Concept Image and Concept Definition in the Calculus: a Comparison between Their Occurrence in History and in the Class

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ABSTRACT

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Based on Vinner's cognitive model of concept image and concept definition for learning a mathematical notion, and on reviewing of the researches on the acquisition of concept image of students when learning some notions in Calculus, and on the conceptual development of the subject, it was found that, on the one hand, the students shared certain naive concept images with mathematicians in ancient times; On the other hand, there was evidence that to some extent certain misconceptions were eliminated due to the results of organizing the concepts in a coherent way. Two questionnaire were formulated in order to study to what extent the inappropriate concept image would be eliminated when the definition was enforced as well as to what extent the students would be convinced by mathematical arguments. Results suggested that the proper use of definition needed a certain mathematical maturity and that the 'naive idea' often dominated the 'correct idea' which was based on mathematical arguments even when both ideas were presented.

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CHAPTER I

THE WORK OF VINNER

1.1 Introduction

Through the study of Piaget's theory of thinking, S. Vinner [31, 32, 33] believes that it is worthy to extend Piaget's notion of mental imagery into the domain of mathematical thinking. Putting an emphasis on the mental image frame of reference, he develops a theory of thinking and then applies it to explain why the definitional approach in mathematics should not be introduced to the uninitiated student. Furthermore, he goes on to construct a simple model for cognitive processes in order to analyse the process of learning some mathematical concepts.

Before presenting the work of Vinner, some terminology which are frequently used will be introduced first.

Mental image

The mental image of certain nouns (or noun phrases) in each individual's mind is defined as the set of all pictures associated with those nouns (or noun phrases). Generally, there are four types of nouns (or noun phrases), namely, 1. concrete-common (e.g. cat); 2. concrete-proper (e.g. my cat); 3. abstract-common (e.g. set); 4. abstract-proper (e.g. Cantor set).

<u>Mental image frame of reference</u>

The <u>mental image frame of reference</u> refers to the concrete world to which the pictures in one's mind are tied. For instance, children identify numbers as the concrete 'things' in their mind. These 'things' then become part of their number realm.

<u>Platonic approach to mathematics</u>

According to Platonism, the mathematical objects exist independent of experience and have a reality of their own. Mathematicians deal with the world of mathematical objects which resembles to the world of 'things'.

<u>Naive Platonic approach</u>

According to the theory of Piaget, children learn to deal with the abstract objects by looking for the similarity in the concrete world. They act as if there exists an analogy between the abstract world and the concrete world even being aware of the difference. For instance, children may think 'five is greater than three' having the same structure as 'Bill is stronger than Steve'. Since such approach matches Plato's theory of ideas, it is named as the naive Platonic approach .

Formalistic approach to mathematics

According to formalism, mathematics just consists of axioms, definitions and theorems. In this structure of mathematics, certain primitive terms (undefined terms) and primitive statements (axioms and postulates) are first given, then every non-primitive term is defined by means of primitive terms and every new statement can be deduced from the primitive statements by rules of inference.

Definitional approach

By definitional approach , Vinner refers to the approach whereby types of mathematical objects are defined according to 'lower' types. For instance, the complex numbers are defined as pairs of reals, the reals are sets of rational numbers satisfying the conditions of Dedekind cuts; the rational numbers as pairs of integers with the condition that the second one is not zero; the integers are pairs of natural numbers; the natural numbers are a model for Peano axiom.

Naive group

Learners consisting of high school students, college freshmen and adults who study mathematics as a minor subject are called the naive group .

Implicit preliminary approach

According to the work of Piaget, learners have their own preliminary approach to the subject matter in consideration. In the learning situation, the learners themselves are often unable to formulate this approach or are not even aware of it. They just react to the situation depending on their previous experience. Since such preliminary approach is implicit, it is said to be the implicit preliminary approach .

Naive philosophy of mathematics

Children will develop their own implicit preliminary approach to science according to their own conceptions of the world. Briefly, in science children learn to discover the relation between objects and facts in the world. Vinner asserts that the way science relates to the concrete world is just like the way mathematics relates to the realm of mathematical objects. The idea that learning about the world of mathematical objects is analogous to learning about the concrete world is called the <u>naive philosophy of mathematics</u> .

1.2 <u>Vinner's approach to mathematical thinking and its educational implication</u>

In Piaget's theory of thinking, the mental image plays an important role. Vinner (1975) puts an emphasis on the mental image frame of reference to claim:

> "Unless the question of the realm of object in consideration is settled, the mental image cannot play its role in thinking. In other words, thinking (both concrete and abstract) can be accomplished only after the question of realm has been settled." [31, p. 340, the words in the bracket are mine]

In the case of concrete thinking (that is, thinking of concrete objects), most people will not have difficulties due to the fact that the question of realm is generally settled. Whereas, at the abstract level, the question of realm is quite complicated and vague. Worse, the realm associated with abstract objects may appear to be self-contradictory. Consequently, the mental image cannot function, thinking cannot be accomplished and so difficulties arise in abstract thinking (that is, thinking of abstract objects).

In mathematics, the learners somehow have to deal with abstract objects. Thus, it is essential to study the mental images of abstract-common and abstract-proper nouns (or noun phrases) as well as their associated realms. Through the work of R. Dubisch [5], and Piaget [17], Vinner observes that, in general, children act as if there is an analogy between the concrete world and the abstract world. Although different children may develop different approaches, a certain kind of analogy does exist in their mind. Due to this observation, Vinner concludes that children's approaches to the domain of abstract objects are basically naive Platonic approach until they become a member of naive group.

On the other hand, as the learning goes on, children develop their own implicit preliminary approach to learning according to their own conception of the world. Just like in science, they learn to discover the relation between objects and facts in the world. Similarly, children will draw the analogy between the concrete world and the world of mathematical objects. From this point of veiw, children's conceptions of learning mathematics are mainly naive philosophy of mathematics

Moreover, the naive philosophy of mathematics and the naive Platonic approach to the realm of mathematical objects are interrelated. On the one hand, naive philosophy of mathematics can only be developed in those people who have the naive Platonic approach to the domain of abstract objects. On the other hand, naive philosophy of mathematics does advance naive Platonic approach because naive philosophy of mathematics presupposes the existence of the world of mathematical objects in learning process. For instance, one can assume that the real numbers exist without starting from Peano axiom.

As a result, Vinner concludes that in the naive group, the learners' approaches to the learning of mathematics will be naive philosophy of mathematics if there is no explicit discussions in the philosophical and methodological aspects. Now if the definitional approach is introduced, the naive group will face the change of their implicit preliminary approach to the realm of mathematical objects. Such change will cause confusions and difficulties because the nature of thinking for the definitional approach and the naive Platonic approach are totally different. For instance, thinking integers as pairs of natural numbers will be different from thinking them as 'things' in the concrete world.

Yet, one may argue that introducing the definitional approach may benefit the naive group in learning mathematics. Vinner points out that the proofs of theorems in algebra or calculus do not really need the definitional approach or that the definitional approach is so important to be taught. In the former case, even when the need for the definitional approach to a theorem is required, the teachers still can present it as a fact in mathematical world instead of the explanation appealed to the definitional approach. With regard to the latter , the definitional approach may be fascinating for mathematicians and some teachers because they are aware of the problem in the foundations of mathematics. However, for the naive group, mathematics itself is rather applicable in comparsion with the philosophy of mathematics.

Using the above arguments as a basis, Vinner asserts that the definitional approach should be avoided at the naive group. As a step further, he recommends that naive Platonic approach should be adopted as an alternative in the teaching situation. Vinner supports his assertions by examining three different aspects, namely, the development of a mathematical concept, the advantages of the naive Platonic approach and a coherent strategy in teaching environments.

In the history of mathematics, a lot of subjects, for instance, the calculus, were invented and enriched long before the mathematicians adopted the definitional approach to the abstract objects. The lack of rigorous approach in developing these subjects suggests that it is not improper to act as if the abstract objects exist.

Secondly, the naive Platonic approach to mathematical objects is closer to the nature of human thinking. It helps the learners not only to construct the realm of mathematical objects but also to overcome the problem of using concrete models as the exemplification of certain mathematical concepts. Often, a new mathematical object is introduced to the students by a concrete model. For instance, the model of the real line exemplifies the real numbers. Yet, the teacher may shift to other models in order to explain certain facts about this mathematical object. Thus it is important to convey that the existence of the mathematical objects is totally independent of the model. Once the learners are aware the difference between the abstract world and the concrete world, they are able to realize that the domain of mathematical thinking is not the same as the concrete model which the teachers use to exemplify the abstract world. Different concrete models can be attached to a given mathematical concept. In this way, the learners will not be tied up to a particular model and so the learning will become more flexible. Consequently, all the pictures created by these concrete models as well as the naive Platonic approach will form the implicit preliminary approach to the realm of mathematical objects.

Finally, Vinner argues that the choice of the naive Platonic approach should be a full choice in the whole curriculum. In the elementary school, teachers themselves are members of naive group. Generally, they do not have a clear idea about the definitional approach Their approach to the realm of mathematical objects in teaching is mainly the naive Platonic approach. Whereas, the mathematical abilities of the teachers of the naive group are sufficient to adopt the definitional approach. Yet, the choice of the definitional approach should be avoided on the ground that the definitional approach cannot provide the naive group with a coherent way of thinking. Hints about it or small fragments only create confusions in the learning process. As a result, he comments:

> "..., the n.P.a. (naive Platonic approach) strategy should be applied to students who have had the n.P.a. (naive Platonic approach) and nothing else. The d.a (definitional approach) strategy should be applied to students who had the d.a (definitional approach) and nothing else." [31, p.349, the words in the brackets are mine]

The definitional approach described by Vinner highlights the hierarchical structure of definitions. In school, most of the math textbooks adopt the formalistic approach to mathematics. Under such setting, definitions play an important role. What, then, one may ask, is the learners' point of view of definitions? This aspect will be discussed at length in the next section.

1.3 Studies on the notion of definition

In order to study the students' notion of definition under the formalistic approach to mathematics, Vinner conducted two studies, namely, (1) on the concept of defining operations [33] and (2) on the concept of definition [32]. I shall briefly describe each study and its finding.

1.3.1 Case study on the concept of defining operations

The purpose of this study was to examine whether students treated exponentiation as the operation a^x defined by means of known operations. Three formulas, namely,

(1)	$a^m = \underbrace{a.a. \; ... \; .a}_{m \text{ times}}$	where a is any real number and m is whole number
(2)	$a^{-m} = 1/a^m$	where a, m are as above with $a \neq 0$
(3)	$a^{n/m} = \sqrt[m]{a^n}$	where a, m are as above and n is a whole number

were distributed among other formulas in a questionnaire. The students were asked to identify whether a given formula was a theorm; a law; a fact about numbers; a definition or an axiom.

The sample was 251 math majors which were composed of 195 freshmen and 56 sophomores and junior at university of California, Berkeley. As the result of the study indicated, only 1/4 of the first group and about 1/2 of the second group could identify all the defining formula.

1.3.2 Case study on the concept of definition

The purpose of this study was to find out what was the students' point of view as to the notion of definition. Seven sentences written in the same grammatical form which hinted the role of definition were given. Five were the 'canonical definition' as in most of math textbooks. The subjects were asked to identify whether the given sentence was an axiom; a postulate; a fact; a definition; a theorem or something else.

Two different groups participated in the study. The first group consisted of 63 pre-calculus students at the university of Missouri - St. Louis. The second group was composed of 129 students in several high schools in St. Louis. As the result of the study indicated, at least 90% of the college students and at least 79% of high school students had neither the formalistic point of view about the structure of mathematics nor a satisfactory understanding of the relative role of definition.

1.3.3 Some proposals by Vinner

From the result of the first study, Vinner concludes that the idea of defining operations and notions in mathematics does not get along with naive Platonic approach. However, the improvement at the upper undergraduate level indicates that the mathematical maturity may be obtained through mathematical experience.

Furthermore, Vinner applies the above result to assert that the definitional approach should be eliminated from the non-major mathematics curriculum. Often, those who teach the definitional approach claim that the requirement for understanding the definitional approach is mathematical maturity, not previous knowledge. Applying the notion of the implicit preliminary approach of the learners to mathematical objects, Vinner argues that in the case of mathematical maturity, the implicit preliminary approach of the learners should be quite close to the definitional approach. Now, the data showed that the naive group does have difficulties with the idea of defining operations in mathematics; they should have even more difficulties in understanding that mathematical objects are hierarchically defined. However, the data also showed that some years of mathematical experience will develop certain mathematical maturity. Consequently, Vinner proposes that the definitional approach should only be taught after the students have reach certain level of mathematical maturity, for instance, at the graduate study.

From the result of the second study, Vinner makes two observations:

(1) Certain definitions, for instance, the absolute value of a number x is x itself in case x is non-negative and it is -x in case x is negative, are far too complicated for the students to understand. Thus, the students will forget them soon after these definitions are mentioned.

(2) Certain definitions, for instance, a rectangle is a quadrangle that has three right angles, cannot help the students to construct a mental images. As a matter of fact, the data showed that some students try to draw quadrangles that have only three right angles.

As a result, Vinner asserts that a good definition should be as simple and as natural as possible, that is, it should be easy to understand as well as helpful to construct the associated mental images.

1.4 The lexical definition and its teaching implication

As the data in both studies showed, the naive group has difficulty understanding the formalistic point of view about the structure of mathematics. In order to develop a consistent methodology of teaching mathematics starting with a naive point of view, Vinner introduces the notion of lexical definition.

In general, lexical definition is a form of history. It refers to the actual way in which some actual words have been used by some persons. In this case, the meaning of a word is what it means to some person or persons. Often, the 'ordinary man' who looks upon the meaning of a word tends to ignore its special meaning to this essential person. Instead, the 'ordinary man' thinks that the meaning of a word is eternal and independent of human beings. Just like everybody agrees that the square root of 100 is and always must be 10. Thus, definition of a noun can be treated as a certain statement about the object in consideration, a statement that can uniquely (or almost uniquely) characterize this object. In doing so, a definition is not necessary 'canonical'; but depends on the situation. For instance, at the defining situation, the sentence 'a house is a building for human habitation' serves as a definition. However, at a different situation, the same sentence can be considered as a statement or a fact.

Applying the notion of lexical definition, Vinner not only conjectures that the naive students' implicit preliminary approach to mathematical definitions is the same as the ordinary man's approach to lexical definitions but also suggests the ordinary man's approach as a teaching strategy. Basically, he proposes that in teaching situation, mathematical definitions can be viewed as lexical definitions, that is, definitions are statements about mathematical objects; statements that are employed to characterize these objects in defining situation.

With the above approach, definitions are not necessary defined hierarchically as in the definitional approach. For instance, the object denoted by the word 'rectangle' is a quadrangle that has four right angles. This way of presentation avoids the question how the rectangle is defined according to primitive terms. Moreover, this teaching strategy may lead the learners to appreciate the relative role of definition in mathematics. Thus, a failure to identify a definition, in Vinner's studies, does not imply a failure to conceive the underlying structure of mathematics, but only a failure to remember the meaning of a given 'terminology'.

On the other hand, as the data of Vinner's studies indicated, the formalistic approach cannot get along with the naive Platonic approach, conflicts will be expected in the learning process if the teacher holds the definitional approach whereas the students hold the naive Platonic approach. By using the naive lexical approach to mathematical definitions, there is a hope to eliminate the conflicts at the communication process.

However, Vinner's view towards mathematical definitions is mathematical rather than psychological. In facing a given task, to what degree do the learners use definitions either to create the associated mental images or to handle this task? In order to answer this question, Vinner constructs a cognitive model to explain the role of definition in the process of learning mathematical concepts.

1.5 Vinner's model for cognitive process

In teaching and learning mathematics, on the one hand, the teachers make efforts to convey mathematical concepts; on the other hand, the students try hard to grasp these concepts. In 1980, Vinner constructed a model by using the notion of <u>concept image</u> and <u>concept definition</u> to analyse the cognitive process for learning a mathemaical concept. In this section, I shall first present how a concept is formulated according to the work of R.R. Skemp [21], then Vinner's notion of concept image and concept definition as well as their roles at the stage of concept formation and that of performance [35]. At the end, the role of Vinner's model in mathematics education will be discussed according to the work of Vinner and D. Tall [24].

1.5.1 <u>Concept image, concept definition and concept formation</u>

According to the work of Skemp, a concept is formulated by abstracting the invariant properties from different experience which have something in common. During the process of forming a concept, different orders of abstraction will be achieved. The concepts which are derived from sensory and motor experience are called primary concepts. For instance, the notion of circle is an abstraction from the experience with round objects. Concepts which are abstracted from some existing concepts are said to be secondary. For instance, the notion of function is composed of three sub-concepts, namely, domain, range and rule of correspondence [15]. In mathematics, concepts are generally secondary and so are hierarchical. Such concepts can hardly be communicated only by definitions, but by contributory examples and non-examples. In other words, the learners have to extract the common properties from those examples which presuppose other concepts. In order to analyse this cognitive process for learning a mathematical concept, Vinner introduces the notion of concept image and concept definition.

Vinner assumes that there are two cells (not biological) which are referred to image cell and definition cell in each individual's cognitive structure. The former consists of each individual's mental picture of a given concept, namely, all visual representations such as graphs, symbols, ..., etc of the concept as well as a set of properties associated with it. This set of properties together with the mental picture is called concept image. Given a concept, each individual may form different concept images, not necessary in a coherent way, according to one's own perception and experience.

On the other hand, the definition cell contains a verbal definition that accurately explains a given concept. In mathematics, a concept definition which is accepted by mathematical community at large is referred as a formal concept definition (or just concept definition). Particularly, a form of words used by each individual to explain his (or her) own evoked concept image is referred to as a personal concept definition. In general, personal concept definition may be quite different from formal concept definition.

In a learning situation, some concepts can be introduced by concept images. For instance, children acquire the concept 'red' from different kind of red objects without a verbal definition. On the other hand, some concepts may be grasped by concept definition. For example, the concept 'forest' can be conveyed by saying: 'Many many trees together are a forest'.

In the case of informal learning, concept may be first acquired by experience or examples. Ostensive or formal definition will given at the end or only mentioned implicitly. However, in formal learning situation, definition may be introduced prior to the forming of the relevant concept image. The image cell is expected to be filled according to the given definition.

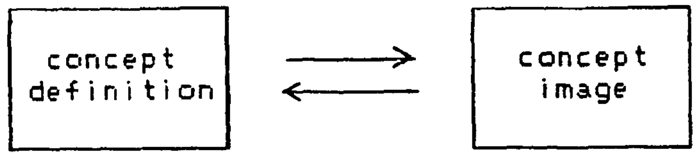
No matter how concepts are learned, three cases will happen when definition (either ostensive or formal) is given:

(1) The image cell will be changed to include (or to be filled) with images reflected by all the aspects of the ostensive or formal definition.

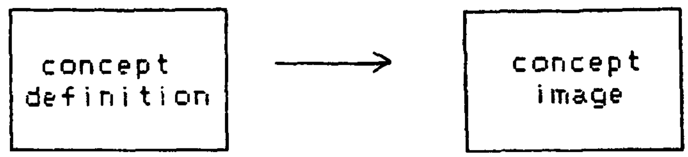
(2) The image cell remains as before but the definition cell will temporarily contain the ostensive or formal definition. In most cases, this definition will be forgotten or distorted after a while.

(3) The image cell remains the same and the definition cell contains the ostensive or formal definition. In this case, the learners react to a given task by evoking their concept image, not concept definition. When the definition is asked directly, the learners respond with ostensive or formal definition.

Vinner concludes that at the stage of concept formation, there should be an interaction between concept image and concept definition (see diagram 1). A one way process (see diagram 2) which is expected by most teachers

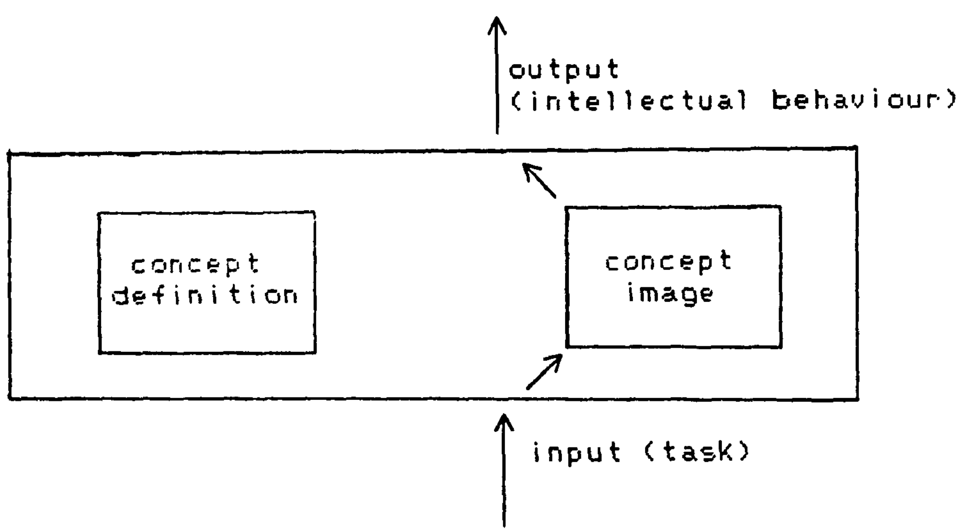


(diagram 1)



(diagram 2)

at the secondary and the collegial levels does not happen very often, that is, just by giving a concept definition, the student does not necessarily evoke an assoicated concept image.



(diagram 3)

As a matter of fact, Vinner model challenges many teachers' beliefs that both concept image and concept definition will be activated when a cognitive task is given to the learners. Vinner goes on to criticize the assumption that concept definition will be necessary in learning of mathematical concepts. He writes:

"There is no way to force a cognitive structure to use definitions, either in order to form concept images or in order to handle a cognitive task. Some definitions are too complicated to deal with. They do not help in creating concept images in students' mind. Hence, they are useless."

1.5.2 <u>The role of Vinner's model and its relevant notions in mathematics education</u>

According to Vinner's model, when the learners are asked to carry out a task, only parts of the image cell will be activated at certain moments. In addition to that, different individuals will evoke different parts of their image cell, the same individual may activate different parts of his (or her) image cell at different moments. Consequently, in order to understand the students' cognitive process, it is essential to study their concept images associated with a given concept. In fact, revealing the concept image of students not only provides us with a better understanding of their intellectual behaviour but also suggests a better approach in teaching. Some educational implications are outlined as following:

(1) Different examples should be given to reflect different aspects of the definition. Implicitly, teachers assume that students will use the concept definition to carry out a cognitive task. Thus, there is no need to give students different examples. However, very often mathematics educators and teachers find that students hold the wrong concept images due to a specific set of examples. For instance, in an introductory course such as a course on functions; pre-calculus and calculus, the concept definition of a mathematical function is given according to Dirichlet-Bourbaki approach, that is:

> "A function is any correspondence between two sets (the domain and the range) which assigns to every element in the domain exactly one element in the range"

Examples given by teachers are often concentrated on algebraic or trigonometric functions. Consequently, the students may have the concept image that a function is given by a formula. Holding such restricted notion, the students may be unable to cope with a broader idea of function in later context.

(2) The notion of concept definition and concept image may be used to explain how conflicts occur in learning a mathematical concept. According to the work of Tall and Vinner [29], in a learning situation, personal concept definition will generate its own concept images and such concept images may be inappropriate to other part of concept images; or to personal concept definition, or even to formal concept definition. Thus, individual's concept image may contain potential conflict factors. At a certain situation, a specific stimulus may activate a particular part of concept image. Now, if the conflicting aspects are simultaneously evoked, possible confusions in the learning processes will result. For instance, in the concept of complex number, the definition of $x + iy$ as an ordered pair of real number (x,y) and the identification of $x + i0 = (x,0)$ as the real number contribute to a potential conflict factor in set-theoretic approach because the element x is different from the ordered pair $(x,0)$ in [illegible] theory of sets.

(3) The notion of concept image is useful for describing the development of understanding of a formal

concept definition. As Vinner's model indicated, students may state or identify correctly the concept definition but develop their own concept image (possibly empty) through experience of manipulating examples or the relevant theory. When a task associated with a given concept is performed, students often react to it with the concept image in their cognitive structures. Thus, studying the students' concept images may provide the teachers with a better understanding of how an imposed concept is learned.

(4) The study of concept image may lead to a teaching strategy which is consistent with the formal theory. As in (3), students react to a task with concept image. Particularly, each individual's intuition for a concept depends directly on his or her own concept image. Tall [24] suggests that a suitably formulating concept definition may lead to a suitably concept image based on the students' intuition. For instance, the notion of continuous function can be introduced by the following definition:

"A function $f:D \to R$ (from a subset D of the real numbers R) is said to be <u>pictorially continuous</u> if over any closed interval $[a,b]$ in D, given $e > 0$, there exists $d > 0$ such that for x, y in $[a,b]$, $|x - y| < d$ implies $|f(x) - f(y)| < e$"

By using such a definition, the teacher can show that: given a pencil which draws a line of any thickness, the graph of a pictorically continuous function can be drawn to any specified scale over a closed interval [a,b] in D without lifting the pencil from the paper. In doing so, the students are expected to create a concept image which is consistent with the formal structure of mathematics.

(5) From the learners' aspect, a suitably development of concept image may lead them to discover the general theory. For instance, having the impression that $1 + 2 = 2 + 1$; $3 + 4 = 4 + 3$; ... etc, the learners may abstract the common properties from these generic examples to arrive at the idea of commutative law.

Overall, the study of concept image provides the mathematics educators a flexible view of mathematics through the eyes of the learners. In learning a mathematical concept, the learners have to construct their concept image through experience for the given concept. In this cognitive development, new information may activate the potential conflict factors and so causes confusion. In order to accommodate this new information to the cognitive structure, the learner have to reconstruct their concept image. This observation leads me to conjecture that a better teaching approach is to guide students to become aware of the difference between the formal theory and their own concept images. Once the difference is recognized, it is possible that they will reconstruct their concept image in order to accommodate the new situation. In such a way, students not only can eliminate conflicts but also experience the development of a mathematical concept.

CHAPTER II

HISTORICAL NOTES ON THE CONCEPTUAL DEVELOPMENT OF CALCULUS

(WITH PARTICULAR REFERENCE TO THE CONCEPT IMAGE)

2.1 Introduction

The aim of this chapter is to summarize the mental images of notions in the calculus as held by the contributors to the subject at their time. As the evolution of the concepts in the calculus took more than two thousand years, their final forms are of great abstractnesses and generalities. Often, these final forms cannot show insight into the motivations which led to their creations. Consequently, the study of the mental images held by the contributors of the subject may provide information about:

(1) the discovery of and difficulty involved with these notions

(2) the most likely students' concept images

As a remark, mathematics educators should not expect that the concept images of the original contributors to the calculus are the same as that of our students. The most obvious difference is due to the fact that mathematicians in ancient time and our present-day students live in different eras and different social environments lead to different experience. For instance, modern technology creates certain concept images (as we shall see later) which cannot have been held by the early contributors to the calculus. Thus, some of the concept images held by past mathematicians are not necessary the same as those held by present-day students.

In this chapter, I shall first present the contributors' views towards those notions in the calculus mainly as it appears in the work of C.B. Boyer [1] and M. Kline [11]. Later, the naive concept evolution versus Vinner's concept formation in the learning situation will be discussed.

The following six phases of the historical development of Calculus are presented according to the classification of Boyer.

2.2 Phase 1: Conceptions in Antiquity

The roots of the notions in calculus can be traced back as early as the ancient Greeks. One of their great discoveries - namely, the incommensurable ratios (The two quantites cannot be measured by a common unit. The incommensurable ratios are expressed in modern mathematics by irrational numbers) - was made by Pythagorean school (c.580 - c.500 B.C.). The Pythagoreans believed that all phenomena in the universe can be reduced to whole numbers or their ratios. As they noted that the ratio of a diagonal to a side of a square cannot be made precise under their concept of number - in other words, they learnt that the process of finding a common unit between a diagonal of a square and its side cannot be terminated in finite steps - they restricted themselves to those ratios which can be expressed by whole numbers and hence failed to conceive the idea of infinite process. Their dismissal of this discovery not only led to their failure to define exactly the notions of lengths, areas, volumes ... etc but also brought on the problem of the relation of the discrete to the continuous.

On the other hand, the notion of the infinitesimal also entered into mathematical thought through the Abderitic school (c.500 - c.300 B.C.). This school asserted that everything is composed of atoms which are indivisible. One of the followers, Democritus (c.460 - 370 B.C.), made use of the idea of indivisible to determine the volumes of the Pyramid, the cylinder as well as the cone. Basically, he considered the volume of a cone as a series of infinitely thin parallel circular laminae.

As we have seen, the Pythagorean theory of proportion could only be applied to commensurable ratio. It was Eudoxus (c.480 - c.355 B.C.) who introduced the notion of a ratio of magnitudes to extend Pythagorean's theory so as to cover both commensurable and incommensurable ratios. Since then, the separation between number and geometry was drawn. Besides, having recognized that the Democritean view of the infinitesimal lacked logical basis, Eudoxus took over the idea suggested by Antiphon, generalized by Bryson later, to establish the area and the volume of curved figures. This method is known as the method of exhaustion. Basically, he sequeezed the area or the volume of the given curve figures between two commensurable ratios. This could be done by applying the following axiom (now called the axiom of Archimedes):

> "if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out" [Boyer p.33]

According to this axiom, the process of squeezing will continue up till the difference become as small as we please. Following by the reductio ad absurdum argument, Eudoxus established the desired area or volume (see Boyer p.34 for detail).

The method of exhaustion was modified later by Archimedes (287 - 212 B.C.). He combined it with infinitesimal considerations - the idea that a figure was made up of its elements - as well as mechanical techniques - the balance point of a figure - to determine the volume of segments of conoids and cylindrical wedges, the area of a parabolic segment ... etc. It is interesting to note that neither Archimedes nor his predecessors included any reference to the infinite and the infinitesimal in the proof.

Another prevailing view in this period was that of monism which stated that all change was illusory. It was said that Zeno (fl. c.450 B.C.) defended this doctrine by formulating a number of paradoxes to challenge both the infinite divisibility and infinite indivisibility. For instance, the Dichotomy paradox asserted that before an object can travel across a given distance, it must first pass over one quarter and so on. Under the assumption of infinite divisibility of space, Zeno argued that the object could never reach its distination within a finite time. Obviously, this contradicted our daily experience. Thus, change was impossible. Aristotle (384 - 322 B.C.), a mathematician, physician and philosopher, refuted Zeno by saying:

> "...there are two senses in which a thing may be infinite: in divisibility or in extent. In a finite time, one can come into contact with things infinite in respect to divisibility, for in this sense, time is also infinite; and so a finite extent of time can suffice to cover a finite length" [Kline p.35]

The refutation itself was not convincing enough so that the Dichotomy paradox still had its position in mathematical thought. In modern view, the failure in answering this paradox was due to the inability of conceiving the limit of infinite convergent series as well as the nature of continuum. However, Aristotle's refutation did express his point of view on infinity. As a matter of fact, he only accepted the 'potential infinite' due to his belief that unknown exists only as a potentiality. Concerning the continuity, he gave a vague assertion like:

> "By continuity, I mean that which is divisible into divisibles that are infinitely divisible" [Boyer p.42]

As the continuum is defined in modern mathematics in terms of the concept of number, it is interesting to examine Aristotle's view of number. His conception of number was under the influence of Pythagoreans, as was the case for most of the contemporary Greek mathematicians. However, such a view of number could not be reconciled with the infinite divisibility of continuous magnitudes. As a comprise, he followed the Eudoxus' assertion:

> "... numbers are discrete quantities and must be distinguished from the continuous magnitudes of geometry" [Kline p.53]

Owing to this comprise, the study of continuous magnitudes became independent of that of number throughout the following centuries.

The paradoxes of Zeno arose from the doctrine that motion was impossible. Beside the above refutation, Aristotle did try to seek a suitable foundation to clarify the nature of motion. He remarked:

> "We can define motion as 'the fulfillment of movable qua movable" [Boyer p.42]

Furthermore, he made a connection between discrete quantities like points and continuous magnitudes like lines by way of motion. In his doctrine, points are like numbers which are indivisible. Thus, points cannot be made up a line for the latter is divisible. However, a line can be generated by a point through movement! On the other hand, he denied the possibility of instantaneous rate of change since it was a perception beyond practical experience. He said:

> "Nothing can be in motion in a present... Nor can anything be at rest in a present." [Boyer p.43]

Besides, both Zeno and Aristotle opposed the doctrine of the infinitesimal. Zeno based his opposition on following arguments:

> "That which, being added to another does not make it greater, and being taken away from another does not make it less, is nothing" [Boyer p.23]

Aristotle argued that there is no clear conept of the infinitesimal derived from sensory experience.

