# CHAOS IN DETERMINISTIC SYSTEMS

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# ABSTRACT

## CHAOS IN DETERMINISTIC SYSTEMS

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The existence of chaos in deterministic systems is introduced by means of physical examples. In this thesis we present sufficient theoretical conditions which guarantee the existence of chaotic functions in one dimension, and in n-dimensions. We also show that chaotic functions are dense and open in the space of continuous functions. The theorems are illustrated by detailed examples.

# ACKNOWLEDGMENTS


Let the honor of your student be as dear to you as your own, the honor of your colleague as the reverence for your teacher, and the reverence for your teacher as the fear of Heaven.

Quotation of
Rabbi Elazar Ben Shammua
in PIRKE AVOT

It is with great pleasure that I convey my gratitude to several people who helped make this thesis possible.

With respect and love I dedicate this thesis to my parents, Rabbi and Mrs. Shmuel Stein, my uncles, Berel and Shi Schamban, my wife Freyda, and my children, Esther, Penina, Yochanon and Mendy. They, each in their own way, have been helpful, supportive, encouraging and an invaluable asset in my endeavours.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Motivation

In 1963, 1964 a meteorologist by the name of E.M. Lorenz showed in [1] and [2] that since one cannot have accurate and complete initial weather observations all over the world at a precise given time, it is impossible to make "long range" weather forecasts. He started by finding numerical solutions to a system of 3 ordinary differential equations designed to represent a convective process, and showed that all solutions, specifically the periodic solutions, were unstable. He also showed that in certain areas, if the initial conditions were just slightly different, the long term solution would be totally different.

In spite of the fact that historically people usually used differential equations to model dynamical systems, there are situations where discrete-time models are more useful. For example, in population growth problems and predator-prey problems, changes occur in discrete jumps, where each new "population" is dependent on the old one, i.e., $X_{n+1} = f(X_n)$, an iterative process.

## 1.2 Chaos in Convective Fluid Flow

In 1916 Rayleigh studied the flow occurring in a layer of fluid of uniform depth $H$ when the temperature difference between the upper and lower surfaces is a constant $\Delta T$. In the case where all motion is parallel to the $X-Z$ plane, and no variations in the direction of the y-axis occur, Saltzman in 1962 (see [1]) simplified the governing equations to

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}$$

$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta t}{H}\frac{\partial\psi}{\partial x} + k\nabla^2\theta ,$$

where $\psi$ is a stream function for 2 dimensional motion,
$\theta$ is the departure of the temperature from that occurring in the state of no convection,
$g$ is the acceleration due to gravity,
$\alpha$ is the coefficient of thermal expansion,
$\nu$ is the kinematic viscosity,
$k$ is the thermal conductivity.

If the upper and lower boundaries are taken to be free, and therefore $\psi$ and $\nabla^2\psi$ vanish there, Rayleigh found that fields of motion of the form

$$\psi = \psi_0 \sin(\frac{\pi a}{H}x)\sin(\frac{\pi}{H}z)$$

$$\theta = \theta_0 \cos(\frac{\pi a}{H}x)\sin(\frac{\pi}{H}z)$$

where a is a parameter, would develop if the quantity $R_a = \frac{g\alpha H^3\Delta T}{\nu k}$, exceeded $R_c = \frac{\pi^4}{a^2}(1 + a^2)^3$. The minimum value of $R_c$, $\frac{27\pi^4}{4}$, occurs when $a = \frac{\sqrt{2}}{2}$.

By expanding $\psi$ and $\theta$ in a double Fourier series and making substitutions,

$$\frac{a\psi}{k(1 + a^2)} = X\sqrt{2}\sin\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right)$$

and

$$\frac{\pi R_a \theta}{R_c \Delta T} = Y\sqrt{2}\cos\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right) - Z\sin\left(\frac{2\pi}{H}z\right),$$

where $X$, $Y$, and $Z$ are functions of time alone, and making suitable approximations, Lorenz obtained

$$\begin{aligned} \dot{X} &= \quad\quad - \sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY \quad\quad\quad - bZ \end{aligned} \tag{1.1}$$

where $\cdot$ represents the derivative with respect to

$\tau = \dfrac{\pi^2(1 + a^2)kt}{H^2}$, and $\sigma = \dfrac{\nu}{k}$, $r = \dfrac{R_a}{R_c}$, $b = \dfrac{4}{1 + a^2}$

Choosing specific values for the above constants, $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$, Lorenz found numerical solutions on the computer for the above convection equations (1.1). The resulting trajectory traces a path which oscillates around either of two equilibrium points, alternating seemingly at random between the two. Because of the very complicated nature of this trajectory, Lorenz attempted to reduce the dimension of the problem by identifying some single feature by which to characterize its behaviour. He let $M_n = \max z(t)$ on the n-th circuit of the trajectory around either of the two equilibrium points. He lists the results for the first 6000 iterations, and also gives the numerical values for the n-th iteration where Z has a relative maximum. Plotting $M_n$ against $M_n$, he obtained a functional relationship of the form shown in Figure 1.1.

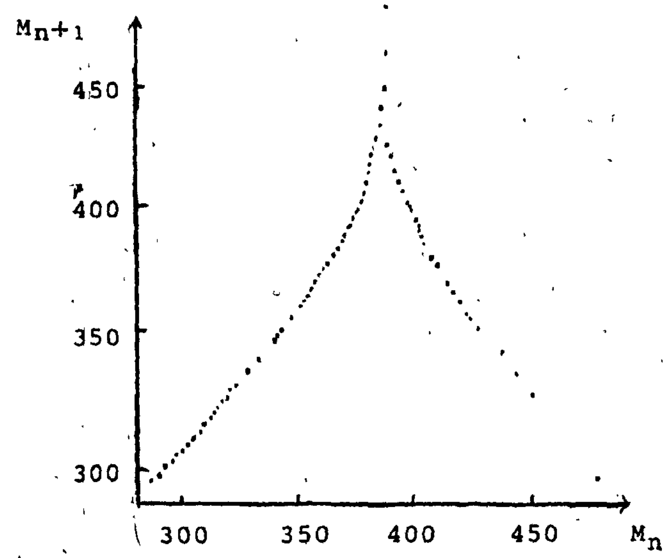


Figure 1.1

With the limits of round-off error on Z, there is a precise two to one relation between $M_n$ and $M_{n+1}$. The initial maximum $M_1 = 483$ is shown as if it had followed a maximum of $M_0 = 385$ since maxima near 385 are preceded by close approaches to the origin, and then by exceptionally large maxima. The sequence $\{M_k\}_{k=0}^{\infty}$ seems to exhibit aperiodic (chaotic) behaviour.

1.3 <u>Analysis of the Convection Equations</u>

In order to study the ramifications and implications of Figure 1.1, Lorenz considered the following idealized 2 - 1 correspondence between successive terms of the sequence $m_0, m_1, m_2, \ldots$ of numbers between 0 and 1, defined as follows:

$$m_{n+1} = 2m_n \quad \text{if} \quad 0 < m_n < \frac{1}{2}$$

$$m_{n+1} \text{ is undefined} \quad \text{if} \quad m_n = \frac{1}{2}$$

$$m_{n+1} = 2 - 2m_n \quad \text{if} \quad \frac{1}{2} < m_n < 1$$

The graph of $m_{n+1}$ vs $m_n$ is shown in Figure 1.2.

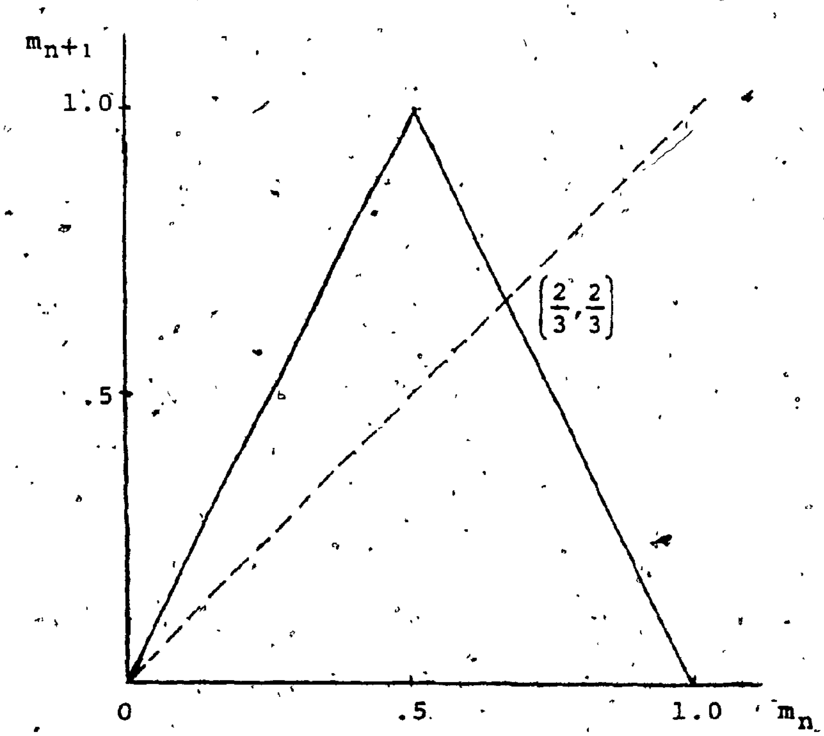


Figure 1.2

It follows from repeated application of the definition that $m_n = e_n \pm 2^n \cdot m_0$, where $e_n$ is an even integer.

To study the above we will consider three possible sequences.

<u>Case 1</u>: Let $m_0 = \frac{u}{2^P}$ where $u$ is an odd positive integer and $P$ any positive integer sufficiently large so that $\frac{u}{2^P} < 1$. This will lead to a sequence of <u>proper</u> fractions whose denominators are $2^P, 2^{P-1}, 2^{P-2}, \ldots\ldots, 2$. Therefore $m_{P-1} = \frac{1}{2}$, the only proper fraction with denominator 2, and the sequence terminates. These sequences form a countable set and correspond to the trajectories which end in a state of no convection.

Case 2: Let $m_0 = \frac{u}{v \cdot 2^P}$ where $u$ and $v$ are relatively prime positive integers, and $P$ is as in Case 1. If $k$ is any positive integer, then $m_{P+1+k} = \frac{u_k}{v}$, where $u_k$ and $v$ are relatively prime and $u_k$ is even. Since for any $v$ the number of proper fractions $\frac{u_k}{v}$ is finite, repetitions must occur, and the sequence is periodic. Again, these sequences form a countable set, and correspond to periodic trajectories.

From the above graph, the only point common with the line $y = x$ is $\left(\frac{2}{3}, \frac{2}{3}\right)$; it is the only point of period one in the domain $\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$.

Similarly by solving the various combinations of

$x = f^2(x)$ and $x = f^3(x)$ we obtain a single sequence of period 2 and two sequences of period 3 ,

i.e. $\left\{\frac{2}{5},\frac{4}{5},\frac{2}{5},\frac{4}{5},\ldots\right\}$ and

$$\left\{\frac{2}{7},\frac{4}{7},\frac{6}{7},\frac{2}{7},\frac{4}{7},\frac{6}{7},\ldots\right\}, \quad \left\{\frac{2}{9},\frac{4}{9},\frac{8}{9},\frac{2}{9},\frac{4}{9},\frac{8}{9},\ldots\right\}$$

Case 3: Let m be irrational. Since $m_n = e_n \pm 2^n \cdot m_0$ where $e_n$ is an even integer, therefore $m_{n+k} \neq m_n$ for all integers k and no repetitions can occur. These sequences are aperiodic and form an uncountable set.

## 1.4 Instability of the Trajectories

Consider two sequences of the above type $\{m_n\}_0^\infty$ and $\{m'_n\}_0^\infty$ where $m'_0 = m_0 + \varepsilon$ . Now $m_n = e_n \pm 2^n \cdot m_0$ and therefore for this n if $\varepsilon$ is sufficiently small $m'_n = e_n \pm 2^n(m_0 + \varepsilon)$ by the piecewise continuity of the transformation. Therefore $m'_n = m_n \pm 2^n \cdot \varepsilon$. We see that all sequences are unstable with respect to small modifications of the initial conditions. In particular, periodic sequences are unstable, and no other sequence can approach them asymptotically.

When the above results concerning instability are applied to the atmosphere, which seems to be non-periodic, they show that long range weather prediction is impossible unless present conditions are known exactly which is obviously not the case.

## 1.5 Applications to Population Growth

There are other areas in nature where the phenomenon of chaos has been observed, particularly in the field of population dynamics. A model which governs the population of a species, where changes can occur only at discrete time intervals, and successive generations do not overlap, may be represented by a first-order n-dimensional difference equation $X_{k+1} = F(X_k)$ $k = 0,1,2,\ldots$ Even when $n = 1$, very complicated dynamics can occur.

Two commonly used models for population growth are $N_{k+1} = N_k\left(1 + r\left(1 - \frac{N_k}{c}\right)\right)$ and $N_{k+1} = N_k e^{r\left(1-\frac{N_k}{c}\right)}$, both of which have been extensively discussed in the mathematical literature (see [3] to [13]). May [3] has shown that as $r$ increases, stable fixed points bifurcate into stable two point cycles. The two-cycle eventually becomes unstable and bifurcates into a stable 4-cycle. This process continues until for some critical value of $r$ is reached, and then cycles of all periods appear and trajectories which are aperiodic.

## 1.6 Other Applications

In [9] May lists many other areas in which the above phenomena occur. For example, in predator-prey problems; genetics; transmission of infectious diseases; economics; and social sciences. In [14] Li and Yorke describe an engineering problem for which the difference equation $x_{n+1} = f(x_n)$ has been used to design the distribution of the points of impact on a spinning bit used for oil drilling in order to avoid uneven wear on its surface.

As May states that even though most of the above process are described in completely deterministic models, they may still exhibit random behaviour.

# CHAPTER 2

# PERIOD THREE IMPLIES CHAOS

## 2.1 Introduction

In the introduction to their paper [14], Li and Yorke give several examples where processes which change in time can be described mathematically by means of difference equations. For example, the insect population which has discrete generations, the size of the $(n + 1)$th generation will be a function of the nth generation:

$$X_{n+1} = F(X_n) \tag{2.1}$$

An important example of the above is

$$X_{n+1} = rX_n\left(1 - \frac{X_n}{k}\right) \tag{2.2}$$

However, even for the above highly simplified model, the dynamics are very complicated. Dividing both sides of (2.2) by $k$, and redefining $X_n = \frac{X_n}{k}$, equation (2.2) simplifies to

$$X_{n+1} = rX_n(1 - X_n) \tag{2.3}$$

For $r = 3.9$ and $X = .500$, we obtain the following:

| i | $x_i$ | i | $x_i$ | i | $x_i$ |
|---|---|---|---|---|---|
| 1 | .500 | 8 | .142 | 15 | .880 |
| 2 | .975 | 9 | .475 | 16 | .412 |
| 3 | .095 | 10 | .973 | 17 | .945 |
| 4 | .335 | 11 | .104 | 18 | .203 |
| 5 | .869 | 12 | .363 | 19 | .632 |
| 6 | .443 | 13 | .902 | 20 | .907 |
| 7 | .962 | 14 | .344 | 21 | .328 |

The above results are shown graphically in Figure (2.1).

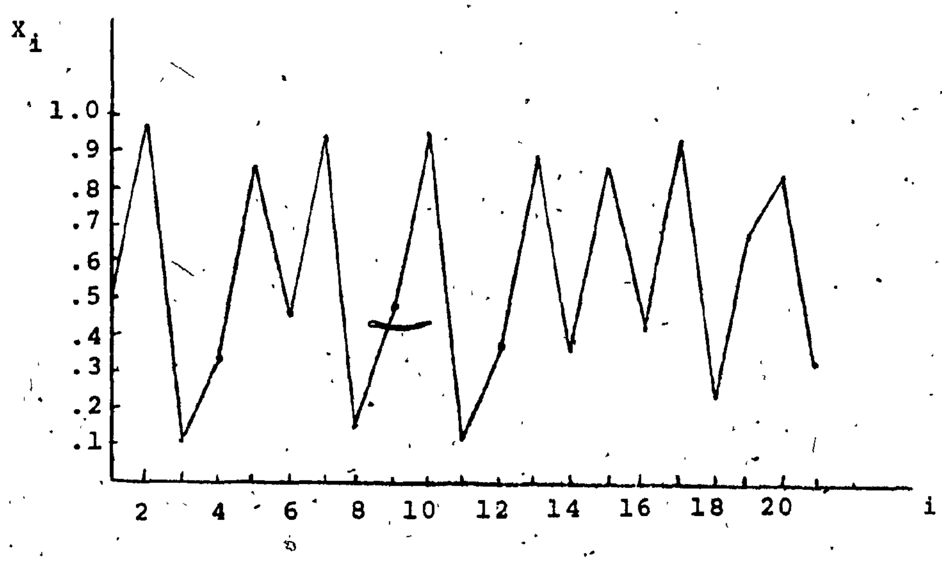


Figure 2.1

The above 21 values do not seem to repeat. Also, in spite of the fact that $X_2 = .975$ and $X_{10} = .973$ are so close together, the behaviour is not periodic with period 8 as $X_{18} = .203$.

An orbit such as the one above, where the sequence $X, F(X); F^2(X), F^3(X), \ldots,$ is not periodic, and does not approach any periodic orbit as $n \to \infty$ in $\{F^n(X)\}$, is called aperiodic, and leads one to suspect chaotic behaviour.

Li and Yorke [14] found a condition which assures the existence of chaotic behaviour. Their main theorem states that if a population of size $X$ grows for two successive generations and then falls to a size $X$ or less, chaotic behaviour will result.

## 2.2 Definitions and Lemmas

To prove the above mentioned theorem, we shall need several definitions and lemmas, and to give a more precise definition of the term "chaos".

Definition 2.1 Let $J$ be any closed interval of the real line. Let $F : J \to J$ be any function. For $x \in J$, let $F^0(x)$ denote $x$ and $F^{n+1}(x)$ denote $F(F^n(x))$, $n = 0,1,2\ldots$

Definition 2.2 We say $p$ is a periodic point with period $n$ if $p \in J$, $p = F^n(p)$, and $p \neq F^k(p)$ for $1 \leq k < n$.

Definition 2.3 We say $p$ is periodic, or is a periodic point if $p$ has period $n$ for some $n \geq 1$.

Definition 2.4 We say $q$ is eventually periodic if for some integer $m$, $P = F^m(q)$ is periodic.

Lemma 2.0 Let $G : I \to R$, where $I$ is an interval, be a continuous function not equal to a constant everywhere on $I$. Then for any compact interval $I_1 \subset G(I)$ there is a compact interval $Q \subset I$ such that $G(Q) = I$.

<u>Proof</u> Let $I = [a,b]$. Let $I_1 = [G(p),G(q)]$.

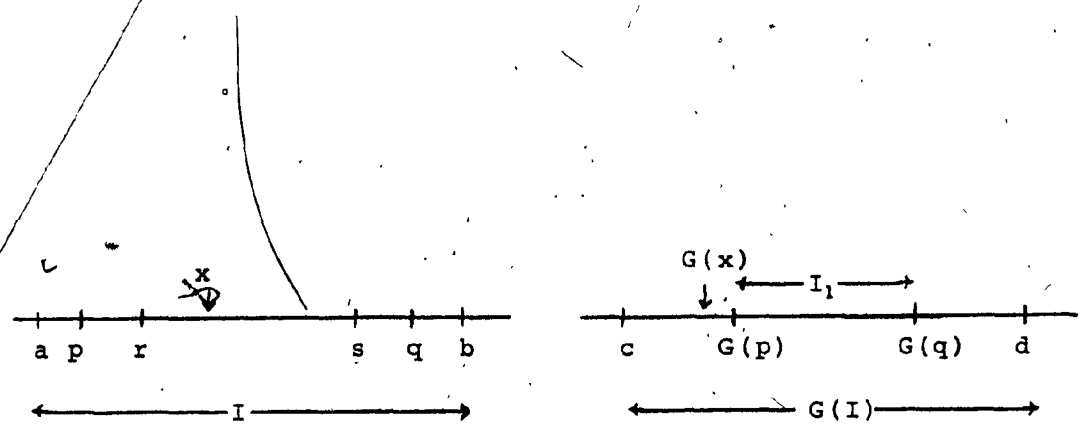


Figure 2.2

Assume $p < q$. Let $r$ be the largest point of $I$ for which $G(r) = G(p)$. Let $s$ be the least point of $I$ for which $s > r$ and $G(s) = G(q)$. Therefore $I_1 = [G(r),G(s)]$. Figure (2.2) represents a possible graph of the above conditions. Let $Q = [r,s]$. We will now show that $G(Q) = I_1$. Since $G$ is continuous, $G$ will assume all values between $G(p)$ and $G(q)$. Therefore $G(Q) \supset I_1$. Now assume $G(Q) \neq I_1$, i.e., there exists an $x \in [r,s]$ such that $G(x) \notin I_1$. Now $G(x) < G(p)$ or $G(x) > G(q)$. We will assume that $G(x) < G(p)$. The other case is similar.

