Computer Orbits of Chaotic Systems

by
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ABSTRACT

Let $\tau: X \to X$, $X = [0,1]$ be a transformation. When τ admits an (ergodic) invariant measure, the Birkhoff Ergodic Theorem describes the asymptotic (statistical) behaviour of the (chaotic) system (X, τ). Among all the invariant measures that are associated with τ, those that are absolutely continuous with respect to the Lebesgue measure (ACIMs) are physically relevant.

When subjected to a numerical or computer experiment, chaotic systems give rise to (inexact) computer-generated orbits. We discuss the following question: to what extent do the computer orbits of a chaotic system reflect the true dynamics of the actual system?

An important tool employed in this study is the shadowing property of the system. The shadowing property has limitations towards computation, e.g. it lacks stability with respect to external perturbations. The other tool utilized is the ACIM. For a large class of transformations τ which admit a (unique) ACIM μ, it is shown that the computer orbits of τ exhibit μ.

ACIMs are stable with respect to a large class of deterministic (and stochastic) perturbations. This result is used to show that there is stability of the shadowing property for certain families of Markov operators. Finally, applications to chaotic dynamics of cellular automata and fractals are discussed.

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CONTENTS

INTRODUCTION

The way various phenomena or processes in nature evolve in time is often described by a non-autonomous system of first-order ordinary differential equations,

$$\frac{dx}{dt} = F(x,t),$$

where $x(t)$ is a vector in the phase-space X describing the state of the system and $F(x,t)$ is a (nonlinear) differentiable function describing the (continuous) time evolution of $x(t)$. When time t is discrete and integer valued (time discretization) we obtain a map,

$$x_{n+1} = \tau(x_n).$$

Maps can arise in continuous time systems in the form of a Poincaré surface of section [223,229]. Such mathematical modelling is usually done on the basis of a time series extracted from an experimental data based on the observed phenomena [249-253].

A phase-space X together with a transformation τ constitute a dynamical system (X, τ). The time evolution of the dynamical system is represented by the orbit $\{x, \tau(x), \tau^2(x), ...\}$, $x \in X$, i.e. given the past it determines the future. The main purpose of the study of dynamical systems is to understand the nature of all orbits. Investigations of orbits of dynamical systems go as far back as the 17th century when Kepler proposed the three laws of motion for the planetary orbits for our solar system.

Almost a century ago it was known that some deterministic dynamical systems exhibit incredibly complex behaviour. Poincaré had found that the general solution of the three-body problem of celestial mechanics [205] can be very complex. In the 1940's, Cartwright-Littlewood [206] and Levinson [207] observed unpredictable long-term behaviour in the solutions of forced Van der Pol equations. In the 1950's, Belousov observed chaotic oscillations in the colour of a mixture of citric and sulphuric acid, potassium bromate and a cerium salt. In the 1960's, such unpredictable or chaotic behaviour was observed for the first time in simple models when Lorenz [208] studied a highly simplified model of Rayleigh-Bénard convection in fluids in connection to the meteorological problem of weather predictions. Related results were observed by Hayashi [209] in his work with nonlinear electrical circuits. With the advent of fast computers since the 1970's, long-term integrations have become a routine facility and chaotic phenomena has been observed in many systems and in various domains of science [210-219]: astronomy, meteorology, ecology, chemistry, fluid dynamics, optics, computer networks, epidimiology, human physiology and quantum mechanics. (See [237] for a historical outline of nonlinear dynamics and chaos.)

Recently there has been a considerable interest in the study of chaotic dynamics of low-dimensional dissipative systems [10] due to the richness of complex behaviours they exhibit. When a system is dissipative, its orbits contract the volume of the phase-space in time. A geometrical picture of chaotic behaviour on the phase-space of a dissipative system was first provided by Smale [220], who showed that the phenomenon of 'stretching and folding' occurs on the phase-space, giving rise to a horseshoe type structure.

Dissipative systems are characterized by the presence of an attractor. An attractor is a compact subset of the phase-space to which the orbits of the system converge as time $t \to \infty$, and the basin of attraction of an attractor is the closure of the set of initial conditions whose orbits converge to the attractor. An attractor could be a fixed point, a set of dimension zero or a closed curve, a set of dimension one. When the attractor is a set of non-integral dimension i.e. a fractal, it is called a strange attractor. The term strange attractor was coined by Ruelle-Takens [221] while investigating the phenomena of fluid turbulence and has become a paradigm of nonlinear dynamics.

There are various characterizations of chaos in a dynamical system: existence of 1) a positive Lyapunov exponent, 2) positive topological entropy, 3) an invariant measure (weakest characterization) and various routes leading to chaos: period doubling, intermittency, crisis. Existence of any of these parameters implies a sensitive dependence on initial conditions on the phase-space: orbits starting closely move apart rapidly. An attractor with this property is called a chaotic attractor. It is typical that a strange attractor is chaotic (although this is not always so [222]).

Our interest lies in the study of attractor(s) which result from the asymptotic motion of a chaotic system and contain all the information regarding the long-term behaviour of the system. The description of geometric structure [198,221] of a chaotic attractor is usually not feasible due to extremely complicated dynamics. In such a case, statistical analysis becomes necessary. A useful tool in this study is ergodic theory along with its measure-theoretic considerations. A basic virtue of ergodic theory is that it allows us to describe long-term behaviour of a system and not to worry about transients.

Whenever a numerical experiment is carried out to study a dynamical system, it is often subjected to unavoidable external perturbations. Since the use of computers in these experiments has played a significant role in elucidating the underlying nature of chaotic systems, it is important to investigate inherent limitation imposed by computer roundoff/truncation errors (space discretization) on such numerical studies. Therefore, it is crucial to understand whether what we see in computer-generated pictures of chaotic attractors are artifacts due to chaos-amplified roundoff/truncation errors or if they represent interesting new phenomenon inherent in the real-world problem being studied. We thus face the following central question:

In what sense and to what extent do numerical or computer experiments with their inherent roundoff/truncation errors reflect the true dynamics of the actual system?

In this thesis, we investigate the above question and study the long-term behaviour of computer-generated orbits of one-dimensional chaotic systems through the application of recent results in ergodic theory. We restrict ourselves to the study of non-invertible mappings on an interval for the following reasons:

1) when difference equations are used as approximations to ordinary differential equations, they often produce non-invertible mappings, and this is frequently the case when a chaotic behaviour occurs [170],

2) they may serve as simple models for physical, biological and various other systems,

3) the study of certain higher dimensional systems (e.g. the Lorenz system) can be reduced to the study of maps of the interval,

4) they can be easily modelled on a computer.

We now present a brief outline of the content of the thesis. Reference to appendices will appear at appropriate places in the chapters.

OUTLINE

CHAPTER 1 ASYMPTOTIC DESCRIPTION OF CHAOTIC SYSTEMS: ROLE OF INVARIANT MEASURES

The first chapter begins with a review of some basic notions from ergodic theory, including the Birkhoff Ergodic Theorem. Thereafter we discuss the significance and physical relevance of invariant measures in the asymptotic description of chaotic systems which appears in experimental and computational work. Using the Frobenius-Perron operator, we then describe the theory of invariant measures, absolutely continuous with respect to Lebesgue measure (ACIMs), which are both theoretically and physically significant measures.

CHAPTER 2 MEASURES ON PERIODIC ORBITS

Let $\tau: [0,1] \to [0,1]$ be a transformation having a unique ACIM μ. It has been observed that computer-generated orbits of τ display μ in the sense of Birkhoff Ergodic Theorem. Since a computer orbit is necessarily periodic, this leads to the question whether the periodic orbits of τ display any of the invariant measures associated with them. In this chapter, we discuss this question for one-dimensional continuous transformations which admit an ergodic invariant measure.

CHAPTER 3 PERTURBATION OF TRUE ORBIT: PSEUDO-ORBIT SHADOWING PROPERTY

Pseudo-orbits arise when noise is introduced in a dynamical system, e.g. while modelling the dynamical system on a computer. For chaotic systems, these pseudo-orbits move apart rapidly from the true orbit. It has been observed that there often exists another true orbit, with slightly different initial point, which stays near the noisy numerical orbit, i.e. shadows it for a long period of time. In this chapter, we discuss this property of chaotic systems, particularly for a large class of piecewise monotonic transformations.

CHAPTER 4 WHY COMPUTER ORBITS LIKE ABSOLUTELY CONTINUOUS INVARIANT MEASURES

Let $\tau: [0,1] \to [0,1]$ be a transformation which admits a unique ACIM. It has often been observed that the histograms of computer simulations of chaotic orbits of τ seem to display the ACIM and the computer orbits exhibit a chaotic behaviour

CHAPTER 4 Cont'd

though the theoretical system and the computer are both completely deterministic. In this chapter, we provide a theoretical justification to this computer phenomena for piecewise monotonic transformations which admit an ACIM.

CHAPTER 5 PERTURBATION OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES

Let $\tau: [0,1] \to [0,1]$ be a transformation which admits an ACIM. During numerical studies of τ, it is subjected to numerous external perturbations. Since an ACIM is a useful parameter for describing the asymptotic behaviour of the system it is important to consider whether it is stable under any such unavoidable perturbations. In this chapter, we show that the ACIMs which a piecewise monotonic transformation admits are stable under a large class of both deterministic and stochastic perturbations.

CHAPTER 6 SHADOWING PROPERTY FOR MARKOV OPERATORS IN THE SPACE OF DENSITIES

In this chapter, we deal with the generalized shadowing property and prove that a class of constrictive Markov operators has this property. Frobenius-Perron operators are an important example of such operators. Shadowing property may not hold for all parameters for a class of maps. We prove that the generalized shadowing property is stable and is valid for all parameters for many families of maps.

CHAPTER 7 APPLICATIONS

Finally, based on several results in the preceeding chapters, we undertake some applications of computer orbits of point transformations to cellular automata and fractals.

CHAPTER 1

ASYMPTOTIC DESCRIPTION OF CHAOTIC SYSTEMS: ROLE OF INVARIANT MEASURES

1.1 INTRODUCTION

Ergodic theory along with its measure-theoretic considerations have played an important role in understanding and giving a statistical description of chaotic dynamical systems. A main advantage in the study of ergodic theory is that it leaves out all transients and describes only the asymptotic behaviour of a system. It is therefore of interest to discuss the concepts which have brought in the significance of such a study.

In this Chapter, we review some fundamental notions from ergodic theory. The significance of invariant measures in describing the long-term behaviour of chaotic dynamical systems is discussed. We evoke Birkhoff Ergodic Theorem in this context. Physical relevance of invariant measures exhibited during experimental and computational work is considered. In terms of Frobenius-Perron operator, we then review the theory of absolutely continuous invariant measures, which are physically significant measures of great importance.

1.2 PRELIMINARIES

Let X be a compact metric space. We shall assume $(X, \mathbf{B}, \mu)$ to be a normalized measure space or a probability space, unless stated otherwise. We begin with stating some basic definitions.

Measurable transformation A transformation $\tau: X \to X$ is measurable if,

$\tau^{-1}(A) \in \mathbf{B}$, for each $A \in \mathbf{B}$, where $\tau^{-1}(A) = \{x: \tau(x) \in A\}$.

Invariant Measure Let $\tau: X \to X$ be a measurable transformation. Then μ is invariant under τ if, $\mu(\tau^{-1}A) = \mu(A)$, for each $A \in \mathbf{B}$, i.e. for one application of the transformation, the amount of mass that leaves a set is equal to the amount that enters, making the transformation in a state of equilibrium with respect to the measure. An invariant measure thus can be considered to be a parameter for describing the equilibrium states of a dynamical system.

Measure Preserving transformation Let $\tau: X \to X$ be a measurable transformation. Then τ is said to be measure preserving if and only if μ is an invariant measure. We then also say that τ is μ-invariant.

Ergodic Measure Let $\tau: X \to X$ be a measurable transformation. Then μ is said to be ergodic if $\tau^{-1}(A) = A$, $A \in \mathbf{B}$ implies $\mu(A) = 0$ or $\mu(X/A) = 0$.

Remark 1 Any invariant measure can be expressed as a sum of component ergodic measures (ergodic decomposition).

Ergodic transformation Let $\tau: X \to X$ be a measurable transformation. Then τ is said to be ergodic if and only if μ is an ergodic measure. We then also say that τ is μ-ergodic.

Absolutely Continuous Measure Let $(X, \mathbf{B})$ be a measure space with two measures υ and μ. We say that υ is absolutely continuous with respect to μ, if $\mu(A) = 0$ implies $\upsilon(A) = 0$, for each $A \in \mathbf{B}$. We then write $\upsilon << \mu$.

If $\upsilon << \mu$, then it is possible to represent υ in terms of μ by means of Radon-Nikodym Theorem [105, Chapter III].

Theorem 1 Let $(X, \mathbf{B})$ be a measure space with two σ-finite measures υ and μ. If $\upsilon << \mu$ then there exists a unique (a.e.) $f \in L_1(X,\mathbf{B},\mu)$ such that,

$$\upsilon(A) = \int_A f \, d\mu, \qquad \text{for each } A \in \mathbf{B}.$$

Remark 2 In a recent paper Wilansky [167] has shown that the assumption of υ being σ-finite can be dropped.

Let $M(X)$ denote the space of measures on $(X, \mathbf{B})$. Let $\tau: X \to X$ be a measurable transformation. τ induces a transformation τ_* on $M(X)$ defined by $(\tau_* \mu)(A) = \mu(\tau^{-1}A)$, $A \in \mathbf{B}$ Since τ is measurable, $\tau_* \mu \in M(X)$ and so τ_* is well defined.

Singular and Non-Singular transformations Let $\tau: X \to X$ be a measurable transformation. Then τ is nonsingular, if $\tau_* \mu(A) = \mu(\tau^{-1}A) = 0$, whenever $\mu(A) = 0, A \in \mathbf{B}$ i.e. $\tau_* \mu << \mu$, otherwise τ is said to be singular.

Proposition 1 Let $(X,\mathbf{B},\mu)$ be a normalized measure space and let $\tau: X \to X$ be nonsingular. Then, if $\upsilon << \mu$, we have $\tau_* \upsilon << \tau_* \mu << \mu$.

Density of a measure Let a set $D(X,\mathbf{B},\mu)$ be defined by,

$$D(X,\mathbf{B},\mu) = \{f \in L_1(X,\mathbf{B},\mu) : f \geq 0, \|f\|_1 = 1\}.$$

A function $f \in D(X,\mathbf{B},\mu)$ is called a density of μ. f is said to be an invariant density if the corresponding measure is invariant. If $f \in D(X,\mathbf{B},\mu)$ then by Radon-Nikodym Theorem the normalized measure, $\mu_f(A) = \int_A f \, d\mu << \mu, A \in \mathbf{B}$.

Support of a density The support of a density f is defined by,

$$\text{supp } (f) = \{x \in X: f(x) > 0\}.$$

Support of a measure The support of a measure μ is defined by,

$$\text{supp } (\mu) = \cap \{A \in \mathbf{B}: A \text{ is closed and } \mu(X \setminus A) = 0\}$$

1.3 BIRKHOFF ERGODIC THEOREM

Let $\tau: (X,\mathbf{B},\mu) \to (X,\mathbf{B},\mu)$ be a measure preserving transformation and $A \in \mathbf{B}$. For $x \in X$ a question of physical interest is:

With what frequency do the elements of the orbit $\{\tau^n(x)\}_{n=0}^{\infty}$ lie in the set A?

Now, $\tau^i(x) \in A$ if and only if $\chi_A(\tau^i(x)) = 1$. Thus the number of elements of the orbit $\{\tau^k(x)\}_{k=0}^{n-1}$ in A is equal to $\sum_{k=0}^{n-1} \chi_A(\tau^k(x))$ and the relative frequency of elements of the orbit $\{\tau^k(x)\}_{k=1}^{n-1}$ in A equals $\frac{1}{n}\sum_{k=0}^{n-1} \chi_A \tau^k(x)$. The interest lies in the study of long-term behaviour of the orbit.

In 1931, Birkhoff proved one of the most important theorems of ergodic theory [107]. The motivation for the ergodic theorem came from the work of Bolzmann and Gibbs on statistical mechanics. The mathematical question (ergodic problem) arising from their work was under what conditions the limit,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k x) \qquad (1)$$

exists and is independent of $x \in X$, where $f: X \to \mathbf{R}$ is a real-valued function on the space X and $\tau:X\to X$ is a transformation. This limit is the average value of the function f along the forward orbit of the transformation τ. Birkhoff proved that the limit (1) exists almost everywhere (a.e.) with respect to μ.

1.3.1 THE ERGODIC THEOREM

A precise statement of the Birkhoff Ergodic Theorem is:

Theorem 2 Let $(X,\mathbf{B},\mu)$ be a measure space and $\tau:X\to X$ be a measure-preserving transformation. If $f \in L_1(X,\mathbf{B},\mu)$ (real) then the limit,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)) = f^*(x)$$

exists for almost all $x \in X$. $f^* \in L_1(X,\mathbf{B},\mu)$. Furthermore, $f^* \circ \tau = f^*$ a.e. with respect to μ and if $\mu(X) < \infty$ then, $\int_X f^*\, d\mu = \int_X f\, d\mu$.

1.3.2 CONSEQUENCES OF THE ERGODIC THEOREM

Corollary 1 If τ is μ-ergodic, then f^* is a constant a.e. and if $\mu(X) < \infty$, then

$f^* = \frac{1}{\mu(X)}\int_X f\, d\mu$ in Theorem 2.

We define the time average of $f \in L_1(X,\mathbf{B},\mu)$ to be, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x))$ and the space-average to be, $\frac{1}{\mu(X)}\int_X f(x)\, d\mu$.

Remark 4 If τ is ergodic, by Corollary 1, the time average equals the space average (ergodic hypothesis). In other words, if we consider τ as a dynamic which occurs every unit of time, then for almost all starting points $x \in X$, the average value of the function

f on the orbit of x, as it evolves through time, exists, and is equal to $\frac{1}{\mu(X)}\int_X f\,d\mu$, the average value of a function f on the space X. Identifying the systems for which the ergodic hypothesis holds still remains one of the most intricate problems in Mathematics and Physics [169].

Remark 5 Let τ be μ-ergodic, $f = \chi_A$, $A \in \mathbf{B}$, and $\mu(X) = 1$. Then by Corollary 1,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A(\tau^k(x)) = \mu(A),\ \mu\text{-a.e.} \tag{2}$$

and the orbit of almost every point of X in an ergodic system enters the set A with asymptotic relative frequency $\mu(A)$. It follows that, if τ admits an ergodic measure μ, then for almost every point $x \in \mathrm{supp}(\mu)$, the orbit $\{\tau^k(x)\}_{k=0}^{\infty}$ exhibits μ in the sense of equation (2).

Remark 6 Let $\tau: X \to X$, $X = [0,1]$ be a transformation with an ergodic measure μ which is absolutely continuous with respect to Lebesgue measure m. Let f be the density of μ. Then by Theorem 1 and Remark 5,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A(\tau^k(x)) = \int_A f\,dm,$$

for a.e. $x \in \mathrm{supp}(\mu)$ where $A \subset X$ is a measurable set. f is known as the density of the orbit $\{\tau^k(x)\}_{k=0}^{\infty}$ and is seen in experiments as a stochastic or chaotic behaviour of orbits.

Let $(\mathbf{X},\mathbf{B},\mu)$ be a normalized measure space.

Corollary 2 Let $\tau: (X,\mathbf{B},\mu) \rightarrow (X,\mathbf{B},\mu)$ be measure preserving. Then τ is ergodic if and only if,

$$\frac{1}{n}\sum_{i=0}^{n-1} \mu(\tau^{-i}A \cap B) \rightarrow \mu(A)\,\mu(B), \text{ as } n \rightarrow \infty, \quad A,B \in \mathbf{B}.$$

Motivated by Corollary 2, we have the following definitions:

Weak and Strong mixing transformations

A transformation $\tau: (X,B,\mu) \rightarrow (X,B,\mu)$ is weakly mixing if for all $A,B \in \mathbf{B}$,

$\frac{1}{n}\sum_{i=1}^{n-1} \left|\mu(\tau^{-i}A \cap B) - \mu(A)\,\mu(B)\right| \rightarrow 0$, as $n \rightarrow \infty$, and τ is strongly mixing if for all $A,B \in \mathbf{B}$, $\mu(\tau^{-n}A \cap B) \rightarrow \mu(A)\,\mu(B)$, as $n \rightarrow \infty$.

Mixing implies coarse-graining of the phase-space.

The concept of mixing was introduced in the early 1950's. The following ergodic property was introduced by Rochlin [255] in 1964.

Exact transformation Let $\tau: (X,\mathbf{B},\mu) \rightarrow (X,\mathbf{B},\mu)$ be measure preserving such that $\tau(A) \in \mathbf{B}$, for each $A \in \mathbf{B}$ and $\lim_{n\rightarrow\infty} \mu(\tau^n A) = 1$, for each $A \in \mathbf{B}, \mu(A) > 0$. Then τ is called exact (or μ-exact).

Remark 7 In general we have, τ exact $\Rightarrow$ τ strongly mixing $\Rightarrow$ τ weakly mixing $\Rightarrow$ τ ergodic.

Remark 8 Physical Interpretation of ergodicity, mixing and exactness

If we start with a set $A \in \mathbf{B}$ of initial condition of nonzero measure then after a large number of iterations of an exact transformation τ the points will have spread and completely filled the space X. τ strongly mixing means that any set $B \in \mathbf{B}$ as it moves under τ becomes asymptotically independent of a fixed set $A \in \mathbf{B}$. τ weakly mixing means that B becomes independent of A if we neglect a finite number of times. τ ergodic means B becomes independent of A on the average.

These ergodic properties guarantee the existence of chaotic behaviour of orbits of τ on the phase space X. They also characterize the evolution of entropy of the system to a state of thermodynamic equilibrium [66].

1.3.3 LIMITATIONS OF THE ERGODIC THEOREM

1) If μ is a continuous measure, then every point of X has μ-measure 0, and therefore no matter what starting point x is used, one cannot be certain that it will exhibit μ in the sense of equation (2). The equation (2) holds for almost every point $x \in \text{supp}(\mu)$, yet it is in general impossible to specify a single point x where the equation actually holds.

2) Since the theorem gives no information about the rate of convergence of the time averages, in general, there is no way of knowing how well a finite segment of a theoretical orbit τ exhibits μ. (Only in special cases, such as of Markov maps [34], the speed of numerically computing the orbit $\{\tau^k(x)\}_{k=0}^{\infty}$ can be assessed).

3) There are computational difficulties in implementing the Birkhoff Ergodic Theorem. In many cases, the exceptional set of μ-measure 0 may be prohibitively large. For

example, for a large class of piecewise-linear transformations τ, the rationals are eventually periodic. But the rationals are the only points with which computations can be performed. Thus for such transformations f^* cannot be found in practice by direct iteration of the difference equation, $x_{n+1} = \tau(x_n)$.

1.4 INVARIANT MEASURES OF CHAOTIC DYNAMICAL SYSTEMS

Experiments with dynamical systems usually exhibit a transient behaviour followed by an asymptotic motion lying on an attractor in the phase space. Since an attractor gives a global picture of the physical long-term behaviour of a dynamical system, one can provide a classification of the different levels of complexity of motions by observing its geometric structure. In many situations, determining the geometric structure is not feasible due to extreme complication of the dynamics. In such cases, statistical analysis becomes very important. Instead of studying the attractor itself, we study the statistical behaviour of the system on the attractor, which is often all that is required.

An attractor, if one exists, is usually equipped with an asymptotic measure i.e. an invariant measure which describes how frequently each part of the attractor are visited by the orbit describing the system. We therefore focus our attention on the study of invariant measures rather than on attractors.

The importance of invariant measures in the study of chaotic dynamical systems results from the following facts:

1) The invariant measures are one of the quantitative measures to characterize the chaotic motion which is generated by one-dimensional (Poincaré) maps [152, Sec. 2.2].

2) The invariant measures leave out all transient effects and therefore describe only

asymptotic equilibrium behaviour of the system.

3) The extraction of relevant information from chaotic dynamical systems requires the measurement of quantities that remain invariant under a smooth change of coordinates. One class of such invariants is of static invariants. Typical strange attractors are characterized by an infinite number of them, though only the low order ones are relevant for experimental applications [149]. The static invariants depend primarily on invariant measures (and their supports). Then invariant measures provide a measure-theoretic description of the attractor, if one exists.

An invariant measure is thus a useful tool for describing the asymptotic equilibrium behaviour of a chaotic dynamical system.

1.5 PHYSICAL MEASURES

In general, a dynamical system can have many invariant measures. The Krylov-Bogoliubov Theorem guarantees the existence of an invariant measure for continuous transformations on compact spaces. For example, tent maps on [0,1] possess an infinite number of ergodic measures [80]. A strange attractor, if it exists, typically comes with uncountably many ergodic invariant measures [32, Sec. I, E]. Not all of these invariant measures are physically relevant. For example, if x is an unstable fixed point of a dynamical system (X,τ), then the Dirac measure at point x is an invariant measure, but it is not observed. In physical experiments and computer simulations, it seems that one invariant measure μ is produced by the time that the system spends in various parts of the phase space X. It appears in many physical systems that the computer-generated orbits have well defined time averages whose histograms are approximately those defined by ergodic invariant measures of the system. Thus there is a selection process of the natural or physical measure μ [32,81,82,84,85,112,148]. We are thus led to the following question:

Which ergodic invariant measure is selected by the numerical or computer-generated orbits of the dynamical system?

A satisfactory answer has been obtained, first in the case of Anosov diffeomorphisms [89], then in the case of an Axiom A diffeomorphisms [127,128], and recently in case of a general form of a diffeomorphism having hyperbolic invariant sets [50]. There the time average for almost all initial conditions with respect to the Lebesgue measure in the neighborhood of an attractor yield the same measure on the attractor.

1.6 KOLMOGOROV AND SRB MEASURES

In general, arguments for selecting a physical measure are:

(1) that it describes physical time averages,

(2) it is smooth along unstable directions, and

(3) it is stable under small stochastic perturbations.

One possibility for a physical measure that satisfies (1), (2) and (3) is the Kolmogorov measure [32]. A physical system is usually affected by small random perturbations ε, whose resulting effect is to smear out asymptotic behaviour and thereby provide the system with one invariant measure μ_ε. If μ_ε converges to a measure μ_K, as $\varepsilon \to 0$, then μ_K is called the Kolmogorov measure. For example, the Axiom-A systems [116] and piecewise-expanding transformations of the unit interval [6] have Kolmogorov measures. However, this approach may have difficulties: an attractor does not always have an open basin of attraction, and thus the added noise may force the system to jump around on several attractors.

Another physical measure is the Sinai-Ruelle-Bowen or the SRB measure [32]. (For hyperbolic systems, the SRB measure is a special Gibbs measure [268].) In general, we define it as follows:

Let $(X,\mathbf{B},\mu)$ be a measure space, $\tau: X \to X$ be a transformation and $f \in C(X)$. Then μ is said to be the SRB measure if μ is invariant and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) = \int_X f \, d\mu$$

not just for μ-almost all x, but for all x in a set of positive Lebesgue measure. (Lebesgue measure corresponds to a more natural 'notion of sampling' than the measure μ which is often singular.)

By definition, the SRB measure μ_{SRB}, if it exists, is unique. It is typical that the SRB measure is stable under small random perturbations [12] (though this may not always be the case under computer observation [4]). It has applications to physical contexts such as in giving a thermodynamic description of fractals [268] and in estimating topological entropy and pressure [22,200]. For some systems such as Axiom A systems, μ_{SRB} exists and is in fact equal to μ_K, though often it is easier to study SRB measures. (We note here that there may be physical measures which are not SRB [32]. Keller [286,287] (cf. [284]) has given examples of maps which have no physical measures.)

1.7 FROBENIUS-PERRON OPERATOR

In this section, we consider the Frobenius-Perron operator which was introduced by Rechard in [122]. Following a random distribution of initial states led to a development of the notion of Frobenius-Perron operator and an examination of its properties as a means of studying the asymptotic properties of flows of densities.

Let us suppose that we have a random variable x on $X = [0,1]$ with probability density function $f(x)$. Then for any measurable set $A \subset [0,1]$, Prob $\{x \in A\} = \int f\, dm$, where m is the Lebesgue measure on [0,1]. Let $\tau: [0,1] \rightarrow [0,1]$ be a transformation. Then $\tau(x)$ is also a random variable. We then have the question:

What is the probability distribution of $\tau(x)$?

We can write: Prob $\{\tau(x) \in A\}$ = Prob $\{x \in \tau^{-1}A\} = \int_{\tau^{-1}A} f\, dm$. To obtain a density function for $\tau(x)$, we must write the last integral as $\int_A \phi\, dm$, for some function ϕ. Obviously, if such a ϕ exists, it will depend on f and the transformation τ.

Let us assume that τ is nonsingular and define, $\mu(A) = \int_{\tau^{-1}A} f\, dm$ where $f \in L_1([0,1])$ and A is an arbitrary measurable set. Since τ is nonsingular, $m(A)=0$ implies $\tau_* m = m(\tau^{-1}A) = 0$ which in turn implies that $\mu(A) = 0$. Hence $\mu << m$. Then, by the Radon-Nikodym Theorem, there $\phi \in L^1$ such that for all measurable sets A, $\mu(A) = \int_A \phi \, dm$. ϕ is unique a.e. and depends on τ and f. We define the Frobenius-Perron operator P_τ for τ by setting $P_\tau f = \phi$. Thus $P_\tau f$ is interpreted as the density function of $\tau(x)$. A precise definition of P_τ is:

Frobenius-Perron operator. Let $\tau: [0,1] \to [0,1]$ be a nonsingular transformation. For any $f \in L_1$, define $P_\tau f$ by $\int_A P_\tau f \, dm = \int_{\tau^{-1}A} f \, dm$ where $A \subset [0,1]$ is a measurable set. P_τ is called the Frobenius-Perron operator corresponding to τ.

Remark 9 By Proposition 1, we also have $\tau_* \mu << m$. We note that f is the density of μ and ϕ is the density of $\tau_* \mu$. Therefore, $P_\tau : D(X,\mathbf{B},\mu) \to D(X,\mathbf{B},\tau_* \mu)$. The operators $\tau_* : M(X) \to M(X)$ and $P_\tau: D(X,\mathbf{B},\mu) \to D(X,\mathbf{B},\tau_* \mu)$ are equivalent, but P_τ acts on L_1 which is often easier to work with than the space of measures.

1.7.1 SIGNIFICANCE OF FROBENIUS-PERRON OPERATOR

Apart from the geometric difficulty in studying the strange attractor, there is a measure-theoretic problem. Although the Birkhoff Ergodic Theorem guarantees that orbits will exhibit the measures on the strange attractor, it is true only for almost all initial states of the system. We may have an orbit starting from an exceptional point. For example, as in the case of $\tau: [0,1] \to [0,1]$ defined by, $\tau(x) = 4x(1-x)$. (See Fig. 1). The worst part about these exceptional behaviours is that we have no a priori way of predicting which initial states will lead to them. Inspite of the sensitivity of orbits to initial states this is not usually reflected in the distribution of states within long orbits.

Furthermore, there are no analytic tools for studying individual orbits. (Birkhoff Ergodic Theorem requires the existence of an invariant measure for τ.) Therefore, the study of individual orbits of a dynamical system is often inconvenient. We then have to resort to an alternative approach to avoid these problems. Instead of studying individual orbits, we then study ensembles of orbits and their evolution in time.

Advantage of Frobenius-Perron Operator

Let $\tau: X \to X$ be a transformation and let $\{x_i\}$ be a collection of starting points in X. Then, each point x_i is transformed to a point $\tau(x_i)$. If we regard the initial states as distributed according to a probability density function $f(x)$, then the collection of points $\{\tau(x_i)\}$ is distributed according to a probability density function $P_\tau f$.

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & \cdots & x_i & & f(x) \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & & \downarrow \\ \tau(x_1) & \tau(x_2) & \tau(x_3) & & \tau(x_i) & & P_\tau f(x) \end{array}$$

This approach treats the dynamical system as a stochastic process. Instead of orbits at a point x in the phase space X, we study the evolution of the probability density functions i.e. $\left\{f, P_\tau f, P_\tau^2 f, \ldots, P_\tau^n f, \ldots\right\}$. Often the limits of such sequences are the densities of measures on the strange attractor which are invariant under τ. One of the main advantages of studying P_τ rather than τ is that, while τ is a nonlinear (and often discontinuous) transformation on X, P_τ is a bounded linear operator on $L_1(X)$. Thus in examining the behavior of $\{P_\tau^n\}$ (i.e. studying the asymptotic behaviour of the dynamical system τ) we can apply the powerful tools of linear functional analysis.

1.8 ABSOLUTELY CONTINUOUS INVARIANT MEASURES

An invariant closed set, such as a periodic orbit supports an invariant measure. Thus a transformation τ will in general have a multitude of such measures. Most of these measures will be trivial in the sense that their support form a finite set, and so the invariant measure does not produce any new information about the dynamics of τ. In order to avoid trivial measures, we require that an invariant measure μ be continuous. Thus it gives a measure zero to any finite set of periodic points of τ.

We are particularly interested in those invariant measures which are physically meaningful i.e. the ones that are observed for large sets of initial starting points. Therefore, the interest lies in an invariant measure whose support is a set of positive Lebesgue measure. Then due to Birkhoff Ergodic Theorem, 1), 2) and 3) of Sec. 1.4 have an additional significance in that they describe dynamical behaviour of orbits on a large set of phase space.

One class of such measures is that of τ-invariant measures which are absolutely continuous with respect to Lebesgue measure (ACIMs). The statistical description here is of great value. The purpose of an ACIM is to describe the statistical properties of orbits: the frequency with which an orbit falls into a set is given by the measure of that set. The density f of a finite ACIM is an L^1 function which can be thought of as the density of the distribution of typical points in the support of that measure. When a map which has an ACIM μ, it also has sensitive dependence on initial conditions [192] (though the converse is not true [142,144]), and therefore it implies that chaotic behaviour exists on a set of positive Lebesgue measure of the phase space. The support of f, where f is the density of μ with respect to the Lebesgue measure, indicates that part of the phase space on which the chaos resides.

The following result gives a necessary and sufficient condition for a measure to be an ACIM, and thus brings in the significance of Frobenius-Perron operator in the theory of absolutely continuous invariant measures.

Theorem 3 [114]. Let $(X,\mathbf{B},\mu)$ be a measure space, $\tau: X \to X$ be a nonsingular transformation and P_τ be the Frobenius-Perron operator associated with τ. Consider a nonnegative function $f \in L_1(X,\mathbf{B},\mu)$. Then the measure μ_f given by,

$$\mu_f(A) = \int_A f(x)\,\mu(dx)$$

is invariant if and only if f is a fixed point of P_τ i.e. $P_\tau f = f$.

Proof. First, assume that μ_f is invariant. Then by definition of an invariant measure,

$$\mu_f(A) = \mu_f(\tau^{-1}A), \qquad \text{for each } A \in \mathbf{B} \text{ so that,}$$

$$\int_A f(x)\,\mu(dx) = \int_{\tau^{-1}A} f(x)\,\mu(dx), \quad A \in \mathbf{B} \tag{3}$$

By definition of a Frobenius-Perron operator,

$$\int_{\tau^{-1}A} f(x)\,\mu(dx) = \int_A P_\tau f(x)\,\mu(dx), \quad A \in \mathbf{B} \tag{4}$$

By (3) and (4), we have, $P_\tau f = f$. Conversely, if $P_\tau f = f$ for some $f \in L_1, f \geq 0$ then from the definition of the Frobenius-Perron operator equation (3) follows and μ_f is invariant.

Remark 10. By Theorem 1 and 3 it follows that an absolutely continuous measure is an ACIM if and only if its density function is a fixed point of the Frobenius-Perron operator.

Remark 11. Though the invariant density of an ACIM for a transformation is the fixed point of the Frobenius-Perron operator induced by the transformation, to determine the density explicitly, one has to solve a complicated set of functional equations. Since this is usually very difficult, one then resorts to approximation techniques [38,64,162,279].

1.9 ONE-DIMENSIONAL MAPS WITH ABSOLUTELY CONTINUOUS INVARIANT MEASURES

There is an abundance of maps which have an ACIM in one-dimension. In this section, we consider some examples of maps $\tau: X \to X$, $X = [0,1]$ (unless stated otherwise) which admit a finite ACIM.

Example 1 Lasota-Yorke Maps

τ is said to be piecewise monotonic and C^v if there exists a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ of X of X such that for each $i = 1,\ldots,n$, $\tau_i = \tau|_{(a_{i-1}, a_i)}$ is a C^v function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^v function. τ need not be continuous at a_i. A differentiable map τ is said to be, $C^{1+\alpha}$ if τ' is a Hölder continuous function with exponent $\alpha > 0$. It was shown by Lasota-Yorke in [57] that if τ is piecewise monotonic, C^2 and $\inf|\tau'| > 1$ (i.e. τ is uniformly expanding) then τ has an ACIM. The number of monotonic segments of τ is shown to be an upper bound for the number of independent ACIMs [103]. The k-adic maps $\tau_k: X \to X$ given by $\tau_k(x) = kx \pmod 1$ considered by Rényi in [123] are of these type. A similar result was proved by Adler in [126] and is known as the Folklore Theorem.

