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Diagnosability and Diagnosis of Sparsely Interconnected Multiprocessor Systems

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ABSTRACT

Diagnosability and Diagnosis of Sparsely Interconnected Multiprocessor Systems

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This thesis is motivated by certain shortcomings of the classical t-diagnosability theory developed by Preparata, Metze and Chien (PMC) when applied to fault diagnosis of large sparsely interconnected multiprocessor systems. This calls for a detailed study of diagnosable systems such as t/s-diagnosable systems which require fewer tests and development of a theory of local diagnosis which permits correct diagnosis of all faulty processors in a large sparsely interconnected system as well as development of diagnosis algorithms which are amenable for distributed implementation on the multiprocessor system itself.

The first part of this thesis (Chapters III and IV) is concerned with a study of t/s-diagnosable systems.

In Chapter III, we present characterizations of t/s-diagnosable systems which generalize those given earlier for t/t-diagnosable systems. We show how the t/s-characterization for the PMC model based on Kohda's t-characterization theorem can be easily modified to arrive at a t/s-characterization for the Barsi, Grandoni and Maestrini (BGM) model as well as characterizations for the sequentially t-diagnosable systems. We also present, in this chapter, certain structural properties of general t/s-diagnosable systems which generalize some of the earlier results. These properties lead to a new $t/t+1$-characterization.

With the objective of determining an efficient test for a vertex v to be in an allowable fault set of size at most t, we first establish in Chapter IV several

properties of allowable fault sets. Using these properties and the characterizations given in Chapter III, we then develop an $O(n^{3.5})$ algorithm for diagnosis of a $t/t+1$-diagnosable system. We also present a $t/t+k$ diagnosis algorithm which runs in polynomial time for each fixed positive integer k. We show how these algorithms can be modified to construct algorithms for sequential t-diagnosis.

In the second part of this thesis (Chapters V and VI), we use the comparison-based approach to develop a theory, wherein instead of a single global constraint, local fault constraints on the number of faulty processors in the neighborhood of each processor in the multiprocessor system are considered.

In Chapter V, we reformulate a result in syndrome decoding reported in the literature under various forms and apply it to regular interconnected multiprocessor systems with very small connectivity. Here a local neighborhood is defined around each processor which consists of its t immediate neighbors and t subsequent neighbors. The faulty or fault-free nature of each processor is then determined as long as no more than t processors are faulty in its corresponding neighborhood. Based on this result, we also present a simple $O(1)$ distributed diagnosis algorithm. We study the application of local fault constraints on a ring of processors. Specifically, we determine if unique diagnosis is possible if p out of any q consecutive processors in the ring are faulty. We develop sequential and distributed algorithms for these systems.

In Chapter VI, we introduce the concept of a t-in-L_k diagnosable system. We first present certain basic results which lead to a sufficient condition for unique diagnosis of a system when certain fault constraints are satisfied in the local domain $L_1(u_i)$ of each processor u_i. In this chapter, we also study the t-in-L_1 diagnosability of certain regular interconnected systems: the closed rectangular, the hexagonal and octagonal grid systems, and the hypercube systems.

We present t-in-L_1 diagnosis algorithms for these regular systems as well as those which satisfy certain other conditions. These diagnosis algorithms can be executed in a distributed manner on the system itself.

TO MY LOVING PARENTS

KALA AND ABANI

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LIST OF SYMBOLS AND ABBREVIATIONS

S	Multiprocessor system
$G(U,E)$	Test interconnection graph
U	Vertex (processor) set
E	Edge (Test) set
$\bigcap$	Set Intersection
$\bigcup$	Set Union
	Set difference
$\oplus$	Symmetric difference
$\|X\|$	Cardinality of set X.
X^c	$U-X$, Complement of set X.
$deg(u_i)$	Degree of vertex u_i.
δ	Minimum degree in G.
$G-X$	Subgraph of G induced on vertex set $U-X$.
$d(u_i,u_j)$	Distance between vertices u_i and u_j.
$\Gamma(u_i)$	Set of Processors tested by u_i.
$\Gamma^{-1}(u_i)$	Set of Processors testing u_i.
VCS	Vertex Cover Set.
MVCS	Minimum Vertex Cover Set.
$\Delta_1(u_i)$	0-descendants of u_i.
$A_0(u_i)$	0-ancestors of u_i.
$H_0(u_i)$	Set representing $A_0(u_i)\bigcup\{u_i\}$.

AFS	Allowable Fault Set.
MAFS	Minimum Allowable Fault Set.
$L(u_i)$	Implied fault set of u_i.
$N_0(u_i)$	Set of neighbors of u_i connected to u_i with a 0-link.
$N_1(u_i)$	Set of neighbors of u_i connected to u_i with a 1-link.
G^*	Implied Fault graph of G.
G_v^*	Graph Representing $G^* - H_0(v)$.
B	Bipartite graph derived from G^*.
B_v	Bipartite graph derived from G_v^*.
t_v	$t_v = t - H_0(v)$.
U_v	$U_v = U - H_0(v)$.
$L_k(u_i)$	Set of processors lying within distance k from u_i.
$\lfloor x \rfloor$	Largest integer less than or equal to x
$\lceil x \rceil$	Smallest integer greater than or equal to x.

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CHAPTER I

INTRODUCTION

Computer applications demand systems which are many orders of magnitude faster than those currently available. Systems which incorporate a large number of processing elements are required to solve a wide range of computationally intensive tasks. Continuing advances in semiconductor technology have now made practical the development of very large systems consisting of hundreds and thousands of processors, and systems consisting of millions of processing elements are being envisaged. As the number of processors in a system increases, the failure of some of these processors during operation becomes more likely. The failure of a few of these processors could be critical and at times even catastrophic. Thus the high reliability and availability of these systems is an area of major concern.

An approach to achieve high reliability and availability is to build fault-tolerant systems: failure of some components can occur in the system but their presence is not allowed to influence the output of the system. Although a fault-tolerant system may suffer a degradation in performance, it continues to generate correct output in the presence of a few faulty components.

A technique involving massive replication, known as N-modular redundancy (NMR), is sometimes used in fault-tolerant systems. In systems using the NMR technique, the computation corresponding to a component are performed by N different elements whose outputs are submitted to a voter. The voter then decides which results ought to be propagated, thereby masking the effects of any faults. Other techniques include:(1) error detecting and/or error correcting codes including self-checking components; (2) validity checks of software and diagnostics of hardware; and (1) watchdog timers and sanity messages.

Techniques such as the ones mentioned above are well-suited for systems comprising of a comparably small number of processors. However, the increased scale of future systems and their radically different architectures will render these techniques less attractive. On the other hand, these developments will provide the basis for alternate strategies such as self-monitoring and self-diagnosis. Clearly systems which incorporate a large number of independent processors could exploit the potential of these processors to monitor one another to detect, diagnose and identify the faulty processors. A procedure to reconfigure the system to comprise of only fault-free processors could be initiated once the faulty processors are identified. System-level fault diagnosis, an area pioneered by the work of Preparata, Metze and Chien [1], provides a framework for this approach for identifying faulty processors in distributed systems.

1.1. Basic Models for System-Level Diagnosis

Several models have been proposed in the literature for diagnosable system design. In this thesis, we consider three basic models: the PMC model, the BGM model and the Comparison model.

1.1.1. PMC Model

The now well-known PMC model introduced by Preparata, Metze and Chien [1] has been extensively studied in the literature. In this model, each processor in the system tests some of the other processors and produces test results. The exact nature of these tests and the manner in which they are conducted are not specified by the model. A test result determines whether the testing processor finds the processor being tested to be faulty or fault-free. Different fault models arise depending on the type of testing, the fault types considered and the interpretations associated with the test results. In the PMC model, also referred to as the **symmetric invalidation model** , faulty units are considered to be permanently faulty and a test result is reliable only if the testing processor is itself fault-free.

The processors in the system and the test interconnection can be represented by a digraph $G(U,E)$ where the vertices represent processors and there is an edge from vertex u_i to u_j if and only if processor u_i tests processor u_j. The digraph is referred to as the connection assignment of the system. A label a_{ij} which represents the corresponding test result is associated with each edge (u_i,u_j). The collection of all test results over the entire system is referred to as a **syndrome.**

1.1.2. BGM Model

A variation of the symmetric invalidation model, formulated by Barsi, Grandoni and Maestrini [2] and known as the BGM or the **asymmetric invalidation model** , arises when a different interpretation is associated with the test results. In the BGM model, in addition to the requirement that a test result is reliable only if the testing processor is fault-free, it is also required that a faulty processor always tests another faulty processor to be faulty. The motivation for this assumption is that processors in the system are considered to be complex and each test is in fact a combination of a sequence of tests. Thus it is highly unlikely that the faults in the testing processor would mask the faults in the processor being tested for the entire sequence of tests. Although the difference between the PMC and the BGM models seems minimal they lead to very different solutions for related problems, some of which seem to be counter-intuitive. For example, Somani, Avis and Agarwal [3] presented a problem which is co-NP-Complete in the BGM model but is solvable in polynomial time in the PMC model whereas in [4], Gupta and Ramakrishnan presented another problem which is solvable in polynomial time in the BGM model but is co-NP-Complete in the PMC model.

1.1.3. Comparison Model

Another model, known as the **comparison model** , was proposed in [5], [6] and [7]. In the comparison-based approach, all the processors in the system are assigned the same task to be performed and the outputs of some pairs of these tasks are compared. The processors in the system and the comparison assignment can be represented by an undirected graph (U, E) where the vertices represent the processors and the edges represent the comparison tests. Chwa and Hakimi proposed that a comparison outcome be considered unreliable only if both processors being compared are faulty. This model of interpretation is clearly in spirit with the symmetric invalidation model for directed testing. It is thus possible to consider an alternative model of test outcome interpretation for the comparison approach also, similar in spirit to asymmetric invalidation.

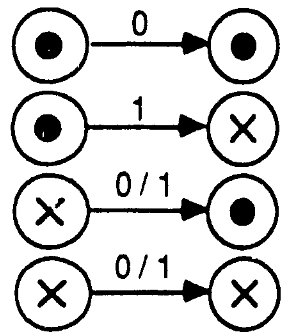
The test outcomes associated with the PMC, BGM and comparison models are shown in Fig. 1.1.

1.2. Review of Literature

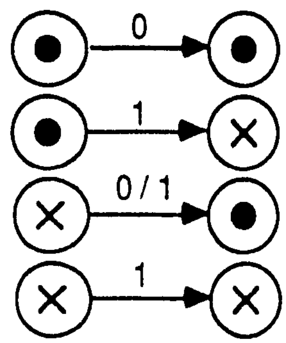
In order to carry out a study of diagnosable systems, assumptions are required regarding the fault sets which could occur in the system. For instance, if all processors in the system are faulty then in the PMC model any possible syndrome could be generated. The classical constraint used is to assume that the number of faulty processors in the entire system is upper bounded by an integer t.

1.2.1. t-Diagnosable Systems

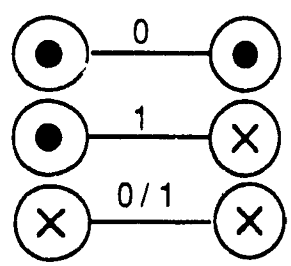
A system is said to be t**-diagnosable** if, given a syndrome, all processors can be correctly identified as faulty or fault-free provided that the number of faulty processors present in the system does not exceed t. Three problems of interest arise in this context.



(a) Test outcomes under the PMC model



(b) Test outcomes under the BGM model



(c) Test outcomes under the comparison model for the symmetric invalidation assumption

: Fault-free unit   : Faulty unit

Fig. 1.1 Test outcomes under the PMC, BGM and the comparison models.

t-characterization problem : Given a positive integer t, determine the necessary and sufficient conditions for the system test assignment to be t-diagnosable.

Hakimi and Amin [8] gave the first complete solution to the t-characterization problem under the PMC model. Other characterizations were given by Allan, Kameda and Toida [9] and by Kohda [10]. A characterization for the BGM model was given in [2]. A t-characterization for the comparison model was given in [6].

t-diagnosability problem : Given a system, determine the largest value of t for which the system is t-diagnosable.

An algorithm to solve the t-diagnosability problem for the PMC model was long considered to be exponential, persuading researchers to restrict their study of diagnosability to various subclasses of digraphs. Using network flow techniques, Sullivan [11] developed a t-diagnosability algorithm based on the t-characterization given by Allan, Kameda and Toida [9]. This algorithm runs in $O(mn^{1.5})$ time where m is the number of tests and n is the number of processors. Subsequently, Raghavan [12] improved upon this result by presenting an algorithm which runs in $O(nt^{2.5})$ time. Narasimhan and Nakajima [13] also presented a diagnosability algorithm based on a characterization which is similar to the t-characterization for the PMC model given by Allan, Kameda and Toida in [9]. Somani [3] presented an $O(n^3)$ t-diagnosability algorithm for the BGM model based on the original characterization given in [2].

***t*-diagnosis problem** : Given a t-diagnosable system, develop an algorithm to locate the faulty units present in the system, using a given syndrome.

Kameda, Toida and Allan [14] were the first to publish a t-diagnosis algorithm with polynomial time complexity. This algorithm as well as the one presented by Madden [15] are backtracking algorithms of time complexity

$O(n^3)$. Dahbura and Masson [16] developed an $O(n^{2.5})$ algorithm for the t-diagnosis problem under the PMC model by transforming it into a problem of finding a maximum matching in a bipartite graph. This algorithm is based on the concept of implied fault sets and other results which they developed in [17]. A t-diagnosis algorithm with complexity $O(|E|+t^3)$ using branch and bound techniques, and based on the algorithm given in [14], was presented by Sullivan [18]. Meyer [19] gave an $O(m)$ t-diagnosis algorithm for the BGM model. Diagnosis algorithms for special classes of graphs have been presented in [20-25].

1.2.2 Variations of the PMC Model

Other variations of the models considered above can arise if the faulty processors can be either permanently faulty or intermittently faulty. Intermittent faults under the symmetric invalidation model were first proposed by Mallela and Masson [26]. The various test outcomes which occur when all combinations of fault-free, permanently faulty and intermittently faulty processors are considered for the symmetric invalidation assumption are shown in Fig. 1.2. Test outcomes under the asymmetric invalidation assumption are defined similarly. A system is said to be t_i**-diagnosable** if given a syndrome which is compatible with a permanent fault situation, the diagnosis will never be incorrect, although it may possibly be incomplete. Mallela and Masson gave a characterization for such systems and subsequently Yang and Masson [27] presented an $O(m)$ diagnosis algorithm. Other variations of this model were reported in [28,29]. Intermittent faults under the asymmetric invalidation model were studied by Somani [3] who presented an $O(n^3)$ diagnosability algorithm. It is obvious that identifying all the faulty processors when intermittent faults are also present is far more difficult than when only permanent faults are present. A generalized theory for system-level diagnosis was proposed by Somani, Davis and Agarwal in [30]. This

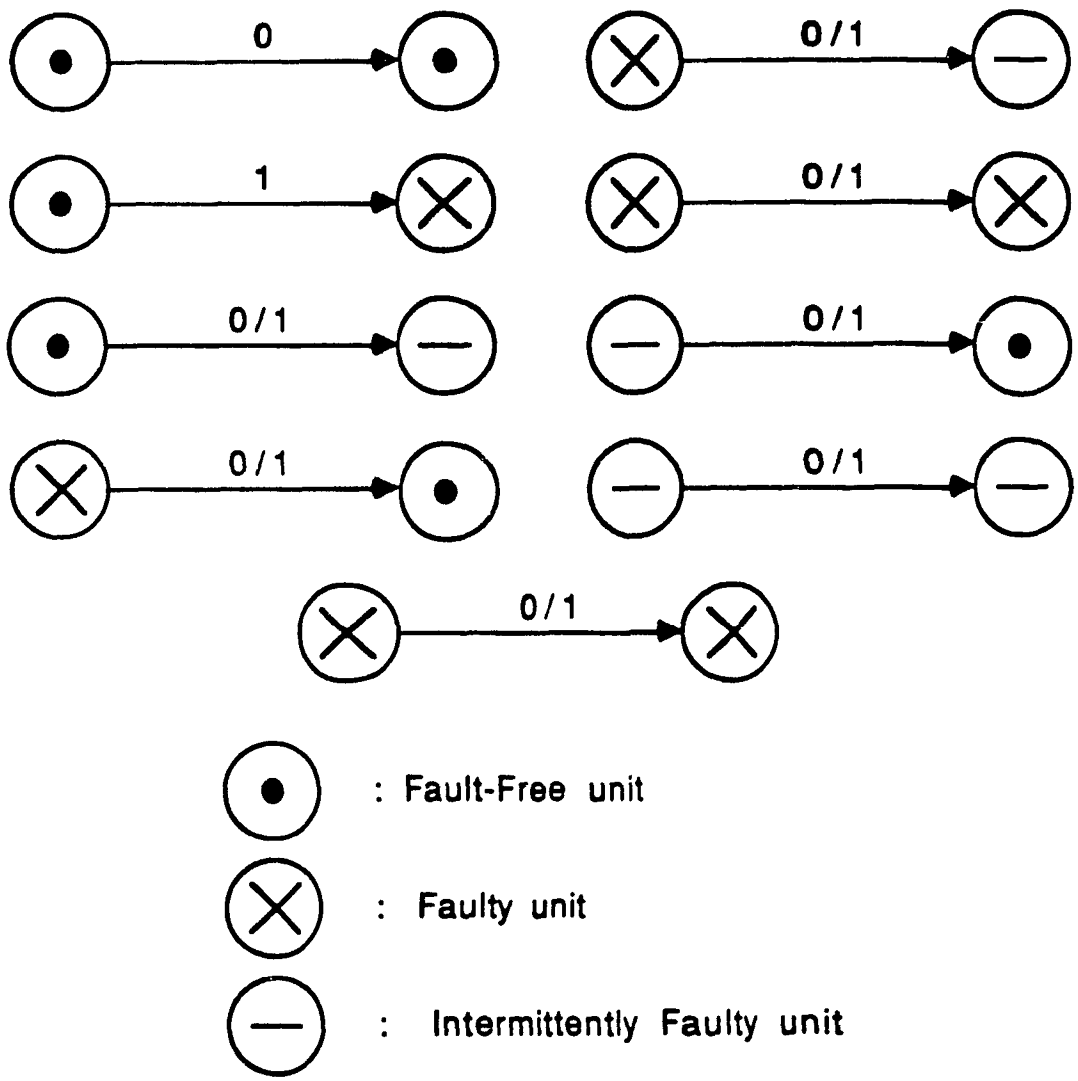


Fig. 1.2 Test outcomes under the PMC model with intermittent faults.

paper presented a generalized characterization theorem which provides the necessary and sufficient conditions for unique diagnosis of fault sets of any size under different fault models.

Maheswari and Hakimi [31] introduced the concepts of probabilistic and weighted diagnosability and established the relationship between them. A probability of failure is assigned to each unit in probabilistic diagnosis. Assuming that processor failures occur independently, the probability of failure of any set of processors can be determined. A system is said to be p-diagnosable if any set of failures which occur with probability at least p can be correctly identified from the test results. This probabilistic model was further studied by Fujiwara and Kinoshita [32]. In the model for weighted diagnosis, a weight is attached to each unit in the system and such a system is said to be t-diagnosable if any set of faulty processors whose weighted sum is at most t can be correctly identified from the test results. Sullivan [33] studied extensively the diagnosability and diagnosis problems for probabilistic and weighted diagnosability models. Dahbura presented in [34] a diagnosis algorithm for the weighted model. Other generalizations of the PMC model have been reported in [35-39]

1.2.3 Other Measures of Diagnosability

The requirement that all the faulty processors in a multiprocessor system be identified exactly is rather restrictive and in many cases may not be satisfied. One such instance is when the number of faulty processors is fairly large with respect to the number of test assignments. In such cases, an inferior quality of fault diagnosis is imposed.

1.2.3.1 Sequential Diagnosis

One way of obtaining a slightly inferior quality of diagnosis is to require that given a syndrome, at least one faulty processor be correctly identified. In such a case fewer tests need to be done. The rationale for this relaxed requirement is that once a faulty processor is correctly identified it can be replaced, hopefully by by another fault-free processor, and another set of tests can be performed to determine the remaining faulty processors. The diagnosis is correct as no fault-free processor is declared faulty. However the diagnosis may be incomplete. Since the system undergoes repairs before the entire set of faulty processors is identified, sequential diagnosis is also referred to as **diagnosis with repair.** A system S is **sequentially t-diagnosable** if and only if, given a syndrome, at least one faulty unit can be correctly identified, provided that the number of faulty units in the system does not exceed t. Sequential diagnosis was first introduced in [1]. Raghavan [12] presented a characterization of sequentially t-diagnosable systems for the PMC model. He also showed that sequential t-diagnosability is co-NP-Complete for both PMC as well as BGM models. More recently Huang et al. [40] presented a characterization of sequentially t-diagnosable systems and presented an approach for the design of sequentially t-diagnosable systems. Although sequential diagnosis algorithms have been presented for various restricted classes of graphs [1,41] the general sequential t-diagnosis problem has not yet been solved satisfactorily.

