CHARMED BARYONS IN A CONSISTENT QUARK MODEL WITH HYPERFINE INTERACTIONS

David Pfeffer

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ABSTRACT

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Masses of the ground state and P-wave baryons containing 1, 2 and 3 charmed quarks are calculated using a confining potential and hyperfine interactions. The prediction of the $\Sigma_c \frac{1}{2}^+$ mass is in close agreement with experiment.

ACKNOWLEDGEMENT

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INTRODUCTION

It has always been one of the major goals of physicists to discover the elementary particles, the indivisible building blocks out of which all matter is formed. So far experiments have revealed the existence of twelve such particles, which are listed in Appendix C. They are divided into two groups, the quarks, which are characterized as having a property known as colour charge, and the leptons, which lack this property and are described as colourless, or white.

Just as in the case of electromagnetism, where only particles with electromagnetic charge feel the effects of electric and magnetic fields, so there exist colour electric and magnetic fields which are felt only by particles, or agglomerations of particles with colour charges.

Since, as was stated previously, the leptons have no colour charge, they do not interact with the colour fields. They will not be discussed further.

While quarks undoubtedly exist, no one has yet been able to find one in isolation. According to present theory, quarks may exist in bound, three particle states called baryons, or bound quark-antiquark states called mesons.*

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* There are indications from experiments that states with six quarks also exist.

It is thought that the colour force between the quarks is so strong in these bound states that the ionization potential is effectively infinite. Just as the bound state of a proton and an electron is electromagnetic charge neutral, so the colour charge of a baryon or a meson is neutral.

This thesis is devoted to calculating the energies of the baryonic states containing 1, 2 or 3 charmed quarks. It is a continuation of work done by C.S. Kalman, who solved the problem for the case of baryons containing strange quarks (see reference 17).

Since the charmed quark is so heavy (approximately 1500 MeV), one can assume that the three quarks in a charmed baryon move with non-relativistic velocities. The full apparatus of field theory is therefore not needed.

The quarks are described by wave functions which are common eigenstates of orbital angular momentum and spin, and move in a two body potential to be described in the next chapter.

CHAPTER 1

1.1 Quantum Chromodynamics

The theory which describes the two body forces between the three pairs of quarks in a baryon is called Quantum Chromodynamics (QCD). In this model, which is very similar in structure to Quantum Electrodynamics (QED), the forces are mediated by the exchange of eight vector bosons called gluons. The interaction Hamiltonian for QCD can be written.[1]

$$H = gJ^{\ell\mu}A^{\ell}{}_{\mu} \quad \text{(summation over } \ell = 1 \text{ to } 8 \text{ implied)} \qquad (1\text{-}1)$$

where $J^{\ell\mu} = \overline{\Psi}\frac{\lambda^{\ell}}{2}\gamma^{\mu}\Psi$

$$\Psi \equiv \begin{pmatrix} R \\ B \\ G \end{pmatrix} \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$$

$A^{\ell}{}_{\mu}$ are the eight gluon vector potentials

λ^{ℓ} are the eight Gellmann SU(3) matrices listed in Appendix A.

R, B, G denote the three colour fermion fields which describe quarks.

Expanding:

$$gJ^{\ell\mu}A^{\ell}_{\mu} = g\sum_{a,b=1}^{3} A_{ba} \cdot J_{ab} \qquad (1\text{-}2)$$

where $J_{ab} = \bar{\Psi}_a \gamma^{\mu} \Psi_b$

$A_{12} = \frac{1}{2}(A_1 - iA_2) \quad A_{21} = \frac{1}{2}(A_1 + iA_2)$

$A_{13} = \frac{1}{2}(A_4 - iA_5) \quad A_{31} = \frac{1}{2}(A_4 + iA_5)$

$A_{23} = \frac{1}{2}(A_6 - iA_7) \quad A_{32} = \frac{1}{2}(A_6 + iA_7)$

$A_{11} = \frac{1}{2}(A_3 + \frac{A_8}{\sqrt{3}}) \quad A_{22} = \frac{1}{2}(-A_3 + \frac{A_8}{\sqrt{3}})$

$A_{33} = \frac{-A_8}{\sqrt{3}}$

The 4-vector subscript is implied

Note that

$$A_{11} + A_{22} + A_{33} = 0 \qquad (1\text{-}3)$$

so that there are still eight independent gluon fields.

Consider a particular term in the expansion:

$$A_{12} \cdot J_{21} \equiv A_{12\mu}\bar{\Psi}_2\gamma^{\mu}\Psi_1 \equiv A_{R\bar{B}\mu}\bar{B}\gamma^{\mu}R \qquad (1\text{-}4)$$

In field theory this term corresponds to a scattering vertex, represented by the Feynman diagram illustrated in Figure 1.

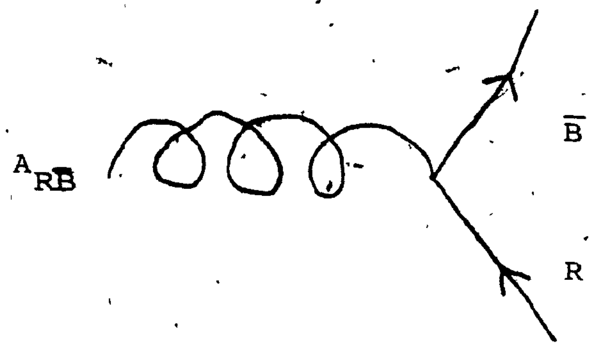

Figure 1. Emission of gluon by R quark

The interaction of an $A_{R\bar{B}}$ gluon with a R quark causes it to change into a B quark.

There are eight conserved charges in QCD[2]:

$$Q^{\ell} = \int d^3\bar{x} \left[J^{\ell 0} + f^{\ell mn} \quad F^{mo\nu} A^{n}_{\ \nu} \right] \qquad (1\text{-}5)$$

where $f^{\ell mn}$ are the SU(3) structure constants (see for example Gibson and Pollard, p. 265).

$$F^{\ell\mu\nu} = \partial^{\mu} A^{\ell\nu} - \partial^{\nu} A^{\ell\mu} + g f^{\ell mn} A^{m\mu} A^{n\nu}$$

$J^{\ell o}$ is the contribution to the charge density of the coloured quarks. The second term arises from the fact that the gluons themselves have colour charge. This situation can be contrasted to the case of QED where the photon has no electromagnetic charge. As a result, the gluons can couple to themselves and form glueballs, which are bound states of gluons in the absence of quarks.