Since $G$ is continuous and $G(x) < G(p)$ and $G(s) = G(q) > G(p)$, there exist $y \in (x,s)$ such that $G(y) =$

$G(p)$ . But $y > r$ , which contradicts the fact that $r$ was largest. Therefore $G(Q) = I_1$ . Similar reasoning applies if $p > q$ . Q.E.D.

<u>Lemma 2.1</u> Let $F : J \to J$ be continuous. Let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subset J$ and $I_{n+1} \subset F(I_n)$ for all $n \geq 0$ . Then there exists a sequence of compact intervals $\{Q_n\}_{n=0}^{\infty}$ such that $Q_{n+1} \subset Q_n \subset I_0$ and $F^n(Q_n) = I_n$ for all $n$ , i.e., for every $x \in \bigcap_{n \geq 0} Q_n$ , $F^n(x) \in I_n$ for all $n \geq 0$ .

<u>Proof</u> The proof is by induction on $n$ . Recall that $F^0(x) = x$ . Let $Q_0 = I_0$ . Then $F^0(Q_0) = I_0$ by definition. Now assume $Q_{n-1}$ has been defined so that

$$F^{n-1}(Q_{n-1}) = I_{n-1} \tag{2.4}$$

Applying $F$ to both sides of (2.4) gives

$$F(I_{n-1}) = F^n(Q_{n-1}) \tag{2.5}$$

Now, by hypothesis, $I_n \subset F(I_{n-1})$ , so $I_n \subset F^n(Q_{n-1})$ by equation (2.5). Applying Lemma 2.0 to the above with $G = F^n$ and $I = Q_{n-1}$ , we obtain the existence of a compact interval $Q = Q_n \subset Q_{n-1}$ such that $F^n(Q_n) = I_n$ . This completes the induction. Q.E.D.

<u>Lemma 2.2</u> Let $G : J \to R$ be continuous. Let $I \subset J$ be a compact interval. Assume $I \subset G(I)$. Then there exists a $p \in I$ such that $G(p) = p$.

<u>Proof</u> Let $I = [\beta_0, \beta_1]$. Since $I \subset G(I)$ choose $\alpha_0, \alpha_1 \in I$ such that

$$G(\alpha_0) = \beta_0 \quad \text{and} \quad G(\alpha_1) = \beta_1$$

Define $H(x) = G(x) - x$.

Now $\alpha_i \geq \beta_0$ and $\alpha_i \leq \beta_1$ for $i = 0,1$ as in Figure 2.3

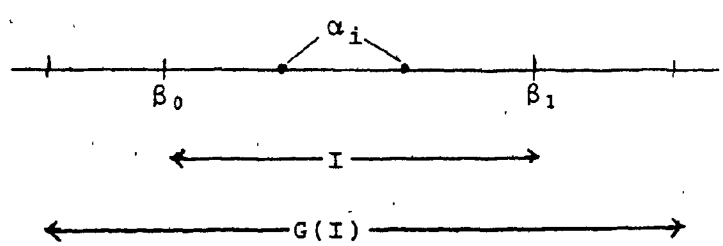


Figure 2.3

Therefore

$$H(\alpha_0) = G(\alpha_0) - \alpha_0 = \beta_0 - \alpha_0 \leq 0$$

and

$$H(\alpha_1) = G(\alpha_1) - \alpha_1 = \beta_1 - \alpha_1 \geq 0$$

Now since $G$ is continuous on $I$, $H$ is continuous on $I$. Therefore there exists $p \in I$ such that $H(p) =$

$G(p) - p = 0$. Hence $G(p) = p$. Q.E.D.

## 2.3 Sufficient Conditions for Chaos

Theorem 2.1 Let $J$ be a closed interval. Let $F : J \to J$ be a continuous function. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$, and $d = F^3(a)$ satisfy

$$d \leq a < b < c \quad \text{or} \quad d \geq a > b > c .$$

Then T1: for every integer $k \geq 1$ there exists a point $p_k \in J$ having period $k$,

and T2: there is an uncountable set $S \subset J$, containing no periodic points which satsifies the following:

(A) For every $p, q \in S$, $p \neq q$

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0 \tag{2.6}$$

and

$$\liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0 \tag{2.7}$$

(B) For every $p \in S$ and periodic point $t \in J$

$$\limsup_{n \to \infty} |F^n(p) - F^n(t)| > 0 \tag{2.8}$$

Any continuous function $F : J \to J$ satisfying (a) and (B) is called chaotic.

Proof We will assume that $d \leq a < b < c$. The proof for $d \geq a > b > c$ is similar.

Proof of T1 Let $K = [a,b]$, $L = [b,c]$, and let $k$ be any positive integer.

If $k > 1$, let $\{I_n\}$ be the sequence of intervals

(2.9) $$I_n = L \quad \text{for} \quad n = 0,1,\ldots,k-2$$

and

(2.10) $$I_n = K \quad \text{for} \quad n = k-1$$

Also define $I_n$ for $n \geq k$ to be periodic, inductively by

(2.11) $$I_{n+k} = I_n \quad \text{for} \quad n = 0,1,2,\ldots$$

If $k = 1$, let

(2.12) $$I_n = L \quad \text{for all} \quad n \geq 0$$

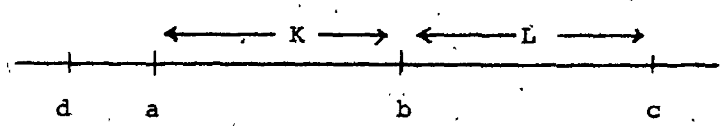


Figure 2.4

Since $F(a) = b$, $F(b) = c$, and $F$ is continuous, it follows by the Mean Value Theorem that $F([a,b]) \supset [b,c]$. i.e., we have

(2.13) $$F(k) \supset L$$

Also, since $F(b) = c$ and $F(c) = d$, we have

$$F([b,c]) \supset [d,c] \supset [b,c]$$

and

$$F([b,c]) \supset [d,c] \supset [a,b],$$

that is,

(2.14) $$F(L) \supset L \quad \text{and} \quad F(L) \supset K$$

Now we have two possible sequences $\{I_n\}_{n=0}^{\infty}$, depending whether $k > 1$ or $k = 1$. Either

(2.15) $$\{I_n\} = \{L,L,\ldots,L,K,L,\ldots,L,K,L,\ldots\},$$

or

(2.16) $$\{I_n\} = \{L,L,L,L,\ldots\}$$

where $k > 1$ or $k = 1$, respectively. However, in either case, as shown by (2.13) and (2.14) $F(I_n) \supset I_{n+1}$ for all $n$. Therefore all the hypothesis of Lemma 2.1 are satisfied.

Let $Q_n$ be the sets in the proof of Lemma 2.1 such that

(2.17) $$Q_{n+1} \subset Q_n \subset I_0 = L, \quad n \geq 0$$

and

(2.18) $$F^n(Q_n) = I_n \text{ for all } n.$$

Therefore by (2.17) and (2.18) and definition of the $\{I_n\}$, we have

(2.19) $$Q_k \subset Q_0 = I_0 = L$$

and

(2.20) $$F^k(Q_k) = I_k = I_0 = L.$$

Therefore by Lemma 2.2, $G = F^k$ has a fixed point $p_k \in Q_k$, i.e.,

(2.21) $$F^k(p_k) = p_k$$

Therefore, $p_k$ has period of at most $k$ for $F$. We will now show that $p_k$ has period $k$.

For $k = 1$, the point $p_1$ has period one.

For $k > 1$, we will now show that $p_k$ cannot have period less than $k$.

Assume $p_k$ has period $i$, where $1 \leq i < k$, so

(2.22) $$F^i(p_k) = p_k$$

Now from (2.9) and (2.10) we have

$$I_n = L \quad \text{for} \quad 0 \leq n \leq k - 2$$

and

$$I_n = K \quad \text{for} \quad n = k - 1.$$

For easier visualization we write out the following chart:

$$I_0 = F^0(Q_0) = L$$
$$I_1 = F^1(Q_1) = L$$
$$I_2 = F^2(Q_2) = L$$
$$\text{-----------------}$$

$$I_{k-2} = F^{k-2}(Q_{k-2}) = L$$

$$I_{k-1} = F^{k-1}(Q_{k-1}) = K$$

$$I_k = F^k(Q_k) = L$$

$$I_{k+1} = F^{k+1}(Q_{k+1}) = L$$

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Now by Lemma 2.1, $Q_{k-1} \supset Q_k$ and $p_k \in Q_k$ ; therefore $p_k \in Q_{k-1}$ and

(2.23) $$F^{k-1}(p_k) \in K .$$

However, since $p_k$ has period $i$ , we have $F^i(p_k) = p_k = F^k(p_k)$ , and therefore $F^{i-1}(p_k) = F^{k-1}(p_k)$ and so

(2.24) $$F^{i-1}(p_k) \in K$$

Now since $i < k$ , Lemma 2.1 implies that $Q_{i-1} \supset Q_i \supset Q_k$ . Further since $p_k \in Q_k$ we have

(2.25) $$p_k \in Q_{i-1}$$

Also since $i < k$ , we have $i - 1 < k - 1$ , so $i - 1 \leq k - 2$ and $I_{i-1} = F^{i-1}(Q_{i-1}) = L$ as seen in the above chart. Therefore, by (2.25)

(2.26) $$F^{i-1}(p_k) \in L$$

Now since $K \cap L = \{b\}$, (2.24) and (2.26) show that $F^{i-1}(p_k) = b$. But since $F^{i-1}(p_k) = F^{k-1}(p_k)$ as stated above, therefore $F^{k-1}(p_k) = b$. Applying $F$ to both sides gives $F^k(P_k) = F(b) = c$. But $F^k(p_k) = p_k$ by (2.21). Therefore $p_k = c$.

By Lemma 2.1 $Q_0 \supset Q_1 \supset \ldots \supset Q_{i-1} \supset Q_i \supset Q_k$. Therefore, since $P_k \in Q_k$, it follows that for all $0 \leq j \leq k$

(2.27) $$p_k = c \in Q_j$$

Also since $i < k$, we have $i + 1 \leq k$, and there are 2 possibilities: (a) $i + 1 \neq k - 1$ or (b) $i + 1 = k - 1$.

Case (a): If $i + 1 \neq k - 1$, $i + 1 \leq k - 2$ or $i + 1 = k$, and in either situation, as seen from the chart and (2.27), we get $F^{i+1}(c) \in L$. But since $p_k = c$ has period $i$, $F^{i+1}(c) = F(c) \in L$, which contradicts $F(c) = d \leq a$ and $d \notin L$.

Case (b): If $i + 1 = k - 1$, then $i = k - 2$ and $i$ and $k$ differ by $2$. But since $p_k$ has period $i$, and $F^k(p_k) = p_k$, $i$ must divide $k$. Therefore $i$ must equal 1 or 2, and $p_k = c$ has period 1 or 2.

But since $p_k = c \in Q_{k-1}$ and $i + 1 = k - 1$, we have $F^{i+1}(c) = F^{k-1}(p_k) \in K$ as seen from the chart, and $F^{i+1}(c) = F(c) = d \leq a$ . Therefore, in this case we are forced to conclude that $d = a$ . But then $c$ has period 3 and not 1 or 2, and again we have a contradiction.

Therefore $i \neq k$, but $i = k$ . Hence, for every $k > 1$, we have a point $p_k$ having period $k$, which proves T1.

Proof of T2: Recall $K = [a,b]$ , $L = [b,c]$ . Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals satisfying the following conditions:

(1) Either $M_n = K$ or $M_n \subset L$, and $F(M_n) \supset M_{n+1}$ . Since $F(K) \supset L$ , $F(L) \supset L$ and $F(L) \supset K$ there exists $M$ such that $F(M_n) \supset M_{n+1}$ for all $n \geq 1$ .

(2) If $M_n = K$, then $n$ is the square of an integer. Note that the converse of condition (2) is false, since $n$ may be the square of an integer but $M_n \neq K$ . Also note that if $n$ is a perfect square, then $n + 1$ and $n + 2$ cannot be perfect squares. Therefore if $M_n = K$, then $M_{n+1}, M_{n+2} \subset L$ .

Let $M \in \mathcal{M}$, and define $P(M,n)$ to be the number of i's for which $M_i = K$ for $1 \leq i \leq n$ , i.e., $P(M,n)$ can assume the value of zero or all integer values up to and possibly including $\sqrt{n}$ . Note that $0 \leq P(M,n^2) \leq n$ and therefore $0 \leq \frac{P(M,n^2)}{n} \leq 1$ .

For each $r \in \left(\frac{3}{4},1\right)$, (the reason for this choice of open interval will be apparent near the end of the proof) select $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

$$\lim_{n \to \infty} \frac{P(M^r, n^2)}{n^2} = r.$$

A possible way of constructing such a sequence is as follows: Consider the set of integers $\{t_n\}_1^{\infty}$, defined by

$$\frac{t_n}{n} \leq r < \frac{t_n + 1}{n}$$

and then $\lim_{n \to \infty} \frac{t_n}{n} = r.$

Define $M_i^r$ by induction: Let $M_1^r \subset L$. Assume $M_i^r$ has been defined for $1 \leq i \leq n^2$, such that $t_n$ (a number) of the $M_i^r$ are $K$ and the remaining $n^2 - t_n$ are contained in $L$. Define

$$M_{(n+1)^2}^r = \begin{cases} K & \text{if } t_{n+1} = t_n + 1 \\ \subset L & \text{if } t_{n+1} = t_n \end{cases}$$

and $M_i^r \subset L$ for $n^2 + 1 < i < (n+1)^2$. For example, if $r = \frac{4}{5}$, we have $t_1 = 0$, $t_2 = 1$, $t_3 = 2$, $t_4 = 3$, (these will always be the first four values when $r \in (\frac{3}{4}, 1)$ ), $t_5 = 4$, $t_6 = 4$, $t_7 = 5$, $t_8 = 6$, $t_9 = 7$, $t_{10} = 8$, $t_{11} = 8$, etc. Note that $t_{n+1} = t_n$ or $t_{n+1} = t_n + 1$. Therefore $M_1^r \subset L$, $M_4^r = K$, $M_9^r = K$, $M_{16}^r = K$, $M_{25}^r = K$, $M_{36}^r \subset L$, $M_{49}^r = K$, etc.

Using the above construction, it is clear that

$$\lim_{n \to \infty} \frac{P(M^2, n^2)}{n} = r$$

Let $\mathcal{M}_0 = \{M^r : r \in (\frac{3}{4}, 1)\} \subset \mathcal{M}$. Then $\mathcal{M}_0$ is an uncountable set since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$ and $(\frac{3}{4}, 1)$ is an uncountable set of numbers.

Now for each $M^r \in \mathcal{M}_0$, by Lemma 2.2, there exists a point $x_r \in \bigcap_n Q_n^r$ such that $F^n(x_r) \in M_n^r$ for all $n$. We can never have $F^k(x_r) = b$, for then $F^{k+2}(x_r) = F^2(b) = d \leq a$. But $F^{k+2}(x_r)$ must be an element of $K$ or $L$, implying that $d = a$ and that $a$ has period 3. Hence, $x_r$ eventually has period 3. This contradicts condition (2) for the following reason: since $a \in K$ and $a \notin L$, we have $M^r_{k+2+3m} = K$, $m = 0,1,2,\ldots,$ and the sequence $M^r = \{\ldots,K,L,L,K,L,L,K,\ldots\}$ (where here $L$ means an interval in $L$), which is impossible since the squares of integers do not occur <u>every</u> third integer, so $F^k(x_r) \neq b$ for any $k$. Thus $x_{r_1} \neq x_{r_2}$ for $r_1 \neq r_2$, for otherwise this would imply that $M^{r_1} = M^{r_2}$ which is not the case.

Let $S = \{x_r : r \in (\frac{3}{4}, 1)\}$; $S$ is uncountable. For $x \in S$, define $P(x,n)$ to be the number of $i$'s in $\{1,2,\ldots,n\}$ for which $F^i(x) \in K$.

It is clear that $\lim_{n \to \infty} \frac{P(x_r, n^2)}{n} = r$, since $F^k(x_r) \neq b$ for any $k$.

We have now shown that $P(x_r, n) = P(M^r, n)$ for all $n$.

Define:

$$\rho(x_r) = \lim_{n \to \infty} \frac{P(x_r, n^2)}{n} = r \tag{2.28}$$

Let $p, q \in S$ with $p \neq q$, and assume that $\rho(p) > \rho(q)$. As can be seen from the way the $M^r$ are defined, $P(p,n) - P(q,n) \to \infty$ as $n \to \infty$, and there must be infinitely many $n$'s such that $F^n(p) \in K$ and $F^n(q) \in L$ or vice versa, since the sequences $M^{\rho(p)}$ and $M^{\rho(q)}$ do not match, at an infinite number of places.

Since $F^2$ is continuous and $F^2(b) = d \leq a$, there exists $\delta > 0$, $\delta < \frac{b-d}{2}$, such that $F^2(x) < \frac{b+d}{2}$ for all $x \in [b - \delta, b] \subset K$ (see Figure 2.5).

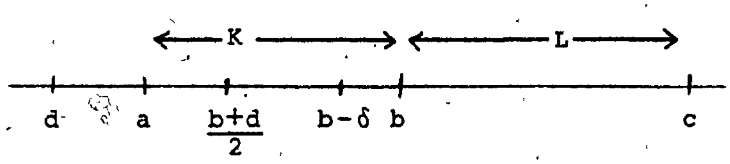



Figure 2.5

Now if $p \in S$ and $F^n(p) \in K$, then condition (2) implies then $F^{n+1}(p) \in L$ and $F^{n+2}(p) \in L$. Thus $F^n(p) \notin$

$[b - \delta, b]$ (otherwise $F^{n+2}(p) < \dfrac{b + d}{2}$ and $F^{n+2}(p) \in K$), and so $F^n(p) < b - \delta$.

Also if $q \in S$ and $F^n(q) \in L$, we have $F^n(q) \geq b$, so $|F^n(p) - F^n(q)| > \delta$. Since there are an infinite number of $n$'s for which $F^n(p) \in K$ and $F^n(q) \in L$, we have $\limsup_{n \to \infty} |F^n(p) - F^n(q)| \geq \delta > 0$, proving (2.6).

The above shows that no matter how close $p$ and $q$ are, there will be an infinite number of values of $n$ for which $F^n(p)$ and $F^n(q)$ will be at least $\delta$ apart.

We also note that $S$ cannot contain any periodic points. Since, if $x \in S$ has period $k$, or eventually has period $k$, there must exist a positive integer $m$ such that $F^m(x) \in K$, $F^{m+k}(x) \in K$, $F^{m+2k}(x) \in K$, which is impossible since $m, m + k, m + 2k$ cannot all be squares of integers.

We will now prove part (B) of the theorem (the technique is similar to the proof of (2.6)), and afterwards complete the proof of part (A) by proving (2.7).

<u>Proof of (B)</u>: Let $t \in J$ be a point having period $k$, and thus $t \notin S$ as shown before. Now we have two possibilities: (a) $t \notin L$ or (b) $t \in L$.

<u>Case (a)</u>: Let $t \notin L$, and consider $F^{mk}(p)$, $m = 1,2,3,\ldots$ where $p \in S$. An infinite number of $F^{mk}(p)$ are in $L$, since, if there would only be a finite number, then from a certain point and onward all of $F^{mk}(p)$ would be in $K$

(since $p \in S$), which is impossible since the squares of integers do not occur every $k$ integers. Also $F^{mk}(t) \notin L$, $m = 1,2,3,\ldots$ since $t$ has period $k$ and $t \notin L$.

Let $\delta = \min(|t - b|, |t - c|)$ (see Figure 2.6).

