ANALYSIS OF CRACK PROBLEMS IN THREE-DIMENSIONAL ELASTIC SOLIDS USING NEW RESULTS IN POTENTIAL THEORY

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ABSTRACT

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The strength and life-time of the structural components and materials in general largely depend on the presence of cracks or other defects. When structural components are subjected to external loads in engineering application, that results in a triaxial state of stress in the member, particularly in the vicinity of a crack. Successful prediction of fracture behaviour depends on the effectiveness of the stress analysis and therefore the solution of the three-dimensional crack problems plays an important role. In view of this, the thesis is aimed at analysis of three-dimensional mechanics of solids containing cracks.

In the area of three-dimensional fracture mechanics, investigations on the possible determination of the complete solution, namely, full space of the elastic field, are very scarce. There are only a few complete solutions, which are mainly based on a very complicated mathematical approach, namely, the use of various integral transforms and special functions expansions. Despite these achievements, there was

no general method for solution established, and each problem would require special consideration. As a consequence, even in those rare cases, when the complete solutions were obtained, the final results for the elastic field often were presented in terms of difficult infinite integrals with Bessel functions, which were of limited practical use and inconvenient for consideration of more complicated problems.

This thesis presents new fundamental complete solutions to different types of problems, which previously would not have been attempted. The method used to obtain these solutions is based on the new results in potential theory for the circular crack geometry, which were recently reported in literature. Along with the applications of those results and their further development, this work presents new method of solution with regard to another type of crack geometry, namely, the half-plane. The main advantage of the method used to obtain these solutions lies in its generality, which makes it possible to consider a vast variety of three-dimensional crack problems, including interaction problems. The exact solutions have been obtained in closed form and in terms of elementary functions, which makes it possible their immediate physical interpretation and practical use. This suggests that the approach used in this thesis is a powerful tool for investigation of three-dimensional crack problems.

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NOMENCLATURE

a	radius of crack
A_{ij}	five elastic constants in transversely isotropic body
c	introduced notation
E	Young's modulus of elasticity
$F_k(z)$	three potential functions (k=1,2,3)
$F(z_k)$	one harmonic function (k=1,2,3)
$g(x)$	function inverse to both ℓ_1 and ℓ_2
$g^*(u)$	function inverse to both $\ell_1^*(y_0)$ and $\ell_2^*(y_0)$
G	shear modulus of elasticity
G_1	transversely isotropic elastic constant
G_2	transversely isotropic elastic constant
h	introduced notation
$h^*(u)$	introduced notation
H	transversely isotropic elastic constant
j	introduced notation
$K(M,N_0)$	Green's function
K_1	mode I stress intensity factor
K_2	mode II stress intensity factor
K_3	mode III stress intensity factor
ℓ_1	distorted length parameter in isotropic body
ℓ_2	distorted length parameter in isotropic body
ℓ_{1k}	distorted length parameter in transversely isotropic body
ℓ_{2k}	distorted length parameter in transversely isotropic body
ℓ_1^*	introduced notation
ℓ_2^*	introduced notation
$\mathcal{L}$	integral operator
$\mathcal{L}^*$	integral operator
m_k	transversely isotropic elastic constants (k=1,2)
$M(\rho,\phi,z)$	coordinate point in space

$N_0(\rho_0,\phi_0)$	coordinate point on the plane $z=0$
P	normal force in the z direction
q	introduced complex length parameter
Q_x	arbitrarily located force in the x direction
Q_y	arbitrarily located force in the y direction
Q_z	arbitrarily located force in the z direction
R	distance between two points on the plane $z=0$
R_0	distance between two points in the half-space $z>0$
s	introduced notation
s^*	introduced notation
t	introduced non dimensional notation
T_x	tangential force in the x direction
T_y	tangential force in the y direction
T	introduced complex tangential force
u	introduced complex tangential displacement
u_x	tangential displacement in the x direction
u_y	tangential displacement in the y direction
w	normal displacement in the z direction
z	coordinate of the point in the half-space $z>0$
z_k	introduced notation
α	transversely isotropic elastic constant
β	transversely isotropic elastic constant
γ_k	transversely isotropic elastic constants (k=1,2,3)
δ	Dirac's delta function
Δ	two-dimensional Laplace's differential operator
ζ	introduced non dimensional notation
$\eta(x)$	introduced notation
$\eta^*(u)$	introduced notation
κ	Green's function
$\lambda(k,\psi)$	introduced notation
$\lambda^*(a,b)$	introduced notation
Λ	one-dimensional complex differential operator
Λ^2	two-dimensional complex differential operator
ν	Poisson's ratio

ξ	introduced non dimensional notation
ρ	cylindrical coordinate
ρ_0	cylindrical coordinate
σ_x	normal stress component in the x direction
σ_y	normal stress component in the y direction
σ_z	normal stress component in the y direction
σ_1	introduced inplane stress component
σ_2	introduced complex inplane stress component
τ_{xy}	tangential stress component
τ_{yz}	tangential stress component
τ_{zx}	tangential stress component
τ_z	introduced complex inplane stress component
ϕ	cylindrical coordinate
ϕ_0	cylindrical coordinate
$\Phi_1(z)$	potential function
$\Phi_2(z)$	potential function
χ_k	complex harmonic functions (k=1,2,3)

CHAPTER 1

INTRODUCTION, BACKGROUND AND OBJECTIVES

1.1 GENERAL

The study of fracture mechanics is based on the assumption that all engineering materials contain preexisting defects and flaws in the form of cracks, voids or inclusions which have a deteriorating effect on their strength and hence a material will fail eventually under excessive applied load. In order to have a correct estimation of the life-time of a structural component under external loading, it is important to know the stress distribution caused by the presence of cracks.

Conventional failure mechanisms can be roughly classified as ductile on the one hand and brittle on the other. For ductile fracture, associated with yielding or plastic flow before breakage, the defects such as dislocations, grain boundary spacings and precipitates tend to distort the crystal lattice planes. Brittle fracture, which takes place before any significant plastic flow occurs, originates at larger defects such as inclusions, sharp notches or cracks. It should be clear that a material may behave in a ductile or brittle manner, depending on the temperature, rate of loading and other variables present. Thus, any material is characterized as ductile or brittle based on the ductile or brittle state of fracture behaviour. Usually, the loading of a cracked body causes inelastic deformation and other nonlinear effects in the neighborhood of the crack tip. In materials, exhibiting brittle behaviour, the amount of inelastic deformation in the vicinity of the crack tip is negligible compared to the crack size and other length parameters of the body. In such cases, the theory of linear elastic fracture mechanics (LEFM) is sufficiently justified to address the problem of stress distribution in the cracked body. The present study is focused on the determination of stress and displacement distribution of some new fundamental problems in LEFM.

The foundation of the contemporary theory of fracture was made by Griffith (1921) [1] in his systematic study of a size effect on the strength of solids. However, long before 1921, a number of results had appeared which gave evidence of the existence of size effect. Even Leonardo da Vinci (1452-1519) made tests to determine the strength of iron wires. He found that for wires of constant diameter the strength had an inverse relationship with the length. In the 19th century a few analogous experiments were conducted and it was established that the strength of a short iron bar is higher than that of a long one of the same diameter. After the First World War a series of tests were conducted on notched-bar specimens at the National Physical Laboratory, Teddington, England. The results obtained showed that the work at fracture per unit volume decreased as the specimen dimensions were increased. A reasonable explanation of these results can be attributed to the fact that all structural components contain flaws which have a deteriorating effects on the strength of the materials. The larger the volume of the material tested, the higher the probability that large cracks exist which reduce the material strength.

To design for structural reliability under these circumstances, an engineer needs to know how and when the crack might grow further and run, with the component ultimately breaking apart. When cracks do suddenly run, the outcome is often dramatic and sometimes catastrophic. A series of examples of serious in-service failures were presented in very recent article by Sinclair (1993) [2]. The collapse of Kings Bridge in Melbourne, Australia, in 1962; the Point Pleasant Bridge in West Virginia in 1967; an oil rig disaster in the North Sea in the late 1970s; ships breaking virtually in two, including a U.S. tank barge in 1972 while in dock in calm seas; train accidents, including an express train derailment in the United Kingdom in the 1970s; and commercial aircraft crashes of de Havilland Comets in the 1950s and of DC-10 in 1979.

Although the frequency of such failures is decreasing as a result of better design, inspection and maintenance, the preventive activity and cost has not diminished. A 1983 survey shows that the expenses associated with fracture only in U.S. are $119 billion per year and they are not likely to decrease due to massive advances in some engineering fields like: aerospace, transportation and nuclear power generation. In all, fracture mechanics is going to play an important role in engineering for years to come.

1.2 BRIEF BACKGROUND AND LITERATURE SURVEY

Crack problems in the mathematical theory of elasticity are of two distinct kinds. The first group of problems concerns the determination of the stress fields governed by the equations of idealized plane strain or plane stress. The first mathematical solution of a stress field in a linear elastic infinite flat plate weakened by an elliptical hole and subjected to uniform tension (Fig.1.1) was done by Inglis [3].

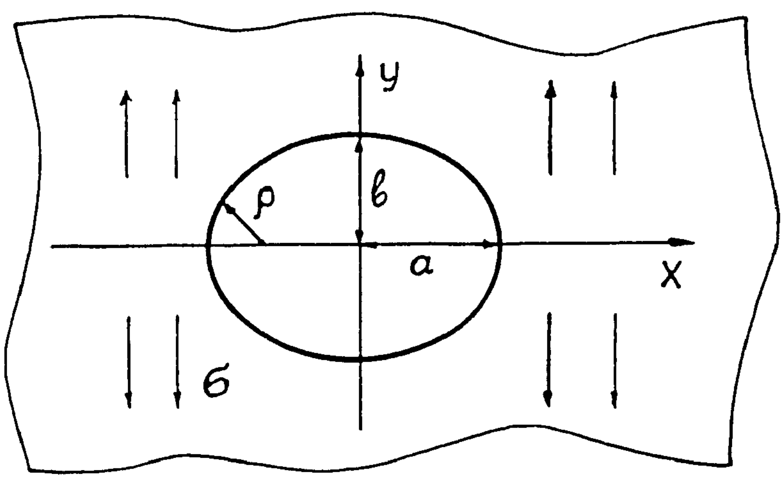

Fig.1.1 Elliptical hole in an infinite plate.

As a result of his solution the maximum stress σ_{max} was given by the product of the applied stress σ, with the stress concentration factor k, i.e.

$$\sigma_{max} = \sigma(1+2\sqrt{a/\rho}) = \sigma k \ , \tag{1.1}$$

where a is the major semi-axis of the ellipse, and ρ is the radius of the curvature.

From the engineering design point of view it should be required that

$$\sigma_{max} < \sigma_y \ , \tag{1.2}$$

where σ_y is the yield stress of the material comprising the plate.

Neuber [4] used methods similar to Inglis' solution to find the stress concentrations for different profiles which would be approximated by elliptic and hyperbolic curves. These solutions were verified by experimental results obtained by methods of photoelasticity and represented valuable information for engineers. This information is used mainly for the consideration of influences of repeated or cyclic loading applied to the shafts with fillets and keyways.

Although the solution given by Inglis was the pioneering work, interest in such calculations appear to stem from Griffith's [1] paper. Griffith made use of Inglis' calculations, where he considered the case in which the minor semi-axis of the ellipse b=0, i.e. when the ellipse degenerates to a straight line. For this reason a crack, which in two-dimensional diagram is represented as a segment of a straight line, is called a Griffith crack (Fig 1.2).

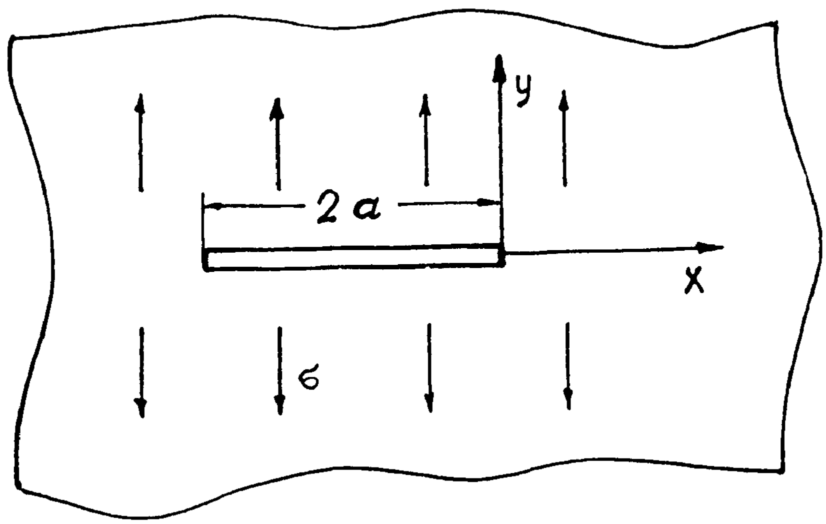

Fig.1.2 Crack in an infinite plate.

In this case $\rho \to 0$, consequently k becomes unbounded and σ_{max} goes to infinity, regardless of any applied stress greater than zero and hence the cracked plate has to fail. However it is not the case, and in order to overcome this difficulty of physically impossible infinite stresses Griffith proposed an energy criterion for fracture consisting of the following. The crack will spontaneously propagate under the action of applied load only if the total energy of the system will be decreasing, namely, when the energy release rate, as a result of the crack extension, will reach or exceed the rate at which work must be done in order to form a unit of new surface. The strain energy release rate of the Griffith crack of length a is defined by:

$$G = \frac{1}{2}\frac{\partial W}{\partial a} \ . \tag{1.3}$$

where W is the strain energy.

The Griffith criterion for failure was that the crack will propagate when the applied tension reaches the value given by the equation

$$\frac{\partial}{\partial a}(W-U)=0 \ , \tag{1.4}$$

where U is the surface energy of the crack. Griffith expressed U in terms of a "surface tension" γ as $U=4\gamma a$. Hence, using (1.3) the Griffith criterion becomes

$$G = 2\gamma \ . \tag{1.5}$$

From Inglis' solution of the problem of the elliptic hole in an infinite plate under tension, Griffith obtained the following expression for the strain energy release rate:

$$G = \frac{\sigma^2 \pi a(1-\nu^2)}{E} \ . \tag{1.6}$$

And the Griffith criterion, namely formula (1.5), leads to the following relation for the critical stress producing fracture:

$$\sigma_c = \left[\frac{2\gamma E}{\pi a(1-\nu^2)}\right]^{1/2} . \tag{1.7}$$

Westergaard [5,6], developed a semi-inverse method based on a complex representation of the Airy stress function suitable for a class of two-dimensional problems including the case of cracks. Williams [7], using a power-series solution, obtained the singular symmetric and antisymmetric crack tip stress field. However, the general application of the singular stress field was first recognized by Irwin [8,9], who introduced the concept of the stress intensity factor (SIF), which measures the strength of the singular stress field. Since then, a vast number of publications have appeared in the literature concerning solutions of crack problems with emphasis on the SIF. Irwin proposed an approach based on the following. If the tensile stress σ_y is exerted on the faces $a \le |x| \le a+\delta a$ of the slightly enlarged crack, then the work done to close up the crack to its original length a must be equivalent to $4\gamma\delta a$ for a brittle solid. He also suggested that in case of a partially brittle solid the plastic strains, which will develop at the tip of the Griffith's crack, will not result in significant loss of accuracy in the calculations. In view of this Irwin replaces 2γ by a constant G_c called *crack driving force*. Thus, the work done by the forces which are applied to the crack edge over an infinitesimal distance of δa equals the energy release rate and is:

$$G_c = \lim_{\delta a \to 0} \frac{1}{\delta a} \int_0^{\delta a} \sigma_y u_y \, dx. \tag{1.8}$$

If the SIF is defined as K_1, then

$$\sigma_y = \frac{K_1}{\sqrt{2x}} . \tag{1.9}$$

The asymptotic behaviour of the crack opening displacement (COD) near the crack tip is

$$u_y = \frac{4(1-\nu^2)}{E} K_1\sqrt{\frac{-x}{2}} , \qquad x<0. \tag{1.10}$$

In the case, when the crack tip has moved through the distance δa the expression in (1.10) can be rewritten as:

$$u_y = \frac{4(1-\nu^2)}{E} K_1\sqrt{\frac{\delta a-x}{2}} . \tag{1.11}$$

The substitution of (1.9) and (1.11) in (1.8) and evaluation of integral, which turns out to be equal $\pi\delta a/2$, results in

$$G_c = \frac{\pi(1-\nu^2)K_1^2}{E} . \tag{1.12}$$

The energy criterion assumes the propagation when G equals a critical material value G_c or equivalently K_1 equals a critical material value K_{1c}. Therefore a criterion for reliable design would be

$$K_1<K_{1c}. \tag{1.13}$$

where K_{1c} is the critical SIF called the material toughness.

The second group of problems deals with the determination of the stress fields in a three-dimensional body in which the crack is in the form of, say, a flat disc.

In the three-dimensional case the strain energy release rate of a disk-shaped crack of radius a is defined by:

$$G = \frac{1}{2\pi a} \frac{\partial W}{\partial a} \ . \tag{1.14}$$

The Griffith criterion, formula (1.4), can be written in the same form, but with the difference that now the surface energy $U=2\pi a^2\gamma$. It can be observed that with this value of U and G given in (1.14), the Griffith criterion will have the same form as in (1.5).

The first work in this regard is associated with the name of Sack [10], who was treating a crack as a limiting case of an oblate spheroid. He obtained the solution for a penny-shaped crack subjected to uniform internal pressure. Sack has shown that in the presence of a crack of radius a the free energy of the solid changes by an amount:

$$W = \frac{8(1-\nu^2)}{3E} \sigma^2 a^3 \ . \tag{1.15}$$

Therefore the strain energy release rate is given by the equation

$$G = \frac{4(1-\nu^2)\sigma^2 a}{\pi E} \ . \tag{1.16}$$

The application of the Griffith criterion (1.5) gives the critical value of the applied tension, namely,

$$\sigma_c = \left[\frac{\gamma E \pi}{2a(1-\nu^2)}\right]^{1/2} . \tag{1.17}$$

Comparison of formulae (1.7) and (1.17) shows that the critical tensile stress of the three-dimensional model differs from the plane strain Griffith result by factor $\pi/2$.

Further, using the theory of Hankel transforms, Sneddon [11] gave the stress and displacement fields around a penny-shaped crack in an infinite solid subjected to a uniform tension normal to the plane of the crack. Segedin [12] has studied a problem, when the solid is loaded by a uniform shear parallel to the crack plane. Green and Sneddon [13] found the stress distribution near an elliptical crack. A thorough study of three-dimensional crack problems is given in Sneddon and Lowengrub [14] and Kassir and Sih [15]. A vast amount of work exists on three-dimensional crack problems and it will be more appropriate to make references later on in forthcoming chapters in direct relation to the present research.

In the present work an account is given of solutions in the mathematical theory of elasticity relating to three-dimensional crack problems. The method used here is due to Fabrikant [16,17] and based on new results in potential theory. The applications of that method in this work have different characters; they vary from the direct use of the already known results to their further development. In general it has allowed the solution to problems which were not considered before, thus, most of the results presented here are new. A brief, but comprehensive description of this method will be presented in Chapter 2.

1.3 OBJECTIVES OF THE PRESENT INVESTIGATION

The main objective of this work is the further development of the new method in application to different configurations of cracks, new types of loading, as well as some studies of contact problems. This purpose will be achieved by:

1) Succinct description of the method used, namely, integral representation for the reciprocal of the distance between two points in polar coordinate space, introduction of the $\mathcal{L}$-operator and investigation of its properties. Combination of the $\mathcal{L}$-operator and Abel's type operator leads to a new two-dimensional integral equation which can be solved exactly and in closed form.

2) Consideration of new problems of penny-shaped crack subjected to linear tangential and normal loading. Particular solutions to be obtained for transversely isotropic and isotropic solids.

3) Obtaining of the Green's functions for the external circular crack problems which has not even been attempted in the literature. It is of great value to express these functions in terms of elementary functions, since this would allow their use as a basis for solving more complicated cases of specific distributed loading, interaction problems *etc.*

4) Using the limiting case of solutions obtained for the internal circular crack to obtain the relevant solutions for

a half-plane crack subjected to normal and tangential point force loading.

5) Development of a new direct method of solution for half-plane crack and punch problems, based on already different integral representation of the reciprocal of the distance between two points in cartesian coordinate space, as well as a new type of $\mathcal{L}^{*}$- operator. Investigation of the properties of $\mathcal{L}^{*}$- operator and indication of its differences from the one introduced previously.

6) Consideration of general weight functions for the penny-shaped crack in transversely isotropic and isotropic bodies. The closed-form solutions for the elastic field of a penny-shaped crack coupled with the reciprocal theorem will be used to derive closed-form expressions in terms of elementary functions for the crack opening displacements and stress intensity factors of a penny-shaped crack loaded by an arbitrarily located force. A general outline will be made for the similar type of problem of interaction between an external circular crack and arbitrarily located point force.

1.4 PRACTICAL IMPORTANCE OF THE PRESENT INVESTIGATION

Even though the problems treated in the present study are idealized, all the results of the problems investigated can be used for the stress analysis of the various bodies with cracks, provided that the crack size and dimensions of the body are in proper relation to each other, namely,

1) For the problems of internal circular crack, which will be considered in Chapter 3, the size of the crack should be small in comparison with the dimensions of the body.

2) For the problems of external circular crack, which will be considered in Chapter 4, the region connecting two half spaces should be small in comparison with the crack faces covering the region $z=0^{\pm}$.

3) For the problems of semi-infinite crack, which will be considered in Chapter 5, the practical significance is in their applicability to any real crack where the distance from the point of application of force to the crack boundary will be much smaller than the radius of curvature of the crack boundary.

The very fact that a great amount of effort has been spent by many prominent researchers on the determination of the stress distribution in three-dimensional bodies with cracks is a clear indication of the fundamental importance of the present investigation. However, despite the success in obtaining analytical solutions to some main stream problems, there were many limitations. In most of the cases, solutions were dealing with the determination of the stress and displacement field in the plane of the crack. Only a few of the problems were solved for the complete field of stresses and displacements, while even in those cases the results were expressed in terms of complicated integrals. The ability to have a complete solution, especially when it can be obtained in closed form and in terms of elementary functions, shows the fundamental importance of the present investigation.

1) By using the complete solution and implementing the reciprocal theorem, the morc complicated problems of interaction between crack and external arbitrarily located loading may be considered. This will be done in Chapter 6.

2) The results of a complete solution can be used for the consideration of other complicated problems such as: interaction between cracks or interaction between punches and cracks.

3) The present investigation is of paramount importance for the future development of the boundary force method which will allow the solution to problems of cracked bodies of finite dimensions. The simple idea with its quite difficult implementation has a very essential precondition, namely, a complete solution to the field of stresses and displacements.

In this regard it will be quite appropriate to mention the article by Atluri [18]. It is shown there, how the systematic generic procedure was developed for the evaluation of the required derivatives of potential function and subsequent determination of the "VNA" complete solution. The implementation of the "VNA" solution in conjunction with the Finite/Boundary element method should allow one to solve the problem of finite body with crack (or multiple cracks) by using the iterative procedure.

Finally, in Chapter 7 the conclusions and recommendations for the future investigation based on the present fundamental work will be given along with the short description of the above-mentioned boundary force method.

CHAPTER 2

DESCRIPTION OF THE METHOD

2.1 INTRODUCTORY REMARKS

The three-dimensional crack problems can be formulated as mixed Boundary Value Problem (BVP) in potential theory for the half-space $z \geq 0$. The term "mixed" is used to distinguish these types of problems from the "uniform" problems when the conditions prescribed over the boundary are of Dirichlet, Neumann or Dirichlet and Neumann type. The exact solution to the uniform BVP of linear elasticity for an infinite layer can be obtained with help of Hankel integral transform.

The mixed BVP are the most difficult to solve, due to the very fact that the conditions prescribed over the boundary are of different type, namely, the potential is prescribed over a part of the boundary and its normal derivative over the remaining part. For example, if the line of division of the boundary conditions is a circle, it is convenient to apply the same Hankel transform for the solution of these problems. However, this method will not anymore result in an exact solution in terms of quadratures as it was in the case of uniform problems. It only allows to bring the problem to the so called "dual integral equations" which are in turn, after special consideration, reduce to a Fredholm integral equation of the second kind, which allows an effective numerical solution.

At the same time, it must be indicated that the mixed BVP are among the most important in various engineering applications, such as problems of contact and fracture mechanics, electromagnetics, diffusion, *etc.*

Basically, two categories of methods for solving mixed BVP can be specified. The first one requires construction of a Green's function, after which each particular solution can be presented in quadratures. The second one comprises various integral transform methods.

Hobson [19], for example, has constructed the Green's function for a circular disc and a spherical bowl by using toroidal coordinates τ, σ, ϕ. The potential function V at the point (τ_0,σ_0,ϕ_0) external to the disc, which on the disc takes the values $\nu(\tau,\phi)$, was expressed by a very complicated integral. He had to use a quite ingenious method in order to find the potential function for ν=const. and $\nu=\mu\alpha$, where μ is constant.

The integral transform method, involving dual integral equations, was originated, probably by Weber (1873) and Beltrami (1881) and continued by others. Significant achievements in the systematic application of the method to various problems are due to Sneddon [20,21]. Some quite remarkable results were obtained by Ufliand [22]. Despite this success, the use of integral transforms generally indicates the inability to solve problems using elementary means.

Thus the Green's function approach is the most general, the main obstacle being the inability to directly derive results, which were instead usually constructed by some ingenious considerations. In contrast, the integral transform method allows a straightforward derivation of the results, but it is the least general, since each particular problem has to be solved from beginning to end. Non-axisymmetric problems involving various interactions (several arbitrarily located charged discs, interaction of punches and cracks, etc.) are extremely difficult to solve by the integral transform method.

The main objectives for the development of the new method were an inconsistency between various solutions to the problems in Potential Theory and the way those solutions have been obtained. That is, the solution was quite elementary, while the method used was very complicated, involving various integral transforms or special functions expansions. Fabrikant [16,17] found a new method, which is in fact able to solve difficult mixed BVP in an elementary way. Not only could this new method more easily solve the problems already solved by other methods, but it also possessed the following advantages and benefits:

1. The method can solve non-axisymmetric problems as easily as axisymmetric ones.
2. Solutions are exact and in closed form and expressed in elementary functions (very easy for numerical calculations).
3. It enables one to treat analytically non-classical

domains.

The major advantages of the method comprise two things. First, the derivation of explicit and elementary expressions for the Green's functions, related to a penny-shaped crack and a circular punch. And second, an investigation of various interactions between cracks, punches and external loading. All this allows the solution to some problems which were not even considered before.

The solution is called complete, when explicit expressions for the field of stresses and displacements are defined in the entire space. The new method allows one to obtain a complete solution (important in the investigation of various interaction problems), while the majority of the crack and indentation problems solved previously were evaluating the elastic field in the plane of the crack only.

2.2 INTEGRAL REPRESENTATION FOR THE RECIPROCAL OF THE DISTANCE BETWEEN TWO POINTS

The integral representation for the reciprocal of the distance between two points located in the plane $z=0$ was given by Fabrikant [16] and it was a crucial starting point for the development of the new method, since this quantity is very important in potential theory. It is given as

$$\frac{1}{R^{1+u}} = \frac{1}{\left[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)\right]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{\min(\rho_0,\rho)} \frac{\lambda\left(\frac{\alpha^2}{\rho\rho_0},\phi - \phi_0\right)\alpha^u d\alpha}{\left[(\rho^2-\alpha^2)(\rho_0^2-\alpha^2)\right]^{(1+u)/2}}, \tag{2.1}$$

where u is a constant and $-1<u<1$.

Here the following notation is introduced,

$$\lambda(k,\psi) = \frac{1 - k^2}{1 + k^2 - 2k\cos\psi}. \tag{2.2}$$

Introducing a new variable

$$\eta(\alpha) = [(\rho^2-\alpha^2)(\rho_0^2-\alpha^2)]^{1/2}/\alpha \quad , \tag{2.3}$$

expression (2.1) may be rewritten as

$$\frac{1}{R^{1+u}} = \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^\infty \frac{\eta^{-u}d\eta}{R^2 + \eta^2}. \tag{2.4}$$

The integral in (2.4) can be evaluated by using the following formula from Gradshtein and Ryzhik [23]

$$\int_0^\infty \frac{\alpha^{m-1}d\alpha}{(p+q\alpha^t)^{n+1}} = \frac{1}{tp^{n+1}}\left(\frac{p}{q}\right)^{m/t} \frac{\Gamma\left(\frac{m}{t}\right)\Gamma\left(1+n-\frac{m}{t}\right)}{\Gamma(1+n)}, \quad \left[0<\frac{m}{t}<n+1, p\neq 0, q\neq 0\right].$$

thus proving the identity in (2.1)

It can be deduced from (2.4) that in the particular case when u=0, the integral in (2.1) can be evaluated as indefinite, resulting in a very important representation

$$\int \frac{\lambda\left(\frac{\alpha^2}{\rho\rho_0}, \phi-\phi_0\right)d\alpha}{(\rho^2-\alpha^2)^{1/2}(\rho_0^2-\alpha^2)^{1/2}} = -\frac{1}{R}\tan^{-1}\left[\frac{(\rho^2-\alpha^2)^{1/2}(\rho_0^2-\alpha^2)^{1/2}}{\alpha R}\right]. \tag{2.5}$$

All the results above are related to the distance between two points in the plane $z=0$. They should be generalized to represent

$$\frac{1}{R_0^{1+u}} = \frac{1}{\left[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2\right]^{(1+u)/2}} . \tag{2.6}$$

The observation shows that representation (2.1) remains valid if ρ and ρ_0 are formally substituted by arbitrary quantities ℓ_1 and ℓ_2. They need to be chosen so that

$$\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2 = \ell_1^2+\ell_2^2-2\ell_1\ell_2\cos(\phi-\phi_0) . \tag{2.7}$$

This leads to two equations,

$$\ell_1\ell_2 = \rho\rho_0 , \qquad \ell_1^2+\ell_2^2 = \rho^2+\rho_0^2+z^2 . \tag{2.8}$$

The solution will then take the form

$$\ell_1(\rho_0,\rho,z)=\frac{1}{2}\left\{[(\rho+\rho_0)^2+z^2]^{1/2}- [(\rho-\rho_0)^2+z^2]^{1/2}\right\}=\frac{r_1-r_2}{2} , \tag{2.9}$$

$$\ell_2(\rho_0,\rho,z)=\frac{1}{2}\left\{[(\rho+\rho_0)^2+z^2]^{1/2}+ [(\rho-\rho_0)^2+z^2]^{1/2}\right\}=\frac{r_1+r_2}{2} . \tag{2.10}$$

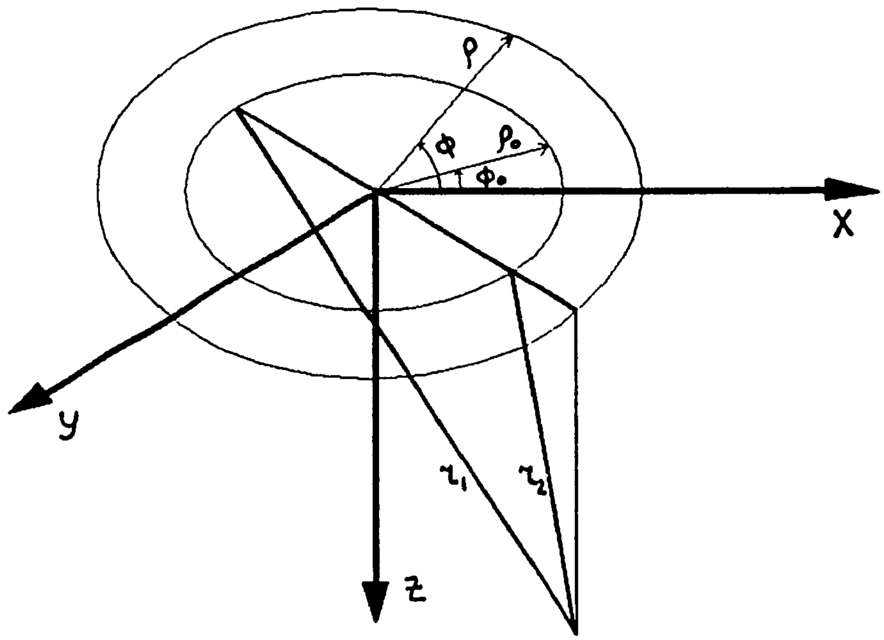

Fig.2.1 Geometric description for ℓ_1 and ℓ_2.

Hereafter the following abbreviations will be used:

$$\ell_1(\alpha) \equiv \ell_1(\alpha,\rho,z) \;, \qquad \ell_2(\alpha) \equiv \ell_2(\alpha,\rho,z) \;. \tag{2.11}$$

$$\ell_1 \equiv \ell_1(a,\rho,z) \;, \qquad \ell_2 \equiv \ell_2(a,\rho,z) \;. \tag{2.12}$$

Note the limiting properties

$$\lim_{z\to 0} \ell_1(\alpha) = \min(\alpha,\rho) \;, \qquad \lim_{z\to 0} \ell_2(\alpha) = \max(\alpha,\rho) \,. \tag{2.13}$$

In view of the properties above, the representation (2.1) can be generalized

$$\frac{1}{R_0^{1+u}} = \frac{1}{\left[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2\right]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2}\int_0^{\ell_1(\rho_0)} \frac{\lambda\left(\frac{\alpha^2}{\rho\rho_0}, \phi - \phi_0\right)\alpha^u d\alpha}{\left[\left(\ell_1^2(\rho_0)-\alpha^2\right)\left(\ell_2^2(\rho_0)-\alpha^2\right)\right]^{(1+u)/2}} . \qquad (2.14)$$

Formula (2.14) simplifies when u=0

$$\frac{1}{R_0} = \frac{1}{\left[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2\right]^{1/2}}$$

$$= \frac{2}{\pi}\int_0^{\ell_1(\rho_0)} \frac{\lambda\left(\frac{\alpha^2}{\rho\rho_0}, \phi - \phi_0\right)d\alpha}{\left[\left(\ell_1^2(\rho_0) - \alpha^2\right)\left(\ell_2^2(\rho_0) - \alpha^2\right)\right]^{1/2}} . \qquad (2.15)$$

Again, it can be noticed that the integral in (2.15) may be evaluated as indefinite:

$$\int \frac{\lambda\left(\frac{\alpha^2}{\rho\rho_0}, \phi - \phi_0\right)d\alpha}{\left[\left(\ell_1^2(\rho_0) - \alpha^2\right)\left(\ell_2^2(\rho_0) - \alpha^2\right)\right]^{1/2}}$$

$$= -\frac{1}{R_0}\tan^{-1}\frac{\left[\left(\ell_1^2(\rho_0) - \alpha^2\right)\left(\ell_2^2(\rho_0) - \alpha^2\right)\right]^{1/2}}{\alpha R_0} . \qquad (2.16)$$

thus, giving another quite important representation in (2.16).

By a simple change of variables, another series of useful formulae can be obtained from those given above, however they will not be discussed here, since the aim is not to give the variety of all very important integral representations, but to make an introduction of an essential part of the new results obtained in potential theory.

2.3 $\mathcal{L}$ - OPERATOR

The $\mathcal{L}$-operator may also be called the Poisson operator, since it was introduced by Poisson for solving the two-dimensional Dirichlet problem for a circle. It is defined as

$$\mathcal{L}(k)f(\phi) = \frac{1}{2\pi}\int_0^{2\pi}\lambda(k,\phi-\phi_0)f(\phi_0)d\phi_0 \ , \tag{2.17}$$

where $\lambda(.,.)$ is defined by (2.2).

The following properties of the $\mathcal{L}$-operator are valid

$$\mathcal{L}(k_1)\mathcal{L}(k_2) = \mathcal{L}(k_1k_2) \ , \qquad \lim_{k\to 1}\mathcal{L}(k)f = f \ . \tag{2.18}$$

These properties are widely used in various transformations and are essential to the method.

2.4 POINT FORCE SOLUTION AND CLASSIFICATION OF MIXED BOUNDARY VALUE PROBLEMS

By giving the point force solution the necessity to use the above integral representations and others which were not presented here, but can be found in Fabrikant [16], can be justified. The suggestion for the classification of BVP will naturally stem from the point force solution. The surface displacement field for point loading of a transversely isotropic half-space $z>0$ can be given as follows:

$$u = \frac{1}{2}G_1 \frac{T}{R} + \frac{1}{2}G_2 \frac{\bar{T}q^2}{R^3} - H\alpha \frac{P}{q}, \tag{2.19}$$

$$w = H\alpha \,\mathcal{Re}\left(\frac{T}{q}\right) + H\frac{P}{R}, \tag{2.20}$$

' here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \tag{2.21}$$

Here P is the normal force, $T=T_x+iT_y$ is an introduced complex tangential force and the overbar indicates its complex conjugate value. Tangential and normal displacements are denoted by $u=u_x+iu_y$ and w respectively. The elastic constants H, α, G_1 and G_2 will be defined later, since they are not the main issue here.

Expressions (2.19) and (2.20) are widely used for the integral equation formulations of various mixed BVP for an elastic half-space.

By suggesting the following classification of mixed BVP two types of internal and external problems can be specified:

Internal problem of type I: the normal displacements are prescribed inside a finite domain S, the normal traction is given outside the domain S, while the tangential tractions are known all over the plane $z=0$.

External problem of type I: the normal traction is prescribed inside S, the normal displacements are given outside, while the tangential tractions are known all over the plane z=0.

Internal problem of type II: the tangential displacements are prescribed inside S, the shear tractions are given outside, while the normal traction is known all over the plane z=0.

External problem of type II: the shear tractions are prescribed inside S, the tangential displacements are given outside, while the normal traction is known all over the plane z=0.

2.5 GENERAL SOLUTION FOR MIXED BOUNDARY VALUE PROBLEMS IN ELASTICITY

The elastic half-space has proven to be a useful mathematical model for the consideration of various contact and crack problems in finite bodies, provided that the domain of contact or the crack size is much smaller than the characteristic dimension of the body. A general solution in terms of three harmonic functions is presented for the case of transverse isotropy.

For a transversely isotropic elastic body, where the plane z=0 is parallel to the plane of isotropy and which is characterized by five elastic constants A_{ij}, the following stress-strain relationships are satisfied:

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x} + (A_{11} - 2A_{66})\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11} - 2A_{66})\frac{\partial u_x}{\partial x} + A_{11}\frac{\partial u_y}{\partial y} + A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x} + A_{13}\frac{\partial u_y}{\partial y} + A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau_{zx} = A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z}\right). \tag{2.22}$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{2.23}$$

Substitution of (2.22) in (2.23) yields:

$$A_{11}\frac{\partial^2 u_x}{\partial x^2} + A_{66}\frac{\partial^2 u_x}{\partial y^2} + A_{44}\frac{\partial^2 u_x}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_y}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0,$$

$$A_{66}\frac{\partial^2 u_y}{\partial x^2} + A_{11}\frac{\partial^2 u_y}{\partial y^2} + A_{44}\frac{\partial^2 u_y}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_x}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0,$$

$$A_{44}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{44} + A_{13})\left[\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z}\right] = 0.$$

$$\tag{2.24}$$

Introduce complex tangential displacements $u=u_x+iu_y$ and $\bar{u}=u_x-iu_y$. This will allow the reduction of the number of equations in (2.24) by one, and one can rewrite these equations in a more compact manner, namely,

$$\frac{1}{2}(A_{11}+A_{66})\Delta u + A_{44}\frac{\partial^2 u}{\partial z^2} + \frac{1}{2}(A_{11}-A_{66})\Lambda^2\bar{u} + (A_{13}+A_{44})\Lambda\frac{\partial w}{\partial z} = 0,$$

$$A_{44}\Delta w + A_{33}\frac{\partial^2 w}{\partial z^2} + \frac{1}{2}(A_{13}+A_{44})\frac{\partial}{\partial z}(\bar{\Lambda}u + \Lambda\bar{u}) = 0. \tag{2.25}$$

Here the following differential operators were used:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad \bar{\Lambda} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \tag{2.26}$$

and the overbar everywhere indicates the complex conjugate value. Note also that $\Delta=\Lambda\bar{\Lambda}$.

It may be verified that equations (2.25) can be satisfied by

$$u = \Lambda\,(F_1+F_2+iF_3), \quad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z}, \tag{2.27}$$

where all three functions F_k satisfy the equation (Elliott [24])

$$\Delta F_k + \gamma_k^2\frac{\partial^2 F_k}{\partial z^2} = 0, \quad \text{for } k=1,2,3. \tag{2.28}$$

Introducing the notation $z_k=z/\gamma_k$, for k=1,2,3, the function $F_k=F(x,y,z_k)$ may be called harmonic. The values of m_k and γ_k are related by the following expressions (Elliott [24])

$$\frac{A_{44}+m_k(A_{13}+A_{44})}{A_{11}} = \frac{m_k A_{33}}{m_k A_{44}+A_{13}+A_{44}} = \gamma_k^2 \ , \ \text{for } k=1,2$$

$$\gamma_3 = \left(\frac{A_{44}}{A_{66}}\right)^{1/2} . \qquad (2.29)$$

Introduce the following inplane stress components:

$$\sigma_1 = \sigma_x+\sigma_y \ , \quad \sigma_2 = \sigma_x-\sigma_y+2i\tau_{xy} \ , \quad \tau_z = \tau_{zx}+ i\tau_{yz} \ . \qquad (2.30)$$

This will simplify expressions (2.22), namely,

$$\sigma_1 = (A_{11}-A_{66})(\bar{\Lambda}u + \Lambda\bar{u}) + 2A_{13}\frac{\partial w}{\partial z}, \quad \sigma_2 = 2A_{66}\Lambda u \ ,$$

$$\sigma_z = \frac{1}{2}A_{13}(\bar{\Lambda}u + \Lambda\bar{u}) + A_{33}\frac{\partial w}{\partial z}, \quad \tau_z = A_{44}\left(\frac{\partial u}{\partial z} + \Lambda w\right) . \qquad (2.31)$$

Now there are only four components of stress, instead of six, as it was in (2.22). The substitution of (2.27) in (2.31) yields:

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\left\{\left[\gamma_1^2 - (1+m_1)\gamma_3^2\right]F_1 + \left[\gamma_2^2 - (1+m_2)\gamma_3^2\right]F_2\right\} ,$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + iF_3) \ ,$$

$$\sigma_z = A_{44}\frac{\partial^2}{\partial z^2}\left[(1+m_1)\gamma_1^2 F_1 + (1+m_2)\gamma_2^2 F_2\right]$$

$$= - A_{44}\Delta\left[(1+m_1)F_1 + (1+m_2)F_2\right] ,$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}\left[(1+m_1)F_1 + (1+m_2)F_2 + iF_3\right] . \qquad (2.32)$$

Here the fact that each F_k satisfies equation (2.28) was used, along with the relation: $A_{11}\gamma_k^2-A_{13}m_k=A_{44}(1+m_k)$, (for

$k=1,2$) which is an immediate consequence of (2.29). Expressions (2.27) and (2.32) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x,y,z)=c_kF(x,y,z_k) \ , \tag{2.33}$$

where $z_k=z/\gamma_k$, and c_k is an as yet unknown complex constant. As will be shown further, this is indeed possible. All the results which will be obtained in this work are valid for isotropic solids, provided that for isotropy

$$\gamma_1=\gamma_2=\gamma_3=1 \ , \quad H=\frac{1-\nu^2}{\pi E} \ , \quad A_{44}=A_{66}=\frac{E}{2(1+\nu)} \ , \quad \beta=\frac{1+\nu}{\pi E} \ , \tag{2.34}$$

where E is the elastic modulus, and ν is Poisson's ratio.

The following identities and elastic constants will be used throughout:

$$m_1m_2=1 \ , \qquad (m_1-1)/(m_1+1)=2\pi A_{44}H(\gamma_1-\gamma_2) \ . \tag{2.35}$$

$$H=\frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)} \ , \qquad \alpha=\frac{(A_{11}A_{33})^{1/2}-A_{13}}{A_{11}(\gamma_1+\gamma_2)} \ ,$$

$$\beta=\frac{\gamma_3}{2\pi A_{44}} \ , \quad G_1=\beta+\gamma_1\gamma_2H \ , \quad G_2=\beta-\gamma_1\gamma_2H \ . \tag{2.36}$$

2.6 DEFINITION OF STRESS INTENSITY FACTOR

The stress intensity factor is one of the indicators of problems leading to failure by fracture, like J-integral, crack opening displacement and strain energy density factor, and it is widely used in fracture mechanics in the design of machine and structural elements. In section 1.2, it was shown how the concept of the SIF with regard to two-dimensional crack problems was gradually developed. Kassir and Sih [15] expressed the local stress field near the crack front in a form analogous to the two-dimensional case in terms of three SIF which are independent of the local coordinates, and are dependent only on the crack geometry, the form of the loading and the location of the point along the crack border. This result is fundamental in analyzing the fracture behavior of cracks and provides uniform expressions for the local stresses under various geometrical and loading conditions where only the values of the SIF differ.

The coefficients K_1, K_2 and K_3 are the opening-mode, sliding-mode and tearing-mode SIF. They can be found from the stress components in terms of the variables ρ and ϕ as follows:

For the case of internal crack

$$K_1 = \lim_{\rho \to a} [(\rho - a)^{1/2} \sigma_z(\rho, \phi, 0)], \tag{2.37}$$

$$K_2 = \lim_{\rho \to a} [(\rho - a)^{1/2} \tau_{z\rho}(\rho, \phi, 0)], \tag{2.38}$$

$$K_3 = \lim_{\rho \to a} [(\rho - a)^{1/2} \tau_{\phi z}(\rho, \phi, 0)]. \tag{2.39}$$

For the case of external crack

$$K_1 = \lim_{\rho \to a} [(a-\rho)^{1/2} \sigma_z(\rho,\phi,0)], \tag{2.40}$$

$$K_2 = \lim_{\rho \to a} [(a-\rho)^{1/2} \tau_{z\rho}(\rho,\phi,0)], \tag{2.41}$$

$$K_3 = \lim_{\rho \to a} [(a-\rho)^{1/2} \tau_{\phi z}(\rho,\phi,0)]. \tag{2.42}$$

For the case of semi-infinite plane crack the stress intensity factors K_1, K_2 and K_3 can be obtained as usual from the appropriate stress components on the plane $z=0$ in terms of the variables x and y as follows:

$$K_1 = \lim_{x \to 0} [\sqrt{x}\, \sigma_z(x,y,0)], \tag{2.43}$$

$$K_2 = \lim_{x \to 0} [\sqrt{x}\, \tau_{zx}(x,y,0)], \tag{2.44}$$

$$K_3 = \lim_{x \to 0} [\sqrt{x}\, \tau_{yz}(x,y,0)]. \tag{2.45}$$

Formulae (2.37-2.45) are widely used for determination of the SIF. However, some alternative formulations for SIF definition will be presented in the next subsection and their importance and convenience will be underlined.

2.6.1 ALTERNATIVE FORMULAE FOR DEFINITION OF STRESS INTENSITY FACTOR

Despite the fact that formulae (2.37-2.45) have already found their true place in applications, it can be noticed that there are some limitations in formulation, since in order to evaluate them the stress components must be known. For some interaction problems, as will be discussed in

Chapter 6, the use of the reciprocal theorem necessitates an alternative formulae for SIF definition. They are defining the SIF directly in terms of displacement components as follows, (Fabrikant [16]):

$$K_1 = \frac{1}{4\pi H} \lim_{\rho\to a} \left[\frac{w(\rho,\phi)}{(a-\rho)^{1/2}}\right], \tag{2.46}$$

$$K_2 + iK_3 = -\frac{a}{\pi(G_1^2-G_2^2)\sqrt{2a}} \lim_{\rho\to a} \left[\frac{G_1 e^{-i\phi}u + G_2 e^{i\phi}\overline{u}}{(a^2-\rho^2)^{1/2}}\right], \tag{2.47}$$

where H, G_1, and G_2 are transversely isotropic elastic constants defined in (2.36), u is an introduced complex tangential displacement and $u=u_x+iu_y$.

The importance of (2.46) and (2.47) stems from the known fact that in the solution of an interaction problem by means of the reciprocal theorem, there is a need to determine the unknown displacements on the plane $z=0$. Subsequently, having at hand the displacement components, the SIF may be evaluated. Formulae (2.46) and (2.47) are also convenient, since they are as simple as the the expressions given in section 2.6 and may be used depending on whether the expressions for displacement components look more simple than the expressions for stress components. And finally, they can be used as an alternative formulae for verifications. It may be noticed that formulae (2.46) and (2.47) are defining the SIF for transversely isotropic solids, but with help of expressions given in (2.34) and (2.36) they can be used as well for the isotropic case.

Since the definition of the τ_z component given in (2.30) contains both x- and y-components, so will the expression for the SIF: $K=K_x+iK_y$. If the expressions for the radial and tangential components are required, the following relationship has to be used

$$\tau_{zx}+ i\tau_{yz}= (\tau_{z\rho}+ i\tau_{\phi z})e^{i\phi} . \qquad (2.48)$$

Hence, according to (2.38) and (2.39) the expression for the combined mode 2 and mode 3 SIF will become

$$K_2+iK_3= \lim_{\rho\to a} [(\rho-a)^{1/2}\tau_z e^{-i\phi}]. \qquad (2.49)$$

Thus, formulae (2.46), (2.47) and (2.49) will be widely used for evaluation of the SIF in the present work.

2.7 SUMMARY

In this chapter a short description of the method of Fabrikant [16] applicable to the solution of three-dimensional crack problems was presented. All its advantages over the previously existing methods were clearly emphasized. The forthcoming Chapters 3, 4, 5 and 6 will be solely devoted to the application and further development of this method in connection with the solution of a wide variety of crack problems. Some attention to the solution of contact problems will be given as well.

CHAPTER 3

INTERNAL CIRCULAR CRACK PROBLEMS

3.1 INTRODUCTORY REMARKS

In this chapter a complete closed form solution in terms of elementary functions for two different internal circular crack problems which are of mixed BVP type are presented. Explicit expressions are derived for the stresses and displacements in both transversely isotropic and isotropic full space weakened by a penny-shaped crack. The first problem will deal with the case of a linear normal load applied to the crack faces. The second problem is a non-axisymmetric linear shear load.

As already mentioned in the introduction, Sack [10] first obtained the solution for a penny-shaped crack subjected to internal pressure. His method could be adopted for the case of a variable internal pressure, but the calculations would probably be rather cumbersome. Even in the case of constant pressure, the expressions obtained for the components of stress do not yield numerical results easily. Also, the choice of an oblate spheroidal coordinate system makes the interpretation of the results somewhat difficult. However, this work has attracted significant attention from scientists. The majority of the crack problems solved subsequently presented the elastic field in the plane $z=0$ only. There are just a few publications with the complete solution for an axisymmetric problem with a penny-shaped crack, where the explicit expressions are given for the stresses and displacements everywhere in the elastic space. For example, Elliott [25], solved the problem of a penny-shaped crack in a transversely isotropic body under uniform pressure. The same problem, but for the isotropic case, was solved by Sneddon [11]. They both used integral transform methods.

A completely different approach, used by Fabrikant [16] and based on the new results in potential theory, allowed him to solve both above-noted problems. As it was indicated, his results were essentially in agreement with those of Elliott and Sneddon, except for some misprints noted by Fabrikant. Some further analysis of the comparison of the later case led to the discovery of an additional misprint in Sneddon's solution, which is noted subsequently.

Westman [26] presented two examples by giving a complete solution to the mixed boundary value problem for a circular crack in an isotropic body, where the conditions were prescribed interior and exterior to a circle and were mixed with respect to uniform shear and tangential displacement.

There has also been some research on the solution for an elliptical crack problem. An example is the remarkable work of Kassir and Sih [27]. Though the final results were given only for the case when $z=0$, this work indeed presents the complete solution to the problem of an elliptical crack in an isotropic body subjected to uniform shear loading and, as the authors indicated in the limiting case for a circle, the results are in agreement with those of Westman [26].

Chen [28] gave a formal method for the solution of an elliptical crack in a transversely isotropic elastic medium under uniform shear stress applied to the crack faces. It was accomplished by using the assumption that the displacement discontinuity at the crack surface is of an elliptical nature. An attempt to extend the same method for the case when the stress distribution at the crack surface is described by a linear polynomial resulted in a rather large amount of manipulative work.

Despite the fact that some axisymmetric problems were successfully considered before, there are no complete closed form solutions in terms of elementary functions to the problems of a penny-shaped crack subjected to a linear normal or shear loading applied to the crack faces. The material presented in this chapter follows the work by Karapetian [29,30].

As was shown in Chapter 2, the expressions (2.27) and (2.32) give the general solution for mixed BVP in elasticity expressed in terms of three harmonic functions. Therefore the solution to the above stated problems comes down to the determination of those functions. However it was mentioned there, that it is possible to express each function F_k through just *one* harmonic function as given in (2.33).

3.2 ELASTIC FIELD OF INTERNAL CIRCULAR CRACK UNDER LINEAR NORMAL LOADING

Consider a transversely isotropic elastic space weakened by an internal circular crack of radius a in the plane z=0 and with linear normal loading applied to the crack faces. The cylindrical coordinates (ρ,ϕ,z) will be adopted to locate points inside the half space z>0. The problem under consideration, due to its symmetry, may be reduced to the external mixed boundary value problem for a half-space and can be reformulated as follows.

Let a transversely isotropic elastic half-space occupy z≥0, (Fig.3.1). The shear traction $\tau_z=0$ over the plane z=0. The normal displacement w=0 outside the circle $\rho=a$, while the linear normal loading $\sigma_z=-p_0(\rho,\phi)$ is given inside the circle.

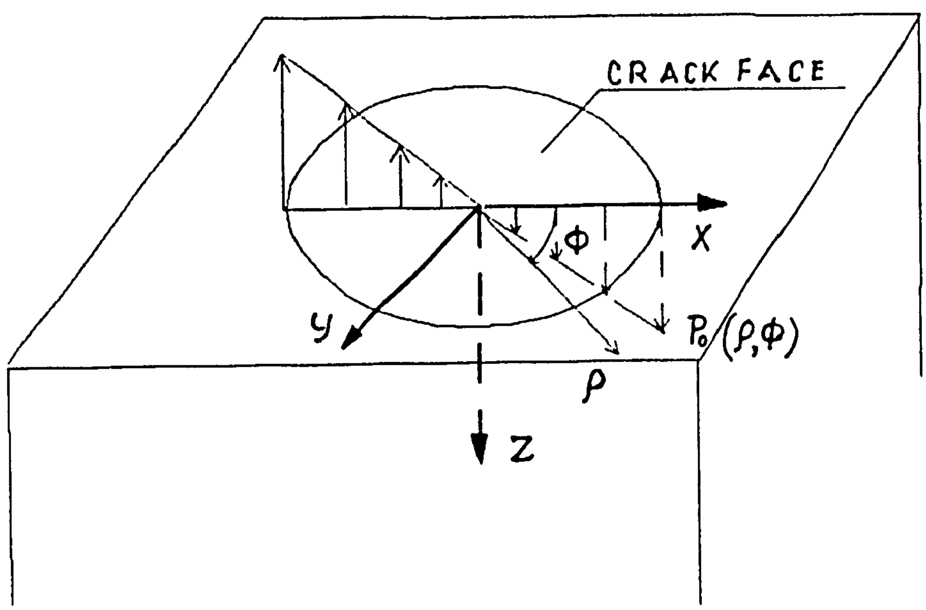

Fig.3.1 Internal circular crack under linear normal loading.

The mathematical statement of the boundary conditions is:

$$\begin{aligned} &\tau_z = 0, && \text{for} \quad 0 \le \rho < \infty , && 0 \le \phi < 2\pi , \\ &w = 0, && \text{for} \quad a \le \rho < \infty , && 0 \le \phi < 2\pi , \\ &\sigma_z = -p_0 \rho \cos\phi, && \text{for} \quad 0 \le \rho < a , && 0 \le \phi < 2\pi . \end{aligned} \tag{3.2.1}$$

3.2.1 POTENTIAL FUNCTIONS

The conditions in (3.2.1) can be satisfied by a representation in terms of one harmonic function. For this type of problem, according to (2.33) the functions are:

$$F_1(z)=c_1F(z_1), \quad F_2(z)=c_2F(z_2), \quad F_3(z)=0. \tag{3.2.2}$$

Expressions of the type $F_1(z)$ and $F(z_1)$, *etc.*, everywhere should be understood as $F_1(x,y,z)$ and $F(x,y,z_1)$ respectively. The substitution of (3.2.2) and the last of expressions (2.32) in the first condition (3.2.1) yields:

$$c_1=-c_2\gamma_1/m_1\gamma_2 , \tag{3.2.3}$$

The function F can be represented as a potential of a simple layer, i.e,

$$F(\rho,\phi,z)\equiv F(z)=\int\int_S \frac{\omega(N)\,dS}{R(M,N)} , \tag{3.2.4}$$

where ω stands for the crack face displacement $w(x,y,0)$, $R(M,N)$ is the distance between the points $M(\rho,\phi,z)$ and

$N(r,\psi,0)$, the integration is taken over the crack domain S.

Expression (3.2.4) satisfies the second condition (3.2.1) identically, due to the well known property of the potential of a simple layer. Inside the crack the same property gives:

$$\frac{\partial F}{\partial z}\Big|_{z=0}=-2\pi\omega=-2\pi w(x,y,0) \ . \tag{3.2.5}$$

Now expressions (3.2.2), (3.2.4), (3.2.5) and (2.27) give the second equation for c_1 and c_2:

$$-m_1c_1/\gamma_1-m_2c_2/\gamma_2=1/2\pi \ . \tag{3.2.6}$$

The constants c_1 and c_2 are determined from (3.2.3) and (3.2.6) as

$$c_1= - \frac{\gamma_1}{2\pi(m_1-1)} \ , \qquad c_2= - \frac{\gamma_2}{2\pi(m_2-1)} \ . \tag{3.2.7}$$

The potential functions will be given by

$$F_1(z)=- \frac{\gamma_1}{2\pi(m_1-1)} F(z_1), \quad F_2(z)=- \frac{\gamma_2}{2\pi(m_2-1)} F(z_2). \tag{3.2.8}$$

The substitution of (3.2.8) and (2.32) in the last condition (3.2.1) leads to the governing integral equation:

$$p(N_0)=- \frac{1}{4\pi^2H}\Delta\int\int_S \frac{\omega(N)\,dS}{R(N_0,N)} \ , \tag{3.2.9}$$

where, as before, $R(N_0,N)$ stands for the distance between two points N_0 and N, and both $N_0, N \in S$. The identities defined in (2.35) were used.

The solution of the governing integral equation in (3.2.9) is as follows, Fabrikant [16]

$$w(\rho,\phi) = 4H\rho \int_{\rho}^{a} \frac{d\alpha}{\alpha^2(\alpha^2-\rho^2)^{1/2}} \int_{0}^{\alpha} \frac{\sigma(\rho_0,\phi)\rho_0^2 d\rho_0}{(\alpha^2-\rho_0^2)^{1/2}} . \tag{3.2.10}$$

It defines the normal displacements inside the crack directly in terms of the prescribed normal loading.

The substitution of the last condition of (3.2.1) in (3.2.10) yields

$$w(\rho,\phi) = \frac{8}{3} Hp_0 \cos\phi \, \rho(a^2 - \rho^2)^{1/2}. \tag{3.2.11}$$

The subsequent substitution of (3.2.11) in (3.2.4) will result in

$$F(\rho,\phi,z) = \frac{8}{3} Hp_0 \int_0^{2\pi}\int_0^{a} \frac{\cos\phi_0(a^2 - \rho_0^2)^{1/2}\rho_0^2 d\rho_0 d\phi_0}{R_0} . \tag{3.2.12}$$

The integral in (3.2.12) was evaluated, and the result for the potential function $F(\rho,\phi,z)$ is

$$F(\rho,\phi,z) = \frac{Hp_0\pi}{3} \rho\cos\phi \; \chi(\rho,z) , \tag{3.2.13}$$

where

$$\chi(\rho,z)=a(\ell_2^2-a^2)^{1/2}\left(15\frac{\ell_1^2}{a^2}-12-2\frac{\ell_1^2}{\ell_2^2}\right)+\sin^{-1}\left(\frac{a}{\ell_2}\right)(4a^2-3\rho^2+12z^2)\,. \tag{3.2.14}$$

The quantities ℓ_1 and ℓ_2 are defined as

$$\ell_1\equiv\ell_1(a,\rho,z)=\frac{1}{2}\left\{[(\rho+a)^2+z^2]^{1/2}-[(\rho-a)^2+z^2]^{1/2}\right\}\,,$$
$$\ell_2\equiv\ell_2(a,\rho,z)=\frac{1}{2}\left\{[(\rho+a)^2+z^2]^{1/2}+[(\rho-a)^2+z^2]^{1/2}\right\}\,. \tag{3.2.15}$$

According to (3.2.8) the three potential functions are:

$$F_1(z) = -\frac{Hp_0}{6}\frac{\gamma_1}{m_1-1}\rho\cos\phi\;\chi(\rho,z_1)\;,$$
$$F_2(z) = -\frac{Hp_0}{6}\frac{\gamma_2}{m_2-1}\rho\cos\phi\;\chi(\rho,z_2)\;,$$
$$F_3(z) = 0\;. \tag{3.2.16}$$

3.2.2 COMPLETE ELASTIC FIELD

The elastic field resulting from the normal loading can be calculated by performing appropriate differentiation of the potential functions (3.2.16), which had to be substituted into the expressions (2.27) and (2.32). The results of the differentiation of $\chi(\rho,z)$, are given in Appendix A3.2. The operators which are defined in (2.26) are initially transformed into cylindrical coordinates and along with the other operator, namely Λ^2, are presented below:

$$\Lambda = e^{i\phi}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) , \quad \bar{\Lambda} = e^{-i\phi}\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}\right) ,$$

$$\Lambda^2 = e^{2i\phi}\left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{2i}{\rho^2}\frac{\partial}{\partial\phi} + \frac{2i}{\rho}\frac{\partial^2}{\partial\rho\partial\phi}\right) ,$$

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} . \tag{3.2.17}$$

The identities defined by (2.35) will also be used. The elastic field is:

$$u=Hp_0\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\Bigg\{\rho\cos\phi\ e^{i\phi}\left[\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)-(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2}\right)\right]$$

$$-\left[a(\ell_{2k}^2-a^2)^{1/2}\left(\frac{5}{2}\frac{\ell_{1k}^2}{a^2}-2-\frac{1}{3}\frac{\ell_{1k}^2}{\ell_{2k}^2}\right)+\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\left(\frac{2}{3}a^2-\frac{1}{2}\rho^2+2z_k^2\right)\right]\Bigg\} , \tag{3.2.18}$$

$$w =\frac{4Hp_0}{3}\rho\cos\phi\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left[(a^2-\ell_{1k}^2)^{1/2}\left(3-\frac{a^2}{\ell_{2k}^2}\right)-3z_k\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right] , \tag{3.2.19}$$

$$\sigma_1=8A_{66}Hp_0\cos\phi\sum_{k=1}^{2}\frac{\gamma_k^2-(1+m_k)\gamma_3^2}{\gamma_k(m_k-1)}\Bigg[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)$$

$$-\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\Bigg] , \tag{3.2.20}$$

$$\sigma_2=-2A_{66}Hp_0e^{2i\phi}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\Bigg\{(e^{-i\phi}-i\sin\phi)\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2}\right)\right.$$

$$\left.-\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right]+\cos\phi\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1-\frac{2a^2}{\ell_{2k}^2}+\frac{8}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)\right.$$

$$- \rho \sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\Bigg]\Bigg\} , \qquad (3.2.21)$$

$$\sigma_z = \frac{2p_0}{\pi}\cos\phi \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} (-1)^{k+1}\gamma_k \left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+ \frac{2}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)\right.$$

$$\left. - \rho \sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right] , \qquad (3.2.22)$$

$$\tau_z = - \frac{2p_0}{3\pi} \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} (-1)^{k+1} \left[2e^{i\phi}\cos\phi (a^2-\ell_{1k}^2)^{1/2} \frac{\ell_{1k}^4}{\rho^2(\ell_{2k}^2-\ell_{1k}^2)}\right.$$

$$\left. + (a^2-\ell_{1k}^2)^{1/2}\left(\frac{a^2}{\ell_{2k}^2} -3\right) + 3z_k \sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right] , \qquad (3.2.23)$$

where the notation

$$\ell_{1k} \equiv \ell_{1k}(a) = \frac{1}{2} \left\{ \left[(\rho+a)^2+z_k^2\right]^{1/2} - \left[(\rho-a)^2+z_k^2\right]^{1/2} \right\} , \qquad (3.2.24)$$

has been introduced along with a similar interpretation for ℓ_{2k}.

3.2.3 ELASTIC FIELD FOR ISOTROPY

To obtain the isotropic solution, a limiting form of the transversely isotropic solution should be taken. For the isotropic case the material parameters $\gamma_1, \gamma_2, \gamma_3, m_1, m_2 \to 1$, while $H = \frac{1-\nu^2}{\pi E}$ and $A_{44} = A_{66} = \frac{E}{2(1+\nu)}$. Thus, the relations for the isotropic limits are as follows:

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{\gamma_k}{m_k-1} f(z_k) = -\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)} ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{m_k}{(m_k-1)} f(z_k) = \frac{2(1-\nu)f(z)-zf'(z)}{2(1-\nu)} ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{\gamma_k^2-(1+m_k)\gamma_3^2}{\gamma_k(m_k-1)} f(z_k) = \frac{(1+2\nu)f(z)+zf'(z)}{2(1-\nu)} ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} (-1)^{k+1}\gamma_k f(z_k) = f(z)-zf'(z) ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} (-1)^{k+1} f(z_k) = -zf'(z) , \qquad (3.2.25)$$

where $f'(z)$ denotes the derivative of $f(z)$.

Application of the above formulae to the results obtained in (3.2.18-3.2.23) yields the following isotropic elastic field:

$$u = \frac{(1+\nu)p_0}{2\pi E}\left\{\rho\cos\phi \; e^{i\phi}\left[(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left((1-2\nu)\left(1+\frac{2}{3}\frac{a^2}{\ell_2^2}\right)\right.\right.\right.$$
$$\left.\left.-\frac{8}{3}\frac{a^2(\ell_1^2-a^2)}{\ell_2^2(\ell_2^2-\ell_1^2)}\right)-(1-2\nu)\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\right]+(1-2\nu)a(\ell_2^2-a^2)^{1/2}\left(\frac{5}{2}\frac{\ell_1^2}{a^2}\right.$$
$$\left.-2-\frac{1}{3}\frac{\ell_1^2}{\ell_2^2}\right)+(1-2\nu)\sin^{-1}\left(\frac{a}{\ell_2}\right)\left(\frac{2}{3}a^2-\frac{1}{2}\rho^2+2z^2\right)$$
$$\left.+z(a^2-\ell_1^2)^{1/2}\left(\frac{4}{3}\frac{a^2}{\ell_1^2}-4\right)+4z^2\sin^{-1}\left(\frac{a}{\ell_2}\right)\right\} , \qquad (3.2.26)$$

$$w = \frac{2(1+\nu)p_0}{3\pi E}\rho\cos\phi\left\{(a^2-\ell_1^2)^{1/2}\left[(1-2\nu)\left(3-\frac{a^2}{\ell_2^2}\right)\right.\right.$$

$$- \frac{2a^2(\ell_1^2-a^2)}{\ell_2^2(\ell_2^2-\ell_1^2)}\Bigg] - (1-2\nu)3z\sin^{-1}\left(\frac{a}{\ell_2}\right)\Bigg\} , \tag{3.2.27}$$

$$\sigma_1=\frac{2p_0}{\pi}\cos\phi\Bigg\{(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\Bigg[(1+2\nu)\left(1+\frac{2}{3}\frac{a^2}{\ell_2^2-\ell_1^2}\right)+\frac{4}{3}\frac{a^2-\ell_1^2}{\ell_2^2-\ell_1^2}$$

$$-\frac{2}{3}\frac{\ell_1^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{4}{3}\frac{a^2(\ell_1^2-a^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{4}{3}\frac{a^2z^2(\ell_2^2+\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}\Bigg]$$

$$-(1+2\nu)\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\Bigg\} , \tag{3.2.28}$$

$$\sigma_2=\frac{p_0}{2\pi}e^{2i\phi}\Bigg\{(e^{-i\phi}-i\sin\phi)\Bigg[(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\Bigg((1-2\nu)\left(1+\frac{2}{3}\frac{a^2}{\ell_2^2-\ell_1^2}\right)$$

$$+\frac{4}{3}\frac{a^2-\ell_1^2}{\ell_2^2-\ell_1^2}-\frac{2}{3}\frac{\ell_1^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{4}{3}\frac{a^2(\ell_1^2-a^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{4}{3}\frac{a^2z^2(\ell_2^2+\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}\Bigg)$$

$$-(1-2\nu)\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\Bigg]+\cos\phi\Bigg[(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\Bigg((1-2\nu)\left(1-2\frac{a^2}{\ell_1^2}\right.$$

$$\left.+\frac{8}{3}\frac{a^2}{\ell_2^2-\ell_1^2}\right)+\frac{8a^2z^2}{\ell_2^2(\ell_2^2-\ell_1^2)}+\frac{8}{3}\frac{a^2(\ell_2^2-\rho^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{32}{3}\frac{a^2z^2\ell_2^2}{(\ell_2^2-\ell_1^2)^3}\Bigg)$$

$$-(1-2\nu)\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\Bigg]\Bigg\} , \tag{3.2.29}$$

$$\sigma_z=\frac{2p_0}{\pi}\cos\phi\Bigg\{(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\Bigg[1-\frac{2}{3}\frac{a^2}{\ell_2^2-\ell_1^2}+\frac{4}{3}\frac{\ell_1^2}{\ell_2^2-\ell_1^2}$$

$$+\frac{2}{3}\frac{\ell_1^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^2}+\frac{4}{3}\frac{a^2(\ell_1^2-a^2)}{(\ell_2^2-\ell_1^2)^2}+\frac{4}{3}\frac{a^2z^2(\ell_2^2+\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}\Bigg]-\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\Bigg\}, \tag{3.2.30}$$

$$\tau_z=\frac{2p_0}{3\pi}\left\{2e^{i\phi}\cos\phi(a^2-\ell_1^2)^{1/2}\frac{\ell_1^4}{\rho^2}\left[\frac{\rho^2-\ell_1^2-2z^2}{(\ell_2^2-\ell_1^2)^2}-\frac{4z^2\ell_2^2}{(\ell_2^2-\ell_1^2)^3}\right]\right.$$

$$\left.-(a^2-\ell_1^2)^{1/2}\left(1-\frac{a^2}{\ell_2^2}\right)\left(3+\frac{2a^2}{\ell_2^2-\ell_1^2}\right)+3z\sin^{-1}\left(\frac{a}{\ell_2}\right)\right\}\ . \qquad (3.2.31)$$

Here ℓ_1 and ℓ_2 are defined by (3.2.15).