1.2 Running Coupling Constant

Consider the QED electrostatic potential between two static electrons of charge e and separated by distance r, where $r << \frac{1}{m}$, and m is the mass of the electron. By keeping terms only up to $\alpha_0{}^2$, where $\alpha_0 = \frac{e^2}{4\Pi}$, in a perturbation expansion[3]:

$$V = + \frac{\alpha}{r} \tag{1-6}$$

where α the coupling constant is given by

$$\alpha = \alpha_0 \left(1 - \frac{\alpha_0}{3\Pi} \ln \frac{L^2}{Q^2}\right) \tag{1-7}$$

where $Q \to O\left(\frac{1}{r}\right)$, $L >> m$

The numerical value of L, the ultraviolet cutoff, cannot be specified. It is a free parameter of the theory. The first term on the right hand side of equation 1-7 is the contribution due to the exchange of a single virtual photon as illustrated in Figure 2a. In the absence of the second term, each source sees the bare charge e of the other. The second term is due to the presence in the vacuum state of a negative energy sea of electrons, as first pointed out by Dirac. The interaction of a negative energy electron with the electromagnetic field causes a virtual transition to a positive energy state. The unoccupied negative energy state that results from such a process behaves like a positively charged particle and is called a positron. In the presence of the static source charges these virtual electron - positron pairs are polarized as illustrated in Figure 2b.

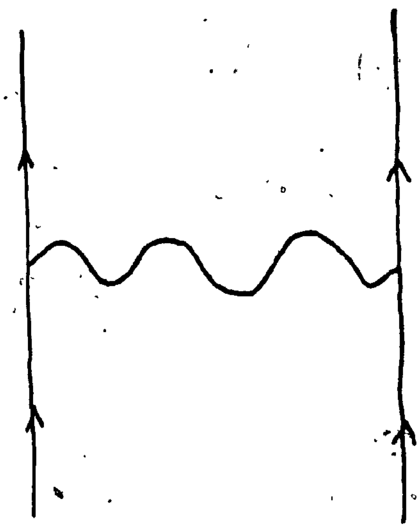

Figure 2a.
One photon exchange

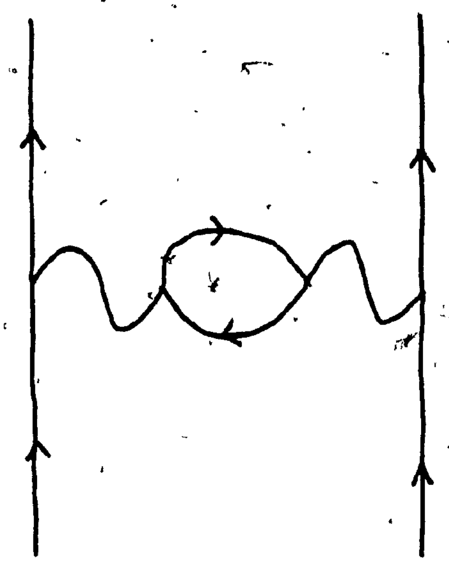

Figure 2b
Vacuum polarization

The bare source charge is partially screened by the electron - positron pairs, leaving a residue, the physical, observable charge, which is what is measured at long ranges.

As the source charges move closer together, they penetrate the positron-electron cloud and see more of the bare charge of the other. When $Q \sim L, \alpha = \alpha_0$ and each electron interacts with the bare charge of the other.

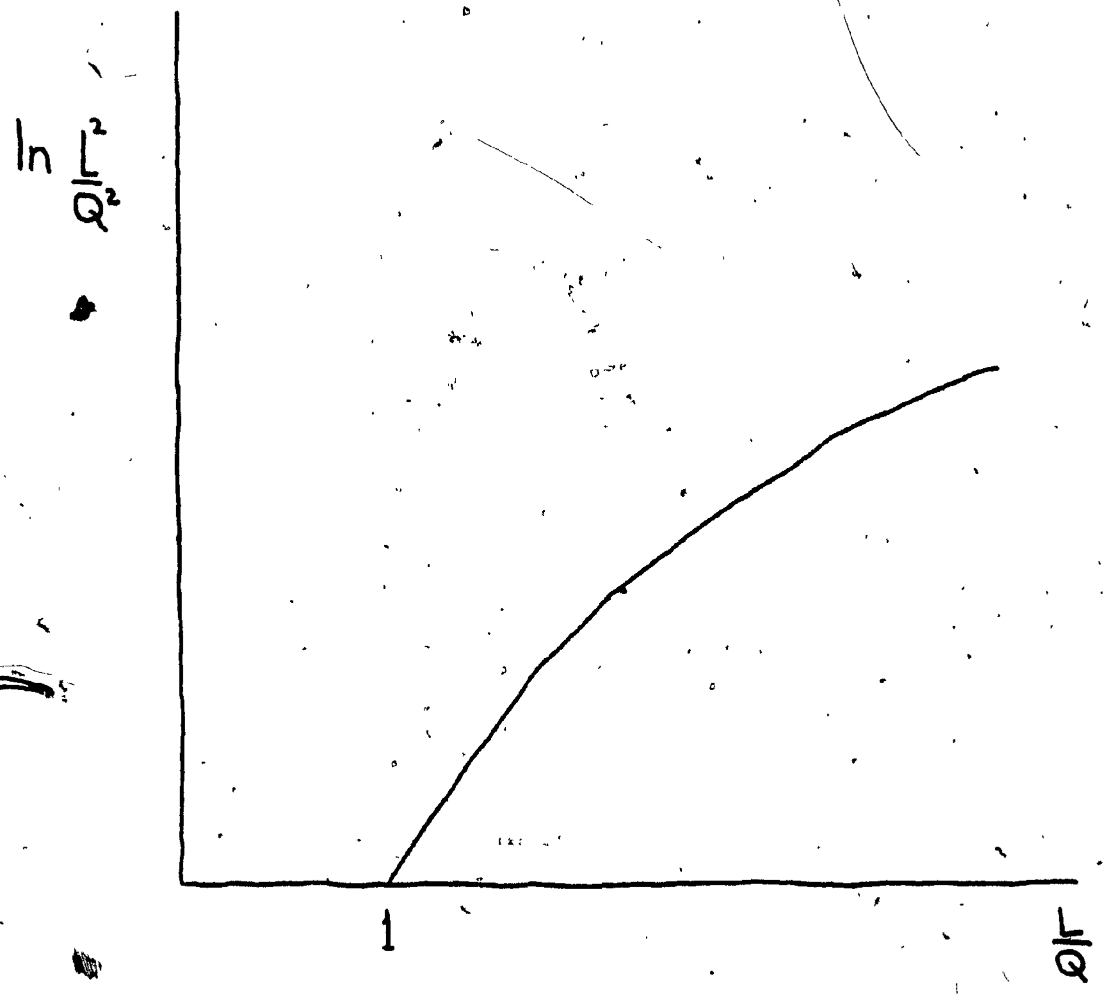

Figure 3. ln $\frac{L^2}{Q^2}$ plotted against $\frac{L}{Q}$ or separation.