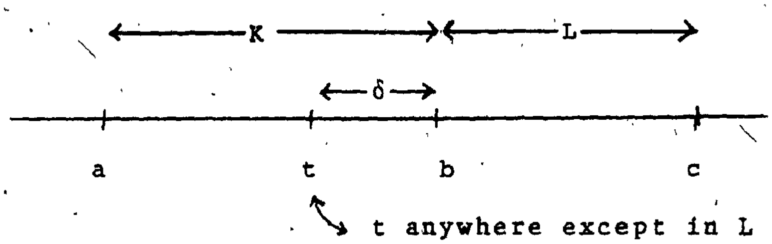



Figure 2.6

Since for an infinite number of $m$ we have $|F^{mk}(p) - F^{mk}(t)| > \delta$, it follows that $\limsup_{n \to \infty} |F^n(p) - F^n(t)| \geq \delta > 0$.

<u>Case (b)</u>: If $t \in L$, consider the orbit $F(t), F^2(t), \ldots, F^{k-1}(t)$. If any of these iterates is not in $L$, that is, for some $r$, $0 < r < k$, $F^r(t) \notin L$, then we have a point $F^r(t) \notin L$ having period $k$, so we have case (a) and the proof is done.

Let all the iterates of $t$ be in $L$. None of them is $b$, since $F^2(b) = d \notin L$. Let

$\delta = \min(t - b, F(t) - b, \ldots, F^{k-1}(t) - b)$ , and $p \in S$ . Since $\rho(p) = r > 0$ , there are an infinite number of $n$ for which $F^n(p) \in K$ , and so we have (2.8)

$$\limsup_{n \to \infty} |F^n(p) - F^n(t)| \geq \delta > 0 .$$

This completes the proof of (B).

<u>Proof of (2.7)</u>: Since $F$ is continuous, $F(b) = c$ , and $F(c) = d \leq a$ , there exists a subset of $L = [b,c]$ , the interval $L^1 = [b^1, c^1]$ , such that $F([b^1,c^1]) \subset [b,c]$ , $F(b^1) = c$ , and $F(c^1) = b$ . Similarly, we may choose intervals $[b^n, c^n]$ , $n = 0,1,2,\ldots$ satisfying the following three conditions (see Figure (2.27)):

(2.29) $[b,c] = [b^0,c^0] \supset [b^1,c^1] \supset [b^2,c^2] \supset \ldots \supset [b^n,c^n] \supset \ldots$ ,

(2.30) $$(b^n,c^n) \supset F[(b^{n+1},c^{n+1})]$$

and

(2.31) $$F(b^{n+1}) = c^n , \quad F(c^{n+1}) = b^n .$$

Let $A = \bigcap_{n=0}^{\infty} [b^n,c^n]$ . Let $b^* = \inf A$ and $c^* = \sup A$ .

Then, by (2.31) we have $F(b^*) = c^*$ and $F(c^*) = b^*$ .

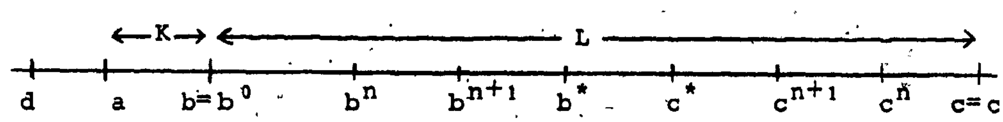



Figure 2.7

We will now be more specific in our choice of sequences $M$. In addition to our previous two requirements on $M \in \mathcal{M}$, we will also assume that <u>if</u> $M_k = K$ for <u>both</u> $k = n^2$ and $k = (n+1)^2$, then

$$M_k = [b^{2n-(2j-1)}, b^*] \quad \text{for} \quad k = n^2 + (2j-1), \quad j = 1, \ldots, n,$$

and

$$M_k = [c^*, c^{2n-2j}] \quad \text{for} \quad k = n^2 + 2j, \quad j = 1, 2, \ldots, n.$$

For <u>all</u> remaining $k$'s that are not squares of integers, assume $M_k = L$.

The following list of sets will aid us in visualizing a typical sequence.

---

$M_{n^2-1} = L$ (or a subset of $L$ if $M_{(n-1)^2} = K$)

$M_{n^2} = K$

$M_{n^2+1} = [b^{2n-1}, b^*]$

$M_{n^2+2} = [c^*, c^{2n-2}]$

$M_{n^2+3} = [b^{2n-3}, b^*]$

$M_{n^2+4} = [c^*, c^{2n-4}]$

---
---

$M_{(n+1)^2-2} = [b^1, b^*]$

$M_{(n+1)^2-1} = [c^*, c^0]$

$M_{(n+1)^2} = K$

$M_{(n+1)^2+1} = L$ (or a subset of $L$ if $M_{(n+2)^2} = K$)

---

Note that conditions (1) and (2) on $M$ are still met for the above sequences since $M_i = K$ or $M_i \subset L$, and conditions (2.29), (2.30), (2.31) guarantee that $F(M_n) \supset M_{n+1}$. Also we have not changed $M_k$ where $k$ is the square of an integer.

The method of selecting $M^r$ to satisfy $\lim_{n \to \infty} \frac{P(M^r, n^2)}{n} = r$ was only dependent on $M_k$ where $k$ was the square of an integer, and thus also remains unaffected by our new requirements.

From the fact that $\rho(x)$ may be thought of as the limit of the fractions of $n$'s for which $F^{n^2}(x) \in K$, (as explained previously, this limit ranges from 0 to 1) it follows that for any $r^*$, $r \in \left(\frac{3}{4}, 1\right)$ there exists infinitely many $n$ such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n + 1)^2$.

To see the above clearly, consider a set $U$ containing 100 elements, and two subsets of $U$, $A$ and $B$, each containing <u>over</u> 75% (i.e., more than 75) of the elements of $U$. Then $A \cap B$ will contain more than 50% (i.e., 51 or more) elements of $U$. Similarly if $A$ and $B$ each had <u>more</u> than 50 elements then $A \cap B$ will have at least 1 element. However, if $A$ and $B$ have 50 elements (or less) it is possible that $A \cap B = \phi$.

Applying this logic to our case, since $r > \frac{3}{4}$ and

and $r^* > \frac{3}{4}$, for each of the sequences $\{M^{r}_{n^2}\}$ and $\{M^{r^*}_{n^2}\}$ over 75% (in the limit) of the elements are K. This means that for over 50% of all perfect squares $n^2$, $M^{r}_{n^2} = M^{r^*}_{n^2} = K$. Since (by construction) for some sufficiently large N, any block of N consecutive squares contains more than $\frac{N}{2}$ matches, there must be at least one pair of successive matches in the block. So there must be an infinite number of n's for which $M^{r}_{k} = M^{r^*}_{k} = K$ for both $k = n^2$ and $k = (n + 1)^2$.

Now since $b^n \to b^*$ as $n \to \infty$, then for every $\varepsilon > 0$ there exists a positive integer N such that $|b^n - b^*| < \varepsilon$ for all $n > N$.

Let $x_r \in S$ and $x_{r^*} \in S$. Thus, for any $n > N$ and $M^{r}_{k} = M^{r^*}_{k} = K$, for both $k = n^2$ and $k = (n + 1)^2$, we have

$$F^{n^2+1}(x_r) \in M^{r}_{n^2+1} = [b^{2n-1}, b^*]$$

and

$$F^{n^2+1}(x_{r^*}) \in M^{r^*}_{n^2+1} = [b^{2n-1}, b^*] .$$

Thus $|F^{n^2+1}(x_r) - F^{n^2+1}(x_{r^*})| < \varepsilon$ .

Since there are infinitely many n's with this property,

$$\liminf_{n \to \infty} |F^n(x_r) - F^n(x_{r^*})| = 0$$

Thus for every $p, q \in S$, $\liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0$.

which proves (2.7) and completes the proof of (A).

This completes the proof of Theorem 2.1. Q.E.D.

## 2.4 Period Five Does Not Imply Period Three

We have now shown that for a continuous function $F : R \to R$ the existence of a point of period three implies the existence of points of all periods. However, existence of a point of period $k$, where $k$ is any positive integer, does not always imply existence of points of all periods, as shown by the following counterexample given by Li and Yorke [14].

Example: Let $F : [1,5] \to [1,5]$ be defined such that $F(1) = 3$, $F(2) = 5$, $F(3) = 4$, $F(4) = 2$, $F(5) = 1$, and on each interval $[n, n+1]$, $1 \leq n \leq 4$, $f$ is linear. The graph of $F$ is drawn in Figure 2.8.

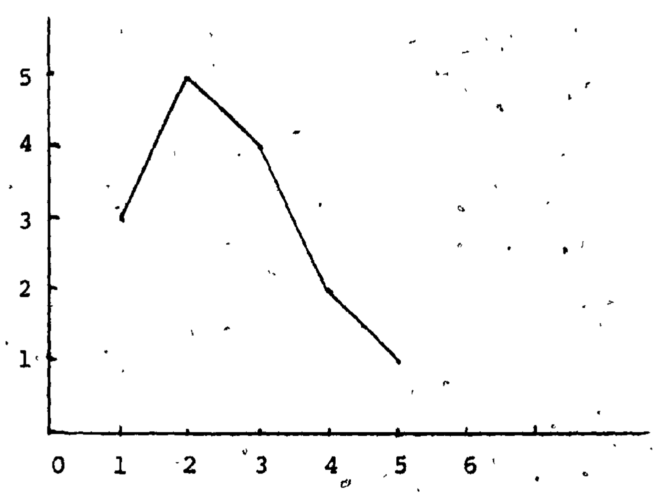



Figure 2.8

Hence, each of the points 1,2,3,4, and 5 has period 5 .

Now $F^3([1,2]) = F^2([3,5]) = F([1,4]) = [2,5]$ , and therefore $F^3$ has no fixed points in $[1,2]$ , since the only point common to $[1,2]$ and $[2,5]$ is 2 , and 2 has period 5 . Similarly, $F^3([2,3]) = [3,5]$ , and $F^3([4,5]) = [1,4]$ , so neither of these two intervals contain a fixed point of $F^3$ .

However, $F^3([3,4]) = F^2([2,4]) = F([2,5]) = [1,5] \supset [3,4]$ ; so by Lemma 2.2, $F^3$ has a fixed point in $[3,4]$ . We shall now show that the fixed point of $F^3$ is unique, and is also the fixed point of $F$ .

Let $p \in [3,4]$ be a fixed point of $F^3$ . Since $F([3,4]) = [2,4]$ , we must have $F(p) \in [2,4]$ . Now, if $F(p) \in [2,3]$ , then $F^2(p) \in [4,5]$ and $F^3(p) \in [1,2]$ (all of which is easily seen by looking at Figure 2.8), which is impossible since $p \in [3,4]$ is a fixed point of $F^3$ .

Hence $F(p) \in [3,4]$ , and $F^2(p) \in [2,4]$ . If $F^2(p) \in [2,3]$ then $F^3(p) \in [4,5]$ , which is also impossible since 4 is not a fixed point of $F^3$ . So $F^2(p) \in [3,4]$ , and therefore $p$ , $F(p)$ and $F^2(p)$ are all in $[3,4]$ .

On $[3,4]$ , $F$ is linear going from (3,4) to (4,2). Thus on $[3,4]$ , $F$ is defined by $F(X) = 10 - 2X$ . Then $F^2(X) = 10 - 2(10 - 2X) = 4X - 10$ , and $F^3(X) = 10 - 2(4X - 10) = 30 - 8X$ , and the unique fixed point of $F^3$ is given by the solution of $X = 30 - 8X$ which is $\frac{10}{3}$ .

But $\frac{10}{3}$ is also the fixed point of $F$, since $F\left(\frac{10}{3}\right) = \frac{10}{3}$, and therefore $\frac{10}{3}$ has period 1. So we have points of period 5, but none having period 3.

## 2.5 Period k Implies Period One

As can be seen from the example of the previous section, the existence of a point of period 5 or of period one does not imply the existence of a point of period 3. However, the existence of a point of period $k$, for all $k$, does imply the existence of a point of period one, as was shown very simply by Straffin [15].

**Theorem 2.2** Let $f$ be a continuous function $f : R \to R$ having a point of period $k \geq 1$, then $f$ has a point of period one

**Proof** Let point $a$ have period $k$. If $k = 1$, the result is trivial. Assume $k > 1$, and consider the orbit $a, f(a), f^2(a), f^3(a), \ldots, f^{k-1}(a), f^k(a) = a$. Let $f(a) > a$. If $f(a) < a$, the proof is similar ($f(a) \neq a$ since $k > 1$). Then there must be a point $b = f^i(a)$ of this orbit such that $f(b) < b$, otherwise the sequence would constantly increase and could not return to $a$.

Let $F(X) = f(X) - X$. Then $F(a) > 0$ and $f(b) < 0$, so by the Intermediate Value theorem there exists a point $c$ between $a$ and $b$ such that $f(c) = c$. Q.E.D.

## 2.6 Related Conclusions

We will now go back to equation (2.3) $F(X) = rX(1 - X)$ and state some results about it. We choose this equation because it is simple, and for certain values of $r$ is a chaotic function. For example, when $r = 3.9$ we have for $X = .142$, $f(X) = .475$, $f^2(X) = .973$ and $f^3(X) = .104$ and thus $f^3(X) < X < f(X) < f^2(X)$.

1) The fixed points of (2.3) are obtained by solving $X = rX(1 - X)$ which gives $\left\{0, \frac{r-1}{r}\right\}$. Note $\frac{r-1}{r} > 0$ only if $r > 1$.

2) If $r \in [0,4]$ then $F : [0,1] \to [0,1]$.

3) For $r \in [0,1]$, $X = 0$ is the only point of period 1. In fact for all $X \in [0,1]$. $F^n(X) \to 0$ as $n \to \infty$.

4) For $r \in (1,3]$ there are two points of period 1, namely 0 and $\frac{r-1}{r} = 1 - r^{-1}$, and for all $X \in (0,1)$, $F^n(X) \to 1 - r^{-1}$ as $n \to \infty$.

5) <u>Definition:</u> For any function $F$ a point $y \in J$ with period $k$ is said to be asymptotically stable if for some interval $I = (y - \delta, y + \delta)$ we have $|F^k(X) - y| < |X - y|$ for all $X \in I$ and $X \neq y$.

A necessary condition that gives this behaviour is that $\left|\frac{d}{dx}F^k(X)\right| < 1$. This can easily be seen from Figure 2.9.

In Figure 2.9 (a) and (b) point $p$ (which occurs where the line $y = X$ intersects the graph) has period 1. If we take iterates of $F$ at the point $X_0$ "near" $p$ in Figure 2.9a, where $\left|\left(\frac{dF}{dX}\right)_p\right| > 1$, we obtain the sequence $X_0, X_1, X_2, \ldots$ which is moving further and further away from

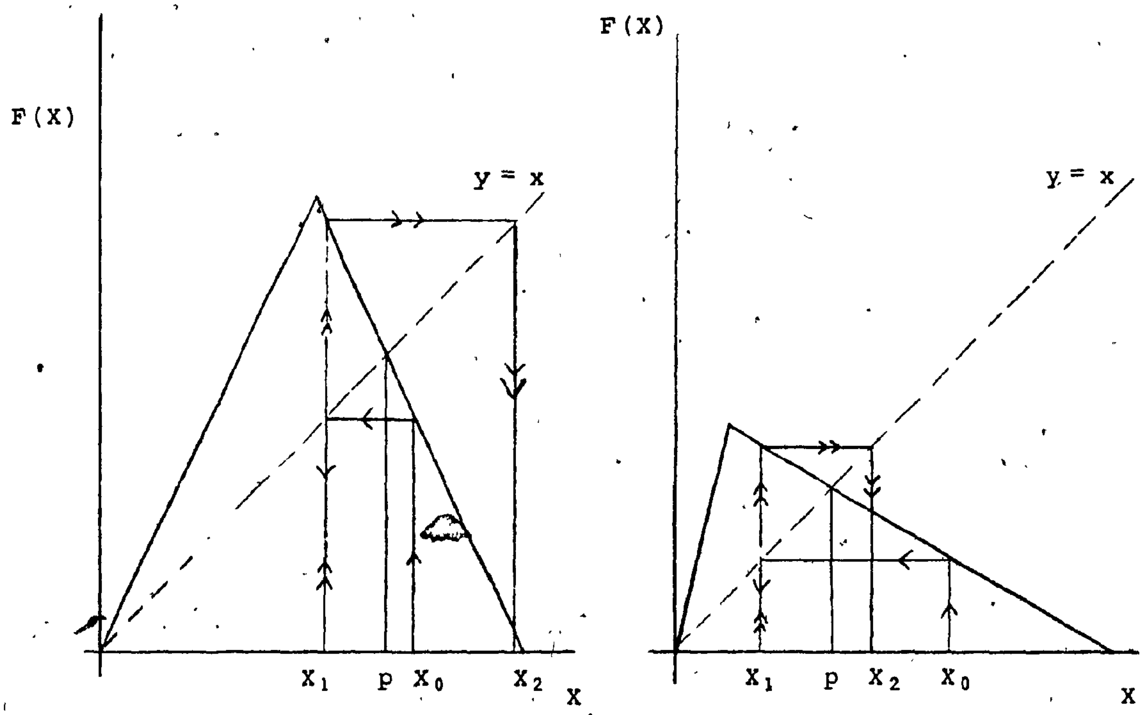



Figure 2.9

p . However, in Figure 2.9b where $\left|\left(\frac{dF}{dX}\right)_p\right| < 1$, $X_0, X_1, X_2, \ldots$ has p as its limit.

Note To obtain the points $X_1, X_2, X_3, \ldots$ graphically, all one needs to do is start from $X_0$ on the x-axis. Draw the ordinate line at that point to get $F(X_0) = X_1$. From the point $(X_0, X_1)$ on the graph move horizontally until the line $y = X$ is intersected. The abscissa of that point is then $X_1$. Repeat the process to get $X_2$ on the x-axis, and then $X_3, X_4, \ldots$ .

Algebraically, the reason for the stability condition is that $|F^k(X) - y| \doteq |X - y|\,|\frac{d}{dX}F^k(y)|$. Thus, when $|\frac{d}{dX}F^k(y)| < 1$ we obtain $|F^k(X) - y| < |X - y|$.

In the example used in Section 1.3, where the slope is $\pm 2$ wherever the derivative exists, we see, as noted there, that every periodic point will be unstable.

6) For equation (2.3), $F'(X) = r - 2rX$. Thus in comment 3 above, where $X = 0$, $F'(0) = r$, so when $r \in [0,1)$, $X = 0$ is a stable point of period $1$.

In comment 4, where $r \in (1,3)$, $X = 0$ is no longer a <u>stable</u> point of period $1$. However, now we have two points of period $1$, the second of which, $X = 1 - r^{-1}$, is a stable point since $|F'(1 - r^{-1})| = |r - 2r(1 - r^{-1})| = |2 - r| < 1$ for $r \in (1,3)$.

7) For $r \in [0,3]$ we only have points of period $1$, listed above and no points of period $k$, $k > 1$.

8) For $r > 3$, this stable cycle of period $1$, also becomes unstable, however, it "bifurcates" into two points $p$ and $q$ near $1 - 3^{-1} = \frac{2}{3}$ which have period $2$, and of course $F(p) = q$ and $F(q) = p$. For $r \in (3, 1 + \sqrt{6} \doteq 3.449)$ and $X \in (0,1)$, $F^{2n}(X)$ converges to either $p$ or $q$, while $F^{2n+1}(X)$ converges to the other, except for those $X$ for which there is an $n$ for which $F^n(X)$ equals the point $1 - r^{-1}$ of period $1$. In this region there are no other periodic points.

9) When $r$ is "slightly" greater than $1 + \sqrt{6}$, the previous two points of period 2 become unstable and they bifurcate into 4 points of period 4, and $F^{4n}(X)$ approaches one of these four points for all $X \in (0,1)$ except for those $X$ for which there is an $n$ such that $F^{n}(X)$ equals one of the points of period 1 or 2 . In this interval there are no other points of any period, i.e., we only have points of period 1, 2, or 4 listed above.

10) The above process continues over smaller and smaller intervals and we obtain points of period $2^m$ . The limiting value of $r$ for which we only have points of period $2^m$ as $m \to \infty$ seems to be near $r \doteq 3.5700$.

11) When $r \doteq 3.6786$ the first point having a period of an odd number appears, and the function will be chaotic. (See Theorem 3.2). When $r \doteq 3.8284$ the first point of period 3 appears, and thus by Li and Yorke [14] all integer periods appear. When $r = 4$ the chaotic region ends since all points then for $r > 4$ tend to $\infty$.

12) The above "bifurcation" process is discussed extensively in articles by May [3,9,13].