1.2.3.2 t/s-Diagnosis

Another way of obtaining an inferior quality of diagnosis is to require that all faulty processors be identified but a few fault-free processors may be declared as faulty. This type of diagnosis is complete but perhaps incorrect. Friedman [42,43] introduced the concept of t/s-**diagnosability** which allowed the possible replacement of fault-free processors, whereas in t-diagnosability only the replacement of faulty processors is considered. A multiprocessor system S is said to be t/s-**diagnosable** if, given a syndrome, the set of faulty processors can be isolated to within a set of at most s processors provided that the number of faulty processors does not exceed t. Allowing some fault-free processors to be possibly identified as faulty permits the system to have far fewer tests in comparison with exact diagnosis. It has been shown that t/t-diagnosable systems with $n * \lceil (t+1)/2 \rceil$ tests can be constructed [44,45]. t/t-diagnosable systems have been studied extensively in the literature. Chwa and Hakimi [46] gave a characterization of t/t-diagnosable systems and presented procedures for designing these systems. Sullivan [33] presented polynomial time algorithms for the t/t- and $t/t+1$-diagnosability problems. He also developed a $t/t+k$-diagnosability algorithm which runs in polynomial time for each fixed integer k. This diagnosability algorithm is based on a characterization of $t/t+k$-diagnosable systems also developed in [33]. In [12], Raghavan has developed, among other things, a characterization of t/s-diagnosable systems. Yang et al. [47] presented an $O(n^{2.5})$ algorithm for the t/t-diagnosis problem. The diagnosis problems for $t/t+1$- and $t/t+k$-diagnosable systems have not been studied in the literature.

1.2.4. Adaptive System-Level Diagnosis

An adaptive system-level diagnosis approach was first introduced by Nakajima [48]. In this approach, the choice of the sequence of tests to be run is made adaptively. The objective is to determine one fault-free unit with a minimum number of tests and then use the fault-free unit to determine the status of the other units. This approach assumes that any unit is capable of testing any other unit. This approach was further studied by Hakimi and Nakajima in [49].

1.2.5. Diagnosis of Sparsely Interconnected Systems

In multiprocessor systems such as those implementable in very large scale integration (VLSI) and wafer-scale integration (WSI), the number of units in a system can be very large [50]. Moreover, the commonly used system interconnection networks such as the rectangular grids are very symmetrical and sparse.

When such a system is analyzed using the classical theory, the number of faulty processors permitted, which is limited by the connectivity of the processor interconnection graph, is very small in comparison to the number of units in the system. This shortcoming motivated the development of some novel theoretical results on sf -diagnosability [51] and the recent works on probabilistic diagnosis algorithms for sparsely interconnected systems [52,53]. In [52], Blough and Sullivan presented a diagnosis algorithm for such systems in which the status of each unit is determined by taking a simple majority vote of all its test outcomes with respect to its immediate neighbors. A probabilistic model for the faults in the system is used to analyze the majority-vote diagnosis algorithm. In [53], Fussell and Rangarajan presented a probabilistic model which allows a processor to perform multiple tests on another processor. They designed probabilistic diagnosis algorithms for these systems which require minimal constraints on the system interconnection.

Most diagnosis algorithms are assumed to be executed on a single, highly reliable supervisory processor which has access to the complete syndrome. A single supervisory processor is a performance bottleneck in a system with a large number of processing elements. Distributed diagnosis algorithms executed on the multiprocessor system itself would be desirable. Such distributed diagnosis algorithms can take on two essentially different flavors: one in which no single processor has the knowledge of the complete syndrome but its decoding of that part of the syndrome available to it may be passed on to its neighbors and the neighbors thereof, and so on; and and the other in which the distributed task is to make the complete syndrome available to every processor of the system and then let each processor perform the diagnosis algorithm to determine all the faulty processors in the system. Work based on the first approach was reported in [51,54]. Work using the second approach has been presented in [55,56].

An excellent overview of the system-level diagnosis area has been presented by Dahbura in [57].

1.3. Scope of the Thesis

The work reported in this thesis is motivated by certain shortcomings of the t-diagnosability theory (mentioned in the previous section) when applied to large sparsely interconnected systems. As we noted before, when such systems are analyzed using the classical theory, the number of faulty processors permitted is very small in comparison to the total number of processors in the system. A reason for this is that in t-diagnosability theory, each syndrome is required to correspond to a unique fault set, a requirement which imposes a large number of tests between processors. A partial solution to this problem is to use the t/s-diagnosability theory, which allows the possibility of identifying certain fault-free units as faulty, thereby reducing to a considerable extent the number of tests required. This calls for a detailed study of t/s-diagnosable systems and t/s-diagnosis algorithms.

Note that the constraint imposed in the classical theory on the number of faulty processors allowed is global in nature. So, we may expect that a theory of fault-diagnosis which is based on fault constraints in the local neighborhood of each processor of a sparsely interconnected system would permit correct diagnosis even when a large number of faulty processors are present in the system. Also, the t- and t/s-diagnosis algorithms are assumed to be executed on a single highly reliable supervisory processor which has access to the complete syndrome. A single supervisory processor is a bottleneck in a system with a large number of processing elements. So, it would be desirable to have distributed diagnosis algorithms executed on the multiprocessor system itself. Thus, we would like to seek a theory of local diagnosis (that is, diagnosis based on local fault constraints) which permits correct diagnosis of a sparsely interconnected system even when there are a large number of faulty processors in the system and which also admits simple diagnosis algorithms which are amenable for distributed implementation on the multiprocessor system itself.

With the above objectives in mind, this thesis is organized as follows.

The first part of this thesis is concerned with a study of t/s-diagnosable systems.

In Chapter II we present certain basic definitions, notations and results.

In Chapter III, we present characterizations of t/s-diagnosable systems which generalize those given in [16,17]. We show how the t/s-characterization for the PMC model based on Kohda's t-characterization can be easily modified to arrive at a t/s-characterization for the BGM model as well as characterizations for the sequentially t-diagnosable systems. We also present, in this chapter, certain structural properties of general t/s-diagnosable systems which generalize some of the earlier results. These properties lead to a new $t/t+1$-characterization.

With the objective of determining an efficient test for a vertex v to be in an allowable fault set of size at most t, we first establish in Chapter IV several properties of allowable fault sets. Using these properties and the characterizations given in Chapter III, we then develop an $O(n^{3.5})$ algorithm for diagnosis of a $t/t+1$-diagnosable system. We also present a $t/t+k$ diagnosis algorithm which runs in polynomial time for each fixed positive integer k. We show how these algorithms can be modified to construct diagnosis algorithms for sequentially t-diagnosable systems. Some results of this chapter have been reported in [58].

In the second part of this thesis, we use the comparison-based approach to develop a theory wherein instead of a single global constraint, local fault constraints on the number of faulty processors in the neighborhood of each processor in the multiprocessor system are considered.

In Chapter V, we reformulate a result in syndrome decoding reported in the literature under various forms [20,21,46] and apply it to regular interconnected multiprocessor systems with very small connectivity. Here a local neighborhood is defined around each processor which consists of its t immediate neighbors and t subsequent neighbors. The faulty or fault-free nature of each processor is then determined as long as no more than t processors are faulty in its corresponding neighborhood. Based on this result, we also present a simple $O(1)$ distributed diagnosis algorithm. We study the application of local fault constraints on a ring of processors. Specifically, we determine if unique diagnosis is possible if p out of any q consecutive processors in the ring are faulty. We develop sequential and distributed algorithms for these systems. The results of this chapter have appeared in [59],[60].

In Chapter VI, we introduce the concept of a t-in-L_k diagnosable system. We first present certain basic results which lead to a sufficient condition for unique diagnosis of a system when certain fault constraints are satisfied in the local domain $L_1(u_i)$ of each processor u_i. In this chapter, we also study the t-in-L_1 diagnosability of certain regular interconnected systems: the closed rectangular, the hexagonal and octagonal grid systems, and the hypercube systems. We present t-in-L_1 diagnosis algorithms for these regular systems as well as those which satisfy certain other conditions. These diagnosis algorithms can be executed in a distributed manner on the system itself. For models of distributed computing [61] may be referred. Results of this chapter have been reported in [60].

The thesis is concluded in Chapter VII.

CHAPTER II

PRELIMINARIES

In this chapter, we present basic definitions and notation used in this thesis relating to graphs and fault diagnosis models. We also give a summary of some important results in system level diagnosis which are used in the remainder of this thesis. A few concepts and notations which are particular to a chapter are introduced in the appropriate chapter.

2.1. Basic Definitions and Notations

Definition 2.1: We use the notations $X \subseteq U$ and $X \subset U$ to say X is a subset of U and X is a proper subset of U, respectively. We say $x \in U$ if x is a member of U. For two sets X and Y, $X-Y$ is the set of elements in X which are not members of Y. $X \ominus Y$ denotes the set $(X-Y)\bigcup(Y-X)$, the **symmetric difference** of X and Y. The set $X \oplus Y$ is also called the **disjoint union of** X and Y. The **cardinality** of a set X is symbolized by $|X|$. The empty set will be denoted by ϕ.

Definition 2.2: A **graph** $G(U,E)$ consists of a finite non-empty set of vertices U and a set of edges E consisting of unordered pairs of vertices in U. An edge incident on vertices u_i and u_j is denoted by (u_i,u_j) or (u_j,u_i) and the vertices u_i and u_j are then said to be **adjacent.**

Definition 2.3: The **degree** of a vertex, denoted as deg(u), refers to the number of edges incident on u.

Definition 2.4: A **directed graph** or **digraph** $G=(U,E)$ consists of a finite non-empty set of vertices U and a set of edges E consisting of ordered pairs of vertices in U. An edge directed from u_i to u_j is denoted by (u_i,u_j) and the vertices u_i and u_j are then said to be **adjacent**.

The symbols n and m denote the cardinalities of the sets U and E respectively.

Definition 2.5: Let $G=(U,E)$ be a graph or a directed graph. For $X \subseteq U$, $G'=(X,E')$ is called the **induced subgraph** of G on the vertex set X if edge $(u_i,u_j) \in E'$ if and only if $u_i,u_j \in X$ and $(u_i,u_j) \in E$. The induced subgraph on $U-X$ will be denoted by $G-X$.

Definition 2.6: A **path** in a graph $G(U,E)$ is a sequence of one or more vertices $u_1,u_2,\ldots,u_k$ such that for all $1 \leq i < k$, $(u_i,u_{i+1}) \in E$.

Definition 2.7: A **directed path** in a digraph $G(U,E)$ is a sequence of one or more vertices $u_1,u_2,\ldots,u_k$ such that for all $1 \leq i < k$, $(u_i,u_{i+1}) \in E$.

Definition 2.8 : The **length** of a path between vertices u_i and u_j refers to the number of edges in the path. The **distance** $d(u_i,u_j)$ between two vertices u_i and u_j denotes the length of a shortest path between u_i and u_j.

Definition 2.9 : Let u_i be a vertex in a digraph $G(U,E)$. Then we define the following subsets:

$$\Gamma(u_i) = \{u_j \mid (u_i,u_j) \in E\}$$

$$\Gamma^{-1}(u_i) = \{u_j \mid (u_j,u_i) \in E\}$$

This definition is extended to sets of vertices in a straight-forward manner. Let X be a subset of vertices in U, then

$$\Gamma(X) = \bigcup_{u_i \in X} \Gamma(u_i) - X$$

$$\Gamma^{-1}(X) = \bigcup_{u_i \in X} \Gamma^{-1}(u_i) - X$$

For what follows, let $G=(U,E)$ denote a general undirected graph.

Definition 2.10 : A subset $K \subseteq U$ is called a **vertex cover set** (VCS) of G if every edge in G is incident on at least one vertex in K. A **minimum vertex cover set** (MVCS) is a VCS of minimum cardinality in G.

Definition 2.11: : A subset $M \subseteq E$ is called a **matching** if no vertex in U is incident on more than one edge in M. A **maximum matching** is a matching of maximum cardinality in G.

Definition 2.12 : A **bipartite graph** , with bipartition (X,Y), is one whose vertex set can be partitioned into two subsets X and Y such that every edge is incident on a vertex in X and a vertex in Y. Finally, for $u_i \in U$, $N(u_i)$ denotes the set of all vertices which are adjacent to u_i.

Definitions of graph-theoretic concepts which have not been discussed in this section may be found in [62].

2.2. Basic Definitions for the PMC Model

A multiprocessor system S consists of n units or processors, denoted by the set $U = \{u_1, u_2, ..., u_n\}$. Each unit is assigned a subset of other units for testing. Thus the test interconnection can be modeled as a directed graph $G=(U,E)$. The **test outcome** a_{ij}, which results when unit u_i tests unit u_j, has value 1 (0) if u_i evaluates unit u_j to be faulty (fault-free). Since all faults considered are permanent, the test outcome a_{ij} is reliable if and only if unit u_i is fault-free. The collection of all test results over the entire system is referred to as a **syndrome.** With respect to the test interconnection graph $G(U,E)$, a syndrome is a function from the set of edges to the set $\{0,1\}$. If $a_{ij} = 0$ (1) then u_i is said to have a **0-link** (**1-link**) to u_j and u_j is said to have a **0-link** (**1-link**) from u_i. If

(u_i, u_j) is an edge in G, then unit u_i is said to **test** unit u_j.

Fig. 2.1 gives an example of a system with 7 nodes.

Definition 2.13 : Given a syndrome, **the disagreement set** $\Delta_1(u_i)$ of $u_i \in U$ is defined as

$$\Delta_1(u_i) = \{u_j \mid a_{ij} = 1 \text{ or } a_{ji} = 1\}.$$

For a subset $W \subseteq U$,

$$\Delta_1(W) = \bigcup_{u_j \in W} \Delta_1(u_j).$$

Definition 2.14 : Given a syndrome, the set of **0-descendents** of u_i is represented by the set

$$D_0(u_i) = \{u_j : \text{there is a directed path of 0-links from } u_i \text{ to } u_j \}$$

and for a set $W \subseteq U$, the **0-ancestors** of W denotes the set

$$A_0(W) = \{u_i : u_j \in D_0(u_i) \text{ and } u_j \in W\}.$$

For $u_i \in U$, $H_0(u_i)$ corresponds to the set $A_0(u_i) \cup \{u_i\}$.

The disagreement set, the 0-descendants, and the 0-ancestors of the unit u_3 in Fig. 2.1 are given below.

$\Delta_1(u_3) = \{u_1, u_2, u_4\}$.

$D_0(u_3) = \{u_5, u_6, u_7\}$.

$A_0(u_3) = \{u_5, u_6, u_7\}$.

Definition 2.15 : Given a system S and a syndrome, a subset $F \subseteq U$ is an **allowable fault set** (AFS) if and only if

c1: $u_i \in (U-F)$ and $a_{ij} = 0$ imply $u_j \in (U-F)$, and

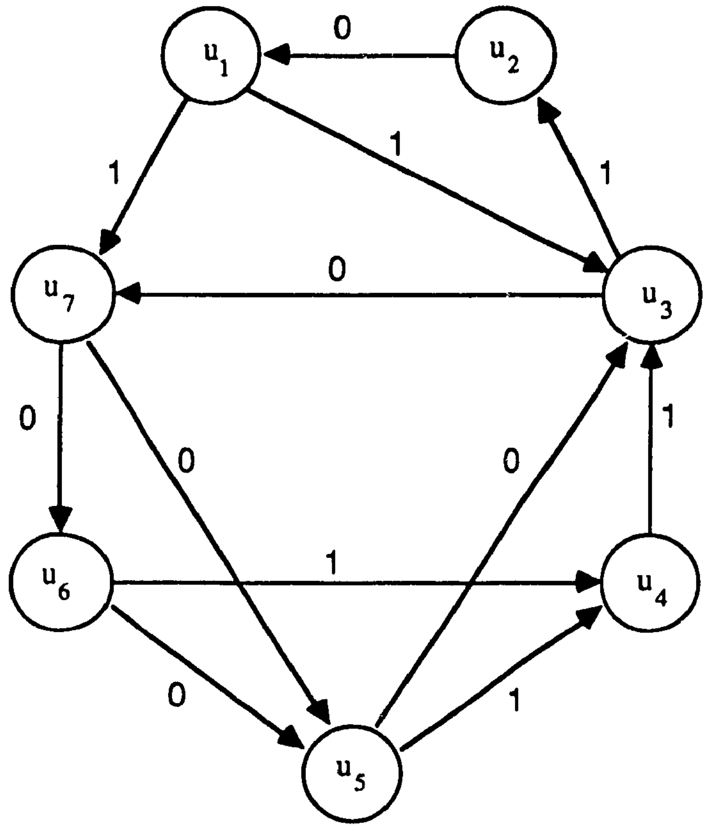


Fig. 2.1 A system with 7 units.

c2: $u_i \in (U-F)$ and $a_{ij} = 1$ imply $u_j \in F$.

In other words, F is an AFS for a given syndrome if and only if the assumption that the units in F are faulty and the units in $U-F$ are fault-free is consistent with the given syndrome. In Fig. 2.1, the subsets $\{u_1, u_2, u_4\}$ and $\{u_3, u_5, u_6, u_7\}$ are allowable fault sets corresponding to the given syndrome.

A **minimum allowable fault set** (MAFS) is an allowable fault set of minimum cardinality for a given syndrome.

Definition 2.16 : Given a system S and a syndrome, the **implied faulty set** $L(u_i)$ of $u_i \in U$ is the set of all units of S that may be deduced to be faulty under the assumption that u_i is fault-free.

It follows that

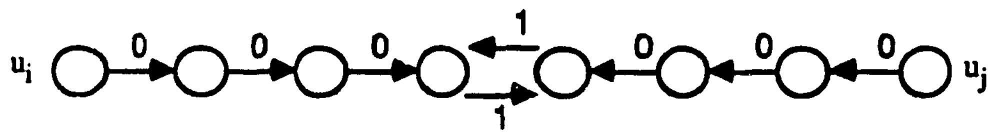
$$L(u_i) = \Delta_1(D_0(u_i)) \bigcup A_0(\Delta_1(D_0(u_i))).$$

Note that if $u_j \in L(u_i)$ then there exists a 1-link (u_k, u_l) or (u_l, u_k) such that there is a directed path of 0-links from u_i to u_k and a directed path of 0-links from u_j to u_l. Such a path will be refered to as an **implied-fault path** between u_i and u_j.

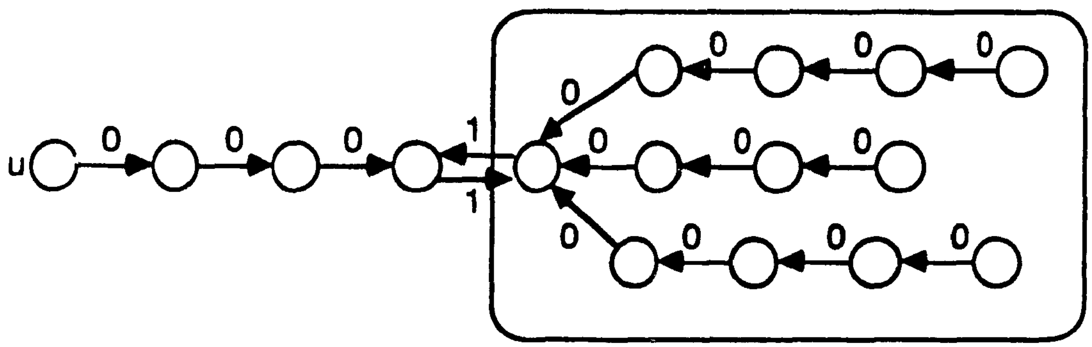
The implied fault path and the implied fault set for a unit u_i are shown in Fig 2.2.

2.3. Basic Definitions for the Comparison Model

A multiprocessor system S under the comparison model consists of n independent processors $U = \{u_1, u_2, \ldots, u_n\}$. In the comparison model of multiprocessor fault diagnosis, all processors in S are assigned to perform the same task. Upon completion, the outputs of certain pairs of these processors are compared. The comparison assignment can be represented by an undirected graph



(a) Implied fault path between two units.



(b) Implied fault set for unit u.

Fig. 2.2 Implied fault path and implied fault set.

$G=(U,E)$ where an edge e_{ij} belongs to E if and only if the outputs of u_i and u_j are compared.

An outcome a_{ij} is associated with each pair of processors whose outputs are compared, where $a_{ij}=0(1)$ if the outputs compared agree(disagree). Since only permanent faults are considered and we make the symmetric invalidation assumption, it follows that $a_{ij}=0$ whenever both u_i and u_j are fault-free; $a_{ij}=1$ if one of u_i and u_j is fault-free and the other faulty; a_{ij} is unreliable if both u_i and u_j are faulty. $N(u_i)$ denotes the set of neighbors of u_i, i.e. the set of all processors adjacent to u_i. An edge that has a 0(1) outcome associated with it is referred to as a 0-link(1-link). $N_0(u_i)$ and $N_1(u_i)$ denote the set of processors adjacent to u_i which are connected with u_i by a 0-link and a 1-link, respectively.

2.4. Diagnosable Systems

Definition 2.17 : A system S is t**-diagnosable** if and only if, given a syndrome, all faulty units can be correctly identified provided that the number of faulty units in the system does not exceed t.

Definition 2.18 : A system S is t/s**-diagnosable** if and only if, given a syndrome, all faulty units can be isolated to within a set of at most s units, provided that the number of faulty units in the system does not exceed t.

