# COHERENT STATES OF THE POINCARÉ GROUP, RELATED FRAMES AND TRANSFORMS

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# ABSTRACT

## Coherent States of the Poincaré Group, Related Frames and Transforms

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We construct here families of coherent states for the full Poincaré group, for representations corresponding to mass $m > 0$ and arbitrary integral or half-integral spin. Each family of coherent states is defined by an affine section in the group and constitutes a frame. The sections, in their turn, are determined by particular velocity vector fields, the latter always appearing in dual pairs.

We discretize the coherent states of Poincaré group in 1−space and 1−time dimensions and show that they form a discrete frame, develop a transform, similar to a windowed Fourier transform, which we call the relativistic windowed Fourier transform. We also obtain a reconstruction formula.

Finally, we perform numerical computations. We evaluate the discrete frame operator numerically and present it graphically for different sections and windows. We also reconstruct some functions, compare reconstructed functions with the original ones graphically. We compare the reconstruction scheme of the relativistic windowed Fourier transform with that of the standard windowed Fourier transform.

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# Introduction

This thesis is based on two related notions— 'coherent states' and 'wavelet transforms' and in the first chapter we describe them in order to set up the background. 'Coherent states' were first introduced by Schrödinger [74] in 1926 as a system of non-orthogonal wave functions to demonstrate the transition from quantum mechanics to classical mechanics. These states are just displaced forms of the ground state of the harmonic oscillator and they minimize the uncertainty relation of Heisenberg, ($\Delta q \, \Delta p = \frac{1}{2}$). The completeness of these states was first noted by von Neumann [60]. Since its discovery by Schrödinger, this idea did not get much attention until 1963, when Glauber [39, 40] and later Klauder [50, 51] rediscovered them. Glauber introduced them in the context of quantum optics during the study of the coherence of light beams and coined the term 'coherent states' (CS), because of their property of maintaining the same pattern over long distances and time. Glauber also surveyed the properties of CS, specially the expansion of arbitrary states in terms of CS. The CS introduced by Schrödinger and Glauber are customarily known as the *canonical coherent states.* Klauder set forth the group theoretical foundation of CS and he stated the resolution of the identity, a central property of coherent states, in the present form [52]. He also parametrized the CS by phase space points.

A connection between the canonical CS and certain types of unitary irreducible group representations was noted by Klauder [50, 51], Gilmore [37, 38] and Perelomov [62]. They observed independently that the construction of the canonical CS is actually associated to the unitary representation $U$ of the Weyl-Heisenberg group $G_{WH}$ and that the CS could be obtained by acting on the harmonic oscillator ground state with $U$. Unlike Klauder, Gilmore and Perelomov did not use the whole group $G_{WH}$ but the coset space $X = G_{WH}/Z$, where $Z$ is the center of $G_{WH}$ (i.e., the set of all elements commuting with every element of $G_{WH}$) and the CS are indexed by the points of the coset space $X$. In addition, the representation $U$ is 'square integrable' in the sense that for $\eta \in \mathcal{H} = L^2(\mathbb{R}, dx)$,

$$\int_X |\langle U(g)\eta \mid \phi \rangle|^2 \, d\nu(x) \; < \; \infty, \; g \in G_{WH}, \; \forall \phi \in \mathcal{H} \;\; (x \equiv gZ),$$

where $\nu$ is the invariant measure on $X$.

Taking any locally compact group $G$, rather than the Weyl-Heisenberg group $G_{WH}$, the following generalization can be made: Let $U$ be a unitary, irreducible representation of $G$ in an abstract Hilbert space $\mathcal{H}$, $\eta \in \mathcal{H}$ and $H_\eta$ the isotropy group of $\eta$. (Let $G$ be a group acting on a set $X$ and let $x \in X$. The isotropy group of $x$, denoted by $G_x$, is the subgroup of all elements of $G$ leaving $x$ fixed, $G_x = \{g \in G \mid gx = x\}$). If $X = G/H_\eta$ is the corresponding coset space the CS system associated with $U$ is the set of vectors

$$S_{\sigma_0} = \{U(\sigma_0(x))\,\eta \mid x = gH_\eta \in X, \; g \in G\}.$$

where $\sigma_0 : X \to G$ is a Borel function defining a section in the group.

Although the method of Gilmore and Perelomov was very suitable for the extraction of CS of most of the groups, it was soon discovered that their method was not capable of handling at least some groups, for example, the Galilei and Poincaré groups [9, 10, 11, 12, 66, 67]. A possible way to overcome the drawbacks of the Gilmore-Perelomov method was suggested by Ali [2] and later developed in a series of papers by Ali, Antoine and Gazeau [3, 4, 5, 6, 7, 8, 14]. Unlike in the Gilmore-Perelomov framework, in the new method, CS are contructed using a homogeneous space $X = G/H$, where $H$ is a suitable closed subgroup of $G$ and $H$ does not necessarily coincide with $H_\eta$ and one needs to find a *section*, $\sigma : X \to G$. Then the CS are defined to be the vectors

$$\eta_{\sigma(x)} = U(\sigma(x))\eta, \; x \in X, \; \eta \in \mathcal{H}.$$

It ought to be noted here that the introduction of sections in the above sense was done implicitly by Prugovečki [66, 12], and in the Gilmore-Perelomov frame-work the section $\sigma(x)$ appears as well, often implicitly.

The history of wavelet anlysis is only about 20 years old, and yet in this short period of time it has become very popular among mathematicians, physicists and engineers alike. The reason behind this popularity is its synthetic nature, i.e., it is a synthesis of a variety of fields, for instance, the Littlewood-Paley decomposition in mathematics, coherent states expansion in quantum mechanics, sub-band coding in signal processing etc. 'Wavelets' were first introduced by Morlet [59] in 1982, as a convenient tool to analyze seismic data. Wavelets are actually coherent states of the affine group, satisfying some admissibilty conditions. After successful applications in seismic study, Morlet and Grossmann [44] studied them extensively and developed a mathematical foundation for wavelet theory. Then Meyer observed a connection between the method of signal analysis and the techniques used in the study of singular integral operators. After that, Daubechies, Grossmann and Meyer [27] constructed a special type of frames, generalizing the concept of a basis in a Hilbert space. The next major breakthrough was due to Mallat [55, 56] and Meyer [58] through their introduction of *multiresolution analysis*, and Daubechies' construction of families of orthonormal wavelets with compact support [29, 30, 26] can be treated as a giant leap in the progress of wavelet analysis. At the heart of wavelet analysis is the wavelet transform (WT) which is defined in terms of wavelets. Because of the presence of a translation and a dilation parameter, the WT is extremely efficient in reconstructing images/signals even at points of discontinuities. Nowadays wavelet analysis is a complete subject in its own right and has applications in many different fields, for instance, signal analysis [53], numerical analysis [21] and physics [20], neural networks [47], fractal image and texture [49], telecommunication [75], etc., and the literature keeps growing .

Many of the techniques used in wavelet analysis have been around for quite a long time but there the term wavelet was not used explicitly. Fourier analysis is a kind of wavelet analysis in the broader sense of the term. The limitation of Fourier analysis is that it offers either an all-time or an all-frequency description of a signal, nothing in between. But for practical purposes, we need both. To overcome this situation, Gabor [34] introduced the 'windowed Fourier transform' (WFT) (also knwon as 'short-time Fourier transform' ). In the WFT a window function is chosen to localize the signal and then the window is shifted repeatedly. The WFT and the wavelet transform are two equivalent transforms, although they are also different. It is possible to switch from one to another without losing any information [33].

Coherent states for the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$, in 1–space and 1–time dimesions have been studied extensively in [6, 8] for unitary irreducible representations (UIR's) corresponding to mass $m > 0$. These states are indexed by the points of the homogeneous space $\Gamma = \mathcal{P}_+^\uparrow(1,1)/T$, where $T$ is the subgroup of time translation. An *affine section* $\sigma : \Gamma \to \mathcal{P}_+^\uparrow(1,1)$ has been defined in order to construct these coherent states. Several types of special sections are mentioned and for each section its *dual* is also defined and there exists a section which is dual to itself. The resulting coherent states constitute a frame and under certain specific situations this frame can be made a tight frame. Consequenly the CS generate analogs of a resolution of the identity. The full Poincaré group $\mathcal{P}_+^\uparrow(1,3)$ was studied earlier and the corresponding CS were constructed in [12, 67] for UIR's corresponding to mass $m > 0$ and spin $s = 0, 1, 2, \ldots$. No idea of sections was used there and in the context of $\mathcal{P}_+^\uparrow(1,1)$ CS, they were related to a particular section, $\sigma = \sigma_0$, the Galilean section. Construction of CS of $\mathcal{P}_+^\uparrow(1,3)$ corresponding to $m > 0$ and spin-$\frac{1}{2}$ representation was reported in [65]. These CS did not form a frame and thus led to no resolution of the identity.

Coherent states of the De Sitter and Poincaré groups were also studied for particular sections in [35].

In chapter 2 we extend the results in [6, 8] for $\mathcal{P}_+^\uparrow(1,1)$ CS to any UIR of $\mathcal{P}_+^\uparrow(1,3)$, for $m > 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Just like in the case of $\mathcal{P}_+^\uparrow(1,1)$, each family of CS is defined in terms of a particular affine section, and it constitutes a frame. Under certain specific conditions this frame can be made tight, thus leading to a resolution of the identity. Since the base space of $\mathcal{P}_+^\uparrow(1,3)$ is much larger than that of $\mathcal{P}_+^\uparrow(1,1)$, the duality of the sections in the case of $\mathcal{P}_+^\uparrow(1,3)$ is better understood. The duality of sections can be interpreted in terms of Lorentz invariant fibrations of Minkowski space. Each affine section can be characterized by a 4-vector field $u(p) = (u_0(p), \mathbf{u}(p))$, where $p$ is a relativistic 4−momentum on the mass hyperboloid $\mathcal{V}_m^+$ and the 4-vector field $u(p)$ maps $\mathcal{V}_m^+$ onto itself. Then the *dual section* $u^*(p)$ which is also a 4-vector field can be obtained by applying the Lorentz boost $\Lambda$ to $\overline{u}(p) = (u_0(p), -\mathbf{u}(p))$, pointwise for all $p$. It should be mentioned here that the CS obtained in this thesis are similar to the vector CS studied in [69]. Vector coherent states are obtained from a set of linearly independent vectors of a representation space of an isotropy subgroup of a group $G$, and each state is written as a linear combination of these vectors. But the situation we consider here is much more general than that in [69] in the sense that the subspace generated by the 'fiducial vector' in our case is not stable under the action of any non-trivial subgroup of $\mathcal{P}_+^\uparrow(1,3)$.

In chapter 3, we discretize the CS of $\mathcal{P}_+^\uparrow(1,1)$ obtained in [6, 8] by periodizing a compactly supported window function and show that this discretized family of CS forms a discrete frame. We calculate the frame operator $\hat{T}$ for different sections and for various conditions on the window function. Under some specific situations, the operator $\hat{T}$ becomes a multiple of the identity operator and consequently the frame becomes tight. We develop a transform, similar to the windowed Fourier transform, which we call the *relativistic windowed Fourier transform.* We also obtain a *reconstruction formula* using the frame operator $\hat{T}$ and the relativistic windowed Fourier transform.

In chapter 4, we perform some numerical computations. We evaluate the operator $\hat{T}$ obtained in chapter 3 for different sections and window functions. Using the reconstruction formula of chapter 3, we reconstruct a function under various assumptions on the sections and the window function, and compare the reconstructed function with the original one both numerically and graphically. It is observed that for a smooth window the reconstruction scheme does a better job than that for a non-smooth window. However different sections play more or less the same role under identical conditions. For comparison, discretizing the CS of the Weyl-Heisenberg group, we obtain the corresponding frame operator and the reconstruction formula. We finish this chapter by reconstructing a function using the reconstruction formula obtained in chapter 3 and that obtained using the Weyl-Heisenberg CS. Comparing these two schemes we see that the reconstructed values are in close agreement.

In the Conclusion, we briefly describe the results we obtained earlier and suggest possible applications. We also indicate possible extensions of some of the results.

# Chapter 1

# Coherent States, Generalized Wavelet Transforms, Frames

In this chapter, we give an overview of 'coherent states', 'generalized wavelet transforms' and 'frames' – notions central to this thesis. In the description of coherent states, we place emphasis on the group-theoretical background, i.e. how coherent states are related to group representations. By the term 'generalized wavelet' (GW) transform, we mean here a specific type of a transform originating from the coherent states of an arbitrary group. Here we shall give a brief description of: i) a *Gabor transform* and ii) a *wavelet transform*, both of which are examples of generalized wavelet transforms; the *Gabor transform* originating from the coherent states of the Weyl-Heisenberg group and the *wavelet transform* related to the affine group. Finally, at the end of this chapter, we give a short description of *frames* and some of their properties.

## 1.1 Coherent States

We begin with the canonical coherent states. The canonical CS are an overcomplete (i.e., the set remains complete upon removal of at least one vector) and non-orthogonal system of Hilbert space vectors. Starting with

$$f(x) = \pi^{-\frac{1}{4}}\, e^{-\frac{x^2}{2}} \in L^2(\mathbb{R}, dx)$$

the canonical CS $f_{q,p}$ are defined by

$$f_{q,p}(x) = e^{-\frac{ipq}{2}}\, e^{ipx}\, f(x-q), \quad \forall x \in \mathbb{R} \tag{1.1}$$

with $\|f\|^2 = 1$. The canonical coherent states have many remarkable properties and here we focus on some of them. Let us define the formal operator

$$A \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, dp \,|\, f_{q,p} \,\rangle \langle\, f_{q,p} \,| \tag{1.2}$$

where $|\, f_{q,p} \,\rangle \langle\, f_{q,p} \,|$ is the projection operator on the state $|\, f_{q,p} \rangle$; by $\langle\, f_{q,p} \,|$ we mean Dirac's 'bra' vector and $|\, f_{q,p} \,\rangle$ the 'ket' vector. The scalar product of a bra vector $\langle\, f_{q,p} \,|$ and a ket vector $|\, h_{q,p} \,\rangle$ will be written $\langle\, f_{q,p} \,|\, h_{q,p} \,\rangle$. Then for $\phi \in \mathcal{H} = L^2(\mathbb{R}, dx)$, and almost for all $x$ we have

$$(A\phi)(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f_{q,p} \,|\, \phi \,\rangle f_{q,p}(x)\, dq\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f_{q,p}(y)}\,\phi(y)\,f_{q,p}(x)\,dy\,dq\,dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x-y)p}\,\overline{f(y-q)}\,\phi(y)\,f(x-q)\,dp\,dy\,dq$$

(Using Fubini's theorem)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-y)\,\overline{f(y-q)}\,\phi(y)\,f(x-q)\,dy\,dq$$

$$= \phi(x)\int_{-\infty}^{\infty}|\,f(y)\,|^2\,dy = \phi(x)\,\|f\|^2 = \phi(x) \tag{1.3}$$

From (1.3) we can write

$$A = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dp\,|\,f_{q,p}\,\rangle\langle\,f_{q,p}\,| = I \tag{1.4}$$

where $I$ is the identity operator on $\mathcal{H}$. This property is known as the 'resolution of the identity'. From (1.4) we observe immediately that the canonical coherent states are linearly dependent, i.e., any canonical coherent state can be written as a linear combination of the rest. That is,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dp\,|f_{q,p}\,\rangle\langle\,f_{q,p}\,|\,f_{q',p'}\,\rangle = |\,f_{q',p'}\,\rangle \tag{1.5}$$

Define the kernel $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ by

$$K(q,p;q',p') = \langle\,f_{q,p}\,|\,f_{q',p'}\,\rangle \tag{1.6}$$

Then

$$\begin{aligned} K(q,p;q,p) &= \int_{-\infty}^{\infty} e^{\frac{ipq}{2}}\, e^{-ipx}\,\overline{f(x-q)}\, e^{-\frac{ipq}{2}}\, e^{ipx}\, f(x-q)\, dx \\ &= \int_{-\infty}^{\infty} |\, f(x-q)\,|^2\, dx \;=\; \|\, f\,\| \;= 1 \;>\; 0, \end{aligned}$$

and $K$ satisfies the properties

$$(i)\quad K(q,p;q,p) \;>\; 0, \quad \forall\,(q,p). \tag{1.7}$$

$$(ii)\quad K(q',p';q,p) \;=\; \langle\, f_{q',p'}\,|\, f_{q,p}\,\rangle = \overline{K(q,p;q',p')}\,; \tag{1.8}$$

$$(iii)\quad K(q,p;q'',p'') \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(q,p;q',p')\, K(q',p';q'',p'')\, dq'\, dp' \tag{1.9}$$

where for $z \in \mathbb{C}$, $\overline{z}$ indicates its complex conjugate. To check $(iii)$, we note that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(q,p;q',p')\, K(q',p';q'',p'')\, dq'\, dp' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, e^{i(x-x')p'}\, e^{-ipx}\, e^{ip''x'} \times \\ &\quad e^{\frac{ipq}{2}}\, e^{-\frac{ip''q''}{2}}\, \overline{f(x-q)}\, f(x-q')\overline{f(x'-q')}\, f(x'-q'') \\ &\quad \text{(Using Fubini's theorem)} \\ &= \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, \delta(x-x')\, e^{-ipx}\, e^{ip''x'} \times \\ &\quad e^{\frac{ipq}{2}}\, e^{-\frac{ip''q''}{2}}\, \overline{f(x-q)}\, f(x-q')\overline{f(x'-q')}\, f(x'-q'') \\ &= \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dx\, e^{-ipx}\, e^{ip''x'}\, e^{\frac{ipq}{2}}\, e^{-\frac{ip''q''}{2}} \times \\ &\quad \overline{f(x-q)}\, f(x-q')\,\overline{f(x-q')}\, f(x-q'') \end{aligned}$$

$$= \|f\|^2 \langle f_{q,p} | f_{q'',p''} \rangle = K(q,p;q'',p'').$$

A kernel with the properties $(i)$,$(ii)$ and $(iii)$ is known as a *reproducing kernel* [15]. As we shall see later (Section 2.1) in a group theoretical context, for an arbitrary $f \in \mathcal{H} = L^2(\mathbb{R}, dx)$, $f_{q,p}(x) = e^{-\frac{ipq}{2}} e^{ipx} f(x-q)$, $\forall x \in \mathbb{R}$, are the coherent states of Weyl-Heisenberg group.

### 1.1.1 Square Integrable Representations

Let $G$ be a locally compact group, $\mathcal{H}$ a Hilbert space (over $\mathbb{C}$) and $g \to U(g)$ a strongly continuous unitary irreducible representation (UIR) of G in $\mathcal{H}$. Let $H \subset G$ be a closed subgroup and

$$X = G/H \tag{1.10}$$

be the left coset space. The elements of $X$ are denoted by $x$, which are cosets $gH$, $g \in G$. Assume that there exists an invariant measure $\nu$ on X. Let $\sigma : X \to G$ be a (global) measurable (Borel) *section* (a map which associates to each $x \in X$, a $\sigma(x) \in G$ such that the coset $\sigma(x)H$ is exactly $x$), $F$ a positive operator on $\mathcal{H}$ with finite rank $n$. Suppose that $F$ has the diagonal representation

$$F = \sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i|, \quad u_i \in \mathcal{H}, \quad \lambda_i > 0\,, \tag{1.11}$$

where

$$\langle u_i | u_j \rangle = \delta_{ij}, \qquad i,j = 1,2,3,.....n, \tag{1.12}$$

and denote by $\mathbb{P}, \mathcal{N}^\perp$ the projection operator and the subspace of $\mathcal{H}$ :

$$\mathbb{P} = \sum_{i=1}^{n} |u_i\rangle\langle u_i| \,, \tag{1.13}$$

$$\mathcal{N}^\perp = \mathbb{P}\mathcal{H}. \tag{1.14}$$

Using the operator $F$ and the section $\sigma$, define the positive operator valued function $F_\sigma : X \to \mathcal{L}(\mathcal{H})^+$ :

$$F_\sigma(x) = U(\sigma(x))FU(\sigma(x))^*. \tag{1.15}$$

**Definition 1.1** The representation $U$ is said to be *square integrable mod*$(H, \sigma)$, if there exists a positive operator $F$, of finite rank, and a bounded positive operator $\mathcal{A}_\sigma$ on $\mathcal{H}$ with the bounded inverse, such that $\{\mathcal{H}, F_\sigma, \mathcal{A}_\sigma\}$ is a *reproducing triple*, that is, one has

$$\int_X F_\sigma(x) d\nu(x) = \mathcal{A}_\sigma \tag{1.16}$$

in the sense of weak convergence. In this case we call $F$ a *resolution generator* and the vectors

$$\eta = F^{\frac{1}{2}}u, \qquad u \in \mathcal{N}^\perp \tag{1.17}$$

*admissible vectors* mod$(H, \sigma)$. We also say that the section $\sigma$ is admissible for the representation $U$. (We say a sequence of vectors $f_k \in \mathcal{H}$ $(k = 1,2,3,\ldots)$ *converges* *weakly* to the vector $f$ if $\lim_{k\to\infty} \langle f_k \mid h \rangle = \langle f \mid h \rangle$, for all $h \in \mathcal{H}$.)

For each $u_i$, $i = 1, 2, \ldots, n$, let $\eta^i = F^{\frac{1}{2}} u_i$, and define

$$\eta^i_x := \eta^i_{\sigma(x)} = U(\sigma(x))\eta^i,\ i = 1, 2, \ldots, n \tag{1.18}$$

If (1.16) holds we call the set (1.18) *a family of covariant coherent states.* Then the 'modified resolution of identity' is given by

$$\sum_i \int_X |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}\,|\,d\nu(x) = \mathcal{A}_\sigma \tag{1.19}$$

We call the relation (1.19) a modified resolution of identity, because the operator $\mathcal{A}_\sigma$ on the right hand side of it is not necessarily an identity operator as in (1.4), but rather a positive bounded operator with a bounded inverse.

At this point, let us show how the canonical CS, Perelomov CS and the vector CS fit into the above definition of covariant CS (CCS). In the definition of CCS, if we replace the invariant subgroup $H$ by the center of $G$, then CCS reduces to Perelomov type of CS. Since the family of canonical CS is a special case of Perelomov CS, the canonical CS also fits into the definition of CCS. Before going on to show how the vector coherent states (VCS) are in conformity with the definition of CCS we briefly recall the construction of VCS. Let $G_0$ be a semisimple Lie group which has a faithful representation and $\mathfrak{g}_0$ be the corresponding Lie algebra. Let $\mathfrak{g}$ be the complex extension of $\mathfrak{g}_0$ and $G$ be the corresponding Lie group. Let $K_0$ be a compact semisimple subgroup of $G_0$ with Lie algebra $k_0$ and $k$ the corresponding extension of $k_0$ with $K$ the corresponding Lie group. Then $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = n_+ + k + n_-$, where $n_+$ and $n_-$ are respectively the spaces of positive and negative roots. Let $P$ be a parabolic subgroup of $G$ with Lie algebra $p = n_+ + k$. Let $U$ be a unitary irreducible representation (UIR) of $G_0$ acting on the Hilbert space $\mathcal{H}$ and $u$ be a UIR of $K_0$ acting on $\mathcal{H}_u \subset \mathcal{H}$. Let $T$ be the extension of $U$ to $G$. Let $e = \{e_i\}$ be a basis for $n_-$. Then an arbitrary vector in $n_-$ can be writtten as $z \cdot e = \sum_i z_i e_i$, where $z_i$ are complex numbers. Treating $z \cdot e$ as a representative of a coset space $P\,exp(z \cdot e) \in G/P$, $z = \{z_i\}$ become the coordinates of $G/P$. If for $|\,\alpha\,\rangle \in \mathcal{H}_u$ and for any non-zero $x \in n_+$, $T(x)|\,\alpha\,\rangle$ is not in $\mathcal{H}_u$, $K_0$ can be treated as an isotropy group that leavs $\mathcal{H}_u$ invariant. If $\{\alpha_i\}$ is an orthonormal basis for $\mathcal{H}_u$, the vector coherent states are defined by

$$\phi(z) = \sum_i \langle\, \alpha_i \,|\, T(exp(z \cdot e))\, \phi \,\rangle\, |\alpha_i\,\rangle, \text{ for } \phi \in \mathcal{H}.$$

Clearly, $\phi : G/K \longrightarrow \mathcal{H}$, is a holomorphic function. The group action is given by

$$g\phi(z) = \sum_i \langle\, \alpha_i \,|\, T(exp(z \cdot e))\, U(g)\, \phi \,\rangle\, |\alpha_i\,\rangle,\ g \in G.$$

What we observe here is that the construction of vector coherent states is a kind of generalization of Perelomov's method, in the sense that in both methods an isotropy subgroup of the group in consideration is used. In Perelomov's case a $1-$dimensional projection operator $F = |\alpha\,\rangle\langle\,\alpha\,|$ is used, whereas in the case of VCS an $n-$dimensional projection operator $F = \sum_{i=1}^{n} |\alpha_i\rangle\langle\alpha_i|$ is used. Thus, if the closed subgroup $H$ in the CCS is an isotropy group that leaves $\mathcal{H}_u$ invariant, the CCS are exactly vector CS.

## 1.2 Generalized Wavelet Transforms

In the previous section we gave a brief description of coherent states and their properties. Here we use coherent states to define generalized wavelet transforms. For example, we use the coherent states of the Weyl-Heisenberg group to define the Gabor transform and those of the affine group for the wavelet transform. A signal (e.g. music, speech etc.) evolves with time and its frequency changes with time. When a signal is represented by its Fourier transform, it gives information only regarding the frequency of the signal and no information concerning its time evolution. But in practice we need both time- and frequency-localization. In that sense the Fourier transform is not a useful way to represent a signal. Fortunately, there are at least two types of transforms which have this desired property. They are i) *Gabor Transform* (also known as the windowed Fourier transform or short-time Fourier transform) and ii) *wavelet transform.* Both of these transforms are related to square integrable group representations in the sense defined above.