For the case of QCD there is an extra term making a contribution to α_s, the coupling constant. The presence of this term is due to the fact, mentioned previously, that since gluons have colour charge, they couple to themselves, and give rise to a term represented by the diagram in Figure 4c[4].

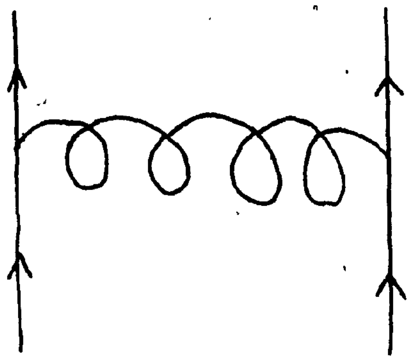

Figure 4a
One gluon exchange

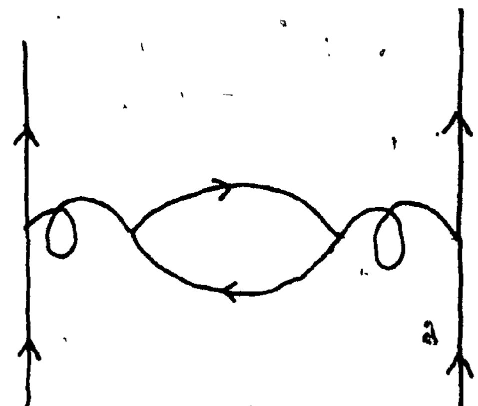

Figure 4b
Vacuum polarization

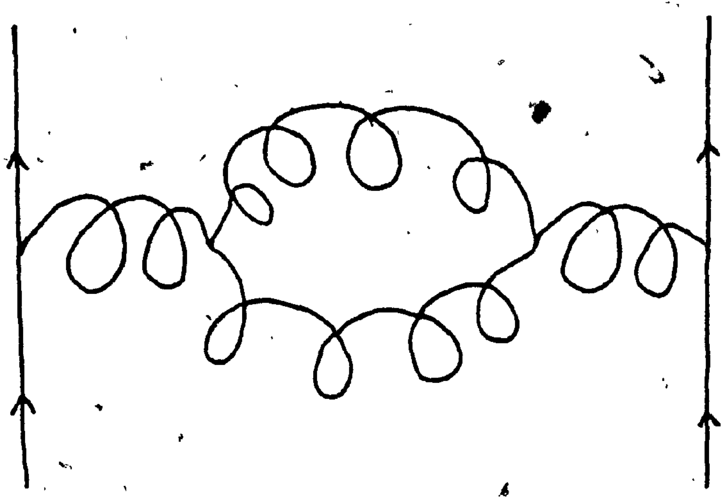

Figure 4c
Virtual gluon-gluon scattering

$$\alpha_s = \alpha_0 \left(1 - \frac{2}{3} \frac{f\alpha_0}{4\pi} \ln \frac{L^2}{Q^2} + \frac{11}{4\pi} \alpha_0 \ln \frac{L^2}{Q^2}\right) \qquad (1\text{-}8)$$

where f = number of flavours , $\alpha_0 = \frac{g^2}{4\pi}$

This additional term more than cancels the term due to $q\bar{q}$ polarization illustrated by Figure 4b. As a result, there is a net antiscreening effect, i.e., the coupling becomes stronger with increasing separation, and tends to zero as the separation decreases. Therefore in the limit as $r \to o$ the quarks behave as free particles. This result, first derived by Gross and Wilczek[5] is known as asymptotic freedom, and is consistent with the experimentally observed phenomenon of scaling.

1.3 Quark Confinement

Theoretical work done by Rebbi[6] and others using QCD on a lattice indicate that with separations in the same range as those associated with the dimensions of the baryon, the potential increases linearly. This result is consistent with the experimental non-observation of free quarks, since at infinite separation, the potential between two quarks becomes infinite in a linear potential.

The lattice method treats space-time as a discontinuous set of points arranged or a cubic lattice with spacing a as illustrated in Figure 5. Quarks can exist only on the lattice points.

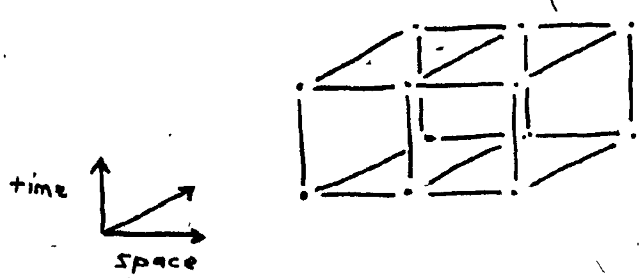

Figure 5. Space time represented by points on a lattice

The colour electrostatic field energy can be calculated for the case of two static quarks located at different lattice sites.

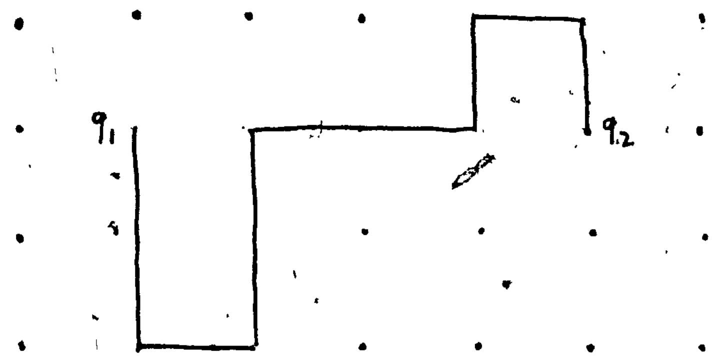

Figure 6. String connecting quarks 1 and 2. The links are represented by the solid lines.

According to Susskind,[7] a string is defined as a set of connecting links with quarks at the end points. Gauss' law as applied to this situation requires a string between the two quarks as illustrated in Figure 6. The energy for each link is given by

$$\varepsilon = \frac{2}{3}\frac{g^2}{a} \tag{1-8}$$

Therefore the minimum electrostatic energy of two quarks separated by a distance r has contributions from $\frac{r}{a}$ links

$$E = \sum_{\substack{\text{links} \\ \text{of string}}} \frac{2}{3}\frac{g^2}{a} = \frac{2}{3}\frac{g^2}{a^2}\, r \tag{1-9}$$

In the limit of continuous spacetime a, g tend to zero. If $\frac{g}{a}$ has a finite limiting value, the electrostatic potential is linear.