3.2.4 DISCUSSION AND NUMERICAL RESULTS

The complete solution for displacements and stresses, as can be seen from the above, is obtained in terms of just two distorted length parameters ℓ_1 and ℓ_2. Their introduction, for the first time, was found in the work of Fabrikant [16], where some frequently used relationships between $\ell_1(a)$ and $\ell_2(a)$ are given. They allow the three-dimensional distance between the point $(\rho_0,\phi_0,0)$ on the surface of a half-space and the interior point (ρ,ϕ,z) to be written in two-dimensional form, as follows:

$$\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2$$
$$= \ell_1^2(\rho_0)+\ell_2^2(\rho_0)-2\ell_1(\rho_0)\ell_2(\rho_0)\cos(\phi-\phi_0)\ ,$$

where, $\ell_1(\rho_0)\ell_2(\rho_0)=\rho\rho_0\ ,\ \ \ell_1^2(\rho_0)+\ell_2^2(\rho_0)=\rho^2+\rho_0^2+z^2. \qquad (3.2.32)$

Since the point force solutions are given in terms of inverse powers of the distance from the point force to an interior point, the need for use of these parameters in developing closed form solutions to half-space problems arises naturally. The solution for distributed loading can be obtained by quadrature. The ability to write the three-dimensional distance in a two-dimensional form allows the integrals for the potential functions to be evaluated in closed form expressions of elementary functions containing $\ell_1(a)$ and $\ell_2(a)$ as parameters [e.g. see eq.(3.2.7), where the argument a arises because of the limits of integration].

The expressions for the elastic field obtained here are in compact complex notation. Their separation, which depends only on the complex exponential $e^{in\phi}$ as a multiplying factor, can be done in a simple manner by using Euler's identity.

It is important to point out that the expressions derived here are easily evaluated anywhere in the half space, for example, for z=0 the identities can be used, i.e.

$$\lim_{z\to 0} \ell_1(a) = \min(a,\rho) , \qquad \lim_{z\to 0} \ell_2(a) = \max(a,\rho) , \qquad (3.2.33)$$

where $\min(a,\rho)$ implies the minimum of the two values and $\max(a,\rho)$ the maximum.

The surface values can be readily determined and be verified with the boundary conditions of the problem. Since the identities in (3.2.33) are equally valid for $\ell_{1k}(a)$ and $\ell_{2k}(a)$, because $z\to 0$ implies $z_k=z/\gamma_k\to 0$ as well, then on the plane $z=0$ the stress and displacement components for isotropic and transversely isotropic cases are the same, except for differences in elastic constants. Therefore the elastic field on the plane z=0 is presented here only for isotropic case. In the results to follow, the first expression of each component corresponds to the case of $\rho<a$, while the second gives the result for $\rho>a$.

$$u=-\frac{p_0(1+\nu)(1-2\nu)}{4E}\left[\rho^2\cos\phi\ e^{i\phi}-\frac{2}{3}a^2+\frac{1}{2}\rho^2\right],$$

$$u=\frac{p_0(1+\nu)(1-2\nu)}{2\pi E}\left\{\rho\cos\phi\ e^{i\phi}\left[(\rho^2-a^2)^{1/2}\frac{a}{\rho}\left(1+\frac{2}{3}\frac{a^2}{\rho^2}\right)-\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right]\right.$$

$$\left.+a(\rho^2-a^2)^{1/2}\left(\frac{1}{2}-\frac{1}{3}\frac{a^2}{\rho^2}\right)+\sin^{-1}\left(\frac{a}{\rho}\right)\left(\frac{2}{3}a^2-\frac{1}{2}\rho^2\right)\right\}, \qquad (3.2.34)$$

$$w=\frac{8p_0(1-\nu)}{3\pi E}\rho\cos\phi(a^2-\rho^2)^{1/2},$$

$$w=0, \qquad (3.2.35)$$

$$\sigma_1=-(1+2\nu)p_0\rho\cos\phi,$$

$$\sigma_1=\frac{2p_0(1+2\nu)}{\pi}\cos\phi\left[(\rho^2-a^2)^{1/2}\frac{a}{\rho}\left(1+\frac{2}{3}\frac{a^2}{\rho^2-a^2}\right)\right.$$

$$\left.-\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right], \qquad (3.2.36)$$

$$\sigma_2=0,$$

$$\sigma_2=\frac{2p_0(1-2\nu)}{\pi}e^{2i\phi}\left\{(e^{-i\phi}-i\sin\phi)\left[(\rho^2-a^2)^{1/2}\frac{a}{\rho}\left(1+\frac{2}{3}\frac{a^2}{\rho^2-a^2}\right.\right.\right.$$

$$\left.-\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right]+\cos\phi\left[(\rho^2-a^2)^{1/2}\frac{a}{\rho}\left(1-2\frac{a^2}{\rho^2}+\frac{8}{3}\frac{a^2}{\rho^2-a^2}\right)\right.$$

$$\left.\left.-\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right]\right\}, \qquad (3.2.37)$$

$$\sigma_z = -p_0 \rho \cos\phi \ ,$$

$$\sigma_z = \frac{2p_0}{\pi}\cos\phi\left[(\rho^2-a^2)^{1/2}\ \frac{a}{\rho}\left(1+\ \frac{2}{3}\ \frac{a^2}{\rho^2-a^2}\right)-\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right]\ , \qquad (3.2.38)$$

$$\tau_z = 0\ , \qquad \text{for } \rho > 0 \qquad (3.2.39)$$

It can be seen that the boundary conditions in (3.2.1) have been identically satisfied. Also note that the first expression in (3.2.35) is equivalent to the one obtained in (3.2.11) (as it should be). And by taking into account that, for the isotropic case $H=\frac{1-\nu^2}{\pi E}$, there will be a complete coincidence.

The evaluation of the opening-mode stress intensity factor K_1 may be done by substituting second of the expressions for σ_z obtained in (3.2.38) into the formula defined in (2.37). It results in

$$K_1 = \frac{4p_0 a^2}{3\pi\sqrt{2a}}\cos\phi\ . \qquad (3.2.40)$$

It is interesting to note that the same result will be obtained by using the alternative expression for the stress intensity factor defined in (2.46) and the first expression for normal displacement component defined in (3.2.35). As may be seen the expression in (3.2.35) looks simpler than the one in (3.2.38). Also it enables one to verify the result in (3.2.40), which is in agreement with the result given in Shah and Kobayashi [31].

At this stage of the discussion, the elastic field for transverse isotropy in the case of a slightly different linear normal loading, namely, $\sigma_z=-p_0\rho\sin\phi$, will be presented. The reason for doing that is quite interesting and important and the explanation will follow.

$$u=Hp_0\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\left\{\rho\sin\phi\ e^{i\phi}\left[\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)-(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2}\right)\right]\right.$$

$$\left.-i\left[a(\ell_{2k}^2-a^2)^{1/2}\left(\frac{5}{2}\frac{\ell_{1k}^2}{a^2}-2-\frac{1}{3}\frac{\ell_{1k}^2}{l_{2k}^2}\right)+\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\left(\frac{2}{3}a^2-\frac{1}{2}\rho^2+2z_k^2\right)\right]\right\},$$

(3.2.41)

$$w=\frac{4Hp_0}{3}\rho\sin\phi\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left[(a^2-\ell_{1k}^2)^{1/2}\left(3-\frac{a^2}{\ell_{2k}^2}\right)-3z_k\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right],$$

(3.2.42)

$$\sigma_1=8A_{66}Hp_0\sin\phi\sum_{k=1}^{2}\frac{\gamma_k^2-(1+m_k)\gamma_3^2}{\gamma_k(m_k-1)}\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)\right.$$

$$\left.-\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right], \qquad (3.2.43)$$

$$\sigma_2=-2A_{66}Hp_0e^{2i\phi}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\left\{(ie^{-i\phi}+i\cos\phi)\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1\right.\right.\right.$$

$$\left.\left.+\frac{2}{3}\frac{a^2}{\ell_{2k}^2}\right)-\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right]+\sin\phi\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1\right.\right.$$

$$\left.\left.\left.-\frac{2a^2}{\ell_{2k}^2}+\frac{8}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)-\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right]\right\}, \qquad (3.2.44)$$

$$\sigma_z=\frac{2p_0}{\pi}\sin\phi\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{k+1}\gamma_k\left[(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2-\ell_{1k}^2}\right)\right.$$

$$- \rho \sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\Bigg] , \qquad (3.2.45)$$

$$\tau_z = -\frac{2p_0}{3\pi}\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{k+1}\Bigg\{2e^{i\phi}\sin\phi(a^2-\ell_{1k}^2)^{1/2}\frac{\ell_{1k}^4}{\rho^2(\ell_{2k}^2-\ell_{1k}^2)}$$

$$+ i\left[(a^2-\ell_{1k}^2)^{1/2}\left(\frac{a^2}{\ell_{2k}^2} - 3\right)+3z_k\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)\right]\Bigg\} . \qquad (3.2.46)$$

Note: To obtain the stress and displacement components for this problem some changes in expressions (3.2.18-3.2.23) had to be done, namely, in real components w, σ_1, σ_z and in the complex component σ_2, everywhere $\cos\phi$ had to be replaced by $\sin\phi$ and $\sin\phi$ by $-\cos\phi$, while in complex components u and τ_z, the same replacements had to be done, followed by the multiplication of each component by i.

Now, when the results for both cases of loading, i.e. for $-p_0\rho\cos\phi$ and $-p_0\rho\sin\phi$ are known, there arises quite reasonable question. Why was the problem not solved from the beginning, say for the linear normal loading such as $\sigma_z=-p_0\rho e^{i\phi}$? It could be done, but the resulting elastic field would have no meaning, since there is no way to achieve (by any means) the separation of the components such as u, σ_2 and τ_z in order to obtain the correct answer. The reason is, because they are complex (e.g. $u=u_x+iu_y$) and the loading $-p_0\rho e^{i\phi}$ is complex itself. What this means can be demonstrated by the following example for the u component. If (3.2.41) multiply by i and then add to it (3.2.18), it will result in

$$u=Hp_0\rho e^{2i\phi}\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}\left[\rho\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)-(\ell_{2k}^2-a^2)^{1/2}\frac{\ell_{1k}}{\ell_{2k}}\left(1+\frac{2}{3}\frac{a^2}{\ell_{2k}^2}\right)\right]. \tag{3.2.47}$$

It is a displacement component u for the case of the loading $\sigma_z=-p_0\rho e^{i\phi}$. After its separation (using Euler's identity) there should be ultimately a complete correspondence between the real part of (3.2.47) and the whole (3.2.18), and the imaginary part of (3.2.47) and the whole (3.2.41). The observation shows that it is not the case at all. Thus the conclusion is, that the solution to the problem with the linear normal loading such as $-p_0\rho e^{i\phi}$ will not give any reasonable answer to the question of stress and displacement components.

As a final point, recall the problem of a penny-shaped crack in an isotropic body under uniform pressure, which was mentioned in the introduction. According to Fabrikant [16] (the notations used by Fabrikant, except for Poisson's ratio ν, were transformed into the ones used by Sneddon) the expressions for radial and tangential stresses on the plane $z=0$ and $\rho>1$ should read:

$$\sigma_r=\frac{2p_0}{\pi}\left\{\frac{1}{(\rho^2-1)^{1/2}}\left[\nu\left(1-\frac{1}{\rho^2}\right)+\frac{1}{2}\left(1+\frac{1}{\rho^2}\right)\right]-(\nu+\tfrac{1}{2})\sin^{-1}\left(\frac{1}{\rho}\right)\right\},$$

$$\sigma_\vartheta=\frac{2p_0}{\pi}\left\{\frac{1}{(\rho^2-1)^{1/2}}\left[\nu\left(1+\frac{1}{\rho^2}\right)+\frac{1}{2}\left(1-\frac{1}{\rho^2}\right)\right]-(\nu+\tfrac{1}{2})\sin^{-1}\left(\frac{1}{\rho}\right)\right\}. \tag{3.2.48}$$

Sneddon [20] obtained: (for $z=0$ and $\rho>1$)

$$\sigma_r = \frac{2p_0}{\pi}\left[\frac{1}{(\rho^2-1)^{1/2}} - \left(\nu+\frac{1}{2}\right)\sin^{-1}\left(\frac{1}{\rho}\right)\right],$$

$$\sigma_\vartheta = \frac{2p_0}{\pi}\left[\frac{2\nu}{(\rho^2-1)^{1/2}} - \left(\nu+\frac{1}{2}\right)\sin^{-1}\left(\frac{1}{\rho}\right)\right]. \tag{3.2.49}$$

The expressions in (3.2.49) are presented in reference [20] at the end of page 495, with the only difference being in the notation for Poisson's ratio. It is done so, in order to avoid any misconceptions, since Sneddon in his notation used σ for both stresses and Poisson's ratio. Sneddon also gives an equation for $[\sigma_r-\sigma_\vartheta]_{z=0}$. After evaluation of the integrals in the brackets, it was found that, when $\rho>1$ the bracket has the value $-\rho^{-2}(\rho^2-1)^{-1/2}$, and not $-(\rho^2-1)^{-1/2}$, as it is given there. With the correct value, namely $-\rho^{-2}(\rho^2-1)^{-1/2}$, it results in the same expressions for σ_r and σ_ϑ as defined by (3.2.48). Also, there is a missing negative sign at the beginning of the right side of the equation for $[\sigma_r-\sigma_\vartheta]_{z=0}$.

Numerical computations were performed using (3.2.19, 3.2.21, 3.2.27 and 3.2.30) in order to illustrate the influence of anisotropy and location on the normal displacement and stress distribution. The numerical values assigned to the elastic constants for a transversely isotropic body were $A_{44}=A_{66}=1$, $A_{11}=150$, $A_{33}=25$, $A_{13}=56.25$. With the help of the identities given in (2.29, 2.35) and the following expressions

$$\gamma^2_{1,2}=M\pm(M^2-A_{33}/A_{11})^{1/2}, \quad M=\frac{(A_{11}A_{33}-A^2_{13}-2A_{13}A_{44})}{2A_{11}A_{44}}, \qquad (3.2.50)$$

the rest of the elastic constants may be easily determined.

Figures (3.2-3.5) illustrate the manner in which the depth and the distance from the periphery of the crack influence the displacements and stresses. Also it may be noted the influence of anisotropy on planes close to the plane z=0, which makes the values of displacement and stress in Figs.3.2 and 3.3 slightly less than the respective values in Figs.3.4 and 3.5. Farther away from the plane z=0 they are almost the same and tend to zero.

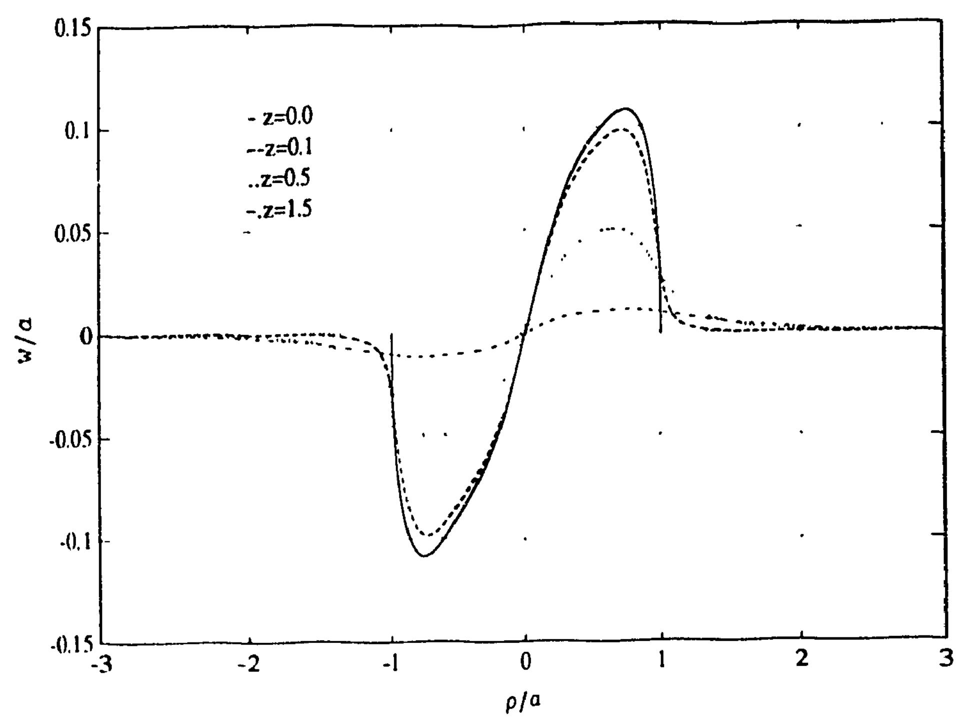

Fig.3.2 Normal displacement distribution in transversely isotropic body for different z: (—z=0.0; --z=0.1; ··z=0.5; ·-z=1.5).

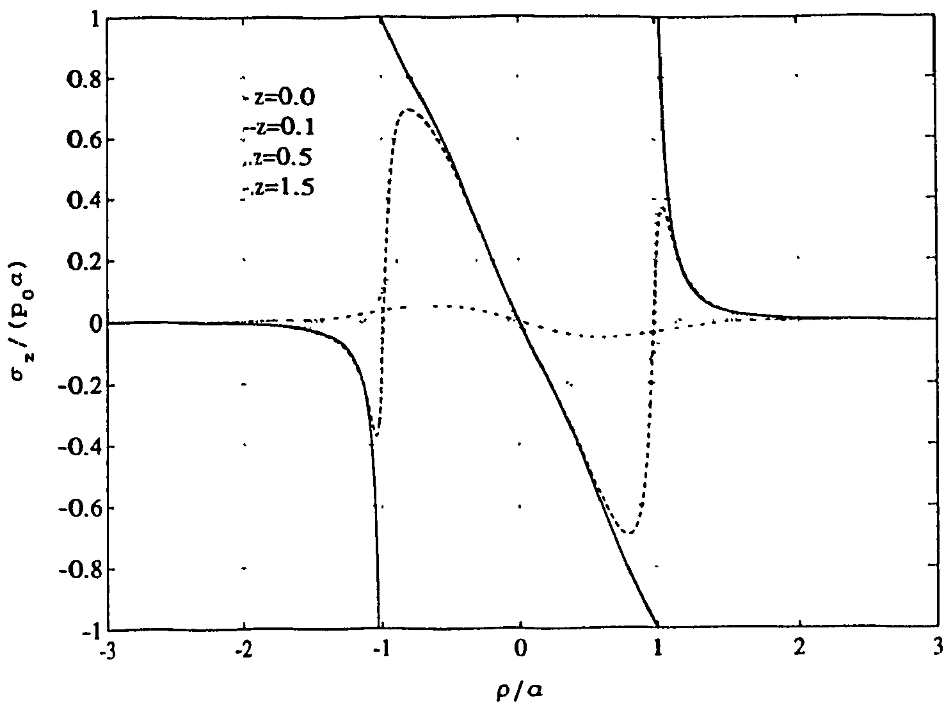

Fig.3.3 Normal stress distribution in transversely isotropic body for different z: (—z=0.0; --z=0.1; ··z=0.5; ·-z=1.5).

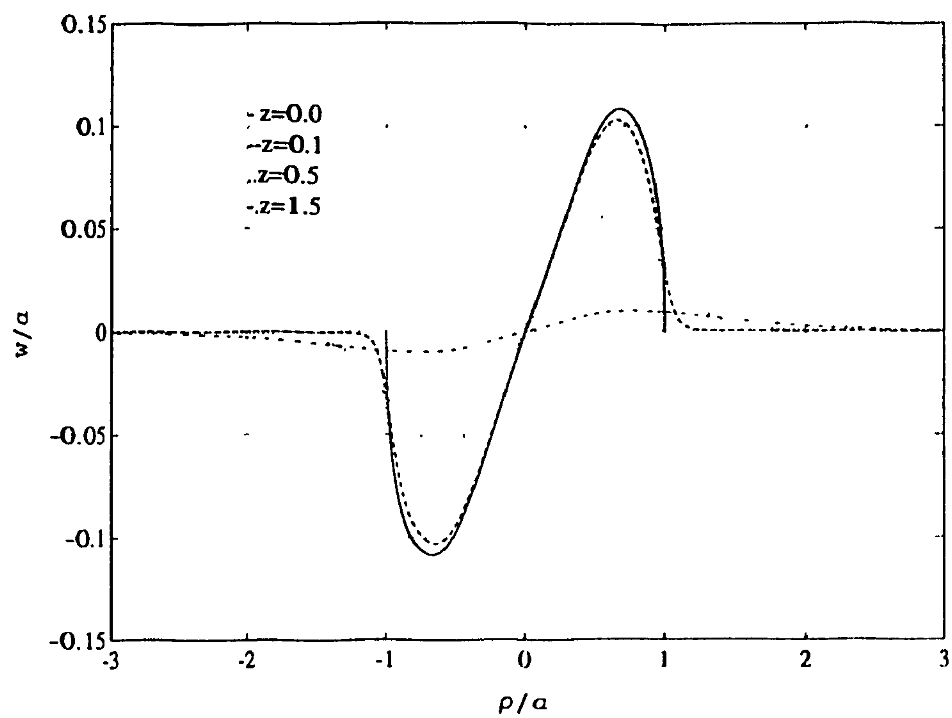

Fig.3.4 Normal displacement distribution in isotropic body for different z: (—z=0.0; --z=0.1; ··z=0.5; ·-z=1.5).

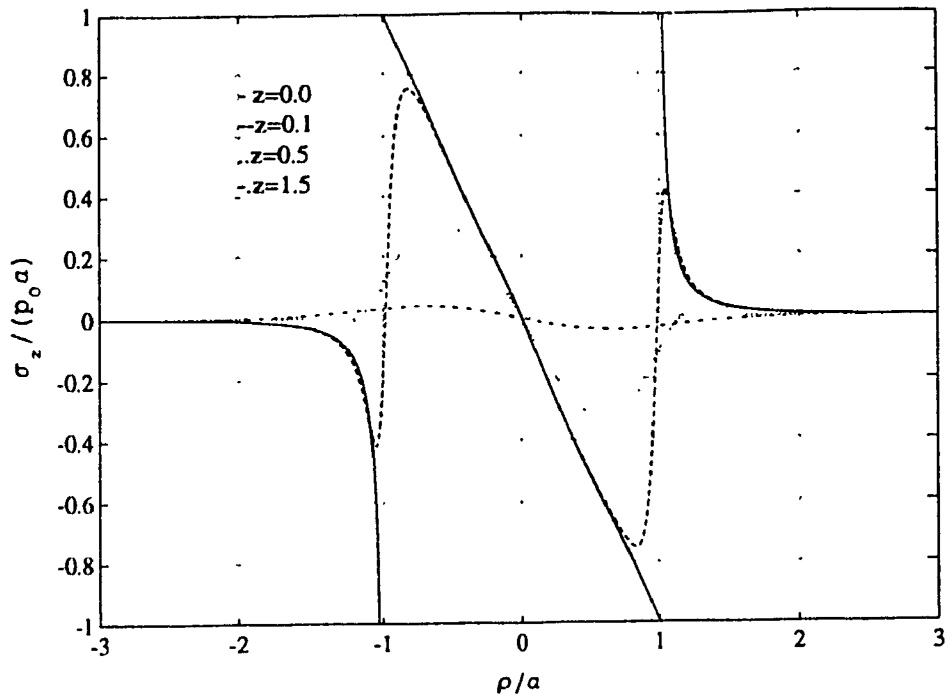

Fig.3.5 Normal stress distribution in isotropic body for different z: (—z=0.0; --z=0.1; ··z=0.5; ·-z=1.5).

3.3 ELASTIC FIELD OF INTERNAL CIRCULAR CRACK UNDER LINEAR SHEAR LOADING

Consider a transversely isotropic elastic space weakened by an internal circular crack of radius a in the plane z=0, with linear shear loading applied to the crack faces. The word "linear" means that the magnitude of loading is a linear function of the coordinates. The cylindrical coordinates (ρ,ϕ,z) are adopted to locate points inside the half-space z>0. The problem under consideration may be classified as an external mixed BVP and can be reformulated for a half-space.

Consider a transversely isotropic elastic half-space $z \geq 0$, with the normal traction $\sigma_z=0$ all over the plane $z=0$. A complex tangential displacement $u=0$ outside a circle $\rho=a$, while the linear shear loading $\tau_z=-\tau^0(\rho,\phi)$ is prescribed inside the circle. The mathematical statement of the boundary conditions is:

$$
\begin{aligned}
&\sigma_z=0\ , && \text{for} \quad 0\leq\rho<\infty\ , \quad 0\leq\phi<2\pi\ , \\
&u=0\ , && \text{for} \quad a\leq\rho<\infty\ , \quad 0\leq\phi<2\pi\ , \\
&\tau_z=-\left(\tau_1^0\rho e^{i\phi}+\tau_{-1}^0\rho e^{-i\phi}\right), && \text{for} \quad 0\leq\rho<a\ , \quad 0\leq\phi<2\pi\ .
\end{aligned} \tag{3.3.1}
$$

Here τ_1^0 and τ_{-1}^0 are any complex constants. Say $\tau_1^0=A+iB$, $\tau_{-1}^0=C+iD$, where A, B, C and D are real.

The vector representation of the above-stated loading is given below. Fig.3.6 represents the axisymmetric part of the loading, i.e. $\tau_1^0\rho e^{i\phi}$ and Fig.3.7 its non-axisymmetric part, i.e. $\tau_{-1}^0\rho e^{-i\phi}$.

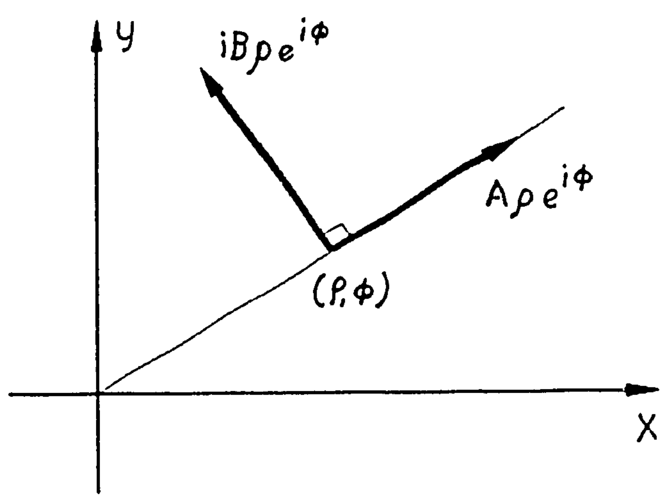

Fig.3.6 Axisymmetric part of loading.

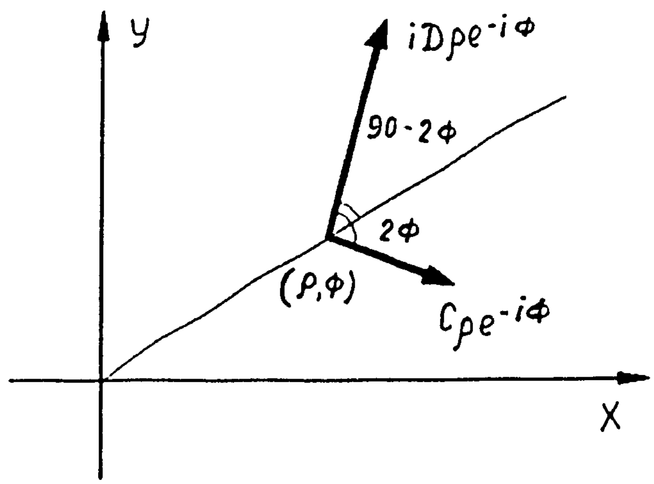

Fig.3.7 Nonaxisymmetric part of loading.

This type of condition may occur as an auxiliary problem when we have torsion of a cracked body, combined with various thermoelastic non-axisymmetric problems of heat flow.

3.3.1 POTENTIAL FUNCTIONS

It is known (Fabrikant [16]) that in the case of a planar crack under arbitrary shear loading, the complete solution can be expressed through the three potential functions,

$$F_1 = -\frac{1}{4\pi(m_1-1)}(\Lambda\bar{\chi}_1+\bar{\Lambda}\chi_1),$$

$$F_2 = -\frac{1}{4\pi(m_2-1)}(\Lambda\bar{\chi}_2+\bar{\Lambda}\chi_2),$$

$$F_3 = \frac{i}{4\pi}(\Lambda\bar{\chi}_3-\bar{\Lambda}\chi_3), \quad \text{where} \quad \chi_k=\chi(\rho,\phi,z_k),\ k=1,2,3. \qquad (3.3.2)$$

Here χ is a complex harmonic function which is represented as:

$$\chi(\rho,\phi,z) = \int_0^{2\pi}\int_0^{a} \ln(R_0+ z)\ \ u(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 \ , \qquad (3.3.3)$$

where $R_0=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}$, and $u(\rho_0,\phi_0)$ is the complex tangential displacement inside the crack.

Assume the existence of the expansion in the form

$$\tau(\rho,\phi) = \sum_{k=-\infty}^{\infty}\tau_k(\rho)e^{ik\phi} \ , \qquad (3.3.4)$$

where τ_k is the kth harmonic.

It has been found convenient to introduce positive harmonics as $\tau_{n+1}(\rho)$, where $n=0,1,2,\ldots$, and zero and negative harmonics as $\tau_{-n+1}(\rho)$, where $n=1,2,3,\ldots$

The solution of integro-differential equation which defines the complex tangential displacements inside the crack directly in terms of the prescribed shear loading is given as (Fabrikant [16])

$$u_{n+1}(\rho) = 2G_1\rho^{n+1}\int_{\rho}^{a} \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}} \int_0^{x} \frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{(x^2-\rho_0^2)^{1/2}}$$

$$+\ 2G_2\rho^{n+1}\int_{\rho}^{a} \frac{dx}{x^{2n+2}(x^2-\rho^2)^{1/2}} \int_0^{x} \frac{2nx^2-(2n+1)\rho_0^2}{(x^2-\rho_0^2)^{1/2}}\,\bar{\tau}_{-n+1}(\rho_0)\rho_0^n d\rho_0$$

for $n=0,1,2,\ldots$

(3.3.5)

$$u_{-n+1}(\rho) = 2G_1\rho^{n-1}\int_{\rho}^{a} \frac{d\alpha}{\alpha^{2n+2}(\alpha^2-\rho^2)^{1/2}} \int_0^{\alpha} \frac{\tau_{-n+1}(\rho_0)\rho_0^n d\rho_0}{(\alpha^2-\rho_0^2)^{1/2}}$$

$$+\ 2G_2\rho^{n-1}\left[\int_{\rho}^{a} \frac{(2n-1)\alpha^2-2n\rho^2}{\alpha^{2n}(\alpha^2-\rho^2)^{1/2}}\, d\alpha \int_0^{\alpha} \frac{\overline{\tau}_{n+1}(\rho_0)\rho_0^n d\rho_0}{(\alpha^2-\rho_0^2)^{1/2}} + K_n(a^2-\rho^2)^{1/2}\right] \tag{3.3.6}$$

where

$$K_n = a^{-2n-1}\int_0^{a} t^n(a^2-t^2)^{1/2}\left[\overline{\tau}_{n+1}(t)-(2n+1)\left(\frac{G_2}{G_1}\right)\tau_{-n+1}(t)\right]dt$$

for $n=1,2,3,...$

$$\tag{3.3.7}$$

Here the complex tangential displacement u is represented in terms of its harmonics as:

$$u(\rho,\phi) = \sum_{n=0}^{\infty} u_{n+1}(\rho)e^{i(n+1)\phi} + \sum_{n=1}^{\infty} u_{-n+1}(\rho)e^{-i(n-1)\phi} \ . \tag{3.3.8}$$

Utilizing the last condition of (3.3.1) in (3.3.5) and (3.3.6) gives for u_1 and u_{-1} harmonics the following results

$$u_1 = \frac{4}{3}(G_1\tau_1^0 - G_2\overline{\tau}_1^0)\rho(a^2-\rho^2)^{1/2} \ ,$$

$$u_{-1} = \frac{4}{3}\,\frac{G_1^2 - G_2^2}{G_1}\,\tau_{-1}^0\rho(a^2-\rho^2)^{1/2} \ . \tag{3.3.9}$$

It is interesting to note that the τ_{-1}^0 harmonic generates not only the corresponding u_{-1} harmonic but also it can, in principle, generate the u_3 harmonic. In this particular case, it turns out to be zero. If substitute

$\bar{\tau}_{-1}=\bar{\tau}^0_{-1}\rho$ in (3.3.5) for n=2, then it yields

$$u_3 = 2G_2\rho^3\int_\rho^a \frac{d\alpha}{\alpha^6(\alpha^2-\rho^2)^{1/2}}\int_0^\alpha \frac{\bar{\tau}^0_{-1}\rho_0^3(4\alpha^2-5\rho_0^2)d\rho_0}{(\alpha^2-\rho_0^2)^{1/2}} = 0\ . \tag{3.3.10}$$

According to (3.3.8), the tangential displacement will be

$$u(\rho,\phi)=\frac{4}{3}\left[(G_1\tau_1^0-G_2\bar{\tau}_1^0)e^{i\phi} + \frac{G_1^2-G_2^2}{G_1}\tau^0_{-1}e^{-i\phi}\right]\rho(a^2-\rho^2)^{1/2}. \tag{3.3.11}$$

The subsequent substitution of (3.3.11) in (3.3.3) will result in

$$\chi(\rho,\phi,z)=\frac{4}{3}\int_0^{2\pi}\int_0^a \ln(R_0+z)\left[(G_1\tau_1^0-G_2\bar{\tau}_1^0)e^{i\phi_0}\right.$$
$$\left. + \frac{G_1^2-G_2^2}{G_1}\tau^0_{-1}e^{-i\phi_0}\right](a^2-\rho_0^2)^{1/2}\rho_0^2 d\rho_0 d\phi_0. \tag{3.3.12}$$

This type of integral obtained in (3.3.12) was already evaluated (Fabrikant [16]) and hence χ becomes

$$\chi(\rho,\phi,z)=\frac{\pi}{6}\left[(G_1\tau_1^0-G_2\bar{\tau}_1^0)e^{i\phi} + \frac{G_1^2-G_2^2}{G_1}\tau^0_{-1}e^{-i\phi}\right]\rho I(\rho,z), \tag{3.3.13}$$

where

$$I(\rho,z)=(a^2-\ell_1^2)^{1/2}\left[\frac{4}{3}a^2+7\rho^2-\frac{19}{3}\ell_1^2-4\ell_2^2+\frac{2(8a^4+4a^2\ell_1^2+3\ell_1^4)}{15\rho^2}\right]$$

$$+ z(4a^2-3\rho^2+ 4z^2)\sin^{-1}\left(\frac{a}{\ell_2}\right) - \frac{16a^5}{15\rho^2} . \qquad (3.3.14)$$

The quantities ℓ_1 and ℓ_2 are defined as in (3.2.15).

According to (3.3.2), after application of the differential operators defined in (3.2.11) the three potential functions will take the form:

$$F_1=-\frac{G_1-G_2}{12(m_1-1)}\left[A\left(2I+\rho\frac{\partial I}{\partial \rho}\right)+\frac{G_1+G_2}{G_1}(C\cos2\phi+D\sin2\phi)\rho\frac{\partial I}{\partial \rho}\right],$$

$$F_2=-\frac{G_1-G_2}{12(m_2-1)}\left[A\left(2I+\rho\frac{\partial I}{\partial \rho}\right)+\frac{G_1+G_2}{G_1}(C\cos2\phi+D\sin2\phi)\rho\frac{\partial I}{\partial \rho}\right],$$

$$F_3=\frac{G_1+G_2}{12}\left[B\left(2I+\rho\frac{\partial I}{\partial \rho}\right)-\frac{G_1-G_2}{G_1}(C\sin2\phi-D\cos2\phi)\rho\frac{\partial I}{\partial \rho}\right]. \qquad (3.3.15)$$

3.3.2 COMPLETE ELASTIC FIELD

The elastic field resulting from the shear loading can be evaluated by performing appropriate differentiation of the potential functions (3.3.15), which had to be substituted into the expressions (2.27) and (2.32). The derivatives of $I(\rho,z)$ are given in Appendix A3.3. The elastic constants and identities defined by (2.35) and (2.36) were used in the results to follow. The elastic field is:

$$u=\frac{4H\gamma_1\gamma_2}{3}e^{i\phi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{Af_1(z_k)+\frac{G_1+G_2}{G_1}\left[(C\cos2\phi + D\sin2\phi)f_2(z_k)\right.\right.$$

$$-i(C\sin 2\phi - D\cos 2\phi) f_3(z_k)\Big]\Big\} - \frac{4\beta}{3} e^{i\phi}\Big\{iBf_1(z_3) + \frac{G_1 - G_2}{G_1}\Big[(C\cos 2\phi$$

$$+ D\sin 2\phi) f_3(z_3) - i(C\sin 2\phi - D\cos 2\phi) f_2(z_3)\Big]\Big\} , \qquad (3.3.16)$$

$$w = H\gamma_1\gamma_2 \sum_{k=1}^{2} \frac{m_k}{(m_k - 1)\gamma_k}\Big[Af_4(z_k) - \frac{G_1 + G_2}{G_1}(C\cos 2\phi + D\sin 2\phi) f_5(z_k)\Big] ,$$

$$(3.3.17)$$

$$\sigma_1 = \frac{16A_{66}H\gamma_1\gamma_2}{3} \sum_{k=1}^{2} \frac{\gamma_k^2 - (1 + m_k)\gamma_3^2}{\gamma_k^2 (m_k - 1)}\Big[Af_6(z_k)$$

$$- \frac{G_1 + G_2}{G_1}(C\cos 2\phi + D\sin 2\phi) f_7(z_k)\Big] , \qquad (3.3.18)$$

$$\sigma_2 = \frac{16A_{66}H\gamma_1\gamma_2}{3} e^{2i\phi} \sum_{k=1}^{2} \frac{1}{m_k - 1} \Big\{Af_7(z_k)$$

$$- \frac{G_1 + G_2}{G_1}\Big[(C\cos 2\phi + D\sin 2\phi) f_8(z_k) - i(C\sin 2\phi - D\cos 2\phi) f_9(z_k)\Big]\Big\}$$

$$- \frac{8}{3\pi\gamma_3} e^{2i\phi}\Big\{iBf_7(z_3) - \frac{G_1 - G_2}{G_1}\Big[(C\cos 2\phi + D\sin 2\phi) f_9(z_3)$$

$$- i(C\sin 2\phi - D\cos 2\phi) f_8(z_3)\Big]\Big\} , \qquad (3.3.19)$$

$$\sigma_z = \frac{4\gamma_1\gamma_2}{3\pi(\gamma_1 - \gamma_2)} \sum_{k=1}^{2} (-1)^{k+1}\Big[Af_6(z_k)$$

$$- \frac{G_1 + G_2}{G_1}(C\cos 2\phi + D\sin 2\phi) f_7(z_k)\Big] , \qquad (3.3.20)$$

$$\tau_z = \frac{\gamma_1\gamma_2}{\pi(\gamma_1 - \gamma_2)} e^{i\phi} \sum_{k=1}^{2} \frac{(-1)^{k+1}}{\gamma_k} \Big\{Af_{10}(z_k)$$

$$+ \frac{G_1+G_2}{G_1}\Big[(C\cos 2\phi + D\sin 2\phi) f_{11}(z_k) - i(C\sin 2\phi - D\cos 2\phi) f_{12}(z_k)$$

$$- \frac{\gamma_3}{\pi} e^{i\phi}\Bigg\{ iBf_{10}(z_3) + \frac{G_1-G_2}{G_1}\Big[(C\cos 2\phi + D\sin 2\phi) f_{12}(z_3)$$

$$- i(C\sin 2\phi - D\cos 2\phi) f_{11}(z_3)\Big]\Bigg\} . \qquad (3.3.21)$$

Here

$$f_1(z) = \rho(a^2 - \ell_1^2)^{1/2}\left(\frac{\ell_1^2}{\rho^2} - 3\right) + 3\rho z \sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.22)$$

$$f_2(z) = \rho(a^2 - \ell_1^2)^{1/2}\left(\frac{1}{2}\frac{\ell_1^2}{\rho^2} - \frac{1}{5}\frac{\ell_1^4}{\rho^4} - \frac{4}{15}\frac{a^2\ell_1^2}{\rho^4} - \frac{8}{15}\frac{a^4}{\rho^4} - \frac{3}{2}\right)$$

$$+ \frac{8}{15}\frac{a^5}{\rho^3} + \frac{3}{2}\rho z \sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.23)$$

$$f_3(z) = \rho(a^2 - \ell_1^2)^{1/2}\left(\frac{1}{2}\frac{\ell_1^2}{\rho^2} + \frac{1}{5}\frac{\ell_1^4}{\rho^4} + \frac{4}{15}\frac{a^2\ell_1^2}{\rho^4} + \frac{8}{15}\frac{a^4}{\rho^4} - \frac{3}{2}\right)$$

$$- \frac{8}{15}\frac{a^5}{\rho^3} + \frac{3}{2}\rho z \sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.24)$$

$$f_4(z) = a(\ell_2^2 - a^2)^{1/2}\left(4 - 6\frac{\ell_1^2}{a^2}\right) - \left(\frac{4}{3}a^2 - 2\rho^2 + 4z^2\right)\sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.25)$$

$$f_5(z) = a(\ell_2^2 - a^2)^{1/2}\left(\frac{\ell_1^2}{a^2} + \frac{2}{3}\frac{\ell_1^2}{\ell_2^2}\right) - \rho^2 \sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.26)$$

$$f_6(z) = (a^2 - \ell_1^2)^{1/2}\left(3 - \frac{a^2}{\ell_2^2 - \ell_1^2}\right) - 3z\sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad (3.3.27)$$

$$f_7(z) = (a^2 - \ell_1^2)^{1/2} \frac{\ell_1^4}{\rho^2(\ell_2^2 - \ell_1^2)} \quad , \tag{3.3.28}$$

$$f_8(z) = (a^2-\ell_1^2)^{1/2}\left(\frac{3}{2} - \frac{1}{2}\,\frac{\ell_1^2(\ell_2^2+\ell_1^2)}{\rho^2(\ell_2^2-\ell_1^2)} - \frac{1}{5}\,\frac{\ell_1^2(3\ell_1^2+4a^2)}{\rho^4}\right.$$
$$\left. -\frac{8}{5}\,\frac{a^4}{\rho^4}\right) + \frac{8}{5}\,\frac{a^5}{\rho^4} - \frac{3}{2}z\sin^{-1}\left(\frac{a}{\ell_2}\right) \quad , \tag{3.3.29}$$

$$f_9(z) = (a^2-\ell_1^2)^{1/2}\left(\frac{3}{2} - \frac{1}{2}\,\frac{\ell_1^2(\ell_2^2+\ell_1^2)}{\rho^2(\ell_2^2-\ell_1^2)} + \frac{1}{5}\,\frac{\ell_1^2(3\ell_1^2+4a^2)}{\rho^4}\right.$$
$$\left. +\frac{8}{5}\,\frac{a^4}{\rho^4}\right) - \frac{8}{5}\,\frac{a^5}{\rho^4} - \frac{3}{2}z\sin^{-1}\left(\frac{a}{\ell_2}\right) \quad , \tag{3.3.30}$$

$$f_{10}(z) = 2\rho\sin^{-1}\left(\frac{a}{\ell_2}\right) - (\ell_2^2-a^2)^{1/2}\,\frac{\ell_1}{\ell_2}\left(2 + \frac{4}{3}\,\frac{a^2}{\ell_2^2-\ell_1^2}\right) \quad , \tag{3.3.31}$$

$$f_{11}(z) = \rho\sin^{-1}\left(\frac{a}{\ell_2}\right) - (\ell_2^2-a^2)^{1/2}\,\frac{\ell_1}{\ell_2}\left(1 - \frac{2}{3}\,\frac{a^2}{\ell_2^2} + \frac{4}{3}\,\frac{a^2}{\ell_2^2-\ell_1^2}\right) \quad , \tag{3.3.32}$$

$$f_{12}(z) = \rho\sin^{-1}\left(\frac{a}{\ell_2}\right) - (\ell_2^2 - a^2)^{1/2}\,\frac{\ell_1}{\ell_2}\left(1 + \frac{2}{3}\,\frac{a^2}{\ell_2^2}\right) \quad , \tag{3.3.33}$$

the notations ℓ_1 and ℓ_2 are defined in (3.2.15).

3.3.3 ELASTIC FIELD FOR ISOTROPY

The expressions for the stress and displacement field for isotropy are now obtained by taking a limiting form of the expressions (3.3.16-3.3.21) for transverse isotropy obtained before. For isotropy $\gamma_1, \gamma_2, \gamma_3, m_1, m_2 \to 1$ and the limiting forms of the sums for shear loading are given as:

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{1}{m_k-1} f(z_k) = - f(z) - \frac{z}{2(1-\nu)} f'(z) ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{m_k}{\gamma_k(m_k-1)} f(z_k) = \frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)} ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{\gamma_k^2-(1+m_k)\gamma_3^2}{\gamma_k^2(m_k-1)} f(z_k) = \frac{2(1+\nu)f(z)+zf'(z)}{2(1-\nu)} ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} (-1)^{k+1} f(z_k) = - zf'(z) ,$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \frac{1}{\gamma_1-\gamma_2} \sum_{k=1}^{2} \frac{(-1)^{k+1}}{\gamma_k} f(z_k) = -f(z) - zf'(z) , \tag{3.3.34}$$

where $f'(z)$ denotes the derivative of $f(z)$.

Using the expressions (3.3.34) along with the isotropic limits $\beta=\frac{1+\nu}{\pi E}$, $H=\frac{1-\nu^2}{\pi E}$, $A_{44}=A_{66}=\frac{E}{2(1+\nu)}$, the isotropic elastic field for linear shear loading becomes:

$$u = \frac{2(1+\nu)}{\pi E} e^{i\phi}\Bigg\{\left[(1-\nu)A + iB + \frac{2(1-\nu)}{2-\nu}(C + iD)e^{-2i\phi}\right]f_1^*(z)$$

$$+ \frac{z}{2}\Bigg[Af_2^*(z) + \frac{2}{2-\nu}\Big((C\cos2\phi + D\sin2\phi)f_3^*(z)$$

$$- i(C\sin2\phi - D\cos2\phi)f_4^*(z)\Big)\Bigg]\Bigg\} , \tag{3.3.35}$$

$$w = \frac{1+\nu}{\pi E}\Bigg\{(1-2\nu)Af_5^*(z) + \frac{2(1-2\nu)}{2-\nu}(C\cos2\phi + D\sin2\phi)f_6^*(z)$$

$$- z\left[Af_7^*(z) + \frac{2}{2-\nu}(C\cos2\phi + D\sin2\phi)f_8^*(z)\right]\Bigg\} \quad , \tag{3.3.36}$$

$$\sigma_1 = \frac{2}{\pi}\Bigg\{(1-\nu)Af_7^*(z) + \frac{2(1+\nu)}{2-\nu}(C\cos2\phi + D\sin2\phi)f_8^*(z)$$

$$- \frac{z}{2}\left[Af_9^*(z) - \frac{2}{2-\nu}(C\cos2\phi + D\sin2\phi)f_{10}^*(z)\right]\Bigg\} \quad , \tag{3.3.37}$$

$$\sigma_2 = \frac{2}{\pi}\, e^{2i\phi}\Bigg\{\Big((1-\nu)A + iB\Big)f_8^*(z) + \frac{2(1-\nu)}{2-\nu}(C + iD)e^{-2i\phi}f_7^*(z)$$

$$+ \frac{z}{2}\left[Af_{10}^*(z) + \frac{2}{2-\nu}\Big((C\cos2\phi + D\sin2\phi)f_{11}^*(z)\right.$$

$$\left.- i(C\sin2\phi - D\cos2\phi)f_{12}^*(z)\Big)\right]\Bigg\} \quad , \tag{3.3.38}$$

$$\sigma_z = \frac{1}{\pi}\, z\left[Af_9^*(z) - \frac{2}{2-\nu}(C\cos2\phi + D\sin2\phi)f_{10}^*(z)\right] \quad , \tag{3.3.39}$$

$$\tau_z = \frac{1}{\pi}\, e^{i\phi}\Bigg\{\left[(A + iB) + \frac{2}{2-\nu}(C + iD)e^{-2i\phi}\right]f_2^*(z)$$

$$- \frac{2\nu}{2-\nu}\Big[(C\cos2\phi + D\sin2\phi)f_4^*(z) - i(C\sin2\phi - D\cos2\phi)f_3^*(z)\Big]$$

$$+ z\left[\left(A + \frac{2}{2-\nu}(C\cos2\phi + D\sin2\phi)\right)f_{13}^*(z)\right.$$

$$\left.- \frac{2}{2-\nu}(C - iD)e^{2i\phi}f_{14}^*(z)\right]\Bigg\} \quad . \tag{3.3.40}$$

Here

$$f_1^*(z) = \rho(a^2-\ell_1^2)^{1/2}\left(2 - \frac{2}{3}\frac{\ell_1^2}{\rho^2}\right) - 2\rho z\sin^{-1}\left(\frac{a}{\ell_2}\right) \quad , \tag{3.3.41}$$

$$f_2^{\bullet}(z)=(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(2+\ \frac{4}{3}\ \frac{a^2}{\ell_2^2-\ell_1^2}\right)-2\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.42}$$

$$f_3^{\bullet}(z)=(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(1-\ \frac{2}{3}\ \frac{a^2}{\ell_2^2}\ +\ \frac{4}{3}\ \frac{a^2}{\ell_2^2-\ell_1^2}\right)-\rho\sin^{-1}\left(\frac{a}{\ell_2}\right), \tag{3.3.43}$$

$$f_4^{\bullet}(z)=(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(1+\ \frac{2}{3}\ \frac{a^2}{\ell_2^2}\right)-\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.44}$$

$$f_5^{\bullet}(z)=a(\ell_2^2-a^2)^{1/2}\left(2-3\frac{\ell_1^2}{a^2}\right)-(\tfrac{2}{3}a^2-\rho^2+2z^2)\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.45}$$

$$f_6^{\bullet}(z)=-a(\ell_2^2-a^2)^{1/2}\left(\frac{1}{2}\ \frac{\ell_1^2}{a^2}\ +\ \frac{1}{3}\ \frac{\ell_1^2}{\ell_2^2}\right)+\ \frac{1}{2}\rho^2\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.46}$$

$$f_7^{\bullet}(z)=(a^2-\ell_1^2)^{1/2}\left(4-\ \frac{4}{3}\ \frac{a^2}{\ell_2^2-\ell_1^2}\right)-4z\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.47}$$

$$f_8^{\bullet}(z)=-(a^2-\ell_1^2)^{1/2}\ \frac{4}{3}\ \frac{\ell_1^4}{\rho^2(\ell_2^2-\ell_1^2)}\ , \tag{3.3.48}$$

$$f_9^{\bullet}(z)=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left[\frac{16}{3}\ \frac{\ell_1^4\ell_2^2}{(\ell_2^2-\ell_1^2)^3}\ -\ \frac{8}{3}\ \frac{a^2\ell_2^2(\ell_1^2+\ell_2^2)}{(\ell_2^2-\ell_1^2)^3}\right.$$

$$\left.+\ \frac{4\ell_1^4}{(\ell_2^2-\ell_1^2)^2}\ -4\right]+4\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.49}$$

$$f_{10}^{\bullet}(z)=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left[\frac{8}{3}\ \frac{\ell_1^2(3\ell_2^2-\ell_1^2)(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}\right.$$

$$\left.-\ \frac{4}{3}\ \frac{\ell_1^4}{(\ell_2^2-\ell_1^2)^2}\right]\ , \tag{3.3.50}$$

$$f^*_{11}(z)=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left[\frac{16}{3}\ \frac{\ell_2^2\ell_1^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}+\frac{4}{3}\ \frac{\ell_1^2(2a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^2}\right.$$

$$\left.+\frac{8}{15}\ \frac{\ell_1^4}{\rho^2(\ell_2^2-\ell_1^2)}+\frac{4}{3}\ \frac{\ell_1^2}{\rho^2}+2\right]-2\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \tag{3.3.51}$$

$$f^*_{12}(z)=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(\frac{16}{3}\ \frac{a^2}{\ell_2^2-\ell_1^2}\ -4\frac{a^2}{\ell_2^2}+2\right)-2\sin^{-1}\left(\frac{a}{\ell_2}\right), \tag{3.3.52}$$

$$f^*_{13}(z)=(a^2-\ell_1^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left[\frac{16}{3}\ \frac{a\ell_2^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}-\frac{4a\ell_1^2}{(\ell_2^2-\ell_1^2)^2}\right]\ , \tag{3.3.53}$$

$$f^*_{14}(z)=\frac{8}{3}(a^2-\ell_1^2)^{1/2}\ \frac{\ell_1}{\ell_2}\ \frac{a^3}{\ell_2^2(\ell_2^2-\ell_1^2)}\ , \tag{3.3.54}$$

the notations ℓ_1 and ℓ_2 are defined in (3.2.15).

3.3.4 DISCUSSION AND NUMERICAL RESULTS

As in the previous problem the complete solution for this non-axisymmetric problem is also presented in terms of just two distorted length parameters ℓ_1 and ℓ_2. The expressions for the elastic field are in compact complex notation and their separation can be done in a simple manner by using Euler's identity. With help of equation (3.2.33) the surface values can be determined and verified with the boundary conditions of the problem. The elastic field on the plane z=0 will be presented here only for the isotropic case. In the results to follow, the first expression of each component corresponds to the case of $\rho<a$, while the second gives the result for $\rho>a$.

$$u = \frac{8(1-\nu^2)}{3\pi E}\left[\left(A + \frac{1}{1-\nu}iB\right)e^{i\phi} + \frac{2}{2-\nu}\,\tau^0_{-1}e^{-i\phi}\right]\rho(a^2-\rho^2)^{1/2} ,$$

$$u = 0 , \tag{3.3.55}$$

$$w = -\frac{(1+\nu)(1-2\nu)}{6E}\left[A(2a^2-3\rho^2) - \frac{1}{2-\nu}(C\cos 2\phi + D\sin 2\phi)3\rho^2\right] ,$$

$$w = -\frac{(1+\nu)(1-2\nu)}{3\pi E}\left\{A\left[3a(\rho^2-a^2)^{1/2} + (2a^2-3\rho^2)\sin^{-1}\left(\frac{a}{\rho}\right)\right]\right.$$

$$\left. - \frac{1}{2-\nu}(C\cos 2\phi + D\sin 2\phi)3\rho^2\sin^{-1}\left(\frac{a}{\rho}\right)\right\} , \tag{3.3.56}$$

$$\sigma_1 = \frac{8(1+\nu)}{3\pi}\left[A\frac{2a^2-3\rho^2}{(a^2-\rho^2)^{1/2}} - \frac{2}{2-\nu}(C\cos 2\phi + D\sin 2\phi)\frac{\rho^2}{(a^2-\rho^2)^{1/2}}\right],$$

$$\sigma_1 = 0 , \tag{3.3.57}$$

$$\sigma_2 = -\frac{8(1-\nu)}{3\pi}\left[\left(A + \frac{1}{1-\nu}iB\right)e^{2i\phi}\frac{\rho^2}{(a^2-\rho^2)^{1/2}} - \frac{2}{2-\nu}\,\tau^0_{-1}\frac{2a^2-3\rho^2}{(a^2-\rho^2)^{1/2}}\right],$$

$$\sigma_2 = 0 , \tag{3.3.58}$$

$$\sigma_z = 0 , \qquad \text{for } \rho>0 \tag{3.3.59}$$

$$\tau_z = -\left(\tau^0_1\rho e^{i\phi} + \tau^0_{-1}\rho e^{-i\phi}\right) ,$$

$$\tau_z = \frac{2}{3\pi}\left\{\left(\tau^0_1 e^{i\phi} + \tau^0_{-1}e^{-i\phi}\right)\left[(\rho^2-a^2)^{1/2}\,\frac{a}{\rho}\left(3 + \frac{2a^2}{\rho^2-a^2}\right)\right.\right.$$

$$\left.\left. - 3\rho\sin^{-1}\left(\frac{a}{\rho}\right)\right] + \frac{2\nu}{2-\nu}\,\bar{\tau}^0_{-1}e^{3i\phi}\frac{a^5}{\rho^3(\rho^2-a^2)^{1/2}}\right\} . \tag{3.3.60}$$

Thus, the boundary conditions in (3.3.1) have been identically satisfied. Also, it should be noted that the first expression in (3.3.55) is equivalent to (3.3.11), (if the expressions for G_1 and G_2 given in (2.36), and that for the isotropic case $\gamma_1,\gamma_2,\gamma_3\to 1$, $H=\frac{1-\nu^2}{\pi E}$, $A_{44}=A_{66}=\frac{E}{2(1+\nu)}$, are taken into account).

The SIF K_2 and K_3 can be evaluated by using the second expression for τ_z obtained in (3.3.60) and the expression for the complex SIF for mixed mode II and III introduced in (2.49). The result is

$$K_2= \frac{(2a)^{3/2}}{3\pi}\left[A + \frac{2}{2-\nu}(C\cos2\phi + D\sin2\phi)\right] \quad , \tag{3.3.61}$$

$$K_3= \frac{(2a)^{3/2}}{3\pi}\left[B + \frac{2(1-\nu)}{2-\nu}(D\cos2\phi - C\sin2\phi)\right] \quad , \tag{3.3.62}$$

Here again, as in the previous problem for normal loading, an alternative expression for evaluation of mixed mode SIF can be used. The substitution of the first expression for the tangential displacement component obtained in (3.3.55) into formulae (2.47) will ultimately give the same result as in (3.3.61) and (3.3.62).

In Figs.3.8 and 3.9 are presented the graphs of the variation of the dimensionless stress intensity factor along the periphery of the crack. The value of 1 was assigned to the real constants A,B,C and D. It can be seen that when $\phi=22.5^0$ the sliding mode SIF K_2 has maximum value, and greater the value of Poisson's ratio the greater is SIF.

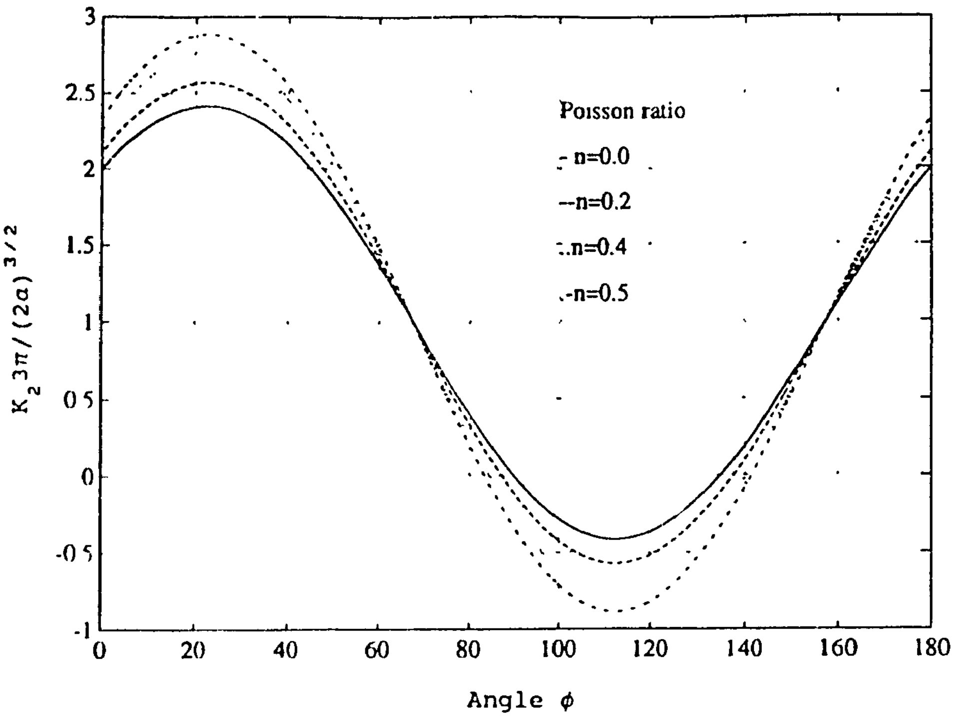

Fig.3.8 Variation of SIF K_2 along crack border for different n: (—n=0.0; --n=0.2; ··n=0.4; ·-n=0.5).

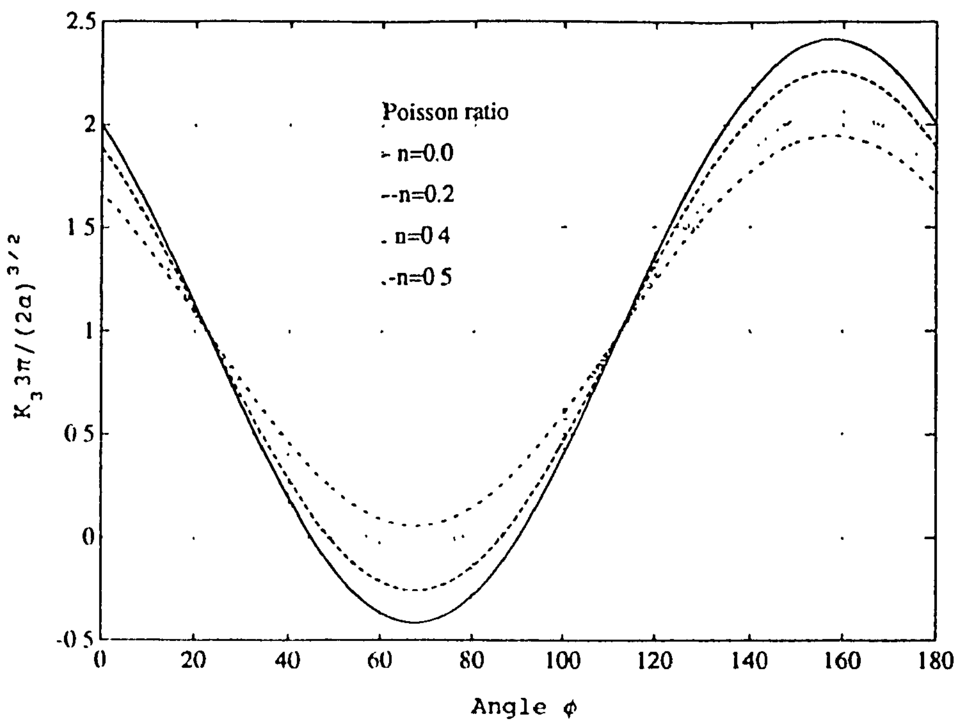

Fig.3.9 Variation of SIF K_3 along crack border for different n: (—n=0.0; --n=0.2; ··n=0.4; ·-n=0.5).

At the same location on the periphery of the crack the tearing mode SIF K_3 has the same value regardless of Poisson's ratio. When $\phi=67.5^0$ the picture is changing, namely, K_2 becomes independent on Poisson's ratio, while K_3 reaches its maximum and the smaller is Poisson's ratio the greater is value for K_3. This phenomenon repeats itself with the period of 180^0.

3.4 SUMMARY

The importance of the results obtained in this chapter consist of the following. First, a complete solution to the problems when the loading prescribed on the crack faces are linear functions (in the case of shear it had both axisymmetric and non-axisymmetric parts) was obtained. Secondly, the results are in closed form and expressed in terms of elementary functions, which makes their numerical evaluation very easy. In early publications (Sneddon [20], Westman [26]), even for the case of constant loading, the results were expressed in terms of integrals containing Bessel functions, making their numerical evaluation difficult. And finally, the ability to have a complete solution plays an indispensable role for the consideration of interaction problems.

The expressions for SIF, which were presented in Chapter 2, have been used and provided an absolute exactness for the final results.

The results obtained in this chapter can be used in the stress analysis of various bodies with cracks subjected to bending and/or torsion.

In Chapter 4, the attention will be focused on the analysis of external circular crack problems.

CHAPTER 4

EXTERNAL CIRCULAR CRACK PROBLEM

4.1 INTRODUCTORY REMARKS

As mentioned earlier, the great majority of solved crack problems deal with the stresses and displacements in the plane $z=0$ only. There are just a few complete solutions published: *e.g.* Sneddon [20], Elliott [25], Westman [26], where explicit expressions are given for the field of displacements and stresses for the simplest axisymmetric problems (a circular punch and penny-shaped crack). The explicit expressions for the field of displacements due to an elliptical crack can be found in Kassir and Sih [15]. Knowledge of complete solutions is indispensable for consideration of more complicated problems of crack interactions, influence of external loads on cracks, etc. By using the reciprocal theorem, many new results can be obtained, like, for example, the stress intensity factors due to an arbitrarily located force.

In Chapter 3, solution for two problems of a penny-shaped crack was given. In Chapter 4 a new fundamental solution to the problems of an external circular crack will be presented. That is, all the relevant Green's functions will be given explicitly in terms of the elementary functions to the problems of an external circular crack under arbitrary normal and shear loading. A complete closed form solution, with formulae for the field of all stresses and displacements, to the problem of external circular crack under arbitrary shear loading has become possible since the recent discovery of a method of continuity solutions (Fabrikant [32]). It was based on the use of the reciprocal theorem to derive the continuation formula for the direct relationship between the tangential stresses in the crack neck in terms of the prescribed tractions τ. This formula allows one to obtain an exact closed-form solution in terms of elementary functions to the governing integral equation of an external circular crack in a transversely isotropic elastic body. The solution to the governing integral equation will be given by two different methods.

For the first time, a complete solution in terms of elementary functions will be given to the two problems of a transversely isotropic elastic space weakened by an external circular crack subjected to an arbitrary normal and shear loading. A complete field of displacements and stresses due to a concentrated normal loading applied symmetrically to crack faces is given for both transversely isotropic and purely isotropic cases. The case of the isotropic body weakened by an external circular crack is solved as a limiting case of the transversely isotropic one. Some of the results are given in a graphical form. In the case of a concentrated shear loading applied antisymmetricaly to crack faces a complete field is given only for the transversely isotropic body. Part of the material presented in this chapter follows the paper by Fabrikant, Rubin and Karapetian [33]. No similar results seem to have ever been reported in the literature, even in the case of an isotropic body.

4.2 EXTERNAL CIRCULAR CRACK UNDER NORMAL LOAD: A COMPLETE SOLUTION

Consider a transversely isotropic elastic space weakened by a flat crack S in the plane z=0, with arbitrary pressure p applied to the crack faces, Fig.4.1.

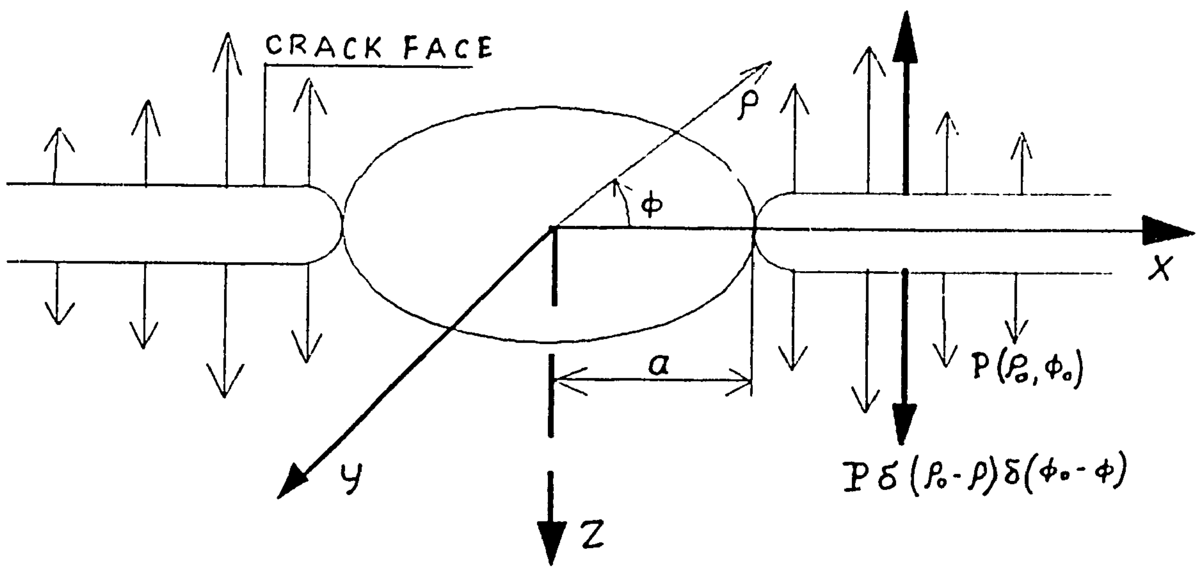

Fig.4.1 External circular crack under arbitrary normal load.

