability.

Example 2.1 In order to illustrate the ideas so far presented, consider the following pension plan:

The plan:

Starting date of operation: January 1st, 2002.

Date of evaluation ($t = 0$): January 1st, 2002.

Retirement age: $r = 65$ (This means that R takes two values: $r = 65$ and $r = \infty$).

Actual fund balance or assets: $(Fa)_0 = 30{,}000$.

Benefit formula: 2% of the sum of all salaries at retirement, that is:

$$b_t = 0.02\,[\text{past+projected salaries}]\,.$$

Assume the plan has only one active member:

Birth date: January 1st, 1957.

Date of entry: January 1st, 1997.

Age at entry: $a = 40$.

Past salaries: 1997 = 65,000; 1998 = 70,000; 1999 = 74,000; 2000 = 78,000; 2001 = 85,000.

The actuarial assumptions are:

Interest rate: $i = 0.08$.

Salary increase rate: 0.07.

No death or withdrawal before retirement.

$\ddot{a}_{65}^{(12)} = 12$.

Assume the actuary has chosen the cost method given by:

$$m_x = \frac{1}{r-a} = \frac{1}{25} \qquad \text{and} \qquad M_x = \frac{x-a}{r-a} = \frac{x-40}{25}\,, \qquad x = 40, 41, \ldots, 65\,.$$

The functions above satisfy the definitions of accrual functions. This cost method is in fact referred to as the *projected unit credit cost method*, as we will see later.

At time $t = 0$ the member is aged $x = 45$. Total past salaries is 372,000, while future salaries are estimated as follows:

$$\begin{aligned} \text{projected salaries} &= 85{,}000\,(1.07 + 1.07^2 + \ldots + 1.07^{20}) \\ &= 85{,}000\,\ddot{s}_{\overline{20}|0.07} \\ &= 3{,}728{,}540.03\,. \end{aligned}$$

Thus, the annual pension benefit turns to be: $b_0 = 0.02\,[372{,}000 + 3{,}728{,}540.03] = 82{,}010.80$, while the PVFB is:

$$(\Pi a)_0 = b_0\ {}_{20|}\ddot{a}_{45}^{(12)} = 211{,}143.24\,.$$

The actuarial liability and the normal cost are, respectively:

$$\begin{aligned} (ALa)_0 &= M_{45}\,(\Pi a)_0 = 42{,}228.65\,, \\ NC_0 &= m_{45}\,(\Pi a)_0 = 8{,}445.73\,. \end{aligned}$$

Finally, note that the difference between the actuarial liability and the fund balance is $(ALa)_0 - (Fa)_0 = 12,228.65$, that can be interpreted as the deficit of the plan.

The Fund Balance F_t

Before continuing with the basic functions, it is worth mentioning how the fund works. If actuarial assumptions were met, the actuarial liability would be exactly the amount of money needed to face all future pension payments. Abandoning this ideal context, let F_t be the *actual* fund balance of the entire fund. During year t, this fund will be increased by investment returns I_t, by contributions C_t and will be diminished by the payment of pension benefits B_t for members in $\mathcal{R}_t \cup \mathcal{P}_t$, that is

$$F_{t+1} = F_t + I_t + C_t - B_t\,,$$

where $\mathcal{P}_t$ is the set of members already retired by time t. To simplify the analysis of the evolution of quantities such as actuarial liabilities, the fund is divided as

$$F_t = (Fa)_t + (Fr)_t ,$$

where $(Fa)_t$ and $(Fr)_t$ are the funds for the active and retired populations, respectively. Furthermore, these funds present the following evolution:

$$\begin{aligned}(Fa)_{t+1} &= (Fa)_t + (Ia)_t + C_t - PP_t \\ (Fr)_{t+1} &= (Fr)_t + (Ir)_t + PP_t - B_t ,\end{aligned}$$

where PP_t represents that amount of money withdrawn from the active fund and transferred to the retired fund to "purchase" pensions benefits of members of set $\mathcal{R}_t$ (new retirees). The investment return for actives is given by

$$(Ia)_t = i'(Fa)_t + I^c_{t@i'} - I^{PP}_{t@i'} , \qquad (2.9)$$

where $I^c_{t@i'}$ is the interest earned on the contributions and $I^{PP}_{t@i'}$ is the same for annuity purchases. These quantities are calculated using i', the actual interest rate observed during year t.

Unfunded Liability $(ULa)_t$

The normal costs were defined as the cost of the plan (to the contributor) in normal circumstances, in the sense that our actuarial assumptions reflect reality, something that does not occur exactly in practice. Winklevoss [9, p. 96] and Dufresne [3, pp. 68–74] give reasons why plan assets will not be equal to actuarial liabilities: experience variations, assumptions changes, benefit changes or past service accrual.

Definition 2.5 The deviation of the accrued liability from the real fund level:

$$(ULa)_t := (ALa)_t - (Fa)_t , \qquad t \in \mathbb{N} , \qquad (2.10)$$

is known as the *unfunded liability of the active population.*

If $(ULa)_t < 0$, it is called a *surplus* and a *deficit* if $(ULa)_t > 0$. However, the term *unfunded* is used, regardless of its sign. If actuarial assumptions were met then $(Fa)_t = (ALa)_t$ and there would not be a need for an unfunded liability.

It is worth studying the evolution of $(ULa)_t$ in a particular year. From our analysis about the evolution of the actuarial liability and Definition 2.3 of normal cost, we have:

$$\begin{aligned}
(ULa)_{t+1} &= (ALa)_{t+1} - (Fa)_{t+1} \\
&= [(ALa)_t + NC_t](1+i) + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}\Big[(ALa)^j_{t+1} - (\widetilde{ALa})^j_{t+1}\Big] \\
&\quad -\Big[\sum_{j\in\mathcal{E}_t}(\widetilde{ALa})^j_{t+1} - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}(\widetilde{ALa})^j_{t+1}\Big] + \sum_{j\in\mathcal{N}_t}(ALa)^j_{t+1} \\
&\quad -\Big[(Fa)_t + (Ia)_t + C_t - PP_t\Big] \\
&= (ULa)_t(1+i) - [C_t + I_t^c - NC_t(1+i)] \\
&\quad -[(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}] + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}\Big[(ALa)^j_{t+1} - (\widetilde{ALa})^j_{t+1}\Big] \\
&\quad -\Big[\sum_{j\in\mathcal{E}_t}(\widetilde{ALa})^j_{t+1} - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}(\widetilde{ALa})^j_{t+1}\Big] \\
&\quad +\sum_{j\in\mathcal{N}_t}(ALa)^j_{t+1} + [PP_t + I_t^{PP}]\,, \qquad (2.11)
\end{aligned}$$

where I_t^c is the expected interest earned on the contributions C_t between years t and $t+1$ and I_t^{PP} is the same for annuity purchases. The later have been introduced motivated by (2.9) (see Anderson [1, p. 12].

Actuarial Gain $(Ga)_t$.

If the assumed interest rate equals the actual interest rate, that is, if $i = i'$, then $(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP} = 0$ [see (2.9)]. On the other hand

$$\sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}\Big[(ALa)^j_{t+1} - (\widetilde{ALa})^j_{t+1}\Big] = \sum_{j\in\mathcal{A}_{t+1}\mathcal{A}_t}\sum_{k=x\vee\underline{r}}^{\overline{r}} M_{x+1}\,(b^j_{t+1} - b^j_t)\;{}_{k-x|}\ddot{a}_x^{(12)}\, q_k^{(r)}(1+i) = 0\,,$$

since $b^j_{t+1} = b^j_t$ if no unexpected changes are experienced in the benefit amount during year t. With regard to the fifth term in (2.11), recall that $\mathcal{E}_t = \mathcal{T}_t \cup \mathcal{H}_t \cup \mathcal{D}_t \cup \mathcal{R}_t$, in consequence, if our service table is correct, all the components in

$$\left[\sum_{j \in \mathcal{E}_t} (\widetilde{ALa})^j_{t+1} - \sum_{j \in \mathcal{A}_t} q^{(\tau)}_x (\widetilde{ALa})^j_{t+1}\right]$$

will be zero. That is, the expected release of liability on account of termination (by any cause) of employment before getting retired will exactly offset the actual amount of accrued liability released on account of employees who actually did leave $\mathcal{A}_t$ (members of set $\mathcal{E}_t$). Finally, the liability produced by new entrants $\sum_{j \in \mathcal{N}_t} (ALa)_{t+1}$, will represent another source of deviation.

To sum up, if all our actuarial assumptions are met, then the third, fourth and fifth terms in (2.11) measures the deviation of $(ALa)_{t+1}$ from the ideal fund balance $(Fa)_{t+1}$ due to deviations from actual to *expected experience.* The sum of these terms is called the actuarial gain.

Remark 2.1 One could be tempted to say that $C_t + I^c_t - NC_t(1+i) = 0$ if contributions are collected as expected. Notice that the unfunded actuarial liability can shrink only if collected contributions are larger than the normal costs. That excess is called *supplemental cost*, and then, we say that

$$C_t = \begin{pmatrix} \text{normal costs} \\ \text{to fund } AL_t \end{pmatrix} + \begin{pmatrix} \text{supplemental costs} \\ \text{to fund } UL_t \end{pmatrix},$$

where AL_t and UL_t are the actuarial liability and the unfunded for the entire plan. Therefore, if there exists an unfunded liability, the difference between the contributions and the normal costs can not be considered as a component of the gain: by definition, a "gain component" must come from the difference between the expected and observed experiences. Such difference will diminish the unfunded liability. There exists, however, a cost method where such difference is in fact a gain component. In that method, called Net Cost Method (abbreviated NET), the unfunded liability is forced to be zero at all times. Supplemental costs are not treated in this thesis; the reader is referred to Winklevoss [9, p. 96].

Definition 2.6 The *actuarial gain of the active population* is given by

$$(Ga)_t := (ULa)_t(1+i) - \left[C_t + I_t^c - NC_t(1+i)\right] - (ULa)_{t+1}\,, \qquad t \in \mathbb{N}\,. \quad (2.12)$$

Usually, if the actuarial gain is negative, it is called an *actuarial loss*. The term *actuarial gain* is used, regardless if its sign. Likewise, we could define $(Ga)_t$ as the summation of individual components (analysis of gain by source). This approach splits the actuarial gain in its components due to interest ($G_t^{(i)}$), termination by any reason ($G_t^{(\tau)}$), benefit changes ($G_t^{(s)}$), and new entrants ($G_t^{(n)}$), as follows:

$$(Ga)_t = G_t^{(i)} + G_t^{(\tau)} + G_t^{(s)} + G_t^{(n)}\,, \quad (2.13)$$

where for $t = 0, 1, 2, \ldots$:

$$\begin{aligned}
G_t^{(i)} &:= (Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}\,,\\
G_t^{(\tau)} &:= \sum_{j\in\mathcal{E}_t} (\widetilde{ALa})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)} (\widetilde{ALa})_{t+1}^j\,,\\
G_t^{(s)} &:= -\sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} \left[(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j\right],\\
G_t^{(n)} &:= -\sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j\,.
\end{aligned}$$

As $\mathcal{E}_t = \mathcal{D}_t + \mathcal{T}_t + \mathcal{H}_t + \mathcal{R}_t$, the gain source $G_t^{(\tau)}$ can be broken down in its components due to mortality ($G_t^{(d)}$), withdrawal ($G_t^{(w)}$), disability ($G_t^{(h)}$) and retirement ($G_t^{(r)}$). The latter is of particular importance. For a member $j \in \mathcal{R}_t$, $M_{x+1} = 1$. Also, in the sum that defines the individual actuarial liability [see (2.3)], all the terms are zero, except when $k = x + 1$. In particular, the expected actuarial liability turns to be

$$(\widetilde{ALa})_{t+1}^j = b_t^j \ddot{a}_{x+1}^{(12)}\,.$$

In other words, the retirement component can be written as

$$G_t^{(r)} = \sum_{j\in\mathcal{R}_t} b_t^j \ddot{a}_{x+1}^{(12)} - \sum_{j\in\mathcal{A}_t} q_x^{(r)} b_t^j \ddot{a}_{x+1}^{(12)}\,. \quad (2.14)$$

Example 2.2 Revisiting Example 2.1, assume that one year has passed since the first evaluation.

New information:

Date of evaluation ($t = 1$): January 1st, 2003.

Salary in 2002: 92,046 (instead of the 90,950 projected).

The evolution of the fund through 2002 was:

Fund balance of last year:	$(Fa)_0$	$=$	30,000.
Collected contributions:	C_0	$=$	10,000.
Interest returns:	$(Ia)_0$	$=$	2,500.
Actual fund balance:	$(Fa)_1$	$=$	42,500.

Similarly as before, total past salaries is 464,046, while future salaries are re–estimated as follows:

$$\text{projected salaries} = 92{,}046\ \ddot{s}_{\overline{19}|0.07} = 3{,}681{,}425.09\ .$$

The annual pension benefit becomes $b_1 = 0.02\,[464{,}046 + 3{,}681{,}425] = 82{,}909.42$. In particular, the (unexpected) increase in benefit is $b_1 - b_0 = 898.62$. The PVFB is:

$$(\Pi a)_1 = b_1\ {}_{19|}\ddot{a}_{46}^{(12)} = 230{,}533.36\ .$$

The actuarial liability and the normal cost are, respectively:

$$(ALa)_1 = M_{46}(\Pi a)_1 = 55{,}328.01$$

$$NC_1 = m_{46}(\Pi a)_1 = 9{,}221.33\ .$$

The unfunded liability is $(ULa)_1 = (ALa)_1 - (Fa)_1 = 12{,}828.01$. In particular, the gain for the first year of operation, $(Ga)_0$, can be obtained from (2.12):

$$(Ga)_0 = (ULa)_0(1+i) - \left[C_0 + I_0^c - NC_0(1+i)\right] - (ULa)_1 = -1{,}299.68\ .$$

Consider the gain by components. There are two sources of deviation from our prior assumptions: the unexpected increase in salary and interest returns less than expected. All the remaining terms in (2.13) are zero.

$$G_0^{(s)} \;=\; -\left[M_{46}\,(b_1 - b_0)\;{}_{20|}\ddot{a}_{45}^{(12)}\,(1+i)\right] = -599.68\;,$$
$$G_0^{(i)} \;=\; (Ia)_0 - i(Fa)_0 - I_0^c = -700\;.$$

Taking the sum, $(Ga)_0 = -1{,}299.68$, as before.

The negative gain is interpreted as an actuarial loss. Historically salary gains have been high compared with the other gain components. Pension actuaries have defined a cost method, for which the normal cost absorbs the salary gain component. This method is called *Individual Level Premium* [see Anderson [1, p. 25].

Comparison of Methods

Given two individual cost methods, it is interesting to determine which one will produce larger actuarial liabilities. The problem is reduced to a comparison of their respective accrual functions, as in the next lemma.

Lemma 2.1 Let M_1 and M_2 be two distinct accrual functions and $D : [a,\infty) \to [-1,1]$ be defined as $D(x) = M_{1x} - M_{2x}$. If $D'(a) > 0$ and $D'(x) = 0$ has a unique solution in (a,r), then $AL_{1t} > AL_{2t}$.

