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An Inquiry into Students' Obstacles (Cognitive, Epistemological and Ontological) Affecting the Understanding of Mathematical Infinity.

Elaine Landry

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ABSTRACT

An Inquiry into Students' Obstacles (Cognitive, Epistemological and Ontological) Affecting the Understanding of Mathematical Infinity.

Elaine M. Landry

This research considers the identification, investigation and analysis of undergraduate mathematics students' obstacles when faced with the idea of mathematical infinity. In assuming a developmental similarity between an idea's historical/philosophical acceptance and the individuals' understanding of it, this study seeks to create a learning situation wherein students' inappropriate beliefs regarding the nature of infinity are both recognized and challenged.

The identification of obstacles (cognitive, epistemological and ontological) and acts of understanding is the aim of the 'model for understanding'; this model is principally constructed by using the historical account of infinity to determine the requisites for its conceptual acceptance. Four case studies, in the form of clinical interviews and teaching experiments, are used for the investigation of students' views; obstacles, conflicts and successes. The analysis of these views is thus obtained through a comparison of the results of the interviews and experiments with the aforementioned model of understanding.

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CHAPTER 1.

Justification of the Investigation and Analysis: *Students' Obstacles Related to the Understanding of Mathematical Infinity*

§1.1 Introduction

If it can be said that there exists an idea which fundamentally characterizes mathematics then this would be infinity: to mention but a few aspects, its existence is fundamental for understanding the Calculus, Analysis, Fractals, Set Theory, and Geometry. For all its power, however, infinity has retained its reputation as an idea which is, essentially, beyond grasp; i.e., it remains that which is psychologically characterized as being mentally and physically, mathematically and/or philosophically obscure and unattainable. With respect to these varying beliefs, this investigation will deal specifically with those *inappropriate* views, i.e., comprehensive yet not suited to serve as a conceptual/mathematical basis for some context(s), of students which present themselves as obstacles towards understanding mathematical infinity. These obstacles are worthy of attention and analysis since, when they remain unaccounted for, they serve to inhibit understanding, both of the nature of infinity and of those aforementioned aspects of mathematics for which it plays a definitive role [Love:1989], [Robertson: 1973], [Fischbein, Tirosh & Hess: 1979].

Mathematics students are among those still clinging to the view that infinity, mathematical or other, is a non-existent, non-palpable, non-sensical and/or meaningless, idea [Sierpinska; 1987a]. Their lack of distinction between physical and/or metaphysical infinity and mathematical infinity allows them to continuously view it as ontologically (essentially) potential and therefore 'objectively' non-existent. Thus the task at hand is to, in someway, get students to make the distinction between mathematical and non-mathematical infinity: to grasp its sense, they must see a need for a purely mathematical notion; to grasp its abstract sense they must accept infinity as a mathematically defined and existent object; to grasp its formalism, they must see that its epistemological and ontological meaning or status is ascribed according to context. It is to be my claim, however, that in order to distinguish and accept mathematical infinity students must grant this idea object status, or existence, as determined by constraints of mathematical reality, only then will they see potentiality as a characteristic as opposed to seeing it as that which defines its essence.

The question now becomes: what is required to enable students to see the need to transform their views so that they become more mathematically appropriate; preceding this, one must first determine how it is that students are able to maintain their mistaken beliefs. My hypothesis is that there is an implicit inadequacy in providing only those mathematical contexts (limits, sequences, repeating decimals.....) in which infinity is characterized as mathematically potential; in doing so we enable students to maintain the belief that philosophical, physical and mathematical infinity are one and the same, in that they are both non-existent, excepting perhaps that mathematical infinity has a symbol (∞). To counter this view, then, we must provide a context where it is not possible to think of infinity other than mathematically. Unless students are given a context in which infinity is both existent and mathematically defined (actual) they will maintain their intuitive beliefs which, at best, can only give to them a mathematical sense of infinity. That is, they will have mathematized their intuitive beliefs without regard for the mathematical meaning of infinity. Related to this, I will show why the acceptance of infinity as an actual object can be used to establish a necessary, though not sufficient, condition for the acceptance of an existent and meaningful infinity.

These aims and assumptions are to be described and considered within five chapters. Chapter one provides the justification of why and how this investigation is to be carried out; it includes also an analysis of how these claims measure against the relevant literature. The theoretical framework of this dissertation will be the focus of chapter two: the main focus being the development of a model of understanding from which one can determine the requirements for understanding mathematical infinity. Preceding the development of this model, particularly the identification and explanation of related obstacles, will be an analysis of the historical perspective and treatment of mathematical infinity. The identification and analysis of students' obstacles and successes will, in this manner, be in accordance with Bachelard's [Herscovics:1989] belief in the 'interrelatedness between both its (the object's) historical and individual development' which considers; a) the tendency to rely on deceptive intuitive experiences, b) the tendency to generalize which may hide particularity and c) the obstacles caused by natural language. Regarding the nature and role of epistemological obstacles, however, I, like Sierpinska [Sierpinska: 1987b], will part ways with Bachelard and consider them as both positive and necessary.

Chapter three will attempt to report on four undergraduate mathematics students' views, obstacles and conflicts when faced with the object of mathematical infinity. The selection of students will be made from respondents to a questionnaire designed to determine their pre-existing views. The actual identification of students' obstacles is to be found by way of individual clinical interviews. The focus of chapter four will be a teaching experiment designed to bring to the fore the conflicts between a necessarily appropriate mathematical view of infinity and those which are other. The aim of this chapter will be to contrast the preceding identification and analysis of students' obstacles towards mathematical infinity with those requirements established for its understanding (Chapter 2). Chapter five will present a summary of the results and suggest the possible pedagogical/philosophical and mathematical implications of this study. It should be here noted that the goal of this thesis is not to overcome students' obstacles, rather, it is hoped that this investigation will delineate and distinguish those considerations which must precede such a task.

§ 1.2 Preliminary Discussion

Though this inquiry shall be restricted to those considerations which are mathematical and thus rigorously defined; the notions of both philosophical and physical infinity will likewise require thoughtful attention. The need to account for these non-mathematical notions arises from a belief that students' intuitions of infinity are often times founded on physical and/or philosophical considerations [Sierpinska:1987a], [Waldegg:1991], [Tall:1980], [Fischbein, Tirosh & Hess:1979]. That is, in order to unite those ideas which are strictly mathematical we must discuss and distinguish those which are other. The basis for such a distinction will be obtained through the identification of those very same obstacles (cognitive, epistemological and ontological) that result when such a distinction is not enacted. That is, through the determination of what presents itself as an obstacle we can, through negation, likewise determine what is necessary, though not sufficient, for understanding mathematical infinity; e.g., if students believe that mathematics occurs in space-time this will then lead to the view that infinity occurs in time and thus can only be potential. Theretore one of the requirements of understanding is that students accept mathematical reality as being outside space-time.

The above directive for the creation of a model of understanding may sound odd but appears clearer when we assume that it is the same history or culture, composed of psychological and philosophical considerations, that shapes both mathematical and individual development [Sierpinska:1987b], [Herscovics:1989], [Robertson:1973], [Waldegg:1991]. In this manner, obstacles are seen not solely as individual inadequacies but are more taken as a reflection of the object's mathematical development and definition. Though the individual, himself, is unique in his understanding, the object, itself, carries with it its own history and included in this is the obstacles that have preceded its very actualization (or object acceptance). In accordance then, with the aforementioned assumptions, we may also assume that students' obstacles may be, likewise, realized and challenged by those same 'historical' ideas. Thus not only can we use the historical development of mathematical infinity as a basis for distinguishing and explaining obstacles but also we can use it as a basis which would enable students to question, if not also overcome, them [Síerpinska:1987b], [Robertson:1973], [Waldegg:1991]. Respecting these assumptions I propose to: *investigate* the historical development of mathematical infinity and state what it means to *understand* mathematical infinity; *identify* the implicit and explicit obstacles of understanding which are coincidental to its development; and assuming students' understanding reflects a similar evolution of ideas, *create* a learning situation wherein their intuitive ideas of the sense of infinity run contrary to the required mathematical idea of its meaning. It is hoped that this process will impose the conflict that may push students to see the need to question and analyse their ideas about infinity; to see the inadequacies and to accept infinity into their mathematical structure. For those who have not yet seen the need to distinguish between mathematical reality and, say, 'natural' reality this indeed will be a difficult, though necessary, task.

The resolution of this problem then, involves the construction of a model of understanding mathematical infinity which would provide a basis from which one can: a) distinguish between those beliefs which are appropriate and indeed necessary from those which are neither; b) determine, through clinical interviews, the source(s) from which both relevant and mislaid views arise; and, c) create or expand students' existing conceptual structure to include, through teaching experiments, a mathematical structure where mathematical infinity and any other view of infinity are both distinguished and united. Students must come to see that the nature of mathematical infinity is determined by the context in which it occurs; that the potentiality so often ascribed to its essence is but a characteristic and in doing so realize that there indeed exists an actual mathematical object called infinity. In this manner, it is not my intention to overcome student's obstacles relating to the notion of infinity, it is rather to bring students to a conflict in such a way that they become aware of their own views and the obstacles they entail [Sierpinska:1987b].

The particular context which I have chosen to demonstrate the existence of a mathematical infinity is Cantor's Set Theoretic notion of cardinal infinities. The reason for this choice is the relative lack of specialized knowledge that is prerequisite; it requires a general knowledge of the number systems and sets, of one-to-one correspondence, and of the power and methods of mathematical proof. The latter requirement is one which, in all fairness, limits the choice of subjects and thus lessens both the applicability and generalizability of any results. That is, because it is essential that students understand the proof for the uncountability of **R**, I shall, for experimental reasons, restrict the choice of subjects to those enrolled in undergraduate Analysis courses. Withstanding this bound, cardinal infinity remains an appropriate context in that it demands the conceptualization of infinity as a mathematically defined, meaningful, and therefore actually existent object. No longer can students intuitively think of infinity as, say, the largest value when it is shown there exists one infinity (**c**) greater than another ($\aleph_0$)

§ 1.3 Literature Review

The purpose of this section is to provide a summary of those researches which have sought to identify and explain students' obstacles and acts that relate to the understanding of mathematical infinity. I shall first concern myself what is said about the nature of these obstacles and acts in and of themselves. Next, I shall consider those researches that advocate a link between historical and individual development. This assumption will be shown to serve two purposes; it allows for the identification of those individual obstacles and acts that have their parallel in historical ideas, and it underlies the methodological belief that the same historical ideas can be used to challenge students' assumptions.

This section will then conclude with the identification of those sources which are claimed to explain students' obstacles and acts. These sources will be delineated in the following way; those concerned with students' a) mathematical and/or structural biases (or attitudes), b) age and/or developmental level and c) contextual/representational biases. Though, undoubtably, these sources are inter-related, they will be distinguished in the following manner. Mathematical/structural biases will be those which consider students' beliefs regarding the nature and status of mathematical knowledge and existence. The attributation of obstacles to age will include those studies which link students' conceptual beliefs with their stage of developmental. Contextual biases will be those in which students' are said to maintain mathematical beliefs in contexts other than those in which the belief is justified.

It should be noted that these researches have allowed me, through analysis, synthesis and criticism of the presented information, to consider and construct most, if not all, of the ideas and hypotheses which underly this thesis. Accordingly, whenever possible I shall, while reviewing the literature, relate to it those ideas which have become fundamental to this thesis.

Nature of obstacles and acts of understanding

Before discussing her actual research, Sierpinska, in <u>*Humanities Students And Epistemological Obstacles Related To Limits.*</u> [Sierpinska:1987a] first deals with epistemological obstacles in and of themselves and further states how they may be overcome. Specifically, "if an obstacle is to be overcome, a mental conflict is necessary" (p.371) and "if the presence of an epistemological obstacle in a student is linked with a conviction of some kind then overcoming this obstacle does not consist in replacing this conviction by an opposite one...It rather means the student will have to rise above his convictions, to analyse from outside the means he had used to solve problems in order to formulate the hypotheses he had admitted tacitly so far, and become aware of rival hypotheses." (p.374). While I concede that one does not simply want to replace convictions with those which are opposite, I believe that in putting students in situations where the appropriate and required ideas are contrary to his convictions we enable the necessary conflict and re-evaluation.

Returning to her agenda of epistemological obstacles related to limits Sierpinska, in _On The Relativity Of Errors_ [Sierpinska:1987b], again has much to say on the notion of obstacles, specifically of those related to infinity within a potential context. Here she defines both the nature and role of epistemological obstacles; "the least one can say is that he (Bachelard) considered epistemological obstacles as a negative phenomena in the development of science... today we think that epistemological obstacles are inevitable and helpful....epistemological obstacles are not a kind of vitiation, of error that impedes the development of knowledge but are rather the very condition of this development." (p.1). Epistemological obstacles are said to be concerned with the fundamental nature and meaning of scientific knowledge. The resolution of the problems that surround the attributes of knowledge result by way of the construction of a philosophy (structure) wherein these obstacles are cognitvely justified. In this manner, to overcome an obstacle one has to be aware of his structure, question its scope of applicability, and be prepared to make the alterations needed for either extension or intensification. "It is like living in a closed world. To overcome an obstacle one has to go beyond this world, to see other possible worlds and relations between them." (p.3)

Sierpinska further states that in overcoming an obstacle one has to ascribe meaning; that is, one has to distinguish within their structure between physical reality, ideal reality, and mathematical reality, only then can one appreciate the respective epistemological and ontological considerations. "From one such 'world' to another, words and symbols change meaning; these meanings have to be explicated and compared. Things important in one 'world' become secondary in another; points of view have to be exchanged and negotiated." (p.3) She suggests how students' obstacles may be challenged if the relation between mathematics and reality is questioned ; "one should from time to time ask one self such fundamental questions as 'what is truth', 'what is a theory', 'what is the relation of theory to the reality', 'what is reality' and so on" [Sierpinska:1987a, p.396].

An Historical Perspective

The necessity of addressing the historical perspective is brought out in Robertson's article, *Another Challenge in the Classroom* [Robertson: 1973]. The claim here is that unless we attempt to make mathematics 'the human endeavor' through the exposition of its 'heritage' we will allow students to maintain "the increasingly prevalent judgement that mathematics is a rigid, mechanical, nonhumanistic discipline" (p.49). That is, students, when faced with the logical presentation of the textbook, are left to feel that this represents the development of the idea at hand. This assumption affects their understanding in that they are lead to believe that their individual development must follow the same path. If we can show them that their problems were also those of some very great mathematicians perhaps they would not feel so disconnected.

The example Robertson chooses to demonstrate the link between historical and individual development is the mathematical idea of infinity. He sums up this history by presenting the paradoxes of Zeno, the objections of Gauss and Poincare, the suggested resolution of Cantor, the praises of Hilbert, and the challenge brought on by the Banach-Tarski Paradox. The inclusion of the Banach-Tarski is, I feel, most important; it says to students that while one conception

(Cantor's) may provide the needed contextual definition and meaning it likewise creates other problems within mathematics itself. In this way students may come to see that mathematics is dynamic, changing and relative as opposed to its being a static, finished and absolute structure in which they have no active role.

In reading this article I was further struck with the idea that should we chose not to respect the parallel between an individual's understanding and an object's mathematical development we lead students to believe that there is no room for the 'why' questions which are needed for both the construction/enhancement of their mathematical structure and the confrontation of their obstacles. These questions demand an historical perspective not a lesson in logic. This feeling is here summed by Dantzig "the systematic exposition of a textbook in mathematics is based on logical continuity and not historical sequence...and therefore leaves the student under the impression that the historical evolution...proceeded in the order in which it occurs in the text" (p.49).

This article also stresses the need to emphasize the relation between mathematics and natural reality. "The general impression of the layman that mathematics is directly and flawlessly tied to the physical reality needs to be replaced by the fact that mathematics is a human guess about the real world." (p.53) While I agree that mathematics allows us to hypothesize about nature, I think that it should have been made more clear that this 'guess' is but a function of mathematics and does not define or describe mathematics in and of itself. When we speak of the function of mathematics in describing the natural object we are bound to the ontology and epistemology of the object under consideration; whilst, when we consider mathematics, itself, we are bound by the ontological and epistemological demands of rigor. I am not here trying to make a case for strict formalism; rather I believe that some sort of formal structure is required to make mathematical sense of the conceptual/intuitive idea.

Related to the above I think that when considering the ideas of Gauss, Cantor, and others it should also be mentioned why these mathematicians held these views: that is, the philosophical assumptions which motivated, or at least affected, these claims should also be explicated. Since these same philosophical ideas greatly determine students' beliefs regarding the nature and function of both mathematical and natural reality they are thus needed for them to question their epistemological and ontological assumptions. Robertson seems to suggest this, but does not explain his reasons: "Maybe mathematics history, philosophy and foundations courses need to be emphasized and appropriate mathematics evolution material incorporated into content courses." (p. 53).

Tall's closing remarks, in *The Notion of Infinite Measuring Number and Its Relevance in the Intuition of Infinity* [Tall:1980] further attest to the similarity between historical and individual development: "This illustrates a common occurrence in the development of mathematical ideas, both in the history of mathematics and the development of the individual. Working in a given context certain 'facts' arise which hold in the given context but break down when the theory is broadened" (p.282). Thus, it is not that we must demand that students abandon their intuitions of infinity only that they see that the potentiality that they ascribe to its essence is but a characteristic determined by the particular context.

Sierpinska's experimental method also assumes this similarity [Sierpinska:1987a]. She seeks to overcome some of students' obstacles related to scientific knowledge, infinity and real numbers by using an historical perspective of the context of infinite series. "My hope was based on the historical fact that the development of calculus was tightly linked with that of infinite series and with the debate about the paradoxical behavior of divergent series." (p.374) The methodological value of an historical perspective, is further advocated in Waldegg's, *The Conceptual Evolution Of Actual Mathematical Infinity* [Waldegg: 1991], where he attempts "to show the existence of well-differentiated stages in the historical development of infinity, in order to relate them to the cognitive development of the individual" (p.211).