Due to symmetry, the problem can be formulated as follows: find the solution to the set of differential equations (2.24) for a half-space $z\geq 0$, subject to the mixed boundary conditions on the plane $z=0$:

$$\sigma_z=-p(x,y),\quad \text{for}(x,y)\in S;\quad w=0,\quad \text{for}(x,y)\notin S;$$
$$\tau_z=0,\quad \text{for } -\infty<(x,y)<\infty. \tag{4.2.1}$$

4.2.1 GOVERNING EQUATIONS

The conditions (4.2.1) can be satisfied by a representation in terms of one harmonic function. For this type of problem, according to (2.33) the functions are:

$$F_1(z)=c_1F(z_1),\quad F_2(z)=c_2F(z_2),\quad F_3(z)=0. \tag{4.2.2}$$

Expressions of the type $F_1(z)$ and $F(z_1)$, *etc.*, everywhere should be understood as $F_1(x,y,z)$ and $F(x,y,z_1)$ respectively. The substitution of (4.2.2) and the last of expressions (2.32) in the third condition (4.2.1) yields:

$$c_1=-c_2\gamma_1/m_1\gamma_2 , \tag{4.2.3}$$

The function F can be represented as a potential of a simple layer, i.e,

$$F(\rho,\phi,z)\equiv F(z)=\int\!\!\int_S \frac{\omega(N)dS}{R(M,N)} , \tag{4.2.4}$$

where ω stands for the crack face displacement $w(x,y,0)$, $R(M,N)$ is the distance between the points $M(\rho,\phi,z)$ and $N(r,\psi,0)$, the integration is taken over the crack domain S.

Expression (4.2.4) satisfies the second condition (4.2.1) identically, due to the well known property of the potential of a simple layer. Inside the crack the same property gives:

$$\frac{\partial F}{\partial z}\bigg|_{z=0} = -2\pi\omega = -2\pi w(x,y,0) \ . \qquad (4.2.5)$$

Now expressions (4.2.2), (4.2.4), (4.2.5) and (2.27) give the second equation for c_1 and c_2:

$$-m_1 c_1/\gamma_1 - m_2 c_2/\gamma_2 = 1/2\pi \ . \qquad (4.2.6)$$

The constants c_1 and c_2 are determined from (4.2.3) and (4.2.6) as

$$c_1 = -\frac{\gamma_1}{2\pi(m_1-1)}, \qquad c_2 = -\frac{\gamma_2}{2\pi(m_2-1)} \ . \qquad (4.2.7)$$

The potential functions will be given by

$$F_1(z) = -\frac{\gamma_1}{2\pi(m_1-1)} F(z_1), \qquad F_2(z) = -\frac{\gamma_2}{2\pi(m_2-1)} F(z_2) \ . \qquad (4.2.8)$$

The substitution of (4.2.8) and (2.32) in the first condition (4.2.1) leads to the governing integral equation:

$$p(N_0) = -\frac{1}{4\pi^2 H}\Delta \int\!\!\int_S \frac{\omega(N)\,dS}{R(N_0,N)} , \qquad (4.2.9)$$

where, as before, $R(N_0,N)$ stands for the distance between two points N_0 and N, and both $N_0, N \in S$. The identities defined in (2.35) were used.

4.2.2 GREEN'S FUNCTIONS FOR AN EXTERNAL CIRCULAR CRACK

Let a be the interior radius of the crack. The exact solution of (4.2.9) can be found in Fabrikant [16] as

$$\omega=\frac{2}{\pi}H\int_{0}^{2\pi}\int_{a}^{\infty}\frac{p(\rho_0,\phi_0)}{R}\tan^{-1}\left(\frac{\eta}{R}\right)\rho_0 d\rho_0 d\phi_0\ , \tag{4.2.10}$$

where

$$R=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2},\quad \eta=(\rho^2-a^2)^{1/2}(\rho_0^2-a^2)^{1/2}/a\ . \tag{4.2.11}$$

The function $F(\rho,\phi,z)$, as defined by (4.2.4), must be called the main potential function since both functions F_1 and F_2 are available, once F is found. The substitution of (4.2.10) in (4.2.4) allows to express the main potential function as follows:

$$F(\rho,\phi,z)=\frac{2}{\pi}H\int_{0}^{2\pi}\int_{a}^{\infty}K(\rho,\phi,z;\rho_0,\phi_0)p(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\ , \tag{4.2.12}$$

where the Green's function K reads:

$$K(M,N_0)=K(\rho,\phi,z;\rho_0,\phi_0)$$

$$=\int_{0}^{2\pi}\int_{a}^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left[\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right]\frac{r dr d\psi}{R(M,N)}\ . \tag{4.2.13}$$

Here $R(\cdot,\cdot)$ denotes the distance between respective points: $M(\rho,\phi,z)$, $N(r,\psi,0)$, and $N_0(\rho_0,\phi_0,0)$. Although the integral in (4.2.13) is not computable in elementary functions, all its derivatives can be expressed in elementary functions, due to the fundamental integrals established in section 1.6 of Fabrikant [16]. It can be written:

$$\frac{\partial K}{\partial z}=-\frac{2\pi}{R(M,N_0)}\tan^{-1}\left[\frac{j}{R(M,N_0)}\right], \tag{4.2.14}$$

where

$$j=\sqrt{\ell_2^2-a^2}\sqrt{\rho_0^2-a^2}/a , \tag{4.2.15}$$

and the contractions ℓ_1 and ℓ_2 everywhere stand for $\ell_1(a)$ and $\ell_2(a)$ respectively, as they are defined by

$$\ell_1(t)=\frac{1}{2}\{[(\rho+t)^2+z^2]^{1/2}-[(\rho-t)^2+z^2]^{1/2}\},$$
$$\ell_2(t)=\frac{1}{2}\{[(\rho+t)^2+z^2]^{1/2}+[(\rho-t)^2+z^2]^{1/2}\}. \tag{4.2.16}$$

Note that j tends to η, as defined by (4.2.11), for $z\to 0$ and $\rho>a$. Expressions (4.2.12) and (4.2.14) allow one to write:

$$\frac{\partial F}{\partial z}=-4H\int_0^{2\pi}\int_a^{\infty}\frac{1}{R(M,N_0)}\tan^{-1}\left[\frac{j}{R(M,N_0)}\right]p(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 . \tag{4.2.17}$$

The integral in (4.2.17), although looks difficult to compute even for $p=\mathrm{const}/\rho^2$, can be expressed in elementary functions for any polynomial loading. Using the equivalent representations through the $\mathcal{L}$-operator (Fabrikant [16])

$$\frac{\partial F}{\partial z}=-8\pi H\int_{\ell_1}^{\rho}\frac{dx}{(x^2-\rho^2)^{1/2}}\int_{g(x)}^{\infty}\frac{\rho_0 d\rho_0}{[\rho_0^2-g^2(x)]^{1/2}}\mathcal{L}\left(\frac{x^2}{\rho\rho_0}\right)p(\rho_0,\phi) \quad , \qquad (4.2.18)$$

Here

$$g(x)=x\left[1+\frac{z^2}{\rho^2-x^2}\right]^{1/2}. \qquad (4.2.19)$$

Using the change of variables $x=\ell_1(t)$, $t=g(x)$, expression (4.2.18) can be rewritten as follows:

$$\frac{\partial F}{\partial z}=-8\pi H\int_{a}^{\infty}\frac{d\ell_1(t)}{[\rho^2-\ell_1^2(t)]^{1/2}}\int_{t}^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-t^2)^{1/2}}\mathcal{L}\left(\frac{\ell_1^2(t)}{\rho\rho_0}\right)p(\rho_0,\phi). \qquad (4.2.20)$$

Since the function F vanishes at infinity, it can be written from (4.2.20) in the form

$$F(\rho,\phi,z)=-8\pi H\int_{\infty}^{z}dz\int_{a}^{\infty}\frac{d\ell_1(t)}{[\rho^2-\ell_1^2(t)]^{1/2}}\int_{t}^{\infty}\frac{\rho_0 d\rho_0}{(\rho_0^2-t^2)^{1/2}}\mathcal{L}\left(\frac{\ell_1^2(t)}{\rho\rho_0}\right)p(\rho_0,\phi).$$

(4.2.21)

One can proceed now with the remaining derivatives of the Green's function K, defined by (4.2.13). Differentiation of (4.2.13) yields:

$$\Lambda K(\rho,\phi,z;\rho_0,\phi_0)=-\int_0^{2\pi}\int_a^{\infty}\frac{\rho e^{i\phi}-re^{i\psi}}{R^3(M,N)}\tan^{-1}\left[\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right]\frac{rdrd\psi}{R(N,N_0)} \; . \tag{4.2.22}$$

This integral can be computed by the following method:

$$\Lambda K = \int_{\infty}^{z} \Lambda\frac{\partial K}{\partial z}\, dz \; . \tag{4.2.23}$$

Thus the two-dimensional integral was replaced by a one-dimensional integral. The integral in (4.2.23) can be computed as indefinite, with the result

$$\int\Lambda\frac{\partial K}{\partial z}dz=\frac{2\pi}{\bar{q}}\left[\frac{z}{R_0}\tan^{-1}\frac{j}{R_0} - \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\tan^{-1}\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right] \; . \tag{4.2.24}$$

Substitution of the upper and lower limits in (4.2.24), will result in

$$\Lambda K(\rho,\phi,z;\rho_0,\phi_0)=\frac{2\pi}{\bar{q}}\left[\frac{z}{R_0}\tan^{-1}\frac{j}{R_0} -\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\tan^{-1}\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right.\right.$$
$$\left.\left.-\tan^{-1}\frac{\bar{s}}{a}\right)-\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}\right] \; , \tag{4.2.25}$$

where Λ is given by (2.26), j is defined by (4.2.15), and

$$\bar{q} = \rho e^{-i\phi}-\rho_0 e^{-i\phi_0} \; , \quad \bar{s} = (\rho\rho_0 e^{-i(\phi-\phi_0)}- a^2)^{1/2} \; ,$$
$$R_0=R(M,N_0)=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2} \; . \tag{4.2.26}$$

The other derivatives, which will be needed for the complete solution, are:

$$\frac{\partial^2}{\partial z^2}\,K(\rho,\phi,z;\rho_0,\phi_0)=2\pi\left\{\frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{j}{z(R_0^2+j^2)}\left[\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2}-\frac{z^2}{R_0^2}\right]\right\}, \tag{4.2.27}$$

$$\frac{\partial}{\partial z}\Lambda K(\rho,\phi,z;\rho_0,\phi_0)=2\pi\left\{\frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)\right.$$

$$\left.+\frac{j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^2}-\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\right]\right\}, \tag{4.2.28}$$

$$\Lambda^2 K(\rho,\phi,z;\rho_0,\phi_0)=2\pi\left\{\frac{(\rho_0^2-a^2)^{1/2}}{\bar{q}\,\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]-\frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{(\rho_0^2-a^2)^{1/2}(a^2-\ell_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\,\bar{s}^2[\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]}$$

$$-\frac{zj}{R_0^2+j^2}\left[\frac{q}{\bar{q}R_0^2}-\frac{\rho^2 e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right]$$

$$\left.+\frac{2}{\bar{q}^2}\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}+\frac{(\rho_0^2-a^2)^{1/2}a}{\bar{q}\,\bar{s}^2\rho e^{-i\phi}}\right\}. \tag{4.2.29}$$

This concludes the general solution to the problem of an external circular crack subjected to an arbitrary pressure. Formulae (4.2.14) and (4.2.25-4.2.29) are the main results of this section.

4.2.3 POINT FORCE LOADING OF AN EXTERNAL CIRCULAR CRACK

Consider an external circular crack opened by two equal concentrated forces P applied in opposite directions at the points $(\rho_0,\phi_0,0^{\pm})$, $\rho_0>a$ (Fig.4.1). Formulae (2.27), (2.32), (4.2.8), (4.2.14), and (4.2.25-4.2.29) give a complete solution for the field of displacements and stresses in elementary functions, namely,

$$u=\frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1(z_1)+\frac{\gamma_2}{m_2-1}f_1(z_2)\right], \tag{4.2.30}$$

$$w=\frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2(z_1)+\frac{m_2}{m_2-1}f_2(z_2)\right], \tag{4.2.31}$$

$$\sigma_1=\frac{2P}{\pi^2(\gamma_1-\gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1+1)\gamma_3^2}-\frac{1}{\gamma_1}\right]f_3(z_1)-\left[\frac{\gamma_2}{(m_2+1)\gamma_3^2}-\frac{1}{\gamma_2}\right]f_3(z_2)\right\}, \tag{4.2.32}$$

$$\sigma_2=\frac{4}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1-1}f_4(z_1)+\frac{\gamma_2}{m_2-1}f_4(z_2)\right], \tag{4.2.33}$$

$$\sigma_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[\gamma_1 f_3(z_1)-\gamma_2 f_3(z_2)\right], \tag{4.2.34}$$

$$\tau_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[f_5(z_1)-f_5(z_2)\right], \tag{4.2.35}$$

where

$$f_1(z)=-\frac{1}{q}\left[\frac{z}{R_0}\tan^{-1}\frac{j}{R_0}-\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\tan^{-1}\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}-\tan^{-1}\frac{\bar{s}}{a}\right)\right.$$

$$-\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}\Bigg] \,, \qquad (4.2.36)$$

$$f_2(z)=\frac{1}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right) \,, \qquad (4.2.37)$$

$$f_3(z)=-\left\{\frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{j}{z(R_0^2+j^2)}\left[\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2}-\frac{z^2}{R_0^2}\right]\right\} \,, \qquad (4.2.38)$$

$$f_4(z)=-\left\{\frac{(\rho_0^2-a^2)^{1/2}}{\bar{q}\;\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]\right.$$

$$-\frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{(\rho_0^2-a^2)^{1/2}(a^2-\ell_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\;\bar{s}^2[\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]}-\frac{zj}{R_0^2+j^2}\left[\frac{q}{\bar{q}R_0^2}\right.$$

$$\left.\left.-\frac{\rho^2e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right]+\frac{2}{\bar{q}^2}\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}+\frac{(\rho_0^2-a^2)^{1/2}a}{\bar{q}\;\bar{s}^2\rho e^{-i\phi}}\right\} \,, \qquad (4.2.39)$$

$$f_5(z)=-\left\{\frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^2}-\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\right]\right\} .$$

(4.2.40)

It must be noted that $R_0=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}$. The expression (4.2.34) for σ_z simplifies when $z=0$ and $\rho<a$, namely,

$$\sigma_z=\frac{P}{\pi^2}\,\frac{(\rho_0^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]} \,. \qquad (4.2.41)$$

Using the definition for SIF given in (2.40) the following result may be obtained from (4.2.41)

$$K_1 = \frac{P}{\pi^2 (2a)^{1/2}} \frac{(\rho_0^2 - a^2)^{1/2}}{a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)} . \tag{4.2.42}$$

It can be written for an arbitrarily distributed pressure:

$$K_1 = \frac{1}{\pi^2 (2a)^{1/2}} \int_0^{2\pi} \int_a^{\infty} \frac{(\rho_0^2 - a^2)^{1/2} p(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0}{a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)} , \tag{4.2.43}$$

which corresponds to the well known result from Cherepanov [34].

4.2.4 CONCENTRATED LOAD OUTSIDE A CIRCULAR CRACK

Consider a transversely isotropic space weakened by an external circular crack of radius a in the plane z=0. Let a concentrated force P be applied at an arbitrary point (ρ, ϕ, z) in the Oz direction. The crack faces are stress-free. The crack opening displacement and the opening mode SIF K_1 has to be found.

Consider the second system in equilibrium: two unit concentrated forces Q applied normally to the crack faces in opposite directions at the points $(\rho_0, \phi_0, 0^{\pm})$. Denote the normal displacement in the space due to the forces Q by w_Q, while w_P is the crack opening displacement due to force P. Note that the term "crack opening displacement" is used here to denote the difference between the normal displacements of the crack faces. Application of the reciprocal theorem to the two systems yields

$$Qw_P = Pw_Q \;, \tag{4.2.44}$$

which gives the crack opening displacement

$$w_P(\rho_0,\phi_0) = \frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2(z_1) + \frac{m_2}{m_2-1}f_2(z_2)\right] \;, \tag{4.2.45}$$

with f_2 defined by (4.2.37). The SIF can be determined by

$$\begin{aligned} K_1(\phi_0) &= \frac{1}{8\pi H}\lim_{\rho_0\to a}\frac{w_P(\rho_0,\phi_0)}{(a-\rho_0)^{1/2}} \\ &= \frac{P}{2(2a)^{1/2}\pi^2}\left[\frac{m_1}{m_1-1}f_6(z_1) + \frac{m_2}{m_2-1}f_6(z_2)\right], \end{aligned} \tag{4.2.46}$$

where

$$f_6(z) = (\ell_2^2-a^2)^{1/2}/r_a^2 \;, \quad r_a^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z^2 \;, \tag{4.2.47}$$

The SIF vanishes as z tends to zero for $\rho\le a$.

In the case of an isotropic body, expression (4.2.45) transforms into

$$\begin{aligned} w_P(\rho_0,\phi_0) &= \frac{P}{\pi^2\mu}\left\{\frac{1-\nu}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)\right. \\ &\left. - \frac{1}{2}\left[\frac{j}{R_0^2+j^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right) - \frac{z^2}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)\right]\right\} \;, \end{aligned} \tag{4.2.48}$$

Here μ is the shear modulus, and ν is Poisson's ratio.

4.2.5 SOLUTION FOR ISOTROPIC BODY

The solution for an isotropic body can be obtained as a limiting case of the transversely isotropic solution, subject to the conditions in (2.34). In the case of an isotropic body, using the results from equation (3.2.25) and the identities in (2.35), the formulae (4.2.30-4.2.40) transform into

$$u=\frac{P(1+\nu)}{\pi^2 E}\left\{(1-2\nu)\frac{1}{q}\left[\frac{z}{R_0}\tan^{-1}\frac{j}{R_0} - \tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}\right.\right.$$

$$\left. - \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\tan^{-1}\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}} - \tan^{-1}\frac{\bar{s}}{a}\right)\right] + \frac{z}{q}\left[\frac{R_0^2-z^2}{R_0^3}\tan^{-1}\frac{j}{R_0}\right.$$

$$\left.\left. + \frac{j}{R_0^2+j^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right) - \frac{\ell_1(\rho_0^2-a^2)^{1/2}(\rho^2-\ell_1^2)^{1/2}}{(\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}\right]\right\}, \tag{4.2.49}$$

$$w=\frac{P(1+\nu)}{\pi^2 E}\left\{\left[\frac{2(1-\nu)}{R_0}+\frac{z^2}{R_0^3}\right]\tan^{-1}\frac{j}{R_0} - \frac{j}{R_0^2+j^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right)\right\}, \tag{4.2.50}$$

$$\sigma_1=-\frac{P}{\pi^2}\left\{(1+2\nu)\left[\frac{z}{R_0^3}\tan^{-1}\frac{j}{R_0} - \frac{j}{z(R_0^2+j^2)}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right)\right]\right.$$

$$+ z\left[\frac{R_0^2-3z^2}{R_0^5}\tan^{-1}\frac{j}{R_0} + \frac{j}{R_0^2(R_0^2+j^2)}\left(\frac{2(\ell_2^2-\rho^2)}{\ell_2^2-\ell_1^2} - \frac{3z^2}{R_0^2} + 1\right)\right.$$

$$- \frac{j\rho^2}{(R_0^2+j^2)(\ell_2^2-\ell_1^2)^2}\left(\frac{2(\rho^2+z^2-a^2)}{\ell_2^2-\ell_1^2} - 1\right)$$

$$+ \frac{2j^3}{z^2(R_0^2+j^2)^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} + \frac{z^2}{j^2}\right)\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right)\Bigg]\Bigg]\Bigg\} , \tag{4.2.51}$$

$$\sigma_2 = \frac{P}{\pi^2}\Bigg\{(1-2\nu)\Bigg[\frac{(\rho_0^2-a^2)^{1/2}}{\bar{q}\ \bar{s}}\left(\frac{2}{\bar{q}} + \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left(\tan^{-1}\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right.$$

$$\left. - \tan^{-1}\frac{\bar{s}}{a}\right) - \frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\frac{j}{R_0} + \frac{2}{\bar{q}^2}\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}$$

$$+ \frac{(\rho_0^2-a^2)^{1/2}(a^2-\ell_1^2)\rho_0 e^{i\phi_0}}{\bar{q}\ \bar{s}^2(\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)})} - \frac{zj}{R_0^2+j^2}\left(\frac{q}{\bar{q}R_0^2} - \frac{\rho^2 e^{2i\phi_0}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right)$$

$$+ \frac{(\rho_0^2-a^2)^{1/2}a}{\bar{q}\ \bar{s}^2\rho e^{-i\phi}}\Bigg] + z\Bigg[\frac{2j^3}{(R_0^2+j^2)^2}\left(\frac{\ell_2-\rho}{\ell_2^2-\ell_1^2} + \frac{z^2}{j^2}\right)\left(\frac{q}{\bar{q}R_0^2} - \frac{\rho^2 e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right)$$

$$+ \frac{(\rho_0^2-a^2)^{1/2}}{\bar{q}}\left(\frac{2}{\bar{q}} - \frac{2\rho_0 e^{i\phi_0}}{\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)}}\right)\frac{\ell_1(\rho^2-\ell_1^2)^{1/2}}{(\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}$$

$$- \frac{3(R_0^2-z^2)^2}{\bar{q}^2R_0^5}\tan^{-1}\frac{j}{R_0} - \frac{j}{R_0^2+j^2}\Bigg[\frac{4R_0^2-2z^2}{\bar{q}^2R_0^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right) + \frac{(R_0^2-z^2)^2}{\bar{q}^2R_0^4}$$

$$- \frac{\rho^2 e^{2i\phi}}{(\ell_2^2-\ell_1^2)^2} + \frac{4\ell_1^2(\ell_2^2-\rho^2)e^{2i\phi}}{(\ell_2^2-\ell_1^2)^3}\Bigg]\Bigg]\Bigg\} , \tag{4.2.52}$$

$$\sigma_z = -\frac{P}{\pi^2}\Bigg\{\frac{z}{R_0^3}\tan^{-1}\frac{j}{R_0} - \frac{j}{z(R_0^2+j^2)}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right) - z\Bigg[\frac{R_0^2-3z^2}{R_0^5}\tan^{-1}\frac{j}{R_0}$$

$$+ \frac{j}{R_0^2(R_0^2+j^2)}\left(\frac{2(\ell_2^2-\rho^2)}{\ell_2^2-\ell_1^2} - \frac{3z^2}{R_0^2} + 1\right)$$

$$- \frac{j\rho^2}{(R_0^2+j^2)(\ell_2^2-\ell_1^2)^2}\left(\frac{2(\rho^2+z^2-a^2)}{\ell_2^2-\ell_1^2} - 1\right)$$

$$+ \frac{2j^3}{z^2(R_0^2+j^2)^2}\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} + \frac{z^2}{j^2}\right)\left(\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right)\Bigg]\Bigg\}, \qquad (4.2.53)$$

$$\tau_z = \frac{P}{\pi^2}\left\{\frac{jq}{R_0^2(R_0^2+j^2)}\left[\frac{2(\ell_2^2-\rho^2)}{\ell_2^2-\ell_1^2} - \frac{3z^2}{R_0^2}\right]\right.$$

$$- \frac{2j^3}{(R_0^2+j^2)^2}\left[\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} + \frac{z^2}{j^2}\right]\left(\frac{q}{R_0^2} - \frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\right)$$

$$\left. - \frac{j\rho e^{i\phi}}{R_0^2+j^2}\left[\frac{\ell_2^2-\rho^2}{(\ell_2^2-\ell_1^2)^2} - \frac{2z^2(\ell_1^2+\ell_2^2)}{(\ell_2^2-\ell_1^2)^3}\right] - \frac{3z^2q}{R_0^5}\tan^{-1}\frac{j}{R_0}\right\}, \qquad (4.2.54)$$

This completes the solution to the problem of an external circular crack under normal load.

4.2.6 NUMERICAL RESULTS

Numerical computations were performed for the field of normal displacements and normal stresses, with a Poisson ratio of ν=0.3. The field of normal displacements due to a pair of concentrated forces applied at the crack faces in opposite directions at the points $(1.5a,0,0^+)$ and $(1.5a,0,0^-)$, is given in Fig.4.2 as a function of ρ/a for different values of z. Similar data for the normal stresses are presented in Fig.4.3.

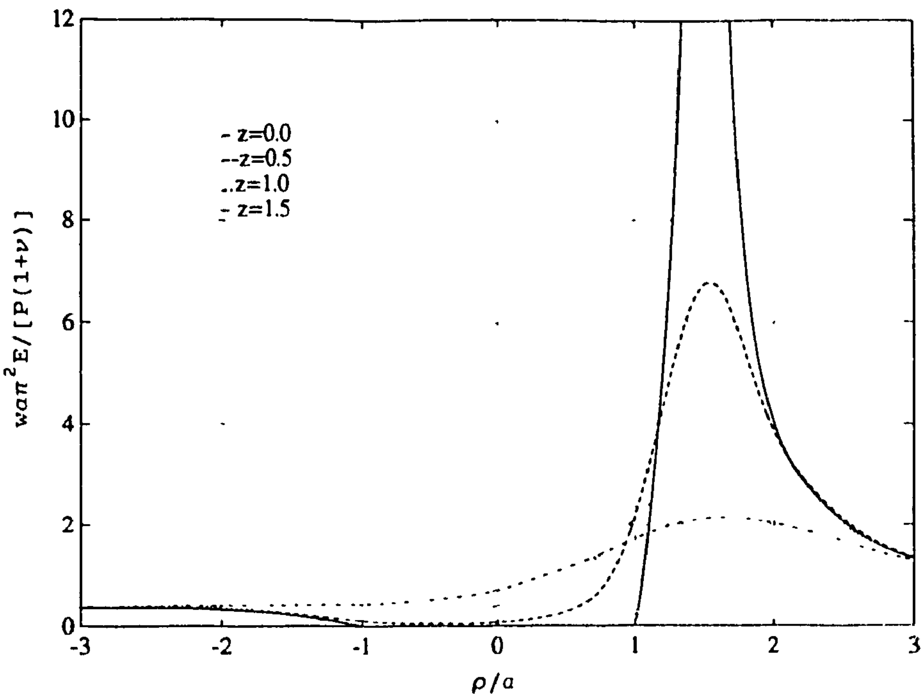

Fig.4.2 Normal displacement distribution in isotropic body for different z: (—z=0.0; --z=0.5; ··z=1.0; ·-z=1.5).

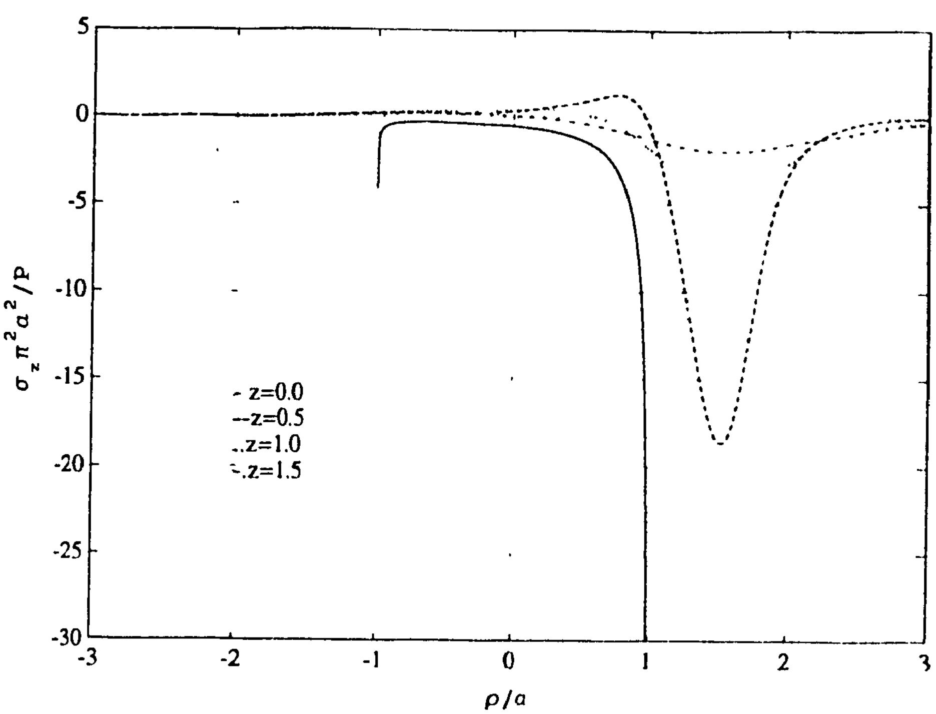

Fig.4.3 Normal stress distribution in isotropic body for different z: (—z=0.0; --z=0.5; ··z=1.0; ·-z=1.5).

4.3 SOLUTION TO THE GOVERNING INTEGRAL EQUATION OF AN EXTERNAL CIRCULAR CRACK UNDER ARBITRARY SHEAR LOADING

Here an exact closed form solution in terms of elementary functions to the governing integral equation of an external circular crack in a transversely isotropic elastic body will be presented. The crack is subjected to arbitrary tangential loading applied antisymmetricaly to its faces. The solution to the governing integral equation gives the direct relationship between the tangential displacements of the crack faces and the applied loading. This makes it possible to have a complete solution to the problem of an external circular crack, with formulae for the field of all stresses and displacements which will be given in the next section 4.4.

4.3.1 GOVERNING INTEGRAL EQUATION

Consider a transversely isotropic elastic space, weakened by external crack $\rho \ge a$, $z=0$. The crack faces are subjected to arbitrary tangential loading τ. It is necessary to find the displacements of the crack faces. Due to geometrical symmetry, the problem can be formulated as a mixed one for an elastic transversely isotropic half-space $z \ge 0$, with the following conditions at the plane $z=0$:

$$\tau_z \equiv \tau_{zx} + i\tau_{yz} = -\tau(\rho,\phi) , \quad \text{for } \rho \ge a , \quad 0 \le \phi < 2\pi ,$$

$$\sigma_z = 0 \,, \qquad \text{for } 0 \le \rho < \infty \,,\; 0 \le \phi < 2\pi \,,$$

$$u = u_x + iu_y = 0 \,, \qquad \text{for } 0 \le \rho \le a \,,\; 0 \le \phi < 2\pi \,. \qquad (4.3.1)$$

A direct relationship between the tangential stresses in the crack neck $\tau^{(1)}$ in terms of the prescribed tractions τ reads:

$$\tau^{(1)}(\rho,\phi) = -\frac{1}{\pi^2\sqrt{a^2-\rho^2}}\left\{\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}}{R^2}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right.$$

$$\left.+\frac{G_2}{G_1}\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}\;\bar{\tau}(\rho_0,\phi_0)(1+\bar{\zeta})e^{2i\phi_0}}{\rho_0^2(1-\bar{\zeta})^2}\rho_0 d\rho_0 d\phi_0\right\}, \qquad (4.3.2)$$

where

$$R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0) \,, \qquad \zeta = \frac{\rho}{\rho_0}e^{i(\phi-\phi_0)} \,. \qquad (4.3.3)$$

Here G_1 and G_2 are elastic constants defined in (2.36) and the overbar everywhere indicates complex conjugate value.

Assuming that the following expansions exist

$$\tau^{(1)}(\rho,\phi) = \sum_{k=-\infty}^{\infty}\tau_k^{(1)}(\rho)e^{ik\phi} \,, \qquad \tau(\rho,\phi) = \sum_{k=-\infty}^{\infty}\tau_k(\rho)e^{ik\phi} \,, \qquad (4.3.4)$$

expression (4.3.2) can be rewritten in a series form as

$$\tau_{n+1}^{(1)}(\rho) = -\frac{2}{\pi}\frac{\rho^{n+1}}{\sqrt{a^2-\rho^2}}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}\,\tau_{n+1}(\rho_0)\rho_0 d\rho_0}{\rho_0^{n+1}(\rho_0^2-\rho^2)} \,, \qquad \text{for } n=0,1,2,\ldots$$

$$\tau^{(1)}_{-n+1}(\rho)=-\frac{2}{\pi}\frac{\rho^{n-1}}{\sqrt{a^2-\rho^2}}\left[\int_a^\infty\frac{\sqrt{\rho_0^2-a^2}\,\tau_{-n+1}(\rho_0)\rho_0 d\rho_0}{\rho_0^{n-1}(\rho_0^2-\rho^2)}\right.$$

$$\left.+(2n-1)\frac{G_2}{G_1}\int_a^\infty\frac{\sqrt{\rho_0^2-a^2}\,\bar{\tau}_{n+1}(\rho_0)}{\rho_0^{n+1}}\rho_0 d\rho_0\right],\quad \text{for } n=1,2,3,\ldots \tag{4.3.5}$$

The governing integral equation of the crack problem has been derived in (Fabrikant [16]) as

$$-\frac{1}{2\pi^2(G_1^2-G_2^2)}\left[G_1\Delta\iint_S\frac{u}{R}dS+G_2\Lambda^2\iint_S\frac{\bar{u}}{R}dS\right]=\tau\ . \tag{4.3.6}$$

Here S is the domain of the crack, and

$$u=u_x+iu_y\ ,\quad \tau=\tau_{zx}+i\tau_{yz}\ ,\quad \Delta=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\ ,\quad \Lambda=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\ . \tag{4.3.7}$$

The main value of the expressions (4.3.2) and (4.3.5) is in the fact that the tangential stresses are now known all over the plane z=0, so the solution of (4.3.6) can formally be written as (Fabrikant [16])

$$u(\rho,\phi)=\frac{1}{2}G_1\int_0^{2\pi}\int_0^a\frac{\tau^{(1)}(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0+\frac{1}{2}G_2\int_0^{2\pi}\int_0^a\frac{q\bar{\tau}^{(1)}(\rho_0,\phi_0)}{\bar{q}\ R}\rho_0 d\rho_0 d\phi_0$$

$$+\frac{1}{2}G_1\int_0^{2\pi}\int_a^\infty\frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0+\frac{1}{2}G_2\int_0^{2\pi}\int_a^\infty\frac{q\bar{\tau}(\rho_0,\phi_0)}{\bar{q}\ R}\rho_0 d\rho_0 d\phi_0. \tag{4.3.8}$$

Here

$$q=\rho e^{i\phi}-\rho_0 e^{i\phi_0}\ , \qquad R^2=q\bar{q}\ . \tag{4.3.9}$$

In the next subsections will be given a solution of (4.3.6) by two different methods.

4.3.2 FIRST METHOD OF SOLUTION

Equation (4.3.6) can be solved by direct substitution of (4.3.2) in the first two terms of (4.3.8), then interchange of the order of integration and computation of the relevant integrals. It is interesting that all the integrals are computable in terms of elementary functions. There are four different integrals to compute. The first is

$$I_1=\int_0^{2\pi}\int_0^{a}\frac{r\,dr\,d\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}\,\frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}\,\frac{1}{\rho_0^2+r^2-2\rho_0 r\cos(\phi_0-\psi)}\ . \tag{4.3.10}$$

This integral has been computed in (Fabrikant [16], f.1.6.31), and the result is

$$I_1=\frac{\pi^2}{R}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\sqrt{\rho^2-a^2}\sqrt{\rho_0^2-a^2}}{aR}\right]\ . \tag{4.3.11}$$

The next integral to compute is

$$I_2=\int_0^{2\pi}\int_0^a \frac{\rho e^{i\phi}-re^{i\psi}}{\rho e^{-i\phi}-re^{-i\psi}} \frac{rdrd\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}} \frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}$$

$$\times \frac{1}{\rho_0^2+r^2-2\rho_0 r\cos(\phi_0-\psi)} \ . \tag{4.3.12}$$

Using the following integral representation

$$\frac{\rho e^{i\phi}-re^{i\psi}}{\rho e^{-i\phi}-re^{-i\psi}} \frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}$$

$$=\frac{2}{\pi}\left\{\sum_{n=0}^{\infty}\frac{e^{2i\phi}}{\rho^2}\left(\frac{e^{i(\phi-\psi)}}{\rho r}\right)^n\int_0^r\frac{(2n+1)\rho^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}\ \alpha^{2n}d\alpha\right.$$

$$-\ \frac{e^{i(\phi+\psi)}}{\rho r}\int_0^r\frac{\alpha^2 d\alpha}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}$$

$$\left.+\sum_{n=0}^{\infty}\frac{e^{2i\phi}}{r^2}\left(\frac{e^{-i(\phi-\psi)}}{\rho r}\right)^n\int_0^r\frac{(2n+1)r^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}\ \alpha^{2n}d\alpha\right\}. \tag{4.3.13}$$

Also making use of the expansion,

$$\frac{1}{\rho_0^2+r^2-2\rho_0 r\cos(\phi_0-\psi)} = \frac{1}{\rho_0^2-r^2}\sum_{k=-\infty}^{\infty}\left(\frac{r}{\rho_0}\right)^{|k|}e^{ik(\phi_0-\psi)} \ . \tag{4.3.14}$$

Substitution of (4.3.13) and (4.3.14) in (4.3.12), change of the order of integration and following integration with respect to ψ and r yields

$$I_2 = 2\pi\left\{ \int_0^a \sum_{n=0}^{\infty} \frac{e^{2i\phi}}{\rho^2}\left(\frac{e^{i(\phi-\phi_0)}}{\rho\rho_0}\right)^n \frac{(2n+1)\rho^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}\,\alpha^{2n}d\alpha\right.$$

$$+ \int_0^a \sum_{n=0}^{\infty} \frac{e^{2i\phi_0}}{\rho_0^2}\left(\frac{e^{-i(\phi-\phi_0)}}{\rho\rho_0}\right)^n \frac{(2n+1)\rho_0^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}\,\alpha^{2n}d\alpha$$

$$\left.- \frac{e^{i(\phi+\phi_0)}}{\rho\rho_0}\int_0^a \frac{\alpha^2 d\alpha}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}\right\}$$

$$- 2\pi\sum_{n=0}^{\infty} (2n+1)\sqrt{\rho_0^2-a^2}\int_0^a \frac{e^{2i\phi_0}}{\rho_0^2}\left(\frac{e^{-i(\phi-\phi_0)}}{\rho\rho_0}\right)^n \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}} \ . \qquad (4.3.15)$$

The change of the order of integration was made according to the scheme

$$\int_0^a dr \int_0^r d\alpha = \int_0^a d\alpha \int_\alpha^a dr \ , \qquad (4.3.16)$$

and the following integral was used

$$\int_\alpha^a \frac{r dr}{(\rho_0^2-r^2)\sqrt{a^2-r^2}\sqrt{r^2-\alpha^2}} = \frac{\pi}{2\sqrt{\rho_0^2-a^2}\sqrt{\rho_0^2-\alpha^2}} \ . \qquad (4.3.17)$$

Comparison of the term in curly brackets in (4.3.15) with a similar term in (4.3.13) indicates total identity, except for the limit of integration. Now performing the summation in (4.3.15). The result is

$$I_2=2\pi\int_0^a\left\{e^{2i\phi}\left[\frac{1+\xi-2(\alpha^2/\rho^2)}{(1-\xi)^2}\right]+e^{2i\phi_0}\left[\frac{1+\bar{\xi}-2(\alpha^2/\rho_0^2)}{(1-\bar{\xi})^2}\right]\right.$$

$$\left.-e^{2i\phi_0}\xi\right\}\frac{d\alpha}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}\ -2\pi\sqrt{\rho_0^2-a^2}\ \frac{e^{2i\phi_0}}{\rho_0^2}\int_0^a\frac{(1+\bar{\xi})d\alpha}{\sqrt{\rho^2-\alpha^2}(1-\bar{\xi})^2}. \qquad (4.3.18)$$

Here

$$\xi = \frac{\alpha^2}{\rho\rho_0}e^{i(\phi-\phi_0)} \ . \qquad (4.3.19)$$

The interesting feature of the first integral in (4.3.18) is that it is computable as indefinite:

$$\int\left\{e^{2i\phi}\left[\frac{1+\xi-2(\alpha^2/\rho^2)}{(1-\xi)^2}\right]+e^{2i\phi_0}\left[\frac{1+\bar{\xi}-2(\alpha/\rho_0)}{(1-\bar{\xi})^2}\right]-e^{2i\phi_0}\xi\right\}\frac{d\alpha}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}$$

$$= -\frac{q}{\bar{q}R}\tan^{-1}\frac{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}{\alpha R}$$

$$+\left[e^{2i\phi}\ -\ \frac{2i\rho e^{i\phi_0}\sin(\phi-\phi_0)}{\bar{q}(1-\bar{\xi})}\right]\frac{\alpha\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}{\rho^2\rho_0^2(1-\xi)}\ . \qquad (4.3.20)$$

By using (4.3.20), I_2 can be expressed

$$I_2=\ 2\pi\left\{\frac{q}{\bar{q}R}\left(\frac{\pi}{2}\ -\tan^{-1}\frac{\eta}{R}\right)+\left[e^{2i\phi}\ -\ \frac{2i\rho e^{i\phi_0}\sin(\phi-\phi_0)}{\bar{q}(1-\bar{t})}\right]\frac{a^2\eta}{\rho^2\rho_0^2(1-t)}\right\}$$

$$- 2\pi\sqrt{\rho_0^2-a^2}\ \frac{e^{2i\phi_0}}{\rho_0^2}\int_0^a \frac{(1+\bar{\xi})d\alpha}{\sqrt{\rho^2-\alpha^2}(1-\bar{\xi})^2}\ , \tag{4.3.21}$$

where

$$\eta = \frac{\sqrt{\rho^2-a^2}\sqrt{\rho_0^2-a^2}}{a}\ , \qquad t = \frac{a^2}{\rho\rho_0}e^{i(\phi-\phi_0)}\ . \tag{4.3.22}$$

The second integral in (4.3.21) is also computable in elementary functions, but it will not be done because it will cancel out, anyway, with yet another integral to be computed.

The next integral to be computed is

$$I_3 = \int_0^{2\pi}\int_0^a \frac{rdrd\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}\left[\frac{\sqrt{\rho_0^2-a^2}\ e^{2i\phi_0}\left(1+\frac{r}{\rho_0}e^{-i(\psi-\phi_0)}\right)}{\sqrt{a^2-r^2}\ \rho_0^2\left(1-\frac{r}{\rho_0}e^{-i(\psi-\phi_0)}\right)^2}\right]. \tag{4.3.23}$$

Using the integral representation given in (2.1) for its particular case when u=0, namely,

$$\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}} = \frac{2}{\pi}\int_0^r \frac{\lambda(\alpha^2/\rho r,\phi-\psi)}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}\,d\alpha\ , \tag{4.3.24}$$

were λ is defined as in (2.2).

Substitution of (4.3.24) in (4.3.23), with the series expansion of the term in brackets, leads to

$$I_3=\frac{2}{\pi}\int_0^{2\pi}\int_0^{a}\frac{rdrd\psi}{\sqrt{a^2-r^2}}\int_0^{r}\frac{\lambda(\alpha^2/\rho r,\phi-\psi)}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}d\alpha\left[\frac{\sqrt{\rho_0^2-a^2}}{\rho_0^2}e^{2i\phi_0}\right.$$

$$\left.\times\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{\rho_0}e^{-i(\psi-\phi_0)}\right)^n\right]$$

$$=4\frac{\sqrt{\rho_0^2-a^2}}{\rho_0^2}e^{2i\phi_0}\int_0^{a}\frac{d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^{a}\frac{rdr}{\sqrt{a^2-r^2}\sqrt{r^2-\alpha^2}}\sum_{n=0}^{\infty}(2n+1)\left(\frac{\alpha^2}{\rho\rho_0}e^{-i(\phi-\phi_0)}\right)^n$$

$$=2\pi\sqrt{\rho_0^2-a^2}\;\frac{e^{2i\phi_0}}{\rho_0^2}\int_0^{a}\sum_{n=0}^{\infty}(2n+1)\left(\frac{\alpha^2}{\rho\rho_0}e^{-i(\phi-\phi_0)}\right)^n\frac{d\alpha}{\sqrt{\rho^2-\alpha^2}}$$

$$=2\pi\sqrt{\rho_0^2-a^2}\;\frac{e^{2i\phi_0}}{\rho_0^2}\int_0^{a}\frac{(1+\bar{\xi})d\alpha}{\sqrt{\rho^2-\alpha^2}(1-\bar{\xi})^2}\;. \qquad (4.3.25)$$

Thus I_3 is equal and of opposite sign to the last term in (4.3.21), and they will cancel out in the final substitution.

The last integral to be computed is

$$I_4=\int_0^{2\pi}\int_0^{a}\frac{\rho e^{i\phi}-re^{i\psi}}{\rho e^{-i\phi}-re^{-i\psi}}\;\frac{rdrd\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}$$

$$\times\left[\frac{\sqrt{\rho_0^2-a^2}e^{-2i\phi_0}\left(1+\frac{r}{\rho_0}e^{\;i(\psi-\phi_0)}\right)}{\sqrt{a^2-r^2}\rho_0^2\left(1-\frac{r}{\rho_0}e^{i(\psi-\phi_0)}\right)^2}\right]. \qquad (4.3.26)$$

Since the series expansion of the term in square brackets of (4.3.26) has only positive harmonics in ψ, only the relevant negative harmonics in (4.3.13) are needed:

$$I_4=\frac{2}{\pi}\int_0^{2\pi}\int_0^{a}\frac{r\,dr\,d\psi}{\sqrt{a^2-r^2}}\left\{\sum_{n=0}^{\infty}\frac{e^{2i\phi}}{\rho^2}\left(\frac{e^{i(\phi-\psi)}}{\rho r}\right)^n\int_0^{r}\frac{(2n+1)\rho^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}\sqrt{r^2-\alpha^2}}\,\alpha^{2n}d\alpha\right\}$$

$$\times\left[\frac{\sqrt{\rho_0^2-a^2}}{\rho_0^2}e^{-2i\phi_0}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{\rho_0}e^{i(\psi-\phi_0)}\right)^n\right]$$

$$=2\pi\frac{\sqrt{\rho_0^2-a^2}}{\rho_0^2e^{2i\phi_0}}\int_0^{a}\sum_{n=0}^{\infty}\frac{e^{2i\phi}}{\rho^2}\left(\frac{e^{i(\phi-\phi_0)}}{\rho\rho_0}\right)^n(2n+1)\frac{(2n+1)\rho^2-(2n+2)\alpha^2}{\sqrt{\rho^2-\alpha^2}}\,\alpha^{2n}d\alpha$$

$$=2\pi\frac{a\sqrt{\rho_0^2-a^2}}{\rho^2\rho_0^2}e^{2i(\phi-\phi_0)}\sum_{n=0}^{\infty}\left(\frac{a^2e^{i(\phi-\phi_0)}}{\rho\rho_0}\right)^n(2n+1)\sqrt{\rho^2-a^2}=2\pi\frac{t^2(1+t)}{a^2(1-t)^2}\eta\ . \tag{4.3.27}$$

Now all four needed integrals are computed, and (4.3.2) can be substituted into (4.3.8) and the result rewritten as follows:

$$u(\rho,\phi)=\frac{1}{2}G_1\left[-\frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty}I_1\tau(\rho_0,\phi_0)\rho_0d\rho_0d\phi_0\right.$$

$$\left.-\frac{G_2}{\pi^2G_1}\int_0^{2\pi}\int_a^{\infty}I_3\bar{\tau}(\rho_0,\phi_0)\rho_0d\rho_0d\phi_0\right]$$

$$
+\frac{1}{2}G_2\left[-\frac{1}{\pi^2}\int_0^{2\pi}\int_a^{\infty} I_2\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right.
$$

$$
\left.-\frac{G_2}{\pi^2 G_1}\int_0^{2\pi}\int_a^{\infty} I_4\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right]
$$

$$
+\frac{1}{2}G_1\int_0^{2\pi}\int_a^{\infty}\frac{\tau(\rho_0,\phi_0)}{R}\rho_0 d\rho_0 d\phi_0
$$

$$
+\frac{1}{2}G_2\int_0^{2\pi}\int_a^{\infty}\frac{q}{\bar{q}R}\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 \,. \tag{4.3.28}
$$

Substitution of (4.3.11), (4.3.22), (4.3.25) and (4.3.27) in (4.3.28) yields, after obvious simplifications

$$
u(\rho,\phi)=\frac{G_1}{\pi}\int_0^{2\pi}\int_a^{\infty}\left[\frac{1}{R}\tan^{-1}\frac{\eta}{R}-\frac{G_2^2}{G_1^2}\frac{t^2(1+t)}{a^2(1-t)^2}\eta\right]\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0
$$

$$
+\frac{G_2}{\pi}\int_0^{2\pi}\int_a^{\infty}\left[\frac{q}{\bar{q}R}\tan^{-1}\frac{\eta}{R}+\frac{a^2\eta}{\rho^2\rho_0^2(1-t)}\left(\frac{2i\rho e^{i\phi_0}\sin(\phi-\phi_0)}{\bar{q}(1-\bar{t})}\right.\right.
$$

$$
\left.\left.-e^{2i\phi}\right)\right]\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 \,. \tag{4.3.29}
$$

Here η and t are defined by (4.3.22).

4.3.3 SECOND METHOD OF SOLUTION

Since the main result of the previous subsection

(4.3.29) was never reported in literature, it is very important to be sure of its correctness, so it makes sense to rederive it by a different method. For this purpose the series expansion of (4.3.8) and (4.3.5) can be used. Such an expansion of (4.3.8) was given in (Fabrikant [16]), and is

$$u_{n+1}(\rho)=J_1+J_2+J_3+J_4 \quad , \qquad \text{for } \rho\geq a \quad , \tag{4.3.30}$$

where

$$J_1= 2G_1\rho^{n+1}\int_\rho^\infty \frac{d\alpha}{\alpha^{2n+2}\sqrt{\alpha^2-\rho^2}}\int_a^\alpha \frac{\tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0}{\sqrt{\alpha^2-\rho_0^2}} \quad ,$$

$$J_2= 2G_2\rho^{n+1}\int_\rho^\infty \frac{d\alpha}{\alpha^{2n+2}\sqrt{\alpha^2-\rho^2}}\int_a^\alpha \frac{2n\alpha^2-(2n+1)\rho_0^2}{\sqrt{\alpha^2-\rho_0^2}}\bar{\tau}_{-n+1}(\rho_0)\rho_0^n d\rho_0 \quad ,$$

$$J_3= \frac{2G_1}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a \frac{\tau_{n+1}^{(1)}(\rho_0)d\rho_0}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}} \quad ,$$

$$J_4= \frac{2G_2}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a \frac{2n\rho^2-(2n+1)\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{-n+1}^{(1)}(\rho_0)d\rho_0 \quad . \tag{4.3.31}$$

Now the expressions (4.3.5) has to be substituted in J_3 and J_4. The first substitution yields

$$J_3= \frac{2G_1}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a\left[-\frac{2}{\pi}\frac{\rho_0^{n+1}}{\sqrt{a^2-\rho_0^2}}\int_a^\infty \frac{\sqrt{t^2-a^2}\tau_{n+1}(t)tdt}{t^{n+1}(t^2-\rho_0^2)}\right]\frac{d\rho_0}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}$$

$$=-\frac{2G_1}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_a^\infty \frac{\tau_{n+1}(t)tdt}{t^{n+1}\sqrt{t^2-\alpha^2}} . \tag{4.3.32}$$

First, by changing the order of integration, J_1 can be transformed as follows

$$J_1=2G_1\rho^{n+1}\int_a^\infty \tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0 \int_{\max(\rho,\rho_0)}^\infty \frac{d\alpha}{\alpha^{2n+2}\sqrt{\alpha^2-\rho^2}\sqrt{\alpha^2-\rho_0^2}}$$

$$=2G_1\rho^{n+1}\int_a^\infty \tau_{n+1}(\rho_0)\rho_0^{n+2}d\rho_0 \int_0^{\min(\rho,\rho_0)} \frac{\alpha^{2n+2}d\alpha}{(\rho\rho_0)^{2n+2}\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}. \tag{4.3.33}$$

Here α was formally substituted by $\rho\rho_0/\alpha$.

Now the following scheme of change of the order of integration will be employed

$$\int_a^\infty d\rho_0 \int_0^{\min(\rho,\rho_0)} d\alpha = \int_a^\rho d\rho_0 \int_0^{\rho_0} d\alpha + \int_\rho^\infty d\rho_0 \int_0^\rho d\alpha = \int_a^\rho d\rho_0\left(\int_0^a d\alpha + \int_a^{\rho_0} d\alpha\right)$$

$$+\int_\rho^\infty d\rho_0\left(\int_0^a d\alpha + \int_a^\rho d\alpha\right) = \int_0^a d\alpha \int_a^\infty d\rho_0 + \int_a^\rho d\alpha \int_\alpha^\infty d\rho_0 . \tag{4.3.34}$$

This means that (4.3.33) can be rewritten as

$$J_1=\frac{2G_1}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_a^\infty \frac{\tau_{n+1}(\rho_0)\rho_0 d\rho_0}{\rho_0^{n+1}\sqrt{\rho_0^2-\alpha^2}}$$

$$+\frac{2G_1}{\rho^{n+1}}\int_a^\rho \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^\infty \frac{\tau_{n+1}(\rho_0)\rho_0 d\rho_0}{\rho_0^{n+1}\sqrt{\rho_0^2-\alpha^2}} \;. \tag{4.3.35}$$

It can be noted that the first term in (4.3.35) will cancel out with J_3 in (4.3.32).

The transformation of J_4 with the help of (4.3.5) is given as

$$J_4=\frac{2G_2}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a \frac{2n\rho^2-(2n+1)\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\left\{-\frac{2}{\pi}\frac{\rho_0^{n-1}}{\sqrt{a^2-\rho_0^2}}\right.$$

$$\left.\times\left[\int_a^\infty \frac{\sqrt{t^2-a^2}\,\bar{\tau}_{-n+1}(t)\,t\,dt}{t^{n-1}(t^2-\rho_0^2)}+(2n-1)\frac{G_2}{G_1}\int_a^\infty \frac{\sqrt{t^2-a^2}\,\tau_{n+1}(t)}{t^{n+1}}t\,dt\right]\right\}d\rho_0$$

$$=-\frac{2G_2}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}}[2n\rho^2-(2n+1)\alpha^2]\left\{\int_a^\infty \frac{\sqrt{t^2-a^2}}{t^n}\bar{\tau}_{-n+1}(t)\right.$$

$$\left.\times\left[\frac{1}{\sqrt{t^2-a^2}\sqrt{t^2-\alpha^2}}+\frac{1}{a\alpha}\right]dt+\frac{2n-1}{a\alpha}\frac{G_2}{G_1}\int_a^\infty \frac{\sqrt{t^2-a^2}\,\tau_{n+1}(t)\,dt}{t^n}\right\}$$

$$=-\frac{2G_2}{\rho^{n+1}}\int_0^a \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}}[2n\rho^2-(2n+1)\alpha^2]\left\{\int_a^\infty \frac{\bar{\tau}_{-n+1}(t)\,dt}{t^n\sqrt{t^2-\alpha^2}}\right.$$

$$\left.+\frac{1}{a\alpha}\int_a^\infty \frac{\sqrt{t^2-a^2}}{t^n}\left(\bar{\tau}_{-n+1}(t)+(2n-1)\frac{G_2}{G_1}\tau_{n+1}(t)\right)dt\right\} \;. \tag{4.3.36}$$

J_2 can be transformed in the manner similar to that used in

(4.3.33-4.3.34). The result is

$$J_2 = \frac{2G_2}{\rho^{n+1}} \int_0^a \frac{\alpha^{2n} d\alpha}{\sqrt{\rho^2-\alpha^2}} [2n\rho^2-(2n+1)\alpha^2] \int_a^\infty \frac{\bar{\tau}_{-n+1}(\rho_0) d\rho_0}{\rho_0^n \sqrt{\rho_0^2-\alpha^2}}$$

$$+ \frac{2G_2}{\rho^{n+1}} \int_a^\rho \frac{\alpha^{2n} d\alpha}{\sqrt{\rho^2-\alpha^2}} [2n\rho^2-(2n+1)\alpha^2] \int_\alpha^\infty \frac{\bar{\tau}_{-n+1}(\rho_0) d\rho_0}{\rho_0^n \sqrt{\rho_0^2-\alpha^2}} . \qquad (4.3.37)$$

Again it may be noted that the first term in (4.3.36) cancels out with the first term in (4.3.37).

Finally, expressions (4.3.32), (4.3.35), (4.3.36) and (4.3.37), after substitution in (4.3.30) give

$$u_{n+1}(\rho) = \frac{2G_1}{\rho^{n+1}} \int_a^\rho \frac{\alpha^{2n+2} d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_\alpha^\infty \frac{\tau_{n+1}(\rho_0) d\rho_0}{\rho_0^n \sqrt{\rho_0^2-\alpha^2}}$$

$$+ \frac{2G_2}{\rho^{n+1}} \int_a^\rho \frac{\alpha^{2n} d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_\alpha^\infty \frac{2n\rho^2-(2n+1)\alpha^2}{\rho_0^n \sqrt{\rho_0^2-\alpha^2}} \bar{\tau}_{-n+1}(\rho_0) d\rho_0$$

$$-2G_2\sqrt{\rho^2-a^2}\ \frac{a^{2n-1}}{\rho^{n+1}} \int_a^\infty \frac{\sqrt{t^2-a^2}}{t^n} \left[\bar{\tau}_{-n+1}(t) + (2n-1)\frac{G_2}{G_1}\tau_{n+1}(t)\right] dt.$$

(4.3.38)

The first two terms in (4.3.38) are valid for n=0,1,2,..., while the third term is used only for n=1,2,3,...

Now an expression will be derived, similar to (4.3.38), for the negative harmonics.

$$u_{-n+1}(\rho)=B_1+B_2+B_3+B_4\ ,\tag{4.3.39}$$

where

$$B_1=2G_1\rho^{n-1}\int_\rho^\infty\frac{d\alpha}{\alpha^{2n-2}\sqrt{\alpha^2-\rho^2}}\int_a^\alpha\frac{\tau_{-n+1}(\rho_0)\rho_0^n d\rho_0}{\sqrt{\alpha^2-\rho_0^2}}\ ,$$

$$B_2=2G_2\rho^{n-1}\int_\rho^\infty\frac{(2n-1)\alpha^2-2n\rho^2}{\alpha^{2n}\sqrt{\alpha^2-\rho^2}}d\alpha\int_a^\alpha\frac{\bar{\tau}_{n+1}(\rho_0)\rho_0^n d\rho_0}{\sqrt{\alpha^2-\rho_0^2}}\ ,$$

$$B_3=\frac{2G_1}{\rho^{n-1}}\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a\frac{\tau_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-\alpha^2}}\ ,$$

$$B_4=\frac{2G_2}{\rho^{n-1}}\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^a\frac{(2n-1)\rho_0^2-2n\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{n+1}(\rho_0)d\rho_0\ .\tag{4.3.40}$$

Substitution of (4.3.5) in B_3 of (4.3.40) yields

$$B_3=-\frac{2G_1}{\rho^{n-1}}\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\left[\int_a^\infty\frac{\tau_{-n+1}(t)dt}{t^{n-2}\sqrt{t^2-\alpha^2}}+(2n-1)\frac{G_2}{G_1}\int_a^\infty\frac{\sqrt{t^2-a^2}}{t^n}\bar{\tau}_{n+1}(t)dt\right].\tag{4.3.41}$$

The expression for B_1 can be transformed, as above,

$$B_1=\frac{2G_1}{\rho^{n-1}}\left\{\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_a^\infty\frac{\tau_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-\alpha^2}}+\int_a^\rho\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^\infty\frac{\tau_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-\alpha^2}}\right\}\ .\tag{4.3.42}$$

Again, it can be seen that the first term in (4.3.41) and

(4.3.42) will cancel out.

Substitution of (4.3.5) in the expression (4.3.40) for B_4, will result in

$$B_4=-\frac{2G_2}{\rho^{n-1}}\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_a^\infty\left(\frac{(2n-1)t^2-2n\alpha^2}{\sqrt{t^2-\alpha^2}}-(2n-1)\sqrt{t^2-a^2}\right)\frac{\bar{\tau}_{n+1}(t)dt}{t^n}\ .$$

(4.3.43)

Relevant transformation of B_2 in (4.3.40) gives

$$B_2=\frac{2G_2}{\rho^{n-1}}\left\{\int_0^a\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_a^\infty\frac{(2n-1)\rho_0^2-2n\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{n+1}(\rho_0)d\rho_0\right.$$

$$\left.+\int_a^\rho\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^\infty\frac{(2n-1)\rho_0^2-2n\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{n+1}(\rho_0)d\rho_0\ .\right. \qquad (4.3.44)$$

Now, substitution of (4.3.41-4.3.44) into (4.3.39) yields, after obvious simplifications

$$u_{-n+1}(\rho)=\frac{2G_1}{\rho^{n-1}}\int_a^\rho\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^\infty\frac{\tau_{-n+1}(\rho_0)d\rho_0}{\rho_0^{n-2}\sqrt{\rho_0^2-\alpha^2}}$$

$$+\frac{2G_2}{\rho^{n-1}}\int_a^\rho\frac{\alpha^{2n-2}d\alpha}{\sqrt{\rho^2-\alpha^2}}\int_\alpha^\infty\frac{(2n-1)\rho_0^2-2n\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{n+1}(\rho_0)d\rho_0,\quad \text{for } n=1,2,3,\ldots$$

(4.3.45)

For convenience, formula (4.3.38) is repeated again

$$u_{n+1}(\rho)= \frac{2G_1}{\rho^{n+1}} \int_a^{\rho} \frac{\alpha^{2n+2}d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{\tau_{n+1}(\rho_0)d\rho_0}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}$$

$$+ \frac{2G_2}{\rho^{n+1}} \int_a^{\rho} \frac{\alpha^{2n}d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{2n\rho^2-(2n+1)\alpha^2}{\rho_0^n\sqrt{\rho_0^2-\alpha^2}}\bar{\tau}_{-n+1}(\rho_0)d\rho_0$$

$$- \frac{2G_2}{\rho^{n+1}}a^{2n-1}\sqrt{\rho^2-a^2}\int_a^{\infty} \frac{\sqrt{\rho_0^2-a^2}}{\rho_0^n}\left[\bar{\tau}_{-n+1}(\rho_0)+(2n-1)\frac{G_2}{G_1}\tau_{n+1}(\rho_0)\right]d\rho_0.$$

(4.3.46)

Expressions (4.3.45-4.3.46) give the series expansion of (4.3.29). They are also useful for direct evaluation of the integrals since the integrands are simpler than those in (4.3.29).

The summation of (4.3.45) and (4.3.46) will be done in stages. First of all, the sum of all terms with G_1, will be computed

$$H_1=\frac{G_1}{\pi}\int_a^{\rho} \frac{d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_0^{2\pi} \int_{\alpha}^{\infty}\sum_{n=1}^{\infty} e^{-i(n-1)(\phi-\phi_0)}\left(\frac{\alpha^2}{\rho\rho_0}\right)^{n-1} \frac{\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\sqrt{\rho_0^2-\alpha^2}}$$

$$+ \frac{G_1}{\pi}\int_a^{\rho} \frac{d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_0^{2\pi} \int_{\alpha}^{\infty}\sum_{n=0}^{\infty} \left(\frac{\alpha^2}{\rho\rho_0}e^{i(\phi-\phi_0)}\right)^{n+1} \frac{\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\sqrt{\rho_0^2-\alpha^2}}$$

$$=\frac{G_1}{\pi}\int_0^{2\pi}\int_a^{\rho} \frac{\lambda(\alpha^2/\rho\rho_0,\phi-\phi_0)d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_{\alpha}^{\infty} \frac{\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\sqrt{\rho_0^2-\alpha^2}} . \qquad (4.3.47)$$

Interchange of the order of integration in (4.3.47) and subsequent integration with respect to α yields

$$H_1 = \frac{G_1}{\pi} \int_0^{2\pi} \int_a^{\infty} \frac{1}{R} \tan^{-1}\left(\frac{\eta}{R}\right) \tau(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0 \ . \tag{4.3.48}$$

Summation of the second term in (4.3.45) and (4.3.46) leads to

$$H_2 = \frac{G_2}{\pi} \int_0^{2\pi} e^{2i\phi_0} \, d\phi_0 \int_a^{\rho} \frac{\lambda(\alpha^2/\rho\rho_0, \phi-\phi_0) \, d\alpha}{\sqrt{\rho^2-\alpha^2}} \int_{\alpha}^{\infty} \Bigg[1$$

$$+ \frac{4iq\alpha^2 e^{-i\phi_0} \sin(\phi-\phi_0)}{\rho\rho_0^2 (1-\xi)(1-\bar{\xi})} \Bigg] \frac{\bar{\tau}(\rho_0, \phi_0) \rho_0 d\rho_0}{\sqrt{\rho_0^2-\alpha^2}} \ . \tag{4.3.49}$$

It is reminded that ξ was defined in (4.3.19).

The order of integration in (4.3.49) can be interchanged and it can be integrated with respect to α. The integral in question, though looking quite formidable, can be integrated as indefinite. Indeed,

$$\int \frac{\lambda(\alpha^2/\rho\rho_0, \phi-\phi_0)}{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}} \left[1 + \frac{4iq\alpha^2 e^{-i\phi_0} \sin(\phi-\phi_0)}{\rho\rho_0^2 (1-\xi)(1-\bar{\xi})} \right] d\alpha$$

$$= -e^{-2i\phi_0} \frac{q}{\bar{q}R} \tan^{-1} \frac{\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}{\alpha R}$$

$$- \frac{2ie^{-i\phi_0}\sin(\phi-\phi_0)\alpha\sqrt{\rho^2-\alpha^2}\sqrt{\rho_0^2-\alpha^2}}{\bar{q}\rho\rho_0^2(1-\xi)(1-\bar{\xi})} \quad . \tag{4.3.50}$$

Utilization of (4.3.50) in (4.3.49) gives

$$H_2=\frac{G_2}{\pi}\int_0^{2\pi}\int_a^{\infty}\left[\frac{q}{\bar{q}R}\tan^{-1}\frac{\eta}{R}+\frac{2i\rho e^{-i\phi_0}a^2\eta\sin(\phi-\phi_0)}{\bar{q}\rho^2\rho_0^2(1-t)(1-\bar{t})}\right]\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 . \tag{4.3.51}$$

And finally, summation of the last line in (4.3.46) yields

$$H_3=-\frac{G_2}{\pi}\sqrt{\rho^2-a^2}\int_0^{2\pi}\int_a^{\infty}\sqrt{\rho_0^2-a^2}\left[\frac{e^{2i\phi}a\bar{\tau}(\rho_0,\phi_0)}{\rho^2\rho_0^2(1-t)}\right.$$

$$\left. +\frac{G_2}{G_1}\frac{ae^{2i(\phi-\phi_0)}(1+t)}{\rho^2\rho_0^2(1-t)^2}\tau(\rho_0,\phi_0)\right]\rho_0 d\rho_0 d\phi_0 \quad . \tag{4.3.52}$$

The closed-form solution is

$$u(\rho,\phi) = H_1+ H_2+ H_3 \ , \tag{4.3.53}$$

and substitution of (4.3.48), (4.3.51) and (4.3.52) in (4.3.53) proves it to be identical with (4.3.29).

The correctness of (4.3.29) can also be verified by the computation SIF.

If the complex SIF is defined similar to the one defined in (2.49), namely,

$$K(\phi) = \lim_{\rho\to a} [\sqrt{a-\rho}\,\tau(\rho,\phi)] = e^{i\phi}(K_2+iK_3) , \tag{4.3.54}$$

then it can be deduced from (4.3.2) that

$$K_2+iK_3 = -\frac{e^{-i\phi}}{\pi^2\sqrt{2a}} \left\{ \int_0^{2\pi}\int_a^{\infty} \frac{\sqrt{\rho_0^2-a^2}\,\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)} \right.$$

$$\left. + \frac{G_2}{G_1}\int_0^{2\pi}\int_a^{\infty} \frac{\sqrt{\rho_0^2-a^2}\,e^{2i\phi_0}\left(1 + \frac{a}{\rho_0}e^{-i(\phi-\phi_0)}\right)}{\rho_0^2\left(1 + \frac{a}{\rho_0}e^{-i(\phi-\phi_0)}\right)^2}\,\overline{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 \right\} , \tag{4.3.55}$$

According to formula (2.47), the same result can be obtained through the displacement as

$$K_2+iK_3 = -\frac{a}{\pi(G_1^2-G_2^2)\sqrt{2a}} \lim_{\rho\to a} \left[\frac{G_1 u(\rho,\phi)e^{-i\phi} + G_2\overline{u}(\rho,\phi)e^{i\phi}}{\sqrt{\rho^2-a^2}}\right] . \tag{4.3.56}$$

Substitution of (4.3.29) in (4.3.56) should ultimately give (4.3.55).

Thus the main result of this section, namely, formula (4.3.29), was derived by two different methods. As it will be seen in the next section this important expression will play an essential role to obtain a complete solution of an external circular crack problem under a shear load.

4.4 EXTERNAL CIRCULAR CRACK UNDER SHEAR LOAD: A COMPLETE SOLUTION

Consider a transversely isotropic elastic space weakened by an external circular crack of radius a in the plane $z=0$. Let arbitrary shear τ_z be applied to the crack faces, as in Fig.4.4.