Fig. 2.3 gives an example of a 2-diagnosable system and a 2/3-diagnosable system is shown in Fig. 2.4.

Definition 2.19 : A system S is **sequentially** t**-diagnosable** if and only if, given a syndrome, at least one faulty unit can be correctly identified, provided that the number of faulty units in the system does not exceed t.

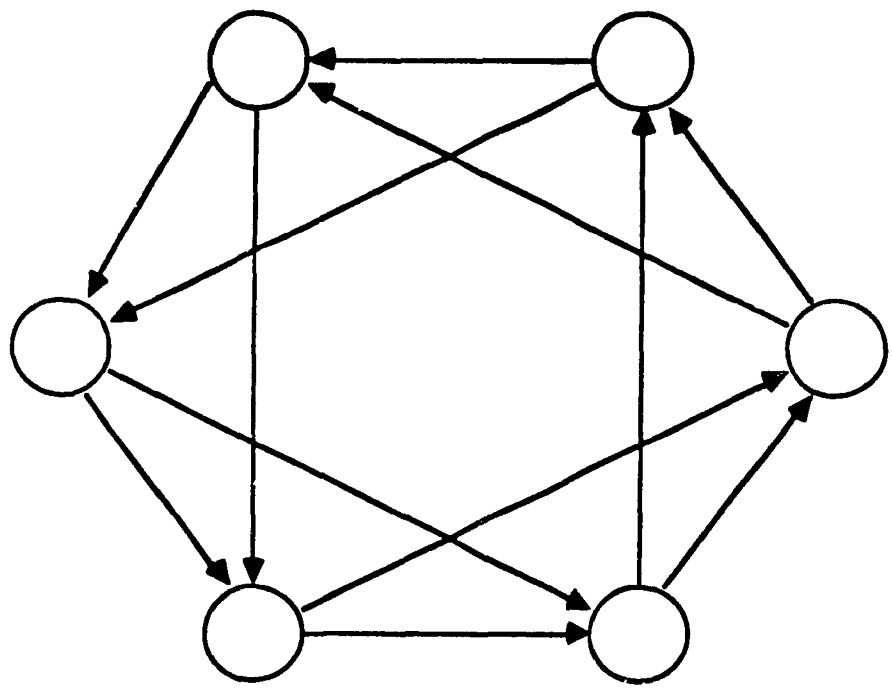


Fig. 2.3 A 2-diagnosable system

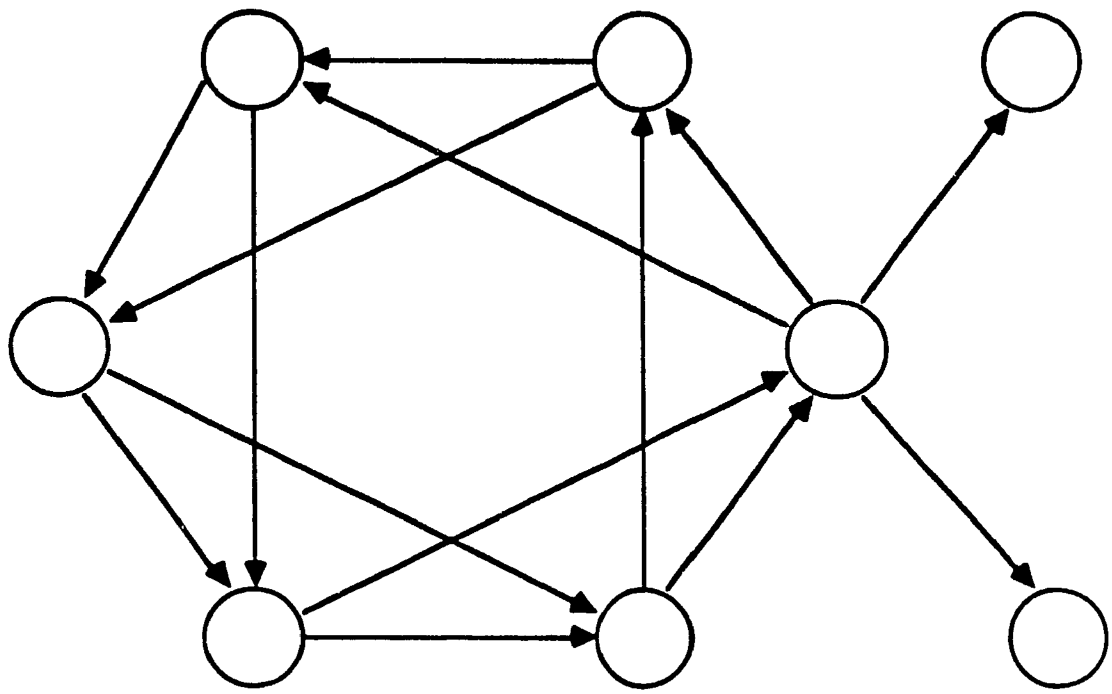


Fig. 2.4 A 2/3 diagnosable system

2.5. Basic Results in System-Level Diagnosis

The following lemmas determine a few properties of AFS's and implied faulty sets.

Lemma 2.1 [16]: Given a system S and a syndrome, each of the following statements holds:

(1) for $u_i, u_j \in U$, $u_i \in L(u_j)$ if and only if $u_j \in L(u_i)$.

(2) for u_i, $u_j \subset U$, if $a_{ij} = 0$ then $L(u_j) \subseteq L(u_i)$.

(3) If $F \subseteq U$ is an AFS, then $\bigcup_{u_i \in U-F} L(u_i) \subseteq F$. //

Lemma 2.2 [33]: Given a system S and a syndrome, if F_1 and F_2 are AFS's then so is $(F_1 \bigcup F_2)$. //

Lemma 2.3 : Given a system S and a syndrome, let $F \subseteq U$ be an AFS containing $u_i \in U$. Then $H_0(u_i) \subseteq F$.

Proof : Suppose that $u_j \in H_0(u_i)$ is not a member of F. Since $u_j \in H_0(u_i)$ there exists a directed path of 0-links from u_j to u_i. Since $u_j \in U-F$ and $u_i \in F$, there exists a 0-link from $U-F$ to F on this path, contradicting the assumption that F is an AFS. //

The following lemmas give alternate definitions of sequential t-diagnosable systems and t/s diagnosable systems. Here a t-fault situation refers to the case when the number of faulty processors is at most t.

Lemma 2.4 [12]: A system S is sequentially t-diagnosable if and only if for every syndrome occuring in a t-fault situation, the intersection of allowable fault sets of size at most t is non-empty. //

Lemma 2.5 [12]: A system S is t/s-diagnosable if and only if for every syndrome occuring in a t-fault situation, the union of allowable fault sets of size at most t is less than or equal to s. //

The following theorems give some important characterization results in system-level diagnosis which we will refer to in the remainder of this thesis.

Theorem 2.1 [8]: A system S with test interconnection graph $G(U,E)$ under the PMC model is t-diagnosable if and only if the following conditions are satisfied:

(i) $n \geq 2t+1$

(ii) For all $u_i \in U$, $|\Gamma^{-1}(u_i)| \geq t$

(iii) For $0 \leq p < t$ and for all $X \subseteq U$ with $|X| = n - 2t + p$, $|\Gamma(X)| > p$.

Theorem 2.2 [10]: A system S with test interconnection graph $G(U,E)$ under the PMC model is t-diagnosable if and only if for all distinct, non-empty subsets $X_i, X_j \subseteq U$, $|X_i| \leq t$, $|X_j| \leq t$, there is a test from $U - X_i - X_j$ into $X_i \oplus X_j$. //

Theorem 2.3 [9]: A system S with test interconnection graph $G(U,E)$ under the PMC model is t-diagnosable if and only if for all $Z \subseteq U$ with $Z \neq \phi$,

$$|Z|/2 + |\Gamma^{-1}(Z)| > t. \ //$$

Theorem 2.4 [16]: A system S with test interconnection graph $G(U,E)$ under the PMC model is t/t-diagnosable if and only if for $0 \leq p < t$ and for all $X \subseteq U$ with $|X| = n - 2t + p$, $|\Gamma(X)| > p$. //

Theorem 2.5 [17]: A system S with test interconnection graph $G(U,E)$ under the PMC model is t/t-diagnosable if and only if for all distinct, non-empty subsets $X_i, X_j \subseteq U$, $|X_i| = t$, $|X_j| = t$, there is a test from

U X_i X_j into $X_i \oplus X_j$. //

Theorem 2.6 [46]: A system S with test interconnection graph $G(U,E)$ under the PMC model is $t/t+\delta$-diagnosable, only if for $0 \leq p < t - \delta$ and for all $X \subseteq U$ with $|X| = 2(t - p)$, $|\Gamma^{-1}(X)| > p$. //

CHAPTER III

STRUCTURE AND CHARACTERIZATIONS OF t/s- AND SEQUENTIALLY t-DIAGNOSABLE SYSTEMS

As stated in Chapter I, Friedman [42] introduced the concept of t/s-diagnosability which allowed the possible replacement of fault-free units in system repair. We recall that a system S is said to be t/s**-diagnosable** if, given a syndrome, the set of faulty processors can be isolated to within a set of at most s processors provided that the number of faulty processors does not exceed t. The problem of characterizing these systems has received considerable attention in the literature, in view of its usefulness in designing these systems as well as developing efficient diagnosability and diagnosis algorithms. In [46], Chwa and Hakimi studied extensively t/t-diagnosable systems and presented ,among other things, characterizations of these systems. In [47], Yang and Masson presented certain new characterizations of t/t-diagnosable systems and used them in developing an efficient t/t-diagnosis algorithm. Sullivan [33] gave the first characterization of t/s-diagnosable systems. He used it to show that the t/s-diagnosability problem is co-NP-Complete and to develop a $t/t+k$-diagnosability algorithm which runs in polynomial time for each fixed positive integer k. He also derived simpler characterizations in the cases when $s = t$ or $s = t+1$ which lead to polynomial diagnosability algorithms for these special cases. Most recently, Raghavan [12] presented characterizations for t/s- and sequentially t-diagnosable systems and showed that the sequential t-diagnosability problem is co-NP-Complete. He also derived simpler characterizations for t/t-diagnosable systems presented earlier in [46] and [47] as well as certain necessary conditions for general t/s-diagnosable systems. Raghavan also showed how his method for characterizing t/s-diagnosable systems can be used to characterize sequentially t-diagnosable systems. All these results in the literature relating to t/s-diagnosable systems assume the PMC model.

Many of the t/s-characterizations presented in the literature either resemble, in part, or seem to have been motivated by, characterizations for t-diagnosable systems. For example, the characterizations given in [46], [47], and [33] bear resemblance to the t-characterizations given by Hakimi and Amin [8], Kohda [10], and Allan et al [9], respectively (See Theorems 2.1, 2.2 and 2.3). Interestingly, the characterizations based on the one given by Allan et al [9] have proved useful in developing efficient algorithms for t- and t/s-diagnosability [11], [33].

In this chapter, we investigate further the t/s-characterization problem. We first present a characterization of a general t/s-diagnosable system based on Kohda's characterization (Theorem 2.2) of t-diagnosable systems. This characterization which assumes the PMC model generalizes the one presented earlier in [47] for t/t-diagnosable systems (See Theorem 2.5). We show that this characterization can easily be modified to yield a characterization in the BGM model as well as characterizations for sequentially t-diagnosable systems. We also present a characterization which generalizes one of the t/t-characterizations (Theorem 2.4) presented by Chwa and Hakimi [46]. We draw attention to certain difficulties one encounters in establishing such a generalization. Using our t/s-characterizations, we investigate the structure of t/s-diagnosable systems. This leads to a necessary condition for a system to be t/s-diagnosable. Given a syndrome we also present a property of allowable fault sets of a t/s-diagnosable system. Again, this property generalizes the one given in [47] for a t/t-diagnosable system. Starting from this property, we develop a new necessary and sufficient condition for a system to be $t/t+1$-diagnosable.

3.1. t/s-Characterizations: PMC Model

In the PMC model, two distinct fault sets F_1 and F_2 cannot generate a common syndrome if there is a test from the outside into the disjoint union (symmetric difference) of the two sets. For instance, Fig. 3.1 shows two subsets of a test interconnection graph $G(U,E)$ in which there is a test from outside into the disjoint union of F_1 and F_2. The outcome $a_{ij} = 0$ in the presence of fault set F_1, and $a_{ij} = 1$ in the presence of fault set F_2. Thus F_1 and F_2 cannot generate a common syndrome. This condition is both necessary and sufficient to ensure that two distinct subsets do not generate a common syndrome. This observation led to the following characterization by Kohda [10] for t-diagnosable systems. This result stated in Theorem 2.2 is repeated below for easy reference.

Lemma 3.1 : A system S with test interconnection graph $G(U,E)$ under the PMC model is t-diagnosable if and only if for all distinct, non-empty subsets $X_i, X_j \subseteq U$, $|X_i| \leq t$, $|X_j| \leq t$, there is a test from $U - X_i - X_j$ into $X_i \oplus X_j$. //

This approach can in fact be used to determine the unique diagnosability of an arbitrary family of fault sets.

Lemma 3.2 : Let S be a system with test interconnection graph $G = (U,E)$ and P, a collection of non-empty subsets of U. Then the system S is uniquely diagnosable with respect to the fault-class P if and only if for all distinct, non-empty subsets $X_i, X_j \subseteq P$, there is a test from $U - X_i - X_j$ into $X_i \oplus X_j$. //

The above approach for characterizing fault sets which generate a common syndrome is used in this chapter to develop characterizations for t/s- and

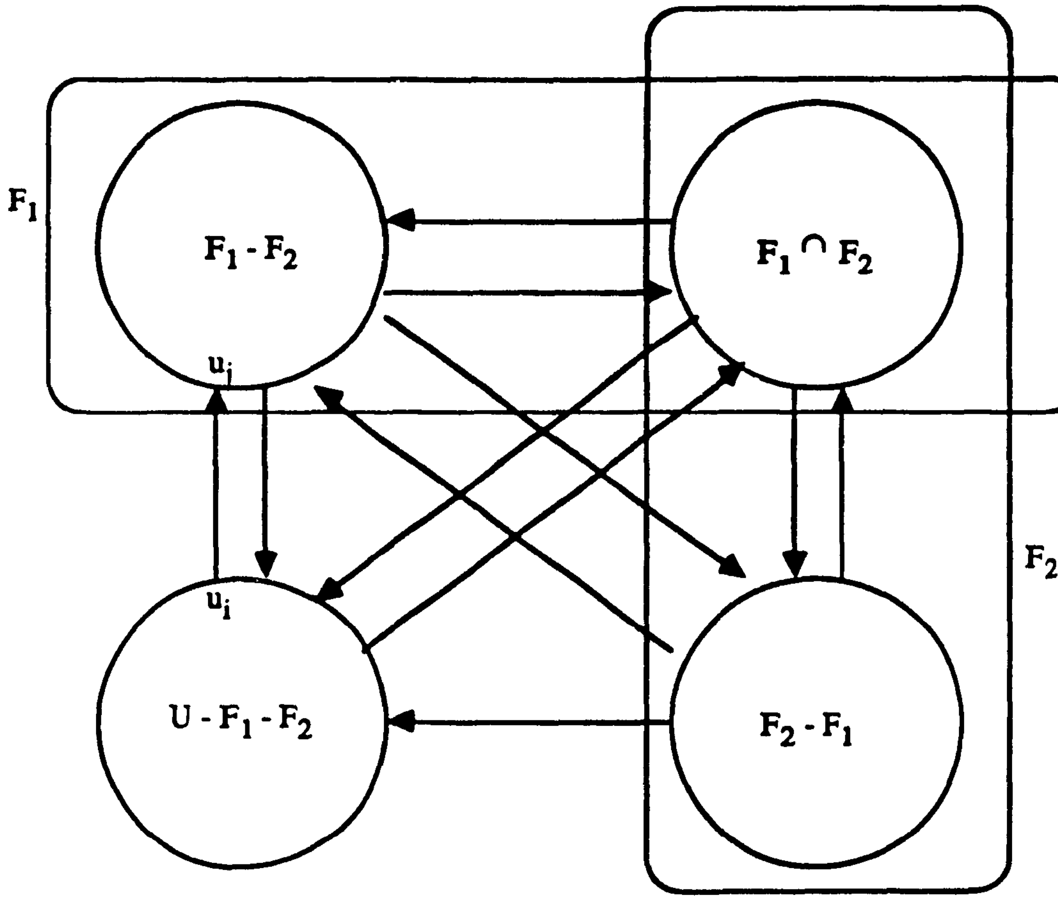


Fig. 3.1 Two fault sets which cannot generate a common syndrome under the PMC model.

sequentially t-diagnosable systems.

From Definition 2.18 of Chapter II, it follows that $t \leq s$ for a system to be t/s-diagnosable. It should be observed that a system is trivially t/s-diagnosable if $n = s$. Thus, in this thesis, it is required that $0 < t \leq s < n$. It should also be noted that under these conditions $n \geq 2t+1$ for t/s-diagnosable systems.

Definition 3.1 : Given a system S and a subset $X \subseteq U$, a set $A = \{X_1, \ldots, X_r\}$ is said to be a **t-decomposition** of X if and only if $\bigcup_{1 \leq i \leq r} X_i = X$ and $0 < |X_i| \leq t$ for $1 \leq i \leq r$. The set P_X is the collection of all t-decompositions of X.

Theorem 3.1 : A system S under the PMC model with test interconnection graph $G = (U, E)$ is t/s-diagnosable if and only if for all $X \subseteq U$, $|X| > s$, and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$.

Proof : (Necessity) Assume that the system S is t/s-diagnosable and that the condition of the theorem does not hold. Then there exists $X \subseteq U$ with $|X| > s$ and $A \in P_X$ such that for all $X_i, X_j \in A$ there is no test from $U - X_i - X_j$ to $X_i \oplus X_j$. Consider now the syndrome where for each edge $(u_k, u_l) \in E$ the outcome is defined as follows :

Case 1. $u_k \in U - X$ or $u_l \in U - X$

1.1. $u_k, u_l \in U - X$; then set $a_{kl} = 0$.

1.2. $u_k \in U - X$ and for all $X_i \in A$, $u_l \in X_i$; then set $a_{kl} = 1$.

1.3. $u_k \in X$ and $u_l \in U - X$; then set $a_{kl} = 0$.

Case 2. $u_k, u_l \in X$

2.1. $u_k, u_l \notin X_i$ for some $X_i \in A$; then set $a_{kl} = 0$.

(Note: In this case there is no subset $X_j \in A$ such that $u_l \in X_j$ and $u_k \notin X_j$. For otherwise, there would be a test from $U-X_i-X_j$ to $X_i \oplus X_j$.)

2.2. For all $X_i \in A$ either $u_k \in X_i$ or $u_l \in X_i$; then set $a_{kl} = 1$.

Let X_i be a member of A. We show that X_i is an AFS of S for the above syndrome.

Let (u_k, u_l) be an edge in G with $u_k, u_l \in U-X_i$.

(i) If $u_k, u_l \in U-X$ then condition 1.1 applies and $a_{kl} = 0$.

(ii) If $u_k \in X$ and $u_l \in U-X$, then condition 1.3. applies and $a_{kl} = 0$.

(iii) If $u_k, u_l \in X$ then by condition 2.1, $a_{kl} = 0$.

Thus, for an edge (u_k, u_l) with $u_k, u_l \in U-X_i$ $a_{kl} = 0$.

Next let (u_k, u_l) be an edge in G with $u_k \in U-X_i$ and $u_l \in X_i$.

(i) If $u_k \in U-X$ and $u_l \in X_j$ for all X_j, then by condition 1.2 $a_{kl} = 1$.

(ii) If $u_k \in X$ then, as we have noted before, condition 2.1 does not occur and so by condition 2.2, $a_{kl} = 1$.

Thus, for an edge (u_k, u_l) with $u_k \in U-X_i$ and $u_l \in X_i$, $a_{kl} = 1$.

From the above, it follows that X_i is an allowable fault set.

Since X_i is an arbitrary member of A, it follows that, for the above syndrome, each $X_i \in A$ is an allowable fault set of size less than or equal to t. Since the union of all these allowable fault sets is X and $|X| > s$, no subset of units of U of size at most s can isolate the faulty units for the above syndrome. Hence S is not t/s-diagnosable, a contradiction.

(Sufficiency). Proof is given by using a contrapositive argument. Suppose S is not t/s-diagnosable. Then there exists a syndrome, say θ, and subsets $X_1, X_2, \ldots, X_r$ of size at most t such that these subsets are allowable fault sets with respect to θ and that $|X| > s$, where X is the union of $X_1, X_2, \ldots, X_r$.

Suppose that for some pair X_i, X_j, $1 \leq j < k \leq r$ there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$. Let (u_k, u_l) be such a test edge. Without loss of generality, let $u_l \in X_i$. If X_i is the fault set then the test outcome $a_{kl} = 1$; if X_j is the fault set then $a_{kl} = 0$. This contradicts the assumption that both X_i and X_j are allowable fault sets for θ. This shows that the condition of the theorem is not satisfied. //

We wish to note that the following syndrome can also be used in the proof of the necessity part of the above theorem:

$$a_{kl} = 0 \text{ , if } u_k, u_l \notin X_i \text{ for some } X_i \in A$$
$$= 1 \text{ , otherwise.}$$

Definition 3.2 : A t-decomposition $A = \{X_1, \ldots, X_r\}$ of X is said to be a non-trivial t-decomposition of X if the following additional conditions are satisfied:

(i) for all $X_i \in A$, $X_i - \bigcup_{j \neq i} X_j \neq \emptyset$

(ii) for all $X_i, X_j \in A$, $i \neq j$ implies $|X_i \bigcup X_j| > t$.