The result of Lasota-Yorke was extended to a larger class of maps by Wong in [100], who showed the existence of ACIMs for maps which are piecewise monotonic, C^1, $\inf |\tau_i'| > 1$ and $\frac{1}{|\tau_i|}$ is of bounded variation on $[a_{i-1}, a_i]$ for each $i = 1,\dots, n$. It has been shown by Gora-Schmitt in [150] that the last condition cannot be dispensed with. Another extension is due to Rychlik in [166] who has shown the existence of an ACIM for maps which are piecewise monotonic, $C^{1+\alpha}$ and expanding.

Example 2 Misiurewicz Maps

Let A be a finite subset of an interval X containing its endpoints. Let $\tau : X \setminus A \to X$ be a continuous map, strictly monotone on each component of $X \setminus A$, which satisfies the following conditions:

(1) τ is of class C^3 on $X \setminus A$

(2) $\tau'(x) \neq 0$, for $x \in X \setminus A$

(3) τ has a non-positive Schwarzian derivative [96] i.e.

$$S\,\tau(x) = \frac{\tau'''(x)}{\tau'(x)} - \frac{3}{2}\left(\frac{\tau''(x)}{\tau'(x)}\right)^2 \leq 0 \text{ on } X \setminus A$$

(4) If $\tau^k(x) = x$ then $|D\tau^k(x)| > 1$

(5) There is a neighbourhood U of A such that for every $a \in A$ and $n \geq 0$, $\tau^n(a) \in A \cup (X \setminus U)$

(6) For every $a \in A$ there is a neighbourhood W_a of a and constants

$C_1, C_2 > 0$, $u \geq 0$ such that
$C_1 |x - a|^u \leq |\tau(x)| \leq C_2 |x - a|^u$, for $x \in W_a$.

(7) $|\tau'(x)| > 1$, for $x \in J \setminus U$

(8) If $a \in A$ is a periodic point for τ, then it is a fixed point for τ.

That is, a map τ satisfies Misiurewicz conditions if it has nonpositive Schwarzian derivative, no sinks and orbits of critical points stay away from critical points. Since τ may have critical points, it is not uniformly expanding. It was shown by Misuirewicz in [67] that τ has an ACIM. It has further been shown by Benedicks-Misiurewicz in [141] that if τ satisfies conditions (1)-(5) and we do not allow critical points to be mapped onto critical points, then a necessary and sufficient condition for τ to have an ACIM is, $\int_X \log |\tau'(x)| \, dx < -\infty$.

Ulam-Von Neumann in [124] had shown the existence of an ACIM for the logistic map $\tau(x) = 1 - 2x^2$, which satisfies the conditions (1)-(8).

Misiurewicz showed the existence of an ACIM for $\tau(x) = \lambda x(1 - x)$ for uncountably many parameter values λ, and this result was extended by Yacobson in [125] for a set $U \subset [2,4]$ of λ's with positive Lebesgue measure. Alternative approaches to Yacobson's result are obtained by Benedicks-Carleson [143], Guckenheimer [154], Johnson [201] and Rychlik [283]. Chernov [196] has derived a characteristic condition for λ to belong to U in terms of kneading theory and Farmer [228] has derived a scaling law which gives a precise estimate for the λ's that lead to chaos.

Example 3 Collet-Eckmann Maps

Let $\tau: [-1,1] \rightarrow [-1,1]$ be a map satisfying the following conditions:

(1) τ is defined on $Y = [\tau(1), 1]$ and takes values in Y. It is strictly increasing on $[\tau(1), 0]$ and strictly decreasing on X. $\tau(0) = 1$

(2) τ is of class C^1

(3) The function τ' is Lipschitz continuous, and $|\tau'|^{-\frac{1}{2}}$ is convex on $[\tau(1), 0]$ and on X.

(4) $\limsup_{x \to 0} \left|\frac{\tau'(x)}{x}\right| < \infty, \ \inf_{x \in Y} \left|\frac{\tau'(x)}{x}\right| > 0$

(5) There is a $c_1 > 0$ and $\theta > 0$ such that

(a) $|D\tau^n(1)| \geq c_1 \exp(n\theta)$, for all $n \geq 0$

(b) If $\tau^m(z) = 0$ for some $m > 0$, then

$|D\tau^m(z)| \geq c_1 \exp(m\theta)$.

The conditions (3) and (4) can be replaced by the condition that τ has a non-positive Schwarzian derivative. It is shown by Collet-Eckmann in [21] that τ has an ACIM.

Remark 12 The condition $Sf \leq 0$ on the Schwarzian derivative, although very powerful, has the disadvantage of being too restrictive (e.g. it is not invariant under C^∞ change of coordinates). This condition can be replaced by smoothness conditions [97]. It has been shown by van Strien [90] and Nowicki-van Strien in [160] that we can have the existence of ACIMs for Misiurewicz maps and for Collet-Eckmann maps (using the results of [258]) respectively while dispensing the condition on the Schwarzian derivative.

Example 4 Markov Maps

A map $\tau: X \to X$ is said to be C^2 Markov if there exists a countable family $\{I_j\}$ of disjoint open intervals in X such that,

(1) $\mu_L (X - \cup_j I_j) = 0$, μ_L is the Lebesgue measure,

(2) for every j, there is a set K of indices such that,

$$\tau(I_j) = \bigcup_{k\in K} I_k \ (\text{mod } 0),$$

(3) for every $x \in \cup I_j$, τ' exists and $|\tau'(x)| \geq \alpha$ for some fixed $\alpha > 0$,

(4) there exists a $\beta > 1$ and $n_0 > 0$ such that, if

$\tau^m(x) \in \bigcup_j I_j$ for all $0 \leq m \leq n_0 - 1$, then $|D\tau^{n_0}(x)| \geq \beta$,

(5) there exists $m > 0$ such that $\mu_L(\tau^{-m}(I_j) \cap I_i) \neq 0$ for every i, j,

(6) there exist $C > 0$ and $0 < Y < 1$ such that,

$$\left|\frac{\tau'(x)}{\tau'(y)} - 1\right| \leq C|x - y|^Y.$$

It was shown by Adler in [126] and Bowen in [111] that τ has a unique ACIM (see [113], Chapter III). The result has been extended to $C^{1+\alpha}$ Markov maps by Bose in [11].

<u>Example 5</u> <u>Random Maps</u>

A random map of [0,1] is a stochastic process $\tau(x)$ specified by a finite collection of measurable functions $\tau_i : [0,1] \to [0,1]$, $i = 1,\ldots,n$, and a probability vector $p = (p_1,\ldots,p_n)$ i.e. $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$.

We define $\tau(x) = \tau_i(x)$ with probability p_i, and assume that the selection of functions is an independent identically distributed process so that

$$\tau^m(x) = \tau_{i_m} \circ \tau_{i_{m-1}} \circ \ldots \circ \tau_{i_1}(x) \quad \text{with probability} \quad \prod_{j=1}^{m} p_{i_j}.$$

A measure μ on [0,1] is said to be τ-invariant if $\mu(A) = \sum_{i=1}^{n} p_i\,\mu(\tau_i^{-1}(A))$ for each measurable set A. It was shown by Pelikan in [161] that if each τ_i is piecewise monotonic and C^2, and

$$\sum_{i=1}^{m} \frac{p_i}{|\tau_i'(x)|} < 1,$$

then τ has an ACIM. The existence of ergodic ACIMs for certain random maps of [0,1] has also been shown by Morita in [147,184].

Some other existence results for ACIMs for one-dimensional piecewise monotonic maps are established in [58,59,61,73,87,88,142,261-267, 280,284,285]. Results of existence of ACIMs for higher dimensional maps are obtained in [177,178,180,273-276,278].

1.10 ABSOLUTELY CONTINUOUS INVARIANT MEASURE AS A PHYSICAL MEASURE

For many transformations τ, the existence of μ_K and μ_{SRB} is consistent with the observed fact that it is the ACIM which appears in numerical and computer experiments [13,20]. That is, of the uncountable number of ergodic invariant measures available to τ, the computer-generated orbits of τ select the ACIM which is then also the physical measure.

A problem of interest is then to characterize the systems for which the SRB measure is absolutely continuous with respect to the Lebesgue measure. We have the following result in that direction:

Theorem 4 Let (X,B,μ) be a normalized measure space and $\tau: X \to X$ be a measurable transformation. If μ is the unique ACIM for τ, then μ is the SRB measure if and only if τ is μ-exact.

Proof Let m denote the Lebesgue measure on X. The SRB measure of τ is defined as the weak limit of $\{\tau_*^n m\}$ as $n \to \infty$, if it exists. Let f be the density of μ. Now, $\tau_*^n m = (P_\tau^n 1)\, dm$, where P_τ is the Frobenius-Perron operator corresponding to τ.

Since $\{P_\tau^n 1\}$ is a weakly compact set in $L_1(X,m)$, it follows that $\{P_\tau^n 1\}$ converges to f if and only if τ is exact [114, Sec. 4.4]. Therefore, it follows that if τ has a unique ACIM μ, then μ is the SRB measure if and only if τ is μ-exact.

Remark 13 A similar characterization of SRB measures for piecewise expansive Markov maps was obtained in [23].

Statistical Stability Let $\tau : X \to X$ be a nonsingular transformation and P_τ be the Frobenius-Perron operator corresponding to τ. Then τ is said to be statistically stable if there exists a unique $f_* \in D$ such that $P_\tau f_* = f_*$ and

$$\lim_{n \to \infty} \| P_\tau^n f - f_* \| = 0, \text{ for each } f \in D.$$

We then have,

Proposition 2. Let $(X,\mathbf{B},m)$ be a normalized measure space and $\tau : X \to X$ be an m-invariant transformation, where m is the Lebesgue measure. If τ has the SRB measure μ which is absolutely continuous with respect to m, then τ is statistically stable.

Proof By Theorem 4 it follows that μ is the unique invariant measure with respect to which τ is exact. Furthermore, since τ is m-invariant, we have from [193] that τ is statistically stable.

1.10.1 EXAMPLES

Example 11 Let $\tau_\lambda = [0,1] \to [0,1]$, $\lambda > 1$, be a triangle map (Fig. 2) defined by,

$$\tau_\lambda(x) = \begin{cases} \lambda x, & 0 \le x \le \frac{1}{\lambda} \\ \frac{\lambda}{\lambda - 1}(1 - x), & \frac{1}{\lambda} \le x \le 1 \end{cases}$$

Let P_λ be the Frobenius-Perron operator corresponding to τ_λ. Then,

$$P_\lambda f(x) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right) + \left(\frac{\lambda-1}{\lambda}\right) f\left(\frac{\lambda}{\lambda-1}(1-x)\right), \quad f \in L_1([0,1], m)$$ where m is the

Lebesgue measure. Since $P_\lambda 1 = 1$, m is τ_λ-invariant. τ_λ is piecewise monotonic, expanding, C^2 and has only one turning point. So τ_λ is a Lasota-Yorke map with the unique ACIM m. Furthermore, it is shown in [129] that τ_λ is m-exact. Hence by Theorem 4 for any $\lambda > 1$, the Lebesgue measure m is the SRB measure for τ_λ, and hence the physical measure.

Example 12 Let $\tau: [0,1] \to [0,1]$ be the quadratic map $\tau(x) = 4x(1 - x)$. τ is a non-expanding Misiurewicz map and has the unique ACIM,

$$\mu(x) = \frac{1}{\pi\sqrt{x(1-x)}} m(x),$$

where m is the Lebesgue measure. Furthermore, since τ is topologically conjugate to the tent map τ_2 of Example 11, and exactness preserves topological conjugacy, τ is μ-exact. Therefore, by Theorem 4, μ is the SRB measure for τ.

Example 13 Let $\tau : [0,1] \to [0,1]$ be the Gauss transformation (Fig. 3) defined by,

$$\tau(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $[x]$ denotes greatest integer $\leq x$. τ is one of the most frequently studied examples of chaotic systems [248]. τ is countably piecewise expanding and preserves the Borel measure on [0,1] given by

$$\mu_G(A) = \frac{1}{\ln 2} \int_A \frac{1}{1+x}\, dm \ ,$$

where $A \subset [0,1]$ is a Borel set and m is the Lebesgue measure on [0,1]. μ_G is known as the Gauss measure and is the unique ACIM. Furthermore, by Theorem 1.2 of Chapter III of [113], τ is μ_G-exact. Hence, by Theorem 4, μ_G is the SRB measure for τ.

Remark 14 In general, if $\tau : [0,1] \to [0,1]$ is any Markov transformation, there exists a unique τ-invariant probability measure μ on the Borel σ-algebra of [0,1] which is absolutely continuous with respect to Lebesgue measure and τ is exact with respect to μ [113]. Then by Theorem 4, μ is the SRB measure.

CHAPTER 2

MEASURES ON PERIODIC ORBITS

2.1 MOTIVATION

For a dynamical system (X,τ), we are interested in the study of asymptotic behaviour of the orbit $\{\tau^k(x)\}_{k=0}^{\infty}$. The utility of periodic-orbit description is well-known [3,16,28,130,151,239]: the set of all periodic orbits can be used to characterize a strange attractor [28,151] or a fractal chaotic attractor [130]. Periodic orbits are topological invariants i.e. any change of coordinates will not change the periodicity of an orbit. Thus changing the point of observation or the variable that is being observed in an experiment will not change the cycle structure. This is important since the characterization of the attractor should be robust. It is then of interest to study the nature of the distribution of periodic orbits of τ. It was Bowen [7] who first considered this problem and obtained measures with maximum entropy as the limit of measures concentrated on periodic orbits. In [104,115] generic properties of invariant measures defined on periodic orbits of Axiom A diffeomorphisms were studied.

Suppose $X = [0,1]$ and $\tau: X \to X$ has an ACIM μ. Computer simulation of the iterated system, $x_{n+1} = \tau(x_n)$, $n = 0,1,2,...$ often show that the computer-generated orbit have histograms which are close to f, the density of μ, i.e. they exhibit f in terms of Birkhoff Ergodic Theorem. Since a computer is a finite-state machine, a computer orbit necessarily must be periodic. This has led to the following question:

Do the (true) periodic orbits of τ exhibit any of the invariant measures associated with it? If so, which ones?

In [49], the asymptotic distribution of periodic orbits in one- dimensional chaotic systems with measures absolutely continuous with respect to Lebesgue measure, is investigated and rigorous limit theorems are proved, which support the heuristic claim that the empirical distribution along a 'typical' periodic orbit is close to the 'observed' invariant distribution of the transformation.

In this Chapter, we consider the above question for a large class of one-dimensional transformations. We begin with the discussion of the question for a simple tent map.

2.2 MEASURES ON PERIODIC-ORBITS OF THE TENT MAP

Consider the tent map τ_2: $[0,1] \to [0,1]$ defined by,

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

which is a prototype of one-dimensional expanding maps. It has periodic orbits of all periods and aperiodic orbits which are chaotic. We shall prove the existence of long periodic orbits of τ_2 which exhibit Lebesgue measure, the unique ACIM.

2.2.1 CONSTRUCTION OF A SET OF PERIODIC ORBITS OF THE TENT MAP

Consider the set, $D = \{0 < \frac{2x}{p^N} < 1, (x,p) = 1\}$, p is an odd prime and N is a positive integer. Let $\tau = \tau_2|_D : D \to [0,1]$. Then $\tau : D \to D$. We then have the following:

Theorem 1 [92]. The map $\tau : D \to D$ is an isomorphism. Let $k = k\,(p^N)$ be the minimum integer ≥ 1 such that $p^N \mid 2^k \pm 1$. Then the number of points in each orbit is k and, the number of periodic orbits is $\frac{[(p-1)\,p^{N-1}]}{2k}$ i.e. τ induces a partition of

D into disjoint periodic orbits of equal length.

Proof Since,

$$\tau\left(\frac{2x}{p^N}\right) = \begin{cases} \dfrac{2\,(\text{even integer})}{p^N}, & \text{if } \dfrac{2x}{p^N} < \dfrac{1}{2} \\[2ex] \dfrac{2\,(\text{odd integer})}{p^N}, & \text{if } \dfrac{2x}{p^N} > \dfrac{1}{2} \end{cases}$$

τ is an isomorphism.

If $\frac{2x}{p^N}$ has a period 1, then $2^1\left(\frac{2x}{p^N}\right) \pm c = \frac{2x}{p^N}$, $c \in Z$, so that, $p^N \mid 2^1 \pm 1$, since k is minimal, 1 must be a multiple of k.

Let $\lambda(p^N)$ be the order of $2\,(\text{mod}\, p^N)$ ie. $\lambda(p^N)$ is the minimum integer ≥ 1 such that $p^N \mid 2^{\lambda(p^N)} - 1$. Now, $2^{2k} - 1 = (2^k - 1)(2^k + 1)$. So, $p^N \mid 2^{2k} - 1$ and we have that $k \leq \lambda(p) \leq 2k$ ie. $1 \leq \frac{2k}{\lambda(p)} \leq 2$. Since $\frac{2k}{\lambda(p)}$ is an integer, $\lambda(p^N) = k, 2k$.

Thus, $k = \lambda(p^N), \frac{\lambda(p^N)}{2}$. Clearly, if $\lambda(p^N)$ is odd the first case occurs and if $\lambda(p^N)$ is even, the second case occurs.

We now show that, if $\frac{2x_1}{p^N} \in D$, then $\tau_2^k\left(\frac{2x_1}{p^N}\right) = \frac{2x_1}{p^N}$. If $x \in \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right]$ ie. $\frac{i}{2^m} \leq x \leq \frac{i+1}{2^m}$ then $i \leq 2^m x \leq i+1$ Thus, $0 \leq 2^m x - i \leq 1$ or $0 \leq -2^m x + i + 1 \leq 1$ according to as i is even or odd.

We therefore write,

$$2^m x - i, \qquad x \in \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right], \text{ for } i \text{ even}$$

$\tau_2^m(x) =$ (1)

$$-2^m x + i + 1, \qquad x \in \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right], \text{ for } i \text{ odd}$$

Let $x = \frac{2x_1}{p^N}$ Then, $\frac{i}{2^m} < \frac{2x_1}{p^N} < \frac{i+1}{2^m}$, ie. $ip^N < 2^{m+1}x_1 < (i+1)\,p^N$

or $0 < \frac{2^{m+1}x_1}{p^N} - i < 1$.. Since i is an integer,

$$i = \left[\frac{2^{m+1}x_1}{p^N}\right], \quad [\ ] \text{ is the greatest integer} \leq x \tag{2}$$

By (1) and (2) we then have,

$$\tau_2^k\left(\frac{2x_1}{p^N}\right) = \frac{2^{k+1}x_1}{p^N} - \left[\frac{2^{k+1}x_1}{p^N}\right], \qquad \text{if } \left[\frac{2^{k+1}x_1}{p^N}\right] \text{ is even} \tag{3}$$

$$\tau_2^k\left(\frac{2x_1}{p^N}\right) = -\frac{2^{k+1}x_1}{p^N} + \left[\frac{2^{k+1}x_1}{p^N}\right] + 1, \qquad \text{if } \left[\frac{2^{k+1}x_1}{p^N}\right] \text{ is odd} \tag{4}$$

We can write, $\frac{2^{k+1}x_1}{p^N} = \frac{(2^{k+1} \pm 2)x_1}{p^N} = \frac{2x_1}{p^N}$.

Now, if $p^N | 2^{k+1}$, then $\left[\frac{2^{k+1}x_1}{p^N}\right] = \frac{(2^{k+1} + 2)x_1}{p^N} - 1$ is odd, and so by (4), we

obtain, $\tau^k\left(\frac{2x_1}{p^N}\right) = \frac{2x_1}{p^N}$. Similarly, if $p^N | 2^k - 1$, $\left[\frac{2^{k+1}x_1}{p^N}\right]$ is even, and so by (3),

$\tau^k\left(\frac{2x_1}{p^N}\right) = \frac{2x_1}{p^N}$.

Thus, the period of $\frac{2x_1}{p^N}$ is exactly k. Since Card (D) = $\left(\frac{p-1}{2}\right)p^{N-1}$, so the number of periodic orbits is $\frac{\text{Card (D)}}{k} = \frac{\left[(p-1)p^{N-1}\right]}{2k}$.

Remark 1. Theorem 1 has been generalized to a piecewise linear map $\tau : [0,1] \to [0,1]$ with an infinite number of partition points in [233], where it has been shown that if τ is restricted to certain domains, then an explicit bound can be obtained for the number of periodic orbits of τ in that domain.

2.2.2 LONG PERIODIC ORBITS OF THE TENT MAP EXHIBIT LEBESGUE MEASURE

We shall need the following

Lemma 1. Let $p \mid 2^{\lambda(p)} - 1$, where $\lambda(p) \geq 1$ is minimal and assume $p^{m'} \| 2^{\lambda(p)} - 1$ ie. the division is exact. Then for any $N \geq m$, $p^N \mid 2^{\lambda'} - 1$ if and only if λ' is a multiple of $\lambda(p)\, p^{N-m'}$ and the minimum value of $\lambda' > 0$ such that $p^N \mid 2^{\lambda'} - 1$ equals $\lambda(p)\, p^{N-m'}$.

Proof For N = m', the result is obvious by hypothesis ($p^{m'} \| 2^{\lambda(p)} - 1$ since $\lambda(p)$ is minimal). Assume the result, for $N = l > m'$.

Then, $p^{l+1} \mid 2^{\lambda'} - 1$ which implies, $p^l \mid 2^{\lambda'} - 1$ so that $\lambda' = z.\lambda\,(p)\, p^{l-m'}$.

Therefore, $$p^{l+1} \mid 2^{z\lambda(p)\, p^{l-m'}} - 1. \qquad (5)$$

Now, $$2^{z\lambda(p)\, p^{l-m'}} - 1 = (1 + yp^{m'})^{zp^{l-m'}}, \quad (y,p) = 1$$
$$= zyp^l + \text{multiple of } p^{l+1} \qquad (6)$$

By (5) and (6), we have that p | z, and so the lemma follows by induction.

We now prove our main result of the section, Theorem 2 [92]. The proportion of points in any fixed periodic orbit of $\tau_2|_D$ which lies in an interval [a,b] approaches m([a,b]) = b-a, where m is the Lebesgue measure, as $p^{N-m'} \to \infty$, i.e.

$$\lim_{\substack{p^{N-m'}\to\infty \\ x\in D}} \frac{1}{k}\sum_{i=0}^{k} \chi_{(a,b)}(\tau^i(x)) = \int_0^1 \chi_{(a,b)}(x)\, dx \qquad (7)$$

Proof We can have the representation,

$$2^{m+1}x_1 = q_m p^N + r_m, \qquad q_m, r_m \in Z, \quad 0 < r_m < p^N \text{ and } \frac{2x_1}{p^N} \in D.$$

Let m vary from 1 to 2k. Then, we have, $\frac{2^{m+1}x_1}{p^N} = q_m + \frac{r_m}{p^N}$ and by (3) and (4)

$$\tau^m\left(\frac{2x_1}{pN}\right) = \begin{cases} \frac{r_m}{p^N}, & r_m \text{ even} \\ \frac{p^N - r_m}{p^N}, & r_m \text{ odd} \end{cases}$$

Now, Card $\left\{m: \tau^m\left(\frac{2x_1}{p^N}\right) \in (a,b),\ 1 \le m \le 2k\right\}$ by definition,

$$= \text{Card}\left\{m: \frac{r_m}{p^N} \in (a,b), r_m \text{ even}; \left(\frac{p^N - r_m}{p^N}\right) \in (a,b), r_m \text{ odd}\right\}$$

$$= \text{Card}\left\{m: p^N a < r_m < p^N b, r_m \text{ even}; p^N a < p^N - r_m < p^N b, r_m \text{ odd}\right\} \qquad (8)$$

By Lemma 1, $k = k'.\, p^{N-m'}$, $k' = k(p)$ and $N \ge m'$. Let $r'_1, \ldots, r'_{k^*}$ be the residue classes of $r_i \pmod p$ i.e. the residue classes when N=1. Then, $k^* = k'$ or $2k'$ and $r_m = p^{m'}i + r'_j$, $0 \le i < p^{N-m'}$, $1 \le j \le 2k'$ for $N \ge m'$. So, the RHS of (8) is,

$$\text{Card}\left\{p^N a < p^{m'} i + r'_j < p^N b,\ i + r'_j \text{ even};\ p^N a < p^N - p^{m'} i - r'_j < p^N b,\ i + r'_j \text{ odd}\right\}$$

$$+ \sum_{j=1}^{2k'} \text{Card}\left\{i:\ p^N a + r'_j - p^N < -p^{m'} i < p^N b + r'_j - p^N,\ i + r'_j \text{ odd}\right\}$$

$$= \sum_{j-1}^{2k'} \text{Card}\left\{i:\ p^N a - r'_j < p^{m'} i < p^N a - r'_j\ ,\ \ i + r'_j \text{ even}\right\}$$

$$= \sum_{j=1}^{2k'} \left[\frac{p^{N-m'}(b-a)}{2} + O(1)\right] + \sum_{j=1}^{2k'} \left[\frac{p^{N-m'}(b-a)}{2} + O(1)\right],$$ (since the number of even

(odd) integers in an interval (a,b) is $\frac{(b-a)}{2} + O(1))$,

$= 2k'p^{N-m'}(b-a) + O(k')$.

Dividing throughout by $2k = 2k'\,p^{N-m'},\ N \geq m'$, we obtain

$$\text{Card}\left\{m:\ \tau^m\left(\frac{2x_1}{p^N}\right) \in (a,b),\ 1 \leq m \leq k\right\} = (b-a) + O\left(\frac{1}{p^N \text{-} m'}\right).$$

Therefore, for $x \in D, \frac{1}{k}\sum_{i=1}^{k} \chi_{[a,b]}(\tau^i(x)) \to b - a = m([a,b])$, as $p^{N-m'} \to \infty$

i.e. $$\lim_{\substack{p^{N-m'} \to \infty \\ x \in D}} \frac{1}{k}\sum_{i=1}^{k} \chi_{[a,b]}(\tau^i(x)) = \int_0^1 \chi_{[a,b]}(x)\,dx$$

This ends the proof of the theorem.

2.2.3 REMARKS

Remark 2 The points of D for which equation (7) holds are known, thereby lending useful meaning to the a.e. statement in the Birkhoff Ergodic Theorem. Therefore, Theorem 2 is also known as One-Dimensional Practical Ergodic Theorem. A two-dimensional analogue of Theorem 2 has been obtained in [93].

Remark 3 The result of Theorem 2 carries over to maps topologically conjugate to τ_2, e.g. $\tau(x) = 4x(1-x)$.

Remark 4 Theorem 2 is a number theoretic result, and hence has its limitations. To extend the result to a larger class of transformations we have to use different techniques.

2.3 PERIODIC ORBIT MEASURE (POM) PROPERTY FOR CONTINUOUS MAPS

Let (X,d) be a compact metric space and $\tau : X \to X$ be a continuous map. It is shown in [115] that if τ has the specification property [91], then any invariant measure can be approximated by a sequence of measures supported on periodic orbits. We discuss this in detail in Appendix A.

Let $\tau: X \to X$, $X= [0,1]$ admit a τ-ergodic measure μ. In this section, we prove that if τ is continuous and piecewise expanding transformation which takes intervals of its defining partition eventually onto all of the space, then long periodic orbits of τ will display the measure μ. We need the following definitions:

Periodic Orbit Measure (POM) property Let $\tau: X \to X$, and μ be any τ-ergodic measure. If for any such measure there exists a sequence of periodic orbits

$$\gamma_n = \{x, \tau(x), \ldots, \tau^{p(n)}(x) = x\} \text{ such that, } \mu_{\gamma_n} = \frac{1}{p(n)} \sum_{i=0}^{p(n)-1} \delta_{\tau^i(x)} \to \mu \text{ weakly as } n \to \infty,$$

where δ is the Dirac measure, then we say that τ has the POM property.

Eventually onto transformation A transformation $\tau: X \to X$, is called eventually onto, if for any non-trivial interval $K \subset X$ there exists an integer M such that, $\tau^M(K) = X$.

Uniformly eventually onto transformation The map $\tau: X \to X$ is called uniformly eventually onto, if and only if $\forall\, \delta > 0\ \exists\ N(\delta)$ such that $\forall\, K \subset X,\ m(K) \geq \delta$, we have $\tau^{N(\delta)}(K) = X$, K is any non-trivial interval.

We then have the following:

Theorem 3 [36]. Let $\tau: X \to X$, $X = [0,1]$ be continuous and eventually onto. Then τ has the POM property.

Proof If τ is continuous and eventually onto, it follows by definition that it is uniformly eventually onto. Then for any given $\varepsilon > 0\ \exists$ and $N(\delta) \in \mathbb{N}$ such that, if an interval $K \subset X$ and $m(K) \geq \delta$, then $\tau^{N(\delta)}(K) = X$.

Let μ be a τ-ergodic measure. Without loss of generality, we assume that μ is a continuous measure. Let R be the set of all μ-regular points

i.e. $$R = \left\{ x \in [0,1]: \forall\, g \in C[0,1], \frac{1}{n}\sum_{k=0}^{n-1} g(\tau^k x) \to \mu(g), \text{ as } n \to \infty \right\}.$$

Then $\mu(R) = 1$. Let $\{g_n\}_{n=1}^{\infty} \subset C[0,1]$ be a sequence of functions which is dense in $C[0,1]$.

Let $M_n = \max_{1 \leq i \leq n} \{\| g_i \|\}$ and ω_n be the common modulus of continuity for $g_1, \ldots, g_n$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence, such that $\varepsilon_n \downarrow 0$ as $n \to \infty$.

Construction of the periodic orbits γ_n

Fix $n \in N$. Let $\delta > 0$ be a small number such that

$$\omega_n(\delta) < \frac{\varepsilon_n}{4}. \tag{9}$$

Let us fix $x_n \in R$ and then fix N_n such that $\forall\, N > N_n$

$$\left|\frac{1}{N}\sum_{k=0}^{N-1} g_i\left(\tau^k(x_n)\right) - \mu(g_i)\right| < \frac{\varepsilon_n}{4} \text{ for each } 1 \le i \le n \qquad (10)$$

Choose an integer $T_n > N_n$ such that,

$$\frac{N(\delta).\, M_n}{\tau_n} < \frac{\varepsilon_n}{4} \qquad (11)$$

Let K be a closed interval of diameter δ such that

$$\tau^{T_n}(x_n) \subset K \qquad (12)$$

Consider all the intervals $\tau_{i_1}^{-1} \circ \tau_{i_2}^{-1} \circ \ldots \circ \tau_{i_n}^{-1}(K)$, where

$$\tau_{i_j} = \frac{\tau}{[c_{i_j} - 1, c_{i_j}]}, \qquad j = 1, \ldots, T_n \text{ and } i_j = 1, \ldots, n_o \text{, for some } n_o.$$

Then by (11) x_n belongs to one of these intervals, say L. Now, $\tau^{T_n}(L)$ is some interval in K.

We can, if required make L smaller to have, $m(\tau^k(L)) \le \delta$ for $k = 0, 1, \ldots, T_n$.

If $\tau^{T_n}(L)$ has diameter less than δ, we must increase it. Since τ is eventually onto, there exists a smallest positive integer t such that, $\bar{K} = \tau^{T_n + t}(L)$ has diameter greater or equal to δ. Thus we have,

$$\tau^k(x_n) \in \tau^k(L) \text{ and } m(\tau^k(L)) \le \delta, \qquad k = 0, 1, \ldots, \tau_n + t - 1.$$

Furthermore, $\tau^{N(\delta)}(\bar{K}) = [0,1]$, Since τ is eventually onto, so that, $\tau^{T_n + t + N(\delta)}(L) = [0,1] \supset L$ which implies that, $\tau^{T_n + t + N(\delta)}$ has a fixed point, say y_n, in L (since $\tau^{T_n + t + N(\delta)}$ is continuous). Thus, $\tau^k(y_n) \in \tau^k(L)$, for each $k = 0, 1, 2, \ldots$ Define γ_n to be the orbit of y_n under τ.

POM property of τ

We have, $\left|\mu_{\gamma_n}(g_i) - \mu(g_i)\right| = \left|\frac{1}{T_n + t + N(\delta)} \sum_{k=0}^{T_n+t+N(\delta)-1} g(\tau^k(y_n)) - \mu(g)\right|$

$$\leq \left|\frac{1}{T_n + t + N(\delta)} \sum_{k=0}^{T_n+t-1} g_i\left(\tau^k(y_n)\right) - g_i\left(\tau^k(x_n)\right)\right|$$

$$+ \left|\frac{1}{T_n + t + N(\delta)} \sum_{k=T_n+t}^{T_n+t+N(\delta)-1} \left[g_i\left(\tau^k(y_n)\right) - g_i\left(\tau^k(x_n)\right)\right]\right|$$

$$+ \left|\frac{1}{T_n + t + N(\delta)} \sum_{k=0}^{T_n+t+N(\delta)-1} \left[g_i\left(\tau^k(x_n)\right) - \mu(g_i)\right]\right|$$

$$\leq \omega_n(\delta) + \frac{1}{T_n} 2.N(\delta).M_n + \frac{\varepsilon_n}{4},$$

$$\leq \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{4}, \qquad \text{by (9), (10), and (11)}$$

$$= \varepsilon_n. \qquad (13)$$

So now we have, $\mu_{\gamma_n}(g) \to \mu(g)$, $\forall\ g \in C[0,1]$. Let $\{g_{n_k}\}$ be a subsequence of $\{g_n\}$ such that,

$$\|g_{n_k} - g\| \to 0 \qquad \text{as } n_k \to \infty \qquad (14)$$

We then have, for any $n_k \leq n$,

$$\left|\mu_{\gamma_n}(g) - \mu(g)\right| \leq \left|\mu_{\gamma_n}(g) - \mu_{\gamma_n}(g_{n_k})\right| + \left|\mu_{\gamma_n}(g_{n_k}) - \mu(g_{n_k})\right| + \left|\mu(g_{n_k}) - \mu(g)\right|$$

by (13) and (14), $< \|g - g_{n_k}\| + \varepsilon_n + \|g_{n_k} - g\| \to 0$, as $n \to \infty$

Hence τ has the POM property.

Remark 5 Let $\tau: X \to X$ be a map with a unique ACIM μ. Then μ is τ-ergodic and Theorem 3 is applicable.

Remark 6 The map τ_2 considered in Sec. 2.2 is continuous and eventually onto, hence by Theorem 3 it has the POM property.

2.3.1 EXAMPLE

Consider the partition of [0,1] given as follows:

$$0 < \frac{1}{2} < \frac{3}{4} < \frac{7}{8} < \ldots < \frac{2^n - 1}{2^n} = a_n < \ldots 1$$

Define a map τ which satisfies,

(a) $\tau(a_{2n}) = 1$, n=1,2,3,... ,

(b) $\tau(a_{2n+1}) = a_{2n-1}$, n=1,2,3,... ,

(c) $\tau\left(\frac{1}{2}\right) = 0, \quad \tau(1) = 1$,

(d) τ is linear on each of the subintervals $I_n = [a_n - 1, a_n]$ of partition.

τ is shown in Fig. 4. Then τ is continuous and $\tau|_{I_n}$ is an expansive homeomorphism with expansive constant,

$$\lambda = \inf_n \inf_{x \in I_n} |\tau'(x)| = \frac{\dfrac{2^{n-1} - 1}{2^{n-1}} - \dfrac{2^{n-2} - 1}{2^{n-2}}}{\dfrac{2^n - 1}{2^n} - \dfrac{2^{n-1} - 1}{2^{n-1}}} = 2$$

We now show that the map τ is eventually onto. Let J be any interval in [0,1]. If J lies completely in T_n for some n then its length is expanded by a factor $\lambda = 2$. Similarly for $\tau(J), \tau^2(J), \ldots$ Thus eventually $\tau^k(J)$ contains a_i for some i. So we can assume it to be true for J.

Now, if $a_n \in J$ for n even, then $1 \in \tau(J)$. If $a_n \in J$ for n odd, then by (b), $a_1 \in \tau^k(J)$, for some k. By (c), $0 \in \tau^{k+1}(J)$ and $1 \in \tau^{k+2}(J)$. Therefore, in either case, we can assume that $1 \in J$.

The image of an interval around 1 must contain an interval of the form $[a_{2n+1}, 1]$ for some n (by (b)). Then, again some subsequent image under τ must contain $[a_1, 1]$ for some n. Again, some subsequent image under τ must contain $[a_1, 1]$ and the next image must contain [0,1], so that τ is eventually onto. Thus by Theorem 3, τ has the POM property.

2.3.2 A GENERALIZATION

We can have transformations which may not be eventually onto.

Example 1 Let τ: $[0,1] \rightarrow [0,1]$ be a piecewise monotonic and onto map such that $\tau(0) = 0$, but $\tau(x) > 0, \forall x \neq 0$ For example, as in Fig. 5. For any interval K not containing 0, $0 \notin \tau^i(K)\ \forall$ i. Hence τ is not eventually onto and Theorem 3 does not hold.