Proof. $D(a) = D(r) = 0$ since M_1 and M_2 coincides at these two points. It is clear that $D(x) > 0$ if $x \in (a,r)$. Then:

$$(ALa)_{1t} - (ALa)_{2t} = \sum_{j\in\mathcal{A}_t}\sum_{k=x\vee\underline{r}}^{\overline{r}} M_{1x}\; b_t^j\;{}_{k-x|}\ddot{a}_x^{(12)}\, q_k^{(r)} - \sum_{j\in\mathcal{A}_t}\sum_{k=x\vee\underline{r}}^{\overline{r}} M_{2x}\; b_t^j\;{}_{k-x|}\ddot{a}_x^{(12)}\, q_k^{(r)}$$
$$= \sum_{j\in\mathcal{A}_t}\sum_{k=x\vee\underline{r}}^{\overline{r}} D(x)\; b_t^j\;{}_{k-x|}\ddot{a}_x^{(12)}\, q_k^{(r)} > 0\;.$$

□

Examples of Individual Methods

Any function M satisfying the definition of an accrual function characterizes a cost method. However, only a few are used in practice. These can divided in two families:

the *benefit allocation methods*, and the *cost allocation methods.*

Projected Unit Credit (PUC) and the Benefit Allocation Methods

The accrual portion M_x is directly related to the accrued benefit that a participant has acquired at age x under provisions of the plan. From this family we choose the method defined by

$$m_x = \frac{1}{r-a} \qquad \text{and} \qquad M_x = \frac{x-a}{r-a}\,, \qquad x \in [a,r]\,. \tag{2.15}$$

This method is called *projected unit credit with a constant dollar benefit allocation.* Hereon, we will refer to it simply as the *projected unit credit* method and abbreviated as PUC.

Another member is given by:

$$m_x = \frac{b_x}{B_r} \qquad \text{and} \qquad M_x = \frac{B_x}{B_r}\,, \qquad x \in [a,r]\,,$$

called *accrued benefit* or *unprojected unit credit.* Here b_x is the benefit amount at age x and $B_x := \sum_{y=a}^{x-1} b_y$. Finally, the method given by:

$$m_x = \frac{s_x}{S_r} \qquad \text{and} \qquad M_x = \frac{S_x}{S_r}\,, \qquad x \in [a,r]\,,$$

is called *projected unit credit with a constant percent of salary benefit allocation*, where s_x refers to the salary scale, and $S_x := \sum_{y=a}^{x-1} s_y$.

Entry–age Normal (EAN) and the Cost Allocation Methods

In the family of cost allocation methods, the projected benefit is funded by a level contribution, from entry age to retirement. If the increase in salaries is not taken into account, then

$$m_x = \frac{{}_{x-a}p_a^{(\tau)}\, v^{x-a}}{\ddot{a}_{a:\overline{r-a}|}} \qquad \text{and} \qquad M_x = \frac{\ddot{a}_{a:\overline{x-a}|}}{\ddot{a}_{a:\overline{r-a}|}}\,, \qquad x \in [a,r]\,, \tag{2.16}$$

where the temporary annuity was defined in Section 1.1. This cost method is known *entry–age with constant dollar cost allocation method.*

If a salary increase is assumed, then:

$$m_x = \frac{s_x}{s_a} \frac{{}_{x-a}p_a^{(\tau)} v^{x-a}}{{}^s\ddot{a}_{a:\overline{r-a}|}} \qquad \text{and} \qquad M_x = \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}}, \qquad x \in [a, r], \tag{2.17}$$

where the salary–based temporary annuity was defined in Section 1.1, as well. This method is known as the *entry–age with constant percent of salary assumption cost allocation method.* This method will be referred as the EAN method.

For a newly hired employee age 30, Figure 2.1 presents the graph of m_x, the percentage of projected retirement benefits allocated to each age under various individual cost methods. Figure 2.2 shows the correspondent cumulative accrual functions. Notice that the entry–age methods yield larger cumulative benefit allocations than those of unit credit methods. For instance, 50% of the projected benefit is allocated at age 40 (just 10 years later) under the EAN method (constant percent version). This is in contrast to the accrued benefit method, which allocates that quantity by age 57. In both figures, the notation is as follows:

$EAN1$	Entry–age, constant dollar method ,
$EAN2$	Entry–age, constant percent method EAN ,
$PUC1$	Projected unit credit, constant dollar method PUC ,
$PUC2$	Projected unit credit, constant percent method ,
AB	Accrued benefit or unprojected unit credit .

From Figure 2.2, the following inequalities hold:

$$0 \le \frac{b_x}{B_r} \le \frac{S_x}{S_r} \le \frac{x-a}{r-a} \le \frac{{}^s\ddot{a}_{a:\overline{x-a}|}}{{}^s\ddot{a}_{a:\overline{r-a}|}} \le \frac{\ddot{a}_{a:\overline{x-a}|}}{\ddot{a}_{a:\overline{r-a}|}} \le 1, \qquad a \le x \le r, \tag{2.18}$$

where $a = 30$ and $r = 65$. If salaries are assumed to increase, Lemma 3.2 can be used to prove (2.18).

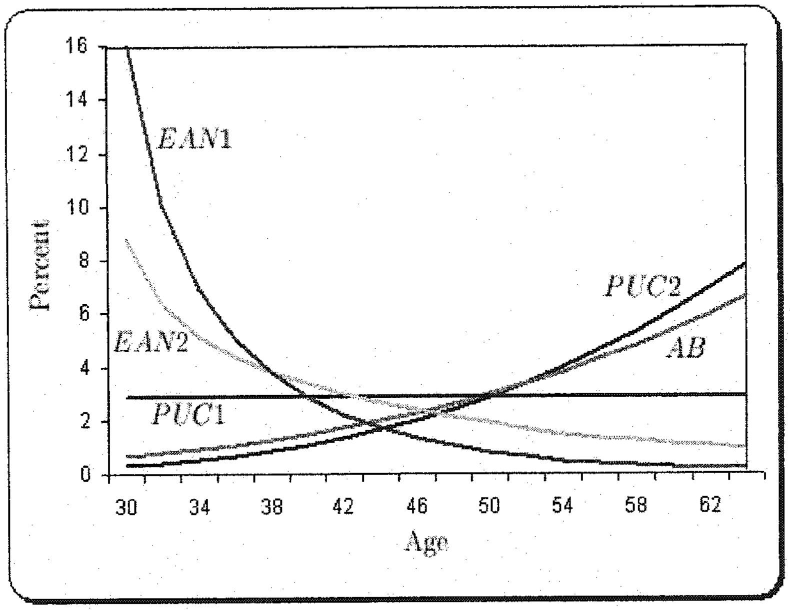

Figure 2.1: Mass accrual function for a newly hired person age 30 [Source: Winklevoss [9, p. 92, Figure 6–2].]

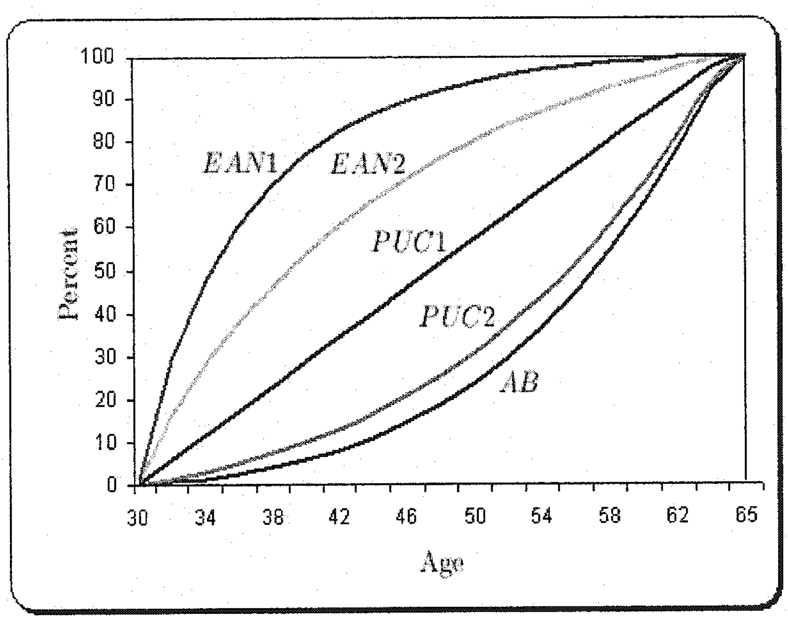

Figure 2.2: Cumulative accrual function for a newly hired person age 30 [Source: Winklevoss [9, p. 93, Figure 6–3]].

2.2 The Retired Population

This section reviews the mathematics that describes the retired population.

From year t to $t+1$, the set of pensioners $\mathcal{P}_t$ will experience an increase due to the new retirees $\mathcal{R}_t$ coming from the active population, and decreases by those who die during the year $\mathcal{D}_t$. Symbolically:

$$\mathcal{P}_{t+1} = \mathcal{P}_t + \mathcal{R}_t - \mathcal{D}_t \, .$$

Notice that the set $\mathcal{D}_t$ in the retired population is different from the one in the active population, though the same symbol is used. For the retired population, by *benefit function* we mean the amount of the pension payment. The amount of money needed to face future pension payments is called the *actuarial liability* or *present value of future benefits.*

The Actuarial Liability $(ALr)_t$

As mentioned in Chapter 1, a pensioner faces only one cause of decrement, namely death, which is the only uncertain event. In particular, the calculations use a mortality table (through the functions q_x and p_x) instead of the service table.

Definition 2.7 The *actuarial liability for the retired group*, denoted $(ALr)_t$, is given by:

$$(ALr)_t := \sum_{j \in \mathcal{P}_t} b_t^j \, \ddot{a}_x^{(12)} \, . \qquad (2.19)$$

The term $b_t^j \, \ddot{a}_x^{(12)}$ is interpreted as the actuarial liability of the retired member $j \in \mathcal{P}_t$. Compared with the active group, the study of the retired population is simpler, since no additional functions are required.

The backward equation [see (1.11)]:

$$\ddot{a}_{x+1} = (\ddot{a}_x - 1)\frac{(1+i)}{p_x} \, , \qquad x \in [r, \omega - x] \, , \qquad (2.20)$$

forms the basis of the study of gains for the retired population. Also, from (1.6) we have:

$$\ddot{a}_x = \left(\ddot{a}_x^{(12)} + \frac{i - i^{(12)}}{i^{(12)} d^{(12)}}\right)\left(\frac{i^{(12)} d^{(12)}}{id}\right), \qquad x \in [r, \omega - x]. \tag{2.21}$$

Putting together (2.20) and (2.21), it easily follows that:

$$\ddot{a}_{x+1}^{(12)} = \ddot{a}_x^{(12)}(1+i) - \left(\frac{i}{d^{(12)}} - \frac{(i - i^{(12)})}{i^{(12)} d^{(12)}} q_x\right) + q_x\, \ddot{a}_{x+1}^{(12)}. \tag{2.22}$$

Equation (2.22) does not coincide with that of Anderson [1, (2.10.4), p. 48]. Anderson's equation is based on the approximation $\ddot{a}_x^{(12)} \approx \ddot{a}_x - 11/24$ (see Bowers et al.[2, p. 151], for its derivation and comments), while (2.21) is based on the UDD assumption instead. Furthermore, note that

$$\frac{i}{d^{(12)}} \approx 1 + \frac{13}{24} i \qquad \text{and} \qquad \frac{(i - i^{(12)})}{i^{(12)} d^{(12)}} \approx \frac{11}{24}, \tag{2.23}$$

which are the approximations used by Anderson.

We are now in position to study the evolution of $(ALr)_t$ over time:

$$\begin{aligned}
(ALr)_{t+1} &= \sum_{j \in \mathcal{P}_{t+1}} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)} \\
&= \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{R}_t} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)}, \quad \text{since} \quad \mathcal{P}_{t+1} - \mathcal{R}_t = \mathcal{P}_{t+1} \cap \mathcal{P}_t \\
&= \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} b_t^j\, \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b_t^j\, \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{R}_t} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)} \\
&= \sum_{j \in \mathcal{P}_t} b_t^j \underbrace{\ddot{a}_{x+1}^{(12)}}_{\text{Use (2.22)}} - \sum_{j \in \mathcal{D}_t} b_t^j\, \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b_t^j\, \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{R}_t} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)} \\
&= (ALr)_t(1+i) - \sum_{j \in \mathcal{P}_t} b_t^j \left(\frac{i}{d^{(12)}} - \frac{(i - i^{(12)})}{i^{(12)} d^{(12)}} q_x\right) \\
&\quad - \left[\sum_{j \in \mathcal{D}_t} b_t^j \ddot{a}_{x+1}^{(12)} - \sum_{j \in \mathcal{P}_t} q_x b_t^j \ddot{a}_{x+1}^{(12)}\right] + \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b_t^j \ddot{a}_{x+1}^{(12)} + \sum_{j \in \mathcal{R}_t} b_{t+1}^j\, \ddot{a}_{x+1}^{(12)},
\end{aligned}$$

where $\Delta b_t^j := b_{t+1}^j - b_t^j$ is the change in benefit amount. In analogy with the active population, the term $b_t^j \ddot{a}_{x+1}^{(12)}$ is interpreted as the *expected* actuarial liability for retiree $j \in \mathcal{P}_t$.

Recall that

$$(Fr)_{t+1} = (Fr)_t + (Ir)_t + PP_t - B_t,$$

describes the evolution of the fund balance for retirees. Further, the amount B_t is divided as

$$B_t = NB_t + OB_t ,$$

where NB_t denotes the total of pension payments to members of set $\mathcal{R}_t$, who in general will have retired sometime during the year and received a few pension payments before the end of the year t. Similarly, OB_t is the total of pension payments paid to members of set $\mathcal{P}_t$ during year t.

