Applications of Renewal Theory to Risk Modelling in an Inflationary Environment

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ABSTRACT

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The expected value and the variance of the aggregate discounted claims of an insurance portfolio are considered. An approximation is given in both cases using renewal theory. The compound Poisson special case is studied in more detail.

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To Corinne, Christian and Francis.

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INTRODUCTION

One of the main problems of interest in classical insurance risk theory is the evaluation of the aggregate claims distribution of a risk business portfolio. From this distribution, the first and the second moments of the aggregate claims are derived in order to evaluate the premium that should be charged to a contract. Traditional models have usually assumed that claim severities are independent and identically distributed random variables and that they are independent of the number of claims recorded over the period of time considered, primarily for mathematical tractability. A major difficulty with this assumption is that the claim severity is independent of the time at which the claim occured. If the time period of interest is short and the inflation rate low, then this is not a serious drawback, but may be a problem in times of higher inflation or for contracts with long terms.

Several authors such as Taylor (1979), Waters (1983), Delbaen and Haezendonck (1987) have considered inflation in the evaluation of ruin probabilities. Bühlmann (1970) and Willmot (1989) have also considered inflation in the evaluation of the distribution of total claims. In this thesis, we focus on the expected value and the variance of the total claims when the number of claims over a fixed period of time is assumed to follow a renewal process.

Renewal theory is used here as a tool to obtain the mean and the variance of the aggregate discounted claims. The first chapter is thus a review of the fundamental results of renewal theory. Those results will then be applied to a risk model with inflation in chapter 2. The limit behaviour of the aggregate mean and variance is then studied in chapter 3 where some numerical tables are given to illustrate the accuracy of the results obtained in chapter 2.

Chapter 1

Renewal Theory

1.1 Ordinary Renewal Processes

A renewal counting process $\{N(t),\ t \geq 0\}$ is a nonnegative integer-valued stochastic process that registers the successive occurences of an event during the time interval (0,t]. For ordinary renewal processes the time durations between consecutive "events" are assumed positive, independent and identically distributed random variables (i.i.d.r.v.).

In an insurance portfolio, the number of claims in a given period can be seen as a renewal process (under the assumption that two claims can not occur at the same time). More details on renewal processes can be found in Karlin and Taylor (1975).

Definition: A renewal process $N = \{N(t) : t \geq 0\}$ is such that

$$N(t) = \max\{n \geq 0 : T_n \leq t\} \quad \text{where} \quad T_0 = 0\,,$$

$$T_n = \tau_1 + \cdots + \tau_n \tag{1.1}$$

and $\{\tau_k\}_{k\geq 1}$ is a sequence of i.i.d.r.v.'s with common distribution function (d.f.) F.

The τ_k's are usually called the interarrival times. We shall speak of T_n as the time of the n^{th} arrival. Its distribution function is denoted by F_n.

Another way of defining $N(t)$ is by using counting functions and by writting $N(t)$ as

$$N(t) = \sum_{k=0}^{\infty} I_{\{T_k \leq t\}} \tag{1.2}$$

where $I_{\{T_k \leq t\}}$ is the indicator function of the event $\{T_k \leq t\}$.

A first remark derived from the definition of a renewal process is that if the τ_k's are exponentially distributed with the same parameter, the renewal process reduces to the well known Poisson process.

Proposition 1.1: The event $\{N(t) \geq n\}$ is equivalent to $\{T_n \leq t\}$, i.e. if the number of outcomes in the time interval (0,t] is at least n then the n^{th} arrival time is at most t.

Proposition 1.2: Let F_n be the d.f. of T_n, then

$$F_1 = F \quad \text{and} \quad F_{k+1}(x) = \int_0^x F_k(x-y)dF(y) \quad \text{for } k \geq 1 \ . \tag{1.3}$$

Proof: F_1 = F and $T_{k+1} = T_k + \tau_{k+1}$ by definition. The proposition is obtained by writing the distribution function of the sum of two independent random variables as the convolution of their distribution functions (the product of convolution is studied in more details in section 1.2). Using (1.1), $P(T_{k+1} \leq x) = P(T_k + \tau_{k+1} \leq x)$ since τ_{k+1} is independent of T_k. Thus

$$\begin{aligned} P(T_{k+1} \leq x) &= \int_0^x P(T_k + \tau_{k+1} \leq x | \tau_{k+1} = y)dF(y) \\ &= \int_0^x P(T_k \leq x - y | \tau_{k+1} = y)dF(y) \quad \text{by independence of the } \tau_k\text{'s} \\ &= \int_0^x P(T_k \leq x - y)dF(y) = \int_0^x F_k(x-y)dF(y) \ . \end{aligned}$$

•

For a fixed value of t, we are interested in the probability density function (p.d.f.) of the r.v. $N(t)$, which is obtained in the following proposition.

Proposition 1.3: Let $p_k(t) = P\{N(t) = k\}$, then

$$p_k(t) = F_k(t) - F_{k+1}(t) \quad \text{for } k \geq 1 \tag{1.4}$$

and $p_0(t) = P\{N(t) = 0\} = P\{\tau_1 > t\} = 1 - F(t)$.

Proof: $P\{N(t) = k\} = P\{N(t) \geq k\} - P\{N(t) \geq k+1\}$. By **Proposition 1.1**, we have that

$$\begin{aligned} p_k(t) &= P(T_k \leq t) - P(T_{k+1} \leq t) \\ &= F_k(t) - F_{k+1}(t)\ . \end{aligned}$$

•

We are also interested in the mean number of renewals in the time interval (0,t] generally called the renewal function, described below.

1.1.1 The Renewal Function

Definition: The renewal function $m(t)$ is defined as $m(t) = E[N(t)]$ for $t \geq 0$.

Since the p.d.f of $N(t)$ depends on the distribution function of the interarrival times an expression of $N(t)$ can be derived as follows.

Proposition 1.4: $m(t) = \sum_{k=1}^{\infty} F_k(t)$ for $t \geq 0$ and where $F_k(0) = 0$.

Proof: The proof can be found in any standard text, see e.g. Grimmett and Stirzaker (1992). For t fixed, $N(t)$ is a discrete random variable with probability density function $p_k(t) = F_k(t) - F_{k+1}(t)$ therefore

$$m(t) \quad = \quad E[N(t)] = \sum_{k=1}^{\infty} k\, p_k(t)$$

$$
\begin{aligned}
m(t) &= \sum_{k=1}^{\infty} k\,[P\{N(t) \geq k\} - P\{N(t) \geq k+1\}] \\
&= \sum_{k=1}^{\infty} P\{N(t) \geq k\} = \sum_{k=1}^{\infty} P\{T_k \leq t\} \text{ by } \textbf{Proposition 1.1} \\
&= \sum_{k=1}^{\infty} F_k(t)\,.
\end{aligned}
$$

•

Thus the knowledge of the $F_k(t)$ is sufficient to obtain the renewal function $m(t)$. But in most cases the k^{th} convolution $F_k(t)$ is difficult to obtain in an explicit form (see section 1.2). A numerical approach is used when F is a distribution closed under convolution. However, a different way of deriving an expression for $m(t)$ is possible. It uses a functional equation approach as given below.

Proposition 1.5 : The renewal function $m(t)$ satisfies the following equation

$$
m(t) = F(t) + \int_0^t m(t-x)\,dF(x)\,. \tag{1.5}
$$

Proof: Consider the following expression for $m(t) = E[N(t)]$=$E[E\{N(t)|\tau_1\}]$. Since

$$
E[N(t)|\tau_1 = x] = \begin{cases} 0 & \text{if } x > t, \text{ i.e. the first arrival occurs after t} \\ 1 + E[N(t-x)] & \text{if } x \leq t. \end{cases}
$$

Thus

$$
\begin{aligned}
m(t) &= \int_0^{\infty} E[N(t)|\tau_1 = x]dF(x) \\
&= \int_0^t dF(x) + \int_0^t E[N(t-x)]dF(x) \\
&= F(t) + \int_0^t m(t-x)dF(x)\,.
\end{aligned}
$$

Therefore m satisfies a functional equation, called a renewal equation. We will study renewal equations in more details in **Section 1.3.** However, we need some results on convolutions of distributions before we do that.

1.2 The Product of Convolution

The definitions given here can be found in Feller (1971). We might have simplified some notations when considering special cases of Feller's definitions.

Definition: Let f and F be two real, locally bounded functions vanishing on $(-\infty, 0)$. The product of convolution of f and F, denoted $f * F$ is defined as follows:

$$f * F(x) = \int_{0-}^{x+} f(x-y)\, dF(y) \quad \text{for } x \geq 0 . \tag{1.6}$$

The above integral is interpreted in a Lebesgues-Stieltjes sense. It can easily be shown that this product of convolution posseses the following properties.