For such cases, we have the following more general result, which applies to transformations that are not eventually onto. We state the result without proof.

Theorem 4 [36]. Let $\tau: I \rightarrow I$, $I = [0,1]$ be a continuous transformation satisfying the conditions:

(1) There exists a finite family of closed intervals $I = \{I_1,...,I_n\}$ such that $I_k \cap I_l, k \neq l$ consists of at most one point, $\bigcup_{i=1}^{N} I_i = I$ and for any interval $K \subset I$ there exists a positive integer t and $1 \leq i \leq N$ with $\tau^t(K) \supset I_i$.

Let μ be a probability τ-ergodic measure satisfying one of the following conditions:

(2) $\mu(I_i) > 0$, for i = 1,..., N

(3) If $I_{j_1},...,I_{j_p} \in I$, $\mu(I_{j_s}) = 0$, $\forall$ s = 1,...,p then the set, $I \setminus \bigcup_{k \geq 0} \bigcup_{s \geq 1} \tau^{-k}(I_{j_s})$ includes some interval L, with $\mu(L) > 0$.

Then the measure μ can be approximated by measures with supports concentrated on periodic orbits.

CHAPTER 3

PERTURBATION OF TRUE ORBIT: PSEUDO-ORBIT SHADOWING PROPERTY

3.1 INTRODUCTION

Computers are often employed in studying the asymptotic behaviour of chaotic systems. During such a numerical or computer experiment these systems are subjected to unavoidable external perturbations. Since chaotic systems are characterised by their sensitive dependence on initial condition [40], these perturbations will be amplified and different orbits starting close together will move apart rapidly. For example, for chaotic systems such as the logistic map, the distance between two nearby orbits on the average grow geometrically on every iterate. We are therefore led to the following fundamental question:

To what extent does a perturbed orbit (or a pseudo-orbit) have the (unpredictable) properties of a true (or an exact) orbit of a chaotic system?

For example, in [185] it was shown that the mixing property of the Chebyshev map $\tau(x) = x^2 - 2$ is preserved by the computer-generated orbits of τ.

A tool which has been used frequently while investigating the above question is the shadowing property of the system. While the pseudo-orbit will diverge exponentially from the true orbit with the same initial point, there often exists a different true orbit with a slightly different initial point which stays near the noisy orbit i.e. shadows it for a long period of time. It is then of interest to know the conditions under which such a shadowing orbit exists.

If the shadowing orbits exist, and are in some sense typical of the entire family of true orbits, the pseudo-orbits will reflect this same typical behaviour (though it is still an open question [165, Sec. 4.11] whether the typical properties of the entire family of true orbits, if any, are retained by the family of shadowing orbits).

In this Chapter, we study the question of shadowing property for some chaotic systems, in particular that of piecewise monotonic maps on an interval.

3.2 BASIC DEFINITIONS

Let (X,τ) be a dynamical system. In general, we assume X to be the unit interval with the Euclidean metric, unless stated otherwise. The dynamical study of τ is largely concerned with the 'orbits' of τ:

Orbit A sequence $\{\tau^n(x)\}_{n=0}^{\infty}$ is called the (true) orbit of the point $x \in X$.

If $\tau_1: X \to X$ is a map with $|\tau_1 - \tau| < \delta$ (i.e. τ_1 is a (localized) perturbation of τ), then each orbit $\{\tau_1^n(x)\}_{n=0}^{\infty}$ of τ_1 is almost an orbit of τ in the sense that $|\tau(\tau_1^n(x)) - \tau_1^{n+1}(x)| < \delta$, for $n=0,1,2,\ldots$. This can be used to motivate the definition of a pseudo-orbit of τ.

Pseudo-Orbit Let a $\delta > 0$ be given. A δ-pseudo orbit for $\tau: X \to X$ is a sequence $\{x_n\}_{n=0}^{\infty}$, $x_n \in X$, $n=0,1,2,\ldots$ such that, $|\tau(x_n) - x_{n+1}| \leq \delta$, for each $n=0,1,2,\ldots$.

Remark 1 The concept of the pseudo orbit goes back at least to Birkhoff [131] (cf. [25]). Pseudo-orbits arise when noise is introduced in a dynamical system e.g. during computer simulation. This gives rise to small deterministic perturbations in the form of roundoff/truncation errors, resulting in computer-generated δ-pseudo-orbits, where δ denotes the maximum magnitude of perturbation.

Remark 2 Pseudo-orbit description has been found useful in giving a precise mathematical definition of attractors [83, 168] and in the study of asymptotic behaviour of invariant measures of small random perturbations of dynamical systems [83, 146]. The utility of pseudo-orbits has also been found in characterization of topological entropy of a given system [156,157].

For τ to be 'stable' we would like each pseudo-orbit for τ to be closely related to a true orbit of τ. The following definition was introduced by Bowen in [108]:

Shadowing Given an $\varepsilon > 0$, a δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$ for τ is ε-shadowed by an $x \in X$ if, $|\tau^n(x) - x_n| \leq \varepsilon, \ \forall \ n=0,1,2,\ldots$

Pseudo-Orbit Shadowing Property τ is said to have the pseudo-orbit shadowing property (POSP or just shadowing property) if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that each δ-pseudo-orbit for τ is ε-shadowed by some point of X.

The shadowing property has been discussed in various contexts. For Anosov diffeomorphisms in [89,110,238], for Axiom-A diffeomorphisms in [8,108,109], for planar diffeomorphisms in [54], for C^1-diffeomorphisms of $\mathbb{R}^k$ in [30], for expansive and generic homeomorphisms in [76.101,132,133,197,231,247], for subshifts of finite type in [101], for hyperbolic flows in [35], for first return maps of real flows in [232], for non-invertible smooth maps in [95], for continuous maps on isolated invariant sets in [172, 173], for expanding maps of compact Riemannian manifolds in [116], for piecewise monotonic maps in [17,19,25,27,45,52,74,134, 181,256,272], and more generally for a sequence of C^1-maps of a Banach space in [26]. Shadowing property for some nonhyperbolic conservative systems has been established in [236].

Remark 3. Applications of Shadowing Property

The concept of shadowing property has found applications in various contexts. Structural stability was obtained in [101] for expansive homeomorphisms with shadowing property. It is used in [50,89] to show the stability of invariant measures of Axiom A systems with support on stochastic attractors. Shadowing property (of planar diffeomorphisms on hyperbolic sets) has been employed in [230,240,241] to give an alternative proof of Smale's Homoclinic Theorem [220]. Topological entropy has been estimated in [153,155,157,158,179] for expansive maps with shadowing property. We shall discuss this in detail in Appendix C. Recently, the shadowing property has also been applied to the problem of noise reduction for data generated by chaotic dynamical systems [269,270].

3.3 CHARACTERIZATIONS OF SHADOWING PROPERTY

In this section, we consider some characterizations of the shadowing property, which shall be applicable in the sequel. It is clear that the shadowing property of τ does not depend on the choice of metric on X. We assume $X = [0, 1]$ for all the results discussed in this section, unless stated otherwise.

Theorem 1 Let $\tau: X \to X$, be a continuous mapping. Then for every integer $N \geq 1$, τ has the shadowing property if and only if τ^N does.

Proof The proof is similar to that of Lemma 2, Sec. 6.3, Chapter 6.

A powerful method for the extension of results of dynamical systems with stochastic behaviour is a topological conjugation of such systems [1,2]. This motivates the following result:

Theorem 2 [17] Let a mapping $\tau : X \to X$, be topologically conjugated by means of a homeomorphism $h : X \to X$ to a mapping $\hat{\tau} = h^{-1}\tau h$, having the shadowing property. Then τ also has the shadowing property.

Proof Let $\{\hat{x}_n\}_{n=0}^{\infty}$ be a $\hat{\delta}$-pseudo orbit of the map $\hat{\tau}$ i.e. $|\hat{\tau}(\hat{x}_n) - \hat{x}_{n+1}| \leq \hat{\delta}$, for each n=0,1,2,.... Set $x_n = h^{-1}\hat{x}_n$ and consider the sequence $\{x_n\}_{n=0}^{\infty}$. By definition, $\hat{\delta} \geq |\hat{x}_{n+1} - \hat{\tau}(\hat{x}_n)| = |h\, x_{n+1} - h\, h^{-1}\tau\, h\, x_n| = |h\, x_{n+1} - h\,\tau\, x_n|$. But since h is a homeomorphism, it is continuous and hence uniformly continuous as a map of the compact set X into itself. Therefore for any $\delta > 0$ there exists a $\hat{\delta} > 0$ such that $|x - y| < \hat{\delta} \Rightarrow |hx - hy| < \delta$. Thus the sequence $\{x_n\}_{n=0}^{\infty}$ is a δ-pseudo orbit of the map τ.

As the map $\hat{\tau}$ possesses the shadowing property, for any $\varepsilon > 0$ there exists

$\delta > 0$ such that for any $\hat{\delta}$-pseudo orbit $\{\hat{x}_n\}_{n=0}^{\infty}$ of the map $\hat{\tau}$, there exists a point $\hat{x} \in X$ such that, $\left|\hat{\tau}^n \hat{x}_n - \hat{x}_n\right| < \varepsilon$, for each n=0,1,2,.... If we set, $x = h\hat{x}$, then, $\varepsilon > \left|\hat{\tau}^n \hat{x} - \hat{x}_n\right| = \left|h^{-1}h\,\hat{\tau}^n h^{-1}hx - h^{-1}h\,\hat{x}_n\right| = \left|h^{-1}\tau^n x - h^{-1}x_n\right|$. Now applying again the uniform continuity of the map h, we obtain, $\left|\tau^n(x) - x_n\right| < \varepsilon$, for each n=0,1,2,... i.e. the orbit $\{\tau^n(x)\}_{n=0}^{\infty}$ shadows the δ-pseudo orbit of the map τ. This completes the proof.

Let $X_n = X \times X \times ...$ n times. A metric d_n on X_n is defined by,

$d_n(x,y) = \max_{1 \le i \le n} |x_i - y_i|$, $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in X_n$, $x_i, y_i \in X$.

τ induces the mapping τ_n of X_n defined by, $\tau_n(x) = (\tau x_1,\dots,\tau x_n)$, where

$x = (x_1,\dots,x_n) \in X_n$, $x_i \in X$.

Following [51], we then have,

Theorem 3 If $\tau: X \to X$ has the shadowing property, then $\tau_n : X_n \to X_n$ also has the shadowing property.

Proof Let an $\varepsilon > 0$ be given and $\delta > 0$ be a number determined by the shadowing property of τ; corresponding to ε. Let $\{x^k\}_{k=0}^{\infty}$ be a δ-pseudo orbit of τ_n. If we denote $x^k = (x_1^k,\dots,x_n^k)$, k=0,1,2,... then $\{x_i^k\}_{k=0}^{\infty}$, $1 \le i \le n$ is a δ-pseudo orbit of τ since, $\left|\tau x_i^k - x_i^{k+1}\right| \le d_n(\tau_n x^k, x^{k+1})$ for each i=1,2,...,n and k=0,1,2,... .

Now, since τ has the shadowing property, there is a $y_i \in X$ $1 \le i \le n$, with $\left|\tau^k y_i - x_i^k\right| \le \varepsilon$, k=0,1,2,... .Set $y = (y_1,\dots,y_n)$. Then,

$$d_n(\tau_n^k(y), x^k) = \max_{1 \le i \le n} \left|\tau^k y_i - x_i^k\right| \le \varepsilon$$

i.e. $y \in X_n$ ε-shadows the δ-pseudo orbit $\{x^k\}_{k=0}^{\infty}$. This completes the proof.

We need the following definitions for our next result:

Tracing The dynamical system (X,τ) is tracing if for a sequence $\{x_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} |\tau(x_n) - x_{n+1}| = 0$, there is an $x \in X$ with $\lim_{n\to\infty} |\tau^n x - x_n| = 0$.

In general, (X,τ) is tracing does not imply that τ has the shadowing property [51].

τ-Connectedness The dynamical system (X,τ) is said to be τ-connected if

$\forall\ x,y \in X$ and every $\delta > 0$ there are δ-pseudo orbits $\{x_n\}_{n=0}^{a}$ and $\{y_n\}_{n=0}^{b}$ so that $x_o = x = y_b$ and $y_o = y = x_a$.

Theorem 4 [51] If (X,τ) is tracing and τ-connected, then τ has the shadowing property.

Proof Assume that the conclusion is not true. Then there is an $\varepsilon > 0$ such that for each $k \geq 1$, there is a $\frac{1}{k}$-pseudo orbit, $\{x_1^k,\ldots,x_{N_k}^k\}$ of τ such that there is no $z \in X$ with, $|\tau^j z - x_j^k| \leq \varepsilon$, for each $j=1,\ldots,N_k$. By τ-connectedness, there is a $\frac{1}{k}$ -pseudo orbit $\{z_o^k,\ldots,z_{L_k}^k\}$ with $z_o^k = x_{N_k}^k$ and $z_{L_k}^k = x_j^{k+1}$, $k=1,2,3,\ldots$.

Rewriting the indices of the sequence,

$\{\ldots,\ x_1^k,\ldots,x_{N_k}^k,\ z_1^k,\ldots,\ z_{N_{k-1}}^k,\ x_1^{k+1},\ldots\ \}$ we have a sequence $\{x_n\}_{n=0}^{\infty}$ such that,

$$|\tau(x_n) - x_{n+1}| \leq \frac{1}{n} \quad \text{i.e.} \quad \lim_{n\to\infty} |\tau\, x_n - x_{n+1}| = 0 \qquad \text{(i)}$$

Since (X,τ) is tracing, there is a $z \in X$ with,

$$\lim_{n\to\infty} |\tau^n z - x_n| = 0. \qquad \text{(ii)}$$

Therefore, by (i) and (ii), $\{x_j^k\}_{j=1}^{N_k}$ is ε-shadowed for some $k > 0$, which is a contradiction. This completes the proof of the theorem.

Let $B(X,\varepsilon) = \{\ y \in X : |x - y| \leq \varepsilon\ \}$ be the closed ε-ball about $x \in X$. We then have the following generalities [25]:

Theorem 5 The map $\tau: X \to X$ has the shadowing property if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$B(\tau x, \varepsilon + \delta) \subseteq \tau\,[B(x,\varepsilon)] \qquad (i)$$

holds for all $x \in X$.

Proof Let an $\varepsilon > 0$ be given and [illegible] be a δ-pseudo orbit of τ. Assume, that the condition (i) holds. Define sets [illegible]$_1$, ... as follows:

$W_0 = B(x_0, \varepsilon)$, $W_k = W_{k-1} \cap \tau$ [illegible] $B(x_k, \varepsilon)]$, $k \geq 1$. Then $\tau^0(W_0) = B(x_0, \varepsilon)$,

and $\tau(W_1) = \tau(W_0) \cap B(x_1,\varepsilon) = \tau\left[B(x_0,\varepsilon)\right] \cap B(x_1,\varepsilon)$. By condition (i),

$B(\tau(x_0), \varepsilon+\delta) \subseteq \tau\left[B(x_0,\varepsilon)\right]$. Also $\left|\tau(x_0) - x_1\right| \leq \delta$. Hence, $\tau(W_1) = B(x_1,\varepsilon)$. By induction, it follows that $\tau^k(W_k) = B(x_k,\varepsilon)$ for each $k \geq 0$. Thus $W_k \neq \varnothing$, $\forall\, k \geq 0$. By construction, $W_k \supset W_{k+1}$, $k \geq 0$. Therefore, $W = \bigcap_{k=0}^{\infty} W_k \neq \varnothing$.

We note that $W_k = \left\{x \in X: \left|x - x_0\right| \leq \varepsilon, \left|\tau(x) - x_1\right| \leq \varepsilon, \ldots, \left|\tau^k(x) - x_k\right| \leq \varepsilon\right\}$. So that W is precisely the set of points which ε-shadow the δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$. Thus τ has the shadowing property.

Remark 4. Consider the family of tent maps τ_s: $[0,2] \rightarrow [0,2]$, $1 < s \leq 2$, given by,

$$\tau_s = \begin{cases} sx, & 0 \leq x \leq 1 \\ s(2-x), & 1 \leq x \leq 2 \end{cases}$$

The condition of Theorem 5 is satisfied by the tent map τ_2 but is not satisfied by any other tent map τ_s, $s \neq 2$. It fails at the critical point $c = 1$.

Theorem 6 $\tau: X \rightarrow X$ has the shadowing property if there is a constant $\lambda \geq 1$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ and a positive integer N such that for each $x \in X$, there is a positive integer $n = n(x) \leq N$ satisfying

$$\tau\,[\,B(\tau^n(x), \varepsilon + \delta)\,] \subseteq \{\tau^{n+1}(y): |x - y| \leq \varepsilon, |\tau^i x - \tau^i y| \leq \lambda\varepsilon,\ 1 \leq i \leq n\}. \qquad (i)$$

Proof Let an $\varepsilon > 0$ be given and $\{x_n\}_{n=0}^{\infty}$ be a δ-pseudo orbit of τ. Assume that a constant $\lambda \geq 1$ exists such that condition (i) is satisfied. For $x \in X$ and n a positive integer, write $A(x, n) = \{ y: |x - y| \leq \varepsilon, |\tau^i(x) - \tau^i(y)| \leq \lambda\varepsilon,\ 1 \leq i \leq n\}$. Then we can rewrite (i) as,

$$\tau\left[B(\tau^n(x), \varepsilon+\delta)\right] \subseteq \tau^{n+1}[A(x,n)]. \qquad \text{(ii)}$$

Define integers m_k and n_k, and sets W_k, $k \geq 0$ as follows:

$$m_0 = 0,\ n_0 = n(x_0),\ W_0 = A(x_{m_0}, n_0) \text{ and for } k \geq 1$$

$$m_k = m_{k-1} + n_{k-1} = n_0 + \ldots + n_{k-1}$$

$$n_k = n(x_{m_k})$$

$$W_k = W_{k-1} \cap \tau^{-m_k - 1}(\tau[A(x_{m_k}, n_k)])$$

We claim that,

$$\tau^{m_k+1}(W_k) = \tau\left[A(x_{m_k}, n_k)\right], \text{ for each } k \geq 0. \qquad \text{(iii)}$$

We show that (iii) holds, by induction on k. By definition, (iii) holds for $k = 0$. Suppose that $k \geq 0$ and (iii) holds for k. Now, since

$\tau^{m_{k+1}+1}(W_{k+1}) = \tau^{m_{k+1}+1}(W_k) \cap \tau\left[A(x_{m_{k+1}}, n_{k+1})\right]$, it is sufficient to show that

$\tau[A(x_{m_{k+1}}, n_{k+1})] \subseteq \tau^{m_{k+1}+1}(W_k)$.

We have, $\tau[A(x_{m_{k+1}}, n_{k+1}) \subseteq \tau[B(x_{m_{k+1}}, \varepsilon)] \subseteq \tau[B(\tau^{n_k}(x_{m_k}), \varepsilon + \delta)]$, by (ii),

$\subseteq \tau^{n_k+1}[A(x_{m_k}, n_k)] = \tau^{n_k}(\tau[A(x_{m_k}, n_k)]) = \tau^{n_k}\left[\tau^{m_k+1}(W_k)\right] = \tau^{m_{k+1}+1}(W_k)$.

Now, fix an $i \geq 0$. If $i=0$, then $y \in W_0 \subseteq B(x_0, \varepsilon)$. Thus, $\left|\tau^0(y) - x_0\right| \leq \varepsilon$. For $i \geq 1$, write $i = m_k + j$, where $k \geq 0$ and $0 < j \leq n_k$. Then,

$$\left|\tau^i(y) - x_i\right| \leq \left|\tau^{j+m_k}(y) - \tau^j(x_{m_k})\right| + \left|\tau^j(x_{m_k}) - x_{m_k+j}\right|$$

Now, $\tau^{j+m_k}(y) = \tau^{j-1}(\tau^{m_k+1}(y)) \in \tau^{j-1}[\tau^{m_k+1}(W_k)] = \tau^j[A(x_{m_k}, n_k)]$, by (iii),

and therefore $\left|\tau^{j+m_k}(y) - \tau^j(x_{m_k})\right| \leq \lambda\varepsilon$. Also, we let $|\tau^j(x_{m_k}) - x_{m_k+j}| \leq \delta$. Hence,

$\left|\tau^i(y) - x_i\right| \leq \varepsilon$, if $\delta \leq (\lambda-1)\varepsilon$.

As we have seen in Theorem 5, $W = \bigcap_{k=0}^{\infty} W_k \neq \varnothing$ and is the set of points y which ε-shadow the δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$. Thus τ has the shadowing property. This completes the proof.

Theorem 7 [179]. Let τ be a continuous map of a compact metric space (X, d) into itself. If τ is an isometry and X is totally disconnected, then τ has the shadowing property.

Proof. Let $\varepsilon > 0$ be given. Since X is compact and totally disconnected there exists a partition $P = \{A_1,\ldots,A_n\}$ of X such that each A_i is closed and $\operatorname{diam}(A_i) \leq \varepsilon$, $1 \leq i \leq n$. Let $d(A,B) = \inf\{d(a,b): a \in A, b \in B\}$ for $A,B \subset X$. Set $\delta = \min_{i \neq j} d(A_i, A_j)$. Then $\delta > 0$. Furthermore, since τ is an isometry we have,

$$d(\tau^k(A_i), \tau^k(A_j)) = d(A_i, A_j) \geq \delta$$

for $(i, j) \in \{1,\ldots,n\}$, $i \neq j$ and $k \geq 0$. Let $\{x_k\}_{k=0}^{\infty}$ be a δ-pseudo-orbit of τ.

Suppose that $x_0 \in A_{i_0}$. Then by induction $x_k \in \tau^k(A_{i_0})$, for each $k \geq 0$. Therefore any point $x \in A_{i_0}$, ε-shadows $\{x_k\}_{k=0}^{\infty}$. Hence τ has the shadowing property and the proof is complete.

Remark 5. A generalization of Theorem 7 is obtained in form of Theorem 1, Sec. 7.8, Chapter 7.

3.4 EXAMPLES

In this section, we test some maps for the shadowing property.

Example 1 Consider the class of mappings τ_a: $[0,1] \to [0,1]$, $0 < a < 1$, defined by,

$$\tau_a(x) = \begin{cases} \frac{x}{a}, & \text{if } 0 \leq x < a \\ \frac{x-a}{1-a}, & \text{if } a \leq x < 1. \end{cases}$$

See Fig. 6. τ_a is similar to Bernoulli shifts which are a prototype of one-dimensional chaotic maps [152]. We show that τ_a has the shadowing property. Fix an $\varepsilon > 0$ and a δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$. Let, $K_m = \bigcap_{n=0}^{m} \{y \in [0,1]: | \tau_a^n (y) - x_n| \leq \varepsilon \}$ and let

$\lambda = \min \tau_a' = \min \{\frac{1}{a}, \frac{1}{1-a}\} > 1$. Then if $\delta = (\lambda - 1)\varepsilon$, it follows by induction that K_m is nonempty for each $m \geq 0$. Also $K_m \supset K_{m+1}$ and K_m is compact for each $m \geq 0$. Therefore, $K = \bigcap_{m=0}^{\infty} K_m$ is non empty. Since K is precisely the set of points that ε-shadow the given δ-pseudo orbit, τ has the shadowing property.

Example 2 Consider the class of mappings τ_α: $[0,1] \to [0,1]$ defined by,

$$\tau_\alpha (x) = \alpha x, \; 0 < \alpha < 1.$$

We show that τ_α has the shadowing property. Let $\varepsilon > 0$. For a given r let $\{x_0, \dots, x_n\}$ be a δ-pseudo orbit of τ_α i.e.

$|\tau_\alpha(x_k) - x_{k+1}| \leq \delta$ for each $k=0,1,\dots, n-1$. Set, $\bar{x}_j = \tau_\alpha^j (y)$, $j = 0,1,\dots,n$, $y \in [0,1]$. Then, $|\bar{x}_{j+1} - x_{j+1}| = |\tau_\alpha \bar{x}_j - x_{j+1}| \leq |\tau \bar{x}_j - \tau x_j| + |\tau x_j - x_{j+1}|$

$\leq \alpha |\bar{x}_j - x_j| + \delta$. By induction it follows that, $|\bar{x}_j - x_j| \leq \frac{\delta}{1-\alpha}$, for each j. Therefore n being arbitrary, if we let $\delta = (1-\alpha)\varepsilon$, then τ_α has the shadowing property.

Remark 6 It is easily seen from the analysis of Example 2, that any contracting mapping of a compact metric space into itself has the shadowing property [121].

Example 3 Consider the tent map, $\tau_{\sqrt{2}}: [0,1] \to [0,1]$ defined by,

$$\tau_{\sqrt{2}}(x) = \begin{cases} \sqrt{2}\,x & , 0 \le x \le \frac{1}{2} \\ \sqrt{2}\,(1-x) & , \frac{1}{2} \le x \le 1 \end{cases}$$

We show that $\tau_{\sqrt{2}}$ does not have the shadowing property [116, Chapter IV, 4.1]. We note that $c = 2 - \sqrt{2}$ is a repelling fixed point of $\tau_{\sqrt{2}}$ and $\tau^3_{\sqrt{2}}(\frac{1}{2}) = c$. Take the δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$ of $\tau_{\sqrt{2}}$ given by,

$x_0 = \frac{1}{2}$, $x_1 = \tau_{\sqrt{2}}(x_0)$, $x_2 = \tau_{\sqrt{2}}(x_1)$, $x_3 = c$, $x_4 = c + \delta$, $x_5 = \tau_{\sqrt{2}}(x_4), \dots,$

$x_{k+1} = \tau_{\sqrt{2}}(x_k)$, $k \ge 4$.

For $p = \frac{1}{4}(3 - 2\sqrt{2})$, note that $\tau^2_{\sqrt{2}}(-p + \frac{1}{2}) = \frac{1}{2} = \tau^2_{\sqrt{2}}(p + \frac{1}{2})$.

Consider the interval, $I = \{x: |x - \frac{1}{2}| \le \frac{1}{4}(3 - 2\sqrt{2})\}$. Then $\tau^3_{\sqrt{2}}(I)$ is the interval whose left endpoint is c and so $\tau^4_{\sqrt{2}}(I)$ is the interval whose right endpoint is c.

Hence, if $y \in I$ then, $\tau^4_{\sqrt{2}}(y) \le c < c + \delta = x_4$. For, $\frac{\sqrt{2}}{2} \le x_k \le \frac{1}{2}$ and $\frac{\sqrt{2}}{2} \le \tau^k_{\sqrt{2}}(y) \le \frac{1}{2}$, we have, $\left| x_{k+1} - \tau^{k+1}_{\sqrt{2}}(y) \right| = \sqrt{2}\left| x_k - \tau^k_{\sqrt{2}}(y) \right|$, for each $k \ge 4$.

Therefore, when δ is small enough, the orbit of y cannot shadow the δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$. Thus $\tau_{\sqrt{2}}$ does not have the shadowing property.

Example 4 Let $\tau_a : [0,1] \to [0,1]$, $0 < a < 1$, be the map given by,

$$\tau_a(x) = \begin{cases} \frac{x}{a}, & 0 \le x < a \\ \frac{x-a}{1-a}, & a \le x \le 1 \end{cases}$$

and τ_2 be the tent map, given by,

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

By Theorem 1 of [2], $\tau_{1/2}$ is topologically semi-conjugate to τ_2. Therefore, from Theorem 2 and Example 1, it follows that τ_2 has the (almost) shadowing property. Since τ_2 is topologically semi-conjugate to the quadratic map $\sigma: [0,1] \to [0,1]$ given by $\sigma(x) = 4x(1 - x)$, this in turn implies the (almost) shadowing property for σ.

Remark 7. Let $\tau_\mu : [0,1] \to [0,1]$, $\mu > 1$ be a tent map given by,

$$\tau_\mu(x) = \begin{cases} \mu x, & 0 \le x \le \frac{1}{2} \\ \mu(1 - x), & \frac{1}{2} \le x \le 1. \end{cases}$$

It is shown in [194] that $\tau^2_{\sqrt{\mu}}$ restricted to the interval $[\frac{\sqrt{\mu} - 1}{\mu - 1}, \frac{\mu - \sqrt{\mu}}{\mu - 1}]$ is topologically conjugate to τ_μ on [0,1]. Then from Theorem 2, and Examples 3 and 4, it follows that even though $\tau_{\sqrt{2}}$ does not have the shadowing property, $\tau^2_{\sqrt{2}}$ restricted to the interval $[\sqrt{2} - 1, 2 - \sqrt{2}]$ has the shadowing property.

3.5 SHADOWING PROPERTY FOR HYPERBOLIC SYSTEMS

The shadowing property was originally discussed for a restricted class of maps, namely for Anosov diffeomorphisms (which are everywhere hyperbolic) [108,110]. For maps with a chaotic attractor, this means essentially that each point x in the attractor must have a stable manifold and an unstable manifold: under the map τ, infinitisimal displacements in the stable direction decay exponentially while infinitisimal displacements in the unstable direction grow exponentially. A point where the stable and unstable manifolds intersect is called a homoclinic intersection; if the manifolds are parallel, this is called a homoclinic tangency. To be everywhere hyperbolic it is required that the system does not have homoclinic tangencies i.e. the angle between the stable and unstable directions be bounded away from zero. If these requirements are satisfied it is possible to show that a true orbit can be found near the noisy numerical orbit for arbitrary long period of times. An important result obtained in such a case is the shadowing lemma.

3.6 SHADOWING LEMMA

We state the lemma as follows:

Let $\tau: X \rightarrow X$ be a continuous map for which X is a hyperbolic invariant set. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ of τ in X is ε-shadowed by a point $x \in X$.

Various proofs of the shadowing lemma have been given. Some of them are by Bowen [8], Conley [242], Robinson [243], Newhouse [244], Ekeland [245], Lanford [121], Shub [246], and Palmer [241].

3.6.1 LIMITATION OF THE SHADOWING LEMMA

Since the shadowing property for pseudo orbits was given for (everywhere) hyperbolic dynamical systems, there has been a considerable research work, investigating the shadowing property and its applications to both theoretical and applied problems. The proof of the shadowing lemma depends on the fact that the stable and unstable manifolds are never parallel when they intersect. Although such hyperbolic systems were initially believed to be generic, we now know that quite the opposite is true: most dynamical systems of interest have homoclinic tangencies. Therefore they do not have the requisite uniform hyperbolicity and the shadowing lemma does not apply. Therefore to extend this result to, say a class of piecewise continuous dynamical systems, we have to use different techniques.

3.7 THEORETICAL RESULTS OF SHADOWING FOR NON-HYPERBOLIC SYSTEMS

We shall now restrict ourselves to the study of shadowing property for non-hyperbolic systems for the rest of the chapter. An important class of such dynamical systems is that of piecewise expanding maps.

3.7.1 SHADOWING PROPERTY FOR PIECEWISE EXPANDING MAPS

We have shadowing property for piecewise expanding maps which take their intervals of partition to the whole space.

Theorem 8 [17]. Let $\tau: X \to X$, $X = [0,1]$ be a piecewise monotonic, $C^{1+\alpha}$ and expanding map with expanding constant $\lambda > 1$, such that $\tau X_i = X$, where X_i is any interval of partition described in the definition of τ. Then τ has the shadowing property.

Proof Let $Y = \delta_i X_i$ where δ denotes the boundary, $\overline{Y} = \bigcup_{n=0}^{\infty} \tau^{-n} Y$, and $\overline{X} = X \setminus \overline{Y}$. By construction of the set $\overline{X}$, it is dense in X and $\tau(\overline{X} \cap X_i) = \overline{X}$ for each i. For any point $x \in X_i$, we denote by $\lambda(x)$ the set X_i.

Fix an $\varepsilon > 0$ and choose a $\delta \in (0, \varepsilon(1 - \frac{1}{\lambda}))$. Let $x,y \in X$ such that $|\tau x - y| < \delta$, say $x \in X_i$ Set $y' = \tau_i^{-1} y$. Then by the expanding property, we obtain,

$$\delta > |\tau x - y| = \left|\tau\tau_i^{-1}\tau x - \tau\tau_i^{-1} y\right| \geq \lambda \left|\tau_i^{-1}\tau x - \tau_i^{-1} y\right| = \lambda\, |x - y'|$$

i.e. there exists a point $y' \in X(x) \cap \overline{X}$ such that $\tau y' = y$ and $|x - y'| < \frac{\delta}{\lambda}$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ is a δ-pseudo orbit of τ. Choose an arbitrary positive integer N and set $y_N = x_N$. As $|\tau x_{N-1} - y_N| \leq \delta$ there exists a $y_{N-1} \in X(x_{N-1}) \cap \overline{X}$

such that, $|x_{N-1} - y_{N-1}| \leq \frac{\delta}{\lambda}$ and $\tau y_{N-1} = y_N$. Now,

$|\tau x_{N-2} - y_{N-1}| < |\tau x_{N-2} - x_{N-1}| + |x_{N-1} - y_{N-1}| \leq \delta + \frac{\delta}{\lambda} = \delta\left(1 + \frac{1}{\lambda}\right) = \varepsilon$. Therefore there exists a $y_{N-2} \in X(x_{N-2}) \cap \overline{X}$ such that, $\tau y_{N-2} = y_{N-1}$ and

$|x_{N-2} - y_{N-2}| \leq \delta\left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right) < \varepsilon$. Continuing this construction, we have for arbitrary $n \leq N$ that there exists a $y_n \in X(x_n) \cap \overline{X}$ such that, $\tau y_n = y_{n+1}$ and

$|x_n - y_n| \leq \delta\left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \dots + \frac{1}{\lambda^{N-n}}\right) < \delta\left(\frac{1}{1 - \frac{1}{\lambda}}\right) < \varepsilon$. In this way, we construct the segment of the orbit $\{y_n\}_{n=0}^{\infty}$ of the system τ, shadowing the δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$ on the time interval [0,N]. But since N can be chosen arbitrarily large, the result follows.

3.7.2 REMARKS

Remark 8 For one-dimensional representation of the Lorenz system a similar result as of Theorem 8 was obtained in [55].

Remark 9 The condition $\tau X_i = X$ in the statement of Theorem 8 is sufficient [164] (*cf.* [17]) but not necessary. The family of tent maps in Remark 4 doesn't satisfy this condition but have the shadowing property for almost all parameters $s \in (1, 2]$ [25]. We therefore have to use different methods to test the shadowing property for expanding maps which don't satisfy this condition.

We now discuss the shadowing property in the family of tent maps, which are a prototype for the one-parameter family of expanding unimodal maps. It is well known that these maps exhibit a chaotic behaviour (see e.g. [77]).

3.8 SHADOWING PROPERTY IN THE FAMILY OF TENT MAPS

We consider the one-parameter family of tent maps (See Fig. 7) i.e. the piecewise linear maps $\tau_S: [0,2] \to [0,2]$, $\sqrt{2} \le s \le 2$ defined by,

$$\tau_S(x) = \begin{cases} sx & , 0 \le x < 1 \\ s(2-x) & , 1 \le x \le 2 \end{cases}$$

We note that $c=1$ is the common critical point of all the tent maps. We shall test the tent maps for the shadowing property in terms of the behaviour of the iterates their critical point c. To do that we shall need some basic concepts of kneading theory developed in [71]. The idea is to associate with each point a sequence of symbols describing on which side of the critical point its successive iterates lie.

<u>Kneading sequence</u> Let $v = v(s) = v_1 v_2 \ldots$ be the kneading sequence of τ_s i.e. the (extended) itinerary of the critical value c defined as follows:

$v_n = L$, if $\tau_s^n(c) < c$

$v_n = C$, if $\tau_s^n(c) = c$

$v_n = R$, if $\tau_s^n(c) > c$.

<u>Signature Sequence</u> A signature sequence $\sigma = \sigma(s) = \sigma_1 \sigma_2 \ldots$ is defined by,

$\sigma_1 = -1$,

$\sigma_{n+1} = \sigma_n$, if $v_n = L$,

$\sigma_{n+1} = -1$, if $v_n = C$,

$\sigma_{n+1} = -\sigma_n$, if $v_n = R$,

Denote by $\langle x,y \rangle$ the interval $[x,y]$ or $[y,x]$, $x \ne y$ whichever makes sense. We shall use the following lemma [25] which we state without proof.

<u>Lemma 1</u>. Given a parameter s, if $\varepsilon > 0$ is sufficiently small then,

a) If $|x - c| \le \varepsilon$, then $\tau_s[B(x,\varepsilon)] = B(\tau_s(x), s\varepsilon)$,

b) If $z \in \langle x,y \rangle$ and $c \notin \text{int} \langle x,y \rangle$, then $\tau_s(z) \in \langle \tau_s(x), \tau_s(y) \rangle$ and

$\left|\tau_s(x) - \tau_s(y)\right| = s\,|x - y|$,

c) If x=y, then $c \in \text{int} <\tau_s^k(x), \tau_s^k(y)>$ for some $k \geq 0$.

Remark 10. By Lemma 1, it follows that given a parameter s, a positive integer n and $\varepsilon > 0$ sufficiently small, the endpoints of $\tau_s^n[c - \varepsilon, c + \varepsilon]$ are $\tau_s^n(c)$ and $\tau_s^n(c) \pm s^n\varepsilon$, where the sign is given by the signature sequence.