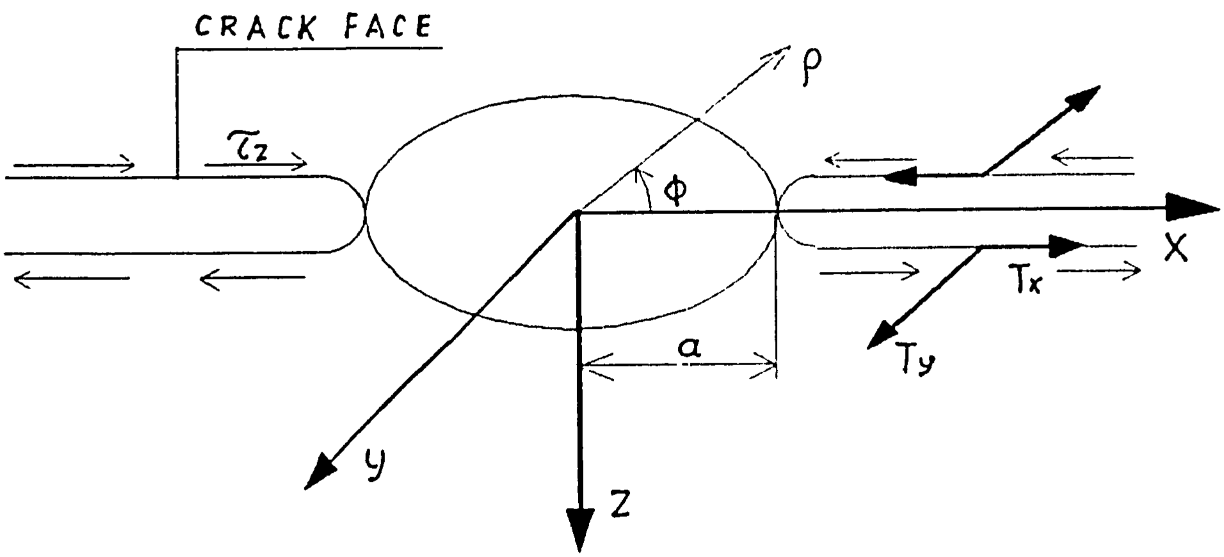

Fig.4.4 External circular crack under arbitrary shear load.

Due to symmetry, the problem can be reduced to the mixed BVP for an elastic half-space $z\geq0$, subject to the following conditions on the plane $z=0$:

$$\tau_z=-\tau(\rho,\phi)\ , \qquad \text{for } a\leq\rho<\infty\ , \quad 0\leq\phi<2\pi\ ,$$

$$u=0\ , \qquad \text{for } 0\leq\rho\leq a\ , \quad 0\leq\phi<2\pi\ ,$$

$$\sigma_z=0\ , \qquad \text{for } 0\leq\rho<\infty\ , \quad 0\leq\phi<2\pi\ . \qquad (4.4.1)$$

4.4.1 FORMULATION OF THE PROBLEM AND SOLUTION OF THE GOVERNING INTEGRAL EQUATION

The general solution through three potential functions F_k was already defined in (3.3.2) where $\chi_k(x,y,z)$ was understood as $\chi(x,y,z_k)$, and $z_k=z/\gamma_k$. As it may be noticed from (3.3.2), the complete solution is expressed through just <u>one</u> complex harmonic function $\chi(x,y,z)$. This function is related to crack surface displacements u by formula

$$\chi(\rho,\phi,z)=\int_0^{2\pi}\int_a^{\infty}\ln\left[\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)+z^2}+z\right]u(r,\psi)\,r\,dr\,d\psi\ . \tag{4.4.2}$$

The governing integro-differential equation, which relates the unknown crack face displacements u to the prescribed shear loading τ, was given in (4.3.6). It is presented here again with detailed description of its parameters

$$\frac{1}{2\pi^2(G_1^2-G_2^2)}\left[G_1\Delta\iint_S \frac{u(N)}{R(N,N_0)}dS_N + G_2\Lambda^2\iint_S \frac{\overline{u}(N)}{R(N,N_0)}dS_N\right]=-\tau(N_0)\ . \tag{4.4.3}$$

Here points N and N_0 have the polar cylindrical coordinates $(r,\psi,0)$ and $(\rho_0,\phi_0,0)$ respectively, $R(N,N_0)$ stands for the distance between the two points, G_1 and G_2 are the elastic constants defined in (2.36), Δ and Λ are the operators defined in (2.26), S is domain of the crack, in this particular case it is the exterior of the circle $\rho=a$.

The solution of (4.4.3) was also given in section 4.3 (f.4.3.29). However, that same formula is given here again, presenting the second integrand in a slightly different way. It reads

$$u(\rho,\phi)=\frac{G_1}{\pi}\int_0^{2\pi}\int_a^{\infty}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\,\frac{t^2(1+t)}{a^2(1-t)^2}\eta\right]\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0$$

$$+\frac{G_2}{\pi}\int_0^{2\pi}\int_a^{\infty}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{\bar{q}}\left(\frac{te^{i\phi}}{\rho(1-t)} - \frac{\bar{t}e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\ , \tag{4.4.4}$$

where η and t are defined in (4.3.22).

The substitution of (4.4.4) in (4.4.2) makes it possible to compute the main potential function χ which, in turn, defines all three functions F_k in (3.3.2); and, finally, the substitution of F_k in (2.27) and (2.32) will give the complete solution for the field of displacements and stresses respectively. Due to the complexity of the integrals involved, the procedure ahead is very non-trivial, and it will be described in detail in the next subsection.

4.4.2 THE COMPLETE SOLUTION

It does not seem possible at first sight to directly substitute (4.4.4) into (4.4.2), interchange the order of integration, and compute the relevant integrals. Certain properties of the integrands in (4.4.4) should be pointed out which will prove useful in the computation of the integrals involved.

Noting the following property

$$\Lambda\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{\bar{t}^2(1+\bar{t})}{a^2(1-\bar{t})^2}\right]$$

$$= -\bar{\Lambda}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{\bar{q}}\left(\frac{te^{i\phi}}{\rho(1-t)} - \frac{\bar{t}e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right], \tag{4.4.5}$$

and introducing the notation:

$$B_1(N,N_0) = \frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right),$$

$$B_2(N,N_0) = \frac{a\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}\,(r\rho_0 e^{i(\psi-\phi_0)}+a^2)}{r\rho_0 e^{i(\psi-\phi_0)}(r\rho_0 e^{i(\psi-\phi_0)}+a^2)^2},$$

$$B_3(N,N_0) = \frac{re^{i\psi}-\rho_0 e^{i\phi_0}}{(re^{-i\psi}-\rho_0 e^{-i\phi_0})R(N,N_0)}\tan^{-1}\left(\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right)$$

$$+\frac{a\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{re^{-i\psi}-\rho_0 e^{-i\phi_0}}\left[\frac{e^{i\psi}}{r(r\rho_0 e^{-i(\psi-\phi_0)}-a^2)} - \frac{e^{i\phi_0}}{\rho_0(r\rho_0 e^{i(\psi-\phi_0)}-a^2)}\right]. \tag{4.4.6}$$

Here the points N and N_0 are characterized by the polar cylindrical coordinates $(r,\psi,0)$ and $(\rho_0,\phi_0,0)$ respectively.

The following property of symmetry holds

$$B_1(N,N_0) = B_1(N_0,N) \ , \qquad B_2(N,N_0) = \overline{B}_2(N_0,N) \ ,$$
$$B_3(N,N_0) = B_3(N_0,N) \ . \tag{4.4.7}$$

Let $R(M,N)$ denote the distance between the points $M(\rho,\phi,z)$ and $N(r,\psi,0)$. By using (4.4.5-4.4.6), it may be written

$$\iint\limits_S \Lambda[B_1(N,N_0)-B_2(N,N_0)]\frac{dS_N}{R(M,N)} =-\iint\limits_S \overline{\Lambda}B_3(N,N_0)\frac{dS_N}{R(M,N)} \ . \tag{4.4.8}$$

Here S is the domain of the crack. Integration by parts in (4.4.8) leads to a very important property,

$$\iint\limits_S [B_1(N,N_0)-B_2(N,N_0)]\Lambda\left(\frac{1}{R(M,N)}\right)dS_N=-\iint\limits_S B_3(N,N_0)\overline{\Lambda}\left(\frac{1}{R(M,N)}\right)dS_N \ . \tag{4.4.9}$$

Two more properties can be obtained by application of Λ and $\overline{\Lambda}$ to both sides of (4.4.9), namely,

$$\iint\limits_S [B_1(N,N_0)-B_2(N,N_0)]\Lambda^2\left(\frac{1}{R(M,N)}\right)dS_N=-\iint\limits_S B_3(N,N_0)\,\Delta\left(\frac{1}{R(M,N)}\right)dS_N \ ,$$
$$\iint\limits_S [B_1(N,N_0)-B_2(N,N_0)]\Delta\left(\frac{1}{R(M,N)}\right)dS_N=-\iint\limits_S B_3(N,N_0)\overline{\Lambda}^2\left(\frac{1}{R(M,N)}\right)dS_N \ . \tag{4.4.10}$$

Integration of both sides in (4.4.9) and (4.4.10) with respect to z will lead to similar properties for $\ln[R(M,N)+z]$ integrand. These properties make it possible to avoid computation of integrals involving B_3, which look very formidable, and compute instead the integrals involving expressions B_1 and B_2, which are more simple.

It can be inferred from (3.3.2) that it will be useful to introduce the notation:

$$U = \Lambda\bar{\chi} + \bar{\Lambda}\chi \ , \qquad V = \Lambda\bar{\chi} - \bar{\Lambda}\chi \ . \tag{4.4.11}$$

The complete solution, given by (3.3.2), (2.27) and (2.32) will depend only on the first and second derivatives of U and V. Since there is no need to evaluate integrals involving B_3, due to the properties (4.4.8-4.4.10), all the derivatives of U and V can be expressed through the two fundamental functions, namely,

$$L_1(M,N_0) = \iint\limits_S B_1(N,N_0)\ln[R(M,N)+z]dS_N \ ,$$

$$L_2(M,N_0) = \iint\limits_S B_2(N,N_0)\ln[R(M,N)+z]dS_N \ . \tag{4.4.12}$$

Formula (4.4.4) can be rewritten in the new notation as

$$u(N) = \frac{G_1}{\pi}\iint\limits_S \left[B_1(N,N_0) - \frac{G_2^2}{G_1^2}\,\bar{B}_2(N,N_0)\right]\tau(N_0)dS_{N_0}$$

$$+ \frac{G_2}{\pi}\iint\limits_S B_3(N,N_0)\bar{\tau}(N_0)dS_{N_0} . \tag{4.4.13}$$

The substitution of (4.4.13) and (4.4.2) in (4.4.11) and use of the properties (4.4.8-4.4.10), will lead to the following results

$$U(M) = \frac{G_1 - G_2}{\pi} \left\{ \bar{\Lambda} \iint\limits_S \left[L_1(M,N_0) + \frac{G_2}{G_1} \bar{L}_2(M,N_0) \right] \tau(N_0)\, dS_{N_0} \right.$$

$$\left. + \Lambda \iint\limits_S \left[L_1(M,N_0) + \frac{G_2}{G_1} L_2(M,N_0) \right] \bar{\tau}(N_0)\, dS_{N_0} \right\} ,$$

$$V(M) = \frac{G_1 + G_2}{\pi} \left\{ -\bar{\Lambda} \iint\limits_S \left[L_1(M,N_0) - \frac{G_2}{G_1} \bar{L}_2(M,N_0) \right] \tau(N_0)\, dS_{N_0} \right.$$

$$\left. + \Lambda \iint\limits_S \left[L_1(M,N_0) - \frac{G_2}{G_1} L_2(M,N_0) \right] \bar{\tau}(N_0)\, dS_{N_0} \right\} . \qquad (4.4.14)$$

In order to find the field of displacements, only the Λ and z - derivatives of U and V has to be evaluated; the field of stresses will be completely defined by the second Λ, z and mixed derivatives. All these derivatives can be expressed in elementary functions, as it will be shown in the next subsection.

4.4.3 THE GREEN'S FUNCTIONS

The results of previous section can be applied to solving the problem of a tangential point force loading of an external circular crack. The solution will give all the Green's functions related to the case.

Consider an infinite transversely isotropic solid weakened in the plane $z=0$ by an external circular crack $\rho\geq a$. Let two equal and oppositely directed tangential forces of magnitude $T=T_x+iT_y$ be applied to the crack faces at the points $(\rho_0,\phi_0,0^{\pm})$ as in Fig.4.4. Here it will be shown in some detail computation of the tangential displacement u, which is defined by the first formula in (2.27), with the functions F_k given in (3.3.2). From (2.27), (3.3.2), (4.4.11) and (4.4.14) it can be deduced that only ΔL_1, ΔL_2, $\Lambda^2 L_1$ and $\Lambda^2 L_2$ need to be computed. Since both L_1 and L_2 are harmonic functions of (ρ,ϕ,z), computation of Δ can be replaced by computation of $-\partial^2/\partial z^2$. The functions F_k defined by (3.3.2) can be rewritten in terms of U and V as follows

$$F_1= -\frac{U_1}{4\pi(m_1-1)}, \quad F_2= -\frac{U_1}{4\pi(m_2-1)}, \quad F_3=\frac{iV_3}{4\pi}. \qquad (4.4.15)$$

Here U_k and V_k are understood as $U(M_k)$ and $V(M_k)$, and the point M_k has the coordinates (ρ,ϕ,z_k), with $z_k=z/\gamma_k$, for k=1,2,3. From (2.27) and (4.4.14-4.4.15) it may be concluded that

$$u = \frac{G_1-G_2}{4\pi^2}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\frac{\partial^2}{\partial z_k^2}\left[L_1(M_k,N_0) + \frac{G_2}{G_1}\bar{L}_2(M_k,N_0)\right]T\right.$$

$$\left. - \Lambda^2\left[L_1(M_k,N_0) + \frac{G_2}{G_1}L_2(M_k,N_0)\right]\bar{T}\right\}$$

$$- \frac{G_1+G_2}{4\pi^2} \left\{ \frac{\partial^2}{\partial z_3^2} \left[L_1(M_3,N_0) - \frac{G_2}{G_1} \bar{L}_2(M_3,N_0) \right] T \right.$$

$$\left. + \Lambda^2 \left[L_1(M_3,N_0) - \frac{G_2}{G_1} L_2(M_3,N_0) \right] \bar{T} \right\} . \qquad (4.4.16)$$

The second z - derivatives of L_1 and L_2 are computed in formulae (A4.4.37) and (A4.4.11) of Appendix A4.4, the quantities of $\Lambda^2 L_1$ and $\Lambda^2 L_2$ are given in (A4.4.34) and (A4.4.42) respectively. So, utilization of (A4.4.37), (A4.4.11), (A4.4.34) and (A4.4.42) in (4.4.16) gives the complete field of tangential displacements in the whole space weakened by an external crack and subjected to a pair of tangential forces T applied at the points N_0 of the crack faces. All the remaining quantities can be computed in a similar manner, with all the necessary derivatives of L_1 and L_2 presented in Appendix A4.4. The final results are

$$u = \frac{G_1-G_2}{2\pi} \sum_{k=1}^{2} \frac{1}{m_k-1} \left\{ - \left[g_2(z_k) + \frac{G_2}{G_1} \bar{g}_7(z_k) \right] T + \left[g_{16}(z_k) + \frac{G_2}{G_1} g_8(z_k) \right] \bar{T} \right\}$$

$$+ \frac{G_1+G_2}{2\pi} \left\{ \left[g_2(z_3) - \frac{G_2}{G_1} \bar{g}_7(z_3) \right] T + \left[g_{16}(z_3) - \frac{G_2}{G_1} g_8(z_3) \right] \bar{T} \right\} ,$$

$$(4.4.17)$$

$$w = \frac{2}{\pi} H \gamma_1 \gamma_2 \mathcal{R}e \left\{ \sum_{k=1}^{2} \frac{m_k}{(m_k-1)\gamma_k} \left[\bar{g}_1(z_k) + \frac{G_2}{G_1} \bar{g}_9(z_k) \right] T \right\} , \qquad (4.4.18)$$

$$\sigma_1 = \mathcal{R}e \left\{ \frac{2\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)} \sum_{k=1}^{2} (-1)^{k+1} \left[\frac{1}{\gamma_3^2(m_k+1)} - \frac{1}{\gamma_k^2} \right] \left[\bar{g}_5(z_k) + \frac{G_2}{G_1} \bar{g}_{10}(z_k) \right] \right\} ,$$

$$(4.4.19)$$

$$\sigma_2=-\frac{2}{\pi}A_{66}H\gamma_1\gamma_2\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[g_5(z_k)+\frac{G_2}{G_1}\bar{g}_{13}(z_k)\right]T\right.$$

$$\left.+\left[g_{11}(z_k)+\frac{G_2}{G_1}\;g_{12}(z_k)\right]\bar{T}\right\}-\frac{1}{\pi^2\gamma_3}\left\{\left[-g_5(z_3)+\frac{G_2}{G_1}\bar{g}_{13}(z_3)\right]T\right.$$

$$\left.+\left[g_{11}(z_3)-\frac{G_2}{G_1}\;g_{12}(z_3)\right]\bar{T}\right\},\qquad(4.4.20)$$

$$\sigma_z=\mathcal{R}e\left\{\frac{\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left[\bar{g}_5(z_k)+\frac{G_2}{G_1}\bar{g}_{10}(z_k)\right]T\right\},\qquad(4.4.21)$$

$$\tau_z=\frac{\gamma_1\gamma_2}{2\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}\frac{(-1)^k}{\gamma_k}\left\{\left[g_3(z_k)+\frac{G_2}{G_1}\bar{g}_{14}(z_k)\right]T\right.$$

$$\left.+\left[-g_4(z_k)+\frac{G_2}{G_1}\;g_{15}(z_k)\right]\bar{T}\right\}+\frac{1}{2\pi^2}\left\{\left[g_3(z_3)-\frac{G_2}{G_1}\bar{g}_{14}(z_3)\right]T\right.$$

$$\left.+\left[g_4(z_3)+\frac{G_2}{G_1}\;g_{15}(z_3)\right]\bar{T}\right\}.\qquad(4.4.22)$$

Here $\mathcal{R}e$ stands for the real part of the expression to follow, the elastic coefficients are defined in (2.29) and (2.36) and the functions g_k are given by (for details see Appendix A4.4)

$$g_1(z)=\frac{1}{\bar{q}}\left\{\tan^{-1}\frac{(\rho_0^2-a^2)^{1/2}}{a}-\frac{z}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)\right.$$

$$\left.+\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]\right\},\qquad(4.4.23)$$

$$g_2(z) = \frac{1}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right) \ , \tag{4.4.24}$$

$$g_3(z) = -\ \frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) + \frac{j}{z(R_0^2+j^2)}\left[\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right], \tag{4.4.25}$$

$$g_4(z) = \frac{1}{\bar{q}}\left\{\frac{z(3R_0^2-z^2)}{\bar{q}\ R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) - \frac{2}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right) + \frac{zj}{R_0^2+j^2}\left[\frac{q}{R_0^2}\right.\right.$$

$$\left. - \frac{\bar{q}\rho^2 e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right] - \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\frac{2}{\bar{q}} + \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)\right.$$

$$\left.\left. -\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right] + \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}^2}\left[\frac{(a^2-\ell_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-\ell_1^2} - \frac{a e^{i\phi}}{\rho}\right]\right\} \ , \tag{4.4.26}$$

$$g_5(z) = -\ \frac{q}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) + \frac{j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2} - \frac{q}{R_0^2}\right] \ , \tag{4.4.27}$$

$$g_7(z) = \frac{z}{a^3}(\rho_0^2-a^2)^{1/2}\left\{\frac{3\bar{t}^{5/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right.\right.$$

$$\left.\left. -\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right] + \frac{\bar{t}^2}{(1-\bar{t})^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}} - 2 - \bar{t} + \frac{a(1+\bar{t})}{(a^2-\ell_1^2)^{1/2}}\right]\right\} \ , \tag{4.4.28}$$

$$g_8(z) = \frac{\bar{t}^2}{a^2}\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}\left[\frac{a}{(a^2-\rho^2\bar{t})^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right.$$

$$\left. - \frac{\bar{t}\ell_1(\rho^2-\ell_1^2)^{1/2}}{(a^2-\ell_1^2\bar{t})(a^2-\rho^2\bar{t})}\right] \ , \tag{4.4.29}$$

$$g_9(z) = -\frac{\bar{t}^2}{a^3}\rho e^{i\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{1}{1-\bar{t}}\left[1 - \frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}}\right]\right.$$

$$\left.+\frac{\bar{t}^{1/2}}{(1-\bar{t})^{3/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]\right\}, \tag{4.4.30}$$

$$g_{10}(z) = \frac{j\rho e^{i\phi}\bar{t}^2\ell_1^2(a^2+\ell_1^2\bar{t})}{a^2(a^2-\ell_1^2\bar{t})^2(\ell_2^2-\ell_1^2)}, \tag{4.4.31}$$

$$g_{11}(z) = \frac{1}{\bar{q}}\left\{\frac{3R_0^4+6z^2R_0^2-z^4}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) - \frac{8z}{\bar{q}^2}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[-\frac{3(1-\bar{\zeta})^{1/2}}{\bar{q}^2}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right) + \frac{z}{\bar{s}}\left(\frac{8}{\bar{q}^2} + \frac{4\rho_0e^{i\phi_0}}{\bar{q}\;\bar{s}^2}\right.\right.$$

$$\left.+\frac{3\rho_0^2e^{2i\phi_0}}{\bar{s}^4}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{s}}{a}\right)\right] - \frac{z}{\bar{s}^2}\left(\frac{2}{\bar{q}}\right.$$

$$\left.\left.+\frac{\rho_0e^{i\phi_0}}{\bar{s}^2}\right)\left(\frac{(a^2-\ell_1^2)^{1/2}\rho_0e^{i\phi_0}}{\rho\rho_0e^{-i(\phi-\phi_0)}-\ell_1^2} - \frac{ae^{i\phi}}{\rho}\right)\right] + \frac{2jae^{i\phi}}{\rho\bar{s}^2}\left(\frac{1}{\bar{q}} + \frac{e^{i\phi}}{\rho}\right.$$

$$\left.\left.+\frac{\rho_0e^{i\phi_0}}{\bar{s}^2}\right)\left(a-(a^2-\ell_1^2)^{1/2}\right) + \frac{j}{R_0^2+j^2}\left[\frac{\rho e^{3i\phi}\bar{q}}{\ell_2^2-\ell_1^2}+\frac{z^2\bar{q}}{R_0^2\bar{q}}+\frac{e^{i\phi}(\rho^2-\ell_1^2)}{\bar{q}\rho} - 2e^{2i\phi}\right]\right\}, \tag{4.4.32}$$

$$g_{12}(z) = \frac{\bar{t}^3}{a}\rho^3e^{3i\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{(\ell_2^2-a^2)^{1/2}}{(a^2-\ell_1^2\bar{t})(a^2-\rho^2\bar{t})}\left[\frac{a^2+2\rho^2\bar{t}}{\ell_2^2(a^2-\rho^2\bar{t})}\right.\right.$$

$$\left.\left.+\frac{a^2+\rho^2\bar{t}}{\ell_2^2\bar{t}(\ell_2^2-\ell_1^2)}+\frac{2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)(a^2-\ell_1^2\bar{t})}\right]-\frac{3}{(a^2-\rho^2\bar{t})^{5/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right\}, \tag{4.4.33}$$

$$\bar{g}_{13}(z)=-zt^2\rho e^{i\phi}\frac{(\rho_0^2-a^2)^{1/2}}{a^3}\left\{\frac{\rho^4(\ell_2^2+\rho^2 t)(\ell_2^2-a^2)}{\ell_2(\ell_2^2-\rho^2 t)^2(\ell_2^2-\rho^2)^{3/2}(\ell_2^2-\ell_1^2)}\right.$$

$$-\frac{15t^{1/2}}{\rho^2(1-t)^{7/2}}\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]$$

$$-\frac{1}{(1-t)^2}\left[\frac{2(1+t)}{\rho^2}+\frac{6+9t}{\rho^2(1-t)}\right]+\frac{a}{(a^2-\ell_1^2)^{1/2}(1-t)^2}\left[\frac{2(1+t)}{\rho^2}\right.$$

$$\left.\left.+\frac{6+9t}{\rho^2(1-t)}-\frac{1+t}{\ell_2^2-\rho^2}-\frac{3}{\ell_2^2-\rho^2 t}\right]\right\}, \tag{4.4.34}$$

$$g_{14}(z)=-\bar{t}^2\frac{(\rho_0^2-a^2)^{1/2}}{a^3}\left\{\frac{3\bar{t}^{1/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]-\frac{1}{(1-\bar{t})^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}}-2-\bar{t}\right.$$

$$\left.\left.+\frac{a(1+\bar{t})}{(a^2-\ell_1^2)^{1/2}}\right]+\frac{\rho^4 z(\ell_2^2+\rho^2\bar{t})(\ell_2^2-a^2)^{1/2}}{(\ell_2^2-\rho^2)(\ell_2^2-\rho^2\bar{t})^2(\ell_2^2-\ell_1^2)}\right\}, \tag{4.4.35}$$

$$g_{15}(z)=\frac{\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}(a^2-\ell_1^2)^{1/2}\bar{t}^2(a^2+\ell_1^2\bar{t})}{a^2(\ell_2^2-\ell_1^2)(a^2-\ell_1^2\bar{t})^2}, \tag{4.4.36}$$

$$g_{16}(z)=\frac{1}{\bar{q}}\left\{\frac{R_0^2+z^2}{\bar{q}R_0}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2z}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)\right.$$

$$\left.\left.-\frac{e^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right]+\frac{jae^{i\phi}}{\rho\bar{s}^2}\left(a-(a^2-\ell_1^2)^{1/2}\right)\right\}. \tag{4.4.37}$$

The absence of the function $g_6(z)$ from the list above is in order to preserve in (4.4.17–4.4.22) the form of solution used in (Fabrikant [16]) for a penny-shaped crack, where the equivalent notation f_6 was used elsewhere.

Remember that the notations $\zeta, q, R, t, s, R_0, j$ are defined in (4.3.3), (4.3.9), (4.3.22), (A4.4.29) respectively. The identities should also be noted:

$$\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}} = \frac{a}{s}, \quad \frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}} = \frac{(a^2-\ell_1^2)^{1/2}}{s}. \tag{4.4.38}$$

This means that the trigonometric functions which were introduced in various formulae in different manner, are in fact the same, for example

$$\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)$$

$$= \tan^{-1}\left(\frac{s}{(a^2-\ell_1^2)^{1/2}}\right) - \tan^{-1}\left(\frac{s}{a}\right), \tag{4.4.39}$$

Yet another example:

$$\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right) = \tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right). \tag{4.4.40}$$

Every function g_i depends on the coordinate of the field point (ρ,ϕ,z) and the coordinates $(\rho_0,\phi_0,0)$ of the point of application of the force T. The notation $g_i(z)$ was used just to emphasize the fact, that z_k $(k=1,2,3)$ should be substituted instead of z when using formulae (4.4.17-4.4.22).

The expressions (4.4.17-4.4.22) simplify significantly on the plane z=0. The results are: (the first corresponds to the case when $\rho<a$, while the second to the case when $\rho>a$)

$$u=0 ,$$

$$u = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\,\frac{t^2(1+t)\eta}{a^2(1-t)^2}\right]T$$

$$+ \frac{G_2}{\pi}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{\bar{q}}\left(\frac{te^{i\phi}}{\rho(1-t)} - \frac{\bar{t}e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{T} , \qquad (4.4.41)$$

$$w = \frac{2}{\pi}H\alpha\mathcal{R}e\left\{\left\{\frac{1}{q}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right) + \frac{(\rho_0^2-a^2)^{1/2}}{qs}\left(\tan^{-1}\left(\frac{s}{(a^2-\rho^2)^{1/2}}\right)\right.\right.\right.$$

$$\left.\left. - \tan^{-1}\left(\frac{s}{a}\right)\right) + \frac{G_2}{G_1}\,\frac{a(\rho_0^2-a^2)^{1/2}}{\rho_0 e^{i\phi_0}}\left[\frac{a}{\bar{s}^3}\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)\right.$$

$$\left.\left.\left. - \frac{1}{\bar{s}^2}\left(1 - \frac{(a^2-\rho^2)^{1/2}}{a(1-\zeta)}\right)\right]\right\}T\right\} ,$$

$$w = \frac{2}{\pi}H\alpha\mathcal{R}e\left\{\left[\frac{1}{q}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right) + \frac{(\rho_0^2-a^2)^{1/2}}{qs}\tan^{-1}\left(\frac{a}{s}\right)\right.\right.$$

$$\left.\left. + \frac{G_2}{G_1}\,\frac{a(\rho_0^2-a^2)^{1/2}}{\rho_0 e^{i\phi_0}}\left(\frac{a}{\bar{s}^3}\tan^{-1}\left(\frac{a}{\bar{s}}\right) - \frac{1}{\bar{s}^2}\right)\right]T\right\} , \qquad (4.4.42)$$

$$\sigma_1=0 ,$$

$$\sigma_1 = \frac{2}{\pi^2}\mathcal{R}e\left\{\left[\left(2\pi HA_{66}\gamma_1\gamma_2 - \frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\right)\left(\frac{1}{qR}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{\eta a^2}{s^2\bar{s}^2}\left(\frac{\rho e^{-i\phi}}{\rho^2-a^2} - \frac{1}{q}\right)\right)\right.\right.$$

$$\left.\left. - \frac{G_2}{G_1}\frac{\eta\rho e^{-i\phi}t^2(1+t)}{a^2(1-t)^2(\rho^2-a^2)}\right]T\right\} , \qquad (4.4.43)$$

$$\sigma_2 = \frac{1}{\pi^2}\left\{\left(2\pi A_{66}H\gamma_1\gamma_2 + \frac{1}{\gamma_3}\right)\left[g_5(0)T + \frac{G_2}{G_1}g_{12}(0)\bar{T}\right]\right.$$

$$\left. + \left(2\pi A_{66}H\gamma_1\gamma_2 - \frac{1}{\gamma_3}\right)\left[g_{11}(0)\bar{T} + \frac{G_2}{G_1}\bar{g}_{13}(0)T\right]\right\} , \qquad (4.4.44)$$

$$\sigma_z = 0 , \qquad \text{for } \rho>0 , \qquad (4.4.45)$$

$$\tau_z = \frac{1}{\pi^2}\frac{(\rho_0^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}}\left[\frac{T}{R^2} + \frac{G_2}{G_1}\frac{\rho^2 e^{2i\phi}\bar{t}^2(a^2+\rho^2\bar{t})}{a^2(a^2-\rho^2\bar{t})^2}\bar{T}\right]$$

$$= \frac{1}{\pi^2}\frac{(\rho_0^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}}\left[\frac{T}{R^2} + \frac{G_2}{G_1}\frac{e^{2i\phi_0}(1+\bar{\zeta})}{\rho_0^2(1-\bar{\zeta})^2}\bar{T}\right] ,$$

$$\tau_z = -T\delta(\rho-\rho_0)\delta(\phi-\phi_0) . \qquad (4.4.46)$$

The second and third mode SIF can be obtained by using the expression similar to the one defined in (2.49). From (4.4.46) it will be obtained

$$K_2+iK_3 = \frac{(\rho_0^2-a^2)^{1/2}}{\pi^2(2a)^{1/2}}\left[\frac{Te^{-i\phi}}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)}\right.$$

$$\left. + \frac{G_2}{G_1}\frac{e^{-i(\phi-\phi_0)}(\rho_0 e^{-i\phi_0} + ae^{-i\phi})}{\rho_0(\rho_0 e^{-i\phi_0} - ae^{-i\phi})^2}\bar{T}\right] . \qquad (4.4.47)$$

In the case of a distributed loading, the SIF are given by

$$K_2+iK_3=\frac{e^{-i\phi}}{\pi^2(2a)^{1/2}}\left\{\int_0^{2\pi}\int_a^{\infty}\frac{(\rho_0^2-a^2)^{1/2}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)}\right.$$

$$\left.+\frac{G_2}{G_1}\int_0^{2\pi}\int_a^{\infty}\frac{(\rho_0^2-a^2)^{1/2}(\rho_0+ae^{-i(\phi-\phi_0)})\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0}{\rho_0(\rho_0 e^{-i\phi_0}-ae^{-i\phi})^2}\right\}. \qquad (4.4.48)$$

It should be noted that (4.4.48) is in agreement with (4.3.2).

This completes the solution to the problem of external circular crack under shear load. Formulae (4.4.17-4.4.37) are the main new results of this section.

4.5 SUMMARY

In this chapter the fundamental solutions to the problems of external circular crack under normal and shear load have been presented. The complete solution obtained here is of great value because it makes it possible to solve easily many complicated problems which were not even attempted before. For example, in Chapter 6 interaction between an arbitrarily located horizontal force Q and an external circular crack of radius a will be considered. The solution will be obtained in an elementary way using the reciprocal theorem.

Chapter 5 will deal with a new type of crack problems, namely, semi-infinite crack where the infinite straight line delineates the boundary conditions. It means that the polar cylindrical coordinate system which was used so far will be replaced by the cartesian coordinate system. By means of new developed concepts for the reciprocal of the distance it will be possible to obtain all the relevant Green's functions to the half-plane contact and crack problems.

CHAPTER 5
HALF-PLANE CRACK PROBLEMS

5.1 INTRODUCTORY REMARKS

In this chapter an intensive study of the problems of a half-plane crack in a transversely isotropic elastic space subjected to arbitrary normal and tangential loading will be made. An exact closed form-solution will be obtained in terms of elementary functions to the complete field of stresses and displacements due to a point force loading. Both transversely isotropic and purely isotropic cases are considered. The transversely isotropic solution has not been reported in the literature. The isotropic case was considered by Ufliand [22] who used the integral transform approach. A comparison with his results for the case of normal loading shows an exact correspondence. Explicit formulae are also given for the stresses and displacements in the plane of the crack. For the case of shear loading a comparison was made with the results presented in Kassir and Sih [15], who gave an explicit expressions for stresses in the plane of the crack and mixed mode SIF.

Though the semi-infinite crack in an elastic space might look like a very artificial model to an engineer, this is not so, and the range of applications of this model are quite wide. Indeed, the model may be applied to any case when the stress distribution in a cracked body is at question, with the distance from the loading to the crack edge being small as compared to the crack edge curvature. The practical importance of such problems to engineering is the main reason why their solution is presented here.

The solution for the case of an isotropic body was obtained by Ufliand [22], who used a very complicated Kontorovich-Lebedev integral transform, which does not seem to be applicable to the case of transverse isotropy. The same problem was later solved by Kit and Khai [35], who used the two-dimensional Fourier transform, with subsequent reduction to the Riemann-Hilbert problem.

Also, for the problem of a half-plane crack under normal load the same results will be obtained by two alternative methods.

The material presented in the forthcoming sections 5.2 and 5.3 follows the work by Fabrikant, Rubin and Karapetian [36,37].

5.2 HALF-PLANE CRACK UNDER NORMAL LOAD: A COMPLETE SOLUTION

The principal idea used in this work is as follows. A half-plane crack can be obtained from a circular one by a limiting procedure where the radius of the circular crack tends to infinity. Thus, if there is a complete solution to a penny-shaped crack problem, a complete solution to a half-plane crack can be obtained. As simple as the idea might look, its implementation is not trivial.

5.2.1 SOLUTION FOR A CIRCULAR CRACK

Consider a penny-shaped crack opened by two equal concentrated forces P applied in opposite directions at the points $(\rho_0, \phi_0, 0^{\pm})$, $\rho_0 < a$. A complete solution for the field of displacements and stresses in elementary functions, is (Fabrikant [16]):

$$u=\frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1(z_1) + \frac{\gamma_2}{m_2-1}f_1(z_2)\right], \tag{5.2.1}$$

$$w=\frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2(z_1) + \frac{m_2}{m_2-1}f_2(z_2)\right], \tag{5.2.2}$$

$$\sigma_1=\frac{2P}{\pi^2(\gamma_1-\gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1+1)\gamma_3^2} - \frac{1}{\gamma_1}\right]f_3(z_1)-\left[\frac{\gamma_2}{(m_2+1)\gamma_3^2} - \frac{1}{\gamma_2}\right]f_3(z_2)\right\}, \tag{5.2.3}$$

$$\sigma_2=\frac{4}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1-1}f_4(z_1) + \frac{\gamma_2}{m_2-1}f_4(z_2)\right], \tag{5.2.4}$$

$$\sigma_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[\gamma_1 f_3(z_1) - \gamma_2 f_3(z_2)\right], \tag{5.2.5}$$

$$\tau_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[f_5(z_1) - f_5(z_2)\right], \tag{5.2.6}$$

where

$$f_1(z)=\frac{1}{\bar{q}}\left[\frac{(a^2-\rho_0^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)-\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right], \tag{5.2.7}$$

$$f_2(z)=\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) \quad , \tag{5.2.8}$$

$$f_3(z)=\left\{-\frac{z}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)+\frac{h}{z(R_0^2+h^2)}\left[\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}-\frac{z^2}{R_0^2}\right]\right\} \quad , \tag{5.2.9}$$

$$f_4(z)=\frac{(a^2-\rho_0^2)^{1/2}}{\bar{q}\;\bar{s}}\left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}-\frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)$$

$$+\frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)-\frac{(a^2-\rho_0^2)^{1/2}(\ell_2^2-a^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\;\bar{s}^2[\ell_2^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]}$$

$$+\frac{zh}{R_0^2+h^2}\left[\frac{q}{\bar{q}R_0^2}-\frac{\rho^2e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\ell_2^2-\rho^2)}\right] \quad , \tag{5.2.10}$$

$$f_5(z)=-\left\{\frac{q}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)+\frac{h}{R_0^2+h^2}\left[\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2}+\frac{q}{R_0^2}\right]\right\} \quad . \tag{5.2.11}$$

Here the following notation was used:

$$R_0=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2} \quad , \qquad q=\rho e^{i\phi}-\rho_0 e^{i\phi_0} \quad , \tag{5.2.12}$$

$$\ell_1(a,\rho,z)\equiv\ell_1=\frac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}-[(\rho-a)^2+z^2]^{1/2}\} \quad ,$$

$$\ell_2(a,\rho,z)\equiv\ell_2=\frac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}+[(\rho-a)^2+z^2]^{1/2}\} \quad , \tag{5.2.13}$$

$$h=\sqrt{a^2-\ell_1^2}\sqrt{a^2-\rho_0^2}/a \quad , \qquad s=\sqrt{a^2-\rho\rho_0 e^{i(\phi-\phi_0)}} \quad . \tag{5.2.14}$$

5.2.2 SOLUTION FOR A HALF-PLANE CRACK

Let a Cartesian system of coordinates (x,y,z) be introduced. It is related to the original system of polar cylindrical coordinates (ρ,ϕ,z) by the relationships:

$$\rho\cos\phi=x+a\ ,\quad \rho\sin\phi=y\ ,\quad z\equiv z\ , \tag{5.2.15}$$

The transformation (5.2.15) shifts the coordinate system origin to the edge of the crack as in Fig.5.1. Then if the limit is taken as $a\to\infty$, a solution to the problem of a half-plane crack in an infinite transversely isotropic body will be obtained.

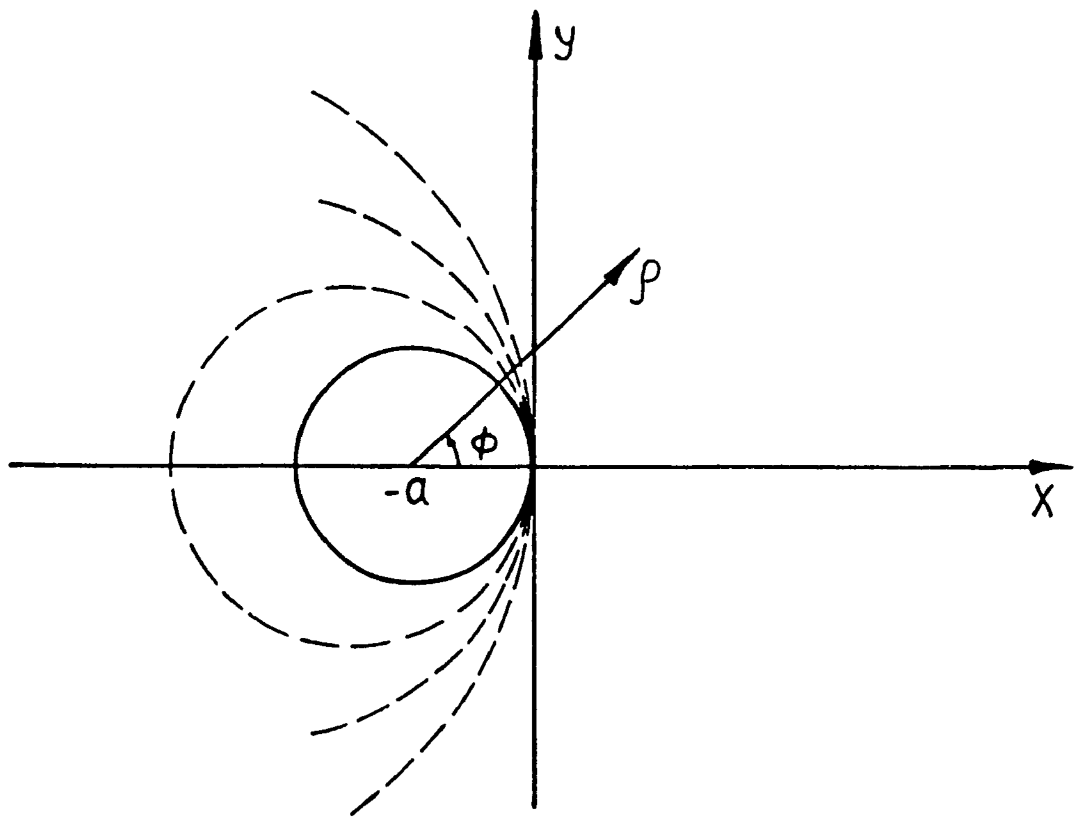

Fig.5.1 Generation of half-plane crack from circular crack.

The crack is being opened by two equal forces P applied in opposite directions at the points $(x_0,y_0,0^{\pm})$, $x_0<0$ as shown in Fig.5.2.

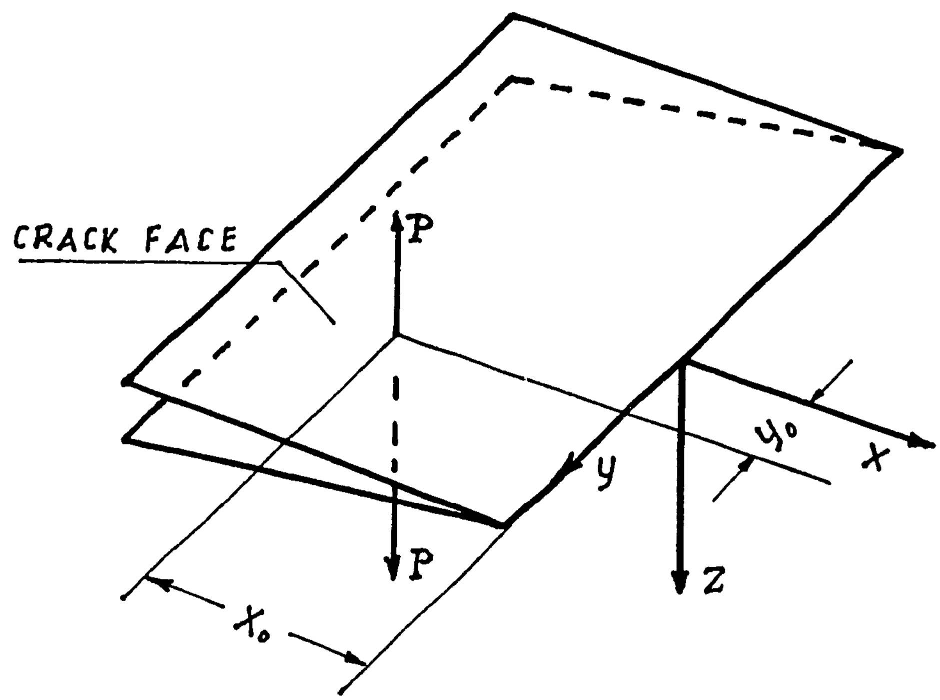

Fig.5.2 A half-plane crack under normal load.

Equations (5.2.1-5.2.6) give the field of stresses and displacements at the point (ρ,ϕ,z) due to a concentrated loading at the point (ρ_0,ϕ_0). In the limiting case $a\to\infty$, both ρ and ρ_0 tend to ∞, while both ϕ and ϕ_0 tend to zero. This makes computation of the required limits non-trivial, and a certain diligence is required in order to single out the right expressions which would have finite limits. After several trials and errors, the right combinations have been found as follows:

$$\lim_{a\to\infty}\left[\frac{a^2-\ell_1^2}{a}\right]=\lim_{a\to\infty}\frac{1}{a}\left[a^2-\frac{1}{2}\left(a^2+(x+a)^2+y^2+z^2-\{a^2+(x+a)^2+y^2+z^2\right.\right.$$

$$+2a[(x+a)^2+y^2]^{1/2}\}^{1/2}\{a^2+(x+a)^2+y^2+z^2-2a[(x+a)^2+y^2]^{1/2}\}^{1/2}\Big)\Big]$$

$$=\lim_{a\to\infty}\left\{-x-\frac{1}{2a}\left[x^2+y^2+z^2-\{[a^2+(x+a)^2+y^2+z^2]^2-4a^2[(x+a)^2+y^2]\}^{1/2}\right]\right\}$$

$$=\lim_{a\to\infty}\Bigg\{-x-\frac{1}{2a}\Big[x^2+y^2+z^2-\{[(x+a)^2+y^2-a^2]^2+z^4$$

$$+2z^2[(x+a)^2+y^2+a^2]\}^{1/2}\Big]\Bigg\}=\lim_{a\to\infty}\left\{-x-\frac{x^2+y^2+z^2}{2a}+\frac{1}{2}\left[\frac{(x^2+y^2+2ax)^2}{a^2}+\frac{z^4}{a^2}\right.\right.$$

$$\left.\left.+\frac{2z^2(x^2+y^2+2ax+2a^2)}{a^2}\right]^{1/2}\right\}=\sqrt{x^2+z^2}-x\equiv\ell_1^{\bullet}\ . \tag{5.2.16}$$

Using the same procedure, one may derive

$$\lim_{a\to\infty}\frac{\ell_2^2-a^2}{a}=\sqrt{x^2+z^2}+x\equiv\ell_2^{\bullet}\ . \tag{5.2.17}$$

The remaining limits are more simple to compute:

$$\lim_{a\to\infty}\left\{\frac{a^2-\rho_0^2}{a}\right\}=\lim_{a\to\infty}\left\{\frac{a^2-(x_0+a)^2-y_0^2}{a}\right\}=-2x_0\ , \tag{5.2.18}$$

$$\lim_{a\to\infty}\frac{s^2}{a}=\lim_{a\to\infty}\frac{a^2-[(x+a)+iy][(x_0+a)-iy_0]}{a}=-(x+x_0)-i(y-y_0)\equiv s^{\bullet}. \tag{5.2.19}$$

By using (5.2.16-5.2.19), it may be deduced that

$$\lim_{a\to\infty}h=\sqrt{-2x_0[(x^2+z^2)^{1/2}-x]}\equiv h^{\bullet}, \tag{5.2.20}$$

$$\lim_{a\to\infty} \frac{\sqrt{a^2-\rho_0^2}}{\bar{s}} = \sqrt{\frac{2x_0}{(x+x_0)-i(y-y_0)}} \equiv c \ . \tag{5.2.21}$$

Taking into consideration that $\rho e^{i\phi}=(x+a)+iy$ and $\rho_0 e^{i\phi_0}=(x_0+a)+iy_0$ it may be concluded that R_0 and q remain invariant, namely,

$$R_0=\sqrt{(x-x_0)^2+(y-y_0)^2+z^2} \ , \qquad q=(x-x_0)+i(y-y_0) \ . \tag{5.2.22}$$

And now the general solution for the field of stresses and displacements in a transversely isotropic elastic space, weakened by a semi-infinite flat crack occupying the region $x\le 0$, will take the form

$$u=\frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1^*(z_1)+\frac{\gamma_2}{m_2-1}f_1^*(z_2)\right] \ , \tag{5.2.23}$$

$$w=\frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2^*(z_1)+\frac{m_2}{m_2-1}f_2^*(z_2)\right] \ , \tag{5.2.24}$$

$$\sigma_1=\frac{2P}{\pi^2(\gamma_1-\gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1+1)\gamma_3^2} - \frac{1}{\gamma_1}\right]f_3^*(z_1)-\left[\frac{\gamma_2}{(m_2+1)\gamma_3^2} - \frac{1}{\gamma_2}\right]f_3^*(z_2)\right\} \ , \tag{5.2.25}$$

$$\sigma_2=\frac{4}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1-1}f_4^*(z_1)+\frac{\gamma_2}{m_2-1}f_4^*(z_2)\right] \ , \tag{5.2.26}$$

$$\sigma_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[\gamma_1 f_3^*(z_1) - \gamma_2 f_3^*(z_2)\right] \ , \tag{5.2.27}$$

$$\tau_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[f_5^*(z_1) - f_5^*(z_2)\right] \ , \tag{5.2.28}$$

where

$$f_1^*(z)=\frac{1}{\bar{q}}\left[c\ \tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}-\frac{z}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right]\ , \tag{5.2.29}$$

$$f_2^*(z)=\frac{1}{R_0}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)\ , \tag{5.2.30}$$

$$f_3^*(z)=-\frac{z}{R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{h^*}{z[R_0^2+(h^*)^2]}\left[\frac{\ell_2^*}{\ell_1^*+\ell_2^*}-\frac{z^2}{R_0^2}\right]\ , \tag{5.2.31}$$

$$f_4^*(z)=\frac{c}{\bar{q}}\left(\frac{1}{\bar{s}^*}-\frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}+\frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)$$

$$-\frac{\sqrt{-2x_0\ell_2^*}}{\bar{q}\ \bar{s}^*(\ell_2^*+\bar{s}^*)}+\frac{zh^*}{R_0^2+(h^*)^2}\left[\frac{q}{\bar{q}R_0^2}-\frac{1}{\ell_1^*(\ell_1^*+\ell_2^*)}\right]\ , \tag{5.2.32}$$

$$f_5^*(z)=-\left\{\frac{q}{R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{h^*}{R_0^2+(h^*)^2}\left[\frac{1}{\ell_1^*+\ell_2^*}+\frac{q}{R_0^2}\right]\right\}\ . \tag{5.2.33}$$

Formulae (5.2.23-5.2.33) are the main new results obtained here.

5.2.3 ISOTROPIC SOLUTION

The solution for an isotropic body can be obtained as a limiting case of the transversely isotropic solution, subject to the conditions in (2.34). In the case of an isotropic body, using the results from equations (3.2.19) and the identities in (2.35), the formulae (5.2.23-5.2.33) transform into

$$u=-\frac{P(1+\nu)}{\pi^2 E}\left\{\frac{(1-2\nu)}{\bar{q}}\left[c\ \tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}-\frac{z}{R_0}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right]\right.$$

$$-\frac{z}{\bar{q}}\left[\frac{z\sqrt{-2x_0}}{(\ell_2^*)^{1/2}(\ell_2^*+\bar{s}^*)(\ell_1^*+\ell_2^*)}+\frac{R_0^2-z^2}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right.$$

$$\left.\left.-\frac{z^2}{R_0^2+(h^*)^2}\left(\frac{x_0}{h^*(x^2+z^2)^{1/2}}+\frac{h^*}{R_0^2}\right)\right]\right\}, \tag{5.2.34}$$

$$w=\frac{P(1+\nu)}{\pi^2 E}\left\{\left[\frac{2(1-\nu)}{R_0}+\frac{z^2}{R_0^3}\right]\tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{z^2}{R_0^2+(h^*)^2}\left(\frac{x_0}{h^*(x^2+z^2)^{1/2}}+\frac{h^*}{R_0^2}\right)\right\}, \tag{5.2.35}$$

$$\sigma_1=-\frac{P}{\pi^2}\left\{\left[2(1+\nu)-\frac{3z^2}{R_0^2}\right]\frac{z}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right)-2\nu\frac{h^*}{z[R_0^2+(h^*)^2]}\left(\frac{\ell_2^*}{\ell_1^*+\ell_2^*}-\frac{z^2}{R_0^2}\right)\right.$$

$$-\frac{z^3}{R_0^2[R_0^2+(h^*)^2]}\left[\frac{x_0}{h^*(x^2+z^2)^{1/2}}+\frac{h^*}{R_0^2}\right]-\frac{z}{[R_0^2+(h^*)^2]^2}\left[\frac{x_0[(h^*)^2-R_0^2]}{h^*(x^2+z^2)^{1/2}}\right.$$

$$\left.\left.-2h^*\right]\left(\frac{\ell_2^*}{\ell_1^*+\ell_2^*}-\frac{z^2}{R_0^2}\right)+\frac{h^*z}{R_0^2+(h^*)^2}\left[\frac{x}{2(x^2+z^2)^{3/2}}+\frac{2(R_0^2-z^2)}{R_0^4}\right]\right\}, \tag{5.2.36}$$

$$\sigma_2=\frac{P}{\pi^2}\left\{(1-2\nu)\left[\frac{c}{\bar{q}}\left(\frac{2}{\bar{q}}-\frac{1}{\bar{s}^*}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}-\frac{z(3R_0^2-z^2)}{R_0^3\bar{q}^2}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right.\right.$$

$$\left.+\left(\frac{\ell_2^*}{\bar{s}^*}\right)^{1/2}\frac{c}{\bar{q}(\ell_2^*+\bar{s}^*)}-\frac{zh^*}{R_0^2+(h^*)^2}\left(\frac{q}{\bar{q}R_0^2}-\frac{1}{\ell_1^*(\ell_1^*+\ell_2^*)}\right)\right]-z\left[\frac{3q^2}{R_0^5}\tan^{-1}\left(\frac{h^*}{R_0}\right)\right.$$

$$-\frac{2z\sqrt{-2x_0}(\bar{q}-\ell_2^*-\bar{s}^*)}{\bar{q}^2(\ell_2^*)^{1/2}(\ell_2^*+\bar{s}^*)^2(\ell_1^*+\ell_2^*)} - \frac{z^2(3R_0^2-z^2)}{\bar{q}^2R_0^2[R_0^2+(h^*)^2]}\left(\frac{x_0}{h^*(x^2+z^2)^{1/2}} + \frac{h^*}{R_0^2}\right)$$

$$+\left(\frac{z^2x_0[(h^*)^2-R_0^2]}{h^*[R_0^2+(h^*)^2]^2(x^2+z^2)^{1/2}} - \frac{2z^2h^*}{[R_0^2+(h^*)^2]^2} + \frac{h^*}{R_0^2+(h^*)^2}\right)\left(\frac{q}{\bar{q}R_0^2}\right.$$

$$\left.-\frac{1}{\ell_1^*(\ell_1^*+\ell_2^*)}\right) - \frac{2z^2h^*}{R_0^2+(h^*)^2}\left(\frac{q}{\bar{q}R_0^4} + \frac{2x}{(\ell_1^*)^2(\ell_1^*+\ell_2^*)^3} - \frac{2}{(\ell_1^*)^2(\ell_1^*+\ell_2^*)^2}\right)\Bigg]\Bigg\},$$

$$\tag{5.2.37}$$

$$\sigma_z = -\frac{P}{\pi^2}\left\{\frac{3z^3}{R_0^5}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \frac{2h^*}{z[R_0^2+(h^*)^2]}\left(\frac{\ell_2^*}{\ell_1^*+\ell_2^*} - \frac{z^2}{R_0^2}\right)\right.$$

$$+\frac{z^3}{R_0^2[R_0^2+(h^*)^2]}\left[\frac{2x_0}{h^*(\ell_1^*+\ell_2^*)} + \frac{h^*}{R_0^2}\right] + \frac{2z}{[R_0^2+(h^*)^2]^2}\left[\frac{x_0[(h^*)^2-R_0^2]}{h^*(\ell_1^*+\ell_2^*)} - h^*\right]$$

$$\left.\times\left(\frac{\ell_2^*}{\ell_1^*+\ell_2^*} - \frac{z^2}{R_0^2}\right) - \frac{h^*z}{R_0^2+(h^*)^2}\left[\frac{4x}{(\ell_1^*+\ell_2^*)^3} + \frac{2(R_0^2-z^2)}{R_0^4}\right]\right\}, \tag{5.2.38}$$

$$\tau_z = -\frac{P}{\pi^2}\left\{\frac{3qz^2}{R_0^5}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{qz^2}{R_0^2[R_0^2+(h^*)^2]}\left[\frac{x_0}{h^*(x^2+z^2)^{1/2}} + \frac{h^*}{R_0^2}\right]\right.$$

$$-\left[\frac{x_0z^2[(h^*)^2-R_0^2]}{h^*[R_0^2+(h^*)^2]^2(x^2+z^2)^{1/2}} - \frac{2z^2h^*}{[R_0^2+(h^*)^2]^2}\right]\left(\frac{q}{R_0^2} + \frac{1}{\ell_1^*+\ell_2^*}\right)$$

$$\left.+\frac{2z^2h^*}{R_0^2+(h^*)^2}\left[\frac{q}{R_0^4} + \frac{2}{(\ell_1^*+\ell_2^*)^3}\right]\right\}. \tag{5.2.39}$$

This completes the solution to the problem of a half-plane crack under normal loading.

5.2.4 DISCUSSION AND NUMERICAL RESULTS

Numerical computations were performed for the field of normal displacements and normal stresses, with the Poisson coefficient $\nu=0.3$. The field of normal displacements due to a pair of concentrated forces applied at the crack faces in opposite direction at the points $(-a,0,0^+)$ and $(-a.0,0^-)$, is given in Fig.5.3 as a function of x/a for a set of different values of z. Similar data for the normal stresses are presented in Fig.5.4.

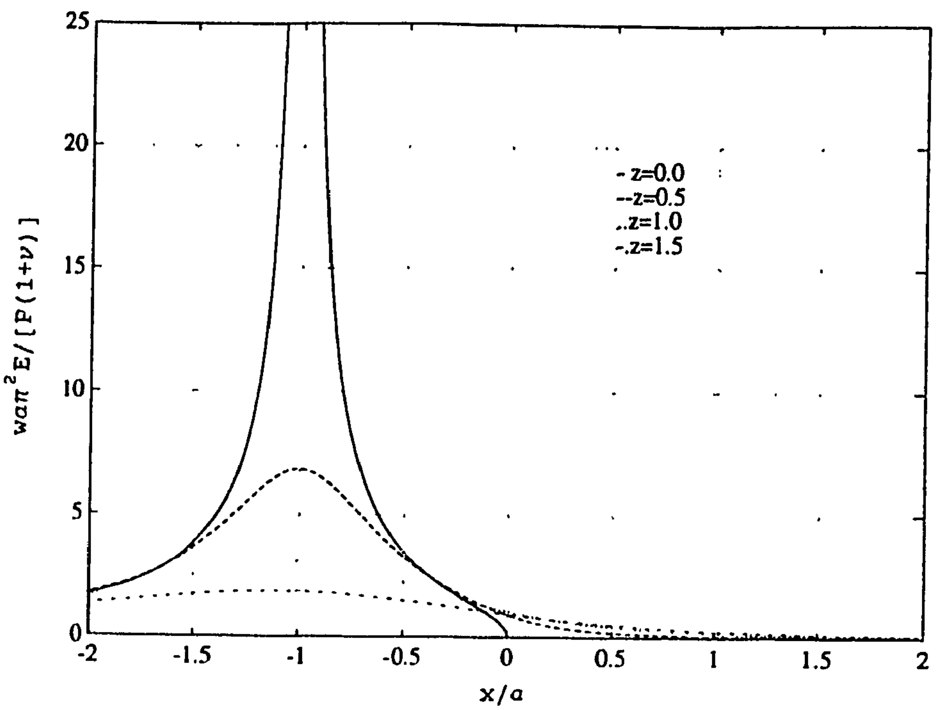

Fig.5.3 Normal displacement distribution in isotropic body for different z: (—z=0.0; --z=0.5; ··z=1.0; ·-z=1.5).

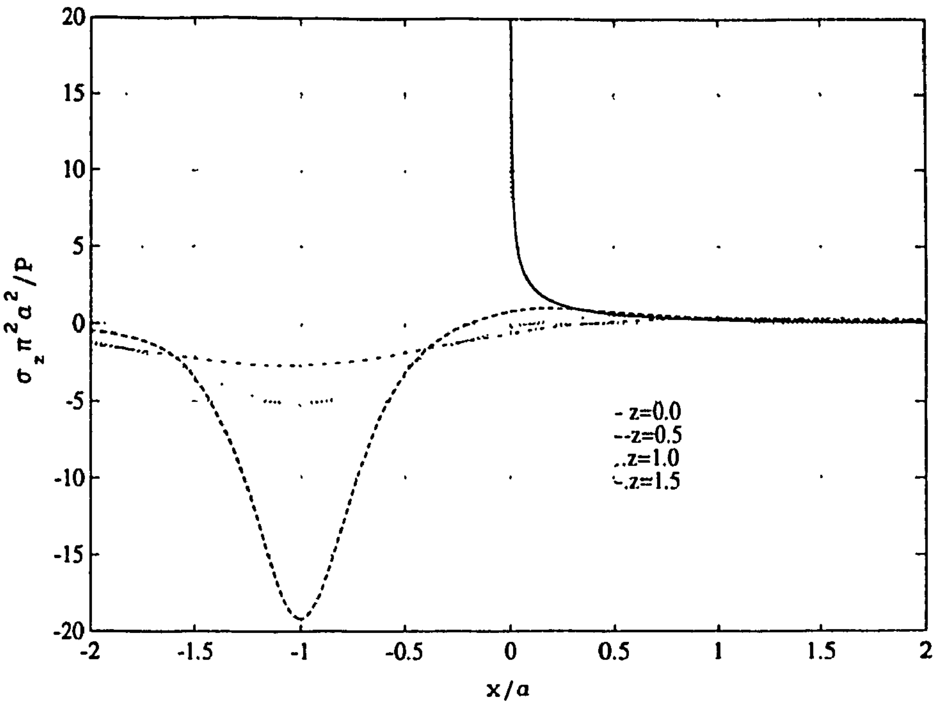

Fig.5.4 Normal stress distribution in isotropic body for different z: (—z=0.0; --z=0.5; ··z=1.0; ·-z=1.5).

The problem of a half-plane flat crack in the case of isotropy was first solved by Ufliand by means of Kontorovich-Lebedev integral transform. He outlined the general solution, but neither he nor any researcher after him gave explicit expressions in terms of elementary functions for the field of stresses and displacements. So, full comparison of all results obtained here with his cannot be done. It is shown below that the results of Ufliand, which are available for comparison, are in exact agreement with the ones obtained in this work.

According to Ufliand [22] (f.84.19), the Green's function is:

$$\Phi_2 = \frac{N}{\pi^2 \rho} \tan^{-1} \frac{2\sqrt{ar}\,\sin\frac{\vartheta}{2}}{\rho}, \tag{5.2.40}$$

where N is the applied force, $\rho=\sqrt{(x+a)^2+y^2+z^2}$, $r=\sqrt{x^2+y^2}$, $x=r\cos\vartheta$, $y=r\sin\vartheta$.

Formula (5.2.40) can be rewritten in the notations used here as follows:

$$\Phi_2 = \frac{P}{\pi^2}\frac{1}{R_0}\tan^{-1}\frac{\sqrt{2a[(x^2+z^2)^{1/2}-x]}}{R_0}. \tag{5.2.41}$$

The system of coordinates used here does not correspond to the one used by Ufliand, so y and z has to be interchanged everywhere.

The normal displacement component w has to be found from the relation

$$2Gw = -2(1-\nu)\Phi_2+z\frac{\partial\Phi_2}{\partial z}. \tag{5.2.42}$$

Here ν is Poisson's coefficient and G is shear modulus, $G=\frac{E}{2(1+\nu)}$.

Substitution of (5.2.41) in (5.2.42) will result in

$$w = -\frac{P(1+\nu)}{\pi^2 E}\left\{\left[\frac{2(1-\nu)}{R_0}+\frac{z^2}{R_0^3}\right]\tan^{-1}\frac{\sqrt{2a[(x^2+z^2)^{1/2}-x]}}{R_0}\right.$$

$$-\frac{z^2}{R_0^2+2a[(x^2+z^2)^{1/2}-x]}\left[\frac{a}{\sqrt{2a[(x^2+z^2)^{1/2}-x]}\sqrt{x^2+z^2}}\right.$$

$$- \frac{\sqrt{2a[(x^2+z^2)^{1/2}-x]}}{R_0^2}\Bigg]\Bigg\} . \qquad (5.2.43)$$

Now, it is not difficult to show that the expression (5.2.35) is in agreement with (5.2.43).

The correspondence between the complex tangential displacements given in (5.2.34) and the tangential displacements, which can be obtained by utilizing formula (84.22, Ufliand [22]) can also be shown. After some simplifications the complex tangential displacements according to Ufliand reads

$$u+iv = -\frac{1}{2G}\left[\frac{\partial\Phi_0}{\partial x} + i\frac{\partial\Phi_0}{\partial y} + \frac{1}{1-2\nu}\, z\frac{\partial}{\partial z}\left(\frac{\partial\Phi_0}{\partial x} + i\frac{\partial\Phi_0}{\partial y}\right)\right] , \qquad (5.2.44)$$

where

$$\Phi_0=(1-2\nu)\int_{\infty}^{z}\Phi_2 dz + \text{const} . \qquad (5.2.45)$$

Since the integral in (5.2.45) is not computable Ufliand resorted to differentiation of function Φ_0 with respect to x and y.

$$\frac{\partial\Phi_0}{\partial x} + i\frac{\partial\Phi_0}{\partial y} = \frac{P}{\pi^2}\left\{\frac{(1-2\nu)}{(x+a-iy)}\left[\frac{\sqrt{a/2}}{\sqrt{x-a-iy}}\ \ln\frac{\sqrt{r+x} + \sqrt{x-a-iy}}{\sqrt{r+x} - \sqrt{x-a-iy}}\right.\right.$$

$$\left.\left. - \frac{z}{R_0}\tan^{-1}\frac{\sqrt{2a(r-x)}}{R_0}\right]\right\} . \qquad (5.2.46)$$

Here $r=\sqrt{x^2+z^2}$.

Utilizing formulae (5.2.23), (2.34) and the first formula of (3.2.19), the expression for complex tangential displacements will read

$$u=u_x+iu_y=-\frac{P(1+\nu)}{\pi^2 E}\left[(1-2\nu)f_1^{\bullet}(z)+z\frac{\partial f_1^{\bullet}(z)}{\partial z}\right], \qquad (5.2.47)$$

where $f_1^{\bullet}(z)$ is defined by (5.2.29).

The structure of (5.2.47) is exactly the same, as that of (5.2.44), provided that it can be proven that

$$\Lambda\Phi_0=\frac{P}{\pi^2}(1-2\nu)f_1^{\bullet}(z)\ . \qquad (5.2.48)$$

This would mean that Ufliand's result (5.2.44) is exactly equal to the one obtained in (5.2.34). Indeed, it can be shown that the first term in (5.2.34), which is nothing else but $(P/\pi^2)(1-2\nu)f_1^{\bullet}(z)$ is equal to (5.2.46). Remembering that $x_0=-a$, $y_0=0$, it can be written

$$\frac{P}{\pi^2}(1-2\nu)f_1^{\bullet}(z)=\frac{P}{\pi^2}\frac{(1-2\nu)}{(x+a-iy)}\left\{\frac{\sqrt{-2a}}{\sqrt{x-a-iy}}\tan^{-1}\frac{\sqrt{-(x-a)+iy}}{\sqrt{(x^2+z^2)^{1/2}+x}}\right.$$

$$\left.-\frac{z}{\sqrt{(x+a)^2+y^2+z^2}}\tan^{-1}\frac{\sqrt{2a[(x^2+z^2)^{1/2}-x]}}{\sqrt{(x+a)^2+y^2+z^2}}\right\}. \qquad (5.2.49)$$

Using the relationship between $\tan^{-1}$ and ln, namely,

$$\tan^{-1}t=\frac{1}{2i}\ln\frac{1+it}{1-it}, \qquad (5.2.50)$$

it becomes an easy task to show that (5.2.49) is the same as (5.2.46).

The expression for the normal stress σ_z on the plane $z=0$ due to the above-stated pair of concentrated forces can also be presented here. By looking at the expression (5.2.38) for σ_z, it may be noted that some of the terms have an indeterminate form as $z\to 0$, namely z/h^*. In order to evaluate an indeterminate form, all these terms must be multiplied and divided by $[(x^2+z^2)^{1/2}+x]^{1/2}$. Once some obvious simplifications are made and limit is taken it yields

$$\sigma_z = \frac{P}{\pi^2}\sqrt{\frac{a}{x}}\,\frac{1}{(x+a)^2+y^2}\,, \tag{5.2.51}$$

which again corresponds to the result obtained by Ufliand [22].

Unfortunately, neither Ufliand nor any other author after him, have given any other explicit expressions which could be compared with the present results.