Also owing to Greek, perhaps starting from Thales (c.640 -c.546 B.C.), mathematics have been developed as an independent body of knowledge based on deductive reasoning which guarantees the truth of what is deduced (if the starting axioms are consistent). Thus, those ideas without logical basis - for instance, the infinitesimal, infinity, instantaneous rate of change - were frequently rejected by mathematicians. Even those who developed the method of exhaustion omitted the reference to the notion of infinity which directly related to the limiting process. In fact, the Greek mathematicians never considered the process as being carried out to an infinite number of steps, as we do nowadays in passing to the limit! For them there was a quantity, no matter how small, left over and hence they failed to comprehend this process. Overall, their images towards those notions in calculus were very vague. In this aspect, the paradoxes of Zeno did raise a very fundamental question - can change and variation be discussed in mathematics? From the developmental point of view, neither Greek conception of number nor their symbolic algebra could provide a clear answer. Yet, these paradoxes encouraged not only the ideas of infinitesimal, infinity, limiting process ... etc. but also the search for a rigorous basis of these ideas.

2.3 Phase 2: Medieval contributions

The contributions in the Middle Ages could be summarized into three different groups, namely, the Hindus of India, the Arabs as well as western and central Europe.

The Hindus introduced the principle of positional notation, the zero and its operation (including the indeterminate forms), negative number as a debit and the operations on the irrational numbers. Among these operations, the most significant were the procedures to operate on irrational numbers. For instance, the addition of two irrational numbers $\sqrt{a}$ and $\sqrt{b}$ can be calculated by

$$\sqrt{(a + b) + 2\sqrt{ab}}\ .$$

This not only freed arithmetic from geometrical representation but also promoted the algorithmic procedures in the later developement of algebra and that of Calculus.

Basically, the Hindus bothered neither with Zeno's paradoxes and the relevant notions nor with the logical basis of incommensurability. In fact, the notion of infinitesimal only appeared when Brahmagupta (fl. 625) found the difficulties in operating with the number zero. This led him 'to regard zero as an infinitesimal quantity which ultimately reduced to nought' [Boyer p.62]

On the other hand, the Arabs not only combined both Hindus' arithmetic and Greeks' geometric demonstration but also transmitted those works to Europe. Mainly, they avoided the problem of incommensurability, continuity, indivisibility and infinity as the Hindus did. However. they did add more results derived from the idea of infinitesimal. For example, the summations of the cubes and the fourth powers of the positive integers.

The successor of the Arab civilization was that of Europe. In the beginning of European civilization, mathematical thought was dominated by the doctrine of Aristotle. As a result, the nature of infinity and infinitesimal caused much disputations on philosophical rather than mathematical aspects. Aristotle had distinguished two kinds of infinity - a potential and actual infinity. He rejected the existence of the latter and restricted the former to the cases of infinitely small continuous magnitudes and infinitely large number. Along this line, a categorematic infinite (a quantity without end) and a syncategorematic infinite (a quantity which is so great but can still be made greater) were asserted by Petrus Hispanus in the thirteenth century. Even as these two kinds of infinity were generally recognized in this period, philosophers held different views concerning their existence. For instance, William of Occam (1300 - 1349) denied the categorematic infinite. Around the same time, Gregory of Rimini argued that:

> "...there is in thought no self-contradiction involved in the idea of an actual infinity" [Boyer p.69]

From the mathematical point of view, the comment made by Richard Suiseth in the fourteenth century on the infinity was interesting. He said:

> "All sophisms regarding infinite ... would imply that any part, when added to the whole would not change it in magnitude" [Boyer p.70]

Later, while more of Greek works, especially the work of Archimedes, were transmitted by the Arabs, the ideas of infinity and infinitesimal entered into mathematics. For instance, Nicholas of Cusa (1401 - 1464) gave the definition of infinitely large (that which cannot be made greater) as well as that of infinitely small (that which cannot be made smaller). He viewed that the triangle and the circle were the polygons with the smallest and the greatest number of sides. Besides, he believed in the actual infinity and upheld that the infinity can be only approached by going through the finite.

It was also said that Leonardo da Vinci (1452 - 1519), who was influenced by both the work of Nicholas and that of Archimedes, employed the idea of infinitesimal to find 'the center of gravity of tetrahedron by thinking of it as made up of infinite number of planes' [Boyer p.93]

Concerning the notion of continuum, Thomas Bradwarline (c.1290 - 1349) asserted that a continuous magnitude is composed of an infinite number of indivisible elements, but it is not made up of such elements. William of Occam, who denied actual infinity, believed that a line does consist of points. In a similar manner, Nicholas of Cusa regarded the line as the unfolding of a point.

Also in this period, the study of motion and variation arose in mathematics. Discussions on the latitude of form (the variability of qualities) including latitudo uniform (uniform rate of change), latitudo difforms (nonuniform rate of change), latitudo uniformiter difformis (uniform rate of change of rate of change) and latitudo difformiter difformis (nonuniform rate of change of rate of change) appeared. The result of these notions led to the developement of infinite series. For instance, Richard Suiseth considered the following problem in his work known as Calculator (or the Liber Calculationum , 1337):

"If throughout half of a given time interval a variation continues at certain intensity, throughout the next quarter of the interval at double this intensity, throughout the following eighth at triple this, and so ad infinitum, then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or doubt the initial intensity" [Boyer p.76]

Having realized that the sum of the above infinite series $1/2 + 2/4 + 3/8 + \ldots + n/2^n + \ldots$ is finite (= 2), Suiseth arrived at a paradoxical result which was similar to that of Zeno. Suiseth wondered how can an infinite rate of change produce a finite average rate of change? Finally, he appealed to tedious verbal arguments based on the intuition of uniform rate of change to demonstrate the convergence of the infinite series (For detail, see Boyer p.77 - p.78).

On the other hand, Nicole Oresme (1323 - 1382) associated the physical change with the geometrical interpretation. He introduced a horizontal line (longitude) to represent the time and the vertical line (latitude) to represent an instantaneous velocity. Having studied this graphical representation and the problem of Suiseth, he explained the convergence by saying that 'the total distance covered would be four times that covered in the first half of the time' [Boyer p.86]. Furthermore, he was confused by the problem of the indivisible and continuum and agreed with Aristotle that every velocity presisted for a time.

Throughout the Middle Ages, discussions concerning infinite, infinitesimal and continuity emphasized philosophical considerations rather than geometric rigor. However, these works were discursive and dialectic. This phenomonon could be explained in terms of the different images that were created by different mathematicians, according to their own conceptions of the world. These new formed images were not clear enough to develop rigorous reasonings. However, the study of motion and its graphical representation not only led to the ideas of variability and functionality but also stimulated the creation of algebraic symbols in mathematics.

2.4 Phase 3: A century of Anticipation

This period started at the end of the Middle Ages and ended with Newton and Leibniz. The three major contributions in this period were:

(1) The generalization of the concept of number

(2) The improvement of algebraic notations

(3) Using mathematics as a tool to solve real world problem

Although operations on irrationals were carried out freely at this period, people were still confused as to whether irrationals were really numbers. For instance, Michael Stifel (1486 - 1567), after expressing irrationals in the decimal notation, said:

> "Since, in proving geometrical figures, when rational numbers fail us, irrational numbers take their place and prove exactly those things which rational numbers could not prove... we are moved and compelled to assert that they truly are numbers, compelled that is, by the results which follow from their use - results which we perceive to be real, certain, and constant. On the other hand, other considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to numeration [decimal representation]... we find that they flee away perpetually, so that not one of them can be apprehended precisely in itself... . Now that cannot be called a true number which is of such a nature that it lacks precision... . Therefore, just as an infinite number is not a number, so an irrational number is not a true number, but lies hidden in a kind of cloud of infinity" [Kline p.251]

As a matter of fact, there were different images of irrationals. Blaise Pascal (1623 - 1662) and Isaac Barrow (1630 - 1677) asserted that the existence of irrationals should depend on the continuous geometrical magnitude and those operations applied on irrationals should be justified by Eudoxian theory of magnitude. Simon Stevin (1548 - 1620) accepted irrationals as numbers which could be approximated by rationals. John Wallis (1616 -1703) and Pene Descartes (1596 - 1650) admitted irrationals as abstract numbers which could be used to represent continuous magnitudes.

Besides, negative and imaginary numbers were also recognized at that time. The former were known in Europe through the Arab texts and the latter were obtained by extending the arithmetic operations by method of completing square. Although some of the contemporary mathematicians did not consider these quantities as legitimate numbers, they did accept the operational procedures. The emphasis on the operational methods led to the development of Algebra.

Prior to the sixteenth century, algebra was purely rhetorical, except that of Diophantus (fl. c.250). In response to the rapidly expanding scientific demands on mathematics, better symbolic notations were introduced by Vieta (1540 - 1603) as well as many other mathematicians like Thomas Harriot (1560 - 1621), Descartes, ... etc. Undoubtedly, an ease of notation not only made the mathematical work more easy to communicate but also led to new discoveries.

First, many mathematicians employed the idea of infinitesimal and modifications of the Eudoxian method of exhaustion to arrive at some significant results. For example, $\int_0^a x^n\,dx = a^{n+1}/(n+1)$, for all positive integral value n, was obtained by Bonaventura Cavalieri (1598 - 1647), Giles Persone de Roberval (1602 - 1675) and Blaise Pascal independently. Moreover, this result was generalized by Pierre de Fermat (1601 - 1665) and Evangelista Torricelli (1608 - 1647), separately, for all rational values of $n \neq -1$. As we shall see later, the methods these mathematicians used to derive the above result were different and so were their images of the underlying notions.

Secondly, as the pressures for the mathematization of science increased, there was a need to discuss change and motion in mathematics. The search for a quantitative statement to describe motion gave the birth of the idea of function. Galileo Galilei (1564 - 1642) was the first mathematician to express his idea of functional relationships in terms of the language of proportion. He said:

"The spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time intervals employed in traversing these distances" [Kline p.338]

During this period, the study of function was equivalent to that of curve. Mathematicians like Roberval, Barrow regarded a curve as the path of a moving point. Owing to the analytic geometry of Descartes and Fermat, a curve could also be represented by an algebraic equation. Yet, neither the idea of function was clearly formed nor the distinction between algebraic and transcendental functions was drawn. In the work of Descartes, he viewed a transcendental function as a mechanical curve. Towards the end of this period, James Gregory (1638 - 1675) gave explicitly the definition of function as 'a quantity obtained from other quantities by a succession of algebraic operations or by any other opertaion imaginable' [Kline, p.339]. The last phrase of Gregory referred to the operation of 'passage to the limit'.

In order to study these mathematicians' images towards the notions of numbers, limit, function, ... etc., a short summary is edited as following:

- Simon Stevin employed the idea of the Eudoxian method to rediscover some results of Archimedes (which were not yet known at that time) without adding the reductio ad absurdum arguments. Instead, he used a method called 'demonstration by numbers' (For detail, see Boyer p.102 - p.103) to argue that if the difference between two numbers could be made as small as desired, there could be no difference. In this method, Stevin tried to establish that numbers can be infinitely divisible as geometrical magnitudes. From the point of modern view, his work marks a significant step towards the notion of number and limit. Yet, he neither thought of a sequence of numbers obtained by his method as carried out to an infinite number of terms nor did consider his method as a mathematical demonstration.

- Luca Valerio (1552 - 1618) anticipated the geometrical form of the limit concept through methodizing the Eudoxian method. He assumed without proof that if the inscribed and circumscribed figures of a given curve differed by any given quantity, then there could be no difference between the area of the curve and the area of the inscribed or circumscribed figure. However, he did not regard the area of the curve as the limit of the area of either the inscribed or circumscribed figure. Basically, his images towards the notion of limiting process are geometrical.

- In order to abridge the gap between the curvilinear and rectilinear, Johnnas Kepler (1517 - 1630) considered continuous magnitudes as composed of an infinite number of infinitesimal elements of the same or lower dimension. For instance, the sphere consists of an infinite number of infinitestimal cones whose vertices are the center of the sphere and whose bases make up the surface; the cone is made up of an infinite number of infinitely thin circular laminae. Having resorted to a vague idea of 'law of continuity', he asserted that there is no essential difference between an infinitesimal area and a line, or between the finite and infinite.

- Galileo Galilei admitted the possibility of categorematic infinity. Yet, he felt that the notion of infinity was hard to grasp intuitively. In his <u>The Dialogue on Two New Science</u> , he observed:

"We have one line segment longer than another, each containing an infinite number of points"

Furthermore, he questioned whether the infinite number of points on the longer line segment could be compared with those in the shorter segment. At the other time, by establishing the difference between the rules for the finite and those for the infinite, he shifted his attention from infinite magnitudes to infinite aggregates. Owing to this shift, he thought of continuous magnitudes as the aggregation of indivisibles in the state of fluids.

Besides, Galileo employed the infinitesimal considerations and the geometrical demonstrations of Oresme to discover the functional relationships between distance and time of a body falling from the rest with a uniformily accelerated motion. He also tried to answer Zeno's paradoxes by regarding rest as an infinite slowness.

- Bonaventura Cavalieri employed the notion of indivisible to develop a geometrical method of demonstration (For detail, see Boyer p.119 - p.120) which led to the result

$$\int_0^a x^n dx = a^{n+1} / (n+1)$$

for all positive integral value n. Under the influence of

Galileo, he regarded continuous magnitudes as generated by the flowing of indivisibles. He expressed that indivisibles could be considered as having no thickness, but also could be small elements of area and volume if one wished. His attitude towards infinity was described as 'agnosticism'. For him, infinity was an auxiliary notion which appeared only at certain stages of the procedures. Throughout his work, the idea of limit was completely concealed.

- Evangelista Torricelli fully recognized that Cavalieri's method of indivisible was useful for discovering new results but lacked mathematical rigor. As a compromise, he added a geometrical demonstration originated from Valerio to the results obtained from the method of indivisibles. By doing this, he also improved Valerio's idea concerning the notion of limit. He asserted that if the inscribed and circumscribed figures differed by any given quantity, then the difference between the area of the curve and either the inscribed or circumscribed figure would be smaller than the given quantity. Basically, his view towards indivisibles resembled to that of Democritus. He asserted:

"... in the case of unequal lines, the number of points on each was the same, but that the points themselves were unequal" [Boyer p.134]

Besides, Torricelli employed the idea of instantaneous direction to develop a dynamic conception of tangents. He determined the tangent of a curve at a given point by considering the resultant direction of horizontal and vertical velocities at that point. Yet, his notion of tangent was based on the classical Greek, that is, a line touches a curve at a single point. He neither used the idea of limit to define instantaneous velocity nor considered the tangent as the limit of secants.

- Gregory of St. Vincent (1584 - 1667), who was familiar with the work of Stevin and Valerio, asserted that a continuous magnitudes could be exhausted by allowing a <u>varying</u> subdivision to continue to <u>infinity</u> . This dynamic consideration led not only to advance a step towards the idea of variability but also to the notion of the limit of an infinite geometrical progression. He wrote:

> "The terminus of a progression is the <u>end</u> of the series to which the progression <u>does not attain</u> , even if continued to infinity, but to which it can approach more closely than by any given interval." [Boyer p.137, my underline]

Moreover, he was also aware that the paradoxes of Zeno could be explained in term of the sum of infinite series. However, he failed to give a satisfactory resolution.

- Andreas Tacquet (1612 - 1660) asserted that continuous magnitudes are composed of homogena - that is, part of the like dimension. For instance, a solid consists of small solids. Thus, he considered neither that a line is made up of points nor that it is generated by way of motion.

- Giles Persone de Roberval worked out his method of indivisibles by using infinitesimal consideration as well as arithmetical manipulation. He asserted that a line, a surface, a solid is actually made up of an infinite number of little lines, surfaces and solids respectively. After dividing a given figure into small parts, he continued to decrease in magnitudes and performed all the calculations arithmetically by establishing an association between numbers and the little lines. However, when the moment came to draw conclusions, he appealed to his geometrical intuition. He argued that a line has no ratio to a cube because an infinity of lines only make a square, and a cube has an infinity of squares; thus, adding or subtracting a single square has no effect. Consequently, he neglected higher order infinitesimal in his calculations.

Besides, he considered a curve as the path of moving point and asserted that direction of motion will give that of the tangent. Although the motions involved in his method of tangent were different from that of Torricelli, they both employed the idea of composition of movements.

- The mathematical work of Blaise Pascal was accomplished in two periods which were characterized by different views on the nature of infinitesimal.

In the first period, he applied the theory of infinitesimal to arithmetic triangle to obtain results similar to that of Cavalieri. In his work, he neglected the quantities in lower dimension by arguing that a single point added nothing to the length of line because the former is indivisible with respect to the latter. Furthermore, he compared his indivisible of geometry to the zero of arithmetic (that is, he regarded the indivisible quantities as nothing but zeros) and defended his omission of infinitely small quantities with theological arguments.

In the second period of Pascal's mathematical activity, he not only tended to avoid arguments involving infinitely small quantities but also changed his view towards infinitesimal. Prehaps, under the influence of Roberval, he also asserted that continuous magnitudes are composed of the like elements. Besides, he appealed to the mysterious notion - the infinitely great and small which are considered to be complmentary (for instance, 100,000 and 1/100,000) as well as to be incomprehensible, in order to establish the transition between finite and infinite.

Overall, Pascal's work was dominated by the theory of numbers and classical geometry. Making a connection between numbers and geometrical magnitudes, he arrived at many results in integral calculus. However, he failed to i terpret them as the limit of infinite series due to his underestimation of the value of algebraic technique.

- Following the algebraic methods of Viete and the graphical representation of variables by Oresme, Pierre de Fermat associated each curve with an equation and assumed that every line segment corresponded to some number. Under such an assumption, he used the idea of geometrical and numerical infinitesimal to develop a method for finding ma imum and minimum values (For detail, see Boyer p.155 - p.156). As an application of this method, he also determined the tangent of a parabola at a given point. The idea employed in his method was the following: he first changed the variable slightly in the begining and then let the change become zero. Using the modern notation in calculus, Fermat first considered $x + \Delta x$, then he let $\Delta x = 0$ (not as nowadays where we let $\Delta y \rightarrow 0$). Although his method worked out nicely, he failed to explain what is the relation between the method of maxima and that of tangent. As Boyer commented, the reason is that Fermat was thinking in terms of equations and the infinitely small rather than of functions and limiting idea.

- The idea of infinitesimal appeared only in the early work of Rene Descartes. For example, he used the phrase 'first instant of its movements' to describe the force drawing a body. After the publicaiion of Geometrie in 1637, he had a tendency to replace the idea of infinitesimal by algebraic and mechanical conceptions. In criticizing Fermat's method of tangent, he established his own method of tangent which is purely algebraic. Moreover, there was no notion of limit and infinitesimal involved. However, from a modern point of view, his procedures could be interpreted in terms of defining the tangent as the limit of secants.

- Having realized the powerful idea of infinity and infinitesimal employed in the work of Cavalieri and Fermat, John Wallis sought the independence of arithmetic from geometry. He regarded a plane as an infinite number of infinitely small parallelograms which could be considered as lines. He further associated numbers with these magnitudes and performed arithmetic operations. Besides, he introduced the symbol "∞" for infinity and "$1/\infty$" for infinitely small quantity. More than this, he not only allowed to use categorematic infinity in arithmetic but also manipulated it as a number throughout his work <u>De sectionibus Conicis</u> . He also drew an analogy between the properties of the infinite and those of finite. For instance, from

$$(0 + 1) / (1 + 1) = 1/3 + 1/6;$$
$$(0 + 1 + 4) / (4 + 4 + 4) = 1/3 + 1/12;$$
$$(0 + 1 + 4 + 9) / (9 + 9 + 9 + 9) = 1/3 + 1/18;$$
... and so on

he observed that the greater the number of terms, the closer the ratio approximated to 1/3. He argued that as the procedure continues to infinity, the difference between 1/3 and the ratio will completely vanish. Thus, the ratio for an infinite number of terms is 1/3. Here, the notion of limit of infinite sequence was indicated vaguely[1]

- Through studying the problem of quadratures, James Gregory obtained a convergent series by constructing a sequence of inscribed and circumscribed polygons to a circle. He asserted that the limit of this convergent series could be considered as <u>the last polygon</u> of each series and this last polygon will give the area of the circle. Considering that the area of a circular sector is a function of the radius and the chord, he upheld that the 'passage to the limit' is an independent arithmetical operation in the definition of function. Besides, he also noted that this new arithmetical operation may create new numbers as well as transcendental functions.

- Thomas Hobbes (1588 - 1679) criticized Wallis' work due to his own belief that mathematics is an idealization of sensory perception. He upheld that the infinitely small magnitude is the smallest possible line, plane or solid. In the connection with the motion, he introduced the concept of the <u>conatus</u> to clarify the idea of motion at a point as the beginning of the motion in an infinite small interval which is less than any given interval. Besides, he also tried to answer Zeno's paradoxes by arguing that 'when the time interval has disappeared, the tendency toward motion remains' [Boyer p.179].

Unfortunately, none of these attempts succceeded mathematically. As Boyer commented, the notions involved here were intellectual rather than empirical.

- Another critic of Wallis was Isaac Barrow. Barrow maintained that the conception of numbers should be based on the geometrical interpretation of continuous magnitudes. He viewed continuous magnitudes either as continuous flow of one instant or point; or as an aggregation of instants or points. He expressed that there is no difference in regarding a line as composed of points or of indefinitely small linelets.

Furthermore, he thought of a tangent of a curve not only as the prolongation of one of the infinitely many lineal elements of which the curve might be assumed to be composed but also as the directior of motion of a point which, by moving, generated the curve.

In summary, mathematicians in the 'century of anticipation' developed a lot of methods to solve calculus problems. Athough most of them employed the idea of infinitesimal, their images towards the underlying notions were considerably diversed. This can be seen through their explanations of the subject.

The evolution of the notion of limit and tangent could be traced from the work of Stevin and Valerio to that of Gregory, from the method of Torricelli to that of Descartes respectively. On the other hand, the notion of function was far from being formulated. Yet, from Galileo to James Gregory, the definition of function proceeded towards a greater abstraction.

Another feature of this period was that ad hoc methods were widely used to discover new results. Now and then, the neglect of rigorous foundations of these methods caused much controversies. However, these methods did give a satisfactory preparation to the birth of calculus.

2.5 Phase 4: Newton and Leibniz

Both Sir Isaac Newton (1642 - 1727) and Gottfried Wilhelm Leibniz (1646 - 1716) are credited as being the inventors of the calculus. Their contributions mainly lay in the recognition of the inverse relationships between the problems of quadrature and those of tangent as well as developing the systematic procedures to deal with the similar problems in general. Athough both of them acquired relevant knowledge from their predecessors in order to establish the algorithmic procedures for the subject, their points of view and modes of presentation are quite different.

Newton, who followed the line from Archimedes, Galileo, Cavalieri to Barrow, outdid his teacher, Barrow, in realizing the significance of Wallis' arithmetization. Newton's conception of number included irrational ratios and negative ratios. His mode of thought could be divided into three different stages according his three expositions of the calculus.

As early as 1665 - 66, Newton started to formulate his calculus after he had attended Barrow's lectures and had discovered the binomial theorem. In his monograph (1669), De Analysi per Aequationes Numero Terminorum Infinitas , he determined the area under a curve by solving the inverse problem, that is, he supposed that for the abscissa x and the ordinate y, the area z under the curve is given by $z = a\, x^{n+1} / (n + 1)$ and derived that the curve will be $y = a\, x^{n}$.

In his procedures, he first employed the idea of infinitely small, both geometrically and analytically, to establish the infinitesimal increase in x, denoted by o and the corresponding augmentation in area, denoted by oy. He obtained

$$z + oy = na(x + o)^{(n+m)/n}/(n + m).$$

Next, he applied the binomial theorem to the right hand side, removed the terms without o, divided through by o, neglected all the terms still containing o and finally arrived at the result $y = ax^n$.

Examining Newton's procedures closely, we notice that he not only derived the general method for finding the instantaneous rate of change but also showed that the summation of infinitely small areas could be accomplished by reversing the process of finding rates of change. In such a way, he made the idea of determining rates of change fundamental in his calculus. However, he did not explain clearly the notions involved in his procedures. As Boyer commented:

"It will be noticed that although the work of Newton contains the essential procedures of the calculus, the justification of these is not clear from the explanation he gave. Newton did not point out by what right the terms involving powers of o were to be dropped out the calculation, ... His contribution was that of facilitating the operations, rather than of clarifying the conceptions. As Newton himself admitted in this work, his method is 'shortly explained rather than accurately demonstrated'." [Boyer p.192 - p.193]

In order to remove the harshness from the doctrine of indivisibles, Newton changed his point of view towards infinitely small in his second exposition of calculus, Methodus Fluxionum et Serierum infinitarum , which was written about 1671, but not published until 1736. In this work, he conceived of geometrical magnitudes as generated by continuous motion of points, lines, planes, rather than as aggregates of infinitesimal elements as in the De Analysi . Subject to this change, a variable quantity is called a fluent, denoted by x, y, z and a rate of changed of the fluent is said to be a fluxion, denoted by $\dot{x}$, $\dot{y}$, $\dot{z}$. Due to these new terminologies, the fundamental problem of the calculus was stated much clearly, that is: given the relationship between two fluents, find the relation between their fluxions, and conversely. Moreover, slight modification in the earlier exposition was also given. Basically, he used $\dot{x}o$ and $\dot{y}o$ to represent the indefinitely small increments of fluent x and y respectively. In finding the relation between the fluxion $\dot{x}$ and $\dot{y}$ for $y = x^n$, he first established

$$y + \dot{y}o = (x + \dot{x}o)^n$$

and then proceeded as in his earlier work to obtain

$$y = nx^{n-1}\dot{x}.$$

Although Newton's view of infinitely small changed from the static indivisible of Cavalieri to the dynamic movement of Galileo, he could not offer better clarification for neglecting the terms containing o. His explanation was that the infinitely small increment of quantities could be considered as zero in comparsion with the one retained. After all, that the justification of neglect of infinitely small depended on the notion of limit was not yet clearly formed in his thought.

Besides, Newton also noticed that in his theory of fluxion, the values of fluxions themselves are not as meaningful as their ratio. Prehaps, due to this observation as well as his dissatisfication with the explanation for discarding the terms containing o, Newton formulated another approach - the method of prime and ultimate ratio - in his third exposition, Tractatus de Quadratura Curvarum, written in 1676 but published in 1704. In this treatise, he attempted to abandon all the arguments involving infinitely small. In determining the ratio of fluxions $\dot{y}/\dot{x}$ for $y = x^n$, he first replaced $.^n$ by $(x + o)^n$ where o stands for the increment in y, then obtained the increase of x^n,

$$nox^{n-1} + [(n^2 - n)/2]o^2 x^{n-2} + \ldots$$

by applying binomial theorem to $(x + o)^n$. At this stage, he considered the ratio of o and the increase of x^n instead of the infinitely small argument which led to the omission of the terms containing o. In doing so, he arrived at the ratio 1 to $nx^{n-1} + [(n^2 - n)/2]o\,x^{n-2} + \ldots$. Now, by allowing the increment o to vanish, the resulting ratio 1 to nx^{n-1} would be the ultimate ratio of 'evanescent increment' or the prime ratio of 'nascent augments'.

Yet, the above argument could not successfully protect Newton from the criticism of the infinitesimal quantities. As Confrey [2] commented:

> "The basic argument being put forth was as follows: Prior to vanishing there is no ultimate ratio and once the fluent has vanished there is no ratio at all" [Confrey, p.110]

Newton himself seemed also realized the weakness for he included all three approaches in the <u>Principia Mathematica Philosophiae Naturalis</u> of 1687.

Another contribution of Newton was the use of infinite series in connection with the binomial theorem. With the aid of infinite series, his fluxional method could be applied to a large class of functions in comparsion with tho_e handled by his predecessors. More than this, the series representation of function also led to broaden James Gregory's notion of function in the following century.

However, Newton himself still viewed function as the path of moving point due to his dynamic considerations towards the subjects.

Around 1673, Leibniz, who followed the thought from Democritus, Kepler, Fermat, Pascal to Christian Huygens (1629 - 1695), was also aware of the inverse relationships between tangents and quadratures through studying the work of Pascal. He noticed, on the one hand, that finding the tangents depended on the ratio of the difference in the ordinates and abscissas, as these became infinitely small. On the other hand, determining the quadratures, relied on the sum of infinitely thin rectangles which were formed by the ordinate and the infinitesimal intervals in the abscissas. Owing to this observation, he used symbol dx to represent the difference and $\int x$, later written as $\int x dx$, to represent the sum. These notations did facilitate his discovery of the rules as well as the development of algorithms.

Differently from Newton, Leibniz made the idea of finding 'difference' or 'differential of a quantity' as fundamental in developing his general procedures. Basically, he first established rules for finding 'difference' of xy, written as $d(xy)$. In the beginning, he wondered whether $d(xy) = dxdy$. Later, he considered

$$d(xy) = (x + dx)(y + dy) - xy$$

and arrived at $d(xy) = xdy + ydx + dxdy$. Observing that $dxdy$ was infinitely small in comparsion with xdy and ydx, he arrived at $d(xy) = xdy + ydx$. He went on to extend this result to $d(x^n) = nx^{n-1}dx$, for all positive integer n. By arguing that the summation is the inverse of determining a 'difference', he concluded that area under the curve $y = x^n$ should be $x^{n+1}/(n + 1)$. From this brief description of Leibniz's discovery, it can be seen how strongly Leibniz believed that a correct result would be obtained if the rules were properly applied. Due to this belief, he emphasized the algorithmic nature of his method and tried to define differentials of higher orders.

Yet, the justification of the omission of $dxdy$ in Leibniz's procedures caused much criticisms. In defending this, he appealed to the infinitely small consideration which resembled the doctrine of Robervai and Pascal, that is, a point is nothing in comparsion with a line. What Leibniz did was to translate this doctrine into algebraic form and concluded that by differentials of higher order could be neglected in comparsion with lower order.

In the early work of Leibniz, the notion of differentials were conceived as finite, assignable quantities. The omission of dxdy forced Leibniz to ponder about the transition from finite to infinitesimal and the nature of infinitesimal. Moreover, since the significance of the differentials lay in their ratios and not themselves, Leibniz had to consider the ratios of infinitesimals. In order to settle these questions, he thought of infinitely small differentials variously as inassignables, as qualitative zero, and as auxiliary variables throughout his later work.

In the beginning, he resorted to analogies in order to explain the infinitely small differentials. He considered his differentials as the momentary increments or decrements of quantities in Newton's work; and the infinitely small quantity as resembling a point to the earth. At another point, he considered the differential as a quantity less than any given quantity. In replying whether the differentials were assignables or not, he expressed that if one wished rigor, the 'assignables' could be replaced by 'inassignables'. However, he explained neither how the replacement of assignables, nor how the transition between finite and infinitesimal, could be justified.

Having failed to give clear explanation, Leibniz evoked the naive idea of continuity to abridge the transitions. He stated the law of continuity as following:

"In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included." [Boyer p.217]

Then, he applied this principle to explain his manipulation. He said:

"the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of nonevanescent quantities, and we finally come to the point where we apply our result to the ultimate case" [Boyer p.217 - p.218]

However, his attitude towards infinite and infinitesimal seemed to be somewhat uncertain in his later years. On the one hand, he expressed that all numbers are finite and assignables in his <u>Theodicee</u> . On the other hand, he wrote to Guido Grandi (1671 -1742) that:

"Meanwhile, we conceive the infinitely small not as a simple and abstract zero, but as a relative zero, ... , that is, as an evanescent quantity which yet retains the character of that which is disappearing." [Boyer p.218 - p.219]

Yet, at the other time, he explained that he believed neither truly infinite nor infinitesimal magnitude. Prehaps, the vacillations of views could be explained by the diverse images that were evoked when he tried to settle a resolution.

Leibniz conceived the area under a curve as the sum of an infinite number of infinitely narrow rectangles. Being unable to explain why the sum of an infinite number of infinitesimals might turn out to be finite, he could not answer the paradoxes of Zeno properly. He did use infinite series but he thought that the real goal should be to end up with finite terms. This led him to oppose the expansion of function into series. Maybe owing to this objection, his view towards function was somewhat different from Newton. He considered function as 'quantities that depended on a variable', or as 'any quantity varying from point to point of a curve'. Yet, he failed to distinguish clearly between independent and dependent variables.

As a result, the different views held by Newton and Leibniz may be explained by their manner of working, their mathematical experience as well as their personal tastes.

Newton, an empirical scientist, developed his fluxion in term of motion, whereas, Leibniz, a speculative philosopher, expressed his differential in thought of 'monad'. It is interesting to note that both of the inventors tried hard to reconstruct their images to accommodate the new challenges. Yet both of them failed to clarify the underlying notions in their methods.