Recurrent Point. A point $x \in X$ is τ-recurrent if for every $\varepsilon > 0$ there is a positive integer n such that, $\left|\tau^n(x) - x\right| \leq \varepsilon$.

3.8.1 A NECESSARY AND SUFFICIENT CONDITION FOR THE SHADOWING PROPERTY FOR TENT MAPS

In this section, we obtain necessary and sufficient conditions for a tent map to have the shadowing property. These conditions are that the critical point is recurrent and returns close to itself on the 'correct side', a condition which can be stated in terms of the kneading and signature sequences.

Theorem 9 [25]. Let $s \neq 2$. Then τ_s has the shadowing property if and only if for every $\varepsilon > 0$, there is a positive integer M such that

$$c \in \{\tau_s^M(y) : |\tau_s^i(y) - \tau_s^i(c)| \leq \varepsilon,\ 0 \leq i \leq M\}.$$

Proof Fix the parameter s and denote τ_s by τ.

Part 1: Necessity

We first suppose that the condition holds and prove that τ has the shadowing property. Let an $\varepsilon > 0$ be given. We show that the condition of Theorem 6 holds with $\lambda = s^4$, $\delta = (s\text{-}1)\,\varepsilon$ and $N = M+1$ i.e. for each $x \in [0,2]$ there is a positive integer $n = n(x) \leq M+1$ such that,

$$\tau[B(\tau^n x, s\varepsilon)] \subseteq \{\tau^{n+1}(y) : |x - y| \leq \varepsilon, |\tau^i(x) - \tau^i(y)| \leq s^4\varepsilon,\ 1 \leq i \leq n\} \qquad (i)$$

which is implied by,

$$B(\tau^n x, s\varepsilon) \subseteq \{\tau^n(y) : |x - y| < \varepsilon, |\tau^i(x) - \tau^i(y)| \leq s^4\varepsilon,\ 1 \leq i \leq n\} \qquad \text{(ii)}$$

Suppose first that $\tau^3(c) = c$ i.e. $s = \frac{1+\sqrt{5}}{2}$. Let $\varepsilon > 0$ be small enough so that

$c \notin \tau^i[c - \varepsilon, c + \varepsilon]$ for i=1 or 2 and also $(s + s^3)\,\varepsilon < \frac{1}{2}$. Then equation (ii) holds with, $n(x) = 1$, if $|x - c| \geq \varepsilon$,

$n(x) = 2$, if $\frac{\varepsilon}{s} \leq |x - c| < \varepsilon$,

$n(x) = 3$, if $\frac{\varepsilon}{s^2} \leq |x - c| < \frac{\varepsilon}{s}$.

Now suppose that, $|x - c| \leq \frac{\varepsilon}{s^2}$. Then, $\tau^3(x) \in [c, c + s\varepsilon]$ and

$\left\{\tau^3(y): |x-y| < \varepsilon, \left|\tau^i(x) - \tau^i(y)\right| \leq s^4\varepsilon,\ 1 \leq i \leq 3\right\} = \left[c, \tau^3(x) + s^3\varepsilon\right]$. Since

$\tau\left[B(\tau^3(x), s\varepsilon)\right] = \tau\left[c, \tau^3(x) + s\varepsilon\right]$, so equation (i) holds with $n(x) = 3$. Similarly, the condition of Theorem 6 holds if $\tau^4(c) = c$.

We may therefore suppose that $\tau^3(c) \neq c$ and $\tau^4(c) \neq c$, and $\varepsilon > 0$ is small enough so that $c \notin \tau^i\left[c - s^4\varepsilon, c + s^4\varepsilon\right]$, $1 \leq i \leq 4$. For $n \geq 1$ let,

$C_n = \left\{\tau^n(y): \left|\tau^i(y) - \tau^i(c)\right| \leq \varepsilon,\ 0 \leq i \leq n\right\}$. We may also assume that M is the least positive integer such that $c \in C_M$. Our choice of ε insures that $M \geq 5$. If $|x - c| \geq \varepsilon$, then equation (ii) holds with $n(x)=1$. Now fix x with $|x - c| < \varepsilon$. For $n \geq 1$ let,

$D_n = \left\{\tau^n(y): \left|\tau^i(x) - \tau^i(y)\right| \leq \varepsilon,\ 0 \leq i \leq n\right\}$ and

$E_n = \left\{\tau^n(y): |x-y| \leq \varepsilon, \left|\tau^i(x) - \tau^i(y)\right| \leq s^4\varepsilon,\ 1 \leq i \leq n\right\}$.

Then D_n and E_n are intervals. With this notation, equations (i) and (ii) become,

$$\tau\left[B(\tau^n(x), s\varepsilon)\right] \subseteq \tau(E_n) \qquad \text{(i)'}$$

and $$B(\tau^n(x), s\varepsilon) \subseteq E_n \qquad \text{(ii)'}$$

Now, if

$$|x - c|\, s^k \leq \varepsilon \text{ and } c \notin \operatorname{int} D_1, \ldots, \operatorname{int} D_{k-1} \qquad \text{(iii)}$$

we have, $D_k = \langle \tau^k(c), \tau^k(x) + \sigma_k \varepsilon \rangle$ and $\tau^k(c) = \tau^k(x) - \sigma_k |x-c|\, s^k$. Suppose first that $c \notin D_k$, $1 \leq k \leq M$. We claim that $|x-c|\, s^M > \varepsilon$. If not, then by equation (iii) ,

$D_M = \langle \tau^M(c), \tau^M(c) + \sigma_M \left(|x-c|\, s^M + \varepsilon\right) \rangle$. Since M is the least positive integer such that $c \in C_M$, we have as in equation (iii) that

$C_M = \langle \tau^M(c), \tau^M(c) + \sigma_M \varepsilon \rangle$ and so $c \in C_M \subseteq D_M$, a contradiction. Therefore, $|x-c|\, s^M > \varepsilon$ if $c \notin D_k$, $1 \leq k \leq M$.

Let k be the least positive integer such that $|x-c|\, s^k > \varepsilon$. Thus $1 \leq k \leq M$. From equation (iii) and the fact that $c \notin D_1, \ldots, D_{k-1}$, we get that

$D_{k-1} = \langle \tau^{k-1}(x) - \sigma_{k-1} |x-c|\, s^{k-1}, \tau^{k-1}(x) + \sigma_{k-1} \varepsilon \rangle$ Then since $c \notin D_{k-1}$,

$D_k = \left[\tau^k(x) - \varepsilon, \tau^k(x) + \varepsilon\right]$. Finally since $c \notin D_k$, $\tau(D_k) = \left[\tau^{k+1}(x) - s\varepsilon, \tau^{k+1}(x) + s\varepsilon\right] \subseteq E_{k+1}$; so (ii)' holds with $n(x) = k+1$.

Now suppose that $5 \leq k \leq M$, is the least positive integer such that $c \in D_k$. Using equation (iii) again, we have, $\langle \tau^k(x), \tau^k(x) + \sigma_k \varepsilon \rangle \subseteq D_k$. Suppose, for definiteness that $\sigma_k = +1$. We then consider the cases when $\tau^k(x) \geq c$ and $\tau^k(x) < c$.

<u>Case 1</u> Let $\tau^k(x) \geq c$.

By the minimality of k, $c \notin D_1, \ldots, D_{k-1}$ and so $\tau^{k-1}(x) + \sigma_{k-1} \varepsilon \in D_{k-1}$. Thus $\tau^k(x) + s\varepsilon = \tau(\tau^{k-1}(x) + \sigma_{k-1}\varepsilon) \in \tau(D_{k-1}) \subseteq E_k$. But $c \in E_k$, and so $\left[c, \tau^k(x) + s\varepsilon\right] \subseteq E_k$. Since $c \leq \tau^k(x) \leq c + \varepsilon$ we have,

$\tau\left[B(\tau^k(x), s\varepsilon)\right] = \tau\left[c, \tau^k(x) + s\varepsilon\right] \subseteq \tau(E_k)$ and therefore (i)' holds with $n(x) = k$.

<u>Case 2</u> Let $\tau^k(x) < c$.

We have chosen the ε so small that $k \geq 5$. Since $c \in E_k$, we have $c \notin E_{k-4} \cup \ldots \cup E_{k-1}$. As in Case 1, $\tau^{k-4}(x) + \sigma_{k-4} \varepsilon \in D_{k-4}$,

$\tau^k(x) + s^4\varepsilon = \tau^4(\tau^{k-4}(x) + \sigma_{k-4}\varepsilon) \in \tau^4(D_{k-4}) \subseteq E_k$ and $\left[c, \tau^k(x) + s^4\varepsilon\right] \subseteq E_k$.

But $s + 2 < s^4$, since $s \geq \sqrt{2}$ and so $\tau\left[B(\tau^k(x), s\varepsilon)\right] \subseteq \tau\left[c, \tau^k(x) + (s+2)\varepsilon\right] \subseteq E_k$.

Therefore, equation (i)' holds with $n(x) = k$.

<u>Part 2</u>: <u>Sufficiency</u>

Suppose now that the condition does not hold. We then show that τ can't have shadowing property. If τ_s does not have the shadowing property then there is an $\varepsilon > 0$ such that for every $\delta > 0$, there a δ-pseudo-orbit which cannot be ε-shadowed by any true orbit.

<u>Case 1</u> Suppose that the critical point is not τ-recurrent. Then there is an $\varepsilon > 0$ such that, $|\tau^n(c) - c| > \varepsilon, \ \forall \ n \geq 1.$ (iv)

Then given $0 < \delta \leq \varepsilon$, let $\{x_n\}_{n=0}^{\infty}$ be a δ-pseudo orbit defined as follows:

$$x_0 = c, \ x_1 = \tau(c) + \delta, \ x_k = \tau^{k-1}(x_1) \ , \ k \geq 2.$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ can be ε-shadowed by the orbit of y. Let m be the least positive integer such that $c \in <\tau^m(y), x_m>$. (If no such integer exists, then by Lemma 1 (c), $\tau(y) = x_1$, which is impossible since $x_1 > s$, the maximum value of τ). Then $\tau(c) \in <\tau(y), x_1>$ and $c \notin <\tau(y), x_1>$. Therefore, $\tau^2(c) \in <\tau^2(y), x_2>$. But $x_2 = \tau(x_1)$, so assuming $m > 2$, $c \in <\tau^2(y), x_2>$. Continuing in this way, we obtain that $c, \tau^m(c) \in <\tau^m(y), x_m>$, and so $|\tau^m(c) - c| < \varepsilon$, a contradiction to (iv).

<u>Case 2</u> Suppose that the condition fails for $\varepsilon > 0$ but that the critical point is τ-recurrent. From Part 1, we have $C_n = \left\{\tau^n(y): \ |\tau^i(y) - \tau^i(c)| \leq \varepsilon, \ 0 \leq i \leq n\right\}$.

Since $c \notin C_n$ for all $n \geq 1$, we have $C_n = <\tau^n(c), \tau^n(c) + \sigma_n\varepsilon>$. Thus, if $|\tau^n(c) - c| \leq \varepsilon$, we must have $\sigma_n = +1$ or -1 according to as $V_n = R$ or L.

Given $0 < \delta \leq \varepsilon$, we construct a δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$ as follows. Since c is τ-recurrent, $|\tau^k(c) - c| \leq \delta$ for some $k \geq 1$. Let,

$$x_i = \begin{cases} \tau^i(c), & \text{if } 0 \le i \le k-1 \\ \tau^{i-k}(c), & \text{if } i \ge k. \end{cases}$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ can be ε-shadowed by the orbit of y. Then $\{x_m\}_{m=k}^{\infty}$ can be ε-shadowed by the orbit of $\tau^k(y)$. Hence, $\tau^{k+j}(y) \in C_j$, for every $j \ge 0$. In particular we have

$$\sigma_j = \begin{cases} +1, & \text{if } \tau^{k+j}(y) > \tau^k(c) \\ -1, & \text{if } \tau^{k+j}(y) < \tau^k(c) \end{cases} \qquad \text{(v)}$$

By Lemma 1, $c \in \langle\tau^j(c), \tau^{k+j}(c)\rangle$ for some $j \ge 1$. Since $\langle\tau^j(c), \tau^{k+j}(y)\rangle \subseteq C_j$, we have $\left|\tau^j(c) - c\right| \le \varepsilon$. Thus,

$$\sigma_j = \begin{cases} +1, & \text{if } \nu_j = R \\ -1, & \text{if } \nu_j = L. \end{cases} \qquad \text{(vi)}$$

Since $c \in \langle\tau^j(c), \tau^{k+j}(y)\rangle$, both (v) and (vi) cannot hold simultaneously. This completes the proof.

We then have the following corollary [25] of Theorem 9, which we state without proof.

Corollary 1. Let $s \ne 2$. Then τ_s has the shadowing property if and only if for every $\varepsilon > 0$, there is a positive integer n such that, $\left|\tau_s^n(c) - c\right| \le \varepsilon$ and, either $\nu_n = C$ or $\sigma_n = +1$ or -1, according to as $\nu_n = L$ or R.

3.8.2 GENERAL REMARKS

Remark 11. Corollary 1 states that for $s \neq 2$, τ_s has the shadowing property if and only if the critical point $c=1$ is τ_s-recurrent and for every $\varepsilon > 0$ there is a positive integer n such that $|\tau_s^n(c) - c| \leq \varepsilon$ and either $\tau_s^n(c) = c$ or $|\tau_s^n(x) - c|$ has a local maximum at $x=c$. In particular, τ_s has the shadowing property if the critical point is periodic (e.g. if $s = \frac{1+\sqrt{5}}{2}$), and doesn't if the critical point is preperiodic (i.e. has a periodic point in its orbit), but not periodic (e.g. if $s = \sqrt{2}$).

Remark 12. Restricting the parameters to lie in the interval $[\sqrt{2}, 2]$ is a technical convenience. For $1 < s \leq \sqrt{2}$, τ_s^2 restricted to an appropriate interval is topologically conjugate to τ_{s^2} [194]. Hence, Theorem 9 holds for the parameter interval (1,2].

Remark 13. It is shown in [25] that τ_s has the shadowing property for almost all parameters $s \in (1,2]$ and the set of parameters for which τ_s does not have the shadowing property is locally uncountable.

Remark 14. If for some fixed parameter, we don't have the shadowing property for a map, it is of interest to know whether the shadowing occurs for a slightly different parameter. In fact, we have the following question:

Can a noisy orbit be approximated by a true orbit for slightly greater parameter value (increased parameter shadowing)?

The tent maps τ_s are shown in [25,74] to have the increased parameter shadowing property for all parameters. Similar results are established for a family of quadratic maps in [74] and for a family of one-dimensional Poincaré maps corresponding to the Lorenz system in [256].

3.8.3 A GENERALIZATION

In this section, we present a generalization of the main result of Sec. 3.8.1. We need the following basic definitions:

Uniformly Piecewise linear map A map $\tau: X \to X$, $X = [0,1]$ is said to be uniformly piecewise linear if there are $\alpha_1, \ldots, \alpha_m$ and $s > 1$ such that, $\tau(x) = \alpha_i \pm sx$, for $x \in [a_k, a_{k+1}]$ where $0 = a_0 < a_1 < \ldots < a_m = 1$ are the turning points of τ and the sign depends on $i = 1, \ldots, m$.

Linking Property. Let $\tau: X \to X$ and an $\varepsilon > 0$ be given. A point $x \in X$ is ε-linked to a point $y \in X$ by τ, if there exists an integer $m \geq 1$ and a point $z \in B(x,\varepsilon)$ such that $\tau^m(z) = y$ and $\left|\tau^j(x) - \tau^j(z)\right| \leq \varepsilon$, for $0 \leq j \leq m$. We say that $x \in X$ is linked to $y \in X$ by τ if x is ε-linked to y by τ for each $\varepsilon > 0$. A subset C of X is linked by τ, if every point $c \in C$ is linked to one of the points in C by τ.

We then have the following:

Theorem 10 [27]. Suppose τ is a map that is conjugate to a continuous uniformly piecewise linear map of a compact interval to itself. Then τ has the shadowing property if and only if the set of all turning points of τ is linked by τ.

Example 5. Let τ_s, $1 < s \leq \sqrt{2}$ be the family of tent maps considered in Sec. 3.8. Then $c = 1$ is the critical point for τ_s for each s. When $s \neq 2$, then $\tau_s(1) \in \left[\tau_s^2(1), \tau_s(1)\right]$, for each $j \geq 0$ where $\left[\tau_s^2(1), \tau_s(1)\right]$ is a proper subinterval of (0,2). Therefore, by definition, 1 is linked to neither 0 nor 2 by τ_s. So for 1 to be linked to 0 or 2, we must have s=2 i.e. by Theorem 10, τ_s has the shadowing property if and only if either s=2 or 1 is linked to itself.

Remark 15. In Theorem 9, the condition for shadowing property for the tent map τ_s, $s \neq 2$ is that, for any given $\varepsilon > 0$, there exists an integer $M \geq 1$ such that, $1 \in \left\{\tau^M(y) : \left|\tau^j(y) - \tau^j(1)\right| \leq \varepsilon,\ 0 \leq j \leq M\right\}$ i.e. 1 is linked to itself by τ_s. Thus by Example 5, Theorem 10 generalizes Theorem 9.

3.9 SHADOWING PROPERTY IN THE FAMILY OF QUADRATIC MAPS

In Example 4 we have seen that the quadratic map $\tau(x) = 4x(1 - x)$ which is topologically conjugate to the tent map τ_2 has the shadowing property. This motivates the question whether we have shadowing property for other quadratic maps. Consider the family of quadratic maps $\tau_\lambda: [0,1] \to [0,1]$ given by $\tau_\lambda(x) = \lambda x(1 - x)$, $1 < \lambda \leq 4$, which are a prototype for the one-parameter family of non-expanding unimodal maps with negative Schwarzian derivative. These maps are well known for mimicking various biological models [62, 69] and exhibiting a chaotic behaviour [77] as the parameter λ is varied. Shadowing property for τ_λ is shown in [52] for a set of chaotic parameters λ of positive Lebesgue measure obtained in [125]. τ_λ are also shown in [74] to have the increased parameter shadowing property for almost all parameters, when they have attracting periodic orbits.

3.10 LIMITATIONS OF THE SHADOWING PROPERTY

Though shadowing property is useful in studying the reliability of numerical data obtained from pseudo-orbits during experiments with chaotic systems, there exist inherent limitations of the shadowing property towards computer implementation:

1) A map may not have the shadowing property for all parameters. (See Remark 13.) Thus there is a lack of continuity in the shadowing property. A small change in the parameter may result in a loss of the shadowing property: this can have important consequences during computer simulation of the system under study.

2) Even though a map may have the shadowing property with measure one, it is not necessarily reflected during computation: computation takes place on a set of measure zero.

3) A computer necessarily has a finite accuracy. Therefore, for a given δ-pseudo-orbit, there may not be a necessary ε in the computer memory, and hence an ε-shadowing orbit.

4) When fixed-precision arithmetic is employed during computation, the numbers generated (and hence the initial conditions) are all rational. Therefore, even though the system under study may be chaotic, the shadowing orbits are unstable periodic, rather than chaotic [78,137,202]. (For systems such as Bernoulli shifts, the unstable periodic orbits mimic chaotic orbits on binary computers [203], but reasons for this have nothing to do with shadowing property.)

To avoid some of the above problems during the computer simulation of a chaotic system, we resort to special methods to ensure that there exists a shadowing orbit near the pseudo-orbit which shadows it for a long period of time. One such method is discussed in Appendix B.

CHAPTER 4

WHY COMPUTER ORBITS LIKE ABSOLUTELY CONTINUOUS INVARIANT MEASURES

4.1 INTRODUCTION

Computer experiments have played a very important role in understanding the nature of chaotic phenomena and in suggesting directions for theoretical analysis [29,99,102]. However, care should be taken in the interpretation of computer data. Computer simulation of chaotic dynamical systems results in the generation of a chaotic computer orbit which diverges from the true chaotic orbit rapidly (e.g. in the case of Lorenz system [65]). Since the theoretical system and the computer are completely deterministic, it leads to the following questions:
How and why does the computer produce chaotic orbits and, in what sense are the computer orbits chaotic?

In this Chapter, we explain this computer phenomena for a large class of one-dimensional transformations.

It has long been observed that the histogram of computer simulation of chaotic orbits seem to display the invariant measure that is absolutely continuous with respect to Lebesgue measure. In [15,33] this was given a theoretical justification and it was shown that the computer orbits of piecewise monotonic transformations of the form kx (mod 1) display Lebesgue measure, the unique ACIM. Unfortunately, the number-theoretic methods used have limitations and there are deficiencies in the random perturbation model $x_{n+1} = \tau(x_n) + W$, W is a random variable, used for computer simulation. We obtain analogous results for general piecewise monotonic transformations on the unit interval using different methods. This leads to the following question which was also raised in [49,137] while studying the distributional properties of long-periodic orbits of maps with an ergodic invariant measure:

Why do computer-generated orbits (mostly) give the statistics of the theoretical invariant density?

We discuss this question later on in the chapter. We now proceed to prove that if a transformation has a unique ACIM, then the histograms of sufficiently long computer orbits of the transformation approximate the histogram obtained from the density of the ACIM. This result justifies the use of a computer to predict long-term behaviour of a system which has an ACIM.

4.2 DESCRIPTION OF THE COMPUTER SPACE

Let $\tau: X \to X$, $X = [0,1]$ be a piecewise monotonic transformation which has a unique ACIM μ. For example, τ may be in the class of piecewise expanding Lasota-Yorke maps or in the class of non-expanding Misiurewicz maps.

We begin with describing the framework and the set of numbers with which a computer works i.e. the computer space. For a fixed precision, the computer distinguishes only a finite number of points in the interval [0,1]. Since a computer is a finite-state machine, it discretizes the continuous phase space into finite number of cells. Let M denote the number of phase cells, C denote the computer space and c a computer point. Then C is a finite space. Let N = Card (C). Any $c \in C$ can be identified with a small interval $I_c^{(N)} \subset X$, consisting of all abstract points which are treated by the computer as c.

In our computer analysis, we use the floating-point arithmetic [135, Sec. 4.2]. Therefore the distribution of computer points in X may not be uniform. We represent a real number as an ordered pair (e,f), where e is the integral part and

f is the fractional part. We have, $-E_1 \leq e \leq E_2$ and $\bar{f}$ is represented as a binary string

$$\bar{f} = \{f_1, \ldots, f_F\}, \quad f = \sum_{i=1}^{F} \frac{f_i}{2^i}, \quad f_i = 0 \text{ or } 1.$$

For fixed E_1, E_2 and F the computer 'sees' the real numbers as $(e, \bar{f}) = 2^e \cdot f$, so that in any 'computer interval' $[2^e, 2^{e+1}]$, $e = -E_1, -E_1 + 1, \ldots, -1$, we have exactly 2^F computer distinguishable points.

To obtain a high precision, we should have large ranges of e and f: a large range for e produces points closer to zero and a large range for f produces an increase in the density of available computer points. Therefore, in our case, we shall let $E_1, F \to \infty$.

We have, $N = E_1 2^F$ and the smallest computer interval has the length, $\frac{(2^{-E_1+1} - 2^{-E_1})}{F}$. Thus, we assume that,

$$|I_c^{(N)}| \geq \frac{1}{SN}, \quad \text{for each } N, \tag{i}$$

where, $S = \frac{1}{E_1(2^{-E_1+1} - 2^{-E_1})}$. When the computer points are uniformly distributed we have $|I_c^{(N)}| = \frac{1}{N}$. Now, having described the computer space, we give a computer model for the theoretical transformation τ.

4.3 A COMPUTER MODEL FOR THE THEORETICAL TRANSFORMATION

A computer model is a variant of the theoretical transformation with which a computer works. We denote the computer model by $\overline{\tau_N}: C \to C$. τ and $\overline{\tau_N}$ are related by the equation,

$$\lim_{N\to\infty} \sup_{x \in I_c^{(N)}} \left| \tau(x) - \overline{\tau}_N(c) \right| = 0 \qquad \text{(ii)}$$

τ acts on a continuous phase space while $\overline{\tau}_N$ acts on a discrete finite set. On application of $\overline{\tau}_N$ any point in the space C is either periodic or eventually periodic and there are finite number of periodic orbits. For chaotic systems, we wish to know the asymptotic statistical behavior of the periodic orbits of $\overline{\tau}_N$. For that, we have to relate the computer transformation $\overline{\tau}_N$ to the theoretical transformation τ.

4.4 AN APPROXIMATION FOR THE THEORETICAL TRANSFORMATION

We define τ_N, N=1,2,3,..., a transformation which relates $\overline{\tau}_N$ and τ, as follows:

For any $c \in C$ and any $x \in I_c^{(N)}$, we have;

$$\tau_N \big|_{I_c^{(N)}} : I_c^{(N)} \xrightarrow{\text{onto}} I^{(N)}_{\overline{\tau}_N(c)} \text{ linearly and } I^{(N)}_{\overline{\tau}_N^k(c)} = I^{(N)}_{\tau_N^k(x)}, \quad k=0,1,2,\ldots$$

where $I^{(N)}_{\tau_N^k(x)}$ is the computer interval containing the theoretical point $\tau_N^k(x)$.

Then, 1) the histogram of $\left\{\overline{\tau}_N^k(c)\right\}_{k=0}^{L}$ and $\left\{\tau_N^k(x)\right\}_{k=0}^{L}$ are the same and

2) $\tau_N \to \tau$ uniformly. (iii)

We can now describe the statistical behavior of computer orbits using the above construction.

4.5 DISTRIBUTION OF COMPUTER ORBITS

The distribution of a computer orbit $\left\{\overline{\tau}^k(x)\right\}_{k=0}^{L}$ is determined by its histogram obtained as follows:

Let **P** be a uniform partition of [0,1] and let every computer interval

$I_c \subset I \in \mathbf{P}$ If then Card (I) is the number of points of $\left\{\bar{\tau}_N^k(c)\right\}_{k=0}^{L}$ which lie in every

interval $I \in \mathbf{P}$, the vector $\frac{1}{L+1}$(Card (I_1), Card (I_2),...), $I_1, I_2, \ldots \in \mathbf{P}$ is the required histogram.

Now, sin⌒, τ has a unique ACIM μ, its density function f, as a consequence of Birkhoff Ergodic Theorem, can be regarded as the statistical description of the long term behaviour of the orbits starting at almost every point in supp (μ). Following Remark 1) of Sec. 1.3.3 we have to find starting points c such

that the distribution of the computer orbit $\left\{\bar{\tau}_N^k(c)\right\}_{k=0}^{L}$ for large L approximates the ACIM μ. To do that, we introduce the following definition:

Free Computer Orbit A finite segment of a computer orbit $\left\{\bar{\tau}_N^k(c)\right\}_{k=0}^{L}$ is 'free' if $\bar{\tau}_N^{k+1}(c) \neq \bar{\tau}_N^k(c)$, $\forall\ k \leq L$.

If the transformation $\bar{\tau}_N$ admits long periodic orbits, it also has long free orbits:

if c is a periodic point with period L+2, then $\left\{\bar{\tau}_N^k(c)\right\}_{k=0}^{L}$ is the free orbit of length L.

We shall prove that for such starting points the histograms induced by the sequence of corresponding free orbits approximates the histogram of the transformation τ induced by the ACIM.

4.6 MAIN RESULT: ASYMPTOTIC BEHAVIOUR OF COMPUTER ORBITS

To prove our desired result, we need the following lemma.

Lemma 1 [41] Let $\nu_N = \left\{\bar{\tau}_N^k(c)\right\}_{k=0}^{L_N}$ be a finite segment of the $\bar{\tau}_N$-orbit and let,

$$\mu_N(\nu_N) = f_N \cdot m = \frac{1}{L_N+1}\left[\sum_{k=0}^{L_N}\left(\left|I_{\tau_N^k(c)}\right|\right)^{-1}\chi_{I_{\tau_N^k(c)}}\right]m$$

where m is the Lebesgue measure and χ is the characteristic function. Then,

a) μ_N is almost an ACIM,

b) for $\frac{L_N}{N} \geq T > 0$, N=1,2,3,... the family $\{f_N\}_{N=1}^{\infty}$ of densities is weakly precompact in L^1.

c) $f_N \to \bar{f}$ as $N \to \infty$, weakly in L^1, where $\bar{f}$ is the density of the unique ACIM μ for the theoretical transformation τ.

Proof Denote $I_c = I_c^{(N)}$.

(a) It is obvious that $\mu_N(\nu_N)$ is absolutely continuous with respect to Lebesgue measure. To prove that it is almost τ_N-invariant it is sufficient to prove that,

$$\left|\int g\, d\mu_N(\nu_N) - \int (g o \tau_N)\, d\mu_N(\nu_N)\right| \leq \frac{2}{L_N+1}\sup|g|, \text{ for any } g \in C[0,1].$$

We have, $\left|\int g(x)\, d\mu_N(\nu_N) - \int (g o \tau_N)\, d\mu_N(\nu_N)\right| =$

$$\frac{1}{L_N+1}\left[\sum_{k=0}^{L_N}\left(\left|I_{\tau_N^k(c)}\right|\right)^{-1}\int_{I_{\tau_N^k(c)}} g(x)\, dm(x) - \sum_{k=0}^{L_N}\left(\left|I_{\tau_N^k(c)}\right|\right)^{-1}\int_{I_{\tau_N^k(c)}} g(\tau_N(x))\, dm(x)\right]$$

By the definition of τ_N we have,

$$\int_{I_{\tau_N^k(c)}} g(\tau_N(x))\, dm(x) = \frac{\left|I_{\tau_N^{k+1}(c)}\right|}{\left|I_{\tau_N^k(c)}\right|}\int_{I_{\tau_N^{k+1}(c)}} g(y)\, dm(y)$$

Thus, $\left|\int g(x)\, d\mu_N(\gamma_N) - \int (g o \tau_N)\, d\mu_N(\gamma_N)\right| \leq$

$$\frac{1}{L_N+1}\left|\frac{1}{|I_c|}\int_{I_c} g(x)\, dm(x) - \frac{1}{I_{\tau_N^{L_N+1}(x)}}\int_{I_{\tau_N^{L_N+1}}} g(x)\, dm(x)\right| \leq \frac{2\sup|g|}{L_N+1}$$

Therefore, if L_N is sufficiently large, the measure $\mu_N(\nu_N)$ is almost τ_N-invariant.

(b) Since, v_N is a free orbit, we have for any point c which is the computer representation of the point x,

$$|\mu_N(v_N)| = |f_N(x)| \le \frac{1}{L_N+1} \sum_{k=0}^{L_N} \left| \left(\left| I_{\tau_N^{k}(c)} \right| \right)^{-1} \chi_{I_{\tau_N^{k}(c)}} \right| = \frac{1}{L_N+1} \left(|I_c| \right)^{-1}$$

$\le \frac{SN}{L_N+1}$, by (i)

$\le \frac{S}{T}$, by hypothesis.

i.e. the family $\{f_N\}_{n=1}^{\infty}$ is uniformly bounded. Hence it is weakly precompact in L^1 [105].

(c) Let $f_N \to \bar{f}$ weakly in L_1 as $N \to \infty$. We prove that $\bar{f}\,.\,m$ is τ-invariant. For any $g \in C[0,1]$, we have

$$\left| \int g \bar{f}\, dm(x) - \int (go\tau) \bar{f}\, dm(x) \right| \le \left| \int g\bar{f}\, dm(x) - \int g f_N dm(x) \right|$$

$$+ \left| \int g f_N dm(x) - \int (go\tau_N) f_N\, dm(x) \right|$$

$$+ \left| \int (go\tau_N) f_N\, dm(x) - \int (go\tau) f_N dm(x) \right|$$

$$+ \left| \int (go\tau) f_N\, dm(x) - \int (go\tau) \bar{f}\, dm(x) \right|$$

$= A + B + C + D$ (say).

Since $f_N \to \bar{f}$ weakly in L_1, A and D $\to 0$, as $N \to \infty$.

Furthermore, $B \le \frac{2 \sup |g|}{L_N+1}$, by (a)

$\le \frac{2 \sup |g|}{MN+1} \to 0$, as $N \to \infty$, by (b).

Let ω_g denote the modulus of continuity of g. Then since $\tau_N \to \tau$ uniformly, (by (iii)) and the function f_N are uniformly integrable (by (b)), we have,

$C < \omega_g (\sup |\tau_N - \tau|) \int f_N \, dm(x) \to 0$, as $N \to \infty$. Therefore, $f_N \to \bar{f}$, as $N \to \infty$ weakly in L^1. This completes the proof of the lemma.

We can now prove our main result.

Theorem 1 [41]. Let $v_N = \left\{\tau_N^k(c)\right\}_{k=0}^{L_N}$ be a sequence of long free computer orbits satisfying,

$$\frac{L_N}{N} \geq \alpha > 0 \quad , \text{ for } \quad N=1,2,3,\ldots$$

Then for any fixed partition $\mathbf{P}$ of [0,1], the sequence of histograms induced by $\{v_N\}_{N=1}^{\infty}$ on $\mathbf{P}$ approaches the histogram on $\mathbf{P}$ induced by the ACIM of the transformation τ.

Proof. By the Lemma 1 (c), $f_N \to f$ weakly in L^1, as $N \to \infty$. So for any interval $I \in \mathbf{P}$, we have, $\int_I f_N \, dm \to \int_I f \, dm$, as $N \to \infty$. Since any theoretical interval $I \in \mathbf{P}$ is the union of computer intervals, we have, $\int_I f_N \, dm = \frac{\text{Card}\,\{v_N \cap I\}}{L_N + 1}$. Hence, $\frac{\text{Card}\,\{v_N \cap I\}}{L_N + 1} \to \int_I f \, dm$, as $N \to \infty$. Thus the histogram of $\{v_N\}$ approaches the histogram of the ACIM μ for τ.

4.6.1 REMARKS

Remark 1 A similar result as of Theorem 1 has been obtained by Blank [17] under the assumption that the dynamical system (X,τ) has a globally attracting stochastic attractor $(\wedge, \mu)$ [94]. It is also shown that, if τ has the shadowing property, then for almost all computer orbits, the true orbits shadowing these computer orbits display the invariant measure μ in the sense of Birkhoff's Ergodic Theorem.

Remark 2 For a system τ which has the shadowing property and admits a unique ACIM μ, almost all computer orbits starting from a point in supp (μ), exhibit μ in the sense of Birkhoff Ergodic Theorem (e.g. in case of Anosov Systems [19]). The significance of Theorem 1 comes from the fact that long computer orbits exhibit μ even though we may not have any information regarding the shadowing property of τ

Remark 3 Numerical evidence is presented in [14] that some discrete nonlinear systems obey a scaling law of, $<L> \sim M^{\varepsilon}$ where $<L>$ is the average period-length of a computer orbit and M is the number of phase cells. Generically, the scaling exponent ε appeared to be smaller than 1. For example, for the logistic map $\tau(x) = 1 - 2x^2, \varepsilon = \frac{1}{2}$. Thus, for long-periodic computer orbits of such systems which have an ACIM, Theorem 1 holds if $M^{\varepsilon} \geq N\alpha$.

4.7 CHAOTIC BEHAVIOUR OF COMPUTER ORBITS

By Theorem 1, we conclude that, if there exists long periodic computer orbits or long non-periodic orbits which occupy a significant portion of the computer space for all precision, then the measures derived from computer simulation must approach the ACIM of the theoretical transformation under consideration i.e. for all the systems which have an ACIM, the process of discretization of the phase space forces the computer orbits to display only the ACIM. Since the ACIM is supported on intervals (which form the chaotic attractor), this gives the computer orbit a chaotic behaviour. These computer orbits are chaotic in the sense that there doesn't exist any finite algorithm which can describe all the iterates of the orbit.

4.8 APPLICATION TO PIECEWISE LINEAR TRANSFORMATIONS

Let $\mathbf{P}$ be the partition of [0,1] given by, $\mathbf{P} = \left\{ I_i : \left(\frac{i-1}{n}, \frac{i}{n} \right), i = 1, \dots, n \right\}$. Let

τ: [0,1] $\to$ [0,1] be defined by,

$$\tau_i(x) = \tau|_{I_i}(x) = s_i x + \frac{d_i}{n},$$

where $s_i \in S = \{p^j : j = 1, ..., s, (p,2) = 1\}$, the set of all possible slopes of τ and d_i is an integer. For example, τ can be the map shown in Fig. 8. Now, define a set

$D_N = \{\frac{a}{n2^N} : 1 \le a \le n2^N$, a odd, $N \ge 2\}$. Then D_N is a finite set with Card $(D_N) = n2^{N-1}$. Let $x \in D_N$. Then, for some j and an integer d_j,

$$\tau(x) = p^j \frac{a}{n2^N} + \frac{d_j}{n} = \frac{b}{n2^N} \in D_N.$$

Thus $\tau: D_N \to D_N$ is well defined. It is shown in [41] that the minimum length of the periodic orbit of $\tau|_{D_N}$ is $2^{N-2}/s$. Since the computer orbit v_N consists of points of the form $\frac{a}{n2^N}$, a computer with precision greater than $n2^N$ will recognize the orbit, making it possible to apply Theorem 1.