1.2.1 Properties of the Product of Convolution

(i) **Commutativity:** $f * F = F * f$ if F is a d.f. on $[0, \infty)$.

(ii) **Associativity:** Let G be another distribution function on $[0, \infty)$,

$$f * (F * G) = (f * F) * G.$$

(iii) If f is **non-negative** then so is $f * F$.

The proofs are straight forward applications of the definition.

We also define the n^{th} power of convolution of a single distribution function F as:

$$F^{*n} = F^{*(n-1)} * F(x) \quad \text{for n} \geq 1 \tag{1.7}$$

with the understanding that $F^{*1} = F$ and $F^{*0}(x) = I_{[0,\infty)}(x)$ the indicator function of the set $[0, \infty)$.

Under this definition F^{*n} is seen as the distribution function of the sum of n i.i.d.r.v.'s with d.f. F. The following proposition gives additional properties of this power of convolution.

Proposition 1.6: If F and G are two distribution functions with support on $[0,\infty)$ then

(i) $F*G(x) \leq F(x)G(x)$ for any $x \geq 0$ and $F*G(\infty) = F(\infty)G(\infty)$,

(ii) $F^{*(m+n)}(x) \leq F^{*m}(x)$ for any m,n ≥ 1 and any $x \geq 0$ and

(iii) if $F(0) < 1$ then $\sum_{k=0}^{\infty} F^{*k}(x) < \infty$ for any $x \geq 0$.

Proof: The proof properties (i) and (ii) can be found in any standard text book.

(i) $F*G(x) = \int_{0-}^{x+} F(x-y)\,dG(y)$ but F is a non-decreasing function therefore $F(x-y) \leq F(x)$ for any $x, y \geq 0$. An upper bound of the convolution is thus given by

$$\begin{aligned} F*G(x) &\leq \int_{0-}^{x+} F(x)\,dG(y) \\ &\leq F(x)\int_{0-}^{x+} dG(y) \\ &\leq F(x)G(x)\,. \end{aligned}$$

The second part follows by taking the limit as $x \rightarrow \infty$.

(ii) At a given x,

$$\begin{aligned} F^{*(m+n)}(x) &= F^{*m}(x) * F^{*n}(x) \\ &\leq F^{*m}(x)F^{*n}(x) \end{aligned}$$

by property (i). Using this inequality iteratively gives

$$F^{*n}(x) \leq F^{n}(x)$$

for any $n \geq 1$. Finally $F(x) \leq 1$ (by definition of a ditribution function) and thus

$$F^{*(m+n)}(x) \leq F^{*m}(x)\,.$$

(iii) The proof of this property is due to DeVylder and can be found in Garrido (1983). From property (i), we know that $F^{*k}(x) \leq [F(x)]^k$ for any $k \geq 0$ and also that $F(x) \geq F(0)$ for a fixed $x \geq 0$. Thus for a fixed x such that $F(x) < 1$ we have

$$\sum_{k=0}^{\infty} F^{*k}(x) \leq \sum_{k=0}^{\infty}[F(x)]^k = \frac{1}{1-F(x)} \, .$$

If x is such that $F(x) = 1$ then we need another upper bound of $F^{*k}(x)$. Consider the event $(X_1 > \frac{x}{n}, \cdots, X_n > \frac{x}{n})$. For any n, it is included in the event $(X_1 + \cdots + X_n > x)$. The X_i's are i.i.d with d.f. F, thus we have that

$$P(X_1 > \frac{x}{n}, \cdots, X_n > \frac{x}{n}) \leq P(X_1 + \cdots + X_n > x) \, .$$

Hence,

$$[P(X_1 > \frac{x}{n})]^n \leq [1 - F^{*n}(x)] \, ,$$

which implies,

$$F^{*n}(x) \leq \underbrace{1 - [1 - F(\frac{x}{n})]^n}_{u_n} \, . \qquad (1.8)$$

In addition we can rewrite

$$\begin{aligned} \sum_{k=0}^{\infty} F^{*k}(x) &= F^{*0}(x) + F^{*1}(x) + \cdots + F^{*(n-1)}(x) \\ &\quad + F^{*n}(x) + F^{*(n+1)}(x) + \cdots + F^{*(2n-1)}(x) + \cdots \end{aligned}$$

Since $F^{*k}(x) \leq F^{*(k-1)(x)}$

$$\begin{aligned} \sum_{k=0}^{\infty} F^{*k}(x) &\leq nF^{*0}(x) + nF^{*n}(x) + nF^{*2n}(x) + \cdots \\ &\leq n\sum_{k=0}^{\infty} F^{*kn}(x) \end{aligned}$$

where $F^{*n}(x) \leq u_n$ for any $n \geq 1$, u_n being defined above. The fixed real x is such that $F(x) = 1$, but there exists an integer $n \geq 1$ such that $F(\frac{x}{n}) < 1$, therefore $u_n < 1$ for any $n \geq 1$. By **Proposition 1.6(i)** and (1.8) the following bound results:

$$\sum_{k=0}^{\infty} F^{*k}(x) \leq n\sum_{k=0}^{\infty} F^{*kn}(x) \leq n\sum_{k=0}^{\infty}[F^{*n}(x)]^k \leq n\sum_{k=0}^{\infty}[u_n]^k \, .$$

Since $u_n < 1$ the right hand side sum above is a convergent geometric series for any fixed integer n. •

1.3 The General Form of the Renewal Equation

The renewal equation defined in (1.5) is a particular form of the general renewal equation given by Feller (1971). This equation in f is given by

$$f = g + f * F \tag{1.9}$$

where F is a distribution function such that $F(0) < 1$, and g is a locally bounded real valued function.

As pointed out at the end of **Section 1.1** our interest is to derive the renewal function $m(t)$ of a renewal process. **Proposition 1.4** gives us an expression of $m(t)$ that uses the p.d.f. of the number of renewals $N(t)$, but for certain processes this method is complicated. The next paragraphs illustrate two methods to solve equation (1.9) and hence give an expression of $m(t)$. The first method is given by Feller (1971) and the second one uses the Laplace Transform technique.

1.3.1 The Convolution Method

Under the above conditions on F and g the renewal equation (1.9) has a unique solution, f_0, among the functions that are bounded on bounded intervals. This solution is

$$f_0 = g * \sum_{k=0}^{\infty} F^{*k} . \tag{1.10}$$

Proof: A proof is given by Feller (1971). We give here a different proof taken from Garrido (1983). Clearly f_0 is a solution of (1.9) since

$$\begin{aligned} g + f_0 * F &= g + (g * \sum_{k=0}^{\infty} F^{*k}) * (F) \\ &= g + g * [(\sum_{k=0}^{\infty} F^{*k}) * F] \text{ by associativity} \\ &= g + g * [\sum_{k=0}^{\infty} F^{*k} - F^{*0}] \end{aligned}$$

$$\begin{aligned} g + f_0 * F &= g + g * \sum_{k=0}^{\infty} F^{*k} - g \\ &= \sum_{k=0}^{\infty} F^{*k} = f_0 . \end{aligned}$$

Now to prove the uniqueness of f_0, assume that there is another function, f_1, satisfying (1.9). Then

$$f_0 = g + f_0 * F \text{ and } f_1 = g + f_1 * F .$$

Taking the difference we have

$$\begin{aligned} f_0 - f_1 &= f_0 * F - f_1 * F = (f_0 - f_1) * F \\ &= (f_0 - f_1) * F^{*n} \quad \text{for any n} \geq 1 \end{aligned}$$

The idea is to prove that as $n \to \infty$, $f_0 - f_1 \to 0$. Consider

$$\begin{aligned} |f_0(x) - f_1(x)| &= |(f_0 - f_1) * F^{*n}(x)| \quad \text{for any } x \geq 0 \\ &= |\int_0^x (f_0 - f_1)(x - y)\, dF^{*n}(y)| \\ &\leq \int_0^x |(f_0 - f_1)(x - y)\, dF^{*n}(y)| \\ &\leq \sup_{0 \leq y \leq x} |f_0(y) - f_1(y)| \int_0^x dF^{*n}(y) \\ &\leq \sup_{0 \leq y \leq x} |f_0(y) - f_1(y)| F^{*n}(x) . \end{aligned}$$

Two cases have to be considered:

(i) For the first case, assume that the fixed $x \geq 0$ is such that $F(x) < 1$. By **Proposition 1.6(i)**, this leads to the following upper bound:

$$|f_0(x) - f_1(x)| \leq \sup_{0 \leq y \leq x} |f_0(y) - f_1(y)| [F(x)]^n .$$

As $n \to \infty$, for a fixed x such that $F(x) < 1$, $[F(x)]^n \to 0$ which implies that $|f_0(x) - f_1(x)| \to 0$.