2.6 Phase 5: The period of Indecision

Throughout the whole of eighteenth century, the foundation of the method of fluxions and the differential calculus was questioned by many mathemacicians and philosophers. Although they might not have contributed to the subject directly, their criticisms did have a positive impact. Thus, it is interesting to study their images towards the subject as well. Another characteristics of this period was a number of various resolutions proposed for the rigorous foundation of the calculus.

The most significant attack was made by the philosopher, George Berkeley (1685 - 1753), in his The Analyst in which the subtitle is: Or a Discourse Addressed to an Infidel Mathematician Wherein It Is Whether the

Object, Principles and Inference of Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, Than Religious Mysterious and Points of Faith. 'First Cast the Beam Out of Thine Own Eye: and Then Shalt Thou See Clearly to Cast Out the Mote Out of Thy Brother's Eye.'

In this work, Berkeley did accept the validity of the results but cri''cized the lack of legitimate arguments in the methods of Newton and of Leibniz. Basically, Berkeley argued that the confusion came from Newton first considering the increment not to be zero and latter allowing it to be zero, in order to arrive at the results. On the other hand, Leibniz's method for finding the tangent by means of differentials actually gave the secant rather than the tangent. The valid results were obtained only because of the twofold mistakes. Moreover, Berkeley himself expressed that mathematics should deal with conceivable ideas. This view led him to think of geometrical magnitudes as composed of a finite number of indivisible 'minima sensibilia' and reject instantaneous velocity as a mathematical abstraction.

In England, James Jurin (1684 - 1750) refuted Berkeley's attack by resorting to the idea of infinitesimal and of unassignable magnitude. He upheld that the magnitude of a variable quantity was not all determined but perpetually changing till it became zero. Moreover, the ultimate ratios of vanishing quantities was the last ratio. In modern terminology, Jurin asserted that the limit of a sequence s_n is the last term of s_n.

Dissatisfied with Jurin's argument, Benjamin Robins (1707 - 1751) made his own attempt to defend the doctrine of Newton. He upheld that the variable quantities in fluxional method should be interpreted in terms of prime and ultimate ratios. Contrary to Jurin, he asserted that variable quantities need not attain the limit, but approach it within any degree of nearness through a perpetual augmentation or diminution.

Beside the Jurin - Robins controversy, there were many other attempts to clarify the foundation of the calculus in England. Colin Maclaurin (1698 - 1746) defended Newton's fluxional method by way of geometrical rigor. He banished the infinitely small as inconceivable and felt that the idea of instantaneous velocity might be introduced in geometry. On the other hand, Brook Tayor (1685 - 1731) sought rigor in terms of an arithmetical procedure. He dealt with finite increments but could not explain their transition to fluxions.

Overall, the British mathematicians failed to give the rigorous interpretation of the calculus, partly due to the confusion between fluxions and moments and partly because of their vague idea of the limit and function concept. As Boyer commented:

"... the confusion in the interpretation of the limit concept was due to the lack of a clear distinction between questions of geometry and those of arithmetic, and to the absence of the formal idea of a function." [Boyer p.235 - p.236]

Meanwhile on the continent, the criticisms of Leibniz's work came from Bernard Nieuwentijdt (1654 - 1718). Mainly, he questioned about the nature of infinitely small and the convergence of an infinite series. Fortunately, the continental mathematicians paid little attention to the lack of sound foundation. On the other hand, there was a tendency to link the calculus with the concepts of limit and of function. The results of this trend were: (1) the emphasis on arithmetical and algebraic manipulation in the procedures; (2) the employment of the concepts of limit of function as a base to develop the theory of calculus. Under such an environment, the differential calculus grew more rapidly in the continent than in England. A summary of continental mathematicians' images of the subject is as follows:

- Although James Bernoulli (1655 - 1705) gave a caution not to use the pseudo-infinitesimal, his attitude towards infinitely small differential was uncertain. He thought of differentials as variables and as imaginary symbolism. He was aware of the Euclidean axiom - 'If equals are taken away from equals, the results are equal' - failed to hold for infinitely small quantities.

- John Bernoulli (1667 - 1748), James' brother, asserted the existence of infinitesimal by using the reciprocal relationship between the indefinitely small and indefinitely great. Since the sequence of natural numbers is infinite, the infinitesimal exists. Moreover, he put forth Leibniz's view about the notion of function. He defined a function as quantities formed by algebraic and transcendental expressions with variables and constants.

- Under the influence of Wallis and Bernoulli, Leonhard Euler (1707 - 1751) was dissatisfied with the geometrical interpretations in the calculus. Thus, he sought the formal theory of function as the basis of the subject.

Besides the introduction of the notation $f(x)$, he defined a function as a <u>single</u> analytical expression formed in any manner from a variable quantity and constants. Soon, he found through the study of the vibrating-string problem that this definition was insufficient. Consequently, he generalized his own conception of function as:

> "If some quantities depend on others in such a way as to undergo variation when the latter one varied, then the former are functions of the latter" [Kline p.506]

Under this new definition, he admitted that one function may have different expressions in different domains. It is also interesting to remark that Euler thought that a function is discontinuous at points where the expression changes form. For him and his contemporaries, continuous functions have to be analytic expressions except for an occasional discontinuity.

Furthermore, he not only distinguished between algebraic and transcendental functions; between explicit and implicit functions; and between single-valued and multiple-valued functions but he also developed a systematic method to study their differentials and integrals.

On the other hand, he asserted that an infinitely small or evanescent quantity was simply zero and so were the differentials dx and dy. Thus, the methods in calculus were simply the process of finding the value of the expression 0/0 which represented the ratio of evanescent increments dy/dx. In such a way, he accepted the existence of zero quantities whose ratios are finite number without further justification. Also owing to this view, he regarded the integral as the inverse of the differential rather than a process of summmation.

With respect to the infinite, he admitted ∞ as a number and asserted that there were different orders of infinity. For instance as $dx = 0$, $a/dx = \infty$, hence $a/(dx)^2$ will be infinite of the second order, and so on.

- Meanwhile, Jean le Rond D'Alembert (1717 - 1783) asserted that the basis of the calculus lay in the idea of limit. He defined a quantity to be the limit of another if the latter could approach the former within any degree of nearness so that the difference between them wa. osolutely inassignable. Under this definition, the varying quantity would <u>never reach</u> its limit. As a matter of fact, he was only interested in those variables which were <u>monotonic</u> within the neighbourhood under consideration. Besides, he denied the existence of both the infinity and infinitesimal and regarded function as a single analytic expression formed by the process of algebra and the calculus.

- Another approach towards the foundation of the calculus was proposed by Joseph Louis Lagrange (1736 - 1813). Not satisfied with the use of infinitesimal and limit, he defined differentials and fluxions in terms of the coefficients of the Tayor series under the assumption that every function could be represented by a series expansion. In doing so, he introduced the notion of the derived function and its notation f′x. Under such consideration, he viewed a function as ′any expression useful for calculation in which the variable entered in any manner whatsoever′. In such a way, he broadened the notion of function by accepting power series as legitimate function. Yet, he failed to recognize the problem of convergence. In his mind, the expression did not really go on for ever!

- Simon L′Huilier (1750 - 1840) agreed with D′Alembert′s view towards the calculus. He made the idea of limit fundamental. His notion of limit was based on the ideas which were presented in the method of exhaustion. Consequently, he came up with the conclusion that the variable is always less or greater than its limit but does not oscillate. Furthermore, he asserted:

"If a variable quantity at all stages enjoys a certain property, its limit will enjoy the same property " [Boyer p.256, my underline]

Besides, L'Huilier considered the ratio of increments as a single variable whose limit is a single number (the derivative). Thus, in his work, the derivative, not the differential, becomes essential. Moreover, he shared D'Alembert's rejection of the existence of both the infinity and infinitesimal.

- Louis Arbogast (1759 - 1803) supported Lagrange's views towards the calculus. Through the study of whether any function can be represented by series expansion, he made the distinction between continuity and contiguity, discontinuity and discontiguity. His views were described by Grattan-Guinness [9] as:

"'Discontinuous' curves are so because they are defined by different laws (or functions) or by no law at all, but they are continuous to the modern view as they do join up. Our conception of discontinuous functions, functions with jumps, is called 'discontiguous', and for Arbogast continuity is broadened to include contiguity, connectedness - including, one presumes, vertical lines joining jumps." [Grattan-Guinness, p.104]

- Sylvestre-Francois Lacroix (1765 - 1843), on the one hand, tried to interpret the method of series in terms of the method of limit. On the other hand, he used the idea of infinitesimal whenever the method of limit could not provide a rigorous argument. Also under the influence of Leibniz, he confused the coefficient of the Tayor series with a quotient of zero. Besides, he introduced a broader notion of a function. Basically, every quantity whose value depends on one or several others is called a function of the latter. Explicitly, he also pointed out that it was not necessary to know the operations between those quantities. This was, in fact, a significant improvement for he viewed functionality as a matter of relationship, not as a formal representation.

Throughout this period, many other mathematicians searched for the foundations of the subject. Mainly, their works were based on that of Euler, of D'Almbert and of Lagrange. Questioning the validity of Lagrange's assumption that every continuous function (in modern sense) can have a series expansion, brought up the connection between the method of series and that of limit. The gradual evolution of these notions gave a suitable preparation for the next period.

Furthermore, the applications of the calculus enriched the subject itself and new images were created through these applications. Consequently, accommodating these new images into the existing notions led to another level of abstraction.

2.7 Phase 6: The rigorous formation

The need for the formalization of the notions of function, limit, infinity and continuity grew continually not only through studying the calculus but also other subjects like the theory of wave and heat. Mainly, the questions were: (1) What is the relationship between 'prime and ultimate ratios' and 0/0? (2) What is meant by a function in general and by a continuous function in particular? (3) What functions could be represented by the infinite series? Mathematicians like Bernhard Bolzano (1781 - 1848), Augustin-Louis Cauchy (1789 - 1857) and Karl Weierstrass (1815 - 1897) made efforts to answer these questions.

Not satisfied that non-trivial proofs in the calculus, for instance, the intermed'ate value theorem, depended on the considerations derived from spatial intuition, Bernhard Bolzano formulated the definition of continuity of a function of one real variable. Describing th: relationships between limit and continuity of a function, he wrote [18]:

> "According to a correct definition, the expression that a function f(x) varies acc Jing to the law of continuity for all values of x inside or outside certain limits means just that: if x is some suc.ı value, the difference f(x + w) - f(x) can be made smaller than any given quantity provided w can be taken as small as we please." [Russ p.162]

Then, Bolzano interpreted the derivative in terms of limits of ratios of finite differences which basically resembled that of L'Huilier. Viewing the quotient of finite differences as a single function, he went on to explain that this function might have a limiting value at a point where the value of the function itself was 0/0. Besides, he was aware that the continuity did not guarantee the existence of derivative.

By pointing out that Lagrange's method of series neglected the question of convergence, Bolzano gave the criterion of the (pointwise) convergence of an infinite series by using the idea of limit. Even though his idea of convergence was significant with respect to the modern definition of real numbers, he failed to distinguish between continuum and denseness. For instance, the rational numbers posses the property of denseness but do not form a continuum.

Bolzano's views on infinity resembled those which had been used by other contemporary mathematicians, that is, he rejected both the existence of infinitely large and infinitely small magnitudes. Owing to this view, he regarded, on the one hand, that the integral was the inverse of derivative; on the other hand, he remarked that there might be an actual infinity with respect to aggregation. However, the impact of his work was very small becasue it remained unnoticed for more than a half century.

In Bolzano's work, he gave a description of the nature of limit rather than a formal definition. It was Augustin-Louis Cauchy who gave a clear arithmetical definition. He started with the definitions of variable and of function which were:

"One calls a quantity which one considers as having to successively assume many values different from one another a variable" [Kline p.950]

"When variable quantities are so joined between themselves that, the value of one of these being given, one may determine the values of all the others, one ordinarily conceives these diverse quantities expressed by means of the one among them, which then takes the name independent variable; and the other quantities expressed by means of independent variable are those which one calls functions of this variable" [Kline p.950]

In order to integrate the notion of function into other branches of mathematics, for instance, the theory of heat, Cauchy admitted explicitly that an infinite series could be used to represent a function. However, a function might not be an analytic expression. Moreover, he asserted that a function has to be single-value. Besides putting an emphasis on the relations between variables in the definition of function, Cauchy also gave the definitions of limit as well as infinitesimal in terms of variables. He defined both of these notions as following:

"When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this _last_ is called the limit of all the others." [Kline p.951, my underline]

"One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge to the limit 0"

[Kline p.951]

A variable quantity possesses the above quantity was called infinitesimal. He continued to define the infinitely large as 'a variable quantity becomes infinitely large when its numerical value increases indefinitely in such a manner as to converge to the limit ∞ ' [KLine p.951]

It is interesting to note that Cauchy tried to clarify both the notion of infinitely small and large in terms of his notions of limit and variability. Owing to this veiw, he accepted only the potential infinity.

Having broadened the notion of function and established the notions of limit, infinitesimal and infinitely large, Cauchy started to formulate the concept of continuity and that of derivative. Since Euler, the continuity of a function had depended on whether it could be expressible by means of a single equation. Disagreeing with this approach, Cauchy interpreted the notion of continuity in terms of limiting idea. He said:

"The function $f(x)$ is continuous within given limits if between these limits an infinitely small increment i in the variable x produces always an infinitely small increment, $f(x + i) - f(x)$, in the function itself" [Boyer p.277]

Basically, his idea of continuity resembled that of Bolzano. However, it is interesting to note that each of their formulation came from different motivations. Yet, they both ended up with a definition in terms of local consideration.

Cauchy also defined the discontinuity of a function as follows: a function is discontinuous at a point if it is not continuous in every interval around that point.

As Bolzano, Cauchy defined the derivative as the last ratio of infinitely small increments [f(x + i) - f(x)]/i when i approaches zero. He then expressed the differential in terms of the derivative. Although Cauchy formulated the precise definition of both derivative and continuity, he failed to make the distinction between them. He believed that continuous function must be differentiable except at isolated point such as x = 0, y = 1/x.

Since the invention of calculus, two different views on integral existed, namely, as a process of summation and as the inverse of the differential (or fluxion). It was

Cauchy who introduced the notion of the (definite) integral as a limit of a sum. Thus, the derivative and the integral become two independent concepts. He was also the first mathematician who presented a rigorous demonstration of the fundamental theorem of calculus.

After the notion of limit became essential, Cauchy went on to attack the problem of convergence. In his view, a series was convergent 'if, for increasing values of n, the sum s_n approaches indefinitely a certain limit s, the limit s in this case being called the sum of the series' [Boyer p. 281]. Cauchy pointed out that an infinite series could have a sum only if its limit existed and that the solution of Zeno's paradoxes lay in the concept of limit.

Yet, two of Cauchy's misconceptions are worthy to look at. First, he shared a similar view with L'Huilier on that the limit preserved the properties possessed by its variable quantities. Thus he thought that the limit of a sequence of continuous functions is continuous. Secondly, there was a circular reasoning in his theory of real numbers. Basically, he defined the irrational numbers in terms of the limiting idea, that is, as the limits of sequence of rational numbers. The circularity comes from the fact that the notion of limit presupposes the notion of real numbers.

Unhappy with the phrases 'approach indefinitely', 'as little as one wishes', 'last ratios of infinitely small increments' in Cauchy's dynamic approach, Karl Weieratrass tried to remove all traces of motion by formulating his 'static' approach. First, he defined variable as letter designing any one of a set of numerical values. Then he went on to define continuous variables, continuity and limit of a function. Mainly, his definitions are the same as the modern $\epsilon - \delta$ definitions. It is interesting to take a look at his definition of limit of function:

"A function has a limit L at $x = x_0$ if given any positive number ϵ, there exists a δ such that for all x in the interval $|x - x_0| < \delta$, $|f(x) - L| < \epsilon$"

In this definition, the idea of approaching is not involved. Instead, the presupposition of the limiting value L opens the static approach towards the notion of limit. Consequently, the problem of Zeno's paradoxes loses its meaning in Weierstrass theory of limit.

Besides, Weierstrass was aware of the need for formulating a definition of irrational numbers independently of the limiting idea. Briefly, starting with the existence of natural numbers, he considered other numbers as composed by aggregating different elements. For example, the number 1.41 is made up of 1a, 4b, 1c where a is its principle unit and b, c are its 'aliquot' parts. Under the condition that the sum of any finite number of elements is always less than a certain rational number, he proved that $\sqrt{2}$ is the limit of the variable sequence 1a; 1a, 4b; 1a, 4b, 1c; ... instead of treating it as a definition.

Around the same time, Charles Meary (1835 - 1911), Eduard Heine (1821 - 1881), Georg Cantor (1845 - 1918) applied a similar idea to give the definition of irrational numbers. On the other hand, Richard Dedekind (1831 - 1916) adopted a somewhat different approach. By noticing that denseness is not continuity, he suggested to define an irrational number by considering the division of a line into two disjoint subsets. This division he called a cut. For instance, he considered two sets of rational numbers, the first one consisting of those whose square are less than 2, and the second consisting of all the others. Noting that this cut is not determined by a rational number, Dedekind identified it as a new number, that is, the irrational $\sqrt{2}$.

In doing so, every real number can be completely defined by such a cut. The continuity of real numbers follows from the continuity of the line.

By this stage, the controversies in calculus were basically resolved. The arithmetical approach to calculus led to the study of set theory. Consequently, the foundation of the number system was accomplished by Giuseppe Peano (1858 - 1932) who created a model for natural numbers. On the other hand, the study of infinitesimal approach led to another form of calculus, known as non-standard analysis which was proposed by Abraham Robinson (1918 - 1974).

2.8 a remark on the notion of function

As a remark, it is interesting to note that there is a gap between Cauchy's definition of function and Dirichlet-Bourbaki's definition, that is:

> "A function is any correspondence between two sets (the domain and the range) which assigns to every element in the domain exactly one element in the range."

The Dirichlet-Bourbaki definition is widely employed in today's calculus text. In order to give a more complete picture of the evolutional process, I briefly describe the development of function after Cauchy.

Studying the problem related to heat conduction, more precisely, the convergence of 'Fourier series', Peter Gustav Lejeune Dirichlet (1805 - 1859) realized that in order that for each x, the series converges to a real number which is the value of a given function f at x, the notion of function must be independent of mathematical operations. Consequently, he re-defined the definition of function as:

> "y is a function of x when to each value of x in a given interval there corresponds a unique value of y" [Kline p.950]

Yet, the notion of function were only defined over the real numbers. After Cantor developed his set theory, he implicitly expressed that given two sets N and M, 'each element n of N a definite element of M is corresponded, whereby one and the same element of M may be used repeatedly. The element corresponded with n from M is a single-value function of n.' [Grattan-Guinness p.208] Once again, the notion of function was broadened, that is, independent of real numbers. With the birth of the notions of metric and topology, the study of the properties of function led to the nctions of domain and range [13].

2.9 The naive concept evolution versus Vinner's concept formation

By naive concept evolution, I mean the historical process of forming a concept. During this process, the contributors to the concept do not have any idea of its final form. However, through working with a particular set of problems, they roughly have a 'feeling' for it. At the moment when they could extract common properties from their problems, they are likely to describe this abstraction by a definition. Lat r, when a new problem or situation appears, either they try to generalized their original idea in such a way that the new information is accommodated; or they just ignored this information. This can be seen through the development of the notion of function as well as the question whether the infinity should have its place in mathematics. My intention here is to point out that the evolution of a mathematical concept depends on the concept images held by its contributors.

Having this assumption in mind, I examine closely Vinner's model of concept formation. If the learners only react to a given task according to their concept image, why is the process of learning a mathematical concept, in general, shorter than the process of forming it in the first place? Of course, the answer to this question involves many factors, for instance, a coherent way of presenting the subject, a good teaching strategy, ... etc. Among all of these, I shall add that learners, in fact, experience implicitly the final form of the concept, for the materials they learn from are carefully built up around the formal concept definition which describes precisely the underlying notion. Under such setting, certain concept images which are consistent with the formal definition are expected to be created through these materials. From this point of view, the learners may not construct their concept images directly from the concept definition, but indirectly from it. Thus, the concept definition should have its role in teaching and learning mathematics. After all, it is the responsibility of the teachers and mathematics educators to make a good use of definition!

2.10 Educational implications

Through this brief study of the development of the calculus, some educational implications will be made, namely, that:

(1) The concept definitions of function, limit, continuity which are presently taught to students lack of motivations which led to their creations and so do not help creating suitable concept images. On the one hand, they are too abstract to understand; on the other hand, their final forms are far removed from the original idea. Consequently, if we present these notions by using formal definitions, difficulties for students to create associated concept images will be expected.

However, if we present the material based on intuitive ideas, confusions will likely occur in later learning if those ideas are not consistent with the concept definition. A better approach might be based on the concept images of the learners, complemented by those images of the contributors of the subject, in order to develop a teaching strategy which will be consistent with the concept definition.

(2) Different approaches should be encouraged in teaching situation. As we have seen, mathematicians have used different approaches to arrive at the same result. Thus, given a task, there maybe more than one approach to handle it. Being aware of this, the teachers should

encourage students to use students' own approach to a given task. In doing so, the students will learn to discover their own methods rather than memorize the given methods. Of course, teachers should also guide their students to reconstruct their concept images if their approach does not get along with the formal theory.

(3) A great discovery or invention often came from the dissatisfication with the previous approaches; or the awareness of common properties of a set of problems. Thus, the intuitive approach should be adopted before the students are aware of the need of formalistic approach. For instance, the formal definition of continuity should be given after presenting certain examples in which the intuitive approach fails.

CHAPTER III

LITERATURE REVIEW

3.1 Introduction

In this chapter, studies which reveal the student's concept images of some mathematical concepts in the calculus such as the notion of number, the notion of function, the notion of limiting process, the notion of continuous function and the notion of tangent will be reported.

In the past many years, lots of studies have examined students' views towards the basic notions in the calculus. Some of these notions, for instance, the notion of number and the notion of function, are generally taught in the high school. Yet, other notions such as the limiting processes, though not taught explicitly in the high school, can be conceived through the intuitive approach before taking the calculus. Hence, in order to give a more complete picture of students' concept images, some studies which report those concept images held by high school students are also included in this review.

In chapter 2, I have alrealy investigated the concept images held by the contributors to the calculus. In this chapter, I further compare the concept images held by the students and by these contributors. Some observations will be discussed.

3.2 The notion of number

Under the formalistic approach, the underlying notions in the calculus are based on the concept of number. J. Confrey (1980) [2] adopted the theory of conceptual change to assert that there is no single number concept to resolve different problems and different dilemmas in the calculus. According to Vinner's model, this can be explained by the different concept images that are evoked under different stimuli. Consequently, in order to examine the students' views about number, or more specific, about real numbers, different tasks which are expected to evoke different images should be given. In this section, two main issues, namely, the density aspect and the decimal representation of real numbers will be discussed according to the work of I. Kidron and S. Vinner.

3.2.1 The density aspect of the rational numbers

As we have seen in chapter 2, B. Bolzano confused continuum with denseness and so failed to give a satisfactory resolution on the cocept of real numbers. On the other hand, R. Dedekind succeeded in defining real numbers by noticing the difference between the two concepts. What, then, are the students' views about the density aspect of the rational numbers?

The study

In 1983, Kidron and Vinner [10] studied the kind of images held by high school students about the density aspect of rational numbers by asking the following question:-

"Students were told that in an algebra lesson on rational numbers a teacher had written down two rational numbers and had asked his students whether there were more rational numbers between them and if they were - how many. One of the students to whom this was told claimed that the answer depends on the two given numbers. A second student claimed that there are always numbers between two given rationals but the number of these numbers depends on the two numbers. A third student claimed that there are always numbers between two given rationals and the number of these numbers does not depend on the given numbers. Who is right and why?"

The sample of this study consisted of 91 grade 10 and 85 grade 11 students at the high school in Jerusalem.

Results

The results indicated four different views. They were summarized as follows:

(1) 20% of the sample held that given two rationals, there was a finite number of rationals between them and that the number was dependent on the given rationals. For example, some of them explained that 'it is impossible to write down numbers in between two consecutive numbers. For instance, 1/2 and 2/3'.

(2) Around 15% out of the total thought that there was always another rational between two given rationals. Some expressed that no matter how close the given rationals were, it was possible to find numbers between them. For instance, 59/1740 lay between 1/29 and 1/30. However, this group did not answer how many such numbers could be found.

(3) About 1% considered that given two rationals, there was an infinite number of rationals in between though the kind of infinity depended on the given rationals. For instance, one students wrote:

"The distance on the number line determines the number of numbers between two given numbers; if the distance is greater there are more numbers"

This kind of view resembled the idea of mathematicians such as Galileo (see 2.4) around 16 and 17 century. The difference between Galileo and this student was that this student boldly made his (or her) comparsion without further justification.

(4) Around 60% of the subjects believed that given two rationals, there was an infinite number of rationals in between independently of the given rationals. Some of the students appealed to the general arguments to support this view. For instance, the process of finding a new rational between two given rationals can go on for ever.

Few students related their explanation to irrational numbers. For instance, one student wrote:

"The student who claimed that the answer depends on the number is right, since it is possible that the two given numbers are <u>consecutive</u> and there are no rational numbers between them; for example: two numbers between which lies the number π which is irrational." [my underline]

Kidron and Vinner considered that this image indicated 'a beginning of an accommodation process', that is, the student tried to find some way to link together the concept of rationals and that of irrationals.

Moreover, some students viewed the denseness as a property of real numbers rather than the rationals. For instance, one student explained:

> "The third student is right since between two numbers, there are infinitely many numbers and therefore probably some of them are rationals"

<u>Comments</u>

Overall, what kind of intuitions or beliefs are held by the students when the density aspect is discussed? Some comments are made as follows:

(1) The image that numbers were <u>consecutive</u> caused difficulty to conceive the idea of denseness. As the data showed, the idea of consecutive numbers was constantly evoked. This might come from the experience of natural numbers and integers. Often, in the elementary algebra text, the phrases like 'find two consecutive numbers', 'find two consecutive even numbers' were frequently used. Of course,

the word 'numbers' here stood for 'integers' not rational numbers or irrational numbers. However, the students might not be aware of the difference and so the image that numbers (including rationals and irrationals) were 'consecutive' was created. Furthermore, 1/2 and 2/3 were 'consecutive' might come from that 1 and 2 in the numerators as well as 2 and 3 in the denominators were consecutive.

(2) The image of rational as the quotient of two integers may cause confusion. In Kidron and Vinner's study, some students failed to conceive the denseness of rationals due to this conception. For instance, one student wrote:

> "Between 1/2 and 1/3, there are no rational numbers because 1/2.5 is not rational since it is not presented as an integers over an integers"

Besides the incorrect operations in finding the mean of 1/2 and 1/3, the answer showed that an inappropriate image was created by considering a rational number as the quotient of two integers. After all, 1/2.5 could be rewritten as 2/5. Moreover, a remark on the incorrect operations was that the image of rational was 1/n.

(3) Another interesting observation in Kidron and

Vinner's study was that no student even mentioned the decimal representation of rationals. Prehaps, this might be explained by that the decimal representation of rationals needed further mathematical reasonings as well as some direct stimuli. Reminding ourselves that Michael Stifel (see 2.4) could not decide whether irrationals were really numbers, after studying the decimal representation, it was worth to look at the students' points of view about the decimal representation of numbers.

3.2.2 The decimal representation of real numbers

In 1985, Vinner and Kidron [38] examined whether the students recognized the decimal representation of real numbers. Two non-standard questions were posed to 91 tenth grader and 97 eleventh grader at a high school in Jerusalem. The first question addressed the repeating decimal representation of rational numbers and the second one aimed at the infinite non-repeating decimal representation of irrational numbers by asking how an infinite decimal was formed.

Question 1 (Rationals as repeating decimals)

A teacher asked his students what is the 50th digit in the decimal equal to 1/7. One student claimed that it is too much work to carry out 50 division steps. Another student claimed that it is possible to find the answer in less than 50 steps. Who is right? Please, explain!

Results

As the answers to this question indicated, three different views were invoked:

(1) 10% of the sample thought that the decimal representation of 1/7 was not repeating. The students in this group usually stopped their long division calculations before the decimal began to repeat.

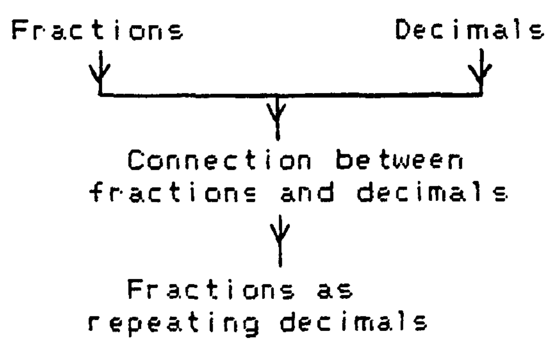
(2) 12% of the sample did not know whether 1/7 was repeating or, at least, not in advance. Most of the students in this group proposed carrying out the detail computations.

(3) 74% of the total believed that 1/7 was repeating. Among them, only 3% argued the repeating of 1/7 by mathematical reasonings instead of actually dividing 1 by 7. For instance, the possible remainders were investigated. 21% gave no explanation and 49% found out the repeating decimal by performing certain steps of division.

Vinner and Kidron concluded that 3% and 21%, totally 24%, of students recognized that any fraction has a decimal representation which may be finite or infinite repeating.

<u>Comments</u>

The data also suggested that different images may be created even under the same stimulus. Usually, the notion of fractions, the notion of decimals as well as the connection between fractions and decimals are taught prior to the notion of fractions as repeating decimals. The hierarchical structure of these notions can be presented in the following diagram:



In the question of Vinner and Kidron, the students had enough stimulus to [illegible] the connection between fractions and decimals. As the data showed, at least 10% + 12% + 49%, totally 71% of students used division as the starting step. However, only 49% were able to establish the relation between fractions and decimals. Here, we see that diverse images are formed from one level to another in this simple hierarchical structure. My point is that to be aware of fractions as repeating decimals one needs not only to recognize the patterns from known information; to predict what will happen if the division is carried out forever, but one also needs to advance thoughts into the abstract domain. I quote one of the students to support this point:

> "The second student is right because in this particular case 1/7 is a repeating decimal. It is $.\dot{1}4285\dot{7}$. This I have discovered by dividing 1 by 7. After getting this number (.142857) I saw that again I should divide 1 by 7. I can assume that there will be no change in the result when I start the division again and so on till infinity"

<u>Question 2</u> (Irrationals as infinite non-repeating decimals)

A teacher asked his students to give him an example of an infinite decima'.
Dan: I'll look for two whole numbers such that when I divide them I won't get a finite decimal; for instance, 1 and 3.
Ron: I'll write down in a sequence digits that occur to me arbitarily, for instance: 1.236418...
Dan: Such a number does not exist because what you write down is not a result of a division of two whole numbers.

Ron: Who told you what you write down must be the result of a division of two whole numbers?
Who is right? Please explain!

Results

Among all the subjects, over 77% claimed that they had learnt about irrational numbers. Yet around 55% believed that any finite and infinite decimal can only be identified by a rational number. On the other hand, 20% of tenth graders and 43% of eleventh graders suggested that an infinite decimal can be obtained not only by dividing two whole numbers. For instance, one can write down an arbitrary sequence of digits. Moreover, among those 77% who had learnt about irrational numbers, at least 32% did not know the infinite decimal representation of irrationals.

Comments by Vinner and Kidron

The answers to this question indicated again that different images were evoked under same stimulus. Vinner and Kidron explained these phenomena by the notion of imagination act [39], that is, the mental activity whereby certain people can abstract their experience to construct an imaginary world. In this case, those who failed to construct or concieve an infinite decimals out of nothing showed a lack of imagination act. Yet, the difference between tenth graders and eleventh graders in the data showed that the imagination acts may be developed through mathematical experience. When certain mathematical maturations are reached, the students are expected to imagine an infinite procedure of writing down digits.

3.2.3 Summary

Vinner and Kidron's studies indicated that neither the notion of denseness nor that of the infinite decimal representation of real numbers are intuitive to the students. After all, both of these notions are intellectual rather than empirical. In certain sense, the learners are expected to build up certain imagination acts which are shared in mathematics community. However such imagination acts may not be consistent with the learners' personal experience. As Tall and Schwarzenberger [28] pointed out, the existence of an infinite decimal representation of real numbers contradicts daily experience which is subject to the limited accuracy of a practical drawing, measurement, ..., etc. No matter how large a scale is adopted, we can only experience finite decimal places of accuracy. Thus, in practice, one can hardly distinguish between a real number and its rational approximation. The failure of conceiving the infinite decimal representation will occur if the learners are unable to distinguish the difference between the theoretical requirement for infinite decimals and practical experience of finite decimals. The difficulty involved here may also explain why, historically, Stifel could not decide whether irrationals are really numbers as well as why some students in Vinner and Kidron's study could not conceive the idea of denseness and that of infinite decimal representation of numbers.