4.9 GENERALIZATIONS

1) A transformation may have more than one ACIM. The result of Theorem 1 applies also to this situation provided the different ACIMs are separated by a distance larger than the computer precision. Then the orbit starting from a point in the support or in the basin of attraction of a particular ACIM displays the histogram of that particular measure.

2) We have assumed for τ to have a finite partition in our analysis. With the results of [38], the histograms of computer orbits $\{\tau_N^k(c)\}_{k=0}^{L}$ can be used to approximate the invariant density of a map τ with an infinite number of pieces on the interval, e.g. for the Gauss transformation.

3) In this chapter, we restricted ourselves to maps in one-dimension. The result of Theorem 1 is valid in higher dimensions, though the problem of determining the number of independent ACIMs is a difficult one in higher dimensions. This is due to the question of the support of a function of bounded variation in higher dimensions. Whereas in one-dimensional case, the support of such functions are finite union of intervals, a function of bounded variation in higher dimensions may have support with no interior [37].

4.10 GENERAL REMARKS

Remark 4 We have employed floating-point arithmetic throughout our computational analysis in this Chapter. It is stated in [137,288], that floating-point arithmetic introduces errors that cannot easily be controlled, especially when the system is chaotic. It has been suggested that Turing's theory of computable numbers [16] can correctly formulate chaos theory in the context of computation. It would be then of interest to know whether we can reformulate our results in terms of this theory.

Remark 5 Even with arbitrarily high precision of computation, the asymptotic properties of a system modelled on a computer can differ qualitatively from those of the original system. It is shown in [16] that the method of computing the density of an ACIM of a dynamical system (X,τ), for which the set of preimages of periodic points of τ is dense in X, by the histogram of the computer orbit, may lead to significant errors. We therefore have to be careful while applying Theorem 1 in such cases.

CHAPTER 5

PERTURBATION OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES

5.1 MOTIVATION

Physical systems are usually affected by a number of small external fluctuations (e.g. due to external noise or due to roundoff/truncation errors in computation). We are then concerned with the question of stability in the presence of noise which can be described as recovering parameters of dynamical systems from the study of their deterministic or stochastic perturbations. The parameter which can be obtained in this way can be considered as stable under perturbations and thus has physical sense. Since an ACIM is a parameter which describes the asymptotic statistical behavior of dynamical systems for typical points in the phase space, it is of interest to discuss the above question for dynamical systems which admit an ACIM.

It was Kolmogorov who first discussed the question of stability of invariant measures. Khasminski [183] considered this problem in case of diffusion processes, and Sinai [89], and Kifer [50,204], Young [277] and Collet [254] in case of hyperbolic dynamical systems. Using a convolution form of Frobenius-Perron operator, the stability has been studied for piecewise expanding maps by Jablonski [289] and Boyarsky [6,18], and for some non-expanding unimodal maps by Boyarsky [9] and Collet [24], which admit a unique ACIM. For piecewise monotonic maps considered in [57,100,166], stability results were obtained by Gora in [42] for various types of stochastic perturbations. Using a general random perturbation model, Kifer [53,116,146] has also obtained such results for piecewise monotonic maps. Blank [63,182,234,257] has obtained stability results for higher dimensional transformations and for various types of deterministic and stochastic perturbations.

Lasota-Mackey [195] have obtained stability results which have found application to construction of fractals by iterated function systems.

In this Chapter, we discuss the stochastic stability of ACIMs of piecewise monotonic maps based on the properties of a stochastic operator proposed by Keller in [48]. The definition of the stochastic operator can be applied to both deterministic and randomly perturbed systems. In Appendix D, we shall also briefly consider a more general random perturbation model described by Kifer in [116]. We begin with developing the necessary background for proving the desired results.

5.2 A THEORY OF STOCHASTIC STABILITY

Let $X = [0,1]$ and μ be the Borel-probability measure on X. Let $(L_1, \| \|_1)$ be the space of equivalence classes of μ-integrable, real valued functions on X. For a function $\bar{f}: X \to \mathbb{R}$ and any partition $P = \{0 = a_0 < a_1 < \dots < a_n = 1\}$ we define,

$$V(\bar{f}) = \sup \sum_{i=1}^{n} |\bar{f}(a_i) - \bar{f}(a_{i-1})|,$$

and for an equivalence class $f \in L_1$ let, $V(f) = \inf\{V(\bar{f}) : \bar{f} \in f\}$. Denote,

$BV = \{f \in L_1 : V(f) < \infty\}$ and define,

$$\|f\|_v = V(f) + \|f\|_1.$$

Then, $\| \|_v$ is a norm on BV, which makes $(BV, \| \|_v)$ into a Banach space [46]. BV is a dense linear subspace of $(L_1, \| \|_1)$ and $\{f \in BV: \|f\|_v \leq 1\}$ is a $\| \|_1$ compact subset of L_1[46]. In the sequel we shall not distinguish between a function and its equivalence class. We also assume that all the identities hold almost everywhere with respect to μ. We begin with some basic definitions:

Linear Stochastic Operator We say that an operator $P: L_1 \to L_1$ is a linear stochastic operator if it satisfies the following conditions:

(1) $P(BV) \subset BV$

(2) there exist constants $\alpha > 1, C > 0$ and $\mathbb{N}$ such that,

$\|P\|_v < \infty$, and $\|P^k f\|_v \leq \frac{1}{\alpha}\|f\|_v + C\,\|f\|_1$, $f \in BV$

(3) P is stochastic i.e. $P \geq 0$ and $\int Pf\,d\mu = \int f\,d\mu$, $f \in L_1$, so that $\|P\|_1 = 1$.

Notation Let S denote the class of all linear stochastic operators and let $S(\alpha,C)$ be a subclass of S which satisfies the condition (2) for a fixed α and C.

We then have the following properties [46]:

The operators in S are quasi-compact as operators on $(BV, \|\ \|_v)$, so that by Ionescu-Tulcea and Marinescu Theorem [47] we have

Property 1

(a) P has only finitely many eigenvalues $\lambda_1,\ldots,\lambda_p$ of modulus 1,

(b) the set $\{\lambda_1,\ldots,\lambda_p\}$ is fully cyclic and hence contained in a finite subgroup of the circle and

(c) the corresponding eigenspaces E_i are finite-dimensional subspaces of BV.

Property 2 We have the following spectral decomposition for P:

(a) $P = \sum_{i=1}^{p} \lambda_i \Phi_i + Q$ where the Φ_i 's are projections onto the E_i's, $\|\Phi_i\|_1 \leq 1$,

$\Phi_i \circ \Phi_j = 0, i \neq j$ and $Q : L_1 \to L_1$ is a linear operator with,

$\sup_n \|Q^n\|_1 \leq p + 1$, $Q(BV) \subseteq BV$, $\|Q^n\|_v \leq Mq^n$ for some $0 < q < 1$, $M > 0$ and $Q \circ \Phi_i = \Phi_i \circ Q = 0$, for each i.

(b) For each $\lambda \in \mathbb{R}$, $|\lambda| = 1$ and $f \in L_1$, the limit

$\Phi(\lambda,P)(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\lambda P)^j(f)$ exists in L_1 and

$$\Phi(\lambda,P) = \begin{cases} \Phi_i , & \lambda=\lambda_i \\ 0 , & \text{otherwise} \end{cases}$$

For operators, $P: BV \to L_1$, we introduce the norm,

$$|||P||| = \sup \{ \|Pf\|_1 : f \in BV, \ \|f\|_v \leq 1 \}.$$

S-Bounded Sequence A sequence $\{P_n\}_{n=1}^{\infty}$ is called S-bounded, if there are $\alpha > 1$ and C>0 such that $P_n \in S(\alpha,C)$, $\forall\ n = 1,2,3,\ldots$.

Stochastically Stable Operator An operator $P \in S$ is stochastically stable if,

$$|||P_n - P||| \to 0, \text{ as } n \to \infty, \text{ implies } \|\Phi(1,P_n) - \Phi(1,P)\|_1 \to \infty, \text{ as } n \to \infty$$

for each S-bounded sequence $\{P_n\}_{n=1}^{\infty}$.

Remark 1. Since Φ is continuous, the above definition can be interpreted as of stochastic stability of P under perturbations of initial conditions.

We need the following lemmas:

Lemma 1 [48]. If $P,R \in S$, $P = \sum_{i=1}^{p} \lambda_i \Phi_i + Q$, $\|Q^n\|_v \leq Mq^n$ then for each $\lambda \in \mathbb{R}$ with $|\lambda| = 1$:

(1) $A(\lambda) = \sum_{\substack{i=1 \\ \lambda_i \neq \lambda}}^{p} \frac{\lambda\, \lambda_i}{\lambda - \lambda_i} \Phi_i + (1-\lambda)\, \Phi(\lambda,P) + (\lambda Id - Q)^{-1}$ is a bounded linear operator on BV.

(2) $A(\lambda) = (\lambda Id - (P - \Phi(\lambda,P)))^{-1}$

(3) $(\Phi(\lambda,P) - Id)\ \Phi(\lambda,R) = A(\lambda)\,(P - R)\,\Phi(\lambda,R)$.

Lemma 2 [48]. If $P \in S$, $P = \sum_{i=1}^{p} \lambda_i \Phi_i + Q$, $\|Q^n\|_v \leq Mq^n$, $R \in S(\alpha,C)$ and $\lambda \in \mathbb{R}$, $|\lambda| = 1$, then there are constants B_p (depending on P only),

$$B_{p_1\lambda} = \sum_{\lambda_i \neq \lambda} \frac{1}{|\lambda - \lambda_i|} + |1 - \lambda| . \|\Phi(\lambda,P)\|_1 \text{ and } \Gamma = C\left(\frac{\alpha}{\alpha - 1}\right)$$

such that, if $|||P - R||| \leq 1$, then

$$\|\Phi(\lambda,P) - Id\| . \Phi(\lambda,R)\|_1 \leq (B_p + B_{p_1\lambda}) . \Gamma\, |||P - R||| \left\{2 + \frac{\ln |||P - R|||}{\ln q}\right\} = \Delta.$$

Proof By Definition 1 of a linear stochastic operator, we have, for $f \in BV$,

$$\|\Phi(\lambda,R) f\|_v \leq \lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=0}^{n-1} \|R^j f\|_v \leq \max_{j=0,\dots,k-1} \limsup_{n\to\infty} \|R^{nk+j} f\|_v \leq \max_{j=0,\dots,k-1} C\left(\frac{\alpha}{\alpha - 1}\right) \|R^j f\|_1$$

$$\leq \Gamma . \|f\|_1 \qquad \text{(i)}$$

Also for any $N \in \mathbb{N}$,

$$(\lambda.Id - Q)^{-1} = \lambda \sum_{j=0}^{N-1} (\lambda Q)^j + (\lambda Q)^N . (\lambda.Id - Q)^{-1} \qquad \text{(ii)}.$$

Hence for any $f \in L_1$, we have,

$$\|\Phi(\lambda,P) - Id)\, \Phi(\lambda,R) f\|_1 \leq \|A(\lambda) . (P-R) . \Phi(\lambda,R) f\|_1, \text{ by Lemma 1 (3)}$$

$$\leq (B_{p_1\lambda} + N . (p+1) . \|P-R\| . \Phi(\lambda,R) f\|_1 + M_q^N \frac{1}{1 - q} . \|(P - \lambda.Id)\, \Phi(\lambda,R) f\|_v,$$

by Lemma 1 (1), equation (ii) and Property 2,

$$\leq ((B_{p_1\lambda} + N. (p+1)) . |||P - R||| + M_q^n . \frac{1}{1 - q} . (\|P\|_v + 1).\Gamma.\|f\|_1$$

$$\leq (B_p + B_{p_1\lambda}) . (N . |||P-R||| + q^N) . \Gamma . \|f\|_1, \text{ by equation (i), for suitable } B_p.$$

Choosing $N = \left[\frac{\ln |||P-R|||}{\ln q}\right] + 1$, the lemma follows.

We then have the following corollary:

Corollary 1 Let, $\Delta_\lambda (P,R) = \inf_{P,R} \Delta$. Then, if $P \in S$ and $\{P_n\}_{n=1}^{\infty}$ is S-bounded then,

$\Delta_\lambda (P,P_n) = O(|||P - P_n|||. \,|\ln |||P - P_n|||\,|)$.

Ergodic Operator The operator P is ergodic, if $\dim(E_1) = \dim(\Phi(1,P)(L_1)) = 1$.

Mixing Operator The operator P is mixing, if P is ergodic and 1 is the only eigenvalue of P of modulus 1.

We then have the following necessary conditions for ergodicity and mixing:

Lemma 3 [48] Let $P,R \in S$.

(a) If P is ergodic, then

(1) $\|\Phi(1,P) - \Phi(1,R)\|_1 \leq \Delta_1(P,R)$ and

(2) $\Phi(1,P) - \Phi(1,R) = (Id - (P - \Phi(1,P)))^{-1} . (P-R)\,\Phi(1,R)$

(b) If P is mixing, then

$\lim_{N\to\infty} \sup \|(P^N - R^N)f\|_1 \leq \Delta_1(P,R) + \sum_{j=2}^{p_R} \|\Phi(\lambda_j,R)\|_1$ for all $f \in L_1$, $\|f\|_1 \leq 1$, where $1 = \lambda_1,\ldots,\lambda_{p_R}$ are the eigenvalues of modulus 1 of R.

Proof (a) If P is ergodic, then

$$\Phi(1,P)(f) = \int f\, d\mu \,.\, \Phi(1,P)(1).$$

Hence,
$$\Phi(1,P)(\Phi(1,R)(f)) = \int \Phi(1,R)(f)\, d\mu \,.\, \Phi(1,P)(1) = \int f\, d\mu \,.\, \Phi(1,P)(1) = \Phi(1,P)(f)$$

and so by Lemma 1, (1) follows. Also, (2) follows from Lemma 2 and Corollary 1.

(b) Now, if P is mixing,

$$\|(P^N - R^N)(f)\|_1 \leq \|\Phi(1,P) - \Phi(1,R)\|_1 + \sum_{j=2}^{p_R} \|\Phi(\lambda_j,R)(f)\|_1 + \|(Q(P)^N - Q(R)^N(f)\|_1$$

by (a), $\leq \Delta_1(P,R) + \sum_{j=2}^{p_R} \|\Phi(\lambda_j,R)(f)\|_1 + \|Q(P^N) - Q(R^N)f\|$. Therefore, by Property 2,

$$\lim_{n\to\infty} \sup \|(P^N - R^N)(f)\|_1 \leq \Delta_1(P,R) + \sum_{j=2}^{p_R} \|\Phi(\lambda_j R)(f)\|_1.$$

5.3 A SUFFICIENT CRITERIA FOR STOCHASTIC STABILITY

In this section, we prove the following criteria for stochastic stability:

Theorem 1 Let $P \in S$, $\{P_n\}_{n=1}^{\infty}$ be S-bounded and $\lim_{n\to\infty} |||P - P_n||| = 0$.

Then,

(a) 1) If P is ergodic then, $\|\Phi(1,P) - \Phi(1,P_n)\|_1 = O(|||P - P_n|||.\ln|||P - P_n|||) \to 0$ as $n \to \infty$ i.e. P is stochastically stable.

Furthermore,

2) P_n is ergodic for $\Delta_1 (P,P_n) < 1$.

(b) If P is mixing, then $\lim_{n\to\infty} \lim_{N\to\infty} \sup \|(P^N - P_n^N) f\|_1 = 0$, for each $f \in L_1$ and P_n is mixing for sufficient large n.

Proof (a) 1) Since P is ergodic, we have by Lemma 3(a) that, $\|\Phi (1,P) - \Phi (1,P_n)\|_1 \le \Delta_1 (P,P_n)$. By Corollary 1,

$\Delta_1 (P,P_n) = 0\,(|||P - P_n|||\,.\,|\ln |||P - P_n|||\,|)$. Since $\lim_{n\to\infty} |||P - P_n||| = 0$, $\Delta_1 (P,P_n) \to 0$, as $n \to \infty$.

2) Now, if P_n is not ergodic, there exists an $h \neq 0 \in \{h: P_n h = h\}$ such that, $\int h\, d\mu = 0$. As $\Phi (1,P) (h) = \int h\, d\mu\, .\, \Phi (1,P) (1) = 0$, we have,

$$0 < \|h\|_1 = \|\Phi (1,P) (h) - \Phi (1,P_n) (h)\|_1 \le \|\Phi (1,P) - \Phi (1,P_n)\|_1 \|h\|_1 ,$$

by Cauchy-Schwarz inequality,

$$< \|h\|_1 ,$$ a contradiction.

(b) If P_n is not mixing, then by Lemma 3(b). P_n has an eigenvalue of modulus 1 different from 1. Since the spectrum of P_n is fully cyclic, by Property 1(b), there exists an eigenvalue $\lambda^{(n)}$ of P_n, with $|\lambda^{(n)}| = 1$ and $|\lambda^{(n)} - 1| > 0$.

Now, P is mixing, so from Lemma 2, we have

$$B_{P_1\lambda} = \sum_{\lambda_i \neq \lambda} \frac{1}{|\lambda - \lambda_i|} + |1 - \lambda|\, .\, \|\Phi (\lambda, P)\|_1 = \frac{1}{|1 - \lambda^{(n)}|} .$$

Furthermore,

$$\|\Phi(\lambda^{(n)}, P_n)\|_1 \leq \left(\frac{1}{|1-\lambda^{(n)}|} + B_p\right) . \Gamma . |||P-P_n||| . \left(2 + \frac{\ln |||P - P_n|||}{\ln q}\right) \to 0, \quad \text{as } n \to \infty.$$

But, $\|\Phi(\lambda^{(n)}, P_n)\|_1 = 1$, for each n = 1,2,3,..., which is a contradiction. Hence, it follows from Lemma 3(b), that both P and P_n are mixing. This completes the proof of the theorem.

We now consider the applications of the model discussed in this section.

5.4 DETERMINISTIC PERTURBATIONS OF PIECEWISE EXPANDING MAPS

Let X = [0,1] with the Lebesgue measure m. We denote the class of all piecewise expanding Wong maps by C. The Frobenius-Perron operator, $P_\tau : L_1 \to L_1$ associated with τ is given by,

$$\int g . P_\tau f \, dm = \int (g \circ \tau) f \, dm , \quad \forall f \in L_1 \text{ and } g \in L_\infty.$$

It can be shown, by the change of variable formula that,

$$P_\tau f(x) = \sum_{i=1}^{N} f(\tau^{-1}|_{I_i}(x)) \; . \; |\tau^{-1}|_{I_i}(x)| \; . \; \chi_{\tau(I_i)}(x) = \sum_{y \in \tau^{-1}(x)} f(y) \; . \; |\tau'(y)|^{-1} .$$

On the class C of piecewise expanding maps, we introduce the following:

<u>Skorokhod metric</u>. $d(\tau_1,\tau_2) = \inf \{\varepsilon > 0: \exists A \subseteq [0,1]$ and $\exists \sigma : [0,1] \to [0,1]$ such that $m(A) > 1 - \varepsilon$, σ is a diffeomorphism,

$$\tau_1|_A = \tau_2 \circ \sigma|_A \text{ and } \forall \; x \in A, \; |\sigma(x) - x| < \varepsilon, \; \left|\frac{1}{\sigma'(x)} - 1\right| < \varepsilon\}.$$

We can now define what it means by small deterministic perturbations of a map τ.

<u>Small Deterministic Perturbations</u>. Let $\{\tau_n\}$ be a family of maps from the space X into itself and $\tau: X \to X$ be a map. If $d(\tau_n, \tau) \to 0$, as $n \to \infty$, then τ_n are said to be <u>small deterministic perturbations of the map τ</u>.

We shall need the following characterization of variation:

Define the integral $\int(\Phi)$, of a function $\Phi \in L_1$ by,

$$\int(\Phi)(z) = \int_{x \leq z} \Phi(x)\, d\mu(x).$$

Property 3 Let $f \in BV$ and $\Phi \in L_1$. Then,

$$|\int f \Phi\, d\mu| \leq V(f) \,.\, \| \int(\Phi) \|_\infty + | \int \Phi\, d\mu | \,.\, \|f\|_\infty \leq 2 \,.\, \|f\|_v \,.\, \| \int(\Phi) \|_\infty.$$

Proof. Let $J_1, \ldots, J_M$ be a partition of X into subintervals such that $J_1 \leq \ldots \leq J_M$

and assume that Φ is constant on each J_i. Denote $G = \int(\Phi)$.

Then,

$$\left|\int f \,.\, \Phi\, d\mu\right| = \left|\sum_{i=1}^{M} \int_{J_i} f \,.\, \Phi\, d\mu\right| = \left|\sum_{i=1}^{M} u_i \int_{J_i} \Phi\, d\mu\right|, \text{ for some } u_i \in \overline{\mathrm{co}\, f(J_i)}$$

$$= \left|\sum_{i=1}^{M} u_i \,[G(b_i) - G(a_i)]\right|, \text{ where } a_i, b_i \text{ are endpoints of } J_i,$$

$$\leq \sum_{i=2}^{M} |u_{i+1} - u_i| \,.\, \|G\|_\infty + |G(0) \,.\, u_1| + |G(1) \,.\, u_M|, \text{ where}$$

$0 = \inf X$, and $1 = \sup X$,

$$\leq V(f) \,.\, \|G\|_\infty + |G(1)| \,.\, \|f\|_\infty,$$

$$\leq 2\, \|f\|_v \,.\, \|G\|_\infty.$$

Now for a general Φ, the required inequality follows by approximation.

Property 4 For $f \in L_1$, $V(f) = \sup \left|\int f \Phi\, d\mu\right|$, where the supremum extends

over all $\Phi \in L_1$ with $\| \int(\Phi) \|_\infty \leq 1$ and $\int \Phi\, d\mu = 0$.

Proof By Property 3, it follows that

$$V(f) \geq \sup \left| \int f . \Phi \, d\mu \right| .$$

Therefore, we prove the reverse inequality. Let $S = \sup_{\Phi} \left| \int f . \Phi \, d\mu \right|$ the supremum as in the statement of the Property; and assume that $S < \infty$. Choose a sequence $\{I_n\}$ of finite partitions of X, into subintervals, I_{n+1} finer than I_n which generates the σ-algebra on X. Then the expectation $E_m[f \mid I_n] \rightarrow f$ a.e. with respect to μ. This implies that for each version $\bar{f}$ of f, $\frac{1}{\mu(I_n(x))} \int_{I_n(x)} f \, d\mu \rightarrow \bar{f}$, everywhere except for a set $N(\bar{f})$ of zero μ measure, where $I_n(x)$ denotes the element of I_n containing x. Now, the sums of the type $\sum_{i=1}^{k} \left| \bar{f}(a_i) - \bar{f}(a_{i-1}) \right|$ with $a_i \notin N(\bar{f})$, can be approximated by the integrals $\int f . \Phi \, d\mu$, with Φ as required.

The $\sup_{\substack{a_0 < \ldots < a_k \\ a_i \notin N(\bar{f})}} \sum_{i=1}^{k} | \bar{f}(a_i) - \bar{f}(a_{i-1}) | \leq S < \infty$. That is, $\bar{f}|_{X / N(\bar{f})}$ is of bounded variation and can be extended to a function $\bar{\bar{f}}$ on the whole of X (by using one-sided limits), such that $\sup_{a_0 < \ldots < a_k} \sum_{i=1}^{k} \left| \bar{\bar{f}}(a_i) - \bar{\bar{f}}(a_{i-1}) \right| \leq S$. Since $\bar{\bar{f}}$ is also a version of f, we finally have $V(f) \leq S$. This completes the proof.

Denote $P_n = P_{\tau_n}$. We then have,

Lemma 4 [48]. If P_1, P_2 are the Frobenius-Perron operators corresponding to $\tau_1, \tau_2 \in C$, then $|||P_1 - P_2||| \leq 12 . d(\tau_1, \tau_2)$.

Proof Let $f \in BV$, $g = \frac{|P_1 f - P_2 f|}{(P_1 f - P_2 f)}$. Then,

$$\int \left| P_1 f - P_2 f \right| dm = \int g . (P_1 f - P_2 f) \, dm = \int f . (g \circ \tau_1 - g \circ \tau_2) \, dm$$

$$\leq 2\ \|f\|_v \,.\, \sup_z \left| \int_0^z (go\tau_1 - go\tau_2)\, dm \right|, \text{ by Property 3,}$$

$$\leq 2\,.\, \|f\|_v \,.\, \sup_z \left(4\,\varepsilon + \left| \int_0^z (go\tau_2 o\sigma - go\tau_2)\, dm \right|\right)$$

for any ε, σ and A in the definition of d,

$$\leq 2\,.\, \|f\|_v \,.\, \sup_z (4\varepsilon + \varepsilon + \varepsilon),$$

$$= 12\,.\, \varepsilon\,.\, \|f\|_v$$

Therefore, $|||P_1 - P_2||| \leq 12 d(\tau_1, \tau_2)$.

We then have the following result of stability for piecewise expanding Wong maps from Lemma 4.

Theorem 2 Let $\tau, \tau_n \in C$ and P, P_n be their corresponding Frobenius-Perron operators. If $\{P_n\}_{n=1}^{\infty}$ is S-bounded and $d(\tau, \tau_n) \to 0$, as $n \to \infty$, then Theorem 1 holds. In particular, if τ is ergodic, then τ_n is ergodic for large n, and the unique invariant densities of τ_n converge in L_1 to that of τ.

5.4.1 REMARKS

Remark 2 The existence and conditions of uniqueness of absolutely continuous invariant measures for the maps of class C were shown in [100]. By Theorem 2, it follows that such ACIMs are stable under the class of deterministic perturbations considered.

Remark 3 A similar result as of Theorem 2 was obtained by Kowalski in [56] for Lasota-Yorke maps and by Blank in [182] for various types of deterministic perturbations.

5.5 STOCHASTIC PERTURBATIONS OF PIECEWISE MONOTONIC MAPS

Let $X = [0,1]$ and m be the Lebesgue measure on X. Let $\tau : X \to X$ be a piecewise monotonic map. Let P denote the Frobenius-Perron operator associated with τ. From [87], P is given by,

$$Pf(x) = \sum_{y \in \tau^{-1}(x)} f(y)\, g(y),$$

$f \in L_1$, $g \in L_\infty$, $\mathrm{Var}(g) < \infty$, and it is shown in [46] that $P \in S$. We now add stochastic perturbations to the process $x_{n+1} = \tau(x_n)$, $n = 0,1,2,...$ described by the transformation τ.

Small Stochastic Perturbations. Let $K(x,y): X \times X \to \mathbb{R}$ be a stochastic kernel. Consider the Markov process X_n, $n = 0,1,2,...$ described by the transition probability $p(x,y) = K(\tau(x),y)$. This means that a particle jumps from x to $\tau(x)$ and then disperses randomly nearly $\tau(x)$ with the distribution $K(\tau(x),y)$. It can thus be interpreted as a stochastic perturbation of the transformation τ. The time evolution of the densities of the process X_n is given by,

$$P_K : L_1 \to L_1,\ P_K f(x) = \int f(u)\, p(u,x)\, dm(u) = \int P f(u)\, K(u,x)\, dm(u).$$

We shall need the following result from [46].

Lemma 5 Let $\mu = \Phi(1,P)(1)\, m$. Then,

(1) τ is m-ergodic iff P is ergodic, and

(2) τ is μ-exact iff τ is μ-weakly mixing iff P is strongly mixing.

We then have the following stability result for piecewise monotonic maps.

Theorem 3 [48] Let $P \in S$ be the Frobenius-Perron operator of a piecewise monotonic map $\tau: X \to X$, which satisfies (2) of the definition of the stochastic operator for $k = 1$. For $z \in X$, set,

$$K_z(y) = \int_{x \le z} K(x,y)\, dm(x),\ B_z = \{(x,y) : x \le z < y \text{ or } y \le z < x\},$$

$$\ldots) = \int\!\!\int_{B_z} K(x,y)\, dm(x)\, dm(y)\ , \quad c(K) = \sup_{z \in X} b(z).$$

Then for P_K defined as above, we have,

(a) $\|P_K\|_1 \le \|P\|_1 = 1$

(b) $V(P_K f) \le \sup_{z \in X} V(K_z) \,.\, V(Pf)$

(c) $K_z(y_1) \ge K_z(y_2),\ y_1 \le y_2$ implies that P and P_K are in the same class $S(\alpha, C)$.

(d) $|||P - P_K||| \le c(K) \,.\, \|P\|_v$.

<u>Proof</u> (a) Since K is stochastic, the result follows from the definition of the stochastic operator.

(b) Take $\phi \in L_1$ with $\| \int (\phi) \|_\infty \le 1$ and $\int \phi \, dm = 0$ then,

$$\int P_K f \phi \, dm = \int P f \bar{\phi} \, dm \qquad \text{(i)}$$

where $\phi(x) = \int K(x,y)\, \phi(y)\, dy$ such that $\int \bar{\phi} \, dm = \int \phi \, dm = 0$ and

$\left| \int (\bar{\phi})(z) \right| = \left| \int K_z - \phi dm \right| \le V(K_z)$, by Property 4.

Applying Property 4 to equation (i) again, we obtain, $V(P_K f) \le V(Pf) \sup_{z \in X} V(K_z)$

(c) Since $K_z(y)$ is monotonically decreasing and $0 \le K_z \le 1$ we have, $V(K_z) \le 1, \ \forall \ z \in X$

Therefore, by (b),

$$V(P_K f) \le V(Pf). \qquad \text{(ii)}$$

By assumption, for P, we have,

$$\|Pf\|_v \le \alpha^{-1} \|f\|_v + c\|f\|_1. \qquad \text{(iii)}$$

Now,

$$\|P_K f\|_v = V(P_K f) + \|P_K f\|_1$$

$\leq V(Pf) + \|f\|_1$, by (ii) and since $\|P_K f\|_1 = \|f\|_1$

$= (\|Pf\|_v - \|f\|_1) + \|f\|_1 = \|Pf\|_v \leq \alpha^{-1} \|f\|_v + C\|f\|_1$, by (iii).

Hence P and P_K are in the same class $S(\alpha,C)$

(d) For $f \in BV$, set $F = Pf - P_K f$ and $\phi = \frac{|F|}{F}$.

Then, $\int |Pf - P_K f|\, dm = \int F \cdot \phi\, dm = \int P f(y) \cdot (\phi(y) - \int K(y,x)\, \phi(x) \cdot dm(x))\, dm(y)$

Furthermore,

$$\left| \int_{\{y \leq z\}} (\phi(y) - \int K(y,x)\, \phi(x)\, dm(x))\, dm(y) \right|$$

$$= \left| \int \phi(y) \cdot \chi_{\{y \leq z\}}(y) - \int_{\{x \leq z\}} K(x,y)\, dm(x)\, dm(y) \right|$$

$$\leq \int \left| \chi_{\{y \leq z\}}(y) - \int_{\{x \leq z\}} K(x,y)\, dm(x) \right| dm(y)$$

$$= \int_{B_z}\int K(x,y)\, dm(x)\, dm(y),$$

by the definition of $K(x,y)$ and conditions (ii) and (iv)

$= b(z)$.

Hence, $\int |Pf - P_K f|\, dm \leq c(K) \cdot V(Pf) \leq c(K) \cdot \|P\|_v \cdot \|f\|_v \cdot \leq c(K) \cdot \|P\|_v$.

Therefore, $|||P - P_K||| \leq c(K)\, \|P\|_v$. This completes the proof.

Remark 4 By Theorems 1, 3 and Lemma 5, we conclude that if τ is ergodic then small stochastic perturbations of the transformation τ cause only small perturbations of the invariant density. If τ is mixing, then even the way of convergence is not much affected by the perturbation.

5.6 GENERAL REMARK

The significance of invariant measures which are stable with respect to random perturbations, has been underlined by Ruelle [81-86,163,171] in connection with the mathematical models for the phenomenon of hydrodynamic turbulence. Then it is natural to assume that physical relevant measures, such as ACIMs, which may describe turbulence must be stable under any perturbations.

CHAPTER 6

SHADOWING PROPERTY FOR MARKOV OPERATORS IN THE SPACE OF DENSITIES

6.1 INTRODUCTION

The shadowing property for certain dynamical systems was studied in Chapter 3. We saw that the family of tent maps have the shadowing property for almost all parameter values, although they fail to have the shadowing property for an uncountable dense set of parameters.

In this Chapter, we study the question of shadowing property for an operator on the family of maps. We propose a generalized shadowing property for linear operators and show that certain Markov operators $P: L_1 \to L_1$ have this property on a weakly compact subsets of the space of probability density functions. An important class of such operators are the Frobenius-Perron operators. We shall prove that unlike the situation in the space X itself, the generalized shadowing property is valid for all parameters in families of maps. Thus there is continuity with respect to the generalized shadowing property.

We now give the necessary background to prove the above stated results.

6.2 PREREQUISITES

We have the following basic definitions:

Generalized shadowing property Let X be a nonempty subset of a linear space with metrics d_1 and d_2 and let an operator $T: (X,d_1) \to (X,d_2)$ linearly. We say that T has the (δ,ε)-generalized shadowing property with respect to d_1 and d_2 (or simply the generalized shadowing property) if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit (in d_1) can be ε-shadowed by a true orbit (in d_2).

Let $(X,\mathbf{B},\mu)$ be a finite measure space and let $L_1 \equiv L_1(X,\mathbf{B},\mu)$ with the L_1-norm $\| \|_1$. Let D_1 denote the space of densities on X. Then (D_1,σ) is a metric space, where σ is the metric induced by the L_1-norm $\| \|_1$.

Markov operator A linear operator $P: L_1 \to L_1$ is called Markov if $P(D_1) \subset D_1$.

Remark 1 A Markov operator P describes the evolution of densities in dynamical systems. The Frobenius-Perron operator is an important example of a Markov operator. The study of asymptotic properties of Markov operators provides a natural generalization to the analogous results obtained in ergodic theory, where Frobenius-Perron operators are studied (and in the theory of Harris operators [259]). For a recent survey in the asymptotic theory of Markov (and related) operators see [235].

The notion of constrictiveness in the theory of Markov operators was introduced by Lasota.

Constrictive Markov Operator A Markov operator $P: L_1 \to L_1$ is weakly (strongly) constrictive if there exists a weakly (strongly) precompact set $A \subset L_1$ such that,

$$\lim_{n \to \infty} \inf_{g \in A} \|P^n f - g\|_1 = 0 \text{ , for } f \in D_1.$$

The Markov operator $P: L_1 \to L_1$ is quasi-constrictive if we have a set $A \in \mathbf{B}, \mu(A) < \infty$ and constants $0 \leq \varepsilon < 1$, $\delta > 0$ satisfying the following condition: For every $f \in D_1$, there exists an integer $N = N(f)$ such that

$$\int_{E \cup (X \setminus A)} P^n f \, d\mu \leq \varepsilon \text{ , for } n \geq N, \mu(E) \leq \delta.$$

The importance of the above condition comes from the realization that it can be considered as a generalization of the Doeblin condition [106] in the theory of Markov processes. The quasi-constrictiveness of Markov operators is a substantial refinement of their quasi-compactness.

Following [186] we call the set A as an attractor for the iterates of densities under P. From [175] we have,

Lemma 1. In the class of Markov operators, the notions of weakly constrictive, strongly constrictive and quasi-constrictive operators are equivalent.

The significance of Lemma 1 comes from the fact that for many operators appearing in applied problems the weak constrictiveness is much easier to verify.

In the sequel, we delete the adjectives strong, weak or quasi for constrictive Markov operators.

Remark 2 Spectral decomposition and asymptotic periodicity of constrictive Markov operators has been discussed in [62, 138, 175, 176] under gradually weakened conditions concerning the existence of the attractor. Applications of constrictive Markov operators have been considered in [60, 114].

Vague Convergence We say that a sequence $\{f_n\}$ in D_1 converges vaguely to f, if for any $h \in C(X)$, we have,

$$\int_X h(x)\, f_n(x)\, \mu(dx) \to \int_X h(x)\, f(x)\, \mu(dx) \text{ , as } n \to \infty.$$

Topology of Vague and Weak convergence Let $\{\phi_n\}$ be a countable dense subset of $C(X)$ in the supn.-m topology. Let $\beta_n = \sup_{x \in X} |\phi_n(x)| > 0$ and let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \alpha \le 1$. Define a seminorm $\| \|$ on L_1 by,

$$\| f \| = \sum_{n=1}^{\infty} \alpha_n \left| \int_X \phi_n(x)\, f(x)\, \mu(dx) \right| .$$

Then $\| \|$ defines the topology of vague convergence on D_1 and $\| f \| \le \alpha \| f \|_1$. Let ρ be the metric induced by $\| \|$. For a weakly compact set $D \subset D_1$, the weak topology of L_1 restricted to D, is defined by $\| \|$.

6.3 PRELIMINARY RESULTS

Let (X,ρ) be a compact metric space. For $A \subset X$, closed let

$A_\varepsilon = \{x \in X: \inf_{y \in A} \rho(x,y) < \varepsilon\}$ be an ε neighbourhood of A. We shall need the following asymptotic stability result.