The Gabor transform [34] of a signal $f \in L^2(\mathbb{R})$ is defined by

$$(T_{Gab}f)(\omega, t) = \int_{\mathbb{R}} ds\, f(s) g(s-t) e^{-i\omega s} \tag{1.20}$$

where $g \in L^2(\mathbb{R})$ is a fixed function, known as a *window-function* or *mother wavelet.* In the literature (1.20) is known as a *continuous Gabor transform.* Signal analysts use its discrete version, where t and $\omega$ are discretized and written as $t = nt_0$, $\omega = m\omega_0$, where $m$ and $n$ are integers, and $\omega_0$, $t_0 > 0$ are fixed. The discrete version of (1.20) is given by

$$(T_{Gab}f)_{m,n} = \int_{\mathbf{R}} ds\, f(s)g(s - nt_0)e^{-im\omega_0 s} \tag{1.21}$$

On the other hand, the continuous wavelet transform of a signal $f \in L^2(\mathbb{R})$, for the mother wavelet $\psi$ is defined by

$$(T_{wav}f)(a, b) = a^{-\frac{1}{2}} \int_{\mathbf{R}} dt\, f(t)\psi\left(\frac{t - b}{a}\right) \tag{1.22}$$

where $a\,(> 0)$, $b \in \mathbb{R}$. A possible discretization of $a$ and $b$ is written as $a = {a_0}^m$, $b = nb_0\, {a_0}^m$, where $m$ and $n$ are integers and $a_0 > 1$ and $b_0 > 0$ are fixed. Then the discretized version of wavelet transform is given by

$$(T_{wav}f)_{m,n} = {a_0}^{-\frac{m}{2}} \int_{\mathbf{R}} dt\, f(t)\psi({a_0}^{-m}t - nb_0) \tag{1.23}$$

In both cases the mother wavelet $\psi$ must satisfy the *admissibility condition*:

$$\int_{\mathbf{R}} dt\, \psi(t) = 0 \tag{1.24}$$

The functions

$$g_{\omega,t}(s) = e^{i\omega s} g(s-t) \tag{1.25}$$

and

$$\psi_{a,b}(s) = a^{-\frac{1}{2}} \psi\left(\frac{s-b}{a}\right) \tag{1.26}$$

define coherent states of the Weyl-Heisenberg group and of the affine group respectively, as we will now demonstrate.

### 1.2.1 The Weyl-Heisenberg Group

Let $G_{WH}$ be the Weyl-Heisenberg group, $G_{WH} = T \times \mathbb{R} \times \mathbb{R}$, where $T$ is the set of complex numbers of modulus one. The group multiplication is given by

$$(t;q,p)(t';q',p') = (e^{i\frac{(pq'-p'q)}{2}} tt'; q+q', p+p') \tag{1.27}$$

with identity element $(1;0,0)$ and inverse $(t;q,p)^{-1} = (t^{-1}; -q, -p)$.

A unitary irreducible representation $U$ of $G_{WH}$, acting on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}, dx), \tag{1.28}$$

is given by

$$(U(g)f)(x) = te^{-\frac{ipq}{2}} e^{ipx} f(x-q) \tag{1.29}$$

where $g = (t; q, p) \in G_{WH}$. The center of $G_{WH}$ is $Z = (t; 0, 0)$ and the left coset space $X$ is defined by

$$X = G_{WH}/Z \approx \mathbb{R}^2 \tag{1.30}$$

An arbitrary element $g \in G_{WH}$ has, according to (1.30), the following coset decomposition:

$$g = (t; x, y) = (1; x, y)(t; 0, 0) \tag{1.31}$$

If $(q, p) \in \mathbb{R}^2$ are the global coordinates of $X$, then since

$$(t;\ x,\ y)\,(1;\ q,\ p) = (1;\ x+q,\ y+p)\,(e^{i(qy-px)}\,t;\ 0,\ 0), \tag{1.32}$$

the action of $G_{WH}$ on $X$ is given by

$$g \cdot (q, p) = (x+q,\ y+p) \tag{1.33}$$

and the invariant measure on $X$, under this action, is $dq\,dp$.

The section $\sigma_0 : X \rightarrow G_{WH}$ is defined by

$$\sigma_0(q, p) = (1; q, p). \tag{1.34}$$

Then the coherent states $f_{q,p}$ of the Weyl-Heisenberg group are given by

$$f_{q,p}(x) = (U(\sigma_0)(q,p)f)(x) = e^{-\frac{ipq}{2}} e^{ipx} f(x-q) \quad (\text{using}\,(1.29)). \tag{1.35}$$

### 1.2.2 The Affine Group

The affine group is the set

$$G_{aff} = \{(a, b) | a > 0,\ b \in \mathbb{R}\} \tag{1.36}$$

and has the natural action $x \mapsto ax + b$ on $\mathbb{R}$. The group multiplication law is given by

$$(a, b)(a', b') = (aa',\ b + ab') \tag{1.37}$$

with identity element $(1, 0)$ and the inverse $(a,\ b)^{-1} = (a^{-1},\ -a^{-1}b)$. This group is not unimodular and the left Haar measure is $a^{-2}\, da\, db$, the right Haar measure $a^{-1}\, da\, db$ [43]. The unitary irreducible representation $U$ of $G_{aff}$, acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$, is given by [16, 17]

$$(U(a, b)\psi)(x) = a^{-\frac{1}{2}}\, \psi\left(\frac{x - b}{a}\right), \quad (a > 0 \quad b \in \mathbb{R}) \tag{1.38}$$

The square integrability of the representation $U$ and the admissibility of a vector $\psi \in L^2(\mathbb{R}, dx)$ in this case reduces to the condition:

$$C_\psi = \int_{G_{aff}} |\langle U(a, b)\psi | \psi\rangle|^2 \, \frac{da\, db}{a^2} < \infty \tag{1.39}$$

It is connected to our definition of square integrability of a representation and admissibility of a vector in the sense that here $F = |\,\psi\,\rangle\,\langle\,\psi\,|$ $(n = 1)$ is a multiple of

1−dimensional projection operator on $\mathcal{H} = L^2(\mathbb{R}, dx)$, and the trivial section $\sigma(g) = g$ may be used, because the representation is square integrable with respect to the whole group. The condition (1.39) can also be written as

$$C_\psi = 2\pi\|\psi\|^2 \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty \tag{1.40}$$

where $\hat{\psi}$ is the Fourier transform of $\psi$. From (1.40) we observe that if $\hat{\psi}(0) \neq 0$, the integral does not converge, so it is necessary to assume $\hat{\psi}(0) = 0$. That is,

$$\psi \quad \text{is admissible if} \quad \hat{\psi}(0) = 0 \tag{1.41}$$

or equivalently,

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \tag{1.42}$$

i.e., the mean of $\psi$ is zero. Now we see that the coherent states of the affine group $G_{aff}$ are the vectors:

$$\psi_{ab}(x) = (U(a,b)\psi)(x) \tag{1.43}$$

that is,

$$\psi_{ab}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \quad (a > 0, \quad b \in \mathbb{R}) \tag{1.44}$$

with $\psi$ an admissible vector.

## 1.3 Frames

The concept of a *frame* was first introduced by Duffin and Schaeffer [31] in 1952, in the context of non-harmonic Fourier analysis and has also been reviewed in [77]. Here we will define it and give a brief description of some its properties, for later use, specially in chapter 3. Let $\mathcal{H}$ be a separable Hilbert space (over $\mathbb{C}$) and $X$ be a locally compact space, $\nu$ a regular Borel measure on $X$ with support equal to $X$.

**Definition 1.2** A set of vectors $\{\eta_x^i\}_{i=1}^N$ in $\mathcal{H}$ is a *frame* if for all $x \in X$ the vectors $\{\eta_x^i\}$, $i = 1, 2, \dots N$, are linearly independent, and there exist two numbers $A, B > 0$ such that for all $\phi \in \mathcal{H}$ one has

$$A\|\phi\|^2 \leq \sum_{i=1}^{N} \int_X |\langle \eta_x^i | \phi \rangle|^2 \, d\nu(x) \leq B\|\phi\|^2 \tag{1.45}$$

It is appropriate to note here that the definition of frame given above is much more general than the one given in [31, 28, 48, 45, 25, 46]. If $N = 1$ and $X$ is a discrete space with $\nu$ a counting measure i.e., $\nu(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$, where $n$ is an integer, it reduces to the usual definition of frame used in the literature and we call it *a discrete frame*, otherwise, it will be called *a continuous frame*. The numbers A, B are called the *frame bounds*. The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence. In general, the set of vectors $\{\eta_x^i\}_{i=1}^N$ is not an *orthogonal basis*, even a tight frame may not be an orthonomal basis but an orthonomal basis is a tight frame with $A = B = 1$.

Let $\{\phi_i, i = 1, 2, \ldots \infty\}$ be a discrete frame and $\phi$ be any vector in $\mathcal{H}$. Then an operator $S$ defined by

$$S\phi = \sum_{i=1}^{\infty} \langle \phi_i | \phi \rangle \phi_i \tag{1.46}$$

is known as the *frame operator*. It can be shown [31] that

i) S is a bounded linear operator with

$$AI \leq S \leq BI, \quad \text{(I is the identity operator)} \tag{1.47}$$

ii) S is invertible with

$$B^{-1} I \leq S^{-1} \leq A^{-1} I \tag{1.48}$$

iii) $\{S^{-1}\phi_i\}$ is a frame, called the *dual frame or reciprocal frame* of $\{\phi_i\}$.

iv) Every $\phi \in \mathcal{H}$ can be written as

$$\phi = \sum_{i=1}^{\infty} \langle \phi | S^{-1} \phi_i \rangle \phi_i = \sum_{i=1}^{\infty} \langle \phi | \phi_i \rangle S^{-1} \phi_i \tag{1.49}$$

which we call the *reconstruction formula*.

v) The decomposition of $\phi \in \mathcal{H}$ is, in general, not unique in the sense that if we write $\phi = \sum_{i=1}^{\infty} \alpha_i \phi_i$, where $\alpha_i = \langle \phi | S^{-1} \phi_i \rangle$, then it is possible to find a set of complex numbers $\beta_i$ such that $\phi = \sum_{i=1}^{\infty} \beta_i \phi_i$, where

$$\sum_{i=1}^{\infty} |\beta_i|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2 + \sum_{i=1}^{\infty} |\alpha_i - \beta_i|^2 \tag{1.50}$$

**Example 1.1** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then

i) $\{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$ is a tight inexact frame with bounds $A = B = 2$ (since the eigenvalues of the frame operator are always 2), but is not an orthonormal basis; although it contains one.

ii) $\{e_1, e_2/2, e_3/3, \ldots\}$ is a complete orthogonal sequence; but not a frame (since, in this case, the eigenvalues of the frame operator lie between 0 and 1 and are not bounded away from 0 and consequently, the inverse of the frame operator is not bounded).

iii) $\{2e_1, e_2, e_3, \ldots\}$ is a non-tight exact frame with bounds $A = 1$, $B = 2$ (since, the minimum eigenvalue of the frame operator is 1 and the maximum eigenvalue is 2).

In practice, the reconstruction of a signal / image using the reconstruction formula, could be highly efficient, in the sense that the sum converges very rapidly and can be truncated after a few terms, if $|B/A-1| \ll 1$. It is a matter of fact that the frames used in wavelet analysis (we mean both Gabor wavelets and affine wavelets) are obtained by discretizing a set of coherent states of the respective group. We have already shown in the previous section of this chapter that Gabor wavelets are associated to the coherent states of the Weyl-Heisenberg group and affine wavelets or simply wavelets with those of the affine group.

# Chapter 2

# The $\beta$-Duality And Spin Coherent States

In the previous chapter, we gave a brief description of coherent states, along with some examples. Here our aim is to construct the coherent states of the Poincaré group $\mathcal{P}_+^\uparrow(1,3)$, in 3-space and 1-time dimensions, for a unitary irreducible representation $U_W^s$, corresponding to a positive mass $m$ and arbitrary spin $s$, by adopting the same technique as used in extracting the coherent states of the Weyl-Heisenberg group in the previous chapter. Then we will show that the resulting coherent states form a frame, which we call a relativistic frame. In section 2.1, we give a brief description of Minkowski space and some notational conventions, we construct coherent states of the full Poincaré group and discuss $\beta$-duality respectively in section 2.2 and section 2.3. Finally, in section 2.4, we show that the coherent states of the full Poincaré group form frames and analyse the frames for various known sections.

## 2.1 Notational Conventions

We denote the coordinates of a point of the four-dimensional space-time continuum $\mathcal{R}_{1,3}$, known as Minkowski space, by $x^\mu = (x^0, \mathbf{x})$, with

$$x^0 = ct\,, \quad \text{the time coordinate} \tag{2.1}$$

$$\mathbf{x} = (x^1 = x,\ x^2 = y,\ x^3 = z), \quad \text{the spatial coordinates} \tag{2.2}$$

In what follows we shall assume $\hbar = c = 1$. We use the Greek indices to denote the components of four-vectors taking values 0, 1, 2, 3, and Roman indices to denote the components of ordinary space vectors taking values 1, 2, 3. We write covariant vectors with subscripts and contravariant vectors with superscripts. Thus $a^\mu$ is a contravariant vector and the corresponding covariant vector $a_\mu$ is obtained by

$$a_\mu = \sum g_{\mu\nu} a^\nu \tag{2.3}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.4}$$

is the space-time *metric tensor*. (2.3) gives $a_0 = a^0 \quad a_k = -a^k, \quad k = 1, 2, 3.$ Here the convention of summing over repeated indices is to be understood.

The Minkowski scalar product of two four-vectors $a^\mu$ and $b^\mu$ is defined by

$$a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{2.5}$$

and the Minkowski norm of $a^\mu$ is

$$a^\mu a_\mu = (a^0)^2 - \mathbf{a} \cdot \mathbf{a} \tag{2.6}$$

Because of the negative signs in the metric tensor (2.4), the scalar product $a^\mu a_\mu$ is no longer positive definite, it can be positive, negative or zero. The vectors in Minkowski space are classified into three categories, depending on whether the norm of $a^\mu$ is positive, negative or zero. The vector $a^\mu$ is said to be *space-like* if its norm is negative; it is called *light-like* if the norm is zero; and it is called *time-like*, if the norm is positive.

An affine transformation in Mikowski space is defined by

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.7}$$

with the $\Lambda^\nu_\mu$ satisfying

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}, \tag{2.8}$$

Here $\Lambda^\mu_\nu$ is the matrix of the Lorentz transformation and the vector $a^\mu$ represents a simple translation of the space-time axes. (2.7) is known as an *inhomogeneous Lorentz* *transformation* and if $a^\mu = 0$, it is called a *homogeneous Lorentz tranformation.* The homogeneous Lorentz transformation leaves the scalar product invariant

$$x'y' = g_{\mu\nu}x'^\mu y'^\nu = g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma = xy \tag{2.9}$$

and forms a group known as the **Lorentz group** (we denote it by $\mathcal{L}$).

Topologically the matrices of the Lorentz group $\mathcal{L}$ consist of four disconnected pieces. From (2.8) we have $g_{\mu\nu}\Lambda^\mu_0 \Lambda^\nu_0 = g_{00} = 1$ which implies

$$(\Lambda^0_0)^2 = 1 + \sum_{k=1}^{3}(\Lambda^k_0)^2 \geq 1 \tag{2.10}$$

Then we have either

$$\Lambda^0_0 \geq 1 \quad or \quad \Lambda^0_0 \leq -1 \tag{2.11}$$

Taking the determinant of both sides of (2.8) we get

$$det(\Lambda) = \pm 1 \tag{2.12}$$

Thus we have the following decomposition of the Lorentz group into four pieces [72] :

$\mathcal{L}^\uparrow_+$ : $det(\Lambda) = +1$, sign $\Lambda^0_0 = +1$. This one itself forms a group, known as the **proper, orthochronous Lorentz group.**

$\mathcal{L}_-^\uparrow$ : $det(\Lambda) = -1$, sign $\Lambda_0^0 = +1$. This piece has an element $I_s$ defined by

$$I_s x = (x^0, -x^1, -x^2, -x^3)$$

and is called *space inversion operation.*

$\mathcal{L}_-^\downarrow$ : $det(\Lambda) = -1$, sign $\Lambda_0^0 = -1$. This piece has an element $I_t$, known as a *time reversal operation*, is defined by

$$I_t x = (-x^0, x^1, x^2, x^3)$$

$\mathcal{L}_+^\downarrow$ : $det(\Lambda) = +1$, sign $\Lambda_0^0 = -1$. This one has an element $I_{st} = I_s I_t$.

From these four pieces we can build the following three subgroups of the Lorentz group:

$\mathcal{L}^\uparrow = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow$, the orthochronous Lorentz group (direction of time unchanged).

$\mathcal{L}_+ = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow$, the proper Lorentz group.

$\mathcal{L}_0 = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\downarrow$, the orthochronous Lorentz group (spatial directions unchanged).

The inhomogeneous Lorentz transformations form a group, known as the **Poincaré group**. We denote this group by $\mathcal{P}_+^\uparrow(1,3)$ (in 1-time and 3-space dimensions) which can be identified as $T^4 \oslash \mathcal{L}_+^\uparrow(1,3)$, where $T^4 \simeq \mathbb{R}_{1,3}$ is the group of space-time translations and $\oslash$ denotes the semi-direct product. Since $\mathcal{L}_+^\uparrow$ and $SL(2,\mathbb{C})$ are physically equivalent (as they are $2-1$ homomorphic [70]) here we use $SL(2,\mathbb{C})$, the universal covering group of $\mathcal{L}_+^\uparrow$ instead to include the half integral spins [76]. Here by 'universal covering group' we mean: Let $G$ be an admissible topological group and $(\tilde{G},\, \pi)$ a covering of the topological space $G$. By introducing a multiplication into $\tilde{G}$, it is possible to show that $\tilde{G}$ itself is a topological group and $\pi : \tilde{G} \to G$ is a continuous homomorphism with discrete kernel

$$N = \{\tilde{g} \in \tilde{G} \,|\, \pi(\tilde{g}) = e\}$$

which is a discrete invariant subgroup and $G = \tilde{G}/N$. In this case $\tilde{G}$ is called a *covering group* of $G$. Given any admissible group $G$, there exists a simply connected covering group of $G$. It is determined up to isomorphism and is known as the *universal covering group* of $G$ [64]. We write [13]

$$\mathcal{P}_+^\uparrow(1,3) = T^4 \oslash SL(2,\mathbb{C}) \tag{2.13}$$

Elements of $\mathcal{P}_+^\uparrow(1,3)$ will be denoted by

$$(a, A),\; a = (a_0, \mathrm{a}) \in \mathbb{R}_{1,3}, \quad A \in SL(2,\mathbb{C}) \tag{2.14}$$

The multiplication law is

$$(a, A)(a', A') = (a + \Lambda a', AA') \tag{2.15}$$

where $\Lambda \in \mathcal{L}_+^\uparrow(1,3)$ (the proper, orthochronous Lorentz group) is the Lorentz tranformation corresponding to A:

$$\Lambda^\mu_\nu = \frac{1}{2} Tr[A\sigma_\nu A^\dagger \sigma_\mu], \quad \mu, \nu = 0, 1, 2, 3, \tag{2.16}$$

where: $Tr[M]$ denotes the trace of the matrix $M$,

$$\sigma = (\sigma^0, \boldsymbol{\sigma}); \quad \sigma^0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. We take the following specific realization for the Pauli matrices:

$$\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.17}$$

Let

$$\mathcal{V}_m^+ = \{k = (k_0, \mathbf{k}) \in \mathbb{R}_{1,3} \,|\, k^2 = {k_0}^2 - \mathbf{k}^2 = m^2, \ k_0 > 0\} \tag{2.18}$$

be the **forward mass hyperboloid.** It is noted that the invariant measure under the action of $\Lambda \in \mathcal{L}_+^\uparrow(1,3)$ on $\mathcal{V}_m^+$ is $\frac{d\mathbf{k}}{k_0}$ and $\Lambda$ acts on $\mathcal{V}_m^+$ in the following manner:

$$k' = \Lambda k \Rightarrow \sigma \cdot k' = A\sigma \cdot kA^\dagger \tag{2.19}$$

where $\sigma \cdot k = \sigma^\mu k_\mu = k_0 \mathbb{I}_2 - \mathbf{k} \cdot \boldsymbol{\sigma}$

If $\hat{T}^4$ is the dual group of $T^4$, then the orbits of an arbitrary $\hat{n} = (\hat{n}_0, \hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{T}^4$ under the action of $SL(2,\mathbb{C})$ are of the form,

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \tag{2.20}$$

here we identify $m$ with mass. In total there are six different orbits (2 for $m^2 > 0$, 1 for $m^2 < 0$, and 3 for $m = 0$) [19]. Correspondingly, there are six different classes of unitary irreducible representations of $\mathcal{P}_+^\uparrow(1,3)$ [19, 54]. For our purposes we need the representations corresponding to the orbit of $\hat{n} = (m, 0, 0, 0)$, $m > 0$. The corresponding representations denote particles of mass $m > 0$ and spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. These representations $U_W^s$ are carried by the Hilbert spaces [76].

$$\mathcal{H}_W^s = \mathbb{C}^{2s+1} \otimes L^2\left(\mathcal{V}_m^+, \frac{d\mathbf{k}}{k_0}\right) \tag{2.21}$$

of $\mathbb{C}^{2s+1}$-valued functions $\phi$ on $\mathcal{V}_m^+$, which are square integrable in the sense:

$$\int_{\mathcal{V}_m^+} \phi(k)^\dagger \phi(k) \frac{d\mathbf{k}}{k_0} = \|\phi\|^2 = \langle \phi | \phi \rangle < \infty \tag{2.22}$$

The representation is defined by

$$(U_W^s(a,A)\phi)(k) = e^{ik\cdot a}\mathcal{D}^s(h(k)^{-1}Ah(\Lambda^{-1}k))\phi(\Lambda^{-1}k),\ k\cdot a = k_0a_0 - \mathbf{k}\cdot\mathbf{a} \qquad (2.23)$$

where $\mathcal{D}^s$ is the $(2s+1)$-dimensional irreducible spinor representation of $SU(2)$ (carried by $\mathbb{C}^{2s+1}$) and

$$k \to h(k) = \frac{m\mathbb{I}_2 + \sigma\cdot\overline{k}}{\sqrt{2m(k_0+m)}}, \quad (\overline{k} = (k_0\,,-\mathbf{k})) \qquad (2.24)$$

is the image in $SL(2,\mathbb{C})$ of the Lorentz boost $\Lambda_k$, which brings the four-vector $(m,\mathbf{0})$ to the 4-vector $k$ in $\mathcal{V}_m^+$:

$$\Lambda_k(m,\mathbf{0}) = k \Leftrightarrow h(k)m\mathbb{I}_2h(k) = m[h(k)]^2 = \sigma\cdot\overline{k} = \begin{pmatrix} k_0+k_z & k_x - ik_y \\ k_x+ik_y & k_0-k_z \end{pmatrix} \qquad (2.25)$$

The matrix form of the Lorentz boost is

$$\Lambda_k = \begin{pmatrix} \frac{k_0}{m} & \frac{k_x}{m} & \frac{k_y}{m} & \frac{k_z}{m} \\ \frac{k_x}{m} & 1+\frac{k_x{}^2}{m(k_0+m)} & \frac{k_xk_y}{m(k_0+m)} & \frac{k_xk_z}{m(k_0+m)} \\ \frac{k_y}{m} & \frac{k_xk_y}{m(k_0+m)} & 1+\frac{k_y{}^2}{m(k_0+m)} & \frac{k_yk_z}{m(k_0+m)} \\ \frac{k_z}{m} & \frac{k_xk_z}{m(k_0+m)} & \frac{k_yk_z}{m(k_0+m)} & 1+\frac{k_z{}^2}{m(k_0+m)} \end{pmatrix} = \Lambda_k^\dagger \qquad (2.26)$$

which could be written as

$$\Lambda_k = \frac{1}{m}\begin{pmatrix} k_0 & \mathbf{k}^\dagger \\ \mathbf{k} & mV_k \end{pmatrix} \qquad (2.27)$$

where $V_k$ is the $3 \times 3$ symmetric matrix

$$V_k = \mathbb{I}_3 + \frac{\mathbf{k} \otimes \mathbf{k}^\dagger}{m(k_0 + m)} = V_k{}^\dagger = V_{\overline{k}} \tag{2.28}$$

It is noted that $\Lambda_k$ and $V_k$ have following useful properties which can be easily verified:

$$\det \Lambda_k = 1\,, \quad \det(V_k) = \frac{k_0}{m}\,, \tag{2.29}$$

and

$$(\Lambda_k p)_0 = \frac{1}{m}(k_0 p_0 + \mathbf{k} \cdot \mathbf{p}) = \frac{k \cdot \overline{p}}{m}, \quad (\underline{\Lambda_k p}) = \frac{1}{m}\mathbf{k} p_0 + V_k \mathbf{p}, \tag{2.30}$$

the underline denoting the spatial part of a 4-vector, while

$$k_0(\underline{\Lambda_k p}) - \mathbf{k}(\Lambda_k p)_0 = k_0 V_k^{-1} \mathbf{p}, \tag{2.31}$$

from which it follows that

$$\|k_0(\underline{\Lambda_k p}) - \mathbf{k}(\Lambda_k p)_0\| \leq k_0 \|\mathbf{p}\|. \tag{2.32}$$

Also, since

$$\Lambda_k^{-1} = \Lambda_{\overline{k}} \tag{2.33}$$

we have,

$$h(k)^{-1} = h(\overline{k}). \tag{2.34}$$

## 2.2 Phase Space For Massive, Spin-s Particles

In general, a *phase space* of a system with $n$ degrees of freedom is a $2n$-dimensional space. The notion of phase space goes back to the Hamiltonian formulation of classical mechanics [41], where a dynamical state of a system at a given instant is completely determined by its $n$ postion coordinates $q_1$, $q_2$, $\cdots$ $q_n$, and the $n$ corresponding conjugate momenta $p_1$, $p_2$, $\cdots$ $p_n$. Then the $2n$-dimensional space $\Gamma$ whose points have the coordinates $(q_1, q_2, \cdots q_n; p_1, p_2, \cdots p_n)$ is known as phase space. The phase space $\Gamma$ for a massive relativistic particle with arbitrary spin $s$ corresponding to a Wigner representation can be written as [1]

$$\Gamma = \mathcal{P}_+^\uparrow(1,3)/(T \times SU(2)), \tag{2.35}$$

where $T$ denotes the subgroup of time translations. For $A \in SL(2,\mathbb{C})$, let

$$A = h(k)\mathcal{R}(k), \quad \mathcal{R}(k) \in SU(2), \tag{2.36}$$

where $h(k)$ is defined in (2.24), be its Cartan decomposition. An arbitrary element $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$ has the left coset decomposition,

$$(a, A) = ((0, \mathbf{a} - \frac{a_0\mathbf{k}}{k_0}), h(k))((\frac{ma_0}{k_0}, \mathbf{0}), \mathcal{R}(k)) \tag{2.37}$$

according to (2.36). Thus , elements in $\Gamma$ have the global coordinatization, i.e., any point on $\Gamma$ has the coordinates $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^6$: with

$$\mathbf{q} = \mathbf{a} - \frac{a_0 \mathbf{k}}{k_0}, \quad \mathbf{p} = \mathbf{k} \tag{2.38}$$

In terms of these variables, the action $\theta : \mathcal{P}_+^\uparrow(1,3) \times \Gamma \rightarrow \Gamma$ of $\mathcal{P}_+^\uparrow(1,3)$ on $\Gamma$ is, according to (2.37), given by

$$\begin{aligned}
\theta(g,((0,\mathbf{q}),\Lambda_p)) &= g((0,\mathbf{q}),\Lambda_p), \quad g=(a,\Lambda_k) \\
&= (a,\Lambda_k)((0,\mathbf{q}),\Lambda_p) \\
&= (a+\Lambda_k(0,\mathbf{q}),\Lambda_k\Lambda_p) \\
&= ((a+\Lambda_k(0,\mathbf{q}),\Lambda_{\Lambda_k p}) \\
&= (a+\Lambda_k(0,\mathbf{q}),\Lambda_{p'}), \quad p'=\Lambda_k p \\
&= \left((a_0+\frac{\mathbf{k}\cdot\mathbf{q}}{m}, \mathbf{a}+\mathbf{q}+\frac{\mathbf{k}\cdot\mathbf{q}}{m(m+k_0)}\mathbf{k}), \Lambda_{p'}\right) \qquad (2.39)
\end{aligned}$$

Then using (2.38) and writing the new coordinates as $(\mathbf{q}',\mathbf{p}')$, we get

$$\left.\begin{aligned}
\mathbf{q}' &= \mathbf{a}+\mathbf{q}+\tfrac{\mathbf{k}\cdot\mathbf{q}}{m(m+k_0)}\mathbf{k} - \tfrac{\mathbf{p}'}{p_0{}'}(a_0+\tfrac{\mathbf{k}\cdot\mathbf{q}}{m}) \\
\mathbf{p}' &= \underline{\Lambda_k p}
\end{aligned}\right\} \tag{2.40}$$

which can be written as

$$
\left.\begin{aligned}
\mathbf{q}' &= \tfrac{1}{p_0{}'}(p_0{}'[\mathbf{a} + \underline{\Lambda_k(0,\mathbf{q})}]) - \mathbf{p}'[a_0 + \{\Lambda_k(0,\mathbf{q})\}_0] \\
\mathbf{p}' &= \underline{\Lambda_k p}\,, \quad p = (\sqrt{m^2+\mathbf{p}^2},\, \mathbf{p})
\end{aligned}\right\} \tag{2.41}
$$

where $\Lambda \in \mathcal{L}_+^\uparrow(1,3)$ is related to $A$ by (2.16) and $p_0{}' = (\Lambda_k p)_0$.