The Unfunded Liability $(ULr)_t$

The difference between the actuarial liability and the real fund balance is also known as the unfunded liability:

$$(ULr)_t := (ALr)_t - (Fr)_t .$$

In ideal conditions, $(ALr)_t = (Fr)_t$, and there is no unfunded liability. Its evolution between instants t and $t+1$ is as follows:

$$\begin{aligned}
(ULr)_{t+1} &= (ALr)_t(1+i) - \sum_{j\in\mathcal{P}_t} b_t^j\Big(\frac{i}{d^{(12)}} - \frac{(i-i^{(12)})}{i^{(12)}d^{(12)}}q_x\Big) \\
&\quad - \Bigg[\sum_{j\in\mathcal{D}_t} b_t^j\,\ddot{a}_{x+1}^{(12)} - \sum_{j\in\mathcal{P}_t} q_x\, b_t^j\,\ddot{a}_{x+1}^{(12)}\Bigg] + \sum_{j\in\mathcal{P}_{t+1}\cap\mathcal{P}_t} \Delta b_t^j\,\ddot{a}_{x+1}^{(12)} + \sum_{j\in\mathcal{R}_t} b_{t+1}^j\,\ddot{a}_{x+1}^{(12)} \\
&\quad -[(Fr)_t + (Ir)_t + PP_t - NB_t - OB_t] \\
&= (ULr)_t(1+i) - \big((Ir)_t - i(Fr)_t - I_t^{PP} + I_t^B\big) \\
&\quad + \sum_{j\in\mathcal{P}_{t+1}\cap\mathcal{P}_t} \Delta b_t^j\,\ddot{a}_{x+1}^{(12)} - \Big[\sum_{j\in\mathcal{D}_t} b_t^j\,\ddot{a}_{x+1}^{(12)} - \sum_{j\in\mathcal{P}_t} q_x b_t^j\,\ddot{a}_{x+1}^{(12)}\Big] \\
&\quad -\Big[(PP_t + I_t^{PP}) - \sum_{j\in\mathcal{R}_t} b_{t+1}^j\,\ddot{a}_{x+1}^{(12)} - (NB_t + I_t^{NB})\Big] \\
&\quad - \Bigg[\sum_{j\in\mathcal{P}_t} b_t^j\Big(\frac{i}{d^{(12)}} - \frac{(i-i^{(12)})}{i^{(12)}d^{(12)}}q_x\Big) - (OB_t + I_t^{OB})\Bigg]
\end{aligned}$$

If all actuarial assumptions are met, all the terms above are zero. Therefore, it makes sense to define the actuarial gain for the retired population as:

$$(Gr)_t = (ULr)_t(1+i) - (ULr)_{t+1} , \qquad t \in \mathbb{N} . \tag{2.24}$$

By components, the gain is written as

$$(Gr)_t = G_t^{(i)} + G_t^{(b)} + G_t^{(mr)} + G_t^{(r)} + G_t^{(p)} \ ,$$

standing for the gains due to interest, benefit changes, retiree mortality, newly retirees and old pensioners (members in $\mathcal{P}_t$), defined, respectively, by:

$$\begin{aligned}
G_t^{(i)} &:= (Ir)_t - i(Fr)_t - I_t^{PP} + I_t^B \ , \\
G_t^{(b)} &:= -\sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b_t^j \, \ddot{a}_{x+1}^{(12)} \ , \\
G_t^{(mr)} &:= \Big[\sum_{j \in \mathcal{D}_t} b_t^j \, \ddot{a}_{x+1}^{(12)} - \sum_{j \in \mathcal{P}_t} q_x b_t^j \, \ddot{a}_{x+1}^{(12)} \Big] \ , \\
G_t^{(r)} &:= (PP_t + I_t^{PP}) - \Big(\sum_{j \in \mathcal{R}_t} b_{t+1}^j \, \ddot{a}_{x+1}^{(12)} + (NB_t + I_t^{NB}) \Big) \ ,
\end{aligned}$$

and

$$G_t^{(p)} := \Big[\sum_{j \in \mathcal{P}_t} b_t^j \Big(\frac{i}{d^{(12)}} - \frac{(i - i^{(12)})}{i^{(12)} d^{(12)}} q_x \Big) - (OB_t + I_t^{OB}) \Big] \ .$$

In practice, the benefit changes rarely, so the gain due to this source, $G_t^{(b)}$, is usually zero.

It is worth mentioning that Anderson's expression for $G_t^{(p)}$ is

$$G_t^{(p)} = \sum_{j \in \mathcal{P}_t} b_t^j \Big[1 + \frac{13}{24} i - \frac{11}{24} q_x \Big] - (OB_t - I_t^{OB}) \ ,$$

since he uses a different approximation for $\ddot{a}_{x+1}^{(12)}$ [see (2.23)].

2.3 Aggregate Cost Methods

In the above sections the basic functions for the individual cost methods were studied. The normal cost, in particular, was calculated as the sum of individual normal costs [see (2.3)]. In aggregate cost methods, on the other hand, the normal cost is defined for the entire group, that is, we do not produce individual normal costs. A unified approach is presented, which follows the ideas of Taylor [8].

Definition 2.8 Under the aggregate cost method, the *normal cost*, denoted NC_t, is given by:

$$NC_t := \beta(t)\big[\Pi_t - AL_t\big] \,, \tag{2.25}$$

where the parameter $\beta(t)$ measures the portion of future normal costs to be paid at the beginning of year t, and Π_t denotes the present value of future benefits for *the entire group*. It is given by

$$\Pi_t := \sum_{j\in\mathcal{A}_t} \sum_{k=x\vee\underline{r}}^{\overline{r}} b_t^j \;{}_{k-x|}\ddot{a}_x^{(12)} q_k^{(r)} + \sum_{j\in\mathcal{P}_t} b_t^j \;\ddot{a}_x^{(12)} \,. \tag{2.26}$$

Finally, AL_t is the parameter that represents the actuarial liability. The choice of these two parameters determine particular aggregate methods.

Before giving some possible choices for function β, a new random variable is needed. Let R_2 be the discrete random variable representing the retirement age *given* that the person does retire. The difference between R_2 and R is the fact that R_2 has a *conditional* distribution, with support on $\{x \vee \underline{r}, \ldots, \overline{r}\}$, and given by

$$p(k) := \Pr[R_2 = k] = \begin{cases} \frac{{}_{k-x}p_x^{(\tau)} q_k^{(r)}}{\sum_{k=x\vee\underline{r}}^{\overline{r}} {}_{k-x}p_x^{(\tau)} q_k^{(r)}}, & \text{if } k = x\vee\underline{r},\ldots,\overline{r}; \\ 0, & \text{elsewhere.} \end{cases} \tag{2.27}$$

In particular, if the plan is such that it is mandatory to retire at only one age, say r, then $p(k) = 1$ if $k = r$ and zero otherwise.

In practice, two possible choices for the parameter β in (2.25) are considered. If no salary increase is taken into account, then:

$$\beta(t) = \frac{n_t}{\sum_{j\in\mathcal{A}_t}\sum_{k=x}^{\overline{r}} \ddot{a}_{x:\overline{k-x}|}\, p(k)} = \frac{\text{number of employees in } \mathcal{A}_t}{\text{present value over future working years}} \,. \tag{2.28}$$

Notice that, if there is only one possible age of retirement r, then:

$$\beta(t) = \frac{\sum_{x=a}^{r-1} \ell_x}{\sum_{x=a}^{r-1} \ell_x \ddot{a}_{x:\overline{r-x}|}} \,, \qquad t\in\mathbb{N} \,,$$

which can be interpreted as the reciprocal of an average temporary annuity, weighted by ℓ_x. The second version assumes an increase of salaries, and it is of particular importance. The function β is given by:

$$\beta(t) = \frac{S_t}{FS_t} = \frac{\sum_{j\in\mathcal{A}_t} S_x^j}{\sum_{j\in\mathcal{A}_t}\sum_{k=x}^{\bar{r}} S_x^j \, {}^s\ddot{a}_{x:\overline{k-x}|}\, p(k)}\,, \qquad t \in \mathbb{N}\,, \tag{2.29}$$

and is the ratio of the sum of all the salaries and the present value of future salaries. If the plan allows to retire at only one age r, then (2.29) becomes:

$$\beta(t) = \frac{\sum_{x=a}^{r-1} \ell_x \, s_x}{\sum_{x=a}^{r-1} \ell_x \, s_x \, {}^s\ddot{a}_{x:\overline{r-x}|}}\,,$$

which can be interpreted as the reciprocal of an average salary–based temporary annuity weighted by $\ell_x s_x$.

It is not straightforward to obtain a general expression for the gain, unless the parameter β is known in advance. Hereon, we will take β as given by (2.29), since (2.28) can be seen as a special case, where $S_x^j = 1$ for all $j \in \mathcal{A}_t$.

A very common but erroneous assertion in the pension literature, is that for aggregate cost methods, the gain is zero (see Anderson [1, p. 34]). Another example can be found in Ramsay [6, p. 376]: "... assuming the gain is always zero, ...". However, he continues to define the formula [see (55)] to calculate the gain! Certainly any cost method has to have a mechanism to measure the deviations between expected and real experience, the aggregate family being no exception. The confusion can be clarified with the following assumption.

Assumption 2.1 *In aggregate cost methods, the gain for year t (if any), is spread into future normal costs.*

In terms of the normal cost, Assumption 2.1 means that NC_t already includes G_t. This gives us the framework to define NC_t in aggregate methods, and the relationship between its analogue in individual methods:

$$\begin{pmatrix} \text{normal cost in} \\ \text{aggregate methods} \end{pmatrix} = \begin{pmatrix} \text{normal cost} \\ \text{in the sense of} \\ \text{individual methods} \end{pmatrix} + \begin{pmatrix} \text{gains and losses} \\ \text{in the sense of} \\ \text{individual methods} \end{pmatrix}.$$

Recall that for the individual cost methods, the unfunded of year $t+1$ satisfies the recursive relation [see (2.12)]

$$(ULa)_{t+1} = (ULa)_t(1+i) - \left[C_t + I_t^c - NC_t(1+i)\right] - (Ga)_t\,. \tag{2.30}$$

In aggregate methods (2.30) takes the form

$$(ULa)_{t+1} = (ULa)_t(1+i) - \left[C_t + I_t^c - NC_t(1+i)\right]\,,$$

which can be rewritten as

$$\begin{aligned}(ALa)_{t+1} = (ALa)_t(1+i) - \left[(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}\right] \\ + (PP_t + I_t^{PP}) - NC_t(1+i)\,.\end{aligned} \tag{2.31}$$

An additional function is required to define G_t.

Definition 2.9 The *unit normal cost percentage* at instant t, is given by

$$U_t := \frac{\Pi_t - AL_t}{FS_t}\,, \qquad t \in \mathbb{N}\,. \tag{2.32}$$

In particular the normal cost satisfies $NC_t = U_t S_t$. Since the concepts of the PVFB for the retired group and the actuarial liability for the retired population coincides, symbolically, $(\Pi r)_t = (ALr)_t$, an equivalent definition of U_t would be

$$U_t = \frac{(\Pi a)_t - (ALa)_t}{FS_t}\,. \tag{2.33}$$

The idea is to study the evolution of the unit normal cost percentage. In doing so, the components in (2.33) are analysed separately. Firstly,

$$\begin{aligned}(\Pi a)_{t+1} &= \sum_{j\in\mathcal{A}_{t+1}} (\Pi a)_{t+1}^j \\ &= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (\Pi a)_{t+1}^j + \sum_{j\in\mathcal{N}_t} (\Pi a)_{t+1}^j\,, \quad \text{since} \quad \mathcal{A}_{t+1} - \mathcal{N}_t = \mathcal{A}_{t+1}\cap\mathcal{A}_t \\ &= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (\widetilde{\Pi} a)_{t+1}^j + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} \left[(\Pi a)_{t+1}^j - (\widetilde{\Pi} a)_{t+1}^j\right] + \sum_{j\in\mathcal{N}_t} (\Pi a)_{t+1}^j\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathcal{A}_t}\sum_{k=(x+1)\vee\underline{r}}^{\overline{r}} b_t^j \underbrace{{}_{k-x-1|}\ddot{a}_{x+1}^{(12)}}_{\text{Use (2.4)}} q_k^{(r)} - \sum_{j\in\mathcal{E}_t}(\widetilde{\Pi a})_{t+1}^j \\
&\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(\Pi a)_{t+1}^j - (\widetilde{\Pi a})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(\Pi a)_{t+1}^j \\
&= (\Pi a)_t\,(1+i) - \left[\sum_{j\in\mathcal{E}_t}(\widetilde{\Pi a})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}(\widetilde{\Pi a})_{t+1}^j\right] \\
&\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(\Pi a)_{t+1}^j - (\widetilde{\Pi a})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(\Pi a)_{t+1}^j\,, \qquad (2.34)
\end{aligned}
$$

where the tildes denote values calculated using the expected benefit at $t+1$, that is, they are computed as thought the increase in the benefit $\Delta b_t^j = b_{t+1}^j - b_t^j$ is zero.

The present value of future salaries on the other hand satisfies

$$
\begin{aligned}
FS_{t+1} &= \sum_{j\in\mathcal{A}_{t+1}}(FS)_{t+1}^j \\
&= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}(FS)_{t+1}^j + \sum_{j\in\mathcal{N}_t}(FS)_{t+1}^j\,, \quad \text{since} \quad \mathcal{A}_{t+1} - \mathcal{N}_t = \mathcal{A}_{t+1}\cap\mathcal{A}_t\,, \\
&= \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}(\widetilde{FS})_{t+1}^j + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(FS)_{t+1}^j - (\widetilde{FS})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(FS)_{t+1}^j \\
&= \sum_{j\in\mathcal{A}_t}(\widetilde{FS})_{t+1}^j - \sum_{j\in\mathcal{E}_t}(\widetilde{FS})_{t+1}^j + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(FS)_{t+1}^j - (\widetilde{FS})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(FS)_{t+1}^j \\
&= \sum_{j\in\mathcal{A}_t}\sum_{k=x+1}^{\overline{r}} S_t^j\,\frac{s_{x+1}}{s_x}\,{}^{s}\ddot{a}_{x+1:\overline{k-x-1}|}\,p(k) - \sum_{j\in\mathcal{E}_t}(\widetilde{FS})_{t+1}^j \\
&\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(FS)_{t+1}^j - (\widetilde{FS})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(FS)_{t+1}^j \\
&= FS_t\,(1+i) - S_t(1+i) - \left[\sum_{j\in\mathcal{E}_t}(\widetilde{FS})_{t+1}^j - \sum_{j\in\mathcal{A}_t}(\widetilde{FS})_{t+1}^j\right] \\
&\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t}[(FS)_{t+1}^j - (\widetilde{FS})_{t+1}^j] + \sum_{j\in\mathcal{N}_t}(FS)_{t+1}^j\,. \qquad (2.35)
\end{aligned}
$$

In the above, identity

$$
{}^{s}\ddot{a}_{x+1:\overline{k-x-1}|} = ({}^{s}\ddot{a}_{x:\overline{k-x}|} - 1)(1+i)\frac{s_x}{s_{x+1}} + q_x^{(\tau)};\,{}^{s}\ddot{a}_{x+1:\overline{k-x-1}|}
$$

was used, while $(\widetilde{FS})_{t+1}^j$ is the *expected* present value of future salaries of $j\in\mathcal{A}_t$, calculated by assuming that salaries increase according to the salary–scale function, that is,

$$(\widetilde{FS})^j_{t+1} = \sum_{k=x+1}^{\bar{r}} S^j_t \, \frac{s_{x+1}}{s_x} \, {}^s\ddot{a}_{x+1:\overline{k-x-1}|} \, p(k) \, .$$