(ii) For the second case we assume that the fixed x is such that $F(x) = 1$. We still have

$$|f_0(x) - f_1(x)| \leq \sup_{0 \leq y \leq x} |f_0(y) - f_1(y)| F^{*n}(x)$$

for any integer $n \geq 1$, thus we also have

$$|f_0(x) - f_1(x)| \leq \sup_{0\leq y\leq x} |f_0(y) - f_1(y)| F^{*nk}(x)$$

for any integer $k \geq 1$. Since x is such that $F(x) = 1$, x is non-zero (the case $x = 0$ is excluded) by **Proposition 1.6(i)** and (1.8), we have an upper bound as following:

$$\begin{aligned} F^{*nk}(x) &\leq [F^{*n}(x)]^k \\ &\leq [u_n]^k \end{aligned}$$

where $u_n < 1$ as defined in (1.8). Using the preceding upper bound of $F^{*nk}(x)$, we have

$$|f_0(x) - f_1(x)| \leq \sup_{0\leq y\leq x} |f_0(y) - f_1(y)| [u_n]^k$$

We are interested in the limit of $|f_0(x) - f_1(x)|$ as $k \to \infty$. For x and n fixed, $u_n < 1$. Therefore $[u_n]^k \to 0$ as $k \to \infty$ and the conclusion follows. •

The general renewal equation given by Feller still has f_0 as unique solution even if F is a defective distribution bounded by 1. The proof remains the same.

Equation (1.5) gives the renewal function m and the expression is obtained using the p.d.f. of the number of renewals $N(t)$. The same expression can be obtained with a renewal equation and results in (1.10). From (1.5) we have

$$m(t) = F(t) + \int_0^t m(t - x)dF(x) = F(t) + m * F(t),$$

a renewal equation whose solution is given by (1.10) as

$$\begin{aligned} m(t) &= F * [\sum_{k=0}^{\infty} F^{*k}(t)] \\ &= \sum_{k=1}^{\infty} F^{*k}(t) \,. \end{aligned}$$

Notice that we get the same value of $m(t)$ as in (1.5).

1.3.2 The Laplace Stieltjes Transform Method

The renewal function m satisfies the renewal equation (1.5). To get a solution of the renewal equation (1.9), another useful technique, frequently used in physics to solve differential equations, is known as the Laplace transform method.

Definition: Let F be a d.f. defined on $[0, \infty)$, the Laplace Transform of F is the function defined by

$$L_F(\theta) = \int_0^\infty e^{-\theta x}\, dF(x) \text{ where } \theta \geq 0 .$$

The above integral is interpreted in a Lebesgues-Stieltjes sense. It is understood that the interval of integration is closed and may be replaced by $(-\infty, \infty)$. But in our application F vanishes on $(-\infty, 0)$ and θ is a positive real value. Feller (1971) extends the definition of the above Laplace transform for defective d.f.'s (i.e. the function is bounded by 1). With the usual notation for expectations,

$$L_F(\theta) = E[e^{-\theta X}] \text{ where X is a random variable with d.f. F.} \tag{1.11}$$

Proposition 1.7: Let K, G, H be three d.f's such that $K(x) = G * H(x)$, then

$$L_K(\theta) \;=\; L_G(\theta) L_H(\theta) \;. \tag{1.12}$$

Proof: Let Z, Y, X be three random variables with d.f.'s K, G, H respectively. Since K is the convolution of G and H, therefore Z~Y+X where X and Y are independent. Thus using (1.11) $L_K(\theta) = E[e^{-\theta Z}] = E[e^{-\theta Y}]E[e^{-\theta X}] \Rightarrow L_K(\theta) = L_G(\theta)L_H(\theta)$. For others details see Feller (1971). •

We will use (1.12) to get the solution of the general renewal equation (1.9) in which F and g are given monotone right continuous functions vanishing for $t < 0$. We can consider them improper distribution functions and can assume that F is not concentrated at the origin. By (1.12)

$$L_f(\theta) \;=\; L_g(\theta) + L_f(\theta) L_F(\theta)$$

or equivalently

$$L_f(\theta) \;=\; \frac{L_g(\theta)}{1 - L_F(\theta)}\,.$$

Using the inverse of the Laplace Transform, we can get f in certain cases.

In particular, if we let $f = m$ and $g = F$, it leads us to the renewal equation in (1.5). Its corresponding Laplace Transform thus satisfies

$$L_m(\theta) \;=\; \frac{L_F(\theta)}{1 - L_F(\theta)}\,.$$

Proposition 1.8: For any $n \geq 1$ and $\theta > 0$

$$P\{N(t) \geq n\} \leq e^{\theta t}\, L_F^n(\theta) \tag{1.13}$$

where F is the common distribution function of the independent interarrival times τ_i.

Proof (Jeulin 1992): First recall Markov's inequality.

Let $\epsilon, \alpha > 0$ be two reals, and X a random variable. If X accepts an absolute moment of order α, then

$$P(|X| \geq \alpha) \leq \frac{1}{\epsilon^\alpha} E|X|^\alpha\,. \tag{1.14}$$

We use the following relation between events $\{N(t) \geq n\}$=$\{T_n \leq t\}$ = $\{t - T_n \geq 0\}$ to get

$$\begin{aligned} P\{N(t) \geq n\} &= P\{\theta(t - T_n) \geq 0\ \} \text{ for } \theta > 0 \\ &= P\{e^{\theta(t-T_n)} \geq 1\} \\ &\leq E[e^{\theta(t-T_n)}] \text{ by Markov's inequality.} \end{aligned}$$

Since $T_n = \tau_1 + \cdots + \tau_n$, we have

$$P\{N(t) \geq n\} \;\leq\; E[e^{\theta t} \prod_{i=1}^{n} e^{-\theta \tau_i}]$$

$$
\begin{aligned}
P\{N(t) \geq n\} &\leq e^{\theta t} \prod_{i=1}^{n} \underbrace{E[e^{-\theta T_i})]}_{L_F(\theta)} \\
&\leq e^{\theta t} L_F^n(\theta) .
\end{aligned}
$$

As a consequence, the number of outcomes $N(t)$ in [0,t) is a renewal process that accepts moment of any order, i.e. $E[N(t)^r] < \infty$ for any $r \geq 0$, since

$$
\begin{aligned}
E[N(t)^r] &= \sum_{n=1}^{\infty} n^r P\{N(t) = n\} \\
&\leq \sum_{n=1}^{\infty} n^r P\{N(t) \geq n\} \\
&\leq e^{\theta t} \sum_{n=1}^{\infty} n^r L_F^n(\theta) \text{ by } (1.13) .
\end{aligned}
$$

Since F is a distribution function (for any $\theta > 0$, $L_F(\theta) < 1$) and since the radius of convergence of the above series is one, the expression on the right hand side is finite.

Chapter 2

Applications to a Compound Process with Inflation

2.1 The Classical Collective Risk Model

The collective risk model presented here can be found in the text by Bowers et al. (1986). We will review some of the results developed for this model.

The basic concept of the collective risk model is that for an insurance portfolio of policies, two factors have to be taken into account:

(i) the number of claims, N, in a given period and

(ii) the amount of the i^{th} claim, X_i.

Then

$$S_N = X_1 + \cdots + X_N \tag{2.1}$$

represents the aggregate claims generated by the portfolio for the period under study. The number of claims N is a random variable associated with the frequency of claims. In addition, the individual claim amounts $X_1, X_2, ...$ are also random variables that measure the severity of claims.

In order to make this model tractable, two assumptions had to be made. These assumptions are:

(i) $X_1, X_2, \ldots$ are independent and identically distributed random variables (i.i.d.r.v).

(ii) $N, X_1, X_2, \ldots$ are mutually independent.

The following proposition recalls some basic results obtained with this model.

Proposition 2.1:

(i) $E(S_N) = E(N)E(X_1)$.

(ii) $Var(S_N) = E[Var(S_N|N)] + Var[E(S_N|N)] = E(N)Var(X_1) + [E(X_1)]^2 Var(N)$.

(iii) $M_{S_N}(t) = E[e^{tS_N}] = M_N[log M_{X_1}(t)]$.

All the proofs can be found in Bowers et al (1986) but are reproduced here for completness.

Proof: We will prove only (iii), parts (ii) and (i) are easily derived from (iii).

$$\begin{aligned} M_{S_N}(t) &= E[e^{tS_N}] \\ &= E[E(e^{tS_N}|N)] \\ &= E[\{E(e^{tX_1})\}^N] \\ &= E[\{M_{X_1}(t)\}^N] \\ &= M_N[log M_{X_1}(t)]\,. \end{aligned}$$

•

When N is a Poisson r.v. (respectively Negative Binomial) S_N in (2.1) is called a Compound Poisson (respectively Compound Negative Binomial) r.v.. Different distributions can be fitted to the claim severities, e.g. the Log-Normal, the Gamma or the Log-Gamma distributions.