Sources of obstacles and acts of understanding

a) Mathematical/Structural biases

Fischbein, Tirosh and Hess make explicit the dichotomy between intuitive infinity and mathematical infinity in, *The Intuition of Infinity* [Fischbein, Tirosh and Hess: 1979]. "It (in a psychological sense) is, of course, a pure construct; no direct experience may be invoked in order to support it. It is not even a hypothesis. No conceivable test is able to support or reject infinity. On the other hand, we have to agree that infinity is a meaningful, mathematical (ideal) concept; if we are able to prove-such as Cantor did-that infinity is a non-contradictory concept, consistent with the totality of the other mathematical concepts, we may accept its mathematical reality." (p.3) This concise statement does much in identifying students' obstacles towards infinity and in defining the requirements for its understanding. It makes a case for the separation of natural reality and mathematical reality, it marks a line between the epistemologies respective to each structured reality and it recognizes the importance of the role of Cantor's ideas in establishing an ontological basis for infinity.

As further noted by these researchers, one of the problems underlying students' obstacles lies both in the recognition and grasp of the intuitively 'contradictory' nature of infinity. More often than not students seek resolution to this dichotomy, in the way of Aristotle, by granting infinity only a potential existence. Thus the problem arises that while this view appears to satisfy one's intuition it is not always mathematically appropriate. This article then, seeks to understand the effects that these intuitions have in three contexts; one that requires potential infinity (infinite divisibility) and two requiring actual infinity (geometrical comparison of cardinalities). It further considers the affects that age (grade 5-9), mathematical knowledge and general school achievement have on the reliance upon intuition.

The results are indeed interesting. Students' responses are most appropriate at age 12 (Piaget's formal operational stage) and afterwards begin to both decline and lose consistency. Both instructional level and mathematical knowledge appear to inhibit, in some instances, students' success in actual contexts. That is, instruction appears to develop a concept of infinity (whose truth is based on logical consistency) without affecting students' intuitions, and mathematical knowledge seems to restrict students to the "use of formal logical schemes which are naturally adapted to finite objects.... When a conflictual situation is generated the finitist interpretation tends to prevail even in well trained students" (p.38).

What this says to me is that if students are faced with a conflicting situation (the demand for actual infinity in equality of cardinality), where no learned 'rule' appears to hold, they will either call into play another rule (the whole is bigger than the part), which deals with finite objects, or failing all else return to their, physically based, intuitions. This explains the relative success in the infinite divisibility problems; the process is potential and they are aware of the rule that a point is infinitesimal. It also explains why students do much better on the algebraic situations than the geometric [Waldegg:1991]. In the former, the point becomes a finite number and the rule is simply finding a function, i.e., one need not consider the actuality of infinity. In the latter, geometric context, the 'concreteness' of the representation places students in a situation were the view of a line as a *set* is replaced with the view of a line as being finitely measurable and thus open to that rule dealing with the relation between the whole and its parts.

My response to this is that we have to get students to question their intuitions of infinity by providing situations of conflict where no view, other than that which includes a belief in the mathematical existence of infinity, is required. Those situations where the concept of a set is not explicit (geometrical) and where the idea of an infinite number of elements is replaced by an equation (algebraic) are simply not suitable.

Tirosh, Fischbein and Dor, continue the investigation of students' intuitions of infinity in, *The Teaching of Infinity* [Tirosh, Fischbein and Dor: 1985]. The assumption motivating this research is that "primary intuitive attitudes" (p.501) will inevitably inhibit the solutions of problems dealing with the equality of cardinality among infinite sets. In this way their objective are twofold; 1) To identify the inner conflicts in the intuitive understanding of the various aspects of the notion of actual infinity, and 2) To try to improve high school students' intuitive understanding of the notion of actual infinity through the systematic instruction of Set Theoretic notions. The procedure used, to achieve these objectives, was the 'conflict teaching approach' which would "cause a state of inner disequilibrium which is the optimum time for creating new modified concepts" (p.501).

The results of the pre-test indicated that two beliefs do most to inhibit understanding: 1) the belief that the whole is bigger that its parts and 2) the belief that only one kind of infinity exists. These beliefs can, and were, positively affected by the instruction, "only 10% still used primary intuitive attitudes.... 20% were still in a state of disequilibrium... 70% of students were made aware of their own intuitive conflicts" (p.506).

Although I am encouraged by these positive results I cannot help but wonder how it became known that the successful students were, in fact, made aware of their intuitive conflicts. Though it is mentioned that each student was individually interviewed, without explicit information on the interview questions and results, I can assume here that the students, like Agnes in the Sierpinska [Sierpinska:1989] article, simply 'got use to' the 1-1 correspondence as a rule without understanding its consequences. Thus when it is claimed that "An unanticipated achievement of the instruction was that students' awareness of the inner conflicts in their intuitive ways of thinking produced in them a much deeper understanding of the need and importance of formal mathematical proofs in contrast with their biased primary intuitive evaluations" (p.506). I question whether these authors see the result of students 'awareness of inner conflicts' as being sufficient for 'seeing the need' for formalization. Personally, I feel that while a formal grasp of the meaning is necessary for understanding mathematical infinity it is not sufficient, that is, students must also link this with the mathematical sense of infinity; else, it may just be the case that they simply 'got used to' the formal methods.

In, [Sierpinska:1987a], we find evidence that students' attitudes towards mathematics in general does much to affect their understanding of infinity. That is, students must have an appropriate mathematical structure wherein mathematical objects and knowledge are placed outside those considerations of 'natural' reality. The generalization and identification of students' attitudes is based on their reactions and discussions of the result that .999...... =1. One preliminary point to be made here is that the students were "not discussing the mathematical validity of the result but its truth value." (p.379). In other words, students were not judging their knowledge within the constraints of a mathematical reality: "Arithmetically or algebraically, its all right, but in reality...". (p.378) Regarding this, Sierpinska identifies the following attitudes towards mathematical knowledge that were found in these students: a) intuitive empiricist (Aristotle)- mathematics is an empirical science, i.e.. that axioms of a mathematical theory should be indisputable facts, intuitively acceptable, or conforming to results of scientific research, b) discursive formalist (Russell)- mathematics is a formal game of symbols, devoid of any meaning, and c) discursive empiricist (Lakatos)- mathematics is a hypothetico-deductive science but its development rests in problems, bold hypotheses, the verification of these and applications in the real world and in mathematics itself (p. 382).

Continuing in her analysis of students' discussions she also identifies their attitudes towards infinity: a) intuitive definitist- all sequences are finite, b) discursive definitist- all bounded sequences are finite, c) intuitive indefinitist- all sequences are finite but sometimes it is impossible to determine the number of terms, and d) discursive indefinitist- all bounded sequences are finite but sometimes it is impossible to determine the number of terms. It is important to note a particular consequence of these attitudes: "the continuity of the sets of real numbers is rejected; R is finite or at most countable" (p.384-385).

Sierpinska's conclusion that attitudes towards mathematical knowledge are at the root of the epistemological obstacles regarding limits, and infinity is indeed important. What I find troublesome, however, is the attributation of non-mathematical attitudes to the fact that the subjects were humanities students; "proving things using mathematical tools was a sensible activity for mathematics students, while it was something perfectly futile for (the group)." (p.396). Although she does not mention the mathematical experience of these students, other than stating that it is less significant than that of the mathematics and physics students, it appeared that they were quite knowledgeable; e.g., they spoke of asymptotes of hyperboles, exponentials, epsilon differences, boundedness...etc.

Even if we accept that the humanities students were not as experienced, how can we then conclude that these attitudes are not also held by mathematics students. Perhaps mathematics students more easily accept how we can prove that .999..... =1 but this does not mean that they understand why the proof holds or even accept the value of proof in general. Beyond this, however, I think that these students' attitudes were not so much influenced by mathematics in general but were mostly affected by the idea of infinity. The situation of .999....=1 presents more a representational problem than it does an epistemological one; students see .999..... as a 'dynamic' process while 1 is seen as a 'static' number.

Furthermore, these beliefs are more affected by their view of the spatio-temporal, therefore potential, components of infinite processes than they are affected by their assumptions regarding the role of proof or the status of knowledge in mathematics. Thus while I agree that students' attitudes towards mathematical knowledge affect their understanding of limits; their ontological assumptions regarding the potential characteristic of infinity does more to affect this understanding by influencing the students' epistemological attitudes. In this manner, another aspect of the research that sets it apart from my aim is its restriction, within the context of limits, to only potential infinity.

In *How and When Attitudes Towards Mathematics And Infinity Become Constituted into Obstacles in Students ?* [Sierpinska: 1989] Sierpinska brings out one very interesting point; that, in understanding mathematical infinity, not only must students consider its epistemological status but also its ontological status. That is, their mathematical structure must be such that they be able distinguish between the nature of the formal object, of the representational object, and of the procedural object. With regards to infinity, for each of these epistemological considerations there is a corresponding ontological claim; the object is, respectively, formally actual, ideally actual, or ideally potential. Thus we must not only consider students' cognitive and epistemological obstacles, but we must also be aware of the ontological issues that surround mathematical infinity.

b) Age and/or Developmental Level

We may note changes in the scope of Sierpinska's investigation; here she includes two students' attitudes towards, specifically, mathematics and infinity within the context of cardinality [Sierpinska: 1989]. Mentioned here is the necessity of separating out potential, intuitively based, infinity in order to accept the required actual infinity, "one has to be able to reason against one's intuitions, discursively and formally, and to accept it." (p.167). It is assumed, by the directives of Piaget and Inhelder, that because of the subjects age (10 & 14) and corresponding developmental stage that they will not understand this notion.

To question this assumption, this research looks at students' reaction to the existence of a 1-1 correspondence between sets as a criterion for them having 'as many' elements. The choice of this context appears to be based on the belief that in viewing actual infinity students will have a basis for distinguishing mathematical from non-mathematical. "Infinity has a rich meaning outside mathematics: it is part of our culture, of beliefs concerning the structure of matter, size of the Universe, time... Now one cannot accept this notion be reduced to the 1-1 correspondence criterion without coming to think, maybe, mathematics is not a discipline describing some kind of reality." (p.167)

To test her hypotheses students were provided with the definition that two sets have as many elements if their elements can be paired off, that is, if every element from the first has a pair in the second and every element in the second has a pair in the first. According to this definition students were taken through six steps: Step1. To determine the 'equality' among two finite sets of green and yellow 'counters', Steps2-6. to determine the 'equality' among drawn line segments, Step7. To determine whether the sets of natural and even numbers are 'equal', as a summary students were then asked what they thought infinity was and how they imagine it to be.

Interestingly enough, the younger student, Agnes, had no problems in Steps2-6, while the older, Martha, agreed on the equality in Step5 only. Both students attempted to formulate an answer and method of proof, Martha was held back by the "impossibility of actually performing the procedure" (p.169). When contradictions occur this student, instead of altering her view, rejects that 1-1 correspondence establishes 'as many'. While Martha is aware that a line is a mental construct composed of an infinity of infinitesimal and mental points this is not a stable belief; the degree of concreteness or abstractness varies according to each situation. This duality was not seen in Agnes, although she seems to shift from a 'physical' point to one which is 'unimaginably' tiny, she maintained a stable concrete view.

One must question here; If it is not the concreteness that is holding either student back then what is the source of Martha's problems? A possible explanation is Martha's operational attitude towards mathematics which "may be characteristic of the transitional period between concrete and formal operational stages" (p.171). For Agnes mathematics is a game and the 1-1 correspondence criterion is but a rule in this game, for Martha this criterion is an act to be performed. Thus it would appear that it is not so much the concrete view that hinders the acceptance of equality of cardinality, but rather it is that the criterion must be considered formally and not actually. What it is concluded is that students' attitudes towards mathematical theory may do more to affect their acceptance of infinity than does their level of maturation.

In accepting this result I can't help but feel that their lack of mathematical experience and the knowledge required to understand equality of cardinality coupled with the almost extensive use of drawn segments restricts the subjects to a supposedly physical situation where infinity must retain its potential characteristic. Perhaps it is this idea, of spatially potential infinity, that is pushing Martha to consider whether this pairing of infinite elements can actually be done. Furthermore, while ability to abstract processes may be directly related to their attitudes towards mathematics, the belief that infinity is essentially potential may not be.

c) Representational and /or contextual bias

Tall claims that students' difficulties with cardinal infinities arise from the fact that numbers may be used for counting as well as measuring [Tall·1980], not from inherent conceptual biases or inappropriate intuitions [Fischbein, Tirosh and Hess: 1979]. That is, when students are presented with geometrical representation of the equality of cardinality between a set an a proper subset they rely more on the measuring aspect than the counting. Instead of "challenging this view, and re-educating" (p.272) the student, Tall suggests that one demonstrate this mislaid belief by use of infinitesimals within the hyperreal number field used in non-standard analysis. The reasoning being that "Such logical sleight of hand (one-to-one correspondence) does not always satisfy the cognitive psychological requirements of the learner". (p.272)

To this I respond that the introduction of a hyperreal field, although it may make sense students' intuitions, does not address the fact that students are using inappropriate intuitions to grasp cardinalities and/or existent infinity in general. Even if the above objection does not hold, one must again question whether a geometrical representation is even a suitable context for the introduction or explanation of cardinality. As previously noted, in such a situation students place more emphasis on the 'concreteness' of the lines than they do in viewing them as sets. It is the geometrical context which is pushing students into a measuring mode as opposed to a counting mode. Even so, this measuring view demands that the student see the point as mathematically infinitesimal.

The author recognizes this and claims that this can be assumed since a boy of eight sees the infinite divisibility of a line therefore he must be able to conceive of a point as infinitesimal. This same boy, however, when questioned as to whether this point will get too small to divide responds "No, not if you look through a microscope" (p.273). To me this signifies that the point is not abstracted; it is physically infinitesimal not mathematically infinitesimal. (Interestingly enough I found, in my pilot study, that even students at an undergraduate level maintain this 'atomistic' view of a point).

The conclusion for Tall is that if students accept the infinite nature of a point then their intuitions make sense within some context thus "By this process we can at least realize the *relative* nature of our interpretation" (p.282). So what are we to do? Explain to the student that in viewing lines or planes as composed infinitesimal points we may say, in a hyperreal sense, there may be an inequality and in seeing them as sets they may be equal. Just because we can mathematically make sense of students' intuitions does not mean that they can be made any more mathematically appropriate for the context at hand; it only says that the context of geometrical representations allows for confusion between counting and measuring.

Tall continues his investigation into the value of students' intuitions, in _Intuitions of Infinity_ [Tall:1981]. Here it is again claimed that although students do not have a mathematical idea of cardinal their intuitions, based on their experience with asymptotes, repeating decimals and limits, are in someway well founded and sensible. The particular questions which he seeks to answer are; "Why do students rely on these intuitive beliefs? and, In what sense is the mathematical definition of infinity better?" (p.30). As regards the latter question Tall restates his previous claim; that is, he brings out the fact that it is the interpretation of the context which determines the appropriateness of students' intuitions. As a result of this relativity one cannot claim that the mathematical view of cardinal infinity is better. "The formal mathematical definition is perfectly alright in a context where number means a comparison of the size of sets and a cardinal number gives a theoretical extension to the counting concept. However, in other contexts, such as limiting processes, the cardinal concept is singularly inappropriate to explain intuitions of infinity which arise." (p.31) Tall then goes on to construct yet another context, that of the

'Superrational Numbers', wherein, he claims, students intuitions make sense.

With respect to his initial question, regarding the sources of students' intuitions, Tall has some new ideas. Here he conjectures that these intuitions begin with the belief in the *potentiality* of N, in that the process of counting never ends. Furthermore, this intuitive base is reinforced by "the dynamic way in which limits are expressed, i.e., 'f(x) tends to L' as 'x tends to a' leads to a cognitive belief that limits are approached but not actually reached." (p.33). This hypothesis strengthens my belief that not only are we not doing students any favors by giving them informal definitions but when these actions allow them to extend intuitive beliefs to mathematical ideas we simply fail them. It does not matter if these intuitions are appropriate in some context, what is at issue is that they hinder the understanding of those contexts in which the idea of an existent infinity is required.

One of the problems I have with this article is found at the beginning where Tall claims that "infinity is an extrapolation of our finite experience" (p.30). This statement, itself, is fraught with many of those same ideas which were discussed above and which serve to motivate many of students' obstacles. The first fault lies in the absence of what is meant by experience and its role in mathematics; must we experience any mathematical object to know it, or its negation? Again, mathematics, through the requisite of rigor, effected a separation of itself from the empirical demands of natural sciences and has thus a reality which is of itself; unless this experience is to occur within this mathematical reality I don't see its relevance. Another assumption which I find hard to accept is that our experience needs be finite, perhaps we can only experience potential infinity within the constraints of space-time but mathematics is outside such considerations: Can't we mathematically experience infinity?, Isn't that what we do even in the contexts of asymptotes and limits?. Students *are* extrapolating their mathematical experience of the infinite but, unfortunately, are doing so on the basis of its potentiality and without regard for context.

Thus while 'ordinary', informal, experience may be such that it only gets one to an intuitive sense of potential infinity, it must be recognized that a mathematical meaning can be had provided one begins with a more formal experience of actual infinity. Thus, if Tall is speaking merely of students' intuitions of infinity, mathematical or other, then these statements are more comprehensible; but, if he is considering the requisites for understanding mathematical infinity, I do not think that 'extrapolating from finite experience' will provide an appropriate basis.

One statement which I found to be interesting was that students, in considering infinity within potential contexts, "often learn not to 'understand' these arguments but they do 'get used to' them" (p.30). My question here is how are students to understand these mathematical arguments when all of their mathematical experience of infinity has occurred within potential contexts; they get used to accepting infinity as a mathematically empty symbol, whose essence is potential, and as a result they cannot ascribe meaning to the related arguments. Tall, however, seems to advance the belief that this 'getting used to' together with experience can permit the student to move ahead. "In a cognitive sense more students at university are no longer aware of the intuitive notion of potential infinity, they believe in the actual infinity of the set N." (p.33) Not only do I disagree that all university students were ever aware of having an intuitive sense of infinity but I have found (in a pilot study) that some do not believe in or accept actual infinity, though they may indeed use it.

The importance of context is brought out in Love's, Infinity: The Twilight Zone of Mathematics [Love:1989], which focuses on using Cantor's 'Cardinalities of Sets' as auxiliary material for 'interested' students. "Unless some ideas involving mathematical infinities are included in their instruction, students will never develop a true understanding of the various number systems introduced in their algebra classes nor will they understand how these sets relate to points, lines, and planes introduced in their geometry classes." (p.284) The problem here is that Love does not explain why or how this acceptance of actual infinity is necessary for understanding infinity itself. It is not that understanding cardinalities will allow us to then understand its uses, i.e., in number systems and the like, but that it will aid in the understanding of infinity itself; only then we can turn to its contextual characterizations and/or uses.