3.3 The notion of function

> "A function is any correspondence between two sets (the domain and the range) which assigns to every element in the domain exactly one element in the range"
>
> [Dirichlet-Bourbaki]

As the Dirichlet-Bourbaki approach to the notion of function gradually became one of the main topic in modern mathematics curriculum, lots of studies have been done in the past few decades to investigate different aspects of this notion under such an approach. For instance, K. Lovell (1971) [12] examined the growth of the function concept among 12- to 17- year old students; S. Wagner (1981) [40] applied Piaget's methodology to study 10- to 15- year old students' understanding of relations. In this section, I report of findings in mathematics education literature which reveal mainly the concept images of students as to the concept definition of the notion of function at the high school and college level. The work of I. A. Maryanskii (1965); S. Vinner (1983); T. Dreyfus and S.Vinner (1982, 1989) as well as that of Z. Markovits, B. S. Eylon and M. Bruckheimer (1983, 1986) will be presented.

3.3.1 The work of Maryanskii

During the period 1958-1962, I. A. Maryanskii [14] conducted studies to determine the psychological difficulties involved in students' mastery of the notion of function.

The study

The samples of the study consisted of 132 grades 8 - 10 students in five schools at Rovno. Some written questions were first distributed in order to reveal the students' images towards the notion of function. Then a follow-up discussion was held between teacher and students in the classroom.

Results

The results of this study were reported according to the notions of variable quantity, set and functional relationship.

(1) The notion of quantity was conceived as 'what is measured' or 'that which is large or small'.

(2) The notion of set was understood by 'very many' or 'an aggregate'. Moreover, it is interesting to note that 43 out of 46 students could not give a positive answer to the question which stated 'Can a set contain only one member?'.

(3) The common images concerning the functional relationship were: 'the features of the function were not the one-to-one correspondence between the values of two quantities, but the changeability of the quantities and the presence of a general or a causal connection between them' or 'a function cannot assume identical values'. Due to the

latter conception of function, students frequently rejected the 'constant function' as a legitimate candidate of function.

As the overall results indicated, there was no significant improvement between the tenth graders who knew lots of functions and eighth graders who just began studying elementary functions.

From the above observations, Maryanskii concluded that the psychological difficulty in mastering the notion of function lay in the fact that this notion exceeded the bounds of what was ordinary for students, that is , this notion did not remind the students of any familiar concepts. Maryanskii further explained that introducing the notion of function by the variable quantity was not sufficient to develop a familiar environment to assimilate the Dirichlet-Bourbaki definition of function. As a matter of fact, emphasizing the aspect of variability might lead to neglect the essential features of the function such as the single-valued part and the nature of the set on which the functional relationship is defined.

Comments

Maryanskii's study was carried out during 1958-1962 when the Dirichlet-Bourbaki approach to function in school mathematics was at its infancy. Maryanskii's study points to the psychological difficulties involved in mastering the Dirichlet-Bourbaki definition based on the variability approach to the notion of function.

3.3.2 The work of Vinner and Dreyfus

The aim of Vinner and Dreyfus' two studies was to reveal the concept images of function evoked by the Dirichlet-Bourbaki definition.

The first study

Vinner (1978) [35] carried out his first study in order to reveal the students' concept images of function. The samples in this study included 65 grade 10 and 81 grade 11 students in two high schools at Jerusalem. A questionnaire consisted of five questions was handed out to these two groups. The first four questions asked to identify or construct a function from a given sentence or graph. For instance, one sample question was: 'Is there a function that admits integral values for non-integral numbers and admits non-integral values for integral numbers?' The students were told to answer each question by choosing 'yes' or 'no' as well as giving their explanations. The fifth question addressed to students' opinions about function. The grade 10 students answered the questionnaire after several months when they had learnt the Dirichlet-Bourbaki approach to function while the 11 graders who took the test had learnt this notion of function at grade 10.

Results on the concept definition

The result of this study showed that among the 34% of the subjects gave the Dirichlet-Bourbaki definition, only 20% acted according to this definition. The students' opinions about the definition of function could be summarized into four categories:

(1) 57% of students gave Dirichlet-Bourbaki definition or the mixture of this definition and some personal concept images. For instance, a 10th grade student wrote:

"Function in my opinion is that every x has one number or one object in y but not vice versa"

(2) 14% thought that the function was 'a rule of correspondence', 'a relation', 'a dependence between variables' or 'a law'. For instance, an 11th grader expressed that:

"A function is a relation in which every element in one set is related to a single element from another set according to a certain law . We have to discover the law but it always must exist."

(3) 14% expressed that the function was an algebraic term, a formula, an equation and arithmetical operations. For instance, a 10th grader wrote:

"A function is a set of numbers such that when doing to them a certain arithmetical operation we obtain another set of numbers which is functional to the first set"

(4) 7% identified the function as some elements in a mental picture. For instance, the graphical representation or the form $y = f(x)$.

Since the definition of function was not asked directly in the questionnaire, the opinions about the definition did partially reflect the concept images held by students.

Results on the concept image

Concept images were exposed extensively when the students were asked to identify whether a given relation is a function or not. As a matter of fact, the answers to the first four questions not only showed the lack of operationality of the concept definition, as Vinner's model described, but also indicated certain inconsistencies between students' concept images and the Dirichlet-Bourbaki definition. The common concept images concerning the notion of function could be summarized as follows:

(1) A function should be given by one rule. For instance, the function which assigns each positive number the number 1, each negative number the number -1, and 0 the number 0 is not considered as a member of the 'function family'. Some of the students who held this view explained that there were different rules.

(2) A function cannot have exceptional point. For instance, the function which assigns each number except zero

its square and 0 the number -1 was rejected due to the reason that there is no number whose square is negative.

(3) A graph of a function should be reasonable. For instance, a function must be symmetrical; always increasing or always decreasing.

Furthermore, certain inappropriate images were generated by failing to recall, perhaps to understand precisely the Dirichlet-Bourbaki definition. For instance, 'a function is a one-to-one correspondence'. Yet, some other images indicated the confusion between one-to-many and many-to-one. For instance, 'for every y in the range there is only one x in the domain that corresponds to it'.

The second study

Besides the above study, Vinner and Dreyfus (1982,1989) [37,4] conducted a similar study. The aim of this study was to examine the kind of images held by the college students after they had learnt the Dirichlet-Bourbaki approach to function in high school, but before this notion was re-introduced to them.

The samples were 271 first year college students in two Israeli institutions and 36 junior high school mathematics teachers. The numbers of subjects reported here were those who gave enough information to reveal part of their concept images. Moreover, according to the students' mathematical training, Vinner and Dreyfus divided the college students into four levels, namely, low level (33); intermediate level (67); high level (113) and mathematics majors (58).

A questionnaire resembled that given in Vinner's first study was handed out to the students. It consisted of seven questions. However, in this second study, the subjects had the option to choose 'I do not know' as one of their answers in the first six questions.

Results on the conept definition

The results of th s study indicated that 2 low level; 12 intermediate level; 17 high level; 26 mathematics majors students as well as 25 junior high school teachers (totally 27%), gave the Dirichlet-Bourbaki definition of function. Among them, 2 out of 2 low level; 12 out of 12 intermediate level; 12 out of 17 high level; 6 out of 26 mathematics majors and 14 out 25 junior high school teachers did not use this definition to answer the first six questions, that is, only 12% of the total acted according the Dirichlet-Bourbaki definition.

The opinions about the definition of function in the second study resembled to those in the first study.

Results on the concept image

The data in the second study reflected some additional aspects of concept images generated by the Dirichlet-Bourbaki definition. For instance, one-valuedness - that is, 'if a correspondence assigns exactly one value to every element in its domain, then it is a function' - was used to justify whether a given relation is a function or not. Another aspects like 'discontinuity' - a gap in the graph - as well as 'split domain' - the subdomain where different rules are defined - were also evoked as a criterion of justification.

Observations on both studies

Some observations made by Vinner and Dreyfus were as follows:

question 'Does there exist a function whose value for integral numbers are nonintegral and whose value for nonintegral numbers are integral?' was asked in both studies, the subjects tended to find an algebraic rule like $y = 1/x$ to justify the answer. This kind of answer, as Vinner and Dreyfus explained, might come from the observation that $1/y$ is a nonintegral number if x is an integer greater than one.

Moreover, in the second study, when the subjects were asked to construct a function whose values are equal to each other, the answers included a general expression like $y = c$; a specific example like $y = 5$ as well as the formula like $y = x/x$, $y = x^0$ and even $y = x$.

A. Sfard (1987) [19] also reported a similar observations in her research. Briefly, she asked 60 16- to 18- year old secondary school students to choose either

(1) Function is a computational process which produces some value of one variable (y) from the given value of another variable (x);

or (2) Function is a kind of a (possibly infinite) table in which to every value of one variable (x) corresponds certain value of another variable (y).

Around 70% of the subjects preferred case (1).

Moreover, the operational aspects of functions were more popular among the older students.

(4) In the second study, the number of the subjects who gave the Dirichlet-Bourbaki definition increased with the level of the mathematical training, that is, 6% in the low level, 18% in the intermediate level, 15% in the high level, 45% in the group of mathematics majors and 70% in the group of junior high school teachers. Vinner and Dreyfus explained this phenomenon by arguing that the use of the Dirichlet-Bourbaki definition needed a strong mathematical orientation. However, Vinner and Dreyfus also remarked that the high precentage in the teacher group might come from that only half of the teachers provided sufficient information to analyse the answers. Yet, another possibility was that the teachers were well aware of the role of definitions in mathematics.

I would like to make three comments. First, according to Vinner and Dreyfus' data, among those teachers who gave the Dirichlet-Bourbaki definition, only 44% acted according to this definition. This leads me to wonder if some teachers recalled the definition properly due to their repetitive presentation of the same material rather than the awareness of the role of definition in mathematics.

Secondly, Vinner and Dreyfus did not explain the decreasing precentage (15%) in the high level group. Reviewing how Vinner and Dreyfus divided their subjects, I find that the high level group mainly consisted of students majoring in chemistry, biology and technological education. The scientific and technological background of this group may lead them to regard a function as a dependence between two variables just like the relation between scientific data. As a matter of fact, 32% of this group did express this view.

Thirdly, the percentage giving the Dirichlet-Bourbaki definition in the first study (34%) is higher than in the second one (27%). This may be explained by the fact that the students in the second group had learnt the definition of function a long time ago in comparsion with the students in the first group.

(5) From the historical point of view, certain aspects of function such as 'discontinuity', 'different rules',..., etc. were often used by mathematicians (for instance, Euler) as a ground to reject a given example as a

function. Taking into consideration that the difficulties exhibited in both studies as well as the difficulties involved in the historical evolution of the concept of function, Vinner and Dreyfus suggested that mathematics educators should reconsider seriously whether the Dirichlet-Bourbaki approach to function should be adopted in those courses where such definition is not particularly useful. Furthermore, they suggested that a better teaching strategy is to introduce different aspects of function based on the learners' previous experience.

Comments

On the one hand, Vinner and Dreyfus did not explain how the transition between the naive approach, that is, the approach based on one's experience, and the Dirichlet-Bourbaki approach to the notion of function, can be made psychologically. Moreover, what is the consequence if the naive approach cannot accommodate with the Dirichlet-Bourbaki approach, in the later learning. On the other hand, from a more positive point of view, the historical difficulties inherent in the notion itself as well as the findings that the concept images of the learners resembled those of the contributors of the subject, provide mathematics educators with more information as to how to develop a better approach towards the teaching of functions.

3.3.3 The work of Markovits, Eylon and Bruckheimer

The aim of the work of Z. Markovits, B. S. Eylon and M. Bruckheimer [15, 16] was to investigate students' understanding of different aspects of the notion of function such as domain, range, rule of correspondence, algebraic and graphical representation after they had learnt the Dirichlet-Bourbaki approach under the Israeli curriculum.

The study

In the Israeli curriculum, the sequence of presenting the function concept begins with the three sub-concepts, namely, domain, range and rule of correspondence; then proceeds by different representations of function such as arrow diagrams, graphical and algebraic forms were discussed. Finally, there are activities to translate a given function from one representation to another. After studying how the function concept was built in the curriculum, Markovits, Eylon and Bruckheimer came up with a large variety of problems which could be divided into four categories, namely, (1) The ability to identify and construct examples of functions; (2) The ability to identify and find the preimages, images or pairs of preimage and image for a given function; (3) The ability to identify function as well as to transfer a given function from one representation to another; (4) The ability to construct examples of functions satisfying certain constraints. These problems were handed out to about 400 grade 9 students in Rehovot, Israel. Since the study addressed students' understanding of the notion of function, the students' answers revealed their concept images under different stimuli created by the questions. In the following summary, the relevant concept images with particular reference to the graphical and algebraic representation of function will be reported, according to the four categories suggested by MarKovits, Eylon and Bruckheimer.

(1) <u>The ability to identify and construct examples of functions</u>

The questions posed in this category resembled those in Vinner and Dreyfus' studies and so did the respondents' concept images. However, the common concept image that 'every preimage has more than one image' was frequently used to reject a given example as function, when the algebraic representation of a piecewise or a constant function were given. Moreover, most of the students answered correctly when the graphical representation was used.

In the construction task, most of the students confused many-to-one relation with one-to-many relation. I remark here that this phenomenon also appeared in Vinner and Dreyfus' studies, when the students were asked to write down their opinions of function.

(2) <u>The ability to identify and find the preimages, images or pairs of preimage and image for a given function</u>

When the graphical representation of function was given, more than half of students had difficulty in locating preimages and images. Many of them held that 'the preimages or images were on the curve, corresponding to the given points on the axes'. However, they did recognize that points on the graph represented pairs of preimage and image, and vice versa.

More difficulties arose when a function was given in an algebraic form. Overall, the case of determining whether a given number was a preimage of a given function was relatively easy in comparsion with the case of determining an image. In the former case, the students only needed to check whether the given number belonged to the domain. As a matter of fact, most of the students did check it correctly. However, in the latter case, checking whether the given number belonged to the range, involved performing certain calculations and checking whether the resulting number was the legitimate member of the domain. For instance, given a function from natural numbers to natural numbers which is defined by $f(x) = 4x+6$. Determining whether 8 is an image of f, involves checking whether 8 is a natural number, finding whether the preimage of 8 is a natural number. In answering this question, half of the students only checked whether 8 was a natural number. The majority of those who performed the calculations neglected to check whether the pre-image 1/2 belonged to the domain.

I remark that this phenomenon can be explained as follows: during the solution process, an evoked idea does not need to remain throughout the process. Often, when another idea is evoked, the previous one is soon forgetten. In identifying a preimage, the idea of checking dominates the whole process, whereas in identifying an image, the idea of finding the corresponding preimage takes over that of checking.

(3) <u>The ability to identify identical function as well as to transfer a given function from one representation to another</u>

In identifying identical functions, the concept images that 'a function is defined by the rule of correspondence only' and 'a graph of function must be continuous' were evoked.

Given a function f from natural numbers to natural numbers defined by $f(x) = 4x+6$ and a function g from real numbers to real numbers defined by $g(x) = 4x+6$, the students were asked whether f and g were identical. More than half thought that f and g were the same. On the other hand, most of the students believed that f was not identical with the function h defined by $h(x) = 2x+3$ with the natural numbers as domain and range. From this result, Markovits, Eylon and Bruckheimer suggested that the students held the idea that 'a function is defined by the rule of correspondence only'.

When Markovits, Eylon and Bruckheimer asked whether the algebraic representation of the above f and its graphical representation were identical, some students gave the reason such as 'the points must be connected'. Here, the concept image that a graph of function must be continuous is evoked. I remark that this idea may come from plotting a graph by finding a set of discrete points and then joining these points together as is done in the scientific experiments as well as when learning to draw some elementary functions such as straight line and parabola.

In the tasks of transfering a given function from graph-to-algebra and vice versa, the students often ignored the given domain and range. For instance, when they were asked to draw the graph of f defined by $f(x) = 3$ with the natural numbers as domain and range, they replaced the natural numbers by real numbers.

Furthermore, Markovits, Eylon and Bruckheimer also noted that in a familiar setting such as linearity, algebra-to-graph was easier to master than graph-to-algebra.

(4) <u>The ability to construct examples of functions satisfying certain constraints</u>

Different constraints imposed on function were under investigation. For instance, constructing a function which increases over part of the domain and is constant over the remainder; drawing a function passing through given points,..., etc. The aim was to study what kind of images associated with the function concept would be evoked under these constraints. Besides, Markovits, Eylon and Bruckheimer tried to examine to what extent the students were aware that there might be infinitely many different functions satisfying the given conditions.

As the results indicated, the linear image of function and once again, the idea that a piecewise function was not a single function were evoked. Besides, very few students were aware that there might be infinitely many functions satisfying the given constraints. Overall, the answers showed that the students preferred to give examples in graphical form.

When the students were asked to construct a function which increases over part of the domain and decreases over the remainder, the parabolic function was given. However, when they were asked to give an example of a function which increases over part of the domain and is constant over the remainder, most of the students failed to give an example. Markovits, Eylon and Bruckheimer explained that the students did not think of piecewise function as a single function.

In another question, the students were asked to find a function passing through two given points. The answers indicated that the subjects tended to use linear functions as examples. As the number of given points increased, especially the given points could not be joined by a single linear function, for instance, points like (3,4), (6,7) and (8,13), more students failed to answer the question. Some of them even mentioned that a function passing through (3,4), (6,7) and (8,13) did not exist. Markovits, Eylon and Bruckheimer explained that the linear image of function might come from the experience of geometry such as two points determine a line as well as the fact that the linearity was the simplest form of function. Perhaps, the linear image also led to the image that there was only one function satisfying given constraints.

In an overall summary, Markovits, Eylon and Bruckheimer concluded that the students did reach certain understanding of the Dirichlet-Bourbaki definition since a reasonable percentage identified successfully whether a given example was a function or not; as well as gave a suitable example of function. Taking into consideration that the Dirichlet-Bourbaki definition can be applied in non-numerical contexts like transformations in geometry, they suggested that the use of this definition is worthwhile. Yet, they also proposed downgrading some components of the function concept, for instance, the domain and range, due to the difficulties found in their study.

Comments

Two comments are made as follows:

(1) From the results of identifying identical functions, Markovits, Eylon and Bruckheimer suggested that the students held the idea that 'a function is defined by the rule of correspondence <u>only</u>'. By using the notion of concept image, I view that in the case of identifying identical functions, the question itself stimulates the concept image that a function is defined by the rule of correpondence. However, this does not mean that the other images of function do not exist. On the one hand, the idea of comparing the domain and range of f and g, where f is defined by $f(x) = 4x + 6$ with natural numbers as domain and range and g is defined by the same expression but with real numbers as domain and range, relates to the conception of

numbers, that is, the students need to pay attention to the difference between the domain and range of f and that of g. On the other hand, neglecting to compare the domain and range may come from their previous experience of functions. This point can be further explained by some personal teaching experience.

Few years back, I taught an elementary algebra course in Macao. My students were 12- to 13- year old who attended the first year of secondary school. Their mathematical training included a little bit of modern mathematics, for instance, the notion of sets, the set of natural numbers, integers, ... ,etc; some algebra, for instance, the notion of real variables, basic algebraic operations as well as solving linear equations. After the topic of solving linear inequalities such as $x + 2 > 3$, where x was real variable, was introduced, I assigned some similar questions as home assignments. When the time I marked the returned assignments, I was shocked to learn that more than half of the class, including some good students, wrote:

$$\begin{aligned} x + 2 \quad &> 3 \\ x + 2 - 2 &> 3 - 2 \\ x \quad &> 1 \end{aligned}$$

<u>Thus, x are 2, 3, 4, ...</u>

At the first glance, I thought that the use of number line to demonstrate the solutions of inequalities led my students to choose the labels which are natural numbers as answers. Having this in mind, I asked my students whether they noticed that there were many other numbers between two natural numbers. For instance, 5/2 between 2 and 3. Almost all the students answered loudly, 'Yes, 5/2 is between 2 and 3 but you cannot count it as a solution of x + 2 > 3.' I questioned why 5/2 could not be a solution of x + 2 > 3. One good students answered with <u>confidence</u> , 'We considered only the natural numbers as solutions.' Well, this was the exact point to which I wanted to draw my students' attentions. So I asked, 'Why do you only consider the natural numbers? He said, 'that is the way to write the solutions of inequalities.' I was completely surprised by this answer. After more discussions, I finally found out the real reason: my students had learnt to solve the linear inequalities in the course of modern mathematics where the variable was considered to be <u>natural numbers only</u> .

In my case, it was hard to know to what extent my students were aware that a variable can be used in different context, such as real numbers and natural numbers, for they did accept the numbers other than natural numbers as solutions of linear equations. My point here is that my students' previous experience had led them to evoke the concept image that the solution of an inequality has to be natural numbers. In the study of Markovits, Eylon and Bruckheimer, there was no information as to why the students thought that f and g were identical. Carelessness, previous experience, ... or some other factors might be there. Furthermore, according to their working experience of function in the elementary level, I would argue that the difference between f and h, where h is defined by $h(x) = 2x+3$ with the natural numbers as domain and range, is more obvious in comparsion with that between f and g. Thus, the conclusion that 'a function is defined by the rule of correspondence <u>only</u>' in fact needs further justifications.

(2) The work of Markovits, Eylon and Bruckheimer does show the complexity of mastering different aspects of function. However, the idea of downgrading the importance of domain and range leads to abandon one of the main features of Dirichlet-Bourbaki approach. If students have the difficulty mastering the idea that different kind of numbers

can serve as the domain and the range of functions, I wonder how we, as mathematics educators and teachers, hope that our students understand the idea of defining a function on the set of geometrical objects? On the other hand, from the historical evolution of the notion of function, the idea of domain and range not only highlights a step of abstraction, but also integrates the function concept into other branch of mathematics such as topology, making the notion of function centeral in modern mathematics. Hence, it is not clear to me how downgrading the components of domain and range can coexist with the Dirichlet-Bourbaki approach.

3.3.4 Summary

In the work of Maryanskii, the work of Vinner and Dreyfus as well as that of Markovits, Eylon and Bruckheimer, lots of concept images associated with different aspects of function are revealed. Among these concept images, some of them resemble to those held by mathematicians in ancient time. For instance, 'a function is an operational process' resembles to the idea of Jame Gregory (see 2.4); 'a function is a dependence between variables' resembles to that of Leibniz and some of his followers (see 2.5 and 2.6). In justifying whether a given example as function, the ideas such as 'discontinuity', 'different rules', 'split domain', 'exceptional points' resemble to that of past mathematicians in the period of indecision (see 2.6). In certain degree, the resemblance do provide us (mathematics educators and teachers) with information about what kind of difficulties will be expected in the learning situation. Besides, the gradual evolution of the notion of function often came from the obstacles in applying such notion in different context. From this observation, I suggest that mathematics educators and teachers should create the environment in which the learners will experience the need to 'broaden', if necessary, the notion of function. For instance, in the calculus course, the series representation of function will show the insufficience of conceiving a function as an operational process.

Furthermore, from these three studies, different researchers gave different perspectives concerning the underlying difficulty in the notion of function. Maryanskii emphasized the psycholoical difficulties caused in the variability approach; Vinner and Dreyfus suggested the reconsideration of using the Dirichlet-Bourbaki approach; Markovits, Eylon and Bruckheimer thought that the use of Dirichlet-Bourbaki approach was worthwhile but some components such as domain and range should be considerable reduced.

At the first glance, all these three perspectives seem to be diverse. However, they all indicate a dilemma: whether the Dirichlet-Bourbaki approach to function should be adopted in the elementary level? If the Dirichlet-Bourbaki approach is adopted, the psychological difficulties involved in mastering the notion of function will be expected; whereas if the Dirichlet-Bourbaki is abandoned, difficulties and conflicts may occur in mastering different mathematics concepts based on the notion of function in the later learning. Although Vinner and Dreyfus, Markovits, Eylon and Bruckheimer held two different opinions about the value of the Dirichlet-Bourbaki approach in school mathematics, they all agreed with one point: downgrading some aspects of function in the elementary level. Thus, the question now becomes to what extent the components such as domain and range could be reduced? Moreover according to Marvanskii's suggestion - there is a need to develop a familiar environment in order to assimi'ate the Dirichlet-Bourbaki approach - as well as Vinner's approach to the theory of mathematical thinking. I wonder, what kind of environment will provide our students an opportunity to change their implicit preliminary approach towards the

Dirichlet-Bourbaki approach to the notion of function? This leads to my further study in chapter 4.

3.4 The notion of limiting processes

In the elementary calculus, the notion of limit is often introduced by different kinds of iterative processes, for instance, tangent as the limit of secants; area under a curve as the limit of a sum. It is worth studying what kind of concept images held by students as to the notion of limit under such settings. The work of D. Tall and S. Vinner (1981) as well as that of R. B. Davis and S. Vinner (1986) did give pratially answers to this question. As well, in exploring the possibilities of elaborating didactical situation that would help the students overcome epistemological obstacles related to limits, A. Sierpinka (1987) also uncovered certain concept images of students.

On the other hand, as I have already discussed in chapter 1, the learners have their own preliminary approach towards mathematical concepts. Hence, it is also interesting to examine what kind of beliefs they possess towards the notion of limiting processes, before taking the calculus. E. Fischbein, D. Tirosh, p. Hess and U. Melamed (1979) carried out some studies to examine this aspect.

In this section, I shall concentrate on part of the above studies which directly relate to the concept images of limiting processes. Some educational implications will also be discussed.

3.4.1 The preliminary approach to the limiting processes

The aim of this section is to report the students' beliefs towards the notion of limiting processes before taking the calculus. Two studies which carried out by Fischbein, Tirosh, Hess and Melamed would be selectively presented.

The first study

The first study was conducted by E. Fischbein, D. Tirosh and P. Hess [7]. The aim of this study was to determine the resistance of the intuition of infinity through age and teaching influences, as well as the relation between the intuitive interpretation of infinite and the school achievement level of the subjects. The term intuition here was used for direct, global, self-evident forms of knowledge. For instance, the statement: 'If A > B and B > C, then A > C' may be considered as an intuitively accepted truth.

At the first glance, this study did not deal with the notion of limiting processes. However, from the development of the calculus (see chapter 2), the notion of infinity and that of limiting processes were tied up closely until Cantor who formulated the modern theory of cardinal infinities. With this connection, the work of Fischbein, Tirosh and Hess did reflect part of the concept images addressed to the notion of limiting processes.

The samples consisted of 470 primary and junior high school students - 46 grade 5, 58 grade 6 (19 out of 58 had participated some supplementary course in modern mathematics), 152 grade 7, 104 grade 8 and 110 grade 9 at Tel-Aviv. A questionnaire which included items such as the divisibility of segments and limits was handed out to the subjects.

<u>The second study</u>

The second study was carried out by E. Fischbein, D.Tirosh and U. Melamed [8]. The aim of this study was to investigate the degree of intuitive acceptance of a certain solution or interpretation given by the subjects.

In selecting the tasks, Fischbein, Tirosh and Melamed favoured the questions related to the notion of infinity. They explained that infinity exceeded the bound of daily experience and so only the intuitive leap could endow the meaning of this notion. Due to this assumption, a questionnaire consisted of some questions which directly related to the notion of limiting processes was formulated.

In order to measure the degree of intuitive acceptance of a certain solution, the subjects were asked to answer, in addition to each question, two group of 'check questions' which mainly measured the subjects' level of confidence and their feeling of obviousness as to their own solutions. For instance, one check question for the level of confidence was 'Have you doubted with respect to the correctness of your answer?'

The samples in this study included 108 grade 8 and 9 students who enrolled in different high schools situated in 4 different regions of Israel.

Questions related to the notion of limiting processes

In the following summary, I shall concentrate on those questions which directly related to the limiting processes.

Question 1

We divide the segment AB into two equal parts. Points H is the midpoint of the segment. Now we divide AH and HB. Points P and Q represent the midpoints of the segments AH and HB, repectively. We continue dividing in the same manner. With each division, the fragments become smaller and smaller. Question: Will we arrive at a situation such that the fragments will be so small that we will be unable to divide further? Explain you answer.

This question was posed in the first study. The answers could be divided into two main categories:

(1) 41% of the samples claimed that the process was infinite. The main explanations were that one could always continue to divide a segment and that there was an infinite number of points in every segment. Besides, some of them used the concrete form of explanations such as: With perfect instruments one could always go on to divide a segment.

(2) 55% of the subjects claimed that the process

was finite. Almost half of them explained that the segment was limited. The others suggested that practically, the imperfect instruments gave the end of the process or that somehow, everything would come to an end.

Furthermore, the 6th graders with better training in mathematics and those students with high level of school achievement favoured that the process was infinite. Whereas, more than 80% of those with low level of school achievemnt were classified as 'finitists'. Moreover, starting from grade 7, the percentage of 'infinitist' answers increased from grade 7 to grade 8, decreased from grade 8 to grade 9.

From the above observations, Fischbein, Tirosh and Hess commented that the two opposite categories of answers could not be completely explained as an effect of teaching or as a result of personal interpretation. For if the teaching process influenced the answers, there should be a tendency to improve in the upper grade. Whereas if the personal interpretation of the question (for instance, the successive divisions of a segment might be interpreted either as an ideal, mathematical process which could be go on for ever or as a concrete process which would stop at some point) influenced the answers, the 'finitists' would favour the concrete form of explanations and the 'infinitists' would address to the 'purist' arguments. However, the explanations given by both groups did not support this claim. Thus, additional factors such as the inconsistencies between the 'finitist' character of one's intellectual schemes and the notion of infinity itself must be taken into consideration. As we shall see, in the result of question 2, the inconsistence between intuitive approach and formal approach to the limiting process will be explored extensively.

<u>Question 2</u>
(a) Given a segment AB = a meter. Let us add to AB a segment BC = 1/2 meter. Let us continue in the same way adding segments, as described above, come to an end?
(b) Let us consider part (a). What will be the sum of the segments AB + BC + CD ... etc?

This question was asked in the second study. 84% of the total thought that the process would never come to an end in part (a). The justifications for the answers based on two typical arguments:

(a) A line segment could be divided indefinitely. For instance, one student wrote:

"A line consists of an infinity of points and each segment could be divided an infinity of times. Therefore, we can continue to add segments"

(2) A line segment could be extended indefinitely. For instance, one student expressed:

"It is always possible to add a line segment which will have the half of the length of the former line segment."

Moreover, the belief that the process was infinite had a high level of intuitive acceptance.

With regard to the answer in part (b), only 5.6% of all the subjects expressed that the sum of the segments AB + BC + CD + ... was 2. Half of this group refered to the reason that 2 was the sum of 1 + 1/2 + 1/4 + ... Yet, in general, the subjects in this group thought that the level of intuitive acceptance for this answer was low.

Around 17% claimed that the sum would be less than 2. The typical arguments in this category were:

"The sum tends to 2 meters. No matter how much we shall continue to add segments, we shall never reach 2 meters"

One subject even justified his (or her) answer by saying:

"$S = 2 - 1/\infty$ because there is no end to the sum of the segments."

Overall, these types of answers were considered to have the highest intuitive acceptance in comparsion with other answers to part (b).

Half of the subjects expressed confidently that the sum of the segments was infinite due to the fact that the process was infinite. The justifications included: 'A line can be extended endlessly'; 'The process can be continued endlessly'; 'There will be an infinity of segments'.

Fischbein, Tirosh and Melamed commented that the high percentage (84%) of correct answer in part (a) indicated that most of the subjects did anticipate the notion of infinite process. However, the subjects' notion of infinity was generally inconsistent with the formal theory (Cantor's approach to infinity) as the data shown in part (b). Fischbein, Tirosh and Melamed explained that for the intuitive understanding, infinity was only conceived as a potentiality. Whereas, in the formal theory, infinity was an actuality and did not have any intuitive interpretation. Hence, the intuitive interpretation of infinity would certainly cause inconsistence in formal theory. Finally, I quoted here a remark of D. Tall [25] in order to give a different perspective of this result in terms of the notion of concept image.

"Even before the limiting process had been discussed the concept image intuitively alights on the infinite nature of the process rather than the numerical limit."

<u>Question 3</u>
C is an arbitrary point somewhere on segment AB. We divide and subdivide segment AB as we did in question 1. Question: Will we arrive at a situation such that one of the points of division will coincide with point C? Explain your answer.