Now we want to show that the invariant measure under this action is $d\mathbf{q}\,d\mathbf{p}$: The Jacobian matrix $J$ of the transformation (2.40) is given by

$$
J = \begin{pmatrix} \frac{\partial \mathbf{q}'}{\partial \mathbf{q}} & \frac{\partial \mathbf{q}'}{\partial \mathbf{p}} \\ \frac{\partial \mathbf{p}'}{\partial \mathbf{q}} & \frac{\partial \mathbf{p}'}{\partial \mathbf{p}} \end{pmatrix} = \begin{pmatrix} A = V_k - \frac{1}{m p_0{}'}(\mathbf{p}' \otimes \mathbf{k}^\dagger) & \frac{\partial \mathbf{q}'}{\partial \mathbf{p}} \\ O_3 & B = V_k + \frac{1}{m p_0}(\mathbf{k} \otimes \mathbf{p}^\dagger) \end{pmatrix} \tag{2.42}
$$

where $V_k$ is defined in (2.28) and $O_3$ is a $3 \times 3$ zero matrix. Since $J$ in (2.42) is an upper triangular block matrix, the determinant $\det(J) = \det(A) \cdot \det(B) = 1$ which implies

$$
d\mathbf{q}'\,d\mathbf{p}' = d\mathbf{q}\,d\mathbf{p} \tag{2.43}
$$

Hence the invariant measure $\nu$ on $\Gamma$, in the variables $(\mathbf{q},\mathbf{p})$, is $d\mathbf{q}\,d\mathbf{p}$.

Next, in terms of these variables let us define the basic section,

$$
\sigma_0 : \Gamma \to \mathcal{P}_+^\uparrow(1,3) \quad \text{such that} \tag{2.44}
$$

$$
\sigma_0(\mathbf{q},\mathbf{p}) = ((0,\mathbf{q}), h(p)) \tag{2.45}
$$

which we call the the *Galilean section*, if $h(p) = I$. Let $\Pi : \mathcal{P}_+^\uparrow(1,3) \rightarrow \Gamma$ defined by $\Pi(g) = g \cdot (T \times SU(2)) \in \Gamma$ such that $\Pi(\sigma_0(\mathbf{q},\mathbf{p})) = (\mathbf{q},\mathbf{p})$, then any other section $\sigma = \sigma_0 \circ \Pi : \Gamma \rightarrow \mathcal{P}_+^\uparrow(1,3)$ is defined by:

$$\sigma(\mathbf{q},\mathbf{p}) = \sigma_0(\mathbf{q},\mathbf{p})((f(\mathbf{q},\mathbf{p}),\mathbf{0}),\mathcal{R}(\mathbf{q},\mathbf{p})) \tag{2.46}$$

where $f : \mathbb{R}^6 \rightarrow \mathbb{R}$ and $\mathcal{R} : \mathbb{R} \rightarrow SU(2)$ are smooth functions. We work with *affine sections*, for which the function $f$ is taken to be of the form

$$f(\mathbf{q},\mathbf{p}) = \varphi(\mathbf{p}) + \mathbf{q} \cdot \boldsymbol{\vartheta}(\mathbf{p}) \tag{2.47}$$

where, $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$, and $\boldsymbol{\vartheta} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are smooth functions of $\mathbf{p}$ alone. In calculating the frame operator, we need to evaluate an integral, (see (2.120)) where it is necessary for $f$ to have the above form. We will later see that as far as the construction of CS is concerned, $\varphi$ only introduces an inessential phase and it does not make any difference even if we set $\varphi = 0$ (see(2.120)). Moreover, we also impose the restriction that $\mathcal{R}(\mathbf{q},\mathbf{p}) = \mathcal{R}(\mathbf{p})$ be a function of $\mathbf{p}$ alone (we need this for the evaluation of the integral (2.120) below). Thus writing,

$$\sigma(\mathbf{q},\mathbf{p}) = (\hat{q}, h(p)\mathcal{R}(\mathbf{p})), \quad \hat{q} = (\hat{q}_0, \hat{\mathbf{q}}) \in \mathbb{R}_{1,3} \tag{2.48}$$

we see that

$$\hat{q}_0 = \frac{p_0}{m}\boldsymbol{\vartheta}(\mathbf{p}) \cdot \mathbf{q} \tag{2.49}$$

$$\hat{\mathbf{q}} = M(\mathbf{p}, \boldsymbol{\vartheta})\mathbf{q} \tag{2.50}$$

where $M(\mathbf{p}, \boldsymbol{\vartheta})$ is the $3 \times 3$ real matrix

$$M(\mathbf{p}, \boldsymbol{\vartheta}) = \mathbb{I}_3 + \frac{\mathbf{p} \otimes \boldsymbol{\vartheta}(\mathbf{p})^\dagger}{m} \tag{2.51}$$

We shall analyze (2.51) extensively later. Let us simply note here that

$$\det[M(\mathbf{p}, \boldsymbol{\vartheta})] = 1 + \frac{\mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})}{m} \tag{2.52}$$

so that if $\det[M(\mathbf{p}, \boldsymbol{\vartheta})] \neq 0$,

$$M(\mathbf{p}, \boldsymbol{\vartheta})^{-1} = \mathbb{I}_3 - \frac{\mathbf{p} \otimes \boldsymbol{\vartheta}(\mathbf{p})^\dagger}{m + \mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})} \tag{2.53}$$

Also, assuming $M(\mathbf{p}, \boldsymbol{\vartheta})$ to be continuous in $\mathbf{p}$ and $\boldsymbol{\vartheta}$, and non-singular for all $(\mathbf{q}, \boldsymbol{\vartheta})$ (i.e., $\mathbf{q} = \mathbf{0} \Leftrightarrow \hat{\mathbf{q}} = \mathbf{0}$ in (2.50)) and since $\det[M(\mathbf{0}, \boldsymbol{\vartheta})] = 1$, it follows that

$$\det[M(\mathbf{p}, \boldsymbol{\vartheta})] > 0\,, \quad \forall\,(\mathbf{q}, \boldsymbol{\vartheta}). \tag{2.54}$$

## 2.3 The $\beta-$ Duality and Space-Like Sections

Since the matrix $M(\mathbf{p}, \boldsymbol{\vartheta})$ in (2.50) has an inverse, we easily obtain from there,

$$\hat{q}_0 = \boldsymbol{\beta}(\mathbf{p}) \cdot \hat{\mathbf{q}}, \tag{2.55}$$

where $\boldsymbol{\beta}(\mathbf{p})$ is the 3$-$vector field

$$\boldsymbol{\beta}(\mathbf{p}) = \frac{p_0 \boldsymbol{\vartheta}(\mathbf{p})}{m + \mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})}. \tag{2.56}$$

Solving for $\boldsymbol{\vartheta}(\mathbf{p})$, this gives

$$\boldsymbol{\vartheta}(\mathbf{p}) = \frac{m\boldsymbol{\beta}(\mathbf{p})}{p_0 - \mathbf{p} \cdot \boldsymbol{\beta}(\mathbf{p})}. \tag{2.57}$$

Let us also introduce the *dual* vector fields $\boldsymbol{\beta}^*$ and $\boldsymbol{\vartheta}^*$,

$$\boldsymbol{\beta}^* = \frac{\mathbf{p} - m\mathbf{V}_p \boldsymbol{\beta}(\mathbf{p})}{p_0 - \mathbf{p} \cdot \boldsymbol{\beta}(\mathbf{p})}. \tag{2.58}$$

$$\boldsymbol{\vartheta}^* = \frac{m\boldsymbol{\beta}^*(\mathbf{p})}{p_0 - \mathbf{p} \cdot \boldsymbol{\beta}^*(\mathbf{p})}, \tag{2.59}$$

where $\mathbf{V}_p$ is the matrix defined in (2.28). The significance of these dual quantities will shortly become clear. First note that

$$\boldsymbol{\beta}^{**} = \boldsymbol{\beta}, \qquad \boldsymbol{\vartheta}^{**} = \boldsymbol{\vartheta} \tag{2.60}$$

and

$$\boldsymbol{\vartheta}(\mathbf{p}) = \frac{1}{m}[\mathbf{p} - m\mathbf{V}_p\boldsymbol{\beta}^*(\mathbf{p})] = \frac{\mathbf{p} - p_0\mathbf{V}_p^{-1}\boldsymbol{\vartheta}^*(\mathbf{p})}{m + \mathbf{p}\cdot\boldsymbol{\vartheta}^*(\mathbf{p})}. \tag{2.61}$$

Let us try to get a better understanding of the $\boldsymbol{\beta} - \boldsymbol{\beta}^*$ duality [13] as defined in (2.56) and (2.58). Since in order to satisfy the positivity condition of the Jacobian $J_X(\mathbf{k})$ (see (2.124)) we will need $\|\boldsymbol{\beta}(\mathbf{p})\| < 1$ (see again (2.131)), let us define the relativistic 4-velocity $n(p)$ by

$$n(p) = (n_0(\mathbf{p}), \mathbf{n}(\mathbf{p})), \quad n_0(\mathbf{p}) = [1 - \|\boldsymbol{\beta}(\mathbf{p})\|^2]^{-\frac{1}{2}}, \quad \frac{\mathbf{n}(\mathbf{p})}{n_0(\mathbf{p})} = \boldsymbol{\beta}(\mathbf{p}). \tag{2.62}$$

Then, by (2.55), the point $\hat{q} = (\hat{q}_0, \hat{\mathbf{q}}) \in T^4$ satisfies

$$n(p)\cdot\hat{q} = 0, \tag{2.63}$$

that is, $\hat{q}$ lies on the space-like hyperplane with normal vector $n(p)$, determined by $\boldsymbol{\beta}(\mathbf{p})$. Let us denote this hyperplane by $\Sigma_p^\beta$. In particular, for the *Galilean section*,

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_0(\mathbf{p}) = \mathbf{0}, \quad \sigma = \sigma_0, \quad \text{and} \quad \Sigma_p^\beta = \Sigma_p^0 = \{(0,\mathbf{q}) \,|\, \mathbf{q} \in \mathbb{R}^3\} \tag{2.64}$$

while for the *Lorentz section*,

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_\ell(\mathbf{p}) = \frac{\mathbf{p}}{p_0}, \quad \sigma = \sigma_\ell, \quad \text{and} \quad \Sigma_p^\beta = \Sigma_p^\ell = \{\hat{q} \in T^4 \,|\, \frac{1}{m} p\cdot\hat{q} = 0\} \tag{2.65}$$

Note that these two sections are related by the duality of (2.58), i.e., $\boldsymbol{\beta}_0{}^*(\mathbf{p}) = \boldsymbol{\beta}_\ell(\mathbf{p})$. From (2.62) and (2.63), we see that, with $\|\boldsymbol{\beta}\| < 1$, $\forall\mathbf{p}$, we can associate to $\boldsymbol{\beta}(\mathbf{p})$ the time-like 4-vector field $u(p) = mn(p)$ which is normal to the space-like hyperplane $\Sigma_p^\beta$ in $T^4$. More generally, if $\mathbf{p} \to \boldsymbol{\beta}(\mathbf{p})$ is a $\mathbf{p}$-dependent 3-vector field, we can associate to it a ray $[u(p)]$ of $p$-dependent relativistic 4-vector fields $u(p)$ in the following way:

$$u(p) = (u_0(\mathbf{p}), \mathbf{u}(\mathbf{p})), \quad p = (p_0, \mathbf{p}), \quad u_0(\mathbf{p}) > 0, \quad \frac{\mathbf{u}(\mathbf{p})}{u_0(\mathbf{p})} = \boldsymbol{\beta}(\mathbf{p}) \tag{2.66}$$

with

$$u(p) \cdot u(p) = u_0(\mathbf{p})^2 - \|\mathbf{u}(\mathbf{p})\|^2. \tag{2.67}$$

The 4-vector $u(p)$, and hence the ray $[u(p)] = \mathbb{R}^+ u(p)$ is time-like, light-like or space-like according as $\|\boldsymbol{\beta}(\mathbf{p})\| < 1$, $\|\boldsymbol{\beta}(\mathbf{p})\| = 1$, or $\|\boldsymbol{\beta}(\mathbf{p})\| > 1$, respectively. Under a Lorentz transformation $\Lambda$ , $u(p) \to \Lambda u(p)$ and

$$\boldsymbol{\beta}(\mathbf{p}) = \frac{\mathbf{u}(\mathbf{p})}{u_0(\mathbf{p})} \to \boldsymbol{\beta}'(\mathbf{p}) = \frac{\underline{\Lambda u(p)}}{(\Lambda u(p))_0}. \tag{2.68}$$

Of course, such a transformation preserves the property of $u$ being time-like, light-like or space-like, and hence of the equivalent properties of $\|\boldsymbol{\beta}(\mathbf{p})\|$ being $<$, $=$, or $> 1$. Now let $\Lambda_k$ be a Lorentz boost. To any $u(p) \in [u(p)]$, let us associate its $k$-conjugate 4-vector field under $\Lambda_k$:

$$u^{*,k}(p) = \Lambda_k \overline{u(p)}, \quad \overline{u(p)} = (u_0(\mathbf{p}), -\mathbf{u}(\mathbf{p})). \tag{2.69}$$

Clearly,

$$u(p) = \Lambda_k \overline{u^{*,k}(p)} \Rightarrow (u^{*,k}(p))^{*,k} = u(p). \tag{2.70}$$

Let $\boldsymbol{\beta}^{*,k}(\mathbf{p})$ be the $k$-conjugate 3-velocity associated to $[u^{*,k}(p)]$, i.e.,

$$\boldsymbol{\beta}^{*,k}(\mathbf{p}) = \frac{\mathbf{u}^{*,k}(\mathbf{p})}{u_0^{*,k}(\mathbf{p})}. \tag{2.71}$$

Then, using (2.30) and (2.31),

$$\boldsymbol{\beta}^{*,k}(\mathbf{p}) = \frac{\mathbf{k} - m\mathbf{V}_k\boldsymbol{\beta}(\mathbf{p})}{k_0 - \mathbf{k}\cdot\boldsymbol{\beta}(\mathbf{p})}, \qquad \boldsymbol{\beta}(\mathbf{p}) = \frac{\mathbf{k} - m\mathbf{V}_k\boldsymbol{\beta}^{*,k}(\mathbf{p})}{k_0 - \mathbf{k}\cdot\boldsymbol{\beta}^{*,k}(\mathbf{p})}. \tag{2.72}$$

In particular, the 4-vector field

$$u^{*}(p) = \Lambda_p \overline{u(p)} \tag{2.73}$$

depends on $p$ only, (like $u(p)$ itself). We call $u^{*}(p)$ the *dual* of the 4-vector field $u(p)$. Thus, for arbitrary $u(p) \in [u(p)]$,

$$u(p) = \Lambda_p \overline{u^{*}(p)}, \qquad u^{**}(p) = u(p) \tag{2.74}$$

$$\boldsymbol{\beta}(\mathbf{p}) = \frac{\mathbf{u}(\mathbf{p})}{u_0(\mathbf{p})}, \qquad \boldsymbol{\beta}^*(\mathbf{p}) = \frac{\mathbf{u}^*(\mathbf{p})}{u_0^*(\mathbf{p})} \tag{2.75}$$

and $\boldsymbol{\beta}(\mathbf{p})$, $\boldsymbol{\beta}^*(\mathbf{p})$ satisfy the duality relationship in (2.58). Taking the dot product of $\boldsymbol{\beta}(\mathbf{p})$ with $\mathbf{p}$ on both sides of (2.58), using the explicit form for $\mathbf{V}_p$ in (2.28), and rearranging, we get the relation

$$(p_0 - \mathbf{p} \cdot \boldsymbol{\beta}(\mathbf{p}))\ (p_0 - \mathbf{p} \cdot \boldsymbol{\beta}^*(\mathbf{p})) = m^2 \tag{2.76}$$

Similarly, one may verify the matrix relation

$$\left(m\mathbf{V}_p - \mathbf{p} \otimes \boldsymbol{\beta}(\mathbf{p})^\dagger\right) \left(m\mathbf{V}_p - \mathbf{p} \otimes \boldsymbol{\beta}^*(\mathbf{p})^\dagger\right) = m^2\mathbb{I}_3, \tag{2.77}$$

Note also, that $u^*(p)$ is time-like if $u(p)$ is time-like and vice-versa. Hence,

$$\|\boldsymbol{\beta}^*(\mathbf{p}))\| < 1 \qquad \Leftrightarrow \qquad \|\boldsymbol{\beta}(\mathbf{p})\| < 1. \tag{2.78}$$

Physically, to each ordinary 3-vector velocity $\boldsymbol{\beta}(\mathbf{p})$, $u(p)$ associates a relativistic 4-velocity $n(p)$ ($= u(p)/[u(p)\cdot u(p)]^{\frac{1}{2}}$, if $u(p)\cdot u(p) \neq 0$ and $= u(p)/u_0(p)$ if $u(p)\cdot u(p) = 0$), while $\boldsymbol{\beta}^{*,k}(\mathbf{p})$ is the velocity obtained by relativistically adding the 3-velocity $-\boldsymbol{\beta}(\mathbf{p})$ to the 3-velocity associated to the boost $\Lambda_p$.

Below are details of some particular space-like affine sections and a light-like limiting section, all of which have interesting physical interpretations.

1. *The Galilean section $\sigma_0$:*

As noted in (2.64), for this section

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_0(\mathbf{p}) = 0, \qquad \boldsymbol{\vartheta}(\mathbf{p}) = \boldsymbol{\vartheta}_0(\mathbf{p}) = 0 \tag{2.79}$$

$$\boldsymbol{\beta}_0^*(\mathbf{p}) = \frac{\mathbf{p}}{p_0}, \qquad \boldsymbol{\vartheta}_0^*(\mathbf{p}) = \frac{\mathbf{p}}{m} \tag{2.80}$$

Here, $\|\boldsymbol{\beta}_0(\mathbf{p})\| < 1$, $\|\boldsymbol{\beta}_0^*(\mathbf{p})\| < 1$, $\forall \mathbf{p}$.

2. *The Lorentz section $\sigma_\ell$:*

This time (see (2.65))

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_\ell(\mathbf{p}) = \boldsymbol{\beta}_0^*(\mathbf{p}), \qquad \boldsymbol{\vartheta}(\mathbf{p}) = \boldsymbol{\vartheta}_\ell(\mathbf{p}) = \boldsymbol{\vartheta}_0^*(\mathbf{p}), \tag{2.81}$$

in other words, the Galilean and Lorentz sections are duals to each other.

3. *The symmetric section $\sigma_s$:*

This section is self-dual, being given by

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_s(\mathbf{p}) = \boldsymbol{\beta}_s^*(\mathbf{p}) = \frac{\mathbf{p}}{m + p_0}, \quad \boldsymbol{\vartheta}(\mathbf{p}) = \boldsymbol{\vartheta}_s(\mathbf{p}) = \boldsymbol{\vartheta}_s^*(\mathbf{p}) = \frac{\mathbf{p}}{m + p_0}. \tag{2.82}$$

Again, $\|\boldsymbol{\beta}_\ell(\mathbf{p})\| < 1$, $\forall \mathbf{p}$. Note that in a sense $\sigma_s$ *lies half-way* between $\sigma_0$ and $\sigma_\ell$.

4. *The limiting sections $\sigma_\pm$:*

These sections are duals of each other and are both light-like, being given by

$$\boldsymbol{\beta}(\mathbf{p}) = \boldsymbol{\beta}_+(\mathbf{p}) = \frac{\mathbf{p}}{\|\mathbf{p}\|}, \quad \boldsymbol{\vartheta}(\mathbf{p}) = \boldsymbol{\vartheta}_+(\mathbf{p}) = \frac{m\mathbf{p}}{\|\mathbf{p}\|(p_0 - \|\mathbf{p}\|)} \tag{2.83}$$

$$\boldsymbol{\beta}_-(\mathbf{p}) = \boldsymbol{\beta}_+^*(\mathbf{p}) = -\frac{\mathbf{p}}{\|\mathbf{p}\|}, \quad \boldsymbol{\vartheta}_-(\mathbf{p}) = \boldsymbol{\vartheta}_+^*(\mathbf{p}) = -\frac{m\mathbf{p}}{\|\mathbf{p}\|(p_0 + \|\mathbf{p}\|)} \tag{2.84}$$

In this limiting situation, $\|\boldsymbol{\beta}_+(\mathbf{p})\| = \|\boldsymbol{\beta}_-(\mathbf{p})\| = 1, \forall \mathbf{p}$.

At this point we state and prove the following proposition (for later usages).

**Proposition 2.1** *The following conditions are equivalent:*

1. *The 4-vector $\hat{q} = (\hat{q}_0, \hat{\mathbf{q}})$ is space-like, i.e.,*

$$|\hat{q}_0|^2 - \|\hat{\mathbf{q}})\|^2 < 0. \tag{2.85}$$

2. *The matrix*

$$S(\mathbf{p}, \boldsymbol{\vartheta}) = \mathbb{I}_3 + \left[\boldsymbol{\vartheta} \otimes \left(\frac{\mathbf{p}}{m} - \frac{\boldsymbol{\vartheta}}{2}\right)^\dagger + \left(\frac{\mathbf{p}}{m} - \frac{\boldsymbol{\vartheta}}{2}\right) \otimes \boldsymbol{\vartheta}^\dagger\right] \tag{2.86}$$

*is positive definite for all $p \in V_m^+$, or equivalently,*

$$1 + \frac{1}{m}\mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p}) > \|\boldsymbol{\vartheta}\| \|\frac{\mathbf{p}}{m} - \frac{\boldsymbol{\vartheta}}{2}\| + \frac{1}{2}\|\boldsymbol{\vartheta}\|^2, \tag{2.87}$$

*for all* $p \in \mathcal{V}_m^+$.

*3. For all unit vectors* $\hat{\mathbf{e}} \in \mathbb{R}^3$ *and for all* $\mathbf{p} \in \mathbb{R}^3$,

$$|\hat{\mathbf{e}} \cdot \left(\boldsymbol{\vartheta}(\mathbf{p}) - \frac{\mathbf{p}}{m}\right)| < \frac{1}{m}[(\hat{\mathbf{e}} \cdot \mathbf{p})^2 + m^2]^{\frac{1}{2}}. \tag{2.88}$$

*4. For all* $p \in \mathcal{V}_m^+$,

$$p_0 \|\boldsymbol{\vartheta}(\mathbf{p})\| < m + \mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p}) = |m + \mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})|. \tag{2.89}$$

*5. For all* $\mathbf{p} \in \mathbb{R}^3$, *the 3-vector field* $\boldsymbol{\beta}$ *obeys*

$$\|\boldsymbol{\beta}(\mathbf{p})\| < 1. \tag{2.90}$$

*6. For all* $\mathbf{p} \in \mathbb{R}^3$, *the 3-vector field* $\boldsymbol{\beta}^*$ *obeys*

$$\|\boldsymbol{\beta}^*(\mathbf{p})\| < 1. \tag{2.91}$$

*Note that (2.88) shows, in particular, that*

$$\hat{q} \text{ space-like} \Rightarrow \|\boldsymbol{\vartheta}(\mathbf{p}) - \frac{\mathbf{p}}{m}\| < \frac{p_0}{m}, \tag{2.92}$$

*for all* $p \in \mathcal{V}_m^+$.