Lemma 3.3 : Given a system S, the following statements are equivalent:

(I) For all $X \subseteq U$ with $|X| > s$ and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$.

(II) For all $X \subseteq U$ with $|X| > s$ and for all non-trivial t-decompositions A' of X, there exist subsets $X_i', X_j' \in A'$ such that there is a test from $U - X_i' - X_j'$ to $X_i' \oplus X_j'$.

Proof : Given a subset $X \subseteq U$ such that $|X| > s$, the set of all non-trivial decompositions is contained in P_X. Hence (I) implies (II). Now assume (I) does not hold. Then there exists $Y \subseteq U$ with $|Y| > s$ and there exists $A_1 \in P_Y$ such that for all pairs $Y_i, Y_j \in A_1$ there is no test from $U-Y_i-Y_j$ to $Y_i \oplus Y_j$. Consider the set $A_1 = \{Y_1, \ldots, Y_r\}$. Now we construct from A_1 a non-trivial t-decomposition of Y which does not satisfy condition II.

A set $A_2 = \{W_1, \ldots, W_q\}$ is first constructed from A_1 by merging subsets in A_1 whenever their union has at most t elements. It follows that for any two subsets $W_i, W_j \in A_2$ there is no test from $U-W_i-W_j$ to $W_i \oplus W_j$; for otherwise one can find two elements $Y_k, Y_l \in A_1$ such that there is a test from $U-Y_k-Y_l$ to $Y_k \oplus Y_l$. The set A_2 satisfies condition (II) of Definition 3.2. From A_2, a non-trivial t-decomposition $A_3 = \{Z_1, \ldots, Z_w\}$ is next constructed as follows. An element of A_2 is first chosen arbitrarily and placed in A_3. Recursively another element W_k of A_2 is added to A_3 such that W_k contains at least one unit which is not a member of any of the subsets added previously to A_3. This procedure is continued until no further elements of A_2 can be added to A_3. Clearly A_3 satisfies all the conditions of a non-trivial t-decomposition of Y. Since A_3 is a subset of A_2, for all $Z_i, Z_j \in A_3$ there is no test from $U-Z_i-Z_j$ to $Z_i \oplus Z_j$. This shows the equivalence of (I) and (II). //

From Theorem 3.1 and Lemma 3.3, we obtain the following.

Theorem 3.2 : Given a system S and $0 < t \leq s < n$, the following statements are equivalent.

1. S is t/s-diagnosable.

2. For all $X \subseteq U$ with $|X| > s$ and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that there is a test from $U-X_i-X_j$ to $X_i \oplus$

X_j.

3. For all $X \subseteq U$ with $|X| > s$ and for all non-trivial t-decompositions A' of X, there exist subsets $X_i', X_j' \in A'$ such that there is a test from $U - X_i' - X_j'$ to $X_i' \oplus X_j'$. //

Corollary 3.2.1 : A system S with test interconnection graph $G = (U, E)$ is t/t-diagnosable if and only if for all $X \subseteq U$ with $|X| > t$ and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$. //

We remark that given a syndrome for a t/s-diagnosable system which corresponds to a fault set of size at most t, there exists a non-trivial t-decomposition $\{X_1, X_2, \ldots, X_k\}$ of X such that $k \leq s - t + 2$, X_i, $1 \leq i \leq k$ is an allowable fault set, and every allowable fault set of size at most t is contained in $\bigcup_{1 \leq i \leq k} X_i$.

Combining the above Theorem with Lemma 3 of [47], we obtain the following theorem which summarizes all the characterizations of t/t-diagnosable systems based on Kohda's t-characterization.

Theorem 3.3 :

1. S is t/t-diagnosable.

2. For all $X \subseteq U$ with $|X| > t$ and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$.

3. For any two sets of units $X_1, X_2 \subseteq U$ where $|X_1| = |X_2| = t$ and $X_1 \neq X_2$, there is a test from $U - X_1 - X_2$ to $X_1 \oplus X_2$.

4. For any two sets of units $X_1, X_2 \subseteq U$ where $|X_1|, |X_2| \leq t$, $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$, there is a test from $U-X_1-X_2$ to $X_1 \oplus X_2$. //

We next proceed to develop a characterization which generalizes the one given by Chwa and Hakimi in [46] (See Theorem 2.4 of Chapter II).

Theorem 3.4 : A system S is t/s-diagnosable if and only if for all p, $0 \leq p < t$ and for each $X \subseteq U$ with $|X| = n-(s+t)+p$ one of the following is true:

(i) $\Gamma(X) > p$

(ii) $\Gamma(X) = q \leq p$ and $G-X-\Gamma(X)$ is $(t-q)/(s+t-(p+q+1))$-diagnosable.

Proof : (necessity) Assume that the condition does not hold. Then for some p, $0 \leq p < t$ and for some $X \subseteq U$ with $|X| = n-(s+t)+p$, we have $\Gamma(X) = q \leq p$ and $G-X-\Gamma(X)$ is not $(t-q)/(s+t-(p+q+1))$-diagnosable. Let $Y = U-X-\Gamma(X)$. Then $|Y| = (s+t)-(p+q)$. Since $G-X-\Gamma(X)$ is not $(t-q)/(s+t)-(p+q+1)$-diagnosable and $|Y| = (s+t)-(p+q)$, by Theorem 3.1., there exists a $(t-q)$-decomposition $A = \{Y_1, Y_2, \ldots, Y_k\}$ of Y such that for all $Y_i, Y_j \in A$, there is no test from $Y-Y_i-Y_j$ into $Y_i \oplus Y_j$. Consider the t-decomposition of $Z = Y \bigcup \Gamma(X)$ given by $B = \{Z_1, Z_2, \ldots, Z_k\}$ where $Z_i = Y_i \bigcup \Gamma(X)$. Then

$$\begin{aligned} |Z| &= |Y| + |\Gamma(X)| \\ &= s+t-p \\ &> s \end{aligned}$$

By construction, for all $Z_i, Z_j \in B$, $i \neq j$, there is no test from $U-Z_i-Z_j$ into $Z_i \oplus Z_j$. As in the case of the proof of Theorem 3.1, we can construct a syndrome such that the subsets Z_i, $1 \leq i \leq k$, are all allowable fault sets of size at most t for this syndrome. Since the union of these subsets is of size greater than s, it follows that the system S is not t/s-diagnosable.

(Sufficiency) Now assume that the system is not t/s-diagnosable. Then, by Theorem 3.1., there exists $Z \subseteq U$ with $|Z| > s$ and a t-decomposition $A = \{Z_1, \ldots, Z_k\}$ of Z such that for all $Z_i, Z_j \in A$ there is no test from $U - Z_i - Z_j$ to $Z_i \oplus Z_j$. Let r be an integer with $1 < r \leq k$ such that

$$| \bigcup_{1 \leq i \leq r-1} Z_i | \leq s$$

and

$$| \bigcup_{1 \leq i \leq r} Z_i | > s .$$

Let $Y = \bigcup_{1 \leq i \leq r} Z_i$, $X = U - Y$ and $|\Gamma(X)| = q$. Then, since $|Z_r| \leq t$, we have

$$|Y| = s + t - b$$

where $0 \leq b < t$ and $|X| = n - (s+t) + b$. Furthermore, since

$$\bigcup_{1 \leq i \leq r} Z_i - \bigcup_{1 \leq i \leq r-1} Z_i$$

contains at least $t-b$ units and $|Z_r| \leq t$, it follows that

$$| \bigcap_{1 \leq i \leq r} Z_i | \leq b .$$

Moreover $\Gamma(X) \subseteq \bigcap_{1 \leq i \leq r} Z_i$.

Thus

$$|\Gamma(X)| = q \leq b .$$

Since $A = \{Z_1, \ldots, Z_k\}$ is a t-decomposition of Z such that for all $Z_i, Z_j \in A$ there is no test from $U - Z_i - Z_j$ to $Z_i \oplus Z_j$, the set $B = \{W_1, \ldots, W_k\}$ where $W_i = Z_i - \Gamma(X)$, $1 \leq i \leq k$, is a t-q-decomposition of $U - X - \Gamma(X) = Y - \Gamma(X)$ such that for all $W_i, W_j \in B$ there is no test from

$(Y - \Gamma(X))$ $W_i - W_j$ to $W_i \oplus W_j$. By Theorem 3.1, it follows that $G - X - \Gamma(X)$ is not $(t-q)/s+t-(p+q+1)$-diagnosable where $p = b$. From the above, it follows that for $p = b$ and for the set X where $|X| = n \ (s+t)+p$ neither (i) nor (ii) is satisfied. //

It can be shown that for all s and t with $t \leq s < n$ we can find a value of p such that $|X| \geq 0$. When $|X| = 0$, condition (i) of the above theorem is not satisfied. In that case, condition (ii) needs to be tested. But this condition for the case $|X| = 0$ is essentially a restatement of the condition of Theorem 3.1 for appropriate values of s, t and n.

Note that the characterization given in Theorem 3.4 itself involves the Kohda-type characterization presented in Theorem 3.1.

3.2. t/s-Characterization: BGM Model

In the BGM model, to determine whether two distinct fault sets, F_1 and F_2, can generate a common syndrome, the tests lying within the disjoint union of the two sets have to be considered in addition to the tests coming from outside into the disjoint union of the two sets. For instance, in Fig. 3.2, if either of the tests (u_i, u_j) or (u_k, u_l) is present then the fault sets F_1 and F_2 cannot give rise to a common syndrome; a_{ij} and a_{kl} have value 1 in the presence of fault set F_1 and have value 0 in the presence of fault set F_2. This leads to the following t-characterization under the BGM model.

Lemma 3.4 : A system S with test interconnection graph $G(U,E)$ under the BGM model is t-diagnosable if and only if for all distinct, non-empty subsets $X_i, X_j \subseteq U$, $|X_i| \leq t$, $|X_j| \leq t$, at least one of the

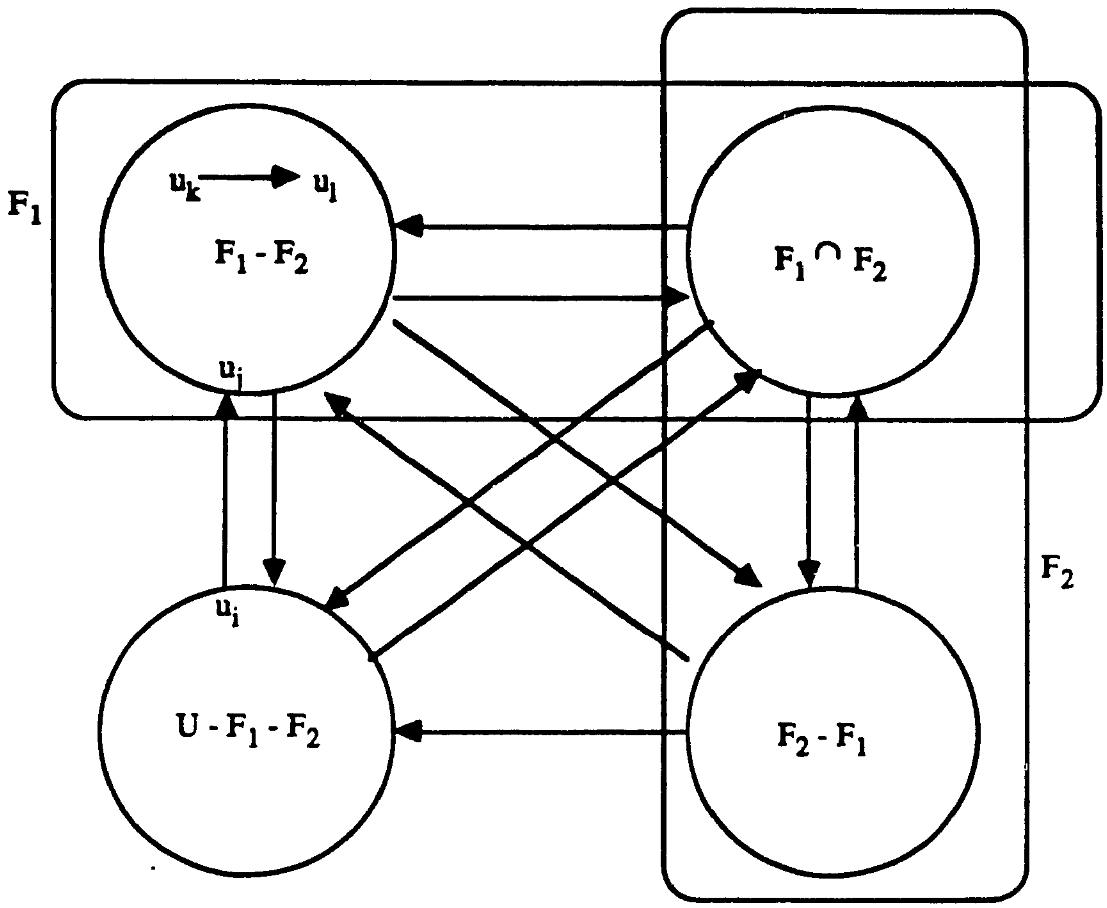


Fig. 3.2 Two fault sets which cannot generate a common syndrome under the BGM model.

following is satisfied:

(i) there is a test from $U-X_i-X_j$ into $X_i \oplus X_j$.

(ii) there is a test between units lying entirely within X_i-X_j or within X_j X_i. //

Lemma 3.5 : Let S be a system with test interconnection graph $G = (U,E)$ under the BGM model and P, a collection of non-empty subsets of U. Then the system S is uniquely diagnosable with respect to the fault-class P if and only if for all distinct, non-empty subsets $X_i, X_2 \subseteq P$, at least one of the following is satisfied:

(i) there is a test from $U-X_i-X_j$ into $X_i \oplus X_j$.

(ii) there is a test between units lying entirely within X_i-X_j or within X_j X_i. //

We now present a characterization of t/s-diagnosable systems for the BGM model.

Theorem 3.5 : A system S under the BGM model with test interconnection graph $G=(U,E)$ is t/s-diagnosable if and only if for all $X \subseteq U$, $|X| > s$, and for all t-decompositions $A \in P_X$, there exist subsets $X_i, X_j \in A$ such that at least one of the following is satisfied:

(1) there is a test from $U-X_i-X_j$ to $X_i \oplus X_j$.

(2) there is a test between units lying entirely within X_i-X_j or X_j-X_i.

Proof : (Necessity) Assume that the system S is t/s-diagnosable and that the condition of the theorem does not hold. Then there exists $X \subseteq U$ with $|X| > s$ and $A \in P_X$ such that for all $X_i, X_j \in A$ we have the following:

(a) there is no test from $U-X_i-X_j$ to $X_i \oplus X_j$.

(b) there is no test between units lying entirely within $X_i - X_j$ or within $X_j - X_i$.

Consider now the syndrome where for each edge $(u_k, u_l) \in E$ the outcome is defined as follows :

Case 1. $u_k \in U-X$ or $u_l \in U-X$

1.1. $u_k, u_l \in U-X$; then set $a_{kl} = 0$

1.2. $u_k \in U-X$ and for all $X_i \in A$, $u_l \in X_i$; then set $a_{kl} = 1$

1.3. $u_k \in X$ and $u_l \in U-X$; then set $a_{kl} = 0$

Case 2. $u_k, u_l \in X$, then set $a_{kl} = 1$

Let X_i be a member of A. We show that X_i is an AFS of S for the above syndrome.

Let (u_k, u_l) be an edge in G with $u_k, u_l \in U-X_i$.

(i) If $u_k, u_l \in U-X$ then condition 1.1 applies and $a_{kl} = 0$.

(ii) If $u_k \in X$ and $u_l \in U-X$, then condition 1.3. applies and $a_{kl} = 0$.

(iii) Suppose that $u_k, u_l \in X$. Then since $u_k, u_l \notin X_i$ condition (a) or condition (b) is violated for some pair of members of A. Hence, this case cannot occur.

Thus, for an edge (u_k, u_l) with $u_k, u_l \in U-X_i$, $a_{kl} = 0$.

Next let (u_k, u_l) be an edge in G with $u_k \in U-X_i$ and $u_l \in X_i$.

(i) If $u_k \in U-X$ and $u_l \in X_j$ for all $X_j \in A$, then by condition 1.2 $a_{kl} = 1$.

(ii) If $u_k \in X$, then by condition 2, $a_{kl} = 1$.

Thus, for an edge (u_k, u_l) with $u_k \in U - X_i$ and $u_l \in X_i$, $a_{kl} = 1$.

Also for each edge $(u_k, u_l) \in X_i$, condition 2 applies and hence in this case $a_{kl} = 1$.

From the above, it follows that X_i is an allowable fault set.

Since X_i is an arbitrary member of A, it follows that, for the above syndrome, each $X_i \in A$ is an allowable fault set of size at most t. Since the union of all these allowable fault sets is X and $|X| > s$, no subset of units of U of size at most s can isolate the faulty units for the above syndrome. Hence S is not t/s-diagnosable, a contradiction.

(Sufficiency). Proof is given by using a contrapositive argument. Assume S is not t/s-diagnosable. Then there exists a syndrome, say θ, and subsets $X_1, X_2, \ldots, X_m$ of size at most t such that these subsets are allowable fault sets with respect to θ and that $|X| > s$, where X is the union of $X_1, X_2, \ldots, X_r$.

Suppose that for some pair X_i, X_j, $1 \leq j < k \leq r$, (1) is satisfied. Let (u_k, u_l) be such a test edge. Without loss of generality, let $u_l \in X_i$. If X_i is the fault set then the test outcome $a_{kl} = 1$; if X_j is the fault set then $a_{kl} = 0$. This contradicts the assumption that both X_i and X_j are allowable fault sets for θ.

Now suppose that for some pair X_i, X_j, $1 \leq j < k \leq r$, (2) is satisfied. Let (u_k, u_l) be such a test edge. Without loss of generality, let $u_k, u_l \in X_i - X_j$. If X_i is the fault set then the test outcome $a_{kl} = 1$; if X_j is the fault set then $a_{kl} = 0$. This contradicts the assumption that both X_i and X_j are allowable fault sets for θ.

This shows that the condition of the theorem is not satisfied. //

Theorems 3.1 and 3.5 show that a characterization given for the PMC model can easily be modified to arrive at a characterization for the BGM model. The usefulness of this approach is further illustrated in section 3.4, where we present characterizations of sequentially t-diagnosable systems under the PMC and the BGM models.

3.3. Structural Properties of $t/t+k$-Diagnosable Systems

The following results give some insight regarding the structure of $t/t+k$-diagnosable systems.

Theorem 3.6 : Let S be a $t/t+k$-diagnosable system where $0 \leq k < t$. Then for every integer p with $0 \leq p < t-k$ and for all X with $|X| = n-2t+p$, $|\Gamma(X)| > p$.

Proof : Assume the contrary. Let p be an integer with $0 \leq p < t-k$ and let X be a subset of U with $|X| = n-2t+p$ such that $|\Gamma(X)| = q \leq p$. Let $Y = U-(X \bigcup \Gamma(X))$. Then $|Y| = 2t-(p+q)$ and there is no test from X into Y. Note that Y is non-empty since both p and q are less than t. Partition Y into two subsets Y_1 and Y_2 such that $| \, |Y_1| - |Y_2| \, | \leq 1$. Clearly both Y_1 and Y_2 are of size at most $t-\lfloor (p+q)/2 \rfloor$. Now consider the subsets $X_1 = Y_1 \bigcup \Gamma(X)$ and $X_2 = Y_2 \bigcup \Gamma(X)$.

$$|X_1| \leq t - \lfloor (p+q)/2 \rfloor + q \leq t$$

and

$$|X_2| \leq t - \lfloor (p+q)/2 \rfloor + q \leq t.$$

Moreover, since $p < t-k$,

$$\begin{aligned} |X_1 \bigcup X_2| &= 2t - (p+q) + q \\ &= 2t - p \\ &> t + k \end{aligned}$$

Thus $|X_1| \leq t$, $|X_2| \leq t$, $|X_1 \bigcup X_2| > t+k$ and there is no test from $U - X_1 - X_2$ into $X_1 \oplus X_2$ which violates the condition of Theorem 3.1. //

This property of a $t/t+k$-diagnosable system bears resemblance to a property presented in [46].

The following theorem shows an interesting relationship between allowable fault sets of size at most t in a t/s-diagnosable system.