Around 80% of the total subjects in both studies answered affirmatively that the point C would be reached. The percentage of this answer was also high among the high achievers (75%) and was even higher among the 6th graders (91%) in the first study. Half of the 80% gave the reason such as: 'The segment was limited'; 'The division was infinite'. On the other hand, very few subjects, about 8% in the first study and 6.5% in the second one, claimed that the solution would depend on the location of the point C. Moreover, the measurment for the intuitive acceptance in the second study indicated that even those 6.5% who gave the correct answer did not believe strongly in their own solution.

Fischoein [6] pointed out that in Israel, the students started to learn about the irrational numbers in the grade 7. As the results of this question indicated, most of the students were not influence by this knowledge. By using the notion of concept image, I remark that the question did not give any stimulus addressed to the notion of irrational numbers. Whereas the concept image that every point could be reached by continually dividing a segment was called up intuitively. As one of the university students expressed:

> "Intuitively, I feel that the point C will be reached because the process of division is infinite. On the other hand, I know, from mathematics, that it is possible that the point should not be reached. Surely it will not reached if the point is an irrational one."

<u>Comments</u>

Two comments addressed to the notion of concept images are made in order:

(1) The results of Fischbein's studies showed that the subject's preliminary approach to the limiting processes resembles to that of mathematicians in ancient time. For instance, the two categories of answers to question (1) could be anticipated from the controversy between infinite divisibility and infinite indivisibility of a geometrical magnitude throughout the development of calculus; the suggestion that the sum of segments is infinite in part (b) of question 2 resembles to that of R. Suiseth. He wondered how an infinite series could have a finite sum (see 2.3); even those subjects who were aware that the finite sum less than 2 might be obtained appealed to either Zeno's argument or the arithmetic operations on infinity such as sum = $2 - 1/\infty$ which was suggested by J. wallis and his contemporaries (see 2.4). Furthermore, the inability to distinguish between the infinite divisibiltiy of a line segment and the nature of continuum in question 3 reminded me of the confusion of B. Bolzano (see 2.7). Besides, Fischbein's data in the second study indicated that most of the above concept images have a relatively high intuitive acceptance. These intuitive but incorrect views held by the subjects and mathematicians in ancient time do provide

mathematics educators with evidences that the formalistic approach to the notion of limiting processes is highly inconsistent with those concept images which are built from the 'finite' experience.

(2) The results in question 1 and part (a) of question 2 are diverse. In question 1, the 'finitists' and 'infinitists' views were split almost equally among the subjects. Whereas in answers to part (a) of question 2 showed a high precentage of 'infinitists'. Besides the fact that different samples were tested (question 1 was chosen in the first study and question 2 in the second one), this phenomenon may be explained by the fact that in the former case, the infinite process was carried out within a bounded line segment and so activated the idea that the process somehow would come to an end. On the other hand, in the latter case, the continually adding called up the idea that the line segment would be extended forever. From this observation, I remark that the concept images of the 'infinite process' depend not only on the iterative processes but also on the feature of the question itself.

3.4.2 <u>The work of Tall and Vinner</u>

The aim of this study was to examine whether the concept definition of the limit of a sequence can be integrated into the concept images which were generated under the British curriculum setting. Tall and Vinner (1981) [29] first explained how the concept images of the limit of a sequence were built practically in the Curriculum, then applied the notions of concept images and concept definitions to analyse the data which were obtained by Tall (1977) [22]. Some potential conflict factors and teaching implications proposed by Schwarzenberger and Tall (1978) [28] would also be discussed.

The British curriculum

In British curriculum, the notion of a sequence was introduced with reference to the notion of function, that is, a sequence was defined as a function having for domain the set of natural number or non-integers. Examples of sequences such as arithmetic and geometric progessions were given. Later, the use of different iterative processes to approximate a given number was discussed. Finally, the notion of convergence of a sequence was formally introduced.

The questionnaire

A questionnaire concerning the notion of the limit of a sequence was handed out to 36 first year mathematics students at Warwick University, England. It consisted of four questions.

<u>Question 1</u>
Have you been taught the concept of the limit of a sequence $s_1, s_2, s_3, \ldots$ in school?
A. with precise definition
B. informally
C. not at all
(tick the most appropriate)

<u>Question 2</u>
Find the limit (if it exist) of
$$\lim_{n\to\infty} (1 + 9/10 + 9/100 + \ldots + 9/10^n)$$

<u>Question 3</u>
If you know the definition of the limit of a sequence, write it down:
$s_n \to s$ as $n \to \infty$ means:

<u>Question 4</u>
Is $0.\dot{9}$ (nought point nine recurring) equal to one, or is it just less than one? Explain the reason behind your answer.

<u>Results on question 1</u>

10 out of 36 students claimed that they had met the precise definition of limit and only 7 out of 10 gave a suitable definition in question 3. On the other hand, 21 answered that they learned the informal definition and only

1 gave a definition.

Results on question 2

29 out of 36 wrote down

$$\lim_{n \to \infty} (1 + 9/10 + 9/100 + \ldots + 9/10^n) = 2$$

Results on question 4

	Number of students	Remarks
$0.\dot{9} = 1$	14	only 1 student in this group claimed that s/he knew the precise definition of the limit
$0.\dot{9} < 1$	20	
uncertain	2	favour $0.\dot{9} = 1$

Some of the students who claimed that they knew a precise definition of the limit of a sequence not only believed that $0.\dot{9} < 1$ but also gave reasons such as:

"Just less than one, because even at infinity the number through close to one is still not technically equal to one"

"It is just less than one, but the difference between it and one is infinitely small."

Tall and Vinner commented that the concept images of infinity as a limiting process and that of infinitesimal as arbitarily small were evoked respectively. Even one of those who claimed $0.9 \doteq 1$ gave the following explanation:

> "I think $0.9 \doteq 1$ because we could say '$0.\dot{9}$ reaches one at infinity' although infinity does not actually exist, we use this way of thinking in calculus, limits, etc."

Besides, there was a conflict between 0.999..9 to a finite number of places and the infinite expansion. I remark here that this confusion may come out of the experience of decimal operations as well as that of using calculator. Often, students learn how to read and how to operate on finite decimal places prior to any notion of infinite decimals. Even when the notion of infinite decimals is introduced through converting fractions into decimals, finite decimals are frequently used to approximate infinite decimals in the science course. For example, 0.3333 may be used to approximate 1/3 up to the accuracy of four deicmal places. In the beginning, the symbol '$\doteq$' which stands for 'is approximately equal to' is emphasized. As time goes on, nobody bothers to put one dot above the equal sign.

Moreover, as the calculators become widely available, students get their answers from pressing the keys instead of carryi ig out the tedious calculations. Unfortunately, the symbol '≑' is not on the keyboard. The equivalences between '≑' and '=' are drawn and so the concept image that any real numbers can rewritten as a finite number of decimal places is generated.

Remarks on the results concerning question 2 and question 4

In mathematics, the computation of

$$\lim_{n \to \infty} (1 + 9/10 + 9/100 + \ldots + 9/10^n)$$

resemble the problem of determining whether $0.\dot{9}$ is equal to or just less than one. However, 14 students in Tall's samples claimed that

$$\lim_{n \to \infty} (1 + 9/10 + 9/100 + \ldots + 9/10^n) = 2$$

but $0.\dot{9} < 1$. Tall and Vinner explained that question 2 and question 4 evoked different parts of the concept images of the limiting processes. In order to activate the suitable concept image to question 4, a follow up test was given by Schwarzenberger. In a lecture, he said:

"I will write down a number of decimals and ask you to express them as fractions in the simplest form. Example 0.5 = 1/2."

He then wrote on the blackboard:

1) 0.25
2) 0.05
3) 0.3
4) $0.\dot{3}$ = .333...
5) $0.\dot{9}$ = .999...

and pointed out that $0.\dot{3}$ and $0.\dot{9}$ are recurring.

In this test, 24 students answered either $0.9 \doteq 1$ or $0.9 \doteq 1/1$. Among them, 13 were those who said

$$\lim_{n\to\infty} (1 + 9/10 + 9/100 + \ldots + 9/10^n) = 2$$

but $0.\dot{9} < 1$ in answering the questionnaire. Furthermore, the answers to (4) and (5) showed that some of those students who claimed earlier $0.\dot{9} < 1$ experienced conflict in their mind. For instance:

Student		Answer	The comment of Tall
A	(4)	$0.\dot{3} = 1/3$	Student A saw the conflict and responded with exasperation
	(5)	$0.\dot{9} = 3 \times 0.\dot{3}$ $= 3 \times 1/3$ = rubbish	
B	(4)	0.333... = 1/3 no fraction (this answer was crossed out)	According to the follow up discussion, student B came to the conflicts in
	(5)	0.999... = no fracion	(5) and this

		forced him back to cross out his answer in (4)
C	(4) $0.\dot{3} = 1/3$ (5) $0.\dot{9} = 1$ or (none exists)	Student C visualized two possible alternative, but found it difficult to express
D	(4) $0.\dot{3} = 1/3$ (5) $0.\dot{9} = 0.999$	Student D switched from $0.\dot{3}$ as an infinite decimal to $0.\dot{9}$ as a finite decimal
E	(4) $0.\dot{3} = 1/3$ (5) $0.\dot{9} \doteq 1$	Student E switched from $0.\dot{3}$ as an infinite decimal to $0.\dot{9}$ as a number approximately equal to one

I remark that in the above answers, there was a tendency to accept $0.\dot{3} = 1/3$ but not $0.\dot{9} = 1$. Prehaps, the cause for this tendency came from the fact that $0.\dot{9} = 1$ could not be verified by long division. The representation $0.\dot{3}$ of 1/3 could be obtained from dividing 1 by 3. However, dividing 1 by 1 gave $1.\dot{0}$ not $0.\dot{9}$. As Tall and Schwarzenberger [23] pointed out that pschologically, two different representations $1.\dot{0}$ and $0.\dot{9}$ contradicted the belief that there was only one infinite decimal representation for each real number.

Moreover, some similar findings were reported by A. Sierpinska (1987) [20]. In her studies, she described the reactions of her six 17 year old humanities students after the following method was used to obtain $0.\dot{9} = 1$:

$$\begin{aligned} \text{Let} \qquad x &= 0.\dot{9} \\ 10x &= 9.\dot{9} \\ 10x - x &= 9.\dot{9} - 0.\dot{9} \\ 9x &= 9.0 = 9 \\ x &= 9/9 = 1 \end{aligned}$$

Only one student accepted that $0.\dot{9} = 1$ and the above mathematical procedures but gave no explanation. Some accepted the result but looked for different explanation. For instance:

"It cannot equal to one unless ... unless we assume that it really goes on to the very infinity... . Then it may equal to one. Because these small difference also get smaller and smaller in an unlimited way."

"$0.\dot{9} = 999.../999... = 1$"

Yet, some other rejected both the result and the mathematical procedures. For instance, one student expressed:

"Oh, but here it doesn't hold! ... Because it's infinity ..."

Half of them held the view that the result $0.\dot{9} = 1$ was not reasonable but the mathematical procedure was valid. For instance, one girl said:

"Arithmetically or algebraically, it's all right, but in reality ... It will be close to one but not equal to one. There will be a slight, very slight difference,..."

Sierpinska's study did show that the concept image generated from experience would take over those concept images created from mathematical context when suitable interpretation needed to be sought. After all, the intellectual and counter-intuitive features of the representation $0.\dot{9}$ for one led to the answer $0.\dot{9} < 1$.

<u>Results on question 3</u>

In the results on question 3, Tall and Vinner found that one of the popular concept images of '$s_n \to s$ as $n \to \infty$' is 's_n apprcaches s, but never actually reaches there'. The causes of such a concept image were examined by Tall, Vinner and Schwarzanberger. A summary was edited as follows:

(1) The verbal definition of the limit of a sequence calls up this concept image. Informally, $s_n \to s$ as $n \to \infty$ means that as n gets very large, s_n gets close to s. The colloquial phrase 'gets close to' in the verbal definition helps generating the concept image that s_n is not allowed equal to s. As a matter of fact, the word 'close' has the meaning near but not coincident with in daily experience. As one student wrote:

"$s_n \to s$ means s_n get close to s as n gets large, but does not actually reach s until infinity."

(2) The verbal definition of the limit loses the precision with reference to the formal definition:

"A sequence (s_n) of real numbers tends to a limit s is: given any positive real number $\epsilon > 0$, there exists N (which may depend on ϵ) such that

$$|s_n - s| < \epsilon$$

for all $n > N$."

An informal verbal translation - as n gets very large, s_n gets close to s - specifies neither how close, how large nor the relationship between the ϵ and N. The loss of precision will enforces of the concept image generated by the intuitive interpretation of the notion of limit.

(3) The curriculum itself does not provide enough examples to eliminate such concept image. For instance, most of examples of the limiting process is either $s_n = 1/n$ or $s_n = 1 + r + \dots + r^{n-1}$, $-1 < r < 1$. All these sequences have the property that s_n does not equal to its limit. When Tall presented the example

$$s_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1/2n & \text{if } n \text{ is even} \end{cases}$$

to a group of four students, the students claimed that (s_n) was not one sequence but two and so it could not be considered as a sequence some of whose terms equalled the limit!

A proposal by Tall and Schwarzenberger

Tall and Schwarzenberger proposed that to avoid the concept image that 's_n does not actually reach s until infinity', the stress should be placed on that 's_n and s are practically indistinguishable' and not on 'n very large'. The phrase 'practically indistinguishable' could be demonstrated by the limited accuracy of practical measurement of a real number. For instance, consider the sequence $k_n = 0.999...9$ with n 9s, then $1 - k_n = 1/10^n$. For any desired degree of accuracy ϵ satisfying that $1/10^n$ is less than ϵ, k_n is indistinguishable from 1 to within ϵ. In other words, the limit is reached within the degree of accuracy ϵ.

Comments

The existence of the concept image that the terms of a sequence cannot reach its limit is not accidental. Often, the process of finding the limiting value s of a sequence, s_1, s_2, ..., s_n, ... suggests the idea that the limit is never complete. In order to deal with the infinite situation, the learners often seek for the similarities in the 'finite' situation and so come to the conclusion that the limit s will enjoy the same property of the terms. For instance, $0.9 < 1$, $0.99 < 1$, $0.999 < 1$, ... and so the limit of this sequence should be less than one. As a matter of fact, this argument was used by S. L'Hulier to claim that the terms of a sequence was always less or greater than, but not equal to its limit (see 2.6). Moreover, the assertion of Cauchy that the limit of a sequence of continuous function is continuous suggested the similar idea (see 2.7). My point is that the concept image that the terms of a sequence never reach its limit has its own root in making the transition from 'finite' to 'infinite'.

Besides, more evidences can be found in the historical development of calculus. This concept image not only was held by lots of mathematicians such as Gregory of St. Vincent (see 2.4 and 2.6) but also caused much discussions, for instance, the Jurin-Robins controversy in England (see 2.6). From this observation, the proposal by Tall and Schwarzenberger may not solve the problem completely due to the fact that the limiting value s has to be known in advance. My point is that the explanations addressed to the question:'How can the limiting value be found?' may also lead to such misconception. Perhaps, a <u>constant</u> enforcement stressed on the examples such as 0, 1, 1/2, 1/3, ... may help in eliminating this incorrect concept image.

3.4.3 The work of Davis and Vinner

The aim of Davis and Vinner's study (1986) [3] was to examine the students' views towards the notion of limiting processes under a pedagogical approach which was based on the idea of teaching with understanding. Analyses were done by applying the notion of concept image and concept definition. Some naive conceptualizations that might lead to confusions in the formal theory of limit were also discussed.

Problem background

With reference to the work of Tall and Vinner (1981), Davis and Vinner criticized that one could not really tell the causes of those concept images such as 's_n approaches s, but never reaches it' or 's_n gets close to s as n gets large, but does not actually reach s until infinity'. That is, the answer to the questions such as 'Did this phenomenon come from the bad pedagogy?' or 'Was it inherent in the concept itself?' would not be clear if one did not take the teaching strategy into considerations.

A pedagogical approach begins with uderstanding

In order to take the pedagogical approach into considerations, Davis designed a 2-year calculus course which based on the idea of 'reasonable reponses to reasonable challenges'. A group of 15, aged 15 - 16, capable 12th graders attended this course at University high school, Urbana-Champaign.

In this course, the notion of the limit of a sequence was introduced by finding the work done for compressing a spring. A sequence of refined work done was formulated when the constant force (the maximum force and the minimum force) was assumed over shorter and shorter intervals. A general mathematical formula which represented such sequence would be derived. More examples of sequences were given through finding the area under $y = x$, $y = x^2$,..., etc. Some specific sequences like

$$(1 + 1)^{1/1}, \quad (1 + .1)^{1/.1}, \quad (1 + .01)^{1/.01}, \ldots$$
$$(\sin\pi)/\pi, \quad (\sin\pi/10)/(\pi/10), \quad (\sin\pi/100)/(\pi/100), \ldots$$

were also discussed.

As the course went on, the students were asked to formulate sequence of new types and establish a definition of the limit of a sequence. During this construction, different versions of definitions were suggested by examining different examples. For instance, the definition like 'the limit L is the number that the term s_n in the sequence are approaching' will be challenged by the following arguments: Consider the sequence 0.9, 0.99, 0.999,..., the term $s_n = 0.999...9$ with n 9's is approaching 1066, but s_n and 1066 do not get close to each other.

After some more discussions between definitions and examples went on for a while, three definitions were arrived:

(1) An informal form of definition:

The number L is the limit of the sequence s_1, s_2, ..., s_n, ... if $s_n = L$ within a tolerant error, for the terms large enough.

(2) A geometric form

Consider the graph of a sequence s_1, s_2, ..., s ... if L is the limit of the sequence, then for any positive tolerance, a cut-point N (depends on the size of the tolerance) can be founded so that all the term s_n will be lie in the shaped strip (see Figure 1) except for a finite number of terms.

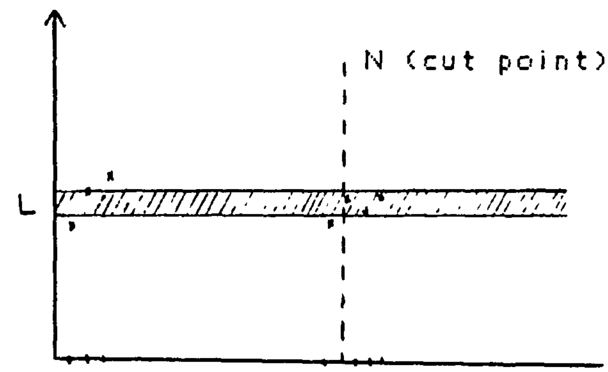


Figure 3.1

(3) The usual formal definition:

The number L is the limit of the sequence s_1, s_2, ..., s_n, ... if given any $\epsilon > 0$, there exists a positive integer N such that

$$L - \epsilon \leq s_n \leq L + \epsilon \qquad \text{for all } n > N$$

Examples such as 1, 0, 1, 0,... with $\epsilon = 2/3$ and $L = 1/2$ were used to clarify the meaning of the word 'any' in the formal definition.

After the students acquired the formal definition of limit, they started to learn the proofs of some basic theorems such as 'the uniqueness of a limit', the limits of sums, ..., etc. Moverover, the distinction between 'limits' and 'approximations' was demonstrated by the uniqueness of limit. The weakness of intuitive ideas was made clear by considering the set of rationals, Cantor's diagonal counting, the set of reals etc. Axioms like law of Trichotomy, the Archimedean Postulate and the existence of least upper bounds were also introduced.

According to the way which the material was presented and the results of the written tests during the academic year, Davis and Vinner claimed that the student did understand limits in such pedagogical settings.

<u>A written test</u>

After the summer vacation of that academic year, the students were asked to write a few paragraphs about the notion of limit of a sequence including:

1) A description of a limit of a sequence in intuitive or informal terms

2) A precise formal definition

The results of the test indicated that some common naive misconceptions did exist. They were:

(1) A sequence must not reach its limit. For

example, one student wrote:

"... the limit of a sequence is never reached by that sequence."

This finding resembles that of Tall and Vinner (see 3.4.2). Not surprisingly, the emphasis on mathematical context cannot eliminate such concept image due to the fact that the concept images generated from intuitive idea often take over those concept images generated from the formal theory, as A. Sierpinska's study indicated (see 3.4.2).

(2) There is an implication that a sequence is monotonic. Davis and Vinner explained that this concept image might be identified by the phrase 'going toward a limit'. Besides, this concept image might also come from the fact that the montonic sequences were used to introduce the notion of limiting processes in the beginning of the course. I remark here that this idea was also held by J. D'Alembert and some of his followers (see 2.6).

(3) The limit of a sequence must be an upper or lower bound for its terms. For instance:

"The limit is a boundary beyond which the sequence cannot go."

"The limit is the point past which the sequence does not go ... "

This idea resembles that of Nicholas of cusa. He asserted that the triangle and the circle were the polygons with the smallest and the greatest number of sides (see 2.3). Similiar ideas can also be found among other mathematicians throughout the development of calculus.

(4) A sequence has a 'last' term. For instance:

"The limit of a sequence is the end of the sequence, ... "

"The limit of a sequence is the end point for a list of numbers..."

Again, this idea resembles that of Gregory of St. Vnicent and lots of other mathematicians (see 2.4 and 2.6).

(5) One can somehow 'go through infinitely many terms of a sequence'. For instance,

"The limit of a sequence is something that the limit approaches. It is the number or value of what you would get if you continued the sequence infinitely ..."

This explanation is similar that of Nicholas of Cusa. He expressed that 'the infinity can be approached by going through the finite' (see 2.3).

(6) There is a confusion between the value of the function at x_0 and the value of $f(x)$ as x 'goes' to x_0. Davis and Vinner suggested that the confusion indicated a potential overlap between the limit of a sequence and the limit of function.

(7) A sequence has some obvious, consistent pattern. For example,

1, 2, 3, 1, 1/2, 1/3, 1/4, 1/5, 1/6,...

will be excluded. This view was also held by some subjects in the study of Tall and Vinner (see 3.4.2).

Besides the above naive misconceptions, some students neglected the sequential order, for instance, first ϵ, then N, in the definition of limit. As one student wrote:

"The limit of sequence is the number from which all the term in the sequence, after a certain point, vary only by a limit number ϵ."

Also, there was confusion between the fact that 'n does reach infinity' and the question of 'whether s_n may possibly 'reach' its limit'.

Sources of naive misconception

In order to answer the question: 'Where do these naive misconceptions come from?', Davis and Vinner listed five sources:

(1) The infuence of language

The word 'limit' and the phrase 'n goes to infinity' lead to those mental repsentations that should not be part of the notion of limit. In daily usage, the word 'limit' means a bound that may not or cannot be passed. For instance, the speed limits; the city limit (the boundary of the city); ... etc. The different meaning of limit in daily experience may distort the mathematical notion.

Davis and Vinner remarked that in the 2-year calculus course, the word 'limit' was not introduced until the notion of limit seemed to be well established. Instead, the terminology 'associated number' was used. For instance, the sequence 1, 1/2, 1/3, 1/4, ... has the associated number zero. However, it is not yet clear whether this setting will reduce the confusion.

(2) Assembling mathematical representations from pre-mathematical fragments

Those ideas which come from the experience may also affect one's mental representations. For example, the time sequence of occurrence may effect the idea of choosing ϵ before determining N.

(3) Building concepts within mathematics

Besides the meaning of the word outside mathematics and the ideas from daily experience, Davis and Vinner commented that the concept images within mathematics itself have to be established gradually. For instance, the notion of limit is so complex to put into a single idea. The learners cannot master the whole idea immediately. Thus, parts of the idea may get adequate representation before other parts will.

(4) The influence of specific examples

Sometimes, examples may lead to misconceptions. For instance, the notion of limit is introduced and then elaborated by way of monotonic sequences in the above 2-year calculus course. It is not surprising to find out that the monotonicity became part of the concept images of limit as the result of the written test showed.

(5) Misinterpreting one's own experience

Misconceptions may also come from misinterpreting one's own experience. For instance, finding the limit of $s_n = 2n/(n+1)$ by rewriting s_n as $2/(1+1/n)$ may get the idea that the availability of a simple algebraic formula for s_n is an essential part of the theory of limit.

<u>A suggestion</u>

Davis and Vinner remarked that the misconceptions seemed to be part of the students' ideas about limit. Hence, it is worth to provoke a more explicit confrontation between a student's correct idea and his or her incorrect ideas in order to probe into his or her understanding. For instance, the teacher may ask the question such as 'Some people say that the terms in a sequence are not equal to the limit. Other people disagree. What would you say to these people to reconcile this disagreement.'

<u>Comments</u>

The study of Davis and Vinner provides us with more information about the students' ideas towards the notion of limit. As the data indicated, some of the misconceptions were inherent in the notion itself; some came from personal experience or specific examples. Among all those revealed misconceptions, many of them resemble the ideas held by the contributors to the calculus. The resemblance does reflect one point - the struggle for abridgment between 'finite' and 'infinite'.

Moreover, the result of this study also showed that even a pedagogical approach with strong mathematical training did not eliminate some misconceptions. The concept images which come from the intuitive ideas tend to take over the concept definition as well as those concept images which are purely built in the mathematical context. The subjects in Davis and Vinner's study might understand the idea of limit during the academic year. However, this 'understanding' will soon be forgotten if no suitable enforcement is added.

3.4.4 Summary

The studies of Fischbein, Tall and Davis did show that the concept images generated from the intuitive ideas create difficulty in acquiring the notion of limit due to the inconsistence between the intuitive interpretations and the formal theory. In order to overcome these difficulties, new teaching approach should be sought. Tall (1981) [23] asserted that the theory of 'superreal numbers' which is considered closer to the students' intuitive ideas may be used to clarify the notion of infinity. However, the operations in the 'superreal number' system are far too complex for the students to handle. Tirosh, Fischbein and E. Dor (1985) [30] suggested that a conflict-teaching approach may make the students aware of the conflicts between the intuitive idea and the formal theory. Similar suggestion was proposed by A. Sierpinska (1987) [20] as well. However, there is no solid evidence to back this approach. The results of Davis' study showed the insufficience in his approach. He further upheld that putting more emphasis on the conflicts between students' correct ideas and their incorrect ideas may provide us with more information about the students' ideas towards a given notion. From all the suggestions, it seems to me that in order to justify the conflict-teaching approach, it is worth to investigate how students will react to a situation in which both a correct idea and an incorrect idea are given. This leads to my further study in chapter 5.

3.5 The notion of continuous function

The aim of this section is to present the work of D. Tall and S. Vinner (1981) as well as that of S. Vinner (1987) concerning the concept image and concept definition of continuous function at college level.

In elementary calculus, the notion of continuous functions are always taught intuitively. The students are told that the graph of a function is continuous if it can be drawn without lifting the pen from the paper. Later on, the students may encounter the limit definition, that is:

"f is said to be continuous at a if $\lim_{x \to a} f(x) = f(a)$"

or the $\epsilon - \delta$ definition, that is:

"f is said to be continuous at a if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, whenever $|x-a| < \delta$".

In the work of Tall and Vinner (1981) as well as that of Vinner (1987), they tried to reveal what kind of concept images the students held with reference to the notion of continuity in such setting. In the former work, the author also discussed some conflicts between the concept images formed by the intuition and the formal definition in the cognitive process.

3.5.1 The work of Tall and Vinner

Tall and Vinner (1981) [29] examined the concept images of continuous function at Warwick university, England.

The study

According to School Mathematics Project Advanced Level Texts in England, the notion of continuity is taught intuitively with a light discussion on the behaviour near a discontinuity. At the end of the course, the limit definition and $\epsilon - \delta$ definition are formally introduced. This study was carried out when the students had just arrived at the university.

41 students participated the study. Their

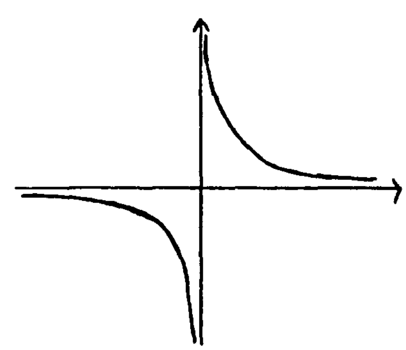
mathematical background was considered to be satisfactory. Most of them got an A or B grade in A-level mathematics. A questionnaire which consisted of five questions and four of them had pictures was handed out to them. In this questionnaire, the students were asked:

"Which of the following functions are continuous? If possible, give your reason for your answer."

<u>Results</u>

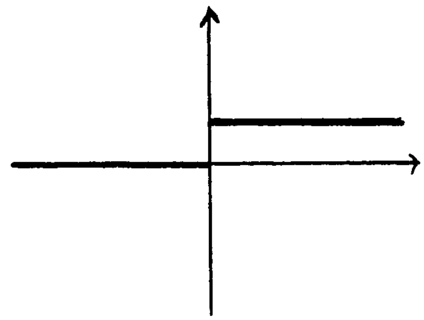
The results indicated that the common concept images were 'all in one piece' and 'no gaps'. However, some correct answers were justified with wrong reasons. For instance, the function

$f(x) = 1/x,\ x \neq 0$



is continuous 'because it is given by only one formula' or the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



is discontinuous 'beacuse it is not given by one formula'.

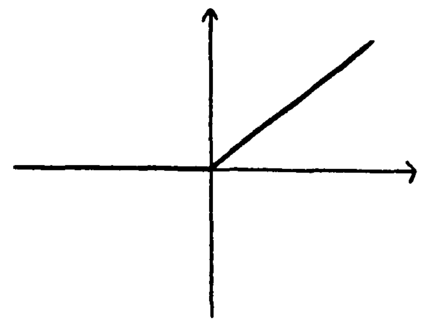
Sometimes, the students used some non-relevant argument to justify their claim. For example, when the function

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

is given, the reason such as 'It has a continuous pattern of definition' is used to justify the continuity whereas the reason such as 'It is impossible to draw' is used to claim the discontinuity.

Furthermore, there was also a confusion between continuty and differentiability. For instance, one student claimed the discontinuity of the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$



by using the reason like 'There is a sudden change in gradient'

<u>Comments by Tall and Vinner</u>

Tall and Vinner commented that the common concept images - 'all in one piece' and 'no gaps'- were correct in global sense. As the notion of continuity is defined locally, the mental pictures associated with the above concept images may lead to confusion in the development of formal theory. For instance, f : rationals → rationals given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x^2 < 2 \\ 1 & \text{if } x > 0 \text{ and } x^2 > 2 \end{cases}$$

has a graph with a gap but is continuous over its domain. Moreover, the function given by:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1 - x & \text{if } x \text{ is irrational.} \end{cases}$$

has a graph with more than one piece but is continuous at

1/2.

3.5.2 The work of Vinner

Vinner (1987) [36] studied the concept image and reasoning of continuous function at Hebrew university, Jerusalem.

The study

A group of 406 science students whose mathematical background was relatively strong were examined. Most of them had taken a short calculus course in high school. When they arrived at the university after their military service (3 years for men and 2 years for women), it was expected that they had forgetten a great amount of content. A questionnaire was handed out to them a few weeks after the notion of continuous functions was taught. These students learned mainly the intuitive definition. Some also learned the limit definition, the $\epsilon - \delta$ definition or the intermediate value definition, that is:

"A function $f(x)$ is continuous on $[a,b]$ if for every x, y such that $a \leq x < y \leq b$ and for any intermediate value c between $f(x)$ and $f(y)$, there exists d, $x < d < y$

such that f(d) = c".

A questionnaire consisted of two parts was handed to them. The first part had seven graphs and the second part had five functions defined by formulae without any pictures. The students are asked to determine whether the given graphs of functions or functions are continuous. Explanations were encouraged.

Results

The results of this study indicated that most of the students got the correct answers. However, some of them gave the wrong reasons. Five categories of the explanations were found:

(1) 40% of the subjects considered that continuity was the same as being defined and discontinuity was as being undefined. For instance, 'The function is continuous because it is defined for every x' or 'The function is discontinuous because it is not defined for every x'.

Vinner explained that this confusion might come from the false contraposition of 'If f(x) is not defined at certain point, then it is discontinuous', that is, if f(x) is defined at every x, then it is continuous.

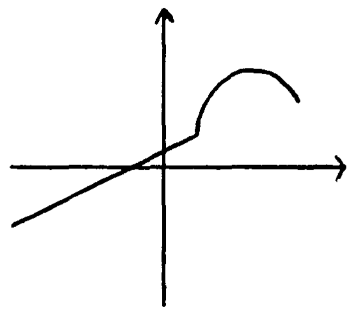
(2) Continuity or discontinuity are related to the graph. For instance, 'The function is continuous because its graph can be drawn in one stroke'; 'The graph has no jump'; 'It is in one piece'; 'The function is discontinuous because its graph has two parts which do not meet' or 'There is a gap in the graph'.