Let

$$\begin{aligned}
N &:= \big((\Pi a)_t - (ALa)_t\big)(1+i) \, , \\
\Delta N &:= -\Big[\sum_{j\in\mathcal{E}_t}(\Pi a)^j_{t+1} - \sum_{j\in\mathcal{A}_t} q^{(\tau)}_x (\widetilde{\Pi}a)^j_{t+1}\Big] + \sum_{j\in\mathcal{N}_t}(\Pi a)^j_{t+1} \\
&\quad + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(\Pi a)^j_{t+1} - (\widetilde{\Pi}a)^j_{t+1}] - [(Ia)_t - i(Fa)_t - I^c_t + I^{PP}_t] \\
&\quad + [PP_t + I^{PP}_t] - NC_t(1+i) \, , \\
D &:= FS_t(1+i) \, , \\
\Delta D &:= -S_t\,(1+i) - \Big[\sum_{j\in\mathcal{E}_t}(\widetilde{FS})^j_{t+1} - \sum_{j\in\mathcal{A}_t} q^{(\tau)}_x (\widetilde{FS})^j_{t+1}\Big] + \sum_{j\in\mathcal{N}_t}(FS)^j_{t+1} \, .
\end{aligned}$$

Putting together (2.31), (2.34) and (2.35), and the facts that $N/D = U_t$ and $D + \Delta D = FS_{t+1}$, the unit normal cost percentage at instant $t+1$ can be expressed as

$$\begin{aligned}
U_{t+1} &= \frac{N + \Delta N}{D + \Delta D} \\
&= \frac{N}{D} - \frac{1}{D + \Delta D}\Big(-\Delta N + \frac{N}{D}\Delta D\Big) \\
&= U_t - \frac{1}{FS_{t+1}}(-\Delta N + U_t\,\Delta D) \\
&= U_t - \frac{1}{FS_{t+1}}\Big\{[(Ia)_t - i(Fa)_t - I^c_t + I^{PP}_t] \\
&\quad + \Big[\sum_{j\in\mathcal{E}_t}[(\widetilde{\Pi}a)^j_{t+1} - U_t\,(\widetilde{FS})^j_{t+1}] - \sum_{j\in\mathcal{A}_t} q^{(\tau)}_x\,[(\widetilde{\Pi}a)^j_{t+1} - U_t\,(\widetilde{FS})^j_{t+1}]\Big] \\
&\quad - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(\Pi a)^j_{t+1} - U_t\,(FS)^j_{t+1}] - [(\widetilde{\Pi}a)^j_{t+1} - U_t\,(\widetilde{FS})^j_{t+1}] \\
&\quad - \sum_{j\in\mathcal{N}_t}[(\Pi a)^j_{t+1} - U_t\,(FS)^j_{t+1}] - [PP_t + I^{PP}_t]\Big\} \, , \qquad (2.36)
\end{aligned}$$

since $U_t\ S_t = NC_t$. There is something interesting about (2.36) that suggests the definition of the "individual actuarial liability" in aggregate methods. Let

$$\begin{aligned}
(ALa)^j_{t+1} &:= (\Pi a)^j_{t+1} - U_t\,(FS)^j_{t+1} \\
(\widetilde{ALa})^j_{t+1} &:= (\widetilde{\Pi}a)^j_{t+1} - U_t\,(\widetilde{FS})^j_{t+1} \, .
\end{aligned}$$

These definitions are consistent with (2.33) and the fact that the actuarial liability for the entire group is the sum of a lot of actuarial liabilities. As a consequence, (2.36) can be rewritten as

$$\begin{aligned} U_{t+1} &= U_t - \frac{1}{FS_{t+1}} \{[(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}] \\ &\quad + \Big[\sum_{j\in\mathcal{E}_t} (\widetilde{ALa})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)} (\widetilde{ALa})_{t+1}^j\Big] \\ &\quad - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] \\ &\quad - \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j - [PP_t + I_t^{PP}]\Big\} . \end{aligned} \tag{2.37}$$

Once again, under normal circumstances all the items within braces in (2.37), except $-[PP_t + I_t^{PP}]$ will be zero. Thus, it makes sense to define the gain as follows.

Definition 2.10 At instant t, the *actuarial gain for the active population*, denoted $(Ga)_t$, is given by:

$$(Ga)_t = (U_t - U_{t+1})\, FS_{t+1} , \qquad t \in \mathbb{N} . \tag{2.38}$$

The gain components are (definition of the gain by source):

$$(Ga)_t = G_t^{(i)} + G_t^{(\tau)} + G_t^{(s)} + G_t^{(n)} , \tag{2.39}$$

where for $t = 0, 1, 2, \ldots$:

$$\begin{aligned} G_t^{(i)} &:= (Ia)_t - i(Fa)_t - I_t^c + I_t^{PP} , \\ G_t^{(\tau)} &:= \sum_{j\in\mathcal{E}_t} (\widetilde{ALa})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)} (\widetilde{ALa})_{t+1}^j , \\ G_t^{(s)} &:= - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] , \\ G_t^{(n)} &:= - \sum_{j\in\mathcal{N}_t} (ALa)_{t+1}^j . \end{aligned}$$

Notice that both, aggregate and individual cost methods have the same expression of the gain by components, [compare (2.39) and (2.13)]. Again, writing down $\mathcal{E}_t = \mathcal{T}_t + \mathcal{H}_t + \mathcal{D}_t + \mathcal{R}_t$, the gain due to departures from the active status can be decomposed into the sum of four components. Expressions for these four components are all similar.

Certainly, the parameters β and AL_0 in the aggregate methods can be chosen arbitrarily, though only a few choices are recognized in practice. In this work, we assume the parameter β given by (2.29), that is, salaries are assumed to vary with time. Thus it all reduces to assigning the value for AL_0.

Examples of Aggregate Cost Methods

Three methods are considered.

Frozen Initial Liability (FIL) The actuarial liability is given recursively by (2.31), where the initial liability AL_0 (starting value) is computed by using the so–called aggregate entry–age cost method (see Anderson [1, p. 41]), where

$$AL_0 = \Pi_0 - \frac{\Pi W_0}{FSW_0} FS_0 \ , \tag{2.40}$$

and ΠW_0 is the present value of future benefits calculated at age of hire a, while

$$FSW_0 := \sum_{j \in \mathcal{A}_t} \sum_{k=\underline{r}}^{\overline{r}} S_0^j \ \frac{s_a}{s_x} \ {}^s\ddot{a}_{a:\overline{k-a}|} \ p(k) \ ,$$

is the present value of future salaries discounted to age at hire. Finally, FS_0 is the present value of future salaries discounted at valuation date. Formula (2.40) is not used by Ramsay, since it leads to a contradiction; see Anderson [1, p. 41] for comments.

Attained Age Normal (AAN) The starting value in (2.31) is the actuarial liability of the projected unit credit method computed at instant $t = 0$ [see (2.15)], that is:

$$AL_0 = \sum_{j \in \mathcal{A}_0} \sum_{k=x_0 \vee \underline{r}}^{\overline{r}} \left(\frac{x_0 - a}{k - a} \right) b_0^j \ {}_{k-x_0}|\ddot{a}_{x_0}^{(12)} q_k^{(r)} + (ALr)_0 \ ,$$

where x_0 is the age of member $j \in \mathcal{A}_0$.

Net Method (NET) Here, for any time $t \in \mathbb{N}$,

$$AL_t = F_t \ ,$$

which implies that there is no unfunded liability, and it is the only method with such characteristic. This fact introduces an important difference with the other aggregate methods. Furthermore, the evolution of U_t between t and $t+1$ turns to be

$$\begin{aligned} U_{t+1} &= U_t - \frac{1}{FS_{t+1}} \Big\{ [(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}] \\ &\quad + \Big[\sum_{j \in \mathcal{E}_t} (\widetilde{ALa})_{t+1}^j - \sum_{j \in \mathcal{A}_t} q_x^{(\tau)} (\widetilde{ALa})_{t+1}^j \Big] \\ &\quad - \sum_{j \in \mathcal{A}_{t+1} \cap \mathcal{A}_t} [(ALa)_{t+1}^j - (\widetilde{ALa})_{t+1}^j] \\ &\quad - \sum_{j \in \mathcal{N}_t} (ALa)_{t+1}^j - [PP_t + I_t^{PP}] - [C_t + I_t^c - NC_t(1+i)] \Big\} \ . \end{aligned} \tag{2.41}$$

In comparing with (2.37), an extra term appears in the decomposition of the gain, namely,

$$G_t^{(c)} := C_t + I_t^c - NC_t(1+i) \ ,$$

due to excess in contributions.

Recall Remark 2.1 about the above term. We argued why such term can not be considered as a gain component, since the difference goes to amortize any unfunded liability. But NET there is no unfunded liability, and therefore, no supplemental costs. In other words, the normal cost is an estimate value of what we think the *total* contribution should be. Any variation (contributions less than expected) should be considered as a gain (or loss) component.

It is worth mentioning, that traditionally this method is referred to as the *Aggregate* or *Global* cost method. But this can be confusing, since the Frozen Initial Liability and Attained Age Normal are also aggregate methods. The choice of the name comes from the fact that the normal cost not only will absorb unexpected deviations (gains and losses), but also the unfunded is forced to be included. In other words, there is zero unfunded liability, since the actuarial liability is *defined* to be the real fund balance.

Chapter 3

Percentile Cost Methods

The primary objective of any pension plan is to ensure the financial stability of current employees after retirement, by means of funding of deferred annuities throughout their working life. In particular, it is desirable to maintain a high likelihood of receiving such benefits.

In Chapter 2 the basic valuation functions were defined. These are based on the present value of future benefits, which contains $\ddot{a}_x^{(12)}$, a mean value term. Despite the wide use of this approach, it is natural to ask if this mean value is an appropriate estimator of the amount needed to pay future pensions, specifically when dealing with small plans with large experience variabilities. Percentile cost methods may provide the answer to this problem, replacing the mean value term $\ddot{a}_x^{(12)}$ with a percentile ${}_\alpha\ddot{\xi}_x^{(12)}$ value. Reconsider the formulas of Chapter 2 integrating this change. To distinguish them, append the subscript α before any particular function; e.g. the normal cost for a percentile method is denoted ${}_\alpha NC_t$.

We formalize the ideas discussed above as follows.

Definition 3.1 An *α−percentile cost method* is an actuarial cost method that funds promised benefits so that the ideal fund balance for retirees is the lump-sum amount such that there is a $100\alpha\%$ chance of paying the promised lifetime benefits if no further contributions are paid into the fund.

3.1 Individual Percentile Cost Methods

In percentile cost methods, the annuity factor $\ddot{a}_x^{(12)}$ is replaced by the confidence function ${}_\alpha\ddot{\xi}_x^{(12)}$. All the formulas of Chapter 2 remain the same, except for this change. However the interpretation of the formulas shifts to a percentile approach, instead of mean values.

An analogue to the deferred annuity is introduced in order to reduce notation. Let

$$ {}_{k-x|\alpha}\ddot{\xi}_x^{(12)} := {}_{k-x}p_x^{(\tau)} v^{k-x} {}_\alpha\ddot{\xi}_k^{(12)} , \qquad k \geq x , $$

be the *deferred percentile function.* The analogue to (2.4) then becomes

$$ {}_{k-x-1|\alpha}\ddot{\xi}_{x+1}^{(12)} = {}_{k-x|\alpha}\ddot{\xi}_x^{(12)}(1+i) + q_x^{(\tau)}\, b_{t+1}^j \, {}_{k-x-1|\alpha}\ddot{\xi}_{x+1}^{(12)} . $$

Definition 3.2 At instant $t \in \mathbb{N}$, the formulas for the plan's valuation are defined as:

$$
\begin{aligned}
{}_\alpha(\Pi a)_t &:= \sum_{j\in\mathcal{A}_t} {}_\alpha(\Pi a)_t^j = \sum_{j\in\mathcal{A}_t} \sum_{k=x\vee\underline{r}}^{\overline{r}} b_t^j \, {}_{k-x|\alpha}\ddot{\xi}_x^{(12)} q_k^{(r)} , \\
{}_\alpha(ALa)_t &:= \sum_{j\in\mathcal{A}_t} {}_\alpha(ALa)_t^j = \sum_{j\in\mathcal{A}_t} M_x \, {}_\alpha(\Pi a)_t^j , \\
{}_\alpha NC_t &:= \sum_{j\in\mathcal{A}_t} {}_\alpha NC_t^j = \sum_{j\in\mathcal{A}_t} m_x \, {}_\alpha(\Pi a)_t^j , \\
{}_\alpha FNC_t &:= {}_\alpha(\Pi a)_t - {}_\alpha(ALa)_t = \sum_{j\in\mathcal{A}_t} (1-M_x) \, {}_\alpha(\Pi a)_t^j \qquad \text{and} \\
{}_\alpha(ULa)_t &:= {}_\alpha(ALa)_t - (Fa)_t .
\end{aligned}
$$

The quantities with the superscript j are interpreted as related to employee $j \in \mathcal{A}_t$.

Definition 3.3 The α–actuarial gain is given by

$$ {}_\alpha(Ga)_t := {}_\alpha(ULa)_t(1+i) - \left[C_t + I_t^c - {}_\alpha NC_t(1+i)\right] - {}_\alpha(ULa)_{t+1} , \tag{3.1} $$

and can be decomposed as:

$$ {}_\alpha(Ga)_t = {}_\alpha G_t^{(i)} + {}_\alpha G_t^{(\tau)} + {}_\alpha G_t^{(s)} + {}_\alpha G_t^{(n)} , \tag{3.2} $$

where for $t \in \mathbb{N}$,

$$\begin{aligned}
{}_{\alpha}G_t^{(i)} &:= (Ia)_t - i(Fa)_t - I_t^c + I_t^{PP} ,\\
{}_{\alpha}G_t^{(\tau)} &:= \sum_{j\in\mathcal{E}_t} {}_{\alpha}(\widetilde{ALa})_{t+1}^j - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}\, {}_{\alpha}(\widetilde{ALa})_{t+1}^j ,\\
{}_{\alpha}G_t^{(s)} &:= -\sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} \left[{}_{\alpha}(ALa)_{t+1}^j - {}_{\alpha}(\widetilde{ALa})_{t+1}^j \right] ,\\
{}_{\alpha}G_t^{(n)} &:= -\sum_{j\in\mathcal{N}_t} {}_{\alpha}(ALa)_{t+1}^j .
\end{aligned}$$

Again, as $\mathcal{E}_t = \mathcal{D}_t + \mathcal{T}_t + \mathcal{H}_t + \mathcal{R}_t$, the gain source ${}_{\alpha}G_t^{(\tau)}$ can be broken down in their components due to mortality (${}_{\alpha}G_t^{(d)}$), withdrawal (${}_{\alpha}G_t^{(w)}$), disability (${}_{\alpha}G_t^{(h)}$) and retirement (${}_{\alpha}G_t^{(r)}$). In particular, since

$${}_{\alpha}(\widetilde{ALa})_{t+1}^j = b_t^j\, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} ,$$

for a member on set $\mathcal{R}_t$, the retirement component can be written as

$${}_{\alpha}G_t^{(r)} = \sum_{j\in\mathcal{R}_t} b_t^j\, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \sum_{j\in\mathcal{A}_t} q_x^{(r)}\, b_t^j\, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} .$$

Examples of Individual Percentile Cost Methods

In the traditional approach, individual cost methods were divided in two classes: the benefit and the cost allocation methods. An individual percentile cost method is still defined through the choice of an accrual function. Two individual percentile cost methods can be distinguished:

Percentile projected unit credit (αPUC): given by

$$m_x = \frac{1}{r-a} \quad \text{and} \quad M_x = \frac{x-a}{r-a} , \qquad x \in [a, r] . \tag{3.3}$$

Percentile entry-age normal (αEAN): defined by

$$m_x = \frac{s_x\; {}_{x-a}p_a^{(\tau)} v^{x-a}}{s_a \quad {}^{s}\ddot{a}_{a:\overline{r-a}|}} \quad \text{and} \quad M_x = \frac{{}^{s}\ddot{a}_{a:\overline{x-a}|}}{{}^{s}\ddot{a}_{a:\overline{r-a}|}} , \qquad x \in [a, r] . \tag{3.4}$$

3.2 The Retired Population: Individual Approach

The development of the formulas are quite similar to the analogue of the traditional approach.