Theorem 3.7 : Given a syndrome for a $t/t+k$-diagnosable system S, let F_1 and F_2 be two allowable fault sets of size at most t. Then

$$|F_1 \oplus F_2| < 2(k+1).$$

Proof : The result is obvious for $k \geq t$. So assume $0 \leq k < t$. Suppose the result is not true in this case. Let F_1 and F_2 be two allowable fault sets of size at most t for the given syndrome such that $|F_1 \oplus F_2| \geq 2(k+1)$. Since S is $t/t+k$-diagnosable, $|F_1 \bigcup F_2| \leq t+k$. Let A be a subset of $U - F_1 - F_2$ such that $|A| \bigcup |F_1 \bigcup F_2| = t+k+1$. Let $Z = A \bigcup (F_1 \bigcap F_2)$. Then

$$\begin{aligned} |Z| &= |A \bigcup F_1 \bigcup F_2| - |F_1 \oplus F_2| \\ &\leq (t+k+1) - 2(k+1) \\ &\leq t - (k+1) \end{aligned}$$

Let $p = t - (k+1)$ and $X = U - (A \bigcup F_1 \bigcup F_2)$. Then

$$\begin{aligned} |X| &= n - (t+k+1) \\ &= n - 2t + (t - (k+1)) \\ &= n - 2t + p. \end{aligned}$$

Thus $0 \leq p < t-k$ and $|X| = n - 2t + p$. Hence, by Theorem 3.6, $|\Gamma(X)| > p$. Since $|Z| \leq p$, it follows that there is a test (u_i, u_j) from X into $F_1 \oplus F_2$. Without loss of generality, assume $u_j \in F_1 - F_2$. Then $a_{ij} = 1$ in the presence of fault set F_1 and $a_{ij} = 0$ in the presence of fault set F_2, contradicting the assumption that F_1 and F_2 share a common syndrome. //

We note that the above result is a generalization of Lemma 5 in [17]. We also observe that this condition is not sufficient even for $t/t+1$-diagnosable systems. A system which satisfies the conditions of Theorem 3.7 but is not $t/t+1$ diagnosable is shown in Fig. 3.3. We obtain necessary and sufficient conditions for $t/t+1$-diagnosable systems if in addition to the above condition, we also impose the following:

If, for all $Y \in U$ with $|Y| = 3$, there is no test among units in Y, then $|\Gamma^{-1}(Y)| > t-1$.

This leads to the following result.

Theorem 3.8 : A system S with test interconnection graph $G = (U, E)$ is $t/t+1$-diagnosable if and only if the following conditions are satisfied:

(i) If, for all $X_i, X_j \subseteq U$ with $X_i \neq X_j$ $|X_i| \leq t$ and $|X_j| \leq t$, there is no test from $U - X_i - X_j$ into $X_i \oplus X_j$, then $|X_i \oplus X_j| < 4$.

(ii) If, for all $Y \in U$ with $|Y| = 3$, there is no test among units in Y, then $|\Gamma^{-1}(Y)| > t-1$.

Proof : (necessity) The necessity of condition (i) follows from Theorem 3.7 . Now suppose condition (ii) is not true. Let $Y = \{u_1, u_2, u_3\}$ be a subset of U such that there no test between units in Y and $|\Gamma^{-1}(Y)| \leq t-1$. Let $W \subseteq U - Y$ be a set with $|\Gamma^{-1}(Y)| \subseteq W$ and $|W| = t-1$. Now consider the subsets $Y_i = \{u_i\} \bigcup W$, $1 \leq i \leq 3$. For any pair Y_i, Y_j, $1 \leq i, j \leq 3$ there is no test from $U - Y_i - Y_j$ into $Y_i \oplus Y_j$ and $|\bigcup_{1 \leq l \leq 3} Y_l| > t+1$. Hence by Theorem 3.1 the system S is not $t/t+1$-

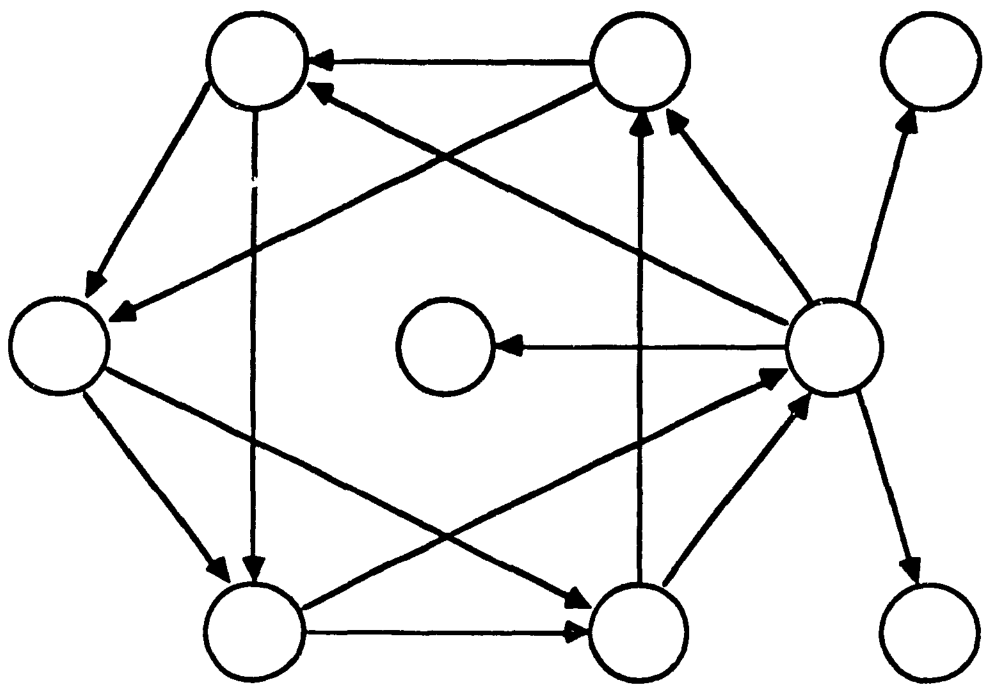


Fig. 3.3 A system satisfying the conditions of Theorem 3.7 which is not 2/3-diagnosable.

diagnosable.

(Sufficiency) Now assume that the system is not $t/t+1$-diagnosable. Then, by Theorem 3.2, there exists $Z \subseteq U$ with $|Z| > s$ and a non-trivial t-decomposition $A = \{Z_1,...,Z_k\}$ of Z such that for all $Z_i, Z_j \in A$ there is no test from $U-Z_i-Z_j$ to $Z_i \oplus Z_j$. Let r be an integer with $1 < r \leq k$ such that

$$|\bigcup_{1 \leq i \leq r-1} Z_i| \leq t+1$$

and

$$|\bigcup_{1 \leq i \leq r} Z_i| > t+1.$$

Note that $2 \leq r \leq 3$.

case 1 : $r = 2$.

Since $|Z_1| \leq t$, $|Z_2| \leq t$ and $|Z_1 \bigcup Z_2| \geq t+2$, it follows that $|Z_1 \bigcap Z_2| \leq t-2$. Hence $|Z_1 \oplus Z_2| \geq 4$. Since there is no test from $U-Z_1-Z_2$ to $Z_1 \oplus Z_2$, condition (1) is not satisfied for these two sets.

case 2 : $r = 3$.

Since $\{Z_1,...,Z_k\}$ is a non-trivial decomposition, it follows that the union of any two subsets Z_i, Z_j, $1 \leq i < j \leq 3$ contains at least $t+1$ units. Consider the subsets Z_1, Z_2 and Z_3. Assume condition (1) is satisfied. Clearly each of these subsets has at least one unit which does not belong to any of the other subsets. Also, each of these subsets contains at most two units which do not belong to any of the other subsets. We claim that no subset has two units which do not belong to the other subsets. Suppose that this is not true. Without loss of generality, assume Z_3 has two units which do not belong to the union of Z_1 and Z_2. Then

$$|(Z_1 \bigcup Z_2) \bigcap Z_3| \leq t-2.$$

Since $|Z_1 \bigcup Z_2| \geq t+1$, it follows that one of these two subsets contains two units which do not belong to Z_3, contradicting our assumption that condition (i) is satisfied. Thus each of Z_1, Z_2 and Z_3 has exactly one unit which is not contained in the other two subsets. Now let Y be the subset which contains these three units and $Z' = Z_1 \bigcup Z_2 \bigcup Z_3$. Then

$$|Z'| = |Z_1 \bigcup Z_2 \bigcup Z_3| = t+2.$$

Since there is no test from $U-Z_i-Z_j$ into $Z_i \oplus Z_j$, $1 \leq i < j \leq 3$, it follows that there is no test between units in Y and $\Gamma^{-1}(Y) \subseteq Z' - Y$. But

$$|Z' - Y| = t+2-3$$
$$= t-1.$$

So $|\Gamma^{-1}(Y)| \leq t-1$, violating condition (ii). //

Given a syndrome for a $t/t+1$-diagnosable system, if the intersection of allowable fault sets is empty, then it can be shown using the above theorem that the allowable fault sets can be partitioned into two fault sets F_1 and F_2 such that one of the subsets contains exactly one unit and the other contains one or two units.

3.4. Sequential t-Diagnosable Systems: Characterizations

The characterization for t/s-diagnosable systems given in Theorem 3.1. can easily be modified to obtain a characterization for sequential t-diagnosable systems. We note if a system is sequentially diagnosable, then for any given collection of subsets of size at most t, the subsets have at least one unit in common (See Lemma 2.4) or these subsets cannot generate a common syndrome i.e. they satisfy condition (i) of Theorem 3.1 for the PMC model and for the BGM model they satisfy condition (i) or (ii) of Theorem 3.5. It is easy to show (as in the proofs of Theorems 3.1 and 3.5) that these conditions are also sufficient. Thus we have the following results.

Theorem 3.9 : A system S under the PMC model with test interconnection graph $G=(U,E)$ is sequentially t-diagnosable if and only if for all $X \subseteq U$, and for all t-decompositions $A = \{X_1,\ldots,X_k\}$ of X at least one of the following is satisfied:

(i) $\bigcap_{1 \leq i \leq k} X_i \neq \Phi$

(ii) there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$. //

The above characterization is the same as the one given in [40].

Theorem 3.10 : A system S under the BGM model with test interconnection graph $G=(U,E)$ is sequentially t-diagnosable if and only if for all $X \subseteq U$, and for all t-decompositions $A = \{X_1,\ldots,X_k\}$ of X at least one of the following is satisfied:

(i) $\bigcap_{1 \leq i \leq k} X_i \neq \Phi$

(ii) there exist subsets $X_i, X_j \in A$ such that there is a test from $U - X_i - X_j$ to $X_i \oplus X_j$.

(iii) there exist subsets $X_i, X_j \in A$ such that there is a test between units lying entirely within $X_i - X_j$ or within $X_j - X_i$. //

3.5 Summary

In this chapter we have studied the structure of t/s-diagnosable systems and have presented several results generalizing previously known results derived for special cases. First we have presented characterizations (under the PMC model) of t/s-diagnosable systems which generalize those given earlier in [46], [47] for t/t-diagnosable systems. Interestingly, the characterization given in Theorem 3.4 itself involves the characterization given in Theorem 3.1. The reason for this is as follows. It can be seen from Theorem 3.3 that characterization of a t/t-diagnosable system involves an arbitrary collection A of exactly two subsets X_i and X_j each of size at most t. To test for the condition stated in this theorem we need to check for the presence of tests into $X_i \oplus X_j$ from $U - X_i - X_j$. On the other hand, it can be seen from Theorem 3.1 that characterization of a t/s-diagnosable system involves an arbitrary collection $A = \{X_i, X_2, ..., X_k\}$ of subsets of U such that each X_i is of size at most t and $|X| = |\bigcup_{1 \leq i \leq k} X_i| > s$. To test for the condition in Theorem 3.1, we need to check the presence of tests into $X_i \oplus X_j$ (for every pair X_i and X_j) from $U - X_i - X_j$. Thus, in addition to the tests from outside X we should also examine the tests lying entirely within X, since in general $k \geq 2$.

We have shown that the characterization based on Kohda's t-characterization theorem can easily be modified to arrive at t/s-characterizations for the BGM model as well as characterizations for the sequential t-diagnosable systems. We note that no study of the t/s-characterization problem has been reported in the literature for the BGM model.

We have also established certain properties of t/s-diagnosable systems. Whereas the property given in Theorem 3.6 resembles the one given in [46] by Chwa and Hakimi, the property of allowable fault sets given in Theorem 3.7 generalizes the one given by Yang et al [47] for the case of a t/t-diagnosable system. The latter property has helped us develop a new characterization of $t/t+1$-diagnosable systems.

CHAPTER IV

DIAGNOSIS OF t/s-DIAGNOSABLE SYSTEMS

In this chapter we investigate the problem of designing efficient algorithms to isolate all faulty units in a t/s-diagnosable system. The only work reported in the literature dealing with this problem was due to Yang, Masson and Leonetti [47]. These authors presented an $O(n^{2.5})$ diagnosis algorithm for a t/t-diagnosable system. This work is based on certain properties of allowable fault sets (AFS's) of a t/t-diagnosable system and the t-diagnosis algorithm presented by Dahbura and Masson [16]. Our main contributions in this chapter are: (i) an $O(n^{3.5})$ algorithm for a $t/t+1$-diagnosable system, and (ii) a $t/t+k$-diagnosis algorithm which runs in polynomial time for each fixed positive integer k. We also show how these algorithms can be modified to determine the set of all units which lie in every allowable fault set of cardinality at most t. These units can be correctly identified as faulty. This modified algorithm can thus be used for diagnosis of a sequentially t-diagnosable system.

Our approach in this chapter is first to develop an efficient test to determine whether a vertex v of a t/s-diagnosable system is in an AFS of cardinality at most t. The set of all vertices which satisfy this property will be the required set isolating faulty units. This is ensured from Lemma 2.5. With this objective in mind we first present in the following section several properties of AFS's in a t/s-diagnosable system. These properties lead to the algorithms developed in the subsequent sections. Our investigations in this chapter are based on the notions of implied-fault set and the implied-fault graph used by Dahbura and Masson [17] in their study. Definitions of most of the concepts and symbols used in this chapter may be found in Chapter II.

4.1 Basic Properties of Allowable Fault Sets

In this section we establish certain properties of allowable fault sets with respect to a given syndrome. Our study is directed towards investigating conditions for a vertex v to be in an allowable fault set of cardinality at most t. For this purpose we use the notion of implied-fault set and the implied-fault graph used by Dahbura and Masson [17] in their study.

Given a syndrome for a system S, define the **implied-fault graph** $G^* = (U^*, E^*)$ to be an undirected graph such that $U^* = U$ and $E^* = \{(u_i, u_j) : u_i \in L(u_j)\}$. Recall that $L(u_i)$ is the set of all units of S that may be deduced to be faulty under the assumption that u_i is fault-free. Also $H_0(u_i)$ corresponds to the set $A_0(u_i) \bigcup \{u_i\}$ where $A_0(u_i)$ is the set of 0-ancestors of $\{u_i\}$. For $u \in U$, let G_u^* denote the subgraph of G^* obtained after all units in $H_0(u)$ and all edges incident on these units have been removed from G^*. Let K_u represent a MVCS (minimum vertex cover set) of G_u^* and let $G - H_0(u)$ denote the subgraph of G where all vertices in $H_0(u)$ along with all edges incident on these vertices have been removed from G. These concepts are illustrated in Fig. 4.1 and Fig 4.2 for a particular test interconnection graph $G(U,E)$ and a given syndrome.

Note that if $u_i \in L(u_i)$ then u_i can immediately be identified as faulty. Thus we assume that $u_i \notin L(u_i)$ for any $u_i \in U$. This means that G^* has no self-loops.

The results and proof of the following lemma can be found in [16]. We present this lemma for the sake of completeness.

Lemma 4.1 : Given a syndrome for a system S, we have the following:

(1) Every AFS of G is a VCS of G^*.

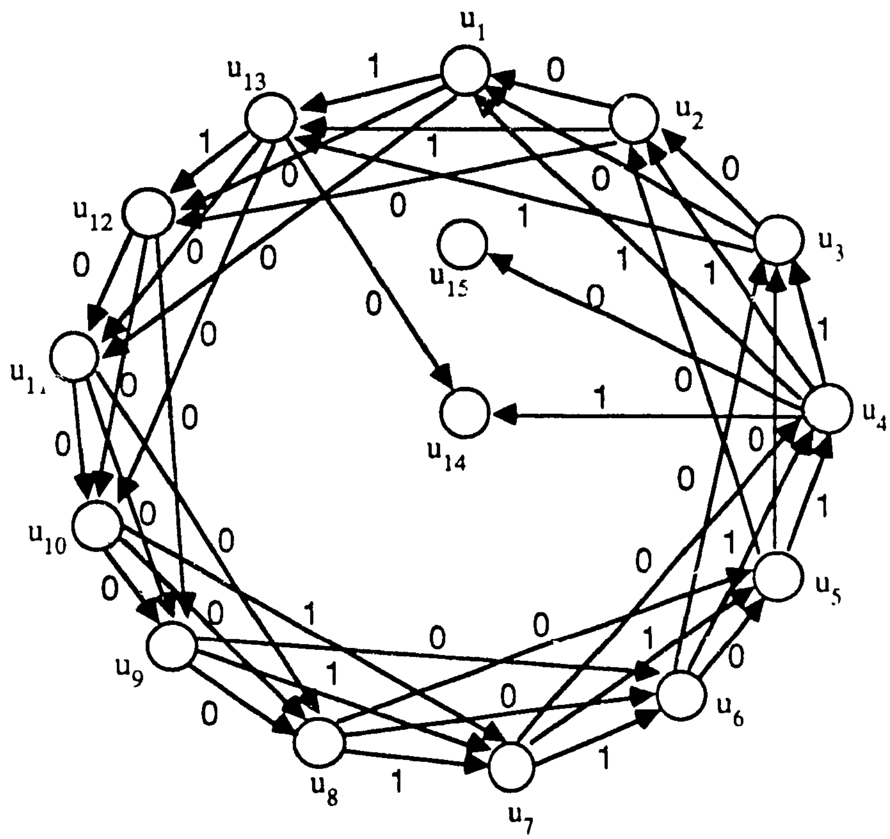


Fig. 4.1 A syndrome for a 3/4 diagnosable system

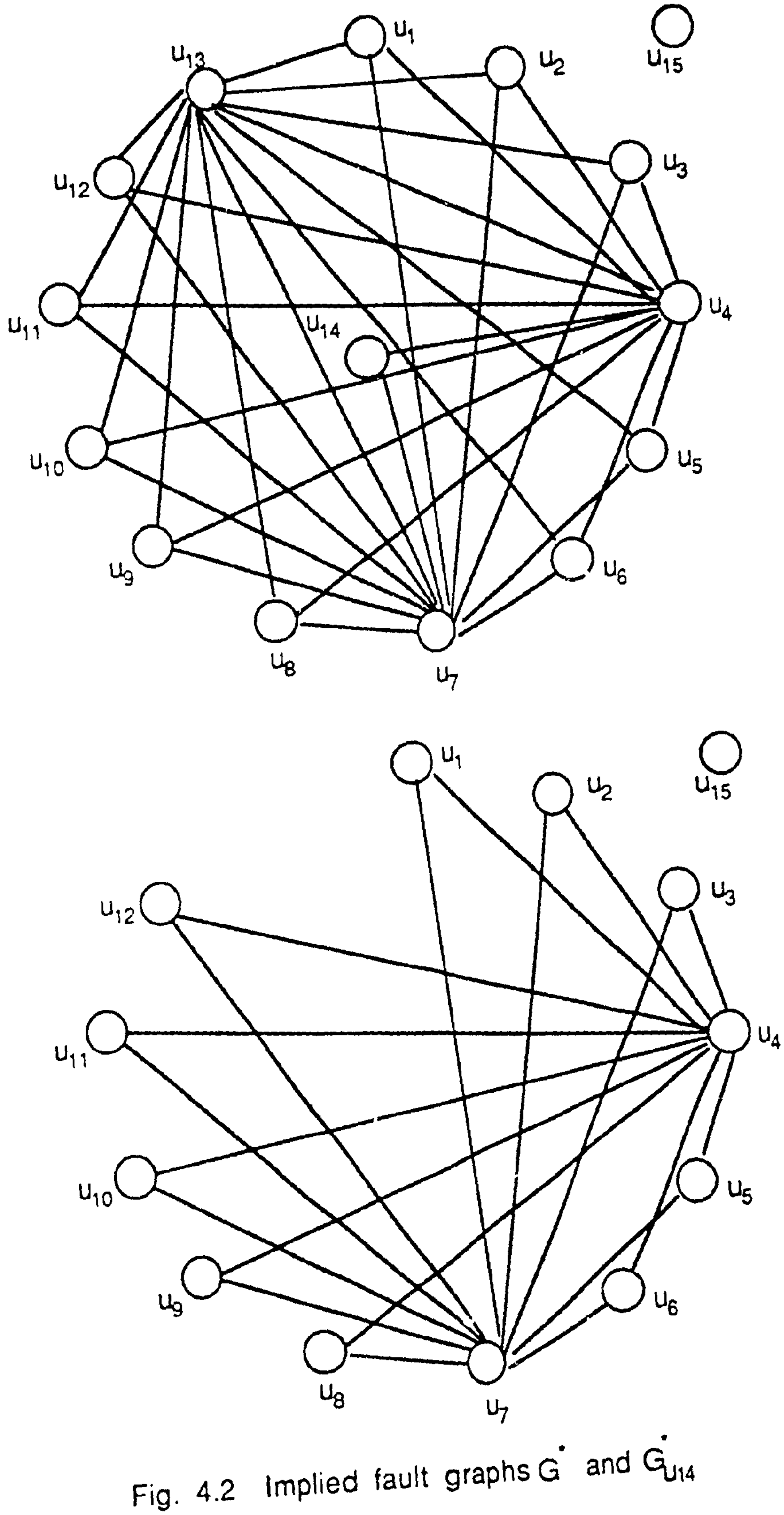


Fig. 4.2 Implied fault graphs G^* and G^*_{u14}

(II) If $F \subseteq U$ is a minimal VCS of $G*$, then F is an AFS of G.