(3) There is certain reference to the concept of limit. For instance, 'The function is continuous because it tends to a limit for every x'. Some students wrote down 'f is continuous because $\lim_{x \to x_0} f(x) = f(x_0)$' or 'f is discontinuous because $\lim_{x \to x_0} f(x) \neq f(x_0)$' without any specification of x_0 in the particular questions.

(4) No explanation. Some answers were just statements that the given graphs of functions or functions are continuous or discontinuous without any explanation.

(5) Less than 3% of the subjects gave reasons like 'a continuous function is one to one'; 'a graph or a formula is discontinuous because it consists of two functions'. For

instance, the graphs of functions such as the one given by:



were considered to be discontinuous. Others used non-relevant reasons such as that 'a function is continuous because it has no inflection point' to claim continuity. Besides, there was a confusion between continuity and differentiability.

Furthermore, only few students tried to graph the function when the formulae were given.

As the above data indicated, wrong reasons were often used to justify the answers. Vinner explained this phenomenon by the assertion that a student might identify successfully examples or non-examples of a given mathematical concept concept without knowing why. After all, the act of identification did not have to follow any process of mathematical thinking. When students were forced to give explanations, they tended to justify their answers based on their concept images or some non-relevant but mathematical arguments.

Remarks on B4 and B5

Among all the questions, the correct percentages to the questions

$$\text{B4:} \quad y = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\text{and B5:} \quad y = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

were relatively low. Vinner took one step further to analyse the answers to these two questions.

The results indicated that 25.5% in B4 and 23% in B5 used the reason 'The function is defined for every x' to justify for continuity. On the other hand, 7% in B4 and 6.5% in B5 claimed the discontinuity by explaining that the function is not defined for every x. Vinner commented that these 7% and 6.5% showed the inability to understand the formulae since both functions were defined for every x.

Besides, 22% in B4 used the limit concept to justify for the discontinuity. Vinner remarked that the use of limit concept in B4 was accidental because of the relative low percentages to the similar situation in B5.

Comments

My comments on both studies are as follows:

(1) Comparing the results of the two studies, I find that the common concept image of a continuous function in the former study is 'all in one piece' and in the latter one is 'defined for every x'. The cause of such a diversity may be explained by different emphases in different curriculum settings, different teaching strategies as well as the questionnaire itself. However, no conclusion can be drawn without further analysis on the first two aspects. For instance, the students may avoid to use the intuitive arguments if examples contradictory to intuition are discussed in the class. Also, inability to understand or to operate on the formal definition may lead to no explanation or non-relevant explanation. A remark of Tall and Vinner in the first study that:

> "The notion of continuity is rarely alluded to as a formal definition, but a concept image is built up from informal usage of the term"

may give us the insight as to why the students in England are more comfortable to justify their answers by intuitions.

Furthermore, in the first study, both function and its graph were given in most of the questions. Whereas, in the second study, either function or graph was given. In answering the questions only involved formula, the concept image related to the graph of function may not have been evoked and so has led to different arguments in both studies.

(2) Studying the distribution of the explanation for correct answers to part I of the questionnaire in Vinner's study (see table 3.1), I observe that in A3 and A6, the numbers of

<u>Distribution of explanation for correct answers to questions A1 - A7</u>

(the number indicate percentage out of correct answers)

Question Categories	A1	A2	A3	A4	A5	A6	A7
1	46	44	20	32	47	22	50
2	22	22	33	21	20	35	21
3	15	15	17	13	15	13	4
4	10	14	12	25	14	25	19
5	7	15	18	9	4	5	6

(Table 3.1)

students giving reasons in category 1 decreases and those in category 2 increases. Vinner did not explain this phenomenon. Examining closely the questions in part I, I find that A3 has a jump in the graph and A6 has a gap. All the other questions in part I did not have these properties. My point is that if the confusion in the category 1 comes from the false contraposition of 'If f(x) is not defined at certain point, then it is discontinuous', that is, if f(x) is defined at every x, then it is continuous, how can one explain the fact that the percentages of category 1 go down in A3 and A6 where both functions are defined for every x? Hence, the confusion in category 1 needs further justification. For instance, what is the students' conception of the phrase 'defined for every x'.

According to the information at hand, I do not know whether the 33% of category 2 in A3 and 35% in A6 used the explanations in category 1 for other questions. Yet, I argue that the increase in number of students in category 2 indicated, that at the moment of decision making, the feature of a given task also contributed to the evocation of concept images. In this case where the graph highlights a jump or a gap, the concept images as those in category 2 become stronger. Moreover, this may also be elaborated by the low percentages of category 2 in B1 - B5 (see table 3.2).

When a formula is given, the concept images related to the

<u>Distribution of explanation for correct answers to questions B1 -B5</u>

(the number indicate percentage out of correct answers)

Question Categories	B1	B2	B3	B4	B5
1	82	69	60	25	46
2	2	8	4	7	3
3	4	4	4	22	8
4	8	13	25	30	31
5	4	6	7	16	12

(Table 3.2)

graph, or the connection between the formula and its graph cannot be evoked. Instead, the concept image related to the mathematical terms such as category 1 or category 3 play the dominant role.

(3) Vinner commented that the 22% in B4 using the concept of limit to justify for the discontinuity, is accidental. It is not clear to me in which way the students used the concept of limit, whether they just stated the limit definition or whether they attempted some calculations. However, I remark that even though B4 and B5 are similar, the process of evaluating $\lim_{x \to 0} \sin (1/x)$ and $\lim_{x \to 0} x^2 \sin (1/x)$ are different. Some trial values or

examining the behaviour of sine function may lead to the conclusion that $\lim_{x\to 0} \sin(1/x)$ does not exist. Thus, the function in B4 is discontinuous. On the other hand, more steps are needed to justify the continuity of B5, namely, showing that $\lim_{x\to 0} x^2 \sin(1/x)$ exists and its limit equals to the value of the function at $x = 0$. Both of the existence and evaluation of $\lim_{x\to 0} x^2 \sin(1/x)$ are not straight forward. Even worse, the graph of $y = x^2 \sin(1/x)$ is not easy to draw. All these may also affect the percentages in category 3 for B5.

(4) A small number of students in the first study and very few students in the second study justified the continuity by 'a single formula' and the discontinuity by 'not a single formula'. This idea resembles that of Euler, Arbogast and their contemporaries (see 2.6). However, the low percentage of this misconception may be viewed as the result of teaching and learning. In school, the notion of function is taught prior to the notion of continuity. According to the Dirichlet-Bourbaki definition of function (see 3.3), the student learns that a function can have more than one 'rule' or 'formula'. Besides, the connection

between function and its graph is well established before any notion of continuity is introduced. I argue that under this setting, the background knowledge of continuity (at least the intuitive one) is built up carefully and so the students do have better preparation to eliminate certain misconceptions. Of course, I do not exclude the possibility that some other misconceptions may be created. At the time of Euler, mathematicians still fought for a suitable meaning for both 'function' and 'continuity'. The images they held represented the struggle for seeking a criterion to clarify these notions and, so, were considerably incoherent. Whereas, a clear presentation of the notion of function in present-day curriculum eliminates certain misconception. Hence present students' concept images are not likely to be the same as those of past mathematicians.

Furthermore, the confusion between continuity and differentiability may be identified with that of Lagrange, who thought that every continuous function can have a series expansion (see 2.6). The percentages of this misconception in both studies are also low.

3.5.3 Summary

Both studies showed that the students justified their answers by using their concept images or even some non-relevant but mathematical arguments. The diverse images shown in both studies suggested that different concept images will be formed or evoked under different environments. However, most of the revealed concept images do not correspond to the formal definition of continuity. Hence, in order to develop a coherent teaching strategy which is consistent with or which eliminates those concept images, more studies have to be done on the following two aspects: (1) Revealing more concept images in different settings and (2) exploring the conceptual structure of the notion of continuity.

3.6 The notion of tangent

The aim of this section is to report two studies which reveal the students' concept images about the notion of tangent after they have learned about this notion in the calculus course.

At early stage of mathematical studies, students experienced the tangent in circle geometry. They learn that the tangent is a line that touches the graph at one point and does not cross it. Whereas, in calculus course, two approches are used to introduce the notion of tangent, namely, tangent as the limit of secants or as intuitive concept. In the latter approach, the emphasis is put on calculating the slope of the tangent which is tied up to the notion of derivative. Hence, it is interesting to study whether the notion of tangent in calculus contributes to some significant change to the notion of tangent in circle geometry? S. Vinner (1982) conducted a study to examine this problem. Later, D. Tall (1987) carried out another study to compare the students' ideas about the notion of tangent through using computer to draw a line through very close points on the graph, with the traditional strategy, that is, assuming an intuitive knowledge of the meaning of a tangent. Both of these studies will be reported briefly. Some observations will also be discussed.

3.6.1 The work of Vinner

The aim of Vinner's study (1982) [34] was to investigate whether the generalized approach to the notion of tangent as in calculus would replace the intuitive approach in circle geometry after the former approach was introduced.

The study

278 first year college students who had learned about derivatives and tangents in calculus course participated this study. A questionnaire which included five questions was handed out to them. The first three questions addressed to the notion of tangent where those concept images generated in circle geometry would lead to a conflict situation. For instance, the case where the tangent crosses the curve, such as the tangent at the inflection point; the case where the tangent will coincide with part of the graph such as the tangent at the origin of the function:

$$y = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

In each of the first three questions, the function and its graph as well as the point p at which the tangent was expected to be drawn were given. The students were asked to choose and follow the instructions in one of the following cases:

(A) Through P it is possible to draw exactly one tangent to the curve (draw it).

(B) Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinity). Draw all of them in case their number is finite and some of them in case it is infinite.

(c) It is impossible to draw through P a tangent to the curve.

Furthermore, in the last two questions, the students were asked to write down the definition of tangent if they recalled one; otherwise, tried to define one according to their own experience.

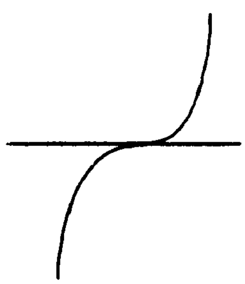
I remark here that in the case (B), the statement itself is confusing. If there is more than one tangent, the choice of one tangent should be excluded. From this observation, the choice of (A) and (B) cannot really distinguish the students' ideas towards the notion of tangent without examining their drawings. Hence, I shall only report the results according to the drawings.

The drawing results

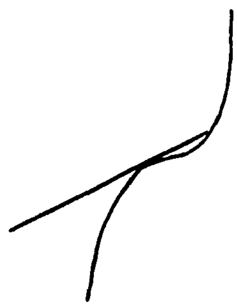
The results showed that very few students held the generalized tangent concept as it was taught in the calculus. At least one third of the students still held the intuitive tangent concept in circle geometry. Some other really experienced conflicts between the old and the new tangent concept. Overall, 21% in question 1, 29% in question 2, 15% in question 3 chose 'C' , that is, no tangent as answers, but 28% in question 1; 42% in question 2, 27% in question 3 did not provide any drawings. A summary of the students' drawings were edited as follows:

Question 1

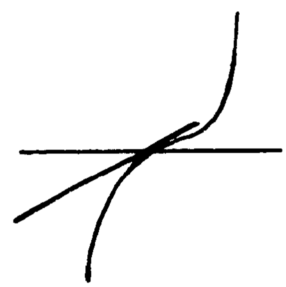
$y = x^3$ and $P(0,0)$



(a) 18%



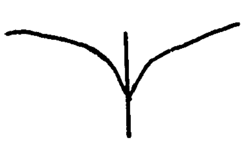
(b) 38%



(c) 6%

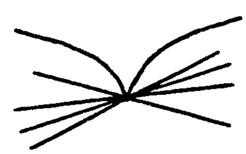
Question 2

$y = \sqrt{|x|}$ and $P(0,0)$

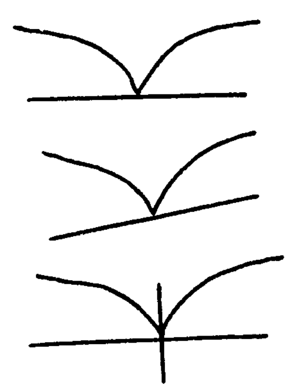


(a) 8%

(b) 18%



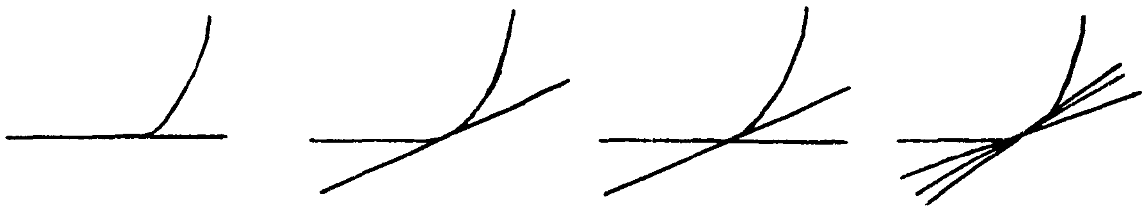
(c) 18%



(d) 14%

Question 3

$$y = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \text{ and } P(0,0)$$



(a) 12%   (b) 33%   (c) 16%   (d) 7%

The precentages indicates the distribution of students' answers. Vinner considered that those students who drew the tangent as in the cases (a) held the correct idea towards the concept of tangent in the calculus. However, as Tall (1987) commented later that in the case of question 2, there were controversies even within mathematics community. Some mathematicians consider that there is no tangent in this case because at the origin, the right hand derivative is positive infinite; whereas, the left right hand derivative is negative infinite.

Furthermore, 36% of the total gave the tangent definition as in geometry and 41% as in the calculus.

Comments

Observing Vinner's data, I first divide the students' ideas towards the notion of tangent into three categories, namely, (1) the generalized tangent concept, that is, the tangent concept in the calculus; (2) the geometrical tangent concept, that is, the tangent concept in circle geometry; and (3) a mixed tangent concept, that is, the combination of the generalized and geometrical tangent concept. Then, I apply the notion of concept image to make the following comments:

(1) There is a tendency to create a situation in which the intuitive tangent concept can be applied. For instance: in the case (b) of question 1, 38% of the total tried to draw the tangent a little bit off the origin in order to obtain the tangent satisfying the properties that a tangent touches the graph at only one point and does not cross the graph. Similar idea can be found in the answers to question 2 and question 3. In the former case, the tangent which looks as if it is a 'balance' is drawn in the first two parts of case (d). With regard to the latter case, the tangent which only touches the graph at one point is drawn as in the case (b). These answers showed that the concept

images of the generalized tangent concept were not evoked. As I have mentioned in 3.4.2, this can be explained by noting that the concept images formed by intuitive arguments often dominate those formed by mathematical contexts.

(2) Some students really experienced conflicts when the concept images generated by the mixed tangent concept were evoked. Some of them presented the conflict by including both notions in their answers. For instance, case (c) in question 1, the third part of case (d) in question 2, as well as case (c) in question 3. Yet, others showed the conflict through the idea of drawing infinitely many 'tangents' near the given point P. For instance, case (c) in question 2 and case (d) in question 3. Perhaps, this group of students applied the concept images of tangents as the limit of secants, to construct the situation in which the tangent should touch the graph at one point. Furthermore, it is interesting to note that no student constructed infinitely many tangents in question 1. Perhaps, this comes from the 'unpleasant' situation that any line passing through the inflection point has to cross the graph and this directly violates to the geometrical tangent concept.

(3) The two tangents obtained from case (b) in the

question 2 may suggest the impact of examples. In the calculus, similar graphs are often used to demonstrate the situation where the derivative does not exist at point P. This demonstration may lead to the confusion that at point P, there are two tangents. Moreover, even if we agree that the vertical tangent is undirected, it is really hard to tell whether there is a vertical tangent, without carrying out all the detail calculations. From this observation, I also suggest that in facing a task, the students tend to use their concept images rather than mathematical calculations to justify their answers.

3.6.2 The work of Tall

The aim of Tall's study (1987) [26] was to test the impact of computer packages in teaching and learning situation as to the notion of tangent. In traditional teaching situation, the dynamic approach to the tangent as the limit of secants was considered to be tedious due to the number of calculations. By using the aid of modern technology, it was hoped that students might acquire the generalized tangent concept through handling appropriate examples and non-examples. In Tall's study, he tried to examine whether using a computer package to investigate different examples and non-examples would help enrich and develop a more coherent concept images of tangent.

The study

Two groups of students participated in this study. 41 students in the experimental group learned to negotiate the meaning of a tangent through using computer to draw a line passing two very close points on the graph, in an introduction course of calculus. On the other hand, 65 students in the control group acquired the meaning of a tangent through a traditional teaching strategy, that is, an assumption of the intuitive knowledge of tangent.

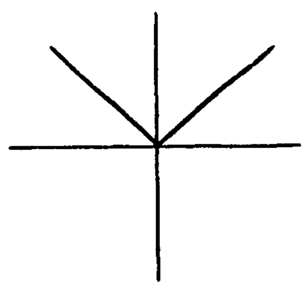
A questionnaire which consisted of six questions was handed to them after the notion of tangent was studied in detail. In each question, the function and its graph were both given and the students were asked whether the graph had a tangent at the origin. Furthermore, they were asked to sketch the tangent if their answer was 'yes'; otherwise, to give explanations. In order to compare the performance of the experimental group with more mathematical mature students, this questionnaire was also distributed to a group of 47 first year university mathematics students.

Results

The results indicated that the performance of the experimental group was at least as good as the university group. Although the concept images generated by the intuitive tangent concept persisted in all three groups, there was a significant improvement among the experimental group. The answers to some of the questions were as follows:

Question 1

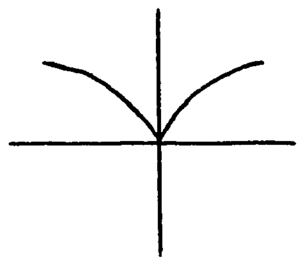
$y = |x|$



78% of the total in the experimental group as well as the university group answered that there was no tangent. However, only 45% in the control group gave this answer. Furthermore, 15% in the experimental group claimed that there were two tangents by discussing 'left' and 'right' tangent. More students in the control group (26%) drew a 'balance' tangent along x-axis than in the experimental group (less than 3%).

Question 2

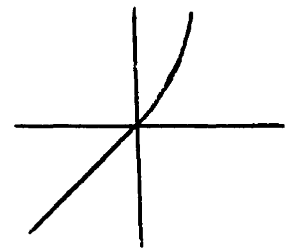
$y = \sqrt{|x|}$



Most of the students (88%) answered that there was no tangent. Whereas, only 37% in the control group and 40% in the university group suggested the same answer. On the other hand, 7% in the experimental group, 35% in the control group and 49% in the university group asserted that there was a vertical tangent. Besides, less than 3% in the experimental group, around 15% in the control group and 4% in the university group drew a 'balance' tangent along x-axis.

Question 3

$$y = \begin{cases} x & \text{if } x \leq 0 \\ x + x^2 & \text{if } x \geq 0 \end{cases}$$

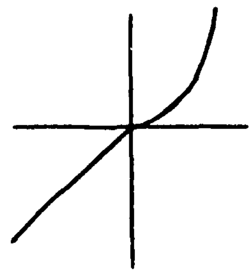


A high percentage (76%) of subjects in the experimental group gave that the tangent at the origin was along the line $y = x$. Whereas, only 34% in the control group and 62% in the university group gave the above answers. Moreover, 23% in the control group claimed that there was no tangent by using the arguments such as 'because at $x = 0$ is where two functions meet'. Tall remarked that throughout the experimental course, only those functions with a single formulae were discussed. The above data suggested that the experience of the experimental group helped the subjects enchance their abilities to transfer the acquired knowledge to a new context.

Since the tangent in this question coincided with the curve to the left of the origin, the conflicts between the generalized tangent concept and the geometrical tangent concept would have been expected as the results of Vinner's study indicated. However, only 20% in the experimental group but 38% in the control group and 30% in the university group drew the tangent a little off the curve so that the tangent seemed to touch the curve only at one point.

Question 4

$$y = \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$



This question resulted in responses such as 'many' and 'infinite number' of tangents touching the concern of the graph; a 'balance' line above the concern etc. However, 90% of the subjects in the experimental group explained that there was no tangent. Whereas, only 58% in the control group and 81% in the university group gave the same explanation.

<u>Comments</u>

Tall commented that in acquiring the notion of the tangent, the use of a computer package not only helped the students to develop a more coherent concept image but also enchanced the student's ability to transfer their knowledge to new context. For instance, the experimental group did better in determining whether the tangent existed at a point where the formulae changed. Hence, there was a hope to develop a curriculum in which the essential properties of a new concept were presented by handling examples and non-examples in an appropriate complex context.

3.6.3 Summary

In both studies, there is evidence that the concept images generated from the intuitive tangent concept still exist after the generalized tangent concept is introduced. In order to be aware of the inadequacy of the intuitive tangent concept, more examples which directly cause conflicts should be given in the teaching situation.

Moreover, comparing the results in both studies, I observe that in Tall's study, the image of 'infinite number' of tangents is not as frequently evoked as in comparsion with that of Vinner's. This suggests that in Vinner's study, the evocation of 'infinite number' of tangents may come from the question itself. Of course, other factors such as the teaching strategies should also be considered.

Finally, no student employed explicitly the idea of composition of movements to describe the notion of tangent in either studies since the questionnaire addressed mathematical rather than physical contexts. As modern mathematics becomes independent of the science, most of the notions in calculus can be defined independently of space and motion. Hence, the idea of movement is not called up when the question is posed in the mathematical context.

CHAPTER VI

A QUESTIONNAIRE ON THE NOTION OF FUNCTION

4.1 Introduction

A questionnaire was given with the aim of examining how students reacted to a given task according to the given concept definition of a function.

In chapter 1, I have already discussed Vinner's notion of concept image and concept definition for learning of a mathematical concept. In chapter 3, some of the studies which were conducted by Vinner and other researchers, to reveal the student's concept images with reference to the notions in the Calculus were reported. The results of these studies indicated that the concept image, not concept definition played a dominant role during the stage of performance.

Analysing the studies on the notion of function, I observed that many students gave inappropriate explanations due to their lack of recall of the Dirichlet-Bourbaki definition. As a matter of fact, some of them did justify their answers according to their own definition. On the one hand, the Dirichlet-Bourbaki approach to the notion of function was neither easy to teach nor to master due to its abstractness and generality. Often, teachers give the definition and then restrict themselves to those examples which will be useful later on. Under such a setting, the students' conceptions of functions mainly come from their experience of examples. Hence, it is reasonble for the students to use part of their concept images instead of concept definition when a task is given.

Furthermore, there is lack of motivation for the students to acquire of the Dirichlet-Bourbaki definition. In the courses such as Functions or Calculus, the students are rarely asked to identify whether a given example is a function or not. Instead, the character of some particular functions like parabola, sine, ... etc are discussed in the former course; properties such as continuity and differentiability are examined in the latter one. There is no reinforcement of the Dirichlet-Bourbaki definition itself. Not surprisingly, the concept definition will soon be forgetten.

From the above observations, some questions may be asked: What is the role of definition in student's mind? To what extent do students react to a given task according to the concept definition? Will the situation become different if a definition is given together with the task? Moreover, what kind of defintion can be used as a lexical definition (see 1.4) for learning the function concept at senior high or college level? These questions become interesting only if we are convinced that the good use of definition will help mathematical thinking as well as organizing different mathematical concepts in a coherent way. In this chapter, I do not intend to answer all these questions thoroughly. Rather, the analogy between classifying a mathematical object according to a definition and justifying steps according to logical reasonings leads me to study the kind of concept images that will be evoked if a definition is reinforced.

4.2 Selection of the tasks, population and research method

The aim of this section is to report the criterion of selecting the questions in the questionnarie, th population as well as the reseach method.

Often, the learners fail in conceiving the Dirichlet-BourabKi approach to the function concept due to their experience with some particular functions. However, this failure does not mean the failure in mathematical reasoning. Skemp [21] quoted the work of Vygotsky:

"One child was told to call a dog by the word 'cow'. He w s then asked 'Has a cow horns?' Child: 'Yes, it has'. Experimenter: 'But this cow is really a dog.' Child: 'Of course, if a dog is a cow, if it is called a cow, then there must be horns. Such a dog which is called a cow must have little horns'."

The previous concept image of a (real) cow not the present concept definition (a cow is a dog) was evoked while the child was facing to make his decision. Yet, the child did react to the task according to his own concept definition. I bring this example up here to point out that a good definition should not be too artifical, it should be close to the learners' experience. My conjecture is that the learners will act more logically if the given definition can stimulate a familiar idea.

4.2.1 <u>The questionnaire</u>

Under the above conjecture, it is handy to extend the study of Vinner and Dreyfus (see 3.3) because their studies provide a lot of information concerning the learners' experience. Basically, I used this information as well as the historical development of the function concept to formulate the so called 'definition' in my questionnaire.

Three definitions were arrived as follows:

Definition I
A function is a single rule which assigns, to each real number, some real numbers.

Definition II
A function is a rule between two sets (the domain and range) which assigns to every element in the domain exactly one element in the range.

Definition III
A function is a rule which assigns, to each real number, a single real number computed by a mathematical formula.

Attached to each definition, there are six questions which include four rules and two graphs. They are:

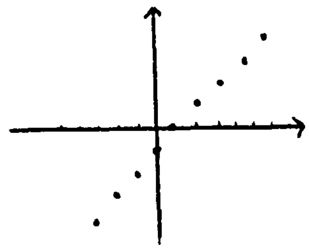
(1) The rule which assigns to each real number x the number $y = (x^2 - 1)/(x^2 + 1)$.

(2) The rule which assigns to each real number x the number 2.

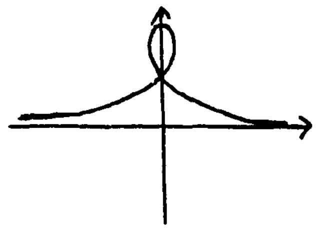
(3) The rule which assigns to each real number x the number x if x is greater than 2 and the number x-1 if x is less than or equal to 2.

(4) The rule which assigns to each student the student's ID number.

(5)



(6)



The guiding princple for choosing the graphs and rules was based on the difficulties found in the work of Maryanskii, the work of Vinner and Dreyfus as well as that of Markowits, Evlon and Bruckheimer. Some remarks are as follows:

(1) Rule 3 is chosen because the broken function is considered to be difficult for the learners in the above-mentioned studies and because this rule voilates definition I but not definition II and III.

(2) The notion of function is often introduced graphically, that is, the students are told that if a vertical line intersects the graph at one point, the graph is a graph of a function. This concept image of function is insufficient as to the Dirichlet-Bourbaki definition. Rule

1, rule 4 and the graph 6 are chosen because the graph of the function given by rule 1 is hard to draw; in the case of rule 4, there is no graphical presentation; graph 6 fails the vertical line test but still is a 'function' according to the defintion I.

(3) Rule 2 is chosen because the constant function is also found to be difficult and because the number 2 is given explicitly, not necessary computed by a mathematical formula.

(4) Another aspect that emerged from the above studies is that learners often ignore the domain of function, when a rule or a graph is given. Rule 4 has the set of students as its domain and graph 5 has the set of integers as its domain, both of them are legitimate candidates of definition II but not definition I and III.

The students are asked to answer each question by 'Yes', 'No' or 'I do not know' and also to give reason for their choice. At the end of the questionnaire, the students are asked which of the three definitions best fits their idea of function?

4.2.2 Population and the relevant information

The sample included 64 Math 203, 59 Math 205 and 67 Math 262 students at Concordia university. Only those students who either answered each question in the questionnarie, or gave at least 2 reasons to explain their answers were considered.

Math 203, Math 205 and Math 262 are Calculus I, II and III respectively. Students in these courses are mainly arts and science students. The mathematical background, especially in Math 203, are considered to be diverse. However, the precentage of science students increases according to the level of the course. A brief description for the role of function in these courses is summarized as follows:

Math 203 is Calculus I. In this course, the students review the Dirichlet-Bourbaki definition in the first week. Later on, properties and behaviour of functions such as continuity, increasing, ... etc are discussed.

Math 205 is Calculus II. In the first part of this course, the students mainly learn applications and techniques of integration.

Math 262 is calculus III. Functions with several variables are studied. On the one hand, the notion of function is reviewed and applied to a more general setting such as the element in the domain can be ordered pair, order triple, ..., etc.

4.2.3 The method

The questionnarie was supposed to be distributed among the students during the class. However, some instructors were reluctant to do so due to time constraint and it was agreed that it would be given at the end of class to be completed at home. As a result, only one Math 203 and eleven Math 262 students returned the take home questionnaire. Those who answered in class took about 15 minutes to do so. 21 students answered it prior to their first class test.

Before the questionnarie was handed out to the whole group, a pilot study was conducted with 20 Math 203, 21 Math 205 and 9 Math 262 students. Among them, 41 answered the questionnaire during the class and 9 at home. Some small changes were made after this pilot study. They were:

(1) In the pilot study, rule 3 was stated as:

"The rule which assigns to each real number x the number x if x is greater than 2; and the number x − 1 is less than or equal to 2."

One student thought that ';' was the complex number 'i' because the ditto copy was not clear enough, so ';' was taken off in the final version.

(2) In the pilot study, the last question was stated as:

"Will you accept any of the above definitions as the definition of function?"

Some students answered that they accepted all of them, others that they did not accept any one of them. Consequently, this question was changed to:

"which of the above definitions best fits your ideas of a function?"

The final questionnaire was handed out in the middle of the fall term at the academic year 88. At this time students, especially Math 203, would have acquired quite a lot of working experience of function.

4.3 Analysis of the data

The aim of this section is to report and analyse the results of the questionnaire on the notion of function.

Before the detail analysis is given, some remarks must be made:

(1) Throughout the analysis, the distribution of the answers for each question is presented by five subgroups, namely, (A) yes; (B) no; (C) I do not know; (D) no answer and (E) answers with reasons.

(2) The reasons given by the respondents are grouped into different categories, ten in definition I and III; nine in definition II, according not only the common aspects of the three given definitions, for instance, a rule of correspondence between two sets, or features such as 'a single rule', 'some real numbers', but also according to the students' responses. Also, categories 'no explanation' and

'miscellaneous' are used. The former category refers to the reason such as 'Satisfies the definition' or 'It is a function'. Whereas, the latter refers to those reasons which are incomprehensive; lack of information; or hard to put into any specific category.

(3) As I have mentioned in (2), the three given 'definitions' in the questionnarie share the common structure, that is, a rule correspondence between two sets. Definition II is the Dirichlet-Bourbaki definition, these two sets are specified as 'domain' and 'range'. However, both sets in definition I and III are only given as the set of real numbers. In order to pin-point the target set to which I refer in my analysis, the terms 'the first set' and 'the second set' are used to represent the role of 'domain' and 'range' in these two definitions.

(4) In order to examine which aspects of concept images will likely be evoked according to a given task and a given definition, a given reason may be classified into more than one category. Examples will be given in 4.3.1.

In order to deal with the data from different perspectives, some statistical observations will first be reported according to the results of each question and each definition. Analysis will be made on responses to all questions within a given definition, as well as, responses to a particular question under different definitions. Moreover, some miscellaneous responses will also be discussed.

4.3.1 Statistical results

In this section, two statistical tables of the distribution of answers and of the distribution of reasons within different categories according to a given definition, will be presented based on the results of each question and each definition.

Categories for definition I

The reasons given by the respondents under definition I are grouped into ten categories, namely:

Category 1

Reacts in some way according to the common structure of the given definitions. that is, a rule of correspondence between two sets

Category 2

Considers the property of the elements in the first set or the feature of the first set

Category 3

Considers the property of the elements in the second set or the feature of the second set

Category 4

Emphasize the idea of correpondence

Category 5

Considers whether a given rule is 'a single rule' or not

Category 6

Considers the meaning of the word 'some' or the phrase 'some real numbers

Category 7

Reacts according to the working experience with functions

Category 8

Appeals to some mathematical observations or calculations

Category 9

No explanation

Category 10

Miscellaneous

Under this classification, reason such as 'This assigns only one real number (y) to each real number' is considered to fall into three categories, namely, 1, 2 and 3. Also, reason such as 'No function can make a loop' is put in category 7.