Such devices as treating QCD on a lattice are very artificial and much more work must be done in order to get firm results applicable to the physical case. Unfortunately no one has been able to use QCD to solve the quark system the way QED has been used to calculate the hydrogen spectrum. Therefore it is necessary to proceed phenomenologically.

1.4 The Breit Interaction

So far we have restricted our analysis to static electric charges. However, quarks are in motion. In addition, they have static magnetic dipole momentum due to spin. Using the one gluon exchange approximation

$$\tilde{V}(\vec{p}_i\vec{p}_j\vec{q}) = \alpha_s \langle\lambda_i \cdot \lambda_j\rangle J^\mu(\vec{p}_i) J_\mu(\vec{p}_j) \frac{1}{q^2} \qquad (1\text{-}10)$$

$\tilde{V}$ is the two body potential in momentum space represented by Figure 7.

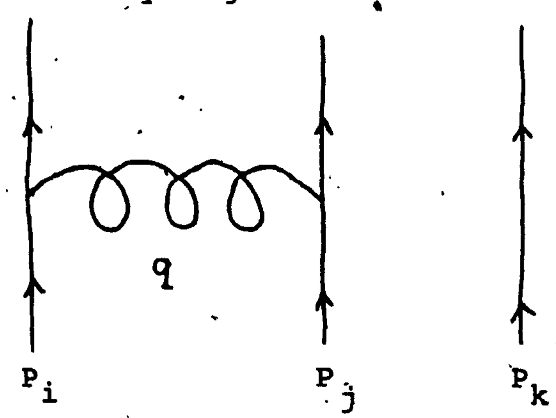

Figure 7. Two quarks undergoing one gluon exchange process, while third quark is spectator.

$J^\mu(\vec{p}_i)$ is the current of the ith quark

$\vec{p}_i$ is the momentum of the ith quark

$\langle\lambda_i\cdot\lambda_j\rangle$ the colour charge factor equals $-\frac{2}{3}$ for baryons.

$\vec{q}$, the momentum transfer, is the momentum conjugate to $\vec{r}$, the separation between the quarks. In other words:

$$V(\vec{p}_i,\vec{p}_j,\vec{r}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\vec{p}_i,\vec{p}_j,\vec{q})\, e^{-i\vec{q}\cdot\vec{r}} d^3\vec{q} \qquad (1\text{-}11)$$

If the quarks are taken as non-relativistic $V(\vec{p}_i\vec{p}_j\vec{r})$ can be expanded, using equation 1-10, to first order in $\frac{v}{c}$. The result is the well known Breit interaction.[8]

$$V(\vec{p}_i,\vec{p}_j,\vec{r}) = -\frac{2}{3}\frac{\alpha_s}{r} + \text{spin independent terms}$$

$$+V_{\text{spin orbit}} \qquad + V_{\text{hyperfine}} \qquad (1\text{-}12)$$

where $V_{\text{spin orbit}} = \frac{1}{2}\alpha_s \frac{1}{r^3}\left(\vec{r}\times\left(\frac{\vec{p}_i}{m_i}-\frac{\vec{p}_j}{m_j}\right)\cdot\left(\frac{\vec{s}_i}{m_i}+\frac{\vec{s}_j}{m_j}\right)\right)$

$$-\frac{1}{6}\alpha_s \frac{1}{r^3} \vec{r}\times\left(\frac{\vec{p}_i}{m_i}+\frac{\vec{p}_j}{m_j}\right)\cdot\left(\frac{\vec{s}_i}{m_i}-\frac{\vec{s}_j}{m_j}\right) \qquad (1\text{-}13)$$

$$V_{\text{hyperfine}} = +\frac{2}{3}\frac{\alpha_s}{m_i m_j}\left(\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\,\delta^3(\vec{r})\right.$$

$$\left.+\frac{1}{r^3}\left(3\vec{S}_i\cdot\hat{r}\,\vec{S}_j\cdot\hat{r}-\vec{S}_i\cdot\vec{S}_j\right)\right) \qquad (1\text{-}14)$$

where $\vec{r} = \vec{r}_i - \vec{r}_j$; m_i, $\vec{S}_i$ the mass and spin of the ith quark.

1.5 The Hamiltonian

Equation 1-10 accurately describes the interaction between quarks only when α_s is small. This region of asymptotic freedom occurs for $r < 1(\mathrm{GeV})^{-1}$. Typical quark separations in baryons are about $5(\mathrm{GeV})^{-1}$. Therefore, the quarks are mostly to be found well outside the asymptotic region, in what is called the confinement region, where the linear potential is expected to dominate.

This situation is taken into account by introducing into the Hamiltonian a phenomenological harmonic oscillator confinement potential, between quark pairs. The reason for using a harmonic oscillator potential instead of the theoretically expected linear potential has to do with calculational convenience and is explained in the next chapter.

Isgur and Karl [9] have shown that although the confining potential is spin independent, it will still contribute terms to the spin-orbit interaction due to the effect of Thomas precession. Their calculations have shown that these Thomas precession terms cancel almost exactly the spin-orbit terms due to one gluon exchange. This is similar to the atomic case, where the Thomas precession cancels one half the coulombic spin-orbit interaction. In the Isgur-Karl model, the spin-orbit interaction is neglected entirely.

In order to compensate for the discrepancy between the harmonic oscillator and the true physical confining potential at large distances, a two body potential $U(r)$ is introduced.

It is expected that for small separation (where $\frac{1}{2}kr^2 \sim 0$),

$U(r)$ is approximately Coulombic.

The complete Hamiltonian for a baryonic three quark system in the Isgur-Karl model is [10]

$$H = \sum_i m_i + H_{HO} + U + V_{hyperfine} \qquad (1\text{-}15)$$

where $H_{HO} = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i>j} \frac{1}{2} k r_{ij}^2$

$$U = \sum_{i>j} U(r_{ij})$$

$r_{ij} = |\vec{r}_i - \vec{r}_j|$, $\vec{p}_i, m_i$ are the momentum and mass of the ith quark, $\sum_i$ is a summation over the three quarks of the baryon.

$U + V_{hyperfine}$ is considered a perturbation on the harmonic oscillator term H_{HO}.

CHAPTER 2

2.1 The Harmonic Oscillator

The Hamiltonian H, described in the last chapter, can be used to calculate the masses of baryons containing 1, 2 and 3 charmed quarks by perturbation theory. Matrix elements of H are taken between eigenfunctions of the unperturbed Hamiltonian H_{HO}. These matrix elements form the mass matrices, whose eigenvalues make up the expected physical mass spectrum.