Let $P(x)$ denote the common d.f. of the i.i.d.r.v.'s X_i, then the distribution function of the aggregate claim F_{S_N} is obtained by using the law of total probability as follows:

$$\begin{aligned} F_{S_N}(x) &= P(S_N \leq x) = \sum_{n=0}^{\infty} P(S_N \leq x|N = n)P(N = n) \\ &= \sum_{n=0}^{\infty} P(X_1 + \cdots + X_n \leq x)P(N = n) \end{aligned}$$

$$F_{S_N}(x) = \sum_{n=0}^{\infty} P^{*n}(x)P(N = n)$$

where $P^{*n}(x)$ is the n^{th} convolution of $P(x)$.

For long term insurance contracts or for insurance companies operating in an economic environment with a high inflation rate, the preceding model does not produce adequate premiums. Factors as interest or inflation have then to be taken into account. These factors allow the dependence among claim severities or the dependence between claim severities and claim occurence times; see for example **Proposition 2.2**. In the following section, we study a compound model with inflation.

2.2 A Risk Model under Inflation

The model presented here as well as **Proposition 2.2** and **Proposition 2.3** are taken from Garrido (1986).

Let $\{T_k\}_{k\geq 1}$ denote the claim occurence times recorded by a risk business. The T_k's are r.v.'s over a probability space (Ω, A, P). We also denote by

$$\tau_k = T_{k+1} - T_k \text{ for } k \geq 1 \tag{2.2}$$

the claim inter-occurence times. The τ_k's are positive i.i.d.r.v.'s with common d.f., say F, and common moment generating function (m.g.f.) Φ_F. We assume that the m.g.f. Φ_F exists over a set M included in the real line $\Re$ and that $E(\tau_i) < \infty$ for $i \geq 1$.

In conjunction, let $\{Y_k\}_{k\geq 1}$ denote the corresponding claim severities recorded at times T_k and be r.v.'s defined on (Ω, A, P).

If $N(T)$ denotes the number of claims recorded over the time interval $(0, T]$, $T > 0$, and $\beta > 0$ the constant instantaneous rate of interest earned by the risk business, then

$$Z(T) = \sum_{k=0}^{N(T)} e^{-\beta T_k} Y_k \text{ where } T_0 = 0 \text{ and } Y_0 = 0 \tag{2.3}$$

defines the aggregate discounted value at time 0 of all claims recorded over $(0, T]$.

We also assume that the risk business operates in an inflationary economic environment where the instantaneous rate of inflation α is constant throughout the period $(0, T]$. Inflation is used here in its broad sense and can either mean claim cost escalation, growth in policies face values or exogeneous inflation (or combinations of such). The r.v.

$$X_k = e^{-\alpha T_k} Y_k \text{ for } k \geq 1 \tag{2.4}$$

denotes the deflated amount of the k^{th} claim in currency units of time point 0. Our key simplifying assumptions are:

(i) $\{X_k\}_{k \geq 1}$ are i.i.d.r.v.'s.

(ii) $E(X_1) = \mu < \infty$.

(iii) $\{X_k, \tau_k\}_{k \geq 1}$ are mutually independent.

Together, assumptions (i) and (iii) imply that dependence among claim severities or between claim severities and claim occurence times is through inflation only. Once deflated, the claim amounts X_k are considered independent, as we can confidently assume that time does not affect claim amounts anymore.

Note that here only the X_k's are assumed i.i.d.. As it is proved in the following proposition, the claim severities $\{Y_k\}_{k \geq 1}$ are not necessarily independent nor identically distributed nor does Y_k need to be independent of T_k, $k \geq 1$.

Proposition 2.2: In general, the r.v's $\{Y_k\}_{k \geq 1}$ are not mutually independent and the $\{Y_k, T_k\}_{k \geq 1}$ are not pairwise independent.

Proof: Consider the special case where $N(T)$ is a Poisson process, then T_k has a Gamma distribution with parameter (k, λ), $\lambda > \alpha$. By definition

$$E(Y_k) = E(X_k e^{\alpha T_k}) \text{ for } k \geq 1 .$$

Since $\{X_k, \tau_k\}$ are pairwise independent

$$E(Y_k) = E(X_k) E(e^{\alpha T_k})$$

$$E(Y_k) = \mu(\frac{\lambda}{\lambda - \alpha})^k$$

and

$$\begin{aligned}
E(Y_k Y_{k+1}) &= E(X_k X_{k+1} e^{\alpha(T_k + T_{k+1})}) \\
&= \mu^2 E[e^{2\alpha T_k}] E[e^{\alpha \tau_{k+1}}] \\
&= \mu^2 (\frac{\lambda}{\lambda - 2\alpha})^k (\frac{\lambda}{\lambda - \alpha}) \\
&= \mu^2 \frac{\lambda^{k+1}}{(\lambda - 2\alpha)^k (\lambda - \alpha)} .
\end{aligned}$$

Thus $E(Y_k Y_{k+1}) \neq E(Y_k)E(Y_{k+1})$ which proves the first statement of the proposition.

Now consider the pair $\{Y_k, T_k\}_{k \geq 1}$ and

$$E(T_k Y_k) = E(T_k X_k e^{\alpha T_k}) = \mu E(T_k e^{\alpha T_k}) . \tag{2.5}$$

Define the function $g(\alpha, t) = e^{\alpha t}$ and apply Lebesgue's theorem for derivatives under the expectation. The function $g(\alpha, t)$ is bounded by $e^{\alpha T}$ since $t \in (0, T]$. Moreover for any $t \in (0, T]$, $g(\alpha, t)$ is integrable. We have

$$E(\frac{\partial}{\partial \alpha} e^{\alpha T_k}) = \frac{\partial}{\partial \alpha} E(e^{\alpha T_k})$$

which implies,

$$\begin{aligned}
E(T_k e^{\alpha T_k}) &= \frac{\partial}{\partial \alpha} (\frac{\lambda}{\lambda - \alpha})^k \\
&= \frac{k \lambda^k}{(\lambda - \alpha)^{k+1}}
\end{aligned}$$

Substituting $E(T_k e^{\alpha T_k})$ from above into (2.5) we get $E(T_k Y_k) = \mu \frac{k\lambda^k}{(\lambda-\alpha)^{k+1}}$. Since

$$E(T_k)E(Y_k) = \frac{k}{\lambda} \mu (\frac{\lambda}{\lambda - \alpha})^k \neq E(T_k Y_k)$$

we see that $\{T_k, Y_k\}$ are not pairwise independent. •

2.2.1 Premium Calculations

The object of this section is to approximate the premium (or the expected aggregate claims loss) over a period of length $(0, T]$.

Equation (2.3) gives us the aggregate discounted value at time 0 of all claims recorded over $(0,T]$ as

$$Z(T) = \sum_{k=0}^{N(T)} e^{-\beta T_k} Y_k \text{ where } T_0 = 0 \text{ and } Y_0 = 0.$$

Substituting Y_k, defined in (2.4) as $Y_k = e^{\alpha T_k} X_k$ for $k \geq 1$, into (2.3) gives us

$$Z(T) = \sum_{k=0}^{N(T)} e^{-(\beta-\alpha)T_k} X_k \text{ where } T_0 = 0 \text{ and } X_0 = 0 .$$

We denote by $\delta = \beta - \alpha$ the net instantaneous rate of interest. The aggregate discounted value at time 0 of all claims is now defined as

$$Z(T) = \sum_{k=0}^{N(T)} e^{-\delta T_k} X_k \text{ where } T_0 = 0 \text{ and } X_0 = 0 . \tag{2.6}$$

The net single premium paid at time 0 for a contract of duration $T > 0$ is given by

$$\Pi_0(T) = E[Z(T)] = E[\sum_{k=0}^{N(T)} e^{-\delta T_k} X_k] \text{ where } T_0 = 0 \text{ and } X_0 = 0. \tag{2.7}$$

Since $\{X_k, \tau_k\}_{k\geq 1}$ are mutually independent we may simplify $\Pi_0(T)$ by using conditional expectations. Let $\Im$ be the σ-field generated by the claim occurence times up to $T_{N(T)+1}$,

$$\Im = \sigma\{T_1, \cdots, T_{N(T)+1}\} . \tag{2.8}$$

The σ-field $\sigma(X_k)$ generated by the X_k's is independent of $\Im$. Thus consider the following conditional expectations:

$$\begin{aligned} \Pi_0(T) &= E[E(\sum_{k=0}^{N(T)} e^{-\delta T_k} X_k | \Im)] \\ &= E[\sum_{k=0}^{N(T)} E(e^{-\delta T_k} X_k | \Im)] \\ &= E[\sum_{k=0}^{N(T)} e^{-\delta T_k} E(X_k)] \\ &= \mu E(\sum_{k=0}^{N(T)} e^{-\delta T_k}) . \end{aligned} \tag{2.9}$$

By **Proposition 1.1** (of chapter 1) the random variables T_k and $N(T)$ are dependent. Therefore the usual iterated conditional expectation of **Proposition 2.1(i)** used in classical risk theory can not be applied to obtain an exact value of $\Pi_0(T)$.