**Proof:**

We start with 1. The condition rewritten as

$$\|\hat{\mathbf{q}}\|^2 - |\hat{q}_0|^2 > 0,$$

implies by (2.49) and (2.50) that

$$\mathbf{q} \cdot \left[ M(\mathbf{p}, \boldsymbol{\vartheta})^{\dagger} M(\mathbf{p}, \boldsymbol{\vartheta}) - \frac{p_0^2}{m^2} \boldsymbol{\vartheta}(\mathbf{p}) \otimes \boldsymbol{\vartheta}(\mathbf{p})^{\dagger} \right] \mathbf{q} > 0, \quad \forall \mathbf{q} \in \mathbb{R}^3,\ \mathbf{q} \neq \mathbf{0}. \tag{2.93}$$

The expression within the square brackets is easily seen to be the matrix $S$. In other words,

$$\mathbf{q} \cdot S(\mathbf{p}, \boldsymbol{\vartheta}) \mathbf{q} > 0, \quad \forall \mathbf{q} \in \mathbb{R}^3,\ \mathbf{q} \neq \mathbf{0}. \tag{2.94}$$

which is equivalent to 2. Next note that $S$ is a matrix of the type

$$A = \mathbb{I}_3 + \mathbf{a} \otimes \mathbf{b}^{\dagger} + \mathbf{b} \otimes \mathbf{a}^{\dagger}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3.$$

Such a matrix has eigenvectors

$$\mathbf{e} = \frac{\mathbf{b} \times \mathbf{a}}{\|\mathbf{b} \times \mathbf{a}\|}, \quad \mathbf{e}_+ \quad \text{and} \quad \mathbf{e}_-, \quad \mathbf{e}_+ \cdot \mathbf{e}_- = 0,$$

where $\mathbf{e}_+$ and $\mathbf{e}_-$ are linear combinations of $\mathbf{a}$ and $\mathbf{b}$. Moreover $\mathbf{e}$ has eigenvalue 1, while $\mathbf{e}_\pm$ have eigenvalues $1 + \mathbf{a} \cdot \mathbf{b} \pm \|\mathbf{a}\| \|\mathbf{b}\|$, respectively. Since $\|\mathbf{a}\| \|\mathbf{b}\| \geq \mathbf{a} \cdot \mathbf{b}$, the matrix $A$ is positive definite if and only if $1+\mathbf{a}\cdot\mathbf{b}-\|\mathbf{a}\|\|\mathbf{b}\|>0$. Applying this result to $S$ yields (2.87) and hence 2. Going back to (2.93) take $\mathbf{q}=\hat{\mathbf{e}}$, an arbitrary unit vector in $\mathbb{R}^3$, and rearrange to obtain

$$(\hat{\mathbf{e}}\cdot\boldsymbol{\vartheta}(\mathbf{p}))^2-\frac{2}{m}(\hat{\mathbf{e}}\cdot\mathbf{p})(\hat{\mathbf{e}}\cdot\boldsymbol{\vartheta}(\mathbf{p}))-1<0,$$

which can be factorized to yield

$$\begin{aligned}&\left(\hat{\mathbf{e}}\cdot\boldsymbol{\vartheta}(\mathbf{p})-\frac{1}{m}\{(\hat{\mathbf{e}}\cdot\mathbf{p})+[(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}\}\right)\\ &\times\left(\hat{\mathbf{e}}\cdot\boldsymbol{\vartheta}(\mathbf{p})-\frac{1}{m}\{(\hat{\mathbf{e}}\cdot\mathbf{p})-[(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}\}\right)<0.\end{aligned}$$

Since $(\hat{\mathbf{e}}\cdot\mathbf{p})+[(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}>0$, and $(\hat{\mathbf{e}}\cdot\mathbf{p})-[(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}<0$, the above inequality holds if and only if

$$\frac{1}{m}\{(\hat{\mathbf{e}}\cdot\mathbf{p})-[(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}\}<\hat{\mathbf{e}}\cdot\mathbf{p}<\frac{1}{m}\{[(\hat{\mathbf{e}}\cdot\mathbf{p})+(\hat{\mathbf{e}}\cdot\mathbf{p})^2+m^2]^{\frac{1}{2}}\},$$

and this is the same as 3. To see that 1. and 4. are equivalent, use (2.53) to rewrite (2.49) as

$$\hat{q}_0=\frac{p_0\boldsymbol{\vartheta}(\mathbf{p})\cdot\hat{\mathbf{q}}}{m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})}.$$

Then,

$$|\hat{q}_0|^2=\frac{p_0^2}{|m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})|}\hat{\mathbf{q}}\cdot[\boldsymbol{\vartheta}(\mathbf{p})\otimes\boldsymbol{\vartheta}(\mathbf{p})^{\dagger}]\hat{\mathbf{q}}. \tag{2.95}$$

In this equation, for fixed $\|\hat{\mathbf{q}}\|$, if we choose $\hat{\mathbf{q}}$ to lie along $\boldsymbol{\vartheta}(\mathbf{p})$, we get

$$|\hat{q}_0|^2 = \frac{p_0^2\|\boldsymbol{\vartheta}(\mathbf{p})\|^2\|\hat{\mathbf{q}}\|^2}{|m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})|^2}. \tag{2.96}$$

Thus, for any $\|\hat{\mathbf{q}}\|$, we can find a $\hat{\mathbf{q}}$ for which

$$\frac{|\hat{q}_0|^2}{\|\hat{\mathbf{q}}\|^2} = \frac{p_0^2\|\boldsymbol{\vartheta}(\mathbf{p})\|^2}{|m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})|^2}. \tag{2.97}$$

Now, $\hat{q}$ is space-like $\Leftrightarrow$ $|\hat{q}_0|^2 < \|\hat{\mathbf{q}}\|^2 \Leftrightarrow |\hat{q}_0|^2/\|\hat{\mathbf{q}}\|^2 < 1$, and in general, from (2.95)

$$\frac{|\hat{q}_0|^2}{\|\hat{\mathbf{q}}\|^2} < \frac{p_0^2\|\boldsymbol{\vartheta}(\mathbf{p})\|^2}{|m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})|^2}.$$

Hence, if

$$\frac{p_0^2\|\boldsymbol{\vartheta}(\mathbf{p})\|^2}{|m+\mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p})|^2} < 1, \tag{2.98}$$

then $\hat{q}$ is space-like. Conversely, if $\hat{q}$ is space-like, then $|\hat{q}_0|^2/|\hat{\mathbf{q}}\|^2 < 1$, and since for fixed $\|\hat{\mathbf{q}}\|^2$ there is a always a $\hat{\mathbf{q}}$ for which (2.97) holds, it follows that $\hat{q}$ space-like $\Rightarrow$ (2.98). Since by (2.52) and (2.54), $m + \mathbf{p}\cdot\boldsymbol{\vartheta}(\mathbf{p}) > 0$, we see that 1. $\Leftrightarrow$ 4. The equivalence of 4. and 5. follows from the definition of $\boldsymbol{\beta}$ in (2.56). Finally (2.78) shows that 5. $\Leftrightarrow$ 6.

**q.e.d.**

We end this section by proving the stability of the class of affine sections under the action of $\mathcal{P}_+^\uparrow(1,3)$. If $\sigma : \Gamma \rightarrow \mathcal{P}_+^\uparrow(1,3)$ is any section then, as shown in [6], for arbitrary $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$, $\sigma_{(a,A)}$ is again a section where,

$$\sigma_{(a,A)}(\mathbf{q},\mathbf{p}) = (a,A)\sigma((a,A)^{-1}(\mathbf{q},\mathbf{p})) = \sigma(\mathbf{q},\mathbf{p})h((a,A),(a,A)^{-1}(\mathbf{q},\mathbf{p})), \tag{2.99}$$

$(a,A)^{-1}(\mathbf{q},\mathbf{p})$ being the translate of $(\mathbf{q},\mathbf{p})$ by $(a,A)^{-1}$ under the action (2.40) and

$$h((a,A),(a,A)^{-1}(\mathbf{q},\mathbf{p})) = \sigma(\mathbf{q},\mathbf{p})^{-1}(a,A)\sigma((a,A)^{-1}(\mathbf{q},\mathbf{p})) \in T \times SU(2). \tag{2.100}$$

Moreover, if $\sigma$ defines the frame $\mathcal{F}\{\eta^i_{\sigma(\mathbf{q},\mathbf{p})}, A_\sigma, 2s+1\}$ $(\eta^i_{\sigma(\mathbf{q},\mathbf{p})} = U^s_W(\sigma(\mathbf{q},\mathbf{p}))\eta^i)$, then $\sigma_{(a,A)}$ defines the frame $\mathcal{F}\{\eta^i_{\sigma_{(a,A)}(\mathbf{q},\mathbf{p})}, A_{\sigma_{(a,A)}}, 2s+1\}$ where

$$A_{\sigma_{(a,A)}} = U^s_W(a,A)A_\sigma U^s_W(a,A)^*.$$

Let $\mathcal{U}$ denote the class of all affine space-like sections, defined by (2.47) - (2.50) and satisfying the conditions of Proposition 2.1, but with $\varphi$ not necessarily assumed to be zero. Then we have the result:

**Proposition 2.2** *If $\sigma \in \mathcal{U}$, then $\sigma_{(a,A)} \in \mathcal{U}$, for all $(a,A) \in \mathcal{P}_+^\uparrow(1,3)$.*

**Proof:**

If $\varphi$ is included in the definition of the section, the relations (2.49) and (2.50) generalize to

$$\hat{q}_0 = \frac{p_0}{m}(\varphi(\mathbf{p}) + \boldsymbol{\vartheta}(\mathbf{p}) \cdot \mathbf{q}), \tag{2.101}$$

$$\hat{\mathbf{q}} = \frac{\mathbf{p}}{m}\varphi(\mathbf{p}) + M(\mathbf{p}, \boldsymbol{\vartheta})\mathbf{q}. \tag{2.102}$$

From this it follows that

$$\hat{q}_0 = \frac{p_0}{m}\varphi(\mathbf{p}) + \boldsymbol{\beta}(\mathbf{p}) \cdot (\hat{\mathbf{q}} - \frac{\mathbf{p}}{m}\varphi(\mathbf{p})), \tag{2.103}$$

i.e.,

$$n(p) \cdot \hat{q} = \frac{n(p) \cdot p}{m}\varphi(\mathbf{p}), \tag{2.104}$$

with $n(p)$ given by (2.62). Next, write $\Lambda = \Lambda_k \rho$, where $\rho$ is a rotation. Then

$$\Lambda^{-1} = \Lambda_{\rho^{-1}\overline{k}}\rho^{-1},$$

so that writing

$$(\mathbf{q}', \mathbf{p}') = (a, A)^{-1}(\mathbf{q}, \mathbf{p}) \tag{2.105}$$

in (2.99), we get (see (2.40))

$$\mathbf{q}' = -\frac{a_0}{m}\rho^{-1}\mathbf{k} + \rho^{-1}\mathbf{V}_k(\mathbf{q} - \mathbf{a}), \tag{2.106}$$

$$\mathbf{p}' = \underline{\Lambda^{-1}p}, \tag{2.107}$$

with $\mathbf{V}_k$ as in (2.28). Thus, if $\sigma$ is the affine section corresponding to the quantities $\boldsymbol{\beta}$ and $\varphi$, and $(\hat{q}', p') = \sigma(\mathbf{q}', \mathbf{p}')$, then

$$n(p') \cdot \hat{q}' = \frac{n(p') \cdot p'}{m} \varphi(\mathbf{p}'). \tag{2.108}$$

Let

$$(\hat{q}'', h(p)) = (a, A)\sigma(\mathbf{q}', \mathbf{p}') = (a, A)(\hat{q}', h(p')) = (a + \Lambda\hat{q}', h(p)),$$

and $n'(p) = \Lambda n(p)$. Clearly, $\boldsymbol{\beta}' = \underline{\Lambda n}/(\Lambda n)_0$ satisfies $\|\boldsymbol{\beta}'\| \leq 1$ if $\|\boldsymbol{\beta}\| \leq 1$. Furthermore,

$$\begin{aligned} n'(p) \cdot \hat{q}'' &= n'(p) \cdot (a + \Lambda\hat{q}') \\ &= \frac{n'(p) \cdot p}{m} \left[ \frac{n(p) \cdot (m\Lambda^{-1}a + \varphi(\mathbf{p})p)}{n'(p) \cdot p} \right]. \end{aligned} \tag{2.109}$$

Thus, $\sigma_{(\mathbf{a},A)}(\mathbf{q}, \mathbf{p})$ is again an affine section corresponding to the quantities $\boldsymbol{\beta}'$ and $\varphi'$, with

$$\boldsymbol{\beta}'(\mathbf{p}) = \frac{\underline{\Lambda n(p)}}{(\Lambda n(p))_0} \tag{2.110}$$

$$\varphi'(\mathbf{p}) = \frac{n(p) \cdot (m\Lambda^{-1}a + \varphi(\mathbf{p})p)}{n'(p) \cdot p} \tag{2.111}$$

**q.e.d.**

In view of this result, starting with any family of coherent states $\mathcal{G}_\sigma = \{\boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})}\}$, we may generate an entire class of covariantly translated families $\mathcal{G}_{\sigma(a,A)}$ of other coherent states, using the natural action (2.99) of $\mathcal{P}_+^\uparrow(1,3)$ on the space of sections. If $\sigma$ is characterized by $\boldsymbol{\beta}$ and $\varphi$, then $\sigma(a, A)$ is characterized by $\boldsymbol{\beta}'$ and $\varphi'$, the relationship between $\varphi$ and $\varphi'$ being given by (2.111) above.

## 2.4 Relativistic Frames and Coherent States

We now take an arbitrary affine section $\sigma$, and going back to the Hilbert space $\mathcal{H}_W^s$ in (2.21), choose a set of vectors $\boldsymbol{\eta}^i$, $i = 1, 2, \ldots, 2s+1$, in it to define the formal operator (see (1.16) and (2.23)):

$$A_\sigma = \sum_{i=1}^{2s+1} \int_{\mathbb{R}^6} |\boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})}\rangle\langle\boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})}|\, d\mathbf{q}\, d\mathbf{p}, \quad \boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})} = U_W^s(\sigma(\mathbf{q},\mathbf{p}))\boldsymbol{\eta}^i. \tag{2.112}$$

From the general definition (1.18), in order for the set of vectors

$$\mathcal{G}_\sigma = \{\boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})} | (\mathbf{q},\mathbf{p}) \in \mathbb{R}^6,\ i = 1, 2, \ldots, 2s+1\} \subset \mathcal{H}_W^s \tag{2.113}$$

to constitute a family of coherent states for the representation $U_W^s$, the integral in (2.112) must converge weakly, and define $A_\sigma$ as a bounded operator with a bounded inverse.

To study the convergence properties of the operator integral in (2.112) we have to determine the convergence of the ordinary integral

$$I_{\phi,\psi} = \sum_{i=1}^{2s+1} \int_{\mathbf{R}^6} \langle \phi | \eta^i_{\sigma(\mathbf{q},\mathbf{p})} \rangle \langle \eta^i_{\sigma(\mathbf{q},\mathbf{p})} | \psi \rangle d\mathbf{q}\, d\mathbf{p} \tag{2.114}$$

for arbitrary $\phi, \psi \in \mathcal{H}^s_W$. In (2.48) set

$$\hat{A}(p) = h(p)\mathcal{R}(\mathbf{p}) \quad \text{and} \quad \hat{\Lambda}(p) = \Lambda_p\, \rho(p) \tag{2.115}$$

where $\hat{\Lambda}(p)$ and $\rho(p)$ are the matrices in the Lorentz group $\mathcal{L}^{\uparrow}_{+}(1,3)$ which correspond to $\hat{A(p)}$ and $\mathcal{R}(\mathbf{p})$, respectively. Then.

$$\eta^i_{\sigma(\mathbf{q},\mathbf{p})}(k) = \exp\{-i\mathbf{X}(\mathbf{k}) \cdot \mathbf{q}\} \mathcal{D}^s(v(k,p)) \eta^i(\hat{\Lambda}(p)^{-1} k), \tag{2.116}$$

where

$$\mathbf{X}(\mathbf{k}) = -\frac{k_0 p_0}{m} \boldsymbol{\vartheta}(\mathbf{p}) + M(\mathbf{p}, \boldsymbol{\vartheta})^{\dagger} \mathbf{k}, \tag{2.117}$$

is a one-to-one function of $\mathbf{k}$ having the property that:

$$\mathbf{X}(\mathbf{k}) = \mathbf{X}(\mathbf{k}') \quad \text{implies} \quad \mathbf{k} = \mathbf{k}' \textit{ and } k_0 = k'_0 \tag{2.118}$$

and

$$v(k,p) = h(k)^{-1} \hat{A}(p) h(\hat{\Lambda}(p)^{-1} k) \in SU(2). \tag{2.119}$$

Substituting into (2.114) yields

$$I_{\phi,\psi} = \sum_{i=1}^{2s+1} \int_{\mathbf{R}^6 \times \mathcal{V}_m^+ \times \mathcal{V}_m^+} \exp\left[i(k-k') \cdot \frac{p}{m} \varphi(\mathbf{p}) + i\left\{\mathbf{X}(\mathbf{k}) - \mathbf{X}(\mathbf{k}')\right\} \cdot \mathbf{q}\right] \phi(k)^\dagger$$
$$\times \mathcal{D}^s(v(k,p)) \boldsymbol{\eta}^i(\hat{\Lambda}(p)^{-1} k) \, \boldsymbol{\eta}^i(\hat{\Lambda}(p)^{-1} k')^\dagger \mathcal{D}^s(v(k',p))^\dagger \psi(k')$$
$$\times \frac{d\mathbf{k}}{k_0} \frac{d\mathbf{k}'}{k_0'} \, d\mathbf{q} \, d\mathbf{p} \qquad (2.120)$$

In order to perform the $\mathbf{k}$, $\mathbf{k}'$ integrations in (2.120), we need to change variables: $\mathbf{k} \to \mathbf{X}(\mathbf{k})$. At this point we compute the Jacobian $\mathcal{J}_{\mathbf{X}}(\mathbf{k})$ of this transformation from (2.117). To this end, using (2.51) for $M(\mathbf{p}, \boldsymbol{\vartheta})$ we write the components of $\mathbf{X}(\mathbf{k})$ as

$$X_i(\mathbf{k}) = k_i - \frac{1}{m}\left(k_0 p_0 - \mathbf{k} \cdot \mathbf{p}\right) \boldsymbol{\vartheta}_i(\mathbf{p}), \; i = 1,\, 2,\, 3. \qquad (2.121)$$

where $\boldsymbol{\vartheta}_i(\mathbf{p})$, $i = 1, 2, 3$; are the components of $\boldsymbol{\vartheta}(\mathbf{p})$. Then

$$\frac{\partial X_i}{\partial k_j}(\mathbf{k}) = \delta_{ij} - \frac{1}{m}\left(\frac{k_j p_0}{k_0} - p_j\right) \boldsymbol{\vartheta}_i(\mathbf{p}) \quad \text{and}$$

$$\mathcal{J}_{\mathbf{X}}(\mathbf{k}) = \left(\frac{\partial X_i}{\partial k_j}(\mathbf{k})\right) = \mathbb{I}_3 + \frac{1}{m k_0} \boldsymbol{\vartheta}(\mathbf{p}) \otimes [k_0 \mathbf{p} - \mathbf{k} p_0]^\dagger, \qquad (2.122)$$

which has the determinant

$$\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] = 1 + \frac{1}{m k_0} \boldsymbol{\vartheta}(\mathbf{p}) \cdot [k_0 \mathbf{p} - \mathbf{k} p_0]. \qquad (2.123)$$

Since at $\mathbf{k} = \mathbf{p} = \mathbf{0}$, $\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] = 1$, and since in order to change variables we need

$\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] \neq 0$, we must impose the condition that

$$\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] > 0, \quad \forall(\mathbf{k}, \mathbf{p}). \tag{2.124}$$

We shall show that the condition(2.124) holds for all $\mathbf{k}$, $\mathbf{p} \in \mathbb{R}^3$ if and only if the 4-vector $\hat{q} = (\hat{q}_0, \hat{\mathbf{q}})$ is space-like. To this end we first rewrite (2.124) in terms of $\boldsymbol{\beta}^*$. Let $\rho(k \to \overline{p})$ be the rotation matrix defined by

$$\rho(k \to \overline{p}) = \Lambda_p^{-1}\Lambda_k\Lambda_p\Lambda_{\overline{k}}^{-1}, \tag{2.125}$$

( it is a rotation, since the determinant of the matrix is 1 and for any 4-vector $a = (1, \mathbf{0})$, $\left(\Lambda_{\overline{p}}^{-1}\Lambda_k\Lambda_p\Lambda_{\overline{k}}^{-1}\, a\right)_0 = 1$ ) which could be written as

$$\rho(k \to \overline{p})\Lambda_k\Lambda_{\overline{p}}^{-1} = \Lambda_p^{-1}\Lambda_k, \tag{2.126}$$

and acting on the vector $(m, \mathbf{0})$ with both sides of the equation we obtain,

$$\rho(k \to \overline{p})\Lambda_k\overline{p} = \Lambda_p^{-1}k = \Lambda_{\overline{p}}k \tag{2.127}$$

Then using (2.61) we can write,

$$k_0 + \frac{1}{m}(k_0\mathbf{p} - \mathbf{k}p_0)\cdot\boldsymbol{\vartheta}(\mathbf{p}) = k_0 + \frac{1}{m}(k_0\mathbf{p} - \mathbf{k}p_0)\cdot\frac{1}{m}[\mathbf{p} - m\mathbf{V}_p\boldsymbol{\beta}^*(\mathbf{p})] \tag{2.128}$$

and plugging in the explicit form of $\mathbf{V}_p$ from (2.28) and simplifying we get,

$$\begin{aligned} k_0 + \frac{1}{m}(k_0\mathbf{p} - \mathbf{k}p_0)\cdot\boldsymbol{\vartheta}(\mathbf{p}) &= \frac{p_0}{m^2}k\cdot p - \frac{p_0}{m}[\frac{k_0\mathbf{p}}{m} - \mathbf{V}_p\mathbf{k}]\cdot\boldsymbol{\beta}^*(\mathbf{p}) \\ &= \frac{p_0}{m}[(\Lambda_k p)_0 + (\underline{\Lambda_{\overline{p}}k})\cdot\boldsymbol{\beta}^*(\mathbf{p})], \end{aligned} \tag{2.129}$$

by virtue of (2.30). So

$$\begin{aligned} \det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] &= 1 + \frac{1}{mk_0}[k_0\mathbf{p} - \mathbf{k}p_0]\cdot\boldsymbol{\vartheta}(\mathbf{p}) \\ &= \frac{1}{k_0}\left[k_0 + \frac{1}{m}[k_0\mathbf{p} - \mathbf{k}p_0]\cdot\boldsymbol{\vartheta}(\mathbf{p})\right] \\ &= \frac{p_0}{k_0 m}\left[(\Lambda_k p)_0 + (\underline{\Lambda_{\overline{p}}k})\cdot\boldsymbol{\beta}^*(\mathbf{p})\right]. \end{aligned} \tag{2.130}$$

which implies

$$\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] = \frac{p_0(\Lambda_k p)_0}{mk_0}\left[1 + \frac{(\underline{\Lambda_k\overline{p}})}{(\Lambda_k p)_0}\cdot\rho(k\to\overline{p})\boldsymbol{\beta}^*(\mathbf{p})\right], \tag{2.131}$$

after making the use of (2.127).

Thus, it would appear that the positivity of $\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})]$, as required by (2.124), would be ensured if the second term within the square brackets in (2.131) did not exceed 1 in magnitude, in other words, (2.124) would seem to require that $\|\boldsymbol{\beta}^*(\mathbf{p})\| < 1$, $\forall\,\mathbf{p}$.

**Proposition 2.3** *The condition (2.124),* $\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] > 0$, *holds for all* $\mathbf{k}, \mathbf{p} \in \mathbb{R}^3$ *if and only if the 4-vector* $\hat{q} = (\hat{q}_0, \hat{\mathbf{q}})$ *is space-like, i.e., if and only if any one of the equivalent conditions in Proposition* 2.1 *is satisfied.*

**Proof:**

Suppose that $\hat{q}$ is space-like. Then by Proposition 2.1, $\|\boldsymbol{\beta}^{\bullet}\| < 1$. Hence, since $\rho(k \to \overline{p})$ is a rotation matrix,

$$\|\frac{(\Lambda_k \overline{p})}{(\Lambda_k \overline{p})_0} \cdot \rho(k \to \overline{p})^{\dagger} \boldsymbol{\beta}^{\bullet}(\mathbf{p})\| < 1,$$

so that

$$1 + \frac{(\Lambda_k \overline{p})}{(\Lambda_k \overline{p})_0} \cdot \rho(k \to \overline{p})^{\dagger} \boldsymbol{\beta}^{\bullet}(\mathbf{p}) > 0,$$

i.e., $\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] > 0$ (2.131)).

Conversely, assume that $\det[\mathcal{J}_{\mathbf{X}}(\mathbf{k})] > 0$. Then by (2.122),

$$1 + \frac{\mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})}{m} > \frac{p_0}{m} \frac{\mathbf{k}}{k_0} \cdot \boldsymbol{\vartheta}(\mathbf{p}).$$

Taking $\mathbf{k}$ in the direction of $\boldsymbol{\vartheta}(\mathbf{p})$ and letting $\|\mathbf{k}\| \to \infty$, the above inequality implies that

$$1 + \frac{\mathbf{p} \cdot \boldsymbol{\vartheta}(\mathbf{p})}{m} > \frac{p_0}{m} \|\boldsymbol{\vartheta}(\mathbf{p})\|,$$

which, in view of condition 4. of Proposition 2.1, implies that $\hat{q}$ is space-like.