The unperturbed Hamiltonian is written

$$H_{HO} = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i>j} \frac{1}{2} k r_{ij}^2 \tag{2-1}$$

where: with only one c quark $m_1 = m_2 = m_u$, $m_3 = m_c$

with two c quarks $m_1 = m_2 = m_c$, $m_3 = m_u$

with three c quarks $m_1 = m_2 = m_3 = m_c$

where m_u is the mass of the u quark and m_c is the mass of the c quark.

In order to separate out the coordinates of the center of mass, a new set of spatial coordinates is chosen in place of $\vec{r}_1, \vec{r}_2, \vec{r}_3$. These are $\vec{\rho}, \vec{\lambda}, \vec{R}$ defined by:

$$\vec{R} = \frac{m_1(\vec{r}_1 + \vec{r}_2) + m_3\vec{r}_3}{2m_1 + m_3} \tag{2-2a}$$

$$\vec{\rho} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2) \tag{2-2b}$$

$$\vec{\lambda} = \frac{1}{\sqrt{6}} (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \quad (2\text{-}2c)$$

The physical significance of the new coordinates can be inferred from Figure 2-1. $\sqrt{2}\vec{\rho}$ is the separation between q_1 and q_2 for the two body system which they form. $\frac{\sqrt{6}}{2} \vec{\lambda}$ is the coordinate of q_3 with respect to the centre of mass of the q_1, q_2 system.

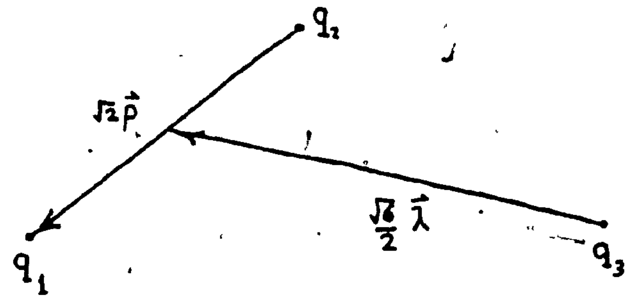

Figure 2-1. Spatial coordinates of the three quark system.

In the rest frame of the baryon, $p_R = 0$. Equation 2.1 is rewritten

$$H_{HO} = \frac{p_\rho^2}{2m} + \frac{3}{2} k\rho^2 + \frac{p_\lambda^2}{2m_\lambda} + \frac{3}{2} k\lambda^2 \quad (2\text{-}3)$$

where $m = m_1$ $\quad m_\lambda = \dfrac{3m_1m_3}{2m_1+m_3}$ $\quad , \vec{p}_\rho = \frac{1}{i}\nabla_\rho$ etc.

Therefore the unperturbed Hamiltonian H_{HO} decouples into two independent oscillators, a ρ oscillator with frequency $\omega = \left(\frac{3k}{m}\right)^{\frac{1}{2}}$ and a λ oscillator with frequency $\omega_\lambda = \left(\frac{3k}{m_\lambda}\right)^{\frac{1}{2}}$. This decoupling is not possible with a linear potential and is the reason a harmonic oscillator is used.

When the oscillators are quantized, the eigenstates are characterized by n_ρ and n_λ, the number of quanta in each oscillator.

The eigenvalues are

$$E_{n_\rho n_\lambda} = \omega(n_\rho + \tfrac{3}{2}) + \omega_\lambda(n_\lambda + \tfrac{3}{2}) \quad (2\text{-}4)$$

The eigenfunctions of low lying eigenstates are listed in Figure 8.

Figure 8. Oscillator eigenfunctions for low-lying states[11]

	n_ρ	n_λ	Parity	wave function
ground state $\ell=0$	0	0	+1	$\Psi^{S}_{00} = \Psi$
P wave $\ell = 1$	1	0	-1	$\Psi^{\rho}_{11} = \alpha_\rho \rho_+ \Psi$
	0	1	-1	$\Psi^{\lambda}_{11} \quad \alpha_\lambda \lambda_+ \Psi$
radially excited $\ell = 0$	2	0	+1	$\Psi^{\rho\rho}_{00} = \left(\frac{2}{3}\right)^{\frac{1}{2}} \alpha_\rho^{2} \left(\rho^{2} - \frac{3}{2\alpha_\rho^2}\right)\Psi$
	0	2	+1	$\Psi^{\lambda\lambda}_{00} = \left(\frac{2}{3}\right)^{\frac{1}{2}} \alpha_\lambda^{2} \left(\lambda^{2} - \frac{3}{\alpha_\lambda^2}\right)\Psi$
	1	1	+1	$\Psi^{\rho\lambda}_{00} = \left(\frac{1}{3}\right)^{\frac{1}{2}} \alpha_\rho \alpha_\lambda \; 2\vec{\rho}\cdot\vec{\lambda}\Psi$

where $\rho_+ = \rho_x + i\rho_y$ etc., $\alpha_\rho = (3km)^{\frac{1}{4}}$ $\alpha_\lambda = (3km_\lambda)^{\frac{1}{4}}$, ℓ is the orbital angular momentum quantum number.

$$\Psi = \frac{\alpha_\rho^{3/2}}{\pi^{3/2}} \alpha_\lambda^{3/2} e^{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)} \quad (2\text{-}5)$$

There are four flavour wave functions relevant to this work, they are

$$\phi_{\Lambda_c} = \frac{1}{\sqrt{2}} (ud - du)c \tag{2-5a}$$

$$\phi_{\Sigma_c} = \frac{1}{\sqrt{2}} (ud + du)c \tag{2-5b}$$

$$\phi_{\Xi_c} = ccu \tag{2-5c}$$

$$\phi_{\Omega_c} = ccc \tag{2-5d}$$

satisfying

$$\langle \phi_i \mid \phi_j \rangle = \delta_{ij} \tag{2-6}$$

Quarks are spin ½ particles, let $\vec{S}_1$, $\vec{S}_2$, $\vec{S}_3$ denote the spins of the three quarks of a baryon.

$$\vec{S}_{12} = \vec{S}_1 + \vec{S}_2 \tag{2-7a}$$

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \tag{2-7b}$$

It is possible to form eigenstate of S^2, m_s and S_{12}^2 and these are listed in Figure 2-2.