Definition 3.4 The α–actuarial liability for the retired group, denoted ${}_{\alpha}(ALr)_t$, is given by:

$$
{}_{\alpha}(ALr)_t := \sum_{j\in\mathcal{P}_t} b_t^j \; {}_{\alpha}\ddot{\xi}_x^{(12)} \; . \tag{3.5}
$$

As before, the term $b_t^j \; {}_{\alpha}\ddot{\xi}_x^{(12)}$ can be interpreted as the α–actuarial liability of the retired member $j \in \mathcal{P}_t$.

For the percentile cost methods, the analogue of formula (2.20) is

$$
{}_{\alpha}\ddot{\xi}_{x+1}^{(12)} = {}_{\alpha}\ddot{\xi}_x^{(12)} - \left(\ddot{s}_{\overline{1}|}^{(12)} + {}_{\alpha}\ddot{\theta}_x^{(12)}(1+i)\right) + q_x \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} \; , \tag{3.6}
$$

which follows directly from (1.13). Thus, the evolution is found in a way similar to the traditional approach:

$$
\begin{aligned}
{}_{\alpha}(ALr)_{t+1} &= {}_{\alpha}(ALr)_t(1+i) + \sum_{j\in\mathcal{P}_{t+1}\cap\mathcal{P}_t} (\Delta b_t^j) \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} \\
&- \left[\sum_{j\in\mathcal{D}_t} b_t^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \sum_{j\in\mathcal{P}_t} q_x \; b_t^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)}\right] + \sum_{j\in\mathcal{R}_t} b_{t+1}^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \sum_{j\in\mathcal{P}_t} b_t^j \; \left(\ddot{s}_{\overline{1}|}^{(12)} + {}_{\alpha}\ddot{\theta}_x^{(12)}(1+i)\right) .
\end{aligned}
$$

The evolution of the α–unfunded liability ${}_{\alpha}(ULr)_t = {}_{\alpha}(ALr)_t - (Fr)_t$ between instants t and $t+1$ is as follows:

$$
\begin{aligned}
{}_{\alpha}(ULr)_{t+1} \;\; = \;\; & {}_{\alpha}(ULr)_t(1+i) - \left[(Ir)_t - i(Fr)_t - I_t^{PP} + I_t^B\right] \\
& + \sum_{j\in\mathcal{P}_{t+1}\cap\mathcal{P}_t} \Delta b_t^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \left[\sum_{j\in\mathcal{D}_t} b_t^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \sum_{j\in\mathcal{P}_t} q_x \; b_t^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)}\right] \\
& - \left[(PP_t + I_t^{PP}) - \sum_{j\in\mathcal{R}_t} b_{t+1}^j \; {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - (NB_t + I_t^{NB})\right] \\
& - \left[\sum_{j\in\mathcal{P}_t} b_t^j\left(\ddot{s}_{\overline{1}|}^{(12)} + {}_{\alpha}\ddot{\theta}_x^{(12)}(1+i)\right) - (OB_t + I_t^{OB})\right] .
\end{aligned}
$$

Therefore, the gain can be defined as

$$
{}_{\alpha}(Gr)_t = {}_{\alpha}(ULr)_t \; (1+i) - {}_{\alpha}(ULr)_{t+1} \; ,
$$

or by components:

$$ {}_\alpha(Gr)_t = {}_\alpha G_t^{(i)} + {}_\alpha G_t^{(b)} + {}_\alpha G_t^{(mr)} + {}_\alpha G_t^{(r)} + {}_\alpha G_t^{(p)} \ , $$

where for $t \in \mathbb{N}$:

$$ \begin{aligned} {}_\alpha G_t^{(i)} &:= (Ir)_t - i(Fr)_t - I_t^{PP} + I_t^B \ , \\ {}_\alpha G_t^{(b)} &:= - \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b_t^j \ {}_\alpha\ddot{\xi}_{x+1}^{(12)} \ , \\ {}_\alpha G_t^{(mr)} &:= \left[\sum_{j \in \mathcal{D}_t} b_t^j \ {}_\alpha\ddot{\xi}_{x+1}^{(12)} - \sum_{j \in \mathcal{P}_t} q_x \ b_t^j \ {}_\alpha\ddot{\xi}_{x+1}^{(12)} \right] , \\ {}_\alpha G_t^{(r)} &:= (PP_t + I_t^{PP}) - \left(\sum_{j \in \mathcal{R}_t} b_{t+1}^j \ {}_\alpha\ddot{\xi}_{x+1}^{(12)} + (NB_t + I_t^{NB}) \right) , \\ {}_\alpha G_t^{(p)} &:= \left[\sum_{j \in \mathcal{P}_t} b_t^j \left(\ddot{s}_{\overline{1}|}^{(12)} + {}_\alpha\ddot{\theta}_x^{(12)}(1+i) \right) - (OB_t + I_t^{OB}) \right] . \end{aligned} $$

3.3 Aggregate Percentile Cost Methods

In traditional cost methods (individual and aggregate versions), as well as in the individual percentile cost methods, the PVFB is defined as the sum of individual PVFB's [see (2.2), (2.26) and Definition 3.2]. In aggregate percentile cost methods, however, we do not produce individual PVFB's. Instead, the PVFB is defined as the amount required to fund the projected retirement annuities for all participants with a specified overall probability α.

Definition 3.5 Given a level of confidence $0 \le \alpha \le 1$, the *α–PVFB for the entire group* at instant t, is the quantity ${}_\alpha\Pi_t$ such that:

$$ \Pr[X_t \le {}_\alpha\Pi_t] = \alpha \ , \qquad t \in \mathbb{N} \ , \tag{3.7} $$

where $X_t := (Xa)_t + (Xr)_t$ is the random variable of the PVFB for the entire group,

$$ (Xa)_t := \begin{cases} \sum_{j \in \mathcal{A}_t} b_t^j \ v^{R-x} \ \ddot{Y}_R^{(12)}, & \text{if } R \ne \infty; \\ 0, & \text{otherwise} \end{cases} , \tag{3.8} $$

$$ (Xr)_t := \sum_{j \in \mathcal{P}_t} b_t^j \ \ddot{Y}_R^{(12)} \ , \tag{3.9} $$

are the random variables of the PVFB for the active and retired populations, respectively.

Note that $\mathbb{E}[(Xa)_t] = (\Pi a)_t$ and $\mathbb{E}[(Xr)_t] = (ALr)_t$ are the traditional measures of the PVFB's for the active population and retired populations, respectively.

In order to evaluate the probability in (3.7), the distribution of X_t, which is a sum of independent (though not necessarily identically distributed) random variables, must be approximated. The well–known normal approximation is our first step.

Approximation 1: Normal. If X is a random variable with mean μ and standard deviation σ, then:

$$\Pr[X \leq x] \approx \Phi\Big(\frac{x-\mu}{\sigma}\Big) , \qquad x \in \mathbb{R} , \tag{3.10}$$

where

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-y^2}}{\sqrt{2\pi}} dy ,$$

is the standard normal distribution function. In particular, the (approximated) solution of the equation

$$\Pr[X \leq x] = \alpha$$

is

$$x \approx \mu + z_\alpha \, \sigma , \tag{3.11}$$

where z_α is the $100\alpha^{th}$ percentile point of the $N(0,1)$ distribution. The accuracy of the normal approximation depends upon the skewness of X, which in turn depends, in our particular case, on the number of active participants. Then, when the skewness is not close to zero (usually in small plans with large experience variations), a more accurate approximation is required.

Proposition 3.1 Haldane's Type A Approximation. If X is a random variable with mean μ, standard deviation σ and index of skewness γ, then

$$\Pr[X \leq x] \approx \Phi\left(\frac{(1+sz)^h - \psi(h,s)}{\varphi(h,s)}\right) , \qquad x \in \mathbb{R} , \tag{3.12}$$

where Φ is the standard normal distribution function and

$$z := (x-\mu)/\sigma\,, \qquad s := \sigma/\mu\,, \qquad h := 1-\tfrac{1}{3s}\gamma\,,$$
$$\psi(h,s) := 1-\tfrac{1}{2}h(1-h)\big[1-\tfrac{1}{4}(2-h)(1-3h)s^2\big]s^2 \qquad \text{and}$$
$$\varphi(h,s) := hs\sqrt{1-\tfrac{1}{2}(1-h)(1-3h)s^2}\,.$$

Proof. Can be found in Pentikäinen [5]. □

Notice that if the skewness is zero, Haldane's approximation becomes the normal approximation.

In this case, the solution of the equation

$$\Pr[X \le x] = \alpha$$

is approximated by

$$x \approx \mu + \frac{\sigma}{s}\Big[\big(\psi(h,s) + z_\alpha \varphi(h,s)\big)^{1/h} - 1\Big]\,. \tag{3.13}$$

To apply Haldane's approximation, the condition $1-\frac{1}{2}(1-h)(1-3h)s^2 \ge 0$ is required. In terms of the central moments, that means

$$\gamma^2 - 2\frac{\sigma\gamma}{\mu} \le 6\,.$$

Remark 3.1 There exist others possible approximations. For instance, the so–called Haldane Type B approximation, requires the computation of the kurtosis. In this case, the kurtosis of $\ddot{Y}_x^{(m)}$ is easily obtained:

$$\ddot{\zeta}_x^{(m)} = \frac{{}^4A_x^{(m)} - 4\,{}^3A_x^{(m)}\,A_x^{(m)} + 6\,{}^2A_x^{(m)}\,(A_x^{(m)})^2 - 3(A_x^{(m)})^4}{\Big[{}^2A_x^{(m)} - (A_x^{(m)})^2\Big]^2}\,,$$

that would be used to compute the kurtosis of $(Xa)_t$.

Still another one is the Wilson–Hilferty approximation, that requires the first three central moments. The approximation can be written in operative way as follows:

$$\Pr[X \le x] \approx \Phi\left(c_1 + c_2\Big(z + c_3\Big)^{\frac{1}{3}}\right)\,,$$

where

$$c_1 = \frac{\gamma}{6} - \frac{6}{\gamma}\,, \qquad c_2 = 3\left(\frac{2}{\gamma}\right)^{\frac{2}{3}} \qquad c_3 = \frac{2}{\gamma} \qquad \text{and} \qquad z = \frac{x-\mu}{\sigma}\,.$$

In this case,

$$x \approx \mu + \sigma\left[\left(\frac{z_\alpha - c_1}{c_2}\right)^3 - c_3\right]\,.$$

Haldane type–A is chosen, because it seems to be the most accurate, according to Pentikäinen's results (see [5]).

The following property will be useful.

Proposition 3.2 If $X_1, X_2, \ldots, X_n$ are n independent random variables, while σ_j^2 and γ_j are, respectively, the variance and the index of skewness of X_j, $j = 1, 2, \ldots, n$, then the index of skewness of $X = \sum_{j=1}^n X_j$ is given by

$$\gamma[X] = \frac{1}{\sigma_X^3}\sum_{j=1}^n \sigma_j^3\gamma_j\,, \qquad \text{where} \qquad \sigma_X^2 = \sum_{j=1}^n \sigma_j^2\,.$$

Proof. By induction over n; the result is trivial for $n = 1$. Assume that it also holds true for arbitrary n and let $Y = \sum_{j=1}^{n+1} X_j = X + X_{n+1}$. Since X and X_{n+1} are independent, then:

$$\begin{aligned}
\gamma[Y] &= \frac{1}{\sigma_Y^3}\mathbb{E}\{[Y - \mathbb{E}(Y)]^3\} = \frac{1}{\sigma_Y^3}\mathbb{E}\{[(X + X_{n+1}) - \mathbb{E}(X + X_{n+1})]^3\} \\
&= \frac{1}{\sigma_Y^3}\mathbb{E}[(X - \mathbb{E}(X))^3] + \mathbb{E}[(X_{n+1} - \mathbb{E}(X_{n+1}))^3] \\
&= \frac{1}{\sigma_Y^3}(\sigma_X^3\,\gamma[X] + \sigma_{n+1}^3\,\gamma_{n+1})\,, \\
&= \frac{1}{\sigma_Y^3}\left(\sum_{j=1}^n \sigma_j^3\,\gamma_j + \sigma_{n+1}^3\,\gamma_{n+1}\right) = \frac{1}{\sigma_Y^3}\sum_{j=1}^{n+1}\sigma_j^3\gamma_j\,,
\end{aligned}$$

and $\sigma_Y^2 = \sum_{j=1}^{n+1}\sigma_j^2$, by independence. □

To calculate the variance and index of skewness of X_t, first we work with its components, $(Xa)_t$ and $(Xr)_t$.

Proposition 3.3 The variance $(\sigma a)_t^2 := \text{Var}[(Xa)_t]$ and index of skewness $(\gamma a)_t := \gamma[(Xa)_t]$ of $(Xa)_t$ are given by

$$(\sigma a)_t^2 = \sum_{j \in \mathcal{A}_t} \left[(\sigma a)_t^j\right]^2 = \sum_{j \in \mathcal{A}_t} \left\{ \sum_{k=x \vee \underline{r}}^{\overline{r}} (b_t^j \, v^{k-x})^2 [(\ddot{\sigma}_k^{(12)})^2 + (\ddot{a}_k^{(12)})^2]_{k-x} p_x^{(\tau)} q_k^{(r)} - \left[(\Pi a)_t^j\right]^2 \right\} \tag{3.14}$$

and

$$(\gamma a)_t = \frac{1}{(\sigma a)_t^3} \sum_{j \in \mathcal{A}_t} \left[(\sigma a)_t^j\right]^3 (\gamma a)_t^j \,, \tag{3.15}$$

where

$$\left[(\sigma a)_t^j\right]^3 (\gamma a)_t^j = \sum_{k=x \vee \underline{r}}^{\overline{r}} (b_t^j \, v^{k-x})^3 [\ddot{\gamma}_k^{(12)} (\ddot{\sigma}_k^{(12)})^3 + (\ddot{a}_k^{(12)})^3 + 3\ddot{a}_k^{(12)} (\ddot{\sigma}_k^{(12)})^2] \,_{k-x} p_x^{(\tau)} q_k^{(r)}$$
$$- 3 \left[(\sigma a)_t^j\right]^2 \left[(\Pi a)_t^j\right] - \left[(\Pi a)_t^j\right]^3 \,, \tag{3.16}$$

that is, $(\gamma a)_t^j = \gamma[(Xa)_t^j]$ denotes the index of skewness of $(Xa)_t^j$.