It is only by examining a multitude of situations that we can appreciate the nature of infinity as a mathematical object as opposed to considering a specific contextual characteristics as defining this nature. To understand we must distinguish its epistemological and ontological imports for the purpose of then uniting them to grasp its nature. Thus while I agree with Love's claim that ideas of cardinal infinities will improve students' understanding of infinity, I believe, this is due more to it being seen as actual, mathematically meaningful, than that it has yet another particular use. Furthermore, if we claim that this view of infinity is necessary for students to understand mathematical infinity, I do not believe that this particular application should be restricted to 'interested' students.

Waldegg, in [Waldegg:1991], is unique in his claim that students' 'response schemes' are similar to those given by mathematicians up to and including the time of Bolzano. This assumption is the first I've found which assumes that those same ideas which differentiate Bolzano from Cantor can be used to confront the problems of accepting actual infinity.

It is here said that to reach the 'Bolzano' stage one must be able to "conceive infinity as an attribute of a collection and not as a noun or adverb and give infinity a sort of 'world' of its own" (p.213). Personally, I find this description somewhat contradictory; if infinity is to be merely an attribute of a set then the created world would belong to the set and not its attribute. If infinity is just a descriptor what is to distinguish an infinite line from an infinite set? The author's other 'Bolzano' directive is more agreeable, that is, students must see "the need for the (mathematical) definition of the term infinity and leave metaphysics behind, place himself in the mathematical realm so that its existence is based mainly upon its non-contradictory nature" (p.214). Furthermore, in order to do this successfully one must be able "to conceive of a set as a whole, without any need to think separately of each element" (p.215). Overall students in this stage must see that infinity attains "a permanent position as an object of study with its own operativity" (p.215). Respecting these ideas and directives I will, in my Teaching Experiments, introduce what I term 'the arithmetic of infinity'.

Moving beyond the 'Bolzano' stage to the Cantorian acceptance of actual infinity is said to require: a) a comparison criterion based on external relationships, as opposed to Bolzano's which was based on an inclusion relationship, b) the verification system must be rigorously defined, as opposed to Bolzano's empirical system based on geometrical properties, and, c) conservation (in the Piagetian sense) or recognition of the invariance of cardinality, as opposed to Bolzano's belief that a transformation will affect the number of elements in the set of points in a line.

Readdressing his initial hypothesis, regarding historical and individual development, Waldegg identifies some of students' problems in accepting actual infinity. The first to be discussed is the conflict of having to choose between criterion of comparison; students place more emphasis on the intra-objectal relation of the whole to its part, as opposed to the inter-objectal relation of a bijection between the two sets. Another source of students' obstacles is their 'constructive' conception of a set as a sequence which "implicitly carries with it the inability to complete the process" (p.218): the set must be conceived as a syncratic object. Waldegg seeks to generate these conflicts in three questionnaires dealing, respectively, with numerical sets, geometrical sets and algebraic sets. To determine their stage he classified responses according to three aspects; The sets are infinite, B is a proper subset of A (Bolzano), and, the two sets are equipotent (Cantor).

In both the numerical and geometrical situations students relied more on the subset relation (38% & 56% respectively). In the algebraic contexts, however, 75% of students justified their responses by the bijection relation. Regarding these results Waldegg concludes that students' responses, in the geometrical contexts, are "anchored to the *perception* of the geometrical representation...which seems to prevent access to higher levels of conceptualization" (p.225) while in algebraic contexts, although they "seemed to facilitate the access to the inter-objectal stage by allowing the existence of an operativity independent from meaning....this level does not solve the paradoxes, it only hides them" (p.226).

Summary

Research has thus shown that, within those contexts where infinity is characterized as potential and/or geometrical, students' inappropriate views of mathematical infinity remain unchallenged; though they may be identified. Students' reliance on an intuitive sense of infinity, which however adequate for potential and/or measuring contexts, cannot be taken as mathematically appropriate since it cannot provide a sufficient basis for accepting infinity as an existent mathematical object. Recalling our assumption of the link between historical development and individual understanding we should not be surprised that, as determined by Cantor, without a rigorous formalization of infinity it can not be accepted into a mathematical structure.

Thus, while investigations in potential and/or geometrical contexts can provide a situation for the identification of students' inappropriate ideas these contexts cannot move them forward: these instances can serve to mathematize students' intuitions of infinity but they cannot make them mathematically formal. It appears, then, that while there is no correct view of mathematics, there is an appropriate view; and, furthermore, this view must represent a middle ground between a purely physical or purely ideal conception, it must establish and justify both existence and meaning. The problem with infinity, however, is that the same term has opposing characteristics even within the mathematical structure; one which is more related to the physical realm (in being potential) and one more linked to the ideal realm (in being actual). What we require then is a context were infinity is shown to be both mathematically existent and mathematically meaningful; one in which the inappropriate views of students will conflict with the mathematical requirements, both epistemological and ontological. Only then will students see the need to question their structure, mathematical or other.

CHAPTER 2.
The Theoretical Framework

This chapter seeks to present a systematization of those assumptions which will underly my investigation, identification and analysis of students' obstacles and successes. Thus, my aim here is the creation of a 'model of understanding' which will consider what is necessary, though not sufficient, for the claim that students understand mathematical infinity. This model will also account for the obstacles (epistemological, ontological and cognitive) which prevent, or inhibit, understanding. The justification of this model will be based on 1) the historical development of infinity, 2) several philosophical and/or pedagogical theories of understanding and 3) the belief in a relation of similarity between historical and individual development.

§2.1 An Historical Account of Mathematical Infinity

The overall aim of this section is to make explicit those ideas which have served to influence, positively or negatively, the historical development of mathematical infinity. These philosophical and mathematical ideas will further be seen as being those which at present, and in like manner, hinder and/or enable the individual's acceptance of a mathematical object called infinity. Those ideas which have lead to a positive determination of infinity will be taken as *acts of understanding*; those which have had a negative effect will be seen as *obstacles to understanding.*

Acts of understanding are characterized by:

Assent to consider an object or idea; to see the object as that which is worthy of inquiry or explanation.

Discrimination between the mathematical object and one's accepted belief system.

Appreciation of sense, meaning and context via generalization and synthesis.

Communication of the reciprocity between the objects sense and meaning.

[c.f. Sierpinska: 1990]

Obstacles to Understanding are characterized as

Cognitive: (Piaget, Brousseau, Herscovics) that which is related to the psychological/ conceptual development of the individual (and the idea). More specifically these obstacles are associated with the process of assimilation-the integration of things to be know into some existing cognitive (philosophical/mathematical) structure and accommodation-changes in the cognitive (philosophical, mathematical) structure necessitated by the acquisition of new ideas. [Herscovics: 1989]

Epistemological: (Bachelard, Sierpinska, Herscovics) that which is related to the development of scientific knowledge; specifically, the

a). the tendency to rely on deceptive intuitive experiences.

b). the tendency to generalize which may hide the particularity.

c). the obstacles caused by natural language.

In assuming a relation between the historical and individual development/acceptance of mathematical knowledge similarities can likewise be drawn among their respective successes and obstacles. "The historical study of any discipline or specific concept always uncovers epistemological obstacles that had to be overcome for *any* growth." [Herscovics: 1989, pg. 82] I will part ways with Herscovics, however, in that I will not make the distinction to be one of individual/cognitive and historical/epistemological. Rather, I shall assume the relation to be structural/cognitive and knowledge/epistemological; that is, those obstacles which relate to questions of structure I shall call *cognitive* and those which relate to questions of knowledge I shall call *epistemological.*

The above has two consequences: it enlarges the scope of cognitive obstacles to include those assimilations (intensification of the structure) and accomodations (extensification of the structure) which are philosophical/mathematical as well as individual; it also, expands the scope of epistemological obstacles in that all structural convictions, conflicts and/or changes that come by way of questions related to the nature and status of knowledge, mathematical or other, are termed epistemological. However, the specification of epistemological, to questions related to knowledge, means that we also must give definition to those structural questions which consider existence; these I call *ontological.*

Ontological: those structural questions which are related to the nature or status of the object, mathematical or other.

In light of the above we must now consider what we mean by the characterization obstacle.

Obstacle: (Bachelard, Sierpinska, Byers & Erlwanger, Herscovics) a point of mental stagnation, tension or confusion which enhances curiosity, frustration, or both.

* Note: All obstacles will be abbreviated, hereafter, as: cognitive obstacle **(CO)**, epistemological obstacle **(EO)** and ontological obstacle **(OO)** and act of understanding as **(U)**.

It should be recalled here that the idea of infinity has meaning within three distinct, yet dynamically interrelated, realms; the physical, the philosophical (metaphysical and/or ideal) and the mathematical. Because it is my belief that most students' problems in understanding begin with a lack of distinction between these three conceptions of infinity, I feel it necessary to examine the sources of these intuitions. Perhaps we can then determine the reasons they are maintained. Thus, while this investigation concerns the understanding of mathematical infinity the influence and impact of other notions of infinity must also be recognized and analyzed.

The Greek philosophers acknowledged the idea of infinity (i.e., apeiron), the problem was that this idea was not accepted into the structure of either philosophy or mathematics. That is, because "Apeiron was a negative, even pejorative, word. The original chaos out of which the world was formed was apeiron." [Rucker: 1982, p.2] and since the aim of both philosophers and mathematicians was to put order into chaos of 'natural' reality, the idea of infinity was given up in name of systematization **(EO)**. "There was no place for the apeiron in the universe of Pythagoras and Plato. Pythagoras believed that any given aspect of the world could be represented by a finite arrangement of natural numbers. Plato believed that even his ultimate form, the Good, must be finite and definite." [Rucker:1982, p.3]

Aristotle, in acknowledging the seeming endless and infinite divisibility of both space and time, could not so readily dismiss the apeiron. The problem that faced him was how to accept this idea without having to dismantle the system which, itself, denied infinity any 'real' status. The solution was thus to be found in an ontological separation between actual existence and potential existence; wherein the former relates to what is real and the latter to what is possible, i.e., would be actual if not for the constraints of space and time **(OO)**. "In order to avoid these actual infinities that seemed to threaten the orderliness of his a-priori system, Aristotle invented the notion of the potentially infinite as opposed to the actually infinite." [Rucker:1982, p.3] The problem that occurs here is the extension of this potentiality to the very essence of infinity **(EO)**; in considering only the contexts of mathematical process (e.g., Zeno's paradoxes), and in assuming that all processes are subjected to space-time constraints **(CO)**, the epistemological and ontological considerations of a mathematical idea are subjugated to those of the physical (EO). "Actual infinity is non-existent to Aristotle because the definition of infinity says that infinite is that which, regarded as a quantity, can always be increased.....This second kind of infinity (the mathematical) never becomes actual infinity either, because mathematically the process of bisection continues forever." [Sinnige:1968, p.150]

One philosopher who appears not to have combined mathematical existence within a physical domain is Anaxagorus. "It was Anaxagorus who occupied himself with the problem of formulating a more correct and consistent concept of infinity." [Sinnige:1968, p.129] This philosopher spoke of infinity in and of itself not as a property of reality (i.e., Anaximander's apeiron) or of process (i.e., Zeno and Aristotle). Two fundamental ideas allowed him to consider infinity in the manner; the choice of context **(U)**, of an infinite set, and the distinction between counting numbers, which define the relations to finite orderings, and multiples, which define the relations between infinite orderings **(U)**. In light of this I feel that Anaxagorus is truly the forerunner of Cantor; he says, "The sum total of the elements of an infinite set is not a bit smaller nor greater, for it is not practical that there should be more than all, but the sum total is always equal to itself.... there are just as many parts in the great as the small taken as a multiple.... Nor is there a smallest part even of a small quantity, but there is always a smaller one. This quantity is equal to the smaller quantity in multiple (power in Cantor's sense)... The theorem implies the use of the concept of number in two different meanings. In the case of a finite quantity number is always a counting number, in the case of an infinite quantity the concept should rather be denoted by a term such as a multiple." [Sinnige:1968, p.132]

How nice it would have been if the development of infinity was based on Anaxagorus' investigation into the nature of infinity as an existing object rather than on Aristotle's claim regarding the characterization of it as a potential object. As it turns out the latter's distinction was to set the stage, that of both metaphysics and mathematics. What is required to move from this mind set to that which is mathematically appropriate is the consideration of the ontological relation between mathematical reality and natural reality **(U)**. One must recognize that "what passes for actual existence in mathematical discourse is merely possible existence in ontological discourse... The actual object of mathematical discourse is merely possible existence in ontological discourse." [Benardete:1964, p.29]. The first step in this process begins with the question; If infinity is outside reality then where are we to look to guarantee or rationalize its existence?

The above question was not the direct concern of the Scholastic philosophers but they ultimately played a role in its resolution. In understanding this it must be noted that while the goal of the Greeks was to place order on reality, the goal of the Scholastics was to seek reason through faith, that is to extract the order which is established through God. One claim, resulting from the belief in the omnipotence of God, is that if God determines all, including mathematics, he must be outside of it's constraints. Assuming this Plotinus came to accept the infinite, as unlimited, into his structure. "Absolutely One (God) , it has never known measure and stands outside number, and so is under no limit either in regard to anything external or internal." [Rucker:1982, p.3] Thus we see how infinity is moved out of the considerations of the physical realm and again placed in the metaphysical; but this time it is with a difference in that it guarantees the existence of the needed, logically prior, actual infinity **(U)**, i.e., God's infinite power and providence permit it's metaphysical actualization.

The possibility of extending the existence of infinity into the mathematical was supposed by St. Augustine. Because God is outside of time he is not only infinite but is capable of infinite thoughts, and since mathematics is one of God's thoughts it also includes the infinite (note here that God's knowledge = existence). "Such as say that things infinite are past God's knowledge may just as well leap head long into the pit of impiety, and say that God knows not all numbers... What madman would say so? (St. Augustine)" [Rucker:1982, p.3] Again how nice it would have been if St. Augustine's mathematical ideas were maintained, together of course with the realization that mathematics, like God, is outside time. But the idea that was to dominate was that we, as all other 'real' objects, are bounded in space-time **(OO)** so we could not know the infinite even if it were actual **(EO)**. "The infinite is fully intelligible, but incomprehensible to finite human minds...". [Leclerc:1972, p.62]

What needs be realized here is that the criterion for reasonableness differs between mathematical reality and natural reality; that the former requires rigor while the latter requires evidence **(U)**. Nicolas Cusanus marks the first attempt to make infinity reasonable, both metaphysically and mathematically; he does this by replacing the negative view of infinity, as unlimited, unending and unbounded, by a positive view, as that which can be seen as a unity (set?) **(U)**. "The infinite is an absolute unity which is the integration of all opposites and diversities... indeed in mathematics we now have a singularly good way to the positive understanding of the concept of infinity.(Nicolas Cusanus)" [Leclerc:1972, p.74]

The philosophical treatment of infinity may be thus summarized: it was subjugated to a potential object by that Greek belief that order needs be finite; it was accepted as metaphysically actual, by the Scholastics, since order was God's dominion it may take on his characteristics. Thus the problem is no longer ontological, reality, mathematical or other, may include the infinite; but, How do we know this object?, that is, the problem now becomes essentially epistemological. Here, then, it needs be recognized that mathematics defines it own reality, be it within the providence of God or not, it is not subject to the same epistemological requirements as are objects within a spatio-temporal reality **(U)**. "Mathematics is completely free in its development and only bound by the self-evident consideration that its concepts must be both consistent in themselves and stand in an orderly relation fixed through definition to the previously formed concepts already present and tested. (Cantor)" [Hallett:1984, p.16]

While still unable to rationalize infinity's existence, mathematicians allowed it into their structure. During the seventeenth century it was given both symbolization (by Wallis in 1616) and necessity (by Leibniz and Newton in their development of the Calculus). "The actual infinite enters mathematics thematically for the first time ....under the auspices of applied, rather than pure, mathematics..."

[Benardete:1964, p.18] Owing to this strictly methodological usage, that is, because there existed no theoretical underpinning to guarantee its existence, infinity was then returned to its potential status **(EO)**; "it is expelled from mathematics, by Gauss and Cauchy, not on ontological grounds, not through any specific concern with nature and the world, but expressly in the name of rigor." [Benardete:1964, p.18]. Indeed it is paradoxical that the very epistemological tool (rigor) that is required to accept infinity is that which is used to deny it.

It should be noted here that much of the debate, both philosophical and mathematical, regarding the status of infinity was greatly influenced by the the epistemological claims of the British empiricists, especially Locke, Hume, Hobbes and Berkely. "Riemann's argument like Gauss' before him hinged on the inability of our sense to transcend the finite **(EO)**; we simply do not know what happens at infinity." [Maor:1987,p.126] Thus, because of the choice of context i.e., limits, for which infinity is characterized as potential (as would be any other finite number even if its existence were not already accepted) coupled with the belief that there was neither mathematical need nor proof for its existence; infinity is now rejected on epistemological/empirical rather than on ontological grounds.

The underlying problem is that the inconsistency, between mathematical epistemology and ontology, is seen as resulting from the nature of infinity and not as that which results by way of the chosen context **(OO)**. In viewing infinity within the limit context infinity loses its status as a mathematically existent object and becomes, as Gauss states, a way of characterizing unending processes: "The infinite is but a façon de parler; an abridged form for the statement that the limit exists which certain ratios may approach as closely as we desire..." [Benardete:1964, p.13] This attitude was further reinforced by Cauchy's limit definition in which "the expression 'infinity' could be entirely banished from the vocabulary of mathematics.." [Benardete:1964, p.13]. As a consequence, with Riemann's reformulation of the Calculus in terms of limits, it seemed that infinity was simply no longer necessary. (Interestingly, however, while Riemann was perhaps successful in doing away with the mathematical need for infinity, it was also he whom was able to conceive of a mathematical representation of it; as the point corresponding to the north pole on a Riemann surface.)

The above epistemological/empirical refutation of mathematical infinity could only be answered when it is realized that the empirical component is not requisite for mathematical ontology **(U)**. The empirical belief that we can only know the perceptive nature of the object may be viable within a system where objects are independent of us, but in considering mathematics as a system whose objects arise by way of our defining them, even if the logical/deductive relations we adhere to are themselves independent, it remains that we can know the objective nature of mathematical objects. The problem is that without a rigorous and consistent definition of infinity the mathematical knowledge of its existence is not given necessity and unfortunately, this means that at most it must remain as ontologically potential.