Results on question 1

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	140	33	9	8	107
Math 203	42	15	4	3	28
Math 205	47	7	2	3	29
Math 262	51	11	3	2	50

88 out of 140 students in subgroup A and 17 out of 33 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		48	25	31	5	2	14	9	29	5	14
Math 203	A	7	6	4	1	0	1	1	10	1	3
	B	4	2	2	0	0	4	2	0	0	1
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	1	0	0	0	0
Math 205	A	13	7	8	2	2	0	0	8	0	3
	B	0	0	0	0	0	0	1	2	0	0
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	21	9	14	1	0	4	4	7	4	5
	B	3	1	3	1	0	3	0	2	0	2
	C	0	0	0	0	0	1	1	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 2

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	107	56	19	8	105
Math 203	32	25	4	3	30
Math 205	31	14	10	4	24
Math 262	44	17	5	1	51

65 out of 107 students in subgroup A and 35 out of 56 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		31	19	32	2	0	20	11	25	5	10
Math 203	A	5	5	3	1	0	1	2	8	0	2
	B	4	3	3	0	0	7	1	3	1	1
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	5	4	7	0	0	0	1	4	2	1
	B	0	0	1	0	0	1	1	3	1	0
	C	0	0	0	0	0	1	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	15	7	14	0	0	0	5	5	1	3
	B	1	0	2	1	0	9	1	2	0	1
	C	1	0	1	0	0	1	0	0	0	1
	D	0	0	1	0	0	0	0	0	0	0

Results on question 3

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	99	49	28	14	88
Math 203	30	18	10	6	26
Math 205	26	16	13	4	20
Math 262	43	15	5	4	42

56 out of 99 students in subgroup A and 25 out of 49 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		22	17	15	0	6	13	6	34	4	16
Math 203	A	3	4	2	0	1	0	1	7	0	2
	B	2	2	2	0	1	5	0	4	0	2
	C	0	0	0	0	0	0	0	0	0	2
	D	0	0	0	0	0	0	0	0	0	1
Math 205	A	3	4	2	0	0	0	1	4	2	3
	B	1	0	0	0	1	1	1	2	0	2
	C	0	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	12	6	8	0	1	2	3	12	2	2
	B	0	1	1	0	2	4	0	4	0	0
	C	1	0	0	0	0	1	0	1	0	0
	D	0	0	0	0	0	0	0	0	0	1

Results on question 4

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	88	72	23	7	101
Math 203	32	24	5	3	29
Math 205	28	16	11	4	24
Math 262	28	32	7	0	48

41 out of 88 students in subgroup A and 51 out of 72 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		24	38	14	9	0	12	5	6	1	19
Math 203	A	3	0	2	1	0	0	0	2	0	5
	B	3	7	1	1	0	4	1	0	0	3
	C	0	1	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	2
Math 205	A	7	3	2	2	0	1	1	0	1	2
	B	0	4	1	0	0	2	0	1	0	0
	C	0	1	0	0	0	0	0	0	0	2
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	9	3	4	2	0	0	2	1	0	2
	B	2	18	3	3	0	5	1	2	0	2
	C	0	1	1	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0

Results on question 5

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	124	60	4	2	102
Math 203	42	20	1	1	30
Math 205	43	14	1	1	26
Math 262	39	26	2	0	46

53 out of 124 students in subgroup A and 48 out of 60 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		24	22	12	6	1	15	25	23	1	15
Math 203	A	2	1	1	1	0	1	11	5	0	2
	B	5	3	2	1	0	7	1	0	0	2
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	4	2	2	1	1	0	6	4	0	1
	B	2	1	1	0	0	3	1	5	0	3
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	5	4	4	2	0	0	4	5	1	4
	B	6	9	2	1	0	4	1	13	0	3
	C	0	0	0	0	0	0	1	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 6

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	96	69	17	8	92
Math 203	24	28	9	3	25
Math 205	26	24	5	4	24
Math 262	46	17	3	1	43

52 out of 96 students in subgroup A and 35 out of 69 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		25	16	15	2	0	31	23	19	4	14
Math 203	A	3	3	1	0	0	7	1	2	1	1
	B	0	0	0	0	0	1	10	6	0	1
	C	0	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	7	5	4	0	0	5	2	1	0	1
	B	1	1	0	0	0	1	5	1	0	3
	C	0	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	12	7	10	1	0	14	1	6	3	3
	B	1	0	0	1	0	2	4	3	0	2
	C	1	0	0	0	0	1	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	1

Categories for definition II

The reasons given by the respondents under definition II are grouped into nine categories, namely:

Category 1

Reacts in some way according to the common structure of the given definitions, that is, a rule of correspondence between two sets

Category 2

Considers the property of the elements in the domain or the feature of the domain

Category 3

Considers the property of the elements in the range or the feature of the range

Category 4

Emphasizes the idea of the correspondence

Category 5

Considers the meaning of the phrase 'exactly one element in the range

Category 6

Reacts according to the working experience with functions

Category 7

Appeals to some mathematical observations or calculations

Category 8

No explanation

Categroy 9

Miscellaneous

Results on question 1

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	126	46	10	8	95
Math 203	37	19	4	4	25
Math 205	45	7	3	4	22
Math 262	44	20	3	0	48

63 out of 126 students in subgroup A and 31 out of 46 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		23	7	6	8	22	3	44	6	10
Math 203	A	6	0	0	0	4	1	3	2	3
	B	0	0	0	2	2	0	6	0	1
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0
Math 205	A	5	4	3	1	3	1	6	2	1
	B	0	1	1	0	0	0	5	0	0
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0
Math 262	A	12	2	2	4	11	1	9	2	4
	B	0	0	0	1	2	0	14	0	1
	C	0	0	0	0	0	0	1	0	0
	D	0	0	0	0	0	0	0	0	0

Results on question 2

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	106	62	12	10	100
Math 203	33	21	6	4	27
Math 205	22	27	4	6	29
Math 262	51	14	2	0	44

63 out of 106 students in subgroup A and 33 out of 62 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		30	7	10	3	29	6	46	4	8
Math 203	A	6	1	1	1	5	1	8	1	0
	B	0	1	0	0	0	1	6	1	0
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	1
Math 205	A	4	1	1	0	6	0	3	1	1
	B	3	2	2	1	2	1	6	0	3
	C	0	0	0	0	0	0	0	0	0
	D	1	0	0	[illegible]	1	0	1	0	0
Math 262	A	14	2	4	[illegible]	14	3	15	1	2
	B	2	0	2	0	1	0	6	0	1
	C	0	0	0	0	0	0	1	0	0
	D	0	0	0	0	0	0	0	0	0

Results on question 3

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	101	43	29	17	73
Math 203	33	17	7	7	18
Math 205	25	15	11	8	17
Math 262	43	11	11	2	38

51 out of 101 students in subgroup A and 16 out of 43 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		23	4	5	5	25	3	37	5	4
Math 203	A	6	0	0	2	6	0	2	1	0
	B	0	1	1	0	0	0	2	0	0
	C	0	0	0	0	0	0	1	0	1
	D	0	0	0	0	0	0	0	0	1
Math 205	A	2	1	3	0	4	1	5	2	0
	B	1	1	0	0	1	0	4	0	0
	C	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0
Math 262	A	14	1	1	2	13	2	16	2	0
	B	0	0	0	1	1	0	6	0	0
	C	0	0	0	0	0	0	1	0	0
	D	0	0	0	0	0	0	0	0	1

Results on question 4

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	135	18	23	14	83
Math 203	42	9	6	7	20
Math 205	37	2	14	6	17
Math 262	56	7	3	1	46

72 out of 135 students in subgroup A and 6 out of 18 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		38	10	8	10	35	1	11	2	14
Math 203	A	8	1	0	4	8	0	3	0	0
	B	0	0	0	0	0	0	0	0	2
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	1	0	2
Math 205	A	8	2	2	2	8	1	1	1	2
	B	0	0	0	0	0	0	0	0	0
	C	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0
Math 262	A	22	6	5	4	19	0	6	1	4
	B	0	1	1	0	0	0	0	0	3
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0

Results on question 5

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	153	20	4	13	82
Math 203	52	8	1	3	20
Math 205	40	9	2	8	21
Math 262	61	3	1	2	41

73 out of 153 students in subgroup A and 8 out of 20 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		35	11	9	7	30	3	18	6	7
Math 203	A	8	2	1	3	7	1	4	1	1
	B	0	0	0	0	0	0	1	0	1
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0
Math 205	A	9	3	2	0	8	1	4	1	0
	B	1	0	1	0	0	0	1	0	0
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0
Math 262	A	17	5	4	4	15	0	7	4	3
	B	0	0	0	0	0	1	1	0	2
	C	0	1	1	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0

Results on question 6

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	32	134	11	13	101
Math 203	10	45	4	5	24
Math 205	11	36	5	7	26
Math 262	11	53	2	1	51

11 out of 32 students in subgroup A and 88 out of 134 in subgroup B gave reasons.

The distribution of reasons according to nine categories within the first four subgroups

		1	2	3	4	5	6	7	8	9
Total		22	6	7	2	53	5	29	3	11
Math 203	A	0	0	0	0	0	0	0	0	0
	B	5	0	0	0	13	1	6	0	4
	C	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0
Math 205	A	1	2	2	0	0	1	2	0	3
	B	4	1	2	0	11	0	6	0	0
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0
Math 262	A	3	3	3	0	1	0	1	0	0
	B	9	0	0	2	28	3	14	3	3
	C	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0

Categories for definition III

The reasons given by the respondents under definition III are grouped into ten categories, namely:

Category 1

Reacts in some way according to the common structure of the given definitions, that is, a rule of correspondence between two sets

Category 2

Considers the property of the elements in the first set or the feature of the first set

Category 3

Considers the property of the elements in the second set or the feature of the second set

Category 4

Emphasizes the idea of the correspondence

Category 5

Considers the phrase 'a single real number'

Category 6

Considers whether the rule is given as a mathematical formula

Category 7

Reacts according to the working experience with functions

Category 8

Appeals to some mathematical observations or calculations

Category 9

No explanation

Category 10

Miscellaneous

Results on question 1

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	135	30	13	12	69
Math 203	42	13	5	4	19
Math 205	40	5	7	7	14
Math 262	53	12	1	1	36

54 out of 135 students in subgroup A and 14 out of 30 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		18	2	2	0	10	21	2	29	8	2
Math 203	A	3	1	0	0	2	7	0	5	1	1
	B	1	0	0	0	0	1	0	3	0	0
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	1	1	0	0	0
Math 205	A	6	0	0	0	0	2	1	4	1	1
	B	0	0	0	0	0	1	0	2	0	0
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	9	1	2	0	8	9	0	8	6	0
	B	0	0	0	0	0	0	0	7	0	0
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 2

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	78	76	23	13	73
Math 203	23	24	11	6	19
Math 205	20	25	8	6	15
Math 262	35	27	4	1	39

34 out of 78 students in subgroup A and 39 out of 76 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		8	2	0	0	5	37	2	39	1	4
Math 203	A	1	0	0	0	1	3	0	6	0	0
	B	0	1	0	0	0	5	0	3	0	1
	C	0	0	0	0	0	2	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	2	1	0	0	0	2	0	2	0	0
	B	1	0	0	0	1	5	1	4	0	0
	C	0	0	0	0	0	1	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	4	0	0	0	3	5	1	17	0	1
	B	0	0	0	0	0	13	0	7	1	2
	C	0	0	0	0	0	1	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 3

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	95	45	32	18	57
Math 203	28	18	12	6	14
Math 205	25	13	12	9	7
Math 262	42	14	8	3	36

32 out of 95 students in subgroup A and 20 out of 45 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		7	1	0	0	5	25	0	35	3	4
Math 203	A	1	0	0	0	1	3	0	2	0	0
	B	0	1	0	0	0	5	0	5	0	1
	C	0	0	0	0	0	1	0	1	0	0
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	1	0	0	0	0	2	0	3	0	0
	B	0	0	0	0	0	1	0	2	0	0
	C	0	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 262	A	4	0	0	0	3	11	0	17	1	0
	B	1	0	0	0	1	1	0	4	2	1
	C	0	0	0	0	0	1	0	0	0	1
	D	0	0	0	0	0	0	0	1	0	0

Results on question 4

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	46	95	33	16	83
Math 203	19	28	9	8	23
Math 205	16	22	14	7	15
Math 262	11	45	10	1	45

16 out of 46 students in subgroup A and 60 out of 95 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		4	13	1	2	3	40	0	14	2	13
Math 203	A	0	1	0	0	0	0	0	1	0	3
	B	0	2	0	0	0	13	0	3	0	0
	C	0	0	0	1	0	1	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	1
Math 205	A	1	0	0	0	1	2	0	0	0	1
	B	0	2	0	0	0	4	[illegible]	0	0	1
	C	0	1	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	2
Math 262	A	3	0	0	1	2	1	0	2	0	1
	B	0	7	1	0	0	18	0	8	2	3
	C	0	0	0	0	0	1	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 5

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	109	44	17	20	67
Math 203	37	14	5	8	16
Math 205	36	10	6	7	12
Math 262	36	20	6	5	39

37 out of 109 students in subgroup A and 25 out of 44 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		6	4	1	1	5	17	2	31	2	13
Math 203	A	2	0	0	0	2	3	0	4	0	3
	B	0	0	0	0	0	2	0	2	0	0
	C	0	0	0	0	0	0	0	1	0	0
	D	0	0	0	0	0	0	0	1	0	0
Math 205	A	2	0	0	0	2	2	0	4	0	1
	B	0	0	0	0	0	0	0	3	0	1
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	1
Math 262	A	2	0	0	1	1	7	1	8	1	3
	B	0	4	1	0	0	3	1	7	1	4
	C	0	0	0	0	0	2	0	1	0	0
	D	0	0	0	0	0	0	0	0	0	0

Results on question 6

The distribution of answers according to five subgroups

	A	B	C	D	E
Total	39	105	24	22	64
Math 203	15	34	7	8	14
Math 205	10	30	10	9	13
Math 262	14	41	7	5	37

7 out of 39 students in subgroup A and 54 out of 105 in subgroup B gave reasons.

The distribution of reasons according to ten categories within the first four subgroups

		1	2	3	4	5	6	7	8	9	10
Total		1	1	0	0	20	8	3	28	1	6
Math 203	A	1	0	0	0	0	1	0	0	0	1
	B	0	0	0	0	7	0	1	9	0	0
	C	0	0	0	0	0	0	0	0	0	1
	D	0	0	0	0	0	0	0	0	0	0
Math 205	A	0	0	0	0	0	0	0	0	0	0
	B	0	0	0	0	1	2	0	3	0	0
	C	0	0	0	0	0	0	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	1
Math 262	A	0	0	0	0	0	1	1	1	0	1
	B	0	1	0	0	12	3	1	15	1	2
	C	0	0	0	0	0	1	0	0	0	0
	D	0	0	0	0	0	0	0	0	0	0

4.3.2 Responses to all questions within a given definition

In this section, the overall responses to six questions under a given definition will be analysed.

Before studying the detail analysis, we note that throughtout this section, the vector notation (a, b, c, d, e, f) will represent the number of responses to each question, for instance, (15, 8, 2 10, 5, 1) stands for 15 responses in question 1, 8 responses in question 2, ... etc.

Responses to definition I

Besides its common structure with the other definitions, that is, a corespondence between two sets, definition I has its own features, namely: (1) The first set has to be the set of real numbers; (2) The second set is a subset of real numbers; (3) Only 'a single rule' is allowed; (4) Each element in the first set may be assoicated with one or more elements in the second set. These features directly exclude rule 3, 4 and graph 5 as examples of functions. The responses in category 1, 2, 3, 5 or 6 partially reflect the numbers of stuoents who take some of these features into consideration. The general responses to definition I are:

(1) (15, 8, 2, 10, 5, 1) students accepted a given example as a 'function' of definition I, by considering only the idea of correspondence. Examples were:

"Whatever #x is will give y a # by knowing $y = (x^2 - 1)/(x^2 + 1)$" (yes) [Rule 1]

"It sends everything to 2" (yes) [Rule 2]

"Each student is assigned an ID number" (yes) [Rule 4]

or "For every x, there is a y that corresponds to it" (yes) [Rule 1, 3, graph 5 and 6]

In question 4, 3 out 10 students pointed out that each student can in fact have a unique ID. However, overlooking features such as 'single rule' and the property of the first set led to the inappropriate acceptance of rule 3, 4 and graph 5.

(2) (19, 15, 13, 6, 5, 4) students did consider the feature of the first set or the second set or both in addition to the correspondence relationship. Examples were:

"Because x is a real # ∴ y is going to be R # when the value of x are filled in" (yes) [Rule 1]

"Because this rule assign the number two, a real number, to each real number ⇒ $y = 2$" (yes) [Rule 2]

"Again, under this rule, each real no. transformed into another real no., whether this result is equal to x or $x - 1$" (yes) [Rule 3]

"The ID# is a # as a whole but also a collection of real # which is assigned to the student" (yes) [Rule 4]

"The dot maps a coordinate of x to a corrdinate of y, Real → Real, ∴ a function" (yes) [Graph 5]

Within this group, only (1, 0, 0, 1, 1, 0) students rejected one of the given rules or graphs as an example of definition I. For instance:

"If x is a real number then y could not be a real number according to this function" (no) [Rule 1]

"I am a student and I am not a real number although I am assigned one" (no) [Rule 4]

"Assigns integers to integer_ (no) [Graph 5]

As in case (1), the failure to classify rule 3 may come from overlooking the condition 'a single rule'. However, the failure in justifing rule 4 and graph 5 raises some questions such as: What are the students' conception of real numbers? How do the students comprehend the phrase 'for each real number'? Certainly, some responses show that some students do not have a clear idea of real numbers. Some examples are:

"Is a student a real number?!!!" (I do not know) [Rule 4]

"2 is an integer, not a real. 2.0 is real" (no) [Rule 2]

The latter response is likely due to the influence of computer programming. In writing a program with a language such as BASIC and FORTRAN, the declaration of 'integers' and 'real numbers' is very important. From this experience, the students may acquire the images that a real number must have a decimal point.

Furthermore, those students who accepted graph 5 as a graph of 'function' by definition I, pointed out that graph 5 is a graph of 'function' which maps 'real' to 'real'. It is likely that these students fail to comprehend the meaning of 'for each real numbers'.

(3) (6, 12, 2, 26, 5, 1) students considered <u>only</u> the reasons in category 2 or 3 as grounds to justify their responses. They mainly considered features of the first set or the second set. Examples were:

"x will take the value of all real numbers" (yes) [Rule 1]

"(x^2 - 1) and (x^2 + 1) represent real numbers: when divide both, you get a real number" (yes) [Rule 1]

"Because the image is a real value" (yes) [Rule 1]

"Domain $\in$ R" (yes) [Rule 1 and 2]

"2 is a real number" (yes) [Rule 2]

"A student is not a real number" (no) [Rule 4]

"Not all $\in$ R are considered" (no) [Graph 5]

We note that the misuse of notation '$\in$' in the fourth example was not an isolated event. Incorrect use of mathematical notations such as '$\forall$' were also found throughout the questionnarie.

Few students justified their answer by some mathematical observations related to the first or second set. Examples were:

"Domain is positive (f(x))" (yes) [Graph 6]

"The range is always defined" (yes) [Rule 1 and 3]

(4) The phrase 'some real numbers' confused some students who thought that it means 'more than one real

number' and so rejected those rules which assign 'one' real number y to each real number x. Examples were:

"Because it assigns to each real number one other real number (and not several)" [Rule 1, 2, 3]

"This assigns only one real number (2) for each real number x whereas the above definition specifies numbers , indicating (strictly speaking) more than one number" (no) [Rule 2]

"2 is not some real numbers" (no) [Rule 2]

"For every student there is one ID # not more" (no) [Rule 4]

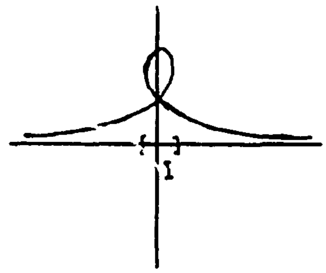
In question 6, 22 students noticed that there are more than one y assoicated with some x and so concluded that the given graph is an example of definition I. For instance:

"According to def. real numbers can be assigned to one real number & there are 2 #'s assigned for one number" (yes) [Graph 6]

"Because = 0 has 2 value of y" (yes) [Graph 6]

Few students argued that graph 6 is not a graph of a function by definition I, because not all real numbers x are assigned some real numbers y . For instance, one student wrote:

"Because some of the numbers in the domain get plotted to only a function value. If you restrict the domain to the interval I indicated above then I would answered Yes!" (no) [Graph 6]



(5) Very few students ((2, 0, 6, 0, 1, 0)) took the feature 'a single rule' into considerations. Examples related to rule 1 and graph 5 were:

"#'s have been assigned by 1 rule to #'s" (yes) [Rule 1, Graph 5]

Six students noticed that question 3 involves more than one rule. One student reponsed to this observation by explaining:

"Yes because to each real number this rule assigns a real number according to a composite rule, but unique" (yes) [Rule 3]

The other five students thought that rule 3 violates the definition I. Some of their explanations were:

"multiple rule" (no) [Rule 3]

"Here you have more than one rule" (no) [Rule 3]

Responses to definition II

Definition II is the Dirichlet-Bourbaki definition. Besides the correspondence relationship between two sets (domain and range), each element in the domain is associated with only one element in the range. Thus, the responses in category 1 and 5 will reflect at least a partial mastery of the definition II. Moreover, as I have pointed out in chapter 3, it is the abstract nature of the domain and range which makes a function a powerful idea in different branches of mathematics. Category 2 and 3 provide us with information as to the extent that a student is aware of the abstract idea of domain and range. The general responses to definition II were:

(1) Overall, (16, 24, 20, 29, 30, 18) students gave reasons which fell into at least both categories 1 and 5. Examples were:

"For all . you can find one value of y in the range" (yes) [Rule 1 and 3]

"For each value of x, there is one value (2) in the range" (yes) [Rule 2]

"For every student has a unique ID" (yes) [Rule 4]

"No two students have the same ID number and every student has an ID #" (yes) [Rule 4]

"For some elements of the domain there is more than one element in the range" (no) [Graph 6]

Most students (16, 21, 19, 29, 30, 1) thought that rule 1 - 4 and graph 5 are examples and Graph 6 is not a example of definition II. Now, some of reasons for excluding rule 2 or rule 3 as examples of Dirichlet-Bourbaki definition were:

"Because every element in the domain corresponds not to one element exaclty in the range, but to the <u>same</u> element" (no) [Rule 2 and 3, my underline]

"y = the number 2.∴it does not to <u>every</u> element in the domain exactly 1 element in the range" (no) [Rule 2]

"This can be rewritten as f(x) = 2 and to <u>each</u> number in the domain it assigns exactly one i.e. two. It is to be noted also that the definition does not include the condition of 1-1 correspondence" (no answer) [Rule 2]

It is interesting to note that the three arguments give different perspectives of the definition II. The first one suggests that there is a difference between 'exaclty one element' and 'same element'. At first glance, this reason conveys the idea that the corresponding elements in the range should be distinct. However, the same reason was also applied to rule 3.

In the second case, besides the inappropriate interpretion of rule 2, the incomplete sentence structure and the highlight of 'every' suggests that this student may be well aware that the 'domain' and 'range' of 'x = 2' are {2} and R (the set of all real numbers) respectively. All s/he did was to formulate reasons according to the definition II, that is, it (the rule which) does not (assign), to every element in the domain, exactly one element in the range.

In the third case, this student gave a suitable interpretion of rule 2. However, the last remark suggests that, for this student, there is a connection between 1-1 correspondence and the idea of functional relationship. In fact, the concept image that a function is a 'one-to-one relationship' is frequently evoked to justify a given example.

(2) As in definition I, some students ((6, 3, 2, 6, 3, 3)) only considered whether there is a correpondence

relationship. Most of them ((6, 2, 2, 6, 2, 3)) concluded that a given rule or graph is an example of definition II, by appealing to the typical reasons such as:

"For every value of x there is a value of y" (yes)

The acceptance of graph 6 as an example of definition II suggests that the mastery of the notion of correspondence relationship is not sufficient for mastering the notion of function.

The one student who excluded graph 5 as a 'function' of definition II explained that 'For only some points of x there is a y" (no) [graph 5], suggesting that this student failed to consider that the set of integers can be served as the domain of 'graph 5'.

(3) Some students justified a given rule or graph by considering whether there is only one element in the range. Especially in the case of graph 6, 34 students pointed out, in some way or another, that for certain x in the domain, there is more than one y in the range. Examples were:

"One x can have more than one y" (no) [Graph 6]

"Here at y = 0, there are 2 elements for the range" (no) [Graph 6]

" Ω give two values" (no) [Graph 6]

Some students interpreted the phrase 'exactly one element in the range' as 'the range can have only one element in it' and so accepted only rule 2 as a function by definition II.

(4) Other students ((5, 9, 7, 9, 11, 7)) justified their answers by giving reasons which fall into category 2 or 3 or both. Some concluded that a given rule or graph is a function by definition II because it 'has a domain and range'. Some of the students in this group did write down the correct domain and range. Yet, others justified that a given rule or graph is not a function by arguing that 'it has no range or domain' or 'there is no range'. The last reason occured only when rule 2 or 3 was given.

(5) Reasons such as 'a real is not a set' or 'a student is not a set' were also used. These responses pointed that the notion of an abstract set was either not understood or simply igorned.

Responses to definition III

The features of definition III are: each real number in the first set corresponds 'a single real number' in the second set and the rule is given as a mathematical formula. Thus, the reasons which fall into the category 1, 2, 3, 5 and 6 are considered to reflect the character of definition III.

(1) Since definition III also embeds the idea of correspondence, certain students justified a given example by appealing to the correspondence relationship, as was the case for definition I and II.

(2) Some students ((8, 5, 5, 2, 4, 0)) used the reasons falling in both category 1 and category 5, such as 'for each real number x, there is only one real number y'. Using such reasoning, only a few ((0, 1, 1, 0, 0, 0)) concluded that rule 1 - 4 or graph 5 is not 'function' by definition III even though rule 2 was not given as a mathematical formula and rule 4 and graph 5 did not take all the real numbers as its first set. Once again, not all aspects of definition III were evoked when a task was

considered.

(3) A group of (1, 0, 0, 0, 0, 20) students <u>only</u> considered whether there is <u>a single real number</u> corresponding to each real number. Examples were:

"$y = (x^2 - 1)/(x^2 + 1)$ will give a single real number once computed" (yes) [Rule 1]

" 2 values for some values of x" (no) [graph 6]

From the distribution of responses within this group, it seems that graph 6 triggered the consideration of the phrase 'a single real number'.

(4) Some students considered the feature of a 'mathematical formula' in addition to the correspondence relationship. Basically, they used 'a mathematical formula' as a ground to accept or reject a given rule or graph as a 'function'. Examples were:

"$x = 2$ is still a mathematical formula" (yes) [Rule 2]

"I do believe that rule 3 is a mathematical formula" (yes) [Rule 3]

"Formula with conditions" (yes) [Rule 3]

"Student's ID number not computed by a mathematical formula" (no) [Rule 4]

"There must be some kind of formula" (yes) [Rule 4]

"A mathematical formula plot for the set of integers (including zero)" (yes) [Graph 5]

Sometimes, a given example was considered as a 'mathematical formula' by one group and rejected by another. Examples were:

"To each number x, we have another real number but not computed by a formula" (no) [Rule 1]

"For each x, we had only one real number (y) computed by a mathematical formula" (yes) [Rule 1]

"No formula $f(x) = 2$" (no) [Rule 2]

"$y = f(x) = 2 \Rightarrow$ formula" (yes) [Rule 2]

Some other students could not decide whether a given example was directly or indirectly related to a 'mathematical formula'. Examples were:

"Is 2 math formula?" (I do not know) [Rule 2]

"I am not sure if the definition of a mathematical formula includes such expression as
$f(x) = x$ if $x > 2$; $f(x) = x - 1$ if $x \leq 2$" (I do not know) [Rule 3]

Furthermore, certain students even tried to invent 'mathematical formula' as in:

"Can be depending if formula always gives 2 i.e. x = (y.0) + 2 " (yes) [Rule 2]

"The numbers are given out in increasing value such as a x + 1 type of formula" (yes) [Rule 4]

"A mathematical formula i.e. y = mx + b exists to calculate y values" (yes) [Graph 5]

"There is a formula implies x - 1 but x must be declared discret:" (I do not know) [Graph 5]

The last response shows that this student noticed that graph 5 fullfilled the requirement of a 'mathematical formula' but not the requirement of 'the first set has to be all real numbers'.

Overall, different definitions called for different criteria of justification, for instance, the 'multi-values' part in definition I, the 'single-value' part in definition II as well as 'a mathemtical formula' in definition III.

4.3.3 Reponses to each question according to different definitions

In 4.3.2, I have discussed the different responses under the same definition. In this section, analysis will be done on the influence of the three given definitions upon a single question. In order to avoid repetitions, I shall focus on those aspects which are not covered in 4.3.2. However, a brief review of results in 4.3.2 may be presented in order to give a complete picture of the data.

The vector notation (a, b, c) will represent the number of responses in each definition, for instance, (1, 16, 6) stands for 1 response in definition I etc.

<u>Responses to question 1</u>

Rule 1 satisfies all three definitions. However, the observation that 'positive and negative x yield the same y' led some students ((1, 16, 6)) to reject it; whereas (3, 2, 0) students accepted it as an example of the given definition. The increase number of rejections in definition II may be explained partly by the confusion between 'one-to-many' and 'many-to-one' relationship. The latter is acceptable as functional realtionship according to definition II while the former is not. Yet, another possibility comes from the concept image that a functional relationship is 'one-to-one'. As one student commented:

"This is not one to one function x = 1 or - 1 if only if y = 0" (no) [definition II]

Responses to question 2

Rule 2 satisfies definition I and II but not defintion III. Of the (22, 33, 17) responses, some wrote 'x = 2', others 'y = 2' or 'f(x) = 2'. First, let us examine the reasons for writing 'x = 2'. This group consisted of (13, 21, 10) students out of which (13, 13, 9) circled 'yes' as the answer. In definition I, most of these students did not give further explanation. However, one did express:

"Since x is itself assigned the #2, then x = 2 and nothing else" (yes) [definition I]

Under definition II, 2 out of 13 explained that 'for x = 2, we get only one value in the range' (yes). Some drew the correct graph of 'y = 2' even though they wrote 'x = 2'. One response showed the inability to understand both definiton II and rule 2:

"The function is x = 2. ∴ x is domain is 2 and range is R" (yes) [Definition II]

Among those who circled 'no', one student wrote:

"x = the number 2, does not <u>every</u> element in the domain exactly 1 element in the range" (no) [Definition II]

Under definition III, some explanations were:

" = 2 still a mathematical formula" (yes) [Definition III]

"x = 2 give a single real number" (yes) [Definition III]

One student wondered whether 'x = 2' is a formula or identity and so circled 'I do not know'.

Overall, the condition 'x = 2' likely stems from an inappropriate interpretation of the word 'assign'. This can be traced to the explanation that 'since x is itself assigned the #2, then x = 2 ...' or 'Because x is only equal to two'. As a matter of fact, few students did write down '2 is assigned to x' instead of 'x = 2'. My point is that the word 'assign' is not interpreted as a 'corresponding' relationship but as an 'assignment'. The interpretation that 'x is assigned the #2' or '2 is assigned to x' means 'x = 2' may come from experience in computer science. For instance, the statement 'assign 2 to the variable X' can be translated as 'X = 2' in writing a computer program.

Among the (9, 12, 7) students who observed that 'y = 2' or 'f( ) = 2', (8, 6, 5) circled 'yes' as an answer. Furthermore, (3, 1, 0) explained that 'y = 2 is a constant function'. Two students using definition II expressed that 'y = 2 is unique' while one using definition III argued that 'f(x) = 2 is a mathematical formula'. The rest gave no further explanation. One student who chose 'no' as an answer under definition II explained:

"can't have y = 2 more than once" (no) [Definition II]

On the other hand, those who rejected rule 2 as an example of definition III considered that 'f(x) = 2 is not a mathematical formula'.

Responses to question 3

Rule 3 satisfies definition II and III but not defintion I. Some students complained that rule 3 was too complicated to understand. Overall, (17, 9, 9) students were able to transform rule 3 into mathematical form. More than half of them were in Math 262. Basically, two types of mathematical observations were given:

$$(1) \quad f(x) = \begin{cases} x & x > 2 \\ x - 1 & x \leq 2 \end{cases}$$

$$(2) \quad x = \begin{cases} x & x > 2 \\ x - 1 & x \leq 2 \end{cases}$$

For the first case, few students provided a graphical form. Some of the students who wrote down expressions (2) got confused so circled 'I do not Know'. Again, I remark here that expression (2) is likely the result of an inappropriate interpretation of the word 'assign' in the given statement.

The following discussions which follows will concentrate on those students who used expression (1).