The field of stresses and displacements in the plane $z=0$, which looks much simpler than (5.2.23-5.2.28) and (5.2.34-5.2.39) and might be handy for practical use, is given below. In results to follow the first expression of each component corresponds to the case of $x<0$, while the second one to the case of $x>0$. Here are the results for transverse isotropy:

$$u = -PH\alpha \, \frac{c}{\bar{q}} ,$$

$$u = -\frac{2PH\alpha}{\pi} \frac{c}{\bar{q}} \tan^{-1}\left(\frac{\bar{s}^{\bullet}}{2x}\right)^{1/2}, \tag{5.2.52}$$

$$w = \frac{2PH}{\pi} \frac{1}{R} \tan^{-1}\left(\frac{2\sqrt{x_0 x}}{R}\right) ,$$

$$w = 0 , \tag{5.2.53}$$

$$\sigma_1 = 0 ,$$

$$\sigma_1 = \frac{2P}{\pi^2}\left[\frac{1}{\gamma_1\gamma_2} - 2\pi A_{66} H\alpha\right]\frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}, \tag{5.2.54}$$

$$\sigma_2 = 2PHA_{66}\alpha\left(\frac{2c}{\bar{q}^2} - \frac{c}{\bar{q}\,\bar{s}^{\bullet}}\right) ,$$

$$\sigma_2 = \frac{4PHA_{66}\alpha}{\pi}\left\{\left(\frac{2c}{\bar{q}^2} - \frac{c}{\bar{q}\,\bar{s}^{\bullet}}\right)\tan^{-1}\left(\frac{\bar{s}^{\bullet}}{2x}\right)^{1/2} + \frac{2\sqrt{-x_0 x}}{R^2\bar{s}^{\bullet}} + \frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}\right\} , \tag{5.2.55}$$

$$\sigma_z = -P\delta(x-x_0)\delta(y-y_0) ,$$

$$\sigma_z = \frac{P}{\pi^2}\frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}, \tag{5.2.56}$$

$$\tau_z = 0 , \qquad \text{for} \quad -\infty<x<\infty . \tag{5.2.57}$$

The isotropic case differs only by elastic constants, namely,

$$u = -\frac{P(1+\nu)(1-2\nu)}{2\pi E}\frac{c}{\bar{q}} ,$$

$$u = -\frac{P(1+\nu)(1-2\nu)}{\pi^2 E}\frac{c}{\bar{q}}\tan^{-1}\left(\frac{\bar{s}^{\bullet}}{2x}\right)^{1/2}, \tag{5.2.58}$$

$$w = \frac{2P(1-\nu^2)}{\pi^2 E} \frac{1}{R} \tan^{-1}\left(\frac{2\sqrt{x_0 x}}{R}\right) ,$$

$$w = 0 , \tag{5.2.59}$$

$$\sigma_1 = 0$$

$$\sigma_1 = \frac{P(1+2\nu)}{\pi^2} \frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}, \tag{5.2.60}$$

$$\sigma_2 = \frac{P(1-2\nu)}{2\pi}\left(\frac{2c}{\bar{q}^2} - \frac{c}{\bar{q}\,\bar{s}^*}\right)$$

$$\sigma_2 = \frac{P(1-2\nu)}{\pi^2}\left\{\left(\frac{2c}{\bar{q}^2} - \frac{c}{\bar{q}\,\bar{s}^*}\right)\tan^{-1}\left(\frac{\bar{s}^*}{2x}\right)^{1/2} + \frac{2\sqrt{-x_0 x}}{R^2\bar{s}^*} + \frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}\right\} ,$$

$$\tag{5.2.61}$$

$$\sigma_z = -P\delta(x-x_0)\delta(y-y_0) ,$$

$$\sigma_z = \frac{P}{\pi^2} \frac{1}{R^2}\left(-\frac{x_0}{x}\right)^{1/2}, \tag{5.2.62}$$

$$\tau_z = 0 , \qquad \text{for} \quad -\infty<x<\infty . \tag{5.2.63}$$

In formulae (5.2.52-5.2.63) $R^2=(x-x_0)^2+(y-y_0)^2$, and the rest of the notations are defined as before.

The opening mode SIF can be obtained by using expression defined in (2.43). Substitution of (5.2.62) in (2.43) will result in

$$K_1 = \frac{P}{\pi^2} \frac{\sqrt{-x_0}}{x_0^2+(y-y_0)^2} , \tag{5.2.64}$$

It is of interest to note that the same result as in (5.2.64) will be obtained by using the alternative expression introduced in (2.46), which in the case of half-plane crack transforms into the following

$$K_1 = \frac{E}{4(1-\nu^2)} \lim_{x\to 0} \frac{w}{\sqrt{-x}} \quad . \tag{5.2.65}$$

It is easy to check that the substitution of (5.2.59) in (5.2.65) gives the same result as in (5.2.64). In Fig.5.5 is presented the formula (5.2.64) in graphical form.

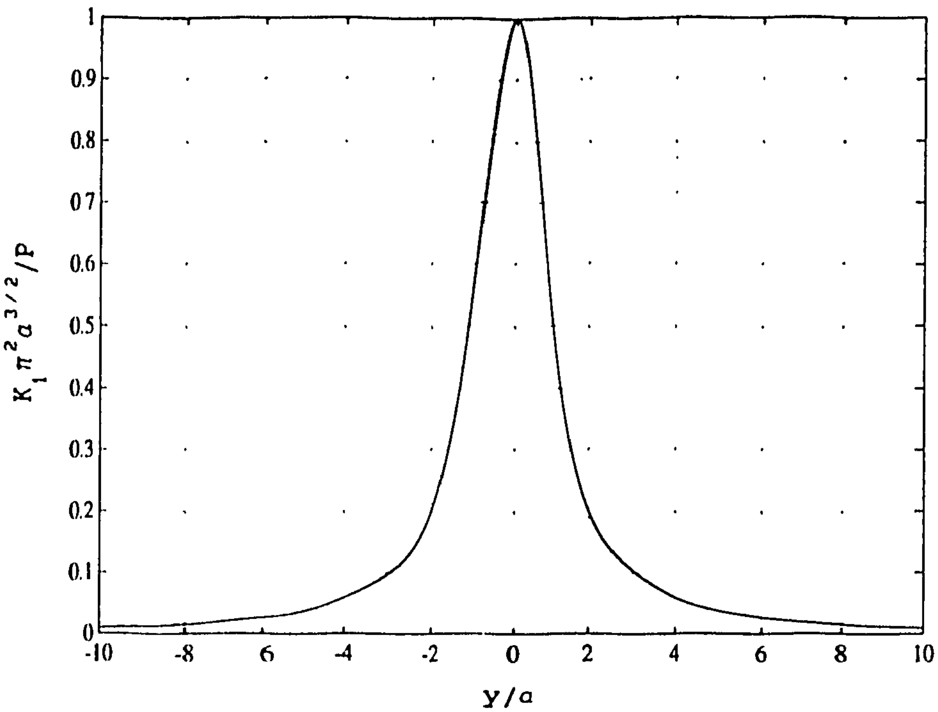

Fig.5.5 Variation of SIF K_1 along crack border.

5.3 HALF-PLANE CRACK UNDER TANGENTIAL LOAD: A COMPLETE SOLUTION

The principal idea for the solution of this problem is the same as in the previous case. The only difference is that here the results from the solution of penny-shaped crack under tangential load must be used.

5.3.1 SOLUTION FOR A CIRCULAR CRACK

Consider a penny-shaped crack subjected to the action of two equal concentrated forces $T=T_x+iT_y$ applied in opposite directions in the xy plane at the points $(\rho_0,\phi_0,0^{\pm})$, $\rho_0<a$. A complete solution for the field of displacements and stresses in elementary functions, is (Fabrikant [16]):

$$u=\frac{H\gamma_1\gamma_2}{\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[f_2(z_k)+\frac{G_2}{G_1}\bar{f}_7(z_k)\right]T+\left[f_{16}(z_k)+\frac{G_2}{G_1}f_8(z_k)\right]\bar{T}\right\}$$

$$+\frac{\beta}{\pi}\left\{\left[f_2(z_3)-\frac{G_2}{G_1}\bar{f}_7(z_3)\right]T+\left[f_{16}(z_3)-\frac{G_2}{G_1}f_8(z_3)\right]\bar{T}\right\}, \qquad (5.3.1)$$

$$w=\frac{2}{\pi}H\gamma_1\gamma_2\mathcal{R}e\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left[\bar{f}_1(z_k)+\frac{G_2}{G_1}\bar{f}_9(z_k)\right]T, \qquad (5.3.2)$$

$$\sigma_1=\mathcal{R}e\left\{\frac{2\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k-1}\left[\frac{1}{\gamma_3^2(m_k+1)}-\frac{1}{\gamma_k^2}\right]\left[\bar{f}_5(z_k)+\frac{G_2}{G_1}\bar{f}_{10}(z_k)\right]\right\}, \qquad (5.3.3)$$

$$\sigma_2=-\frac{2}{\pi}A_{66}H\gamma_1\gamma_2\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[f_5(z_k)+\frac{G_2}{G_1}\bar{f}_{13}(z_k)\right]T\right.$$

$$\left.+\left[f_{11}(z_k)+\frac{G_2}{G_1}f_{12}(z_k)\right]\bar{T}\right\}-\frac{1}{\pi^2\gamma_3}\left\{\left[-f_5(z_3)+\frac{G_2}{G_1}\bar{f}_{13}(z_3)\right]T\right.$$

$$+\left[f_{11}(z_3)-\frac{G_2}{G_1}\,f_{12}(z_3)\right]\overline{T}\Bigg\}\ , \tag{5.3.4}$$

$$\sigma_z=\mathcal{R}e\left\{\frac{\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left[\overline{f}_5(z_k)+\frac{G_2}{G_1}\,\overline{f}_{10}(z_k)\right]T\right\}\ , \tag{5.3.5}$$

$$\tau_z=\frac{\gamma_1\gamma_2}{2\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}\frac{(-1)^k}{\gamma_k}\Bigg\{\left[f_3(z_k)+\frac{G_2}{G_1}\,\overline{f}_{14}(z_k)\right]T$$

$$+\left[-f_4(z_k)+\frac{G_2}{G_1}\,f_{15}(z_k)\right]\overline{T}\Bigg\}+\frac{1}{2\pi^2}\Bigg\{\left[f_3(z_3)-\frac{G_2}{G_1}\,\overline{f}_{14}(z_3)\right]T$$

$$+\left[f_4(z_3)+\frac{G_2}{G_1}\,f_{15}(z_3)\right]\overline{T}\Bigg\}\ . \tag{5.3.6}$$

Here $\mathcal{R}e$ indicates the real part, the elastic constants are defined by (2.36), and

$$f_1(z)=\frac{1}{\overline{q}}\left[\frac{(a^2-\rho_0^2)^{1/2}}{\overline{s}}\tan^{-1}\frac{\overline{s}}{(\ell_2^2-a^2)^{1/2}}-\frac{z}{R_0}\tan^{-1}\frac{h}{R_0}\right]\ , \tag{5.3.7}$$

$$f_2(z)=\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\ , \tag{5.3.8}$$

$$f_3(z)=-\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)+\frac{h}{z(R_0^2+h^2)}\left[\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}-\frac{z^2}{R_0^2}\right]\ , \tag{5.3.9}$$

$$f_4(z)=\frac{(a^2-\rho_0^2)^{1/2}}{\overline{q}\ \overline{s}}\left(\frac{\rho_0e^{i\phi_0}}{\overline{s}^2}-\frac{2}{\overline{q}}\right)\tan^{-1}\left(\frac{\overline{s}}{(\ell_2^2-a^2)^{1/2}}\right)$$

$$+\ \frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)\ -\ \frac{(a^2-\rho_0^2)^{1/2}(\ell_2^2-a^2)^{1/2}\rho_0 e^{i\phi_0}}{\bar{q}\ \bar{s}^2[\ell_2^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]}$$

$$+\ \frac{zh}{R_0^2+h^2}\left[\frac{q}{\bar{q}R_0^2}\ -\ \frac{\rho^2 e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\ell_2^2-\rho^2)}\right]\ , \tag{5.3.10}$$

$$f_5(z)=-\left\{\frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)\ +\ \frac{h}{R_0^2+h^2}\left[\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\ +\ \frac{\rho e^{i\phi}-\rho_0 e^{i\phi_0}}{R_0^2}\right]\right\}\ , \tag{5.3.11}$$

$$f_7(z)=\frac{ha^2}{s^2}\left[\frac{3}{s^2}\ -\ \frac{t}{\ell_2^2-a^2t}\ -\ \frac{3(\ell_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)\right]\ , \tag{5.3.12}$$

$$f_8(z)=\frac{1}{\bar{q}}(a^2-\rho_0^2)^{1/2}\left\{\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\left[\tan^{-1}\left(\frac{1}{\bar{\zeta}-1}\right)^{1/2}-\tan^{-1}\left(\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\right]\right.$$

$$\left.-\ \frac{e^{i\phi}}{\rho}\left[\frac{(a^2-\ell_1^2)^{1/2}}{a}\left(1\ +\ \frac{\rho^2}{\ell_2^2-\rho\rho e^{i(\phi-\phi_0)}}\right)-1\right]\right\}\ , \tag{5.3.13}$$

$$f_9(z)=-\rho e^{i\phi}\ \frac{(a^2-\rho_0^2)^{1/2}}{a^3}\left\{\frac{1}{t}\sin^{-1}\left(\frac{a}{\ell_2}\right)\ +\ \frac{a(\ell_2^2-a^2)^{1/2}}{(1-t)(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}\right.$$

$$\left.-\ \frac{1}{t(1-t)^{3/2}}\ \tan^{-1}\left(\frac{a(1-t)^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right\}\ , \tag{5.3.14}$$

$$f_{10}(z)=-\ \frac{h\rho e^{i\phi}(3\ell_2^2-a^2t)}{(\ell_2^2-\ell_1^2)(\ell_2^2-a^2t)^2}\ , \tag{5.3.15}$$

$$f_{11}(z)=\frac{1}{\bar{q}}\left\{\frac{3R_0^4+6R_0^2z^2-z^4}{R_0^3\bar{q}^2}\tan^{-1}\left(\frac{h}{R_0}\right)-(a^2-\rho_0^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{8}{\bar{q}^2}\ -\ \frac{4\rho_0 e^{i\phi_0}}{\bar{s}^2\bar{q}}\right.\right.\right.$$

$$+\frac{3\rho_0^2 e^{2i\phi_0}}{\bar{s}^4}\Bigg)\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)-\frac{e^{i\phi}}{\rho}\left(\frac{2e^{i\phi}}{\rho}+\frac{3}{\bar{q}}\right)$$

$$-\frac{3(\bar{\zeta}-1)^{1/2}}{\bar{q}^2}\left(\tan^{-1}\frac{1}{(\bar{\zeta}-1)^{1/2}}-\tan^{-1}\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\Bigg]$$

$$+\frac{ha^2e^{i\phi}}{\rho\bar{s}^2}\left[\frac{2\rho_0 e^{i\phi_0}}{\bar{s}^2}-\frac{2e^{i\phi}}{\rho}-\frac{2}{\bar{q}}+\left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}-\frac{2}{\bar{q}}\right)\frac{(\ell_2^2-a^2)\bar{t}}{\ell_2^2-a^2\bar{t}}\right]$$

$$-\frac{h}{R_0^2+h^2}\left[\frac{\bar{q}\rho e^{3i\phi}}{\ell_2^2-\ell_1^2}+\frac{e^{i\phi}(\ell_2^2-\rho^2)}{\rho\bar{q}}-\frac{z^2 q}{R_0^2\bar{q}}+2e^{2i\phi}\right]\Bigg\}\,, \tag{5.3.16}$$

$$f_{12}(z)=\frac{1}{\bar{q}}(a^2-\rho_0^2)^{1/2}\Bigg\{\frac{3(\bar{\zeta}-1)^{1/2}}{\bar{q}^3}\left[\tan^{-1}\left(\frac{1}{\bar{\zeta}-1}\right)^{1/2}-\tan^{-1}\left(\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\right]$$

$$-\frac{e^{2i\phi}(a^2-\ell_1^2)^{1/2}}{a(\ell_2^2-\ell_1^2)}\left[\frac{\ell_2^2+\rho^2}{\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)}}+\frac{2\rho^2(\ell_2^2-a^2)}{(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})^2}+1\right]$$

$$+\frac{e^{i\phi}}{\rho}\left[\frac{3}{\bar{q}}+\frac{2e^{i\phi}}{\rho}-\frac{(a^2-\ell_1^2)^{1/2}}{a}\left(\frac{\ell_2^2+2\rho^2}{\bar{q}(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}+2\left(\frac{1}{\bar{q}}+\frac{e^{i\phi}}{\rho}\right)\right)\right]\Bigg\}, \tag{5.3.17}$$

$$\bar{f}_{13}(z)=-h\Bigg\{\frac{a^2}{\bar{s}^2}\,\rho_0 e^{i\phi_0}\left[\frac{15(\ell_2^2-a^2)^{1/2}}{\bar{s}^5}\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)-\frac{15}{\bar{s}^4}\right.$$

$$\left.+\frac{5}{\bar{s}^2(\ell_2^2-a^2\bar{t})}+\frac{2\bar{t}}{(\ell_2^2-a^2\bar{t})^2}\right]+\frac{\rho e^{i\phi}(3\ell_2^2-a^2\bar{t})}{(\ell_2^2-\ell_1^2)(\ell_2^2-a^2\bar{t})^2}\Bigg\}\,, \tag{5.3.18}$$

$$f_{14}(z)=\frac{(a^2-\rho_0^2)^{1/2}}{a^3(1-t)}\left\{\frac{a(\ell_2^2-a^2)^{1/2}}{(\ell_2^2-\ell_1^2)(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}\left[\frac{3(\ell_2^2-\ell_1^2 t)}{1-t}\right.\right.$$

$$+ \frac{\rho\rho_0 e^{i(\phi-\phi_0)}(2\ell_2^2+\ell_1^2 t-3\rho^2)}{\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)}}\Bigg] - \frac{3}{(1-t)^{3/2}} \tan^{-1}\left(\frac{a(1-t)^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\Bigg\} ,$$

(5.3.19)

$$f_{15}(z)=\frac{\rho^2 e^{2i\phi}[(a^2-\rho_0^2)(\ell_2^2-a^2)]^{1/2}(3\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}{\ell_2^2(\ell_2^2-\ell_1^2)(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})^2}, \qquad (5.3.20)$$

$$f_{16}(z)=\frac{1}{\bar{q}}\Bigg\{\frac{R_0^2+z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h}{R_0}\right)+(a^2-\rho_0^2)^{1/2}\Bigg[\frac{z}{\bar{s}}\Bigg(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}$$

$$-\frac{2}{\bar{q}}\Bigg)\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right) + \frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\Bigg(\tan^{-1}\frac{1}{(\bar{\zeta}-1)^{1/2}}$$

$$-\tan^{-1}\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\Bigg)+\frac{e^{i\phi}}{\rho}\Bigg]-\frac{e^{i\phi}ha^2}{\rho\bar{s}^2}\Bigg\} . \qquad (5.3.21)$$

Here the notations defined in (5.2.12-5.2.14) were used along with the following

$$t=\frac{\rho\rho_0}{a^2}e^{i(\phi-\phi_0)} , \qquad \zeta=\frac{\rho}{\rho_0}e^{i(\phi-\phi_0)} . \qquad (5.3.22)$$

5.3.2 SOLUTION FOR A HALF-PLANE CRACK

Let two equal concentrated forces $T=T_x+iT_y$ being applied to the crack faces in opposite directions at the points $(x_0,y_0,0^{\pm})$, $x_0<0$ as shown in Fig.5.6.

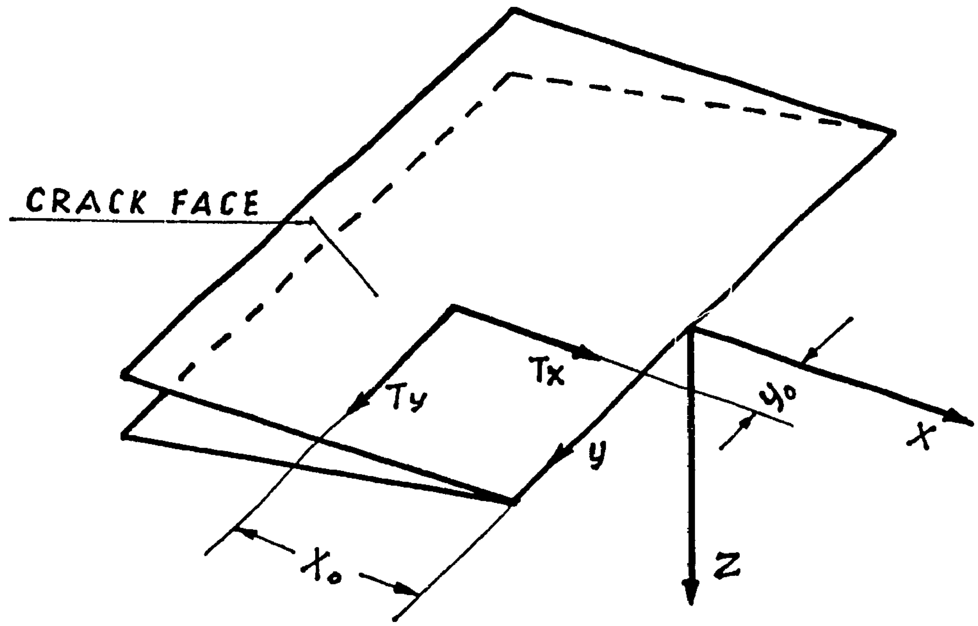

Fig.5.6 A half-plane crack under shear load.

By using the same arguments as in the previous problem the general solution for the field of stresses and displacements in a transversely isotropic elastic space, weakened by a semi-infinite flat crack occupying the region x≤0, will take the form

$$u=\frac{H\gamma_1\gamma_2}{\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[f_2^*(z_k)+\frac{G_2}{G_1}\,\overline{f}_7^*(z_k)\right]T+\left[f_{16}^*(z_k)+\frac{G_2}{G_1}\,f_8^*(z_k)\right]\overline{T}\right\}$$

$$+\frac{\beta}{\pi}\left\{\left[f_2^*(z_3)-\frac{G_2}{G_1}\,\overline{f}_7^*(z_3)\right]T+\left[f_{16}^*(z_3)-\frac{G_2}{G_1}\,f_8^*(z_3)\right]\overline{T}\right\}\ , \tag{5.3.23}$$

$$w=\frac{2}{\pi}H\gamma_1\gamma_2\mathcal{R}e\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left[\overline{f}_1^*(z_k)\ +\ \frac{G_2}{G_1}\,\overline{f}_9^*(z_k)\right]T\ , \tag{5.3.24}$$

$$\sigma_1=\mathcal{R}e\left\{\frac{2\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k-1}\left[\frac{1}{\gamma_3^2(m_k+1)}-\frac{1}{\gamma_k^2}\right]\left[\overline{f}_5^*(z_k)+\frac{G_2}{G_1}\,\overline{f}_{10}^*(z_k)\right]T\right\}, \tag{5.3.25}$$

$$\sigma_2=-\frac{2}{\pi}A_{66}H\gamma_1\gamma_2\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\left[f_5^*(z_k)+\frac{G_2}{G_1}\,\bar{f}_{13}^*(z_k)\right]T\right.$$

$$\left.+\left[f_{11}^*(z_k)+\frac{G_2}{G_1}\,f_{12}^*(z_k)\right]\bar{T}\right\}-\frac{1}{\pi^2\gamma_3}\left\{\left[-f_5^*(z_3)+\frac{G_2}{G_1}\,\bar{f}_{13}^*(z_3)\right]T\right.$$

$$\left.+\left[f_{11}^*(z_3)-\frac{G_2}{G_1}\,f_{12}^*(z_3)\right]\bar{T}\right\}\ , \tag{5.3.26}$$

$$\sigma_z=\mathcal{R}e\left\{\frac{\gamma_1\gamma_2}{\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^{k+1}\left[\bar{f}_5^*(z_k)+\frac{G_2}{G_1}\,\bar{f}_{10}^*(z_k)\right]T\right\}\ , \tag{5.3.27}$$

$$\tau_z=\frac{\gamma_1\gamma_2}{2\pi^2(\gamma_1-\gamma_2)}\sum_{k=1}^{2}\frac{(-1)^k}{\gamma_k}\left\{\left[f_3^*(z_k)+\frac{G_2}{G_1}\,\bar{f}_{14}^*(z_k)\right]T\right.$$

$$\left.+\left[-f_4^*(z_k)+\frac{G_2}{G_1}\,f_{15}^*(z_k)\right]\bar{T}\right\}+\frac{1}{2\pi^2}\left\{\left[f_3^*(z_3)-\frac{G_2}{G_1}\,\bar{f}_{14}^*(z_3)\right]T\right.$$

$$\left.+\left[f_4^*(z_3)+\frac{G_2}{G_1}\,f_{15}^*(z_3)\right]\bar{T}\right\}\ , \tag{5.3.28}$$

where functions $f_1^*(z)-f_{16}^*(z)$ are given in Appendix A5.3.

Formulae (5.3.23-5.3.28) are the main new results of this section.

5.3.3 ISOTROPIC SOLUTION

The results for isotropy can be obtained as in the previous problems by using the necessary limiting forms defined in (3.3.34) along with the isotropic limits defined in (2.34). Some additional limiting forms needed for isotropic solution are as follows

$$\lim_{\gamma_1\to\gamma_2\to 1}\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{2}(-1)^{k+1}\left[\frac{1}{\gamma_3^2(m_k+1)}-\frac{1}{\gamma_k^2}\right]f(z_k)$$

$$=(1+\nu)f(z)+\frac{z}{2}f'(z)\,, \qquad (5.3.29)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{2}\frac{(-1)^k}{\gamma_k}f(z_k)=f(z)+zf'(z)\ . \qquad (5.3.30)$$

The final results are

$$u=\frac{1+\nu}{\pi^2 E}\left\{f_{17}^*(z)T+f_{18}^*(z)\overline{T}+\frac{z}{2}\left[\left(f_{27}^*(z)+f_{28}^*(z)\right)T-\left(f_{29}^*(z)+f_{30}^*(z)\right)\overline{T}\right]\right\},$$

$$(5.3.31)$$

$$w=\frac{1+\nu}{\pi^2 E}\mathcal{R}e\left\{\left[f_{19}^*(z)-z\left(f_{31}^*(z)+f_{32}^*(z)\right)\right]T\right\}\ , \qquad (5.3.32)$$

$$\sigma_1=\frac{1}{\pi^2}\mathcal{R}e\left\{\left[f_{20}^*(z)+z\left(\overline{f}_{33}^*(z)+f_{34}^*(z)\right)\right]T\right\}\ , \qquad (5.3.33)$$

$$\sigma_2=\frac{1}{\pi^2}\left\{\left(f_{21}^*(z)+f_{22}^*(z)\right)T+\left(f_{23}^*(z)+f_{24}^*(z)\right)\overline{T}\right.$$

$$\left.+z\left[\left(f_{33}^*(z)+f_{35}^*(z)\right)T+\left(f_{36}^*(z)+f_{37}^*(z)\right)\overline{T}\right]\right\}\ , \qquad (5.3.34)$$

$$\sigma_z=-\frac{1}{\pi^2}\mathcal{R}e\left\{z\left[f_{33}^*(z)+f_{34}^*(z)\right]T\right\}\ , \qquad (5.3.35)$$

$$\tau_z=\frac{1}{\pi^2}\left\{f_{25}^*(z)T+f_{26}^*(z)\overline{T}+\frac{z}{2}\left[\left(f_{38}^*(z)+f_{39}^*(z)\right)T+\left(f_{40}^*(z)+f_{41}^*(z)\right)\overline{T}\right]\right\},$$

$$(5.3.36)$$

where functions $f_{17}^*(z)-f_{41}^*(z)$ are given in Appendix B5.3.

This completes the solution of the problem of a half-plane crack under shear load.

5.3.4 DISCUSSION AND NUMERICAL RESULTS

In order to obtain K_2 and K_3 SIF it is necessary to evaluate the values of $\tau_{zx}(x,y,0)$ and $\tau_{yz}(x,y,0)$ stress components, because

$$K_2 = \lim_{x\to 0} (2x)^{1/2}\tau_{zx} , \qquad K_3 = \lim_{x\to 0} (2x)^{1/2}\tau_{yz} . \qquad (5.3.37)$$

The obtained expression (5.3.36) for τ_z component provides these values, since $\tau_z=\tau_{zx}+i\tau_{yz}$. Moreover, it should be noted that the notation for the tangential load used here was in compact complex form, namely $T=T_x+iT_y$, where T_x is the shear load normal to crack edge and T_y is the shear load parallel to crack edge. Hence for the case of the shear load normal to crack edge and applied at the points $(-a,0,0^{\pm})$ it gives

$$\tau_z(x,y,0) = \frac{T_x}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\left[\frac{1}{(x+a)^2+y^2} + \frac{2\nu}{2-\nu}\,\frac{1}{[(x+a)-iy]^2}\right] . \qquad (5.3.38)$$

The separation of real and imaginary parts in (5.3.38) will result in the stress components which are required for determination of SIF, namely,

$$\tau_{zx}(x,y,0) = \frac{T_x}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\frac{1}{(x+a)^2+y^2}\left[1 + \frac{2\nu}{2-\nu}\,\frac{(x+a)^2-y^2}{(x+a)^2+y^2}\right] , \qquad (5.3.39)$$

$$\tau_{yz}(x,y,0)=\frac{T}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\frac{4\nu}{2-\nu}\frac{y(x+a)}{[(x+a)^2+y^2]^2}\quad . \tag{5.3.40}$$

Thus, according to (5.3.37), (5.3.39) and (5.3.40) the result will be:

$$K_2=\frac{\sqrt{2}\ T_x}{\pi^2 a^{3/2}}\frac{1}{1+d^2}\left[1+\left(\frac{2\nu}{2-\nu}\right)\frac{1-d^2}{1+d^2}\right], \tag{5.3.41}$$

$$K_3=\frac{\sqrt{2}\ T_x}{\pi^2 a^{3/2}}\frac{4\nu}{2-\nu}\frac{d}{(1+d^2)^2}, \tag{5.3.42}$$

where $d=y/a$.

For the case of the shear load parallel to crack edge and applied at the points $(-a,0,0^{\pm})$ formula (5.3.36) gives

$$\tau_z(x,y,0)=\frac{iT_y}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\left[\frac{1}{(x+a)^2+y^2}-\frac{2\nu}{2-\nu}\frac{1}{[(x+a)-iy]^2}\right]. \tag{5.3.43}$$

The separation of real and imaginary parts in (5.3.43) results in

$$\tau_{zx}(x,y,0)=\frac{T_y}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\frac{4\nu}{2-\nu}\frac{y(x+a)}{[(x+a)^2+y^2]^2}, \tag{5.3.44}$$

$$\tau_{yz}(x,y,0)=\frac{T_y}{\pi^2}\left(\frac{a}{x}\right)^{1/2}\frac{(2-3\nu)(x+a)^2+(2+\nu)y^2}{(2-\nu)[(x+a)^2+y^2]^2}\quad . \tag{5.3.45}$$

And the results for SIF are:

$$K_2 = \frac{\sqrt{2}\ T_y}{\pi^2 a^{3/2}} \frac{4\nu}{2-\nu} \frac{d}{(1+d^2)^2}\ , \tag{5.3.46}$$

$$K_3 = \frac{\sqrt{2}\ T_y}{\pi^2 a^{3/2}} \frac{1}{1+d^2}\left[1 - \left(\frac{2\nu}{2-\nu}\right)\frac{1-d^2}{1+d^2}\right]\ . \tag{5.3.47}$$

The formulae (5.3.39-5.3.42) and (5.3.44-5.3.47) are in agreement with the results presented in Kassir and Sih [15]. It is noted that formulae (5.69b), (5.108a), (5.108b) in Kassir and Sih [15] contain some misprints. Graphical representations of K_2 and K_3 SIF for different values of Poisson ratio are given in Figs.5.7-5.9.

It is interesting to note that with an increase of Poisson ratio the maximum of the curve in Fig.5.9 switches to a minimum. By calculation of the maximum of the function in formula (5.3.47) it may easily be concluded that when $\nu<\frac{2}{7}$ we have only one maximum at d=0, and when $\nu>\frac{2}{7}$ this maximum turns into a minimum and two other maxima appear. Their split increases with ν, namely,

$$d = \pm\sqrt{\frac{7\nu-2}{\nu+2}}\ . \tag{5.3.48}$$

This phenomenon is illustrated in Fig.5.9, where for a transitional value of Poisson ratio, namely $\nu=\frac{2}{7}$, the curve has generated an expected plateau-like maximum.

Another feature depicted in Figs.5.7 and 5.9 is that at d=±1 the values of SIF are independent of ν and equal to

0.5. The justification for it can be found in formulae (5.3.41) and (5.3.47).

From formulae (5.3.42) and (5.3.46), which are given in Fig.5.8, it becomes quite evident that K_3 SIF due to shear T_x normal to the crack edge is the same as K_2 SIF due to shear T_y parallel to the crack edge, regardless on value of Poisson's ratio. That is, some kind of reciprocity holds. However due to the effect of the same Poisson's ratio the SIF attains double peaks at d=±0.577 and vanishes at d=0.

The phenomenon of reciprocity works for the other coupled cases illustrated in Figs.5.7 and 5.9 only when Poisson ration ν=0.

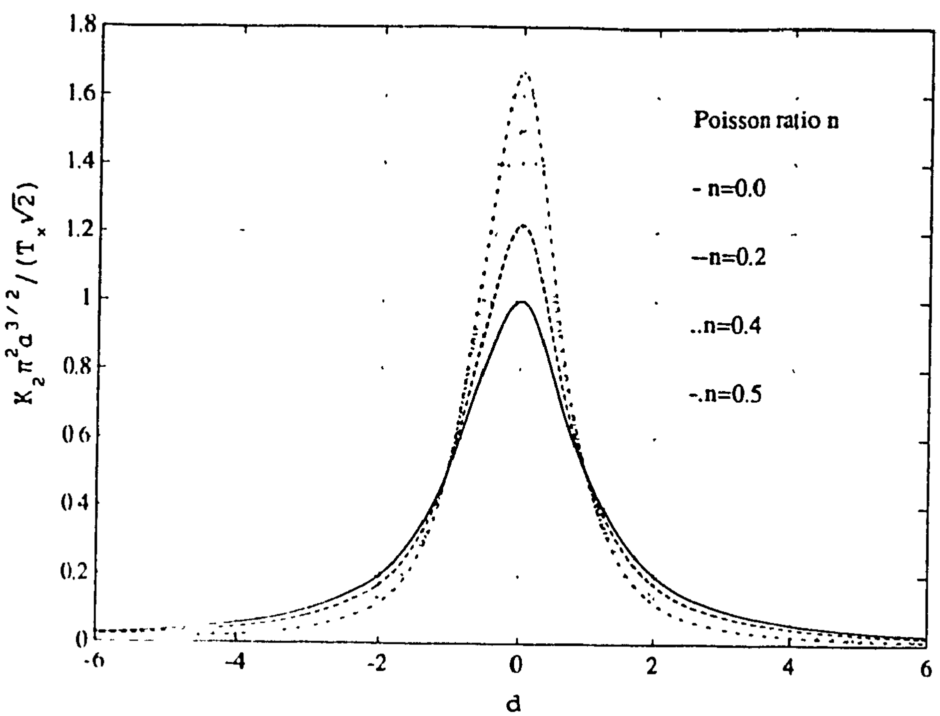

Fig.5.7 Variation of SIF K_2 along crack border due to shear T_x for different n: (—n=0.0; --n=0.2; ··n=0.4; ·-n=0.5).

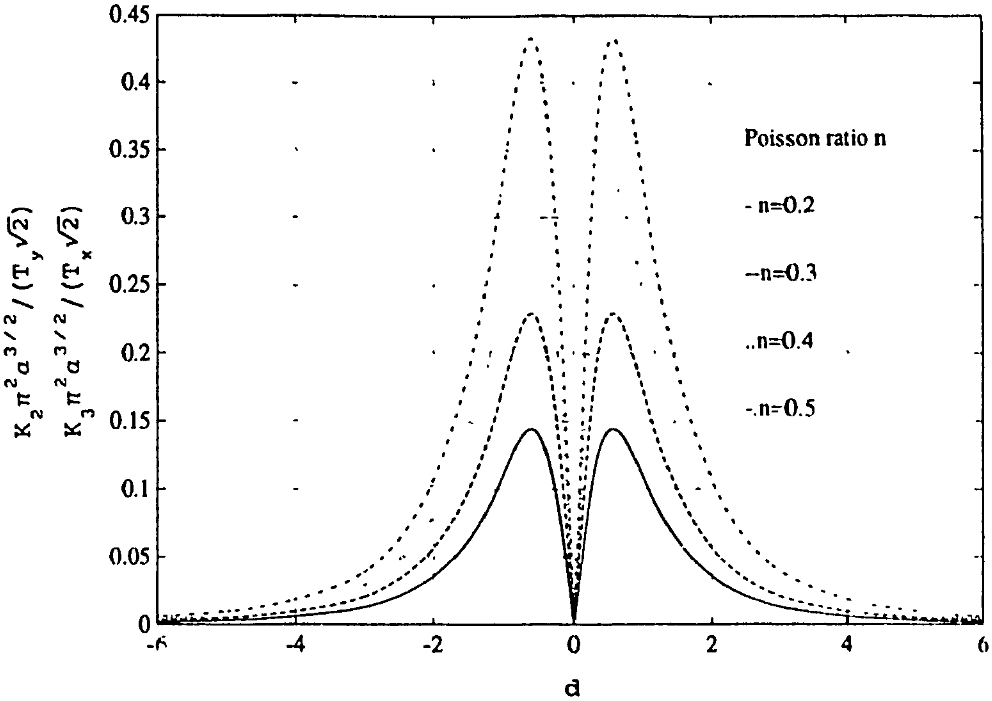

Fig.5.8 Variation of SIF K_3 along crack border due to shear T_x, and variation of SIF K_2 along crack border due to shear T_y for different n: (—n=0.2; --n=0.3; ··n=0.4; ·-n=0.5).

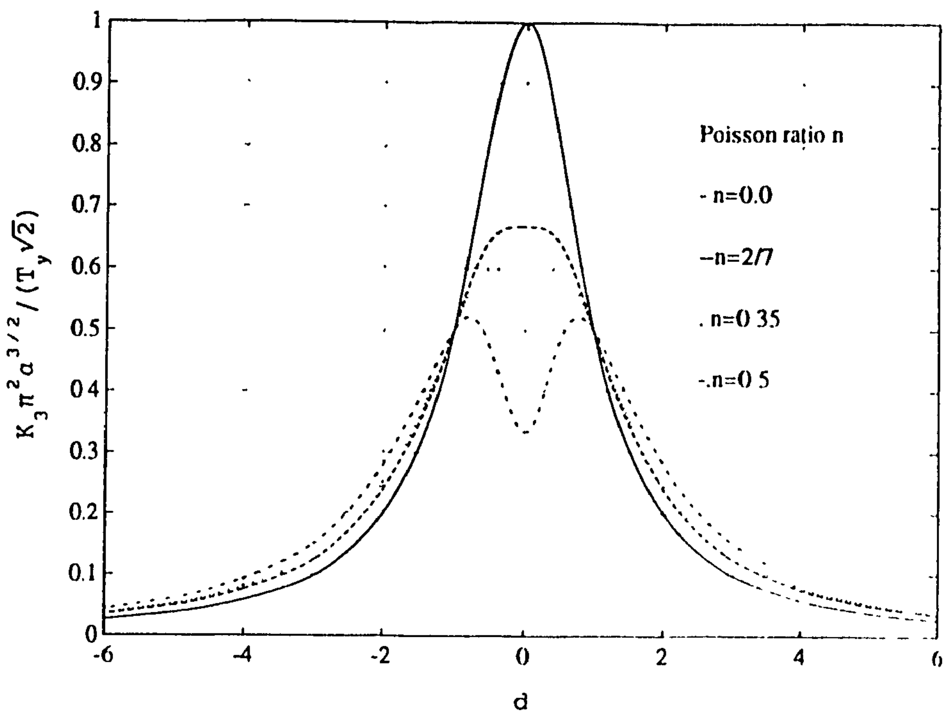

Fig.5.9 Variation of SIF K_3 along crack border due to shear T_y for different n: (—n=0.0; --n=2/7; ··n=0.35; ·-n=0.5).

The results obtained have shown that the theory developed in (Fabrikant [16]) can be used in its limiting case for solving the half-plane crack problem. The available analysis of comparison has indicated that the results are in perfect agreement. It is of interest to see whether a new direct method for solving this type of problems can be developed.

A separate mathematical approach can in fact be developed for solving relevant mixed BVP of potential theory. This new method is based on a different integral representation for the reciprocal of the distance between two points and will be presented in the coming section.

5.4 NEW METHOD FOR SOLVING MIXED B.V.P. OF POTENTIAL THEORY, WITH APPLICATION TO HALF-PLANE CONTACT AND CRACK PROBLEMS

A new method is presented for exact solution in closed form of a mixed BVP of potential theory when the potential is prescribed on one half-plane (say, $y \geq 0$, $z=0$) and the charge density distribution is prescribed on the other half-plane ($y<0$, $z=0$). The method is based on a new integral representation for the reciprocal of the distance between two points. Its substitution into the simple layer distribution leads to an integral equation which can be solved exactly, with no integral transforms or series expansions involved. The general results are applied to solving a relevant punch problem and a half-plane crack problem. A complete solution for the fields of stresses and displacements is given in closed form and in terms of elementary functions. The work presented in this section follows the paper by Fabrikant and Karapetian [38].

5.4.1 INTRODUCTION AND DESCRIPTION OF THE NEW METHOD

Consider the following problem: find a function $V(x,y,z)$, harmonic in the space and vanishing at infinity, subject to the boundary conditions on the plane $z=0$:

$$V(x,y,0)=\omega(x,y) \qquad \text{for } y\geq 0\ ,\ -\infty<x<\infty\ ,$$
$$\left.\frac{\partial V}{\partial z}\right|_{z=0}=-2\pi\sigma(x,y) \qquad \text{for } y<0\ ,\ -\infty<x<\infty\ . \qquad (5.4.1)$$

As was mentioned before, this kind of problem has been solved by Ufliand [22], who used the integral transform technique.

A new method proposed here is based on the following representation for the reciprocal of the distance between two points $N(x,y)$ and $N_0(x_0,y_0)$, namely,

$$\frac{1}{R}=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=\frac{2}{\pi}\int_{\max(y,y_0)}^{\infty}\frac{\lambda^*(2u-y-y_0,x-x_0)\,du}{[(u-y)(u-y_0)]^{1/2}}\ . \qquad (5.4.2)$$

Here

$$\lambda^*(a,b)=\frac{a}{a^2+b^2}, \tag{5.4.3}$$

which stems from the work of Rubin [39] on fractional integrals.

Introduce a new variable

$$\eta^*=2\sqrt{u-y}\sqrt{u-y_0}\ . \tag{5.4.4}$$

It is easy to show that

$$\lambda^*(2u-y-y_0,x-x_0)=\frac{2u-y-y_0}{(2u-y-y_0)^2+(x-x_0)^2}=\frac{\eta^*}{2[R^2+(\eta^*)^2]}\frac{d\eta^*}{du}\ . \tag{5.4.5}$$

Substitution of (5.4.5) into (5.4.2) yields

$$\frac{1}{R}=\frac{2}{\pi}\int_0^{\infty}\frac{d\eta^*}{R^2+(\eta^*)^2}, \tag{5.4.6}$$

thus proving (5.4.2).

Everywhere in this section parameters with asterisk are used for two purposes: 1) to emphasize the analogy between the method used by Fabrikant [16,17] for the geometry of a circle; 2) to show that in the case of a half-plane there are certain differences as well.

The integral representation (5.4.2) is convenient for solving problems in the upper half-plane $y\geq0$. In the half-plane $y<0$, the following equivalent can be established:

$$\frac{1}{R} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \frac{2}{\pi}\int_{-\infty}^{\min(y,y_0)} \frac{\lambda^*(y+y_0-2u,x-x_0)\,du}{[(y-u)(y_0-u)]^{1/2}} . \qquad (5.4.7)$$

Now it is necessary to establish an integral representation for the reciprocal of the distance R_0 between $M(x,y,z)$ and $N_0(x_0,y_0)$, namely,

$$\frac{1}{R_0} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}} . \qquad (5.4.8)$$

Since the quantities y and y_0 in (5.4.2) are arbitrary they can be formally replaced by, say, $\ell_1^*(y_0,y,z)$ and $\ell_2^*(y_0,y,z)$, such that

$$\frac{1}{R_0} = \frac{1}{\sqrt{(x-x_0)^2+[\ell_1^*(y_0,y,z)-\ell_2^*(y_0,y,z)]^2}} , \qquad (5.4.9)$$

which requires that

$$\ell_2^*(y_0,y,z)-\ell_1^*(y_0,y,z)= \sqrt{(y_0-y)^2+z^2} . \qquad (5.4.10)$$

According to (5.4.2) and (5.4.7), it is necessary to define $\ell_1^*(y_0,y,z)$ and $\ell_2^*(y_0,y,z)$ in such a way that

$$\ell_1^*(y_0,y,0) = \min(y,y_0), \quad \ell_2^*(y_0,y,0) = \max(y,y_0). \qquad (5.4.11)$$

It can also be required that λ^* in (5.4.5) stayed invariant, namely, $\ell_1^*(y_0,y,z)+\ell_2^*(y_0,y,z)=y_0+y$. All these requirements can be satisfied by putting

$$\ell_1^*(y_0) \equiv \ell_1^*(y_0,y,z) = \frac{1}{2}\left[y+y_0-\sqrt{(y-y_0)^2+z^2}\right] ,$$

$$\ell_2^*(y_0) \equiv \ell_2^*(y_0,y,z) = \frac{1}{2}\left[y+y_0+\sqrt{(y-y_0)^2+z^2}\right] . \qquad (5.4.12)$$

One can also define function $g^*(u)$, which is inverse to both $\ell_1^*(y_0)$ and $\ell_2^*(y_0)$, in such a way that $g^*[\ell_1^*(y_0)]=g^*[\ell_2^*(y_0)]=y_0$, namely,

$$g^*(u) = u - \frac{z^2}{4(u-y)} . \qquad (5.4.13)$$

Hereafter ℓ_1^* is understood as $\ell_1^*(0,y,z)$ and $\ell_1^*(y_0)$ denotes $\ell_1^*(y_0,y,z)$, similarly for ℓ_2^*, as they are defined in (5.4.12).

As before, it can be easily verified that

$$\frac{1}{R_0} = \frac{2}{\pi}\int_{\ell_2^*(y_0)}^{\infty}\frac{\lambda^*(2u-y-y_0,x-x_0)\,du}{[(u-\ell_1^*(y_0))(u-\ell_2^*(y_0))]^{1/2}}$$

$$= \frac{2}{\pi}\int_{-\infty}^{\ell_1^*(y_0)}\frac{\lambda^*(y+y_0-2u,x-x_0)\,du}{[(\ell_1^*(y_0)-u)(\ell_2^*(y_0)-u)]^{1/2}} . \qquad (5.4.14)$$

In fact, the integrals in (5.4.14) can be computed as indefinite, namely,

$$\int\frac{\lambda^*(2u-y-y_0,x-x_0)\,du}{[(u-\ell_1^*(y_0))(u-\ell_2^*(y_0))]^{1/2}} = \frac{1}{R_0}\tan^{-1}\left(\frac{h^*(u)}{R_0}\right) , \qquad (5.4.15)$$

where

$$h^*(u) = 2\sqrt{u-\ell_1^*(y_0)}\sqrt{u-\ell_2^*(y_0)} \quad . \tag{5.4.16}$$

The integral representations (5.4.2), (5.4.7) and (5.4.14) make it possible to formulate and solve various problems, as shown below.

5.4.2 PROBLEM OF THE FIRST TYPE

Let the boundary conditions on the plane $z=0$ be

$$V(x,y,0)=\upsilon(x,y) \quad , \qquad \text{for } y\geq 0 \ , \ -\infty<x<\infty \ ,$$
$$\frac{\partial V}{\partial z}\Big|_{z=0}=0 \quad , \qquad \text{for } y<0 \ , \ -\infty<x<\infty \ . \tag{5.4.17}$$

The question is to find the charge density distribution σ for $y>0$, and the potential $V(x,y,z)$ in the whole space.

The potential may be presented as a simple layer

$$V(x,y,z) = \int_{-\infty}^{\infty} dx_0 \int_{0}^{\infty} dy_0 \frac{\sigma(x_0,y_0)}{R_0} \ . \tag{5.4.18}$$

Substitution of the first representation (5.4.14) in (5.4.18) yields

$$V(x,y,z) = 2\int_{\ell_2^*}^{\infty} \frac{du}{(u-y)^{1/2}} \int_{0}^{g^*(u)} \frac{\mathcal{L}^*(2u-y-y_0)\sigma(x,y_0)\,dy_0}{[g^*(u)-y_0]^{1/2}} \ . \tag{5.4.19}$$

Here the $\mathcal{L}^*$-operator is introduced as

$$\mathcal{L}^*(k)\sigma(x,\cdot) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k\sigma(x_0,\cdot)\,dx_0}{k^2+(x-x_0)^2}\,, \qquad k>0, \tag{5.4.20}$$

and the following rule for the change of the order of integration was used:

$$\int_0^{\infty} dy_0 \int_{\ell_2^*(y_0)}^{\infty} du = \int_{\ell_2^*}^{\infty} du \int_0^{g^*(u)} dy_0 \quad . \tag{5.4.21}$$

It is of interest to note that the $\mathcal{L}$-operator presented in section 2.3 had the property $\mathcal{L}(k_1)\mathcal{L}(k_2)=\mathcal{L}(k_1k_2)$ and $\mathcal{L}(1)=1$, the new operator $\mathcal{L}^*$ is different:

$$\mathcal{L}^*(k_1)\mathcal{L}^*(k_2)=\mathcal{L}^*(k_1+k_2)\ , \qquad \mathcal{L}^*(0)=1\ . \tag{5.4.22}$$

Substitution of boundary conditions (5.4.17) in (5.4.19) leads to the governing equation

$$\int_y^{\infty} \frac{du}{\sqrt{u-y}} \int_0^{u} \frac{\mathcal{L}^*(2u-y-y_0)\sigma(x,y_0)\,dy_0}{\sqrt{u-y_0}} = w(x,y)\ . \tag{5.4.23}$$

Application of the operator

$$\frac{d}{dy_1}\int_{y_1}^{\infty} \frac{dy}{\sqrt{y-y_1}}\,\mathcal{L}^*(y)\ , \tag{5.4.24}$$

to both sides of (5.4.23) gives

$$-2\pi\int_0^{y_1}\frac{\mathscr{L}^*(2y_1-y_0)\sigma(x,y_0)dy_0}{\sqrt{y_1-y_0}}=\frac{d}{dy_1}\int_{y_1}^{\infty}\frac{dy}{\sqrt{y-y_1}}\mathscr{L}^*(y)\upsilon(x,y). \tag{5.4.25}$$

Here the following integral was used:

$$\int_{y_1}^{u}\frac{dy}{\sqrt{u-y}\sqrt{y-y_1}}=\pi\ . \tag{5.4.26}$$

The next operator to apply is

$$\frac{d}{dy_2}\int_0^{y_2}\frac{dy_1}{\sqrt{y_2-y_1}}\mathscr{L}^*(-2y_1)\ , \tag{5.4.27}$$

with the result

$$-2\pi^2\mathscr{L}^*(-y_2)\sigma(x,y_2)$$

$$=\frac{d}{dy_2}\int_0^{y_2}\frac{dy_1}{\sqrt{y_2-y_1}}\mathscr{L}^*(-2y_1)\frac{d}{dy_1}\int_{y_1}^{\infty}\frac{dy}{\sqrt{y-y_1}}\mathscr{L}^*(y)\upsilon(x,y). \tag{5.4.28}$$

The final result is

$$\sigma(x,y)=-\frac{\mathscr{L}^*(y)}{2\pi^2}\frac{d}{dy}\int_0^{y}\frac{dy_1}{\sqrt{y-y_1}}\mathscr{L}^*(-2y_1)\frac{d}{dy_1}\int_{y_1}^{\infty}\frac{dy_0}{\sqrt{y_0-y_1}}\mathscr{L}^*(y_0)\upsilon(x,y_0). \tag{5.4.29}$$

The substitution of (5.4.29) back into (5.4.19) makes it possible to express the potential in the whole space through its boundary values.

The first simplification yields

$$V(x,y,z)=-\frac{1}{\pi}\int_{\ell_2^*}^{\infty}\frac{du}{\sqrt{u-y}}\mathcal{L}^*\left(2u-y-2g^*(u)\right)\frac{d}{dg^*(u)}\int_{g^*(u)}^{\infty}\frac{\mathcal{L}^*(y_0)\varphi(x,y_0)dy_0}{\sqrt{y_0-g^*(u)}}\ . \tag{5.4.30}$$

Introducing a new variable t by the relationship $u=\ell_2^*(t,y,z)$, namely $t=g^*(u)$, expression (5.4.30) will take the form

$$V(x,y,z)=-\frac{1}{\pi}\int_{0}^{\infty}\frac{\sqrt{\ell_2^*(t)-y}\ dt}{\ell_2^*(t)-\ell_1^*(t)}\mathcal{L}^*\left(2\ell_2^*(t)-y-2t\right)\frac{d}{dt}\int_{t}^{\infty}\frac{\mathcal{L}^*(y_1)\varphi(x,y_0)dy_0}{\sqrt{y_0-t}}\ . \tag{5.4.31}$$

Here $\ell_2^*(t)$ is used as an abbreviation of $\ell_2^*(t,y,z)$, and the following formula of differentiation was used:

$$\frac{d\ell_2^*(t)}{dt}=\frac{\ell_2^*(t)-y}{\ell_2^*(t)-\ell_1^*(t)}\ . \tag{5.4.32}$$

The change of the order of integration in (5.4.31) yields

$$V(x,y,z)=\frac{1}{\pi}\int_{0}^{\infty}dy_0\left\{\mathcal{L}^*(y_0)\frac{d}{dy_0}\int_{0}^{y_0}\frac{dt}{\sqrt{y_0-t}}\left[\frac{\mathcal{L}^*(2\ell_2^*(t)-y-2t)}{\ell_2^*(t)-\ell_1^*(t)}\right.\right.$$

$$\left.\left.\times\ \sqrt{\ell_2^*(t)-y}\right]\right\}\varphi(x,y_0)\ . \tag{5.4.33}$$

The integral in curly brackets can be computed exactly (see Appendix A5.4), and the final result is

$$V(x,y,z)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}dx_0\int_{0}^{\infty}\frac{z}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]\sigma(x_0,y_0)\,dy_0\ , \qquad (5.4.34)$$

where h(t) is defined in (A5.4.2) and h is an abbreviation for h(0), namely,

$$h = 2\sqrt{y_0\ell_2^*}\ . \qquad (5.4.35)$$

In the case of $y<0$ and $z\to 0$, formula (5.4.35) gives the potential in the negative half-plane through its values in the positive half-plane as follows

$$V(x,y,0)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}dx_0\int_{0}^{\infty}\sqrt{-\frac{y}{y_0}}\ \frac{\sigma(x_0,y_0)\,dy_0}{R^2}\ , \qquad \text{for } y<0. \qquad (5.4.36)$$

It must be noted that $R^2=(x-x_0)^2+(y-y_0)^2$.

5.4.3 PROBLEM OF THE SECOND TYPE

In this problem the boundary conditions on the plane z=0 are:

$$-\frac{1}{2\pi}\left.\frac{\partial V}{\partial z}\right|_{z=0}=\sigma^+(x,y)\ , \qquad \text{for } -\infty<x<\infty\ , \qquad y>0\ ,$$
$$V(x,y,0)=0\ , \qquad \text{for } -\infty<x<\infty\ , \qquad y\le 0\ . \qquad (5.4.37)$$

It is necessary to find function V(x,y,z) in the whole space and σ^- for y<0. In order to derive the governing integral equation, the second condition (5.4.37) must be used. Repetition of the derivation of (5.4.23) for y<0, results in

$$\vartheta^+(x,y)\Big|_{y<0} = 2\int_0^\infty \frac{du}{\sqrt{u-y}} \int_0^u \frac{\mathcal{L}^*(2u-y-y_0)\sigma^+(x,y_0)dy_0}{\sqrt{u-y_0}} \,. \tag{5.4.38}$$

By using integral representation (5.4.7), the potential in the lower half-plane due to the charge distribution there will take the form

$$\vartheta^-(x,y)\Big|_{y<0} = 2\int_{-\infty}^{y} \frac{du}{\sqrt{y-u}} \int_u^0 \frac{\mathcal{L}^*(y+y_0-2u)\sigma^-(x,y_0)dy_0}{\sqrt{y_0-u}} \,. \tag{5.4.39}$$

Since the second condition (5.4.37) implies that $\vartheta^+ + \vartheta^- = 0$, the governing integral equation will be

$$2\int_0^\infty \frac{du}{\sqrt{u-y}} \int_0^u \frac{\mathcal{L}^*(2u-y-y_0)\sigma^+(x,y_0)dy_0}{\sqrt{u-y_0}}$$

$$= -2\int_{-\infty}^{y} \frac{du}{\sqrt{y-u}} \int_u^0 \frac{\mathcal{L}^*(y+y_0-2u)\sigma^-(x,y_0)dy_0}{\sqrt{y_0-u}} \,. \tag{5.4.40}$$

The left-hand side of (5.4.40) can be transformed by using two representations (5.4.2) and (5.4.7) as follows

$$\int_0^{\infty}\frac{du}{\sqrt{u-y}}\int_0^{u}\frac{\mathcal{L}^*(2u-y-y_0)\sigma^+(x,y_0)\,dy_0}{\sqrt{u-y_0}} = \int_0^{\infty}dy_0\left\{\int_{y_0}^{\infty}\frac{\mathcal{L}^*(2u-y-y_0)\,du}{\sqrt{u-y}\sqrt{u-y_0}}\right\}\sigma^+(x,y_0)$$

$$= \int_0^{\infty}dy_0\left\{\int_{-\infty}^{y}\frac{\mathcal{L}^*(y+y_0-2u)\,du}{\sqrt{y-u}\sqrt{y_0-u}}\right\}\sigma^+(x,y_0)$$

$$=\int_{-\infty}^{y}\frac{du}{\sqrt{y-u}}\int_0^{\infty}\frac{\mathcal{L}^*(y+y_0-2u)\sigma^+(x,y_0)\,dy_0}{\sqrt{y_0-u}}=-\int_{-\infty}^{y}\frac{du}{\sqrt{y-u}}\int_u^{0}\frac{\mathcal{L}^*(y+y_0-2u)\sigma^-(x,y_0)\,dy_0}{\sqrt{y_0-u}}.$$

(5.4.41)

Comparison of the last two expressions of (5.4.41) leads to

$$\int_0^{\infty}\frac{\mathcal{L}^*(y_0-2u)\sigma^+(x,y_0)\,dy_0}{\sqrt{y_0-u}} = -\int_u^{0}\frac{\mathcal{L}^*(y_0-2u)\sigma^-(x,y_0)\,dy_0}{\sqrt{y_0-u}}\ . \qquad (5.4.42)$$

Application of the operator,

$$\frac{d}{d\vartheta}\int_{\vartheta}^{0}\frac{du}{\sqrt{u-\vartheta}}\,\mathcal{L}^*(2u)\ , \qquad (5.4.43)$$

to both sides of (5.4.42) yields

$$\pi\mathcal{L}^*(\vartheta)\sigma^-(x,\vartheta) = -\int_0^{\infty}\sqrt{-\frac{y_0}{\vartheta}}\,\frac{\mathcal{L}^*(y_0)}{y_0-\vartheta}\,\sigma^+(x,y_0)\,dy_0\ , \qquad (5.4.44)$$

and finally,

$$\sigma^-(x,y)\Big|_{y<0} = -\frac{1}{\pi}\int_0^\infty \sqrt{-\frac{y_0}{y}}\,\frac{\mathcal{L}^*(y_0-y)}{y_0-y}\,\sigma^+(x,y_0)\,dy_0 \ . \qquad (5.4.45)$$

Expression (5.4.45) gives the direct relationship between the charge distribution σ^+ in the upper half-plane and σ^- in the lower one. Formula (5.4.45) can be rewritten without the $\mathcal{L}^*$-operator as follows.

$$\sigma^-(x,y) = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} dx_0 \int_0^\infty \sqrt{-\frac{y_0}{y}}\,\frac{\sigma^+(x_0,y_0)\,dy_0}{(x-x_0)^2+(y-y_0)^2} \ . \qquad (5.4.46)$$

Now the charge distribution σ is known all over the plane $z=0$, and the potential can be found directly in terms of the prescribed density σ^+. By utilizing (5.4.7), the following expression for the potential can be presented

$$V^-(x,y,z) = 2\int_{-\infty}^{\ell_1^*} \frac{du}{\sqrt{y-u}} \int_{g^*(u)}^{0} \frac{\mathcal{L}^*(y+y_0-2u)\,\sigma^-(x,y_0)\,dy_0}{\sqrt{y_0-u}} \ . \qquad (5.4.47)$$

Substitution of (5.4.46) in (5.4.47) gives, after simplification,

$$V^-(x,y,z) = -2\int_{-\infty}^{\ell_1^*} \frac{du}{\sqrt{y-u}} \int_0^\infty \frac{\mathcal{L}^*(y+y_0-2u)\,\sigma^+(x,y_0)\,dy_0}{\sqrt{y_0-g^*(u)}} \ . \qquad (5.4.48)$$

The positive counterpart, according to (5.4.19), will take the form

$$V^{+}(x,y,z) = 2\int_{0}^{\infty} dy_0 \left\{ \int_{\ell_2^{*}(y_0)}^{\infty} \frac{\mathcal{L}^{*}(2u-y_0-y)\,du}{\sqrt{u-y}\sqrt{g^{*}(u)-y_0}} \right\} \sigma^{+}(x,y_0)$$

$$= 2\int_{0}^{\infty} dy_0 \left\{ \int_{-\infty}^{\ell_1^{*}(y_0)} \frac{\mathcal{L}^{*}(y+y_0-2u)\,du}{\sqrt{y-u}\sqrt{y_0-g^{*}(u)}} \right\} \sigma^{+}(x,y_0) \quad . \qquad (5.4.49)$$

The order of integration in (5.4.49) can be changed according to the scheme

$$\int_{0}^{\infty} dy_0 \int_{-\infty}^{\ell_1^{*}(y_0)} du = \int_{\ell_1^{*}}^{y} du \int_{g^{*}(u)}^{\infty} dy_0 + \int_{-\infty}^{\ell_1^{*}} du \int_{0}^{\infty} dy_0 \ . \qquad (5.4.50)$$

By taking complete potential as superposition V^{-} and V^{+}, it may be seen that (5.4.48) cancels out with the second term in (5.4.50), so the only term left is

$$V(x,y,z) = 2\int_{\ell_1^{*}}^{y} \frac{du}{\sqrt{y-u}} \int_{g^{*}(u)}^{\infty} \frac{\mathcal{L}^{*}(y+y_0-2u)\,\sigma^{+}(x,y_0)\,dy_0}{\sqrt{y_0-g^{*}(u)}} \ . \qquad (5.4.51)$$

One may also change the order of integration in (5.4.51) and perform the integration with respect to u.

$$V(x,y,z) = 2\int_{0}^{\infty} dy_0 \left\{ \int_{\ell_1^{*}}^{\ell_1^{*}(y_0)} \frac{\mathcal{L}^{*}(y+y_0-2u)\,du}{\sqrt{y-u}\sqrt{y_0-g^{*}(u)}} \right\} \sigma(x,y_0)$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \frac{1}{R_0} \tan^{-1}\left(\frac{2\sqrt{y_0 \ell_2^*}}{R_0}\right) \sigma(x_0, y_0)\, dy_0 \ . \qquad (5.4.52)$$

In (5.4.52) a symbol + with σ was no longer used, because it is obvious from the limits of integration that σ is related to the half-plane $y>0$. In the plane $z=0$ formula (5.4.51) and (5.4.52) simplify as

$$V(x,y,0) = 2\int_0^y \frac{du}{\sqrt{y-u}} \int_u^{\infty} \frac{\mathcal{L}^*(y+y_0-2u)\sigma(x,y_0)\, du}{\sqrt{y_0-u}}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \frac{1}{R} \tan^{-1}\left(\frac{2\sqrt{y_0 y}}{R}\right) \sigma(x_0, y_0)\, dy_0 \ . \qquad (5.4.53)$$

5.4.4 APPLICATION TO THE ELASTIC CONTACT PROBLEMS

Let it be necessary to consider a transversely isotropic elastic half-space $z\geq 0$, characterized by five elastic constants A_{ik}, as described in section 2.5. A semi-infinite smooth punch act on the boundary $z=0$, $y>0$, while the rest of the boundary, namely, $z=0$, $y<0$ is stress-free. Assume that the punch produces normal displacement

$$w=w(x,y) \ , \qquad \text{for} \quad -\infty<x<\infty \ , \quad y\geq 0 \ . \qquad (5.4.54)$$

The other boundary conditions on the plane $z=0$ are

$$\sigma_z = 0 , \qquad \text{for} \quad -\infty < x < \infty , \quad y < 0 ,$$
$$\tau_{yz} = \tau_{zx} = 0 , \qquad \text{for} \quad -\infty < x, y < \infty . \tag{5.4.55}$$

The governing integral equation of the problem is

$$H \int_{-\infty}^{\infty} dx_0 \int_{0}^{\infty} \frac{\sigma(x_0, y_0)}{R} dy_0 = w(x, y) , \qquad -\infty < x < \infty , \ y > 0 . \tag{5.4.56}$$

Here H is an elastic constant defined in (2.36), and $R^2 = (x - x_0)^2 + (y - y_0)^2$. According to (5.4.29), solution of (5.4.56) takes the form

$$\sigma(x, y) = -\frac{\mathscr{L}^*(y)}{2\pi^2 H} \frac{d}{dy} \int_{0}^{y} \frac{dy_1}{\sqrt{y - y_1}} \mathscr{L}^*(-2y_1) \frac{d}{dy_1} \int_{y_1}^{\infty} \frac{dy_0}{\sqrt{y_0 - y_1}} \mathscr{L}^*(y_0) w(x_0, y_0) . \tag{5.4.57}$$

The complete solution to the problem can be expressed through two potential functions

$$F_1(z) = \frac{H\gamma_1}{m_1 - 1} F(z_1) , \qquad F_2(z) = \frac{H\gamma_2}{m_2 - 1} F(z_2) . \tag{5.4.58}$$

Here $z_k = z/\gamma_k$, $k = 1, 2$ and m_k and γ_k are elastic constants defined in (2.29) and the main potential function F(z) is defined as

$$F(z) \equiv F(x, y, z) = \int_{-\infty}^{\infty} dx_0 \int_{0}^{\infty} \ln\left[z + \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}\right] \sigma(x_0, y_0) dy_0 . \tag{5.4.59}$$

By substitution of (5.4.57) in (5.4.59), one can easily compute $\partial F/\partial z$, which will coincide with (5.4.34). Further integration of both sides with respect to z (see Appendix B5.4) will give the main potential function F as

$$F(x,y,z)=\frac{1}{\pi^2 H}\int_{-\infty}^{\infty}dx_0\int_{0}^{\infty}K(x,y,z;\ x_0,y_0)w(x_0,y_0)dy_0\ , \qquad (5.4.60)$$

where

$$K(x,y,z;x_0,y_0)=-\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)-\frac{1}{\sqrt{2y_0}}\ 2\mathcal{R}e\left[\frac{1}{\sqrt{iq}}\tan^{-1}\left(\frac{iq}{2\ell_2^{\bullet}}\right)^{1/2}\right]. \qquad (5.4.61)$$

Here $\mathcal{R}e$ indicates the real part, $q=(x-x_0)+i(y-y_0)$ and the other parameters are defined by formulae (5.4.12) and (5.4.35).

The field of displacements and stresses was given through the main potential function defined in (5.4.58) by formulae (2.27) and (2.32).

In order to find a complete solution, it is necessary to have the following derivatives of K, namely,

$$\frac{\partial K}{\partial z}=\frac{z}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]\ , \qquad (5.4.62)$$

$$\Lambda K=\frac{q}{R_0^3}\left[\frac{R_0}{h}+\tan^{-1}\left(\frac{h}{R_0}\right)\right]-\frac{1}{h\bar{q}}\left[1-\left(\frac{2i\ell_2^{\bullet}}{\bar{q}}\right)^{1/2}\tan^{-1}\left(\frac{\bar{q}}{2i\ell_2^{\bullet}}\right)^{1/2}\right]\ , \qquad (5.4.63)$$

$$\frac{\partial^2 K}{\partial z^2} = \frac{1}{R_0^3}\left(1 - \frac{3z^2}{R_0^2}\right)\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] + \frac{1}{h(R_0^2+h^2)}\left[\frac{z^2}{R_0^2} - \frac{\ell_1^*}{\ell_1^*-\ell_2^*}\right] ,$$

(5.4.64)

$$\frac{\partial}{\partial z}\Lambda K = - \frac{3zq}{R_0^5}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] + \frac{z}{h(R_0^2+h^2)}\left[\frac{i}{2(\ell_1^*-\ell_2^*)} + \frac{q}{R_0^2}\right] ,$$

(5.4.65)

$$\Lambda^2 K = - \frac{3q^2}{R_0^5}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] + \frac{1}{h(R_0^2+h^2)}\left[\frac{q^2}{R_0^2} + \frac{\ell_1^*-y}{\ell_1^*-\ell_2^*}\right]$$

$$+ \frac{3}{\bar{q}^2}\left[\frac{1}{h} - \left(\frac{i}{2y_0\bar{q}}\right)^{1/2}\tan^{-1}\left(\frac{\bar{q}}{2i\ell_2^*}\right)^{1/2}\right] - \frac{1}{\bar{q}h(2i\ell_2^*+\bar{q})} . \qquad (5.4.66)$$

The substitution of (5.4.62-5.4.66) and (5.4.60) in (2.27) and (2.32) gives the complete solution.