Nevertheless, an approximation can be given if we assume that the insurance portfolio is large enough so that adding one more claim to the actual number $N(T)$ has little effect on the premium. This approximation can also be justified for smaller portfolios if T is large enough. The following proposition gives us an approximate value of $\Pi_0(T)$.

Proposition 2.3:

$$K_\delta(T) = E[\sum_{k=0}^{N(T)+1} e^{-\delta T_k}] = \Phi_F(-\delta)\sum_{k=0}^{\infty} H_\delta^{*k}(T) \tag{2.10}$$

where $H_\delta(T) = \int_0^T e^{-\delta s}\, dF(s)$ and Φ_F is the moment generating function of the interoccurence time τ_1.

Proof: In the calculation of $K_\delta(T) = E[\sum_{k=0}^{N(T)+1} e^{-\delta T_k}]$, we condition on the σ-field generated by the occurence time of the first claim T_1:

$$K_\delta(T) \;=\; E[E\{\sum_{k=0}^{N(T)+1} e^{-\delta T_k}|\sigma(T_1)\}].$$

Since

$$E(\sum_{k=0}^{N(T)+1} e^{-\delta T_k}|T_1 = s) \;=\; \begin{cases} e^{-\delta s} & \text{if } s > T \\ e^{-\delta s} + e^{-\delta s}E(\sum_{k=0}^{N(T-s)+1} e^{-\delta T_k}) & \text{if } s \le T\,, \end{cases}$$

$$K_\delta(T) \;=\; \int_T^\infty e^{-\delta s}\, dF(s) + \int_0^T [e^{-\delta s} + e^{-\delta s} K_\delta(T-s)]\, dF(s)$$

and therefore

$$K_\delta(T) \;=\; \int_0^\infty e^{-\delta s}\, dF(s) + \int_0^T e^{-\delta s} K_\delta(T-s)\, dF(s)\,,$$

where from, the following renewal equation results:

$$K_\delta(T) = \Phi_F(-\delta) + K_\delta * H_\delta(T) \tag{2.11}$$

for $H_\delta(T) = \int_0^T e^{-\delta s}\, dF(s)$, $\delta > 0$. Note that H_δ is a defective distribution function (since $H_\delta(\infty) = \Phi_F(-\delta) \le 1$).

The solution of this renewal equation is thus given by (1.10) and can be written as

$$K_\delta(T) \;=\; \Phi_F(-\delta) * \sum_{k=0}^{\infty} H_\delta^{*k}(T)$$

provided that H_δ is bounded. Clearly here

$$\begin{aligned} H_\delta(T) &\le H_\delta(\infty) = \int_0^\infty e^{-\delta s}\, dF(s) = \delta \int_0^\infty e^{-\delta s}\, F(s) ds \\ &\le \delta \int_0^\infty e^{-\delta s}\, ds \\ &\le 1 \,. \end{aligned}$$

Hence H_δ is bounded and thus (2.11) has a unique solution. Since $\Phi_F(-\delta)$ is a constant, we can rewrite $K_\delta(T)$ as

$$K_\delta(T) = \Phi_F(-\delta) \sum_{k=0}^{\infty} H_\delta^{*k}(T)\,. \tag{2.12}$$

•

The preceding theorem establishes the basis of the approximation of $\Pi_0(T)$. Substituting (2.12) for the right hand term in (2.9) we obtain the following approximation:

$$\Pi_0(T) \approx \mu K_\delta(T) = \mu \Phi_F(-\delta) \sum_{k=0}^{\infty} H^{*k}(T)\,. \tag{2.13}$$

The accuracy of this approximation will be studied numerically in **Chapter 3**. We compare in that chapter our approximation in the special compound Poisson case (i.e. when $N(T)$ follows a Poisson process) with the exact value of the premium given by Willmot (1989). We will also analyse Willmot's model and see how the renewal approach allows us to study more general models.

2.2.2 The Variance of the Aggregate Discounted Claims

In this section we develop an approximate value for the variance of $Z(T)$ again by solving a renewal equation. Recall equation (2.6) that gives the aggregate discounted claims at time 0

$$Z(T) \;=\; \sum_{k=0}^{N(T)} e^{-\delta T_k} X_k \;\text{ where } T_0 = 0 \text{ and } X_0 = 0\,.$$

Proposition 2.1(ii) gives the variance of a compound sum $S_N = \sum_{i=0}^{N} X_i$ to be

$$Var(S_N) = E(N)Var(X_1) + [E(X_1)]^2 Var(N)$$

for N independent of the claim severities X_i. Consider now the aggregate discounted claims $Z(T)$ and the σ-field $\Im$ defined in (2.8). We have that

$$\begin{aligned} V_0(T) &= Var[Z(T)] \\ &= E\{Var[Z(T)|\Im]\} + Var\{E[Z(T)|\Im]\} \\ &= V_1(T) + V_2(T). \end{aligned} \tag{2.14}$$

(i) Consider the first term $V_1(T) = E\{Var[Z(T)|\Im]\}$. The σ-field $\Im$ contains all the information up to time $T_{N(T)+1}$. Therefore

$$\begin{aligned} V_1(T) &= E[Var(\sum_{k=0}^{N(T)} e^{-\delta T_k} X_k|\Im)] \\ &= E\{\sum_{k=0}^{N(T)} Var(e^{-\delta T_k} X_k|\Im) + 2 \sum_{0\leq i<j\leq N(T)} Cov(e^{-\delta T_i} X_i, e^{-\delta T_j} X_j|\Im)\}\ , \end{aligned}$$

since under $\Im$, $N(T)$ and $\{T_1, \cdots, T_{N(T)+1}\}$ are known. By independence of claim frequency and severities X_i we have

$$\begin{aligned} V_1(T) &= E\{\sum_{k=0}^{N(T)} e^{-2\delta T_k} Var(X_k)\} \\ &\quad +2E\{\sum_{0\leq i<j\leq N(T)} [E(X_i X_j)e^{-\delta T_i} e^{-\delta T_j} - E(X_i)E(X_j)e^{-\delta T_i} e^{-\delta T_j}]\} \end{aligned}$$

where the last term is equal to zero since the r.v's X_k are mutually independent. We thus get that

$$V_1(T) = E\{Var[Z(T)|\Im]\} = E\{\sum_{k=0}^{N(T)} e^{-2\delta T_k} Var(X_k)\}\ .$$

Let us assume that the deflated claim amounts X_k have a moment of order 2 (denote by σ^2 the variance of X_k) and that $N(T)$ is large enough. When T is large, we can approximate the above variance term with

$$V_1(T) \approx \sigma^2 K_{2\delta}(T) \tag{2.15}$$

where $K_\delta(T)$ is defined in (2.12).

(ii) For the second term, we have

$$\begin{aligned} V_2(T) &= Var\{E(\sum_{k=0}^{N(T)} e^{-\delta T_k} X_k | \Im)\} \\ &= Var(\mu \sum_{k=0}^{N(T)} e^{-\delta T_k}) \\ &= \mu^2 Var(\sum_{k=0}^{N(T)} e^{-\delta T_k}) . \end{aligned}$$

Assume that the number of claims $N(T)$ is large enough to replace it in $Z(T)$ by $N(T)+1$, much as we did to get an approximate value of the premium $\Pi_0(T)$. Then

$$V_2(T) = \mu^2 Var(\sum_{k=0}^{N(T)+1} e^{-\delta T_k}) = \mu^2 V(T) \tag{2.16}$$

where $V(T) = Var(\sum_{k=0}^{N(T)+1} e^{-\delta T_k})$. We apply again **Proposition 2.1(ii)** and condition on the σ-field generated by the first occurence time $\sigma(T_1)$. We have

$$V(T) = \underbrace{E\{Var[\sum_{k=0}^{N(T)+1} e^{-\delta T_k} | \sigma(T_1)]\}}_{V_{21}} + \underbrace{Var\{E[\sum_{k=0}^{N(T)+1} e^{-\delta T_k} | \sigma(T_1)]\}}_{V_{22}} \tag{2.17}$$

where

$$\sum_{k=0}^{N(T)+1} e^{-\delta T_k} = \begin{cases} e^{-\delta T_1} & \text{if } T_1 > T \\ e^{-\delta T_1} + e^{-\delta T_1} \sum_{k=0}^{N(T-T_1)+1} e^{-\delta T_k} & \text{if } T_1 \leq T \end{cases}$$

given T_1.