Lemma 1 [12]. Let $P: (X,\rho) \to (X,\rho)$ be continuous map. Assume that there exists a closed set $A \subset X$ such that $PA=A$ and that,

$$\rho(P^i x,A) = \inf_{y \in A} \rho(P^i x,y) \to 0, \text{ as } i \to \infty \qquad (i)$$

uniformly for all $x \in X$. Then for any $\varepsilon > 0$ and any $\delta > 0$ $\exists$ a positive integer $N \ni P^n A_{\varepsilon+\delta} \subset A_\varepsilon$ for $n \geq N$.

Proof From [72], we know that there exists a neighbourhood U of A such that $P(U) \subset U$. We can assume that, $U \subset A_\varepsilon$. Then, it is enough to prove that for some positive integer N, we have $P^N(A_{\varepsilon+\delta}) \subset U$. Let $d = \inf\{P(x,y): x \in A, y \notin U\}$. From equation (i), we have that there exists an N such that for any $n \geq N$ and any $x \in X$, we have $\rho(P^n x,A) < \frac{d}{2}$. Hence $P^N(A_{\varepsilon+\delta}) \subset A_{\frac{d}{2}} \subset U$. This completes the proof of the lemma.

The following result relates the existence of the generalized shadowing property of an operator and its iterates.

Lemma 2 [12]. If P is continuous in the ρ metric and P^N has the (δ,ε)-generalized shadowing property, then P has the $(\frac{\delta}{N},\varepsilon_1)$-generalized shadowing property, where

$$\varepsilon_1 = \max\{\omega(\varepsilon) + \frac{\delta}{N}, \omega(\omega(\varepsilon) + \frac{\delta}{N}) + \frac{\delta}{N}, \dots, \underbrace{\omega(\dots\omega(\omega}_{(N-1)\text{ - times}}(\varepsilon) + \frac{\delta}{N}) + \frac{\delta}{N}\dots) + \frac{\delta}{N}\},$$

and ω is the modulus of continuity of P in the ρ-metric, i.e.

$$\omega(t) = \sup\{\rho(Px, Py) : x, y \in X, \rho(x, y) \leq t\}.$$

If P is continuous in the ρ metric and has the (δ,ε)-generalized shadowing property, then P^N also has the (δ,ε)-generalized shadowing property.

Proof Part 1 Let $\{x_o, x_1, \ldots\}$ be a $\frac{\delta}{N}$-pseudo orbit of P. We take sequence of points $\{x_o, x_N, x_{2N}, \ldots\}$, which form a δ-pseudo orbit for P^N. Since P^N has the (δ,ε)-generalized shadowing property, there exists a point $y \in X$ such that for any positive integer k, we have,

$$\rho(P^{kN}y, x_{kN}) < \varepsilon, \tag{1}$$

We now prove that there exists an ε_1 such that for any k and any $1 \leq j \leq N\text{-}1$,

$$\rho(P^{kN+j}(y), x_{kN+j}) < \varepsilon_1 \tag{2}$$

By (1) and the definition of ω, we obtain,

$$\rho(P^{kN+1}(y), P(x_{kN})) < \omega(\varepsilon) \tag{3}$$

Since, $\{x_n\}_{n=0}^{\infty}$ is a $\frac{\delta}{N}$-pseudo orbit, we obtain,

$$\rho(P^{kN+1}(y), x_{kN+1}) \leq \rho(P^{kN+1}(y), P(x_{kN})) + \rho(P(x_{kN}), x_{kN+1}) < \omega(\varepsilon) + \frac{\delta}{N},$$

by (3). Thus equation (2) holds for $j = 1$. Continuing, in this way we obtain (2) by induction, for $j \leq N\text{-}1$.

Part 2 Let $\{x_o, x_1, \ldots\}$ be a δ-pseudo orbit for P^N. Then

$$\{x_o, P(x_o), \ldots, P^{N-1}(x_o), x_1, P(x_1), \ldots, P^{N-1}(x_1), \ldots\} \tag{4}$$

is a δ-pseudo orbit for P. Since P has the generalized shadowing property, there exists a $y \in X$ such that the orbit $\{y, Py, P^2y, \ldots\}$ approximates the pseudo orbit given by (4) within ε. Therefore, it follows that the orbit $\{y, P^Ny, P^{2N}y, \ldots\}$, approximates the orbit $\{x_o, x_1, \ldots\}$ within ε; which completes the proof of the lemma.

We now proceed to express Lemma 2 more explicitly in case of a Markov operator.

A Special Metric Associated with a Markov Operator

Let (X,Σ,μ) be a measure space with Σ a countably generated σ-algebra of measurable sets. If P is a Markov operator on $L_1 = L_1(X,\Sigma,\mu)$ then there exists a transition function $P(\ ,\ )$, which is a measurable function in the first variable and a measure in the second variable, such that P is the unique operator satisfying,

$$\int (Pf)g\, d\mu = \int f(x)\, \{\int P(x, dy)\, g(y)\}\, \mu(dx)$$

for all $f \in L_1$ and $g \in L_\infty$, i.e. P is adjoint to the operator $Qg(x) = \int P(x,dy)\, g(y)$.

Using this representation of the operator P, we associate a special metric with P. Let $\{\phi_n\}$ be a countable dense subset of $C(X)$. We define the metric ρ^* as follows:

$$\rho^*(f,g) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} c^k \left| \int (\phi_n o Q^k)\, (f - g)\, d\mu \right|,$$

where $0 < c < 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n \leq 1 - c$, $\alpha_n > 0$ and $\beta_n = \sup_{x \in X} |\phi_n(x)|$.

Remark 3. The metric ρ^* gives the weak topology of L_1 on any weakly compact set in L_1.

Remark 4. If $\rho = \rho^*$ is the special metric associated with the operator P in Lemma 2, then

$$\varepsilon_1 = \varepsilon(\frac{1}{c})^{N-1} + (\frac{\delta}{N})\,(1 + \frac{1}{c} + \ldots + (\frac{1}{c})^{N-1})$$

$$= \varepsilon(\frac{1}{c})^{N-1} + \frac{\delta(1 - (\frac{1}{c})^{N-1}}{N(1 - \frac{1}{c})}.$$

The following lemma compares the metrics σ and ρ.

Lemma 3 [12]. Let σ be the norm metric in $L_1 = L_1(X,B,\mu)$ and ρ the metric of weak convergence defined in Sec. 6.2. Then for all $f, g \in D$, we have,

$$\rho(f,g) \leq \sigma(f,g).$$

Proof We have,

$$\rho(f,g) = \sum_{n=1}^{\infty} \alpha_n \left| \int_X \phi_n(x)(f(x) - g(x))\, \mu(dx) \right|$$

$$\leq \sum_{n=1}^{\infty} \alpha_n \beta_n \left| \int_X (f(x) - g(x))\, \mu(dx) \right|$$

$$\leq \alpha . \sigma(f,g)$$

$$\leq \sigma(f,g) \text{ , since } \alpha \leq 1.$$

We now have the requisite background to prove main results of the chapter and discuss their applications.

6.4 SHADOWING PROPERTY FOR CONSTRICTIVE MARKOV OPERATORS IN THE SPACE OF DENSITIES

In this section, we obtain a sufficient condition for a constrictive Markov operator to have the (δ,ε)-generalized shadowing property. Under specific conditions, we also obtain an estimate of δ in terms of ε.

Let D be a compact set of (D_1, ρ). The main result is

Theorem 1 [12]. Let $P: L_1 \to L_1$, $L_1 = L_1(X,B,\mu)$, be a constrictive Markov operator with the attractor A consisting of a single element f^* of a ρ-compact set $D \subset D_1$. Assume $PD \subset D$. If $\lim_{n\to\infty} \|P^n f - A\|_1 = 0$ uniformly for all $f \in D$, then $P: (D,\sigma) \to (D,\rho)$ has the generalized shadowing property (with respect to the metrics σ and ρ).

Proof Fix an $\varepsilon > 0$. By Lemma 1, there exists an integer $N_0 > 0$ such that

$P^{N_o}(D) \subset A_\varepsilon$. Let $\delta = \frac{\varepsilon}{N_o}$. Let N be the smallest positive integer such that,

$$P^N(A_{\varepsilon+\delta}) \subset A_\varepsilon \tag{1}$$

Let $\overline{P} = P^N$ and

$$k = \left[\frac{N_o - 1}{N}\right] + 1 \ , \tag{2}$$

where $[x]$ denotes the greatest integer $\leq x$. Then, we have,

$$k \leq N_o,\ kN \geq N_o \text{ and } \overline{P}^k f_o \in A_\varepsilon \ , \text{ for any } f_o \in D. \tag{3}$$

Consider any δ-pseudo orbit $\{f_o, f_1, f_2, \dots\}$ of $\overline{P}$ starting from the point $f_o \in D$ i.e.

$$\sigma(\overline{P}f_n, f_{n+1}) < \delta \ , \quad \text{for each } n=0,1,2,\dots$$

then, by Lemma 3, we have,

$$\rho(\overline{P}f_n, f_{n+1}) < \delta \ , \quad \text{for each } n=0,1,2,\dots \tag{4}$$

<u>Claim</u>: $\rho(\overline{P}^j f_o, f_j) \leq \varepsilon$, for $j=0,1,2,\dots, k$.

For $j=0$, the result is obvious. For $j=1$, $\rho(\overline{P}^j f_o, f_j) \leq j\delta$, by (4).

Therefore assume the result to be true for j itself. We then have,

$$\sigma(\overline{P}^{j+1} f_o, f_{j+1}) \leq \sigma(\overline{P}(\overline{P}^j f_o), \overline{P}f_j) + \sigma(\overline{P}f_j, f_{j+1}) \leq j\delta + \delta = (j+1)\delta.$$

Hence, by induction, $\sigma(\overline{P}^j f_o, f_j) \leq j\delta$, for each $1 \leq j \leq k$.

By Lemma 3, we get

$$\rho(\overline{P}^j f_o, f_j) \leq k\delta \leq \varepsilon. \tag{5}$$

So, the δ-pseudo orbit (in ρ), $\{f_o, f_1, f_2, \dots\}$ and the true orbit $\{\overline{P}^j f_o\}$ stay close (within ε) to each other for the first k iterates. We prove that this is the case for other iterates also.

By (1), $\overline{P}A_{\varepsilon+\delta} \subset A_\varepsilon$, and by (3), $\overline{P}^k f_0 \in A_\varepsilon$. From (5), we have, $f_k \in A_{\varepsilon+\delta}$ and so again by (1), $\overline{P} f_k \in A_\varepsilon$. Furthermore, (4) implies that $f_{k+1} \in A_{\varepsilon+\delta}$. Then, since $\overline{P}^{k+1} f_0 \in A_\varepsilon$, $f_{k+1} \in A_{\varepsilon+\delta}$ and since A consists of only one element,

$$\rho(\overline{P}^{k+1} f_0, f_{k+1}) < \varepsilon + (\varepsilon + \delta) < 3\varepsilon.$$

By repeating the above argument by induction and combining it with (5), we obtain:

For $j \geq 1$, $\rho(\overline{P}^j f_0, f_j) < 3\varepsilon$ whenever $\sigma(\overline{P} f_j, f_{j+1}) < \delta$ for $j \geq 0$.

Therefore, $\overline{P} = P^N$ has the generalized shadowing property. By Lemma 2, we conclude that P also has this property. This completes the proof.

Remark 5. The Markov operator $P: L_1 \to L_1$, $L_1 = L_1(X, \Sigma, \mu)$ considered in Sec. 6.3 satisfies the condition $P(D) \subset D$ of Theorem 1 for a class of weakly compact sets $D \subset L_1$.

Remark 6. For a constrictive Markov operator P we can have the following spectral decomposition [114, Sec. 5.3]:

$$Pf(x) = \sum_{i=1}^{\nu} \lambda_i(f) g_i(x) + Qf(x)$$

where, $\lambda_i(x) = \int_X f(x)\, k_i(x)\, \mu(dx)$,

for some integer ν, $g_i \in D_1$, $k_i \in L_\infty$ for $i = 1, \ldots, \nu$ and $Q: L_1 \to L_1$. Now, P is exact if and only if $\nu = 1$ [114, Sec. 5.5]. Thus, Theorem 1 gives a sufficient condition of generalized shadowing property for exact Markov operators.

In practical situations where we have exponential convergence of the iterates to the invariant density, it is of interest to know when we can express δ in terms of ε.

Theorem 2 [12]. Suppose that there exist $M > 0$ and $0 < q < 1$ such that for any $f \in D$,

$$\|P^n f - f^*\|_1 < Mq^n. \tag{1}$$

If ρ^* is the metric associated with P, then P has the (δ,ε) generalized shadowing property with

$$\delta \approx \text{constant }(n)\, \varepsilon^{(1+n)}, \text{ as } \varepsilon \to 0, \text{ for some } n > 0.$$

Proof Let N_o be the constant in Theorem 1. We wish to have,

$$\|P^{N_o} f - f^*\|_1 < \varepsilon.$$

Comparing with (1), it is sufficient to take $N_o = \left[\log_q(\frac{\varepsilon}{M})\right] + 1$. Then, by Theorem 1, Lemma 2 and Remark 4, we know that P has the $(\bar{\delta},\bar{\varepsilon})$ generalized shadowing property with

$$\bar{\delta} = \frac{3\varepsilon}{NN_o} \text{ and } \bar{\varepsilon} = 3\varepsilon(\frac{1}{c})^{N-1} + 3\varepsilon\frac{(\frac{1}{c})^N - 1}{NN_o(\frac{1}{c} - 1)}.$$

Since $N < N_o$, and $\frac{1}{c}$ can be chosen arbitrarily close to 1, we have,

$$\bar{\delta} \approx \text{constant}\frac{\varepsilon}{(\log_q(\frac{\varepsilon}{M}))^2} \approx \text{constant } \varepsilon^{(1+n_1)}$$

$$\bar{\varepsilon} \approx \text{constant } \varepsilon.\frac{(\frac{\varepsilon}{M})^{\log_q(\frac{1}{c})}}{(\log_q(\frac{\varepsilon}{M}))^2} \approx \text{constant } \varepsilon^{(1-n_2)}$$

where n_1, n_2 are arbitrarily small positive real numbers. Thus, for $n = \frac{n_1 + n_2}{1 - n_2}$, we obtain, $\bar{\delta} \approx \text{constant }(n)\, \bar{\varepsilon}^{(1+n)}$, $n > 0$. Therefore, P has the $(\bar{\delta}, \bar{\varepsilon})$-generalized shadowing property (in ρ^*). This completes the proof.

Remark 7 Shadowing property of constrictive Markov operators describing the evolution of densities in some stochastically perturbed dynamical systems will be discussed in Appendix E.

6.5 APPLICATION TO FROBENIUS-PERRON OPERATOR

Let $(X,\mathbf{B},\mu)$ be a measure space and $\tau: X \to X$ be a non-singular transformation. Let P_τ be the Frobenius-Perron operator associated with τ. Then P_τ is a Markov operator and the metric associated with P_τ is ρ^*. The following result expresses the relation between the exactness of τ and P_τ.

<u>Lemma 4</u> [114, Sec. 4.4]. Let $(X,\mathbf{B},\mu)$ be a probability space and $\tau: X \to X$ a nonsingular transformation. Assume that there exists a unique $f^* \in D$ such that $P_\tau f^* = f^*$, where P_τ is the Frobenius-Perron operator corresponding to τ. Then τ is μ-exact, where μ is the measure whose density is f^*, if and only if for every $f \in D$, $\lim_{n\to\infty} \|P^n f - f^*\|_1 = 0.$

<u>Remark 8</u>. From Lemma 4 it follows that P_τ is a constrictive Markov operator if τ is μ-exact. The assumption in Lemma 4 for a unique $f^* \in D$ such that $P_\tau f^* = f^*$, conforms with the fact that a constrictive Markov operator has a stationary density [114, Sec. 5.4].

Then from Theorem 1 and Remark 8, we have

<u>Theorem 3</u> Let $(X,\mathbf{B},\mu)$ be probability space, and let $\tau: X \to X$ be μ-exact, where μ is an ACIM with density f^*. Let D be a ρ^*-compact subset of D_1, and assume that $P_\tau D \subset D$. Assume that $\|P_\tau^n f - f^*\|_1 \to 0$ as $n \to \infty$ uniformly with respect to $f \in D$. Then, $P_\tau: (D,\sigma) \to (D,\rho^*)$ has the generalized shadowing property.

6.6 EXAMPLES: PIECEWISE EXPANDING AND RANDOM MAPS

We now consider the application to constrictive Markov operators induced by certain piecewise expanding and random maps of the unit interval X.

Piecewise Expanding Maps Let $\tau \in C$, where C is the class of all Wong maps considered in Sec. 1.9. It is shown in [46] that the Frobenius-Perron operator $P_\tau \in S$, the class of all linear stochastic operators considered in Chapter 5. As in Theorem 1 of [57], there exists a constant K independent of f, such that

$\lim_{n\to\infty}\sup \bigvee_0^1 P_\tau^n f \le K$ for every $f \in D$ of bounded variation. Let $D = \{f \in D_1 : Vf \le K'\}$, where $K' \ge K$. Then D is ρ^*- compact in L_1. Let τ admit a unique ACIM μ with respect to which it is exact. Let f^* be the density of μ. Then, by Property 2 of Chapter 5 it follows that,

$$\|P_\tau^n f - f^*\|_V \le Mq^n$$

i.e. the convergence to f^* is uniform with respect to all $f \in D$, where $M > 0$ and $0 < q < 1$ are both independent of f. Hence, by definition of $\| \, \|_V$, we have,

$$\|P_\tau^n f - f^*\|_1 \le Mq^n \to 0, \text{ as } n \to \infty \text{ uniformly with respect to } f \in D.$$

Therefore, by Theorem 3, P_τ: $(D,\sigma) \to (D, \rho^*)$ has the generalized shadowing property.

Random Maps Let $\tau \in R$, where R is the class of all Pelikan random maps of Sec. 1.9. It is shown in [161] that the Markov operator P_τ defined by $P_\tau = \sum_{i=0}^{n} p_i P_{\tau_i}$ satisfies,

$$\bigvee_0^1 P_\tau f \le \alpha \bigvee_0^1 f + K\|f\|_1,$$

for all $f \in BV$ and some $0 < \alpha < 1$, $K > 0$, both independent of f. Hence,

$$\|P_\tau f\|_v = \bigvee_0^1 P_\tau f + \|P_\tau f\|_1 \le \alpha V f + K\|f\|_1 + \|f\|_1 \le \alpha\|f\|_v + K'\|f\|_1$$

for some $\alpha < 1$ and $K' > 0$ i.e. $P_\tau \in S(\alpha, K')$, a subclass of S.

In [161], sufficient conditions are given for τ to have a unique ACIM μ. Let f^* be the density of μ. Then again by Property 2,

$$\|P_\tau^n - f^*\|_v \leq Mq^n$$

i.e. the convergence to f^* is uniform with respect to all $f \in D$, where $M > 0$ and $0 < q < 1$ are both independent of f. Hence by Theorem 1, $P_\tau: (D,\sigma) \to (D,\rho)$ has the generalized shadowing property.

6.7 CONTINUITY OF SHADOWING PROPERTY

In this section, we prove that for many families of maps the generalized shadowing property continues to hold as the parameter is varied over its range. The desired result is

Theorem 4 [12]. Let $\{\tau_\lambda\}_{\lambda \in \Lambda} \subset C$ be a family of maps, where Λ is a parameter space, which admits a unique ACIM μ_λ on the unit interval X. Assume that the map

$$\lambda \to \tau_\lambda \text{ from } (\Lambda, |\ |) \to (C,d) \text{ is continuous,} \qquad (1)$$

where d is the Skorokhod-metric of Sec. 5.3. If $P_\lambda = P_{\tau_\lambda}$, the Frobenius-Perron operator corresponding to τ_λ, then for each $\lambda_0 \in \Lambda\ \exists$ a neighbourhood N of $\lambda_0\ \ni$ for each $\varepsilon > 0\ \exists$ a $\delta > 0$ and every δ-pseudo orbit (in σ) can be shadowed by a true orbit (in ρ^*) uniformly for all $\lambda \in \Lambda$, i.e. if $\{f_0, f_1, \cdots\}$ satisfies $\sigma(P_{\lambda_0} f_n, f_{n+1}) < \delta$ for any $\lambda_0 \in N$, then $\rho^*(P_\lambda^n f_0, f_n) < \varepsilon$ for all $\lambda \in N$.

Proof Since each τ_λ has a unique ACIM, from Sec. 4.4 of [114], each τ_λ is μ_λ-exact. Therefore, by Remark 8, P_λ is a constrictive Markov operator. From Sec. 6.5.1 it follows that the convergence is uniform for f in the weakly compact set $D = \{f \in D_1: V(f) \leq L\}$, where L is a sufficiently large positive real number i.e., $\lim_{n\to\infty} \|P_\lambda^n f - f_\lambda\|_1 = 0$ uniformly for $f \in D$. We now construct the desired neighbourhood N. Then, by Theorem 1, we will have the generalized shadowing property for P_λ uniformly for $\lambda \in N$.

Following the representation of P_λ in Theorem 1 of [57], there exist constants $\eta > 1$ and $T > 0$ such that,

$$\|P_\lambda^k f\|_V \leq \frac{1}{\eta}\|f\|_V + T\|f\|_1, \; f \in L_1.$$

Also $P_\lambda(BV) \subset BV$, where $BV = \{f \in L_1 : V(f) < \infty\}$, so that $\{P_\lambda\}_{\lambda \in 3} \subset S(\eta, T)$. Then by Theorem 2, Sec. 5.3, Chapter 5 and condition (1), it implies that the map $\lambda \to f_\lambda$ from $(\Lambda, |\,|) \to (D, \sigma)$ is continuous, where $|\;|$ is the absolute value norm. Fix $\lambda_o \in \Lambda$. Then given $\varepsilon > 0$ $\exists$ a neighbourhood $N \subset \Lambda$ of λ_o such that $\lambda \in N$ implies $\|f_\lambda - f_{\lambda_o}\|_1 < \varepsilon$. This completes the proof.

Remark 9. Any family of maps satisfying the statement of Theorem 4 is said to have the stability of shadowing property.

6.7.1 EXAMPLE

Consider the family of tent maps $\tau_\lambda : X \to X$, defined by,

$$\tau_\lambda(x) = \begin{cases} \lambda x, & 0 \leq x \leq \frac{1}{2} \\ \lambda(1 - x), & \frac{1}{2} \leq x \leq 1 \end{cases}$$

where $\lambda \in \Lambda \equiv [1 + p, 2], p > 0$. Then, $|\tau'_\lambda(x)| \geq 1 + p > 1$.

All the maps τ_λ have the same partition: $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. Since each τ_λ has only one turning point in its partition, it has a unique ACIM. Furthermore, $V\left(\frac{1}{\tau'_\lambda(x)|_{I_i}}\right) = 0$, for i=1,2, since τ_λ is piecewise linear. Thus the family $\{\tau_\lambda\} \subset C$. Also, $\lambda \to \tau_\lambda$ is continuous. Hence, the hypothesis of Theorem 4 is satisfied. By Remark 9, the generalized shadowing property is stable for this family of tent maps.

Remark 10. Although the map $\tau: X \to X$ may not have the shadowing property, the

Frobenius-Perron operator corresponding to τ, $P_\tau : L_1 \to L_1$ may have the generalized shadowing property. For example, consider the tent map $\tau : [0,1] \to [0,1]$, defined by,

$$\tau(x) = \begin{cases} \sqrt{2}\,x, & 0 \le x \le \frac{1}{2} \\ \sqrt{2}\,(1 - x), & \frac{1}{2} \le x \le 1. \end{cases}$$

It is shown in Example 3, Sec. 3.4, Chapter 3 that τ does not have the shadowing property. However, by the previous example, we see that P_τ has the generalized shadowing property with respect to the metrics σ and ρ^*.

CHAPTER 7

APPLICATIONS

I APPLICATION TO CELLULAR AUTOMATA

7.1 INTRODUCTION

Pattern formation is a widespread phenomenon of great importance in physical, and especially in biological sciences [5,139,140]. A major question from the modelling point of view is how to describe local interaction of cells. One way to model local interaction is by using the theory of Markov random fields [98]. To use this model, we have to know local conditional probabilities i.e. how a cell's state of health is influenced by its neighbours. These probabilities are, in general, difficult to compute and even if they are known, the method of Markov random fields is computationally very complex.

In this Chapter, we describe a different method of modelling interacting cellular systems. We discuss a model considered in [39] for interacting cellular systems which employs one-dimensional point transformations on configuration (pattern) space of cellular automata. These transformations reflect the interaction of neighbours and has a probabilistic interpretation. From the results of Chapter 4, for certain point transformations there is a theoretical justification for using computer orbits on the configuration space to compute the asymptotic (invariant) measures for the transformation. This will enable us to study pattern formation of cellular automata. The point transformation model reduces to the local deterministic rules used in cellular automata [118]. Various perturbations on the rules can also be considered in a natural way in the model. We shall refer the model as the point model since it is defined by a point transformation on the configuration space.

7.2 A POINT TRANSFORMATION MODEL FOR ONE-CELL SYSTEMS

We start with a single cell and present a heuristic description of how to construct a point transformation on configuration space, which reflect the dynamics of value changes for the cells.

The value of a cell, a discrete site, is described by a real variable x, taking values in the interval [0,1]. For simplicity and physical interpretation, we shall think of a cell as a biological cell, whose value represents its state of health on a scale 0-1. When $x = 0$, we think of the cell as being completely ill, while $x = 1$ means the cell is perfectly healthy. Any intermediate value of x would mean that the cell is infected. How a cell changes its value or health in a unit of time is represented by a transformation $\tau: [0,1] \to [0,1]$. If the cell has value x initially, then its value after n units of time is $\tau^n(x)$.

7.2.1 DEFINITION OF THE POINT-TRANSFORMATION

In order to define τ, we choose a partition of the value (health) set [0,1]. For example, consider the partition $[0,\frac{1}{3}]$, $[\frac{1}{3},\frac{2}{3}]$, $[\frac{2}{3},1]$. Here $[0,\frac{1}{3}]$ denotes poor health, $[\frac{1}{3},\frac{2}{3}]$ denotes dubious health and $[\frac{2}{3},1]$ denotes good health. A cell can assume any value in the continuum of each partition element.

Denote the interval of partition $[0,\frac{1}{3}]$ by 0, $[\frac{1}{3},\frac{2}{3}]$ by 1 and $[\frac{2}{3},1]$ by 2. Now, consider the transformation τ shown in Fig. 9. τ is continuous on each of the three value sets. The parameters $P_{01}, P_{10}, P_{12}, P_{21}$ are the transition probabilities, which represent the probabilities of going from one value set to another. For example, the probability of going from 0 to 1 i.e. from $[0,\frac{1}{3}]$ to $[\frac{1}{3},\frac{2}{3}]$ is,

$$P_{01} = \frac{\left| [0,\frac{1}{3}] \cap \tau^{-1}[\frac{1}{3},\frac{2}{3}] \right|}{\left| [0,\frac{1}{3}] \right|}$$

where $|\ |$ is the absolute value norm. P_{01} is the fraction of $[0,\frac{1}{3}]$ which is transformed into $[\frac{1}{3},\frac{2}{3}]$. Thus τ reflects the probability of a cell changing its value state 0 to state 1. The basic characteristics of the model can be summarized as:

(1) The cells are situated on a discrete set of sites. They do not have to be identical or arranged in a regular array.

(2) The value of each cell is described by a number in the continuum $I = [0,1]$.

(3) The state of each cell in the array is updated in discrete time steps.

(4) The value of each cell at time n depends on its own value at time $n-1$ and the states of health of the cells in a local neighbourhood of the cell at time $n - 1$.

Remark 1 Our model can be considered similar to a continuous analogue of the Hodgepodge machine, a cellular automata described in [31].

7.3 ASYMPTOTIC BEHAVIOUR OF THE POINT MODEL

We are interested in the limiting behaviour of τ on the configuration space. Let τ admit an ACIM μ. The existence of μ reflects the fact that chaotic behaviour exists on the configuration space. If f is the density function of μ with respect to Lebesgue measure, the support of f indicates the portion of the configuration space on which the chaos resides. If for example, τ is a Lasota-Yorke map it is known [103] that support of f must contain a discontinuity point of τ i.e. $\frac{1}{3}$ or $\frac{2}{3}$ in our case. Therefore if support of f is very small, then we know it must be centred at one of the discontinuity points and the dynamics of τ is virtually predictable.

Otherwise, if support of f is large, the chaotic region is also large and it is difficult to have good predictions.

It was shown in Chapter 4 that if a piecewise monotonic transformation τ has a unique ACIM, then the histograms of sufficiently long computer orbits of τ approximate the histogram of the ACIM. Following Remark 1) of Sec. 4.9 the result also holds for transformations with more than one ACIM. These measures are absolutely continuous with respect to Lebesgue measure and are the very ones the computer 'likes' to display. In their ability to reflect local interaction, these measures on the configuration space give meaningful prediction for the long-term behaviour of interacting cellular systems. Thus, by observing a simulated cellular system on a computer, one can predict the true long-term behaviour of the orbits i.e. the computer orbit which is only an approximation to the theoretical dynamics of the cellular system, nonetheless exhibits the true long-term behaviour of the system.

In Appendix F, we present a variety of histograms of orbits of τ for various choices of transition probabilities and transformations, representing the wide range of possible asymptotic behaviour possible for one-cell systems.

7.4 APPLICATION OF THE POINT MODEL TO CELLULAR AUTOMATA

Cellular automata were invented by von Neumann and Ulam while investigating the possibility of constructing self-replicating machines [75]. Since then they have found applications in various nonlinear areas of physical, chemical and biological systems [117]. They are viewed as discrete dynamical systems capable of self-organization [118] and universality [119]. Basic properties and defining characteristics of cellular automata are discussed in [117].

The point model has many common features with cellular automata but there are important differences: in the point model, the value of each cell ranges in a continuum, homogeneity is not required, and the rules for updating the values on the array can vary with location. This facilitates the considerations of boundary conditions, for example. The point transformation, describes the deterministic rule for updating the values of the cells, but can also be interpreted in a probabilistic way.

We can use a point transformation to describe the rules for cellular automata.

7.4.1 MODELLING ONE-CELL CELLULAR AUTOMATA

Define one-dimensional transformations τ as follows:

$$\tau_0(x) = \begin{cases} x, & \text{if } 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

and

$$\tau_1(x) = \begin{cases} x + \frac{1}{2}, & \text{if } 0 \le x \le \frac{1}{2} \\ x, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

as shown in Fig. 10. This corresponds to $P_{01} = 0 = 1 - P_{10}$ and $P_{01} = 1 = 1 - P_{10}$, respectively.

To model one-cell cellular automata on a linear array with rule r, we define the transformation $\tau: X \to X$, $X = [0,1]$ as follows:

Let $x \in X$. To this point, we assign a configuration of the cellular automata, ε, where

$$\varepsilon = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}.$$

The rule r gives the updated configuration which replaces ε; let us denote it by ε'. We now define, $\tau(x) = \tau_{\varepsilon'}(x)$. The transformation τ models the rule r precisely and all information which can be obtained from rule r can be obtained from τ as well. Fig. 11 shows rule 126 for two different initial configurations using a point transformation τ on a 400-cell array.

7.4.2 PERTURBATION OF CELLULAR AUTOMATA

In our framework of point transformations, the problem of small perturbations for the cellular automata can be handled naturally. The behaviour of perturbed cellular automata can be studied in our setting in many different ways. We consider the following cases:

Case 1 The slopes of τ are perturbed so that instead of τ we obtain a point transformation τ_δ with slope $1 + \delta, \delta > 0$. Fig. 12 shows an orbit of τ_δ for $\delta = 0.01$ on a 400-cell array.

Case 2 τ is changed in such a way that τ_0 is replaced by τ_δ and τ_1 is replaced by $\tau_{1-\delta}$ as shown in Fig. 13. Here τ_δ and $\tau_{1-\delta}$ reflect the probability of changing value states. τ_δ is defined by $P_{01} = \delta = 1 - P_{10}$ and $\tau_{1-\delta}$ is defined by $P_{01} = 1 - \delta = 1 - P_{10}$. Fig. 14 shows an orbit for $\delta = 0.01$.

Case 3 τ is perturbed by a small random variable W_δ with range $[-\delta,\delta]$. In this case the stochastic process $x_{n+1} = \tau(x_n) + W_\delta$ describes the dynamics of the cellular automata. Fig. 15 shows the histograms of orbits for $\delta = 0.01$ in a 10-cell linear array.

From [57] for Case 1 and 2 and from [61] for Case 3, we conclude that, for any $\delta > 0$, there exists an ACIM μ_δ for the perturbed cellular automata, and that the weak limit of μ_δ, as $\delta \to 0$, μ, is an invariant measure under τ. We want μ to be an ACIM. Since μ could be a point measure, we therefore require a convergence for μ_δ, which is stronger than weak convergence in the space of measures. Let f_δ be the density of μ_δ, which is approximated by the histogram of the perturbed cellular automata. If the family $F = \{f_\delta\}_{\delta > 0}$ is uniformly bounded, it follows from Theorem 9, Chapter IV. 8 of [105], that F is weakly sequentially compact, and hence that F has a limit point f which is the density of a measure μ, invariant under τ. Then, μ is absolutely continuous with respect to Lebesgue measure, and so is a nontrivial invariant measure.

I I APPLICATION TO FRACTALS

7.5 INTRODUCTION

Fractals were introduced by Mandelbrot [199] to give a precise geometric description of natural objects. A theory which successfully describes the construction of fractals is of iterated function systems. Iterated function systems were introduced in [225] and since then they have found important applications in many fields, especially in the area of computer graphics [188-191].

In this chapter, we discuss the relationship between iterated function systems and dynamical systems. By studying the orbits of 'associated' dynamical systems of the corresponding iterated function systems, we can obtain more information about fractals. These dynamical systems often exhibit a chaotic behaviour on the fractal. When implemented on a computer, a dynamical system τ gives rise to computer orbits. We show that τ has the shadowing property and so the computer orbits can be shadowed arbitrarily close by the true orbits. This explains why Random Iteration Algorithm can successfully generate fractals [187, Sec. 4.8]. When τ admits a unique ACIM, long computer orbits of τ exhibit the ACIM. This justifies the use of a computer in predicting the long-term behaviour of dynamics on a fractal.

7.6 ITERATED FUNCTION SYSTEMS AND DYNAMICAL SYSTEMS

In this section, we develop the necessary background for further study. We shall assume X to be the unit interval [0,1] with the Euclidean metric (though most of the results go over for X a compact metric space).

Iterated Function System (IFS) Let $\tau_n : X \to X$, $n=1,\ldots,N$ be a finite set of contraction mappings with contractivity factors λ_n respectively. Then $\{X; \tau_1,\ldots,\tau_N\}$ is called an (hyperbolic) IFS. $\lambda = \max\{\lambda_1,\ldots,\lambda_N\}$ is called the contractivity of the IFS.

Let $H(X)$ denote the space of all (nonempty) compact subsets of X. Then $H(X)$ is a compact and complete metric space with respect to the metric h defined by,

$$h(A,B) = \max\{|A - B|, |B - A|\},$$

for any $A,B \in H(X)$, where $|A - B| = \sup_{a\in A} \inf_{b\in B} \{|a - b|\}$. The transformation $T: H(X) \to H(X)$ defined by,

$$T(B) = \bigcup_{n=1}^{N} \tau_n(B)$$

for any $B \in H(X)$, is a contraction mapping on $(H(X), h)$ with contractivity factor λ. T has a unique fixed point $A \in H(X)$ satisfies,

$$A = T(A) = \bigcup_{n=1}^{N} \tau_n(A),$$

and is given by, $A = \lim_{n\to\infty} T^n(B)$, for any $B \in H(X)$.

Attractor of the IFS. The set A, which is the fixed point of T is called the attractor of the IFS.

There are a number of algorithms available for approximating the attractor A [224-227].

Code Space associated with the IFS. Let Σ denote the code space on N symbols, i.e. $\Sigma = \{0,1,\ldots,N-1\}$, N is a positive integer. Let d_c be a metric on Σ defined by

$$d_c(\rho, \sigma) = \sum_{n=1}^{\infty} \frac{|\rho_n - \sigma_n|}{(N+1)^n},$$

for all $\rho = (\rho_n)$, $\sigma = (\sigma_n) \in \Sigma$.

For $\sigma \in \Sigma$, $n \in \mathbb{N}$ and $x \in X$, let $\phi(\sigma, n, x) = \tau_{\sigma_1} \circ \tau_{\sigma_2} \circ \ldots \circ \tau_{\sigma_n}(x)$.

Then,

$$\phi(\sigma) = \lim_{n \to \infty} \phi(\sigma, n, x)$$

exists, belongs to A, and is independent of $x \in X$. Furthermore, $\phi : \Sigma \to A$ is continuous and onto.

Address of a point of the Attractor. An address of a point $a \in A$ is any element of the set,

$$\phi^{-1}(a) = \{\sigma \in \Sigma : \phi(\sigma) = a\}.$$

Totally Disconnected IFS. The IFS is said to be totally disconnected, if each point of its attractor possesses a unique address.