5.4.5 APPLICATION TO THE HALF-PLANE CRACK PROBLEM

Consider a transversely isotropic elastic space weakened in the plane $z=0$ by a crack $y>0$. The crack is opened by normal stress σ applied to crack faces in opposite directions. The boundary conditions on plane $z=0$ are

$$\sigma_z = p(x,y) , \qquad -\infty<x<\infty , \qquad y>0 ,$$
$$w = 0 , \qquad -\infty<x<\infty , \qquad y\le 0 ,$$
$$\tau_z = 0 , \qquad -\infty<x,y<\infty . \qquad (5.4.67)$$

The main potential functions in this case are

$$\Phi_1(x,y,z) \equiv \Phi_1(z) = -\frac{\gamma_1}{2\pi(m_1-1)}\Phi(z_1),$$

$$\Phi_2(x,y,z) \equiv \Phi_2(z) = -\frac{\gamma_2}{2\pi(m_2-1)}\Phi(z_2), \tag{5.4.68}$$

where

$$\Phi(z) \equiv \Phi(x,y,z) = \int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \frac{w(x_0,y_0)\,dy_0}{R_0}. \tag{5.4.69}$$

The governing integral equation will take the form

$$p(x,y) = -\frac{1}{4\pi^2 H}\Delta\int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \frac{w(x_0,y_0)\,dy_0}{[(x-x_0)^2+(y-y_0)^2]^{1/2}}. \tag{5.4.70}$$

Here Δ is defined in (2.26).

Integral equation (5.4.70) is inverse to (5.4.53), therefore its solution is

$$w(x,y) = \frac{2}{\pi}H\int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \frac{p(x_0,y_0)}{R}\tan^{-1}\left(\frac{2\sqrt{yy_0}}{R}\right)dy_0. \tag{5.4.71}$$

Substitution of (5.4.71) in (5.4.69) makes it possible to express the main potential function in terms of the prescribed pressure as follows

$$\Phi(x,y,z) = \frac{2}{\pi}H\int_{-\infty}^{\infty} dx_0 \int_0^{\infty} \mathcal{K}(x,y,z;\ x_0,y_0)\,p(x_0,y_0)\,dy_0, \tag{5.4.72}$$

where

$$\mathcal{K}(x,y,z;\ x_0,y_0)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{R}\tan^{-1}\left(\frac{2\sqrt{yy_0}}{R}\right)\frac{dx_0dy_0}{R_0} \quad . \tag{5.4.73}$$

Various derivatives of $\mathcal{K}$ are mainly needed for the fields of displacements and stresses. They are as follows

$$\frac{\partial\mathcal{K}}{\partial z} = -\frac{2\pi}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) , \tag{5.4.74}$$

$$\Lambda\mathcal{K} = \frac{2\pi}{\bar{q}}\left[\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) - c\tan^{-1}\left(\frac{\bar{s}^*}{-2\ell_1^*}\right)^{1/2}\right] , \tag{5.4.75}$$

$$\frac{\partial^2\mathcal{K}}{\partial z^2} = 2\pi\left[\frac{z}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{h}{z[R_0^2+h^2]}\left(\frac{\ell_1^*}{\ell_1^*-\ell_2^*} - \frac{z^2}{R_0^2}\right)\right] , \tag{5.4.76}$$

$$\frac{\partial}{\partial z}\Lambda\mathcal{K} = 2\pi\left[\frac{q}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{h}{R_0^2+h^2}\left(\frac{i}{2(\ell_1^*-\ell_2^*)} + \frac{q}{R_0^2}\right)\right] , \tag{5.4.77}$$

$$\Lambda^2\mathcal{K} = 2\pi\left[\frac{c}{\bar{q}}\left(\frac{i}{\bar{s}^*} + \frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}^*}{-2\ell_1^*}\right)^{1/2} - \frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)\right.$$

$$\left. - \frac{2\sqrt{y_0\ell_1^*}}{\bar{q}\ \bar{s}^*(\bar{s}^*-2\ell_1^*)} - \frac{zh}{R_0^2+h^2}\left(\frac{q}{\bar{q}R_0^2} + \frac{1}{4\ell_2^*(\ell_2^*-\ell_1^*)}\right)\right] . \tag{5.4.78}$$

In expressions (5.4.62-5.4.66) and (5.4.74-5.4.78) the values for ℓ_1^*, ℓ_2^* and h are defined by formulae (5.4.12) and (5.4.35). The other parameters are defined as

$$s^*=(y+y_0)-i(x-x_0) \ , \qquad q=(x-x_0)+i(y-y_0) \ ,$$

$$c = \sqrt{\frac{2y_0}{(y+y_0)+i(x-x_0)}} = \sqrt{\frac{2y_0}{\bar{s}^*}} \;. \tag{5.4.79}$$

and the overbar everywhere indicates the complex conjugate value.

Consider, as an example, the action of a pair of normal concentrated forces $P\delta(x-x_0)\delta(y-y_0)$ applied to the crack faces in the opposite directions at the points $(x_0, y_0, 0^{\pm})$, $y_0>0$. According to formulae (2.27), (2.32), (5.4.72) and (5.4.74-5.4.78), the complete solution for the field of stresses and displacements in a transversely isotropic elastic space is

$$u=\frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1^*(z_1) + \frac{\gamma_2}{m_2-1}f_1^*(z_2)\right] , \tag{5.4.80}$$

$$w=\frac{2}{\pi}HP\left[\frac{m_1}{m_1-1}f_2^*(z_1) + \frac{m_2}{m_2-1}f_2^*(z_2)\right] , \tag{5.4.81}$$

$$\sigma_1=\frac{2P}{\pi^2(\gamma_1-\gamma_2)}\left\{\left[\frac{\gamma_1}{(m_1+1)\gamma_3^2} - \frac{1}{\gamma_1}\right]f_3^*(z_1)-\left[\frac{\gamma_2}{(m_2+1)\gamma_3^2} - \frac{1}{\gamma_2}\right]f_3^*(z_2)\right\} , \tag{5.4.82}$$

$$\sigma_2=\frac{4}{\pi}HA_{66}P\left[\frac{\gamma_1}{m_1-1}f_4^*(z_1) + \frac{\gamma_2}{m_2-1}f_4^*(z_2)\right] , \tag{5.4.83}$$

$$\sigma_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[\gamma_1 f_3^*(z_1) - \gamma_2 f_3^*(z_2)\right] , \tag{5.4.84}$$

$$\tau_z=\frac{P}{\pi^2(\gamma_1-\gamma_2)}\left[f_5^*(z_1) - f_5^*(z_2)\right] , \tag{5.4.85}$$

where

$$f_1^*(z)=\frac{1}{\bar{q}}\left[c\ \tan^{-1}\left(\frac{\bar{s}^*}{-2\ell_1^*}\right)^{1/2}-\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right]\ , \tag{5.4.86}$$

$$f_2^*(z)=\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\ , \tag{5.4.87}$$

$$f_3^*(z)=-\ \frac{z}{R_0^3}\ \tan^{-1}\left(\frac{h}{R_0}\right)\ +\ \frac{h}{z[R_0^2+h^2]}\left(\frac{\ell_1^*}{\ell_1^*-\ell_2^*}\ -\ \frac{z^2}{R_0^2}\right)\ , \tag{5.4.88}$$

$$f_4^*(z)=-\ \frac{c}{\bar{q}}\left(\frac{i}{\bar{s}^*}\ +\ \frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}^*}{-2\ell_1^*}\right)^{1/2}+\ \frac{z(3R_0^2-z^2)}{\bar{q}^2R_0^3}\ \tan^{-1}\left(\frac{h}{R_0}\right)$$

$$+\ \frac{2\sqrt{y_0\ell_1^*}}{\bar{q}\ \bar{s}^*(\bar{s}^*-2\ell_1^*)}\ +\ \frac{zh}{R_0^2+h^2}\left(\frac{q}{\bar{q}R_0^2}\ +\ \frac{1}{4\ell_2^*(\ell_2^*-\ell_1^*)}\right)\ , \tag{5.4.89}$$

$$f_5^*(z)=-\left\{\frac{q}{R_0^3}\ \tan^{-1}\left(\frac{h}{R_0}\right)\ +\ \frac{h}{R_0^2+h^2}\left(\frac{i}{2(\ell_1^*-\ell_2^*)}\ +\ \frac{q}{R_0^2}\right)\right\}\ . \tag{5.4.90}$$

The results obtained in (5.4.80-5.4.85) are valid for isotropic solids as well, provided that (2.34) and (2.35) are used and the relevant limits are computed according to the L'Hospital rule. The scheme which must be used is given in (3.2.25).

5.5 SUMMARY

The main advantage of the new method is its simplicity: no integral transforms or special function expansions are needed. All the analysis, including computation of three-dimensional potentials is performed in closed form and in terms of elementary functions. All the parameters used have physical significance, thus simplifying further investigation of the properties of solutions.

A comparison of the results obtained in this section, namely formulae (5.4.80-5.4.90), with those obtained in section 5.2, namely formulae (5.2.23-5.2.33), shows that they are in perfect correspondence. Some differences in expressions should be attributed to the choice of the coordinate system and the definition of some of the parameters involved.

The conclusion is that the method developed in section 5.4 and applied for the solution of mixed B.V.P. of potential theory for a half space geometry, where an infinite straight line delineates the boundary conditions, in fact is possible and gives an exact, complete solution to the stated problems.

CHAPTER 6

PROBLEMS OF INTERACTION BETWEEN AN ARBITRARILY LOCATED FORCE AND CIRCULAR CRACK

6.1 INTRODUCTORY REMARKS

The concept of "weight functions" in two-dimensional elastic crack analysis was first introduced by Bueckner [40]. His weight functions are elastic displacement fields which equilibrate zero body forces and zero surface tractions but have a stronger singularity at the crack front than normally admissible displacement fields. Subsequent to Bueckner's analysis, Rice [41] showed that weight functions could be evaluated by differentiating with respect to crack length the known displacement solutions for two-dimensional crack problems. Rice [41,42] has also laid the foundation for three-dimensional weight function theory based on displacement field variations cause by a first order variation in position of a crack front. However, in their simplest interpretation, weight functions can be considered as *the stress intensity factors around a crack front caused by an arbitrarily located concentrated force.* Since their inception, weight functions have played an important role in fracture mechanics and a great deal of effort has been aimed at evaluating weight functions for various crack geometries.

In the present study the focus is specifically placed on closed-form solutions to weight functions for three-dimensional geometries. Perhaps the first weight functions evaluated in closed form are the so called "crack face weight functions" which are the stress intensity factors when the concentrated loading acts on the crack faces. Such solutions for half-plane, penny-shaped and circular external cracks in isotropic bodies can be extracted from the work of Galin [43], Ufliand [22], Tada, Paris and Irwin [44] and Kassir and Sih [15]. A few additional solutions not present in this previous literature can be found in more recent work such as Meade and Keer [45] and Fabrikant [16], for example. In these previous solutions, loading was generally either symmetric or anti-symmetric about the crack plane. Thus one can superpose solutions, say for symmetric normal loading and anti-symmetric normal loading, to obtain the solution for concentrated normal loading on one crack face only.

Though many solutions exist for crack face weight functions for the crack shapes noted above, few closed-form solutions were previously given for general weight functions. The books by Tada, Paris and Irwin [44], Kassir and Sih [15] generally summarized the known closed-form results for penny-shaped cracks and circular external cracks when the point forces are located off the crack plane on the axis of symmetry. The first general weight function was probably given by Rice [46] who evaluated the tensile mode weight function for a half-plane crack subjected to an arbitrarily located force. However, the result was given in integral form when the force direction was parallel to the crack plane and analytically when the direction was perpendicular to the plane. Recently, the derivation of the general weight functions for the penny-shaped and the half-plane crack have been given by Bueckner [47] for an isotropic body. The analysis was extended by Gao [48] to the circular external crack. In both of these analyses, explicit expressions for the weight functions were given only when the forces were located on a crack face (the crack face weight functions).

In the present analysis, weight functions for the penny-shaped crack are again considered. Use is made of some recent results by Fabrikant [16] who derived closed-form expressions for the elastic field of a penny-shaped crack in a transversely isotropic body loaded by point forces on its faces. These solutions, coupled with the reciprocal theorem, are used to derive closed-form expressions in terms of elementary functions for the crack opening displacement of a penny-shaped crack in a transversely isotropic body loaded by an arbitrarily located point force. Explicit closed-form expressions are obtained for the general weight functions of a penny-shaped crack in a transversely isotropic body by a limiting procedure. The outline of a similar investigation is done for the external circular crack where use is made of the results obtained in Chapter 4. Such explicit expressions have not been given previously, even for the isotropic case. The work of the next section follows the paper by Karapetian and Hanson [49].

6.2 CRACK OPENING DISPLACEMENTS AND STRESS INTENSITY FACTORS CAUSED BY A CONCENTRATED LOAD OUTSIDE AN INTERNAL CIRCULAR CRACK

In this section an evaluation of crack opening displacements and stress intensity factors in terms of elementary functions for the problem of a concentrated load outside an internal circular crack is given. The results are obtained for both transversely isotropic and purely isotropic cases. A particular case of a concentrated load at a point on the normal axis is obtained and compared with the previous analysis.

6.2.1 POINT FORCE LOADING APPLIED TO A CIRCULAR CRACK

Consider a transversely isotropic space weakened by an internal circular crack of radius a in the plane $z=0$. Let the crack be subjected to the action of two equal normal concentrated forces P applied in opposite directions at the points $(\rho_0,\phi_0,0^{\pm})$, $\rho_0<a$ as shown in Fig.6.1(a). A complete solution for the field of displacements in elementary functions for $z>0$ is (Fabrikant [16])

$$u = \frac{2}{\pi}HP\left[\frac{\gamma_1}{m_1-1}f_1(z_1) + \frac{\gamma_2}{m_2-1}f_1(z_2)\right], \tag{6.2.1}$$

$$w = \frac{2}{\pi} HP \left[\frac{m_1}{m_1 - 1} f_2(z_1) + \frac{m_2}{m_2 - 1} f_2(z_2) \right] , \qquad (6.2.2)$$

where $u = u_x + iu_y$ and u_x, u_y, w are the displacements in the x,y,z directions.

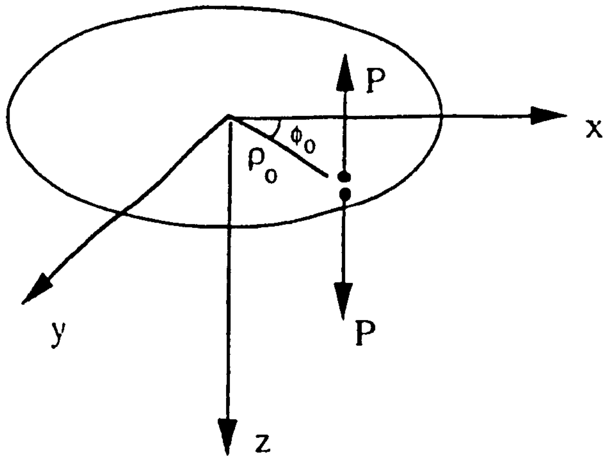

Fig.6.1(a) Point normal loading.

If two equal tangential concentrated forces $T = T_x + iT_y$ are applied to the crack faces antisymmetricaly at the points $(\rho_0, \phi_0, 0^{\pm})$, $\rho_0 < a$, (Fig.6.1(b)) a complete solution for the field of displacements in elementary functions for $z > 0$ is

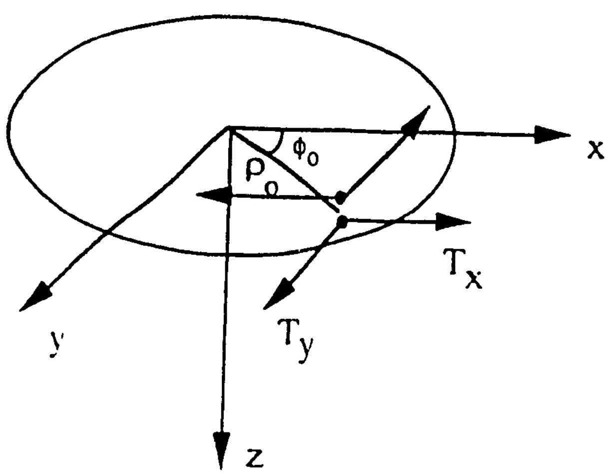

Fig.6.1(b) Point shear loading.

$$u = \frac{H\gamma_1\gamma_2}{\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[f_2(z_k)+\frac{G_2}{G_1}\bar{f}_7(z_k)\right]T + \left[f_{16}(z_k)+\frac{G_2}{G_1}f_8(z_k)\right]\bar{T}\right\}$$

$$+\frac{\beta}{\pi}\left\{\left[f_2(z_3)-\frac{G_2}{G_1}\bar{f}_7(z_3)\right]T + \left[f_{16}(z_3)-\frac{G_2}{G_1}f_8(z_3)\right]\bar{T}\right\}, \qquad (6.2.3)$$

$$w = \frac{2}{\pi}H\gamma_1\gamma_2\mathcal{R}e\left\{\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left[\bar{f}_1(z_k)+\frac{G_2}{G_1}\bar{f}_9(z_k)\right]T\right\}. \qquad (6.2.4)$$

Here $\mathcal{R}e$ indicates the real part and the functions $f_i(z)$ are given as

$$f_1(z)=\frac{1}{\bar{q}}\left[\frac{(a^2-\rho_0^2)^{1/2}}{\bar{s}}\tan^{-1}\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}} - \frac{z}{R_0}\tan^{-1}\frac{h}{R_0}\right], \qquad (6.2.5)$$

$$f_2(z)=\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right), \qquad (6.2.6)$$

$$f_7(z)=\frac{ha^2}{s^2}\left[\frac{3}{s^2} - \frac{t}{\ell_2^2-a^2t} - \frac{3(\ell_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)\right], \qquad (6.2.7)$$

$$f_8(z)=\frac{1}{\bar{q}}(a^2-\rho_0^2)^{1/2}\left\{\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\left[\tan^{-1}\left(\frac{1}{\bar{\zeta}-1}\right)^{1/2}-\tan^{-1}\left(\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)\right]\right.$$

$$\left.-\frac{e^{i\phi}}{\rho}\left[\frac{(a^2-\ell_1^2)^{1/2}}{a}\left(1+\frac{\rho^2}{\ell_2^2-\rho\rho e^{i(\phi-\phi_0)}}\right)-1\right]\right\}, \qquad (6.2.8)$$

$$f_9(z)=-\rho e^{i\phi}\,\frac{(a^2-\rho_0^2)^{1/2}}{a^3}\left\{\frac{1}{t}\sin^{-1}\left(\frac{a}{\ell_2}\right)+\frac{a(\ell_2^2-a^2)^{1/2}}{(1-t)(\ell_2^2-\rho\rho_0 e^{i(\phi-\phi_0)})}\right.$$

$$\left.-\frac{1}{t(1-t)^{3/2}}\tan^{-1}\left(\frac{a(1-t)^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right\}, \qquad (6.2.9)$$

$$f_{16}(z)=\frac{1}{\bar{q}}\left\{\frac{R_0^2+z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h}{R_0}\right)+(a^2-\rho_0^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right.\right.\right.$$

$$\left.-\frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)+\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}}\left(\tan^{-1}\frac{1}{(\bar{\zeta}-1)^{1/2}}\right.$$

$$\left.\left.\left.-\tan^{-1}\frac{(a^2-\ell_1^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right)+\frac{e^{i\phi}}{\rho}\right]-\frac{e^{i\phi}ha^2}{\rho\bar{s}^2}\right\}, \qquad (6.2.10)$$

Here the following notation was used:

$$R_0=[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2]^{1/2}, \qquad q=\rho e^{i\phi}-\rho_0 e^{i\phi_0}, \qquad (6.2.11)$$

$$\ell_1(a,\rho,z)\equiv\ell_1(a)=\frac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}-[(\rho-a)^2+z^2]^{1/2}\},$$

$$\ell_2(a,\rho,z)\equiv\ell_2(a)=\frac{1}{2}\{[(\rho+a)^2+z^2]^{1/2}+[(\rho-a)^2+z^2]^{1/2}\}, \qquad (6.2.12)$$

$$t=\frac{\rho\rho_0}{a^2}e^{i(\phi-\phi_0)}, \qquad \zeta=\frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}. \qquad (6.2.13)$$

$$h=\sqrt{a^2-\ell_1^2}\sqrt{a^2-\rho_0^2}/a, \qquad s=\sqrt{a^2-\rho\rho_0 e^{i(\phi-\phi_0)}}. \qquad (6.2.14)$$

An overbar indicates the complex conjugate value and the transversely isotropic elastic constants are defined in (2.36).

6.2.2 OPENING MODE DISPLACEMENTS AND STRESS INTENSITY FACTORS

Consider two systems in equilibrium: let a concentrated force Q_z be applied at an arbitrary point (ρ,ϕ,z) in the Oz direction as shown in Fig.6.2(a), while in the second system two equal concentrated forces P are applied normal to the crack faces in opposite directions at the points $(\rho_0, \phi_0, 0^{\pm})$ as in Fig.6.1(a). Denote the normal displacement in the space due to the force P by w_P, while w_{Q_z} is the crack opening displacement due to force Q_z. Note that the term "crack opening displacement" is used here to denote the difference between the normal displacements of the crack faces.

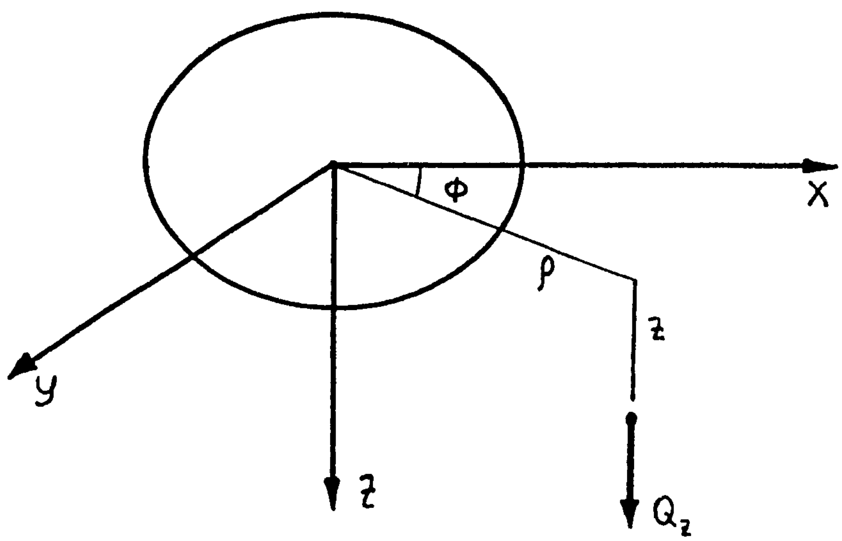

Fig.6.2(a) Arbitrarily located point normal loading.

Application of the reciprocal theorem to the two systems yields

$$Pw_{Q_z} = Q_z w_P \,, \qquad (6.2.15)$$

which gives the crack opening displacement

$$w_{Q_z}(\rho_0,\phi_0)=\frac{2}{\pi}HQ_z\left[\frac{m_1}{m_1-1}f_2(z_1)+\frac{m_2}{m_2-1}f_2(z_2)\right] , \tag{6.2.16}$$

with f_2 defined by (6.2.6).

Similarly, there can be considered two other systems in equilibrium. The displacement w_Q is produced by a concentrated force $Q=Q_x+iQ_y$ applied at an arbitrary point (ρ,ϕ,z) and directed perpendicular to Oz as shown in Fig.6.2(b), while the tangential displacement u_P is due to the force P applied normal to the crack faces in opposite directions at the points $(\rho_0,\phi_0,0^{\pm})$ as in Fig.6.1(a).

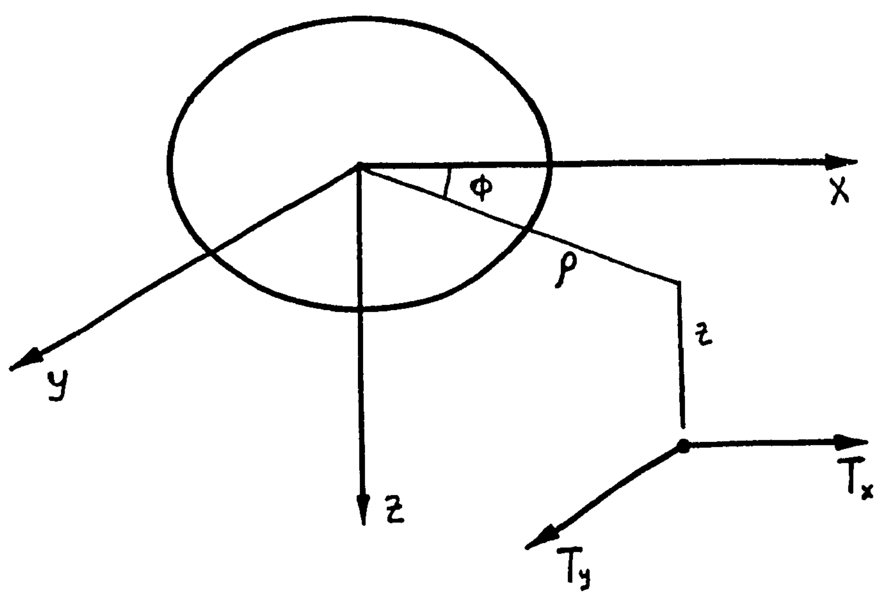

Fig.6.2(b) Arbitrarily located point shear loading.

Application of the reciprocal theorem for Q_x and Q_y separately will give the following crack opening displacement

$$w_{Q_x}(\rho_0,\phi_0)=\frac{2}{\pi}HQ_x\mathcal{R}e\left[\frac{\gamma_1}{m_1-1}f_1(z_1)+\frac{\gamma_2}{m_2-1}f_1(z_2)\right] \quad , \tag{6.2.17}$$

$$w_{Q_y}(\rho_0,\phi_0)=\frac{2}{\pi}HQ_y\mathcal{I}m\left[\frac{\gamma_1}{m_1-1}f_1(z_1)+\frac{\gamma_2}{m_2-1}f_1(z_2)\right] \quad , \tag{6.2.18}$$

with f_1 defined by (6.2.5).

The SIF can be determined by using the formula defined in (2.46), namely,

$$K_1(\phi_0)=\frac{1}{8\pi H}\lim_{\rho_0\to a}\frac{w_Q(\rho_0,\phi_0)}{(a-\rho_0)^{1/2}} \quad , \tag{6.2.19}$$

and obtain K_1 due to Q_z, Q_x and Q_y as follows

$$K_1=\frac{Q_z}{2\pi^2(2a)^{1/2}}\sum_{k=1}^{2}\frac{m_k}{m_k-1}f_2^*(z_k) \quad , \tag{6.2.20}$$

$$K_1=\frac{Q_x}{2\pi^2(2a)^{1/2}}\mathcal{R}e\left[\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}f_1^*(z_k)\right] \quad , \tag{6.2.21}$$

$$K_1=\frac{Q_y}{2\pi^2(2a)^{1/2}}\mathcal{I}m\left[\sum_{k=1}^{2}\frac{\gamma_k}{m_k-1}f_1^*(z_k)\right] \quad . \tag{6.2.22}$$

Here the functions $f_1^*(z)$ are given as

$$f_1^*(z_k)=\frac{1}{q}\left[\frac{a}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(\ell_{2k}^2-a^2)^{1/2}}\right)-\frac{z_k(a^2-\ell_{1k}^2)^{1/2}}{R_k^2}\right] \quad , \tag{6.2.23}$$

$$f_2^*(z_k)=\frac{(a^2-\ell_{1k}^2)^{1/2}}{R_k^2} \quad , \tag{6.2.24}$$

where

$$R_k=[\rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z_k^2]^{1/2},$$

$$q=\rho e^{i\phi}-ae^{i\phi_0}\ , \quad s=\sqrt{a^2-a\rho e^{i(\phi-\phi_0)}}\ , \tag{6.2.25}$$

$$\ell_{1k}=\frac{1}{2}\{[(\rho+a)^2+z_k^2]^{1/2}-[(\rho-a)^2+z_k^2]^{1/2}\}\ ,$$

$$\ell_{2k}=\frac{1}{2}\{[(\rho+a)^2+z_k^2]^{1/2}+[(\rho-a)^2+z_k^2]^{1/2}\}\ . \tag{6.2.26}$$

6.2.3 COMBINED SECOND AND THIRD MODE STRESS INTENSITY FACTORS

The solution for this case still can be obtained in an elementary way by using the reciprocal theorem. However, it will first be shown how the reciprocal theorem can be used in the case of complex forces and displacements.

Consider two systems in equilibrium. The first one is an elastic space weakened by an internal circular crack, with two equal and oppositely directed tangential forces $T=T_x+iT_y$ applied at the points $(\rho_0,\phi_0,0^{\pm})$ of the crack faces (Fig.6.1(b)). The second system is the same space, with the crack faces tractions free, and the horizontal force $Q=Q_x+iQ_y$ applied at the point (ρ,ϕ,z) (Fig.6.2(b)). For simplicity of the transformation to follow, it is better to present (6.2.3) in a generalized form

$$u = A_1T + A_2\overline{T}\ . \tag{6.2.27}$$

Here A_1 and A_2 are the combined factors of T and $\overline{T}$

respectively.

The tangential displacements at the point (ρ,ϕ,z) in the x and y directions due to a couple of forces T_x will be respectively

$$(u_x)_{T_x} = T_x \mathcal{R}e(A_1+A_2) \quad , \qquad (u_y)_{T_x} = T_x Im(A_1+A_2) \quad . \tag{6.2.28}$$

The similar displacements due to a couple of forces T_y are

$$(u_x)_{T_y} = T_y \mathcal{R}e[(A_1-A_2)i] = -T_y Im(A_1-A_2) \quad ,$$

$$(u_y)_{T_y} = T_y Im[(A_1-A_2)i] = T_y \mathcal{R}e(A_1-A_2) \quad . \tag{6.2.29}$$

Denoting the tangential displacement discontinuity in the x direction due to force Q_x as Δ_x. According to the reciprocal theorem, the result is

$$(\Delta_x)_{Q_x} = Q_x \mathcal{R}e(A_1+A_2) \quad . \tag{6.2.30}$$

The remaining three equations are obtained in a similar manner, and they are

$$(\Delta_y)_{Q_x} = -Q_x Im(A_1-A_2) \, , \; (\Delta_x)_{Q_y} = Q_y Im(A_1+A_2) \, , \; (\Delta_y)_{Q_y} = Q_y \mathcal{R}e(A_1-A_2) \, . \tag{6.2.31}$$

The meaning of the notation in (6.2.31) is the same as in (6.2.28-6.2.29). Summation (6.2.30) with the first expression of (6.2.31) multiplied by $i=\sqrt{-1}$ yields

$$(\Delta)_{Q_x} = (\Delta_x)_{Q_x} + i(\Delta_y)_{Q_x} = Q_x[\bar{A}_1+A_2] \quad . \tag{6.2.32}$$

A similar operation with the second and the third expressions of (6.2.31) gives

$$(\Delta)_{Q_y} = (\Delta_x)_{Q_y} + i(\Delta_y)_{Q_y} = Q_y[i\bar{A}_1 - iA_2] \ . \tag{6.2.33}$$

And finally, summation (6.2.32) and (6.2.33) results in

$$\Delta_Q = (\Delta)_{Q_x} + (\Delta)_{Q_y} = \bar{A}_1 Q + A_2 \bar{Q} \ . \tag{6.2.34}$$

A comparison of (6.2.27) and (6.2.34) suggests that the tangential displacement discontinuity at the point $(\rho_0,\phi_0,0)$ due to a tangential force Q applied at the point (ρ,ϕ,z) can be obtained by using the expression for tangential displacements at the point (ρ,ϕ,z) due to a pair of equal and oppositely directed tangential forces T applied at the points $(\rho_0,\phi_0,0^{\pm})$, by way of substituting Q instead of T, and by replacing the coefficient of Q by its complex conjugate. Using this rule, from (6.2.3) may be obtained

$$\Delta_Q = \frac{H\gamma_1\gamma_2}{\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[f_2(z_k) + \frac{G_2}{G_1}f_7(z_k)\right]Q + \left[f_{16}(z_k) + \frac{G_2}{G_1}f_8(z_k)\right]\bar{Q}\right\}$$

$$+\frac{\beta}{\pi}\left\{\left[f_2(z_3) - \frac{G_2}{G_1}f_7(z_3)\right]Q + \left[f_{16}(z_3) - \frac{G_2}{G_1}f_8(z_3)\right]\bar{Q}\right\} \ . \tag{6.2.35}$$

Similar consideration can be made for two other systems in equilibrium. The first system is an elastic space weakened by an internal circular crack, with two equal and oppositely directed tangential forces T applied at the points $(\rho_0, \phi_0, 0^{\pm})$ of the crack faces (Fig.6.1(b)). The second one is the same space, with the crack faces tractions free, and the vertical force Q_z applied at the point (ρ, ϕ, z) (Fig.6.2(a)). And again, for the transformation to follow, present (6.2.4) in a generalized form

$$w = \mathcal{R}e(BT) = \frac{1}{2}(BT + \bar{B}\bar{T}) \ . \tag{6.2.36}$$

Here B is the combined factor of T.

The normal displacement at the point (ρ, ϕ, z) in the z direction due to a couple of forces T_x will be

$$w_{T_x} = T_x \ \frac{1}{2}(B + \bar{B}) = T_x \mathcal{R}e(B) \ . \tag{6.2.37}$$

The respective displacement due to a couple of forces T_y is

$$w_{T_y} = iT_y \ \frac{1}{2}(B - \bar{B}) = -T_y Im(B) \ . \tag{6.2.38}$$

If the tangential displacement discontinuity in the x and y directions due to force Q_z is denoted as Δ_x and Δ_y respectively, then according to the reciprocal theorem, it gives

$$(\Delta_x)_{Q_z} = Q_z \mathcal{R}e(B) \ , \qquad (\Delta_y)_{Q_z} = -Q_z Im(B) \ . \tag{6.2.39}$$

Summation of the first expression of (6.2.39) with the second one multiplied by i results in

$$\Delta_{Q_z} = (\Delta_x)_{Q_z} + i(\Delta_y)_{Q_z} = Q_z[\mathcal{R}e(B) - iIm(B)] = \bar{B}Q_z \tag{6.2.40}$$

A comparison of (6.2.36) and (6.2.40) suggests that the tangential displacement discontinuity at the point $(\rho_0,\phi_0,0)$ due to a normal force Q_z applied at the point (ρ,ϕ,z) can be obtained by using the expression for tangential displacements at the point (ρ,ϕ,z) due to a pair of equal and oppositely directed tangential forces T applied at the points $(\rho_0,\phi_0,0^{\pm})$. So, from (6.2.4) it gives

$$\Delta_{Q_z} = \frac{2}{\pi}Q_z H\gamma_1\gamma_2\sum_{k=1}^{2}\frac{m_k}{(m_k-1)\gamma_k}\left[f_1(z_k)+\frac{G_2}{G_1}f_9(z_k)\right] . \qquad (6.2.41)$$

The SIF of the second and third kind can be expressed through the tangential displacement discontinuity as it was defined in (2.47), namely,

$$K_2+iK_3=-\frac{a}{2\pi(G_1^2-G_2^2)\sqrt{2a}}\lim_{\rho_0\to a}\left[\frac{G_1e^{-i\phi_0}\,\Delta + G_2e^{i\phi_0}\,\bar{\Delta}}{(a^2-\rho_0^2)^{1/2}}\right]. \qquad (6.2.42)$$

Substitution of expressions in (6.2.35) and (6.2.41) in (6.2.42) and subsequent proceeding to the limit will give the desired results for the SIF.

For application of Q_x

$$K_2= -\frac{Q_x a}{4\pi^2\sqrt{2a}}\,\mathcal{Re}\left\{\sum_{k=1}^{2}\frac{1}{m_k-1}\left[\frac{G_1e^{-i\phi_0}}{G_1+G_2}\left(-f_3^{\bullet}(z_k)+f_4^{\bullet}(z_k)\right)\right.\right.$$

$$\left.+\frac{G_2e^{i\phi_0}}{G_1+G_2}\left(-\bar{f}_3^{\bullet}(z_k)+\bar{f}_4^{\bullet}(z_k)\right)\right] + \left[\frac{G_1e^{-i\phi_0}}{G_1-G_2}\left(f_5^{\bullet}(z_3)+f_6^{\bullet}(z_3)\right)\right.$$

$$+ \frac{G_2 e^{i\phi_0}}{G_1 - G_2} \left(\bar{f}_5^*(z_3) + \bar{f}_6^*(z_3)\right)\Bigg]\Bigg\} , \qquad (6.2.43)$$

$$K_3 = - \frac{Q_x a}{4\pi^2\sqrt{2a}} \, Im\left\{ \sum_{k=1}^{2} \frac{1}{m_k - 1} \left[\frac{G_1 e^{-i\phi_0}}{G_1 + G_2} \left(-f_3^*(z_k) + f_4^*(z_k)\right) \right.\right.$$

$$\left. + \frac{G_2 e^{i\phi_0}}{G_1 + G_2} \left(-\bar{f}_3^*(z_k) + \bar{f}_4^*(z_k)\right)\right] + \left[\frac{G_1 e^{-i\phi_0}}{G_1 - G_2} \left(f_5^*(z_3) + f_6^*(z_3)\right) \right.$$

$$\left.\left. + \frac{G_2 e^{i\phi_0}}{G_1 - G_2} \left(\bar{f}_5^*(z_3) + \bar{f}_6^*(z_3)\right)\right]\right\} . \qquad (6.2.44)$$

For application of Q_y

$$K_2 = - \frac{Q_y a}{4\pi^2\sqrt{2a}} \, \mathcal{R}e\left\{ \sum_{k=1}^{2} \frac{1}{m_k - 1} \left[\frac{G_1 i e^{-i\phi_0}}{G_1 + G_2} \left(-f_3^*(z_k) - f_4^*(z_k)\right) \right.\right.$$

$$\left. + \frac{G_2 i e^{i\phi_0}}{G_1 + G_2} \left(\bar{f}_3^*(z_k) + \bar{f}_4^*(z_k)\right)\right] + \left[\frac{G_1 i e^{-i\phi_0}}{G_1 - G_2} \left(f_5^*(z_3) - f_6^*(z_3)\right) \right.$$

$$\left.\left. + \frac{G_2 i e^{i\phi_0}}{G_1 - G_2} \left(-\bar{f}_5^*(z_3) + \bar{f}_6^*(z_3)\right)\right]\right\} , \qquad (6.2.45)$$

$$K_3 = - \frac{Q_y a}{4\pi^2\sqrt{2a}} \, Im\left\{ \sum_{k=1}^{2} \frac{1}{m_k - 1} \left[\frac{G_1 i e^{-i\phi_0}}{G_1 + G_2} \left(-f_3^*(z_k) - f_4^*(z_k)\right) \right.\right.$$

$$+ \frac{G_2 i e^{i\phi_0}}{G_1 + G_2} \left(\bar{f}_3^*(z_k) + \bar{f}_4^*(z_k)\right)\Bigg] + \left[\frac{G_1 i e^{-i\phi_0}}{G_1 - G_2} \left(f_5^*(z_3) - f_6^*(z_3)\right) \right.$$

$$+ \frac{G_2 i e^{i\phi_0}}{G_1 - G_2} \left(-\overline{f}^*_5(z_3) + \overline{f}^*_6(z_3)\right)\Bigg]\Bigg\} . \qquad (6.2.46)$$

For application of Q_z

$$K_2 = - \frac{Q_z a}{4\beta\pi^2\sqrt{2a}} \mathcal{R}e\left\{\sum_{k=1}^{2} \frac{m_k}{(m_k-1)\gamma_k}\Bigg[G_1\overline{f}^*_7(z_k)\right.$$

$$\left. + G_2\left(f^*_7(z_k) + f^*_8(z_k) + \frac{G_2}{G_1}\overline{f}^*_8(z_k)\right)\Bigg]\right\} , \qquad (6.2.47)$$

$$K_3 = - \frac{Q_z a}{4\beta\pi^2\sqrt{2a}} Im\left\{\sum_{k=1}^{2} \frac{m_k}{(m_k-1)\gamma_k}\Bigg[G_1\overline{f}^*_7(z_k)\right.$$

$$\left. + G_2\left(f^*_7(z_k) + f^*_8(z_k) + \frac{G_2}{G_1}\overline{f}^*_8(z_k)\right)\Bigg]\right\} . \qquad (6.2.48)$$

Here the functions $f^*_i(z)$ are given as

$$f^*_3(z_k) = \frac{(a^2-\ell^2_{1k})^{1/2}}{R^2_k a} + \frac{G_2}{G_1}\frac{a(a^2-\ell^2_{1k})^{1/2}}{s^2}\Bigg[\frac{3}{s^2} - \frac{\rho e^{i(\phi-\phi_0)}}{a(\ell^2_{2k}-a\rho e^{i(\phi-\phi_0)})}$$

$$- \frac{3(\ell^2_{2k}-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(\ell^2_{2k}-a^2)^{1/2}}\right)\Bigg] , \qquad (6.2.49)$$

$$f^*_4(z_k) = \frac{1}{\overline{q}}\left\{\frac{R^2_k+z^2_k}{R^2_k\overline{q}}\frac{(a^2-\ell^2_{1k})^{1/2}}{a} - \frac{3z_k}{\overline{s}\,\overline{q}}\tan^{-1}\left(\frac{\overline{s}}{(\ell^2_{2k}-a^2)^{1/2}}\right)\right.$$

$$- \frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}} \tan^{-1}\left(\frac{(a^2-\ell_{1k}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right) - \frac{e^{i\phi}}{\rho}\left(\frac{a(a^2-\ell_{1k}^2)^{1/2}}{\bar{s}^2} -1\right)$$

$$- \frac{G_2}{G_1}\left[\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}} \tan^{-1}\left(\frac{(a^2-\ell_{1k}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right) + \frac{e^{i\phi}\rho(a^2-\ell_{1k}^2)^{1/2}}{a(\ell_{2k}^2-a\rho e^{i(\phi-\phi_0)})}\right.$$

$$\left.\left. + \frac{e^{i\phi}}{\rho}\left(\frac{(a^2-\ell_{1k}^2)^{1/2}}{a} -1\right)\right]\right\} , \qquad (6.2.50)$$

$$f_5^*(z_3) = \frac{(a^2-\ell_{13}^2)^{1/2}}{R_3^2 a} - \frac{G_2}{G_1}\frac{a(a^2-\ell_{13}^2)^{1/2}}{s^2}\left[\frac{3}{s^2} - \frac{\rho e^{i(\phi-\phi_0)}}{a(\ell_{23}^2-a\rho e^{i(\phi-\phi_0)})}\right.$$

$$\left. - \frac{3(\ell_{23}^2-a^2)^{1/2}}{s^3} \tan^{-1}\left(\frac{s}{(\ell_{23}^2-a^2)^{1/2}}\right)\right] , \qquad (6.2.51)$$

$$f_6^*(z_3) = \frac{1}{\bar{q}}\left\{\frac{R_3^2+z_3^2}{R_3^2\bar{q}}\frac{(a^2-\ell_{13}^2)^{1/2}}{a} - \frac{3z_3}{\bar{s}\,\bar{q}} \tan^{-1}\left(\frac{\bar{s}}{(\ell_{23}^2-a^2)^{1/2}}\right)\right.$$

$$- \frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}} \tan^{-1}\left(\frac{(a^2-\ell_{13}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right) - \frac{e^{i\phi}}{\rho}\left(\frac{a(a^2-\ell_{13}^2)^{1/2}}{\bar{s}^2} -1\right)$$

$$+ \frac{G_2}{G_1}\left[\frac{(\bar{\zeta}-1)^{1/2}}{\bar{q}} \tan^{-1}\left(\frac{(a^2-\ell_{13}^2)^{1/2}}{a(\bar{\zeta}-1)^{1/2}}\right) + \frac{e^{i\phi}\rho(a^2-\ell_{13}^2)^{1/2}}{a(\ell_{23}^2-a\rho e^{i(\phi-\phi_0)})}\right.$$

$$\left.\left. + \frac{e^{i\phi}}{\rho}\left(\frac{(a^2-\ell_{13}^2)^{1/2}}{a} -1\right)\right]\right\} , \qquad (6.2.52)$$

$$f_7^{\bullet}(z_k) = \frac{z_k(a^2-\ell_{1k}^2)^{1/2}}{R_k^2 s^2} - \frac{a}{s^3}\tan^{-1}\left(\frac{s}{(\ell_{2k}^2-a^2)^{1/2}}\right) , \tag{6.2.53}$$

$$f_8^{\bullet}(z_k) = \frac{a}{s^3}\tan^{-1}\left(\frac{s}{(\ell_{2k}^2-a^2)^{1/2}}\right) - \frac{1}{a^2}\sin^{-1}\left(\frac{a}{\ell_{2k}}\right)$$

$$- \frac{e^{i(\phi-\phi_0)}\rho(\ell_{2k}^2-a^2)^{1/2}}{s^2(\ell_{2k}^2-\rho a e^{i(\phi-\phi_0)})} , \tag{6.2.54}$$

where R_k, q, s, ℓ_{1k} and ℓ_{2k} are defined as in (6.2.25-6.2.26), while

$$R_3 = [\rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z_3^2]^{1/2}, \quad \zeta = \frac{\rho}{a}\, e^{i(\phi-\phi_0)} ,$$

$$\ell_{13} = \frac{1}{2}\{[(\rho+a)^2+z_3^2]^{1/2} - [(\rho-a)^2+z_3^2]^{1/2}\} ,$$

$$\ell_{23} = \frac{1}{2}\{[(\rho+a)^2+z_3^2]^{1/2} + [(\rho-a)^2+z_3^2]^{1/2}\} . \tag{6.2.55}$$

Thus the expressions (6.2.20-6.2.22) and (6.2.43-6.2.48) are present all three modes of SIF in a transversely isotropic body.

6.2.4 ISOTROPIC SOLUTION

All the results obtained before are valid for isotropic solids subject to the conditions in (2.34) and (2.36), namely,

$$\gamma_1=\gamma_2=\gamma_3=1, \quad H=\frac{1-\nu^2}{\pi E}, \quad \beta=\frac{1+\nu}{\pi E}, \quad G_1=\frac{(2-\nu)(1+\nu)}{\pi E}, \quad G_2=\frac{\nu(1+\nu)}{\pi E}, \tag{6.2.56}$$

where E is the elastic modulus, and ν is Poisson coefficient. The limits were computed according to the L'Hospital rule. The following scheme was used:

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{m_k}{(m_k-1)} f(z_k)= \frac{2(1-\nu)f(z)-zf'(z)}{2(1-\nu)} , \tag{6.2.57}$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{\gamma_k}{m_k-1} f(z_k)= - \frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)} , \tag{6.2.58}$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{m_k}{(m_k-1)\gamma_k} f(z_k)=\frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)} , \tag{6.2.59}$$

$$\lim_{\gamma_1\to\gamma_2\to 1} \sum_{k=1}^{2} \frac{1}{m_k-1} f(z_k)=-f(z)-\frac{z}{2(1-\nu)} f'(z) . \tag{6.2.60}$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1=1 , \qquad \lim_{\gamma_1\to\gamma_2\to 1} \left[\frac{\partial m_1}{\partial \gamma_1}\right]=2(1-\nu) , \tag{6.2.61}$$

and the symbol (') means differentiation with respect to z.

Application of (6.2.57) and (6.2.58) to the expressions (6.2.20-6.2.22) will give K_1 SIF due to Q_z, Q_x and Q_y, namely, For Q_z

$$K_1 = \frac{Q_z (a^2 - \ell_1^2)^{1/2}}{2\pi^2 \sqrt{2a}\, R^2} f_1^{**}(z) \; . \tag{6.2.62}$$

For Q_x

$$K_1 = - \frac{Q_x (a^2 - \ell_1^2)^{1/2}}{4\pi^2 \sqrt{2a}\, (1-\nu)} \mathcal{Re}\, [f_2^{**}(z)] \; . \tag{6.2.63}$$

For Q_y

$$K_1 = - \frac{Q_y (a^2 - \ell_1^2)^{1/2}}{4\pi^2 \sqrt{2a}\, (1-\nu)} Im\, [f_2^{**}(z)] \; . \tag{6.2.64}$$

Here the functions $f_1^{**}(z)$ are given as

$$f_1^{**}(z) = 1 + \frac{1}{1-\nu} \left(\frac{z^2}{R^2} - \frac{\rho^2 - \ell_1^2}{2(\ell_2^2 - \ell_1^2)} \right) , \tag{6.2.65}$$

$$f_2^{**}(z) = \frac{1}{\bar{q}} \left[(1-2\nu) \left(\frac{a}{(a^2-\ell_1^2)^{1/2} \bar{s}} \tan^{-1} \left(\frac{\bar{s}}{(\ell_2^2 - a^2)^{1/2}} \right) - \frac{z}{R^2} \right) \right.$$

$$\left. + \frac{z}{R^2} \left(\frac{2z^2}{R^2} - \frac{\rho^2 - \ell_1^2}{\ell_2^2 - \ell_1^2} - 1 \right) - \frac{z \ell_2^2}{(\ell_2^2 - a^2 + \bar{s}^2)(\ell_2^2 - \ell_1^2)} \right] . \tag{6.2.66}$$

In Fig.6.3 and Fig.6.4 are given graphical representations of K_1 SIF due to Q_z and Q_x forces respectively, which are located at an arbitrary point in space.

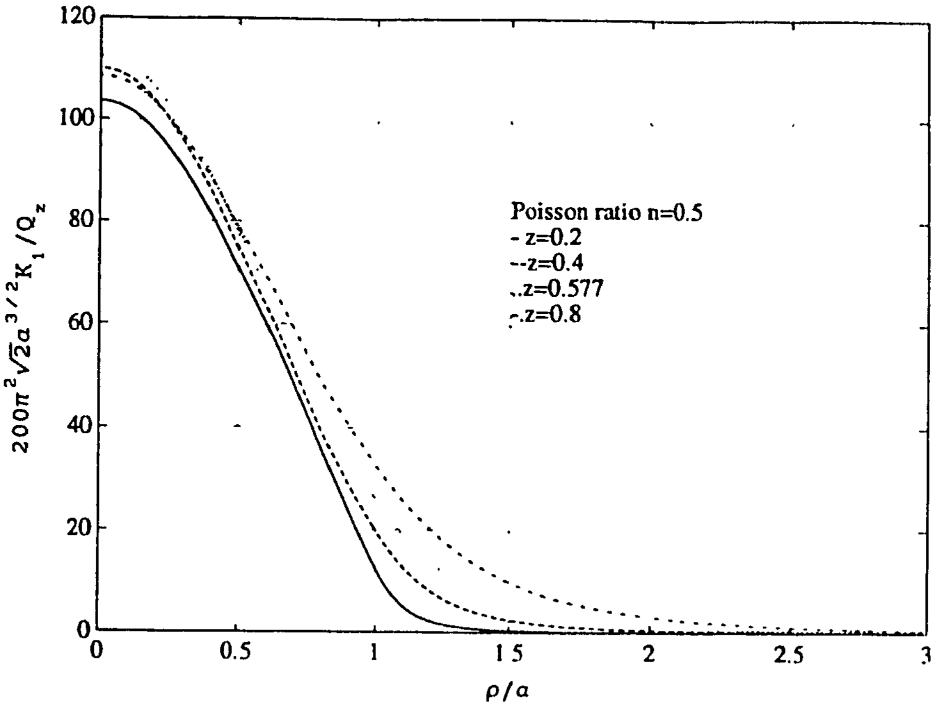

Fig.6.3 K_1 SIF due to force Q_z at arbitrary point in space for different z: (—z=0.2; --z=0.4; ··z=0.577; ·-z=0.8).

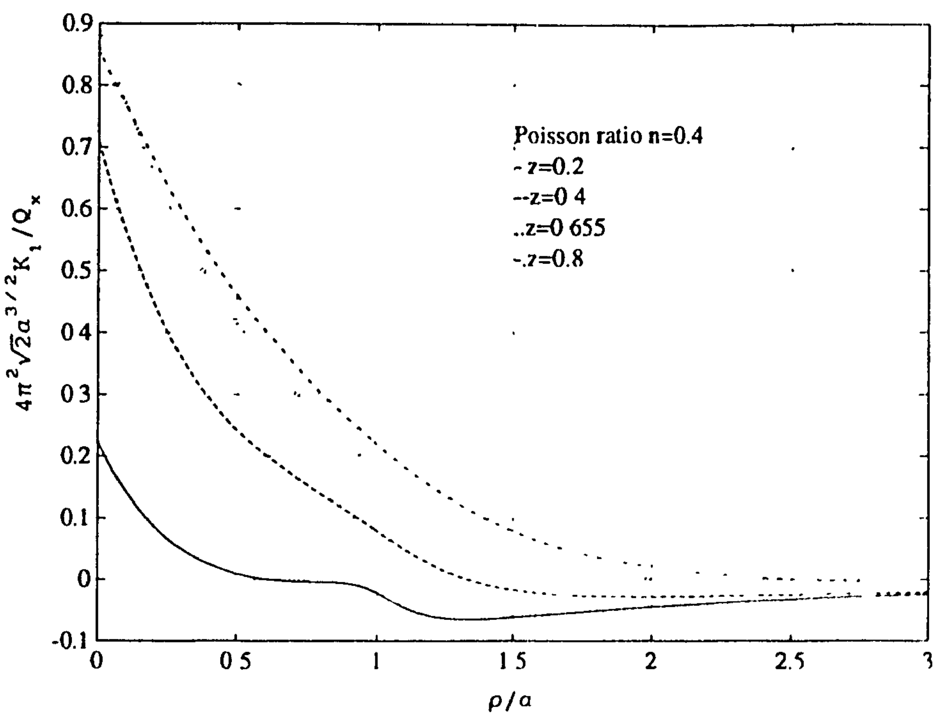

Fig.6.4 K_1 SIF due to force Q_x at arbitrary point in space for different z: (—z=0.2; --z=0.4; ··z=0.655; ·-z=0.8).

If formulae (6.2.59) and (6.2.60) are applied to the expressions in (6.2.43-6.2.48) then K_2 and K_3 SIF due to Q_z, Q_x and Q_y, will be obtained, namely,

For Q_z

$$K_2=-\frac{Q_z(a^2-\ell_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\mathcal{R}e\left[(2-\nu)\overline{f}_3^{**}(z)+\nu f_3^{**}(z)+\nu f_4^{**}(z)+\frac{\nu^2}{2-\nu}\overline{f}_4^{**}(z)\right],\tag{6.2.67}$$

$$K_3=-\frac{Q_z(a^2-\ell_1^2)^{1/2}}{8\pi^2(1-\nu)\sqrt{2a}}\mathcal{I}m\left[(2-\nu)\overline{f}_3^{**}(z)+\nu f_3^{**}(z)+\nu f_4^{**}(z)+\frac{\nu^2}{2-\nu}\overline{f}_4^{**}(z)\right].\tag{6.2.68}$$

For Q_x

$$K_2=-\frac{Q_x a}{4\pi^2\sqrt{2a}}\mathcal{R}e\left\{e^{i\phi_0}\left[\frac{\nu}{2}\left(\overline{f}_5^{**}(z)-\overline{f}_6^{**}(z)\right)+\frac{\nu}{4(1-\nu)}\left(\overline{f}_7^{**}(z)-\overline{f}_8^{**}(z)\right)\right]\right.$$

$$\left.+e^{-i\phi_0}\left[\frac{2-\nu}{2}\left(f_5^{**}(z)-f_6^{**}(z)\right)+\frac{2-\nu}{4(1-\nu)}\left(f_7^{**}(z)-f_8^{**}(z)\right)\right]\right\},\tag{6.2.69}$$

$$K_3=-\frac{Q_x a}{4\pi^2\sqrt{2a}}\mathcal{I}m\left\{e^{i\phi_0}\left[\frac{\nu}{2}\left(\overline{f}_5^{**}(z)-\overline{f}_6^{**}(z)\right)+\frac{\nu}{4(1-\nu)}\left(\overline{f}_7^{**}(z)-\overline{f}_8^{**}(z)\right)\right]\right.$$

$$\left.+e^{-i\phi_0}\left[\frac{2-\nu}{2}\left(f_5^{**}(z)-f_6^{**}(z)\right)+\frac{2-\nu}{4(1-\nu)}\left(f_7^{**}(z)-f_8^{**}(z)\right)\right]\right\}.\tag{6.2.70}$$

For Q_y

$$K_2 = -\frac{Q_y a}{4\pi^2\sqrt{2a}}\mathcal{Re}\left\{-ie^{i\phi_0}\left[\frac{\nu}{2}\left(\bar{f}_5^{**}(z)+\bar{f}_6^{**}(z)\right)+\frac{\nu}{4(1-\nu)}\left(\bar{f}_7^{**}(z)+\bar{f}_8^{**}(z)\right)\right]\right.$$

$$\left.+ie^{-i\phi_0}\left[\frac{2-\nu}{2}\left(f_5^{**}(z)+f_6^{**}(z)\right)+\frac{2-\nu}{4(1-\nu)}\left(f_7^{**}(z)+f_8^{**}(z)\right)\right]\right\},$$

(6.2.71)

$$K_3 = -\frac{Q_y a}{4\pi^2\sqrt{2a}}Im\left\{-ie^{i\phi_0}\left[\frac{\nu}{2}\left(\bar{f}_5^{**}(z)+\bar{f}_6^{**}(z)\right)+\frac{\nu}{4(1-\nu)}\left(\bar{f}_7^{**}(z)+\bar{f}_8^{**}(z)\right)\right]\right.$$

$$\left.+ie^{-i\phi_0}\left[\frac{2-\nu}{2}\left(f_5^{**}(z)+f_6^{**}(z)\right)+\frac{2-\nu}{4(1-\nu)}\left(f_7^{**}(z)+f_8^{**}(z)\right)\right]\right\}.$$

(6.2.72)

Here the functions $f_i^{**}(z)$ are given as

$$f_3^{**}(z) = \frac{a}{s^2}\left[(1-2\nu)\left(\frac{z}{R^2} - \frac{a}{s(a^2-\ell_1^2)^{1/2}}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)\right)\right.$$

$$\left.-\frac{z}{R^2}\left(1+\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}-\frac{2z^2}{R^2}\right)-\frac{z\ell_2^2}{(\ell_2^2-\rho a e^{i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}\right], \qquad (6.2.73)$$

$$f_4^{**}(z) = (1-2\nu)\left\{\frac{1}{a(a^2-\ell_1^2)^{1/2}}\left[\frac{a^3}{s^3}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)-\sin^{-1}\left(\frac{a}{\ell_2}\right)\right]\right.$$

$$\left.-\frac{z\rho a^2 e^{i(\phi-\phi_0)}}{(a^2-\ell_1^2)(\ell_2^2-\rho a e^{i(\phi-\phi_0)})s^2}\right\}+\frac{z\rho e^{i(\phi-\phi_0)}}{(\ell_2^2-\rho a e^{i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}$$

$$+\frac{2z\rho\ell_2^2 e^{i(\phi-\phi_0)}}{(\ell_2^2-\rho a e^{i(\phi-\phi_0)})^2(\ell_2^2-\ell_1^2)}\,, \tag{6.2.74}$$

$$f_5^{\bullet\bullet}(z)=\frac{2-\nu}{1-\nu}\,\frac{(a^2-\ell_1^2)^{1/2}}{R^2 a}-\frac{\nu^2}{(2-\nu)(1-\nu)}\,\frac{a(a^2-\ell_1^2)^{1/2}}{s^2}\left[\frac{3}{s^2}\right.$$

$$\left.-\frac{\zeta}{\ell_2^2-a\rho e^{i(\phi-\phi_0)}}-\frac{3(\ell_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)\right]\,, \tag{6.2.75}$$

$$f_6^{\bullet\bullet}(z)=\frac{\nu}{1-\nu}\,\frac{(a^2-\ell_1^2)^{1/2}}{\bar{q}^2 a}\left[\frac{3(\ell_2^2-a^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right)\right.$$

$$\left.-\frac{2R^2+z^2}{R^2}-\frac{\bar{q}\rho e^{i\phi}}{\ell_2^2-a\rho e^{i(\phi-\phi_0)}}\right]\,, \tag{6.2.76}$$

$$f_7^{\bullet\bullet}(z)=\frac{(a^2-\ell_1^2)^{1/2}}{R^2 a}\left(\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}-\frac{2z^2}{R^2}\right)+\frac{\nu}{2-\nu}\,\frac{(a^2-\ell_1^2)^{1/2}}{s^2 a}\left[\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}\left(\frac{3a^2}{s^2}\right.\right.$$

$$\left.-\frac{a\rho e^{i(\phi-\phi_0)}}{\ell_2^2-a\rho e^{i(\phi-\phi_0)}}\right)-\frac{3a^2(\ell_2^2-a^2)^{1/2}}{s^3}\tan^{-1}\left(\frac{s}{(\ell_2^2-a^2)^{1/2}}\right)$$

$$\left.+\frac{2z^2\ell_2^2 a\rho e^{i(\phi-\phi_0)}}{(\ell_2^2-a\rho e^{i(\phi-\phi_0)})^2(\ell_2^2-\ell_1^2)}+\frac{3z^2\ell_2^2 a^2}{s^2(\ell_2^2-a\rho e^{i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}\right], \tag{6.2.77}$$

$$f_8^{\bullet\bullet}(z)=\frac{(a^2-\ell_1^2)^{1/2}}{\bar{q}^2 a}\left[\frac{2z^2(R^2-z^2)}{R^4}+\frac{R^2+z^2}{R^2}\,\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}\right.$$

$$- \frac{3(\ell_2^2-a^2)^{1/2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{(\ell_2^2-a^2)^{1/2}}\right) + \frac{3z^2\ell_2^2}{(\ell_2^2-a\rho e^{-i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)}$$

$$- \frac{2}{2-\nu}\,\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}\left(\frac{\bar{s}^2}{\ell_1^2-a\rho e^{-i(\phi-\phi_0)}} - \frac{a e^{i(\phi-\phi_0)}}{\rho}\right) - \frac{\nu}{2-\nu}\left(\frac{\rho^2-\ell_1^2}{\ell_2^2-\ell_1^2}\right.$$

$$\left. + \frac{\bar{q}\rho e^{i\phi}(\rho^2-\ell_1^2)}{(\ell_2^2-a\rho e^{i(\phi-\phi_0)})(\ell_2^2-\ell_1^2)} - \frac{2z^2\ell_2^2\bar{q}\rho e^{i\phi}}{(\ell_2^2-a\rho e^{i(\phi-\phi_0)})^2(\ell_2^2-\ell_1^2)}\right)\Bigg]\Bigg] \; . \tag{6.2.78}$$

In Fig.6.5 and Fig.6.6 are given K_2 and K_3 SIF due to force Q_z, while in Fig.6.7 and Fig.6.8 are given K_2 and K_3 SIF due to force Q_x.

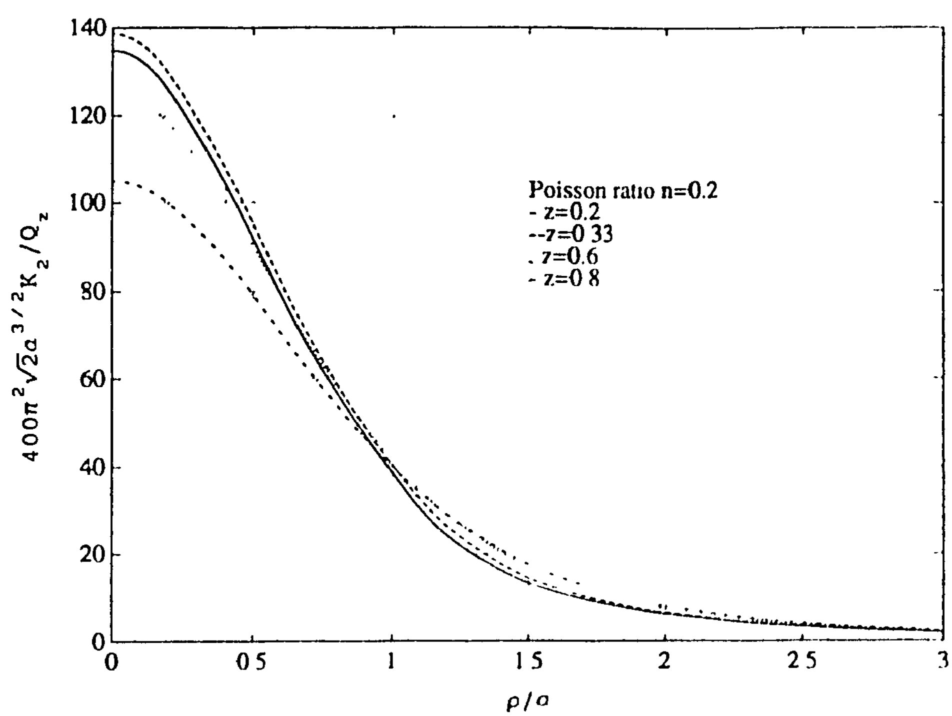

Fig.6.5 K_2 SIF due to force Q_z at arbitrary point in space for different z: (—z=0.2; --z=0.33; ··z=0.6; ·-z=0.8).

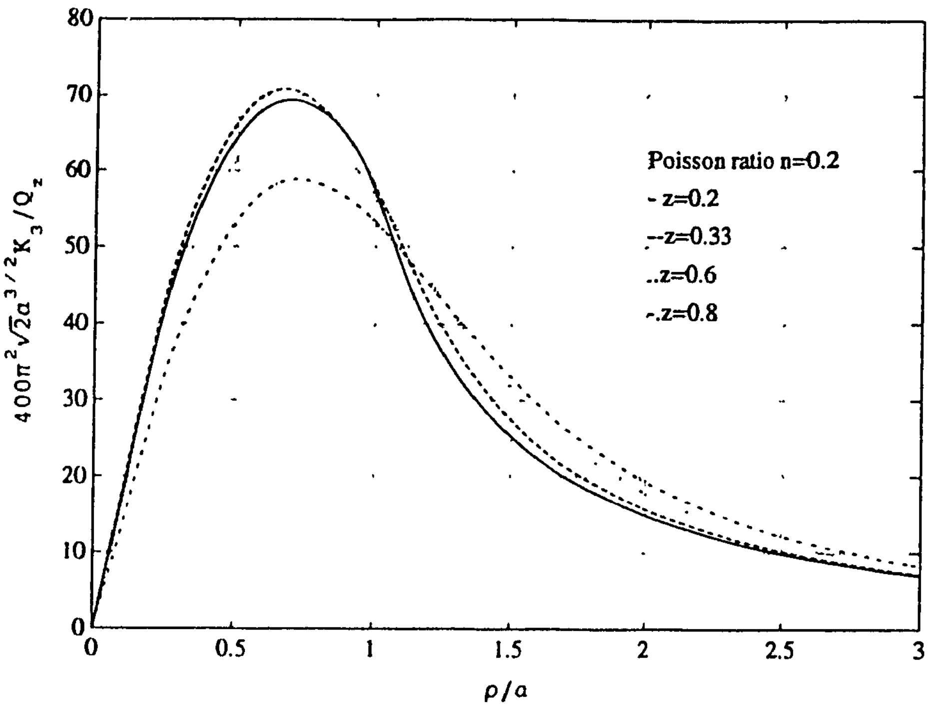

Fig.6.6 K_3 SIF due to force Q_z at arbitrary point in space for different z: (—z=0.2; --z=0.33; ··z=0.6; ·-z=0.8).

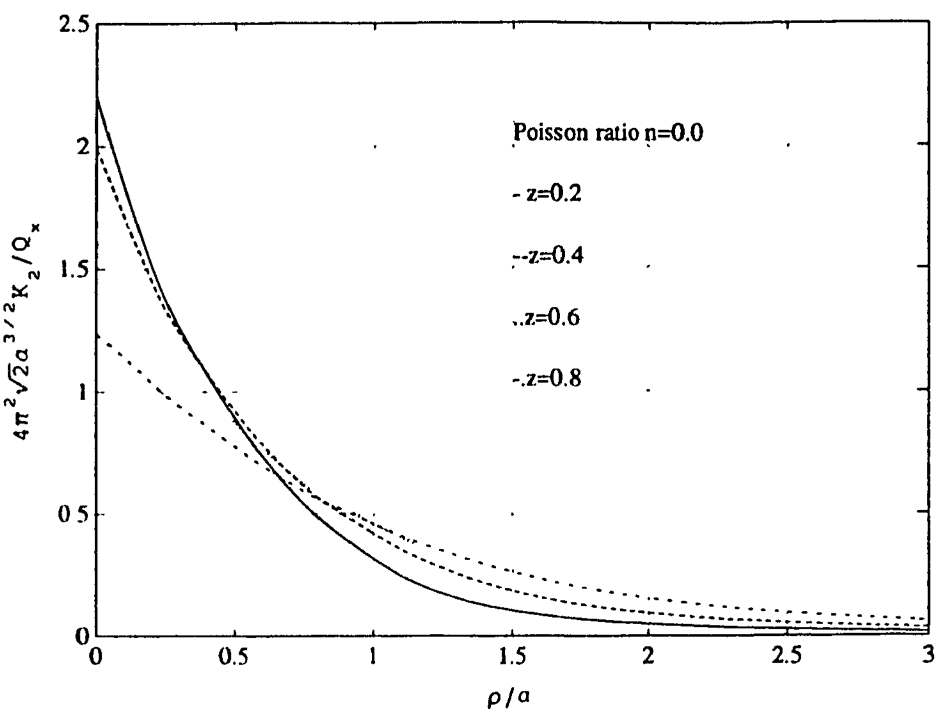

Fig.6.7 K_2 SIF due to force Q_x at arbitrary point in space for different z: (—z=0.2; --z=0.4; ··z=0.6; ·-z=0.8).