**q.e.d.**

From now on, unless otherwise stated, we shall work with space-like affine sections $\sigma$; actually the only exception will be the two limiting sections $\sigma_{\pm}$ in (2.83) and (2.84). Thus we shall assume that the conditions of Proposition 2.1 hold.

Going back to the computation of $I_{\phi,\psi}$ in(2.120), we note that $d\mathbf{q}$ integration yields a $\delta$-measure in $\mathbf{X}$, and hence making the change of variables $\mathbf{k} \to \mathbf{X}$, integrating and rearranging (using (2.131)) we obtain,

$$I_{\phi,\psi} = \int_{\mathcal{V}_m^+ \times \mathcal{V}_m^+} \phi(k)^\dagger \mathcal{A}_\sigma(k,p)\psi(k) \frac{d\mathbf{p}}{p_0}\frac{d\mathbf{k}}{k_0}, \tag{2.132}$$

where $\mathcal{A}_\sigma(k,p)$ is the $(2s+1) \times (2s+1)$-matrix kernel

$$\begin{aligned}\mathcal{A}_\sigma(k,p) &= (2\pi)^3 m \sum_{i=1}^{2s+1} [p_0 + \mathbf{p}\cdot\rho(k \to \Lambda_k^{-1}p)^\dagger \beta^*(-\underline{\Lambda_k^{-1}p})]^{-1} \\ &\quad \times \mathcal{D}^s(v(k,\overline{\Lambda_k^{-1}p}))\eta^i(\rho(\overline{\Lambda_k^{-1}p})^{-1}\rho(k \to \Lambda_k^{-1}p)p) \\ &\quad \times \eta^i(\rho(\overline{\Lambda_k^{-1}p})^{-1}\rho(k \to \Lambda_k^{-1}p)p)^\dagger \mathcal{D}^s(v(k,\overline{\Lambda_k^{-1}p}))^\dagger \end{aligned} \tag{2.133}$$

where $\rho(p)$, $\rho(k \to p)$ and $v(k,p)$ are given by (2.115), (2.125) and (2.119), respectively. Assuming the integral (2.114) to exist for all $\phi$, $\psi \in \mathcal{H}_W^s$, let us write

$$\mathcal{A}_\sigma(k) = \int_{\mathcal{V}_m^+} \mathcal{A}_\sigma(k,p)\,\frac{d\mathbf{p}}{p_0}. \tag{2.134}$$

Then the operator $A_\sigma$ in (2.112) is a matrix -valued multiplication operator:

$$(A_\sigma\phi)(k) = \mathcal{A}_\sigma(k)\phi(k), \quad \phi \in \mathcal{H}_W^s. \tag{2.135}$$

At this point we make two simplifying assumptions on the the nature of the vectors $\eta^i \in \mathcal{H}_W^s$, $i = 1,2,\ldots,2s+1$.

1. Assumption of rotational invariance of the operator $\sum_{i=1}^{2s+1} |\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|$, i.e., $\forall\mathcal{R} \in SU(2)$,

$$\mathcal{D}^s(\mathcal{R}) \left[\sum_{i=1}^{2s+1} |\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|\right] \mathcal{D}^s(\mathcal{R})^\dagger = \sum_{i=1}^{2s+1} |\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|. \tag{2.136}$$

This implies that

$$\left(\sum_{i=1}^{2s+1} |\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|\right)(k) = \mathbb{I}_{2s+1}\,|\eta(k)|^2, \tag{2.137}$$

where $\eta \in L^2(\mathcal{V}_m^+, \frac{d\mathbf{k}}{k_0})$, and thus we may take for $\boldsymbol{\eta}^i \in \mathcal{H}_W^s$ the vectors

$$\boldsymbol{\eta}^i = \hat{\mathbf{e}}_i \otimes \eta, \quad i = 1, 2, \ldots, 2s+1, \tag{2.138}$$

the $\hat{\mathbf{e}}_i$ being the canonical unit vectors in $\mathbb{C}^{2s+1}$ (i.e., $\hat{\mathbf{e}}_i = (\delta_{ij}), j = 1, 2, \ldots, 2s+1$).

2. Assumption of rotational invariance of $|\eta(k)|^2$ in (2.137):

$$|\eta(\rho k)|^2 = |\eta(k)|^2, \quad \forall \rho \in SO(3). \tag{2.139}$$

We shall generally refer to these two assumptions as the *assumption of rotational invariance*. With this assumption, the kernel $\mathcal{A}_\sigma(k,p)$ in (2.133) simplifies to

$$\begin{aligned} \mathcal{A}_\sigma(k,p) &= a_\sigma(k,p)\,|\eta(p)\,|^2\mathbb{I}_{2s+1}, \\ a_\sigma(k,p) &= \frac{(2\pi)^3 m}{p_0 + \mathbf{p}\cdot\rho(k \to \Lambda_k^{-1}p)^\dagger\,\boldsymbol{\beta}^*(-\underline{\Lambda_k^{-1}p})}. \end{aligned} \tag{2.140}$$

On $\mathcal{H}^s_W$ define the operators $(P_0, \mathbf{P})$,

$$(P_\mu \phi)(k) = k_\mu \phi(k). \tag{2.141}$$

We shall also denote the analogous operators on $L^2(\mathcal{V}^+_m, \frac{d\mathbf{k}}{k_0})$ by the same symbols. Note that $P_0^{-1}$ is a bounded operator with spectrum $[0, \frac{1}{m}]$. Then, with the above simplifications (2.134) becomes

$$\mathcal{A}_\sigma(k) = \langle a_\sigma(k, P)\rangle_\eta \mathbb{I}_{2s+1}, \tag{2.142}$$

where $\langle\cdot\rangle_\eta$ denotes the $L^2(\mathcal{V}^+_m, \frac{d\mathbf{k}}{k_0})$ expectation value with respect to the vector $\eta$ in (2.137). Hence for the operator $A_\sigma$ (see (2.135))

$$\|A_\sigma\| = \sup_{k \in \mathcal{V}^+_m} |\langle a_\sigma(k, P)\rangle_\eta|, \tag{2.143}$$

provided this supremum exists. On the other hand, since $\|\beta^*(-\underline{\Lambda_k^{-1} p})\| < 1$ and $\|\rho(k \to \Lambda_k^{-1} p)^\dagger\| = 1$, from (2.140) we get,

$$\frac{1}{(2\pi)^3 m}(p_0 - \|\mathbf{p}\|) < \frac{1}{a_\sigma(k, p)} < \frac{1}{(2\pi)^3 m}(p_0 + \|\mathbf{p}\|) \tag{2.144}$$

The two extreme values in the above inequality are reached for the limiting sections $\sigma_\pm$ (see (2.82)). Thus, we have the result:

**Lemma 2.1** *If $\|\beta(\mathbf{p})\| \leq 1, \forall \mathbf{p}$, then $a_\sigma(k,p)$ is a bounded function satisfying*

$$\frac{(2\pi)^3}{m}(p_0 - \|\mathbf{p}\|) \leq a_\sigma(k,p) \leq \frac{(2\pi)^3}{m}(p_0 + \|\mathbf{p}\|). \tag{2.145}$$

Suppose now that $\eta$ lies in the domain of $P_0^{\frac{1}{2}}$, i.e.,

$$\int_{\mathcal{V}_m^+} |\eta(k)|^2 d\mathbf{k} < \infty, \tag{2.146}$$

and set

$$\langle P_0 \pm \|\mathbf{P}\|\rangle_\eta = \int_{\mathcal{V}_m^+} (p_0 \pm \|\mathbf{p}\|)\, |\eta(k)|^2 \frac{d\mathbf{p}}{p_0}. \tag{2.147}$$

Then (2.132), (2.140 and (2.145) together imply

**Lemma 2.2** *If the assumption of rotational invariance on $\eta^i$, $i = 1, 2, \ldots, 2s+1$ is satisfied, and if $\eta \in Dom(P_0^{\frac{1}{2}})$, then for all $\beta$ such that $\|\beta(\mathbf{p})\| \leq 1$, $\forall \mathbf{p}$,*

$$\frac{(2\pi)^3}{m}\langle P_0 - \|\mathbf{P}\|\rangle_\eta \|\phi\|\,\|\psi\| \leq |I_{\phi,\psi}| \leq \frac{(2\pi)^3}{m}\langle P_0 + \|\mathbf{P}\|\rangle_\eta \|\phi\|\,\|\psi\|. \tag{2.148}$$

As a consequence of this lemma we see that both the operator $A_\sigma$ in (2.112) and its inverse, $A_\sigma^{-1}$, are bounded, with

$$(A_\sigma^{-1}\phi)(k) = [\langle a_\sigma(k,P)\rangle]^{-1}\phi(k), \quad \phi \in \mathcal{H}_W^s. \tag{2.149}$$

Indeed, collecting all these results we obtain,

**Proposition 2.4** *Let $\boldsymbol{\eta}^i$, $i = 1, 2, \ldots, 2s+1$, satisfy the condition of rotational invariance. Then for each $\boldsymbol{\beta}$ satisfying $\|\boldsymbol{\beta}(\mathbf{p})\| \leq 1$, $\forall \mathbf{p}$, the set of vectors $\mathcal{G}_\sigma$ in (2.113) is a family of spin-s coherent states, forming a rank-$(2s+1)$ frame $\mathcal{F}\{\boldsymbol{\eta}^i_{\sigma(\mathbf{q},\mathbf{p})}, A_\sigma, 2s+1\}$, if and only if $\eta \in Dom(P_0^{\frac{1}{2}})$. The operator $A_\sigma$ acts via multiplication by a bounded invertible function $\mathcal{A}_\sigma(k)$ given by (2.142) and $A_\sigma^{-1}$ via multiplication by the function $\mathcal{A}_\sigma^{-1}(k)$. Moreover,*

$$\frac{(2\pi)^3}{m}\langle P_0 - \|\mathbf{P}\|\rangle_\eta \leq \| A_\sigma \| \leq \frac{(2\pi)^3}{m}\langle P_0 + \|\mathbf{P}\|\rangle_\eta, \tag{2.150}$$

*and,*

$$Spectrum(A_\sigma) \subset \frac{(2\pi)^3}{m}[\langle P_0 - \|\mathbf{P}\|\rangle_\eta, \langle P_0 + \|\mathbf{P}\|\rangle_\eta]. \tag{2.151}$$

Note that since we are assuming rotational invariance, we could just as well have done without the restriction, $\mathcal{R}(\mathbf{q},\mathbf{p}) = \mathcal{R}(\mathbf{p})$, in defining the sections $\sigma$ in (2.48). The following construction now emerges for building spin-$s$ coherent states for the representations $U^s_W$ (see (2.23)) of mass $m > 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, of $\mathcal{P}^\uparrow_+(1,3)$:

1. Choose a function $\boldsymbol{\beta} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\|\boldsymbol{\beta}(\mathbf{p})\| \leq 1$, $\forall \mathbf{p}$, or equivalently, a map $u : \mathcal{V}^+_m \to \mathcal{V}^+_m$ as in (2.66); choose an arbitrary measurable function $\mathcal{R} : \mathbb{R}^3 \to SU(2)$ and construct the corresponding affine space-like (or in the limit, the affine light-like) section $\sigma$, using (2.57), (2.48), (2.49) and (2.50).

2. Choose an $\eta \in L^2(\mathcal{V}^+_m, d\mathbf{k}/k_0)$, satisfying (2.139) and (2.146) and form the vectors $\boldsymbol{\eta}^i$ $i = 1, 2, \ldots, 2s+1$, using (2.138).

3. Construct the family, $\mathcal{G}_\sigma$, of coherent states $\eta^i_{(\mathbf{q},\mathbf{p})}$ using (2.112).

While this procedure provides us with a large class of CS and frames, the latter are generally not tight, i.e., $A_\sigma$ is not, in general, a multiple of the identity. A few special cases worked out below will make this statement clearer. For computational purposes, the following expressions prove useful (assuming rotational invariance):

$$a_\sigma(k,p) = \frac{(2\pi)^3 m(\Lambda_k^{-1}p)_0}{mk_0 - [k_0(\underline{\Lambda_k^{-1}p}) + \mathbf{k}(\Lambda_k^{-1}p)_0]\cdot\boldsymbol{\vartheta}(-\underline{\Lambda_k^{-1}p})}, \tag{2.152}$$

and,

$$I_{\phi,\psi} = (2\pi)^3 \int_{V_m^+\times V_m^+} \phi(k)^\dagger \frac{m}{mk_0 - (k_0\mathbf{p} + \mathbf{k}p_0)\cdot\boldsymbol{\vartheta}(-\mathbf{p})} |\eta(\Lambda_k p)|^2 \psi(k)\, d\mathbf{p}\, \frac{d\mathbf{k}}{k_0}. \tag{2.153}$$

1. *The Galilean section* $\sigma_0$ :

From (2.79), (2.80) and (2.152),

$$a_\sigma(k,p) = a_0(k,p) = \frac{(2\pi)^3}{m} \frac{k_0 p_0 - \mathbf{k}\cdot\mathbf{p}}{k_0}, \tag{2.154}$$

and using the rotational invariance of $|\eta(k)|^2$,

$$A_\sigma(k) = A_0(k) = \frac{(2\pi)^3}{m}\langle P_0\rangle_\eta \mathbb{I}_{2s+1}. \tag{2.155}$$

Hence,

$$A_\sigma = A_0 = \frac{(2\pi)^3}{m}\langle P_0\rangle_\eta I, \tag{2.156}$$

so that the frame is tight.

2. *The Lorentz section* $\sigma_\ell$ :

From (2.81) and (2.140),

$$a_\sigma(k,p) = a_\ell(k,p) = \frac{(2\pi)^3 m}{p_0}, \tag{2.157}$$

so that

$$\mathcal{A}_\sigma(k) = \mathcal{A}_\ell(k) = (2\pi)^3 m\langle P_0^{-1}\rangle_\eta \mathbb{I}_{2s+1}. \tag{2.158}$$

Thus,

$$A_\sigma = A_\ell = (2\pi)^3 m\langle P_0^{-1}\rangle_\eta I, \tag{2.159}$$

and once again the frame is tight.

3. *The symmetric section* $\sigma_s$ :

From (2.82) and (2.153),

$$I_{\phi,\psi} = (2\pi)^3 \int_{\mathcal{V}_m^+\times\mathcal{V}_m^+} \phi(k)^\dagger \frac{k_0p_0 + m^2}{m(k_0+p_0)} |\eta(p)|^2 \psi(k)\, \frac{d\mathbf{p}}{p_0}\, \frac{d\mathbf{k}}{k_0}. \tag{2.160}$$

Thus,

$$a_\sigma(k,p) = a_s(k,p) = (2\pi)^3 \frac{k_0 p_0 + m^2}{m(k_0 + p_0)}, \tag{2.161}$$

and

$$\mathcal{A}_\sigma(k) = \mathcal{A}_s(k) = (2\pi)^3 \langle \frac{k_0 P_0 + m^2}{m(k_0 + P_0)} \rangle_\eta \mathbb{I}_{2s+1}. \tag{2.162}$$

The operator $A_\sigma = A_s$ is given by

$$(A_s\phi)(k) = \mathcal{A}_s(k)\phi(k) = (2\pi)^3 \langle \frac{k_0 P_0 + m^2}{m(k_0 + P_0)} \rangle_\eta \phi(k), \quad \phi \in \mathcal{H}_W^s. \tag{2.163}$$

To determine the spectrum of $A_s$, note that the function $f : [m, \infty) \to \mathbb{R}^+$, defined by

$$f(k_0) = (2\pi)^3 \frac{k_0 p_0 + m^2}{m(k_0 + p_0)}, \tag{2.164}$$

is uniformly bounded for all $p_0 \in [m, c]$, for any finite $c > m$. Also $f'(k_0) \neq 0$, for all $k_0, p_0 > m$ (here $f'$ denotes the derivative of $f$) and

$$f(m) = (2\pi)^3, \qquad f(\infty) = (2\pi)^3 \frac{p_0}{m}. \tag{2.165}$$

Thus,

$$(2\pi)^3 \leq f(k_0) \leq (2\pi)^3 \frac{p_0}{m}, \tag{2.166}$$

which, by virtue of (2.162), implies that

$$Spectrum(A_s) = (2\pi)^3 \left[ \|\eta\|^2, \frac{\langle P_0 \rangle_\eta}{m} \right] . \tag{2.167}$$

Hence, in this case, the frame is never tight.

# Chapter 3

# A Relativistic Windowed Fourier Transform

In the previous chapter we constructed the coherent states of the full Poincaré group $\mathcal{P}_+^\uparrow(1,3)$ and showed that they form continuous frames. Here we discretize the coherent states of $\mathcal{P}_+^\uparrow(1,1)$ and show that discretized CS of $\mathcal{P}_+^\uparrow(1,1)$ can also be made into discrete frames and develop a transform, analogous to the windowed Fourier transform, which we call the *relativistic windowed Fourier transform* (RWFT). Finally, we obtain a *reconstruction formula* for any function on the Hilbert space $\mathcal{H} = L^2(\mathcal{V}_m^+, \frac{dk}{k_0})$ using the RWFT.

## 3.1 The Mathematical Formalism and Notation

Unlike the previous chapter (which dealt with $\mathcal{P}_+^\uparrow(1,3)$), here we restrict ourselves to $\mathcal{P}_+^\uparrow(1,1)$) i.e., instead of dealing with a 3-component space vector, we work with a single-component space vector. Most of the relations and statements which were true for $\mathcal{P}_+^\uparrow(1,3)$ in Chapter 2, are also true for $\mathcal{P}_+^\uparrow(1,1)$. Let $G = \mathcal{P}_+^\uparrow(1,1)$ be the Poincaré group in one space and one time dimensions. It acts on the Minkowski space $\mathbb{R}_{1,1}$ whose points we denote by $x = (x_0, \mathbf{x})$, $x_0 = t$, $\mathbf{x} \in \mathbb{R}$ (we assume that $\hbar = c = 1$). The metric is $g = diag(1, -1)$. The elements of $\mathcal{P}_+^\uparrow(1,1)$ are denoted by $(a; \Lambda)$, where $a = (a_0, \mathbf{a}) \in \mathbb{R}^2$ is a space-time translation and $\Lambda$ is a Lorentz boost. Let $\mathcal{V}_m^+$ denote the forward mass hyperbola,

$$\mathcal{V}_m^+ = \left\{p = (p_0, \mathbf{p}) \in \mathbb{R}^2 \,|p_0 > 0\,,\; p_0^2 - \mathbf{p}^2 = m^2\right\} \tag{3.1}$$

for some mass $m$. The matrix $\Lambda$ may be parametrized by a vector $p = (p_0, \mathbf{p}) \in \mathcal{V}_m^+$:

$$\Lambda = \Lambda_p = \frac{1}{m}\begin{pmatrix} p_0 & \mathbf{p} \\ \mathbf{p} & p_0 \end{pmatrix}. \tag{3.2}$$

The group multiplication is defined by the semi-direct product:

$$(a_1; \Lambda_1)(a_2; \Lambda_2) = (a_1 + \Lambda_1 a_2; \Lambda_1 \Lambda_2), \tag{3.3}$$

and the inverse and the identity elements are respectively $(a; \Lambda)^{-1} = (-\Lambda^{-1}a; \Lambda^{-1})$ and $(0, I)$, where I is the $2 \times 2$ identity matrix. The elements $\Lambda_p$ of the Lorentz group act on $\mathcal{V}_m^+$ in the natural manner,

$$k \to k' = \Lambda_p k, \quad k \in \mathcal{V}_m^+. \tag{3.4}$$

This action is transitive and by a straightforward calculation, it can be shown that the invariant measure on $\mathcal{V}_m^+$ is $d\mathbf{k}/k_0$. We work with the unitary irreducible representation $U_W$ of $\mathcal{P}_+^\dagger(1,1)$, on the Hilbert space

$$\mathcal{H}_W = L^2(\mathcal{V}_m^+, d\mathbf{k}/k_0)\,, \tag{3.5}$$

given, for $g = (a;\Lambda_p) \in \mathcal{P}_+^\dagger(1,1)$, by

$$(U_W(g)\phi)(k) = e^{ik.a}\phi(\Lambda_p^{-1}k), \quad \forall\phi \in \mathcal{H}_W\,, \tag{3.6}$$

where $k \cdot a = k_0 a_0 - \mathbf{k}\cdot\mathbf{a}$. We call $U_W$ the *Wigner representation* of $\mathcal{P}_+^\dagger(1,1)$ for the mass $m$. Since for any $\phi \in \mathcal{H}_W$,

$$\begin{aligned}
&\int_{\mathcal{P}_+^\dagger(1,1)} |\langle U_W(g)\,\phi\,|\,\phi\,\rangle|^2\, da_0\, d\mathbf{a}\, d\mathbf{p}/p_0 \\
= \;& \int_{\mathcal{P}_+^\dagger(1,1)}\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+} \exp[i(k'_0\, a_0 - \mathbf{k}'\cdot\mathbf{a} - k_0\, a_0 + \mathbf{k}\cdot\mathbf{a})]\,\overline{\phi(\Lambda_p^{-1}\,k)}\,\phi(k)\,\overline{\phi(k')} \\
&\times \phi(\Lambda_p^{-1}\,k')\,\frac{d\mathbf{k}}{k_0}\,\frac{d\mathbf{k}'}{k'_0}\,da_0\,d\mathbf{a}\,\frac{d\mathbf{p}}{p_0} \\
= \;& 2\pi\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+\times\mathcal{V}_m^+}\int_{-\infty}^{\infty} \delta(k'_0 - k_0)\,\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{a})]\,\overline{\phi(\Lambda_p^{-1}\,k)}\,\phi(k)\,\overline{\phi(k')} \\
&\times \phi(\Lambda_p^{-1}\,k')\,\frac{d\mathbf{k}}{k_0}\,\frac{d\mathbf{k}'}{k'_0}\,d\mathbf{a}\,\frac{d\mathbf{p}}{p_0} \\
= \;& 2\pi\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+}\int_{-\infty}^{\infty} |\phi(\Lambda_p^{-1}\,k)\,|^2\,|\phi(k)\,|^2\,d\mathbf{a}\,\frac{d\mathbf{k}}{k_0^2}\,\frac{d\mathbf{p}}{p_0} = \infty\,,
\end{aligned}$$

we see that the Wigner representation is not square integrable with respect to the whole group $\mathcal{P}_+^\dagger(1,1)$. However, as in the $\mathcal{P}_+^\dagger(1,3)$ case, it is square integrable with respect to the homogeneous space $\Gamma = \mathcal{P}_+^\uparrow(1,1)/T$, where $T$ is the subgroup of time translations. Then $\Gamma$ has global coordinatization $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^2$, and the left invariant measure is $d\mathbf{q}\,d\mathbf{p}$ [6]. The action of $\mathcal{P}_+^\uparrow(1,1)$ on $\Gamma$ corresponding to (2.40) is :

$$(a,\Lambda_k)\cdot(\mathbf{q},\mathbf{p}) = (\mathbf{q}',\mathbf{p}') \tag{3.7}$$

where

$$\mathbf{p}' = \underline{\Lambda_k p} \tag{3.8}$$
$$\mathbf{q}' = \frac{p_0\mathbf{q} + m(\underline{\underline{\Lambda_{p'}^{-1}\,a}})}{p_0{}'} \tag{3.9}$$

In case of $\mathcal{P}_+^\uparrow(1,1)$ the particular section $\sigma_0 : \Gamma \to \mathcal{P}_+^\uparrow(1,1)$ is defined by

$$\sigma_0(\mathbf{q},\mathbf{p}) = ((0,\mathbf{q}),\Lambda_p), \quad p = (\sqrt{|\mathbf{p}|^2+m^2},\mathbf{p}) \tag{3.10}$$

Then any other measurable section reduces to

$$\sigma(\mathbf{q},\mathbf{p}) = \sigma_0(\mathbf{q},\mathbf{p})((f(\mathbf{q},\mathbf{p}),0),I) \tag{3.11}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. The relations (2.48), (2.49) and (2.50) reduce respectively to

$$\sigma(\mathbf{q},\mathbf{p}) = (\hat{q},\Lambda_p) \tag{3.12}$$

$$\hat{q}_0 = \frac{p_0 f(\mathbf{q},\mathbf{p})}{m} \tag{3.13}$$

$$\hat{\mathbf{q}} = \mathbf{q} + \frac{\mathbf{p} f(\mathbf{q},\mathbf{p})}{m} \tag{3.14}$$

The function $f(\mathbf{q},\mathbf{p})$ is defined in (2.47) and in this case $\varphi : \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{\vartheta} : \mathbb{R} \to \mathbb{R}$ are continuous functions of $\mathbf{p}$ alone. As we saw in Chapter 2. that $\varphi$ played an inessential role for our purposes, here we set $\varphi = 0$. It is to be noted here that all the relations from (2.55) to (2.61) also hold true for the case of $\mathcal{P}_+^\uparrow(1,1)$ where $\boldsymbol{\beta}(\mathbf{p})$ has to be treated as a 1−dimensional vector field and $V_p = \frac{p_0}{m}$. Consequently (2.61) reduces to

$$\boldsymbol{\vartheta}(\mathbf{p}) = \frac{1}{m}[\mathbf{p} - p_0\,\boldsymbol{\beta}^*(\mathbf{p})] \tag{3.15}$$

Then the coherent states of $\mathcal{P}_+^\uparrow(1,1)$, for an arbitrary section $\sigma(\mathbf{q},\mathbf{p})$, are the set of vectors

$$\boldsymbol{\eta}_{\sigma(\mathbf{q},\mathbf{p})} = (U_W(\sigma(\mathbf{q},\mathbf{p})\boldsymbol{\eta})(k) = e^{ik.\hat{q}}\boldsymbol{\eta}(\Lambda_p^{-1}k) \tag{3.16}$$

for an $\boldsymbol{\eta}$ in the domain $P_0^{\frac{1}{2}}$ defined in (2.141)). The coherent states in (3.16) satisfy the frame condition

$$\int_\Gamma |\,\boldsymbol{\eta}_{\sigma(\mathbf{q},\mathbf{p})}\rangle\langle\boldsymbol{\eta}_{\sigma(\mathbf{q},\mathbf{p})}\,|\,d\mathbf{q}\,d\mathbf{p} = A_\sigma \tag{3.17}$$

(see (1.19)) where both $A_\sigma$ and $A_\sigma{}^{-1}$ are bounded operators.