Bolzano, in his Paradoxes of the Infinite, appears to recognize that the only way we can accept, or know, mathematical infinity, as opposed to being restricted by its paradoxes, is by first establishing a definition **(U)**. This would grant mathematical/necessary existence and in doing so would further permit the separation of those considerations which are mathematical from those which are not **(U)**. "As is readily understood, however, it would be impossible to recognize the appearance of contradiction in these paradoxes for what it is, namely a mere appearance, unless we first of all became quite clear what precise notion we attached to the term infinite." [Bolzano: 1850, p.75]

Although the aim and context of Bolzano were both philosophically and mathematically justifiable his method was not. That is, while he selected a mathematically appropriate context, that of the cardinality of sets, to establish the objective existence of mathematical infinity he also chose a relation which was not exterior to his context, i.e., it depended on the relation between the set and its proper subset. The reason underlying this selection of criterion was that he, like many students, was unable to let go of the 'whole-part' relation **(CO)**; ".... the two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole." [Bolzano:1850, p.98].

It should further be noted why Bolzano rejected the criterion of 1-1 correspondence, students will probably hold the same argument; "for in this case not only do we who do the counting never arrive at a last term in A, but such a last term is prevented from existing at all by the very force of the definition of an infinite." [Bolzano: 1850, p.99]. Even though Bolzano himself clearly states that the infinite set must be considered as a whole he maintained that this whole was gotten through the sum (aggregate) of its parts. "An aggregate whose basic conception renders the arrangement of its members a matter of indifference...I shall call a set" [Bolzano: 1850, p.77] Thus, in viewing counting as a subjective process rather than an objective relation **(CO)** Bolzano's whole system fails. From this there is a lesson; we must recognize the need to make explicit the fact that mathematical relations, themselves, are outside space-time considerations **(U)**, else the same result may occur.

Cantor also sought to avoid the epistemological/empirical controversy by providing a rigorous definition that would enable one to distinguish those conceptions of infinity which are mathematical from those which are not **(U)**. In this manner there are said to exist three conceptions of infinity, which result from considering each of the possible ontological realities, the physical, the mathematical and the metaphysical (spiritual). His systemization is thus summarized by three principles (of infinity): a) any potential infinity presupposes a corresponding actual infinity, b) the transfinite is on par with the finite and mathematically is to be treated as far as possible like the finite and c) the Absolute infinite cannot be mathematically determined. In this manner we see how the rationalization for actual infinity is made; physical infinity exists potentially since the physical can be described mathematically and transfinite infinity exists actually because of the omnipotence and existence of the Absolute (God).

One may here claim that Cantor has not moved past the Scholastic conception of infinity and has side-swiped the mathematical issue by deferring to the dominion of God. The difference that distinguishes Cantor, and settles the issue, is that he relies on God only to *guarantee* the metaphysical existence of infinity wherein he successfully uses his 'Set Theory' to *prove* its mathematical existence **(U)**. Thus infinity is shown to be actual, not by epistemological or ontological considerations related to the physical or metaphysical, but is restricted to the mathematical requirement of definition and consistency gotten through an exterior and independent relation (1-1 correspondence) **(U)**. Hence, through the ontological separation of infinity Cantor successfully defeated the epistemological claims of both the sense bound empiricists and the spirit bound Scholastics. "In particular one is only obliged with the introduction of new numbers to give definitions of them through which they achieve such a definiteness and possibly such a relation to the older number that in a given case they can distinguish from one another, as soon as a number fulfills all these conditions, it can and must be considered in mathematics as existent and real (Cantor)." [Hallett:1984, p.17] With respect to the preceding we are now in a position to extract, from the historical development, those acts of understanding which served to enable the acceptance of mathematical infinity and likewise, are able to note those obstacles which hindered its acceptance. Of the former we have the realization that context determines the characterization of mathematical infinity; that infinity may be said to exist actually within a set context (Anaxagorus, Bolzano and Cantor). Understanding *actual* infinity further requires that we consider the cardinality of a set to be an external and objective relation (Anaxagorus, Cantor), as opposed to being an internal (Bolzano) or a subjective, counting, relation (Zeno, Aristotle).

The acceptance of infinity as *existent*, requires that we separate the ontological considerations of abstract reality from those of natural reality (Scholastics, Cantor), instead of taking existence to mean that which is within the natural reality of space-time (Aristotle, Empiricists) Similarly, the acceptance of infinity as a *knowable* object demands that we distinguish between the criterion for ideal, a priori, knowledge and that of empirical, a posteriori, knowledge. (Bolzano, Cantor), as opposed to believing that all knowledge is essentially a posteriori (Empiricists). Finally, the acceptance of infinity as a *mathematically existent* and *mathematically knowable* object requires that we see the abstract existence and ideal knowledge of mathematics as arising from itself through the demands of rigor and consistency, i.e., that "mathematics is free" (Cantor).

§ 2.2 A Model for Understanding

The following will represent an attempt to make explicit those assumptions which, I feel, distinguish and define mathematical understanding. Specifically, I will consider understanding's *nature*, its *requisite*, and its *results.* In accordance with the resulting structure, I shall then investigate what it means to understand mathematical infinity. This inquiry will thus culminate in a list of the individual attitudes and mathematical attributes that are required for such an understanding.

Assumptions

Understanding - *Nature* of

- Mathematics is to be here considered as the object of understanding as opposed to a tool, or a set of formally logical symbols, for understanding.
- Understanding will be distinguished from knowledge in that it is considered as a personal and subjective process whose objective product is considered as knowledge.
- When the product of understanding is internally accepted as true, by the individual, it is a conviction; when it is externally established as true, by a relevant culture, it is knowledge.
- Understanding presupposes the existence of a 'belief system' or 'structure'; a socially/culturally/individually constructed view of reality, mathematical or physical.

- Understanding seeks to enhance this structure by assenting to, creating/discovering, that which brings about the greatest consistency and coherence.
- Understanding is an active process which instills a personal sense of ingenuity.

Understanding - *Requisites* for

- It requires some individual 'moving force'; namely curiosity and/or frustration.
- The moving force (curiosity or frustration), motivated by consistency and/or coherence, is mentally represented by a constant shift from generalization (extensification of the structure) to synthesis (intensification of the structure).
- It requires the 'conditional' acceptance, or consideration, of the object.
- It requires the recognition of the object in relation to what understanding (knowledge) is concurrently accepted.
- It requires an attempt to fit the object into this pre-existing structure.
- It requires the delineation of mathematical considerations from those which are epistemologically and ontologically non-mathematical.
- It requires an awareness as to what must be both maintained and rejected, with respect to a 'mathematical' reality.

- It requires the appreciation of the (intuitive) sense and (formal) meaning of the object in relation to the given context.
- It requires an initial intuition, or connection with previous experience, which provides a sense of the object; this must extent itself to a formalization which gives the object its mathematical meaning.

Understanding - *Results* of

- Its outcome must involve the consideration of both the sense -goal or consequences- and its meaning - formal definition and interpretation.
- The intuitive grasp provides the basis for explanation of the causes leading to the goal or consequences.
- The formal grasp provides the basis for the justification of generality and rigor.
- Understanding is not necessarily indicated by an ability to solve, or explain using mathematics; though, this may signify understanding of the object's mathematical sense.
- Understanding is not necessarily indicated by an ability to prove, or formalize using mathematics; though, this may signify understanding of the object's mathematical meaning.
- Understanding may be indicated by an ability to communicate the object's sense in terms of its meaning and conversely, its meaning in terms of its sense.

§ 2.3 Understanding Mathematical Infinity

Q1. What does it mean to understand mathematical infinity?

Q2. Why is it necessary to accept actual infinity?

In seeking resolution to the first question it is necessary to recall that understanding requires both an intuitive and formal grasp of the object. It is to be my claim, however, that a formal grasp of the meaning of infinity requires the acceptance of an actually existing infinity. If we separate out 'being' as either potentially existent or actually existent we can observe the following; students see the latter ontological classification as meaning non-existent. While this may be true within natural reality considerations it is not true within mathematical reality. To counter, or at least challenge, students' obstacles then we must try to get them to accept that infinity is actually existent, then we can concern ourselves with their accepting that it is potentially existent. Only in this way are we able to provide a reason for them to separate potentiality from existence, and to accept infinity as a mathematical object whatever its ontological status is.

There is an hypothesis that states that in order to understand a mathematical object one must first understand its function in process [Dubinski:1990]. That is, only through 'encapsulating' an idea within its process do we then come to know its product (object). The problem with this supposition, in relation to the idea of infinity, is that all of its processes are characterized by potentiality while its product, or object meaning, is essentially taken as existent. Thus while it may be possible, in a constructivist sense, to create an idea of mathematical infinity, this construction cannot be based solely on the contexts of potential processes when potentiality is so tightly linked with non-existence.

In our providing only instances of infinite processes students are lead to the mistaken belief that infinity, itself, is essentially non-existent. While in introducing students, through cardinal infinities, to an actual object called infinity we can establish existence, then we can turn to the distinction between potential and actual existence. Although the methodologically rigorous object of mathematical infinity is mostly demanded at the undergraduate or graduate level; it is necessary to previously provide a formalization of sorts so that its conceptualization is not made only on the basis of those contexts which are strictly procedural (potential) and as such require only an intuitive grasp of its sense; we need a situation which demands an mathematical grasp of it sense. While the understanding of mathematical infinity requires that there be a dynamic relation between the intuitive grasp of the object and the formal grasp of its meaning; the latter, however, requires a definition and/or proof of the object's existence. In this way, a necessary prerequisite for understanding mathematical infinity is the acceptance of its actuality.

The Cognitive, Epistemological and Ontological Requirements and Their Related Obstacles.

Understanding mathematical infinity requires:

$\mathbf{U_1}$: The appropriate belief system which sees mathematics as a structure in and of itself, whose reality (object existence) is established through rigor, definition and proof.

As opposed to seeing mathematics as being in some dependence relation with natural reality ($\mathbf{CO_1}$), whose objects exist within a spatio-temporal domain ($\mathbf{OO_1}$), whose existence is empirically established through observation and/or imagination ($\mathbf{EO_1}$).

$\mathbf{U_2}$: Accepting that mathematical knowledge is justified by consistency; the dynamic of definition and proof.

As opposed to believing that knowledge mathematical or other is justified by physical or causal evidence ($\mathbf{EO_2}$)

U_3: Accepting that mathematical processes and/or relations are independent of natural reality.
As opposed to thinking that all processes or relations our within the space-time constraints of natural reality (**CO_2**).

U_4: Accepting that these relations and processes are independent of having to actually be able to be performed.
As opposed to believing that they require actual performance (**EO_3**), e.g., the limit process requires that we attain ∞ or e.g., that 1-1 correspondence requires that we actually pair up each element.

U_5: The conditional acceptance of an idea called infinity; to gain an intuitive grasp of its sense.
As opposed to the belief that there is no infinity, mathematical or other (**OO_2**).

U_6: An attempt to fit infinity within a mathematical structure; to gain an intuitive grasp of its mathematical sense.
As opposed to seeing physical infinity and mathematical infinity as having both the same sense (**OO_3**) or granting infinity sense in a metaphysical realm only (**EO_4**).

U_7: Accepting that it is the context and the definition within that context that determine the characteristics of a mathematical object.

As opposed to assuming that all of the characteristics of mathematical objects are independent of context (**OO_4**)

U_8: Seeing that it is the process, and not the object itself, that characterizes mathematical infinity as potential; to gain an abstract grasp of its mathematical sense.
As opposed to a seeing infinity, itself, as strictly potentially or actually existent (**OO_5**).

U_9: An attempt to find mathematical meaning for the object called infinity; to see the need for its mathematical existence.
As opposed to viewing infinity as a mere symbol of computation or procedure (**EO_5**) or believing that existence, mathematical or other, can only be granted to 'real' objects (OO_6).

U_{10}: Accepting infinity into this mathematical structure as an object whose existence is actual, both well defined and proved, and potential, in relation to mathematical processes; to gain a formal grasp of its meaning.
As opposed to rejecting infinity, or mathematics itself, as essentially meaningless (**CO_3**).

The Mathematical Requirements

Understanding mathematical infinity requires a(n):

1. Intuitive Sense of Infinity

Being able to distinguish between;

a) Physical infinity; seeing time and space as non-ending, infinitely divisible, continuous, beyond count.

b) Ideal infinity; seeing the Universe, God, as being beyond space-time, as infinitely large or great, beyond precise definition and/or knowledge.

c) Mathematical infinity; seeing lines as infinitely divisible (a point as infinitesimal), infinite sequences beyond count, repeating decimals, as well as the other instances, as non-ending.

2. Abstract Sense of Mathematical Infinity

Seeing infinity as a

a) value, i.e., $\mathrm{Lim}_{x \to a} f(x) = \pm\infty$

b) process, i.e., $\mathrm{Lim}_{x \to \infty} f(x) = L$, $\sum^{\infty} f(x)$

c) tool, i.e., infinitesimal, dx

3. Formalized (Mathematical) Sense of Infinity

Seeing infinity as

a) a point on the Riemann or Poincare sphere

b) as a point in, or value of, the extended Reals, i.e., R U ($\pm\infty$)

c) the cardinality of a given set, i.e., $\aleph_0$, **c**.....

By way of a conclusion I would like to add that the preceding consideration of mathematical understanding, with regard to infinity or other objects of mathematics, is not meant to be definitive or all-inclusive; it is an hypothetical account. In viewing understanding as a process it should further be noted that more times than not this process is implicit. In noting points of discontinuity, conflict and/or stagnation educators seek to uncover those assumptions which inhibit understanding: in these situations we come, though negation, to suggest what is necessary for understanding. Yet since the assumptions of both educators and students are implicit the true nature of understanding must remain uncertain.

CHAPTER 3.

The Clinical Interview: *Identifying and Investigating Obstacles Related to Infinity*

The experimental part of this thesis will implement two methods of inquiry; the *clinical interview* and the *teaching experiment*. The aim of the clinical interview is set at identifying students' views of mathematical infinity within varying contexts. The direction and formation of the interview questions will be based on the ideas found in the literature review and will correspond to the responses given in the preliminary *questionnaire*. It it hoped that these questions will make students' beliefs, regarding infinity, explicit enough so that conflicts, and the obstacles that underly them, can be brought to light. The confrontation of these obstacles will be the aim of the teaching experiment, which seeks not only to assess students' inappropriate views, but also seeks to change them. In this manner, the results of the clinical interviews will allow for the investigation and identification of students' obstacles that is required for the construction of those hypotheses which underly the teaching experiment.

§ 3.1 Hypotheses - *(via Literature Review)*

1. That the epistemological obstacles related to infinity are the result of a lack of distinction between physical infinity and mathematical infinity.

2. That students do not see the need to distinguish between mathematical and physical infinity because they see mathematics as being in a 1:1 correspondence with reality.

3. That students feel that mathematical knowledge and existence are the same as scientific knowledge in that they are empirically verifiable.

4. That the problem students have accepting that .999... =1 is a representational one. That is, the problem may arise from students seeing .999... as a `dynamic' number while 1 is a 'static' number; perhaps if they were asked whether as a process .999... =1 they would accept the equality.

§3.2 The Questionnaire-*Preliminary Investigation and Selection*

1. Given a line segment, AB, how many points are between A and B?

A B

2. Suppose there is another segment, CD, such that CD is contained in AB. How many points are between C and D?

A C D B

2b. Are there more points in [A,B] than in [C,D]? Why or why not?

3. Consider a line segment, say [0,1]. Suppose we divide this segment into half and then we take half of the remaining half......and so on....How often can this process be repeated?

0 1/4 1/2 1

4. Given 0.999.... What do the dots mean?

5. Is 0.999.... < 1? Or is 0.999.... = 1? Why?

6. Given the sequence {0.9, 0.99, 0.999, 0.9999, ... }. If we were to represent the elements of this sequence on a number line would there be any number between the last element of the sequence and the number 1? Why or why not

7. Given that 1/3 = 0.333...... If we multiply both sides by 3 we get that 3/3 = 0.999......, that is, we get that 1= 0.999..... Why?

8. Could we position infinity (∞) on a graph or surface? Why or why not?

9. Could we take the union of a set with infinity (e.g., **R** U ∞)? Why or why not?

10. Could you briefly explain what the word infinity means to you.

11. Why do people study mathematics?

Subjects

Four students (Martin, Luc, Wayne and Jean) were chosen from those enrolled in Math 362, *Introduction to Analysis*, at Concordia University. This selection was based on their mathematical experience; at this level it was assumed that their experience would include that relating both to mathematical infinity (in potential contexts such as limits, sequences and series) and also include implicit knowledge of mathematical proof (especially 'proof by contradiction'). These students should also be aware of the formal definitions of a limits, sequences and series; this may allow them to accept, more readily, that potentiality is a function of the context, as opposed to seeing it as a function of infinity itself. The questionnaire was given to all students in Math 362: the choice of these particular students was based on their willingness to participate and the explicitness of answers given in response to the questionnaire.

§ 3.3 The Clinical Interview

The purpose of the clinical interview was uncover the nature of the assumptions and knowledge, their sources and level of conviction, that were employed by students to explain and justify their responses to the questionnaire. As previously stated, this information will be used to *identify* students obstacles when faced with the idea of mathematical infinity. The *analysis* of these obstacles will then be made though a qualitative measurement of the level of correspondence with the model of understanding set out in Chapter 2.

The Questions-*(of the Clinical Interview)*

1. Given a line segment, AB, how many points are between A and B?

a) if "infinitely many"

1.1 How big is a point?

1.2 Does a point have shape and size?

1.3 Could we ever see a point?

1.4 What does infinitesimal mean?

b) if "finitely many"

1.1- 1.4

1.5 In mathematics a point is an abstract object having neither shape nor size. If we are considering 'mathematical points', how many points are between AB?

c) if "thousands, millions..."

1.6 Is this the same as infinitely many?

if "yes" - Repeat section a)

if "no" - Repeat section b)

2b) Are there more points in [A,B] than in [C,D]? Why or why not?

a) if "same cardinality"

2b.1 Why does [C,D] have the same number of points as [A,B] when [C,D] is part of [A,B]?

2b.2 Is $\infty+1=\infty$? Or is $\infty+1>\infty$?

2b.3 Is $\infty+\infty=\infty$? Or is $\infty+\infty>\infty$?

b) if "not the same cardinality"

2b.4 How many more points are there in [A,B]?