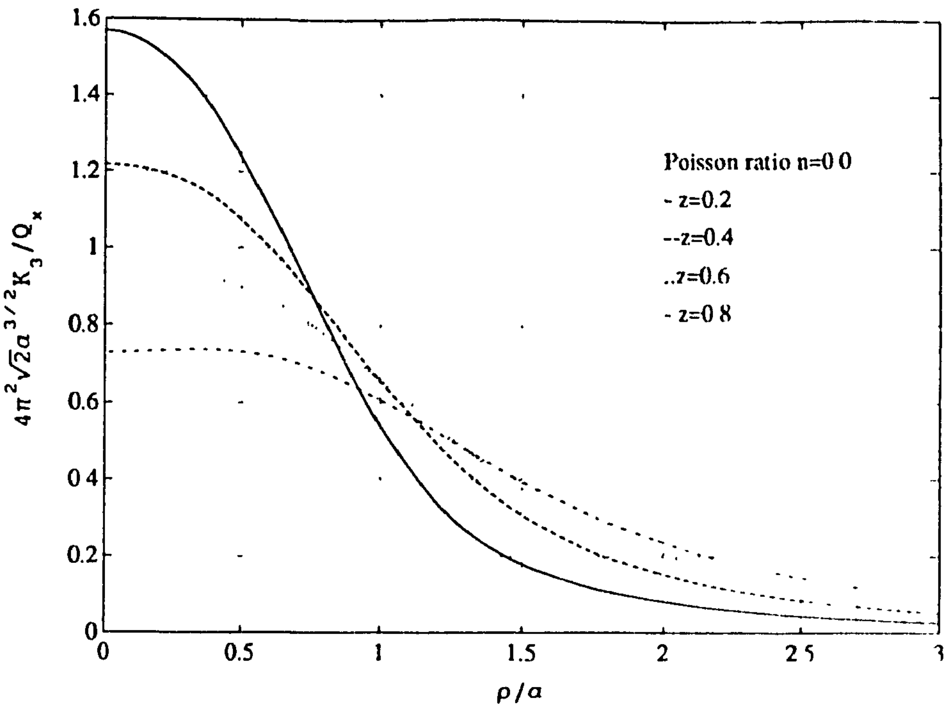

Fig.6.8 K_3 SIF due to force Q_x at arbitrary point in space for different z: (—z=0.2; --z=0.4; ··z=0.6; ·-z=0.8).

6.2.5 AXISYMMETRIC CASE AND COMPARISON WITH THE RESULTS REPORTED IN THE LITERATURE

It is quite interesting to consider the particular case of concentrated forces applied at the point on the vertical axes z, namely the case of axial symmetry when $\rho=0$. In this case the expressions obtained in (6.2.62-6.2.64) and (6.2.67-6.2.72) will drastically simplify and reduce to:

For Q_z

$$K_1 = \frac{Q_z(2a)^{1/2}}{4\pi^2(a^2+z^2)}\left[1 + \frac{1}{1-\nu}\,\frac{z^2}{a^2+z^2}\right], \tag{6.2.79}$$

$$K_2 = -\frac{Q_z}{4\pi^2(1-\nu)(2a)^{1/2}}\left[(1-2\nu)\left(\frac{z}{a^2+z^2} - \frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right)\right) - \frac{2za^2}{(a^2+z^2)^2}\right], \tag{6.2.80}$$

$$K_3 = 0 \,. \tag{6.2.81}$$

For Q_x

$$K_1 = \frac{Q_x\cos\phi_0}{4\pi^2(2a)^{1/2}(1-\nu)}\left\{(1-2\nu)\left[\frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right) - \frac{z}{a^2+z^2}\right] - \frac{2za^2}{(a^2+z^2)^2}\right\}, \tag{6.2.82}$$

$$K_2 = -\frac{Q_x\cos\phi_0}{2\pi^2(1-\nu)(2-\nu)(2a)^{3/2}}\left\{3(1-\nu)(1-2\nu)\left[\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right) - \frac{z^2}{a^2+z^2}\right]\right.$$

$$\left. + \frac{2a^2}{a^2+z^2}\left[2(1-\nu^2) - \frac{(2-\nu)z^2}{a^2+z^2}\right]\right\}, \tag{6.2.83}$$

$$K_3 = \frac{(1-2\nu)Q_x\sin\phi_0}{2\pi^2(2-\nu)(2a)^{3/2}}\left[3 - \frac{3z}{a}\tan^{-1}\left(\frac{a}{z}\right) + \frac{a^2}{a^2+z^2}\right] . \tag{6.2.84}$$

For Q_y

$$K_1 = \frac{Q_y\sin\phi_0}{4\pi^2(2a)^{1/2}(1-\nu)}\left\{(1-2\nu)\left[\frac{1}{a}\tan^{-1}\left(\frac{a}{z}\right) - \frac{z}{a^2+z^2}\right] - \frac{2za^2}{(a^2+z^2)^2}\right\}, \tag{6.2.85}$$

$$K_2 = -\frac{Q_y\sin\phi_0}{2\pi^2(1-\nu)(2-\nu)(2a)^{3/2}}\left\{3(1-\nu)(1-2\nu)\left[\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right) - \frac{z^2}{a^2+z^2}\right]\right.$$

$$\left. + \frac{2a^2}{a^2+z^2}\left[2(1-\nu^2) - \frac{(2-\nu)z^2}{a^2+z^2}\right]\right\}, \tag{6.2.86}$$

$$K_3 = -\frac{(1-2\nu)Q_y\cos\phi_0}{2\pi^2(2-\nu)(2a)^{3/2}}\left[3 - \frac{3z}{a}\tan^{-1}\left(\frac{a}{z}\right) + \frac{a^2}{a^2+z^2}\right] . \qquad (6.2.87)$$

All the results for the particular case, namely when $\rho=0$, are in perfect agreement with the known results given in Kassir and Sih, [12]. In spite of this, the graphical representations for the formulae (6.2.79-6.2.84) given in Kassir and Sih [12] are not quite correct and there is a reason to do them again and indicate some interesting features depicted on the plots. In Figs.6.9-6.13 are given graphical representations for the formulae (6.2.79-6.2.84).

For example the calculation of the maximum of the function in formula (6.2.79) results in the following relationship

$$z = \sqrt{\frac{\nu}{2-\nu}} . \qquad (6.2.88)$$

As it can be seen from (6.2.88), when say $\nu=0.5$ the value of $z=0.577$ is corresponding to the maximum of K_1 SIF as represented in Fig.6.9. It is interesting to note that in Fig.6.3 at the point $\rho=0$, for the same values of ν and z the maximum of K_1 SIF is identical to the one in Fig.6.9.

The calculation of the maximum of the function in formula (6.2.80) have result in the same relationship as in (6.2.88). In Fig.6.10 it can be seen that, when say $\nu=0.2$ the value of $z=0.33$ is corresponding to the maximum of K_2 SIF. And in Fig.6.5 for the same values of ν and z at the point $\rho=0$ the maximum of K_2 SIF is identical to the one in

Fig.6.10. In Fig.6.6 it appears that regardless of the values of ν and z, at the point $\rho=0$, the K_3 SIF is identically zero.

The calculation of the maximum of the function in formula (6.2.82) results in

$$z = \sqrt{\frac{1-\nu}{1+\nu}} \; . \tag{6.2.89}$$

And again, the similar discussion of Fig.6.11 and Fig.6.4 will indicate the correspondence of the maximum of K_1 SIF for the values of say $\nu=0.4$ and $z=0.655$ as a result of (6.2.89).

The analogous conclusions could be done for the rest of the graphical representations.

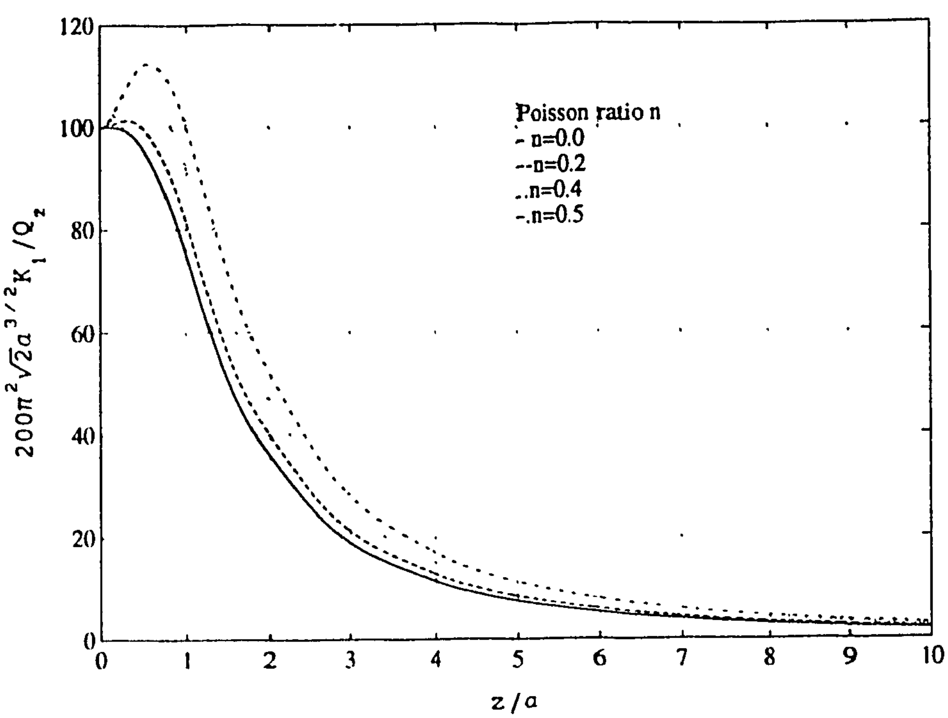

Fig.6.9 K_1 SIF due to force Q_z at a point on the normal axis for different n: (—n=0.0; --n=0.2; ··n=0.4; ·-n=0.5).

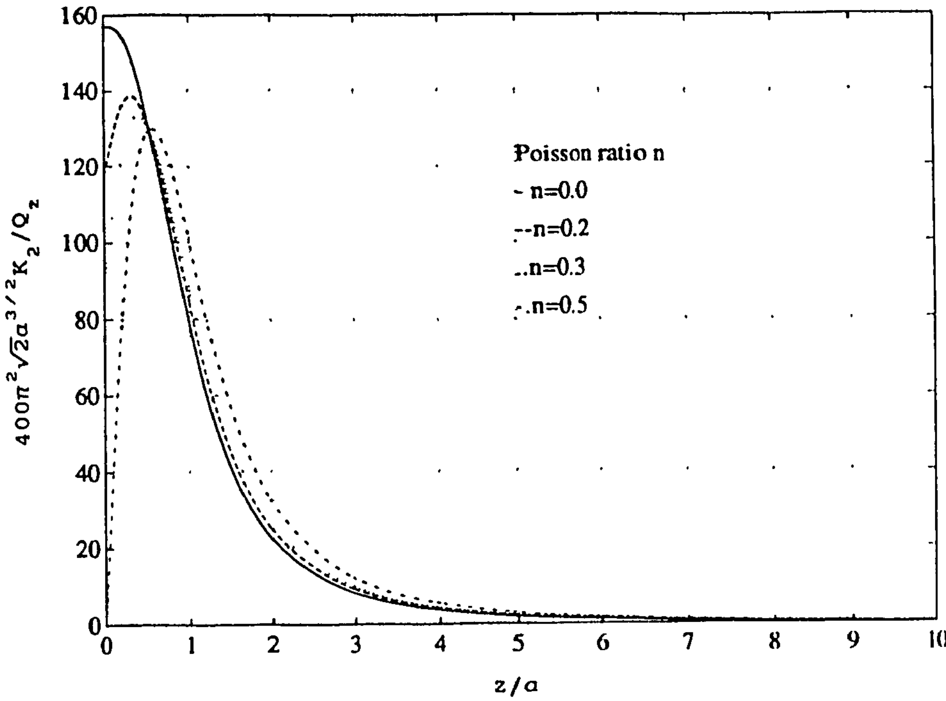

Fig.6.10 K_2 SIF due to force Q_z at a point on the normal axis for different n: (—n=0.0; --n=0.2; ··n=0.3; ·-n=0.5).

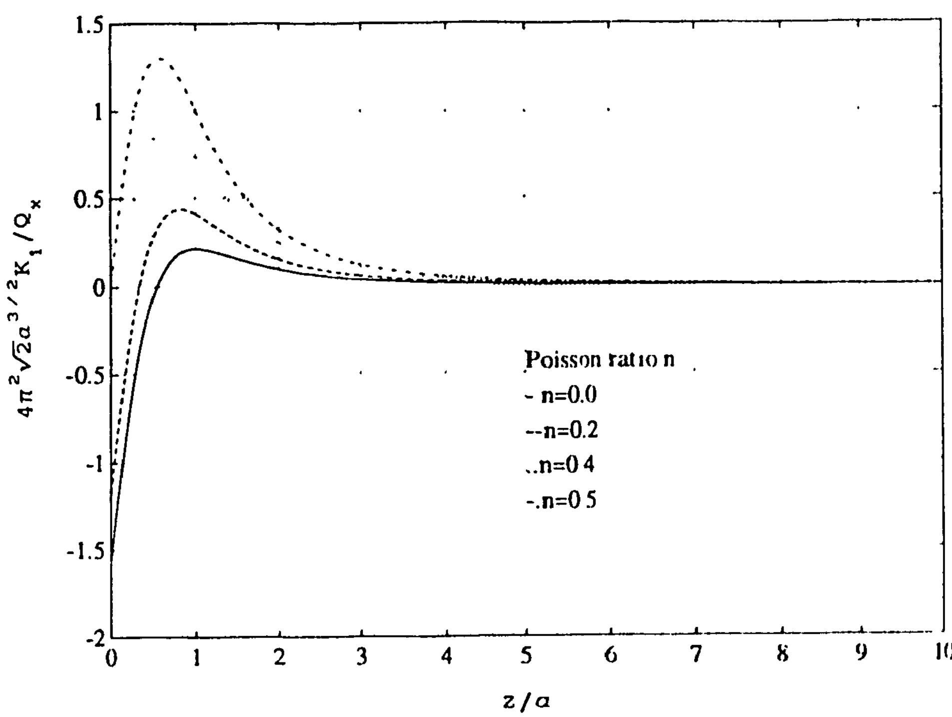

Fig.6.11 K_1 SIF due to force Q_x at a point on the normal axis for different n: (—n=0.0; --n=0.2; ··n=0.4; ·-n=0.5).

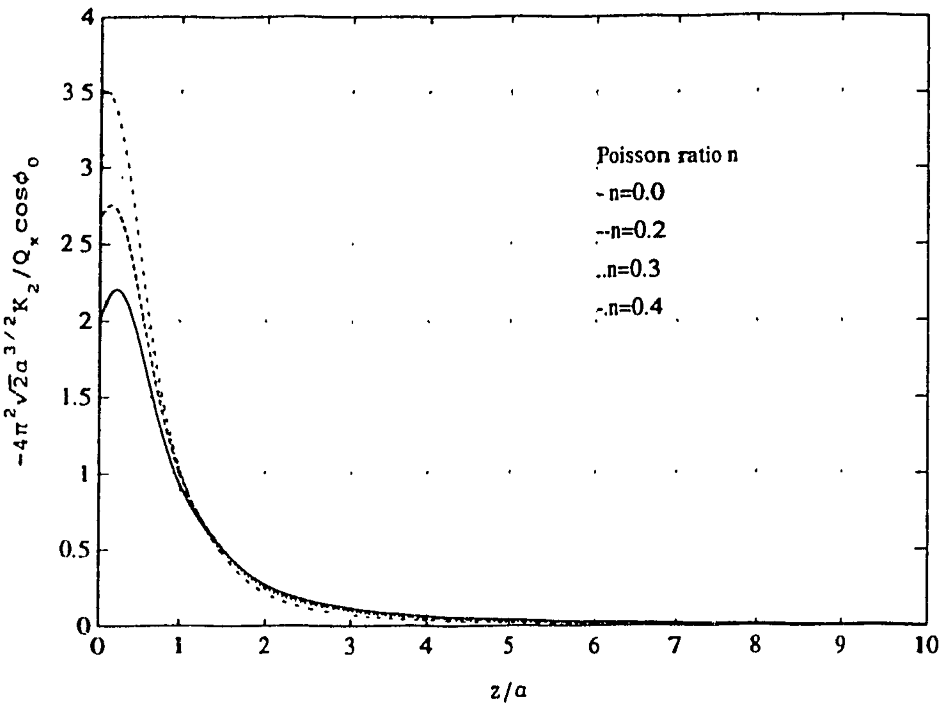

Fig.6.12 K_2 SIF due to force Q_x at a point on the normal axis for different n: (—n=0.0; --n=0.2; ··n=0.3; ·-n=0.4).

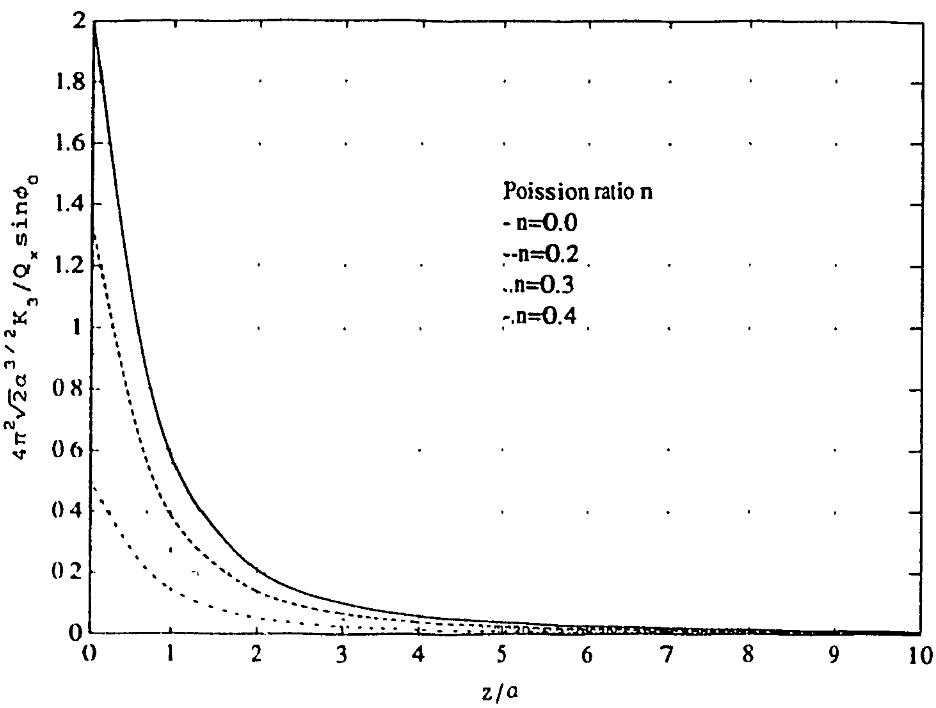

Fig.6.13 K_3 SIF due to force Q_x at a point on the normal axis for different n: (—n=0.0; --n=0.2; ··n=0.3; ·-n=0.4).

6.3 CONSIDERATION OF AN INTERACTION PROBLEM OF AN EXTERNAL CIRCULAR CRACK AND CONCENTRATED LOAD

The complete solution, obtained in Chapter 4 was of great value because it makes possible to solve easily many complicated problems which were not even attempted before. As an example, the interaction between an arbitrarily located horizontal force Q and an external circular crack of radius a. will be considered. It is necessary to find the SIF at the crack boundary.

The solution can be obtained in an elementary way by using the same procedure as in section 6.2. According to the rule obtained in formula (6.2.34) the tangential displacement discontinuity at the point $(\rho_0,\phi_0,0)$ due to a tangential force Q applied at the point (ρ,ϕ,z) will be

$$\Delta_Q= \frac{G_1-G_2}{2\pi}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\left[g_2(z_k)+ \frac{G_2}{G_1}g_7(z_k)\right]Q+\left[g_{16}(z_k)+ \frac{G_2}{G_1}g_8(z_k)\right]\bar{Q}\right\}$$

$$+ \frac{G_1+G_2}{2\pi}\left\{\left[g_2(z_3)- \frac{G_2}{G_1}g_7(z_3)\right]Q+\left[g_{16}(z_3)- \frac{G_2}{G_1}g_8(z_3)\right]\bar{Q}\right\}. \qquad (6.3.1)$$

Here the functions $g_i(z)$ are defined in section 4.4.

The SIF of the second and third kind can be expressed through the tangential displacement discontinuity as in (6.2.42).

The limiting quantities, which need to be computed, are as follows

$$\lim_{\rho_0\to a}\left\{\frac{g_2(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{(\ell_2^2-a^2)^{1/2}}{aR^2}\ , \tag{6.3.2}$$

$$\lim_{\rho_0\to a}\left\{\frac{g_7(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{z}{a^3}\left\{\frac{3\bar{t}^{5/2}}{(1-\bar{t}\ )^{5/2}}\left[\tan^{-1}\left(\frac{(a^2-\ell_1^2)^{1/2}}{s}\right)-\tan^{-1}\left(\frac{a}{s}\right)\right]\right.$$

$$\left.+\frac{\bar{t}^2}{(1-\bar{t}\ )^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}} - 2 - \bar{t} + \frac{a(1+\bar{t}\ )}{(a^2-\ell_1^2)^{1/2}}\right]\right\}\ , \tag{6.3.3}$$

$$\lim_{\rho_0\to a}\left\{\frac{g_7(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = e^{2i\phi_0}\left[\frac{a}{(a^2-\rho^2\bar{t})^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t}\ )^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right.$$

$$\left.-\frac{\bar{t}\ \ell_1(\rho^2-\ell_1^2)^{1/2}}{(a^2-\ell_1^2\bar{t}\ )(a^2-\rho^2\bar{t}\ )}\right]\ , \tag{6.3.4}$$

$$\lim_{\rho_0\to a}\left\{\frac{g_{16}(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{1}{\bar{q}}\left\{\frac{(R^2+z^2)\,(\ell_2^2-a^2)^{1/2}}{\bar{q}\ R^2} - \frac{2z}{\bar{q}\ a} -\right.$$

$$\left[\frac{e^{i\phi_0}}{(a^2-\rho^2\bar{t}\ )^{1/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t}\ )^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right] + \frac{z}{s}\left(\frac{2}{\bar{q}} + \frac{ae^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{a}{s}\right)\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{(a^2-\ell_1^2)^{1/2}}{s}\right)\right] + \frac{(\ell_2^2-a^2)^{1/2}e^{i\phi}}{\rho\bar{s}^2}\left(a-(a^2-\ell_1^2)^{1/2}\right)\right\}. \tag{6.3.5}$$

Here the following notations are introduced

$$R^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z^2, \qquad s = \sqrt{\rho a e^{i(\phi-\phi_0)}-a^2}\ ,$$

$$t = \frac{a}{\rho}e^{i(\phi-\phi_0)}\ , \qquad q = \rho e^{i\phi}-ae^{i\phi_0}\ . \tag{6.3.6}$$

It is reminded that the overbar everywhere denotes the complex conjugate quantity. Substitution of (6.3.2-6.3.5) and (6.3.1) in (6.2.42) gives the required SIF. It would be too cumbersome to write the final expression explicitly. A significant simplification takes place when $z=0$. It can be obtained from formula (4.4.41) of Chapter 4

$$\Delta_Q = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\frac{\bar{t}^2(1+\bar{t})\eta}{a^2(1-\bar{t})^2}\right]Q$$

$$+ \frac{G_2}{\pi}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{\bar{q}}\left(\frac{te^{i\phi}}{\rho(1-t)} - \frac{\bar{t}e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{Q}\ . \qquad (6.3.7)$$

The limit can be computed easily

$$\lim_{\rho_0\to a}\left\{\frac{\Delta_Q}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{G_1}{\pi}\frac{(\rho^2-a^2)^{1/2}}{a}\left[\frac{1}{R_a^2} - \frac{G_2^2}{G_1^2}\frac{\bar{t}_a^2(1+\bar{t}_a)}{a^2(1-\bar{t}_a)^2}\right]Q$$

$$+ \frac{G_2}{\pi}\frac{(\rho^2-a^2)^{1/2}}{a\bar{q}_a}\left[\frac{1}{\bar{q}_a} + \frac{t_a}{\bar{q}_a} - \frac{\bar{t}_a e^{i\phi_0}}{a(1-\bar{t}_a)}\right]\bar{Q}\ . \qquad (6.3.8)$$

and its substitution in (6.2.42) yields

$$K_2+iK_3 = \frac{e^{-i\phi_0}}{2\pi^2\sqrt{2a}}\left[\frac{Q}{R^2} + \frac{G_2}{G_1}\frac{e^{i\phi}(\rho e^{-i\phi} + ae^{-i\phi_0})}{\rho(\rho e^{-i\phi} - ae^{-i\phi_0})^2}\bar{Q}\right]\ , \qquad (6.3.9)$$

with $R^2=\rho^2+a^2-2\rho a\cos(\phi-\phi_0)$. The result (6.3.9) corresponds to the half of the expression (4.4.47), as it should be, since it is one-sided loading of the crack. Further and complete consideration of this problem may be done in a fashion similar to the one of section 6.2.

6.4 SUMMARY

In this chapter the solution of two interaction problems for internal and external circular crack have been presented. Knowledge of complete solution to the internal and external circular crack problems gives a powerful basis for solving more difficult problems of interaction of arbitrarily located forces with the crack. The complete solution plays also an indispensable role for consideration of interactions between cracks, *etc.*

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

7.1 GENERAL

The investigation carried out in the present work was directed towards the determination of fundamental new results in the area of three-dimensional fracture mechanics. The problems which were discussed can be separated into four main types, namely:

1) Internal circular crack problems.
2) External circular crack problems.
3) Semi-infinite crack problems.
4) Problems of interaction.

The method described in Chapter 2 was used to obtain the solution for these problems. However, it must be emphasized that the present work also reflects some extension of that method and even development of the similar new method for half-plane mixed BVP problems with application to the crack and contact problems. The results obtained in the present investigation clearly indicate that there are broad prospects for this method with regard to application and extension for the solution of different new types of problems. The possibilities for future work will be presented in the closing section.

7.2 CONCLUSIONS ON INTERNAL CIRCULAR CRACK PROBLEMS

The problems considered in Chapter 3, namely, internal circular crack under linear normal and linear shear loading, are new. Previous considerations of general loading were limited to a presentation of coupled integral equations with respect to an unknown function still to be determined. Because of certain analytical complications, the researchers would resort to the solution of the problems of a penny-shaped crack under constant loading. But even in those cases the final results were expressed in terms of integrals, while the results given in Chapter 3 for yet more complicated problems of variable loading are expressed in terms of elementary functions. In fact the absence of any results similar to those obtained in Chapter 3 makes it impossible to do an analysis of comparison. However, a few indications of both correspondence and discrepancy of the present results with the already known available results have been made. For example, formula (3.2.34) for the SIF in the problem of linear normal loading and formulae (3.2.42), (3.2.43) for the radial and tangential stresses in the problem of constant normal loading. In the case of linear shear loading, the problem considered comprised both axisymmetric and non-axisymmetric parts of loading. As it has already been mentioned, it is extremely difficult by previously known methods to obtain the solution to non-axisymmetric problems and as a result of it, it is not possible to find any solution similar to one considered here in order to do a comparison.

It must also be noted that the results obtained in Chapter 3 were both for isotropic and for transversely isotropic materials. The elegant limiting procedure has made it possible to obtain the solution for the isotropic case from the transversely isotropic one. This consistence, i.e. consideration of transversely isotropic and isotropic cases, has remained through the rest of the problems presented in this work.

The results obtained in Chapter 3 can be used in the stress analysis of various bodies with cracks subjected to bending and/or torsion.

7.3 CONCLUSIONS ON EXTERNAL CIRCULAR CRACK PROBLEMS

In Chapter 4 problems of external circular crack under normal and shear loading were considered. The results obtained there are new and of fundamental value. Their novelty consists of the fact that the solution for the field of stresses and displacements due to point force normal and shear loading was given for the full space and in terms of elementary functions. The results are of fundamental value because now it is possible to consider even more complicated problems such as: external circular crack under variable normal or shear loading, like in Chapter 3, or the problem of interaction between an external circular crack and arbitrarily located forces, like in Chapter 6, *etc.*

It should be mentioned that the consideration of those problems for the isotropic case was given first by Ufliand [50] and then by Lowengrub and Sneddon [51]. However, their solution was so complicated that it would limit them to obtaining only some of the stress components in the plane of the crack. Also, the approach in their solution was not general and was restricted by the consideration of constant loading, while in the general case would arise the same analytical complications as in the case of a penny-shaped crack. The present results have a general character, since, formulae (4.2.21) or (4.3.45-4.3.46) enable one to consider the problems of arbitrary loading.

Although it was not possible to compare the results obtained, due to unavailability of similar results, one of the main formulae, namely, (4.3.29) was derived by two different methods and was also verified by evaluation of the stress intensity factor.

It is important to indicate that the results presented in Chapter 4 for stress and displacement components are expressed in terms of elementary functions. For the problem of normal loading, the solution was obtained for both transversely isotropic and isotropic cases.

The results presented in Chapter 4 can be used for the stress analysis of the various bodies with cracks, provided that the region connecting two half spaces be small in comparison with the crack covering the region $z=0^{\pm}$.

Finally, it should be mentioned that the results of the section 4.3 are applicable to the contact problems as well.

7.4 CONCLUSIONS ON HALF-PLANE CRACK PROBLEMS

An intensive study of semi-infinite crack problems was made in Chapter 5. As in previous chapters, the results obtained in Chapter 5 are new and have fundamental value. The novelty consists of the complete solution for the field of stresses and displacements due to point force normal and shear loading. The fundamental value of those results was laid in the section 5.4, where the new method for the solution of relevant mixed BVP has been developed.

The complete solution to the elastic field for both normal and shear problems was obtained by making an original consideration of the limiting procedure, using the results of the complete elastic field for internal circular crack. This idea, which gave an exact solution, still may find its application in the consideration of some other problem, which will be mentioned in the very last section of the Chapter 7. The comparison with some of the results available in the literature has shown an exact correspondence. This was given in subsections 5.2.4 and 5.3.4.

It is interesting that the same results were obtained with help of the new development method in section 5.4, namely correspondence of formulae (5.4.80-5.4.90) and (5.2.23-5.2.33).

The solution was obtained for both transversely isotropic and isotropic cases and was expressed in terms of elementary functions. Having a complete solution at hand make it possible to consider the problem of interaction between the crack and arbitrarily located forces.

The method proposed and developed in section 5.4 was based on a new integral representation for the reciprocal of the distance between two points. It led to the solution not only for the crack problem but also for the punch problem. The solution to the governing integral equations are of a general nature, since they make it possible to express the potential in the whole space through its arbitrary boundary value as in formulae (5.4.33) or (5.4.52). All this illustrates that the method enables one to consider some other types of problems, for example, when the loading prescribed on the crack faces is variable.

Finally, the new developed method proves that it is in fact possible to make a consideration of crack problems with geometries other than circular.

7.5 CONCLUSIONS ON INTERACTION PROBLEMS

Chapter 6 has shown how a complete solution makes it possible to consider more complicated problems of interaction. One type of interaction problem, namely, interaction between an internal or external circular crack and arbitrarily located forces, was considered. The solution to those problems was obtained by using the reciprocal theorem. In the case of interaction between an internal circular crack and an arbitrarily located force, there were obtained-closed form expressions in terms of elementary functions for the crack opening displacement and stress intensity factors for all three modes. The results were obtained for both transversely isotropic and isotropic cases. Such explicit expressions have not been given in literature previously. Thus, the results are new and it was not possible to make an analysis of comparison. Only for the axisymmetric case results are available and reported in literature; comparison of present results with those have indicated an exact correspondence. It was given in subsection 6.2.5. Also, was made a thorough graph-analysis of presently obtained results which once again has demonstrated their correctness.

The solution of the interaction problem between an external circular crack and an arbitrarily located force was presented partially. The procedure was outlined and some of the expressions which are necessary for the complete solution were obtained. However, it should be indicated that the complete solution is now readily available due to the results obtained in Chapter 4.

Thus, the results of the problems considered in Chapter 6 have emphasized the fundamental importance of the complete solution of problems investigated in this work and have proven that they are in fact obtainable if the complete elastic field is known.

7.6 IMPLICATIONS

In the previous sections the fundamental importance of all the results of the present investigation in terms of their theoretical implications were emphasized. Now the practical applicability of those results must be underlined. In section 1.4 the practical significance of the problems considered in this work was generally indicated. In addition, it may be said that any further investigation of the fundamental problems also have a practical application. For example, the results of Chapter 3 can be used in stress analysis of various cracked bodies subjected to bending or torsion. Similar results may be obtained for external circular crack under bending or torsion.

The results of stress intensity factors obtained for all the problems considered are of practical significance as well, since they are known as a failure criterion. In that respect, the problems of interaction considered in Chapter 6 are of vast practical importance.

Finally, in section 1.4 it was mentioned that for the development of the boundary force method for the solution of the cracked bodies of the finite dimension, it is necessary to have a complete solution to the three-dimensional infinite crack problems. A brief idea of the method consists of the following: Assume, that the solution to the problem of an infinite cracked body in interaction with arbitrarily located force is known. Then, making imaginary cut along the contour of the finite body from the infinite space and applying to the boundary of that contour the unknown distributed forces so, that to make the boundary stress-free. This will result in an integral equation with respect to those unknown forces. By obtaining its solution, one can solve the problem of finite body with crack.

7.7 DIRECTIONS FOR FURTHER RESEARCH

A number of problems, which are the logical continuation of the present investigation, are given below:

1) For external circular crack the following problems may be considered:

a) External circular crack under variable normal loading.

b) External circular crack under variable shear loading.

c) Crack opening displacements and stress intensity factors caused by a concentrated load outside an external circular crack.

d) External circular crack under antisymme' ic normal loading.

e) External circular crack under symmetric shear loading.

The solution of the problems in d) and e) will make it possible to obtain the results for one-sided loading of the crack face by using the already known results of Chapter 4 and a principle of superposition.

The same type of problems indicated in a)-e) can be considered for the semi-infinite crack.

The fundamental problem may be considered for the interaction of two non-axially symmetric penny-shaped cracks in an infinite solid.

Similar to the problem of Chapter 5, namely the new development method for half-plane mixed BVP, there may be developed new methods for other geometries of crack, based on the different integral representations for the reciprocal of the distance between two points.

The list of problems may be continued, since much remains to be investigated in the area of three-dimensional fracture mechanics with the new approach which makes it possible to determine complete solutions in closed form and in terms of elementary functions.

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APPENDICES

APPENDIX A3.2

The results of differentiation of $\chi(\rho,z)$, defined by the equation (3.2.14), needed for the elastic field are:

$$\chi(\rho,z)=a(\ell_2^2-a^2)^{1/2}\left(15\frac{\ell_1^2}{a^2}-12-2\frac{\ell_1^2}{\ell_2^2}\right)$$

$$+\sin^{-1}\left(\frac{a}{\ell_2}\right)(4a^2-3\rho^2+12z^2)\ , \qquad \text{(A3.2.1)}$$

$$\frac{\partial\chi}{\partial\rho}=6\left[(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(1+\frac{2}{3}\ \frac{a^2}{\ell_2^2}\right)-\rho\sin^{-1}\left(\frac{a}{\ell_2}\right)\right]\ , \qquad \text{(A3.2.2)}$$

$$\frac{\partial^2\chi}{\partial\rho^2}=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(6-12\frac{a^2}{\ell_2^2}+16\frac{a^2}{\ell_2^2-\ell_1^2}\right)-6\sin^{-1}\left(\frac{a}{\ell_2}\right)\ , \qquad \text{(A3.2.3)}$$

$$\frac{\partial\chi}{\partial z}=8\left[(a^2-\ell_1^2)^{1/2}\left(\frac{a^2}{\ell_2^2}-3\right)+3z\sin^{-1}\left(\frac{a}{\ell_2}\right)\right]\ , \qquad \text{(A3.2.4)}$$

$$\frac{\partial^2\chi}{\partial z^2}=24\left[\sin^{-1}\left(\frac{a}{\ell_2}\right)-\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\ \frac{\ell_1}{\ell_2}\left(1+\frac{2}{3}\ \frac{a^2}{\ell_2^2-\ell_1^2}\right)\right]\ , \qquad \text{(A3.2.5)}$$

$$\frac{\partial^2\chi}{\partial\rho\partial z}=16\frac{1}{\rho}(a^2-\ell_1^2)^{1/2}\ \frac{\ell_1^4}{\rho^2(\ell_2^2-\ell_1^2)}\ . \qquad \text{(A3.2.6)}$$

Here ℓ_1 and ℓ_2 are defined by equation (3.2.15).

APPENDIX A3.3

The results of differentiation of the function $I(\rho,z)$, defined by the expression (3.3.14), are given below:

$$I(\rho,z)=(a^2-\ell_1^2)^{1/2}\left[\frac{4}{3}a^2+7\rho^2-\frac{19}{3}\ell_1^2-4\ell_2^2+\frac{2(8a^4+4a^2\ell_1^2+3\ell_1^4)}{15\rho^2}\right]$$

$$+z(4a^2-3\rho^2+4z^2)\sin^{-1}\left(\frac{a}{\ell_2}\right)-\frac{16a^5}{15\rho^2}\,, \qquad \text{(A3.3.1)}$$

$$\frac{\partial I}{\partial\rho}=\rho(a^2-\ell_1^2)^{1/2}\left(6-2\frac{\ell_1^2}{\rho^2}-\frac{4}{5}\frac{\ell_1^4}{\rho^4}-\frac{16}{15}\frac{a^4}{\rho^2\ell_2^2}-\frac{32}{15}\frac{a^4}{\rho^4}\right)$$

$$-6\rho z\sin^{-1}\left(\frac{a}{\ell_2}\right)+\frac{32}{15}\frac{a^5}{\rho^3}\,, \qquad \text{(A3.3.2)}$$

$$\frac{\partial^2 I}{\partial\rho^2}=(a^2-\ell_1^2)^{1/2}\left(6-2\frac{\ell_1^2}{\rho^2}+\frac{12}{5}\frac{\ell_1^4}{\rho^4}+\frac{16}{5}\frac{a^2\ell_1^2}{\rho^4}+\frac{32}{5}\frac{a^4}{\rho^4}\right)$$

$$-6z\sin^{-1}\left(\frac{a}{\ell_2}\right)-\frac{32}{5}\frac{a^5}{\rho^4}\,, \qquad \text{(A3.3.3)}$$

$$\frac{\partial^3 I}{\partial\rho^3}=\frac{1}{\rho}(a^2-\ell_1^2)^{1/2}\left[-16\frac{\ell_1^4}{\rho^2(\ell_2^2-\ell_1^2)}-\frac{48}{5}\frac{\ell_1^4}{\rho^4}-\frac{64}{5}\frac{a^2\ell_1^2}{\rho^4}\right.$$

$$\left.-\frac{128}{5}\frac{a^4}{\rho^4}\right]+\frac{128}{5}\frac{a^5}{\rho^5}\,, \qquad \text{(A3.3.4)}$$

$$\frac{\partial I}{\partial z}=a(\ell_2^2-a^2)^{1/2}\left(15\frac{\ell_1^2}{a^2}-12-2\frac{\ell_1^2}{\ell_2^2}\right)+\sin^{-1}\left(\frac{a}{\ell_2}\right)(4a^2-3\rho^2+12z^2)\,,$$

$$\text{(A3.3.5)}$$

$$\frac{\partial^2 I}{\partial z^2}=8\left[(a^2-\ell_1^2)^{1/2}\left(\frac{a^2}{\ell_2^2}-3\right)+3z\sin^{-1}\left(\frac{a}{\ell_2}\right)\right], \tag{A3.3.6}$$

$$\frac{\partial^3 I}{\partial z^3}=24\left[\sin^{-1}\left(\frac{a}{\ell_2}\right)-\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left(1+\frac{2}{3}\frac{a^2}{\ell_2^2-\ell_1^2}\right)\right], \tag{A3.3.7}$$

$$\frac{\partial^2 I}{\partial\rho\partial z}=\frac{a}{\rho}(\ell_2^2-a^2)^{1/2}\left(6\frac{\ell_1^2}{a^2}+4\frac{\ell_1^2}{\ell_2^2}\right)-6\rho\sin^{-1}\left(\frac{a}{\ell_2}\right), \tag{A3.3.8}$$

$$\frac{\partial^3 I}{\partial\rho\partial z^2}=16\frac{1}{\rho}(a^2-\ell_1^2)^{1/2}\frac{\ell_1^4}{\rho^2(\ell_2^2-\ell_1^2)}, \tag{A3.3.9}$$

$$\frac{\partial^3 I}{\partial\rho^2\partial z}=\frac{1}{\rho}(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left[6-12\frac{a^2}{\ell_2^2}+16\frac{a^2}{\ell_2^2-\ell_1^2}\right]-6\sin^{-1}\left(\frac{a}{\ell_2}\right), \tag{A3.3.10}$$

$$\frac{\partial^4 I}{\partial\rho\partial z^3}=16\frac{1}{\rho^2}(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left[\frac{\ell_1^2(5\ell_1^2-4a^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{2\ell_1^2(a^2-\ell_1^2)(\ell_2^2+\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}\right], \tag{A3.3.11}$$

$$\frac{\partial^4 I}{\partial\rho^2\partial z^2}=16(a^2-\ell_1^2)^{1/2}\left[\frac{a^2}{(\ell_2^2-\ell_1^2)^2}+\frac{3a^2(\ell_2^2-a^2)}{\ell_2^4(\ell_2^2-\ell_1^2)}-\frac{4a^2(\ell_2^2-a^2)}{(\ell_2^2-\ell_1^2)^3}\right], \tag{A3.3.12}$$

$$\frac{\partial^4 I}{\partial\rho^3\partial z}=16\frac{1}{\rho^2}(\ell_2^2-a^2)^{1/2}\frac{\ell_1}{\ell_2}\left[\frac{4\ell_1^2\ell_2^2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^3}+\frac{\ell_1^2(2a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)^2}-\frac{3\ell_1^4}{5\rho^2(\ell_2^2-\ell_1^2)}\right]. \tag{A3.3.13}$$

Here ℓ_1 and ℓ_2 are defined in (3.3.15).

APPENDIX A4.4

The computation of various derivatives of L_1 and L_2 as they are defined in (4.4.12) are presented here. The simplest integral to compute is

$$I_1=\iint_S \frac{B_2(N,N_0)}{R(M,N)}dS_N$$

$$=\int_0^{2\pi}\int_a^{\infty}\frac{ae^{-i(\psi-\phi_0)}(r\rho_0 e^{i(\psi-\phi_0)}+a^2)}{r\rho_0(r\rho_0 e^{i(\psi-\phi_0)}+a^2)^2}\ \frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}\,r\,dr\,d\psi}{[r^2+\rho^2-2r\rho\cos(\phi-\psi_0)+z^2]^{1/2}}\ . \tag{A4.4.1}$$

Use the integral representation from Chapter 2, namely,

$$\frac{1}{R(M,N)}=\frac{1}{[(r^2+\rho^2-2r\rho\cos(\phi-\psi_0)+z^2]^{1/2}}$$

$$=\frac{2}{\pi}\int_{\ell_2(r)}^{\infty}\frac{\lambda(\rho r/\alpha^2,\phi-\psi)\,d\alpha}{[(\alpha^2-\rho^2)(g^2(\alpha)-r^2)]^{1/2}}\ . \tag{A4.4.2}$$

Here

$$\lambda(k,\vartheta)=\frac{1-k^2}{1+k^2-2k\cos\vartheta}=\sum_{k=-\infty}^{\infty}k^{|n|}e^{in\vartheta}\ ,\qquad \text{for } k<1\ , \tag{A4.4.3}$$

$$\ell_2(r)=\frac{1}{2}\left[\sqrt{(r+\rho)^2+z^2}+\sqrt{(r-\rho)^2+z^2}\right],$$

$$g(\alpha)=\alpha\left(1-\frac{z^2}{\alpha^2-\rho^2}\right)^{1/2}. \tag{A4.4.4}$$

It is reminded that the function g is inverse to ℓ_2, so that $g[\ell_2(r)]=r$. I_1 can be transformed by substituting (A4.4.2) in (A4.4.1) and expanding B_2 in Fourier series

$$I_1=\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{a^3}\left[\left(\frac{a^2e^{-i(\psi-\phi_0)}}{r\rho_0}\right)^2\sum_{n=0}^{\infty}(2n+1)\left(\frac{a^2e^{-i(\psi-\phi_0)}}{r\rho_0}\right)^n\right]$$

$$\times\left[\frac{2}{\pi}\int_{\ell_2(r)}^{\infty}\frac{\lambda(\rho r/\alpha^2,\phi-\psi)\,d\alpha}{[(\alpha^2-\rho^2)(g^2(\alpha)-r^2)]^{1/2}}\right]rdrd\psi$$

$$=4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{\ell_2}^{\infty}\frac{d\alpha}{\sqrt{\alpha^2-\rho^2}}\left[\int_a^{g(x)}\frac{\sqrt{r^2-a^2}\,rdr}{\sqrt{g^2(\alpha)-r^2}}\right]\sum_{n=0}^{\infty}(2n+1)\left(\frac{a^2\rho}{\alpha^2\rho_0}e^{-i(\phi-\phi_0)}\right)^{n+2}$$

$$=\pi\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{\ell_2}^{\infty}\frac{g^2(\alpha)-a^2}{\sqrt{\alpha^2-\rho^2}}\left(\frac{a^2\rho}{\alpha^2\rho_0}e^{-i(\phi-\phi_0)}\right)^2\frac{1+\dfrac{a^2\rho}{\alpha^2\rho_0}e^{-i(\phi-\phi_0)}}{\left(1-\dfrac{a^2\rho}{\alpha^2\rho_0}e^{-i(\phi-\phi_0)}\right)^2}d\alpha$$

$$=\pi\bar{\varepsilon}^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{\ell_2}^{\infty}\frac{(\alpha^2-\ell_1^2)(\alpha^2-\ell_2^2)(\alpha^2+\bar{\varepsilon}^2)}{\alpha^2(\alpha^2-\bar{\varepsilon}^2)^2(\alpha^2-\rho^2)^{3/2}}d\alpha\ . \qquad \text{(A4.4.5)}$$

The abbreviation ℓ_2 stands for $\ell_2(r)$, as it is defined in (A4.4.4), and

$$\ell_1\equiv\ \ell_1(r)=\frac{1}{2}\left[\sqrt{(r+\rho)^2+z^2}+\sqrt{(r-\rho)^2+z^2}\right]\ ,\quad \varepsilon=a^2\frac{\rho}{\rho_0}e^{i(\phi-\phi_0)}\ .$$

(A4.4.6)

The following rule of interchange of the order of integration was used in (A4.4.5)

$$\int_a^\infty dr \int_{\ell_2(r)}^\infty d\alpha = \int_{\ell_2(r)}^\infty d\alpha \int_a^{g(x)} dr \; . \tag{A4.4.7}$$

Now the computation of I_1 has been reduced to an elementary single integral. The integrand in (A4.4.5) can be decomposed into simple fractions

$$\frac{(\alpha^2-\ell_1^2)(\alpha^2-\ell_2^2)(\alpha^2+\bar{\varepsilon}^2)}{\alpha^2(\alpha^2-\bar{\varepsilon}^2)^2} = 1 + \frac{a^2\rho^2}{\bar{\varepsilon}^2\alpha^2} + \frac{3\bar{\varepsilon}^4-(\ell_1^2+\ell_2^2)\bar{\varepsilon}^2-a^2\rho^2}{\bar{\varepsilon}^2(\alpha^2-\bar{\varepsilon}^2)}$$

$$+ \frac{2(\bar{\varepsilon}^2-\ell_1^2)(\bar{\varepsilon}^2-\ell_2^2)}{(\alpha^2-\bar{\varepsilon}^2)^2} \; . \tag{A4.4.8}$$

Substitution of (A4.4.8) in (A4.4.5) finally allows to compute

$$I_1 = \pi\frac{\sqrt{\rho_o^2-a^2}}{a^3}\bar{t}^2\left\{\frac{z^2(1+\bar{t})}{(1-\bar{t})^2}\left(1-\frac{a}{(a^2-\ell_1^2)^{1/2}}\right) - \frac{a^2}{\bar{t}}\left(1-\frac{(a^2-\ell_1^2)^{1/2}}{a}\right)\right.$$

$$+ \frac{\rho^2\bar{t}^2-\bar{t}(a^3+\rho^2+z^2)+a^2}{\bar{t}(1-\bar{t})^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\bar{t}\ell_1^2} - 1\right] + \frac{1}{\bar{t}^{1/2}}\left[\frac{a^2-\rho^2\bar{t}}{(1-\bar{t})^{3/2}}\right.$$

$$\left. - \frac{2a^2}{\bar{t}(1-\bar{t})^{1/2}} - \frac{3z^2\bar{t}}{(1-\bar{t})^{5/2}}\right]\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right.$$

$$\left.\left. - \tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\right\} \; . \tag{A4.4.9}$$

It is reminded that t is defined in (4.3.22).

The next integral to be computed is

$$I_2=\iint\limits_S \frac{zB_2(N,N_0)}{R^3(M,N)}dS_N = -\frac{\partial I_1}{\partial z}\ . \tag{A4.4.10}$$

Differentiation of (A4.4.5) with respect to z gives

$$I_2= 2\pi z\bar{\varepsilon}^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{\ell_2}^{\infty}\frac{(\alpha^2+\bar{\varepsilon}^2)\,d\alpha}{(\alpha^2-\bar{\varepsilon}^2)^2(\alpha^2-\rho^2)^{3/2}}$$

$$=2\pi z\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{3\bar{t}^{5/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\right.$$

$$\left.+\ \frac{\bar{t}^2}{(1-\bar{t})^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}} - 2 - \bar{t} + \frac{a(1+\bar{t})}{(a^2-\ell_1^2)^{1/2}}\right]\right\}\ . \tag{A4.4.11}$$

Application of the operator Λ to the complex conjugate of (A4.4.5) yields

$$I_3= \Lambda\bar{I}_1=\pi\varepsilon^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho e^{i\phi}\left\{3\int_{\ell_2}^{\infty}\frac{(\alpha^2-\ell_1^2)(\alpha^2-\ell_2^2)(\alpha^2+\varepsilon^2)}{\alpha^2(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}d\alpha\right.$$

$$\left.-2\int_{\ell_2}^{\infty}\frac{(\alpha^2-a^2)(\alpha^2+\varepsilon^2)\,d\alpha}{\alpha^2(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{3/2}}\right\}=\pi\varepsilon^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho e^{i\phi}$$

$$\times\left\{\int_{\ell_2}^{\infty}\frac{(\alpha^2-\ell_1^2)(\alpha^2-\ell_2^2)(\alpha^2+\varepsilon^2)}{\alpha^2(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}d\alpha\ -2z^2\int_{\ell_2}^{\infty}\frac{(\alpha^2+\varepsilon^2)\,d\alpha}{(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}\right\}$$

$$=\pi\rho e^{i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}t^2\left\{\left(1-\frac{a}{(a^2-\ell_1^2)^{1/2}}\right)\left[\frac{a^2}{\rho^2 t}+\frac{(1+t)(a^2-2z^2-\rho^2)}{(1-t)^2\rho^2}\right.\right.$$

$$+\frac{1}{(1-t)^2\rho^2}\left(9z^2-\frac{15z^2}{1-t}+\frac{a^2-\rho^2t}{t}\right)\Bigg]-\frac{a}{(a^2-\ell_1^2)^{1/2}}\Bigg[\frac{1+t}{(1-t)^2}\left(1-\frac{a^2}{\ell_2^2}\right)$$

$$-\frac{a^2}{\ell_2^2t}+\frac{3z^2}{(1-t)^2(\ell_2^2-\rho^2t)}-\frac{a^2-\rho^2t}{t(1-t)(\ell_2^2-\rho^2t)}\Bigg]$$

$$+\frac{1}{\rho^2\sqrt{t}}\left[\frac{2a^2}{t(1-t)^{3/2}}-\frac{3(a^2-\rho^2t)}{(1-t)^{5/2}}+\frac{15z^2t}{(1-t)^{7/2}}\right]$$

$$\times\left[\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)\right]\Bigg\}\ . \qquad \text{(A4.4.12)}$$

Application of yet another Λ-operator to (A4.4.12) yields

$$I_4=\Lambda^2\bar{I}_1=\pi\varepsilon^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho e^{i\phi}\left\{\Lambda\int_{\ell_2}^{\infty}\frac{(\alpha^2-\ell_1^2)(\alpha^2-\ell_2^2)(\alpha^2+\varepsilon^2)}{\alpha^2(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}d\alpha\right.$$

$$\left.-\,2z^2\Lambda\int_{\ell_2}^{\infty}\frac{(\alpha^2+\varepsilon^2)d\alpha}{(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}\right\}$$

$$=\pi\varepsilon^4\rho^2e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{3\int_{\ell_2}^{\infty}\frac{(\alpha^2-a^2)(\alpha^2+\varepsilon^2)d\alpha}{\alpha^2(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{5/2}}\right.$$

$$\left.-\,15z^2\int_{\ell_2}^{\infty}\frac{(\alpha^2+\varepsilon^2)d\alpha}{(\alpha^2-\varepsilon^2)^2(\alpha^2-\rho^2)^{7/2}}+\frac{2z^2(\ell_2^2+\varepsilon^2)(\ell_2^2-a^2)}{\ell_2(\ell_2^2-\varepsilon^2)^2(\ell_2^2-\rho^2)^{5/2}(\ell_2^2-\ell_1^2)}\right\}\ .$$

(A4.4.13)

The integrals in (A4.4.13) are elementary, for their evaluation can be used the indefinite integrals presented in

Appendix B4.4. The final result is

$$I_4=\Lambda^2\overline{I}_1=\pi\rho^2 e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{2\rho^4 t^2(\ell_2^2+\rho^2 t)(\ell_2^2-a^2)^2}{\ell_2^3(\ell_2^2-\rho^2 t)^2(\ell_2^2-\rho^2)^{3/2}(\ell_2^2-\ell_1^2)}\right.$$

$$-\frac{\ell_2^2 t^2(1+t)(2\rho^2-3\ell_1^2+a^2)}{\rho^2(1-t)^2(\ell_2^2-\rho^2)^{3/2}}+t^2\left(1-\frac{a}{(a^2-\ell_1^2)^{1/2}}\right)\left[\frac{8+9t-2t^2}{\rho^2(1-t)^3}\right.$$

$$\left.+\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}+\frac{z^2(48+87t-38t^2+8t^3)}{\rho^4(1-t)^4}\right]-\frac{3a\ell_1^2 t}{\rho^4(a^2-\ell_1^2)^{1/2}}$$

$$-\frac{15t^2z^2\ell_2^2(a^2-\ell_1^2)^{1/2}}{a(1-t)^4\rho^4(\ell_2^2-\rho^2 t)}-\frac{at}{(a^2-\ell_1^2)^{1/2}}\left[\frac{3(a^2-\rho^2 t)}{\rho^2(1-t)^2(\ell_2^2-\rho^2 t)}\right.$$

$$\left.-tz^2\left(\frac{15}{\rho^4(1-t)^4}+\frac{9+15t-4t^2}{\rho^2(1-t)^3(\ell_2^2-\rho^2)}\right)\right]+\frac{t^{3/2}}{\rho^4(1-t)^{1/2}}\left[\frac{6a^2}{t(1-t)^2}\right.$$

$$\left.-\frac{15(a^2-\rho^2 t)}{(1-t)^3}+\frac{105tz^2}{(1-t)^4}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)\right.$$

$$\left.\left.\left.-\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\right]\right\}\ . \tag{A4.4.14}$$

Integration with respect to z of (A4.4.1) gives

$$L_2(M,N_0)=\iint_S B_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$=\pi\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{(\overline{t}+2)[z^2-(\ell_2^2-a^2)^{3/2}]\overline{t}^2}{3(1-\overline{t})^2}+\overline{t}a\ell_1(\rho^2-\ell_1^2)^{1/2}\left[\frac{\overline{t}(\rho^2-\ell_1^2)}{\ell_1^2(1-\overline{t})^2}-1\right]\right.$$

$$+\overline{t}\left((\ell_2^2-a^2)^{1/2}-z\right)\left(\frac{a^2-\rho^2\overline{t}}{1-\overline{t}}+a^2\right)+a(\rho^2\overline{t}-2a^2)\sin^{-1}\left(\frac{a}{\ell_2}\right)$$

$$+ 2a^2(a^2-\rho^2\bar{t})^{1/2}\sin^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(\ell_2^2-\rho^2\bar{t})^{1/2}}\right) + \frac{z\bar{t}^{3/2}}{(1-\bar{t})^{1/2}}\left(\frac{a^2-\rho^2\bar{t}}{1-\bar{t}} - \frac{2a^2}{\bar{t}}\right.$$

$$\left.- \frac{z^2\bar{t}}{(1-\bar{t})^2}\right)\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\Bigg\}. \qquad \text{(A4.4.15)}$$

Indefinite integrals from Appendix B4.4 were used here.

Application of the Λ-operator to the complex conjugate of (A4.4.11) yields

$$\Lambda\frac{\partial}{\partial z}\iint_S \frac{\bar{B}_2(N,N_0)}{R(M,N)}dS_N$$

$$=2\pi z t^2 \rho e^{i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{\rho^4(\ell_2^2+\rho^2 t)(\ell_2^2-a^2)}{\ell_2(\ell_2^2-\rho^2 t)^2(\ell_2^2-\rho^2)^{3/2}(\ell_2^2-\ell_1^2)}\right.$$

$$- \frac{15t^{1/2}}{\rho^2(1-t)^{7/2}}\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]$$

$$- \frac{1}{(1-t)^2}\left[\frac{2(1+t)}{\rho^2} + \frac{6+9t}{\rho^2(1-t)}\right] + \frac{a}{(a^2-\ell_1^2)^{1/2}(1-t)^2}\left[\frac{2(1+t)}{\rho^2}\right.$$

$$\left.\left. + \frac{6+9t}{\rho^2(1-t)} - \frac{1+t}{\ell_2^2-\rho^2} - \frac{3}{\ell_2^2-\rho^2 t}\right]\right\}. \qquad \text{(A4.4.16)}$$

Yet another z-differentiation of (A4.4.11) results in

$$\frac{\partial^2}{\partial z^2}\iint_S \frac{B_2(N,N_0)}{R(M,N)}dS_N=2\pi\bar{t}^2\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{3\bar{t}^{1/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right.\right.$$

$$- \tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\Bigg] - \frac{1}{(1-\bar{t})^2}\left[\frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}} - 2 - \bar{t}\right.$$

$$\left. + \frac{a(1+\bar{t})}{(a^2-\ell_1^2)^{1/2}}\right] + \frac{\rho^4 z(\ell_2^2+\rho^2\bar{t})(\ell_2^2-a^2)^{1/2}}{(\ell_2^2-\rho^2)(\ell_2^2-\rho^2\bar{t})^2(\ell_2^2-\ell_1^2)}\Bigg\} . \qquad \text{(A4.4.17)}$$

Application of the Λ-operator to the complex conjugate of (A4.4.15) will result in

$$\Lambda\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$=\pi t^2\rho e^{i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\Bigg\{2(\ell_2^2-a^2)^{1/2}\left[\frac{\ell_1^2}{\rho^2 t} - \frac{2(\rho^2-\ell_1^2)}{\rho^2(1-t)^3}\right]$$

$$- \frac{2[z^3-(\ell_2^2-a^2)^{3/2}]}{3\rho^2(1-t)^2}\left[t+4+\frac{6t}{1-t}\right] - 2\left((\ell_2^2-a^2)^{1/2}-z\right)\left[\frac{a^2-\rho^2 t}{\rho^2 t(1-t)^2}\right.$$

$$\left. - \frac{a^2}{\rho^2(1-t)}\right] + \left[\frac{z}{\rho^2} - \frac{(a^2-\ell_1^2)(\ell_2^2-a^2)^{1/2}}{\ell_1^2(\ell_2^2-\rho^2 t)}\right]\left[\frac{a^2-\rho^2 t}{(1-t)^2} - \frac{z^2 t}{(1-t)^3}\right]$$

$$+ \frac{z}{\rho^2[t(1-t)]^{1/2}}\left[\frac{3(a^2-\rho^2 t)}{(1-t)^2} - \frac{2a^2}{t(1-t)} - \frac{5z^2 t}{(1-t)^3}\right]$$

$$\times \left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right) - \tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\Bigg\} . \qquad \text{(A4.4.18)}$$

Application of yet another Λ to (A4.4.18) yields

$$\Lambda^2\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$
\begin{aligned}
&= \pi t^2 \rho^2 e^{2i\phi} \frac{\sqrt{\rho_0^2 - a^2}}{a^3} \left\{ \frac{2(\ell_2^2 - a^2)^{3/2}}{(\ell_2^2 - \ell_1^2)\rho^2 (1-t)^2} \left(4 + t + \frac{6t}{1-t} \right) \right. \\
&+ \frac{8[z^3 - (\ell_2^2 - a^2)^{3/2}]}{3\rho^4 (1-t)^2} \left(t + \frac{8+t}{(1-t)^2} - 2 \right) + \frac{2(\ell_2^2 - a^2)^{1/2}}{\rho^2 (\ell_2^2 - \ell_1^2)} \left[\frac{\ell_1^2}{t} - \frac{2(\rho^2 - \ell_1^2)}{(1-t)^3} \right. \\
&\left. + \frac{a^2}{1-t} - \frac{a^2 - \rho^2 t}{t(1-t)^2} \right] + 4 \frac{(\ell_2^2 - a^2)^{1/2} - z}{\rho^4} \left[\frac{2(a^2 - \rho^2 t)}{t(1-t)^3} - \frac{2a^2}{1-t} - \frac{a^2}{(1-t)^2} \right] \\
&+ \frac{4(\ell_2^2 - a^2)^{1/2}}{\rho^2} \left[\frac{a^2 - \ell_1^2}{\ell_2^2 - \ell_1^2} \left(\frac{1}{t} + \frac{2}{(1-t)^3} \right) - \frac{2\ell_1^2}{\rho^2 t} - \frac{2}{(1-t)^3} + \frac{6(\rho^2 - \ell_1^2)}{\rho^2 (1-t)^4} \right] \\
&+ \frac{z}{\rho^4} \left[1 - \frac{a(a^2 - \ell_1^2)^{1/2}}{a^2 - t\ell_1^2} \right] \left[\frac{6z^2 t}{(1-t)^4} - \frac{4(a^2 - \rho^2 t)}{(1-t)^3} \right] + \left[\frac{a^2 - \rho^2 t}{(1-t)^2} - \frac{z^2 t}{(1-t)^3} \right] \\
&\times \left[\frac{z a (a^2 - \ell_1^2)^{1/2}}{\rho^2 (a^2 - t\ell_1^2)} \left(\frac{1}{\ell_2^2 - \ell_1^2} + \frac{2a^2 - 2\rho^2 t (a^2 - \ell_1^2)(\ell_2^2 - \ell_1^2)}{\rho^2 (a^2 - t\ell_1^2)} \right) - \frac{2z}{\rho^4} \right] \\
&- z \left[\frac{3(a^2 - \rho^2 t)}{(1-t)^2} - \frac{2a^2}{t(1-t)} - \frac{5z^2 t}{(1-t)^3} \right] \left[\frac{1}{\rho^4 (1-t)} - \frac{a(a^2 - \ell_1^2)^{1/2}}{\rho^2 (a^2 - t\ell_1^2)} \left(\frac{1}{\ell_2^2 - \ell_1^2} \right. \right. \\
&\left. \left. + \frac{1}{\rho^2 (1-t)} \right) \right] + \frac{z}{\rho^4 [t(1-t)]^{1/2}} \left[\frac{15(a^2 - \rho^2 t)}{(1-t)^3} - \frac{6a^2}{t(1-t)^2} - \frac{35 z^2 t}{(1-t)^4} \right] \\
&\times \left. \left[\tan^{-1} \left(\frac{t^{1/2} (a^2 - \ell_1^2)^{1/2}}{a(1-t)^{1/2}} \right) - \tan^{-1} \left(\frac{t^{1/2}}{(1-t)^{1/2}} \right) \right] \right\} . \qquad \text{(A4.4.19)}
\end{aligned}
$$

Formula (A4.4.19) looks too long, and a way to simplify it was not found. On the other hand, the same result can be obtained by the integration of (A4.4.14) with respect to *z*. Such an integration can be performed by using the indefinite integrals from Appendix B4.4, and at first glance it is too long as well and includes various trigonometric functions. Since (A4.4.19) contains only $\tan^{-1}$ in the last line, then it may be concluded that the coefficients of all the other trigonometric functions should be zero. This simple idea led to a relatively short result, namely,

$$\Lambda^2 \iint_S \overline{B}_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$=\pi t^2\rho^2 e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{(\ell_2^2-a^2)^{1/2}}{\rho^2}\left[\frac{35t}{2(1-t)^4}+\frac{2(a^2-\rho^2 t)}{t(1-t)^2(\ell_2^2-\rho^2 t)}-\frac{\ell_1^2}{\rho^2}\left(\frac{8}{t}\right.\right.\right.$$

$$\left.\left.+\frac{35}{(1-t)^4}\right)\right]+\left[\frac{2[z^3-(\ell_2^2-a^2)^{3/2}]}{3\rho^2}+(\ell_2^2-a^2)^{1/2}\right]\frac{48+87t-38t^2+8t^3}{2\rho^2(1-t)^4}$$

$$+[z-(\ell_2^2-a^2)^{1/2}]\left[\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}+\frac{8+9t-2t^2}{\rho^2(1-t)^3}\right]$$

$$+\frac{z}{\rho^4[t(1-t)]^{1/2}}\left[\frac{15(a^2-\rho^2 t)}{(1-t)^3}-\frac{6a^2}{t(1-t)^2}-\frac{35z^2 t}{(1-t)^4}\right]$$

$$\left.\times\left[\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)\right]\right\}. \tag{A4.4.20}$$

As it can be seen, (A4.4.20) is much shorter than (A4.4.19), and except for the last line, looks totally different. Direct numerical computations show that (A4.4.19) and (A4.4.20) are identical, but there was not found a way to reduce (A4.4.19) to (A4.4.20) or to reduce both to a third expression which might be even simpler then (A4.4.20).

It is of interest to note that (A4.4.20) was obtained from (A4.4.14) by integration with respect to z. If now differentiate (A4.4.20) with respect to z, it does not result in (A4.4.14), it gives something very different, namely,

$$\Lambda^2\iint\limits_S \frac{\overline{B}_2(N,N_0)}{R(M,N)}dS_N$$

$$=\pi t^2\rho^2 e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{\ell_2(\ell_2^2-\rho^2)^{1/2}}{\rho^2(\ell_2^2-\ell_1^2)}\left[\frac{2(a^2-\rho^2 t)}{t(1-t)^2(\ell_2^2-\rho^2 t)}-\frac{\ell_1^2}{\rho^2}\left(\frac{8}{t}\right.\right.\right.$$

$$\left.\left.+\frac{35}{(1-t)^4}\right)-\frac{35t}{2(1-t)^4}\right]-\frac{z(\ell_2^2-a^2)^{1/2}}{\ell_2^2-\ell_1^2}\left[\frac{4(a^2-\rho^2 t)\ell_2^2}{\rho^2 t(1-t)^2(\ell_2^2-\rho^2 t)^2}\right.$$

$$\left.-\frac{2\ell_1^2}{\rho^4}\left(\frac{8}{t}+\frac{35}{(1-t)^4}\right)\right]+\left[\frac{2z}{\rho^2}\left(z-\frac{\ell_2^2(\ell_2^2-a^2)^{1/2}}{\ell_2^2-\ell_1^2}\right)+\frac{\ell_2(\ell_2^2-\rho^2)^{1/2}}{\ell_2^2-\ell_1^2}\right]$$

$$\times\frac{48+87t-38t^2+8t^3}{2\rho^2(1-t)^4}+\left[1-\frac{\ell_2(\ell_2^2-\rho^2)^{1/2}}{\ell_2^2-\ell_1^2}\right]\left[\frac{8+9t-2t^2}{\rho^2(1-t)^3}\right.$$

$$\left.+\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}\right]-\frac{z(\ell_2^2-a^2)^{1/2}}{\rho^2(\ell_2^2-\ell_1^2)(\ell_2^2-\rho^2 t)}\left[\frac{6a^2}{t(1-t)^2}\right.$$

$$\left.-\frac{15(a^2-\rho^2 t)}{(1-t)^3}+\frac{35z^2 t}{(1-t)^4}\right]+\frac{1}{\rho^4[t(1-t)]^{1/2}}\left[\frac{6a^2}{t(1-t)^2}-\frac{15(a^2-\rho^2 t)}{(1-t)^3}\right.$$

$$\left.\left.+\frac{105z^2 t}{(1-t)^4}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\right\}. \qquad (A4.4.21)$$

Again, numerical computations show that (A4.4.21) is identical to (A4.4.14), but it was not found a way to reduce one to the other or to reduce both to a third expression which would be simpler than both of them.