## 3.2 Periodization of Compactly Supported Functions

Let $Y : \mathbb{R} \to \mathbb{C}$ be a compactly supported function, with support $[a, b]$ which means $Y$ is zero outside the interval $[a, b]$ and the length $L$ of the support is finite, i.e., $L = b - a < \infty$. We want to periodize the function $Y$ in the following manner:

Let $\tilde{Y} : \mathbb{R} \to \mathbb{C}$ be a periodic function with period $L$ such that

$$\tilde{Y}(x) = Y(x), \quad \forall x \in [a, b]. \tag{3.18}$$

$$\tilde{Y}(x + L) = \tilde{Y}(x), \quad \forall x \in \mathbb{R} \tag{3.19}$$

$\tilde{Y}$ has the Fourier series decomposition

$$\tilde{Y}(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{2\pi n x}{L}}, \quad \forall x \in \mathbb{R} \tag{3.20}$$

and,

$$Y(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{2\pi n x}{L}}, \quad \forall x \in [a, b], \tag{3.21}$$

where

$$c_n = \frac{1}{L} \int_a^b \tilde{Y}(x) e^{-i\frac{2\pi n x}{L}} \, dx \tag{3.22}$$

Then

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{L}(x - x')} = \delta(x - x') \tag{3.23}$$

where $\delta(x - x')$ is the Dirac delta function defined, in the sense of distributions, by

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')}\, dk \tag{3.24}$$

It has the property that for a sufficiently smooth function $f(x)$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - x_0)\, dx = f(x_0)\,. \tag{3.25}$$

For an arbitrary section $\sigma(\mathbf{q}, \mathbf{p})$ (see (3.11)) and $\ell, j \in \mathbb{Z}$, where by $\mathbb{Z}$ we denote the set of integers, we write the discretized form of coherent states in (3.16) as

$$\boldsymbol{\eta}_{\ell,j}(k) = e^{-i\mathbf{X}_j(\mathbf{k})\cdot\mathbf{q}_{\ell,j}}\, \boldsymbol{\eta}(\Lambda_j^{-1}k), \quad \mathbf{q}_{\ell,j} = \Delta\mathbf{q}_j\ell\,, \tag{3.26}$$

where, $\Lambda_j$ denotes the discretized version of $\Lambda_p$, for time being we take $\Delta\mathbf{q}_j > 0$ and will fix it later and

$$\begin{aligned} \mathbf{X}_j(\mathbf{k}) &= \mathbf{k} - (\Lambda_j^{-1}k)_0\boldsymbol{\vartheta}(\mathbf{p}_j) \\ &= -k_0p_{j0}\boldsymbol{\vartheta}(\mathbf{p}_j) + \mathbf{k}(1 + \frac{1}{m}\mathbf{p}_j\boldsymbol{\vartheta}(\mathbf{p}_j)) \end{aligned} \tag{3.27}$$

Let $a = (a_0\,, \mathbf{a})$, $b = (b_0\,, \mathbf{b})$, $p_j = (p_{j0}, \mathbf{p}_j)$, and $k = (k_0, \mathbf{k}) \in \mathcal{V}_m^+$ satisfying

$$a_0{}^2 - \mathbf{a}^2 = m^2, \quad a_0 > 0,\ \mathbf{a} \in \mathbb{R} \quad \text{etc.}$$

Let $\boldsymbol{\eta}(k) = 0$ if $\mathbf{k} \notin [\mathbf{b}, \mathbf{a}]$, i.e., the length of the support of $\boldsymbol{\eta}(k)$ is $\mathbf{b} - \mathbf{a}$. Then $\boldsymbol{\eta}(\Lambda_j^{-1} k) = 0$ if

$$\mathbf{k} \notin \left[\frac{a_0\,\mathbf{p}_j + \mathbf{a}\,p_{j0}}{m}, \frac{b_0\,\mathbf{p}_j + \mathbf{b}\,p_{j0}}{m}\right]$$

and the length of the support of $\boldsymbol{\eta}(\Lambda_j^{-1} k)$ is

$$((b_0 - a_0)\mathbf{p}_j + (\mathbf{b} - \mathbf{a})p_{j0})/m.$$

Let $\boldsymbol{\eta}(\Lambda_j^{-1} k) = \tilde{\boldsymbol{\eta}}(\mathbf{X}_j(\mathbf{k}))$, where $\mathbf{X}_j(\mathbf{k})$ is given by (3.27) and

$$\begin{aligned} \tilde{b}_j &= \Lambda_j^{-1}(b) = \left(\frac{b_0\,p_{j0} + \mathbf{b}\cdot\mathbf{p}_j}{m}, \frac{b_0\,\mathbf{p}_j + \mathbf{b}\,p_{j0}}{m}\right) && (3.28)\\ \tilde{a}_j &= \Lambda_j^{-1}(a) = \left(\frac{a_0\,p_{j0} + \mathbf{a}\cdot\mathbf{p}_j}{m}, \frac{a_0\,\mathbf{p}_j + \mathbf{a}\,p_{j0}}{m}\right). && (3.29) \end{aligned}$$

Then the length of the support $L_j$ of $\tilde{\boldsymbol{\eta}}(\mathbf{X}_j(\mathbf{k}))$ is given by

$$L_j = \underline{\mathbf{X}_j(\tilde{b}_j)} - \underline{\mathbf{X}_j(\tilde{a}_j)} = \frac{(b_0 - a_0)\mathbf{p}_j + (\mathbf{b} - \mathbf{a})p_{j0}}{m} - (b_0 - a_0)\vartheta(\mathbf{p}_j). \qquad (3.30)$$

Then, for any $\phi \in \mathcal{H}_W$, we write

$$\langle \boldsymbol{\eta}_{\ell,j} | \phi \rangle = \int_{\mathcal{V}_m^+} e^{i\mathbf{X}_j(\mathbf{k})\cdot\mathbf{q}_{\ell,j}}\, \overline{\boldsymbol{\eta}(\Lambda_j^{-1} k)}\, \phi(k) \frac{d\mathbf{k}}{k_0}, \qquad (3.31)$$

where $\overline{\boldsymbol{\eta}}$ indicates complex conjugate of $\boldsymbol{\eta}$.

We define the formal operator $T$

$$T = \sum_{\ell,j=-\infty}^{\infty} |\eta_{\ell,j}\rangle\langle\eta_{\ell,j}| \tag{3.32}$$

in order to calculate the discretized frame operator. We will study the convergence properties of this operator in the next section.

## 3.3 The Frame Operator

Now we want to study the convergence properties of the operator defined in (3.32). To do this, for arbitrary $\phi$, $\psi \in \mathcal{H}_W$, we consider the formal sum

$$\begin{aligned} I_{\phi,\psi} &= \langle\phi|T\psi\rangle \\ &= \sum_{\ell,j=-\infty}^{\infty} \langle\phi|\eta_{\ell,j}\rangle\langle\eta_{\ell,j}|\psi\rangle \\ &= \sum_{\ell,j=-\infty}^{\infty} \int_{\mathcal{V}_m^+\times\mathcal{V}_m^+} e^{i[\{\mathbf{X}_j(\mathbf{k})-\mathbf{X}_j(\mathbf{k}')\}\cdot\mathbf{q}_{\ell,j}]}\,\overline{\phi(k)}\,\eta(\Lambda_j^{-1}k)\,\overline{\eta(\Lambda_j^{-1}k')}\,\psi(k') \\ &\times \frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'} \end{aligned} \tag{3.33}$$

using (3.31).

Now let $\eta(\Lambda_j^{-1}k) = \tilde{\eta}(\mathbf{X}_j(\mathbf{k}))$ and setting $\Delta\mathbf{q}_j = \frac{2\pi}{L_j}$, where $L_j$ is the length of the support of $\tilde{\eta}(\mathbf{X}_j(\mathbf{k}))$ and defined in (3.30), we can write

$$\begin{aligned} I_{\phi,\psi} &= \sum_{\ell=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_{\mathcal{V}_m^+ \times \mathcal{V}_m^+} e^{i[\{\mathbf{X}_j(\mathbf{k}) - \mathbf{X}_j(\mathbf{k}')\} \cdot \frac{2\pi}{L_j} \ell]} \, \overline{\phi(k)} \, \tilde{\eta}(\mathbf{X}_j(\mathbf{k})) \, \overline{\tilde{\eta}(\mathbf{X}_j(\mathbf{k}'))} \, \psi(k') \\ &\times \frac{d\mathbf{k}}{k_0} \frac{d\mathbf{k}'}{k_0'} \end{aligned} \tag{3.34}$$

Since,

$$d\mathbf{X}_j(\mathbf{k}') = \frac{1}{k_0'} \left[ k_0' - (\underline{\Lambda_j^{-1} k'}) \cdot \boldsymbol{\vartheta}(\mathbf{p}_j) \right] d\mathbf{k}', \tag{3.35}$$

changing variables $\mathbf{k}' \to \mathbf{X}_j(\mathbf{k}')$ yields

$$\begin{aligned} I_{\phi,\psi} &= \sum_{j=-\infty}^{\infty} \int_{\mathbf{R}} d\mathbf{X}_j \int_{\mathcal{V}_m^+} \frac{d\mathbf{k}'}{k_0'} \sum_{\ell=-\infty}^{\infty} e^{i[\{\mathbf{X}_j(\mathbf{k}) - \mathbf{X}_j(\mathbf{k}')\} \cdot \frac{2\pi}{L_j} \ell]} \\ &\times \frac{1}{k_0 - (\underline{\Lambda_j^{-1} k}) \cdot \boldsymbol{\vartheta}(\mathbf{p}_j)} \, \tilde{\eta}(\mathbf{X}_j(\mathbf{k})) \, \overline{\tilde{\eta}(\mathbf{X}_j(\mathbf{k}'))} \, \overline{\phi(k)} \, \psi(k') \end{aligned} \tag{3.36}$$

Using (3.23) we write,

$$\begin{aligned} I_{\phi,\psi} &= \sum_{j=-\infty}^{\infty} \int_{\mathbf{R}} d\mathbf{X}_j \int_{\mathcal{V}_m^+} \frac{d\mathbf{k}'}{k_0'} \, L_j \, \delta(\mathbf{X}_j(\mathbf{k}) - \mathbf{X}_j(\mathbf{k}')) \\ &\times \frac{1}{k_0 - (\underline{\Lambda_j^{-1} k}) \cdot \boldsymbol{\vartheta}(\mathbf{p}_j)} \, \tilde{\eta}(\mathbf{X}_j(\mathbf{k})) \, \overline{\tilde{\eta}(\mathbf{X}_j(\mathbf{k}'))} \, \overline{\phi(k)} \psi(k') \end{aligned} \tag{3.37}$$

Then using $\int f(x)\delta(x - x_0)dx = f(x_0)$ we get,

$$I_{\phi,\psi} = \int_{\mathcal{V}_m^+} \frac{d\mathbf{k}}{k_0} \overline{\phi(k)} \psi(k) \sum_{j=-\infty}^{\infty} \frac{L_j}{k_0 - (\underline{\Lambda_j^{-1} k}) \cdot \boldsymbol{\vartheta}(\mathbf{p}_j)} |\boldsymbol{\eta}(\Lambda_j^{-1} k)|^2, \tag{3.38}$$

which implies

$$T(k) = \sum_{j=-\infty}^{\infty} \frac{L_j}{k_0 - \underline{(\Lambda_j^{-1}k)} \cdot \boldsymbol{\vartheta}(\mathbf{p}_j)} |\boldsymbol{\eta}(\Lambda_j^{-1}k)|^2. \tag{3.39}$$

Let, for $t$, $\theta_j$, $\theta_a$ and $\theta_b$ $\in \mathbb{R}$:

$$\mathbf{k} = m\ \sinh(t), \quad k_0 = m\ \cosh(t); \quad \mathbf{p}_j = m\ \sinh(\theta_j), \quad p_{j0} = m\ \cosh(\theta_j) \tag{3.40}$$

$$\mathbf{a} = m\ \sinh(\theta_a), \quad a_0 = m\ \cosh(\theta_a); \quad \mathbf{b} = m\ \sinh(\theta_b),\ b_0 = m\ \cosh(\theta_b) \tag{3.41}$$

Also,

$$\boldsymbol{\vartheta}(\mathbf{p}_j) = \frac{1}{m}\left[\mathbf{p}_j - p_{j0}\boldsymbol{\beta}^*(\mathbf{p}_j)\right] \quad \text{(see (3.15))} \tag{3.42}$$

Now substituting (3.40), (3.41), (3.42) in (3.39) and writing

$$\hat{T}(t) = T(\cosh(t), \sinh(t)),\ \hat{\boldsymbol{\eta}}(t) = \boldsymbol{\eta}(\cosh(t), \sinh(t)),\ \hat{\boldsymbol{\beta}}^*(\theta_j) = \boldsymbol{\beta}^*(\cosh(\theta_j), \sinh(\theta_j)),$$

we have,

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right) \sum_{j=-\infty}^{\infty} \frac{\cosh((\frac{\theta_b+\theta_a}{2}) + \Phi^*(\theta_j))}{\cosh(t - \theta_j + \Phi^*(\theta_j))} |\hat{\boldsymbol{\eta}}(t - \theta_j)|^2 \tag{3.43}$$

where we have written,

$$\hat{\boldsymbol{\beta}}^*(\theta_j) = \tanh(\Phi^*(\theta_j)). \tag{3.44}$$

Let $\theta_j = j\theta_0$, where $\theta_0 > 0$ (fixed) (we call $\theta_0$ the *step size*), then $\hat{T}$ takes the form:

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right) \sum_{j=-\infty}^{\infty} \frac{\cosh((\frac{\theta_b+\theta_a}{2}) + \Phi^*(j\theta_0))}{\cosh(t - j\theta_0 + \Phi^*(j\theta_0))} |\hat{\eta}(t - j\theta_0)|^2 \qquad (3.45)$$

For any compactly supported function $\hat{\boldsymbol{\eta}}$, with finite length of the support $L$, the sum in (3.45) contains at best $L/\theta_0 + 1$ terms only, i.e., it is a finite sum. We observe that each term in the sum is positive and bounded for any $t \in \mathbb{R}$ and $j \in \mathbb{Z}$, consequently, the operator $\hat{T}$ is strictly positive and bounded. Hence the discretized version $\boldsymbol{\eta}_{\ell,j}$ of the coherent states in (3.16) form a *frame*, more appropriately, a *discrete frame*. Depending upon different situations, this frame could be tight or non-tight. Here we now show it by analysing some particular sections and imposing restrictions on $\hat{\boldsymbol{\eta}}$. Note that the contributions of the sections come from the function $\Phi^*$ in (3.45).

1. *The Galilean section* $\sigma_0$:

   For this section $\boldsymbol{\vartheta}(\mathbf{p}_j) = 0$, that implies $\Phi^*(j\theta_0) = j\theta_0$ (using (3.42) and (3.44)) and

   $$\hat{T}(t) = \frac{2\sinh\left(\frac{\theta_b-\theta_a}{2}\right)}{\cosh(t)} \sum_{j=-\infty}^{\infty} \cosh\left(\frac{\theta_b + \theta_a}{2} + j\theta_0\right) |\hat{\eta}(t - j\theta_0)|^2. \qquad (3.46)$$

   For different values of $t$, $\hat{T}$ is different, so in this case the frame is never tight. By contrast, in the corresponding continuous case the frame was tight [6].

2. *The Lorentz section $\sigma_\ell$:*

For this section $\vartheta(\mathbf{p}_j) = \frac{\mathbf{p}_l}{m}$ that implies $\Phi^*(j\theta_0) = 0$ and

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right)\cosh\left(\frac{\theta_b + \theta_a}{2}\right)\sum_{j=-\infty}^{\infty}\frac{1}{\cosh(t - j\theta_0)}|\hat{\boldsymbol{\eta}}(t - j\theta_0)|^2. \quad (3.47)$$

If $|\hat{\boldsymbol{\eta}}(t - j\theta_0)|^2 = \cosh(t - j\theta_0)$ in the support of $\hat{\boldsymbol{\eta}}$, i.e.,

$$|\hat{\boldsymbol{\eta}}(t - j\theta_0)|^2 = \begin{cases} \cosh(t - j\theta_0) & \text{if } \sinh(\theta_a) \le t - j\theta_0 \le \sinh(\theta_b) \\ 0 & \text{otherwise} \end{cases} \quad (3.48)$$

Then

$$\begin{aligned} \hat{T}(t) &= 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right)\cosh\left(\frac{\theta_b + \theta_a}{2}\right)\sum_{j=\lceil\frac{t-\sinh(\theta_b)}{\theta_0}\rceil}^{\lfloor\frac{t-\sinh(\theta_a)}{\theta_0}\rfloor} \\ &= 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right)\cosh\left(\frac{\theta_b + \theta_a}{2}\right) \\ &\times \left(\left\lfloor\left[\frac{t - \sinh(\theta_a)}{\theta_0}\right]\right\rfloor - \left\lceil\left[\frac{t - \sinh(\theta_b)}{\theta_0}\right]\right\rceil\right) \end{aligned} \quad (3.49)$$

where

$$\lceil n \rceil = \begin{cases} n & \text{if } n \text{ is an integer} \\ \text{integral part of } n + 1 & \text{otherwise}, \end{cases} \quad (3.50)$$

$$\lfloor n \rfloor = \begin{cases} n & \text{if } n \text{ is an integer} \\ \text{integral part of } n & \text{otherwise}, \end{cases} \quad (3.51)$$

is a constant operator and therefore, the frame is tight.

3. *The symmetric section $\sigma_s$:*

For this section $\vartheta(\mathbf{p}_j) = \frac{\mathbf{p}_j}{m+p_{j0}}$ which implies $\Phi^*(j\theta_0) = \frac{j\theta_0}{2}$ and

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right) \sum_{j=-\infty}^{\infty} \frac{\cosh(\frac{\theta_b+\theta_a+j\theta_0}{2})}{\cosh(t - \frac{j\theta_0}{2})} |\hat{\eta}(t - j\theta_0)|^2. \tag{3.52}$$

In this case, $\hat{T}$ is a non-constant operator and so the frame is never tight.

Note that if $\Phi^*$ is a constant function, say for example, $\Phi^*(j\theta_0) = \alpha$ (a constant), then

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right)\cosh\left(\frac{\theta_b + \theta_a}{2} + \alpha\right) \sum_{j=-\infty}^{\infty} \frac{1}{\cosh(t - j\theta_0 + \alpha)}|\hat{\eta}(t - j\theta_0)|^2. \tag{3.53}$$

Then if we have $|\hat{\eta}(t - j\theta_0)|^2 = \cosh(t - j\theta_0 + \alpha)$ in the the support of $\hat{\eta}(t - j\theta_0)$, $\hat{T}(t)$ becomes a constant operator and hence the frame is tight.

Again, substituting (3.40) and (3.41) in (3.27) and (3.30); using (3.42) and (3.44); writing $\hat{\eta}(t) = \eta(\cosh(t), \sinh(t))$ and $\theta_j = j\theta_0$, the discretized version of coherent states in (3.16) takes the form,

$$\hat{\eta}_{\ell,j}(t) = \exp\left[-i\pi\ell\frac{\sinh(t - j\theta_0 + \Phi^*(j\theta_0))}{\sinh(\frac{\theta_b-\theta_a}{2})\cosh(\frac{\theta_b+\theta_a}{2} + \Phi^*(j\theta_0))}\right]\hat{\eta}(t - j\theta_0). \tag{3.54}$$

Also, assuming $\hat{\phi}(t) = \phi(\cosh(t), \sinh(t))$, using (3.54), we can write the scalar product in (3.31) as

$$\begin{aligned}\langle \hat{\eta}_{\ell,j} \,|\, \hat{\phi} \rangle &= \int_{-\infty}^{\infty} \exp\left[ i\pi\ell \frac{\sinh(t - j\theta_0 + \Phi^*(j\theta_0))}{\sinh(\frac{\theta_b-\theta_a}{2})\cosh(\frac{\theta_b+\theta_a}{2} + \Phi^*(j\theta_0))} \right] \\ &\quad \times \overline{\hat{\eta}(t - j\theta_0)}\, \hat{\phi}(t)\, dt, \end{aligned} \tag{3.55}$$

which we call the **discretized relativistic windowed Fourier transform** of the function $\hat{\phi}(t)$ for the window-function $\hat{\eta}(t)$).

Now

$$\hat{T} = \sum_{\ell,j=-\infty}^{\infty} |\hat{\eta}_{\ell,j}\rangle\langle\hat{\eta}_{\ell,j}| \quad (\text{see}(3.32)), \tag{3.56}$$

implying

$$\mathbb{I} = \left[\hat{T}\right]^{-1} \sum_{\ell,j=-\infty}^{\infty} |\hat{\eta}_{\ell,j}\rangle\langle\hat{\eta}_{\ell,j}| \tag{3.57}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}_W$ and $\left[\hat{T}\right]^{-1} = \frac{1}{T}$. Then we can write,

$$\hat{\phi}(t) = \sum_{\ell,j=-\infty}^{\infty} \langle\hat{\eta}_{\ell,j} \,|\, \hat{\phi}\rangle \left[\hat{T}(t)\right]^{-1} |\, \hat{\eta}_{\ell,j}(t)\rangle, \quad \text{for any } \hat{\phi} \in \mathcal{H}_W. \tag{3.58}$$

We call (3.58) the **reconstruction formula**. Substituting (3.45),(3.54), and (3.55) in (3.58), we can reconstruct any function $\hat{\phi} \in L^2(\mathcal{V}_m^+, \frac{d\mathbf{k}}{k_0})$. In the next chapter, we will analyse the reconstruction formula and reconstruct some functions in $\mathcal{H}_W$ numerically.

# Chapter 4

# Numerical Computations Using The Relativistic Windowed Fourier Transform

We devote this chapter to numerically computing the operator $\hat{T}$ and applying the *reconstruction formula* obtained in the previous chapter in various situations. First we evaluate the operator $\hat{T}(t)$ for different values of $t$, various window functions and sections. Then we reconstruct several functions using the reconstruction formula, for different window functions, sections and the step sizes. We also evaluate the original functions at the same values of $t$ as the reconstructed ones and compare them graphically. Finally, we show the effectiveness of the relativistic windowed Fourier transform in comparison with the general windowed Fourier transform (also known as the Gabor transform).

## 4.1 Numerical Evaluations of $\hat{T}$

Here we rewrite the operator $\hat{T}$:

$$\hat{T}(t) = 2\sinh\left(\frac{\theta_b - \theta_a}{2}\right) \sum_{j=-\infty}^{\infty} \frac{\cosh((\frac{\theta_b+\theta_a}{2}) + \Phi^*(j\theta_0))}{\cosh(t - j\theta_0 + \Phi^*(j\theta_0))} |\hat{\eta}(t - j\theta_0)|^2 \tag{4.1}$$

For the Galilean sections (4.1) turns into (see (3.46))

$$\hat{T}(t) = \frac{2\sinh(\frac{\theta_b-\theta_a}{2})}{\cosh(t)} \sum_{j=-\infty}^{\infty} \cosh\left(\frac{\theta_b + \theta_a}{2} + j\theta_0\right) |\hat{\eta}(t - j\theta_0)|^2. \tag{4.2}$$

Unless otherwise stated, from now on we assume:

$$\mathbf{a} = \sinh(\theta_a) = -1,\ \mathbf{b} = \sinh(\theta_b) = 1,\ \theta_0 = 0.025\,, \tag{4.3}$$

which implies

$$\hat{T}(t) = \frac{2}{\cosh(t)} \sum_{j=-\infty}^{\infty} \cosh(0.025j)\,|\hat{\eta}(t - 0.025j)|^2. \tag{4.4}$$

Let the window function $\hat{\eta}$ be compactly supported with the support $[-1, 1]$ and defined by

$$\hat{\eta}(t) = \begin{cases} 1 + t & \text{if } -1 \le t < 0 \\ 1 - t & \text{if } 0 \le t < 1 \end{cases} \tag{4.5}$$

which we call the *triangular window.*

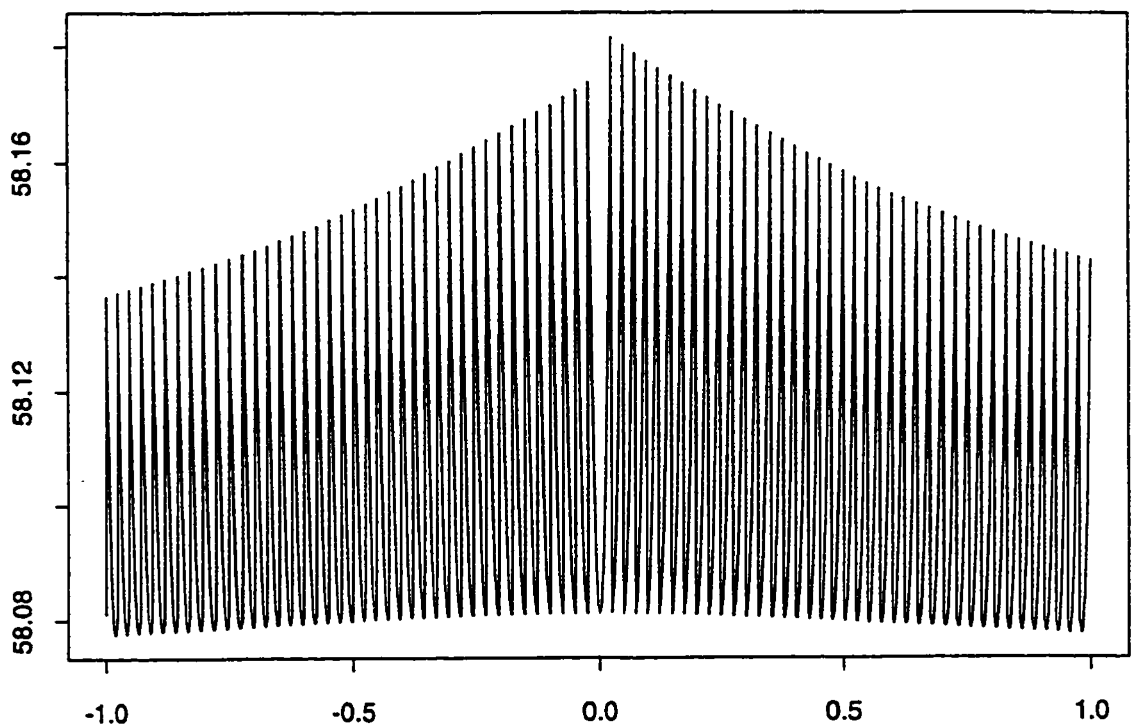


Figure 4.1: Frame operator for Galilean section and triangular window.