2b.2 &2b.3

3. Consider a line segment, say [0,1]. Suppose we divide this segment into half and then take half of the remaining half...and so on...How often can this process be repeated?

a) if "finitely many repetitions"

3.1 When does this process end?

3.2 What is the final value (where does the process end)?

3.3 Is there a final piece?

3.4 Does this piece have a shape or size?

3.5 What if this line is elastic and I stretch it where it ended, can I still repeat this process?

3.6 What is the value of $1/2^n$ as $n\to\infty$?

b)if "infinitely many repetitions"

3.1- 3.4

3.7 Mathematically does this process end?

3.6

4. Given 0.999... What do the dots mean?

4.1 Is it possible to write out all these nines?

a) if "no"

4.2 Does this mean that 0.999... is not a number?

b) if "yes" - will not be chosen.

5. Is 0.999....<1? Or is 0.999... =1? Why?

a) if "0.999... <1"

5.1 What is the number between 0.999... and 1?

(If number is 0.000....1 show -> 0 as n-> ∞.)

5.2 As the limit goes to ∞ is 0.999...=1?

5.3 If we write 0.999... as a geometric series we get 0.999...=1. Why?

6. Given the sequence {0.9, 0 99, 0.999, 0.9999, ...}. If we were to represent the elements of this sequence on a number line would there be any number between the last element of the sequence and the number 1? Why or why not?

a) if "yes"

6.1 What is the number?

6.2 As the limit goes to ∞ is there a number between 0.999... and 1?

6.3 What is the last element of this sequence?

b) if "no"

6.3

7. Given that 1/3 = 0.333.... If we multiply both sides by 3 we get that 3/3 = 0.999..., that is, we get that 1= 0.999... Why?

a) if "1/3 $\neq$ 0.333..." Show 1/3 =0.333... by long division.

b) if "we cannot multiply an infinite number".

7.1 Is it us or the algorithm that defines or determines multiplication?

7.2 Show x= 0.999... so 10x= 9.999.... then 10x-x =9.999...- 0.999...=9

so we get that 9x =9 so x=1 therefore 1= 0.999...

7.3 Why do we keep getting that 0.999...= 1?

7.4 Is it possible that 0.999... $\neq$ 1 in some contexts and that 0.999...=1 in another? Why?

8. Could we position infinity (∞) on a graph or surface? Why or why not?

8.1 Show Riemann Sphere

8.2 Couldn't we say that ∞ is represented by the point at the north pole on this sphere?

8.3 How is this representation possible?

9. Could we take the union of a set with infinity (e.g., **R** U ∞)? Why or why not?

9.1 Suppose we consider a set S composed of the limit values of some functions, that is, L is an element of S if for some function F(x), lim F(x) =L.

9.2 Could we consider ∞ as an element of this set S?

Consider F(x) =1/x as x-> +0 we get an L=∞.

Suppose now we consider S' such that S' is the set of all limit points where L is a real number or L=∞, Isn't this set given by S' = (R U ∞)?

9.3 Does this set S' = (R U ∞) make sense?

10. Could you briefly explain what the word infinity means to you.

10.1 What does ∞ mean?

10.2 Is there a difference when we talk about mathematical infinity, i.e., ∞, and infinity outside mathematics?

10.3 Is infinity a number? A mathematical object?

10.4 What does ∞ mean in a limit?

10.5 Is there a difference between infinity and infinite?

11. Why do people study mathematics?

11.1 How do we know something is true in math?

11.2 How do we know something is true outside mathematics, say, in psychology?

11.3 What is proof in mathematics?

11.4 How do we know something exists, say a set, in mathematics?

11.5 How do we know something exists, say a table, outside mathematics?

11.6 Can we know the properties of a 2000-sided polygon if we've never seen one?

11.7 Can we know mathematical infinity?

11.8 Does mathematical infinity exist?

Justification of Questions

1. The purpose of this question is to test my belief that one will not be able to fully understand mathematical infinity if one sees mathematical objects as existing within a spatio-temporal domain. Specifically this question aims at determining whether students' claim that there are a finite number of points is owing to their viewing a point as a physical entity. In like manner, I wish to ascertain whether those who see an infinite number of points do so because of their belief that a point is an abstract object which is outside of spatial consideration.

2. & 2b. The purpose of these questions is to determine whether students see the quantity of infinite as being like finite quantities in that equality relations can be given. This question will further be used to bring out the conflict between the whole/part relation and cardinal equalities. Although I mentioned earlier that I do not think that graphical context such as these are suitable for testing cardinal understanding they do enable students to question the role context has in

defining relations. It is hoped that they would also see that the rules for finite numbers do not hold for infinite numbers (i.e. cardinal numbers). This type of questioning would also bring conflict to those who see infinity as a large finite number; in this manner, I should be able to tell what they mean when they say infinity.

3. The purpose of this question is to determine whether students see mathematical objects as having spatial qualities and further, if they see mathematical processes as necessarily occurring in time. If students believe this process to be in time then the division can only occur finitely many times. Likewise, if their idea of line is that of a physical object then not only will the process be restricted but there will be a final piece. Thus, if they believe there is a final piece which has a shape and size then the line is physical and the process occurs in time. If they believe the process to be infinitely repeatable, I should like to question how they are able to justify that although the process of bisection continues mathematically this process as a limit reaches 0. Hopefully this will also bring out what the student sees as the relation between mathematics and reality.

4. The purpose of these questions is simply to determine if students realize what notation is used to represent an infinite expansion. This is required so that I am more assured that

students know what the representation of the number 0.999... means. It is my contention that although they claim that 0.999... means the nines repeat without end they still do not view this number as an existing entity, that is, 0.999... is seen as 'never finished' because the nines only potentially repeat without end. [c.f Vong: 1989]

5. The purpose of these questions is to see whether students' problems in accepting that 0.999... =1 is a consequence of i) their not agreeing that the nines are infinitely repeated or ii) that they understand this but, because of some other representational problem, they cannot see the equality. It would appear that the largest obstacle facing students is having to accept that 0.999... is a finished, actual object. Perhaps they simply cannot accept that an infinite number equals a finite number. Thus it is my next task to see of 0.999... presented as a process will lead to an acceptance of this equality.

6. The purpose of these questions is to provide an alternate view of 0.999... , one which includes the idea of a sequential type of process which approaches 1 in a finite number of steps but equals 1 in an infinite number of steps. The aim of this example is twofold; to see if students will assume that this process occurs in time (then one could never get to the last element) and also to see of students will change their answer

to question 5 if an alternate representation which includes the phrase 'last element' instead of 0.999... is used.

7. The purpose of these questions is also to provide another way of considering 0.999... that some what removes the need to consider time and thus may allow students to see the equality more readily. It also involves a type of 'mathematical demonstration' which may bring out students' attitudes towards mathematical knowledge. Overall, however, its use is to determine whether the problems students have in accepting that 0.999... =1 is a result of their seeing 0.999... as a dynamic potential number: if this is the case, they will also hold the belief that 1/3 ≠ 0.333... since 0.333... will also be considered as 'unfinished'.

8. The purpose of these questions is to see whether students will accept infinity as a concrete mathematical idea. That is, before they come to accept infinity as an actual existing object it may be helpful for them to be able to include infinity as a sensible, though perhaps not formally defined, concrete idea (or representational object). This example, of the Riemann Sphere, also allows the student to view infinity within another situation which, hopefully, may enable them to see how context influences the nature of infinity.

9. The purpose of these questions is to provide students with a context where infinity not only makes sense but is given some sort of definition, i.e., as an element of a set of limit values. If they see existence as arising by way of definition this example ought to push students to accept infinity as an existent object within mathematical reality. Moreover, it should be noted that these questions, in total, are meant to represent the varying views that are necessary for the understanding of mathematical infinity (see Chapter 2). By this I mean that questions 1-3 are meant to examine students intuitive sense of infinity, questions 4-7 to explore their abstract sense of infinity, and questions 8&9 investigate whether they are willing to accept a formalized sense of infinity.

10. The purpose of these questions is to determine the extent of students' explicit and general knowledge of mathematics and infinity. Of particular importance is whether they feel the need to distinguish between mathematical and physical infinity. If, however, they are not aware of any difference I shall then try to move them towards thinking that there ought to be. It is hoped that they will come to see the need to accept infinity as an object that exists within a mathematical structure. Also, I would like to know if they separate out the different meanings of infinity and infinite, that is, if they perceive a difference between the abstract mathematical 'number' (i.e., ∞ and the Alefs) and the process which is a-

temporal and non-spatial (i.e., limits at infinity and infinite summations).

11. The purpose of these questions is to determine students' attitudes towards mathematics, particularly, if they see mathematical knowledge or existence as needing empirical verification. If this is believed then one cannot see either the number infinity or the infinite process and thus it is reasonable to come to the conclusion that one cannot 'know' things about these ideas, one must imagine them. That is, one would believe only in potential infinity... 'if it were to exist then it would have these properties'. Yet even if students see mathematical knowledge as coming from proof and existence by way of definition but do not view infinity or infinite processes as 'mathematical' ideas then they might still maintain a potentialist view of infinity. What is required then is that they know how mathematical knowledge is arrived at and that they see infinity or infinite processes as mathematical ideas.

§3.4 Results - *(of the Clinical Interview)*

All participants of the clinical interviews were told that this investigation was part of my thesis which considered students views of mathematical infinity. They were further told that I was not looking for 'correct' answers but rather was interested in the reasons they had for giving a particular response. I then requested that they try to be as explicit as they could in explaining these reasons. The interviews began with their responses given in the questionnaire; the corresponding interview questions were then presented and discussed. The ages of the students were varied; Martin, Jean and Luc are in their twenties while Wayne is in the mid-fifties range. All interviews were given individually, each session last approximately forty-five minutes and was recorded via a tape recorder placed off to the side.

1.Identification

The Problem of 0.999...

Martin, as well as all other subjects, accepts that .999... =1 in a limit, or procedural, context. Martin-(Q3) "Because of notational purposes I mean if you keep putting that down its going to 0 kind of.." Wayne- (Q3) "Well you can say it has an end mathematically because you get this limit business as it tend to 0." Jean- (Q5) " Yes, if you take the limit of this number you get 1." The implicit assumption that underlies this view, that

.999...$\neq$1 in non-procedural contexts, is that .999... is seen as a 'dynamic' number we can only represent it at the nth stage, but in the process we take n to infinity so they would be equal. Luc- (Q4.2) "Its not a rational number, it vassilates (sic.)." Martin-(Q6.3) " ....but I mean could you have an end to this sequence....but you actually couldn't answer that since its going, this goes on (ad) infinitum um.... .999... wouldn't it mean since its all nines it must be another nine. Wayne-(Q5) "Um.. .99999 with another 9....(Q6) "would you have a last element of the sequence....I don't think so....you just keep adding nines as many as you would like to have."

The Cardinality of Line Segments

Only Jean claimed that segment AB had more points than CD; his justification was based on the distances between points in the respective segments and the fact that CD was part of AB -much like Bolzano- (Q2b) "... more like the distance between A and B over the distance between two points is 0, almost, so the number of points between A and B is (A-B)/0 and the number of points between C and D is (C-D)/0, but since A-B is greater than C-D, its going to be more points between A and B." Another result of this student's reliance on the whole-part relation is his belief that $\infty+\infty$ and $\infty+1$ will both be greater than ∞.

The responses to the above questions (2b.1&2b.2) dealing with the 'arithmetic' of infinity yielded much information on these students' views of infinity. Martin justifies the equality in a way which will, later on, affect his acceptance of cardinal infinity " but since we're dealing with infinite numbers you can't really have a greater infinity so I say the same". Wayne shares this opinion "I mean if they're both infinite one can't be bigger than the another" but he does not accept the equality "because infinity is just a notion....it tends towards infinity , but ∞ is not a number like a or b. So you can't say a+b its like dividing by 0 its not a mathematical function...its not a valid operation." This reference to 'valid operations' further allows Wayne to reject any manipulation which involves infinity (Q7.2) "Well I don't buy that because I think the operation is invalid....you're trying to subtract two infinity things again. (Q5.3) I think you're trying to slip another operation in there on something that's infinite. (Q7) Yes but this (1/3) can't be expressed by this (0.333...) fashion this is rational and this is like an irrational limitless number. (Q8.1) No, I would still say that I think that's another way of saying that the infinity sign or infinity is a point and I don't think its a point...and I just don't accept that it is a number, its not an a or a b...it just doesn't exist, its something you head for but never quite make. (Q9.1) I still have a difficult time treating this as some number like L. I just can't do that. I don't know why we couldn't look at it that way....so this still wouldn't be a valid operation.....I think of it as an idea or something we tend to...." For Luc the equality holds almost by default (Q2b) " I think that infinity is undefined so, I mean, ∞-1 is just as undefined. So I would say at this point that they are the same.

2. Analysis of Understanding

A. Attitudes Towards Mathematics

Martin

Martin believes that there is a dependence relation between mathematics and physics **(CO_1)**. (Q10.3) "Well when you consider like um physics and all the rest of the sciences spanned off from mathematics..." Although his claim is that physics depends on mathematics this dependence relation is often times confused and even inverted. This view thus affects his notion of the nature and function of mathematics: (Q11) "Well I just put, to learn to apply formulas to natural phenomena **(OO_1)**...I guess that would encompass physics, but I mean like I said I think mathematics started off and physics stemmed form mathematics so..." His opinions further affect his concept of the nature of mathematical objects: we see that existence is to be established empirically **(EO_1)** (Q11.4) "Its (the object) recognized and in a sense its made visible by observations....".

Related to the above view, no distinction is made between mathematical and non-mathematical knowledge **(EO_2)** (Q11.5) I'd say its basically the same. Its like mathematics is, you know as it is in physics .....Its just I guess like its recognized or its...like for example, take black holes, there's still an argument that they don't exist but according to certain phenomena there is.....I guess we have to go by what we observe and what we think is there.." One consequence of these beliefs is that proof loses its a priori power as that which establishes existence or guarantees knowledge. In this way it is reduced to a, somewhat trivial, a posteriori verification procedure **(EO_3)**. (Q11.3) "....taking what's there and on paper showing you have that this type of thing...like keep getting as basic, simple as possible and then...in a sense verification of the law."

Wayne

This student could be said to have the most appropriate view of mathematics. By this I mean that the way and status of existence and knowledge in mathematics is distinguished from that of reality **(U_1)**. Regarding mathematical knowledge he claims **(U_2)** (Q11.1)"Well we know something is true in mathematics because ....we have to break it down into little pieces of logic... This is true therefore we take this and so on...." This is distinguished from non-mathematical knowledge where (Q11.2) "there's alot of guessing". The way of knowledge is also separated, (Q11.3) " Well the value of proof in mathematics is if you can prove something in the general sense, like I mean you try something and empirically it works for...you shouldn't just jump in and say it works empirically that it works all over the board...but if you can prove it in the abstract sense **(U_4)** and its true for all cases, or all cases within a certain range, then its valid." The way and nature of existence is likewise distinguished; (Q11.4) " (in mathematics) Well we simply declare it to be so. We start from the beginning with some axioms.....if you want something as you're going along you say: define set equals something." outside mathematics, however, (Q11.5) "Well we go by our sense I suppose...you can really get wild and drag in some religion."

Jean

The questions dealing with Jean's mathematical attitudes were lost due to technical difficulties so I will have to surmise from other statements what he assumes. In relation to the status of existence in mathematics Jean appears to hold the view that mathematical objects are the product of our imagination **(U_3)** (Q1.3) " I think that we imagine it (a point)." The way to mathematical knowledge seems to be unconnected with empirical requirement **(U_2)** (Q1.4) "I don't think we ever see it (a point)." Jean does assume that there is a distinction, with respect to limit processes, between mathematically doing and physically doing **(U_4)**. (Q3.5) " Because the limit isn't the same as when you're doing it, because if you're doing it you would have to reach the number n as great as infinity, to be doing it, and you will never be able to reach that number so."

Luc

This students' attitude towards mathematics appears to be one which views mathematics as being on the same level as empirical sciences (**CO_1**). The function of mathematics is thus given as (Q11) "...to explain natural phenomena (**OO_1**) or possible natural phenomena." Regarding the status of mathematical knowledge there is no distinction between necessary and contingent (**EO_2**): skeptism is maintained for both (Q11.1) " How do we know anything is true in anything: I don't think we can......(Q11.2) There is no difference. No I don't think so." With reference to the way of mathematical knowledge, proof, there are some very serious challenges. Luc does not accept the validity of 'proof by contradiction' (**CO_2**) (Q11.3) "That's the thing I have problems with; he (the teacher) always uses something called proof by contradiction and I really, I hate those...it bothers me." Underlying this skeptism is this student's doubt of the validity of the tautology 'p or not p' (Q11.3) " ...but if you have a duality like this..If you take out one then obviously its the other: but like do we always assume in mathematics that you have this duality....But if a 2-dimensional plane has a duality of 2 then does a 3-dimensional plane have a duality of 2, also you know a triality (sic.), a 4-

dimensional does it have 4 possible truths..."

It is my hypothesis that it is his empirical assumptions that give rise to the above claim; in returning to the question of proof Luc, however, seems to find solace in some sort of separation of mathematics from reality **(U_1)** (Q11.3) "I think it may prove a working rule, it proves itself and nothing else.....I don't think that those laws that we come up with in math have anything to do with the universe **(U_3)**....I mean they do in a way that we use them and nothing more so I think they are relative truths...(relative) to themselves." Regarding the way of establishing existence in mathematics he makes the idealist claim that (Q11.4) "the idea always does (exist)": yet, because of his epistemic instability this student is unable to accept that existence can be known **(EO_1)** (Q11.4) " I don't know, I don't know if you could."

B. Attitudes Towards Infinity

Martin

Several of Martin's responses reflect his characterization of infinity as potential **(OO_5)** (Q8) " No because its unattainable I mean you can't really draw infinity it would just...cause infinity, I mean to me, it goes on forever type thing, so unless you are sitting there for a long time **(EO_3)**. (Q9) " I said no because infinity encompasses everything..so..." When asked specifically he states that infinity is (Q10) " A value which cannot be reached!...Its just that when you try and picture the expanding universe its like...you talk about numbers type thing and then you...it and you know...to me its just a value that cannot be reached **(OO_4)**." Martin's view of an inter-dependence relation between mathematics and physics makes it hard to determine whether by infinite he means a large indeterminate finite number **(OO_3)**. (Q10) "Cause that's um...again I'm taking physics, especially astrophysics, which is where we start to see the vastness of, I mean, just taking numbers you know like numbers we use again and again in mathematics, I mean people don't...realize how vast they are type thing.... So I mean that's only a billion so you go on (ad) infinitum to larger and larger numbers I mean it just gives you that a , like it cannot be reached...."