(III) $F \subseteq U$ is a MAFS of G if and only if F is a MVCS of G^*.

Proof:

(I) Let F be an AFS of G for the given syndrome. Assume F is not a VCS of G^*. Then there exist $u_i, u_j \in U - F$ such that (u_i, u_j) is an edge in G^*. Since all directed edges from $U - F$ into F in G are 1-links (F is an AFS of G). It can be seen from the structure of an implied-fault path that all vertices which lie on such a path between u_i and u_j must belong to $U - F$. But this implies that there is a 1-link between two vertices in $U - F$, contradicting the assumption that F is an AFS of G. This shows that (I) holds.

(II) Let F be a minimal VCS of G^*. Assume (II) does not hold. Then at least one of the following conditions is satisfied.

(a) There exist $u_i, u_j \in U - F$ with $a_{ij} = 1$

(b) There exist $u_j \in F$ and $u_i \in U - F$ with $a_{ij} == 0$

Assume (a) holds. Then the edge (u_i, u_j) is in G^*. But this contradicts the fact that F is a VCS of G^* since neither u_i nor u_j is a member of F.

Now assume (b) holds and (a) does not hold. Since F is a minimal VCS of G^* there exists a unit u_k in $U - F$ such that (u_j, u_k) is an edge in G^*; for otherwise $F - \{u_j\}$ will be a VCS contradicting the minimality of F. Hence $u_j \in L(u_k)$. Since $a_{ij} = 0$, it follows that $u_i \in L(u_k)$ and so (u_i, u_k) is an edge in G^*. Since neither u_i nor u_k is a member of F, this contradicts the fact that F is a VCS of G^*.

(III) Statement (III) follows from (I) and (II). //

Lemma 4.2 : F is an AFS in G of minimum cardinality containing unit v if and only if $H=F-H_0(v)$ is a MAFS of $G-H_0(v)$.

Proof : (Necessity) We first show that H is an AFS of $G-H_0(v)$. Since $U-F=(U-H_0(v))-H$ and F is an AFS of G all edges within $(U-H_0(v))-H$ are 0-links and all edges from $(U-H_0(v))-H$ into H are 1-links. Hence H is an AFS of $G-H_0(v)$. To show that H is a MAFS of $G-H_0(v)$, assume H_1 is an AFS of $G-H_0(v)$. Clearly all edges with both vertices incident on vertices in $(U-H_0(v))-H_1$ are 0-links and all edges from $U-H_0(v)-H_1$ into H_1 are 1-links. Now consider edges from $(U-H_0(v))-H_1$ into $H_0(v)$. These edges must all be 1-links; otherwise the vertices incident on these edges would all belong to $H_0(v)$. This shows that the set $H_1 \bigcup H_0(v)$ is an AFS of G. Hence if $|H_1| < |H|$ then $H_1 \bigcup H_0(v)$ would be an AFS of cardinality smaller than F, contradicting the fact that F is an AFS of minimum cardinality containing v. Hence $|H_1| \geq |H|$ and H is an MAFS of $G-H_0(v)$.

(Sufficiency) If $F-H_0(v)$ is an MAFS of $G-H_0(v)$ then as we have shown in the proof of necessity, F is an AFS of G. If F is not an AFS in G of minimum cardinality containing v, then let F_1 be an AFS of G containing v with $|F_1| < |F|$. But then $F_1-H_0(v)$, from the necessity part, would be an AFS of $G-H_0(v)$ of cardinality smaller than $F-H_0(v)$, a contradiction. //

Lemma 4.3 : For $v \in U$, $(G-H_0(v))^* = G_v^*$.

Proof : Since the vertex sets of both graphs are the same, we need only show that the edge sets are identical. Clearly every edge in $(G-H_0(v))^*$ is in G_v^*. Now assume that there is an edge (u_i, u_j) in G_v^* which is not in $(G-H_0(v))^*$. Then every implied-fault path in G between u_i and u_j must contain at least one vertex from $H_0(v)$. But this implies that either u_i or u_j is a member of $H_0(v)$, contradicting the assumption that both vertices are members of $G-H_0(v)$. Hence the two edge sets are also identical. //

Lemma 4.4 : Given a syndrome for a system S, let $F \subseteq U$ be an AFS containing $v \in U$. Then $F-H_0(v)$ is a VCS of G_t^*.

Proof : Let $F_1 = F-H_0(v)$. From Lemma 2.3 and the proof of Lemma 4.2, it follows that F_1 is an AFS of $G-H_0(v)$. Then from Lemma 4.1, F_1 is a VCS of $(G-H_0(v))^*$. Thus, by Lemma 4.3, F_1 is a VCS of G_t^*. //

Theorem 4.1 : Given a syndrome for a system S, F is an AFS of minimum cardinality among all allowable fault sets that contain unit $u \in U$ if and only if $F-H_0(v)$ is a MVCS of G_t^*.

Proof : Proof follows from Lemmas 4.2, 4.3 and 4.4. //

The condition in the above theorem can be used to test if a unit belongs to an AFS of cardinality at most t for a given syndrome. However this condition requires determining a MVCS for a general undirected graph, a problem which is known to be NP-Complete [63]. So, we would like to develop a test which requires determining a MVCS of a bipartite graph. With this objective in mind, we now define a bipartite graph for each vertex v. This bipartite graph is derived from G_v^*. We then relate a MVCS of this graph to an AFS containing vertex v and establish certain properties of this AFS which will be used in the following sections to develop appropriate diagnosis algorithms.

Given a system S and a syndrome, define $B=(U_B,E_B)$ to be the undirected bipartite graph with bipartition (X,Y) where

$$X = \{x_1,....,x_n\} \ , \ Y = \{y_1,....,y_n\}$$

and

$$E_B = \{(x_i, y_j) : u_i \in L(u_j) \text{ in S}\}.$$

For $v \in U$, define the undirected bipartite graph $B_v = (U_v, E_v)$ with bipartition (X_v, Y_v) to be the vertex induced subgraph of B such that

$$X_v = \{x_i : u_i \in U - H_0(v)\},$$

$$Y_v = \{y_i : u_i \in U - H_0(v)\}.$$

Fig. 4.3 illustrates these concepts for the test interconnection graph $G(U, E)$ and the syndrome given in Fig 4.1.

For each vertex v in G, let

$$t_v = t - | H_0(v) |$$

and

$$U_v = U - H_0(v).$$

Theorem 4.2 : Given a syndrome for a system S, a unit $v \in U$ does not belong to any AFS of cardinality at most t if B_v has a MVCS of cardinality greater than $2t_v$.

Proof : Let the cardinality of a MVCS of B_v be greater than $2t_v$. Assume $v \in U$ belongs to an AFS F such that $| F | \leq t$. Let $H = F - H_0(v)$. Clearly $| H | \leq t_v$. Define $B_X(H) = (U_X, E_X)$ to be the vertex induced subgraph of B_v, where

$$U_X = \{x_i : u_i \in H\} \cup \{y_i : u_i \in U_v - H\}.$$

$B_Y(H) = (U_Y, E_Y)$ is defined to be the vertex induced subgraph of B_v, where

$$U_Y = \{x_i : u_i \in U_v - H\} \cup \{y_i : u_i \in H\}.$$

Clearly $F_X = \{x_i : u_i \in H\}$ and $F_Y = \{y_i : u_i \in H\}$ are VCS's of $B_X(H)$ and $B_Y(H)$, respectively. It follows that $F_B = F_X \bigcup F_Y$ is a VCS of $B_X(H) \bigcup B_Y(H)$. Since F is an AFS, in G^* there are no edges connecting vertices of $U - F$. From this it follows that every edge in B_v has end vertices in F_B. Therefore F_B is a VCS of B_v, contradicting our assumption that the

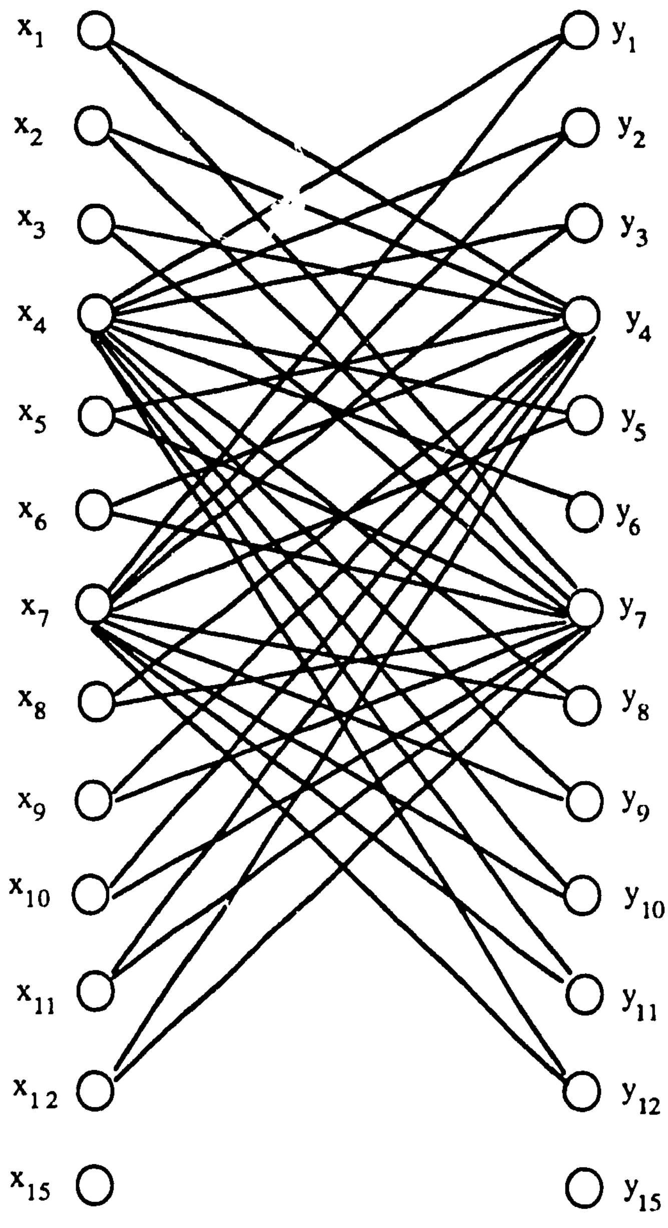


Fig. 4.3 Bipartite graph corresponding to $\dot{G}_{u_{14}}$

cardinality of a MVCS of B_v is greater than $2t_v$. Hence v is not contained in any AFS of cardinality at most t. //

In the following we use $F_B(v)$ to denote a MVCS of B_v. For a given $F_B(v)$ let

$$F_I = \{ u_i \mid x_i \in F_B(v) \text{ and } y_i \in F_B(v) \}$$

and

$$F_v = \{ u_i \mid x_i \in F_B(v) \text{ or } y_i \in F_B(v) \}.$$

We now proceed to establish certain properties of F_v.

Lemma 4.5. $F_I \bigcup H_0(v)$ is an AFS of G.

Proof: Assume the contrary. Then at least one of the following conditions is satisfied.

(a) There exist $u_i, u_j \in U - F_v - H_0(v)$ with $a_{ij} = 1$

(b) There exist $u_j \in F_I \bigcup H_0(v)$ and $u_i \in U - (F_I \bigcup H_0(v))$ with $a_{ij} = 0$

Assume (a) holds. Then the edge (u_i, u_j) is in G_v^*. Hence (x_i, y_j) is an edge in B_v. But this contradicts the fact that $F_B(v)$ is a VCS of B_v since neither x_i nor y_j is a member of $F_B(v)$.

Now assume (b) holds and (a) does not hold. Clearly $u_j \notin H_0(v)$; for otherwise u_i would also belong to $H_0(v)$. Thus $u_j \in F_v$. Hence either x_j or y_j is a member of $F_B(v)$. Without loss of generality let $x_j \in F_B(v)$. Since $F_B(v)$ is a MVCS of B_v, there exists y_k in B_v with $y_k \notin F_B(v)$ such that (x_j, y_k) is an edge in B_v. Hence $u_j \in L(u_k)$. Since $a_{ij} = 0$, $u_i \in L(u_k)$. Hence (x_i, y_k) is an edge in B_v. Since neither x_i nor y_k is a member of $F_B(v)$, this contradicts the fact that $F_B(v)$ is a VCS of B_v. //

Lemma 4.6.

Given a syndrome for a system S and a unit $v \in U$, we have the following:

(i) In G^*, there is no edge (u_i, u_j) with $u_i \in U-(F_v \bigcup H_0(v))$ and $u_j \in F_v - F_I$.

(ii) In G, there is no edge (u_i, u_j) with $u_i \in U-(F_v \bigcup H_0(v))$ and $u_j \in F_v - F_I$.

Proof:

(i) Assume the contrary. Let (u_i, u_j) be be an edge from $U-(F_v \bigcup H_0(v))$ into $F_v - F_I$ in G^*. Then either x_j or y_j is not a member of $F_B(v)$. Thus in B_v either the edge (x_i, y_j) or the edge (x_j, y_i) is not incident on any vertex in $F_B(v)$, contradicting the fact that $F_B(v)$ is a VCS of B_v.

(ii) By Lemma 4.5, the set $F_v \bigcup H_0(v)$ is an AFS of G. Thus every edge from $U-(F_v \bigcup H_0(v))$ into $F_v - F_I$ in G must be a 1-link. So if such an edge (u_i, u_j) exists in G, then (u_i, u_j) is an edge in G^*. Thus from (i) it follows that there is no edge (u_i, u_j) in G with $u_i \in U-(F_v \bigcup H_0(v))$ and $u_j \in F_v - F_I$. //

Lemma 4.7. Every AFS of G contained in $F_v \bigcup H_0(v)$ contains the subset F_I.

Proof: To show that every AFS of G contained in $F_v \bigcup H_0(v)$ contains the subset F_I it suffices to show that every VCS of G^* contained in $F_v \bigcup H_0(v)$ contains F_I. The above assertion holds if every vertex in F_I is incident on some vertex of $U-(F_v \bigcup H_0(v))$ in G^*. Assume the contrary. Let u_k be a vertex in F_I which is not incident on any vertex of the set $U-(F_v \bigcup H_0(v))$. Then let $W_B(v) = \{x_i \mid u_i \in F_v\} \bigcup \{y_i \mid u_i \in F_I - \{u_k\}\}$. From Lemma 4.6(i) and the construction of B_v it follows that in B_v, there is no edge (x_i, y_j) with $u_i \in U-(F_v \bigcup H_0(v))$ and $u_j \in F_v - F_I$. So $W_B(v)$ is a VCS of B_v. But $|W_B(v)| = |F_B(v)| - 1$. This contradicts the assumption that $F_B(v)$ is a MVCS of B_v. Thus every vertex in F_I is incident on some vertex of $U-(F_v \bigcup H_0(v))$ in G^*. This implies that every VCS of G^* contained in $F_v \bigcup H_0(v)$ contains the subset F_I. By Lemma 4.1, it follows that every AFS of

G contained in $F_v \bigcup H_0(v)$ contains the subset F_I. //

4.2 $O(n^{3.5})$ Algorithm for Diagnosis of a $t/t+1$-Diagnosable System

In this section we establish that in the case of a $t/t+1$ diagnosable system, the condition of Theorem 4.2 is both necessary and sufficient for a vertex v to be in an AFS of cardinality at most t. This will lead to an $O(n^{3.5})$ diagnosis algorithm to isolate all faulty units to within at most $t+1$ units in a $t/t+1$-diagnosable system. First we derive a necessary condition for a system to be $t/t+1$-diagnosable.

Lemma 4.8. If S, a multiprocessor system with test interconnection $G=(U,E)$, is $t/t+1$-diagnosable then for all $X_i, X_j \subseteq U$ with $|X_i| > t$, $X_j \not\subseteq X_i$, and $|X_i| + |X_j| \leq 2t$, there exists a test from $U-X_i-X_j$ into $X_i \oplus X_j$.

Proof: Assume S is $t/t+1$-diagnosable but the condition does not hold. Then there exist $X_i, X_j \subseteq U$ with $|X_i| > t$, $X_j \not\subseteq X_i$, $|X_i| + |X_j| \leq 2t$ such that there is no test from $U-X_i-X_j$ into $X_i \oplus X_j$.

Since $|X_i| > t$ and $X_j \not\subseteq X_i$, $|X_i \bigcup X_j| > t+1$, we construct two sets W_i and W_j from X_i and X_j by moving elements from $X_i - X_j$ into $X_j - X_i$ until W_i and W_j have cardinality at most t. Thus we obtain two sets W_i and W_j with $|W_i| \leq t$, $|W_j| \leq t$ and $|W_j \bigcup W_j| > t$ such that there is no test from $U-W_i-W_j$ into $W_i \oplus W_j$. By Theorem 3.1, this contradicts the assumption that S is $t/t+1$-diagnosable. //

Recall from the previous section that $F_B(v)$ is a MVCS of B_v and F_v and F_I are sets derived from $F_B(v)$.

Theorem 4.3 : Given a syndrome for a $t/t+1$-diagnosable system S, a unit $u \in U$ belongs to an AFS of cardinality at most t if and only if

$| F_B(v) | \leq 2t_v$.

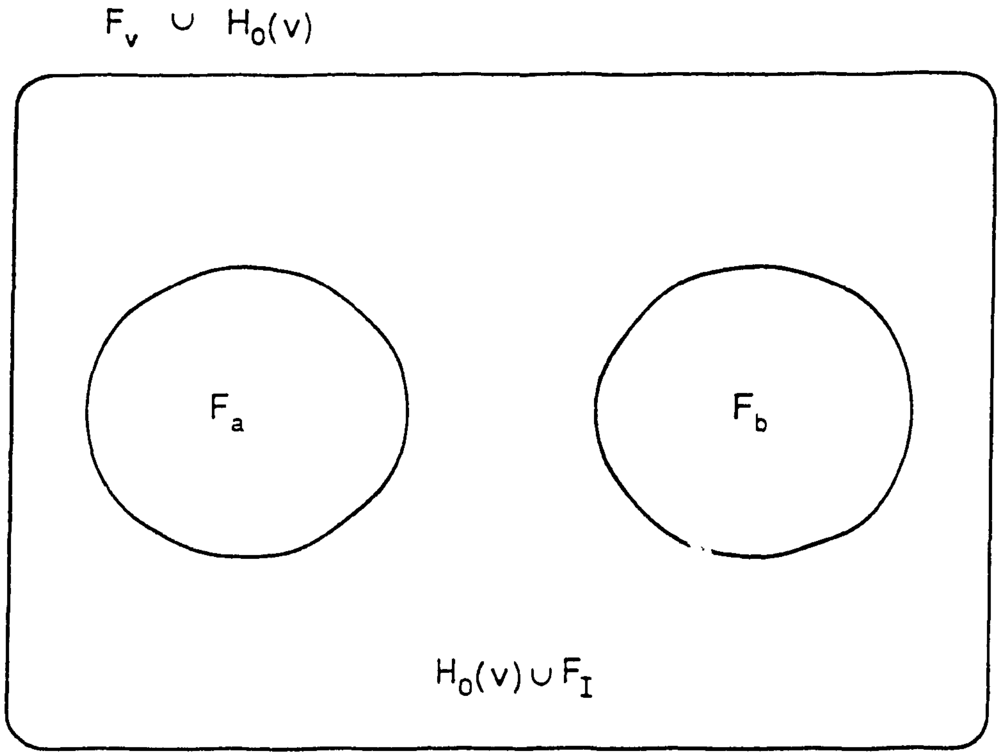
Proof : If $| F_B(v) | > 2t_v$ then, by Theorem 4.2, G does not contain an AFS of cardinality at most t containing the unit v.

Now assume $| F_B(v) | \leq 2t_v$. If $F_v \bigcup H_0(v)$ contains an AFS of G of cardinality at most t containing the unit v then we are through. So assume $| F_B(v) | \leq 2t_v$ and $F_v \bigcup H_0(v)$ does not contain any AFS of G of cardinality at most t containing the unit v. Note that from Lemma 4.5, $F_v \bigcup H_0(v)$ is an AFS of G containing the unit v. If $F_v = F_I$ then $| F_v \bigcup H_0(v) | \leq t$ since $| F_B(v) | \leq 2t_v$. So we further assume that $F_v \neq F_I$. Since G^* does not contain any units with self-loops and $F_v \neq F_I$, the subset $| F_v - F_I | \geq 2$. Let F_α be an AFS of smallest cardinality containing unit v such that $F_\alpha \subseteq F_v \bigcup H_0(v)$. Clearly $| F_\alpha | > t$.

By Lemma 4.7 every AFS of G contained in $F_v \bigcup H_0(v)$ contains the subset F_I, and since $v \in F_\alpha$, it follows that $F_I \bigcup H_0(v) \subseteq F_\alpha$.