Figure 2.2 Spin wave functions for three particles

S_{12}	S	m_s	Spin wave functions
1	$\frac{3}{2}$	$\frac{3}{2}$	$\chi^{s}_{\frac{3}{2}} = \uparrow\uparrow\uparrow$
		$\frac{1}{2}$	$\chi^{s}_{\frac{1}{2}} = \frac{1}{\sqrt{3}}\,(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$
		$-\frac{1}{2}$	$\chi^{s}_{-\frac{1}{2}} = \frac{1}{\sqrt{3}}\,(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$
		$-\frac{3}{2}$	$\chi^{s}_{-\frac{3}{2}} = \downarrow\downarrow\downarrow$
	$\frac{1}{2}$	$\frac{1}{2}$	$\chi^{\lambda}{}_{\frac{1}{2}} = \frac{1}{\sqrt{6}}\,(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\,\uparrow\uparrow\downarrow)$
		$-\frac{1}{2}$	$\chi^{\lambda}{}_{-\frac{1}{2}} = \frac{1}{\sqrt{6}}\,(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\,\downarrow\downarrow\uparrow)$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\chi^{\rho}{}_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\,(\uparrow\downarrow - \downarrow\uparrow)\uparrow$
		$-\frac{1}{2}$	$\chi^{\rho}{}_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}\,(\uparrow\downarrow - \downarrow\uparrow)\downarrow$

where for example

$$\uparrow\uparrow\downarrow = |S_1{=}\tfrac{1}{2}\ m_{s_1}{=}\tfrac{1}{2}\rangle\,|S_2{=}\tfrac{1}{2}\ m_{s_2}{=}\tfrac{1}{2}\rangle\,|S_3{=}\tfrac{1}{2}\ m_{s_3}= -\tfrac{1}{2}\rangle$$

Product wave functions are formed by combining space, spin and flavour wave functions. These product wave functions must satisfy the Pauli exclusion principle in quark indices 1 and 2. If $\vec{L}$ is the orbital angular momentum of the space wave function then $\vec{J} = \vec{L} + \vec{S}$, the total angular momentum or spin of the baryon.

As was stated previously, the Hamiltonian matrix is formed by taking matrix elements of the Hamiltonian between all these product states. Since the Hamiltonian does not mix states of differing J, parity or flavour wave function type, the Hamiltonian matrix is reducable into smaller mass matrices denoted by F^{JP}, where $F = \Lambda, \Sigma, \Xi, \Omega$.

Figure 2-3 lists the product states which mix together in each sector.

Figure 2-3 Product states. Note that a flavour wave function factor is implied for each term in the product state column.

Physical States	Product States		
$\Lambda_c \frac{1}{2}+$	$\psi^s\chi^\rho$	$\psi^{\rho\rho}\chi^\rho$	$\psi^{\lambda\lambda}\chi^\rho$
$\Sigma_c \frac{1}{2}+$	$\psi^s\chi^\lambda$	$\psi^{\rho\rho}\chi^\lambda$	$\psi^{\lambda\lambda}\chi^\lambda$
$\Sigma_c \frac{3}{2}+$	$\psi^s\chi^s$	$\psi^{\rho\rho}\chi^s$	$\psi^{\lambda\lambda}\chi^s$
$\Lambda_c \frac{1}{2}-$	$\psi^\rho\chi^s$	$\psi^\lambda\chi^\rho$	$\psi^\rho\chi^\lambda$
$\Lambda_c \frac{3}{2}-$	$\psi^\rho\chi^s$	$\psi^\lambda\chi^\rho$	$\psi^\rho\chi^\lambda$
$\Lambda_c \frac{5}{2}-$	$\psi^\rho\chi^s$		
$\Sigma_c \frac{1}{2}-$	$\psi^\lambda\chi^s$	$\psi^\lambda\chi^\lambda$	$\psi^\rho\chi^\rho$
$\Sigma_c \frac{3}{2}-$	$\psi^\lambda\chi^s$	$\psi^\lambda\chi^\lambda$	$\psi^\rho\chi^\rho$
$\Sigma_c \frac{5}{2}-$	$\psi^\lambda\chi^s$		
$\Xi_c \frac{1}{2}+$	$\psi^s\chi^\lambda$	$\psi^{\rho\rho}\chi^\lambda$	$\psi^{\lambda\lambda}\chi^\lambda$
$\Xi_c \frac{3}{2}+$	$\psi^s\chi^s$	$\psi^{\rho\rho}\chi^s$	$\psi^{\lambda\lambda}\chi^s$
$\Xi_c \frac{1}{2}-$	$\psi^\lambda\chi^s$	$\psi^\lambda\chi^\lambda$	$\psi^\rho\chi^\rho$
$\Xi_c \frac{3}{2}-$	$\psi^\lambda\chi^s$	$\psi^\lambda\chi^\lambda$	$\psi^\rho\chi^\rho$
$\Xi_c \frac{5}{2}-$	$\psi^\lambda\chi^s$		
$\Omega_c \frac{3}{2}+$	$\psi^s\chi^s$	$\frac{1}{\sqrt{2}}(\psi^{\rho\rho} + \psi^{\lambda\lambda})\chi^s$	
$\Omega_c \frac{1}{2}-$		$\frac{1}{\sqrt{2}}(\psi^\rho\chi^\rho + \psi^\lambda\chi^\lambda)$	
$\Omega_c \frac{3}{2}-$		$\frac{1}{\sqrt{2}}(\psi^\rho\chi^\rho + \psi^\lambda\chi^\lambda)$	

2.2 Matrix Elements

For the case of the ground state baryons, Ψ^S is mixed with a linear combination of radially-excited states $\Psi^{\rho\rho}, \Psi^{\lambda\lambda}$. These linear combination states and their energies were taken from a paper by Copley, Isgur and Karl, and are shown in Figure 2-4.[12]

Figure 2-4. Radially excited states

States	Energy	Composition
$\Lambda_c \frac{1}{2}^+$	2695	$.86\Psi^{\lambda\lambda}\chi^{\rho} + .51\Psi^{\rho\rho}\chi^{\rho}$
$\Sigma_c \frac{1}{2}^+$	2805	$.90\Psi^{\lambda\lambda}\chi^{\lambda} + .43\Psi^{\rho\rho}\chi^{\lambda}$
$\Sigma_c \frac{3}{2}^+$	2875	$.92\Psi^{\lambda\lambda}\chi^{S} + .39\Psi^{\rho\rho}\chi^{S}$

So that for example the mass matrix for the $\Lambda_c \frac{1}{2}^+$ sector is

$$\begin{pmatrix} \langle\Psi^S\chi^\rho|H|\Psi^S\chi^\rho\rangle & .86\langle\Psi^S\chi^\rho|H|\Psi^{\lambda\lambda}\chi^\rho\rangle + .51\langle\Psi^S\chi^\rho|H|\Psi^{\rho\rho}\chi^\rho\rangle \\ .86\langle\Psi^S\chi^\rho|H|\Psi^{\lambda\lambda}\chi^\rho\rangle + .51\langle\Psi^S\chi^\rho|H|\Psi^{\rho\rho}\chi^\rho\rangle & 2695 \end{pmatrix}$$

If a typical product state is written $\Psi_i \chi_j$ then by using equation 1-15.

$$\langle \Psi_i \chi_j | H | \Psi_i \chi_j \rangle = E_i + \langle \Psi_i \chi_j | V_{hyperfine} | \Psi_i \chi_j \rangle \tag{2-8}$$

where E_i depends only on the spatial wave function and is given by

$$E_i = 2m_1 + m_3 + \langle \Psi_i | H_{HO} + U | \Psi_i \rangle \tag{2-9}$$

The non-diagonal terms involve only the hyperfine interaction, i.e.,

$$\langle \Psi_i \chi_j | H | \Psi_k \chi_\ell \rangle = \langle \Psi_i \chi_j | V_{hyperfine} | \Psi_k \chi_\ell \rangle \tag{2-10}$$

In the limit of no hyperfine interaction there are three non-degenerate masses in the c=1 sector, E(s), E(ρ), E(λ). Similarly for the c=2 and 3 sectors.