Just Touching IFS. The IFS is said to be just touching if it is not totally disconnected but its attractor A contains an open set O such that,

(1) $\tau_i(O) \cap \tau_j(O) = \varnothing$, $\forall\ i, j \in \{1,\ldots,N\}$, $i \neq j$.

(2) $\bigcup_{i=1}^{N} \tau_i(O) \subset O$.

Overlapping IFS. The IFS is said to be overlapping if it is neither just touching nor disconnected.

Shift Dynamical System associated with the IFS. Let $\{X; \tau_1,\ldots,\tau_N\}$ be a totally disconnected IFS with attractor A. The associated shift transformation on A. is the transformation $\tau_S: A \to A$ defined by,

$$\tau_S(a) = \tau_n^{-1}(a), \text{ for } a \in \tau_n(A).$$

The dynamical system (A, τ_S) is called the shift dynamical system associated with the IFS.

Remark 2 Let $\{X; \tau_1,\ldots,\tau_N\}$ be a totally disconnected IFS with attractor A. Let (A, τ_S) be the shift dynamical system associated with the IFS. Let Σ be the associated code space of N symbols and let $\tau_\Sigma: \Sigma \to \Sigma$ be defined by,

$$\tau_\Sigma(\sigma_1\sigma_2\sigma_3\ldots) = \sigma_2\sigma_3\sigma_4\ldots \text{ for all } \sigma = \sigma_1\sigma_2\sigma_3\ldots \in \Sigma.$$

Then the dynamical systems (A, τ_S) and (Σ, τ_Σ) are homeomorphic (with respect to ϕ defined above). This gives an advantage of symbolic dynamics. Since topological conjugacy preserves dynamics, we can study the easy-to-deal-with code space Σ to understand the chaotic dynamics on the fractal A.

In the next definition, we consider the IFS of two maps to keep the notations succinct.

Random Shift Dynamical System associated with the IFS. Let $\{X; \tau_1, \tau_2\}$ be an IFS with attractor A. Assume that both $\tau_1: A \to A$ and $\tau_2: A \to A$ are invertible. The associated random shift transformation on A is the transformation $\tau_{RS}: A \to A$ defined by,

$$\tau_{RS}(x_n) = \begin{cases} \tau_1^{-1}(x_n), \text{ if } x_n \in \tau_1(A) \setminus \tau_2(A), \\ \tau_2^{-1}(x_n), \text{ if } x_n \in \tau_2(A) \setminus \tau_1(A), \\ \tau_1^{-1}(x_n) \text{ or } \tau_2^{-1}(x_n), \text{ if } x_n \in \tau_1(A) \cap \tau_2(A) \end{cases}$$

for each $n = 0,1,\ldots$. The dynamical system (A, τ_{RS}) is called the random shift dynamical system associated with the IFS.

7.6.1 EXAMPLES

We now undertake some examples of different types of iterated function systems considered in the previous section.

Example 1. The set $I = \{[0,1]; \tau_1(x) = \frac{1}{3}x, \ \tau_2(x) = \frac{1}{3}x + \frac{2}{3}\}$ is an IFS with contractivity factor $\lambda = \frac{1}{3}$ and the attractor as the classical Cantor set C.I is a totally disconnected IFS and its associated shift dynamical system (C, τ_S) is given by,

$$\tau_S(x) = \begin{cases} 3x, & \text{if } x \in \frac{1}{3}C, \\ 3x - 2, & \text{if } x \in \frac{1}{3}C + \frac{2}{3}. \end{cases}$$

Example 2. The set $I = \{[0,1]; \tau_1(x) = \frac{1}{2}x, \ \tau_2(x) = \frac{1}{2}x + \frac{1}{2}\}$ is an IFS with contractivity factor $\lambda = \frac{1}{2}$ and the attractor $[0,1]$. I is a just touching IFS and its associated random shift dynamical system $([0,1], \tau_{RS})$ is given by,

$$\tau_{RS}(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}], \\ 0 \text{ or } 1, & \text{if } x = \frac{1}{2}, \\ 2x - 1, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Example 3. The set $I = \{[0,1]; \ \tau_1(x) = \frac{1}{2}x, \ \tau_2(x) = \frac{3}{4}x + \frac{1}{4}\}$ is an IFS with contractivity factor $\lambda = \frac{3}{4}$ and the attractor $[0,1]$. I is an overlapping IFS and its associated random shift dynamical system $([0,1], \tau_{RS})$ is given by,

$$\tau_{RS}(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{4}), \\ \frac{4}{3}x - \frac{1}{3}, & \text{if } x \in [\frac{1}{4}, 1]. \end{cases}$$

or

$$\tau_{RS}(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{4}{3}x - \frac{1}{3}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

7.7 CHAOTIC DYNAMICS ON FRACTALS

The orbits of shift dynamical systems associated with an IFS often exhibit a chaotic behaviour. It is known that the shift dynamical system associated with a totally disconnected IFS (of two or more mappings) is chaotic [187, Sec. 4.8, Theorem 1]. The random shift dynamical systems associated with a just touching or overlapping IFSs are usually piecewise linear expanding maps, which admit an ACIM. Hence typical orbits of these dynamical systems are also chaotic and the chaotic behaviour occurs on a large part of the phase space. (Here we note that the attractor of a totally disconnected IFS is often a set of zero Lebesgue measure, and so the shift dynamical system associated with the IFS does not have an ACIM.)

7.8 COMPUTER ORBITS OF SHIFT DYNAMICAL SYSTEMS: SHADOWING PROPERTY ON FRACTALS

Computer simulation of a (chaotic) shift dynamical system will generate a chaotic computer orbit which diverges from the true orbit rapidly. The following theorem says that there always exists a shadowing orbit arbitrarily close to the computer orbit, thereby lending meaning to the computation of shift dynamics.

Theorem 1 [187, Sec. 4.7]. Let $\{X; \tau_1, \ldots, \tau_N\}$ be an IFS of contractivity λ, $0 < \lambda < 1$. Let A be the attractor of the IFS and suppose that each of the transformation $\tau_n: A \to A$ is invertible. Let (A, τ) denote the associated shift or random shift dynamical system according as the IFS is totally disconnected or just touching/overlapping. Then $\tau: A \to A$ has the shadowing property.

Proof. For n=1,2,3,..., choose $\sigma_n \in \{1,...,N\}$ such that $\tau_{\sigma_1}^{-1}, \tau_{\sigma_2}^{-1}, \tau_{\sigma_3}^{-1},...$ is the sequence of inverse maps used to compute $\tau(x_o), \tau^2(x_o), \tau^3(x_o),...$. Let $\phi: \Sigma \to A$ be the code space map associated with the IFS and let $x_o = \phi(\sigma_1\sigma_2\sigma_3...)$. We then compare the true orbit $\{\tau^n(x_o)\}_{n=0}^{\infty} = \{\phi(\sigma_{n+1}\sigma_{n+2}\sigma_{n+3}...)\}_{n=0}^{\infty}$ of the point x_o with the δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$.

Let M be a sufficiently large positive integer. Since $\tau^M(x_o)$ and $\tau(x_{M-1})$ both belong to A, we have,

$$|\tau^M(x_o) - \tau(x_{M-1})| \leq \text{diam } A \leq 1. \qquad (1)$$

Since $\tau^M(x_o)$ and $\tau(x_{M-1})$ are both computed with the same inverse map $\tau_{\sigma_M}^{-1}$ and λ is the contractivity of the IFS, we obtain,

$$|\tau^{M-1}(x_o) - x_{M-1}| \leq \lambda \text{ diam } A \leq \lambda. \qquad (2)$$

Hence, from (1) and (2) it follows that,

$$|\tau^{M-1}(x_o) - \tau(x_{M-2})| \leq |\tau^{M-1}(x_o) - x_{M-1}| + |x_{M-1} - \tau(x_{M-2})|$$

$$\leq \delta + \lambda;$$

and so repeating the above argument, we have,

$$|\tau^{M-2}(x_o) - x_{M-2}| \leq \lambda(\delta + \lambda).$$

By induction, we then obtain,

$$|\tau^{M-2}(x_o) - x_{M-K}| \leq \lambda\delta + \lambda^2\delta + ... + \lambda^{K-1}\delta + \lambda^K.$$

Hence, for any integer n, $0 < n < M$,

$$|\tau^n(x_o) - x_n| \leq \lambda\delta + \lambda^2\delta + ... + \lambda^{M-n-1}\delta + \lambda^{M-n}.$$

Now, since M was arbitrary, we finally have,

$$|\tau^n(x_o) - x_n| \leq \lambda\delta(1 + \lambda + \lambda^2 + ...) = \frac{\lambda\delta}{1-\lambda},$$

for $n=1,2,3,\ldots$. Therefore $\tau: A \to A$ has the shadowing property. This completes the proof.

7.9 ASYMPTOTIC BEHAVIOUR OF COMPUTER ORBITS OF SHIFT DYNAMICAL SYSTEMS ON FRACTALS

We are interested in the long-term behaviour of computer orbits of shift dynamics on fractals. Let $\{X; \tau_1,\ldots,\tau_N\}$ be an IFS with attractor A and associated shift dynamical system (A, τ). If τ admits a unique ACIM μ, the results of Chapter 4 are applicable and sufficiently long computer orbits of τ exhibit the measure μ. We state this as,

Proposition 1. Let $\{X; \tau_1,\ldots,\tau_N\}$ be an IFS with attractor A. Let (A, τ) be a shift dynamical system associated with the IFS, which admits a unique ACIM μ. Let f be the density of μ. Then the histograms of sufficiently long computer orbits of τ approach the histogram of f in the sense of Theorem 1, Sec. 4.6, Chapter 4.

CONCLUSION

We studied piecewise monotonic maps τ on the interval [0,1] as a prototype for one-dimensional chaotic systems. When τ admits an (ergodic) invariant measure, the Birkhoff Ergodic Theorem describes the long-term behaviour of the system. Among all the invariant measures that τ admits, the ACIMs are the ones which appear in experimental and computational work, describing the chaotic behaviour of orbits of τ on a large part of the phase space. When τ admits a unique ACIM μ, is continuous and eventually onto, the (true) periodic orbits of τ exhibit μ (the POM property).

Numerical and computer experiments play a significant role in studying chaotic systems. In such experiments, the system under study is subjected to (unavoidable) external perturbations, giving rise to computer orbits. While analysing the reliability of computer orbits, the shadowing property plays an important role. For nonhyperbolic systems, results of existence of shadowing property have been obtained only for a few classes of maps, such as for tent maps and quadratic maps. Despite its usefulness, the shadowing property has limitations towards computation: it may not hold for all parameters in a family of maps, and thus lacks stability with respect to external perturbations. In higher dimensions, the situation is different and generalized shadowing property for constrictive Markov operators (Frobenius-Perron operators are an important example of such operators) is valid for all parameters in a family of maps.

For a large class of transformations τ which admit an ACIM, the computer orbits of τ exhibit the ACIM. This justifies the use of computer orbits in predicting the long-term behaviour of a system which admits an ACIM. Furthermore, ACIMs are stable with respect to a large class of both deterministic and stochastic perturbations.

Point transformations can be employed to model the rules of cellular automata and associate dynamical systems to iterated function systems. Computer orbits of these point transformations can then be used to study the pattern formation of cellular automata and chaotic dynamics on fractals generated by iterated function systems.

APPENDIX A

MEASURE ON PERIODIC ORBITS OF MAPS WITH SPECIFICATION PROPERTY

In this Appendix, we briefly discuss the case of the invariant measures on periodic orbits of transformations which have the specification property.

Specification Property [115]. The dynamical system consisting of a compact metric space (X,d) and the continuous transformation τ from X onto itself has the specification property if the following holds:

For a given $\varepsilon > 0$ there exists an integer $M(\varepsilon)$ such that for any $k \geq 2$, for any k points $x_1,\ldots, x_k \in X$, for any string of integers, $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_k \leq b_k$, $a_i - b_{i-1} \geq M(\varepsilon)$ for $2 \leq i \leq k$ and for any integer p with $p \geq M(\varepsilon) + b_k - a_1$ there exists a point $x \in X$ with $\tau^p x = x$ such that $d(\tau^n x, \tau^n x_i) \leq \varepsilon$ for $a_i \leq n \leq b_i$, $1 \leq i \leq k$.

Example 1 The shift map on any compact metric state space has the specification property [91]. In particular, the Bernoulli shift $\tau: [0,1] \to [0,1]$ defined by $\tau(x) = 2x \bmod 1$ has the specification property since it can be displayed as factors of the shift on a finite alphabet.

Remark 1 The definition does not depend on the choice of the metric d. For example, when $k=2$, the definition means that whenever there are two 'pieces of orbits' $\{\tau^n x_1 : a_1 \leq n \leq b_1\}$ and $\{\tau^n x_2 : a_2 \leq n \leq b_2\}$, they may be approximated up to ε by one periodic orbit - the orbit of x - provided that the time for switching from the first piece of orbit to the second (i.e. $a_2 - b_1$) and the time for switching back (i.e. $p - (b_2 - a_1)$) are larger than $M(\varepsilon)$, the number $M(\varepsilon)$ being independent of the pieces of orbit. For any arbitrary k the specification property requires that such an approximation is possible for any number k of pieces of orbits, $M(\varepsilon)$ being independent of k.

Let $M_\tau(X)$ denote the space of all measures under $\tau: X \to X$. Let $x \in X$ have the minimal period p. We associate with x the unique measure μ_x which has mass $\frac{1}{p}$ at each of the points $x, \tau(x), \ldots, \tau^{p-1}(x)$. Then μ_x is τ-ergodic. Let $M_\tau(p)$ denote the set of these measures. Then the specification property guarantees that any invariant measure, in particular the ACIM, can be approached by the measures supported on periodic orbits and we have,

Theorem 1 [115]. If (X,τ) satisfies the specification property then, for $n \in \mathbb{N}$, $\bigcup_{p \geq n} M_\tau(p)$ is weakly dense in $M_\tau(X)$, i.e. τ has the POM property.

Remark 2 The proof of Theorem 1 does not require τ to be continuous or onto.

Remark 3 Motivated by Theorem 1 and using a weaker form of the specification property, Hofbauer first proved that the set of measures on periodic orbits of unimodal maps and certain monotonic mod 1 maps is weakly dense in the space of measures invariant under τ [44]. Then he proved that the set of measures on periodic orbits of continuous piecewise monotonic maps is weakly dense in the space of ergodic measures invariant under τ [43]. There is a large class of transformations which are continuous but not necessarily (finitely) piecewise monotonic. For example, the transformation shown in Fig. 4 is of this type. Therefore, to extend the result to transformations which are continuous but not necessarily finitely piecewise monotonic, we have to use different techniques, as seen in Chapter 2.

APPENDIX B

COMPUTATIONAL RESULTS ON SHADOWING PROPERTY

In this Appendix, we discuss a computer-aided method for determining how long the shadowing property applies to orbits of one-dimensional nonhyperbolic systems. The technique is described for the quadratic map $\tau(x) = ax(1 - x)$, which is a prototype for one-dimensional unimodal maps with negative Schwarzian derivative. We also discuss the necessity for restricting the shadowing property to computable numbers.

B.1 SHADOWING FOR A QUADRATIC MAP

The purpose is to be able to generate a computer orbit and then to calculate rigorously how long a true orbit exists near the computer orbit. We can then directly compute how close the true orbit is to the computer orbit. We examine τ for values of a and initial conditions x_o for which the dynamics appears to be chaotic.

Let N be the number of iterates of the map τ until which shadowing occurs for any computer orbit of τ. A true orbit of τ will be given by $\{x_n\}_{n=0}^{N}$ where

$$x_{n+1} = \tau(x_n).$$

While using a computer to iterate τ, numerical roundoff/truncation errors are encountered. Therefore, numerically we are actually computing,

$$y_{n+1} = \tau(y_n) + E_n,$$

where E_n is some small roundoff/truncation error which depends on both the computer and on the algorithm employed. This generates a (noisy or pseudo) computer orbit $\{y_n\}_{n=0}^{N}$. Furthermore, there is some maximum noise amplitude $\delta > 0$ such that,

$$|E_n| = |y_{n+1} - \tau(y_n)| < \delta, \quad \forall\ n = 0,1,\ldots, N.$$

$\{y_n\}_{n=0}^{N}$ is a δ-pseudo orbit for τ if,

$$|y_{n+1} - \tau(y_n)| < \delta, \quad \forall\ n = 0,1,\ldots, N$$

and the true orbit $\{x_n\}_{n=0}^{N}$, ε-shadows $\{y_n\}_{n=0}^{N}$ if

$$|x_n - y_n| < \varepsilon, \quad \forall\ n = 0,1,\ldots, N.$$

B.1.1 CONSTRUCTION OF THE SHADOWING ORBIT

The method we use can be thought of as a form of interval arithmetic. A true orbit $\{x_n\}_{n=0}^{N}$ is selected by finding a sequence of intervals $\{I_n\}_{n=0}^{N}$ such that $x_n \in I_n$, for each $n = 0,1,\ldots,N$. We use the set of intervals $\{I_n\}_{n=0}^{N}$ to bound the location of each x_n without actually knowing the location of x_n within I_n.

The intervals $\{I_n\}_{n=0}^{N}$ are defined by starting with the endpoint interval I_N. Choose $x_N = y_N$ and set I_N to be the one-point interval $[y_N, y_N]$. Given some interval I_n, select I_{n-1} so that

$$\tau(I_{n-1}) \supset I_n \qquad (1)$$

This is known as the <u>nesting condition</u>. The computer must verify this condition for each n=1,2,...,N-1. Given an interval I_n, we show how to construct I_{n-1}.

Let $I_n = [i_n^-, i_n^+]$. The computer verification proceeds first by constructing an interval $\hat{I}_{n-1}$ as a first approximation to I_{n-1}. This is done by first taking the inverse of the endpoints of a broader form of I_n:

$\hat{I}_{n-1} = \hat{\tau}^{-1}(\hat{I}_n)$, where $\hat{I}_n = [i_n^- - c_1, i_n^+ + c_1]$, $0 < c_1 << 1$ is a number which depends on the computer, $\hat{\tau}$ and $\hat{\tau}^{-1}$ are the computer models for τ and τ^{-1} respectively. Define $\tau^{-1}(.)$ by requiring that for each n, the interval I_n must lie on the same side of the critical point c as does y_n:

$$\mathrm{sgn}(i_n^+ - c) = \mathrm{sgn}(i_n^- - c) = \mathrm{sgn}(y_n - c).$$

Next, $\hat{I}_{n-1}$ is further thickened iteratively until the new larger interval $\hat{I}_n$ satisfies, $\hat{\tau}(\hat{I}_{n-1}) \supset \hat{I}_n$. An upper bound for the difference between $\hat{\tau}(\hat{I}_{n-1})$ and $\tau(\hat{I}_{n-1})$ (when $0 < x < 1$ and $0 < a < 4$) can be obtained:

$$\tau(x) - c_2\varepsilon_\mu < \hat{\tau}(x) < \tau(x) + c_2\varepsilon_\mu ,$$

where c_2 is a constant depending on the computer and ε_μ is the computer double-precision machine epsilon. Let $\hat{I}_{n-1} = [\, \hat{i}^-_{n-1} , \hat{i}^+_{n-1} \,]$, and then define,

$$I_{n-1} = [\hat{i}^-_{n-1} - c_3 , \hat{i}^+_{n-1} + c_3] ,$$

c_3 is a constant depending on the computer. If I_{n-1} is chosen sufficiently large, condition (1) can be verified numerically since there is a bound on error size when τ is evaluated at points in [0,1]. On the other hand, if I_{n-1} is chosen too large it may be impossible to define I_{n-2} satisfying (1). Thus the objective at each step is to choose I_{n-1} just large enough that (1) can be guaranteed to hold. While $\{y_n\}_{n=0}^{N}$ is computed using computer single-precision, the intervals $\{I_n\}_{n=0}^{N}$ are computed using computer double-precision arithmetic.

If the intervals $\{I_n\}_{n=0}^{N}$ are successfully determined, then given $x_n \in I_n$, condition (1) implies that $x_n \in \tau(I_{n-1})$. Thus there exists $x_{n-1} \in I_{n-1}$ such that $x_n = \tau(x_{n-1})$, and $\{x_n\}_{n=0}^{N}$ is a true orbit with $x_n \in I_n$, for each $n = 0,1,\ldots,N$.

B.1.2 REMARKS

Remark 1 The reason that this procedure works for chaotic orbits is that the map τ is expanding on average. For small intervals I_n near y_n, we may typically expect that $\tau(I_n)$ is larger than I_n. Thus the key idea of the method is to start with the endpoint y_N rather than y_o, and proceed backwards, since the map is contracting for backward iterates.

Remark 2 The method was originally carried out in [134] with a 14-digit accuracy CRAY-XMP computer with the following data:

$$E_n = 10^{-14},\ a = 3.8,\ c_1 = 2^{-90},\ c_2 = 2^5,\ c_3 = 10^{-25},\ \varepsilon_\mu = 2^{-95} \text{ and } y_o = 0.4.$$

The results obtained were:

$$N = 10^7 \text{ and } \varepsilon = 10^{-8}.$$

i.e. for $N = 10^7$ iterates, the pseudo-orbit $\{y_n\}_{n=0}^{N}$ with $a = 3.8$ and $y_o = 0.4$ is ε-shadowed by a true orbit $\{x_n\}_{n=0}^{N}$ within $\varepsilon = 10^{-8}$.

Remark 3 The technique of the method has been extended to higher dimensions [134].

Remark 4 A technique similar to the one described in this section, has been employed in [248] to prove the shadowing property for countably piecewise expanding maps such as the Gauss transformation, using the theory of continued fractions.

B.2 SHADOWING WITH COMPUTABLE NUMBERS

A computable number [136] is a number whose decimal expansion $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N$ can be generated by an algorithm, to arbitrary high but finite accuracy N by a computer. $\sqrt{2}$, e, π and all numbers whose decimals can be computed, e.g. by continued fractions, are computable.

Computable numbers form only a countable set of the continuum, so that almost all numbers that can be defined to exist can never be computed by any algorithm. All irrationals can be defined as limits of infinite sequences of computable numbers (rationals) but almost all of these limits are noncomputable. Therefore, the properties that are true with measure one (e.g. the shadowing property) are not necessarily reflected in computation; computation takes place on a set of measure zero.

There are countably many algorithms and countably many computable numbers, hence only countably many initial conditions. So almost all pseudo-orbits that can be defined to exist theoretically cannot be computed. Therefore, it is necessary to restrict the shadowing property to pseudo-orbits generated by computable numbers. In fact, the following form of shadowing lemma has been stated by Palmore-McCauley in that direction:

Theorem 1 [78] Let $\tau: X \to X$, $X = [0,1]$ be a piecewise differentiable map for which X an hyperbolic invariant set. Consider a uniform lattice of 2^N points in N-bit precision in X. Let $\{x_n\}_{n=0}^{\infty}$ be a computable pseudo-orbit of these lattice points such that, $|\tau(x_n) - x_{n+1}| < 2^{-N}$, for $n = 0,1,2,\ldots$.

Then,

(1) if $\{x_n\}_{n=0}^{\infty}$ is periodic, then there is a unique computable shadowing orbit $\{y_n\}_{n=0}^{\infty}$ of τ, such that $\{y_n\}_{n=0}^{\infty}$ is periodic and $|y_n - x_n| < 2^{-N}$, for $n = 0,1,2,\ldots$.

(2) if $\{x_n\}_{n=0}^{\infty}$ is non-periodic then there is a unique computable shadowing orbit $\{y_n\}_{n=0}^{\infty}$ which is chaotic [70] and $|y_n - x_n| < 2^{-N}$, for $n = 0,1,2,\ldots$.

Remark 5 Theorem 1 has found applications [271] in the discussion of shadowing property on a (fractal) strange repeller generated by hyperbolic dynamical systems.

APPENDIX C

APPLICATIONS OF SHADOWING PROPERTY

In this Appendix, we consider two applications of the shadowing property.

C.1 MEASURES ON PERIODIC ORBITS OF MAPS WITH SHADOWING PROPERTY

Let τ be a continuous map from a compact metric space X to itself.

It is shown in [8] that a large class C of Axiom A systems (X, τ) have the shadowing property, which in turn implies that they have specification property [115, Chapter 23]. Therefore, from Theorem 1 of Appendix A, we conclude that the class C of the Axiom A systems have POM property.

<u>Topologically Transitive Map</u> τ is said to be (topologically) transitive if there exists a dense orbit of τ in X.

<u>Strongly Transitive Map</u> τ is said to be strongly transitive, if for any open subset $U \subset X$, there is an integer $N \geq 0$ such that $\bigcup_{j=0}^{N} \tau^j (U) = X$.

<u>Lemma 1</u> [68]. Let $\tau : X \to X$, $X = [0,1]$ be a continuous piecewise monotone map. Then transitivity and strong transitivity are equivalent for τ.

It is obvious that if τ is eventually onto, it is strongly transitive. Conversely, we have:

<u>Lemma 2</u> [27]. Let $\tau: X \to X$, $X = [0,1]$ be a (strongly) transitive, continuous and piecewise monotone map with shadowing property. Then τ is eventually onto.

By Lemma 1, 2 and Theorem 3 of Chapter 2, we have

Proposition 1 Let $\tau: X \to X, X = [0,1]$ be a (strongly) transitive, continuous, piecewise monotone map. If τ has the shadowing property, then τ also has the POM property.

It is shown by Parry [79] that every continuous strongly transitive piecewise monotone map of a compact interval is conjugate to a continuous uniformly piecewise linear map of the unit interval onto itself. Furthermore, the shadowing property preserves topological conjugacy (Theorem 2, Chapter 3). We therefore have:

Corollary 1 Let $\tau : X \overset{\text{onto}}{\to} X$, $X = [0,1]$ be a continuous uniformly piecewise linear map. If τ has the shadowing property, it also has the POM property.

C.2 TOPOLOGICAL ENTROPY OF MAPS WITH SHADOWING PROPERTY

We now discuss the relation between shadowing property and topological entropy for a continuous map $\tau: X \to X, X = [0,1]$. Though topological entropy was first defined in [159], the definition due to Bowen in [153] has been found more suitable for maps with shadowing property. We begin with stating the necessary definitions. The entropy of a map τ tells roughly how many different orbits τ has. We can think of $\tau: X \to X$ as representing some real process which one is observing, there will be an error $\varepsilon > 0$ of observation so that one cannot 'see' that states x and y are distinct if $|x - y| < \varepsilon$. Now one sees that the orbits $\{\tau^n(x)\}_{n=0}^{\infty}$ and $\{\tau^n(y)\}_{n=0}^{\infty}$ are different provided that $|\tau^k(x) - \tau^k(y)| > \varepsilon$ for some $k > 0$. From this viewpoint the number of orbits could be finite, countable or uncountable depending on τ. If one now bounds the time of observation by n and observes only the initial segment of the orbit of x, $\{\tau^k(x)\}_{k=0}^{n-1}$ one will see only finitely many orbits (by compactness of X). This motivates the following definition:

(n, ε)-Separated Set. Let an $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. A subset $A \subset X$ is (n, ε)-separated if for each $x, y \in A$, $x \neq y$, there is a k, $0 \leq k < n$, such that $|\tau^k(x) - \tau^k(y)| \geq \varepsilon$.

Let $s(n, \varepsilon)$ denote the maximum cardinality of an (n, ε)-separated set. To measure how many distinct asymptotic behaviours we can observe, we have to look at the asymptotic behaviour of $s(n, \varepsilon)$. Therefore we let,

$$h(\tau, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

Topological Entropy. The topological entropy of the map τ is given by,

$$h(\tau) = \lim_{\varepsilon \to 0} h(\tau, \varepsilon).$$

Remark 1 h is a quantitative measure of topological form of chaotic behaviour of orbits e.g. as observed by Li-Yorke in [198].

Remark 2 For Axiom-A systems, which have shadowing property, the topological entropy was obtained [153] in terms of the growth rate of the number of periodic orbits of period n as $n \to \infty$. Usually these maps may have positive topological entropy. The case of zero topological entropy has been investigated by Shimomura in [155]. Topological entropy of piecewise monotonic maps of an interval is studied in [3,281,282,290].

We can mimic the above definitions for pseudo-orbits as follows: A collection C of δ-pseudo-orbits is (n, ε)-separated if for each $\{x_i\}$, $\{y_i\} \in C$, $x_i \neq y_i$, there exists a k, $0 \leq k < n$ such that $|x_k - y_k| \geq \varepsilon$. Let $s(n,\varepsilon,\delta)$ denote the maximum cardinality of an (n, ε)-separated set of δ-pseudo-orbits. $s(n,\varepsilon,\delta) < \infty$, since $X^n = X \times X \times \ldots$ n-times is compact. Let,

$$h^*(\tau, \varepsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \delta)$$

and

$$h^*(\tau, \varepsilon) = \lim_{\delta \to 0} h^*(\tau, \varepsilon, \delta).$$

Pseudo-Entropy. The pseudo-entropy of τ is given by,

$$h^*(\tau) = \lim_{\varepsilon \to 0} h^*(\tau, \varepsilon).$$

Misiurewicz [156] (*cf.* [157]) considered the calculation of entropy from (not necessarily periodic) pseudo-orbits and obtained the following result:

Theorem 1 The topological entropy $h(\tau)$ is equal to the pseudo-entropy $h^*(\tau)$.

A δ-pseudo-orbit $\{x_i\}$ is periodic if $x_{kn+r} = x_r$ for some $n \geq 1$ and all $k \geq 0$, $0 \leq r < n$. Computer orbits are an important example of periodic pseudo-orbits. Let B denote a set of (n, ε)-separated periodic δ-pseudo-orbits of period n and $|x_k - y_k| \geq \varepsilon$, for some k, $0 \leq k < n$. Let $P(n, \varepsilon, \delta)$ denote the maximum cardinality of such a set. Now set,

$$H^*(\tau, \varepsilon, \delta) = \lim_{n\to\infty} \sup \frac{1}{n} \log P(n, \varepsilon, \delta),$$

$$H^*(\tau, \varepsilon) = \lim_{\delta\to 0} H^*(\tau, \varepsilon, \delta)$$

and $$H^*(\tau) = \lim_{\varepsilon\to 0} H^*(\tau, \varepsilon)$$

then the following is shown by Barge-Swanson in [157]:

Theorem 2. The topological entropy $h(\tau)$ is equal to the growth rate $H^*(\tau)$ of the number of (separated) periodic pseudo-orbits.

Expansive Map The map τ is said to be expansive if there exists an $\varepsilon > 0$ such that, for each $x, y \in X$, $x \neq y$, $|\tau^n(x) - \tau^n(y)| \geq \varepsilon$, for some $n \in \mathbb{N}$.

Then the following lemma is obvious:

Lemma 2. If $\tau: X \to X$ is expansive, then there exists an (n, ε)-separated set $A \subset X$.

From Theorem 2 and Lemma 2 we have the following estimate of topological entropy.

Proposition 2. Let $\tau: X \to X, X = [0,1]$ be a continuous expansive map. If τ has the shadowing property, then

$$h(\tau) = \lim_{n\to\infty} \sup \frac{1}{n} \log F_n(\tau)$$

where $F_n(\tau) = \{x \in X: \tau^n(x) = x\}$.

Remark 3. Proposition 2 has also been stated by Shimomura in [158]. For τ an homeomorphism, Theorem 3 was first proved by Bowen in [153].

Remark 4. Shimomura [179] has also proved that if $\tau: X \to X$, $X = [0,1]$, is a continuous map with dense periodic orbits and shadowing property then $h(\tau) > 0$.

APPENDIX D

A GENERAL RANDOM PERTURBATION MODEL FOR ABSOLUTELY CONTINUOUS INVARIANT MEASURES

In this Appendix, we undertake a random perturbation model which has been described in [53,116]. The stability results have been obtained in the following setting:

Let τ be a continuous map of a metric space X into itself and P(X) denote the space of Borel probability measures on X with the topology of weak convergence. Consider a family $\{Q_x^\varepsilon, x \in X, \varepsilon > 0\} \in P(X)$ such that all maps $Q_\bullet^\varepsilon : X \to \bullet P(X)$, sending x to Q_x^ε are Borel. Also assume that for each bounded continuous function g on X,

$$\lim_{\varepsilon \to 0} \sup_{x \in X} \left| \int_X g(y)\, Q_x^\varepsilon(dy) - g(x) \right| = 0 \qquad \text{(i)}$$

<u>Small Random Perturbations</u>. Let $\{X_n^\varepsilon,\ n=0,1,2...\}$ be a family of Markov chains corresponding to the dynamical system (X,τ) with transition probabilities,

$$P^\varepsilon(x,A) = P\{X_{n+1}^\varepsilon \in A : X_n^\varepsilon = x\} = Q_{\tau(x)}^\varepsilon(A) \qquad \text{(ii)}$$

defined for any $x \in X$ and a Borel set $A \subset X$. Then, X_n^ε are said to be <u>small random perturbations of the map τ</u>.

<u>Invariant Measure for a Markov Chain</u>. A probability measure μ^ε on X is an <u>invariant measure of the Markov chain</u> X_n^ε if for any Borel set $A \subset X$,

$$\int_X P^\varepsilon(x,A)\, d\mu^\varepsilon(x) = \mu^\varepsilon(A) \tag{iii}$$

Example 1. Consider the tent map τ: $[0,1] \to [0,1]$ defined by,

$$\tau(x) = \begin{cases} 2x \ , & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x) \ , & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

The Markov chain, $X_n = \tau(X_{n-1}) + \chi_J(\tau(X_n))\, W_a$, where W_a is an independent random variable supported on a small interval $[-a, a]$ and $J = [a, 1-a]$ was shown to have an ACIM in [19].

Theorem 1 [116]. If the conditions (i) - (iii) are satisfied and $\mu^{\varepsilon_i} \to \mu$ weakly for some subsequence $\varepsilon_i \to 0$, then μ is an invariant measure of the map τ.

Remark 1. Theorem 1 was originally proved by Khasminski in [183].

Remark 2. Our purpose is to specify measures which can be obtained by means of weak limits of invariant measures of Markov chains X_n^ε when $\varepsilon \to 0$. Theorem 1 says that these limiting measures must belong to the space of Borel τ-invariant probability measures on X. If these spaces contain a lot of elements the problem becomes rather complicated. On the other hand, if τ is uniquely ergodic i.e. there exists only one τ-invariant measure $\mu \in P(X)$, then by Theorem 1, invariant measures of the random perturbations must converge weakly to μ, when the parameter of perturbations tends to zero. The measure μ can then be considered to be stable with respect to such perturbations.

Remark 3. Another problem of interest here is to specify conditions on random perturbations, which ensure that weak limits of their invariant measures have their support on attractors, and has been discussed in [146].

In general, neither the Markov chain X_n^ε nor the dynamical system (X,τ) may have any invariant measures at all. Nevertheless, if X is compact there is a sufficient condition for existence of invariant measures.

Theorem 2 [116]. Let X_n, $n=0,1,2,...$ be a Markov chain on a compact space X with transition probabilities,

$$P(x, A) = P\{X_1 \in A : X_0 = x\}.$$

Suppose that the measures $P(x,\bullet) \in P(X)$ depend on x continuously in the topology of weak convergence in $P(X)$. Then the Markov chain X_n has at least one invariant measure in the sense of (iii).

Remark 4. Another assumption called the Doeblin condition [106, Chapter V, Sec. 5] ensures existence and uniqueness of invariant measures for Markov chains. If random perturbation X_n^ε satisfy Doeblin condition then for each ε there is only one invariant measure μ^ε of X_n^ε, and so the limiting behavior of μ^ε as $\varepsilon \to 0$ becomes even more interesting since, usually, there are a lot of τ-invariant measures which enforces the question about the right candidate for a limit of measures μ^ε.

Remark 5 Applications of the above model were considered in [116] for Lasota-Yorke maps and for some quadratic maps of the interval [0,1] which satisfy Misiurewicz conditions. The second application is of special interest. Though there is a stability with respect to random perturbations, in general, there is no stability with respect to deterministic perturbations. This can be explained as follows:

Consider the family of quadratic maps $\tau_\lambda: [0,1] \to [0,1]$ given by $\tau_\lambda(x) = 4\lambda x(1 - x)$ with λ close to 1. Then $\tau_1(x)$ is a Misiurewicz map and hence has a unique ACIM μ. Define $n_\lambda = \min\{n > 1 : \tau_\lambda^n(\frac{1}{2}) \geq \frac{1}{2}\}$. $\tau_1^n(\frac{1}{2}) = 0$, $\forall n > 1$.

Then, if $\tau_\lambda^{n_\lambda}(\frac{1}{2}) > \frac{1}{2}$, by continuity there exists a $\alpha(\lambda)$ such that $1 > \alpha(\lambda) > \lambda$ and $\tau_{\alpha(\lambda)}^{n_\lambda}(\frac{1}{2}) = \frac{1}{2}$. Therefore, $\frac{1}{2}$ is a periodic point of $\tau_{\alpha(\lambda)}$ and the corresponding periodic orbit is an attracting one, since $\tau'_\lambda(\frac{1}{2}) = 0$, for every λ.