We next show that $F_\alpha \neq F_v \bigcup H_0(v)$. Let $u_i \in F_v - F_I$. Then $W = F_v - \{u_i\}$ is a VCS of G_v^* because, by Lemma 4.6(i), in G_v^*, there is no edge (u_i, u_j) with $u_i \in U-(F_v \bigcup H_0(v))$ and $u_j \in F_v - F_I$. Hence by Lemma 4.3, W is a VCS of $(G - H_0(v))^*$. This means, by Lemma 4.1(ii), W contains an AFS of $G - H_0(v)$. Thus $W \bigcup H_0(v)$ has an AFS of G containing unit v and of cardinality less than that of $F_v \bigcup H_0(v)$. Since F_α is an AFS of smallest cardinality containing v such that $F_\alpha \subseteq F_v \bigcup H_0(v)$, it follows that $| F_b | = | (F_v \bigcup H_0(v)) - F_\alpha | > 0$ and $F_\beta \nsubseteq F_\alpha$ (See Fig. 4.4).

$$
\begin{aligned}
\text{Now } | F_\alpha | + | F_\beta | &= 2 | F_I | + 2 | H_0(v) | + | F_a | + | F_b | \\
&= | F_B(v) | + 2 | H_0(v) | \\
&\leq 2t_v + 2 | H_0(v) | \leq 2t .
\end{aligned}
$$



$$F_\alpha = F_a \cup H_0(v) \cup F_I$$

$$F_\beta = F_b \cup H_0(v) \cup F_I$$

$$F_v = F_a \cup F_b \cup F_I$$

$$|F_{B(v)}| = 2|F_I| + |F_a| + |F_b|$$

$$F_\alpha \cap F_\beta = H_0(v) \cup F_I$$

$$F_\alpha \ominus F_\beta = F_a \cup F_b$$

Fig.4.4 Illustration for proof of Theorem 4.3.

Thus we have $|F_\alpha| > t$, $F_\beta \not\subseteq F_\alpha$, $|F_\alpha| + |F_\beta| \le 2t$, and by Lemma 4.6.(ii) there is no test from $U - F_\alpha - F_\beta$ into $F_\alpha \oplus F_\beta$. This, by Lemma 1.8, contradicts our assumption that the system S is $t/t+1$-diagnosable. //

Given a valid syndrome for a $t/t+1$-diagnosable system S and a unit v in S, we have shown that the bipartite graph B_v has a MVCS of cardinality at most $2t_v$ if and only if G has an AFS of cardinality at most t containing the unit v. Thus we have the following algorithm to isolate all faulty units in a $t/t+1$-diagnosable system. Note that vertices in G^* correspond to units which can immediately be identified as faulty. These vertices are removed from the graph and the parameters adjusted accordingly.

Algorithm 4.1: Diagnosis of a $t/t+1$-Diagnosable System

Step 1. Given a $t/t+1$-diagnosable system S with test interconnection graph $G=(U,E)$ and a syndrome arising from a t-fault situation, construct G^* and remove all vertices with self-loops. From the resulting graph, construct the bipartite graph $B=(U_B, E_B)$ with bipartition (X, Y).

Step 2. Set $F = \phi$; for all $v \in U$, label v unmarked.

Step 3. **While** there exists an unmarked $v \in U$

begin

3.1. Label v marked.

3.2. Set $t_v = t - |H_0(v)|$.

3.3 Construct B_v from B.

3.4. Compute a maximum matching K_v of B_v using the Hopcroft/Karp algorithm [64].

3.5. If $|K_v| \le 2t_v$ **then** add v to F.

end

Step 4. F is the required set.

The bipartite graph in Step 1 can be constructed in $O(n^{2.5})$ operations [16]. Step 2 requires $O(n)$ operations. The computation within Step 3 is dominated by the computation of a maximum matching which requires $O(n^{2.5})$ operations [64]. Since Step 3 is performed for each unit in V, the complexity of the entire algorithm is $O(n^{3.5})$.

The condition of Theorem 4.3 can be use o develop an algorithm similar to Algorithm 4.1 to determine all units in a $t/t+1$-diagnosable system which belong to every AFS of cardinality at most t. For every unit v in a $t/t+1$-diagnosable system which belongs to some AFS of cardinality at most t we test whether there exists an AFS of cardinality at most t which does not contain the unit v. Given a valid syndrome for a $t/t+1$-diagnosable system, if an AFS does not contain a unit v then it must contain $L(v)$, the set of units implied faulty when v is assumed to be fault-free. Thus in order to determine if v does not belong to some AFS of G of cardinality at most t we check if the bipartite graph $B_{L(v)}$, the subgraph induced on B when vertices corresponding to $L(v)$ have been removed, contains a MVCS of cardinality at most $2(t - L(v))$. This condition is both necessary and sufficient to ascertain if the unit v does not belong to some AFS of G of cardinality at most t. We note that an MVCS of $B_{L(v)}$ does not contain the vertices corresponding to v since these vertices are isolated vertices in $B_{L(v)}$ if v does not have a self-loop in G^*. Thus we have the following algorithm to determine the set of all units which lie in every AFS of cardinality at most t in a $t/t+1$-diagnosable system.

Algorithm 4.2:

Step 1. Given a $t/t+1$-diagnosable system S with test interconnection graph $G=(V,E)$ and a syndrome arising from a t-fault situation, construct

G^* and remove all vertices with self-loops. From the resulting graph, construct the bipartite graph $B = (V_B, E_B)$ with bipartition (X, Y).

Step 2. Set $F = \emptyset$; for all $v \in V$, label v unmarked.

Step 3. **While** there exists an unmarked $v \in V$

begin

3.1. Label v marked.

3.2. Set $t_{L(v)} = t - |L(v)|$.

3.3 Construct $B_{L(v)}$ from B.

3.4. Compute a maximum matching $K_{L(v)}$ of $B_{L(v)}$ using the Hopcroft/Karp algorithm [64].

3.5. **If** $|K_{L(v)}| > 2t_{L(v)}$ **then** add v to F.

end

Step 4. F is the required set of all units which lie in every AFS of cardinality at most t.

Thus for a $t/t+1$-diagnosable system not only can all faulty units be isolated to within at most $t+1$ faulty units, but also all units which lie in every allowable fault set of cardinality at most t can be identified in $O(n^{3.5})$. The units of the set F thus obtained by Algorithm 4.2 can be correctly identified to be faulty. Note that if a $t/t+1$-diagnosable system is not sequentially t-diagnosable then the set F produced by Algorithm 4.2 may be empty.

4.3 Diagnosis Algorithm for a $t/t+k$-Diagnosable System

Given a $t/t+k$-diagnosable system where $k>1$ and a syndrome the diagnosis algorithm of the preceding section may not isolate all the faulty units to within at most $t+k$ units. In this case, for a unit $v \in V$, even if the the cardinality of the maximum matching K_v determined by the algorithm is less than or equal to $2t_t$, the graph G may not have an AFS of cardinality at most t which contains the unit v. In this section we develop a diagnosis algorithm for such a system.

First we show that the set $F_t \bigcup H_0(v)$ defined in the previous section contains a smallest AFS containing the vertex v.

Theorem 4.4. $F_t \bigcup H_0(v)$ contains an AFS F_1 containing unit v such that for every AFS F_2 of G containing unit v, $|F_1| \leq |F_2|$.

Proof: Let F_1 be an AFS of smallest cardinality containing unit v such that $F_1 \subseteq F_t \bigcup H_0(v)$. Assume F_2 to be an AFS of G containing unit v. Let $F_3 = (F_t - F_1) \bigcup (F_I \bigcup H_0(v))$. We observe that $F_I \bigcup H_0(v) \subseteq F_1$, since every AFS of G containing v in $F_t \bigcup H_0(v)$ contains $F_I \bigcup H_0(v)$. Also $F_1 \bigcup F_3 = F_t \bigcup H_0(v)$ and $F_I \subseteq F_1 \bigcup F_3$.

First we note that since F_1 and F_2 are VCS's of G^*, there are no edges in the subgraphs of G^* induced by the vertex sets $U-F_1$ and $U-F_2$. Also, by Lemma 4.6(i), for every edge in G^* with one vertex in $U-F_1-F_3$, the other vertex is in $F_1 \bigcap F_3$. From these facts we can conclude that $((F_3 \bigcap F_2) - F_1) \bigcup ((F_3 \bigcup F_2) \bigcap F_1)$ is a VCS of G^*. The shaded area in Fig. 4.5(a) corresponds to this VCS which lies entirely within $F_t \bigcup H_0(v)$.

We now claim that the following inequality holds (See Fig. 4.5(b)):

$$|(F_3 \bigcap F_2) - F_1| \geq |F_1 - F_2 - F_3| \qquad (4.1)$$

Assume (4.1) is not true. Then the VCS $((F_3 \bigcap F_2) - F_1) \bigcup ((F_3 \bigcup F_2) \bigcap F_1)$ is of smaller cardinality than F_1 and so by Lemma 4.1(ii) it contains an AFS of G of cardinality smaller than F_1, a contradiction.

By Lemma 4.6(i), in B_t for every edge (x_i, y_j) with $u_i \in U-F_1-F_2-F_3$, the vertex u_j is in $F_1 \bigcap F_3$ since $F_I \bigcup H_0(v) = F_1 \bigcap F_3$. From this and the fact that F_2 is a VCS of G^*, it follows that in B_t for every edge (x_i, y_j) with

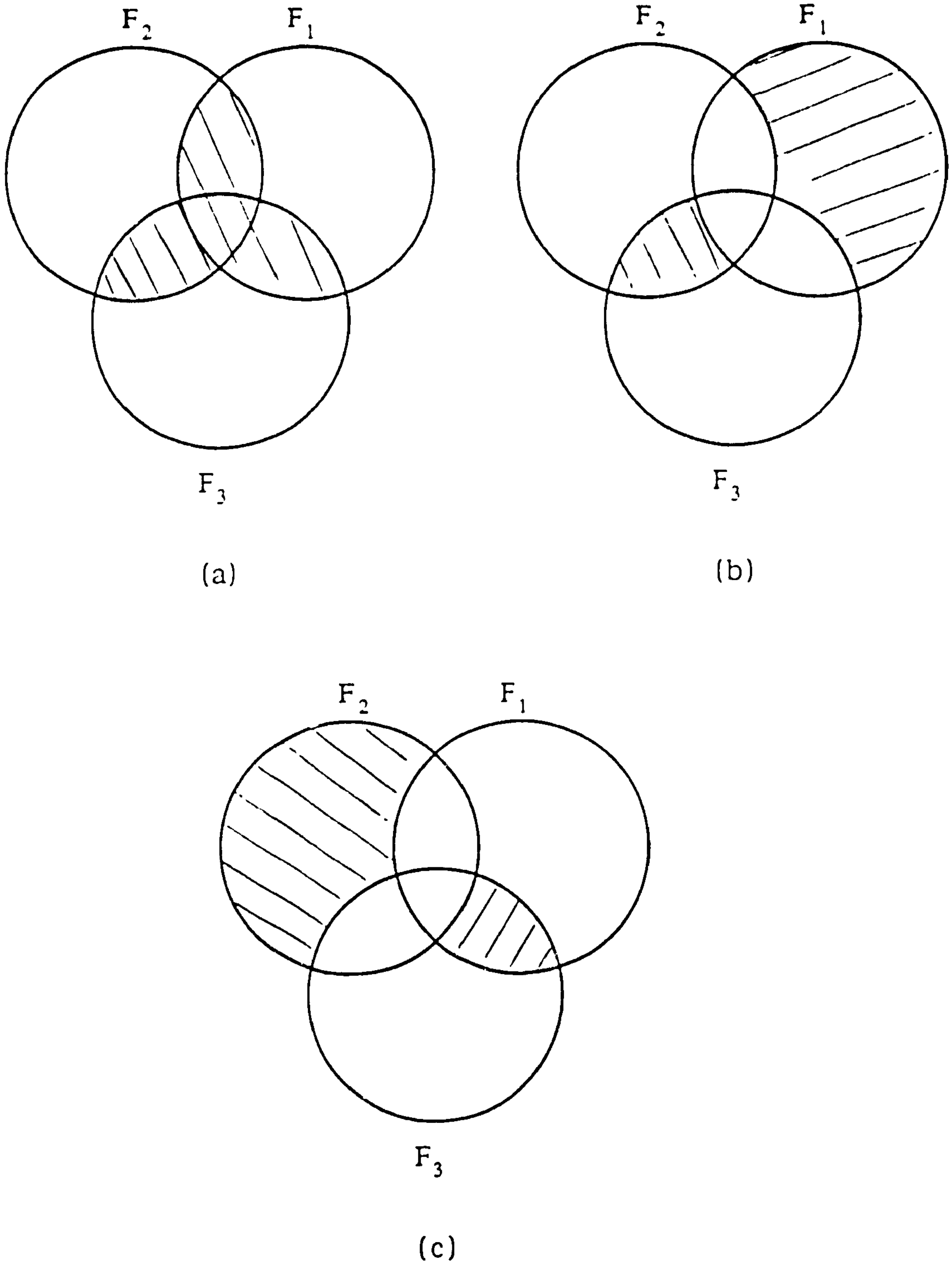
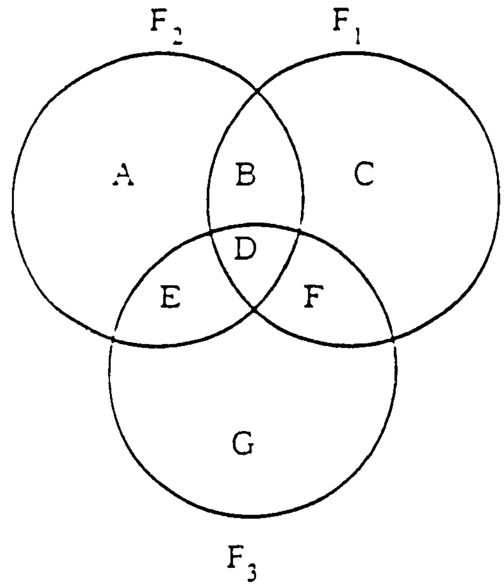


Fig. 4.5 Illustrations for Theorem 4.4.



(d)

$$|F_3| = |A| + |B| + |C| + 2|D| + |E| + |F| + |G| - 2|H_o(v)|$$

$$|F_{B(v)}| = |B| + |C| + 2|D| + |E| + 2|F| + |G| - 2|H_o(v)|$$

Fig. 4.5 (Contd.)

$u_i \in U - F_1 - F_2 - F_3$ the vertex u_j is in $F_1 \cap F_2 \cap F_3 - H_0(v)$. So the set

$$F_a = \{x_i \mid u_i \in (F_1 \cup F_2 \cup F_3) - H_0(v)\} \cup \{y_i \mid u_i \in (F_1 \cap F_2 \cap F_3) - H_0(v)\}$$

is a VCS of B_v.

We claim that the following inequality also holds (See Fig. 4.5(c)):

$$| F_2 - F_1 - F_3 | \geq | (F_3 \cap F_1) - F_2 | \tag{4.2}$$

If (4.2) is not true, then $| F_a | < | F_B(v) |$ because (See Fig. 4.5(d)) $| F_B(v) | - | F_a | = | (F_3 \cap F_1) - F_2 | - | F_2 - F_1 - F_3 | > 0$, contradicting the fact that $F_B(v)$ is a MVCS of B_v.

Thus (4.1) and (4.2) are true. But this implies that $| F_2 | \geq | F_1 |$. It follows that the set F_1 contained in $F_v \cup H_0(v)$ is a smallest AFS containing v. //

Theorem 4.5. Given a syndrome for a $t/t+k$ system S, if $| F_v | > t_v + k$ for some unit $v \in V$, then G does not have an AFS of cardinality at most t containing v.

Proof: Assume the contrary. Let F_1 be an AFS of smallest cardinality contained in $F_v \cup H_0(v)$ of which v is a member. From Theorem 4.4, F_1 is an AFS of smallest cardinality of which v is a member. Thus $| F_1 | \leq t$. Let $F_2 = (F_v \cup H_0(v)) - (F_1 - F_I - H_0(v))$. We claim that $| F_2 | \leq | F_1 |$. For otherwise, the set

$$M = \{x_i \mid u_i \in F_1 - H_0(v)\} \cup \{y_i \mid u_i \in F_1 - H_0(v)\}$$

which is a vertex cover of B_v will have cardinality less than $F_B(v)$ contradicting the minimality of $| F_B(v) |$. Hence $| F_2 | \leq | F_1 | \leq t$.

Consider the two sets F_1 and F_2. $| F_1 | \leq t$, $| F_2 | \leq t$ and $| F_1 \cup F_2 | > t + k$. Since all edges from $U - F_v - H_0(v)$ into $F_v \cup H_0(v)$ are incident on $F_I \cup H_0(v)$, there is no test from $U - F_1 - F_2$ into $F_1 \oplus F_2$. This, by Theorem 3.1, contradicts the assumption that the system is $t/t+k$-diagnosable.

//

Our diagnosis algorithm for isolating all faulty units in a $t/t+k$-diagnosable system is as follows. For each v, we determine a maximum matching K_v of B_v. If $|K_v| > 2t_v$ then, by Theorem 4.2, v does not belong to any AFS of cardinality at most t. Otherwise we determine from K_v a MVCS $F_B(v)$ of B_v. We then construct F_v and F_I. If $|F_v| \leq t_v$ then v is in an AFS of cardinality at most t. If not, we check if $|F_v| \leq t_v + k$. If yes, we check if F_v contains a VCS of cardinality at most t_v. We do so by taking all possible subsets W of F_v of cardinality equal to t_v with $F_I \subseteq W$ and examining if W is a VCS of the subgraph induced on G_v^* by the vertex set F_v.

Algorithm 4.3: Diagnosis of a $t/t+k$-Diagnosable System

S.1 Given a $t/t+k$-diagnosable system S with test interconnection graph $G=(V,E)$ and a syndrome arising from a t-fault situation, construct G^* and remove all vertices with self-loops. From the resulting graph, construct the bipartite graph $B=(V_B, E_B)$ with bipartition (X, Y).

S.2 Set $F = \phi$; for all $v \in V$, label v unmarked.

S.3 **While** there exists an unmarked $v \in V$

begin

3.1 Label v marked.

3.2 Set $t_v = t - |H_0(v)|$.

3.3 Construct B_v from B.

3.4 Compute a maximum matching K_v of B_v using the Hopcroft/Karp algorithm [64].

3.5 **If** $|K_v| \leq 2t_v$ **then**

begin

Compute a MVCS $F_B(v)$ of B_v from K_v using the König Construction Technique [65].

Determine F_v and F_I from $F_B(v)$.

If $|F_v| \leq t_v$ **then** add v to F

else if $|F_v| \leq t_v + k$ **then**

For each subset W with $F_I \subseteq W$ and $|W| = t_v$ check if W is a VCS of the subgraph of G^* induced on the the vertex set F_v. If so, add v to F.

end

end

S.4 F is the required set.

The correctness of Algorithm 4.3 follows from Theorems 4.2, 4.4 and 4.5.

Regarding complexity of this algorithm, the bipartite graph in step S.1 can be constructed in $O(n^{2.5})$ operations [16]. Computations in steps S.3.1 - S.3.4 is dominated by the computaion of a maximum matching in a bipartite graph which is of $O(n^{2.5})$. Step 3.5 may require computing all subsets of F_v of cardinality equal to t_v and testing each of them for the required VCS property. This step may require $|E| \, C_{t_v}^{|F_v|}$ operations. Note that $|F_v| \leq t_v + k$. Since $C_{t_v}^{|F_v|} = O(t_v^k)$ and $t_v < t$, the overall complexity of the algorithm is $O(n^{3.5} + mnt^k)$.

Note that if, in Algorithms 4.1, 4.2 and 4.3, a unit v is included in the set F then all units in $F_I \bigcup H_0(v)$ can also be added to F. So for these units, we do not need to perform step S.3 separately. Though, by doing so, we may reduce the number of computations, it will not improve the overall complexities of the algorithms.

We wish to note that as in the case of the $t/t+1$ diagnosis algorithm of the previous section, Algorithm 4.3 can be modified to identify all units in the $t/t+k$-diagnosable system which belong to every AFS of cardinality at most t. This will involve determining a MVCS of $B_{L(v)}$ for every unit v which is a member of some AFS of cardinality at most t. The following is a formal description of such an algorithm.

Algorithm 4.4:

S.1 Given a $t/t+k$-diagnosable system S with test interconnection graph $G=(V,E)$ and a syndrome arising from a t-fault situation, construct G^* and remove all vertices with self-loops. From the resulting graph, construct the bipartite graph $B=(V_B,E_B)$ with bipartition (X,Y).

S.2 Set $F = \phi$; for all $v \in V$, label v unmarked.

S.3 **While** there exists an unmarked $v \in V$

begin

3.1 Label v marked.

3.2 Set $t_{L(v)} = t - |L(v)|$.

3.3 Construct $B_{L(v)}$ from B.

3.4 Compute a maximum matching $K_{L(v)}$ of $B_{L(v)}$ using the Hopcroft/Karp algorithm [64].

3.5 **If** $|K_{L(v)}| > 2t_{L(v)}$ **then** add v to F.

else

begin

Compute a MVCS $F_B(v)$ of $B_{L(v)}$ from $K_{L(v)}$ using the König Construction Technique [65].