# CHAPTER 3

# PROPERTIES OF CHAOTIC FUNCTIONS

## 3.1 Sharkovsky's Theorem

In 1964, the Ukranian mathematician Sharkovsky [23] proved the following remarkable theorem, which has only recently, after the work of Li and Yorke [14], become more widely known. In it, he has completely answered the question as to when period $k$ implies period $\ell$. For example, we have shown that the existence of points of period five does not imply existence of points of period three, but the converse is true.

Definition 3.1: Let $\rightarrow$ be an order relation defined on the set of natural numbers in the following way: Let $n_1, n_2$ be any positive integers. Then $n_1 \rightarrow n_2$, if the difference equation $x_{n+1} = f(x_n)$ has a point of period $n_2$ whenever it has a point of period $n_1$.

Theorem 3.1: Let $I$ be a compact interval and $f : I \rightarrow I$ be a continuous function. Then the following ordering of the natural numbers holds:

$$
\begin{aligned}
(3.1)\qquad & 3 \rightarrow 5 \rightarrow 7 \rightarrow 9 \rightarrow \ldots\ldots \rightarrow 2\cdot 3 \rightarrow 2\cdot 5 \rightarrow 2\cdot 7 \rightarrow 2\cdot 9 \\
& \rightarrow \ldots\ldots \rightarrow 2^2\cdot 3 \rightarrow 2^2\cdot 5 \rightarrow 2^2\cdot 7 \rightarrow 2\ \cdot 9 \rightarrow \ldots\ldots \\
& \ldots \rightarrow 2^n\cdot 3 \rightarrow 2^n\cdot 5 \rightarrow 2^n\cdot 7 \rightarrow \ldots\ldots \rightarrow 2^n \rightarrow 2^{n-1} \\
& \rightarrow \ldots\ldots\ldots \rightarrow 2^3 \rightarrow 2^2 \rightarrow 2 \rightarrow 1 .
\end{aligned}
$$

(3.1) is called a Sharkovsky ordering.

A proof in English of Theorem 3.1 can be found in [16].

Using 1) the definition of Li and Yorke, that a continuous function $F$ is chaotic if there exists an uncountable set $S$ which does not contain any periodic points nor any asymptotically periodic points such that for every $x, y \in S$, $x \neq y$ we have

$$(3.2) \quad 0 = \liminf_{n \to \infty} |F^n(x) - F^n(y)| < \limsup_{n \to \infty} |F^n(x) - F^n(y)| ,$$

and 2) Theorem 2.1 of [14], and 3) Theorem 3.1, the following theorem of Butler and Pianigiani [17] may be proven.

<u>Theorem 3.2</u>: Let $F: J \to J$ be a continuous function. If $F$ has a point of period $m \cdot 2^n$, where $m$ is an odd integer, $m \geq 3$, and $n$ a non-negative integer, then $F$ is chaotic.

<u>Proof</u>: Since $F$ has a point of period $m \cdot 2^n$, then by Sharkovsky's ordering (3.1), it will also have a point of period $3m \cdot 2^n$. Thus $F^{m \cdot 2^n}$ has a point of period three and therefore by Theorem 2.1 $F^{m \cdot 2^n}$ is chaotic, and by (3.2) $F$ is chaotic. Q.E.D.

It is interesting to note, as is done by Straffin [15], that for the difference equation 2.3, $F(x) = rx(1 - x)$, as $r$ increases, points of longer and longer periods appear, as we have discussed in Section 2.6 in comments 7 through 11. We start with points of period one, then periods 2,4,8,...., $2^n$,.... When $r \doteq 3.5$ there are no new periods of length $2^n$. Then points of other periods appear, all of which are even. The first point of odd period begins when $r \doteq 3.68$, and period three first appears when $r \doteq 3.83$. Thus the order of points of different periods first appearing seems to run exactly backwards through Sharkovsky's ordering as $r$ goes from $3.00$ to $3.83$.

A partial proof of Theorem 3.1 is given by Straffin [15] in a very interesting and elegant way through the use of "digraphs".

## 3.2 Denseness of Chaotic Difference Equations

Kloeden [18] has shown that the set of all chaotic functions is a dense subset of the space of continuous mappings of a compact interval of the real line into itself with the sup norm.

Definition 3.2: Let $X = [a,b]$ be a compact interval and let $C(X)$ denote the space of all continuous functions $f : X \to X$ with the sup norm $\|f\| = \max_{x \in X} |f(x)|$.

Associated with each $f \in C(X)$ there is a difference equation

$$x_{n+1} = f(x_n) \qquad n = 0,1,2,\ldots \tag{3.3}$$

We will show that the subset of all chaotic functions is a dense subset of $C(X)$ by constructing for each $f \in C(X)$ and any $\varepsilon > 0$, a chaotic function in $C(X)$ which is within $\varepsilon$ of $f$ with respect to the sup norm on $C(X)$. Before we start with the main proof we give the following example of a chaotic function which will be used in our proof.

Example 3.1: Let $\eta > 1$, and let $F \in C([0,2\eta])$ be defined as follows:

$$(3.4) \qquad \begin{aligned} F(x) &= 2x & \text{for} \quad & 0 \le x \le \eta \\ F(x) &= -2x + 4\eta & \text{for} \quad & \eta \le x \le 2\eta . \end{aligned}$$

This function has a cycle of order three for every value of $\eta$, i.e., $\frac{4}{7}\eta$, $\frac{8}{7}\eta$ and $\frac{12}{7}\eta$, and is thus a chaotic function by Theorem 2.1 (see Figure 3.1).

Theorem 3.3: Let $X$ be the compact interval $[a,b]$, then the chaotic functions are dense in $C(X)$.

Proof: We wish to show that given any $f \in C(X)$ and any $\varepsilon > 0$ there exists a chaotic function $g \in C(X)$ such that

$$(3.5) \qquad \|f - g\| < \varepsilon .$$

Since $f : X \to X$, then by Lemma 2.2 there exists at least one fixed point $x^*$ of $f$. Assume for the moment that $x^* \neq b$.

Since $f$ is continuous everywhere, it is continuous at $x^*$. Therefore there exists $\delta > 0$, $\delta$ possibly dependent on $x^*$ and $\varepsilon$, such that

$$(3.6) \qquad \delta < \frac{\varepsilon}{2}$$

and

$$(3.7) \qquad |f(x) - f(x^*)| < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in [x^*, x^* + \delta] .$$

Let $\eta = \frac{\delta}{4}$ and define $F^* \in C([x^*, x^* + 2\eta])$ as follows:

$$F^*(x) = 2x - x^*, \qquad x^* \le x \le x^* + \eta$$

(3.8)

$$F^*(x) = -2x + 3x^* + 4\eta, \qquad x^* + \eta \le x \le x^* + 2\eta$$

See Figure 3.2.

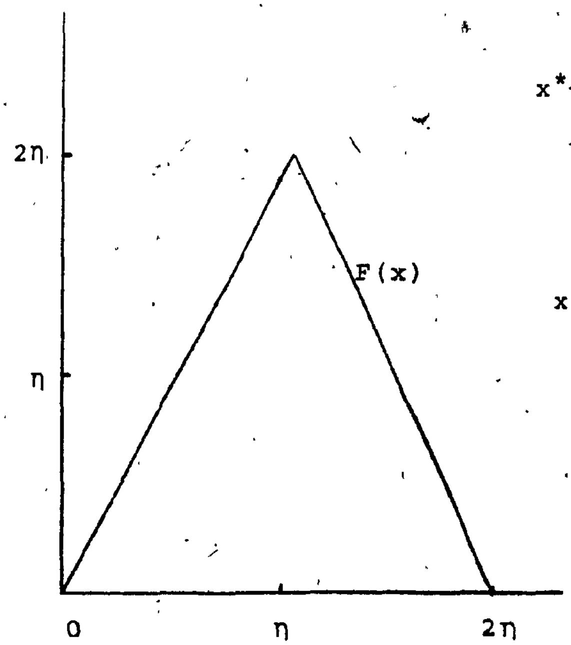



Figure 3.1

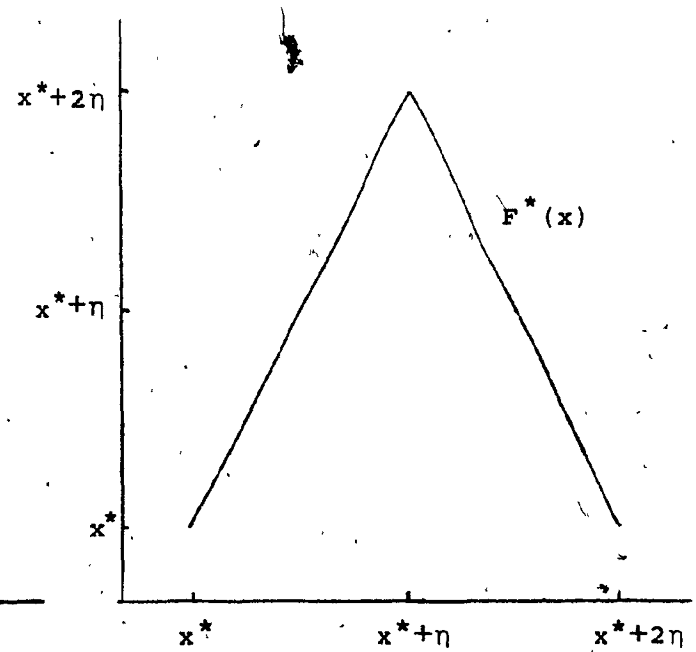



Figure 3.2

We see that the graph of $F^*(x)$ is the same graph as that of $F(x)$ of Example 3.1, but with the origin translated to $(-x^*,-x^*)$.

Now we will define two additional functions.

Let $\gamma : [x^* + 2\eta, x^* + 3\eta] \to X$ be defined by

$$(3.9) \qquad \gamma(x) = x^* + (x - x^* - 2\eta)\left(\frac{f(x^* + 3\eta) - x^*}{\eta}\right),$$

the graph of which is the line segment joining the points $(x^* + 2\eta, x^*)$ and $(x^* + 3\eta, f(x^* + 3\eta))$ in the plane.

Let $I = [x^*, x^* + 2\eta]$ and $J = [x^* + 2\eta, x^* + 3\eta]$. Define a function $g : X \to X$ as follows:

$$(3.10) \qquad g(x) = \begin{cases} f(x), & x \in X - (I \cup J) \\ F^*(x), & x \in I \\ \gamma(x), & x \in J \end{cases}$$

The graph of a possible $g$ is shown in Figure 3.3.

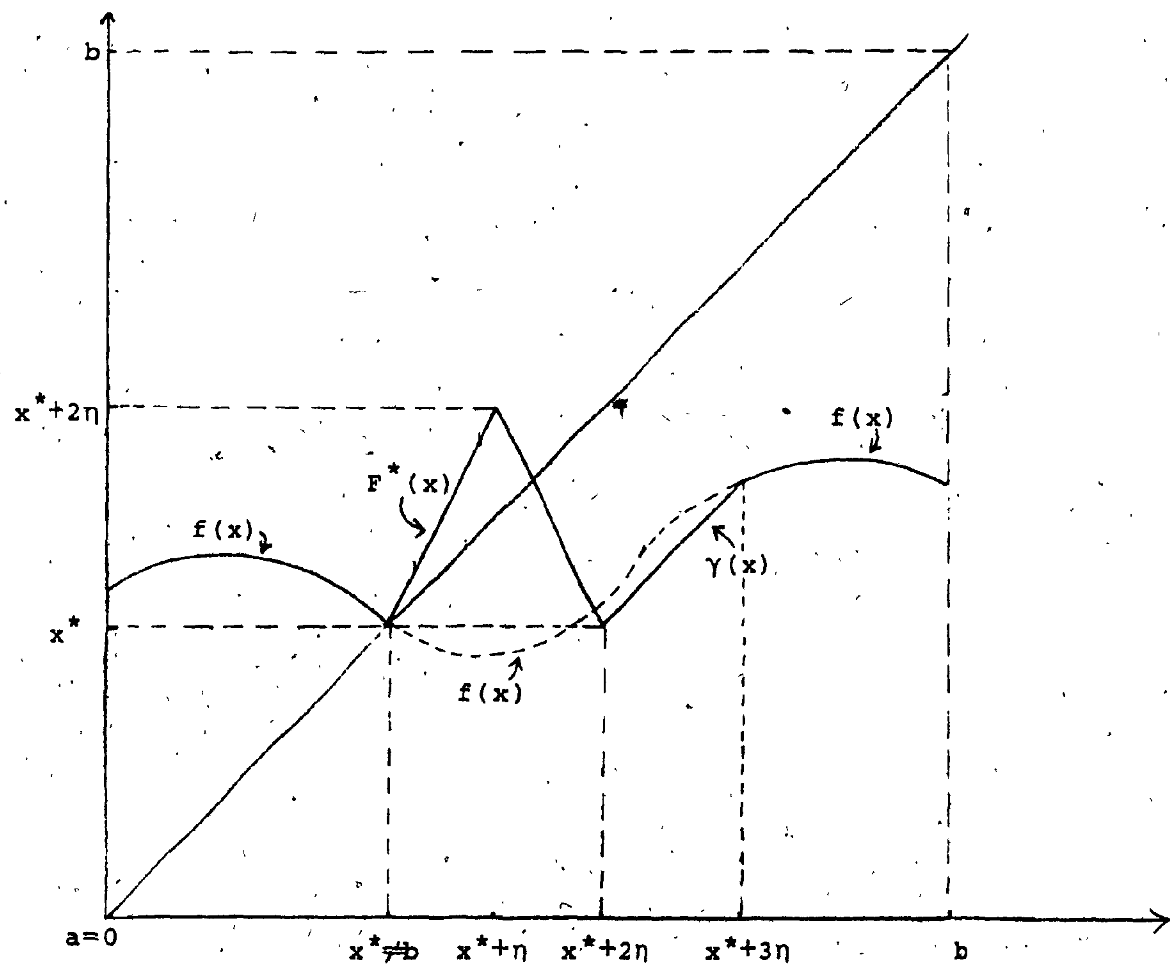



Figure 3.3

Note that $g$ is a continuous function on $X$, and that

$$(3.11)\quad |f(x) - g(x)| = \begin{cases} 0 & x \in X - (I \cup J) \\ |f(x) - F^*(x)|, & x \in I \\ |f(x) - \gamma(x)|, & x \in J \end{cases}$$

Therefore,

$$(3.12)\quad \|f - g\| = \max\left\{ \sup_{x \in I} |f(x) - F^*(x)|, \sup_{x \in J} |f(x) - \gamma(x)| \right\}$$

Now by the triangle inequality, we have

$$(3.13)\quad \sup_{x \in I} |f(x) - F^*(x)| \leq \sup_{x \in I} |f(x) - x^*| + \sup_{x \in I} |x^* - F^*(x)|$$

and

$$(3.14)\quad \sup_{x \in J} |f(x) - \gamma(x)| \leq \sup_{x \in J} |f(x) - x^*| + \sup_{x \in J} |x^* - \gamma(x)|$$

Now $f(x^*) = x^*$ since $x^*$ is a fixed point of $f$, so by (3.7) we have

$$(3.15)\quad |f(x) - x^*| = |f(x) - f(x^*)| < \frac{\varepsilon}{2},$$

$$x \in [x^*, x^* + \delta] = [x^*, x^* + 4\eta]$$

Also from the way the intervals $I$ and $J$, and the functions $F^*$ and $\gamma$ were defined (and as is easily seen from Figure 3.3) we have

$$(3.16)\quad \sup_{x \in I} |x^* - F^*(x)| = |x^* - F^*(x^* + \eta)| = 2\eta = \frac{\delta}{2} < \frac{\varepsilon}{4},$$

and

$$(3.17) \qquad \sup_{x \in J} |x^* - \gamma(x)| = |x^* - f(x^* + 3\eta)| < \frac{\varepsilon}{2}$$

by (3.15). Combining (3.12) through (3.17) we obtain

$$(3.18) \qquad \|f - g\| < \max\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{4}, \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) = \varepsilon .$$

Now $F^*$, and therefore $g$ has a cycle of period three, namely $x^* + \frac{4}{7}\eta$, $x^* + \frac{8}{7}\eta$ and $x^* + \frac{12}{7}\eta$, and so by Theorem 2.1, $g$ is a chaotic function.

The only case now left to consider is where the <u>only</u> fixed point of $f$ is the larger endpoint $b$ of the interval $X$. Here a chaotic function $g$ satisfying $\|f - g\| < \varepsilon$ can be constructed in the same way as above except that the $F^*$ "perturbation" is now the reflection in $(x^*,x^*)$ of that above, as shown in Figure 3.4. Q.E.D.

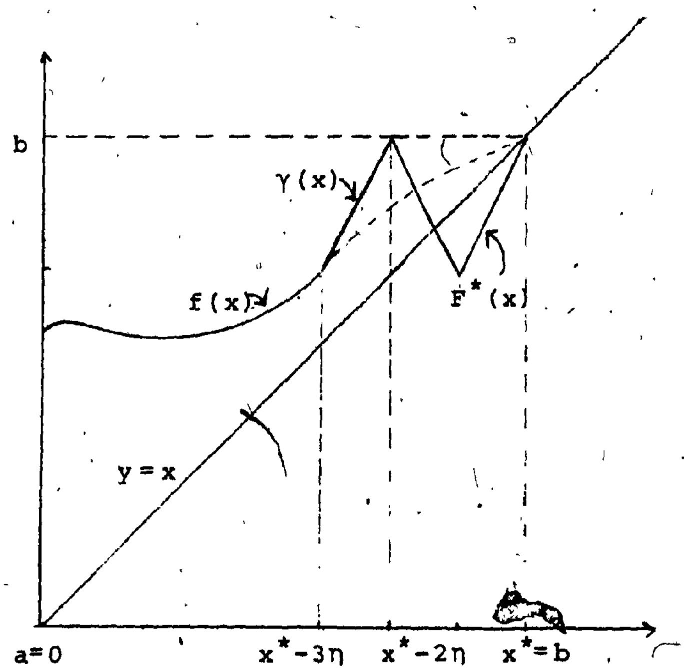



Figure 3.4

In his concluding remarks, Kloeden [18] states that in spite of the fact that the above proof indicates that the non-chaotic difference equations are structurally unstable with respect to the chaotic ones, however, with the sufficiently small bound $\varepsilon$ on the perturbation, "it is unlikely that the behaviour of the perturbed difference equation would differ much from that of the original one outside of the interval $[x^*, x^* + 3\eta]$ in which the perturbations were made. Indeed, the chaotic behaviour of the perturbed difference equation will most likely be restricted entirely to the interval $[x^*, x^* + 2\eta]$ which the perturbation $F^*$ is defined. Iterative sequences entering the interval from outside it will be trapped inside it and then begin to behave chaotically." The reason being that $g([x^*, x^* + 2\eta]) = [x^*, x^* + 2\eta]$. Since the interval is small, and given the limited accuracy of physical measurements, this would probably be mistaken for convergence to the point $x^*$. He also notes that the perturbation $F^*$ is arbitrary, and is not what one would consider as a naturally occurring perturbation.

## 3.3 Stability of the Chaotic Functions

Kloeden [18] has therefore shown that "near" any function $f \in C(X)$ there are chaotic functions $g \in C(X)$ having points of period three. Butler and Pianigiani [17], however, show by means of an example and theorem which we will now discuss, that a small perturbation to a function which does have a point of period three may not result in a function which also has points of period three. This does not indicate, however, that the new function is not chaotic, since they show that every continuous function, sufficiently close to a given function having points of period three, will have points of period five, and, as proven in Theorem 3.2, will be chaotic. Thus the property of being chaotic is stable. We will also be able to conclude that the chaotic functions are not only dense in $C(X)$, but also contain an open dense set.

We will also show, that the special class of chaotic functions, for which we have strict inequality $f^3(a) < a < f(a) < f^2(a)$ forms an open set.

<u>Example 3.2</u>: Period three can be destroyed by arbitrary small perturbations.

<u>Proof</u>: Let $F : [0,1] \to [0,1]$ be defined by

$$(3.19) \qquad F(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Thus the orbit $0, \frac{1}{2}, 1$ has period three. Now consider for any fixed $\varepsilon$, $0 < \varepsilon < \frac{1}{2}$, the function $F$ defined by

$$(3.20) \qquad F_\varepsilon(x) = \begin{cases} \frac{1}{2} + \varepsilon, & 0 \le x \le \varepsilon \\ x + \frac{1}{2}, & \varepsilon \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x \le 1 \end{cases}$$

We have $\| F - F_\varepsilon \| < \varepsilon$, and it is easy to see that $F_\varepsilon$ does not have any point of period three.