Since the process $N(T)+1$ is stationary we have for the first term

$$\begin{aligned} V_{21} &= E\{Var[\sum_{k=0}^{N(T)+1} e^{-\delta T_k} | \sigma(T_1)]\} \\ &= \int_0^T Var[e^{-\delta s} \sum_{k=0}^{N(T-s)+1} e^{-\delta T_k}]\, dF(s) + \int_T^\infty Var(e^{-\delta s})\, dF(s) \\ &= \int_0^T Var(\sum_{k=0}^{N(T-s)+1} e^{-\delta T_k})\, e^{-2\delta s} dF(s) \\ &= V * H_{2\delta}(T) \end{aligned}$$

where $V(T)$ is defined in (2.16) and $H_\delta(T) = \int_0^T e^{-\delta s} dF(s)$.

Similary

$$\begin{aligned} V_{22} &= Var\{E[\sum_{k=0}^{N(T)+1} e^{-\delta T_k} | \sigma(T_1)]\} \\ &= Var[e^{-\delta T_1} I_{\{T_1 > T\}} + e^{-\delta T_1}(1 + K_\delta(T - T_1)) I_{\{T_1 \leq T\}}] \end{aligned}$$

where I is a set indicator function and K_δ is defined in (2.10). Adding all terms together, it follows from (2.17) that:

$$V(T) = G(T) + V * H_{2\delta}(T). \tag{2.18}$$

where $G(T) = Var[e^{-\delta T_1} + e^{-\delta T_1} K_\delta(T - T_1) I_{\{T_1 \leq T\}}]$. We see that $V(T)$ follows a renewal equation as (1.9). If we can find a solution to this renewal equation, we will have an approximate value of the variance of the aggregate discounted claims $V_0(T)$ expressed in (2.14). This approximation would then be

$$V_0(T) \approx \sigma^2 K_{2\delta}(T) + \mu^2 V(T) \tag{2.19}$$

where μ and σ^2 are, respectively, the mean and the variance of the deflated claim amounts X_k.

The difficulty is now to find a solution to the renewal equation (2.18). To verify the existence of such a solution all the conditions of (1.9) must be satisfied, i.e.

- the function $G(T)$ must be locally bounded (it is proved in **Appendix A**) and
- $H_{2\delta}$, as a function of T, is bounded. ($H_{2\delta} \leq 1$).

Given the complexity of the function $G(T) = Var[e^{-\delta T_1} + e^{-\delta T_1} K_\delta(T - T_1) I_{\{T_1 \leq T\}}]$, the renewal equation can not be solved with the convolution method (1.10) since the convolution with $\sum_{k=0}^{\infty} H_{2\delta}^{*k}$ is not tractable. It shows the limits of the convolution method which works only for certain distributions F and functions g of the general renewal equation (1.9).

Nevertheless, we may use the Laplace transform technique to obtain a solution to the renewal equation (2.18). In the **Subsection 1.3.2** on the Laplace transform we only needed to express the Laplace transform of the function g and the Laplace transform of F. Thus, we should first find an expression of the above complex function as follows:

$$G(T) = Var[\underbrace{e^{-\delta T_1}}_{R} + \underbrace{e^{-\delta T_1} K_\delta(T-T_1) I_{\{T_1 \le T\}}}_{U}] \;=\; Var(R) + Var(U) + 2Cov(R,U),$$

$$\begin{aligned} Var(R) &= E(R^2) - E^2(R) \\ &= \Phi_F(-2\delta) - [\Phi_F(-\delta)]^2, \end{aligned}$$

$$\begin{aligned} Var(U) &= E(U^2) - E(U)^2 \\ &= \textstyle\int_0^T e^{-2\delta s} K_\delta^2(T-s)\, dF(s) - [\int_0^T e^{-\delta s} K_\delta(T-s)\, dF(s)]^2 \\ &= K_\delta^2 * H_{2\delta}(T) - [K_\delta * H_\delta(T)]^2 \text{ where } H_\delta(T) = \textstyle\int_0^T e^{-\delta s}\, dF(s), \end{aligned}$$

$$\begin{aligned} Cov(R,U) &= E(RU) - E(R)E(U) \\ &= \textstyle\int_0^T e^{-2\delta s} K_\delta(T-s)\, dF(s) - \Phi_F(-\delta)\; K_\delta * H_\delta(T) \\ &= K_\delta * H_{2\delta}(T) - \Phi_F(-\delta)\; K_\delta * H_\delta(T). \end{aligned}$$

From the preceding expressions we have

$$\begin{aligned} G(T) &= \Phi_F(-2\delta) - [\Phi_F(-\delta)]^2 \\ &\quad + K_\delta^2 * H_{2\delta}(T) - [K_\delta * H_\delta(T)]^2 \\ &\quad + 2[K_\delta * H_{2\delta}(T) - \Phi_F(-\delta) K_\delta * H_\delta(T)] \end{aligned}$$

Considering the renewal equation (2.11), we may rewritte it as following:

$$\begin{aligned} G(T) &= \Phi_F(-2\delta) - [\Phi_F(-\delta)]^2 \\ &\quad + [K_\delta^2 + 2K_\delta] * H_{2\delta}(T) \\ &\quad - [K_\delta - \Phi_F(-\delta)]^2 - 2\Phi_F(-\delta) K_\delta * H_\delta(T) \end{aligned} \tag{2.20}$$

where $K_\delta(T)$ and $H_\delta(T)$ are given in (2.10).

The function $V(T)$ follows the renewal equation

$$V(T) \;=\; G(T) + V * H_{2\delta}(T)\,.$$

Taking the Laplace transform on both sides of the equation we get

$$L_V(\theta) = \frac{L_G(\theta)}{1 - L_{H_{2\delta}}(\theta)} . \tag{2.21}$$

The inverse Laplace transform of $L_V(\theta)$ gives the function $V(T)$ needed to approximate the variance of the aggregate claims $V_0(T)$ as in (2.19).

Many mathematical softwares compute Laplace transforms for certain distributions and also invert these transforms. In the next section we study the special compound Poisson case and calculate the approximation of $\Pi_0(T)$ and $V_0(T)$ using these inverse transforms.

2.3 The Compound Poisson Case

2.3.1 The Premium

This subsection illustrates an example of the model with inflation when the renewal process $N(T)$ is a Poisson process. In this case the distribution of the interoccurence times F is exponential with parameter λ. We will apply all the results found in the preceding sections to get an approximation of the premium $E[Z(T)]$.

• The d.f is $F_\lambda(t) = 1 - e^{-\lambda t}$.

• The m.g.f is $\Phi_F(t) = \frac{\lambda}{\lambda - t}$

• The function $H_\delta(T) = \int_0^T e^{-\delta s}\, dF_\lambda(s) = \frac{\lambda}{\lambda+\delta} F_{\lambda+\delta}(T)$ where δ is the net instantaneous rate of interest.

• The k^{th}-convolution of H, $k \geq 1$, is proportional to a Gamma distribution with paramater $(k, \lambda + \delta)$: $H_\delta^{*k}(T) = (\frac{\lambda}{\lambda+\delta})^k F_{\lambda+\delta}^{*k}(T)$.

The premium $\Pi_0(T)$ is given by (2.13) as

$$\Pi_0(T) \approx \mu \Phi_F(-\delta) \sum_{k=0}^{\infty} H_\delta^{*k}(T) .$$

In the Poisson case, we get the expression

$$\Pi_0(T) \approx \mu(\frac{\lambda}{\lambda+\delta}) \sum_{k=0}^{\infty} (\frac{\lambda}{\lambda+\delta})^k F_{\lambda+\delta}^{*k}(T) .$$

Now let us rewrite the above expression as

$$\Pi_0(T) \approx \frac{\mu\lambda}{\delta}\sum_{k=0}^{\infty}(\frac{\delta}{\lambda+\delta})(\frac{\lambda}{\lambda+\delta})^k F_{\lambda+\delta}^{*k}(T)$$

Consider the discrete r.v N with the following p.d.f.: $p_k = (\frac{\delta}{\lambda+\delta})(\frac{\lambda}{\lambda+\delta})^k$. The premium $\Pi_0(T)$ can be rewritten in the form

$$\Pi_0(T) \approx \frac{\mu\lambda}{\delta}\sum_{k=0}^{\infty} p_k F_{\lambda+\delta}^{*k}(T) \tag{2.22}$$

where the right hand side of the preceding expression is proportional to the distribution function of a compound geometric sum, with $p_k = P(N = k)$ as the distribution of the number of terms in the sum, and an exponential distribution with paremeter $(\lambda + \delta)$ as the distribution of each element in the sum. Therefore we may rewritte (2.22) as

$$\Pi_0(T) = \frac{\mu\lambda}{\delta}F_{S_N}(T) = \frac{\mu\lambda}{\delta}P(S_N \leq T)$$

with

$$F_{S_N}(T) = \sum_{k=0}^{\infty} P(N = k)F_{\lambda+\delta}^{*k}(T)\,. \tag{2.23}$$