When questioned as to the possibility of knowing mathematical infinity Martin does not reject this **(U_5)**, although he says that he has never had to make such a claim since (Q11.7) " ...in the stages that I'm at in mathematics its always like kind of pushed off to the side. Its like don't deal with it its too, its beyond type thing....I've never been taught these things." With regard to the ontological status of infinity this student seems hesitant to admit that infinity could be more than a 'value that cannot be reached' **(EO_5)** but he does accept that it is a required mathematical entity **(U_6)**, if only in name, (Q11.8) "Well basically just by what you've been stating here. I mean you know you could just argue back and forth ...... and argue for and against so I'm, I mean, we have to give it, I mean when we get to these, these values that just never end, I mean we have to give it some type of name you know. So I mean it is there we have to deal with it in the sense just like alot of things mathematics would lead up to infinity." **(U_9)**

Wayne

As noted before, Wayne see infinity more as a directional adjective than a mathematical object **(OO_4)** (Q2b.3) "....because infinity is just a notion, infinity you can say for infinity for example you take the limit as x goes without bounds, it tends to infinity, but infinity is not a number like a or b **(OO_5)**. (Q10) ...so its a mathematical idea and its used to sort of tidy up the math rules or what not **(U_6)**. So you can say its used to say as something increases without bound or goes towards something, its better to write $x \to \infty$, instead of saying as x increases without bounds." In this way it may be said that Wayne accepts infinity as an existing mathematical idea that has directional import but not as an meaningful actual object **(EO_5)** (Q10.3) "I don't consider it as a number I consider it as an idea."

The latter belief, however, is enough to make the appropriate distinction between mathematical and non-mathematical infinity **(U_9)** (Q10) "I think like I said if you're looking for some physical counterpart in the cosmological sense, then I don't think it exists at all, maybe, maybe not, probably not....Well when people think of infinity they start thinking of cosmological or outer space or something.....They start thinking physical things.....But you don't have to worry about that in math you just use it as a notion to prove theorems and functions and good things like that **(U_5)**......(Q11.8) I think that what you want to say (in defining infinity) is this has no bounds, I think if you took out the infinity signs and said without bounds or has no bounds or is boundless, probably you could get along without the infinity sign **(OO_4)**." What is interesting is that while infinity is given existence as an idea (Q11.8)" Oh, mathematical infinity exists, sure.." this idea cannot be known in the way that an actual object (e.g., a number) can be **(CO_3)** (Q11.7) "...I really don't think so...Well we know it and we use it I think as long as, I still think I'm on good ground by thinking its not really a number, its an idea to be used, if we think like that I don't have a problem."

Jean

For Jean infinity is not separated from the physical, or more specifically it is that which characterizes the spatial component of both mathematics and reality **(OO_3)**. (Q8)

"Because its (infinity) ....I mean its the whole universe **(OO_4)**, you can't have a graph as big as the universe. As a result of this belief, in combination with his empirical bias, any strictly mathematical position (via the Riemann Sphere) of infinity becomes meaningless **(CO_3)** (Q8.1) "Well if you're defining this concept. O.k. But to be this it wouldn't mean anything at all.....I don't think it (infinity) looks like this." Jean's spatial/empirical conception appeared to be challenged, however, but the 'extended Reals' problem (Q9)...his original response to whether this (R U ∞) made sense was based on the following; that "(Q9) something small (R) and something big (∞) **(OO_5)** make something big (∞)... (but) When I said this I was thinking that ∞ means everything **(OO_4)**". Jean comes to accept **(U_5)** the demonstration with the disclaimer (Q9.1) "No I never thought it was so."

Luc

Luc characterizes infinity as 'that which is undefined' by this I believe he means that it doesn't have a mathematically spatial location **(OO_3)**. (Q4) "....it vassilates (sic.). It doesn't have a set point.. (Q8) No, because its undefined, its such a large number, and its moving all the time... **(OO_5)** Oh, o.k., I thought you meant like a picture or it like a location....but I thought you meant that *was* infinity....but its only a representation, it doesn't have any comparison to the other points in the graph , its not real **(OO_6)**. (Q3) ...infinity is not defined **(CO_3)**, but I think that at one point you would have to stop." This belief allows him, however, to make the separation between the mathematical and the physical processes **(U_4)** (Q3.5) "You see in mathematical terms you could keep going forever....but I think that in matter or distance ...you come to a, you know, fundamental particle, a fundamental unit of distance." Interestingly enough, after being show the Riemann conception of infinity Luc seems to accept that infinity could be defined **(U_5)** (Q9) " ....but I changed my mind....But you have to put your representation of what it was: the dot, dot, dot is not the only one anymore **(U_7)**."

In continuing on, however, Luc returns to his previous characterization but instead of seeing it in a global sense as an undefined idea he now views it locally as an undefined number **(U_6)** (Q10) " Like I said before it's not defined, not a defined number....It just represents an idea." This idea can only be said meaningful in a mathematical sense (10.2) " Yeah, I do think it's different in mathematics...(infinity is an) infinitely large number that doesn't stop, but I think, I think practically it does stop at a certain point in real space." Viewing infinity as a mathematical idea further allows Luc to grant it existence **(U_9)** (11 8) " I think that at the moment you invented it it began its existence, its an idea you know." In fact, at the end of the interview, Luc appears to change his empirical attitude towards both mathematical existence and knowledge

(U_2) (Q11.7) "So maybe it's not necessary ....maybe the question that should be asked is not whether the number ∞ exists or the whole system that brought it to existence exists, um....maybe the whole thing only exists an an idea on paper maybe that its own validation."

CHAPTER 4.

The Teaching Experiment: *Challenging Obstacles Related to Infinity Through the Instruction of Cantor's Set Theory*

The aim of this chapter is to test whether students' views can be made more appropriate, by the introduction of the cardinal notion of infinity. By 'more appropriate' I mean that their beliefs become based more on mathematical considerations (epistemological and ontological) than those which are of natural reality. A secondary goal of these experiments is that, in providing an analysis and delineation of students' obstacles, the results might aid in a programme which seeks to overcome students' obstacles related to mathematical infinity.

The Teaching Experiments' 'preparatory steps' are given as *preliminary observation* and *ascertaining experiments* [Kieran: 1985]; these initial stages will not be actively carried out but they will be accounted for: the former by the critique and analysis of the relevant literature and by the questionnaire; the latter by the, preceding and corresponding clinical interview. Since the purpose of the aforementioned steps is to suggest the possible problems encountered by students, and then to determine if indeed these can be classified as obstacles, I feel that the required preliminary levels of investigation were sufficiently accomplished. The overall aim, of either methodology, being the construction of hypotheses from which the actual Teaching Experiment will find both its direction and validation.

§ 4.1 Preparatory Steps

Preliminary Observation

With respect to the previously stated literature review I observed, however indirectly, various problems that students have in cognitively/philosophically accepting the notion of mathematical infinity (Chapter 1). The motivation for, and construction of, the questionnaire was thus founded on these implicit observations. Together these two means of inquiry provided the basis for the clinical interviews; they allowed for the suggestion of alternative hypotheses that would in some way permit the identification and explanation of those cognitive, epistemological and ontological obstacles had by students. (Chapter 3).

Ascertaining Experiments

These hypotheses were then tested via clinical interviews with four students enrolled in a first Analysis course. The purpose of the clinical interviews, other than that of validation, were to ascertain both students' attitude and obstacles when faced with the idea of mathematical infinity. The interviews indicated that, overall, epistemological obstacles related to infinity are due to a lack of distinction between potential infinity, both mathematical and physical, and actual- mathematically defined- infinity. Students' views of infinity are based primarily on intuitive notions which lead to a characterization of infinity as a potential mathematical idea (symbol); this further inhibits them from accepting a specifically mathematical characterization of infinity as an existing mathematical object (or idea).

The chief source of their inadequate intuitions is found in their lack of distinction between considerations of mathematical reality and those of non-mathematical reality. If mathematical infinity has to satisfy the conditions of natural reality, the nature and status of this object must be potential and unattainable. Although an appropriate attitude towards mathematics (proof and definition) allowed them to accept a given mathematical demonstration, I did not feel that it affected their understanding. That is, even though Wayne had an appropriate mathematical attitude this did not allow him to accept infinity as defined and existent object. Martin and Luc, however, with their 'confused' attitudes appeared ready to accept the existence of infinity as a mathematical idea. Overall, the clinical interviews allowed for the construction of alternate hypotheses.

§ 4.2 Construction of Hypotheses

Summary of the Results of the Clinical Interviews

1. Students belief in an inter-dependence relation between natural reality and mathematical reality inhibits their accepting infinity as as epistemlogically and/or ontologically sound object, be it mathematical or other.

2. They rely on their intuitive understanding of infinity, that is, they base their answers on the idea of potential (both spatial and temporal) infinity.

3. They do not see that infinity is characterized by the context; infinity, whatever this means, is the same idea, object, in all situations.

4. They do not distinguish between potential and actual infinity.

5. Their attitudes towards mathematics seems to make the acceptance of *how* certain demonstrations can be justified but does not affect their understanding of *why* they are so; e.g., they will accept that .999...=1 in a limit context, not by

referring to the notion of limit, but rather because by the 'rules' it must be the case.

6. They cannot see how a continuous, non-ending, dynamic number (.999... or .333...) can be equal to a solid, finished, static number (1 or 1/3). Rather .999.... is thought to have a next number; it is .999....9. It is like wise with .333.... The equality, however, is accepted in the context of a limit and/or a sequence.

The Teaching Experiment seeks to challenge the preceding five hypotheses: the reason for the specification is that only these results deal with the concept of mathematical infinity, in and of itself, whereas all others depend on other mathematical concepts such as limits, series, and sequences. It is my belief that if we require to know the problems the students are having with the conceptualization of infinite processes we must first determine the problems that they are having with the conceptualization of infinity in and of itself. In relation to the above, the Teaching Experiment will consist of two aims; 1) to make students see that, in the context of cardinality, infinity is an actual and existent (defined and meaningful) mathematical object and 2) that this object is outside of the space-time constraints of physical reality and therefore differs from their intuitive idea.

The Method

The method which I have chosen to reach these aims is the instruction of Cantor's set theoretic notion of actual infinity by way of cardinality. The mathematical justification of this choice is found in the fact that historically the acceptance of actual infinity was granted only through the emergence of Cantor's theory. The pedagogical justification arises out of my belief in the need to adequately separate physical/intuitive infinity from mathematical infinity: otherwise, students seem unable to move from an intuitive, experience or imagination based, concept to a more formal and strictly mathematical concept. Perhaps if we provide the more formal meaning of infinity before we turn to its use in such notions as limits or sequences, students would better prepared accept it as an existing mathematical object.

The subjects of the teaching experiment will be the same Analysis students (Wayne, Martin, Jean and Luc) that were used in my clinical interview. The reason for using the same students is that it establishes some type of control upon which an indication of successful change in conceptualization can be validated.

Hypotheses (of the Teaching Experiment)

A. Mathematical Hypotheses.

1. That the over-reliance on intuitive understanding of infinity can be challenged if a more formal way of encountering it can be established.

2. That the formalization of infinity will better allow students to accept that infinity can be considered as an existing mathematical object.

3. That the meaning established through definition will give infinity some epistemological footing; if we show infinity to exist then we can establish the parameters necessary for knowing this object.

4. That in viewing this mathematization of infinity students will come to realize the need to see the context as that which allows for the distinction between potential (procedural) infinity and actual (set theoretic) infinity.

B. Pedagogical Hypotheses.

1. That cognitively, as well as historically, the structural development and acceptance of mathematical infinity requires that we establish the ontological justification needed for claims of knowledge or understanding.

2. That the shift from intuition-based understanding to mathematical-based understanding, with regards to infinity,

requires the need for a definition of this mathematical object: otherwise students see all contexts, both mathematical and natural, as being those where infinity is characterized as potential. (Bachlard's: tendency to rely on deceptive intuitive experiences).

3. That the very term infinity, through its explication in natural language terms, allows for the maintenance of the supposition that the mathematical and the natural are inter-dependent. Because students' first glimpse of infinity is usually in terms of procedures (i.e., asymptotes, limits and summations) in which we find such phrases as 'going to', 'approaches', and 'large value', students are not cognitively required to distinguish between physical reality and mathematical reality. In this way they are free to assume that we are speaking of infinity as a common place notion. (Bachlard's: obstacle caused by natural language and tendency to generalize which hides particularity).

4. That the formalization of infinity through Cantor's set theoretic notion can and will cause conflict between students' intuitive idea and the required strictly mathematical idea. Furthermore this cognitive conflict may be that which is necessary to challenge their epistemological obstacles related to accepting infinity as existent (either potentially or actually) and meaningful.

§ 4.3 The Teaching Experiment

A. Equality of Cardinality.

1. Given **A** = {1,2,3,4} and **B** = {2,4,6,8}

How do we know that there are the same number of elements in each set?

2. Given **A**= {1,2,3,4,5,6} and **B**= {3,4,5,6,7,8}

How de we know that there are the same number of elements in each set?

*Bring in the method of counting.

3. Given **A** = {1,2,3,4,5,6 ...,14}, **B** = {2,4,6,8,10,12 ...,28}

How do we know that there are the same number of elements in each set?

3.1 e.g., For each n in **A** there exists 2n in **B** <u>and</u> for each n in **B** there exists n/2 in **A**

4. Given **A** = {1,2,3,4,5 ...200}, **B** = {3,5,7 ...401}

How do we know that there are the same number of elements in each set?

*Bring in the method of 1:1 correspondence

4.1 e.g., For each n in **A** there exists 2n+1 in **B** <u>and</u> for each n in **B** there exists (n-1)/2 in **A**

4.2 Defn: A 1-1 correspondence is established between two sets **A** and **B** if every element in **A** corresponds to a unique element in **B** and every element in **B** corresponds to a unique element in **A.**

5a). Can we count the number of elements in an infinite set?

b). How would we determine if there are the same number of elements in two infinite sets?

5.1 Defn: If there exists a 1-1 correspondence between two sets (finite or infinite) then they have the same number of elements, i.e., the same *cardinality*.

6. Consider the set of all natural numbers, **N**={1,2,3,4,5...} and **B**={2,3,4,5,6...}

Do they have the same cardinality? Why?

6.1 *Show for each n in **N** there exists n+1 in **B** and for each n in **B** there exists n-1 in **N**.

7. What about **N**={1,2,3,4,5...} and **B**{-2,-1,0,1,2,...}

Do they have the same cardinality? Why?

7.1 *Show for each n in **N** there exists n-3 in **B** and for each n in **B** there exists n+3 in **N**.

7.2 *Rule: if **N**={1,2,3,4,5...} and **B**={b:b=n±k, k>0}, then N and B have the same cardinality.

8. Consider **N**={1,2,3,4,5,...} and **B**={2,4,6,8,10,...}

Do they have the same cardinality? Why?

8.1 *Show for each n in **N** there exists 2n in **B** and for each n in **B** there exists n/2 in **N**.

9.What about **N**={1,2,3,4,5,...} and **B**={-4,-8,-12,-20,...}

Do they have the same cardinality? Why?

9.1 *Show for each n in **N** there exists -4n in **B** and for each n in **B** there exists -n/4 in **N**.

9.2 *Rule: if **N**={1,2,3,4,5,...} and **B**={b:b=±kn}, then **N** and **B** have the same cardinality.

10. Consider **N**={1,2,3,4,5,...} and **B**={1,3/2,2,5/2,3...}.

Do they have the same cardinality? Why?

10.1 *Show for each n in **N** there exists (n+1)/2 in **B** and for each n in **B** there exists 2n-1 in **N**.

10.2 *Rule: if **N**={1,2,3,4,5,...} and **B**={b:b=±kn ± c for k,c>0}, then **N** and **B** have the cardinality.

11. Consider **N**={1,2,3,4,...} and **Z**= {....-3,-2,-1,0,1,2,3,....}

Do they have the same cardinality? Why?

11.1 *Show 0 <-> 1

-1 <-> 2

1 <-> 3

-2 <-> 4

2 <-> 5

-3 <-> 6

3 <-> 7

That, is for each n odd in **N** $(n-1)/2$ is in **Z**, for each n even in **N** $-n/2$ is in **Z** and for each $n \geq 0$ in **Z** $2n+1$ is in **N** , for each $n<0$ in **Z** $2|n|$ is in **N**.

12. Consider **Q**= {a/b: a,b in **Z**} and **N**={1,2,3,4,5,...}
Do they have the same cardinality?Why?
What could the way to establish a 1-1 correspondence between these sets ?
12.1 *Show Proof
Let us consider the set **P+**= {a/b: a,b in Z+}
We can represent the elements of **P+** by the following matrix

1/1 2/1 3/1 4/1 5/1........

1/2 2/2 3/2 4/2....

1/3 2/3 3/3......

1/4 2/4.....

12.2 So that **P+** is listable in term of **N** even

We can show also that **P-** is listable in term of **N** odd, so that **P+ U P- = Q** is listable in term of **N** even U **N** odd = **N**.

Thus, there is a 1-1 correspondence between **N** and **Q** so that **N** and **Q** have the same cardinality.

13. Do you think the set **N** and **R** have the same cardinality? Why?

What could the way to establish a 1-1 correspondence between these sets ?

13.1 * Show proof for uncountability of **R**.

Consider b_i such that b_i is contained in the interval [0,1] if we can show that there is no 1-1 correspondence between N and [0,1] then N and [0,1] cannot have the same cardinality and so neither can N and R.

Suppose N and R have the same cardinality then there would exist a 1-1 correspondence between N and R, then there would exist a 1-1 correspondence between N and [0,1].