Theorem 3.4: Let $T$ be a real-valued continuous function on $I$, and suppose $T$ has a point of period three. Then there exists $\varepsilon > 0$ such that if $F$ is continuous and $\|F - T\| < \varepsilon$ then $F$ has at least one point having period five.

Proof: Let $x_1, x_2, x_3$ be an orbit of period three for $T$, and assume that $x_1 < x_2 < x_3$. (if $x_1 < x_3 < x_2$ proof is similar).

Now we have two possibilities: (a) $T$ has no fixed points in $[x_1, x_2]$, or (b) $T$ has fixed points in $[x_1, x_2]$. Graphs illustrating the two possibilities are given in Figure (3.5) (a) and (b).

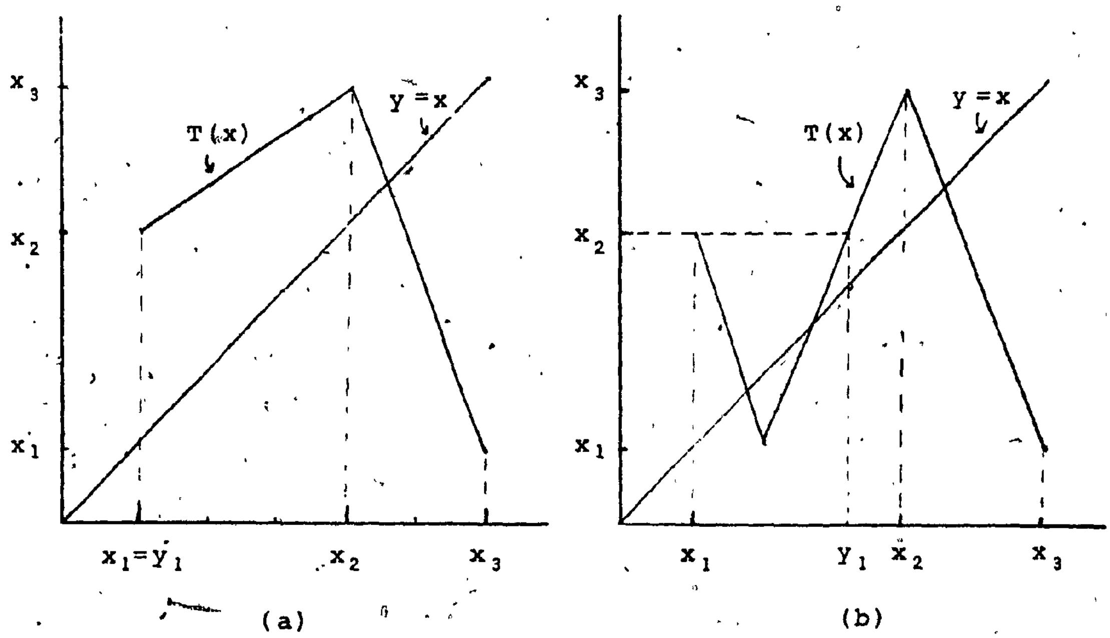



Figure 3.5

In either case, there always exists a point $y_1 \in [x_1, x_2]$ such that $T(y_1) = x_2$ and $T$ has no fixed points in $[y_1, x_2]$ (i.e., the graph of $T(x)$, $y_1 \leq x \leq x_2$, does not intersect the line $y = x$).

Now for the continuous function $T^2$, we have

$$(3.21) \qquad T^2([y_1, x_2]) \supset T([x_2, x_3]) \supset [x_1, x_3]$$

Since $T^2([y_1, x_2]) \supset [y_1, x_2]$, then by Lemma 2.2 there exists $y_2 \in [y_1, x_2]$ such that $T^2(y_2) = x_2$. It is clear that $y_2 \neq y_1$ and $y_2 \neq x_2$, since $T^2(y_2) = x_2$, $T^2(y_1) = x_3$ and $T^2(x_2) = x_1$, and $x_1 < x_2 < x_3$.

Now we have the following:

$$T^3([y_2, x_2]) \supset T([x_1, x_2]) \supset [x_2, x_3]$$

$$T^4([y_2, x_2]) \supset T([x_2, x_3]) \supset [x_1, x_3]$$

$$T^5([y_2, x_2]) \supset T([x_1, x_3]) \supset [x_1, x_3]$$

Hence $T^5$ maps the interval $[y_2, x_2]$ onto a strictly larger interval $[u, v]$ where $u < y_2 < x_2 < v$. By simple continuity arguments, similar to the following Lemma 3.1, there exists $\varepsilon > 0$ such that if $\|T - F\| < \varepsilon$ then $F^5$ maps $[y_2, x_2]$ onto a larger interval than itself. Thus by Lemma 2.2 $F^5$ has at least one fixed point $x_0$. Now since the interval $[y_2, x_2]$ is disjoint from the fixed point set of $T$ (recall how point $y_1$ was obtained), we can also have that $F$ has no fixed points in $[y_2, x_2]$ by taking, if necessary, a smaller value of $\varepsilon$. Therefore, $F$ has at least one point of period five. Q.E.D.

## 3.4 Functions Satisfying $f^3(a) < a < f(a) < f^2(a)$ Form an Open Set

Let $C(I)$ denote the space of continuous functions on the interval $I$ and let it have the sup norm topology.

Lemma 3.1 Let $F: C(I) \to C(I)$, be defined by $F(g) = g \circ g$. Then $F$ is continuous.

Proof We want to show that $F^{-1}(B_\varepsilon(k))$ is open where $k \in C(I)$. Let $h \in F^{-1}(B_\varepsilon(k))$. Then $h \circ h \in B_\varepsilon(k)$. By the compactness of $I$

$$|h\circ h(x) - k(x)| \leq \sigma < \varepsilon$$

Let $\eta = \frac{\varepsilon-\delta}{2}$. There exists $\delta \leq \eta$ such that

$$|x-y| < \delta \text{ implies } |h(x) - h(y)| < \eta,$$

by the uniform continuity of $h$. Consider now $B_\delta(h)$: $\tilde{h} \in B_\delta(h)$ implies

$$\begin{aligned} |h\circ h - \tilde{h}\circ\tilde{h}| &\leq |h\circ h - h\circ\tilde{h}| + |h\circ\tilde{h} - \tilde{h}\circ\tilde{h}| \\ &< \eta + \delta \leq 2\eta = \varepsilon - \sigma. \end{aligned}$$

Therefore,

$$|\tilde{h}\circ\tilde{h} - k| \leq |\tilde{h}\circ\tilde{h} - h\circ h| + |h\circ h - k|$$

i.e., $h \in F^{-1}(B_\varepsilon(k))$ implies that there exists $\delta > 0$ such that $B_\delta(h) \subset F^{-1}(B_\varepsilon(k))$.

Q.E.D.

<u>Lemma 3.2</u> Let $U_\varepsilon(f) = \{g \in C(I) \mid |f^i(x) - g^i(x)| < \varepsilon,$ $i = 1,2,\ldots n\}$. Then $U_\varepsilon(f)$ is open in the sup norm topology.

<u>Proof</u> $F^{-1}(B_\varepsilon(f^i))$ is open. Hence

$$U_\varepsilon(f) = B_\varepsilon(f) \cap F^{-1}(B_\varepsilon(f^2)) \cap \ldots \cap F^{-n+1}(B_\varepsilon(f^n))$$

is open.

<u>Theorem 3.5</u> Let $a \in I$. Then

$$V = \{f \in C(I) \mid f^3(a) < a < f(a) < f^2(a)\}$$

is open in $C(I)$.

<u>Proof</u> If $V = \phi$, there is nothing to prove. Assume $V \neq \phi$ and let $f \in V$. We shall show that $U_\varepsilon(f) \subset V$ for a suitable $\varepsilon$. Let $\varepsilon_1 = \min\{\frac{f(a)-a}{2}, \frac{f^2(a)-f(a)}{2}, \frac{a-f^3(a)}{2}\}$. Choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon_1$, and consider $V_\varepsilon(f)$ (with $n = 3$). Let $g \in V_\varepsilon(f)$.

1. Since $\frac{f(a) - a}{2} \geq \varepsilon_1 > \varepsilon$ we have

$$f(a) > 2\varepsilon + a. \tag{3.27}$$

Also since by (3.24) $|f(a) - g(a)| < \varepsilon$, we have

$$-\varepsilon < f(a) - g(a) < \varepsilon. \tag{3.28}$$

Therefore $g(a) > f(a) - \varepsilon$ from (3.28)

$> 2\varepsilon + a - \varepsilon$ from (3.27)

$= a + \varepsilon > a$

2. From (3.25) we have $-\varepsilon < f^2(a) - g^2(a) < \varepsilon$, thus

$$g^2(a) > f^2(a) - \varepsilon \tag{3.29}$$

But $\dfrac{f^2(a) - f(a)}{2} \geq \varepsilon_1 > \varepsilon$ and so $f^2(a) > 2\varepsilon + f(a)$.
Substituting in (3.29), we obtain $g^2(a) > 2\varepsilon + f(a) - \varepsilon =$ $f(a) + \varepsilon$. But from (3.28) $f(a) + \varepsilon > g(a)$, and so $g^2(a) > g(a)$.

3. From (3.26) we have $-\varepsilon < f^3(a) - g^3(a) < \varepsilon$, and so

$$g^3(a) < f^3(a) + \varepsilon . \tag{3.30}$$

But $\dfrac{a - f^3(a)}{2} \geq \varepsilon_1 > \varepsilon$ and so $f^3(a) < a - 2\varepsilon$. Substituting in (3.30), we obtain $g^3(a) < a - 2\varepsilon + \varepsilon = a - \varepsilon < a$.

Combining the results of 1, 2, and 3 we have

$$g^3(a) < a < g(a) < g^2(a).$$

Thus, $g \in V$. Q.E.D.

Theorems 3.3 and 3.5 show that the chaotic functions contain a dense open set of C(I).

Note that even though the functions satisfying (3.23) form an open set, the functions satisfying $f^3(a) = a < f(a) < f^2(a)$ do not form an open set as seen by Example 3.2, that period three can be destroyed by small perturbations.

# CHAPTER 4

# CHAOS IN N-DIMENSIONS

## 4.1 Introduction

In this chapter, we shall discuss some results pertaining to the existence of chaos in N-dimensions.

Definition 4.1: Let $I$ be a compact subset of $R^N$, $N \geq 1$, and let $f : I \to I$ be a continuous function. Then

$$(4.1) \qquad X_{n+1} = f(X_n), \quad n = 0,1,2,\ldots, \; X_i \in I$$

defines a first order difference equation in $N$ dimensions.

Li and Yorke's period 3 condition for chaos in one dimension does not, in general, hold for $N > 1$ as can be seen from the following example given by Kloeden [19].

Example 4.1: Let $I$ be any closed disk in $R^2$, with center at the origin. Let $f : I \to I$ be defined by

$$(4.2) \qquad f(X) = f((x_1,x_2)) = \left(-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)$$

This mapping is continuous. Using the complex variable $z = x_1 + ix_2$, $f$ can be written as $f(z) = \omega z$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a complex cube root of unity. Thus every point $(x_1,x_2) \in I - \{(0,0)\}$ has period three, and $(0,0)$ has period one. So we see that in general period three does not imply chaos for $N > 1$.

However, it is clear that chaos does exist in N-dimensions by considering the function $F(x_1,x_2,\ldots,x_N) = (f(x_1),0,0,\ldots,0)$ where $f(x)$ is any one-dimensional function exhibiting chaos, but this is a trivial case. What we would like to know is, given an N-dimensional continuous function, $N > 1$, what are the conditions which will ensure the existence of a chaotic set.

In the one-dimensional case, the two most commonly used models to exhibit chaos are the one we used in the discussion of Li and Yorke's paper [14].

$$x_{n+1} = rx_n(1 - x_n) \qquad 0 < r < 4 \tag{2.3}$$

and

$$x_{n+1} = x_n e^{r(1-x_n)} \qquad r > 0 . \tag{4.3}$$

We will now give an example, discussed by Guckenheimer, Oster and Ipaktchi [20], which deals with the per capita birthrates for two age classes. This two-dimensional model exhibits "all" the dynamics of the above one-dimensional models including stable equilibrium, bifurcation of stable cycles, and chaos. This example will be similar to (4.3) since the quadratic case is more difficult to deal with in two-dimensions.

<u>Example 4.2</u>: Let $f(x_1,x_2) = (r(x_1 + x_2)e^{-a(x_1+x_2)},x_1)$. We choose $a = .1$ since this gives population levels for $x_1$ and $x_2$ between 0 and 100. Let $I = (0,100)$ and then

$f : I^2 \to I^2$ .

When $r = 7.5$ and starting with $X_0 = (1,1)$ or any other convenient starting point in $I^2$, after about 50 iterates the population settles to a stable point $X^* = (13.54025, 13.54025)$, i.e., $f(X^*) = X^*$.

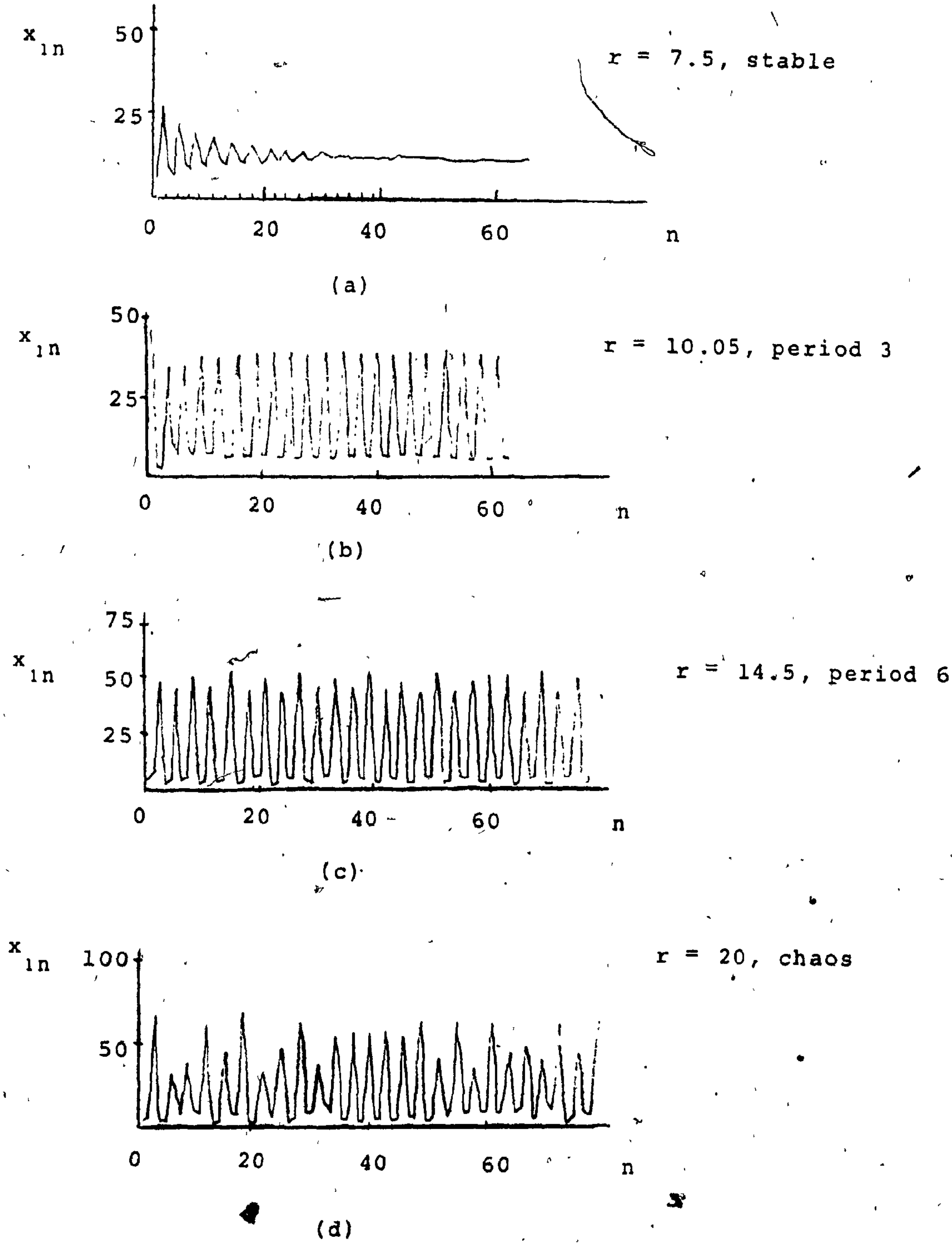
As $r$ is increased to 10.05, the stable equilibrium bifurcates to a cycle of period three, i.e.,

(5.68049,30.30046), (38.30046,5.68049), (5.68049,5.68049).

This differs from (4.3) since there the first bifurcation goes to period two.

At $r = 14$, a cycle of period six appears, and at $r = 16$ a cycle of period twelve appears. As $r$ increases past 16, further bifurcations occur. At $r = 17$, we "seem" to have chaos, with the chaotic behaviour increasing as $r$ increases.

Since the second element of each ordered pair is just the first element of the pre-image, we show the behaviour graphically by plotting the first element of the ordered pair vs. the number of the iterate, i.e., $x_{1n}$ vs. $n$, in Figure 4.1.



$x_{1n}$
50
25
0
20
40
60
n
r = 7.5, stable
(a)
$x_{1n}$
50
25
0
20
40
60
n
r = 10.05, period 3
(b)
$x_{1n}$
75
50
25
0
20
40
60
n
r = 14.5, period 6
(c)
$x_{1n}$
100
50
0
20
40
60
n
r = 20, chaos
(d)


Figure 4.1

## 4.2 Definition of Chaos in N-Dimensions

Definition 4.2: The difference equation (4.1) is said to be chaotic if there exists

(1) A positive integer $n$ such that (4.1) has a point of period $p$ for each $p \geq n$.

(2) A scrambled set of (4.1), i.e., an uncountable set $S$ containing no periodic points of (4.1) such that

(a) $f(S) \subset S$

(b) For every $X_0, Y_0 \in S$, $X_0 \neq Y_0$,

$$\limsup_{k \to \infty} \| f^k(X_0) - f^k(Y_0) \| > 0 .$$

(c) For every $X_0 \in S$ and any periodic point $Q$ of (4.1)

$$\limsup_{k \to \infty} \| f^k(X_0) - f^k(Q) \| > 0$$

(3) An uncountable subset $S_0$ of $S$ such that for every $X_0, Y_0 \in S_0$ $\liminf_{k \to \infty} \| f^k(X_0) - f^k(Y_0) \| = 0$ .

## 4.3 Classes of N-Dimensional Difference Equations Exhibiting the Sharkovsky Ordering and Chaos

I. The following theorem is due to Kloeden [19] which we state without proving. It establishes sufficient conditions for the existence of cycles of all orders.

Theorem 4.1: Let $I$ be a compact subset of $R^N$ of the form $I = \prod_{i=1}^{N} I_i$, where $I_i$ is a compact interval for $i = 1, \ldots, N$. Let $f : I \to I$, $f = (f_1, f_2, \ldots, f_N)$ be a continuous mapping of the form

$$(4.4)\quad f_i(x_1,x_2,\ldots,x_N) = f_i(x_1,x_2,\ldots,x_i) \quad i = 1,2,\ldots,N$$

i.e., the i-th component $f_i$ of $f$ depends only on the first $i$ independent variables $x_1,x_2,\ldots,x_i$. Then for the above class of functions the Sharkovsky ordering (3.1) holds.

Example 4.3: The hypotheses of Theorem 4.1 are satisfied by the "twisted horseshoe" difference equation of [20], $f = (f_1,f_2)$ defined on $I = [0,1]^2$ by

$$(4.5)\qquad f_1(x_1,x_2) = \begin{cases} 2x_1, & 0 \le x_1 \le \frac{1}{2} \\ 2(1-x_1), & \frac{1}{2} \le x_1 \le 1 \end{cases}$$

$$f_2(x_1,x_2) = \frac{x_1}{2} + \frac{x_2}{10} + \frac{1}{4} \qquad (x_1,x_2) \in I$$

Equations (4.5) may be written in more compact form:

$$(4.6)\quad f(x_1,x_2) = \begin{cases} \left(2x_1, \frac{x_1}{2} + \frac{x_2}{10} + \frac{1}{4}\right), & 0 \le x_1 \le \frac{1}{2}, \quad (x_1,x_2) \in I \\ \left(2(1-x_1), \frac{x_1}{2} + \frac{x_2}{10} + \frac{1}{4}\right), & \frac{1}{2} \le x_1 \le 1, \\ & (x_1,x_2) \in I \end{cases}$$

Note: The hypotheses of Theorem 4.1 are met by all one-dimensional difference equations of a continuous function from a compact interval into itself and therefore we have the Sharkovsky ordering in one dimension.