Proof. Expressions (3.14) and (3.15) follow from independence of the $(Xa)_t^j$'s and Lemma 3.2. In (3.16) we use the fact that the index of skewness of a random variable X can be expressed as

$$\gamma(X) = \frac{E(X^3) - 3\text{Var}(X)E(X) - E(X)^3}{\sigma_X^3} \,.$$

To obtain $E[\left((Xa)_t^j\right)^3]$, the third raw moment of the random variable representing the PVFB for the active employee $j \in \mathcal{A}_t$, $(Xa)_t^j$, notice that

$$(\ddot{\sigma}_k^{(12)})^3 \ddot{\gamma}_k^{(12)} = \mathbb{E}[(\ddot{Y}_k^{(12)} - \ddot{a}_k^{(12)})^3] = \mathbb{E}[(\ddot{Y}_k^{(12)})^3] - (\ddot{a}_k^{(12)})^3 - 3\ddot{a}_k^{(12)} (\ddot{\sigma}_k^{(12)})^2 \,,$$

which in turn implies that $\mathbb{E}[(\ddot{Y}_k^{(12)})^3] = (\ddot{\sigma}_k^{(12)})^3 \ddot{\gamma}_k^{(12)} + (\ddot{a}_k^{(12)})^3 + 3\ddot{a}_k^{(12)} (\ddot{\sigma}_k^{(12)})^2$.

Therefore,

$$\begin{aligned} \mathbb{E}[\left((Xa)_t^j\right)^3] &= \mathbb{E}\left[(b_t^j \, v^{R-x})^3 [(\ddot{\sigma}_R^{(12)})^3 \ddot{\gamma}_R^{(12)} + (\ddot{a}_R^{(12)})^3 + 3\ddot{a}_R^{(12)} (\ddot{\sigma}_R^{(12)})^2]\right] \\ &= \sum_{k=x \vee \underline{r}}^{\overline{r}} (b_t^j \, v^{k-x})^3 [\ddot{\gamma}_k^{(12)} (\ddot{\sigma}_k^{(12)})^3 + (\ddot{a}_k^{(12)})^3 + 3\ddot{a}_k^{(12)} (\ddot{\sigma}_k^{(12)})^2] \,_{k-x} p_x^{(\tau)} q_k^{(r)} \,. \end{aligned}$$

□

A particular case arises when the retirement age is assumed to be the same for all active members.

Corollary 3.1 If $\underline{r} = \overline{r} = r$ and $q_k^{(r)} = 1$ if $k = r$ and zero elsewhere, the variance and index of skewness of $(Xa)_t$ are, respectively:

$$\begin{aligned}(\sigma a)_t^2 &= \sum_{j\in\mathcal{A}_t} \left[(\sigma a)_t^j\right]^2 = \sum_{j\in\mathcal{A}_t} \left\{ (b_t^j\, v^{r-x})^2 [(\ddot{\sigma}_r^{(12)})^2 + (\ddot{a}_r^{(12)})^2]\; {}_{r-x}p_x^{(\tau)} - \left[(\Pi a)_t^j\right]^2 \right\} ,\\ (\gamma a)_t &= \frac{1}{\left[(\sigma a)_t\right]^3} \sum_{j\in\mathcal{A}_t} \left[(\sigma a)_t^j\right]^3 (\gamma a)_t^j ,\end{aligned}$$

where

$$\begin{aligned}\left[(\sigma a)_t^j\right]^3 (\gamma a)_t^j = (b_t^j\, v^{r-x})^3 [\ddot{\gamma}_r^{(12)} (\ddot{\sigma}_r^{(12)})^3 + (\ddot{a}_r^{(12)})^3 + 3\ddot{a}_r^{(12)} (\ddot{\sigma}_r^{(12)})^2]\; {}_{r-x}p_x^{(\tau)}\\ - 3\left[(\sigma a)_t^j\right]^2 (\Pi a)_t^j - \left[(\Pi a)_t^j\right]^3 .\end{aligned}$$

Proof. Immediate. □

Remark 3.2 Expressions given in Corollary 3.1 do not coincide with Ramsay's formulas. If the random variable of the present value of future benefits for the active employee $j \in \mathcal{A}_t$, $(Xa)_t^j$ were defined as

$$(Xa)_t^j = b_t^j\; {}_{r-x}p_x^{(\tau)} v^{r-x}\; \ddot{Y}_x^{(12)} , \tag{3.17}$$

expressions for the variance and index of skewness of $(Xa)_t$ would be

$$\begin{aligned}(\sigma a)_t^2 &= \sum_{j\in\mathcal{A}_t} \left(b_t^j\; {}_{r-x}p_x^{(\tau)} v^{r-x}\; \ddot{\sigma}_r^{(12)}\right)^2 ,\\ (\gamma a)_t &= \ddot{\gamma}_r^{(12)} \left(\frac{\ddot{\sigma}_r^{(12)}}{\sigma_t}\right)^3 \sum_{j\in\mathcal{A}_t} \left(b_t^j\; {}_{r-x}p_x^{(\tau)}\; v^{r-x}\right)^3 ,\end{aligned}$$

which are the formulas given in Ramsay[6]. Definition (3.17) is not good, in the sense that the term ${}_{r-x}p_x^{(\tau)}$ should not appear in a random variable representing a *present value.* Assume for a moment, that the benefit function equals to one. Further, that the pension benefit is a lump–sum of one unit payable at retirement age, instead of a whole annuity payable monthly. Following the development of Bowers et al.[2], the proper definition for $(Xa)_t^j$ is $(Xa)_t^j = v^{R-x}$, where $R - x$ is the future "life" within the active group. The term ${}_{r-x}p_x^{(\tau)}$ appears when expectation is taken.

The second step is to work with $(Xr)_t$ from the retired population.

Proposition 3.4 The variance and skewness of $(Xr)_t$ are given, respectively, by

$$(\sigma r)_t^2 = \sum_{j \in \mathcal{P}_t} (b_t^j \, \ddot{\sigma}_x^{(12)})^2 , \tag{3.18}$$

$$(\gamma r)_t = \frac{1}{\sigma_t^3} \sum_{j \in \mathcal{P}_t} (b_t^j \, \ddot{\sigma}_x^{(12)})^3 \ddot{\gamma}_x^{(12)} . \tag{3.19}$$

Proof. It follows by independence and Lemma 3.1. □

Finally, assuming that $(Xa)_t$ and $(Xr)_t$ are mutually independent, it is easy to obtain expressions for the variance and skewness of X_t.

Corollary 3.2 The variance and skewness of X_t are given, respectively, by

$$\sigma_t^2 = (\sigma a)_t^2 + (\sigma r)_t^2 \quad \text{and} \tag{3.20}$$

$$\gamma_t = \frac{1}{\sigma_t^3}[(\sigma a)_t^3 \, (\gamma a)_t + (\sigma r)_t^3 \, (\gamma r)_t] \tag{3.21}$$

If the skewness of X_t is small (say, less than 0.01 in absolute value), then the Normal Approximation can be used to define the percentile version of the PVFB. That is (3.11) implies that:

$${}_\alpha\Pi_t = \Pi_t + z_\alpha \, \sigma_t , \tag{3.22}$$

where z_α is the $100\alpha^{\text{th}}$ percentile of the standard normal distribution. Using Haldane's approximation, ${}_\alpha(\Pi a)_t$ is given as follows:

Definition 3.6 Under percentile aggregate cost methods, the *α-present value of future benefits of the entire plan* at instant t is given by:

$${}_\alpha\Pi_t := \Pi_t + \frac{\sigma_t}{s}\left[\left[\psi(h,s) + z_\alpha \varphi(h,s)\right]^{1/h} - 1\right] , \qquad t \in \mathbb{N} , \tag{3.23}$$

where σ_t and γ_t are given by (3.14) and (3.15), respectively, and the functions $\psi(h,s)$ and $\varphi(h,s)$ are defined in Proposition 3.1.

In particular, if the confidence level is chosen to be $\alpha = 0.50$ (the median), then

$$ {}_{\alpha}\Pi_t = \Pi_t + \frac{\sigma_t}{s}\left[\psi(h,s)^{1/h} - 1\right] . $$

With ${}_{\alpha}\Pi_t$ already defined, proceed as in the traditional aggregate method.

Definition 3.7 Under the aggregate percentile cost method, the *normal cost*, denoted ${}_{\alpha}NC_t$, is given by:

$$ {}_{\alpha}NC_t := \beta(t)\left({}_{\alpha}\Pi_t - {}_{\alpha}AL_t\right) . \tag{3.24} $$

As before, β measures the portion of future costs to be paid at the beginning of year t and AL_t is the parameter that stands for the α–actuarial liability.

Of particular importance is the parameter β given by

$$ \beta(t) = \frac{S_t}{FS_t} , \tag{3.25} $$

where S_t is the sum of all salaries (or covered payroll) and

$$ FS_t = \sum_{j \in \mathcal{A}_t} \sum_{k=x \vee \underline{r}}^{\overline{r}} S_x^j \; {}^{s}\ddot{a}_{x:\overline{k-x}|}\, p(k) $$

is the present value of future salaries at instant t.

Definition 3.8 The *α–unit normal cost percentage* at instant t, is given by:

$$ {}_{\alpha}U_t := \frac{{}_{\alpha}\Pi_t - {}_{\alpha}AL_t}{FS_t} , \qquad t \in \mathbb{N} . \tag{3.26} $$

In particular, the α–normal cost satisfies ${}_{\alpha}NC_t = {}_{\alpha}U_t\, S_t$.

In the traditional aggregate approach, the evolution of the unit normal cost percentage was found by studying the evolution of $(\Pi a)_t$ during the year t, since the difference $(\Pi - AL)$ is the same as $(\Pi a) - (ALa)$. On the other hand, a basic idea is that the PVFB for the entire plan is just the summation of their components of the active and the retired populations. In percentile aggregate methods, on the contrary, things work in the other way round: the first value we obtain is the PVFB for the entire plan, and then split it in two parts. The first part is related with the active population and the other part with the retired population.

Certainly, we can calculate both ${}_\alpha(\Pi a)_t$ and ${}_\alpha(ALr)_t$ by using Haldane's approximation. Unfortunately, when these approximations are summed, the result is not exactly ${}_\alpha\Pi_t$ obtained when using (3.23). A nice way out of this dilemma is given by Ramsay, who define a *proportional adjustment factor*, denoted ${}_\alpha\psi_t$. The idea is to assume that the proportion between ${}_\alpha(\Pi a)_t$ and $(\Pi a)_t$ is the same as the proportion between ${}_\alpha\Pi_t$ and Π_t. In other words, let

$$ {}_\alpha\psi_t := \frac{{}_\alpha\Pi_t}{\Pi_t} , \qquad t \in \mathbb{N} . \tag{3.27} $$

Notice that, ${}_\alpha\psi_t \to 1$ as $\gamma_t \to 0$, for a fixed t.

Definition 3.9 For percentile aggregate cost methods, the *α–present value of future benefits of the active population*, and for employee $j \in \mathcal{A}_t$, denoted ${}_\alpha(\Pi a)_t$, and ${}_\alpha(\Pi a)_t^j$, respectively, at instant t are given by

$$ \begin{aligned} {}_\alpha(\Pi a)_t &:= {}_\alpha\psi_t \, (\Pi a)_t , \\ {}_\alpha(\Pi a)_t^j &:= {}_\alpha\psi_t \, (\Pi a)_t^j , \end{aligned} $$

for $t \in \mathbb{N}$. Expected quantities are defined similarly.

Furthermore, "individual" $\alpha-$actuarial liabilities can consistently be defined as follows:

$$ \begin{aligned} {}_\alpha(ALa)_t^j &:= {}_\alpha\psi_t \, {}_\alpha(\Pi a)_t^j - {}_\alpha U_t \, (FS)_{t+1}^j , \qquad \text{and} \\ {}_\alpha(\widetilde{AL}a)_t^j &:= {}_\alpha\psi_t \, {}_\alpha(\widetilde{\Pi} a)_t^j - {}_\alpha U_t \, (\widetilde{FS})_{t+1}^j . \end{aligned} $$

The formulas for the gain components studied in the traditional aggregate methods can now be used.

Definition 3.10 At instant t, the *α–actuarial gain*, denoted ${}_\alpha(Ga)_t$, is given by:

$$ {}_\alpha(Ga)_t = ({}_\alpha U_t - {}_\alpha U_{t+1})(FS_{t+1}) , \qquad t \in \mathbb{N} . \tag{3.28} $$

By using a similar technique as in the traditional aggregate cost methods, it is easy to obtain the analogue of (2.37), namely,

$$\begin{aligned}
{}_{\alpha}U_{t+1} \;=\; & {}_{\alpha}U_t - \frac{1}{FS_{t+1}}\Big\{\left[(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}\right] \\
& + \Big[\sum_{j\in\mathcal{E}_t} {}_{\alpha}(\widetilde{ALa})^j_{t+1} - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}\, {}_{\alpha}(\widetilde{ALa})^j_{t+1}\Big] \\
& - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} \left[{}_{\alpha}(ALa)^j_{t+1} - {}_{\alpha}(\widetilde{ALa})^j_{t+1}\right] - \sum_{j\in\mathcal{N}_t} {}_{\alpha}(ALa)^j_{t+1} \\
& - [PP_t + I_t^{PP}] + \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (\Delta\ {}_{\alpha}\psi_t)\,(\Pi a)^j_{t+1}\Big\} ,
\end{aligned}$$

where in the last term, $\Delta\ {}_{\alpha}\psi_t := {}_{\alpha}\psi_{t+1} - {}_{\alpha}\psi_t$ is the change in the proportional adjustment factor. In consequence, an additional gain component arises.

By components, G_t turns to be:

$${}_{\alpha}(Ga)_t = {}_{\alpha}G_t^{(i)} + {}_{\alpha}G_t^{(\tau)} + {}_{\alpha}G_t^{(s)} + {}_{\alpha}G_t^{(n)} + {}_{\alpha}G_t^{(\psi)} ,$$

where, for $t = 0, 1, 2, \ldots$;

$$\begin{aligned}
{}_{\alpha}G_t^{(i)} &:= (Ia)_t - i(Fa)_t - I_t^c + I_t^{PP} \quad , \\
{}_{\alpha}G_t^{(\tau)} &:= \sum_{j\in\mathcal{E}_t} {}_{\alpha}(\widetilde{ALa})^j_{t+1} - \sum_{j\in\mathcal{A}_t} q_x^{(\tau)}\, {}_{\alpha}(\widetilde{ALa})^j_{t+1} , \\
{}_{\alpha}G_t^{(s)} &:= - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} \left[{}_{\alpha}(ALa)^j_{t+1} - {}_{\alpha}(\widetilde{ALa})^j_{t+1}\right] , \\
{}_{\alpha}G_t^{(n)} &:= - \sum_{j\in{}_{\alpha}\mathcal{N}_t} {}_{\alpha}(ALa)^j_{t+1} , \\
{}_{\alpha}G_t^{(\psi)} &:= - \sum_{j\in\mathcal{A}_{t+1}\cap\mathcal{A}_t} (\Delta\ {}_{\alpha}\psi_t)\ {}_{\alpha}(\Pi a)^j_{t+1} .
\end{aligned}$$

Finally, as in traditional aggregate cost methods [see (2.31)], the actuarial liability satisfies the recursion formula

$$\begin{aligned}
{}_{\alpha}(ALa)_{t+1} = {}_{\alpha}(ALa)_t(1+i) - \left[(Ia)_t - i(Fa)_t - I_t^c + I_t^{PP}\right] \\
+ (PP_t + I_t^{PP}) - {}_{\alpha}NC_t(1+i) . \quad (3.29)
\end{aligned}$$

Therefore, it all reduces to calculate the starting value ${}_{\alpha}(ALa)_0$.