Substituting (4.5) in (4.4) we get

$$\begin{aligned}
\hat{T}(t) &= 2 + \frac{2}{\cosh(t)} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})-1\rfloor} \cosh(0.025j)\,(1-t+0.025j)^2 \\
&+ \frac{2}{\cosh(t)} \sum_{j=\lceil(\frac{t}{0.025})+1\rceil}^{\lfloor(\frac{1+t}{0.025})\rfloor} \cosh(0.025j)\,(1+t-0.025j)^2 \qquad (4.6)
\end{aligned}$$

where $\lceil n \rceil$ and $\lfloor n \rfloor$ are defined respectively in (3.50) and (3.51). Using (4.6) we obtain the above Figure.

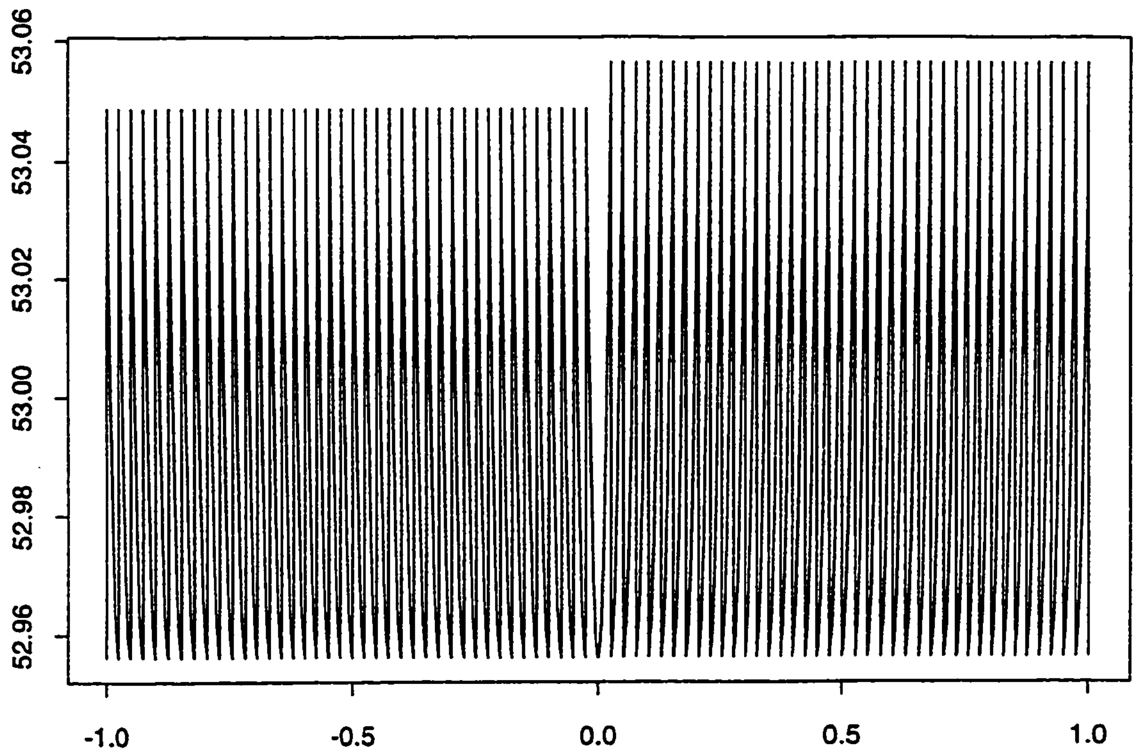


Figure 4.2: Frame operator for the Lorentzian section and triangular window.

For the Lorentzian section, asssuming everything is same as in the case of Galilean section, the operator $\hat{T}(t)$ takes the form (see (3.47)):

$$\begin{aligned}\hat{T}(t) &= 2 + 2 \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})-1\rfloor} \frac{1}{\cosh(t-0.025j)} (1-t+0.025j)^2 \\ &+ 2 \sum_{j=\lceil(\frac{t}{0.025})+1\rceil}^{\lfloor(\frac{1+t}{0.025})\rfloor} \frac{1}{\cosh(t-0.025j)} (1+t-0.025j)^2 \end{aligned} \tag{4.7}$$

From (4.7) we have the above Figure.

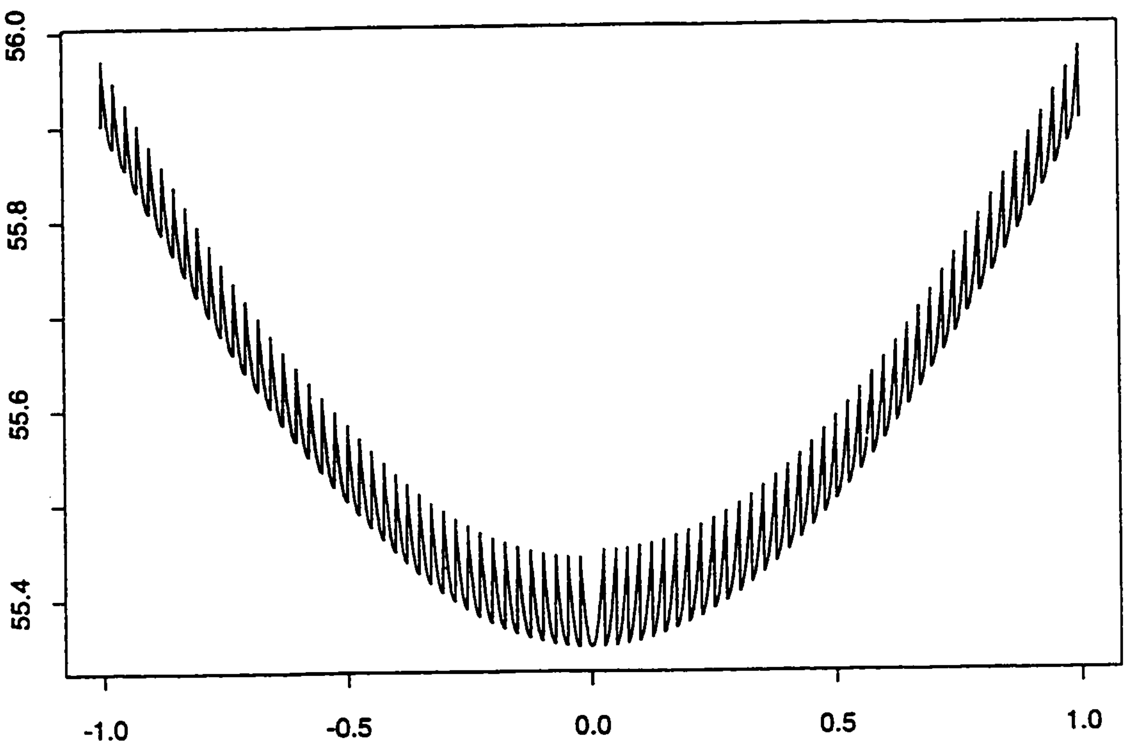


Figure 4.3: Frame operator for the symmetric section and triangular window.

For the symmetric section

$$\begin{aligned}\hat{T}(t) &= 2 + 2 \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})-1\rfloor} \frac{\cosh(0.0125j)}{\cosh(t-0.0125j)}(1-t+0.025j)^2 \\ &+ 2 \sum_{j=\lceil(\frac{t}{0.025})+1\rceil}^{\lfloor(\frac{1+t}{0.025})\rfloor} \frac{\cosh(0.0125j)}{\cosh(t-0.0125j)}(1+t-0.025j)^2 \qquad (4.8)\end{aligned}$$

Using (4.8) we produce the above Figure:

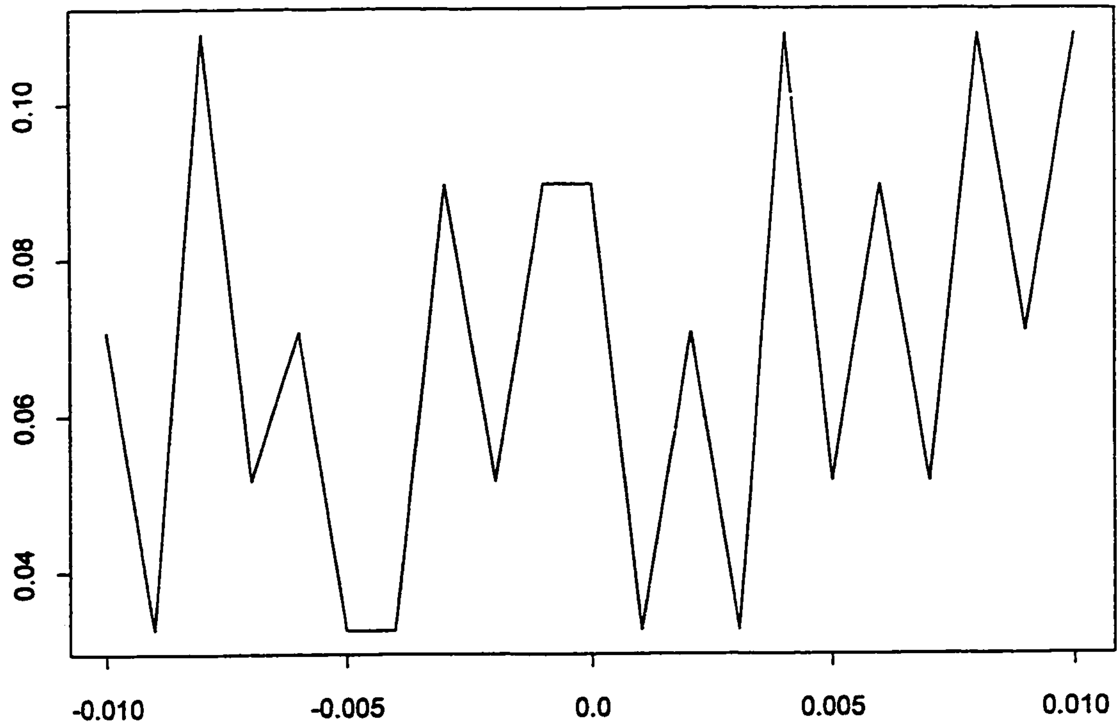


Figure 4.4: Frame operator for the Galilean section and the smooth window.

We now want to calculate $\hat{T}(t)$ for the smooth window

$$\hat{\eta}(t) = \begin{cases} (1-t^2)^{10} & \text{if } -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

For the smooth window (4.9) and Galilean section the operator $\hat{T}(t)$ becomes

$$\hat{T}(t) = \frac{2}{\cosh(t)} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \cosh(0.025j) \left[1-(t-0.025j)^2\right]^{20} \tag{4.10}$$

from which we get the above Figure.

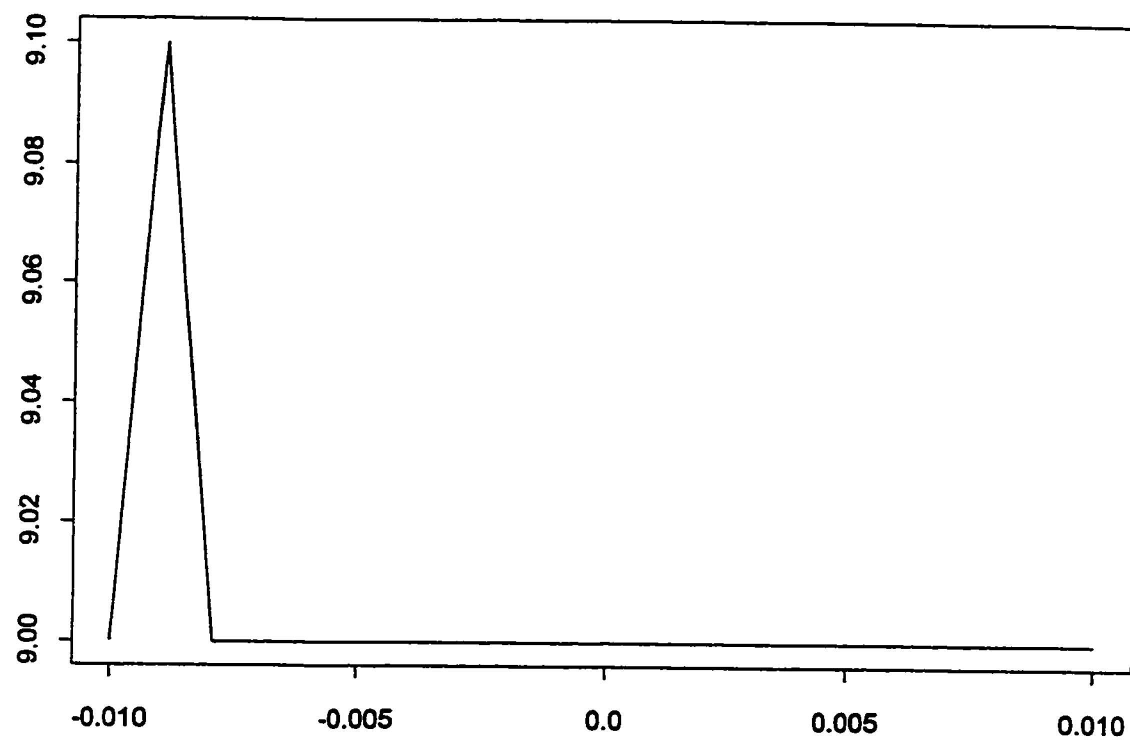


Figure 4.5: Frame operator for the Lorentzian section and the smooth window.

Corresponding to the smooth window and the Lorentzian section $\hat{T}(t)$ takes the form:

$$\hat{T}(t) = 2 \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \frac{1}{\cosh(t - 0.025j)} [1 - (t - 0.025j)^2]^{20} \tag{4.11}$$

which gives the above Figure.

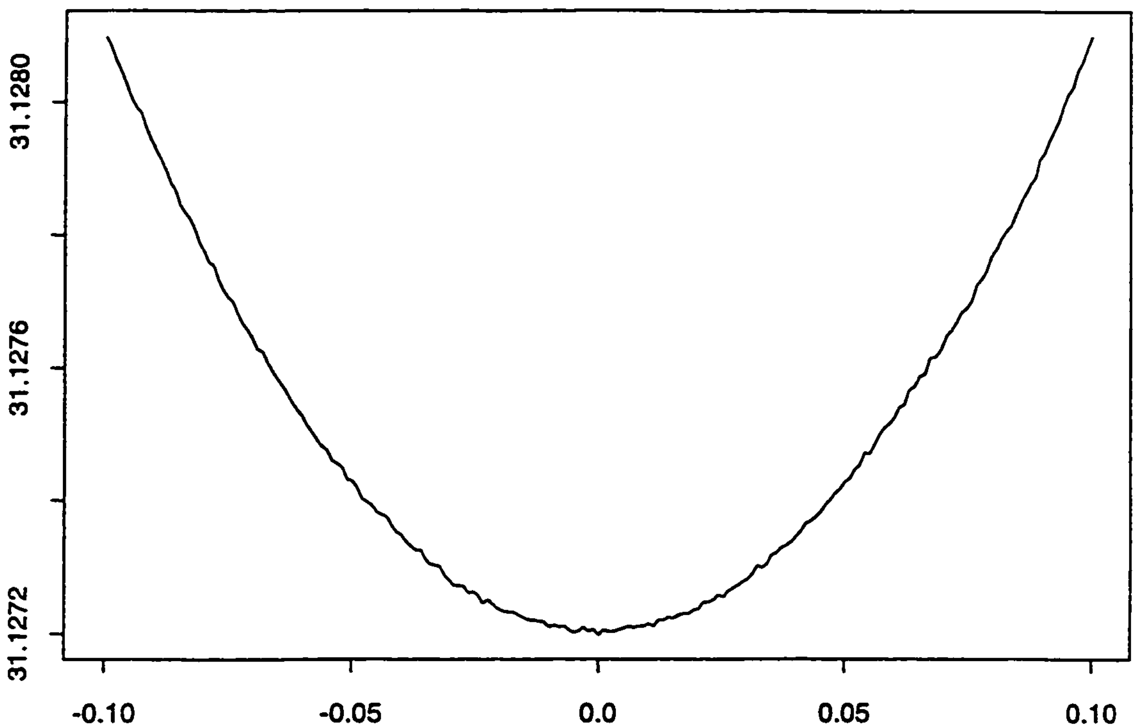


Figure 4.6: Frame operator for the symmetric section and the smooth window.

Finally, for the symmetric section the expression of $\hat{T}(t)$ corresponding to the smooth window becomes

$$\hat{T}(t) = 2 \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \frac{\cosh(0.0125j)}{\cosh(t-0.0125j)} [1-(t-0.025j)^2]^{20} \tag{4.12}$$

From (4.12) we produce the above Figure:

## 4.2 Reconstruction of Functions

In this section, we reconstuct some functions using (3.58). Denoting $\hat{\phi}$, $\left[\hat{T}(t)\right]^{-1}$ and $\hat{\eta}_{\ell,j}$ respectively by $\phi$, $[T(t)]^{-1}$ and $\eta_{\ell,j}$ we can write (3.58) as

$$\phi(t) = \sum_{\ell,j=-\infty}^{\infty} \langle \eta_{\ell,j} \,|\, \phi \rangle \; [T(t)]^{-1} \; | \eta_{\ell,j}(t) \rangle \tag{4.13}$$

where

$$\eta_{\ell,j}(t) = \exp\left[ -i\pi\ell \frac{\sinh(t - j\theta_0 + \Phi^*(j\theta_0))}{\sinh(\frac{\theta_b - \theta_a}{2}) \cosh(\frac{\theta_b + \theta_a}{2} + \Phi^*(j\theta_0))} \right] \eta(t - j\theta_0) \tag{4.14}$$

(see (3.54)) and $T(t)$ is given in (4.1). Let

$$\phi\,(t) = e^{-t^2} \tag{4.15}$$

First we construct $\phi\,(t)$ for the triangular window (non-smooth window) (see (4.5)) and different sections, then we do the same for the smooth window (4.9). For the Galilean section with the assumption (4.3) and (4.15); (4.14) and (3.55) change respectively to:

$$\eta_{\ell,j}(t) = \exp\left[ -i\pi\ell \frac{\sinh(t)}{\cosh(0.025j)} \right] \eta(t - 0.025j) \tag{4.16}$$

and

$$\langle \eta_{\ell,j} \,|\, \phi \rangle = \int_{-\infty}^{\infty} \exp\left[ i\pi\ell \frac{\sinh(t)}{\cosh(0.025j} \right] \overline{\eta(t - 0.025j)}\, e^{-t^2}\, dt \tag{4.17}$$

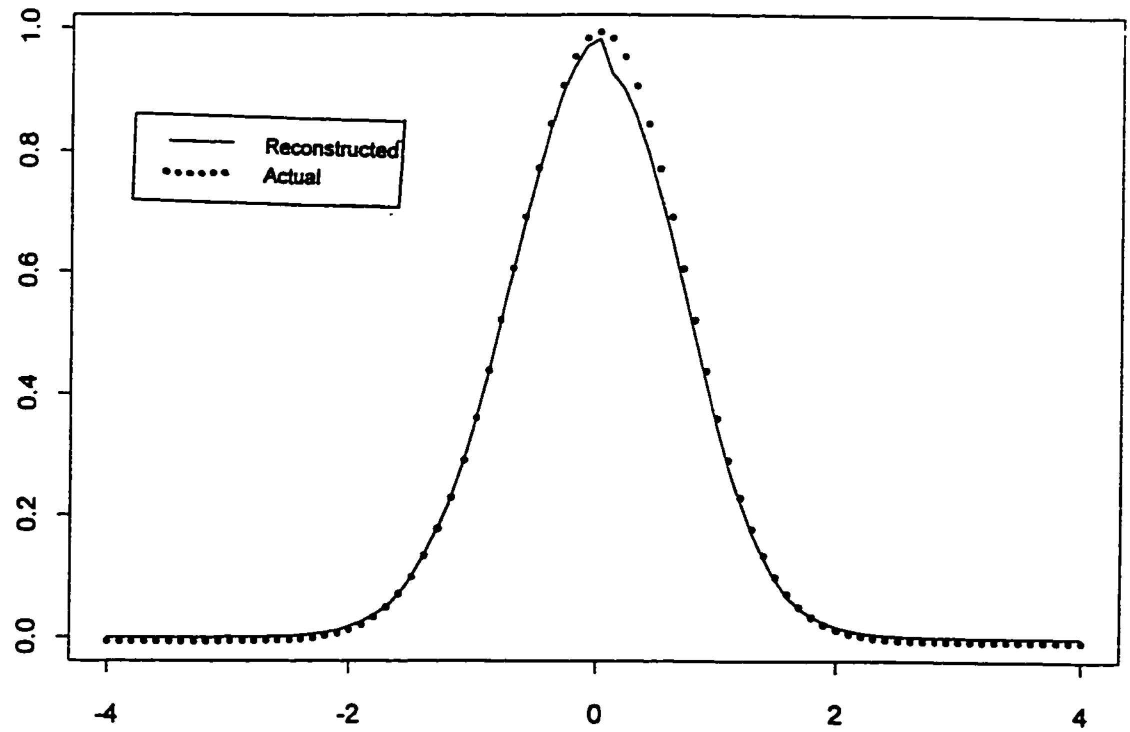


Figure 4.7: Reconstruction of $e^{-t^2}$ for the Galilean section and the triangular window.

Then substituting (4.16), (4.17) and (4.5) in (4.13) we get

$$\begin{aligned}
\phi(t) &= [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})\rfloor} \int_{0.025j}^{0.025j+1} \exp\left[i\pi\ell \frac{\sinh(x)}{\cosh(0.025j)}\right] \\
&\quad \times e^{-x^2} [1 - x + 0.025j]\, dx \\
&\quad + [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \int_{0.025j-1}^{0.025j} \exp\left[i\pi\ell \frac{\sinh(x)}{\cosh(0.025j)}\right] \\
&\quad \times e^{-x^2} [1 + x - 0.025j]\, dx \qquad (4.18)
\end{aligned}$$

where $T(t)$ is given by (4.6). Using (4.18) and (4.15); we have the above Figure:

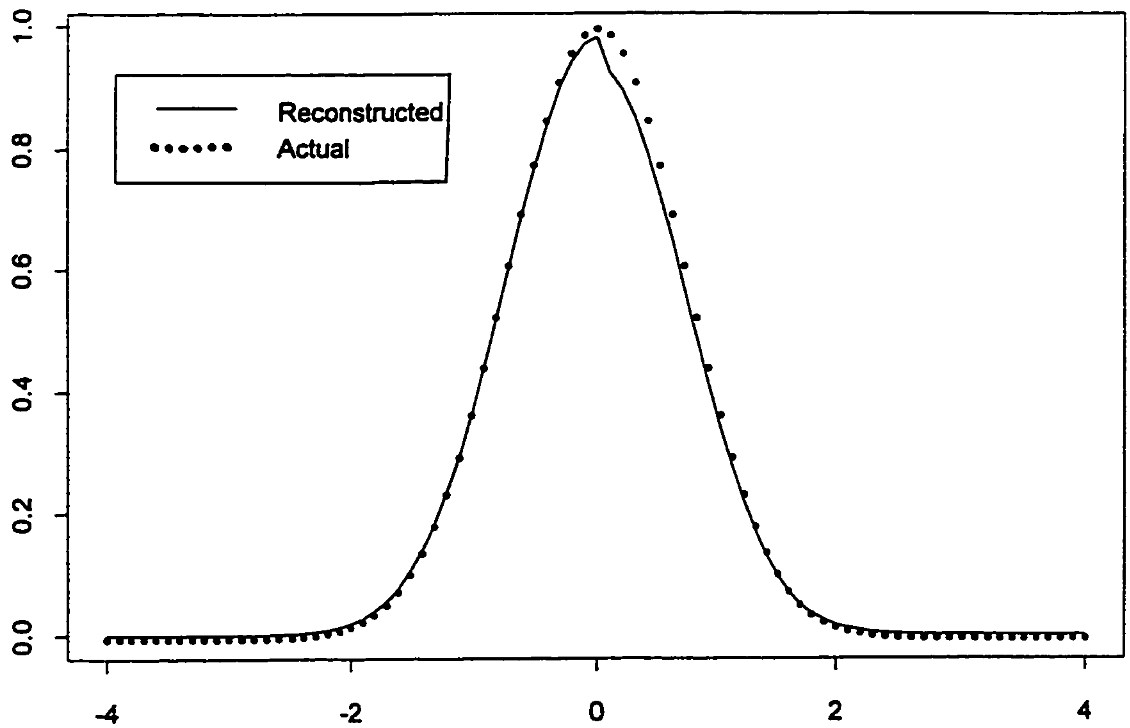


Figure 4.8: Reconstruction of $e^{-t^2}$ for the Lorentzian section and the triangular window.

For the Lorentzian section, the reconstruction formula becomes:

$$\begin{aligned}
\phi(t) &= [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})\rfloor} \int_{0.025j}^{0.025j+1} \exp\left[i\pi\ell \sinh(x-0.025j)\right] \\
&\quad \times e^{-x^2}\left[1-x+0.025j\right] dx \\
&\quad + [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \int_{0.025j-1}^{0.025j} \exp\left[i\pi\ell \sinh(x-0.025j)\right] \\
&\quad \times e^{-x^2}\left[1+x-0.025j\right] dx \qquad (4.19)
\end{aligned}$$

where $T(t)$ is given by (4.7). For (4.19) and (4.15) we have the above Figure.