II. The following theorem is due to Diamond [21] which we also state without proving.

Definition 4.3: Let $A$ be a subset of $R^N$. Let $f : A \to R^N$. A subset $P$ of $A$ is said to be k-periodic if $f^k(P) = P$ and $f^i(P) \cap f^j(P) = \phi$ for $1 \leq i < j < k$.

Theorem 4.2: Let $A$ be a set in $R^N$ and $f : A \to R^N$ be continuous. If there is a non-empty compact set $X$ in $A$ satisfying

(4.7) $$X \cup f(X) \subset f^2(X) \subset A$$

and

(4.8) $$X \cap f(X) = \phi$$

then T1: for every $k = 1,2,\ldots.$ there is in $A$ a k-periodic set.

T2: there is an uncountable set $S$ in $A$ which contains no periodic set, and for which

(i) $f(S) \subset S$

(ii) for all $p, q \in S$, $p \neq q$,

$$\limsup_{k \to \infty} \| f^k(p) - f^k(q) \| > 0$$

(iii) for all $p \in S$ and periodic set $P \subset A$ and $q \in P$

$$\limsup_{k \to \infty} \| f^k(p) - f^k(q) \| > 0 .$$

Note: 1. There is a remarkable resemblance of Theorem 4.2 to Theorem 2.1.

2. The existence of k-periodic sets is a weaker result than the existence of points of period $k$.

3. Conditions (4.7) and (4.8) are compatible with the conditions of Theorem 2.1 for one dimension if

there exists a point $c$ for which $f^3(c) < c < f(c) < f^2(c)$, for the following reason:

By Lemma 2.0 there is an interval $J \subset [c, f(c)]$ such that $f(J) = [f(c), f^2(c)]$. Now we have two possibilities: (a) $f(c) \notin J$ or (b) $f(c) \in J$.

If $f(c) \notin J$, let $X = J$. Therefore $X \cap f(X) = \phi$ and then $f(X) = [f(c), f^2(c)]$ and $f^2(X) \supset [f^3(c), f^2(c)] \supset X \cup f(X)$.

If $f(c) \in J = [a, f(c)]$ where $f(a) = f(c)$, then let $X = [a, f(c) - \varepsilon]$ where $\varepsilon > 0$ is sufficiently small. Then $f(X) = [f(c), c_1]$, where $c_1$ is close to but less than $f^2(c)$ and $X \cap f(X) = \phi$.

Also $f^2(X) \supset [f(c_1), f^2(c)] \supset X \cup f(X)$ since $f(c_1)$ is close to $f^3(c)$ and since $f^3(c) < c$, so by choosing $\varepsilon$ small enough we can have $f^3(c_1) < a$.

Thus, for one-dimension, if $f^3(c) < c < f(c) < f^2(c)$ then the hypotheses of Theorem 4.2 also hold.

III. The previous two cases we have discussed are in reality unsatifactory, since case I only shows that for those types of functions, the Sharkovsky ordering holds, but does not in general show which difference equations are chaotic. Also in case II, in spite of the fact that there is an overlap between the conditions of Diamond and those of Li and Yorke in the one dimensional case, Diamond's Theorem 4.2 applies only to sets and not to specific points.

The following which is due to Marotto [22] gives sufficient conditions for chaos in N-dimensions. He shows that if there exists a cycle which begins sufficiently close to an unstable fixed point of the function and is "repelled" from this point as the number of iterates increases, but then "snaps-back" to the fixed point, that this is sufficient to imply chaos in (4.1). Before we get to his main theorem, we need the following definitions and lemmas.

<u>Definition 4.4</u>: Let $F$ be an N-dimensional differentiable function. Let $DF(X)$ denote the Jacobian matrix of $F$ evaluated at the point $X \in R^N$. Let $B_r(X)$ denote the closed ball in $R^N$ of radius $r$ and center at point $X$. Let $B_r^0(X)$ denote its interior. Let $\|X\|$ be the usual Euclidean norm of $X$ in $R^N$.

<u>Remark 4.1</u>: If $V \subset R^N$ is a subset of the domain of $F$ which satisfies $F(V) \subset V$ then it is clear that choosing a point $X_0 \in V$ will give a trajectory $\{X_k\}_{k=0}^{\infty}$ of (4.1). We will now extend this trajectory uniquely for $k$ negative.

<u>Remark 4.2</u>: Even if $F$ is not 1-1 in its domain, but if there exists a set $U \subset R^N$ for which

(4.9) $\quad F$ is 1-1 in $U$ and $U \subset F(U)$,

then for each $X \in U$ we have $X \in F(U)$ and therefore we have a unique point $Y \in U$ for which $F(Y) = X$.

<u>Definition 4.5</u>: For $U$ and $F$ satisfying (4.9), the inverse of $F$ in $U$, denoted by $F^{-1} = F_u^{-1}$ is the function assigning to each $X \in U$ the unique $Y \in U$ for which

$F(Y) = X$ .

<u>Remark 4.3</u>: If we choose $X_0 \in U$ then we can obtain $\{X_k\}$ for all negative integers $k$ by

$$(4.10) \qquad X_{k-1} = F^{-1}(X_k) \qquad k = 0,-1,-2,\ldots$$

<u>Definition 4.6</u>: Let $F$ be differentiable in $B_r(Z)$ . Then the point $Z \in R^N$ is an expanding fixed point of $F$ in $B_r(Z)$ , if $F(Z) = Z$ , and all the eigenvalues of $DF(X)$ exceed one in norm for all $X \in B_r(Z)$ . If $Z$ is not a fixed point of $F$ , then $Z$ will only be called an expanding point of $F$ .

<u>Remark 4.4</u>: If $Z$ is an expanding fixed point in $B_r(Z)$ , then there exists $\lambda > 1$ for which

$$(4.11) \qquad \|F(X) - F(Y)\| > \lambda\|X - Y\| \quad \text{for all} \quad X \neq Y \, , \quad X , Y \in B_r(Z) .$$

This implies that $F$ is one-to-one in $B_r(Z)$ . Also by letting $Y = Z$ as a special case of (4.11), we obtain that $\|F(X) - F(Z)\| = \|F(X) - Z\| > \|X - Z\|$ for all $X \in B_r(Z)$ . Since $F$ is a homeomorphism in $B_r(Z)$ , it must be that $F(B_r(Z)) \supset B_r(Z)$ . Therefore $F$ satisfies (4.9) with $U = B_r(Z)$ , and $F^{-1}$ exists in $B_r(Z)$ . However (4.11) implies that

$$(4.12) \qquad \|F^{-k}(X) - Z\| < \lambda^{-k}\|X - Z\|$$

so

(4.13) $F^{-k}(X) \to Z$ as $k \to \infty$ for all $X \in B_r(Z)$.

We therefore see that $F^{-1}$ contracts $B_r(Z)$, and $F$ expands $B_r(Z)$.

Remark 4.5: Let $Z$ be an expanding fixed point of $F$ in $B_r(Z)$. If $F$ is not 1-1 in $R^N$, then it is possible that there exists a point $X_0 \in B_r(Z)$, $X_0 \neq Z$, for which $F^M(X_0) = Z$ for some positive integer $M$. Since $F$ is 1-1 on $B_r(Z)$ and $F(Z) = Z$, $M$ must be greater than 1.

Definition 4.7: $B_r(Z)$ is called an expanding neighbourhood of $Z$ if every point of $B_r(Z)$ is an expanding point of $F$. Note that if $Z$ is an expanding point, such an $r$ always exists.

Definition 4.8: Let $Z$ be an expanding fixed point of $F$ on $B_r(Z)$ for some $r > 0$. Then, $Z$ is said to be a snap-back repeller of $F$ if there exists a point $X_0 \in B_r(Z)$, $X_0 \neq Z$, such that $F^M(X_0) = Z$, and the determinant of $DF^M(X_0) \neq 0$ for some positive integer $M$. The purpose of having $|DF^M(X_0)| \neq 0$ is to ensure that the inverse of $F^M$ exists in some neighbourhood of $Z_0$.

Brouwer Fixed Point Theorem: Let $S$ be any closed ball in $R^N$. Let $K$ be a continuous mapping of $S$ into itself. Then $K$ has at least one fixed point, i.e., there exists

$X \in S$ such that $K(X) = X$.

Proof: See [24]. Note this theorem is similar to Lemma 2.2.

Lemma 4.1: Let $Z$ be a snap-back repeller of $F$. Then for some $s > 0$ there exists $Y_0 \in B_s^0(Z)$ and an integer $L$, such that $F^k(Y_0) \notin B_s(Z)$ for $1 \leq k < L$, $F^L(Y_0) = Z$, $|DF^L(Y_0)| \neq 0$, and $Z$ is expanding on $B_s(Z)$.

Proof: Since $Z$ is a snap-back repeller, for some $r > 0$, there exists $X_0 \in B_r(Z)$, $X_0 \neq Z$, such that $F^M(X_0) = Z$, and $|DF^M(X_0)| \neq 0$. Let $X_k = F^k(X_0)$, $k \geq 1$. Now since $0 \neq |DF^M(X_0)| = |DF^{M-k}(X_k)| \cdot |DF^k(X_0)|$, we have

$$|DF^k(X_0)| \neq 0 \quad 1 \leq k < M \tag{4.14}$$

and

$$|DF^{M-k}(X_k)| \neq 0 \quad 1 \leq k < M \tag{4.15}$$

Now since $X_0 \neq Z$ and $X_M = Z$, we can assume without loss of generality that $M$ is minimal, i.e.,

$$X_{M-1} \neq Z. \tag{4.16}$$

For this minimal $M$, we still have $|DF^M(X_0)| \neq 0$ by (4.14).

Now (1) $Z = F(X_{M-1})$ by the definition of $M$,

(2) $Z = F(Z)$ since $Z$ is a fixed point of $F$

(3) $Z \neq X_{M-1}$ by (4.16)

(4) $F$ is 1-1 on $B_r(Z)$ by Remark 4.4.

Therefore $X_{M-1} \notin B_r(Z)$ .

Also, since $X_0 \in B_r(Z)$ there must be an integer $T$ such that

$$(4.17) \qquad X_k \in B_r(Z) , \quad 0 \leq k \leq T < M$$

and

$$(4.18) \qquad X_{T+k} \notin B_r(Z) , \quad 1 \leq k < M - T .$$

i.e., $X_T$ is the last iterate of $X_0$ lying in $B_r(Z)$ before $Z$ is hit. Note that (4.15) implies that

$$(4.19) \qquad |DF^{M-T}(X_T)| \neq 0 .$$

Now let $Y_0 = X_T \in B_r(Z)$ and $L = M - T$ . There are two possibilities: (a) $X_T \in B_r^0(Z)$ or (b) $\|X_T\| = r$ .

(a) If $X_T \in B_r^0(Z)$ , let $s = r$ and the lemma is proved.

(b) Let $\|X_T\| = r$ . Recall $B_r(Z)$ is an expanding neighbourhood of $Z$ : then by the continuity of $DF$ , there exists $\varepsilon_1 > 0$ such that $Z$ is expanding on $B_\omega(Z)$ for all $\omega$ in $r < \omega < r + \varepsilon_1$ . Choose $\varepsilon_2$ small enough so that $X_{T+k} \notin B_\omega(Z)$ for $1 \leq k < L$ and $r < \omega < r + \varepsilon_2$ . This can be done, since, by (4.18), $X_{T+k} \notin B_r(Z)$ , $1 \leq k < M - T$ ; therefore let $\varepsilon_2 = \inf\limits_{1 \leq k < M - T} [\|X_{T-k}\| - r]$ .

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ . Thus, for any $\omega$ satisfying $r < \omega < r + \varepsilon$ , we have $Y_0 = X_T \in B_\omega^0(Z)$ , $F^k(Y_0) = X_{T+k} \notin$

$B_\omega(Z)$ for $1 \leq k < L$. Letting $s = \omega$, the lemma is proved.

Q.E.D.

<u>Lemma 4.2</u>: Let $H : R^N \to R^N$ be continuous. Let $\{C_k\}_{k=0}^{\infty}$ be a sequence of compact sets with $C_k \subset R^N$. If $H(C_k) \supset C_{k+1}$ for all $k \geq 0$, then there exists a non-empty compact set $C \subset C_0$ such that

(4.20) $\quad H^k(X) \in C_k$ for all $X \in C$ and $k \geq 0$

(Note: For $N = 1$, Lemma 4.2 reduces to Lemma 2.1.)

<u>Proof</u>: (Diamond [21]) Let $G_i$ be the restriction of $H$ to $C_i$. Define $Q_1 = G_0^{-1}(C_1)$. Clearly $Q_1$ is compact and contained in $C_0$. Construct inductively a sequence of compact sets, $i = 1,2,\ldots$ by

$$Q_{i+1} = G_0^{-1} G_1^{-1} \cdots \cdots G_i^{-1}(C_{i+1})$$

Then $Q_{i+1} \subset Q_i$ and, by the Cantor intersection theorem, the intersection $\mathring{Q}$ of all the $Q_i$ is non-empty and possesses the desired property because of its manner of construction.

Q.E.D.

<u>Theorem 4.3</u>: Let $F : R^N \to R^N$ be differentiable. If $F$ possesses a snap-back repeller, then the difference equation (4.1) is chaotic.

<u>Proof</u>: Let $Z$ be the snap-back repeller of $F$ and let

$X_0 \in B_r(Z)$, $X_0 \neq Z$, $F^M(X_0) = Z$, and $|DF^M(x_0)| \neq 0$, where $Z$ is expanding in $B_r(Z)$. We will now show that the 3 conditions of Definition 4.2 characterizing chaos hold.

(1) We can assume

$$X_0 \in B_r^0(Z) \tag{4.21a}$$

and

$$F^k(X_0) \notin B_r(Z) \quad 1 \leq k < M. \tag{4.21b}$$

Otherwise replace $X_0$, $r$ and $M$ by $Y_0$, $s$ and $L$ respectively, given by Lemma 4.1, to obtain (4.21), and the following analysis can be done with the new variables.

Since $F^M(X_0) = Z$ and $|DF^M(X_0)| \neq 0$, Remark 4.4 implies that for some $\varepsilon$ satisfying $0 < \varepsilon < r$, there exists a continuous 1-1 function $G$, defined on $B_\varepsilon(Z)$ with $G(Z) = X_0$ and

$$G^{-1}(X) = F^M(X) \quad \text{for all} \quad X \in G(B_\varepsilon(Z)), \tag{4.22}$$

i.e., $G$ is the inverse of $F^M$. Let $Q$ be the compact set defined by $Q = G(B_\varepsilon(Z))$.

Now $X_0 \in Q$ since $Z \in B_\varepsilon(Z)$ and $G(Z) = X_0$. Also since $X_0 \in B_r^0(Z)$ by (4.21a), and $G$ is continuous and 1-1 near $Z$, we can choose $\varepsilon$ sufficiently small so that $Q \subset B_r(Z)$.

Note that

(4.23) $\Rightarrow \quad F^M(Q) = B_\varepsilon(Z)$ .

Now, by (4.21b),

(4.24) $\quad F^m(Q) \subset R^N - B_r(Z) \; , \quad 1 \leq m < M$

Figure 4.2 gives a clearer picture of what is happening.

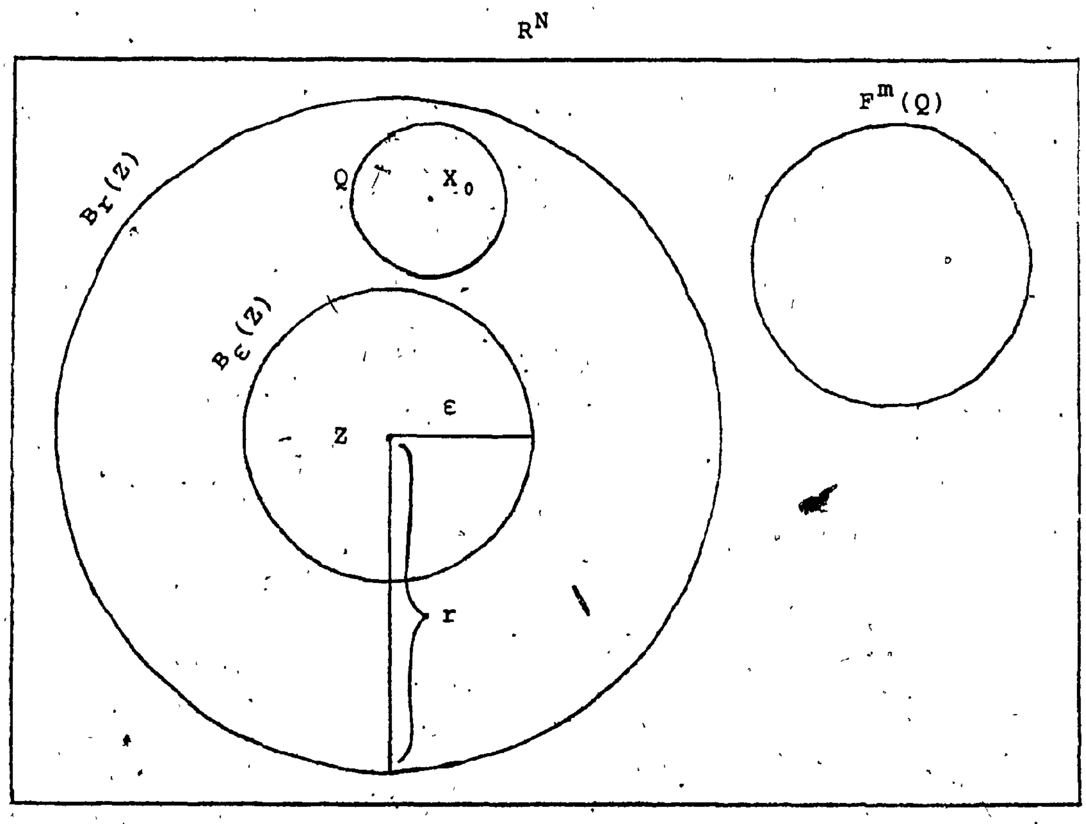



Figure 4.2

Also, since Z is expanding on $B_r(Z)$ , Remark 4.4 implies that $F^{-1}$ exists on $B_r(Z)$ ; thus it follows from $Q \subset B_r(Z)$ that

$$F^{-m}(Q) \subset B_r(Z) \quad \text{for all} \quad m \geq 0 . \tag{4.25}$$

In addition, by Remark 4.4, for any $X \in Q$, $F^{-k}(X) \to Z$ as $k \to \infty$; therefore there exists an integer $J = J(X) \geq 0$ such that $F^{-J}(X) \in B^0_\varepsilon(Z)$, by choosing $J$ large enough. Thus, by continuity, we have a $\delta = \delta(X) > 0$, such that $F^{-J}(B_\delta(X)) \subset B^0_\varepsilon(Z)$. Hence, the set $D = \{B^0_\delta(X) :$ for every $X \in Q\}$ is an open cover of the compact set $A^m$, and it therefore contains a finite subcover $D_0$ of $Q$, where

$$D_0 = \{B_\delta(X_i) : X_i \in Q, \; i = 1,2.,,,.L\}$$

Let $T = \max\limits_{1 \leq i < L} [J(X_i)]$. Therefore,

$$F^{-T}(X) \in B_\varepsilon(Z) \quad \text{for all} \quad X \in Q .$$

Since $\varepsilon > r$, $Z$ is also expanding in $B_\varepsilon(Z)$, and we have

$$F^{-k}(Q) \subset B_\varepsilon(Z) \quad \text{for all} \quad k \geq T . \tag{4.26}$$

Now, for every $k \geq T$, consider the function $F^{-k} \circ G$ defined for all $X \in B_\varepsilon(Z)$. Since $G$ is continuous and 1-1 on $B_\varepsilon(Z)$ and $F^{-k}$ is continuous and 1-1 on $Q = G(B_\varepsilon(Z))$, $F^{-k} \circ G$ is continuous and 1-1 on $B_\varepsilon(Z)$. Thus, from (4.26)

$$F^{-k} \circ G(B_\varepsilon(Z)) \subset B_\varepsilon(Z),$$

and by the Brouwer Fixed Point Theorem $F^{-k} \circ G$ must have a fixed point $Y_k \in B_\varepsilon(Z)$, i.e.,

$$F^{-k} \circ G(Y_k) = Y_k \quad \text{for all } k \geq T.$$

Note, therefore, that

$$(4.27) \qquad F^k(Y_k) = (F^k \circ F^{-k} \circ G)(Y_k) = G(Y_k)$$

Therefore, $F^{M+k}(Y_k) = (F^M \circ F^k)(Y_k)$

$$= (F^M \circ G)(Y_k) \quad \text{by (4.27)}$$

$$= (G^{-1} \circ G)(Y_k) \quad \text{by (4.22)}$$

$$= Y_k$$

Thus, $Y_k$ is a fixed point of $F^{M+k}$. We will now show that $Y_k$ cannot have period less than $M + k$.