Examples of Aggregate Percentile Cost Methods

Percentile Frozen Initial Liability (αFIL)

As its traditional counterpart, [see (2.40)] the starting value in (3.29) is

$$_{\alpha}AL_0 = {_{\alpha}\Pi_0} - \frac{_{\alpha}\Pi W_0}{FSW_0} FS_0 \,, \tag{3.30}$$

where $_{\alpha}\Pi W_0$ is calculated in (3.23) with x replaced by the age at hire a, and FSW_0 is the present value of future salaries discounted to age a:

$$FSW_0 = \sum_{j \in A_t} \sum_{k=\underline{r}}^{\overline{r}} S_0^j \frac{s_a}{s_x} \, {^s\ddot{a}_{a:\overline{k-a}|}} \, p(k) \,.$$

Percentile Attained Age Normal (αAAN)

Here, the starting value $_{\alpha}AL_0$ in (3.29) is calculated by using (3.13), where the benefit function b_t^j is replaced by the accrual function of the projected unit credit method:

$$M_{x_0} = \frac{x_0 - a}{r - a} \,,$$

where x_0 is the age of member j at instant $t = 0$. In other words,

$$_{\alpha}AL_0 = AL_0 + \frac{\sigma_t}{s}\Big[[\psi(h,s) + z_{\alpha}\varphi(h,s)]^{1/h} - 1\Big]$$

and $AL_0 = (ALa)_0 + (ALr)_0$ is the starting value in the traditional attained–age cost method.

Percentile Net Method (αNET)

As its traditional relative, the α–actuarial liability is defined to be exactly the actual fund balance:

$$_{\alpha}AL_t = F_t \,,$$

which in particular implies that, any gain (or loss) is included in the normal cost of the following year. An extra gain component must be included:

$$_{\alpha}G_t^{(c)} = C_t + I_t^c - NC_t(1+i) \,,$$

due to excess in contributions.

3.4 The Retired Population: Aggregate Approach

For traditional methods, the cost of the retired population is obtained in an individual basis. Section 3.2 studied the analogue of the traditional valuation method of this population, namely, the individual approach. This section describes the aggregate approach.

For percentile methods two ways of looking at the calculation of the cost arise. To define the α–actuarial liability ${}_\alpha(ALr)_t$, for the retired population (note that we deliberately use the same symbol as the individual percentile approach), two possibilities should be considered. If the plan assumes that the active and retired populations are separate, then ${}_\alpha(ALr)_t$ is the quantity such that

$$\Pr[(Xr)_t \leq {}_\alpha(ALr)_t] = \alpha\ , \tag{3.31}$$

where

$$(Xr)_t = \sum_{j \in \mathcal{P}_t} b_t^j\, \ddot{Y}_x^{(12)}$$

is the random variable representing the present value of future pension payments, discussed earlier in this chapter. In this case, using Haldane's approximation [see Proposition 3.4 and (3.13)]:

$${}_\alpha(ALr)_t = (ALr)_t + \frac{(\sigma r)_t}{s}\left[\left[\psi(h,s) + z_\alpha \varphi(h,s)\right]^{1/h} - 1\right], \tag{3.32}$$

where

$$\begin{aligned}
(\sigma r)_t^2 &= \sum_{j \in \mathcal{P}_t} (b_t^j\, \ddot{\sigma}_x^{(12)})^2\ ,\\
(\gamma r)_t &= \frac{1}{\sigma_t^3} \sum_{j \in \mathcal{P}_t} (b_t^j\, \ddot{\sigma}_x^{(12)})^3\, \ddot{\gamma}_x^{(12)}\ ,
\end{aligned}$$

while the quantities s, h and the functions $\psi(h,s)$ and $\varphi(h,s)$ are defined in Proposition 3.1.

In practice however, the plan is usually considered as a whole. Under this approach, the α–actuarial liability for the retired population is not the one calculated by (3.32).

Recall that in aggregate percentile methods, the PVFB for the entire plan is given in Definition 3.6. In addition, the proportional adjustment factor ${}_\alpha\psi_t$ is the ratio of ${}_\alpha\Pi_t$ and Π_t. It was used to define the term ${}_\alpha(\Pi a)_t$ for the active population.

Under this approach, we also have

$$ {}_\alpha(ALr)_t = {}_\alpha\psi_t \, (ALr)_t \, . \tag{3.33} $$

Notice that the less skewed the distribution of X_t is, the closer the values of ${}_\alpha(ALr)_t$ given by (3.32) and (3.33). Furthermore, "individual actuarial liabilities" can be consistently defined by

$$ \begin{aligned} {}_\alpha(ALr)^j_{t+1} &:= {}_\alpha\psi_t \, b^j_{t+1} \, \ddot{a}^{(12)}_{x+1} \\ {}_\alpha(\widetilde{ALr})^j_{t+1} &:= {}_\alpha\psi_t \, b^j_t \, \ddot{a}^{(12)}_{x+1} \, , \end{aligned} $$

Using the same technique as the traditional approach, the gain is found to be

$$ {}_\alpha(Gr)_t = {}_\alpha(ULr)_t(1+i) - {}_\alpha(ULr)_{t+1} \, , $$

or by components:

$$ {}_\alpha(Gr)_t = {}_\alpha G^{(i)}_t + {}_\alpha G^{(b)}_t + {}_\alpha G^{(mr)}_t + {}_\alpha G^{(r)}_t + {}_\alpha G^{(p)}_t + {}_\alpha G^{(\psi)}_t \, , $$

where

$$ \begin{aligned} {}_\alpha G^{(i)}_t &:= (Ir)_t - i(Fr)_t - I^{PP}_t + I^B_t \, , \\ {}_\alpha G^{(b)}_t &:= - \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} \Delta b^j_t \; {}_\alpha\psi_{t+1} \, \ddot{a}^{(12)}_{x+1} \, , \\ {}_\alpha G^{(mr)}_t &:= \left[\sum_{j \in \mathcal{D}_t} {}_\alpha(\widetilde{ALr})^j_t - \sum_{j \in \mathcal{P}_t} q_x \; {}_\alpha(\widetilde{ALr})^j_t \right] , \\ {}_\alpha G^{(r)}_t &:= (PP_t + I^{PP}_t) - \left(\sum_{j \in \mathcal{R}_t} {}_\alpha(ALr)^j_{t+1} + (NB_t + I^{NB}_t) \right) , \\ {}_\alpha G^{(p)}_t &:= \left[\sum_{j \in \mathcal{P}_t} b^j_t \left(\ddot{s}^{(12)}_{\overline{1}|} + {}_\alpha\ddot{\theta}^{(12)}_x (1+i) \right) - (OB_t + I^{OB}_t) \right] \qquad \text{and} \\ {}_\alpha G^{(\psi)}_t &:= - \sum_{j \in \mathcal{P}_{t+1} \cap \mathcal{P}_t} (\Delta \, {}_\alpha\psi_t) \, b^j_t \, \ddot{a}^{(12)}_{x+1} \, . \end{aligned} $$

It is worth mentioning that Ramsay's expression for the component ${}_\alpha G^{(p)}_t$ ([6], formula (110), p. 394) is misprinted.

Chapter 4

Numerical Application

This chapter compares both, the traditional and the percentile cost methods by means of an example. The data is of an initial population of 360 actives and 12 retired members, and is the same as in Ramsay [6, p. 399].

4.1 Comparison of Methods

The Plan

- Plan effective date: January 1, 2002 ($t = 0$).
- Retirement age: 65 (R is degenerate at $r = 65$).
- Benefit formula: $b_t^j = 0.015(x_0 - a)S_0^j + 0.015\left[\sum_{k=0}^{t-1} S_k^j + \sum_{z=x}^{r-1} S_t^j \; s_z/s_x\right]$. In other words, the benefit is 1.5% of salary at plan inception times past service plus 1.5% of total future salary.
- Fund balance: $F_0 =$ 2,950,000.

Actuarial assumptions

- Interest rate: 8%.
- Confidence level: $\alpha = 50\%$, the median.

- Salary scale s_x: Table A.1.
- Service table: Table A.1.
- Mortality table: Table A.2.

Table 4.1: Employee data

Active population at instant $t = 0$				Active population at instant $t = 1$			
a	x	#ee's	S_0^j	a	x	#ee's	S_0^j
25	27	90	20,000	25	25	20	20,000
25	39	40	30,000	25	28	89	24,000
25	51	50	35,000	25	40	40	33,000
25	64	10	40,000	25	52	49	36,000
35	39	60	25,000	35	40	59	30,000
35	51	80	30,000	35	52	80	34,000
45	51	30	25,000	45	52	30	28,000

Table 4.2: Retiree data

Retiree population at instant $t = 0$			Retiree population at instant $t = 1$		
x	#ee's	b_0^j	x	#ee's	b_1^j
67	7	12,000	65	9	24,000
70	5	10,000	68	6	12,000
-	-	-	71	5	10,000

Valuation at $t = 0$

Recall that for aggregate percentile methods, the PVFB is calculated by using Haldane's approximation to the distribution function of $X_0 = (Xa)_0 + (Xr)_0$. The following table presents the moments of $(Xa)_0$, $(Xr)_0$ and X_0.

Table 4.3: Moments to compute ${}_{0.5}\Pi_0$

Item	$(Xa)_0$	$(Xr)_0$	X_0
μ_0	10,827,521.23	1,066,954.79	11,894,476.02
σ_0	443,239.02	112,344.43	457,254.96
γ_0	−0.055504	−0.244681	−0.054184

Haldane's approximation gives ${}_{0.5}\Pi_0 = 11{,}898{,}607.74$. The proportionality adjustment coefficient is

$${}_{0.5}\psi_0 = \frac{{}_{0.5}\Pi_0}{\Pi_0} = 1.000347364\ .$$

Once ${}_{0.5}\psi_0$ is obtained, the percentile aggregate versions of the PVFB for the active and retired populations can be computed:

$$\begin{aligned} {}_{0.5}(\Pi a)_0 &= {}_{0.5}\psi_0 \times (\Pi a)_0 = 10,831,282.33\ , \\ {}_{0.5}(\Pi r)_0 &= {}_{0.5}\psi_0 \times (\Pi r)_0 = 1,067,325.41\ . \end{aligned}$$

The quantities for the PVFB are summarized as follows.

Table 4.4: PVFB at instant $t = 0$

		Percentile	
Item	Traditional	Individual	Aggregate
$(\Pi a)_0$	10,827,521.23	11,821,678.09	10,831,282.33
$(\Pi r)_0$	1,066,954.79	1,160,654.36	1,067,325.41
Π_0	11,894,476.02	12,982,332.45	11,898,607.74

Table 4.5: Individual cost methods

Variable	PUC	EAN
NC_0	320,900.91	283,786.70
$(ALa)_0$	5,985,141.57	7,471,216.56
$(ALr)_0$	1,066,954.79	1,066,954.79
AL_0	7,052,096.36	8,538,171.35
${}_{0.5}NC_0$	350,365.26	309,843.31
${}_{0.5}(ALa)_0$	6,534,682.82	8,157,205.63
${}_{0.5}(ALr)_0$	1,160,654.36	1,160,654.36
${}_{0.5}AL_0$	7,695,337.18	9,317,859.99

Recall that in the Frozen Initial Liability (FIL) and Attained Age Normal (AAN), the actuarial liability is calculated recursively. In the traditional FIL method uses the formula

$$AL_0 = \Pi_0 - \frac{\Pi W_0}{FSW_0} FS_0 \, .$$

The AAN method uses the actuarial liability calculated at $t = 0$ using the projected unit credit cost method. On the other hand, percentile aggregate methods require the quantities ${}_{0.5}\Pi W_0$ and ${}_{0.5}AL_0$, to get the starting values in percentile FIL and AAN, respectively. Haldane's approximation is used.

Table 4.6: Information for aggregate methods

Item	ΠW (for FIL)	AL_0 (for AAN)
μ_0	2,059,402.60	7,052,096.36
σ_0	39,600.39	244,307.98
γ_0	−0.073918	−0.187849

By using Haldane's approximation, the following value is obtained:

$$ {}_{0.5}\Pi W_0 = 2,059,890.32 \, . $$

Now it is possible to obtain the starting value ${}_{0.5}AL_0$ for the FIL as follows:

$$ FSW_0 = 74,020,162.92 \, , $$

$$ FS_0 = 123,845,273.07 \, , $$

$$ {}_{0.5}AL_0 = {}_{0.5}\Pi_0 - \frac{{}_{0.5}\Pi W_0}{FSW_0} FS_0 = 8,452,145.74 \, , $$

where FSW_0 is the present value of future salaries discounted at hire age and FS_0 is the present value of future salaries [see (3.30)].

Table 4.7: Frozen Initial Liability (FIL)

Item	Traditional	Percentile
Π_0	11,894,476.02	11,898,607.74
$-AL_0$	$-8,448,830.03$	$-8,452,145.74$
FNC_0	3,445,645.99	3,446,462.00
FS_0	123,845,273.07	123,845,273.07
U_0	2.782218%	2.782877%
S_0	9,800,000.00	9,800,000.00
NC_0	272,657.40	272,721.98

Using Table 4.6, Haldane's approximation gives the starting value ${}_{0.5}AL_0$ for the AAN method as

$$ {}_{0.5}AL_0 = 7,059,723.97 \, . $$

Finally, under the aggregate net method (NET) (traditional and percentile), the actuarial liability at any instant equals the fund balance, $AL_0 = {}_{0.5}AL_0 = F_0$. In other words, the unfunded liability is always zero.

Table 4.8: Attained Age Normal (AAN)

Item	Traditional	Percentile
Π_0	11,894,476.02	11,898,607.74
$-AL_0$	−7,052,096.36	−7,059,723.97
FNC_0	4,842,379.66	4,838,883.77
FS_0	123,845,273.07	123,845,273.07
U_0	3.910024%	3.907201%
S_0	9,800,000.00	9,800,000.00
NC_0	383,182.33	382,905.70

Table 4.9: Net Method (NET)

Item	Traditional	Percentile
Π_0	11,894,476.02	11,898,607.74
$-AL_0$	−2,950,000.00	−2,950,000.00
$PVFNC_0$	8,944,476.02	8,948,607.74
FS_0	123,845,273.07	123,845,273.07
U_0	7.222299%	7.225635%
S_0	9,800,000.00	9,800,000.00
NC_0	707,785.31	708,112.26

Valuation at $t = 1$

The plan at instant $t = 1$ is as follows.