Application of the Λ-operator to I_2 in (A4.4.11) gives the result

$$\Lambda \iint_S \frac{zB_2(N,N_0)}{R^3(M,N)} dS_N = 2\pi \frac{j\rho e^{i\phi}\bar{t}^2\ell_1^2(a^2+\ell_1^2\bar{t})}{a^2(a^2-\ell_1^2\bar{t})^2(\ell_2^2-\ell_1^2)} \ . \tag{A4.4.22}$$

Here j is defined in (A4.4.29), and the property $\Lambda\bar{t}=0$ was used. Yet another application of Λ to (A4.4.20) yields

$$\Lambda^3 \iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$=\pi t^2\rho^3 e^{3i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{(\ell_2^2-a^2)^{1/2}}{\ell_2^2-\ell_1^2}\left[\frac{2(a^2-\rho^2 t)(2a^2-\rho^2 t-\ell_2^2)}{\rho^2 t(1-t)^2(\ell_2^2-\rho^2 t)^2}\right.\right.$$

$$\left.+\frac{(\ell_1^2-2a^2)}{\rho^4}\left(\frac{35}{(1-t)^4}+\frac{8}{t}\right)-\frac{35t}{2\rho^4(1-t)^4}\right]$$

$$+\frac{(\ell_2^2-a^2)^{1/2}}{\rho^4}\left[\frac{140t}{(1-t)^5}-\frac{8(a^2-\rho^2 t)}{t(1-t)^3(\ell_2^2-\rho^2 t)}+\frac{\ell_1^2}{\rho^2}\left(\frac{48}{t}+\frac{280}{(1-t)^5}\right)\right]$$

$$+\left[\frac{(\ell_2^2-a^2)^{1/2}}{\ell_2^2-\ell_1^2}\left(1-\frac{2(\ell_2^2-a^2)}{\rho^2}\right)-\frac{4[z^3-(\ell_2^2-a^2)^{3/2}]}{3\rho^4}\right]\frac{48+87t-38t^2+8t^3}{2\rho^2(1-t)^4}$$

$$+\left[\frac{2[z^3-(\ell_2^2-a^2)^{3/2}]}{3\rho^2}+(\ell_2^2-a^2)^{1/2}\right]\frac{-144-396t+190t^2-86t^3+16t^4}{\rho^4(1-t)^5}$$

$$+[z-(\ell_2^2-a^2)^{1/2}]\left[\frac{48+36t-4t^2}{\rho^4(1-t)^3}+\frac{a^2(-36+4t-36t^2)}{\rho^6 t(1-t)^3}\right]-\left[\frac{(\ell_2^2-a^2)^{1/2}}{\ell_2^2-\ell_1^2}\right.$$

$$\left.+\frac{6[z-(\ell_2^2-a^2)^{1/2}]}{\rho^2}\,\frac{2-t}{1-t}\right]\left[\frac{8+9t-2t^2}{\rho^2(1-t)^3}+\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}\right]$$

$$+\frac{z}{\rho^2}\left[\frac{6a^2}{t(1-t)^2}-\frac{15(a^2-\rho^2 t)}{(1-t)^3}+\frac{35z^2t}{(1-t)^4}\right]\left[\frac{a(a^2-\ell_1^2)^{1/2}}{\rho^2(a^2-t\ell_1^2)}\left(\frac{1}{\rho^2(1-t)}\right.\right.$$

$$\left.\left.+\frac{1}{\ell_2^2-\ell_1^2}\right)-\frac{1}{\rho^4(1-t)}\right]-\frac{z}{\rho^6[t(1-t)]^{1/2}}\left[\frac{30a^2}{t(1-t)^3}-\frac{105(a^2-\rho^2 t)}{(1-t)^4}\right.$$

$$\left.+\frac{315z^2t}{(1-t)^5}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\Bigg\}. \qquad (A4.4.23)$$

Note that

$$\Lambda \ln[R(M,N)+z]=\frac{\rho e^{i\phi}-re^{i\psi}}{R(M,N)[R(M,N)+z]},$$

$$\Lambda^2 \ln[R(M,N)+z]=-\frac{(\rho e^{i\phi}-re^{i\psi})^2[R(M,N)+z]}{R^3(M,N)[R(M,N)+z]^2},$$

$$\Lambda^3 \ln[R(M,N)+z]=(\rho e^{i\phi}-re^{i\psi})^3\,\frac{8R^2(M,N)+9zR(M,N)+3z^2}{R^5(M,N)[R(M,N)+z]^3}. \qquad (A4.4.24)$$

Though $L_1(M,N_0)$ can not be computed, as it is defined in (4.4.12), all its derivatives are computable. The simplest to compute is (Fabrikant [16])

$$J_1=\iint_S \frac{zB_1(N,N_0)}{R^3(M,N)}dS_N$$

$$= \int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{z\,rdrd\psi}{R^3(M,N)}$$

$$= \frac{2\pi}{R(M,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(\ell_2^2-a^2)]^{1/2}}{aR(M,N_0)}\right) \quad . \tag{A4.4.25}$$

The next integral to compute is

$$J_2=\iint_S\frac{\rho e^{i\phi}-re^{i\psi}}{R^3(M,N)}B_1(N,N_0)dS_N \quad . \tag{A4.4.26}$$

The incegral can be expressed through J_1 as follows

$$J_2=\int_{\infty}^{z}\Lambda J_1 dz \quad , \tag{A4.4.27}$$

and it can be computed in the same way as it is done in (Fabrikant [16], Appendix A4.3), with the result

$$J_2=\int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{\rho e^{i\phi}-re^{i\psi}}{R^3(M,N)}rdrd\psi$$

$$=\frac{2\pi}{\bar{q}}\left\{\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)-\frac{z}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)\right.$$

$$\left.+\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]\right\} \quad . \tag{A4.4.28}$$

Here the following notations were introduced

$$R_0=R(M,N_0)=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2}, \qquad s=\sqrt{\rho\rho_0 e^{i(\phi-\phi_0)}-a^2},$$

$$\bar{s}=\sqrt{\rho\rho_0 e^{-i(\phi-\phi_0)}-a^2}, \qquad j=\sqrt{\rho_0^2-a^2}\sqrt{\ell_2^2-a^2}/a\ . \tag{A4.4.29}$$

Integration of (A4.4.28) with respect to z yields

$$\int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{(\rho e^{i\phi}-re^{i\psi})\,rdrd\psi}{R(M,N)[R(M,N)+z]}$$

$$=\frac{2\pi}{\bar{q}}\left\{R_0\tan^{-1}\left(\frac{j}{R_0}\right)-z\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}}{a}\right)+\sqrt{\rho_0^2-a^2}\left[\sqrt{1-\bar{\zeta}}\ \tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left.-\ \frac{z}{\bar{s}}\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]\right]\right\}\ . \tag{A4.4.30}$$

Here the following indefinite integrals were used

$$\int\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)dz\ =\ z\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)$$

$$+\ \bar{s}\left[\sqrt{1-\zeta}\ \tan^{-1}\left(\frac{a(1-\zeta)^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\ -\ \sin^{-1}\left(\frac{a}{\ell_2}\right)\right]\ . \tag{A4.4.31}$$

$$\int\frac{z}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)dz=R_0\tan^{-1}\left(\frac{j}{R_0}\right)+\sqrt{\rho_0^2-a^2}\left[\sqrt{1-\zeta}\ \tan^{-1}\left(\frac{a(1-\zeta)^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right.$$

$$\left.+\sqrt{1-\bar{\zeta}}\ \tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\ -\ \sin^{-1}\left(\frac{a}{\ell_2}\right)\right]\ . \tag{A4.4.32}$$

and the identities

$$\rho_0^2(\ell_2^2-a^2\zeta)(\ell_2^2-a^2\bar{\zeta})=a^2\ell_2^2(R_0^2+j^2)\ ,$$

$$[\ell_1^2-\rho\rho_0 e^{i(\phi-\phi_0)}][\ell_1^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]=\ell_1^2(R_0^2+j^2)\ . \tag{A4.4.33}$$

It is reminded that ζ was defined in (4.3.3). Application of the Λ-operator to (A4.4.30) results in

$$\int_0^{2\pi}\int_a^{\infty}\frac{(\rho e^{i\phi}-re^{i\psi})^2[2R(M,N)+z]}{R^3(M,N)[R(M,N)+z]^2}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{rdrd\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{R_0^2+z^2}{\bar{q}R_0}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2z}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)-(\rho_0^2-a^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{2}{\bar{q}}\right.\right.\right.$$

$$\left.+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)$$

$$\left.\left.-\frac{e^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\right]+\frac{jae^{i\phi}}{\rho\bar{s}^2}\left(a-(a^2-\ell_1^2)^{1/2}\right)\right\}\ . \tag{A4.4.34}$$

The following identities were used here:

$$\Lambda\bar{s} = \frac{\rho_0 e^{i\phi_0}}{\bar{s}}\ ,\qquad \Lambda\sqrt{1-\bar{\zeta}} = -\frac{e^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}}\ ,$$

$$\Lambda\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right) = -\frac{(\ell_2^2-a^2)^{1/2}ae^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}(\ell_2^2-a^2\bar{\zeta})}\left[1-\frac{\bar{q}\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\right]\ ,$$

$$\Lambda\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right) = \frac{(a^2-\ell_1^2)^{1/2}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-\ell_1^2}\left[\frac{\rho_0 e^{i\phi_0}}{\bar{s}}+\frac{\bar{s}\rho e^{i\phi}}{\ell_2^2-\ell_1^2}\right]\ ,$$

$$\Lambda \tan^{-1}\left(\frac{j}{R_0}\right) = \frac{R_0 j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2} - \frac{q}{R_0^2}\right] . \tag{A4.4.35}$$

Application of yet another Λ-operator to (A4.4.34), results in

$$\int_0^{2\pi}\int_a^{\infty}\frac{(\rho e^{i\phi}-re^{i\psi})^3[8R^2(M,N)+9zR(M,N)+3z^2]}{R^5(M,N)[R(M,N)+z]^3}$$

$$\times \tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{rdrd\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{3R_0^4+6z^2R_0^2-z^4}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) - \frac{8z}{\bar{q}^2}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[-\frac{3(1-\bar{\zeta})^{1/2}}{\bar{q}^2}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right) + \frac{z}{\bar{s}}\left(\frac{8}{\bar{q}^2} + \frac{4\rho_0 e^{i\phi_0}}{\bar{q}\ \bar{s}^2}\right.\right.$$

$$\left.+ \frac{3\rho_0^2e^{2i\phi_0}}{\bar{s}^4}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{s}}{a}\right)\right] - \frac{z}{\bar{s}^2}\left(\frac{2}{\bar{q}} + \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)$$

$$\left.\times\left(\frac{(a^2-\ell_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-\ell_1^2} - \frac{ae^{i\phi}}{\rho}\right)\right] + \frac{2jae^{i\phi}}{\rho\bar{s}^2}\left(\frac{1}{\bar{q}} + \frac{e^{i\phi}}{\rho} + \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)$$

$$\left.\times\left(a-(a^2-\ell_1^2)^{1/2}\right) + \frac{j}{R_0^2+j^2}\left[\frac{\rho e^{3i\phi}\bar{q}}{\ell_2^2-\ell_1^2} + \frac{z^2 q}{R_0^2\bar{q}} + \frac{e^{i\phi}(\rho^2-\ell_1^2)}{\bar{q}\rho} - 2e^{2i\phi}\right]\right\}. \tag{A4.4.36}$$

Differentiation of (A4.4.25) with respect to z leads to the integral

$$\frac{\partial}{\partial z}\iint\limits_{S}\frac{zB_1(N,N_0)}{R^3(M,N)}dS_N$$

$$=\int_0^{2\pi}\int_a^{\infty}\left(\frac{1}{R^3(M,N)} - \frac{3z^2}{R^5(M,N)}\right)\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{rdrd\psi}{R(N,N_0)}$$

$$=2\pi\left\{-\frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) + \frac{j}{z(R_0^2+j^2)}\left[\frac{\ell_2^2-\rho^2}{\ell_2^2-\ell_1^2} - \frac{z^2}{R_0^2}\right]\right\} . \tag{A4.4.37}$$

Application of the operator Λ to (A4.4.25) yields

$$\int_0^{2\pi}\int_a^{\infty}\frac{3z(\rho e^{i\phi}-re^{i\psi})}{R^5(M,N)}\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{rdrd\psi}{R(N,N_0)}$$

$$=\frac{q}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) - \frac{j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}}{\ell_2^2-\ell_1^2} - \frac{q}{R_0^2}\right] . \tag{A4.4.38}$$

Yet another application of Λ-operator to (A4.4.28) yields

$$\Lambda J_2=-\int_0^{2\pi}\int_a^{\infty}\frac{3(\rho e^{i\phi}-re^{i\psi})^2}{R^5(M,N)}\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{rdrd\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{z(3R_0^2-z^2)}{\bar{q}\,R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right) - \frac{2}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right) + \frac{zj}{R_0^2+j^2}\left[\frac{q}{R_0^2}\right.\right.$$

$$\left.- \frac{\bar{q}\rho^2e^{2i\phi}}{(\ell_2^2-\ell_1^2)(\rho^2-\ell_1^2)}\right] - \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\frac{2}{\bar{q}} + \frac{\rho_0e^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-\ell_1^2)^{1/2}}\right)\right.$$

$$- \tan^{-1}\left(\frac{\bar{s}}{a}\right)\Bigg] + \frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}^2}\left[\frac{(a^2-\ell_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-\ell_1^2} - \frac{ae^{i\phi}}{\rho}\right]\Bigg\}. \qquad (A4.4.39)$$

The next integral to compute is ΛI_1, and it gives from (A4.4.9)

$$\Lambda\iint_S \frac{B_2(N,N_0)}{R(M,N)}dS_N = 2\pi\frac{\bar{t}^2}{a^3}\rho e^{i\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{1}{1-\bar{t}}\left[1 - \frac{a(a^2-\ell_1^2)^{1/2}}{a^2-\ell_1^2\bar{t}}\right]\right.$$

$$\left. + \frac{\bar{t}^{1/2}}{(1-\bar{t})^{3/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-\ell_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]\right\}.$$

(A4.4.40)

Application of Λ-operator to (A4.4.40), results in

$$\Lambda^2\iint_S \frac{B_2(N,N_0)}{R(M,N)}dS_N = 2\pi\frac{\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}(a^2-\ell_1^2)^{1/2}\bar{t}^2(a^2+\ell_1^2\bar{t})}{a^2(\ell_2^2-\ell_1^2)(a^2-\ell_1^2\bar{t})^2}.$$

(A4.4.41)

Integration with respect to z of both sides of (A4.4.41) yields

$$\Lambda^2\iint_S B_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$= 2\pi\frac{\bar{t}^2}{a^2}\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}\left[\frac{\bar{t}\ell_1(\rho^2-\ell_1^2)^{1/2}}{(a^2-\ell_1^2\bar{t})(a^2-\rho^2\bar{t})}\right.$$

$$- \frac{a}{(a^2-\rho^2\overline{t})^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\overline{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\Bigg] \ . \qquad \text{(A4.4.42)}$$

Application of yet another Λ to (A4.4.42) gives

$$\Lambda^3\iint_S B_2(N,N_0)\ln[R(M,N)+z]dS_N$$

$$= 2\pi\frac{\overline{t}^3}{a}\rho^3 e^{3i\phi}(\rho_0^2-a^2)^{1/2}\Bigg\{\frac{(\ell_2^2-a^2)^{1/2}}{(a^2-\ell_1^2\overline{t})(a^2-\rho^2\overline{t})}\Bigg[\frac{a^2+2\rho^2\overline{t}}{\ell_2^2(a^2-\rho^2\overline{t})}$$

$$+ \frac{a^2+\rho^2\overline{t}}{\ell_2^2\overline{t}(\ell_2^2-\ell_1^2)} + \frac{2(a^2-\ell_1^2)}{(\ell_2^2-\ell_1^2)(a^2-\ell_1^2\overline{t})}\Bigg]$$

$$- \frac{3}{(a^2-\rho^2\overline{t})^{5/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\overline{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right)\Bigg\} \ . \qquad \text{(A4.4.43)}$$

The following identity was used here

$$\Lambda \tan^{-1}\left(\frac{(a^2-\rho^2\overline{t})^{1/2}}{(\ell_2^2-a^2)^{1/2}}\right) = - \frac{\rho e^{i\phi}a^2(\ell_2^2-a^2)^{1/2}}{\ell_2^2(a^2-\rho^2\overline{t})^{1/2}(a^2-\ell_1^2\overline{t})}\left[\overline{t}+ \frac{a^2-\rho^2\overline{t}}{\ell_2^2-\ell_1^2}\right] .$$

(A4.4.44)

APPENDIX B4.4

Here are presented some regular integrals that are used throughout the work in Chapter 4 and are not explicitly present in the tables.

$$\int \frac{dx}{\sqrt{a^2-x^2}\,(b^2-x^2)} = -\frac{1}{b\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}}\right), \tag{B4.4.1}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}\,(b^2-x^2)^2} = -\frac{2b^2-a^2}{2b^3(b^2-a^2)^{3/2}}\tan^{-1}\left(\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}}\right)$$

$$-\frac{x\sqrt{a^2-x^2}}{2b^2(b^2-x^2)(b^2-a^2)}, \tag{B4.4.2}$$

$$\int \frac{dx}{\sqrt{x^2-\rho^2}\,(x^2-\alpha^2)} = \frac{1}{\alpha\sqrt{\rho^2-\alpha^2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right), \tag{B4.4.3}$$

$$\int \frac{dx}{(x^2-\rho^2)^{3/2}(x^2-\alpha^2)} = -\frac{1}{\alpha(\rho^2-\alpha^2)^{3/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$-\frac{x}{\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}, \tag{B4.4.4}$$

$$\int \frac{dx}{(x^2-\rho^2)^{3/2}(x^2-\alpha^2)^2} = \frac{\rho^2-4\alpha^2}{2\alpha^3(\rho^2-\alpha^2)^{5/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$-\frac{x}{(\rho^2-\alpha^2)^2}\left[\frac{\sqrt{x^2-\rho^2}}{2\alpha^2(x^2-\alpha^2)}+\frac{1}{\rho^2\sqrt{x^2-\rho^2}}\right], \tag{B4.4.5}$$

$$\int \frac{dx}{(x^2-\rho^2)^{5/2}(x^2-\alpha^2)} = \frac{1}{\alpha(\rho^2-\alpha^2)^{5/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$+ \frac{x}{3\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}\left(\frac{2}{\rho^2} + \frac{3}{\rho^2-\alpha^2} - \frac{1}{x^2-\rho^2}\right) , \qquad \text{(B4.4.6)}$$

$$\int \frac{dx}{(x^2-\rho^2)^{5/2}(x^2-\alpha^2)^2} = \frac{6\alpha^2-\rho^2}{2\alpha^3(\rho^2-\alpha^2)^{7/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$+ \frac{x}{(\rho^2-\alpha^2)^2\sqrt{x^2-\rho^2}}\left[\frac{\rho^2+4\alpha^2}{2\alpha^2\rho^2(\rho^2-\alpha^2)} - \frac{1}{2\alpha^2(x^2-\alpha^2)} + \frac{2x^2-3\rho^2}{3\rho^4(x^2-\rho^2)}\right],$$

$$\text{(B4.4.7)}$$

$$\int \frac{dx}{(x^2-\rho^2)^{7/2}(x^2-\alpha^2)} = -\frac{1}{\alpha(\rho^2-\alpha^2)^{7/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$+ \frac{x}{15\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}\left[\frac{5}{(\rho^2-\alpha^2)(x^2-\rho^2)} - \frac{3}{(x^2-\rho^2)^2} + \frac{4}{\rho^2(x^2-\rho^2)}\right.$$

$$\left. - \frac{10}{\rho^2(\rho^2-\alpha^2)} - \frac{8}{\rho^4} - \frac{15}{(\rho^2-\alpha^2)^2}\right] , \qquad \text{(B4.4.8)}$$

$$\int \frac{dx}{(x^2-\rho^2)^{7/2}(x^2-\alpha^2)^2} = \frac{\rho^2-8\alpha^2}{2\alpha^3(\rho^2-\alpha^2)^{9/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$

$$- \frac{x\sqrt{x^2-\rho^2}}{2\alpha^2(x^2-\alpha^2)(\rho^2-\alpha^2)^4} + \frac{x}{15\rho^2(\rho^2-\alpha^2)^2\sqrt{x^2-\rho^2}}\left[\frac{10}{(\rho^2-\alpha^2)(x^2-\rho^2)}\right.$$

$$\left. - \frac{3}{(x^2-\rho^2)^2} + \frac{4}{\rho^2(x^2-\rho^2)} - \frac{20}{\rho^2(\rho^2-\alpha^2)} - \frac{8}{\rho^4} - \frac{45}{(\rho^2-\alpha^2)^2}\right]. \qquad \text{(B4.4.9)}$$

Below are presented some indefinite integrals involving ℓ_1 and ℓ_2 which were used in this work as well

$$\int \tan^{-1}\left(c\sqrt{a^2-\ell_1^2}\right)dz = z\tan^{-1}\left(c\sqrt{a^2-\ell_1^2}\right)$$

$$+ \frac{1}{c}\left[\sin^{-1}\left(\frac{a}{\ell_2}\right) - \frac{\sqrt{1+c^2(a^2-\rho^2)}}{\sqrt{1+c^2a^2}}\sin^{-1}\left(\frac{a\sqrt{1+c^2(a^2-\rho^2)}}{\ell_2\sqrt{1+c^2(a^2-\ell_1^2)}}\right)\right]. \qquad \text{(B4.4.10)}$$

Here c does not depend on z.

$$\int \tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-\ell_1^2}}{a\sqrt{1-t}}\right)dz = z\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-\ell_1^2}}{a\sqrt{1-t}}\right)$$

$$+ \frac{a\sqrt{1-t}}{\sqrt{t}}\left[\sin^{-1}\left(\frac{a}{\ell_2}\right) - \frac{\sqrt{a^2-\rho^2 t}}{a}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{\ell_2^2-\rho^2 t}}\right)\right], \qquad \text{(B4.4.11)}$$

$$\int z^2\tan^{-1}\left(c\sqrt{a^2-\ell_1^2}\right)dz = \frac{z^3}{3}\tan^{-1}\left(c\sqrt{a^2-\ell_1^2}\right)$$

$$- \frac{1}{3c}\left\{\sqrt{\rho^2-\ell_1^2}\left[\frac{\ell_1}{2} + \frac{a^2\rho^2c^2}{\ell_1(1+a^2c^2)}\right] + \left[\frac{\rho^2}{2} - \frac{1+c^2(a^2-\rho^2)}{c^2(1+a^2c^2)}\right.\right.$$

$$\left.\left. + \frac{a^2\rho^2c^2}{1+a^2c^2}\right]\sin^{-1}\left(\frac{a}{\ell_2}\right) + \frac{[1+c^2(a^2-\rho^2)]^{3/2}}{c^2(1+c^2a^2)^{3/2}}\sin^{-1}\left(\frac{a\sqrt{1+c^2(a^2-\rho^2)}}{\ell_2\sqrt{1+c^2(a^2-\ell_1^2)}}\right)\right\},$$

(B4.4.12)

$$\int z^2\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-\ell_1^2}}{a\sqrt{1-t}}\right)dz = \frac{z^3}{3}\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-\ell_1^2}}{a\sqrt{1-t}}\right) - \frac{a\sqrt{1-t}}{3\sqrt{t}}\left\{\sqrt{\rho^2-\ell_1^2}\left(\frac{\ell_1}{2}\right.\right.$$

$$+ \frac{\rho^2 t}{\ell_1}\Bigg) + \left(a^2 + \frac{3}{2}\rho^2 - \frac{a^2}{t}\right)\sin^{-1}\left(\frac{a}{\ell_2}\right)$$

$$+ \frac{(a^2-\rho^2 t)^{3/2}(1-t)}{at}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{\ell_2^2-\rho^2 t}}\right)\Bigg\} , \qquad \text{(B4.4.13)}$$

$$\int \frac{dz}{\sqrt{a^2-\ell_1^2}} = -\sin^{-1}\left(\frac{a}{\ell_2}\right) + \frac{\sqrt{\ell_2^2-a^2}}{a} - \frac{\sqrt{a^2-\rho^2}}{a}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right), \qquad \text{(B4.4.14)}$$

$$\int \sqrt{a^2-\ell_1^2}\,dz = \frac{2a^2-\ell_1^2}{a}\sqrt{\ell_2^2-a^2} + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad \text{(B4.4.15)}$$

$$\int \frac{z^2 dz}{\sqrt{a^2-\ell_1^2}} = \frac{(\ell_2^2-a^2)^{3/2}}{3a} + \frac{\ell_1^2\sqrt{\ell_2^2-a^2}}{2a} + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{\ell_2}\right) , \qquad \text{(B4.4.16)}$$

$$\int \frac{\sqrt{a^2-\ell_1^2}}{1-c^2\ell_1^2}dz = -\frac{1}{c^2}\sin^{-1}\left(\frac{a}{\ell_2}\right) + a\sqrt{\ell_2^2-a^2}$$

$$- \frac{1-c^4 a^2\rho^2}{c^2\sqrt{1-c^2\rho^2}}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{a\sqrt{1-c^2\rho^2}}\right), \qquad \text{(B4.4.17)}$$

$$\int \frac{z^2\sqrt{a^2-\ell_1^2}}{b^2-\ell_1^2}dz = \sqrt{\rho^2-\ell_1^2}\left[\frac{\ell_1}{2} - \frac{a^2\rho^2(b^2-a^2)}{b^4\ell_1}\right]$$

$$+ \left(\frac{\rho^2}{2} + a^2 - b^2\right)\sin^{-1}\left(\frac{a}{\ell_2}\right) + \frac{a(\ell_2^2-a^2)^{3/2}}{3b^2}$$

$$+ \frac{(b^2-a^2)(b^4-a^2\rho^2)}{b^5}\sqrt{b^2-\rho^2}\ \sin^{-1}\left(\frac{\ell_1\sqrt{b^2-\rho^2}}{\rho\sqrt{b^2-\ell_1^2}}\right) . \qquad \text{(B4.4.18)}$$

Here b is a quantity which does not depend on z.

$$\int \frac{z^2\sqrt{a^2-\ell_1^2}}{a^2-\ell_1^2 t}dz = \sqrt{\rho^2-\ell_1^2}\left(\frac{\ell_1}{2t} - \frac{\rho^2(1-t)}{\ell_1}\right)$$

$$+ \frac{1}{t}\left(\frac{\rho^2}{2} + a^2 - \frac{a^2}{t}\right)\sin^{-1}\left(\frac{a}{\ell_2}\right) + \frac{(\ell_2^2-a^2)^{3/2}}{3a}$$

$$+ \sqrt{a^2-\rho^2 t}\ \frac{(1-t)(a^2-\rho^2 t^2)}{at^2}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{\ell_2^2-\rho^2 t}}\right), \tag{B4.4.19}$$

$$\int \frac{z^2(\ell_2^2+c^2)(\ell_2^2-a^2)dz}{\ell_2(\ell_2^2-c^2)^2(\ell_2^2-\rho^2)^{5/2}(\ell_2^2-\ell_1^2)} = \left(\frac{c^2-3a^2}{c^4\rho^4} - \frac{2a^2}{c^2\rho^6}\right)$$

$$\times \left[\sqrt{\ell_2^2-a^2} - a\cos^{-1}\left(\frac{a}{\ell_2}\right)\right] - \frac{a^2}{c^2\rho^4}\left[\frac{1}{2a}\cos^{-1}\left(\frac{a}{\ell_2}\right) - \frac{\sqrt{\ell_2^2-a^2}}{2\ell_2^2}\right]$$

$$+\left[\frac{3a^2-c^2}{c^2} - \frac{4(a^2-c^2)}{\rho^2-c^2}\right]\left[\sqrt{\ell_2^2-a^2} - \sqrt{a^2-c^2}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-c^2}}\right)\right]\frac{1}{c^2(\rho^2-c^2)^2}$$

$$+ \frac{2(a^2-c^2)}{c^2(\rho^2-c^2)^2}\left[\frac{\sqrt{\ell_2^2-a^2}}{2(\ell_2^2-c^2)} - \frac{1}{2\sqrt{a^2-c^2}}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-c^2}}\right)\right]$$

$$+\left[2\rho^2-a^2+c^2-2(\rho^2+c^2)(\rho^2-a^2)\left(\frac{1}{\rho^2} + \frac{1}{\rho^2-c^2}\right)\right]\left[\sqrt{\ell_2^2-a^2}\right.$$

$$\left. - \sqrt{a^2-\rho^2}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)\right]\frac{1}{\rho^4(\rho^2-c^2)^2}$$

$$+ \frac{(\rho^2+c^2)(\rho^2-a^2)}{\rho^4(\rho^2-c^2)^2}\left[\frac{1}{2\sqrt{a^2-\rho^2}}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right) - \frac{\sqrt{\ell_2^2-a^2}}{2(\ell_2^2-\rho^2)}\right], \tag{B4.4.20}$$

$$\int \frac{a^3 dz}{\ell_2^2 (a^2 - \ell_1^2)^{3/2}} = \int \frac{\ell_2 dz}{(\ell_2^2 - \rho^2)^{3/2}} = - \frac{a}{\rho^2} \cos^{-1}\left(\frac{a}{\ell_2}\right)$$

$$+ \frac{2a^2 + \rho^2}{2\rho^2 \sqrt{a^2 - \rho^2}} \tan^{-1}\left(\frac{\sqrt{\ell_2^2 - a^2}}{\sqrt{a^2 - \rho^2}}\right) - \frac{\sqrt{\ell_2^2 - a^2}}{2(\ell_2^2 - \rho^2)} , \qquad \text{(B4.4.21)}$$

$$\int \frac{a^3 \ell_1^2 dz}{\ell_2^2 (a^2 - \ell_1^2)^{3/2}} = \int \frac{\ell_2 \ell_1^2 dz}{(\ell_2^2 - \rho^2)^{3/2}} = - \frac{a(4a^2 + \rho^2)}{2\rho^2} \cos^{-1}\left(\frac{a}{\ell_2}\right)$$

$$- \frac{a^2}{2} \sqrt{\ell_2^2 - a^2} \left(\frac{1}{\ell_2^2} + \frac{1}{\ell_2^2 - \rho^2}\right) + \frac{a^2}{\sqrt{a^2 - \rho^2}} \frac{4a^2 - \rho^2}{2\rho^2} \tan^{-1}\left(\frac{\sqrt{\ell_2^2 - a^2}}{\sqrt{a^2 - \rho^2}}\right) ,$$

$$\text{(B4.4.22)}$$

$$\int \frac{\sqrt{a^2 - \ell_1^2}\, \ell_2^2 z^2 dz}{\ell_2^2 - \alpha^2} = a \left\{ \sqrt{\ell_2^2 - a^2} \left(\alpha^2 - \rho^2 + \frac{\ell_1^2 \rho^2}{2\alpha^2} + \frac{\ell_2^2 - a^2}{3}\right)\right.$$

$$- \frac{a\rho^2}{\alpha^2} \left[\frac{a^2 (\alpha^2 - \rho^2)}{\alpha^2} + \frac{\rho^2}{2}\right] \cos^{-1}\left(\frac{a}{\ell_2}\right)$$

$$\left. - \frac{(\alpha^2 - \rho^2)(\alpha^4 - a^2 \rho^2)}{\alpha^4} \sqrt{a^2 - \alpha^2} \tan^{-1}\left(\frac{\sqrt{\ell_2^2 - a^2}}{\sqrt{a^2 - \alpha^2}}\right)\right\} , \qquad \text{(B4.4.23)}$$

$$\int \frac{dz}{\sqrt{a^2 - \ell_1^2}\,(\ell_2^2 - \alpha^2)} = - \frac{1}{a} \left\{ \frac{a}{\alpha^2} \cos^{-1}\left(\frac{a}{\ell_2}\right) + \frac{\sqrt{a^2 - \rho^2}}{\rho^2 - \alpha^2} \tan^{-1}\left(\frac{\sqrt{\ell_2^2 - a^2}}{\sqrt{a^2 - \rho^2}}\right)\right.$$

$$\left. + \frac{\alpha^4 - a^2 \rho^2}{\alpha^2 (\rho^2 - \alpha^2)} \sqrt{a^2 - \alpha^2} \tan^{-1}\left(\frac{\sqrt{\ell_2^2 - a^2}}{\sqrt{a^2 - \alpha^2}}\right)\right\} , \qquad \text{(B4.4.24)}$$

$$\int \frac{z^2 dz}{\sqrt{a^2-\ell_1^2}\,(\ell_2^2-\rho^2)} = \frac{1}{a}\left\{\sqrt{\ell_2^2-a^2}\left(1-\frac{a^2}{2\ell_2^2}\right)+a\left(\frac{1}{2}-\frac{a^2}{\rho^2}\right)\cos^{-1}\left(\frac{a}{\ell_2}\right)\right.$$

$$\left.+\frac{(a^2-\rho^2)^{3/2}}{\rho^2}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)\right\}, \qquad \text{(B4.4.25)}$$

$$\int \frac{\sqrt{a^2-\ell_1^2}\,(a^2+\ell_1^2 t)}{(\ell_2^2-\ell_1^2)\,(a^2-\ell_1^2 t)^2}dz = \frac{a}{(a^2-\rho^2 t)^{3/2}}\tan^{-1}\left(\frac{\sqrt{\ell_2^2-a^2}}{\sqrt{a^2-\rho^2 t}}\right)$$

$$+\frac{t\ell_1\sqrt{\rho^2-\ell_1^2}}{(a^2-\ell_1^2 t)\,(a^2-\rho^2 t)}\ . \qquad \text{(B4.4.26)}$$

All the integrals involving ℓ_1 and ℓ_2 were computed by using the substitutions

$$z = \frac{\sqrt{a^2-\ell_1^2}\sqrt{\rho^2-\ell_1^2}}{\ell_1} \quad \text{or} \quad z = \frac{\sqrt{\ell_2^2-a^2}\sqrt{\ell_2^2-\rho^2}}{\ell_2}\ ,$$

$$dz = -\frac{\ell_2^2-\ell_1^2}{z\ell_1}d\ell_1 \quad \text{or} \quad dz = \frac{\ell_2^2-\ell_1^2}{z\ell_2}d\ell_2\ . \qquad \text{(B4.4.27)}$$

APPENDIX A5.3

The functions referenced in formulae (5.3.23-5.3.28) are listed below.

$$f_1^*(z)=\frac{1}{\bar q}\left[c\ \tan^{-1}\left(\frac{\bar s^*}{\ell_2^*}\right)^{1/2}-\frac{z}{R_0}\tan^{-1}\frac{h^*}{R_0}\right], \tag{A5.3.1}$$

$$f_2^*(z)=\frac{1}{R_0}\ \tan^{-1}\left(\frac{h^*}{R_0}\right), \tag{A5.3.2}$$

$$f_3^*(z)=-\frac{z}{R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{h^*}{z[R_0^2+(h^*)^2]}\left[\frac{\ell_2^*}{\ell_1^*+\ell_2^*}-\frac{z^2}{R_0^2}\right], \tag{A5.3.3}$$

$$f_4^*(z)=\frac{c}{\bar q}\left(\frac{1}{\bar s^*}-\frac{2}{\bar q}\right)\tan^{-1}\left(\frac{\bar s^*}{\ell_2^*}\right)^{1/2}+\frac{z(3R_0^2-z^2)}{\bar q^2R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)$$

$$-\frac{\sqrt{-2x_0\ell_2^*}}{\bar q\ \bar s^*(\ell_2^*+\bar s^*)}+\frac{zh^*}{R_0^2+(h^*)^2}\left[\frac{q}{\bar qR_0^2}-\frac{1}{\ell_1^*(\ell_1^*+\ell_2^*)}\right], \tag{A5.3.4}$$

$$f_5^*(z)=-\left\{\frac{q}{R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{h^*}{R_0^2+(h^*)^2}\left[\frac{1}{\ell_1^*+\ell_2^*}+\frac{q}{R_0^2}\right]\right\}, \tag{A5.3.5}$$

$$f_7^*(z)=\frac{h^*}{s^*}\ \left\{\frac{3}{s^*}\left[1-\left(\frac{\ell_2^*}{s^*}\right)^{1/2}\tan^{-1}\left(\frac{s^*}{\ell_2^*}\right)^{1/2}\right]-\frac{1}{\ell_2^*+s^*}\right\}, \tag{A5.3.6}$$

$$f_8^*(z)=\frac{\sqrt{-2x_0}}{\bar q}\left\{\frac{1}{(\bar q)^{1/2}}\ \tan^{-1}\left(\frac{\bar q}{\ell_1^*}\right)^{1/2}-\frac{\sqrt{\ell_1^*}}{\ell_2^*+s^*}\right\}, \tag{A5.3.7}$$

$$f_9^*(z)=\frac{\sqrt{-2x_0}}{s^*}\left\{\frac{1}{(s^*)^{1/2}}\tan^{-1}\left(\frac{s^*}{\ell_2^*}\right)^{1/2}-\frac{\sqrt{\ell_2^*}}{\ell_2^*+s^*}\right\}, \tag{A5.3.8}$$

$$f^*_{10}(z) = -\frac{h^*}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)^2} \ , \tag{A5.3.9}$$

$$f^*_{11}(z) = \frac{1}{\bar{q}}\left\{\frac{3R_0^4+6R_0^2z^2-z^4}{R_0^3\bar{q}^2}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \sqrt{-2x_0}\left[\frac{z}{(\bar{s}^*)^{1/2}}\left(\frac{8}{\bar{q}^2} - \frac{4}{\bar{q}\ \bar{s}^*}\right.\right.\right.$$

$$\left.+\frac{3}{(\bar{s}^*)^2}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell^*_2}\right)^{1/2} - \left.\frac{3}{(\bar{q})^{3/2}}\tan^{-1}\left(\frac{\bar{q}}{\ell^*_1}\right)^{1/2}\right] + \frac{h^*}{\bar{s}^*}\left[\frac{2}{\bar{s}^*} - \frac{2}{\bar{q}}\right.$$

$$\left.\left.+\left(\frac{1}{\bar{s}^*} - \frac{2}{\bar{q}}\right)\frac{\ell^*_2}{\ell^*_2+\bar{s}^*}\right] - \frac{h^*}{R_0^2+(h^*)^2}\left[\frac{\bar{q}}{\ell^*_1+\ell^*_2} + \frac{\ell^*_1}{\bar{q}} - \frac{z^2\bar{q}}{R_0^2\bar{q}} + 2\right]\right\} \ , \tag{A5.3.10}$$

$$f^*_{12}(z) = \frac{\sqrt{-2x_0}}{\bar{q}}\left\{\frac{3}{(\bar{q})^{3/2}}\tan^{-1}\left(\frac{\bar{q}}{\ell^*_1}\right)^{1/2} - \frac{\sqrt{\ell^*_1}}{\ell^*_2+s^*}\left[\frac{1}{(x^2+z^2)^{1/2}}\right.\right.$$

$$\left.\left.+\frac{\ell^*_2}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)} + \frac{3}{\bar{q}}\right]\right\} \ , \tag{A5.3.11}$$

$$\bar{f}^*_{13}(z) = -\frac{h^*}{\bar{s}^*}\left[\frac{15\sqrt{\ell^*_2}}{(\bar{s}^*)^{5/2}}\tan^{-1}\left(\frac{\bar{s}^*}{\ell^*_2}\right)^{1/2} - \frac{15}{(\bar{s}^*)^2} + \frac{5}{\bar{s}^*(\ell^*_2+\bar{s}^*)}\right.$$

$$\left.+\frac{2}{(\ell^*_2+\bar{s}^*)^2}\right] - \frac{h^*}{(x^2+z^2)^{1/2}(\ell^*_2+\bar{s}^*)^2} \ , \tag{A5.3.12}$$

$$f^*_{14}(z) = \frac{\sqrt{-2x_0}}{s^*}\left\{\frac{\sqrt{\ell^*_2}}{\ell^*_2+s^*}\left[\frac{1}{(x^2+z^2)^{1/2}} + \frac{\ell^*_1}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)} + \frac{3}{s^*}\right]\right.$$

$$\left.-\frac{3}{(s^*)^{3/2}}\tan^{-1}\left(\frac{s^*}{\ell^*_2}\right)^{1/2}\right\} \ , \tag{A5.3.13}$$

$$f^*_{15}(z) = \frac{\sqrt{-2x_0\ell^*_2}}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)^2} \ , \tag{A5.3.14}$$

$$f^*_{16}(z) = \frac{1}{\bar{q}}\left\{\frac{R_0^2+z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \sqrt{-2}\,\bar{x}_0\left[\frac{z}{(\bar{s}^*)^{1/2}}\left(\frac{1}{\bar{s}^*} - \frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}\right.\right.$$

$$\left.\left. + \frac{1}{(\bar{q})^{1/2}}\tan^{-1}\left(\frac{\bar{q}}{\ell_1^*}\right)^{1/2}\right] - \frac{h^*}{\bar{s}^*}\right\} \; . \tag{A5.3.15}$$

APPENDIX B5.3

Here functions which are used in the isotropic solution (5.3.31-5.3.36) are presented.

$$f^{*}_{17}(z)=(2-\nu)\frac{1}{R_0}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)$$

$$-\ \frac{\nu^2}{2-\nu}\ \frac{h^{*}}{\bar{s}^{*}}\ \left\{\frac{3}{\bar{s}^{*}}\left[1\ -\left(\frac{\ell^{*}_2}{\bar{s}^{*}}\right)^{1/2}\tan^{-1}\left(\frac{\bar{s}^{*}}{\ell^{*}_2}\right)^{1/2}\right]\ -\ \frac{1}{\ell^{*}_2+\bar{s}^{*}}\right\}\ , \qquad \text{(B5.3.1)}$$

$$f^{*}_{18}(z)=\frac{\nu}{\bar{q}}\left\{\frac{R_0^2+z^2}{R_0\bar{q}}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)+\sqrt{-2x_0}\left[\frac{z}{(\bar{s}^{*})^{1/2}}\left(\frac{1}{\bar{s}^{*}}\ -\frac{2}{\bar{q}}\right)\tan^{-1}\left(\frac{\bar{s}^{*}}{\ell^{*}_2}\right)^{1/2}\right.\right.$$

$$\left.\left.+\ \frac{\sqrt{\ell^{*}_1}}{\ell^{*}_2+s^{*}}\right]-\ \frac{h^{*}}{\bar{s}^{*}}\right\}\ , \qquad \text{(B5.3.2)}$$

$$f^{*}_{19}(z)=(1-2\nu)\left\{\frac{1}{\bar{q}}\left[\frac{\sqrt{-2x_0}}{(s^{*})^{1/2}}\ \tan^{-1}\left(\frac{s^{*}}{\ell^{*}_2}\right)^{1/2}-\ \frac{z}{R_0}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)\right]\right.$$

$$\left.+\ \frac{\nu}{2-\nu}\ \frac{\sqrt{-2x_0}}{\bar{s}^{*}}\ \left[\frac{1}{(\bar{s}^{*})^{1/2}}\ \tan^{-1}\left(\frac{\bar{s}^{*}}{\ell^{*}_2}\right)^{1/2}-\ \frac{\sqrt{\ell^{*}_2}}{\ell^{*}_2+\bar{s}^{*}}\right]\right\}\ , \qquad \text{(B5.3.3)}$$

$$f^{*}_{20}(z)=-2(1+\nu)\ \left[\frac{\bar{q}}{R_0^3}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)\ +\ \frac{h^{*}}{R_0^2+(h^{*})^2}\left(\frac{\bar{q}}{R_0^2}\ +\ \frac{1}{2(x^2+z^2)^{1/2}}\right)\right.$$

$$\left.+\ \frac{\nu}{2-\nu}\ \frac{h^{*}}{(x^2+z^2)^{1/2}(\ell^{*}_2+\bar{s}^{*})^2}\right]\ , \qquad \text{(B5.3.4)}$$

$$f^{*}_{21}(z)=-(3-2\nu)\ \left[\frac{q}{R_0^3}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)+\ \frac{h^{*}}{R_0^2+(h^{*})^2}\left(\frac{q}{R_0^2}\ +\ \frac{1}{2(x^2+z^2)^{1/2}}\right)\right]\ ,$$

$$\text{(B5.3.5)}$$

$$f^*_{22}(z)=-\frac{\nu(1-2\nu)}{2-\nu}\left\{\frac{h^*}{\bar{s}^*}\left[\frac{15\sqrt{\ell^*_2}}{(\bar{s}^*)^{5/2}}\tan^{-1}\left(\frac{\bar{s}^*}{\ell^*_2}\right)^{1/2}-\frac{15}{(\bar{s}^*)^2}+\frac{5}{\bar{s}^*(\ell^*_2+\bar{s}^*)}\right]\right.$$

$$\left.+\frac{h^*}{(x^2+z^2)^{1/2}(\ell^*_2+\bar{s}^*)^2}\right\}, \qquad \text{(B5.3.6)}$$

$$f^*_{23}(z)=(1-2\nu)\frac{1}{\bar{q}}\left\{\frac{3R_0^4+6R_0^2z^2-z^4}{R_0^3\bar{q}^2}\tan^{-1}\left(\frac{h^*}{R_0}\right)-\sqrt{-2x_0}\left[\frac{z}{(\bar{s}^*)^{1/2}}\left(\frac{8}{\bar{q}^2}-\frac{4}{\bar{q}\ \bar{s}^*}\right.\right.\right.$$

$$\left.\left.+\frac{3}{(\bar{s}^*)^2}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell^*_2}\right)^{1/2}-\frac{3}{(\bar{q})^{3/2}}\tan^{-1}\left(\frac{\bar{q}}{\ell^*_1}\right)^{1/2}\right]+\frac{h^*}{\bar{s}^*}\left[\left(\frac{1}{\bar{s}^*}-\frac{2}{\bar{q}}\right)\frac{\ell^*_2}{\ell^*_2+\bar{s}^*}\right.$$

$$\left.\left.+\frac{2}{\bar{s}^*}-\frac{2}{\bar{q}}\right]-\frac{h^*}{R_0^2+(h^*)^2}\left[\frac{\bar{q}}{\ell^*_1+\ell^*_2}+\frac{\ell^*_1}{\bar{q}}-\frac{z^2\bar{q}}{R_0^2\bar{q}}+2\right]\right\}, \qquad \text{(B5.3.7)}$$

$$f^*_{24}(z)=\frac{\nu(3-2\nu)}{2-\nu}\ \frac{\sqrt{-2x_0}}{\bar{q}}\ \left\{\frac{3}{(\bar{q})^{3/2}}\tan^{-1}\left(\frac{\bar{q}}{\ell^*_1}\right)^{1/2}\right.$$

$$\left.-\frac{\sqrt{\ell^*_1}}{\ell^*_2+s^*}\left[\frac{1}{(x^2+z^2)^{1/2}}+\frac{\ell^*_2}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)}+\frac{3}{\bar{q}}\right]\right\}, \qquad \text{(B5.3.8)}$$

$$f^*_{25}(z)=-\frac{z}{R_0^3}\ \tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{h^*}{z[R_0^2+(h^*)^2]}\left[\frac{\ell^*_2}{\ell^*_1+\ell^*_2}-\frac{z^2}{R_0^2}\right], \qquad \text{(B5.3.9)}$$

$$f^*_{26}(z)=\frac{\nu}{2-\nu}\ \frac{\sqrt{-2x_0\ell^*_2}}{(x^2+z^2)^{1/2}(\ell^*_2+s^*)^2}, \qquad \text{(B5.3.10)}$$

$$f^*_{27}(z)=-\left[\frac{z}{R_0^3}\tan^{-1}\left(\frac{h^*}{R_0}\right)+\frac{x_0z}{h^*(x^2+z^2)^{1/2}[R_0^2+(h^*)^2]}\right.$$

$$\left.+\frac{h^*z}{R_0^2[R_0^2+(h^*)^2]}\right], \qquad \text{(B5.3.11)}$$

$$f_{28}^{*}(z)=-\frac{\nu}{2-\nu}\left\{\frac{z}{\bar{s}^{*}(x^2+z^2)^{1/2}}\left[\frac{3}{\bar{s}^{*}}\left(\frac{x_0}{h^{*}}-\frac{h^{*}}{2(\ell_2^{*}+\bar{s}^{*})}\right)-\frac{1}{\ell_2^{*}+\bar{s}^{*}}\left(\frac{x_0}{h^{*}}\right.\right.\right.$$

$$\left.\left.\left.+\frac{h^{*}}{\ell_2^{*}+\bar{s}^{*}}\right)\right]+\frac{3\sqrt{-2x_0}}{(\bar{s}^{*})^{5/2}}\tan^{-1}\left(\frac{\bar{s}^{*}}{\ell_2^{*}}\right)^{1/2}\right\}, \tag{B5.3.12}$$

$$f_{29}^{*}(z)=\frac{1}{\bar{q}}\left\{\frac{z(3R_0^2-z^2)}{R_0^3\bar{q}}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)-\frac{z(R_0^2+z^2)}{\bar{q}[R_0^2+(h^{*})^2]}\left(\frac{x_0}{h^{*}(x^2+z^2)^{1/2}}\right.\right.$$

$$\left.+\frac{h^{*}}{R_0^2}\right)+\sqrt{-2x_0}\left[\left(\frac{1}{\bar{s}^{*}}-\frac{2}{\bar{q}}\right)\left(\frac{1}{(\bar{s}^{*})^{1/2}}\tan^{-1}\left(\frac{\bar{s}^{*}}{\ell_2^{*}}\right)^{1/2}\right.\right.$$

$$\left.\left.-\frac{z^2}{2(\ell_2^{*})^{1/2}(x^2+z^2)^{1/2}(\ell_2^{*}+\bar{s}^{*})}\right)-\frac{z}{2(\ell_1^{*})^{1/2}(x^2+z^2)^{1/2}(\ell_1^{*}+\bar{q}^{*})}\right]$$

$$\left.+\frac{x_0 z}{\bar{s}^{*}h^{*}(x^2+z^2)^{1/2}}\right\}, \tag{B5.3.13}$$

$$f_{30}^{*}(z)=\frac{\nu}{2-\nu}\;\frac{\sqrt{-2x_0}}{\bar{q}}\;\frac{z}{(x^2+z^2)^{1/2}}\left[\frac{\sqrt{\ell_1^{*}}}{(\ell_2^{*}+s^{*})^2}-\frac{1}{2(\ell_1^{*})^{1/2}(\ell_1^{*}+\bar{q})}\right.$$

$$\left.-\frac{1}{2(\ell_1^{*})^{1/2}(\ell_2^{*}+s^{*})}\right], \tag{B5.3.14}$$

$$f_{31}^{*}(z)=\frac{1}{\bar{q}}\left[\frac{z}{R_0^2+(h^{*})^2}\left(\frac{zx_0}{h^{*}(x^2+z^2)^{1/2}}+\frac{h^{*}z}{R_0^2}\right)\right.$$

$$\left.-\frac{z\bar{c}}{2(x^2+z^2)^{1/2}(\ell_2^{*}+s^{*})}\left(\frac{s^{*}}{\ell_2^{*}}\right)^{1/2}-\frac{R_0^2-z^2}{R_0^3}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)\right], \tag{B5.3.15}$$

$$f_{32}^{*}(z)=-\frac{\nu}{2-\nu}\;\frac{z}{(x^2+z^2)^{1/2}(\ell_2^{*}+\bar{s}^{*})^2}\left(\frac{-2x_0}{\ell_2^{*}}\right)^{1/2}, \tag{B5.3.16}$$

$$f^*_{33}(z) = -\left\{\frac{3zq}{R_0^5}\tan^{-1}\left(\frac{h^*}{R_0}\right) + \frac{zq}{R_0^2[R_0^2+(h^*)^2]}\left[\frac{h^*}{R_0^2} + \frac{x_0}{h^*(x^2+z^2)^{1/2}}\right]\right.$$

$$-\left[\frac{q}{R_0^2} + \frac{1}{2(x^2+z^2)^{1/2}}\right]\left[\frac{zx_0[(h^*)^2-R_0^2]}{h^*[R_0^2+(h^*)^2]^2(x^2+z^2)^{1/2}} - \frac{2zh^*}{[R_0^2+(h^*)^2]^2}\right]$$

$$\left.+ \frac{2zh^*}{R_0^2+(h^*)^2}\left[\frac{q}{R_0^2} + \frac{2}{(\ell_1^*+\ell_2^*)^3}\right]\right\}, \tag{B5.3.17}$$

$$f^*_{34}(z) = \frac{\nu}{2-\nu}\;\frac{z}{(x^2+z^2)(\ell_2^*+\bar{s}^*)^2}\left[\frac{2h^*}{\ell_2^*+\bar{s}^*} + \frac{h^*}{(x^2+z^2)^{1/2}} + \frac{x_0}{h^*}\right], \tag{B5.3.18}$$

$$f^*_{35}(z) = \frac{\nu}{2-\nu}\left\{\frac{30zx_0}{h^*(\ell_2^*)^{1/2}(\bar{s}^*)^{7/2}}\tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2}\right.$$

$$-\frac{zx_0}{h^*\bar{s}^*(x^2+z^2)^{1/2}}\left[\frac{15}{(\bar{s}^*)^2} - \frac{5}{\bar{s}^*(\ell^*+\bar{s}_2^*)} - \frac{2}{(\ell_2^*+\bar{s}^*)^2}\right]$$

$$+\frac{z}{\bar{s}^*(\ell_2^*+\bar{s}^*)(x^2+z^2)^{1/2}}\left[\frac{15}{2(\bar{s}^*)^2} + \frac{5}{\bar{s}^*(\ell^*+\bar{s}_2^*)} + \frac{4}{(\ell_2^*+\bar{s}^*)^2}\right]$$

$$\left.+\frac{z}{(\ell_2^*+\bar{s}^*)^2(x^2+z^2)}\left[\frac{2h^*}{\ell_2^*+\bar{s}^*} + \frac{h^*}{(x^2+z^2)^{1/2}} + \frac{x_0}{h^*}\right]\right\}, \tag{B5.3.19}$$

$$f^*_{36}(z) = \frac{1}{\bar{q}}\left\{\frac{z(15R_0^4-10R_0z+3z)}{R_0^5\bar{q}^2}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \sqrt{\frac{-2x_0}{\bar{s}^*}}\left(\frac{8}{\bar{q}^2} - \frac{4}{\bar{s}^*\bar{q}}\right.\right.$$

$$\left.+\frac{3}{(\bar{s}^*)^2}\right)\tan^{-1}\left(\frac{\bar{s}^*}{\ell_2^*}\right)^{1/2} - \frac{z\sqrt{-2x_0}}{2(x^2+z^2)^{1/2}}\left[\frac{3}{\bar{q}(\ell_1^*)^{1/2}(\ell_1^*+\bar{q})}\right.$$

$$\left.-\frac{z}{(\ell_2^*)^{1/2}(\ell_2^*+\bar{s})}\left(\frac{8}{\bar{q}^2} - \frac{4}{\bar{s}^*\bar{q}} + \frac{3}{(\bar{s}^*)^2}\right)\right]$$

$$- \frac{z}{(x^2+z^2)^{1/2}(\ell_2^*+\bar{s}^*)}\left(\frac{x_0\ell_2^*}{h^*\bar{s}^*} - \frac{h^*}{\ell_2^*+\bar{s}^*}\right)\left(\frac{1}{\bar{s}^*} - \frac{2}{\bar{q}}\right)$$

$$- \frac{zx_0}{h^*\bar{s}^*(x^2+z^2)^{1/2}}\left(\frac{2}{\bar{s}^*} - \frac{2}{\bar{q}}\right) - \frac{z}{[R_0^2+(h^*)^2]^2}\left(\frac{x_0[(h^*)^2-R_0^2]}{h^*(x^2+z^2)^{1/2}} - 2h^*\right)$$

$$\times \left(\frac{\bar{q}}{2(x^2+z^2)^{1/2}} + \frac{\ell_1^*}{\bar{q}} - \frac{zq}{R_0^2\bar{q}} + 2\right) - \frac{zh^*}{R_0^2+(h^*)^2}\left[\frac{1}{\bar{q}(x^2+z^2)^{1/2}}\right.$$

$$\left.- \frac{\bar{q}}{2(x^2+z^2)^{3/2}} - \frac{q^2}{R_0^4} + \frac{2(R_0^2+2z^2)}{\bar{q}^2R_0^2} - \frac{3R_0^4+6R_0^2z^2-z^4}{2\ell_1^*\bar{q}^2R_0^2(x^2+z^2)^{1/2}}\right]\Bigg\} ,$$

(B5.3.20)

$$f_{37}^*(z) = \frac{\nu}{2-\nu}\ \frac{\sqrt{-2x_0}}{\bar{q}}\left\{\frac{3z}{2\bar{q}(\ell_1^*)^{1/2}(x^2+z^2)^{1/2}}\left[\frac{2\ell_1^*}{(\ell_2^*+s^*)^2} - \frac{1}{\ell_2^*+s^*}\right.\right.$$

$$\left.\left.- \frac{1}{\ell_1^*+\bar{q}}\right] + \frac{z\sqrt{\ell_1^*}}{(x^2+z^2)(\ell_2^*+s^*)^2}\left[\frac{\ell_2^*-s^*}{\ell_2^*+s^*} - \frac{4x+s^*}{2\ell_1^*} + \frac{2\ell_2^*+s^*}{(x^2+z^2)^{1/2}}\right]\right\} ,$$

(B5.3.21)

$$f_{38}^*(z) = -\left\{\frac{R_0^2-3z^2}{R_0^5}\tan^{-1}\left(\frac{h^*}{R_0}\right) - \frac{z}{R_0^2[R_0^2+(h^*)^2]}\left(\frac{zh^*}{R_0^2} + \frac{zx_0}{h^*(x^2+z^2)}\right)\right.$$

$$- \left(\frac{zx_0}{2(x^2+z^2)^{1/2}} - \frac{z^2}{R_0^2}\right)\left[\frac{x_0[(h^*)^2-R_0^2]}{h^*[R_0^2+(h^*)^2]^2(x^2+z^2)^{1/2}} - \frac{2h^*}{[R_0^2+(h^*)^2]^2}\right.$$

$$\left.\left.- \frac{h^*}{z[R_0^2+(h^*)^2]}\right] + \frac{h^*}{R_0^2+(h^*)^2}\left[\frac{x}{2(x^2+z^2)^{3/2}} + \frac{2(R_0^2-z^2)}{R_0^4}\right]\right\}, \qquad \text{(B5.3.22)}$$

$$f_{39}^*(z) = \frac{\nu}{2-\nu}\ \frac{h^*}{(x^2+z^2)(\ell_2^*+\bar{s}^*)^2}\left[\frac{1}{2} + \frac{2\ell_1^*}{\ell_2^*+\bar{s}^*} - \frac{2\ell_2^*}{\ell_2^*+\ell_1^*}\right] , \qquad \text{(B5.3.23)}$$

$$f^{*}_{40}(z)=\sqrt{\frac{-2x_0}{\ell^{*}_{2}}}\;\frac{2z(\bar{q}-\ell^{*}_{2}-\bar{s}^{*})}{2\bar{q}^{2}(\ell^{*}_{2}+\bar{s}^{*})^{2}(x^{2}+z^{2})^{1/2}}-\frac{3q^{2}}{R_0^{5}}\tan^{-1}\left(\frac{h^{*}}{R_0}\right)$$

$$+\frac{z^{2}(3R_0^{2}-z^{2})}{\bar{q}^{2}R_0^{2}[R_0^{2}+(h^{*})^{2}]}\left(\frac{h^{*}}{R_0^{2}}+\frac{x_0}{h^{*}(x^{2}+z^{2})^{1/2}}\right)$$

$$-\left[\frac{q}{\bar{q}R_0^{2}}-\frac{1}{\ell^{*}_{1}(\ell^{*}_{1}+\ell^{*}_{2})}\right]\left[\frac{z^{2}x_0[(h^{*})^{2}-R_0^{2}]}{h^{*}[R_0^{2}+(h^{*})^{2}]^{2}(x^{2}+z^{2})^{1/2}}+\frac{h^{*}}{R_0^{2}+(h^{*})^{2}}\right.$$

$$\left.-\frac{2z^{2}h^{*}}{[R_0^{2}+(h^{*})^{2}]^{2}}\right]+\frac{2z^{2}h^{*}}{R_0^{2}+(h^{*})^{2}}\left(\frac{q}{\bar{q}R_0^{4}}+\frac{2x}{(\ell^{*}_{1})^{2}(\ell^{*}_{1}+\ell^{*}_{2})^{3}}\right.$$

$$\left.-\frac{2}{(\ell^{*}_{1})^{2}(\ell^{*}_{1}+\ell^{*}_{2})^{2}}\right)\;, \tag{B5.3.24}$$

$$f^{*}_{41}(z)=\frac{\nu}{2-\nu}\;\frac{h^{*}}{(x^{2}+z^{2})(\ell^{*}_{2}+s^{*})^{2}}\left[\frac{1}{2}-\frac{2\ell^{*}_{2}}{\ell^{*}_{2}+s^{*}}-\frac{2\ell^{*}_{2}}{\ell^{*}_{2}+\ell^{*}_{1}}\right]\;. \tag{B5.3.25}$$

APPENDIX A5.4

Here the derivation of (5.4.34) is presented. By using the rule of differentiation under the integral sign, the expression in curly brackets of (5.4.33) can be rewritten as

$$\frac{\mathcal{L}^{*}(y_0+2\ell_2^{*}-y)\sqrt{\ell_2^{*}-y}}{(\ell_2^{*}-\ell_1^{*})\sqrt{y_0}}+\int_0^{y_0}\frac{dt}{\sqrt{y_0-t}}\,\frac{d}{dt}\left[\frac{\mathcal{L}^{*}(2\ell_2^{*}(t)-y+y_0-2t)}{\ell_2^{*}(t)-\ell_1^{*}(t)}\sqrt{\ell_2^{*}(t)-y}\right]. \tag{A5.4.1}$$

Introduce $h^{*}[\ell_1^{*}(t)]$ as it is defined in (5.4.16) and transform it as follows

$$\begin{aligned}h^{*}[\ell_1^{*}(t)]&=2\sqrt{\ell_2^{*}(y_0)-\ell_1^{*}(t)}\sqrt{\ell_1^{*}(y_0)-\ell_1^{*}(t)}\\&=2\sqrt{yy_0-(z^2/4)-\ell_1^{*}(t)\,(y+y_0)+[\ell_1^{*}(t)]^2}\\&=2\sqrt{y-\ell_1^{*}(t)}\sqrt{y_0-\ell_1^{*}(t)-z^2/\{4[y-\ell_1^{*}(t)]\}}\\&=2\sqrt{\ell_2^{*}(t)-t}\sqrt{y_0-t}\equiv h(t).\end{aligned} \tag{A5.4.2}$$

Here were used the identities

$$\ell_2^{*}(t)-t=y-\ell_1^{*}(t),\quad \ell_1^{*}(t)\ell_2^{*}(t)=yt-\frac{z^2}{4},\quad g^{*}[\ell_1^{*}(t)]=t. \tag{A5.4.3}$$

The derivative

$$\frac{dh(t)}{dt}=-\frac{\sqrt{\ell_2^{*}(t)-t}}{\sqrt{y_0-t}}\,\frac{\ell_2^{*}(t)-\ell_1^{*}(t)+y_0-t}{\ell_2^{*}(t)-\ell_1^{*}(t)}. \tag{A5.4.4}$$

With help of the identity $2\ell_2^*(t)-y+y_0-2t=\ell_2^*(t)-\ell_1^*(t)+y_0-t$, the expression in square brackets of (A5.4.1) can be presented as

$$\frac{[\ell_2^*(t)-\ell_1^*(t)+y_0-t]\sqrt{\ell_2^*(t)-y}}{[\ell_2^*(t)-\ell_1^*(t)]\{[\ell_2^*(t)-\ell_1^*(t)+y_0-t]^2+(x-x_0)^2\}}$$

$$=-\frac{\sqrt{\ell_2^*(t)-y}\sqrt{y_0-t}}{\sqrt{\ell_2^*(t)-t}\,[R_0^2+h^2(t)]}=-\frac{2z(y_0-t)^{3/2}}{h^2(t)[R_0^2+h^2(t)]}\frac{dh(t)}{dt}\ . \qquad \text{(A5.4.5)}$$

Substitution of (A5.4.5) in (A5.4.1) and integration by parts yields (the $\mathscr{L}^*$ operator is replaced by λ^*)

$$\frac{\lambda^*(y_0+2\ell_2^*-y,x-x_0)}{(\ell_2^*-\ell_1^*)\sqrt{y_0}}-2z\int_0^{y_0}\frac{dt}{\sqrt{y_0-t}}\frac{d}{dt}\left[\frac{(y_0-t)^{3/2}h'(t)}{h^2(t)[R_0^2+h^2(t)]}\right]$$

$$=-2z\lim_{t\to y_0}\frac{(y_0-t)h'(t)}{h^2(t)[R_0^2+h^2(t)]}+z\int_0^{y_0}\frac{h'(t)dt}{h^2(t)[R_0^2+h^2(t)]}$$

$$=-2z\lim_{t\to y_0}\frac{(y_0-t)h'(t)}{h^2(t)[R_0^2+h^2(t)]}+\frac{z}{R_0^2}\int_{h(0)}^{h(y_0)}\left(\frac{1}{h^2(t)}-\frac{1}{R_0^2+h^2(t)}\right)dh(t)$$

$$=-2z\lim_{t\to y_0}\frac{(y_0-t)h'(t)}{h^2(t)[R_0^2+h^2(t)]}+\frac{z}{R_0^2}\Bigg|_{h(0)}^{h(y_0)}-\frac{1}{h(t)}-\frac{1}{R_0}\tan^{-1}\left(\frac{h(t)}{R_0}\right)$$

$$=\frac{z}{R_0^2}\left[\frac{1}{h}+\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right)\right]\ . \qquad \text{(A5.4.6)}$$

Here, for the sake of brevity, the notation $h \equiv h(0)$ introduced, as it is defined in (A5.4.2).

Substitution of (A5.4.6) back into (5.4.33) proves (5.4.34).

Note that the limit $(t \to y_0)$ in (A5.4.6) is infinite and cancels out with the next term $z/[R_0^2 h(t)]$ for $t \to y_0$.

There are noted also some other identities which might be useful in various transformations

$$\sqrt{\ell_2^*(t)-y}\sqrt{y-\ell_1^*(t)} = \sqrt{\ell_2^*(t)-t}\sqrt{t-\ell_1^*(t)}$$

$$= \sqrt{\ell_1^*(t)-t}\sqrt{\ell_1^*(t)-y} = \sqrt{\ell_2^*(t)-y}\sqrt{\ell_2^*(t)-t} = \frac{z}{2} \ ,$$

$$y-\ell_1^*(t) = \ell_2^*(t)-t \ , \qquad t-\ell_1^*(t) = \ell_2^*(t)-y \ ,$$

$$\frac{\partial \ell_1^*(t)}{\partial t} = \frac{y-\ell_1^*(t)}{\ell_2^*(t)-\ell_1^*(t)} = \frac{\ell_2^*(t)-t}{\ell_2^*(t)-\ell_1^*(t)} \ . \qquad \text{(A5.4.7)}$$

APPENDIX B5.4

The details of integration, which result in expression given in (5.4.61) are presented here.

The integral to be computed is

$$I = \int \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] dz \ . \qquad \text{(B5.4.1)}$$

Here are some derivatives and identities, which might be useful in various transformations

$$\frac{\partial \ell_1^*}{\partial z} = -\frac{z}{2(\ell_2^* - \ell_1^*)} \ , \qquad \frac{\partial \ell_2^*}{\partial z} = \frac{z}{2(\ell_2^* - \ell_1^*)} \ , \qquad \frac{\partial h}{\partial z} = \frac{y_0 z}{h(\ell_2^* - \ell_1^*)} \ ,$$

$$\frac{d}{dz} \tan^{-1}\left(\frac{h}{R_0}\right) = \frac{R_0}{R_0^2 + h^2}\left[\frac{y_0 z}{h(\ell_2^* - \ell_1^*)} + \frac{z}{h}\right] - \frac{z}{R_0 h} \ ,$$

$$R_0^2 + h^2 = (2\ell_1^* - s^*)(2\ell_1^* - \bar{s}^*) = (2\ell_2^* - i\bar{q})(2\ell_2^* + iq) \ . \qquad \text{(B5.4.2)}$$

Proceed with integration by parts of integral in (B5.4.1). It results in

$$I = \int \frac{z\,dz}{R_0^2 h} - \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \int \frac{dz}{R_0} \frac{d}{dz}\tan^{-1}\left(\frac{h}{R_0}\right) \ . \qquad \text{(B5.4.3)}$$

With help of (B5.4.2) the expression in (B5.4.3) may be rewritten as:

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \int\frac{y_0 z dz}{h(\ell_2^{\bullet}-\ell_1^{\bullet})(R_0^2+h^2)} + \int\frac{zdz}{h(R_0^2+h^2)} . \qquad (B5.4.4)$$

The change of variable of integration with help of the expressions in (B5.4.2) will allow to transform the integrals in (B5.4.4) into the following, respectively

$$\int\frac{y_0 z dz}{h(\ell_2^{\bullet}-\ell_1^{\bullet})(R_0^2+h^2)} = \sqrt{y_0}\int\frac{d\ell_2^{\bullet}}{\sqrt{\ell_2^{\bullet}}(2\ell_2^{\bullet}-i\bar{q})(2\ell_2^{\bullet}+iq)} ,$$

$$\int\frac{zdz}{h(R_0^2+h^2)} = \frac{2}{\sqrt{y_0}}\int\frac{\sqrt{\ell_2^{\bullet}}d\ell_2^{\bullet}}{(2\ell_2^{\bullet}-i\bar{q})(2\ell_2^{\bullet}+iq)} - \frac{y}{\sqrt{y_0}}\int\frac{d\ell_2^{\bullet}}{\sqrt{\ell_2^{\bullet}}(2\ell_2^{\bullet}-i\bar{q})(2\ell_2^{\bullet}+iq)} . \qquad (B5.4.5)$$

After substitution of (B5.4.5) in (B5.4.4) and some simplifications it will become

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{2\sqrt{y_0}}\left[\int\frac{d\ell_2^{\bullet}}{\sqrt{\ell_2^{\bullet}}(2\ell_2^{\bullet}-i\bar{q})} + \int\frac{d\ell_2^{\bullet}}{\sqrt{\ell_2^{\bullet}}(2\ell_2^{\bullet}+iq)}\right] . \qquad (B5.4.6)$$

The integrals in (B5.4.6) are elementary and it gives

$$I = -\frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{1}{\sqrt{2y_0}}\, 2\mathcal{R}e\left[\frac{1}{\sqrt{iq}}\tan^{-1}\left(\frac{2\ell_2^{\bullet}}{iq}\right)^{1/2}\right] . \qquad (B5.4.7)$$

Finally (B5.4.7) allows to write the result of the definite integral as

$$\int_{\infty}^{z} \frac{z}{R_0^3}\left[\frac{R_0}{h} + \tan^{-1}\left(\frac{h}{R_0}\right)\right] dz$$

$$= - \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{1}{\sqrt{2Y_0}}\, 2\mathcal{R}e\left[\frac{1}{\sqrt{iq}}\tan^{-1}\left(\frac{iq}{2\ell_2^{*}}\right)^{1/2}\right]. \qquad \text{(B5.4.8)}$$