By (4.27) we have $F^k(Y_k) = G(Y_k)$, and since $Y_k \in B_\varepsilon(Z)$, and $G(B_\varepsilon(Z)) = Q$ by construction, it follows that

$$(4.28) \qquad F^k(Y_k) \in Q \quad \text{for all } k \geq T.$$

Applying $F^{-k}$ to both sides of (4.28) yields

$$Y_k \in F^{-k}(Q)$$

Now by (4.25), $F^{-k}(Q) \subset B_r(Z)$ for all $k \geq 0$.

Therefore,

$$F^0(Y_k) = Y_k \in B_\varepsilon(Z) \subset B_r(Z)$$

$$F^1(Y_k) \in F^{-k+1}(Q) \subset B_r(Z)$$

$$F^2(Y_k) \in F^{-k+2}(Q) \subset B_r(Z)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F^k(Y_k) \in Q \subset B_r(Z).$$

i.e., $F^n(Y_k) \in B_r(Z)$ for $0 \leq n \leq k$.

Also, by (4.28), we have $F^k(Y_k) \in Q$ for all $k \geq T$.

Therefore, from (4.24) we obtain

$$F^{k+1}(Y_k) \notin B_r(Z)$$

$$F^{k+2}(Y_k) \notin B_r(Z)$$

$$\cdots\cdots\cdots\cdots$$

$$F^{k+M-1}(Y_k) \notin B_r(Z)$$

i.e., $F^n(Y_k) \notin B_r(Z)$ for $k + 1 \leq n < M + k$.

But since $Y_k$ is a fixed point of $F^{M+k}$, we have $F^{k+M}(Y_k) = Y_k$.

Therefore, $Y_k$ cannot have period $s$, where $k + 1 \leq s < M + k$, since $F^s(Y_k) \notin B_r(Z)$ and $Y_k \in B_r(Z)$. Also,

$Y_k$ cannot have period $q$, where $0 < q \leq k$, since then $F^{q-1}(Y_k) = F^{M+k-1}(Y_k)$. But $F^{q-1}(Y_k) \in B_r(Z)$ whereas $F^{M+k-1}(Y_k) \notin B_r(Z)$, again a contradiction. So $Y_k$ cannot have period less than $M + k$. Therefore $Y_k$ has period $M + k$.

Letting $N = M + T$, $p = M + k$ for all $k \geq T$ establishes part (1) of the definition of chaos (Definition 4.2).

(2) Let the integers $M$, $T$, and $N$ be as in (1) and let $U$ and $V$ be two compact sets defined by

$$U = F^{M-1}(Q) \quad \text{and} \quad V = B_\varepsilon(Z)$$

<u>Claim 1:</u> $U \cap V = \phi$

<u>Proof:</u> $U = F^{M-1}(Q) \subset R^N - B_r(Z)$ by (4.24) Therefore, $U \subset R^N - B_\varepsilon(Z)$ since $\varepsilon > r$. But $V = B_\varepsilon(Z)$, and so $U \cap V = \phi$.

<u>Claim 2:</u> $V \subset F^N(U)$

<u>Proof:</u> Since $U = F^{M-1}(Q)$, $F(U) = F^M(Q)$. But, by (4.23), $F^M(Q) = B_\varepsilon(Z)$, and so $F(U) = B_\varepsilon(Z)$. Now $F^N(U) = F^{N-1}(F(U)) = F^{N-1}(B_\varepsilon(Z))$. But $Z$ is expanding in $B_\varepsilon(Z)$, so $F^{N-1}(B_\varepsilon(Z)) \supset B_\varepsilon(Z)$. Therefore, $F^N(U) \supset B_\varepsilon(Z) = V$.

<u>Claim 3:</u> $U \subset F^N(V)$ and $V \subset F^N(V)$.

<u>Proof:</u> $F^N(V) = F^N(B_\varepsilon(Z)) \supset B_\varepsilon(Z)$ since $Z$ is expanding on

$B_\varepsilon(Z)$. Therefore, $F^N(V) \supset V$.

Also let $k = T + 1$ in (4.26). Therefore, $F^{-T-1}(Q) \subset B_\varepsilon(Z)$, and so applying $F^N$ to both sides,

$$F^{N-T-1}(Q) \subset F^N(B_\varepsilon(Z)) = F^N(V) .$$

Now, as defined at the end of part (1) of the proof of Theorem 4.3, $N = M + T$, and so

$$U = F^{M-1}(Q) = F^{N-T-1}(Q) .$$

Therefore, $U \subset F^N(V)$, proving Claim 3.

Now let $H$ be the function defined by

$$H(X) = F^N(X) \quad \text{for all} \quad X \in B_r(Z) .$$

Therefore, by using Claims 1,2 and 3, we have

(4.29) $$\inf\{\|X - Y\| : X \in U , Y \in V\} > 0$$

and

(4.30 $$V \subset H(U) \quad \text{and} \quad U , V \subset H(V) .$$

The remainder of the proof of (2) is essentially identical to the corresponding part of the proof of Theorem 2.1, and we outline it as follows:

Let $A$ be the set of sequences $E = \{E_n\}_{n=1}^{\infty}$, where $E_n$ is either $U$ or $V$, and if $E_n = U$ then

$E_{n+1} = E_{n+2} = V$.

Let $R(E,n)$ be the number of $E_i$'s which equal $U$ for $1 \leq i \leq n$.

For each $\omega \in (0,1)$, let $E^{\omega} = \{E_n^{\omega}\}_{n=1}^{\infty}$ be a sequence in A satisfying $\lim\limits_{n \to \infty} \frac{R(E^{\omega},n^2)}{n} = \omega$.

Let B be defined by $B = \{E^{\omega} : \omega \in (0,1\} \subset A$; then B is uncountable. Also from (4.30), $H(E_n^{\omega}) \supset E_{n+1}^{\omega}$, and so by Lemma 4.2, for each $E^{\omega} \in B$, there is a point $X_{\omega} \in U \cup V$ with $H^n(X_{\omega}) \in E_n^{\omega}$ for all $n \geq 1$. Let $S_H = \{H^n(X_n) : n \geq 0$ and $E^{\omega} \in B\}$, then $H(S_H) \subset S_H$, $S_H$ contains no periodic points of H, and there exists an infinite number of n's such that $H^n(X) \in U$ and $H^n(Y) \in V$ for any $X, Y \in S_H$, with $X \neq Y$.

From the last statement and (4.29), it follows that for every $X, Y \in S_H$, $X \neq Y$, $\limsup\limits_{n \to \infty} |H^n(X) - H^n(Y)| = L_1 > 0$.

Therefore, letting $S = \{F^n(X) : X \in S_H$ and $n \geq 0\}$ and recalling that $H(X) = F^N(X)$, we see that $F(S) \subset S$, S contains no periodic points of F, and for every $X, Y, \in S$, $X \neq Y$

$$\limsup_{n \to \infty} |F^n(X) - F^n(Y)| \geq L_1 > 0$$

We have thus shown (2a) and (2b) of Definition 4.2. Part (2c) can be proven similarly.

(3): First we note that since $Z$ is expanding on $B_\varepsilon(Z)$, if we define

$$D_n = H^{-n}(B_\varepsilon(Z)) \quad \text{for all} \quad n \geq 0 ,$$

then given $\delta > 0$ there exists an integer $J = J(\delta)$ such that $\|X - Z\| < \delta$ for all $X \in D_n$ and $n > J$. Now the proof of (3) again parallels the corresponding part in Theorem 2.1. For the sake of completeness, we present it here.

For any sequence $E^\omega = \{E_n^\omega\}_{n=1}^\infty \in A$, we further restrict the $E_n^\omega$ in the following way: If $E_n^\omega = U$, then $n = m^2$ for some integer $m$. Also, if $E_n^\omega = U$ for both $n = m^2$ and $n = (m+1)^2$, then $E_n^\omega = D_{2m-k}$ for $n = m^2 + k$ where $k = 1,2,\ldots,2m$. For the remaining $n$'s we shall assume $E_n^\omega = V$.

It can easily be checked that the sequences still satisfy $H(E_n^\omega) \supset E_{n+1}^\omega$, and thus by Lemma 4.2 there exists a point $X_\omega$ satisfying $H^n(X_\omega) \in E_n^\omega$ for all $n \geq 0$.

Now, defining $S_0 = \left\{X_\omega : \omega \in \left[\frac{3}{4},1\right)\right\}$, then $S_0$ is uncountable, $S_0 \subset S_H \subset S$, and for any $s, t \in \left[\frac{3}{4},1\right)$ there exists infinitely many $m$'s such that

$$H^n(X_s) \in E_n^s = D_{2m-1} \quad \text{and} \quad H^n(X_t) \in E_n^t = D_{2m-1},$$

where $n = m^2 + 1$.

But as noted above, in the beginning of part (3) of this proof, given $\delta > 0$, $\|X - Z\| < \frac{\delta}{2}$ for all $X \in D_{2m-1}$ and $m$ sufficiently large. Thus, for all $\delta > 0$ there exists an integer $m$ such that

$$\|H^n(X_s) - H^n(X_t)\| < \delta \quad \text{where} \quad n = m^2 + 1 .$$

Since $\delta$ is arbitrary, we have

$$L_2 = \liminf_{n \to \infty} \|H^n(X_s) - H^n(X_t)\| = 0 .$$

Therefore, for any $X, Y \in S_0$

$$\liminf_{n \to \infty} \|F^n(X) - F^n(Y)\| \leq L_2 = 0 .$$

which shows (3) of Definition 4.2, and completes the proof of the Theorem. Q.E.D.

<u>Note</u>: This class of functions which Marotto shows to be chaotic are continuously differentiable, rather than just continuous as has been the case till now.

IV Generalization by P.E. Kloeden

In this section we present a generalization of the foregoing result by Marotto. The theorem, due to Kloeden [25] gives sufficient conditions for a continuous function to have chaotic behaviour. Not only is this generalization applicable to difference equations with snap-back repellers, but also to those with saddle-points.

Before we state and prove Kloeden's theorem, we need the following definitions:

Definition 4.9: $F$ is said to be expanding on a set $A \subset R^N$ if there exists $\lambda > 1$ such that $\|F(X) - F(Y)\| > \lambda \|X - Y\|$ for all $X, Y \in A$. Note that the above condition implies that $F$ is 1-1 on $A$.

Definition 4.10: An $\ell$-ball is a closed ball of finite radius in $R^\ell$. An $\ell$-ball with radius $r$ and center at $Z$ will be denoted by $B_r^\ell(Z)$. An open $\ell$-ball will be denoted by $B_r^{0,\ell}(Z)$.

Theorem 4.4: Let $F : R^N \to R^N$ be a continuous mapping. Let there exist non-empty compact sets $A$ and $B$, and integers $1 \le \ell \le N$, and $n_1, n_2 \ge 1$ such that:

(1) $A$ is homeomorphic to an $\ell$-ball.

(2) $A \subset F(A)$.

(3) $F$ is expanding on $A$.

(4) $B \subset A$.

(5) $F^{n_1}(B) \cap A = \phi$.

(6) $A \subset F^{n_1+n_2}(B)$.

(7) $F^{n_1+n_2}$ is 1-1 on $B$.

Then the difference equation (4.1) is chaotic.

Proof: (1) Since $F$ is continuous and by (6) we have $A \subset F^{n_1+n_2}(B)$, therefore there exists a non-empty compact set $C \subset B$ such that

$$A = F^{n_1+n_2}(C) \tag{4.31}$$

Since by (7), $F^{n_1+n_2}$ is 1-1 on $B$, so $F^{n_1+n_2}$ has a continuous inverse function $g : A \to C$ such that

$$g(F^{n_1+n_2}(X)) = X \quad \text{for all} \quad X \in C. \tag{4.32}$$

Since, by (5) we have $F^{n_1}(B) \cap A = \phi$,

$$F^{n_1}(C) \cap A = \phi. \tag{4.33}$$

From (3) since $F$ is expanding on $A$, we have $F$ is 1-1 on $A$, so $F$ has a continuous inverse $F_A^{-1} : F(A) \to A$. Now since $C \subset B$, and $B \subset A$ by (4), and $A \subset F(A)$ by (2), we have $C \subset F(A)$ and therefore

$$F_A^{-k}(C) \subset A \quad \text{for all} \quad k \geq 0. \tag{4.34}$$

Figure 4.3 shows a Venn Diagram of the sets.

For each $k \geq 0$, the mapping $F_A^{-k} \circ g : A \to A$ is a continuous mapping from a homeomorph of an $\ell$-ball into itself, so by the Brouwer Fixed Point Theorem there exists a point $Y_k \in A$ such that

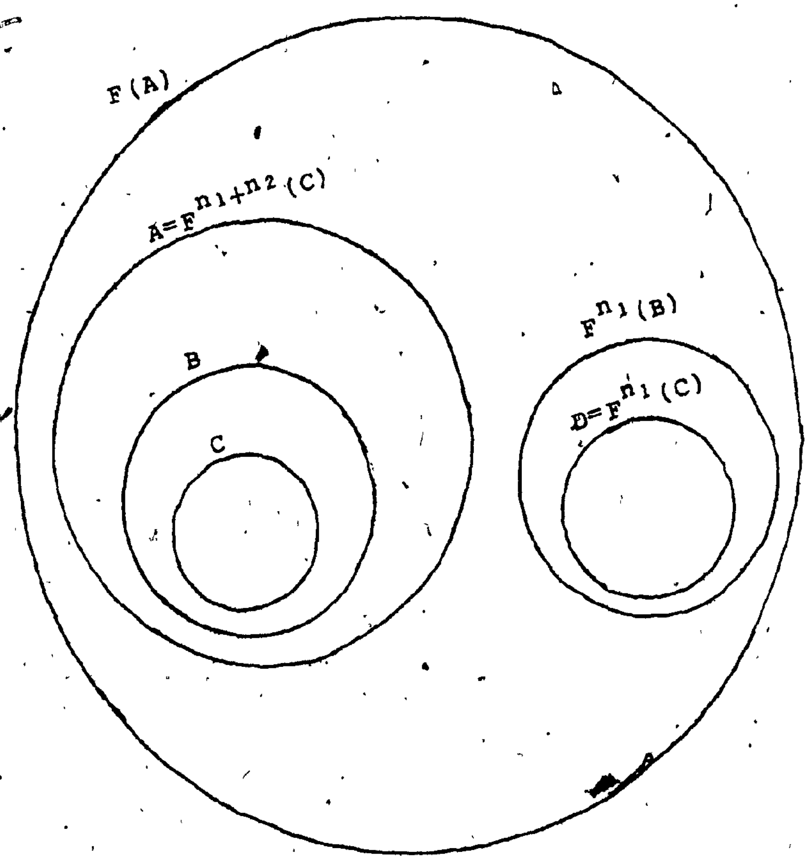



Figure 4.3

(4.35) $$F_A^{-k}(g(Y_k)) = Y_k .$$

Applying $F_A^k$ to both sides of (4.35), we obtain

(4.36) $$g(Y_k) = F_a^k(Y_k) .$$

Now since $g : A \to C$,

(4.37) $$g(Y_k) \in C$$

So

$$F^{n_1+k}(Y_k) = F^{n_1+k}\left(F_A^{-k}(g(Y_k))\right) \quad \text{by (4.35)}$$

$$= F^{n_1}(g(Y_k))$$

$$\in F^{n_1}(C) \quad \text{by (4.37)}$$

Now since we have $F^{n_1}(C) \cap A = \phi$ by (4.35),

(4.38) $$F^{n_1+k}(Y_k) \notin A$$

Also,

$$F^{n_1+n_2+k}(Y_k) = F^{n_1+n_2}(F^k(Y_k))$$

$$= F^{n_1+n_2}(g(Y_k)) \quad \text{by (4.36)}$$

$$= Y_k ,$$

since $F^{n_1+n_2}$ and $g$ are inverse functions. So we have

(4.39) $$F^{n_1+n_2+k}(Y_k) = Y_k .$$

We shall now show that if $k \geq n_1 + n_2$, then $Y_k$ has period $n_1 + n_2 + k$.

Let $Y_k$ have period $p$. Assume first that $i \leq p \leq k$. Now (4.35) and (4.37) imply that

(4.40) $$Y_k \in F_n^{-k}(C) .$$

Applying $F^j$ to both sides of (4.39) gives

$F^j(Y_k) \in F^{j-k}(C) \subset A$ where $1 \leq j \leq k$, since $F_A^{-m}(C) \subset A$ for all $m \geq 0$ by (4.34). Therefore the whole cycle $Y_k$, $F(Y_k), \ldots\ldots, F^{p-1}(Y_k), F^p(Y_k) = Y_k$ belongs to $A$, which contradicts (4.38), that $F^{n_1+k}(Y_k) \notin A$.

$p$ also cannot be between $k$ and $n_1 + n_2 + k$, since by (4.39), $F^{n_1+n_2+k}(Y_k) = Y_k$ and so $p$ would have to divide $n_1 + n_2 + k$ exactly, which is impossible when $k \geq n_1 + n_2$.

Hence (4.1) has a periodic point of period $p$ for each $p \geq N = 2(n_1 + n_2)$, establishing part (1) of the Definition 4.2 of chaos.

(2) Let

$$D = F^{n_1}(C) \tag{4.41}$$

and

$$h = F^N. \tag{4.42}$$

Then $A \cap D = \phi$, by (4.33).

$$\begin{aligned} \text{Also} \quad h(D) &= F^N(D) = F^{2n_1+2n_2}(D) = F^{2n_1+n_2}(F^{n_2}(D)) \\ &= F^{2n_1+n_2}(F^{n_1+n_2}(C)) \quad \text{by (4.41)} \\ &= F^{2n_1+n_2}(A) \quad \text{by 4.31} \\ &\supset A \quad \text{by (2) of hypothesis.} \end{aligned}$$

Also $h(A) = F^N(A) \supset A$ by (2) of hypothesis.

Now since $A \subset F(A)$,

$$A \subset F^j(A) \quad \text{for all } j \geq 0 \tag{4.43}$$

and

$$F_n^{-j}(A) \subset A \quad \text{for all } j \geq 0\,. \tag{4.44}$$

So

$$\begin{aligned} F^{-n_2}(A) &= F^{-n_2}\left(F_A^{-n_1-n_2}(C)\right) \quad \text{by (4.31)} \\ &= F_A^{-n_1-2n_2}(C) \end{aligned}$$

and

$$F_A^{-n_1-2n_2}(C) \subset A \quad \text{by (4.44)}$$

Therefore,

$$\begin{aligned} h(A) = F^N(A) &= F^{2n_1+2n_2}(A) \\ &\supset F^{2n_1+2n_2}\left(F_A^{-n_1-2n_2}(C)\right) \\ &= F^{n_1}(C) = D \quad \text{by (4.41)} \end{aligned}$$

To summarize:

$$A \cap D = \phi \tag{4.45}$$

$$h(D) \supset A \tag{4.46}$$

$$h(A) \supset A \tag{4.47}$$

(4.48) $$h(A) \supset D$$

Therefore,

(4.49) $$\inf\{\|X - Y\| : X \in A , Y \in D\} > 0 .$$

The existence of a scrambled set $S$ follows almost exactly as in Theorems 2.1 and 4.3, which we now again outline.

Let $\mathcal{Y}$ be the set of sequences $E = \{E_n\}_{n=1}^{\infty}$ where $E_n$ is either $A$ or $D$, and $E_{n+1} = E_{n+2} = A$ if $E_n = D$.

Let $R(E,n)$ be the number of sets $E_i$ equal to $D$ for $1 \leq i \leq n$, and for each $\omega \in (0,1)$ let $E^{\omega} = \{E_n^{\omega}\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{Y}$ satisfying

$$\lim_{n \to \infty} \frac{R(E^{\omega}, n^2)}{n} = \omega$$

Let $F_0 = \{E^{\omega} : \omega \in (0,1)\} \subset \mathcal{Y}$. Then $F_0$ is uncountable. Also from (4.46), (4.47) and (4.48) we have $h(E_n^{\omega}) \supset E_{n+1}^{\omega}$ and so by Lemma 4.2 for each $E^{\omega} \in F_0$ there is a point $X_{\omega} \in A \cup D$ such that $h^n(X_{\omega}) \in E_n^{\omega}$ for all $n \geq 1$.