Using the results of Kalman and Hall,[13] Kalman, Hall and Misra.[14]

$$E(s) = 2m_1 + m_3 + \frac{3}{2}(\omega + \omega_\lambda) + \langle U\rangle_s \qquad (2\text{-}11a)$$

$$E(\rho) = 2m_1 + m_3 + \frac{5}{2}\omega + \frac{3}{2}\omega_\lambda + \langle U\rangle_\rho \qquad (2\text{-}11b)$$

$$E(\lambda) = 2m_1 + m_3 + \frac{3}{2}\omega + \frac{5}{2}\omega_\lambda + \langle U\rangle_\lambda \qquad (2\text{-}11c)$$

where

$$\langle U\rangle_s = \frac{1}{3}a(1) + \frac{2}{3}a(r)$$

$$\langle U\rangle_\rho = \frac{2}{9}b(1) + \frac{1}{2}\left(\frac{\alpha_\rho}{\alpha_\lambda}\right)^2 ra(r) + \frac{1}{9}rb(r)$$

$$\langle U\rangle_\lambda = \frac{1}{3}a(1) + \frac{1}{6}ra(r) + \frac{1}{3}\left(\frac{\alpha_\rho}{\alpha_\lambda}\right)^2 rb(r)$$

$$r = \frac{4}{1 + 3\left(\frac{\alpha_\rho}{\alpha_\lambda}\right)^2}$$

where

$$a(t) = \frac{3\alpha^3 t^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \int d^3\rho\, U(\sqrt{2}\rho)\, e^{-t\alpha^2\rho^2}$$

$$b(t) = \frac{3\alpha^5 t^{\frac{5}{2}}}{\pi^{\frac{3}{2}}} \int d^3\rho\, U(\sqrt{2}\rho)\, e^{-t\alpha^2\rho^2}\rho^2$$

$$c(t) = \frac{3\alpha^7 t^{\frac{7}{2}}}{\pi^{\frac{3}{2}}} \int d^3\rho\, U(\sqrt{2}\rho)\, e^{-t\alpha^2\rho^2}\rho^4$$

By expanding a(t), b(t), c(t) about t = 1 in a Taylor expansion up to order $(t-1)^2$

$$a(t) = A + Bt + Ct^2 \qquad (2\text{-}12a)$$

$$b(t) = (3A + Bt - Ct^2)/2 \qquad (2\text{-}12b)$$

$$c(t) = (15A + 3Bt - Ct^2)/4 \qquad (2\text{-}12c)$$

The hyperfine matrix elements used in this work were derived by Isgur and Karl and some of them are listed in the Appendix B.

The matrix elements are written in terms of seven free parameters m_u, m_c, ω, δ, A, B, C.

Kalman[17] calculated the mass matrices for the non-charmed sector. By fitting the eigenvalues of these mass matrices to the experimentally determined mass spectrum he was able to fix the numerical values of six of seven free parameters. He found that

$$m_u = 387,\ \omega = 274,\ \delta = 265,\ A = -198,\ B = -737,\ C = +84.$$

The value of m_c was fixed during the course of this work by fitting the lowest eigenvalue of the $\Lambda c^{\frac{1}{2}+}$ mass matrix to the experimentally determined mass 2285 MeV. As a result m_c was fixed at 1930 MeV. Once the values of the free parameters were found, the numerical values of the matrix could be calculated. The eigenvalues of these matrices are the predicted mass spectrum for charmed baryons.

CONCLUSION

The predicted charmed baryon spectrum is presented in Figure 3-1. The ordering of the states as well as the relative size of the splittings follows the pattern of the non-charmed baryons.

The quark mass term determines the gross features of the spectrum due to the very large charmed quark mass (1930 MeV). In fact there is no overlap in energy between the $c=1$ states (Λ_c and Σ_c) the $c=2$ states (Ξ_c), and the $c=3$ states (Ω_c). Evidence of harmonic oscillator contribution to the baryon mass can be inferred from the energy gap between the ground state and the P-wave baryons, which have a single unit of excitation in the ρ or λ oscillator. The residual splitting is due to the differing relative spin orientations of the various states.

Since the mass of the $\Lambda_c \frac{1}{2}^+$ was fitted to 2285 MeV in order to evaluate the mass of the charmed quark, the prediction of the $\Sigma_c \frac{1}{2}^+$ mass is the only one that can be compared to experiment. The percentage of error is 1.7%. Future experiments should map out the rest of the spectrum, thus providing a more rigorous test of the predictions.

Figure 3-1. Charmed Baryon Spectrum

	Mass (MeV)	
State	Calculation	Experiment
$\Lambda_c \frac{1}{2}+$	2285	2285
$\Sigma_c \frac{1}{2}+$	2427	2460
$\Sigma_c \frac{3}{2}+$	2499	
$\Lambda_c \frac{1}{2}-$	2684	
	2861	
	2937	
$\Lambda_c \frac{3}{2}-$	2685	
	2896	
	2994	
$\Lambda_c \frac{5}{2}-$	2938	
$\Sigma_c \frac{1}{2}-$	2827	
	2862	
	2940	

Figure 3-1 (continued)

State	Calculation	Experiment
$\Sigma_c \frac{3}{2}-$	2841	
	2889	
	2937	
$\Sigma_c \frac{5}{2}-$	2868	
$\Xi_c \frac{1}{2}+$	3582	
$\Xi_c \frac{3}{2}+$	3620	
$\Xi_c \frac{1}{2}-$	4173	
	4331	
	4312	
$\Xi_c \frac{3}{2}-$	4173	
	4323	
	4346	
$\Xi_c \frac{5}{2}-$	4029	
$\Omega_c \frac{3}{2}+$	5317	
$\Omega_c \frac{1}{2}-$	5419	
$\Omega_c \frac{3}{2}-$	5419	