- Plan effective date: January 1, 2003 ($t = 1$).
- Contributions to the fund: $C_0 = 290,000$.
- Interest earned over C_0: $I_0^C = 11,376.84$.
- Fund balance: $F_1 = 3,350,000$.
- Pension payments during year 2002: $B_0 = 34,000$.

During year $t = 0$, the plan showed the following evolution.

Table 4.10: Summary of first-year activity

a	x	Event
25	25	20 new hires ($\mathcal{N}_0$)
25	28	1 termination ($\mathcal{T}_0$)
25	52	1 death ($\mathcal{D}_0$)
25	65	1 death ($\mathcal{D}_0$)
25	65	9 retirements ($\mathcal{R}_0$)
35	40	1 termination ($\mathcal{T}_0$)

Table 4.11: Moments to compute ${}_{0.5}\Pi_1$

	$(Xa)_1$	$(Xr)_1$	X_1
μ_1	10,324,962.81	2,807,363.31	13,132,326.12
σ_1	430,206.32	228,711.68	487,223.27
γ_1	−0.026828	−0.266540	−0.046039

The H–approximation gives ${}_{0.5}\Pi_1 = 13,136,066.88$, which implies that the proportional adjustment coefficient is ${}_{0.5}\psi_1 = 1.000285$. In particular,

$$\begin{aligned} {}_{0.5}(\Pi a)_1 &= {}_{0.5}\psi_1 \times (\Pi a)_1 = 10,327,903.89 \, , \\ {}_{0.5}(\Pi r)_1 &= {}_{0.5}\psi_1 \times (\Pi r)_1 = 2,808,162.99 \, . \end{aligned}$$

For the FIL and AAN methods, the actuarial liability is calculated recursively by

Table 4.12: PVFB at instant $t = 1$

		Percentile	
Item	Traditional	Individual	Grouped
$(\Pi a)_1$	10,324,962.81	11,272,975.96	10,327,903.89
$(\Pi r)_1$	2,807,363.31	3,058,498.88	2,808,162.99
Π_1	13,132,326.12	14,331,474.84	13,136,066.88

Table 4.13: Individual cost methods

Variable	PUC	EAN
NC_1	316,663.18	303,049.61
$(ALa)_1$	4,952,382.38	6,595,569.80
$(ALr)_1$	2,807,363.31	2,807,363.31
AL_1	7,759,745.69	9,402,933.11
${}_{0.5}NC_1$	345,738.43	330,874.90
${}_{0.5}(ALa)_1$	5,407,098.17	7,201,159.09
${}_{0.5}(ALr)_1$	3,058,498.88	3,058,498.88
${}_{0.5}AL_1$	8,465,597.05	10,259,657.97

$$AL_1 = (AL_0 - F_0)(1 + i) - [C_0 + I_0^c - NC_0(1 + i)] + F_1 \, ,$$

where the subindex $\alpha = 0.5$ should be appended in case of percentile quantities. Finally, the NET method defines $AL_1 = F_1$. In particular, an additional gain component arises, namely, the excess in contributions $G_t^{(c)}$.

Table 4.14: Frozen Initial Liability (FIL)

Item	Traditional	Percentile
Π_1	13,132,326.12	13,136,066.88
$-AL_1$	−9,281,829.59	−9,285,480.29
FNC_1	3,850,496.53	3,850,586.59
FS_1	142,702,092.32	142,702,092.32
U_1	2.698276%	2.698339%
S_1	10,950,000.00	10,950,000.00
NC_1	295,461.24	295,468.15

Table 4.15: Attained Age Normal (AAN)

Item	Traditional	Percentile
Π_1	13,132,326.12	13,136,066.88
$-AL_1$	−7,892,724.15	−7,900,663.20
FNC_1	5,239,601.97	5,235,403.68
FS_1	142,702,092.32	142,702,092.32
U_1	3.671706%	3.668764%
S_1	10,950,000.00	10,950,000.00
NC_1	402,051.86	401,729.71

Table 4.16: Net Method (NET)

Item	Traditional	Percentile
Π_1	13,132,326.12	13,136,066.88
$-AL_1$	−3,350,000.00	−3,350,000.00
FNC_1	9,782,326.12	9,786,066.88
FS_1	142,702,092.32	142,702,092.32
U_1	6.855068%	6.857690%
S_1	10,950,000.00	10,950,000.00
NC_1	750,630.00	750,917.04

Analysis of Gains

For individual cost methods, the gain for the first year of operation is calculated as $G_0 = UL_0(1+i) - \left[C_0 + I_0^c - NC_0(1+i)\right] - UL_1$. For aggregate methods, on the other hand, the gain is $G_0 = (U_0 - U_1)FS_1$. Table 4.17 presents the gain for year $t = 0$ for all cost methods. Finally, Tables 4.18 through 4.22 give the detail of the gain by components for every method.

Table 4.17: Gains for individual and aggregate methods

Method	Traditional	Percentile
PUC	65,714.52	86,384.75
EAN	−12,595.25	884.76
FIL	119,787.35	170,277.20
AAN	340,083.81	418,770.74
NET	524,045.78	587,935.66

Table 4.18: Analysis of gains. Projected Unit Credit method

Source	Traditional	Percentile
Interest	2,361.56	2,361.56
Pension payments	–1,432.83	13,401.72
Retiree mortality	72,512.76	79,059.42
Active mortality	202,020.27	220,569.28
Terminations	–113,381.29	–123,791.69
Salary changes	–96,365.95	–105,214.03
Explained	65,714.52	86,386.26
Total(1)	65,714.52	86,384.75
Error	–	–1.51

(1) From Table 4.17

Table 4.19: Analysis of gains. Entry–Age method

Source	Traditional	Percentile
Interest	2,361.56	2,361.56
Pension payments	–1,432.83	13,401.72
Retiree mortality	72,512.76	79,059.42
Active mortality	207,762.17	226,838.40
Terminations	–155,249.20	–169,503.81
Salary changes	–138,549.72	–151,271.02
Explained	–12,595.25	886.27
Total	–12,595.25	884.76
Error	–	–1.51

Table 4.20: Analysis of gains. Frozen Initial Liability

Source	Traditional	Percentile
Interest	2,361.56	2,361.56
Pension payments	−1,432.83	−1,432.83
Retiree mortality	72,512.76	72,537.95
Active mortality	205,470.08	205,477.85
Terminations	−133,160.31	−133,204.57
Salary changes	−99,636.51	−99,675.38
New entrants	73,672.60	73,684.19
Prop. Adj. Coeff.	–	11.52
Explained	119,787.36	119,760.29
Unexplained	0.01	877.25
Total	119,787.35	120,637.55

Table 4.21: Analysis of gains. Attained Age Normal

Source	Traditional	Percentile
Interest	2,361.56	2,361.56
Pension payments	−1,432.83	−1,432.83
Retiree mortality	72,512.76	72,537.95
Active mortality	204,557.64	204,568.23
Terminations	-79,693.61	−79,902.93
Salary changes	−11,159.57	−11,471.57
New entrants	152,937.80	152,704.70
Prop. Adj. Coeff.	0	11.52
Explained	340,083.75	339,376.63
Unexplained	0.05	877.27
Total	340,083.80	340,253.90

Table 4.22: Analysis of gains. Net Method

Source	Traditional	Percentile
Interest	2,361.56	2,361.56
Pension payments	–1,432.83	–1,432.83
Retiree mortality	72,512.76	72,537.95
Active mortality	201,877.86	201,883.47
Terminations	77,333.89	77,416.56
Salary changes	248,690.20	248,861.40
New entrants	385,733.60	385,933.40
Excess in C_t	–463,031.30	–463,384.40
Prop. Adj. Coeff.	–	11.52
Explained	524,045.75	524,188.63
Unexplained	0.04	877.21
Total	524,045.78	525,065.84

4.2 Comments

As the distribution of $\ddot{Y}_x^{(12)}$ is negatively skewed, the percentile ${}_{0.5}\ddot{\xi}_{65}^{(12)}$ is greater than the mean $\ddot{a}_{65}^{(12)}$. Furthermore, $\ddot{a}_{65}^{(12)} = 8.64$, while ${}_{0.5}\ddot{\xi}_{65}^{(12)} = 9.43$, an increase of 9.18%.

At $t = 0$ such difference implies an increase on PVFB of about 9.15% for individual percentile methods, and just 0.03% for aggregate percentile methods. In particular, no significative differences between traditional and percentile aggregate methods, are expected.

The normal cost in PUC and EAN increases by 9.18% for both. For aggregate methods, that increase is 0.05%. At $t = 1$, such differences remain very similar.

At this point, something interesting can be noticed. At least for individual methods, the increase in costs seems to be determined solely by the increase of ${}_{0.5}\ddot{\xi}_{65}^{(12)}$ with respect to $\ddot{a}_{65}^{(12)}$.

The summation of the gain components is sometimes called the *explained gain.* The difference between the explained and the total gain (calculated by other means rather than by components) is called the *unexplained gain.* From a theoretical point of view, the unexplained gain should be zero. However, the numerical application shows this is not the case for aggregate percentile methods. Fortunately, the unexplained gain represents just 0.73% for FIL, 0.26% for AAN and 0.17% for NET, of the total gain. For traditional and individual percentile methods, the unexplained gain comes from rounding errors, while for aggregate percentile methods, it may be due to the proportional assumption when defining the "individual" actuarial liability.

Finally, the gain resulted to be the most sensitive variable. For instance, in the PUC method the gain increased by 31%. For the FIL method, the increase is 42%. If appropriate steps are taken, percentile methods will annihilate the unfunded liability faster than traditional methods do, since contribution premiums will be raised by a greater amount.

Conclusions

This thesis compares the traditional and the percentile cost methods, as suggested by Ramsay. The latter approach replaces the factor $\ddot{a}_x^{(12)}$ (a mean value) by the $100\alpha^{\text{th}}$–percentile ${}_\alpha\ddot{\xi}_x^{(12)}$, where α is a level of confidence. In addition, aggregate percentile methods require the approximation of a the random variable representing the present value of future benefits. This was accomplished by using the Haldane type A approximation.

Of particular importance is the analysis of gains and losses by source. In both individual and aggregate methods, we obtain the same expressions for them. Certainly, this fact coincides with Small [7], who suggests a unified approach. Since both approaches are set on different basis, a separate derivation is better understood.

An important assumption that was relaxed in this work, is the fact that the retirement age is a random variable, rather than a fixed value, as in most of the literature about pension mathematics. As expected, this fact makes formulas a bit more complicated to read. Thus, any expression involving the retirement age R actually requires an expectation over R (introducing a second summation). The numerical application however, uses the data in Ramsay[6], where the retirement age is assumed to be 65. The implementation of multiple retirement ages should be straightforward.

As the distribution of $\ddot{Y}_{65}^{(12)}$ is negatively skewed, for most ages, the value of ${}_\alpha\ddot{\xi}_{65}^{(12)}$ is greater than the mean $\ddot{a}_{65}^{(12)}$. As a consequence, percentile cost methods will result in larger costs. Such differences are expected to decrease as the number of members in the active population increases. In particular, for large public pension schemes, percentile cost methods are expected to produce results quite similar to those of the classical approach. In small plans, on the contrary, the percentile approach produce much higher contributions than the traditional one.

Finally, this work must be extended to the stochastic interest case. Also, disability pension benefits, vested terminations and survivor benefits should be considered. These extensions follow the same reasoning of retirement benefits.

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Appendix A

Table A.1: Service Table

x	$q_x^{(d)}$	$q_x^{(w)}$	s_x	x	$q_x^{(d)}$	$q_x^{(w)}$	s_x
25	0.000464	0.0835	0.0673	45	0.002183	0.0285	0.3579
26	0.000488	0.0800	0.0756	46	0.002471	0.0270	0.3793
27	0.000513	0.0770	0.0848	47	0.002790	0.0255	0.4017
28	0.000542	0.0740	0.0945	48	0.003138	0.0240	0.4251
29	0.000572	0.0710	0.1048	49	0.003513	0.0230	0.4492
30	0.000607	0.0680	0.1159	50	0.003909	0.0215	0.4740
31	0.000645	0.0650	0.1274	51	0.004324	0.0210	0.5005
32	0.000687	0.0615	0.1396	52	0.004755	0.0200	0.5276
33	0.000734	0.0590	0.1523	53	0.005200	0.0195	0.5555
34	0.000785	0.0565	0.1657	54	0.005660	0.0185	0.5844
35	0.000860	0.0530	0.1798	55	0.006131	0.0175	0.6141
36	0.000907	0.0500	0.1945	56	0.006618	–	0.6449
37	0.000966	0.0470	0.2099	57	0.007139	–	0.6771
38	0.001039	0.0445	0.2259	58	0.007719	–	0.7109
39	0.001128	0.0420	0.2425	59	0.008384	–	0.7466
40	0.001238	0.0395	0.2600	60	0.009158	–	0.7838
41	0.001370	0.0370	0.2782	61	0.010064	–	0.8230
42	0.001527	0.0345	0.2971	62	0.011133	–	0.8641
43	0.001715	0.0325	0.3166	63	0.012391	–	0.9074
44	0.001932	0.0300	0.3370	64	–	–	0.9528

Table A.2: Life Table

x	$q_x^{(d)}$	x	$q_x^{(d)}$	x	$q_x^{(d)}$	x	$q_x^{(d)}$
20	0.000377	43	0.001715	66	0.017579	89	0.154859
21	0.000392	44	0.001932	67	0.019804	90	0.166307
22	0.000408	45	0.002183	68	0.022229	91	0.178214
23	0.000424	46	0.002471	69	0.024817	92	0.190460
24	0.000444	47	0.002790	70	0.027530	93	0.203007
25	0.000464	48	0.003138	71	0.030354	94	0.217904
26	0.000488	49	0.003513	72	0.033370	95	0.234086
27	0.000513	50	0.003909	73	0.036680	96	0.248436
28	0.000542	51	0.004324	74	0.040388	97	0.263954
29	0.000572	52	0.004755	75	0.044597	98	0.280803
30	0.000607	53	0.005200	76	0.049388	99	0.299154
31	0.000645	54	0.005660	77	0.054758	100	0.319185
32	0.000687	55	0.006131	78	0.060678	101	0.341086
33	0.000734	56	0.006618	79	0.067125	102	0.365052
34	0.000785	57	0.007139	80	0.074070	103	0.393102
35	0.000860	58	0.007719	81	0.081484	104	0.427255
36	0.000907	59	0.008384	82	0.089320	105	0.469531
37	0.000966	60	0.009158	83	0.097525	106	0.521945
38	0.001039	61	0.010064	84	0.106047	107	0.586518
39	0.001128	62	0.011133	85	0.114836	108	0.665268
40	0.001238	63	0.012391	86	0.124170	109	0.760215
41	0.001370	64	0.013868	87	0.133870	110	1.00000
42	0.001527	65	0.015592	88	0.144073	–	–