Using an argument in Bowers et al (1986), the derivation of $F_{S_N}(T) = P(S_N \leq T)$ for a compound geometric sum can be obtained by using **Proposition 2.1(iii)** ie $M_{S_N}(t) = M_N(log M_X(t))$. In this case N is geometric with parameter $p = \frac{\delta}{\lambda+\delta}$ which leads us to the following expression:

$$M_{S_N}(t) = p + q\frac{pM_X(t)}{1 - qM_X(t)}$$

where $q = 1 - p$ and $M_X(t) = \frac{\lambda+\delta}{\lambda+\delta-t} = \frac{1}{1-\frac{t}{\lambda+\delta}}$. We get for $M_{S_N}(t)$

$$M_{S_N}(t) = p + q\frac{\delta}{\delta - t}\,. \tag{2.24}$$

The distribution function of this compound sum S_N is interpreted as a weighted average of the sum of two distributions: a dirac distribution at 0 and an exponential distribution with paramater δ (the net rate of inflation). Therefore

$$F_{S_N}(x) = p.(1) + q.(1 - e^{-\delta x})1 - qe^{-\delta x} \text{ for } x \geq 0\,.$$

Since we had expressed the premium $\Pi_0(T)$ in (2.23) as the distribution function of a compound/geometric sum at T, we obtained the following result for the premium:

$$\Pi_0(T) \approx \frac{\mu\lambda}{\delta}[1 - \frac{\lambda}{\lambda+\delta}e^{-\delta T}] \text{ for } \delta \neq 0, T > 0 \text{ and } \lambda + \delta > 0. \tag{2.25}$$

Remark: When $\delta = 0$ (the zero net interest) this derivation of the approximation does not hold. We go back to the basic model defined in (2.6) and the results for the mean and the variance of the aggregate discounted claims are straight applications of **Proposition 2.1.**

2.3.2 The Variance of the Aggregate Discounted Claims

We will derive in this section the variance of a compound Poisson process with inflation. (2.19) gives an expression of the variance for the general model. We need to find the function $K_\delta(T)$. Since $\Pi_0(T) = \mu\, K_\delta(T)$ (2.13) we may derive an expression of $K_\delta(T)$ from (2.25) as following:

$$K_\delta(T) = \frac{\lambda}{\delta}[1 - \frac{\lambda}{\lambda+\delta}e^{-\delta T}] \text{ for } \delta \neq 0, T > 0 \text{ and } \lambda + \delta > 0. \tag{2.26}$$

The aggregate claims variance $V_0(T)$ is given by (2.19) as:

$$V_0(T) = \sigma^2 K_{2\delta}(T) + \mu^2 V(T)$$

where $V(T)$ is the solution of the renewal equation

$$V(T) = G(T) + V * H_{2\delta}(T)\,.$$

$G(T)$ is a defective mixed distribution function that has a weight at 0. The Laplace transform of the continous part of $G(T)$ is computed by using the software MAPLE. Refering to result (2.21), the Laplace transform of $V(T)$ is derived in the Poisson case as follows:

$$L_V(\theta) = \frac{\lambda}{2\delta}\{p + (1-p)\frac{2\delta}{2\delta+\theta}\} \tag{2.27}$$

where

$$p = \frac{2\delta^3}{(\lambda + 2\delta)(\lambda + \delta)^2}\,.$$

Equation (2.27) represents the Laplace transform of a mixed distribution: a Dirac distribution at 0 and an exponential distribution with parameter 2δ. Therefore, by linearity, the inverse Laplace transform of (2.27) is given by

$$V(T) = \frac{\lambda}{2\delta}\{1 - qe^{-2\delta T}\} \tag{2.28}$$

where $q = 1 - p$ and p is defined above.

By (2.19) we finally obtain an approximate value of the variance of the aggregate discounted claims at time 0 in the Poisson case as follows:

$$V_0(T) = \sigma^2\{\frac{\lambda}{2\delta}(1 - \frac{\lambda}{\lambda + 2\delta}e^{-2\delta T})\} + \mu^2\{\frac{\lambda}{2\delta}\{1 - qe^{-2\delta T}\} \tag{2.29}$$

where μ and σ^2 are respectively the mean and the variance of the claim severities.

Chapter 3

Limit Properties of the Model

In this chapter, some results on the renewal theorem will be reviewed in order to study the limit behaviour of the premium $\Pi_0(T)$ and the variance $V_0(T)$ as the contract duration T tends to infinity. Numerical values of the premium in the compound Poisson case are also compared to those of Willmot (1989). The following definition can be found in Karlin and Taylor (1975).

Definition: Let F be a d.f. concentrated on $[0,\infty)$ such that $F(0)=0$. A point α of F is called a point of increase if for every positive ϵ, $F(\alpha+\epsilon)-F(\alpha-\epsilon)>0$.

A d.f. F is said to be arithmetic if there exists a positive number λ such that the points of increase of F lie exclusively on the points 0, $\pm\lambda$, $\pm 2\lambda \cdots$ The largest such λ is called the span of F.

The Basic Renewal Theorem: Let F be the d.f. of a positive r.v. with mean μ. Suppose that g is Riemann integrable and that f is the solution of the renewal equation

$$f(t)=g(t)+\int_0^t f(t-x)\,dF(x)\,.$$

(i) If F is not arithmetic, then

$$\lim_{t\to\infty} f(t) = \begin{cases} \frac{1}{\mu}\int_0^\infty g(x)dx & \text{if } \mu < \infty \\ 0 & \text{if } \mu = \infty \end{cases}$$

(ii) If F is arithmetic with span λ, then for all $c > 0$

$$\lim_{n\to\infty} f(t) = \begin{cases} \frac{\lambda}{\mu}\sum_{k=0}^{\infty} g(c+k\lambda) & \text{if } \mu < \infty \\ 0 & \text{if } \mu = \infty \end{cases}$$

Hence the basic renewal theorem is applied to a renewal equation with any positive d.f. F. But in our model, the d.f. F is assumed to be positive and continuous which implies that F is not arithmetic. Moreover the renewal equations of the premium and the variance are defined for defective d.f.'s H_δ and $H_{2\delta}$. Thus to apply the basic renewal theorem we have to modify the structure of those renewal equations. Instead of doing this we will obtain the premium and the variance limit by an alternate direct method. Moreover, the basic renewal theorem can be useful if a non-continuous distribution function is chosen for the claim interoccurence times. But in this chapter only the exponential case is studied for F.

Equation (2.13) gives the premium as being $\Pi_0(T) = \mu\, K_\delta(T)$ where $K_\delta(T)$ follows the renewal equation

$$K_\delta(T) = \Phi_F(-\delta) + K_\delta * H_\delta(T)\ .$$

Taking the limit on both sides of the equality as $T \to \infty$ and using **Proposition 1.6(i)** we obtain

$$K_\delta(\infty) = \frac{\Phi_F(-\delta)}{1 - H_\delta(\infty)}\ .$$

Since $H_\delta(\infty) = \Phi_F(-\delta)$ we have the following value for the premium at infinity

$$\Pi_0(\infty) = \frac{\mu\Phi_F(-\delta)}{1 - \Phi_F(-\delta)}\ . \tag{3.1}$$

Similarly, (2.19) gives the aggregate variance $V_0(T)$ at time 0 as

$$V_0(T) = \sigma^2 K_{2\delta}(T) + \mu^2 V(T)$$

where $V(T)$ is the solution of the renewal equation

$$V(T) = G(T) + V * H_{2\delta}(T).$$

thus taking the limit on both sides of the equality we have

$$V(\infty) = \frac{G(\infty)}{1 - \Phi_F(-2\delta)}.$$

Recall that the value of $G(T)$ is given by

$$\begin{aligned} G(T) &= \Phi_F(-2\delta) - [\Phi_F(-\delta)]^2 \\ &\quad + K_\delta^2 * H_{2\delta}(T) - [K_\delta * H_\delta(T)]^2 \\ &\quad + 2[K_\delta * H_{2\delta}(T) - \Phi_F(-\delta) K_\delta * H_\delta(T)] \end{aligned}$$

and take the limit as $T \to \infty$. Using **Proposition 1.6(i)**,

$$V(\infty) = \frac{A}{B}$$

where

$$\begin{aligned} A &= \Phi_F(-2\delta) - \Phi_F^2(-\delta) + \{\frac{\Phi_F(-\delta)}{1 - \Phi_F(-\delta)}\}^2 \Phi_F(-2\delta) \\ &\quad - \frac{\Phi_F^4(-\delta)}{[1 - \Phi_F(-\delta)]^2} + \frac{2\Phi_F(-\delta)\Phi_F(-2\delta)}{1 - \Phi_F(-\delta)} - \frac{-2\Phi_F^3(-\delta)}{1 - \Phi_F(-\delta)} \end{aligned}$$

and

$$B = 1 - \Phi_F(-2\delta)$$

Φ_F being the moment generating function of the claim interoccurence times τ_i. By (2.19) we obtain the following aggregate claims variance as T tends to infinity:

$$V_0(\infty) = \sigma^2 \frac{\Phi_F(-2\delta)}{1 - \Phi_F(-2\delta)} + \mu^2 \frac{A}{B} \tag{3.2}$$

where $\frac{A}{B}$ is expressed above, μ and σ^2 are, respectively, the mean and the variance of the deflated claim amount X_k.