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Appendix A

Gellman SU(3) Matrices[18]

$$\lambda_1 = \begin{pmatrix} 0 & 1 & \cdot \\ 1 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & \cdot \\ i & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & \cdot \\ 0 & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & \cdot & -i \\ \cdot & \cdot & \cdot \\ i & \cdot & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 0 & 1 \\ \cdot & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 0 & -i \\ \cdot & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Appendix B

Hyperfine Matrix Elements
(taken from reference 9 and 12)

$\Lambda_c \frac{1}{2}^+$

$$\langle \psi^S \chi^\rho | V_{hyp} | \psi^S \chi^\rho \rangle = -\frac{1}{2}\delta$$

$$\langle \psi^S \chi^\rho | V_{hyp} | \psi^{\lambda\lambda} \chi^\rho \rangle = 0$$

$$\langle \psi^S \chi^\rho | V_{hyp} | \psi^{\rho\rho} \chi^\rho \rangle = \frac{\sqrt{6}}{4}\delta$$

$\Sigma_c \frac{1}{2}^+$

$$\langle \psi^S \chi^\lambda | V_{hyp} | \psi^S \chi^\lambda \rangle = -\frac{1}{2}\left(\frac{4\bar{x}-1}{3}\right)\delta$$

$$\langle \psi^S \chi^\rho | V_{hyp} | \psi^{\lambda\lambda} \chi^\lambda \rangle = \frac{\sqrt{6}}{4} x\delta$$

$$\langle \psi \ \chi^\lambda | V_{hyp} | \psi^{\rho\rho} \chi^\lambda \rangle = -\frac{\sqrt{6}}{12}(1-x)\delta$$

$\Sigma_c \frac{3}{2}^+$

$$\langle \psi^S \chi^S | V_{hyp} | \psi^S \chi^S \rangle = \frac{1}{2}\left(\frac{1+2\bar{x}}{3}\right)\delta$$

Appendix B (continued)

$$\langle\psi^{S}\chi^{S}|V_{hyp}|\psi^{\lambda\lambda}\chi^{S}\rangle = -\frac{\sqrt{6}}{8}\, x\,\delta$$

$$\langle\psi^{S}\chi^{S}|V_{hyp}|\psi^{\rho\rho}\chi^{S}\rangle = -\frac{\sqrt{6}}{8}\left(\frac{2+x}{3}\right)\delta$$

$\Lambda_{c\frac{1}{2}}$ -

$$\langle\psi^{\lambda}\chi^{\rho}|V_{hyp}|\psi^{\rho}\chi^{S}\rangle = -\frac{\sqrt{2}}{8}xg\,\delta$$

$$\langle\psi^{\rho}\chi^{S}|V_{hyp}|\psi^{\rho}\chi^{S}\rangle = -\frac{1}{4}\left(\frac{2}{3}+\frac{1}{3}xg\right)\delta + \frac{1}{4}xy^{2}f\delta$$

$$\langle\psi^{\rho}\chi^{\lambda}|V_{hyp}|\psi^{\rho}\chi^{S}\rangle = -\frac{\sqrt{2}}{8}\left(\frac{4}{3}-\frac{1}{3}xg\right)\delta$$

$$\langle\psi^{\lambda}\chi^{\rho}|V_{hyp}|\psi^{\lambda}\chi^{\rho}\rangle = -\frac{1}{2}\delta$$

$$\langle\psi^{\rho}\chi^{\lambda}|V_{hyp}|\psi^{\rho}\chi^{\lambda}\rangle = -\frac{1}{2}xy^{2}f\delta$$

$$\langle\psi^{\rho}\chi^{\lambda}|V_{hyp}|\psi^{\lambda}\chi^{\rho}\rangle = \frac{1}{4}xyf\delta$$

$\Sigma_{c\frac{3}{2}}$ -

$$\langle\psi^{\lambda}\chi^{s}|V_{hyp}|\psi^{\lambda}\chi^{s}\rangle = \frac{1}{4}\left(\frac{2}{3}+\frac{1}{3}xf\right)\delta + \frac{1}{5}xh\delta$$

Appendix B (continued)

$$\langle \psi^\lambda \chi^\lambda | V_{hyp} | \psi^\rho \chi^\rho \rangle = \frac{1}{4} x y f \delta$$

$$\langle \psi^\rho \chi^\rho | V_{hyp} | \psi^\rho \chi^\rho \rangle = 0$$

$$\langle \psi^\lambda \chi^\lambda | V_{hyp} | \psi^\lambda \chi^\lambda \rangle = \frac{1}{6}(1 - xf)\delta$$

$$\langle \psi^\lambda \chi \, | V_{hyp} | \psi^\lambda \chi^\lambda \rangle = -\frac{\sqrt{5}}{40} x h \delta$$

$$\langle \psi^\rho \chi^\rho | V_{hyp} | \psi^\lambda \chi^\lambda \rangle = \frac{\sqrt{5}}{40} x h' \delta$$

where $x = \frac{m_1}{m_3}$ $\qquad y = \left(\frac{2x+1}{3}\right)^{\frac{1}{2}}$

$$\bar{x} = \left(\tfrac{3}{4}y^2 + \tfrac{1}{4}\right)^{-\frac{3}{2}} x$$

$$f = \left(\tfrac{3}{4}y^2 + \tfrac{1}{4}\right)^{-\frac{5}{2}}$$

$$g = y^2 (4 - 2y^2) a \qquad g' = 2ya$$

$$h = 2y^2 a \qquad h' = 2a$$

$$a = \left(\tfrac{3}{4}y^2 + \tfrac{1}{4}\right)^{-\frac{3}{2}} \left(\tfrac{1}{4}y^2 + \tfrac{3}{4}\right)^{-1} (1 + y^2)^{-1}$$

$$\delta = \frac{4\alpha_s}{3} (m_1 \omega)^{\frac{3}{2}} / \sqrt{2\pi} \cdot m_1^{\,2}$$

The rest of the matrix elements can be found in Appendix B of reference 9.

Appendix C

The Elementary Particles

	Electric Charge*	Colour Charge	Mass (MeV)	Existence Confirmed
Quarks				
up	$+\frac{2}{3}$	red, blue or green	~350	yes
down	$-\frac{1}{3}$	red, blue or green	~350	yes
strange	$-\frac{1}{3}$	red, blue or green	~650	yes
charm	$+\frac{2}{3}$	red, blue or green	~1500	yes
bottom	$-\frac{1}{3}$	red, blue or green	~5000	yes
top	$+\frac{2}{3}$	red, blue or green	-	no
Leptons				
electron	-1	white	.511	yes
muon	-1	white	106	yes
tau	-1	white	1784	yes
electon neutrino	0	white	0	yes
muon neutrino	0	white	0	yes
tau neutrino	0	white	**	no

* units of electronic charge
** assumed massless