$b_1 = 0.a_{11}a_{12}a_{13} \cdots a_{1n} \cdots \quad \leftrightarrow 1$

$b_2 = 0.a_{21}a_{22}a_{23} \cdots a_{2n} \cdots \quad \leftrightarrow 2$

$b_3 = 0.a_{31}a_{32}a_{33} \cdots a_{3n} \cdots \quad \leftrightarrow 3$

$b_n = 0.a_{n1}a_{n2}a_{n3} \cdots a_{nn} \cdots \quad \leftrightarrow n$

Now consider c in [0,1] such that $c = 0.d_1d_2d_3.....d_n....$ such that $d_1 \neq a_{11}$, $d_2 \neq a_{22}$, $d_3 \neq a_{33}$ ......$d_n \neq a_{nn}$ .... that is, $d_i \neq a_{ii}$ for all i. No list of correspondence between the natural numbers and the reals in [0,1] can include all the real numbers in the interval [0,1] i.e., c does not correspond with any element in **N**, thus, there is no 1-1 correspondence between **N** and [0,1] so there can be none between **N** and **R**.

As a result would we say that **N** and **R** do not have the same cardinality? Why?

B. Cardinality of N and the 'Arithmetic' of Infinity.

14. Given **N**={1,2,3,4,...}. How many elements are in **N** or what is the cardinality of **N**?

14.1 Defn: A set is said to be *countable* if it can be put into 1-1 correspondence with **N**, we will denote the cardinality (#of elements) of **N** by ∞, so that countable sets are said to have a cardinality = ∞.

15. Given **N**={1,2,3,4,...} and **B**={4,5,6,7,...}

How many elements are in **N**, are in **B**?

15.1 *Show **N**= **B** + {1,2,3} -> ∞= ∞+3

15.2 * Rule ∞=∞ +k

16. Given **N**={1,2,3,4,...}and **B**={2,4,6,8,...}

How many elements are in **N**, are in **B**?

16.1 *Show **N**= **B** + {1,3,5,7,.....} -> ∞= ∞+∞ =2∞

16.2 *Rule: $\infty = k\infty$

17. Recall that the cardinality cf $\mathbf{N}=\{1,2,3,4...\}= \infty$ and that **R**=all real numbers has cannot be put in 1-1 correspondence with **N**, so that the cardinality of R is not r al to ∞, that is, $\#(\mathbf{N}) < \#(\mathbf{R})$, and $\#(\mathbf{R}) \neq k\infty$ or $\infty+k$ since these are both equal to ∞.

17.1 Does this mean that there exists one type of infinity representing the cardinality of **R**, say ∞_1, that is bigger than the regular infinity ∞ ?

C. Evaluation

18. What does ∞ mean?

19. Is there a difference between mathematical infinity and infinity

outside mathematics? What is this difference?

20a). Is infinity a number? Why?

b). Could we treat ∞ as a number when talking about the number of elements in an infinite set? Why?

21. Is there a difference between talking about ∞ with respect to the number of elements in a set and when we say the limit as x goes to infinity or when we say the infinite sum of a series. Why?

Justification of the Experiment and Its Questions

A. Equality of Cardinality.

1&2. The purpose of these examples, and their corresponding questions, is to demonstrate the methods for obtaining the cardinality of a set. In these examples I wish to show that, for small finite sets, we depend on counting.

3&4. The purpose of these examples, and their corresponding questions, is to provide an instance where it is more expedient to use the idea of 1:1 correspondence to establish equality between finite sets. That is, when considering large finite sets, where counting is tedious, we can more readily determine the rule of mapping, than we can count, to decide on equal cardinality.

5. The purpose of this question is to bring students to the conclusion that, when considering infinite sets, we can only use the method of 1:1 correspondence to establish equality of cardinality; this being the case since it is impossible to count the elements. I feel that this is an important distinction because it shows that even though one cannot count the elements of infinite sets, they still can be dealt with mathematically.

6&7. The purpose of these examples, and their corresponding questions, is to see whether the student has both accepted the idea of 1:1 correspondence and its result that we can determine the equality of cardinality between infinite sets. These examples also are to aid in the construction of a general rule for addition, that is, if for every N and B where $B=\{b:b=n\pm k,\ k>0\}$ N and B have the same number of elements. This result will then be used to show that $\infty\pm k=\infty$, that is, will be used in the construction of what I term arithmetic of infinity.

8&9. The purpose of these examples, and their corresponding questions, is to determine whether the student has a working knowledge of the ideas. That is whether they can both answer the question and further provide the rule of multiplication which guarantee 1:1 correspondence and therefore equality of cardinality. This result will further be used to show that if for every N and B where $B=\{b:b=\pm kn,\ k>0\}$, N and B have the same number of elements. This will further be used in the section on the arithmetic of infinity.

10. The purpose of this example, and its corresponding questions, is the same as the above excepting that they will be have to generate the rule which combines the two rules which were previously established.

11&12 The purpose of these examples, and their corresponding questions, is to determine whether I have been successful in leading students to accept the equality of cardinality between two infinite sets, even when these infinite sets are distinct number systems. They also show that in some cases 1:1 correspondence, or the f(n), is not a trivial notion in that sometimes it is difficult to determine. Here I am trying to get away from the need to find a definite rule that establishes correspondence to that which considers whether the set is listable in terms of N. These examples also set the stage for the surprising conclusion that R is not listable in of N.

13. The purpose of this question is two fold; it shows that if there is no rule between two sets then they cannot have the same cardinality, and also, it gives justification for the claim that if N and R do not have the same cardinality and if the cardinality of N is ∞ then there must exist one infinity which is greater than another. For students who take infinity to be the largest value, to be unattainable, or existing only by imagination, this, I believe, will cause such a conflict that some idea will have to be rejected.

B. Cardinality of N and `Arithmetic' of Infinity.

14. The purpose of this example, and its corresponding question, is to bring in the idea of the cardinality of the set N

as being ∞. This idea will be used to establish the 'arithmetic of infinity' which will be used in reference to show that one can treat infinity as a number in that it allows algebraic operations. This definition will also permit me to show that there is an infinity, representing the cardinality of R, which is greater than another infinity (where#N = ∞). Returning to the 'arithmetic' I will also be able to show that **c** is not some operation of $\aleph_0$, that there exist two distinct orders of infinity.

15. The purpose of these examples is to show that addition of any finite cardinal number to the cardinal number ∞ is ∞. This will allow for the extension of the rule that $\infty=\infty\pm k$.

16. The purpose of this example is to show that the multiplication of any finite cardinal number to the cardinal number ∞ is ∞. This will allow for the extension of the rule that $\infty=\pm k\infty$.

17. The purpose of this example, and its corresponding question, is to bring to the fore that indeed we can say that there is one infinity **c** which is greater than another infinity $\aleph_0$. This, as stated repeatedly, will help in the acceptance of an actual mathematical object called infinity.

C. Evaluation.

18-21. All evaluation questions are those used in the clinical interview, and thus are used as an indicator of how successful this Teaching Experiment has been in 1) altering students view of infinity, 2) that this object is outside of physical-temporal considerations, 3) making students see that there is indeed a difference between mathematical and non-mathematical infinity, 4) that infinity can be considered as an existing mathematical object, 5) that it is the context that determines the nature of infinity and, 6) that mathematically infinity may be considered either as potential or actual, depending on the context.

§ 4.4 Results- *(of the Teaching Experiment)*

Before the teaching episode each student was reminded to be as explicit as possible in explaining their thoughts and answers and that I was not interested in 'correct' answers. They were further told that some, if not all, of the material that was to be shown was probably all new to them so that they should not feel intimidated. Again, each student was taught individually; each session lasted about one hour and was recorded via tape recorder. I would like here to make a brief note regarding the eagerness of all of the students; three of the four called me after the clinical interview to inquire as to when the teaching experiment would begin and remind me that they were still interested in participating. Also, after the teaching experiment was completed we continued discussing Cantor's impact on both mathematical infinity and their on views of it, of particular interest was the *continuum hypothesis* and its consequences in physics.

A. Equality of Cardinality

Martin

While Martin accepts that counting is not a viable method for establishing equality of cardinality among infinite sets he is side-swiped in his consideration of the cardinality of the infinite set in itself (Q5.1) " ..well if you're dealing with an infinite set how can you really know how many elements are in that set". Once the distinction is made that we are only looking at whether two infinite sets have the same number of elements, whatever that number may be, he accepts the method of 1-1 correspondence (Q5.1) "I guess by dealing with what you dealt with here". Martin does, however, appear to lapse into the Bolzano mode of considering one set as a subset of the other (Q6) "there's no relationship except that one is more than the other, (Q10) well they correspond to this but the fractions don't come under there (Q11) ...if you take the positives and the negatives it would, but the 0..I can see the 0 posing a problem...I think I'm thinking too much". These assumptions do not, however, stop him from accepting the equality once the 1-1 relation is given. Martin was familiar with the proof for the uncountability of R, (Q13.1) "Isn't this Cantor's method?... That causes a lot of problems I remember that in (Math) 393."

Wayne

Wayne also claims that we cannot know the number of elements in an infinite set , yet, he readily accepts that we can determine the equality of cardinality of two infinite sets by establishing a 1-1 correspondence (Q5b) "....If we could find a correspondence between them we could say there would be the same number of elements...Although we may not know what that number is." (Q6) "O.k. We don't know what it is but they have the same number" Wayne likewise seems to be drawn to the whole-part relation between sets (Q12) ...I'm not sure, just looking at that n, I'm not sure you would have some a/b that would give you n but I don't know how you would look at that n and get back to this..." I think that this problem is the result of my showing numerous examples where there was a definite rule between the sets; when I switched the criterion of correspondence to 'listable in terms of N' the correspondence was accepted (Q12.1) "Like a table to look up, if you could build that table you could find that correspondence.... Yeah, o.k., right". The idea of 'listable' allows Wayne to claim, before the proof, that there would be no correspondence between N and R (Q13) " Well R is not listable....Because you simply can't, suppose you get some point, blah, blah, blah, you know, you can always get another point with one more blah. So you can't really list them."

Jean

Jean, in considering the equality of cardinality between two infinite sets, is also preoccupied with the number of elements in an infinite set but he does not claim that we cannot know, or define, the cardinality of N. (Q5) "I don't know if you ask in the first set is infinity...". To understand what comes next we must recall Jean's response that segment AB contained more points than segment CD was implicitly based on the whole-part relation (explicitly this translated into a difference in the distance between two points in each of the segments). The assumption that this is a global relation, as opposed to a contextual one, effected his acceptance of 1-1 correspondence as a viable condition for establishing equality (Q7.2) "...If you have an element in set B and if you want both of the sets to have the same cardinality then, well its impossible because the number you found is also part of the set N...here I see B more as a subset of set N...that's why". This reliance on the whole-part relation, and rejection of 1-1 correspondence, was challenged by the introduction of the following definition: A set S in infinite iff S has the same number of elements as one of its proper subsets. (Q7.2) " O.k. then...Yes now its good".

The problem is that Jean does not distinguish between a subset and a proper subset and now sees the 'subset' relation as being a rival criterion for establishing equality of cardinality (Q11) "Yes and no. Yes because N is a subset of Z and by the definition you gave me, they have the same cardinality, but no because because there's no 1-1 correspondence between them...(Q12) Well here I really can't see the rule but again, N is a subset of Q so". This assumption does not appear to be much of a problem when there is shown to exist a 1-1 correspondence, it does, however, interfere with the relation between R and N (Q13) "Yes (they have the same number of elements)...Because N is a subset of R and the sets they're both infinite sets". In spite of this Jean does come to accept that it is 1-1 correspondence that establishes equality of cardinality and more importantly accepts that if this criterion is not satisfied then the two sets cannot have the same number of elements (Q13.2) "O.k. I agree. I believe you".

Luc

Luc has no problems distinguishing between those cardinal contexts where it is appropriate to 'count' and those which require 1-1 correspondence. He uses a 'relationship' criterion before the method of 1-1 correspondence is mentioned. (Q3) "O.k. well its the same set multiplied by two, the second set, so there are the same elements". Luc also questions whether we can have an infinite cardinality, (Defn 5.1) "Yeah o.k....can we also say infinite cardinality?". Unlike the others, he is not concerned with the number of elements in N; he readily accepts that, whether the sets be finite or infinite, we can establish equality of cardinality through the 1-1 correspondence criterion (Q5b) "Yeah, if there's, if there's a 1-1 correspondence; if you can find a 1-1 inverse relationship.." One interesting result is that while Luc had no problem stating the correspondence rule between a finite set of natural numbers and a finite set of even numbers, when the same sets were given as infinite he was hesitant (Q8) "... plus 1, plus 2, plus 3....no. No because the first one is 1+1, then its 2+2".

The result that surprised me the most was Luc acceptance of the proof for the uncountability of R. Recalling his rejection of the validity of the method of proof by contradiction I thought that he would similarly reject this proof. One possible explanation is that he had decided (although he changed his decision when I did not recognized his answer as being correct), before the proof, that N and R could not be put into 1-1 correspondence (Q13) "Have the same cardinality, between the two of them? Urn...no, no because they're irrationals, you can't find a relationship p/q like the rationals". It appears that this proof was accepted as an explanation as opposed to a justification: this would be consistent with Luc's previously stated view on the role of proof in mathematics.

B. The Cardinality of N and Arithmetic of Infinity

Martin

Martin maintains his initial skeptism regarding the possibility of knowing the cardinality of N (Q14) "Well there's infinitely many elements in N...um. What are you talking about when you say the cardinality of N?". When he is presented with Defn. 14.1 he readily accepts that we could let ∞ denote the cardinality of N " O.k., no problem". The result that there exists one infinity greater than other does not appear to present a conceptual conflict for Martin, though he is surprised (Q17) "According to this no (there is no problem) um..." In fact this instance allows him to realize that there are different meanings to mathematical infinity, it also appears that the introduction of cardinal infinities permits him to accept infinity as an sensible (legitimate) mathematical object (Q17) "I guess like the previous, like what we went through before is like a, like I said before what I've been brought up to believe infinity is like... having one infinity greater than another infinity when infinity is *kind of a value* like a....but previously... when you see it down on paper, its like a...I guess it does kind of *make sense* now".

Wayne

This student had no difficulty accepting the defining of the cardinality of N as ∞. Recalling Wayne's constant claim of 'invalid operations' when dealing with infinity I assumed that he would have problems in accepting the arithmatization of infinity. This was not he case; in fact, this arithmatization allowed him to see how it is the context which defines the characterization of infinity (Q15.1) "Well I just, $\infty=\infty+3$, but in this context I can see that, yes". This realization is further shown in his reaction to the result that the exists one infinity greater than another (Q17) "....and if we say that N is infinite then, but then you would say that there is something that is bigger than the regular infinity in this context....In these, with these sets of rules with these operations, yes,its o.k.".

Jean

Although Jean initially believed that $\infty \pm k \neq \infty$ and $k\infty \neq \infty$ he had no problems accepting them as being equal in a cardinal context. The reason for latter belief was thus probably due to his previous assumption that the whole must always be greater than the part. The result of an $\infty_1 > \infty$ is not easily accepted by this student (Q17) "We have a big problem". The source of this problem appears to be Jean's belief that mathematical objects are the product of our imagination (Q17) "Well that's just what I'm saying, its hard to imagine something is greater than infinity...What you should say is that the cardinality of R is ∞ and the cardinality of N is smaller than ∞, so if you take the greatest its the cardinality or R, you say that this is ∞, you can always have something smaller than R."

Luc

Luc readily accepts the definition which entails the representation of ∞ as the cardinality of N, but, for whatever reason, he also states that this definition may turn out to be problematic (Q14.1) "No (no problems) but I'm sure there will be later..". This student has no difficulties with the 'arithmetization of infinity', in fact it appears that he has thought a great deal about these results, i.e., that $\infty=\infty+k$ and $\infty=k\infty$. (Q 15.2) "No (no problems). Actually I used this to disprove the existence of God". He claims the one result, however non-mathematical, is that if we connect the good with $+\infty$ and the evil with $-\infty$, and if these are the same $k\infty$, then there can be no God that distinguishes the two.

The mathematical usefulness of providinq this arithmetic was truly demonstrated with Luc. That is, without it, I feel that he would have assumed that ∞_1, the cardinality of R, was simply an operation of ∞, the cardinality of N (Q17) "If you went back to here to $\infty=\infty+k$...um.... I don't know I never though about this before. What if you're just multiplying infinity times a constant?"; recalling the arithmetic he was able to see these cardinalities as distinct (Q17) "Its completely different". Luc, like Martin, is also curious as to what other orders of infinity there could be, that is, if ∞ and ∞_1 are distinct then $\infty x \infty_1$ must be another infinity (Q17) "Yeah I understand that, but what if you were to multiply it ∞ times $\infty 1$ it would be a bigger infinity there would be..but you could get some other number".

C. Attitudes Towards Mathematics and Infinity Re-Assessed

Martin

It is clear that Martin has come to distinguish between the possible different mathematical meanings of infinity **(U_7)**; when asked what infinity means he says (Q18) "Um...According to what we've just seen or like what I thought it was before". In comparison to his seeing infinity as a value that 'cannot be reached' and 'encompasses everything' he now see it as (Q18) " ...its a quantity that can't be larger or smaller...its not necessarily like encompassed..it doesn't necessarily encompass everything." He does, however unsuccessfully, try, in the midst of this answer, to keep his previous view of 'encompassing everything' **(OO_3)** "....if one infinity is greater than another then that could go on and on also so, but um... cause I guess I still visualize it... I guess with my unsculpted mind its like um....I still envision infinity as still encompassing everything, but obviously with what you've just written down on paper I guess its **(U_6)**... I'll have to put more thought into it."

This questioning of beliefs is somewhat inhibited by Martin's assumption of a dependence relation between physics and mathematics **(CO_1)** (Q19) "....cause I mean in the real world the only thing...when you're talking about infinity in the real world its only basically, all you could really go into is physics and stuff like that. I mean...and that in a sense also encompasses everything that we know of so, no I don't think there's much of a difference...one is built on the other so." One interesting result here is that, because of the dependence relation, for Martin, there now has to exist, in the physical realm, one infinity which is greater than another (Q19) "So obviously in quantum mathematics if you have one infinity that's greater than another then I said you'd have to have, it would have to be possible one infinity greater than another... and that really blows your mind." Martin's idea of infinity has become based less on intuition and seem now to be accepted as meaningful, but not strictly mathematically meaningful **(U_5)** (Q19) "See every time you mention infinity I just try to place it in my mind I just think it would say the universe or whatever and its like....hard to visualize **(EO_1)**....but I guess um...since now its down on paper um **(U_2)**...and mathematics is basically the basis for all we know type thing then o.k."