Thus we have found a monotonically increasing sequence $\{\lambda_k\}$, $\lambda_k \to 1$, $k \to \infty$ such that any τ_{λ_k} is an attracting periodic orbit containing $\frac{1}{2}$ and only one point of this orbit can be to the right of $\frac{1}{2}$. The complement of the basin of this periodic orbit has zero Lebesgue measure, by Proposition II 5.7 of [145]. Therefore, the invariant measure μ_{λ_k} supported by this periodic orbit is stable with respect to random perturbations.

On the other hand, the measures μ_{λ_k} do not converge as $\lambda_k \to 1$ to the ACIM μ of τ_1, since these periodic orbits have only one point to the right of $\frac{1}{2}$ and so all weak limits of the corresponding invariant measures have support in the interval $[0, \frac{1}{2}]$. Therefore, we do not have stability with respect to deterministic perturbations. This can have the following physical interpretation:

We can assign a perturbation to some point of the coordinate plane-x-coordinate measure deterministic part of the perturbation and the y-coordinate measures the random part of the perturbation. Then the above argument says that when perturbations approach zero along any straight line passing through zero except the x-axis then the invariant measures of perturbations converge weakly to the corresponding ACIM μ. Otherwise, if perturbations approach zero along a curve which is sufficiently close to the x-axis, the convergence may not take place.

APPENDIX E

SHADOWING PROPERTY FOR CONSTRICTIVE MARKOV OPERATORS OF STOCHASTICALLY PERTURBED SYSTEMS

In this Appendix, we discuss the generalized shadowing property of Markov operators which describe the densities of evolution of stochastically perturbed systems.

E.1 DISCRETE TIME SYSTEMS WITH CONSTANTLY APPLIED STOCHASTIC PERTURBATIONS

Let $\tau: X \to X$, $X = [0,1]$ be a map which generates the discrete dynamical system,

$$x_{n+1} = \tau(x_n), \text{ for } n=0,1,2,\ldots .$$

Consider the stochastically perturbed system,

$$x_{n+1} = \tau(x_n) + W_n, \quad n=0,1,2,\ldots , \tag{1}$$

where the small stochastic perturbations W_n are random variables, which satisfies the following conditions:

(a) τ is (Borel) measurable and $\sup \tau(x) = a < 1$

(b) The initial condition $x_0 \in X$ is independent of the perturbations $\{W_n\}$.

(c) The random variables W_n, $n=0,1,2,\ldots$ are independent (and so called white noise), $0 \leq W_n \leq 1 - a$ and all have the distribution with density g i.e.

$$\text{Prob}\,\{W_n \in B\} = \int_B g(x)\,dx\,, \quad \text{for } n=0,1,2,\ldots,\; B \subset X, B \text{ is a Borel set.}$$

Let D_1 be the space of all densities in X and let $f_n \in D_1$ denote the density of distribution of x_n. By (1), x_{n+1} is the sum of two random variables: $\tau(x_n)$ and

W_n. Since in calculating $x_1,\ldots,x_n$ we require only $W_o,\ldots,W_{n-1}$, $\tau(x_n)$ and W_n are independent. Let $h: X \to R$ be an arbitrary, bounded measurable function. The mathematical expectation of $h(x_{n+1})$ is given from Theorem 10.2.1 of [114] by,

$$E[h(x_{n+1})] = \int_X h(x) f_{n+1}(x) dx \qquad (2)$$

The joint density of (x_n, W_n) is $f_n(y)\, g(z)$. Therefore, we also have,

$$E[h(x_{n+1})] = E[h(\tau(x_n) + W_n)]$$

$$= \int_X\int_X h(\tau(y) + z)\, f_n(y)\, g(z)\, dy\, dz$$

By change of variables, we can then write,

$$E[h(x_{n+1})] = \int_X\int_Y h(x)\, f_n(y)\, g(x - \tau(y))\, dx\, dy \qquad (3)$$

where $Y = \{y \in X: x \geq \tau(y)\}$.

Since h was arbitrary, we obtain by (2) and (3) that,

$$f_{n+1}(x) = \int_Y f_n(y)\, g(x - \tau(y))\, dy\,, \quad \text{for } n=0,1,2,\ldots \qquad (4)$$

Thus given an arbitrary density f_o, the evolution of densities given by (1) is described by the sequence of iterates $\{P^n f_o\}$, where,

$$Pf(x) = \int_Y f(y)\, g(x - \tau(y))\, dy \qquad (5)$$

is a Markov operator from L_1 into itself. It is shown in [60] that P is weakly constrictive. Therefore, by Theorem 1 of Chapter 6, we obtain the following result.

Proposition 1 Let the stochastically perturbed system given by (1) satisfy condition (a) - (c). Let the evolution of densities by the system be described by the Markov operator P given by (5) with the attractor A consisting of a single element f^* of a ρ-compact set $D \subset D_1$. Then if $PD \subset D$ and $\lim_{n\to\infty} \|P^n f - A\|_1 = 0$ uniformly for all $f \in D$, $P: (D,\sigma) \to (D,\rho)$ has the generalized shadowing property.

E.2 DISCRETE TIME SYSTEMS WITH RANDOMLY APPLIED STOCHASTIC PERTURBATIONS

We now consider a nonsingular transformation $\tau: X \to X$, $X = [0,1]$ with the stochastically perturbed system,

$$x_{n+1} = \tau(x_n, W_n), \quad \text{for } n=0,1,2,... \tag{6}$$

where $x_n \in X$ and W_n are random variables. The system evolves according to the transformation $\tau(x_n)$ with following interpretation: At the n^{th}-instant of time, the precise location of x_n is not known, though we know its density $f_n(x_n)$. At the next instant of time $(n + 1)$ the transition $x_n \to \tau(x_n)$ occurs with probability $1 - \varepsilon$, $\varepsilon > 0$. Furthermore, the value of x_{n+1} is uncertain with probability ε (see Fig. 16). If $x_n = y$, then x_{n+1} may be considered as a random variable distributed with a density $K(x,y)$.

Let P_τ be the Frobenius-Perron operator associated with τ. Now, given the density f_n of x_n and a Borel set $A \subset X$, we would like to calculate the probability that $x_{n+1} \in A$.

In our random applied perturbation process, x_{n+1} can be reached in any one of the following ways:

(I) deterministically with probability $1 - \varepsilon$ and

(II) stochastically with probability ε.

Thus, in the deterministic case $x_{n+1} = \tau(x_n)$ and

$$\text{Prob}_I(x_n \in A) = \text{Prob}_I(\tau(x_n) \in A) \tag{7}$$

As the density of $\tau(x_n)$ is $P_\tau f_n$, we have,

$$\text{Prob}_I(\tau(x_n) \in A = \int_A P_\tau f_n(x)\, dx \tag{8}$$

If the stochastic perturbation occurs then,

$$\mathrm{Prob}_{II}(x_{n+1} \in A : x_n = y) = \int_A K(x,y)\,dx$$

Since x_n is a random variable with density f_n, we also have,

$$\mathrm{Prob}_{II}(x_{n+1} \in A) = \int_X \mathrm{Prob}_{II}(x_{n+1} \in A : x_n = y)\, f_n(y)\,dy \qquad (9)$$

Thus by (8) and (9), we obtain,

$$\mathrm{Prob}_{II}(x_{n+1} \in A) = \int_A \Big[\int_X K(x,y)\, f_n(y)\,dy\Big]\,dx \qquad (10)$$

Combining equation (7) and (10), we get,

$$\begin{aligned}\mathrm{Prob}\,(x_{n+1} \in A) &= (1-\varepsilon)\,\mathrm{Prob}_I(x_{n+1} \in A) + \varepsilon \mathrm{Prob}_{II}(x_{n+1} \in A)\\ &= \int_A \Big[(1-\varepsilon)\,P_\tau f_n(x) + \varepsilon \int_X K(x,y)\, f_n(y)\,dy\Big]\,dx.\end{aligned}$$

Since A is arbitrary, this implies,

$$f_{n+1} = (1-\varepsilon)\,P_\tau f_n(x) + \varepsilon \int_X K(x,y)\, f_n(y)\,dy.$$

Thus the expression for the operator P_ε describing the evolution of densities by the process is,

$$P_\varepsilon f(x) = (1-\varepsilon) P_\tau f(x) + \varepsilon \int_X K(x,y)\, f(y)\,dy. \qquad (11)$$

Then, P_ε is a Markov operator from L_1 into itself. Assuming that the stochastic kernel $K(x,y)$ is uniformly integrable it is shown in [60] that P_ε is weakly constrictive. Then again applying Theorem 1 of Chapter 6 we have,

Proposition 2 Let the stochastically perturbed system, corresponding to a nonsingular transformation, be given by (6). Let the evolution of densities by the system be described by the operator P_ε in (11), with attractor B consisting of a single element f^* of a ρ-compact set $D \subset D_1$. Then, if $P_\varepsilon D \subset D$ and

$\lim_{n \to \infty} \|P_\varepsilon^n f - B\|_1 = 0$ uniformly for all $f \in D$, $P_\varepsilon: (D,\sigma) \to (D,\rho)$ has the generalized shadowing property.

APPENDIX F
HISTOGRAMS OF CELLULAR AUTOMATA

For one-dimensional configuration space considered in Chapter 7, Sec. 7.3, we display some point transformations and the associated histograms, which the orbits of the transformations produce. In Fig. 17 we have two disjoint histograms: one has its support centred at the point $\frac{1}{3}$ while the other has its support centred around the point $\frac{2}{3}$. The histogram that appears is determined by the starting point.

In Fig. 18, we have a transformation in which there is much overlap in the middle value interval and it is less likely to have two disjoint histograms, since orbits will now be able to range through last parts of configuration. In this case all starting points will tend to a unique histogram.

Fig. 19 shows a transformation which has a unique histogram and whose support is concentrated in a fairly narrow region in one side of the configuration space.

In Chapter 7, we restricted ourselves to the study of transformations which were piecewise (uniformly) expanding on the configuration space. For transformations τ which are not uniformly expanding on the configuration space,we can have the following interpretation: in case of biological cells, if a cell and all its neighbours are all in very good health, we would expect them all to become completely healthy. This implies that the orbit of τ in the configuration space should tend to be the fixed point 1. Thus, the point 1 is a local attractor. Fig. 20 shows the histogram of such a transformation: all the orbits converge to the attractor $\{1\}$.

FIGURE 1

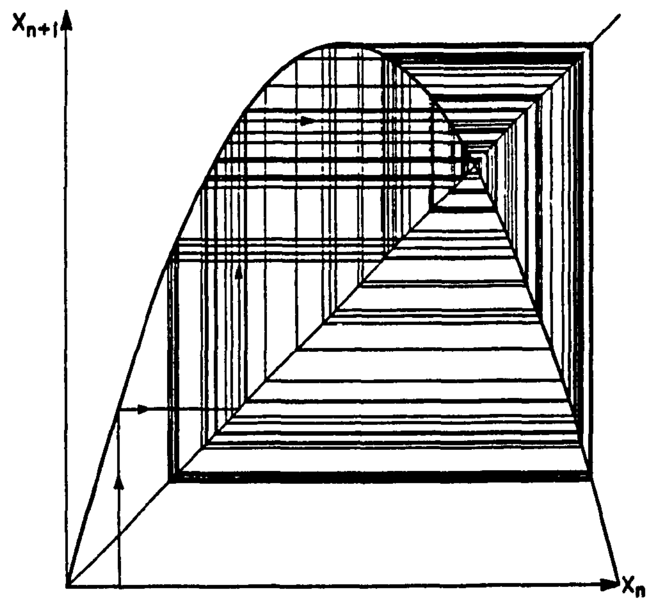


FIGURE 2

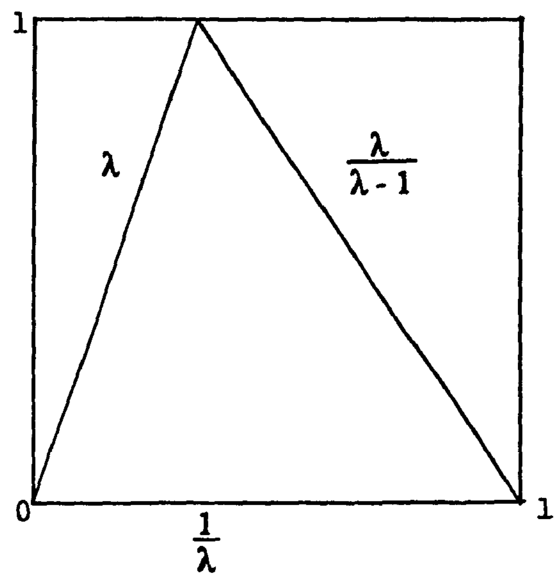


FIGURE 3

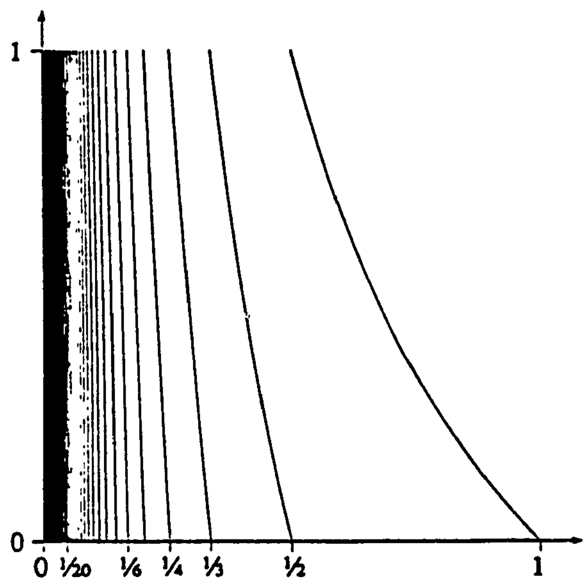


FIGURE 4

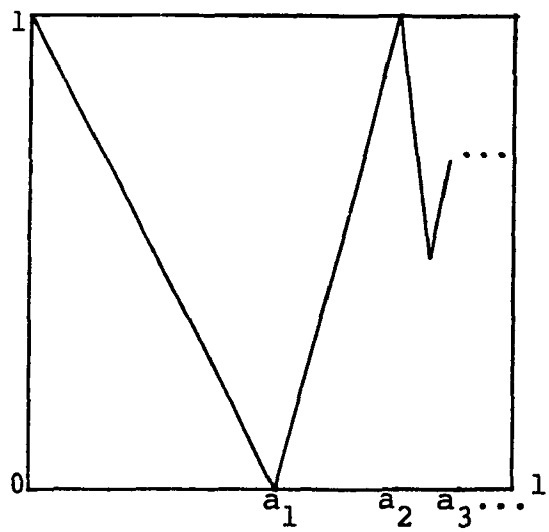


FIGURE 5

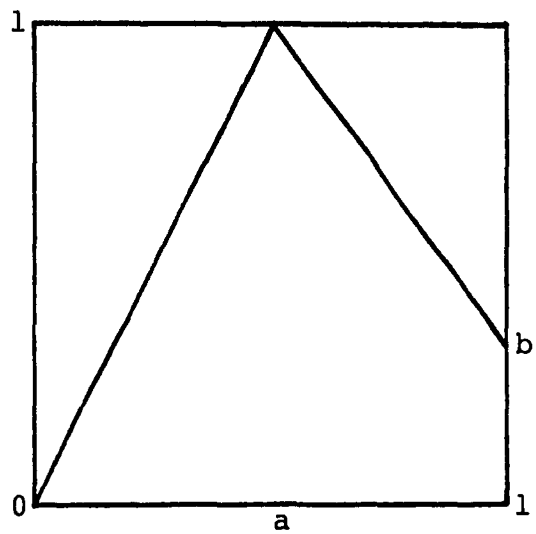


FIGURE 6

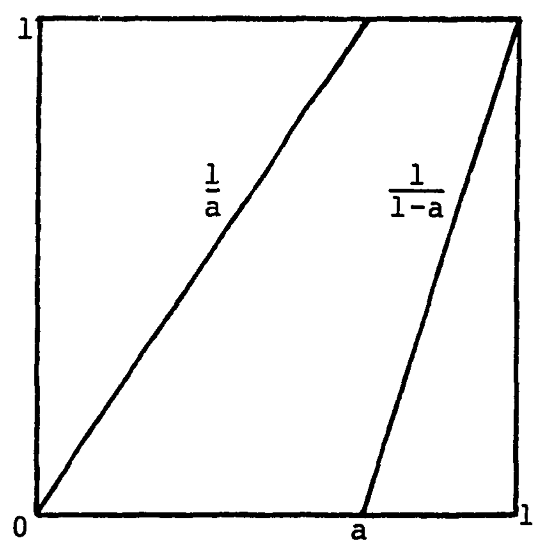


FIGURE 7

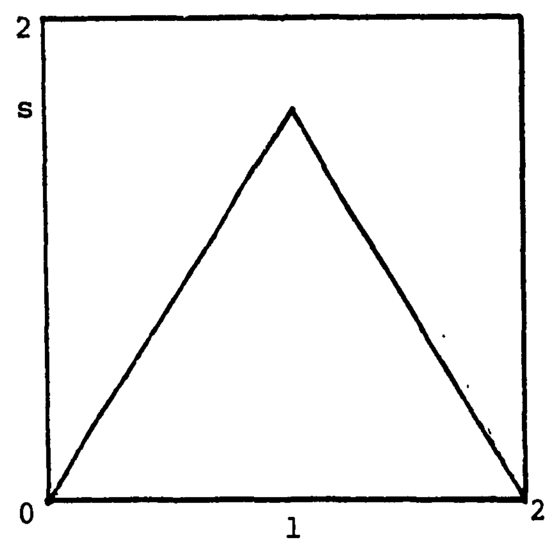


FIGURE 8

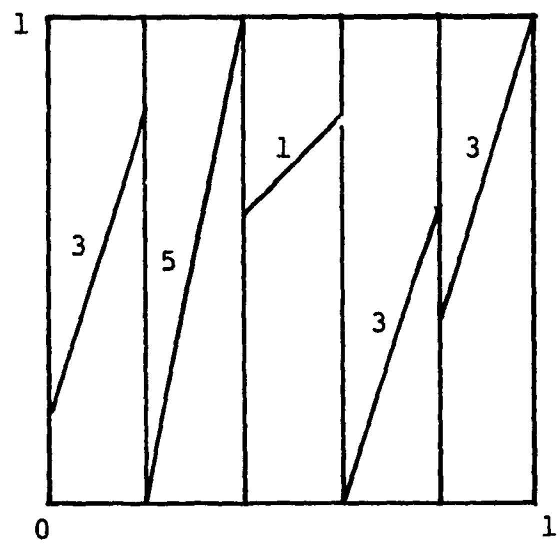


FIGURE 9

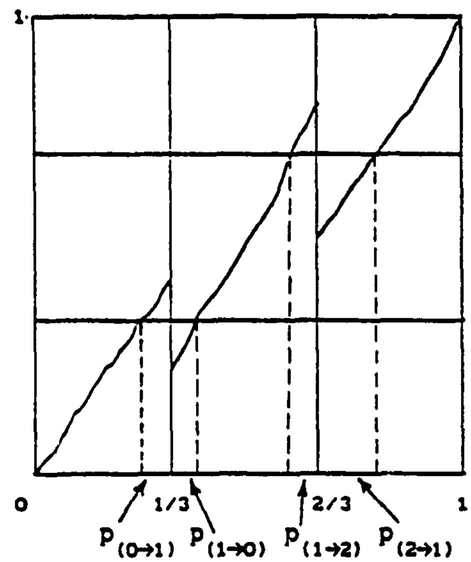


FIGURE 10

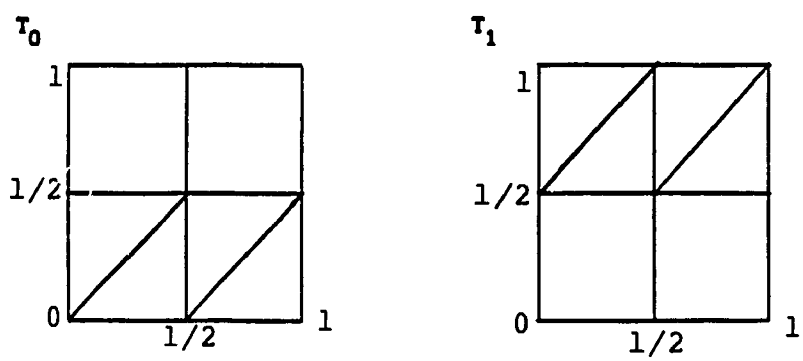
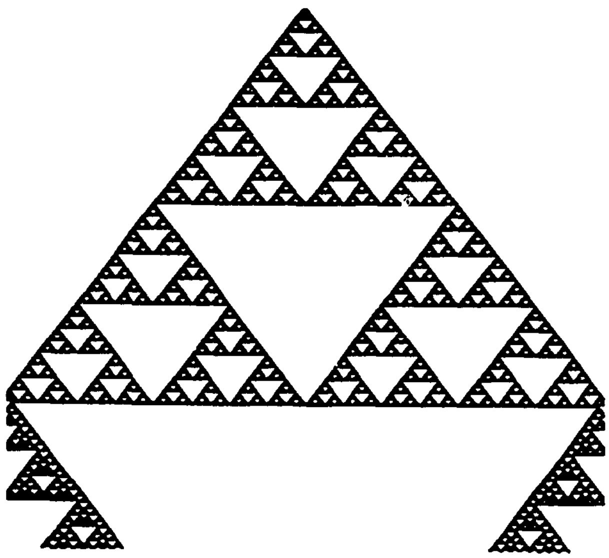
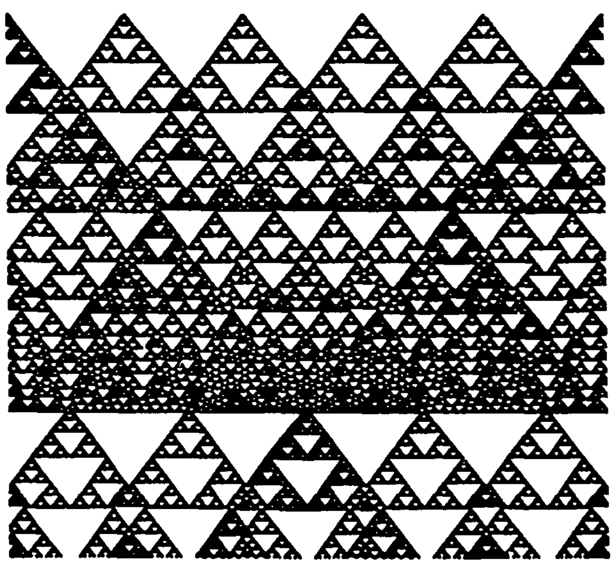


FIGURE 11

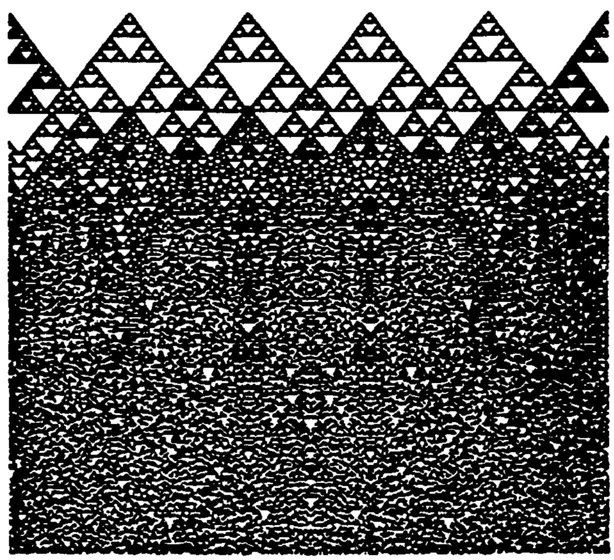


RULE 126
01111110



RULE 126
01111110

FIGURE 12



RULE 126

01111110

FIGURE 13

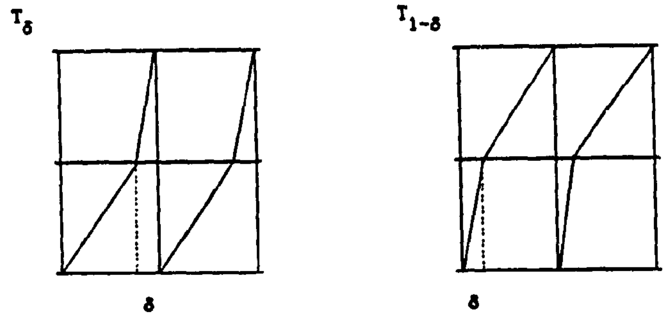


FIGURE 14

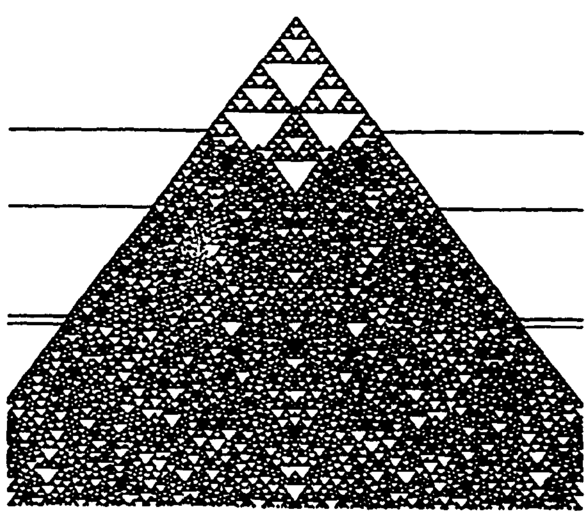
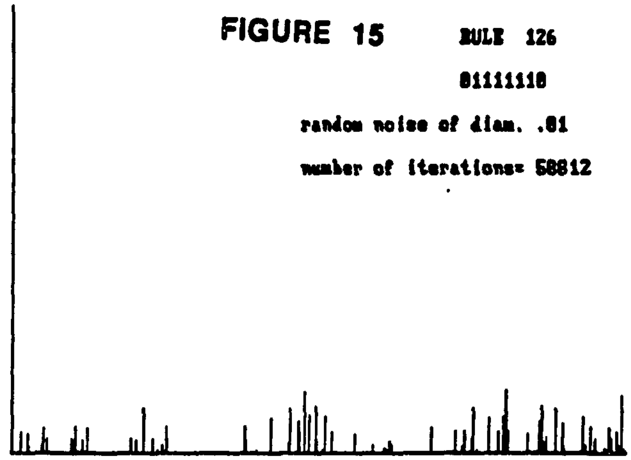
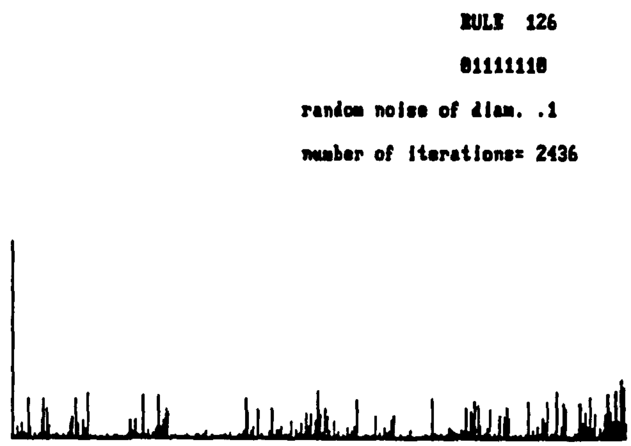


FIGURE 15


RULE 126
01111110
random noise of diam. .01
number of iterations= 50012


RULE 126
01111110
random noise of diam. .1
number of iterations= 2436

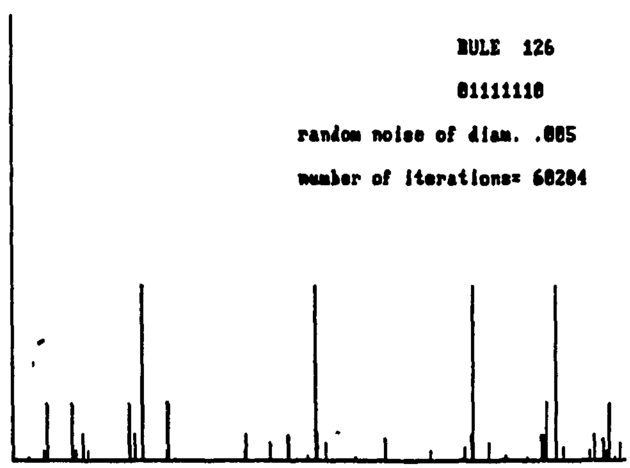

RULE 126
01111110
random noise of diam. .005
number of iterations= 60204

FIGURE 16

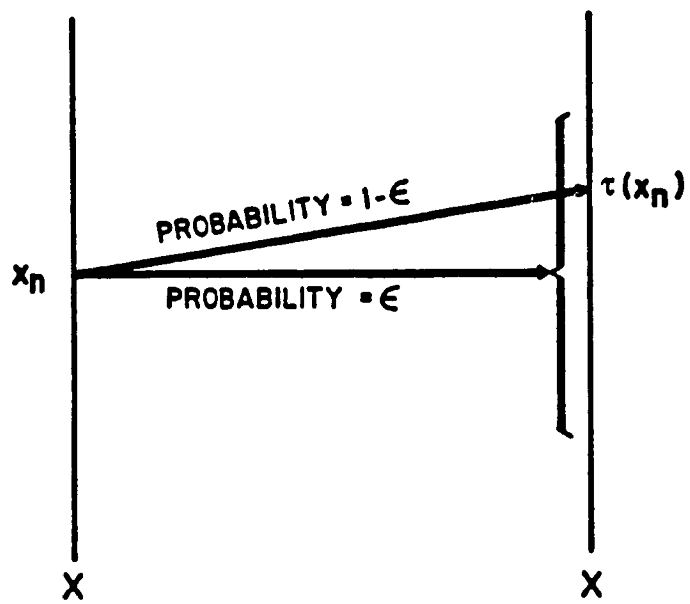
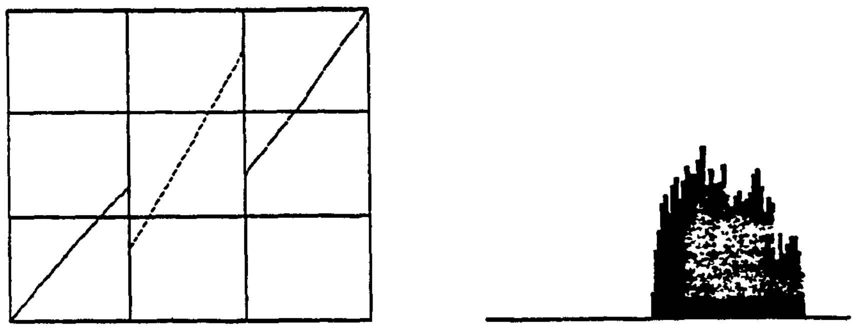


FIGURE 17

p(0-1)= .1 ; p(1-0)= .1 ; p(1-2)= .2 ; p(2-1)= .2
number of iterations = 10000
starting point = .78945



p(0-1)= .1 ; p(1-0)= .1 ; p(1-2)= .2 ; p(2-1)= .2
number of iterations = 10000
starting point = .23456

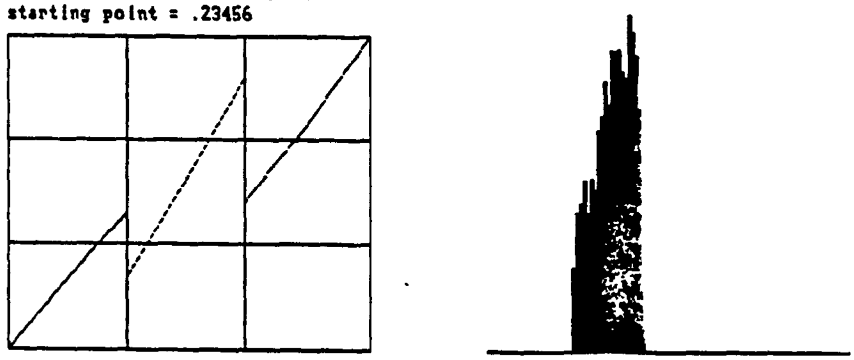
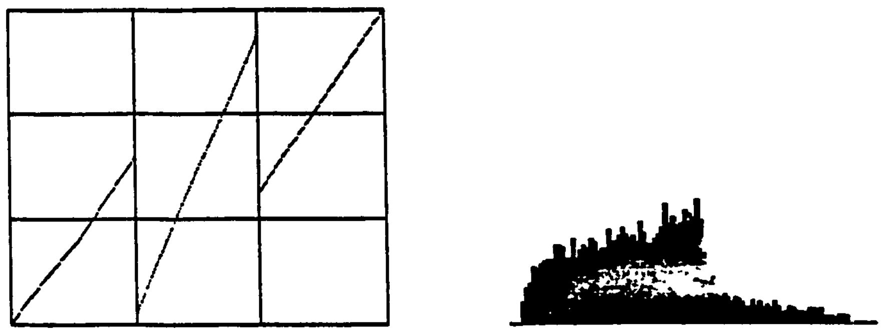


FIGURE 18

p(0-1)= .2 ; p(1-0)= .3 ; p(1-2)= .25 ; p(2-1)= .25
number of iterations = 10000
starting point = .87654



p(0-1)= .2 ; p(1-0)= .3 ; p(1-2)= .25 ; p(2-1)= .25
number of iterations = 10000
starting point = .23456

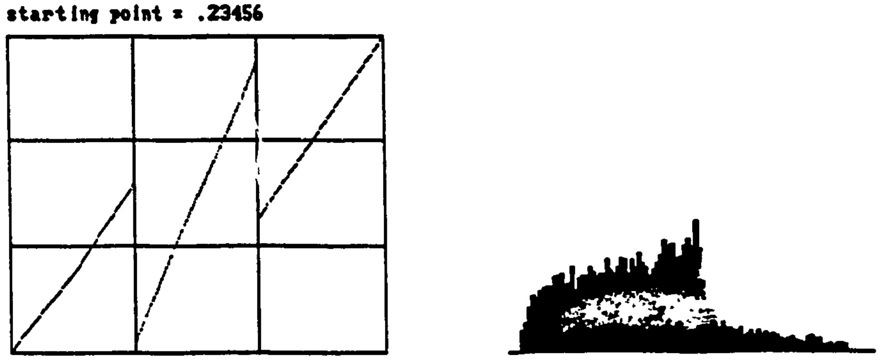
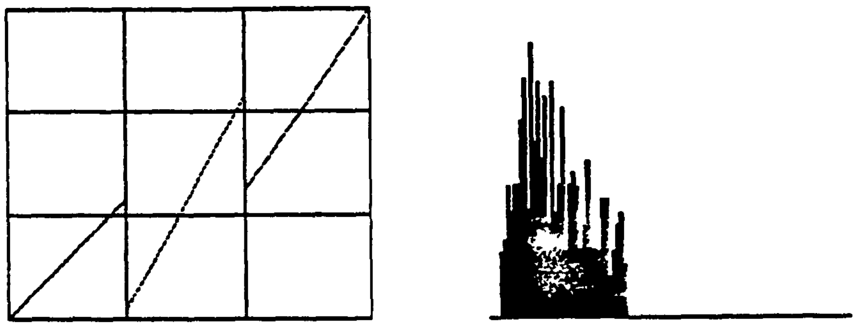


FIGURE 19

p(0-1)= .05 ; p(1-0)= .3 ; p(1-2)= .05 ; p(2-1)= .25
number of iterations = 10000
starting point = .3456



p(0-1)= .05 ; p(1-0)= .3 ; p(1-2)= .05 ; p(2-1)= .25
number of iterations = 10000
starting point = .07654

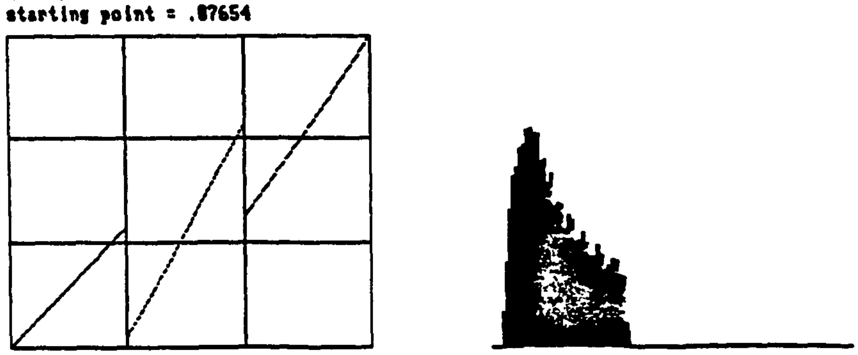
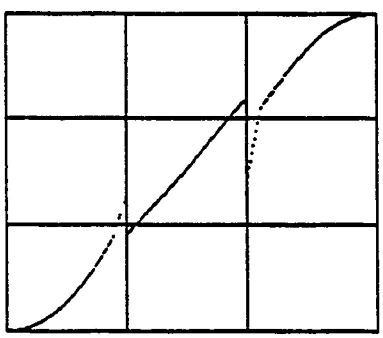


FIGURE 20

p(0- 1)= .1 ; p(1- 0)= .0295 ; p(1- 2)= .065 ; p(2- 1)= .2
number of iterations: 10000 starting point : .71234
left fixed point= .3 ; right fixed point= .7



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