Let $S_h = \{h^n(X_{\omega}) : n \geq 0$ and $E^{\omega} \in F_0\}$. Then $h(S_h) \subset S_h$, $S_h$ contains no periodic points of $h$, and there exists an infinite number of $n$'s such that $h^n(X) \in A$ and $h^n(Y) \in D$ for any $X, Y \in S_h$, $X \neq Y$. Hence, from (4.49) for any $X, Y \in S_h$, $X \neq Y$

$$L_1 = \limsup_{n \to \infty} |h^n(X) - h^n(Y)| > 0$$

Thus letting $S = \{F^n(X) : X \in S_h \text{ and } n \geq 0\}$ it follows that $F(S) \subset S$, $S$ contains no periodic points of $F$, and for any $X, Y \in S$, $X \neq Y$

$$\limsup_{n \to \infty} |F^n(X) - F^n(Y)| \geq L_1 > 0 .$$

This shows that $S$ has properties (2a) and (2b) of Definition 4.2, and similarly (2c) may be shown.

(3) Let $a$ be the fixed point of $F_A^{-1} : A \to A$ given by the Brouwer Fixed Point Theorem.

Since $F$ is expanding on $A$, there exists $\lambda > 1$ such that

(4.50) $\quad |F(X) - F(Y)| \leq \lambda |X - Y|$ for all $X, Y \in A$.

Therefore, for all $k \geq 1$,

(4.51) $$|F_A^{-k}(X) - F_A^{-k}(Y)| \leq \lambda^{-k} |X - y| .$$

In particular, for any $X \in C \subset A$ and $Y = a$

(4.52) $$|F_A^{-k}(X) - a| \leq \lambda^{-k} |X - a| ,$$

so $F_A^{-k}(X) \to a$ as $k \to \infty$ for all $X \in C$. Therefore, for any $\varepsilon > 0$, there exists an integer $j = j(X, \varepsilon)$ such that

(4.53) $$F_A^{-j}(X) \in A \cap B_\varepsilon^n(a) .$$

By continuity, there exists a $\delta = \delta(X, \varepsilon) > 0$ such that

(4.54) $$F_A^{-1}(A \cap B_\delta^{0,n}(X)) \subset A \cap B_\varepsilon^n(a)$$

Now the collection of sets $\{B_\delta^{0,n}(X) : X \in C\}$ constitutes an open cover of the compact set $C$, so there exists a finite subcollection

(4.55) $$C_0 = \{B_{\delta i}^{0,n}(X_i) : X_i \in C , i = 1,2,\ldots,L\}$$

which also covers $C$. Let $T = T(\varepsilon) = \max\{j(X_i,\varepsilon) : i = 1,2,\ldots,L\}$. Then $F_A^{-T}(X) \in B_\varepsilon^n(a) \cap A$ for all $X \in C$, so by (4.52)

(4.56) $$F_A^{-k}(C) \subset B_\varepsilon^n(a) \cap A \quad \text{for all} \quad k \geq T(\varepsilon) .$$

Let $H_k = h_A^{-k}(C)$ for all $k \geq 0$, where $h_A^{-1}$ is the continuous inverse of $h = F^N$ on $A$. Then, for any $\varepsilon > 0$, there exists a $J = J(\varepsilon)$ such that $|X - a| < \frac{\varepsilon}{2}$ for all $X \in H_k$ and all $k > J$.

The rest of the proof again follows Theorem 4.3, which we now outline.

The sequences $E^\omega = \{E_n^\omega\}_{n=1}^\infty \in \mathcal{M}$ will be further restricted as follows: If $E_n^{\omega} = D$ then $n = m^2$ for some integer $m$. If $E_n^{\omega} = D$ for both $n = m^2$ and $n = (m+1)^2$ then $E_n^{\omega} = H_{2m-j}$ for $n = m^2 + j$, $j = 1,2,\ldots,2m$. Finally for the remaining $n$'s, $E_n^{\omega} = A$.

It can easily be checked that the sequences still satisfy $h(E_n^{\omega}) \supset E_{n+1}^{\omega}$, so by Lemma 4.2 there exists a point $X_{\omega}$ such that $h^n(X_{\omega}) \in E_n^{\omega}$ for all $n \geq 0$. Let $S_0 = \left\{X_{\omega} : \omega \in \left[\frac{3}{4}, 1\right]\right\}$. Then $S_0$ is uncountable, $S_0 \subset S_h \subset S$, and for any $s, t \in \left[\frac{3}{4}, 1\right]$ there exists infinitely many $m$'s such that $h^n(X_s) \in E_n^s = H_{2m-1}$ and $h^n(X_t) \in E_n^t = H_{2m-1}$, where $n = m^2 + 1$. But, as shown above, given any $\varepsilon > 0$, $\|X - a\| < \frac{\varepsilon}{2}$ for all $X \in H_{2m-1}$ if $m$ is sufficiently large. Hence, for any $\varepsilon > 0$, there exists an integer $m$ such that

$$\|h^n(X_s) - h^n(X_t)\| < \varepsilon \quad \text{where} \quad n = m^2 + 1 .$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$L_2 = \liminf_{n \to \infty} \|h^n(X_s) - h^n(X_t)\| = 0 .$$

Thus for any $X, Y \in S_0$

$$\liminf_{n \to \infty} \| F^n(X) - F^n(Y) \| \leq L_2 = 0 . \qquad \text{Q.E.D.}$$

4.4 Similarities between Theorems 2.1, 4.3, and 4.4.

The theorems of Li and Yorke [14], Marotto [22], and Kloeden [25] all have the following condition in common: a point (or set) $a$, which under iterations of $f$ gets mapped further away from $a$, and then gets back to or near $a$. This can be seen from the following summary.

Theorem 2.1: The iterates of point $a$ satisfy $f(a) > a$, $f^2(a) > f(a)$, and $f^3(a) = a$ (or $f^3(a) < a$).

Theorem 4.3: The existence of the snap-back repeller $Z$, and the point $X_0 \in B_r(Z)$ for which $f^m(X_0)$, $1 \leq m < M$, is further from $Z$ than $X_0$, and then $f^M(X_0) = Z$.

Theorem 4.4: Existence of the set $C \subset A$, $F^{n_1}(C) \not\subset A$ (i.e., "further" from $A$), and then gets back to $A$ in view of $F^{n_1+n_2}(C) = A$.

We note that each theorem we have presented is a generalization of the previous one. However, it is still difficult, given a continuous (and differentiable) function, to determine whether or not the function will be chaotic. Also, for the N-dimensional case, $N > 1$, the above theorems do not indicate what the dimension of the chaotic set is. It is possible that its dimension is less than the dimension of the set under consideration, as seen from the following example by H. Proppe.

<u>Example 4.4</u>: Let $f : R^2 \to R^2$ be defined by $f(z) = z^2$ where $z = x + iy$.

Let $X = \left\{z : \frac{2\pi}{3} \le \arg z \le \pi\right\} \cap \{z : |z| \ge \varepsilon\}$. Therefore $f(X) = \left\{z : \frac{4\pi}{3} \le \arg z \le 2\pi\right\} \cap \{z : |z| \ge \varepsilon^2\}$, and $f^2(X) = \left\{z : \frac{2\pi}{3} \le \arg z \le 2\pi\right\} \cap \{z : |z| \ge \varepsilon^4\}$. Thus the two conditions of Theorem 4.2 are met, i.e.,

$$X \cap f(X) = \phi$$

and

$$X \cup f(X) \subset f^2(X) \subset R^2 .$$

However, the chaotic behaviour is restricted to the curve $\{z : |z| = 1\}$, since, in the interior of this circle, the iterates tend to the origin, while outside this circle they tend to infinity.

Again, using Example 4.4, we will show that $f$ has a snap-back repeller, and therefore will be chaotic by Theorem 4.3.

Consider the complex number $z = 1$: this is a fixed point of $f$. Also $f(z) = z^2$ may be written as

$$f(x,y) = (x^2 - y^2, 2xy) .$$

Therefore,

$$Df(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

and its eigenvalues, $\lambda$, are obtained from

$$\begin{vmatrix} 2x - \lambda & -2y \\ 2y & 2x - \lambda \end{vmatrix} = 0$$

Thus $\lambda = 2x \pm 2yi$, which, in absolute value, is equal to 2 on $|z| = 1$, and so $z = 1$ is an expanding fixed point of $f$.

Let $X_0 = e^{\left(\frac{2\pi i}{2^n}\right)}$ with $n$ sufficiently large so that $X_0$ is in an expanding neighbourhood of 1. Thus $f^{2^n}(X_0) = 1$. Also $|Df(x,y)| = 4x^2 + 4y^2 = 4$ on $|z| = 1$.

Therefore $X_0$ is a snap-back repeller. However as mentioned above, the chaotic set is restricted to $|z| = 1$.

4.5 Some Examples.

We will now give some examples applying Theorems 2.1, 4.3, and 4.4.

Example 4.5: This will be a one-dimensional example.

Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) Since $\left(\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right)$ forms a cycle of period three (or $\frac{1}{4}, \frac{1}{2}, 1, 0$ are three iterates with $0 = f^3\left(\frac{1}{4}\right) < \frac{1}{4}$), by Theorem 2.1 the difference equation is chaotic.

(b) $f$ is continuous on $[0,1]$, and is also differentiable everywhere except at $x = \frac{1}{2}$, so as long as we stay away from $x = \frac{1}{2}$, we can apply Theorem 4.3, which is a local theorem.

$f$ has two fixed points, $0$ and $\frac{2}{3}$, both of which are snap-back repellers since $\left|\frac{df}{dx}\right| = 2$ everywhere, except at $x = \frac{1}{2}$, and so all points are expanding, and the eigenvalue is $2$. Also, there are two points $\frac{1}{4}, \frac{11}{12} \in [0,1]$ for which $f^3\left(\frac{1}{4}\right) = 0$ and $f^3\left(\frac{11}{12}\right) = \frac{2}{3}$. There is a problem with the point $0$ since the first iterate of $\frac{1}{4}$ is $\frac{1}{2}$, where the function is not differentiable; however, with the point $\frac{2}{3}$ all conditions are satisfied and thus the difference equation is chaotic by Theorem 4.3.

(c) We will now show $f$ is chaotic using Theorem 4.4.

Let $A = \left[\frac{9}{16}, \frac{7}{8}\right]$, $B = \left[\frac{3}{4}, \frac{7}{8}\right]$, $n = \ell = 1$, and $n_1 = n_2 = 1$. These sets and integers satisfy all 7 conditions of Theorem 4.4, since $f(A) = \left[\frac{1}{4}, \frac{7}{8}\right]$, $f(B) = \left[\frac{1}{4}, \frac{1}{2}\right]$, $f^2(B) = \left[\frac{1}{2}, 1\right]$ and thus:

(1) $A$ is homeomorphic to a one-ball.

(2) $A \subset f(A)$

(3) $f$ is expanding on $A$, since for every $x, y \in A$

$$|f(x) - f(y)| = |(2 - 2x) - (2 - 2y)| = 2|x - y|$$

(4) $B \not\subset A$

(5) $f(B) \cap A = \phi$

(6) $A \subset f^2(B)$

(7) Finally $f^2$ is 1-1 on $B$, since for every $x \in B$

$$f^2(x) = 2(2 - 2x) = 4 - 4x.$$

Therefore $f$ is chaotic by Theorem 4.4.

Example 4.6: (Kloeden [25]). Let $f = (f_1, f_2)$ be a continuous mapping defined on the unit square $I^2 \subset R^2$ (see Example 4.3) by

$$f_1(x,y) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$f_2(x,y) = \frac{x}{2} + \frac{y}{10} + \frac{1}{4}$$

which is a twisted horseshoe on $I^2$.

$f$ has a fixed point $(\overline{x}, \overline{y}) = \left(\frac{2}{3}, \frac{35}{54}\right)$. Now, the Jacobian for $f$ is

$$\begin{pmatrix} \pm 2 & 0 \\ \frac{1}{2} & 10 \end{pmatrix}$$

and at $(\overline{x}, \overline{y})$ it is

$$\begin{pmatrix} -2 & 0 \\ \frac{1}{2} & 10 \end{pmatrix}$$

Its eigenvalues $\lambda$ are determined from

$$\begin{vmatrix} -2-\lambda & 0 \\ \frac{1}{2} & \frac{1}{10}-\lambda \end{vmatrix} = 0$$

and so $\lambda = -2, \frac{1}{10}$. Therefore, $(\overline{x},\overline{y})$ is a saddle point, and because $\lambda = \frac{1}{10} < 1$, Theorem 4.3 cannot be used, but Theorem 4.4 does apply.

Let $L_1$ be the line $90x + 378y = 305$, and $L_2$ be the line $90x - 378y = -125$. These are two lines passing through $(\overline{x},\overline{y})$ whose equations have integral coefficients. Let

$$A = \left\{(x,y) : (x,y) \in L_1 ,\ \frac{9}{16} \le x \le \frac{7}{8}\right\}$$

and

$$B = \left\{(x,y) : (x,y) \in L_1 ,\ \frac{3}{4} \le x \le \frac{7}{8}\right\}$$

Thus, to find $f(A)$, we have

$$f_1(x,y) = 2 - 2x \quad \text{since} \quad x \ge \frac{9}{16} > \frac{1}{2}$$

and

$$f_2(x,y) = \frac{x}{2} + \frac{305 - 90X}{10(378)} + \frac{1}{4},$$

and we note that $\left(2 - 2x, \frac{x}{2} + \frac{305 - 90x}{3780} + \frac{1}{4}\right)$ satisfies the equation for $L_1$. Therefore,

$$f(A) = \left\{(x,y) : (x,y) \in L_1, \frac{1}{4} \leq x \leq \frac{7}{8}\right\}.$$

Similarly,

$$f(B) = \left\{(x,y) : (x,y) \in L_1, \frac{1}{4} \leq x \leq \frac{1}{2}\right\}$$

To find $f^2(B)$, we note that $x \leq \frac{1}{2}$ in $f(B)$ and that $\left(2x, \frac{x}{2} + \frac{305 - 90x}{3780} + \frac{1}{4}\right)$ satisfies the equation for $L_2$.

Therefore,

$$f^2(B) = \left\{(x,y) : (x,y) \in L_2, \frac{1}{2} \leq x \leq 1\right\}$$

and

$$f^3(B) = \{(x,y) : (x,y) \in L_1, 0 \leq x \leq 1\}.$$

Thus $f^3(B) \supset L_1 \cap I^2$. Let $\ell = 1$, $n_1 = 1$, and $n_2 = 2$. The above sets and integers satisfy all 7 conditions of Theorem 4.4:

(1) $A$ is homeomorphic to a one-ball

(2) $A \subset F(A)$, shown above

(3) $f$ is expanding on $A$ since for any $(x,y) \in A$ we have

$$f_1(x,y) = 2 - 2x$$

$$f_2(x,y) = \frac{1}{2} \cdot \frac{305 - 378y}{90} + \frac{y}{10} + \frac{1}{4} = \frac{35}{18} - 2y.$$

So, for any two points $(x',y')$, $(x'',y'') \in A$, we have

$$\|f(x',y') - f(x'',y'')\| = \left\|\left(2 - 2x', \frac{35}{18} - 2y'\right) - \left(2 - 2x'', \frac{35}{18} - 2y''\right)\right\|$$

$$= \| (2x' - 2x'', 2y' - 2y'') \|$$

$$= 2 \| (x', y') - (x'', y'') \| .$$

(4) $B \subset A$ as defined.

(5) $f^1(B) \cap A = \phi$ as shown.

(6) $A \subset f^3(B)$ as shown.

(7) Finally, $f^3$ is 1-1 on $B$, since for all $(x,y) \in B$

$$f_1^3(x,y) = 2 - 2[2(2 - 2x)] = 8 - 8x ,$$

$$f_2^3(x,y) = \frac{381}{200}x + \frac{y}{1000} - \frac{249}{400} \quad \text{(straightforward substitution)}$$

which gives the non-singular Jacobian matrix

$$\begin{bmatrix} -8 & 0 \\ \frac{381}{200} & \frac{1}{1000} \end{bmatrix} .$$

Thus the difference equation is chaotic by Theorem 4.4.

## 4.6 Further Similarities between Theorems 4.3 and 2.1

For the one-dimensional case, the existence of a snap-back repeller does not imply the existence of points of period three, since Theorem 4.3 (and Theorem 4.4) only ensures the existence of points of all periods greater than some positive integer $N$. Hence the function will be chaotic, by Theorem 3.2, since there will exist a point whose period is an odd number.

The existence of a point of period $3 \cdot 2^n$, $n = 0,1,2,\ldots,$ also does not imply the existence of a snap-back repeller, as can be seen from the following example.

<u>Example 4.7</u> Let $f$ be the continuous function from $[0,1]$ to $[0,1]$ defined by:

$$f(x) = \begin{cases} \frac{5}{3}x, & 0 \le x \le \frac{3}{5} \\ -\frac{5}{4}x + \frac{7}{4}, & \frac{3}{5} \le x \le 1 \end{cases}$$

Now

$$f^2(.8711) = .9236$$
$$f^2(.9236) = .9925$$
$$f^2(.9925) = .8489,$$

thus if we let $a = .8711$ we have

$$f^6(a) < a < f^2(a) < f^4(a),$$

and therefore $f^2$ has a point of period 3 by Theorem 2.1, and $f$ has a point of period $6 = 2^1 \cdot 3$, and is chaotic. However, the only fixed points of $f$ are zero and $\frac{7}{9}$, neither of which are snap-back repellers, since no point of $(0,1]$ gets mapped into $0$. Also, the only other point which gets mapped to $\frac{7}{9}$ is $\frac{7}{15} < \frac{1}{2}$ and so all the pre-images are less than $\frac{1}{2}$, and therefore they will not be in any ball centered at $\frac{7}{9}$.

Marotto [26] however has shown that we can define a snap-back-repeller with weaker conditions than Definition 4.8, and the conclusion of Theorem 4.3 would still hold, as follows.

Definition 4.11: $Z$ is said to be a snap-back repeller of $F$ if there exists a sequence of compact sets $\{B_k\}_{k=-\infty}^{M}$ (each homeomorphic to the unit ball in $R^N$) which satisfy:

(a) $B_k \to Z$ as $k \to -\infty$.

(b) $F(B_k) = B_{k+1}$.

(c) $F$ is 1-1 in $B_k$ (which may be dropped for the case when $R^N = R$ as shown in [26]).

(d) $B_k \cap B_M = \phi$ for $1 \leq k < M$.

(e) $Z \in B_M^0$

Under the above conditions, Theorem 4.3 will hold with the assumption of continuity of $F$ alone, and the following Theorem may be shown.

Theorem 4.5: Let $F : R \to R$ be a continuous function. Then $F^n$ possesses a snap-back repeller (in the sense of conditions (a), (b), (d) and (e) of Definition 4.11) if and only if $F^m$ has a point of period 3, for some positive integers m and n.

Proof: See [26].

Note that in Example 4.7, $f^2$ has a point of period 3, and 0 is a snap-back repeller according to Definition 4.11.

4.7 Observations

Presently, the methods used to determine whether a one-dimensional function behaves chaotically is (I) to check for points of period $m \cdot 2^n$, where m and n are integers and m is odd, which is generally difficult to do, (2) to find a point $a$ for which $f^{3m}(a) < a < f^{m}(a) < f^{2m}(a)$, which is usually tedious and messy. For n-dimensional functions, chaos is determined by (1) finding a snap-back repeller for $f^n$; or (2) find sets which satisfy the conditions of Theorem 4.4. Both methods are difficult to apply.

It can easily be shown that any continuous function $f$ from $[0,1]$ onto $[0,1]$ for which $f(0) = f(1) = 0$ is a chaotic function since there will exist a point $b \in (0,1)$ such that $f(b) = 1$, and a point $a \in (0,b)$ such that $f(a) = b$, and therefore we have

$$0 = f^3(a) < a < f(a) < f^2(a) = 1 .$$

We would like to have similar conditions applying to other classes of functions, to tell us without testing, whether they are chaotic. For example, it is conjectured that all piecewise continuous functions for which $\inf|f'(X)| > 1$ will be chaotic.

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