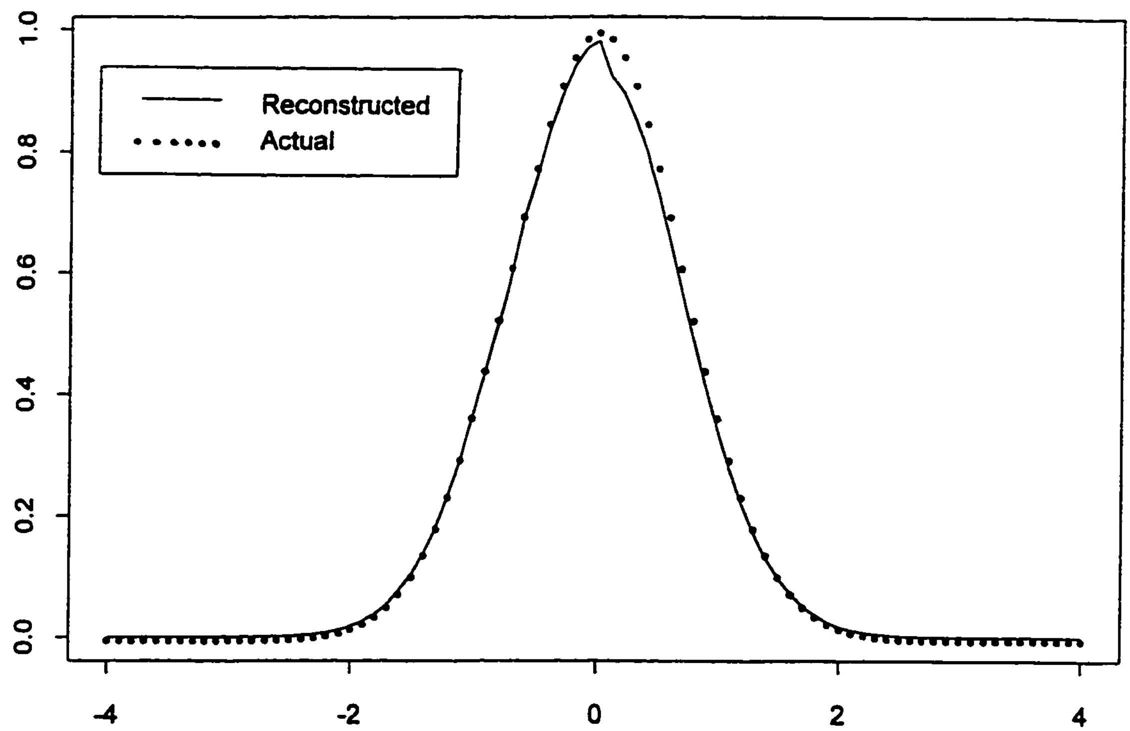


Figure 4.9: Reconstruction of $e^{-t^2}$ for the symmetric section and the triangular window.

The reconstruction formula corresponding to the symmetric section is given by

$$\begin{aligned}\phi(t) \;=\; & [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t}{0.025})\rfloor} \int_{0.025j}^{0.025j+1} \exp\left[i\pi\ell \frac{\sinh(x-0.0125j)}{\cosh(0.0125j)}\right] \\ & \times e^{-x^2}\,[1-x+0.025j]\,dx \\ & + [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \int_{0.025j-1}^{0.025j} \exp\left[i\pi\ell \frac{\sinh(x-0.0125j)}{\cosh(0.0125j)}\right] \\ & \times e^{-x^2}\,[1+x-0.025j]\,dx \qquad (4.20)\end{aligned}$$

where $T(t)$ is given by (4.8). For (4.20) and (4.15) we have the above Figure.

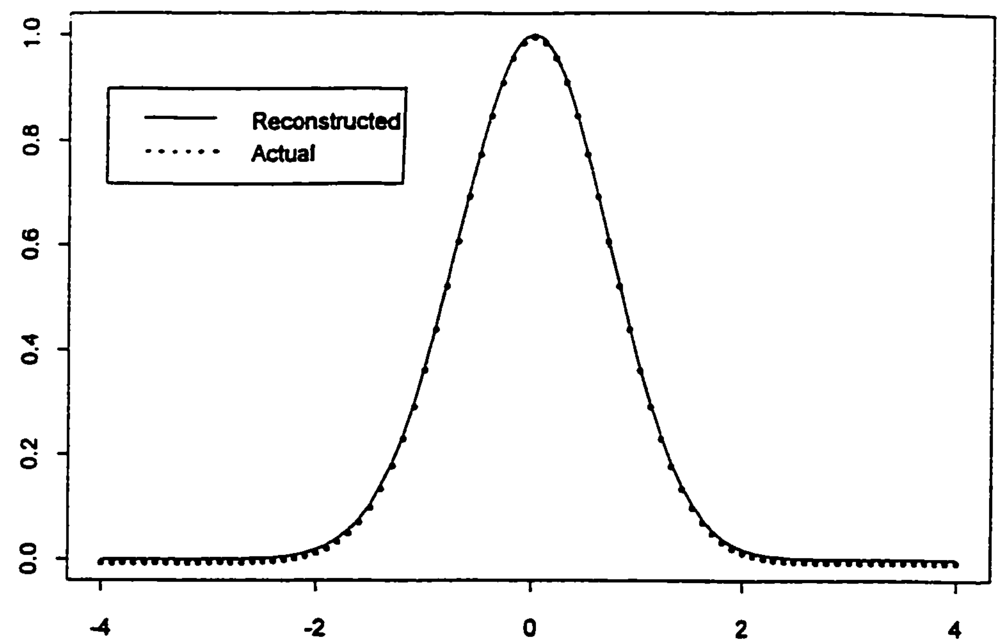


Figure 4.10: Reconstruction of $e^{-t^2}$ for the Galilean section and the smooth window.

**Observation:** We see from the Figure 4.7, Figure 4.8 and Figure 4.9 that the different sections play more or less the same role in the reconstruction scheme and the accuracy of approximation of a function by its reconstructed counterpart is not very high when the triangular (non-smooth) window is used. From now on we use only the Galilean section and some smooth window.

For the smooth window function $\eta$ defined in (4.9), the reconstruction formula for the Galilean section becomes

$$\begin{aligned} \phi(t) &= [T(t)]^{-1} \sum_{\ell=-\infty}^{\infty} \sum_{j=\lceil(\frac{t-1}{0.025})\rceil}^{\lfloor(\frac{t+1}{0.025})\rfloor} \int_{0.025j-1}^{0.025j+1} \exp\left[i\pi\ell \frac{\sinh(x)}{\cosh(0.025j)}\right] \\ &\quad \times e^{-x^2} \left[1-(x-0.025j)^2\right]^{10} dx \end{aligned} \tag{4.21}$$

where $T(t)$ is given by (4.10). Using (4.21) and (4.15) we have the above Figure:

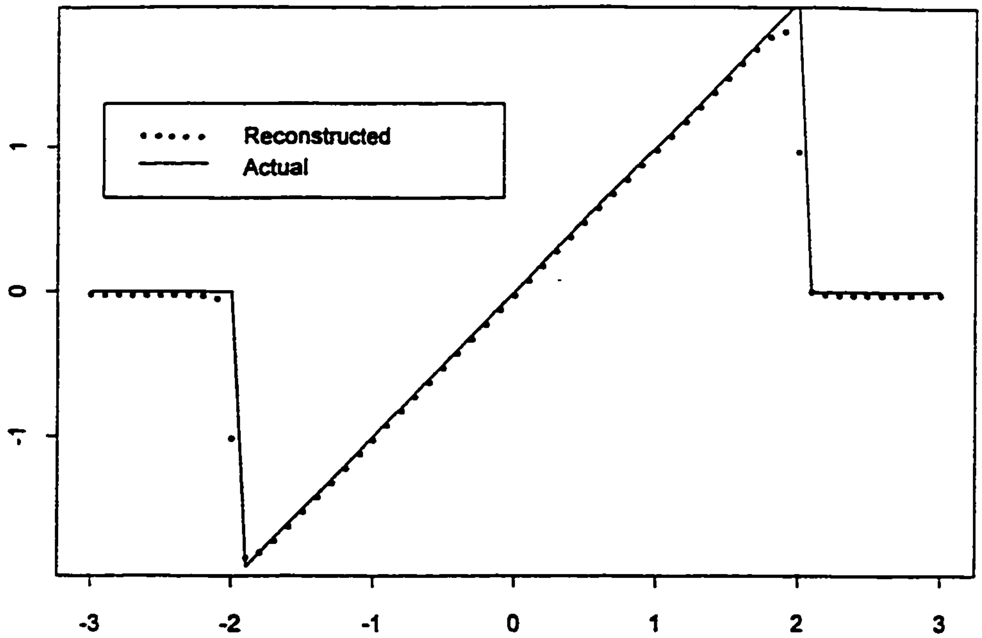


Figure 4.11: Reconstruction of a discontinuous function for the Galilean section and the smooth window.

**Observation:** From the Figure 4.7, Figure 4.8, Figure 4.9 and Figure 4.10, we see that for a better approximation of a function by the reconstruction formula we should use a smooth window.

So far we observed that the reconstruction scheme goes well for smooth functions and smooth windows and at this point we want to see how does it work with discontinuous functions. To this end we take the following discontinuous function:

$$\phi(t) = \begin{cases} t & \text{if } -2 \leq t \leq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

Substituting (4.22) in (4.21) we have the above Figure.

The Figure 4.11 tells us that in the vicinity of $x_0$ where a function $f(x)$ is discontinuous, the reconstructed function starts ringing and the accuracy of approximation is not high. A situation like this in Fourier analysis is known as the *Gibbs' phenomenon* [61]. It also tells us that $f(x_0)$ converges to

$$\frac{f(x_0^+) + f(x_0^-)}{2}, \tag{4.23}$$

where $f(x_0^+)$ and $f(x_0^-)$ are respectively the right- and left-hand limit of $f(x)$ at $x_0$. It is worthwhile mentioning here that the inverse Fourier transform of a function $f$ at a point $x_0$ converges to (4.23) [61].

## 4.3 Comparison with the Windowed Fourier Transform

In this section, we discretize the coherent states of Weyl-Heisenberg group and following the techniques of chapter-3 we come up with the corresponding frame operator and the reconstruction formula. Finally we reconstruct a function using both the relativistic windowed Fourier transform and the usual windowed Fourier transform and compare them.

The discretized version of the coherent states of the Weyl-Heisenberg group is given by:

$$\phi_{m,n}(x) = e^{imp_0x}\,\phi(x - nq_0) \tag{4.24}$$

where $p_0,\ q_0 > 0,\ m,\ n$ are integers and $p_0,\ q_0$ must satisfy the condition

$$p_0 \cdot q_0 \leq 2\pi. \tag{4.25}$$

The condition (4.25) is necessary for $\phi_{m,n}(x)$ to be complete and to form a frame [18, 62, 27]. Let $\phi(x)$ be a compactly supported function with support $L = \frac{2\pi}{p_0}$. The frame operator $T : L^2(\mathbb{R}, dx) \rightarrow \ell^2(\mathbb{Z}^2)$ is defined by

$$T = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\phi_{m,n}\rangle\langle\phi_{m,n}| \tag{4.26}$$

For $\psi \in \mathcal{H} = L^2(\mathbb{R}, dx)$, we consider the formal sum

$$\begin{aligned}
I_{\psi,\psi} &= \langle\psi\,|\,T\psi\rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle\psi\,|\,\phi_{m,n}\rangle\langle\phi_{m,n}\,|\,\psi\rangle \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{imp_0(x-y)}\,\phi(x-nq_0)\,\overline{\phi(y-nq_0)}\,\overline{\psi(x)}\,\psi(y)\,dx\,dy \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{im\frac{2\pi}{L}(x-y)}\,\phi(x-nq_0)\,\overline{\phi(y-nq_0)}\,\overline{\psi(x)}\,\psi(y)\,dx\,dy \\
&= L\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-y)\,\phi(x-nq_0)\,\overline{\phi(y-nq_0)}\,\overline{\psi(x)}\,\psi(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} dx\,|\psi(x)|^2\,L\sum_{n=-\infty}^{\infty} |\phi(x-nq_0)|^2
\end{aligned} \tag{4.27}$$

which implies

$$T(x) = \frac{2\pi}{p_0}\sum_{n=-\infty}^{\infty} |\phi(x-nq_0)|^2 \tag{4.28}$$

and the corresponding reconstruction formula is:

$$\psi(x) = \frac{1}{T(x)} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle \phi_{m,n} | \psi \rangle \, | \phi_{m,n} \rangle \tag{4.29}$$

Let $p_0 = \pi$ and

$$\phi(t) = \hat{\eta}(t) = \begin{cases} (1-t^2)^{10} & \text{if } -1 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

Then we can write

$$T(t) = 2 \sum_{n=\lceil (\frac{t-1}{q_0}) \rceil}^{\lfloor (\frac{t+1}{q_0}) \rfloor} \left[1-(t-nq_0)^2\right]^{10} \tag{4.31}$$

and

$$\begin{aligned} \psi(t) \; &= \; \frac{1}{T(t)} \sum_{m=-\infty}^{\infty} \sum_{n=\lceil (\frac{t-1}{q_0}) \rceil}^{\lfloor (\frac{t+1}{q_0}) \rfloor} \left[ \int_{nq_0-1}^{nq_0+1} e^{-im\pi x} \, [1-(x-nq_0)^2]^{10} \, \psi(x) \, dx \right] \\ & \times e^{im\pi t} \, [1-(t-nq_0)^2]^{10}. \end{aligned} \tag{4.32}$$

Now writing $\cosh(t) \, e^{-\sinh(t)^2}$ for $\phi$ in (4.21) and $\psi$ in (4.32) and setting $\theta_0 = 0.01$ (instead of 0.025) in (4.21), $q_0 = 0.01$ in (4.32), we have the Figure 4.12 (in the next page).

From the Figure 4.12, we observe that the reconstruction scheme goes well both in relativistic windowed Fourier and windowed Fourier techniques and the respective reconstructed values are virtually the same.

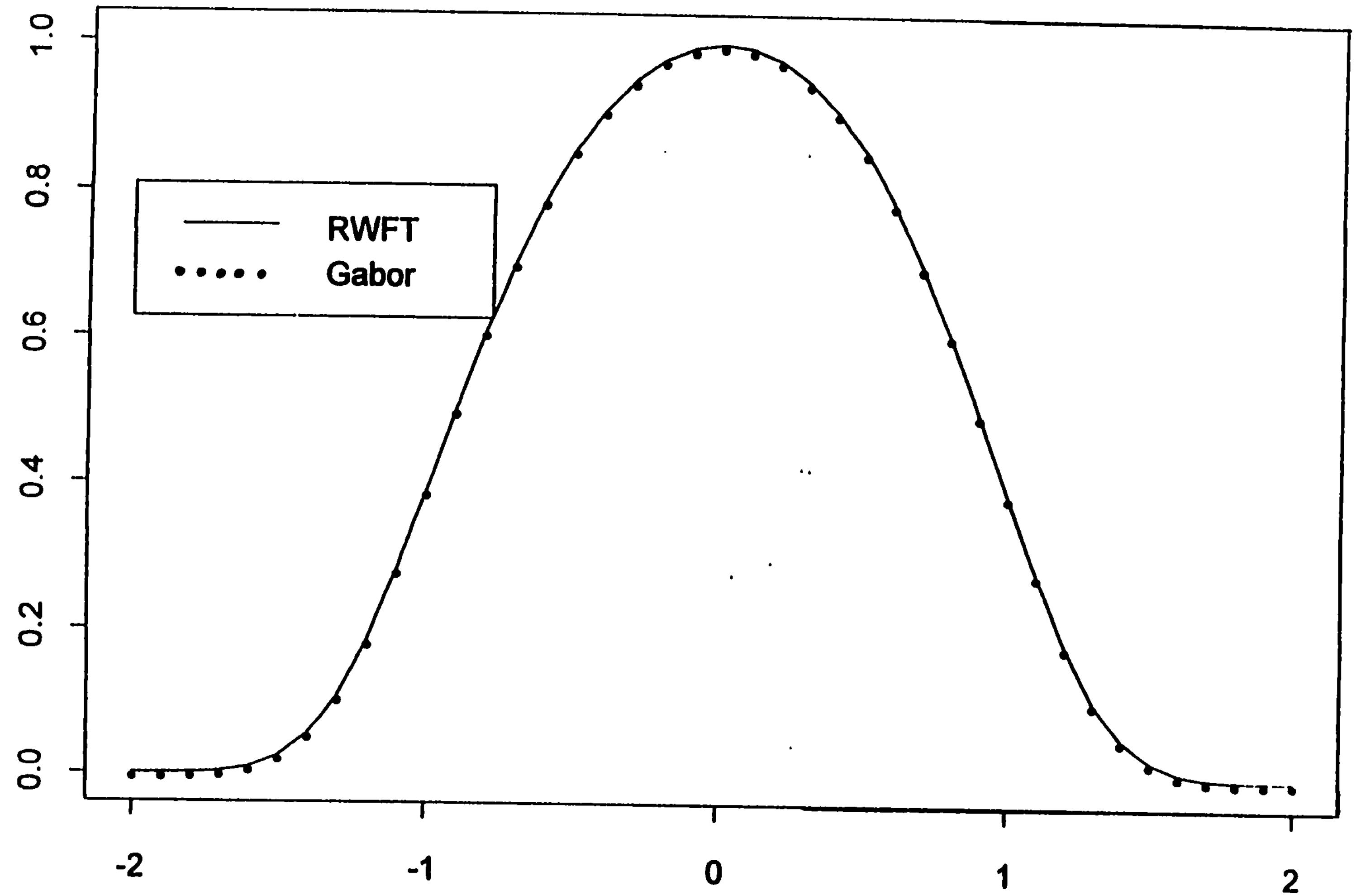


Figure 4.12: Reconstruction of $\cosh(t)\, e^{-\sinh(t)^2}$ using RWFT, WFT and smooth window.

# Conclusion

In chapter 2, we constructed coherent states of the full Poincaré group and we saw that they formed frames. We believe that these coherent states will have applications to spin dependent problems in atomic and nuclear physics, as well as to image reconstruction problems, using the discretized versions of these frames. In chapter 3, we discretized the coherent states of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$ and obtained discrete frames and a reconstruction formula. We observed in chapter 4 that the reconstruction of a signal is much more accurate if we use a smooth window function, instead of a non-smooth window function and the Galilean - , Lorentzian - and symmetric sections all play similar roles in the reconstruction programme. As we also saw that the reconstruction of a signal by relativistic windowed Fourier transform (RWFT) was as good as the reconstruction by the Gabor transform (GT), the RWFT can be used as a substitute of the GT. The RWFT has applications in many fields, including pattern recognition, signal and image reconstruction, detection and extraction of unknown signals etc.

## Some Specific Applications:

States and observables in quantum mechanics can be viewed in a very special way in terms of coherent states. For example, if we consider the hydrogen atom, we can describe its continuum and bound state wave functions, dipole operators, etc., in terms of an overcomplete basis consisting of Galilean coherent states as:

$$\langle \mathbf{r} \,|\, \mathbf{q},\, \mathbf{p} \rangle = \left( U(\sigma_0(\mathbf{q},\, \mathbf{p})) \sum_{n,\ell,m} S_{n,\ell,m} \right) (p_0, \mathbf{r}),$$

where the $S_{n,\ell,m}(p_0, \mathbf{r})$ are the Sturmian functions [68]. The Sturmian functions are solutions of one of the Sturm-Liouville problems and form a complete discrete basis set in the Hilbert space. For computational purposes, for example, matrix elements of multi-photon process in the non-relativistic case [57, 71], the states $\langle \mathbf{r} \,|\, \mathbf{q},\, \mathbf{p} \rangle$ are suitable. This is also true in the intermediate relativistic case, where the Dirac or the Feyman-Gell-Mann equation is able to describe the interaction of a charged spin$-\frac{1}{2}$ particle with the electromagnetic field. Some work in atomic physics in this direction has been done recently in [73]. The spin-Sturmian functions for the Feynman-Gell-Mann equations were obtained in [22]. Using these wave functions one can construct relativistic coherent states in the light of the present work:

$$\langle \mathbf{r} \,|\, \mathbf{q},\, \mathbf{p},\, s \rangle = \left( U_W^s(\sigma(\mathbf{q},\, \mathbf{p})) \sum_{n,\ell,m} S^s_{n,\ell,m} \right) (p_0, \mathbf{r}).$$

In constructing these coherent states one has the freedom in the choice of available sections, in addition to the already existing freedom in the choice of the Sturmian functions. The various sections $\sigma$ also have applications to relativistic statistical mechanics in the computation of distribution functions [36].

We now show, as a specific example, how the RWFT can be used in detection and extraction of unknown signals: In detecting a signal by a radar a 'threshold' is to be set. A threshold is an electronic device that produces an output signal when the receiver output, averaged over the several repitition periods, exceeds a predetermined level. If the threshold voltage level is adjusted to lie well above the rms 'noise' output of the receiver, false alarms caused by noise may be kept to any desired low rate. The use of high threshold setting will also result in failure to note the presence of actual target when their signals are relatively weak. Hence the probability of detection will be a function both of signal-to-noise ratio and the threshold setting. Here by 'noise' we mean very small random fluctuating voltages that unaviodably are present in the input circuit of the receiver. An observed waveform may be a signal mixed with noise or a noise alone. In their work, Chen and Qian [24] used the following steps in detecting an unknown signal:

1) representing the noisy signal in the joint time-frequency domain by using Gabor transforms;

2) determining the mean of the Rayleigh distribution of the background noise in the joint time-frequency domain;

3) thresholding the time-frequency coefficients;

4) defining the time-frequency coefficients which are above the threshold;

5) measuring the time-frequency location, time duration and frequency range from the coefficients above the threshold.

In the step 2) , by the Rayleigh distribution we mean,

$$p(x) = \frac{x}{\mu^2} \exp\left[-\frac{x^2}{2\mu^2}\right] \quad (x \geq 0)$$

where $\mu$ is related to the *mean* value by, $mean = \sqrt{\frac{\pi}{2}}\,\mu$.

After the detection of the signal, it is extracted by using the inverse Gabor transform. The localization of the time and frequency by the Gabor transform makes it possible for de-noising, signal detection, and signal extraction in the time-frequency domain. Finally the extracted signal is reconstructed by a Gabor expansion. We can use the RWFT instead of the Gabor transform in the above procedure to detect, extract and reconstruct an unknown signal.

## Some Possible Extensions of the Present Work:

The immediate extension one can do is to discretize the coherent states of the full Poincaré group to obtain discrete frames and a reconstruction formula, as in the case of $\mathcal{P}_+^\uparrow(1,1)$. These could be applied in the reconstruction of signals and images. Since each transform (e.g., wavelet transform, RWFT, GT etc.) arose from the CS of different groups and the discretization of CS in each case is done in different ways, one transform may be more suitable for a certain class of functions than for another class. So one can also explore the possibility of finding the classes of functions suitable, for the reconstruction formula obtained in chapter 3, so that the reconstruction scheme does a better job, at least for that kind of functions, than other available schemes. It is our intention to proceed in this direction.

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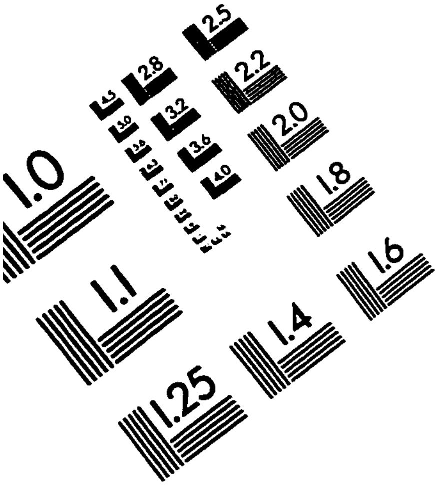


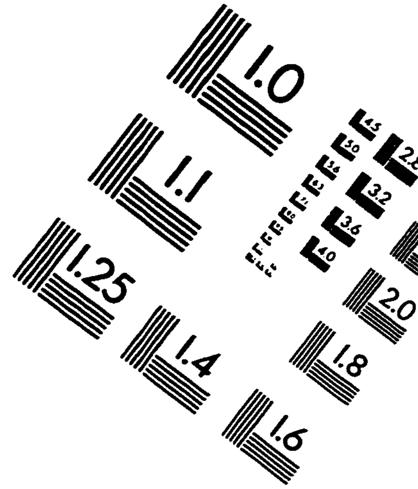


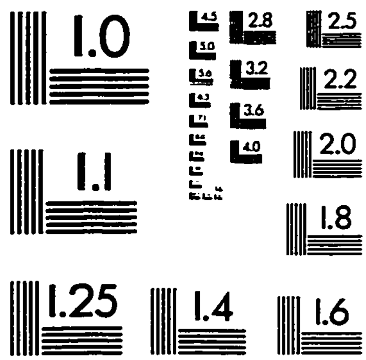


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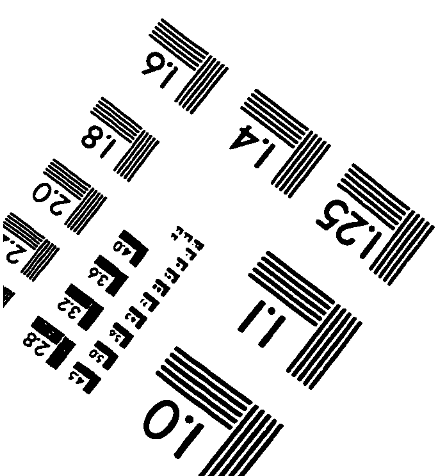

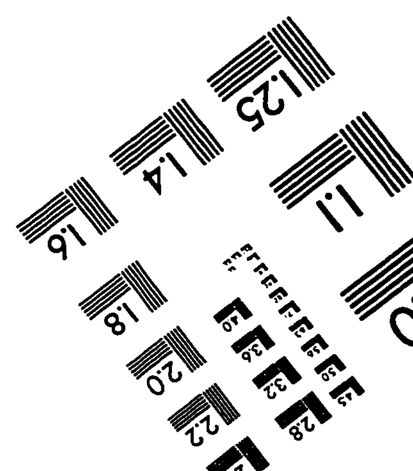

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