3.1 The Compound Poisson Process

3.1.1 The Premium and the Variance Limit

We give the limit of the premium and the variance of the aggregate claim amounts when the distribution function F is exponential with parameter λ (i-e $N(T)$ follows a Poisson process). These results could be applicable for portfolios with long term contracts.

Expression (3.1) gives the limit premium. In the Poisson case this reduces to

$$\Pi_0(\infty) = \frac{\mu\lambda}{\delta} . \tag{3.3}$$

Expression (3.2) gives the limit variance. In the Poisson case we have a simpler expression

$$V_0(\infty) = (\mu^2 + \sigma^2)\frac{\lambda}{2\delta} . \tag{3.4}$$

Both (3.3) and (3.4) reproduce the exact results obtained by Willmot (1989) at $T = \infty$ in the special compound Poisson case.

3.1.2 Numerical Tables

We give here some numerical values of the approximated premium in the compound Poisson case and compare them to the exact premiums given by Willmot(1989).

Willmot studied the distribution of total claims occuring in a fixed period of time on a portfolio of business under inflationary conditions. He assumed that, conditional on $N(T) = n \geq 1$, the times of the n claims are distributed as the order statistics from a sample of size n from the "parent" d.f. (Willmot 1989). In the particular compound Poisson case he obtains the following exact expression for the premium:

$$\Pi_{W_0}(T) = \frac{\mu\lambda}{\delta}(1 - e^{\delta T}) .$$

In our model we are assuming that $N(T)$ follows an ordinary renewal process which allows the claim interoccurence times to have any positive distribution. If we consider the special compound Poisson case, our approximate premium formula in (2.25) becomes

$$\Pi_0(T) = \frac{\mu\lambda}{\delta}(1 - \frac{\lambda}{\lambda+\delta}e^{\delta T}) .$$

The following tables give a numerical comparison of both formulas.

Table 1: Values of the Approximate and Exact Premiums for δ, λ, μ constant, T variable

$\delta = 5\%$, $\mu = 1$ **currency unit,** $\lambda = 100$ **claims per period**

T	5	10	25	50	100	1000
$\Pi_0(T)$	443.18	787.54	1427.28	1835.91	1986.53	2000.00
$\Pi_{W_0}(T)$	442.40	786.94	1426.10	1835.83	1986.52	2000.00

Table 2: Values of the Approximate and the Exact Premiums for T, μ, λ constant, δ variable

$T = 1$ **time unit,** $\mu = 1$ **currency unit,** $\lambda = 100$ **claims per period**

δ	5%	10%	25%	50%	100%	1000%
$\Pi_0(T)$	98.49	96.07	89.26	79.30	63.58	10.00
$\Pi_{W_0}(T)$	97.54	95.16	88.48	78.69	63.21	10.00

Table 3: Values of the Approximate and the Exact Premiums for δ, μ, T constant, λ variable

$\delta = 5\%$, $\mu = 1$ **currency unit,** $T = 1$ **time unit**

λ	100	500	1000	2500	5000	10000
$\Pi_0(T)$	98.49	488.65	976.36	2439.48	4878.00	9755.06
$\Pi_{W_0}(T)$	97.54	487.70	975.41	2438.52	4877.05	9754.11

Similarly for the variance formulas, the exact expression from Willmot (1989) in the compound Poisson case is

$$V_{W_0}(T) = \frac{\lambda}{2\delta}(\mu^2 + \sigma^2)\{1 - e^{-2\delta T}\} .$$

Our approximation formula in (2.29) for the special compound Poisson case is

$$V_0(T) = \sigma^2\{\frac{\lambda}{2\delta}(1 - \frac{\lambda}{\lambda + 2\delta}e^{-2\delta T})\} + \mu^2\{\frac{\lambda}{2\delta}\{1 - qe^{-2\delta T}\} .$$

The following tables give a numerical comparison of both formulas.

Table 4: Values of the Approximate and the Exact Variances for δ, λ, μ constant, T variable

$\delta = 5\%$, $\mu = 1$ currency unit, $\lambda = 100$ claims per period, $\sigma^2 = 25$

T	5	10	25	50	100	1000
$V_0(T)$	10245.35	16444.32	23867.84	25824.98	25998.82	26000.00
$V_{W_0}(T)$	10230.20	16435.13	23865.79	25824.81	25998.82	26000.00

Table 5: Values of the Approximate and the Exact Variances for T, μ, λ constant, δ variable

$T = 1$ time unit, $\mu = 1$ currency unit, $\lambda = 100$ claims per period, $\sigma^2 = 25$

δ	5%	10%	25%	50%	100%	1000%
$V_0(T)$	2496.82	2376.93	2061.13	1652.62	1127.39	130.00
$V_{W_0}(T)$	2474.23	2356.50	2046.04	643.51	1124.06	130.00

Table 6: Values of the Approximate and the Exact Variances for δ, μ, T constant, λ variable

$\delta = 5\%$, $\mu = 1$ currency unit, $T = 1$ time unit, $\sigma^2 = 25$

λ	100	500	1000	2500	5000	10000
$V_0(T)$	2496.82	12393.75	24764.89	61878.29	123733.97	247445.33
$V_{W_0}(T)$	2474.23	12371.13	24742.27	61855.67	123711.35	247422.71

In general, a good approximation of the premium and the variance is obtained as illustrated in the preceding tables. Even at a fixed net rate of interest of 5% and with a low average number of claims λ, the difference between the approximate premium and the exact one does not exceed 1 currency unit. The approximation improves as λ increases. Common insurance portfolios record averages of at least 1000 claims per year. Thus, the approximate values obtained for the premium would even be closer to the exact values in most applications. The same remark holds for the approximate variance.

CONCLUSION

The risk model with inflation discussed in this thesis provides a more realistic study of insurance portfolios than the classical compound Poisson model. The choice of renewal processes to model the number of claims allows the claim interoccurence times to have any positive distribution.

The premium and variance approximate formulas are illustrated here only in the Poisson case because of the exact results available for comparison purposes in that case.

The results obtained in **Chapter 2** can be extended by considering the claim interoccurence times to have distributions such as the Gamma or the Inverse Gaussian.

One could also extend this thesis by studying the limit distribution of the aggregate discounted claims $Z(T)$ when the average number of claims is very large.

Appendix A

We prove in this appendix that the function $G(T)$ defined in equation (2.18) of **Section 2.2.2** is bounded. We need this condition in order for the renewal equation (2.18) to have a solution.

$$\begin{aligned} G(T) &= \Phi_F(-2\delta) - [\Phi_F(-\delta)]^2 \\ &\quad + K_\delta^2 * H_{2\delta}(T) - [K_\delta * H_\delta(T)]^2 \\ &\quad + 2[K_\delta * H_{2\delta}(T)] - 2\Phi_F(-\delta)[K_\delta * H_\delta(T)] \end{aligned}$$

where $K_\delta(T) = \frac{\lambda}{\delta}(1 - \frac{\lambda}{\lambda+\delta}e^{-\delta T})$. Thus, if $\delta \neq 0$ (the case $\delta = 0$ is excluded) then

$$K_\delta(T) \leq \frac{\lambda}{\delta}$$

for any $T > 0$, $\lambda > 0$. Since $H_\delta(T) \leq 1$ and using **Proposition 1.6 (i)**, the following upper bound for $G(T)$ results:

$$\begin{aligned} G(T) &\leq 1 + 1 + K_\delta^2(T)H_{2\delta}(T) + [K_\delta(T)H_\delta(T)]^2 \\ &\quad + 2[K_\delta(T)H_{2\delta}(T)] + 2[K_\delta(T)H_\delta(T)] \\ &\leq 2 + 4\frac{\lambda}{\delta} + 2\frac{\lambda^2}{\delta^2} \end{aligned}$$

for any $T > 0$, $\lambda > 0$ and $\delta \neq 0$. ∙

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