Only few students concluded that rule 3 is not an example of definition I, by considering additional factors such as 'not a single rule' or 'have unique value'. The rest accepted it as a 'function'. Some appealed to arguments such as 'result is a real no.', though most gave no further explanation. On the other hand, using definition II, all believed that rule 3 is a function of the given definition. Some gave explanations such as:

"Again the test of drawing a vertical line to graph will verify that the function plots exactly one number in the range" (yes) [Definition II]

"1 value of y for every value of x" (yes) [Definition II]

Again, using definition III, the students' justification rested on whether expression (1) is a mathematical formula or not.

Besides, the 'discontinuity' of rule 3 also caused some concern. Basically, two types of arguments were involved here: (1) A function must be continuous; (2) A function may be discontinuous. Those who concluded rule 3 as an example of a given definition favoured the latter argument. As a matter of fact, rule 4 and graph 5 are also called up the idea of 'continuity'

<u>Responses to question 4</u>

Rule 4 satisfies only Definition II. As we have discussed in 4.3.2, in addition to the correspondence relationship, the responses in definition I favoured the explanations such as 'a student is not a real number' and those in definition .II often suggested that 'there is no mathematical formula'. Besides there was an increase number of explanations based on the idea of correspondence, especially in definition I and II. A group of (7, 10, 2) students observed that rule 4 is a one-to-one relationship. Among them, (5, 10, 1) concluded that rule 4 is an appropraate example. One, fairly typical response, was:

"Well, I think so, for it is a one-to-one relation which enought to make it a function (I guess, I would have to think more about it!)" (yes) [Definition II]

Another student wrote:

"1 to 1; x - y relationship ? formula used ?' (I do not know) [Definition III]

This answer indicated that this student could not decide whether a one-to-one relationship without a formula can still guarantee rule 4 as a 'function' according to definition III.

Responses to question 5

Again, graph 5 satisfies only definition II. In addition to the correspondence relationship, responses such as 'not all real numbers are considered' were evoked by definition I and III. Some students favoured writing down a mathematical formula for graph 5 in order to fit definition III. Mostly, three types of arguments were used : (1) Appealing to the 'vertical line test', that is, if a vertical line intersects the graph at one point, the graph is a graph of a function. (2) Looking for similarity between graph 5 and some known functions. Reasons such as 'Equation of straight line ; Graph of straight line' (One student even joined the points on the graph by a straight line); It looks like one ; 'It resembles y = x which is a function' were given here; (3) Appealing to the properties of function such as 'continuity'. In general, the number of responses using argument (2) were considerable higher for definition I.

Responses to question 6

Graph 6 satisfies only definition I. The loop feature directly affects the act of identification under different definitions. The reason that 'for some value of x, there is more than one value of y' was used both to accept graph 6 as an example of definition I and to reject it as an example of definition II and III. As we have seen in other questions, definition III promoted the idea of finding a 'mathematical formula'. However, no student wrote down a formula for this graph. Definition I and II called on school experiences with functions resulting with statements such as:

"No function can make a loop" (no) [Definition I]

"Never seen a function like this" (no) [Definition II]

Moreover, the failure in the vertical line test was also used to reject this graph as a graph of function under different definitions. Yet, as in the work of Markovits, Eylon and Bruckheimer [14], some students, but very few, held the concept image that a function must be linear. For instance, one student wrote:

"No because it isn't linear and x can equal more than 1 y" (no) [Definition I]

In summary, different rules and graphs called on different responses under each definition. Besides the images generated by the given definitions, the images created by school experience with functions were also evoked constantly. In 4.4.2, some case studies will be considered for they show that, sometimes, the concept images of function play a dominate role in the whole process of justification.

4.3.4 <u>Miscellaneous responses</u>

The aim of this section is to report some general findings which are not covered in 4.3.2 and 4.3.3.

(1) Students may be well aware that the notions of 'domain' and 'range' are parts of the function concept for the terms 'domain' and 'range' are evoked in those responses under definition I and III. However, some students fail to master these two notions properly. In addition to the observation 'Domain $\in$ R' which I have discussed in 4.3.2., other examples were found throughout the answers. For instance,

"Because for each domain x there is a range y (there is only one for each)" [Rule 1]

"More than 1 range is assigned" [Rule 2]

"x has only range 2" [Rule 2]

"Range is always defined" [Rule 2]

"To each domain there is a range" [Rule 4]

Perhaps, the above confusions may come from school problems on finding the domain and the range of a given function. For instance, in Math 203, the students are asked to find the largest subset of R which serves as the domain of function such as $y = \sqrt{x^2 - 2}$. In order to solve this problem, the students have to determine <u>all the value of x</u> such that $x^2 - 2 > 0$. During this process, the equivalence between finding the domain of $y = \sqrt{x^2 - 2}$ and the value of x is drawn. In time, the idea that the domain is x is formed and so lead to confusion.

(2) The notions of 'variarble' and 'depending relationship' were also evoked. Some examples were:

"The value of y is dependent x and each value of x will yield one and only one value for y according to the rule" [Rule 1]

"If you put a real number in the variable x you will get an answer in y" [Rule 1]

"Each real person is assigned some numbers. However, these numbers cannot change, whereas in a function there is a variable" [Rule 4]

The last response suggests that 'variable' relates to 'change'. This kind of idea can also be found in the study of Marnyanskii (see 3.3.1)

(3) As Vinner [36] remarked: when students are forced to explain their answers, they are likely to use some 'mathematical argument'. In the case of this questionnaire, mathematical terms such as 'solution', 'value' were used in a non-relevant or incomprehensive way. For instance,

"Only integers have solutions" [Graph 5]

"Every x has 1 or 2 real solutions" [Graph 6]

"All real numbers are being assigned values" [Rule 3]

(4) The graphical representation was considered easy under definition II. Most of the students applied the 'vertical line test' to justify their answers. However, lots of students failed to justify the same task under definition I and III. This phenomenon may be explained by remarking that, in the case of definition I and III, the students

cannot apply the 'vertical line test' directly. Furthermore, they have to figure out the connection between the graph and the given definition, in addition to the step of justification.

4.4 Results on the 'best fit' definition and some case studies

In this section, we shall present the results on the last question, that is, 'which of the above definitions best fits your idea of function?'. Moreover, some case studies will be conducted in order to study the following aspect: Is there any connection between the choice of the 'best fit' definition and the responses to the other parts of the questionnaire?

4.4.1 Results on the 'best fit' definition

The aim of this section is to report the students' responses to the question of 'best fit' definition.

The answers of the 'best fit' question are divided into five categories. They are: (1) Definition I; (2) Definition II; (3) Definition III; (4) No answer and (5)

Others. The fifth category refers to those answers which cannot be put into the first four categories. For instance, 'Def I and Def III' or 'none of them'. The distribution among these five categories is:

	Def I	Def II	Def III	No answer	Other
Total	15	75	26	48	26
Math 203	8	24	7	17	8
Math 205	2	17	8	22	10
Math 262	5	34	11	9	8

Overall, around 18% of the whole population explained their answers. Some of these explanations were presented as follows:

(1) 40% claimed that definition II best fits their idea of function. Examples of the answers were:

"No 2. It s general. It does not limit your range and domain and does not involve formula." [Math 203]

"My idea of function is as such defined in #2 where one variable affect another and can be illustrated in a 2 dimensional piane." [Math 205]

"Definition II is a function because for every real number there is only one real number. Irregardless of two real numbers yielding the same real number in the range. Example: $y = x^2$, $x = 1$, $y = 1$; $x = -1$, $y = 1$." [Math 205]

"No. Only definition II. A function is a rule concerning any mathematical object." [Math 262]

"#II; It allows for functions that do not have necessarily the R set for domain or range (i.e. allows for fields not only R)" [Math 262]

From the above explanations, it seems that some students do realize the abstract feature of the domain and range in definition II, that is, the Dirichlet-Bourbaki definition. Yet, the second and third responses suggest that the concept image of function need not follow the concept definition, as Vinner's model describes. After all, the idea of variability is created from the working experience with functions rather than the concept definition. Moreover, the third case also shows that the example $y = x^2$, not the definition II, is used to clarify the idea of function. As a matter of fact, this student not only observed that in rule 1, positive and negative value of x yield the same y but he also changed his choice from 'no' to 'yes' in answering the part of definition II. It is quite possible that he changed his mind after the example $y = x^2$ was evoked.

(2) Only 8% of the subjects chose definition I as the best fit definition. However, the explanations in this group were incomprehensive. For instance, one student wrote:

"(More general) the first definition because the other 2 restrict function which are functions" [Math 262]

Perhaps, the low precentage in this category may come from the fact that the 'multi-values' feature in definition I contradicts deeply the 'single-value' part in the Dirichlet-Bourbaki definition. One student even pointed out that definition I gives only a 'relation' not a 'function'.

(3) Around 14% of the sample thought that mathematics deals with formula and so do 'functions'. Some examples were:

"Yes, definition III has meaning which related to many formulas dealing with Math, Sciences, Computers. and many more systems in our society." [Math 205]

"Definiticn III. Simply put every input (x) has and output (y)." [Math 205]

"Definition III because you can use a formula and with the use of the formula you get a function." [Math 262]

"Definition #3. An equation which can result in a different answer for the same number is not a function. #3 seems to state that the best." [Math 262]

Moreover, one student expressed somewhere in the questionnaire that:

"... give me #'s and tell me what to do with them, but I don't understand all this ..." [Math 203]

(4) Some students either gave a general comment on the three definitions or proposed their 'personal definition'. Examples were:

"Def I: nothing clear to specify. What rule is made up of - MAYBE. Def II: functions can have many y's to 1 x. Def III: x/y relationship doesn't have to be unique, x can have 2 y in a function" [Math 203]

"Yes, we can from diagram to define it is or not a function" [Math 203]

"No, each of the above definitions do not account for variations (even when the error is obvious). Not all functions are found through mathematical computations as in definition #3 and the student's ID is a valid function for definition #1 and #2." [Math 205]

"No, I believe that the definition of a function is a rule that assigns any numbers to a given variable, but each variable only has one value." [Math 205]

"I would accept all of them as loose definitions of the function." [Math 262]

"Definition II is most comprehensive because it yields one element in y (range) for each x in the domain. Definition I has an ambiguous because which y in the range must we choose? Definition III is ambiguous to me because of this case if for all x, y = 2, then, there is no mathematical formula involving x in this case." [Math 262]

The comments made by the students may not be correct. However, some do give us insight into their idea of function. For instance, the third and fourth comments suggest that the idea of variability is very important in their notion of function.

4.4.2 Case studies

In this section, some case studies will be reported in order to study whether the choice of the 'best fit' definition is consistent with other parts of the questionnaire.

Student A (Math 203)

Student A claimed that definition I is closest to his idea of function. He certainly recognized the 'multi-values' and 'real number' features in definition I. However, neglecting the part of 'single rule' led him to accept rule 3 as an example of definition I. Moreover, he excluded rule 1 - 4 as an example of definition II by arguing that neither 'a real' nor 'a student' is a set. The responses in the part of definition III shows uncertainty due to the fact that most of the explanations are erased. Overall, the responses of student A are considered to be consistent. However, this student had difficulty in interpreting definition II due to his vague idea of set.

Student B (Math 203)

Student B claimed that definition II is the best fit definition. However, his justifications rarely appealed to the definition. For instance, rule 1 was considered a function of definition II because 'it looks like one'. On the other hand rule 2 was also considered a legitimate candidate due to the one-to-one property even though rule 2 is not one-to-one. Moreover, the reason 'answers define' was used to accept graph 5 as a function.

Student C (Math 205)

Student C shared the same view with student B on the 'best fit' definition. However, his method of justification was totally different from B. He tried to determine the domain and the range in answering the part of definition II. To make matters worse, none of his calculations or arguments was correct. For instance, C argued:

"Since there are different student ID number, so that the domain is $(-\infty,\infty)$ and the range can be any number in $(-\infty,\infty)$." [Rule 4]

Whereas, he accepted rule 1 as an example of definition I by arguing that if x is a real number, then x is also a real number, he gave no explanation in the rest of the questionnarie.

Student D (Math 262)

Student D accepted none of the three given definitions. Highlighting the 'multi-values' feature led D to reject all the examples as functions of definition I. Although he questioned the meaning of 'exactly one element in the range' in definition II, he answered all the questions correctly by drawing diagrams. Considering the correspondence relationship and the feature of 'single real number', student D failed to identify rule 2, 4 and graph 5 as non-examples of definition III.

Once again, these results show that there are two types of reactions when the task and definition are both given. Student A and D reacted to a given task according to part of the given definition. Yet student B and C, were aware of the definition, but still appealed to their concept image to justify all their responses. Moreover, for the former type of reaction, the responses are considered to be consistent while for the latter, they are incoherent.

4.5 Discussions

Throughout the analysis of the questionnaire, concept images generated from the given definitions and working experience were exposed extensively. As an overall discussion, we make the following points:

(1) The results of the questionnaire suggest, on one hand, that giving definition explicitly does not eliminate inappropriate explanations and, on the other hand, that the given definitions do have impact at the justification stage. However, students often fail to identify all the components of a given definition due to its complexity.

(2) As I have discussed in chapter 3, the evocation of concept image also depends on the feature of the task. The results show not only that different tasks call for different images but also that the same task under

different definitions may lead to different mental representations.

(3) The meaning of the word 'assign', which is used in a particular way in computer science, leads to some mental representations that are independent of the given task and the given definition. This finding suggests that the use of the word 'assign' in the definition of function needs further clarification.

(4) There are differences among the three groups of students. The students in Math 262 not only do better overall but also give more explanations. Many of them are aware of the abstract feature of the Dirichlet-Bourbaki definition. The students in Math 205 not only get the lowest overall score but also favour some non-relevant calculations or arguments. The students in Math 203 do all right when using definition II but not definition I and III. Perhaps, this is because definition II is taught in class but the other definitions are not. This situation suggests that students in this group have difficulty to apply their knowledge in a new situation.

(5) In all three groups, students tend to justify

their answers only according to a part of the definition. For instance, reasons such as '2 is a real number' are used to justify rule 2 as a 'function' of definition I. Strictly speaking, this kind of justification is insufficient, something most of the students fail to realize.

CHAPTER V

A QUESTIONNAIRE ON THE NOTION OF LIMIT

5.1 Introduction

This questionnaire examines the extent to which students are convinced by mathematical arguments.

The study of Tall and Vinner, discussed in chapter 3 showed that the common concept image of $s_n \to s$ is:

"s_n approaches to s, but never reaches it"

One would like to know if this phenomenon comes from the bad pedagogy or if it inherent in the concept itself. The answer is not clear unless we take the teaching strategy into considerations. J. Confrey [2, 1980] comments that:

"... in calculus, the conflict is within the mathematics, as well as in teaching and learning, ..."

Later on, Davis and Vinner [3] confirm Confrey's point of view. They report that misconceptions in learning the notion of limit seem to be unavoidable even a better teaching approach is adopted (see 3.4.3). The 'naive concept' of limit (a sequence must not reach its limit) will be evoked naturally is most case rather than the 'correct concept' (a sequence reaches its limit if the limit exists). The questionnaire attempts to find out how students will react to mathematical arguments especially when both the 'naive' and 'correct' concepts are evoked.

5.2 Selection of the tasks, population and research method

In this section we report on the criterion of selecting the questions in the questionnarie, on the population as well as the reseach method.

The sources of this questionnaire come from different researches including Tall and Schwarzenberger [28, 1979], Confrey [2, 1980] as well as Vinner and Kidron [38, 1985].

The methematical arguments given in the questionnaire follow the technique which Confrey used in her clinical interview. The format of the presentation resembles the questionnaire of Vinner and Kidron.

5.2.1 The questionnaire

This questionnaire consists of two questions concerning the notion of limit. They are:

<u>Question 1</u>

A teacher asked his students: "Is 0.999... (nought point nine recurring) equal to one, or just less than one?

In order to change 0.999... to 1, John did the following:

Let $\quad x = 0.999...$

Then $\quad 10x = 9.999...$

So $\quad 10x = 9.999...$

$\quad\quad x = 0.999...$

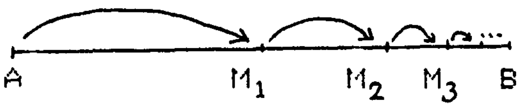
Subtracting $9x = 9$ and $x = 1$

Therefore, $0.999... = 1$

Another student, Peter, claimed that 0.999... is just less than one because the difference between it and one is infinitely small. Who is right? Please, explain? Moreover, could you settle this disagreement?

Question 2

A rabbit starts at point A. On the first hop, the rabbit hops to M , halfway between A and B. On the second hop, the rabbit continues to take hops ahead which are half the length of the previous hop. Does the rabbit reach B?



Mary wrote down $AM_1 = 1/2\ AB$

$$M_1M_2 = 1/4\ AB$$
$$\vdots$$

So the rabbit will hop

$$AM_1 + M_1M_2 + \ldots$$
$$= 1/2\ AB + 1/4\ AB + \ldots$$
$$= (1/2 + 1/4 + \ldots)\ AB$$

Summing up this infinite geometric series with common ratio 1/2, Mary gets:

$$1/2 + 1/4 + \ldots = (1/2) / (1 - 1/2) = 1$$

Thus, the rabbit will reach B

However, Jennifer did not agree. She argued that the rabbit can only get close to B because it keeps hopping half of the previous distance. As the distance it hops becomes smaller, it will never reach B.

In your opinion, who is right? Please, explain. Again, could you settle this disagreement?

5.2.2 Population and the relevant information

This questionnaire was handed out a group of 48 Math 362 students at Concordia University.

Math 362 is Analysis I. The pre-requisite for this course is Calculus I, II and III in which the notion of limit is taught informally. In the textbook of Math 362, the notion of limit is introduced by a provisional definition of the limit of function:

" A function f approaches the limit l near a, if we can make f(x) as close as we like to l by requiring that x be sufficiently close to, but unequal to a"

Next, a series of examples and non-examples focusing on the relationship between 'ϵ' and 'δ' in the formal definition are presented. Finally, the formal definition is arrived:

"The function f approaches the limit l near a means: for every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$."

Hence, it was expected that concept images generated by the notion of limit of function will likely be evoked by the questions in the questionnaire.

5.2.3 The method

A pilot study was carried out with group of 23 students in the middle of the fall term of the 1988 academic year. However, no change was seemed necessary after examining the result of the pilot. Thus, the same questionnaire was distributed to the whole population at the end of the same term, which took about 15 minutes to answer, at the end of the class.

5.3 Analysis of the data

The aim of this section is to report on the results of the questionnaire.

Basically, analysis was done on each question, based on the responses of the students.

Results on question 1

9 students in question asserted that 0.999... = 1 while 33 thought that 0.999... is just less than 1. 6 students were uncertain. In a summary, six kinds of arguments were found:

(1) 8 students argued that 0.999... only 'approaches' very close, but not equal, to 1. Examples were:

"Peter is right by saying that 0.999... is less than 1. It will never reach one but it is approching it slowly"

"Peter is right. Since 0.999... is not 1, 1 is 1 and 0.999... is 0.999..., 0.999... is approaching 1 but is never 1. If we are talking about limits Peter is right, but if we are talking about a simple number we would round it to 1 making John right."

The second explanation suggests that this student failed to conceive the infinite decimal representation of real numbers. One more response will give some insight into the student's idea of decimal representation:

"Peter is right, because 0.999... is an approximation to 1, not equal to 1. Once you have decimal places, it already means an approximation. 1 is an integer number. For example, when someone counts cards, they say, "1, 2, 3" not "1.0, 2.0, ..." because there exact values, in contrast to decimal places which are approximate values."

Under such a conception of decimal representation, it is not surprising to find out that 0.999... and 1 can never equal.

Yet, some student argued that 0.999... and 1 are so close that the error between them can be neglected. One student wrote:

"John is right 0.999...= 1. The recurrence of 9 gives a number which is so close to 1 that the error is extremely small and essentially equals to 0. Therefore the method used by John is correct."

(2) 9 students upheld that although the difference between 1 and 0.999... is infinitely small, the difference still exists and so 0.999... is just less than one. Example in this group was:

"Peter, 0.9 in not 1 even though it is very close to it. Like Peter said the difference between 1 and 0.9 is very small but it exists."

The students in this group basically followed the arguments of Peter in the question. Yet, one student claimed that the difference between 1 and 0.999... can be neglected due to the infinite feature. He wrote:

"x = 0.999...
x = 9/10 + 9/100 + 9/1000 + ...
Since it is an infinite process we could say it equals to 1 because the difference is too small so John is right."

(3) 3 students, who did not view 0.999.. as the sum of geometric series, asserted that 0.999... < 1 due to the infinite expansion. For instance:

"I personally believe that .999 is just less than one and that you could expand indefinitely, never reaching one."

(4) 5 students rewrote 0.999... as the sum of a geometric series with the first term 9/10 and the common ratio 1/10. After finding out the sum, 4 of them concluded that 0.999... = 1. The remaining student claimed that both Peter and John are right. He explained:

"... Both students A and B are right because since .99... is a recurrent series, its limit approaches 1 as the decimal places approach infinity"

Again, this student tried to resolve the disagreement by pointing out the difference between 1 and 0.999... will be eliminated as the digits of 9's goes to infinity.

(5) Some students appealed to some mathematical observations to justify John's arguments. For example:

"$9.999 = 9 \times 10^{0} + 9 \times 10^{-1} + 9 \times 10^{-2} + \ldots$
$0.999 = 9 \times 10^{-1} + 9 \times 10^{-2} + \ldots$
Both are infinite series, and therefore you cannot simply subtract one from the other, as John did. 0.999 tends to 1 but $\neq$ 1"

"...John is wrong by saying that 0.999... = 1. It is like saying 1 = 2 or $1 \times \infty = \infty$, $2 \times \infty = \infty$ implies 1 = 2

$$
\begin{aligned}
A &= A_2 \\
A^2 &= A^2 \\
A^2 - A^2 &= A^2 - A^2 \\
A\,(A - A) &= (A + A)(A - A) \\
A\,(A - A)/(A - A) &= 2A \quad \text{Cannot do this because } A - A = 0 \\
A &= 2A \\
1 &= 2 \quad \text{wrong } 1 \neq 2\text{"}
\end{aligned}
$$

Some other students in this group argued that John just rounded 0.999... to 1.

(6) Some students appealed to their knowlege of limit by, for instance, writing down the formal definition of limit of sequence or function or rewriting 0.999... in terms of function. Examples were:

".999... is less than 1 since without this $\lim_{x \to a}$ of $f(x)$ would necessarily be, at some point, $f(a)$ and this is not the case i.e. $\lim_{x \to 1} f(x)$ say $x = .999...$ is not necessarily $f(1)$"

"John is right. Consider the sequence s

$$
\begin{array}{cccc}
1 & 2 & 3 & \\
.9 & .99 & .999 & \ldots \text{ etc}
\end{array}
$$

$$s(n) = \underbrace{.999...}_{n}$$

$$\lim_{n \to \infty} s(n) = 1$$

i.e. Given $\epsilon > 0$, there exists $N > 0$ such that

$|s(n) - 1| < \epsilon$ whenever $n > N$

But this doesn't mean .999... = 1 it means we can make

.999... arbitrary close to 1"

The last student in fact erased 'Peter' and then wrote down 'John'. His (Her) attempt to settle the disagreement through the formal definition resulted in confusion. As a remark, two other students who attempted to justify their answers by using the definiton of limit of function, also came to a conclusion that 0.999... is just less than 1. In summary, the explanations given by this group showed the influence of the textbook. Generally, they argued that 0.999... tends to, but is not equal to 1, because x tends to, but not equal to 'a' in $\lim_{x \to a} f(x) = 1$.

Overall, most of students tried to settle the disagreement by either justifying that an infinitely small quantity can be neglected or by pointing out that the mathematical argument given by John is not convincing. Perhaps, the vague argument of John led the students to favour Peter. However, even students who appealed to the valid mathematical observation, that is, viewing 0.999... as the sum of a geometric series, still supported Peter.

<u>Results on question 2</u>

Basically, the explanations given to question 2 resembled those of question 1. However, the concept image such as infinitely divisible was evoked, due to the feature of the task. Furthermore, some students tried to settle the disagreement by considering the difference between the theoretical and practical aspect. For instance:

"In theory, the rabbit will never get to B but in practice it will. But if the rabbit keep hopping for ever it will eventually get at a point so close to B that we can say it is at B"

Overall, 15 students thought that the rabbit will reach B while 23 held the opposite view. Taking a closer look at the answers, we note that:

(1) Around a quarter of the whole sample considered the question of how close the rabbit will get to the point B. All except two of these believed that the rabbit will not reach B. Some of them expressed that:

"Jennifer, the rabbit will get infinitely close to B, but never reach B"

"Jennifer is correct. The infinite geometric series is simply a limiting series. i.e. the rabbit will never go beyond B, but it will get infinitely close to B."

"I think that the rabbit will never reach B!!! He will get very very close to B, but will never reach it in theory."

The other two students argued that the rabbit will reach B because it is, in fact, approaching the limit B. For instance:

"I believe that Mary is right, since as he approaches the limit is B."

(2) 8 students considered the difference in distance between where the rabbit is and the point B. 4 of them argued that no matter how small the distance is, the distance exists and so it can always be divided by half. Thus, the rabbit will not reach B. Examples were:

"The rabbit never reach B since we can always divide the distance between B and the rabbit, even though this is very small number."

Yet, another four students thought that the distance between B and the rabbit will eventually tend to zero. Therefore, the rabbit will reach B. One student wrote:

"Mary, as the rabbit hops half the distance between where it is at and B, this distance → 0. So after a while we add zero to our sum."

The remaining student got lost in the argument:

"In a way this is similar to the other question because to say the rabbit has not reached the point there must be a difference it must travel. When it is close and travels 1/2 the remaining difference more what is the difference. It is zero since never reach ..."

(3) 2 students pointed out that the rabbit will only reach B at infinity which is impossible. For instance:

"Jennifer is right. This is well explained in a <u>311</u> math course. The rabbit will reach B at infinity (which is never)"

(4) 3 students argued that the rabbit will reach B due to the convergence of the given geometric series. One student wrote:

"Mary is right (obivously). Jennifer is making the mistake in assuming that because one is summing up an infinite number of distances, the rabbit will take an infinite amount of time. The sum of the infinite series can be finite."

(5) [illegible]r of students tried to relate this question to their knowlege of limit. The arguments involved

using the formal definition resembled those in question 1. Furthermore, two of them appealed to the notion of limit of function to conclude that the rabbit will not reach B. For instance:

"Jennifer is correct. The answer that Mary is giving maybe the limit as the number of jumps goes to infinity, but the existence of such a limit does not necessarily implies that this 'function' will ever take on that value."

Yet, another student claimed that Mary is right by explaining:

"Mary, using limits, at each step he will never get there but taking the limit you will get the distination..."

(6) Some other appealed strongly to their intuitions. Examples were:

"Mary is right because the rabbit does reach its distination. If Jennifer were right, no one (including the rabbit) would ever reach their distination."

"Mary is right because the two points A and B is a finite distance and eventually the rabbit get to point B."

"That all depends on the health of the rabbit if B is physically possible"

These kinds of arguments showed that the increase of support to Mary came from 'intuitive' truth and not from mathematical arguments.

5.4 Case Studies

In this section, some case studies will be discussed in order to examine how the features of a given question affected the answers.

Student A

Student A thought that 0.999... < 1 but the rabbit will reach B. His arguments in both questions were:

"Peter, $0.\dot{9}$ is not 1 even though it is very close to it. Like Peter said the difference between 1 and $0.\dot{9}$ is very small but it exists."

"Mary, as the rabbit hops half the distance between where it is at and B, this distance $\to 0$. So after a while we add zero to our sum"

These two arguments suggest that the same idea led to different choices, due to the feature of the questions. After all, question 1 deals with some abstract idea which is beyond experience, while question 2 can appeal to one's everyday experience.

Student B

Student B answered question 1 by rewriting 0.999... as the sum of the geometric series 9/10 + 9/100 + 9/1000 + ... Noticing the sum of this series is 1, B concluded that 0.999... = 1. However, student B justified question 2 by saying:

> "First student is right. You could argue that the distance still to go becomes infinitesimally small → 0. You reach the [illegible]."

The responses of B show that the 'naive idea' is evoked when the task is not strictly mathematical.

Student C

Student C applied different idea to answer different questions. In question 1, C asserted that:

> "Peter is right because limit of 0.9999999 is approaching 1 but not equal to it. So what Peter said is true and right because the answer has infinite number of recurring 9's"

On the other hand, he agreed with Mary because the distance between A and B is finite. These responses suggested that different images were evoked under different questions.

Student D

Student D was the one who claimed that the rabbit will reach B, by arguing that the sum of an infinite series may be finite (see 5.3). Yet, he thought that 0.999... is just less than 1 because John cannot simply subtract two infinite series (see 5.3). Obviously, he overlooked the fact that 0.999... can also be viewed as a convergent series.

5.5 Discussion

In both questions, very few students were convinced by the abstract mathematical arguments. This suggests that the 'naive' idea of a limit often dominates over the 'correct' idea, even when both ideas are presented. Overall, more than half the students tried to settle the disagreement in both questions by considering whether an infinitely small quantity can be neglected or not. This point of view resembles that of Leibniz and his contemporaries. There is also evidence to the fact that the 'naive' idea is encouraged by the presentation in the textbook. For instance, the introduction of the notion of limit by the limit of function not only causes a confusion between the value of the function at 'a' and the value of f(x) when x 'goes' to 'a', as Vinner and Davis observe (see 3.4.3), but also causes a confusion as whether f(x) actually reaches its limit, since 'x' does not necessarily reach 'a'. Similarly, the formal definition of a limit of a sequence does not indicate whether the terms of sequence can take on the limiting value. This may explain why students who try to use the formal definition often fall into a confusion. All of these observations reinforce my belief that a pedagogical approach emphasizing only the formal mathematical arguments is not enough to convey the correct idea of limit. Perhaps, a set of well-prepared examples addressing to different misconceptions of limit should also be given, in order to make the students aware of these misconceptions.

CHAPTER VI

CONCLUSION

The aim of this chapter is to give an overall conclusion of the results discussed in the previous chapters. Several observations are made as follows:

(1) Vinner's cognitive model of concept image and concept definition (see 1.5) suggests that concept images and not conept definition, will play the dominant role at both stages of concept formation and performance. Comparing the historical concept images held by mathematicians in the development of Calculus, and those held by the learners in the studies of Vinner and others, I argued that, on the one hand, these students share certain naive concept images with mathematicians of the past centuries; On the other hand, there is evidence that, to some extent, certain misconceptions are eliminated as the result of organizing the concepts in a coherent way. From this point of view, definitions do have their impact in teaching situation.

(2) Objecting the formalistic view of definition in teaching mathematics, Vinner proposes that mathematical definitions can be viewed as lexical definitions, that is , definitions are statements about mathematical objects; statments that are employed to characterize these objects in

defining situation (see 1.4). In order to find out the learners' idea of definitions, that is, whether the learners view mathematical definitions as lexical definitions, a questionnarie on the notion of function was formulated. As the results of the questionnaire indicate, some students just rely their concept images in order to carry out the task. Even those students who take a given definition into consideration often fail to classify the task, partly due to inability to handle the complexity of a given definition. Another failure comes from the fact that often, the meaning of certain 'words' in mathematics is not the same as in other subjects or in daily experience. However, the improvement in results by students in the upper level suggests that Vinner's idea of viewing mathematical definitions as lexical definitions also need certain mathematical maturity. While I agree with Vinner that in the learning situation, definitions seem unimportant, I do suggest that the teachers should master definitions which need higher levels of abstraction such as the Dirichlet-Bourbaki definition in the case of function. This will enable teachers to give suitable examples in order to eliminate students' misconceptions, as well as, to organize different concepts in a coherent way. For instance, the use of one-to-one relationship to classify whether a given rule is a function may stem from the popular example such as 'the rule which assigns to each student the student's ID number'. However, teachers ought to construct other examples such as 'the rule which assigns to each student a grade within the range A - F to make clear that a functional relationship is not necessary one-to-one. In such a way, it is hoped that the students will be aware of the difference between their own conceptions and the formal definition, and so acquire a better understanding of a given notion.

(3) Beside the fact that mathematical definitions are not easy to master, mathematical arguments are also not that convincing in the learners' mind. Studying the results on the questionnaire on the notion of limit in which two kinds of arguments were presented - the 'correct' idea was written in mathematical form; the 'naive' misconception in the sentence form - I observed that more than half the students favoured the 'naive' idea, even when this 'naive' idea did not make sense one's real experience. Again, I suggest that the correct idea of certain mathematical notions should be presented through a sequence of refining examples leading towards the formal definition, and complementing mathematical arguments. It is hoped that the concept images generated from these examples will lead to a

better understanding.

Overall, an approach emphasized on examples and non-examples may provide the learners with a better situation for learning mathematical notions.

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