This student hesitantly rejects the global characterization of infinity as a number; he admits that it is a value of some sort but even this is overturned on the basis that it cannot be located **(CO_2)** (Q20a) "..I wouldn't....where to place a number?. I mean you really can't say, you can't really keep writing on an on so they just say (ad) infinitum...**(EO_3)**I guess a limit value, but no." Infinity is accepted as a number in the cardinal sense **(U_6)**; it is capable of some type of location because it is relation to another order of infinity (Q20b) "...if one has to be bigger than another one has to be limited and if its limited I guess you could bring it down to like dealing with numbers in the normal way type thing". This distinction between the characterization of infinity is indicative of the more important separation of infinity as being potential (in that it occurs in a process) or actual (in that is occurs as the result of a definition) **(U_8)**.

These ontological considerations are likewise reflected in a shift in Martin's epistemological claims regarding infinity (Q20b) "I guess like I said before with the background I have I mean infinity has always been pushed to the side type thing, I mean this is the first, I must admit, time I've had to actually deal with infinity....you've made a solid attempt to discern infinity and shown me mathematically what can be done with it **(U_9)**. Like in the past its always been like infinity is infinite in that way it stays type thing and obviously what you've shown me here is kind of changed a little in the sense that you can deal with infinity and stuff like that, so yeah I guess we can know it **(U_1)**".

When explicitly questioned as to whether there is a difference between infinity as it occurs in a process and as it results from a definition it is more evident that Martin is making a distinction between the ontological and epistemological considerations of actual versus potential infinity **(U_{10})** (Q21) " (infinity is more obvious) As an object, as something you can mold, like um...as a mathematical object, its not pushed out to the side like infinity is infinity, like leave it alone type thing...So the other way it was like that's the way I always had it in the past you know. Like dealing with it through limits its like way out there .....there's nothing you can do about it, there's no more ignoring it anymore so... Its kind of scary when someone puts it in front of you and makes you realize it car be handled its like going from high school to university, but when someone puts in front of you its not so bad".

Wayne

Wayne changes his view of infinity and appears ready to separate out the different meanings of mathematical infinity **(U_6)** (Q18) " Well in terms of sets we're using it as the cardinality of sets. In these terms we see that there's different infinities, which is not my answer I had before." This awareness does not seem, however, to be based on a either contextual or potential/actual distinction **(OO_5)** (Q21) "....No I guess not. Cause you normally mean as x goes to infinity, you're presuming some set of numbers so I believe you're taking it as the same infinity.... I suppose you could consider the sequence as a set but I'm not sure".

Two views that appear to be successfully altered were those dealing with the object status **(U_5)** of infinity; initially Wayne claimed that infinity represented a mere directional adjective and thus we could not look at infinity as if it were a object **(EO_5)**. Related to the former shift, instead of 'as x increases without bounds' he says (Q21) "..as x goes to infinity", in relation to the latter he says (Q20a) "...in this context **(U_7)** I would say it is a number, we don't know what it is, but it is a number....I don't know how you could work it like a number, like if I look at this equation here $\infty=\infty+k$ can I just drop one, can I bring it over?....So its not a number in that sense....it can be used like a number for certain comparisons in the number of elements of sets...and then to determine if one comparison is greater than another, but it can't be used as a number in a mathematical equation in that sense."

The above change in the object status of infinity is, I feel, also responsible for Wayne's new epistemological and ontological view of mathematical infinity **(U_9)**. Since, previously, infinity was only granted status as an existing *idea* it was claimed that we could not objectively (mathematically) know this idea **(EO_2)**. Now that infinity is seen as an object, though perhaps not a number, the obstacles related to how we know it (Q21) "I think we've just done that" **(U_7)** and how we establish its existence (Q21) "Yes it exists now" **(U_5)** appear, if not globally resolved, then at least, locally settled. This change in view seems to further allow for the distinction between the characterization of infinity as actual or potential **(U_8)** (Q21) "Well it makes the idea of infinity as a game, as a useful tool or notion clear to me...but the idea of infinity per se, as something being infinite um...we deal with it, we do and we don't".

Jean

Jean characterizes infinity in an intuitive manner and does not seem to distinguish mathematical form non-mathematical infinity **(OO_3)** (Q18) "It means um...well its everything and its unending **(CO_2)**... (Q21)....since we don't know what is infinity its hard to say if its the same." Regarding the epistemological considerations of infinity we simply cannot know infinity **(EO_2)** (Q20a) " well maybe its a number...its greater than the greatest number people know ". There is some confusion as to the ontological status of infinity which arises by way of his desire to keep infinity as an intuitive/potential idea yet still manage the actual cardinal object **(OO_4)**. That is, the actual/potential dichotomy has been made apparent to Jean but he appears to have no 'structural' framework from which to work his way to resolve it (Q17) "I don't think it exists, maybe it exists but um...but nobody will ever count **(CO_2)** the numbers up to these points. (Q21) Yes (it exists) **(U_5)**....Well you just showed me by the definition of cardinality" **(U_1)**.

Luc

Luc's idealism with respect to infinity was somewhat altered by the teaching experiment; he now sees infinity as a mathematical symbol**(EO_5)** as opposed to a global idea.**(U_6)** (Q18) "....Its a symbol for a very large number that represents um...." He still appears, however, to link infinity with some sort of movement **(CO_2)** (Q18) "I don't know if rate comes in here, different rates; one infinity is growing faster, different infinities grow at different rates I guess. The real number rate is bigger than the natural number rate, the real is bigger". By considering the nature of physical entities as being necessarily (fundamentally) finite this student makes the distinction between mathematical and non-mathematical infinity **(U_3)**" I think so (there is a difference), because I think that in mathematics I don't know if there's a notion, I mean like in math, you have less use, or need, of a fundamental particle, of fundamental distance or mass than you do in real life."

Luc does not appear to distinguish between potential (process defined) and actual (object defined) infinity **(OO_5)**"No I think infinity means the same thing". He does seem to accept infinity as a number **(U_9)** "Um...see I, I was wondering I think now you can". Luc is still, however, unsure of its ontological status; he agrees that infinity exists in a mathematical sense "In a mathematical sense I think so" but does not know what existence means **(OO_1)** or how it is to be established **(EO_1)** "I don't think you ...how can you prove infinity?...I don't know....I don't think you can really know that it exists, can you?" Once given a directive for establishing mathematical existence, he does see that the cardinal definition is one such way **(U_1)** " Yeah, o.k. (it exists)". This not only helps him to accept infinity ontologically, but improves his epistemological grasp **(U_2)**" I don't know if you could (know it), I guess we just did that eh?..now I'm contradicting myself.....I guess you could eh".

There is yet another epistemological distinction that results, for Luc; one can know mathematical, but not non-mathematical, infinity **(U_2)** "No I don't think so (that we can know infinity outside mathematics), I really don't think so". In the end Luc does come to accept as an existing and sensible, though perhaps not explicitly meaningful, mathematical object **(U_{10})** "Yeah (its a mathematical object) but less defined then maybe...., but o k.".

CHAPTER 5.
The Conclusion

§ 5.1 Explanations

Before beginning my explanation of the results I should first like to present a table representing the occurances of students' obstacles to and acts of understanding. This information will not only serve as an aid in determining the students' successes but will shed light on the effectiveness and deficiencies of this study. That is, while I feel that the results were indeed positive they also indicated that much more is required to make students distinguish between actual and potential infinity and the role context plays in the determination of these characterizations.

Summary of Obstacles to and Acts of Understanding Infinity

Clinical Interviews:

	CO	OO	EO	U
M	1	1,3,4,5	1,2,3,5	5,6
W	3	4,5	5	1,2,4,5,6
J	3	3,4,5	1	2,4,5
L	1,2,3	1,3,5,6	1,2	1,2,3,4,5,6,7

Teaching Experiments:

	CO	OO	EO	U
M	1,2	3	1,3	1,2,5,6,8,9,10
W		5	2,5	5,6,7,8,9
J	2	3,4,5	2	1,5
L	2	1,5	1,5	1,2,3,6,9,10

In relation to the cognitive obstacles we see by the absence of (CO3) that after the Teaching Experiment the students have come to accept infinity as a meaningful mathematical entity, but they now see more of a need to subject it to some space-time considerations (CO2). This view, however, does not seem to arise from the belief that mathematical objects are the same 'type of things' as physical objects; rather they seem to feel that the object infinity needs a location in the way that, say, 3 can be located. This result is closely linked with their belief that mathematical knowledge is justified by some physical or causal evidence (EO2); in this way we can see the interplay between epistemological obstacles and ontological obstacles: If no spatial (mathematical or physical) location can be found for infinity then how can one know it? The assumption here appears to be; 'I must *find* it before I know it'. (OO2) This I feel is the true problem with the understanding mathematical infinity; one has to grant it ontological status a-priori before one can make claims of knowledge.

Consider for a moment the paradox in the ontological argument for the existence of God; one has to assume His existence to prove it. That is, here, as in the case of infinity, the problem of circularity arises: How can we assume existence when that is the very thing we are trying to prove? At this point, however the analogy wears thin: the method of mathematical proof has not only the power to describe but also to define, that is, the cardinal definition of infinity not only justifies its existence but also causes it to be so. This marks the point of students' difficulties; they accept the describing (epistemological) power of mathematical proof but are either unaware or reject its defining (ontological) power. (An interesting note here is how Cantor managed to side-swipe this issue; he uses God to ontologically guarantee the, a-priori, existence of infinity and uses his *Mengenlehre* to epistemologically establish its existence.)

With respect to these two powers of proof, it appears that the students accept that the cardinal definition epistemologically justifies the existence of infinity but they do not see it as that which determines infinity as an ontologically actual mathematical object (OO5). Thus in relation to the students' ontological obstacles, though clearly there is a decrease in the number of such obstacles, students still assume that mathematical and intuitive infinity have the same sense, though perhaps not the same meaning. They now accept that mathematical infinity has both definition and meaning yet they still rely on their intuition to make sense of it. Thus while the formalization of infinity allowed students to accept infinity as an existing mathematical object it did not make the separations between either intuitive/mathematical or potential/actual explicit enough.

In reference to the separation of the intuitive and mathematical, students appear now to have an intuitively based idea of *mathematical* infinity rather than just a strictly *intuitive* idea. In relation to the potential/actual distinction, however, there appears to be a believed equivalence relation between existence and actuality: now not only does procedural (potential) infinity exist but it appears to be granted the same ontological status as the transfinite (actual), there is no distinction between the mathematically potential and the mathematically actual.

In light of the the preceding, there are two specific observances; students do not clearly understand the nature (status and method) of mathematical knowledge and existence, and they do not realize that it is the context and consistency within that context that determines the epistemological and ontological characteristics which define and describe a mathematical object. In this way the major deficiency of the teaching experiment was that it did not make explicit the difference between existence on the basis of infinity being either procedural (potential) or set theoretic (actual). Students did not see that the cardinal characterization of infinity as actual was contextual, local and not global.

Underlying most of these students' epistemological and ontological obstacles related to infinity is their assumption that there exists some type of dependence relation between mathematical reality and physical reality. While this undoubtably affects their view of the method of mathematics, in that they see proof as necessarily having an empirical component, it more generally affects their view of the nature of mathematics and the nature of mathematical objects. To avoid getting pulled into the constructivist/formalist debate I shall not consider which nature *should* cognitively precede the other: in either case it remains that students' constant cognitive connection of both mathematics and its objects with physical constructions, especially physics, undermines their desire and need to consider the ontological and epistemological status of mathematical infinity as being contextually determined.

That is, before students see the need to distinguish between mathematically potential and mathematically actual infinity, they must see the need to distinguish between mathematics in and of itself and mathematics as it relates to, or functions in, natural reality. It is unclear that these students see mathematics as a ideal (abstract or concrete) reality; the consequence of this being that they are unable to distinguish between the status of intra-structural relations, those which are necessary, and the inter-structural relations, those which are contingent. In the case where mathematical reality and physical reality are taken structurally to be in 1-1 correspondence all relations between them are thus taken as necessary.

Pre-experimentally students placed more emphasis on the correspondence which takes physical considerations (epistemological and ontological) and applies them to mathematical knowledge and objects. The result being that infinity in having to abide by both empirical and space-time restrictions could only exist potentially. Post-experimentally, students seem to have reversed the correspondence, that is, they see the epistemological and ontological set theoretic considerations as those which establish infinity as actually existent, both mathematically and physically. While set theoretic considerations were accepted by the students as a viable means of establishing infinity as an existing mathematical object, the actually that it entailed was not seen as a strictly mathematically and contextual consequence. Hence, the assumption of a 1-1 correspondence between mathematics and reality causes students to believe that the ontological and epistemological status of infinity, be it potential or actual, must necessary be maintained across realities.

One claim against this study may be that it has taken the mathematically inappropriate view that infinity, mathematical or physical, is potential and simply replaced it with a likewise inappropriate view that infinity is actual. To this I would respond that the latter view, while just as inappropriate, provides a better basis from which one can challenge and perhaps even overcome students' obstacles related to the understanding of mathematical infinity. The reason being that more often than not the characterization of potentiality is taken as that which undermines existence; that is, if it is potential it does not exist and if it does not exist we can neither conceptually accept it nor epistemologically grasp it. In this way, it is much more difficult to conceptually accept infinity as an object, mathematical or other, if it is taken to be potential than if it is taken to be actual. In establishing the actual existence of infinity we now stand in a better position to demonstrate that both potentially and actuality are but mere characterizations of an existent mathematical object called infinity. This study has been successful in that it lead to the extensification of students conceptual structure in such a way that now it includes infinity as an actually existent object. What needs to be considered now is the intensification of this structure; that is, it must be shown that existence arises within the constraints of a mathematical structure and actually comes by way of the constraints of set theory.

§ 5.2 Implications

The above explanations carry with them both mathematical and pedagogical implications. Of the former it is clear that students do not have an appropriate view of mathematics, let alone infinity. Students' views of both mathematics and infinity are based on an intuitive belief in a 1-1 correspondence between mathematical and natural (physical) reality. Whether infinity is seen as potential or actual, its existence and knowledge are not seen as arising within the constraints of a mathematical reality. Indeed at the very heart of the matter is the pedagogical separation of the nature of mathematics from the function of mathematics. It would appear that in our attempt to make mathematics accessible we have put more emphasis on how it *functions* in our (natural) reality than we have in how it *is* in mathematical reality; in doing so, we have lost track of its essence. By this I mean that in attempting to make sense of mathematics we have relied far too much on exterior (physical) applications to ascribe meaning; the effect of this being the belief that object meaning and justification are now taken to be considerations which arise within the physical domain.

Pedagogical Implications

We need to consider;

- That many of students' problems in understanding and separating the nature and the function of mathematics are the result of a lack of distinction between mathematical and natural reality.
- That this distinction requires the separation of respective epistemologies. They must accept that the way (axiomatic proof) and status (a-priori) of mathematical knowledge differs from the way (hypothetical proof) and status (a-posteriori) of scientific knowledge.
- That this distinction requires the separation of respective ontologies. They must accept that it is definition which establishes existence and consistency which guarantees it. As opposed to the scientific belief that it is empirical evidence which establishes and experiment which guarantees existence.

- That students must see the mathematical object as being outside the space-time domain and the real object as being bounded by it; only then will they appreciate the difference between the characterizations of necessary or contingent.
- That students must see the need to construct a conceptual structure which would provide the basis necessary to distinguish those considerations which are mathematical from those which are natural.
- This conceptual structure must further be such as to account for the intersection of mathematical and natural reality; to see that the mathematics may be seen as that which is of itself (pure) or as that which functions in natural reality (applied).
-That students must be given the opportunity to consider the epistemology and ontology of pure mathematics, wherein object are free, against that of applied mathematics, wherein objects are bound by space-time.

Mathematical Implications

We need to consider

- That students conception of infinity in primarily based on an intuitive/physical sense of the term; this, combined with a belief in a inter-dependence of physical and natural reality, leads to a global characterization of infinity as potential and therefore non-existent object.

- That students must be given situations were infinity is characterized other than potentially; otherwise they will continue to link potentiality with non-existence.

- That students are not aware that the characterization of mathematical objects is a local, context dependent, matter; they see mathematical characterizations as being global in the sense of extending meaning across all of mathematics and across all of natural reality.

- That students must be shown that it is the particular context that defines the characterization of the object and in doing so limits its applicability as such an object, i.e., that definition not only creates the object but localizes it.

- That students who are shown the set theoretic definition of actual infinity appear to give up their intuitive view of infinity as non-existent, but, since potentiality is so closely linked with non-existence and since they do not appreciate the role of context, they come to see infinity as being globally characterized as actual.

- That students must be shown that actual (set theoretic), concrete (representational- Riemann sphere) and potential (procedural) are but descriptors of the mathematically existent object called infinity.

If one phase could be said to underly the results of this study, its explanations and implications, it would be 'implicit assumptions'. Students are implicitly assuming that mathematical existence and knowledge must conform to an assumed inter-dependent relation between mathematical and physical reality. The result of this being an inappropriate view of the nature and function of both mathematical proof and definition. Together these assumptions serve to inhibit the realization that mathematical meaning is defined solely by its mathematical context, and proved by consistency within that context. Educators implicitly assume that students will, through experience, come to distinguish between the mathematical sense of an object or relation, which may be gotten through reference to non-mathematical contexts, and its mathematical meaning. They assume that students are aware that references to non-mathematical contexts are being used as an heuristic device. Thus, in attempting to close that gap between mathematical sense, which is intuitively physical, and meaning, which is formally rigorous, educators also close the gap between applied and pure mathematics. consequently, students are left to assume that the reality base, for epistemological and ontological considerations, are the same for either.

In conclusion then, I would continue to suggest that the introduction of actual infinity by way set theory is required if we want students to accept infinity as sensible and existent object. The problem remains, however, that should we not deal with the above mentioned implicit assumptions then we will never get students to the point of appreciating and accepting that mathematical infinity, or for that matter any other mathematical object or relation, is meaningful. At this point then, I add the claim that mathematical infinity will continue to be seen as a mathematically meaningless idea unless educators make the following explicit; 1) that mathematics is a reality in and of itself, 2) that mathematical epistemology and ontology are bound only by the constraints (proof, definition and consistency) of this reality, 3) that mathematical meaning and characterizations are context dependent and 4) that this context may have physical applications, but this need not be the case.

"In the use of this method (of infinities) the pupil must be awake and thinking, for when the infinite is employed in an argument by the unskillful, the conclusion is often more absurd." - Elisha S. Loomis [Maor: 1987, p.34]

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