Determine F_t and F_I from $F_B(v)$.

If $| F_v | > t_{L(v)} + k$ **then** add v to F

else if $| F_v | > t_{L(v)}$ **then**

For each subset W with $F_I \subseteq W$ and $| W | = t_{L(v)}$ check if W is a VCS of the subgraph induced on G^* by the vertex set F_v. If not, add v to F.

end

end

S.4 F is the required set of units which lie in every AFS of cardinality at most t.

Thus for a $t/t+k$-diagnosable system not only can all faulty units be isolated to within at most $t+k$ faulty units, but also all units which lie in every AFS of cardinality at most t can be identified in polynomial time for every fixed positive integer k. The units of the set F thus obtained by Algorithm 4.4 can be correctly identified to be faulty. Note that if a $t/t+k$-diagnosable system is not sequentially t-diagnosable then the set F produced by Algorithm 4.4 may be empty.

4.4. On the Diagnosis of a Sequentially t-Diagnosable System

Algorithm 4.4 of the previous section can easily be modified to arrive to arrive at an algorithm for diagnosis of a sequentially t-diagnosable system. The only modification required is to substitute t for k. The algorithm is formally presented below.

Algorithm 4.5: Sequential t-Diagnosis

S.1 Given a sequentially t-diagnosable system S with test interconnection graph $G=(U,E)$ and a syndrome arising from a t-fault situation, construct the bipartite graph $B=(U_B, E_B)$ with bipartition (X, Y).

S.2 Set $F = \phi$; for all $v \in U$, label v unmarked.

S.3 **While** there exists an unmarked $v \in U$

begin

3.1 Label v marked.

3.2 Set $t_{L(v)} = t - | L(v) |$.

3.3 Construct $B_{L(v)}$ from B.

3.4 Compute a maximum matching $K_{L(v)}$ of $B_{L(v)}$ using the Hopcroft/Karp algorithm [64].

3.5 **If** $| K_{L(v)} | > 2t_{L(v)}$ **then** add v to F.

else

begin

Compute a MVCS $F_B(v)$ of $B_{L(v)}$ from $K_{L(v)}$ using the König Construction Technique [65].

Determine F_v and F_I from $F_B(v)$.

If $| F_v | > t_{L(v)}$ **then**

For each subset W with $F_I \subseteq W$ and $| W | = t_{L(v)}$ check if W is a VCS of the subgraph induced on G^* by the vertex set F_v. If not, add v to F.

end

end

S.4 F is the required set of units which lie in every AFS of cardinality at most t.

It is easy to see that the complexity of the above algorithm is $O(n^{3.5} + mnt^t)$.

4.5. Summary

In this chapter we have studied the problem of diagnosing t/s-diagnosable systems. We have presented $t/t+k$-diagnosis algorithms. In the case of $t/t+1$-diagnosable systems, our algorithm runs in $O(n^{3.5})$ time. In the general case of $k > 1$ our algorithm has complexity which is polynomial for each fixed positive integer k. These algorithms are based on certain properties derived from the characterization of t/s-diagnosable systems given in Chapter III and the structure of allowable fault sets of these systems.

We have shown how these algorithms can be modified to design algorithms for identifying all units which lie in every AFS of cardinality at most t of $t/t+k$-diagnosable systems. These units can then be correctly identified as faulty. We then presented an approach for diagnosing a sequentially t-diagnosable system. This approach leads to an algorithm which is of complexity $O(n^{3.5} + mnt^t)$.

The t/t-diagnosis algorithm of Yang, Masson and Leonetti [47] and the $t/t+1$- and $t/t+k$-diagnosis algorithms of this chapter complement the corresponding algorithms developed by Sullivan [33] for the t/t, $t/t+1$- and the $t/t+k$-diagnosability problems.

CHAPTER V

FAULT DIAGNOSIS UNDER LOCAL CONSTRAINTS: A BASIC ALGORITHM AND ANALYSIS OF A RING OF PROCESSORS

In multiprocessor systems such as those implementable in very large scale integration (VLSI) and wafer-scale integration (WSI), the number of units in a system can be very large [50]. Moreover, the commonly used system interconnection networks such as the rectangular grids are very symmetrical and sparse. Two major problems arise in analyzing these systems using the classical approach for system-level diagnosis. First, the value of t, and therefore the largest number of faulty processors that can be diagnosed, is limited by the connectivity of the processor interconnection graph of the system. This shortcoming motivated the recent works on probabilistic diagnosis algorithms for sparsely interconnected systems [52,53]. The other serious problem is that the diagnosis algorithms are assumed to be executed on a single, highly reliable supervisory processor which has access to the complete syndrome. A single supervisory processor is a bottleneck in a system with a large number of processing elements. Distributed diagnosis algorithms executed on the multiprocessor system itself would be desirable. Such distributed algorithms can take on two essentially different flavors: one in which the distributed task is to make the complete syndrome available to every processor in the system and let each processor act as the syndrome decoder to determine the faulty processors; and the other in wh h no single processor has the knowledge of the complete syndrome, but it , decoding of that part of the syndrome available to it may be passed on to its neighbors and the neighbors thereof, and so on. Pioneering work using the first approach was reported in [55,56]. The second approach has not been established firmly;

some early work in this direction has been presented in [51]. More recent work in this area includes [54].

In this and the following chapters, we address both the problems mentioned above. In particular, in this chapter we aim at the application of system-level diagnosis for regular interconnected multiprocessor systems with very small connectivity. The main result is to note a key result in syndrome decoding reported in the literature in various forms [20,21,46] and apply it to regular interconnected systems. Here a local neighborhood is defined around each processor which consists of its t immediate neighbors and t subsequent neighbors. The faulty or fault-free nature of each processor is then determined as long as no more than t units are faulty in its corresponding neighborhood. Based on this result, we also present a simple $O(1)$ distributed diagnosis algorithm.

There are two distinct advantages to this approach for distributed diagnosis. The first is that, since the diagnosis of faulty processors is done based on test results generated by the processors which belong to a small neighborhood, a large number of faulty processors can be diagnosed in the entire system. This number is, of course, bound to be much greater than t, the connectivity of the system. The other advantage is the natural benefit of a distributed algorithm in which the entire diagnosis effort for the whole system is carried out simultaneously in all neighborhoods. Thus, up to one less than half the total number of processors in the system could be faulty, and yet could be diagnosed in $O(1)$ time.

It is, however, clear that not all all possible combinations of faulty units up to this size are expected to be diagnosed, since many of these fault patterns will not satisfy the constraint of having no more than t faulty units in each neighborhood. To partly overcome this problem, we consider in this chapter another diagnosis algorithm in which more than one neighborhood is considered for each processor. A form of majority voting allows us to decode the information generated by each of these neighborhoods about the faulty or fault-free nature of a processor. We present some results of analysis of fault coverage obtained when this approach is applied on a rectangular grid system.

We also study the implication of applying local fault constraints in the diagnosis of a ring of processors.

In this and the following chapters we use the comparison model (described in Chapter II) for our studies.

5.1. Local Diagnosis Algorithm

Let S be a multiprocessor system with test interconnection graph $G=(U,E)$. Note that when the comparison model is used, G is an undirected graph.

Let u_i be a unit in S with t distinct paths of length 2 from u_i. $R_t(u_i)$, called a **neighborhood of order t** around u_i, denotes the set of processors which lie on these paths including u_i.

Fig. 5.1 shows a unit u_i in a system with a hexagonal grid interconnection, and Fig. 5.2(a) defines a local neighborhood of order 6 around a unit u_i. In a system S, there may be more than one way to define a local neighborhood around a unit u_i. This point is illustrated in Fig. 5.2(b). It is possible that a unit which is at a distance 2 from u_i in $R_t(u_i)$ may also be adjacent to u_i. Fig 5.2(c) defines a local neighborhood of order 3 around u_i in a hexagonal grid interconnection system where such a situation arises. It is also easy to observe that in a hexagonal grid interconnection with wraparound it is possible to define a local neighborhood of order 6 around it.

We require the following theorem to prove the results of this chapter. This result helps determine the faulty or fault-free status of any unit using only

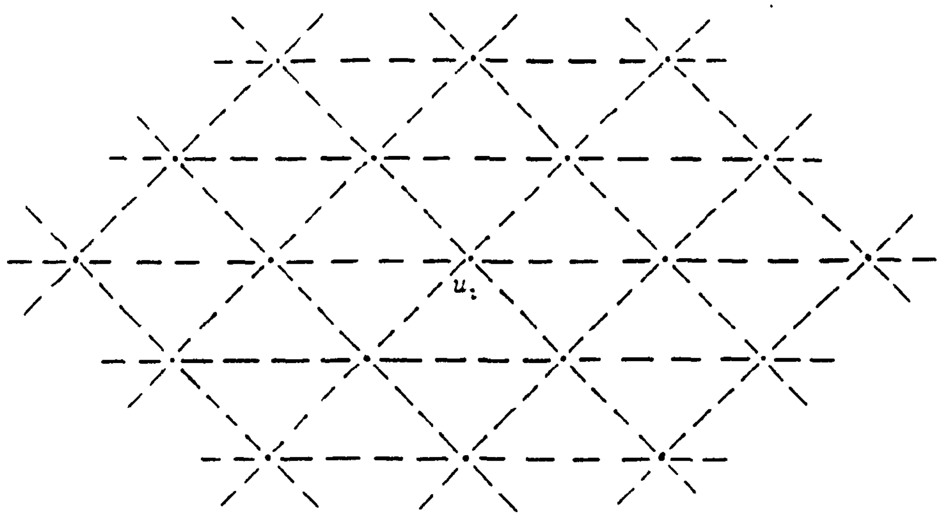


Fig. 5.1 A hexagonal grid interconnection.

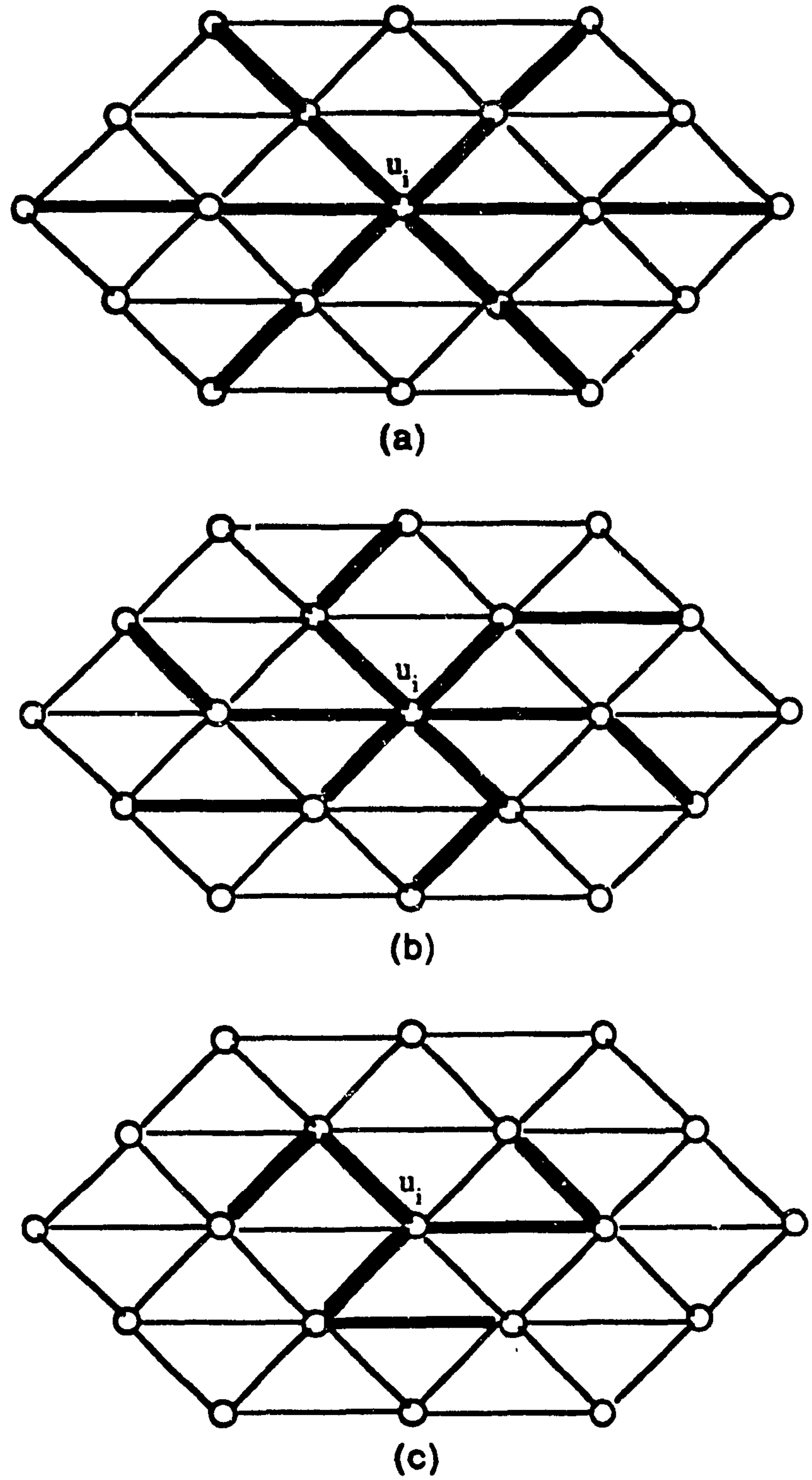


Fig. 5.2 Local neighborhoods around a unit in a hexagonal grid interconnection.

comparison outcomes in its local neighborhood.

Theorem 5.1 : Let u_i be a processor in a system S with t distinct paths of length 2 from u_i. Let $R_t(u_i)$ denote the set of processors which lie on these paths including processor u_i. If at most t processors are faulty in $R_t(u_i)$ then u_i is faulty if and only if

$$| F(u_i) \bigcap R_t(u_i) | > t$$

where $F(u_i)$ is the set of processors which have a 1-link with u_i or can be reached from u_i by a 1-link followed by a 0-link or by a 0-link followed by a 1-link.

Proof. Let $N_0(u_i)$ and $N_1(u_i)$ denote the set of processors which are incident to the processor u_i by a 0-link and a 1-link respectively. Then $F(u_i) = N_1(u_i) \bigcup N_1(N_0(u_i)) \bigcup N_0(N_1(u_i))$. We observe that if u_i is fault-free then $F(u_i) \bigcap R_t(u_i)$ denotes the set of processors which can immediately be declared as faulty. Hence if $| F(u_i) \bigcap R_t(u_i) | > t$ then u_i is faulty, for otherwise it contradicts the fact that at most t processors can be faulty in $R_t(u_i)$.

Now, suppose we assume u_i to be faulty. Consider a pair (u_{j1}, u_{j2}) in $R_t(u_i)$, with u_{j1} adjacent to u_i.

case 1 u_{j1} is fault-free and u_{j2} is faulty: Clearly u_{j1} is in $N_1(u_i)$ and hence belongs to $F(u_i) \bigcap R_t(u_i)$.

case 2. u_{j1} is fault-free and u_{j2} is fault-free: Clearly u_{j1} is in $N_1(u_i)$ and u_{j2} belongs to $N_0(N_1(u_i))$. Thus both u_{j1} and u_{j2} belong to $F(u_i) \bigcap R_t(u_i)$.

case 3. u_{j1} is faulty and u_{j2} is fault-free: In this case, since both u_i and u_{j1} are faulty, u_{j1} may belong to either $N_0(u_i)$ or $N_1(u_i)$. If u_{j1} is in $N_0(u_i)$ then u_{j2} is in $N_1(N_0(u_i))$, and hence also belongs to $F(u_i) \bigcap R_t(u_i)$. On the other hand, if u_{j1} is in $N_1(u_i)$, it itself rather than u_{j2} belongs to $F(u_i) \bigcap R_t(u_i)$.

Thus in all three cases above, we find that if u_i is faulty then for every fault-free processor in $R_t(u_i)$, there exists a corresponding processor in $F(u_i) \cap R_t(u_i)$. Also since at most t processors can be faulty in $R_t(u_i)$, there are at least $t+1$ fault-free processors. As a result if u_i is faulty then $| F(u_i) \cap R_t(u_i) | > t$. This completes the proof. //

The diagnosis result presented above permits correct diagnosis of a unit, as long as a local neighborhood $R_t(u_i)$ of order t can be defined around u_i and it contains at most t faulty units. Clearly, the value of t can be different for different units. Moreover, the neighborhood can be defined in a variety of ways. However, for regular interconnected structures it is convenient to predefine a local neighborhood of the same order around each unit in a uniform way so that an algorithm that works in a distributed manner can be implemented. If the local neighborhood around each unit can be constructed to have the same topology, then each unit can execute a copy of the same local diagnosis algorithm synchronously. Initially, each unit must execute the same job and transmit the result to each of its neighbors so that the results can be compared and the comparison outcomes generated. At this point the comparison outcomes are available in a distributed manner with each bit of the comparison outcome being available at two sites, namely the units involved in that comparison. In other words, each unit has t bits of comparison syndrome, corresponding to the comparison tests with its t neighbors. These are used to compute $N_0(u_i)$ and $N_1(u_i)$. In order to compute $N_0(N_1(u_i))$ and $N_0(N_1(u_i))$, each unit must receive some information from each of its neighbors. But depending on the pairing of the units in $R_t(u_i)$, only one bit of this information from each of the neighbors can be of interest since we are ultimately interested in computing only $F(u_i) \cap R_t(u_i)$. If the local neighborhood has the same topology at each unit the information to be transmitted by a unit to each of its neighbors can easily be evaluated, using a syndrome decoding function.

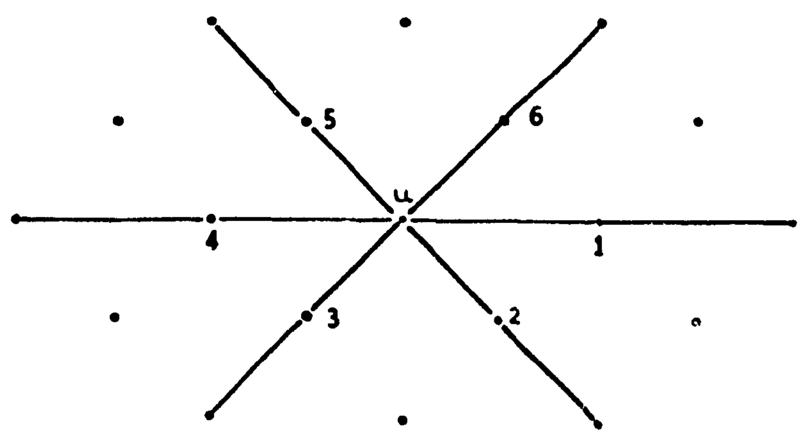
As an example, consider the hexagonal grid interconnection with a local neighborhood as shown in Fig. 5.3. Since the topology of the local neighborhood is the same, and each unit executes a copy of the same algorithm, let the unit under consideration be simply labeled u with the t neighbors of the unit labeled from 1 to t. It is clear that the central unit must route the test outcome corresponding to its neighbor 1 to neighbor 4, the test outcome corresponding to its neighbor 2 to neighbor 5, and so on, as illustrated in Fig. 5.3. In other words, the syndrome routing function f is given by $f(k) = (k+2) \; mod \; 6 + 1$. Clearly, a syndrome routing function will be one-to-one and onto on the set of neighbors and will be dependent on the the relative ordering of the adjacent units and the chosen local neighborhood.

In order to implement the algorithm in a distributed manner, we assume the existence of two t-bit registers A and B at each unit. $A[k]$ contains the comparison outcome available at that unit corresponding to the kth neighbor. In other words, $A[k] = 1$ if and only if k belongs to $N_1(u)$ at unit u. The register B will be used to receive information from the neighbors. Let $B[k]$ correspond to the comparison information sent by the kth neighbor. Now, observe that $A[k] \; XOR \; B[k] = 1$ if and only if the corresponding neighbor of u's kth neighbor belongs to the intersection of $N_0(N_1(u)) \bigcup N_1(N_0(u))$ and $R_t(u)$. Thus the computation of $|F(u) \bigcap R_t(u)|$ can be carried out very efficiently through simple logical operations at each unit. The algorithm performed by any unit can be presented formally as follows.

Algorithm 5.1: Local Diagnosis

begin

0. Perform the same job; send the result to all neighbors.



$$f(1) = 4\ , \quad f(2) = 5\ , \quad f(3) = 6\ ,$$

$$f(4) = 1\ , \quad f(5) = 2\ , \quad f(6) = 3\ .$$

Fig. 5.3 A local neighborhood of order 6 in a hexagonal grid and a syndrome routing function.

1. For all $1 \leq k \leq t$, $A[k] := 0$ if the result computed matches that of the kth neighbor; else $A[k] := 1$.

2. For all $1 \leq k \leq t$, send in parallel $A[k]$ to neighbor $f(k)$. For all $1 \leq k \leq t$, receive in parallel $B[k]$ from neighbor k.

3. For all $1 \leq k \leq t$, $B[k] := A[k]$ XOR $B[k]$

4. SUM := Number of 1's in register A + Number of 1's in register B.

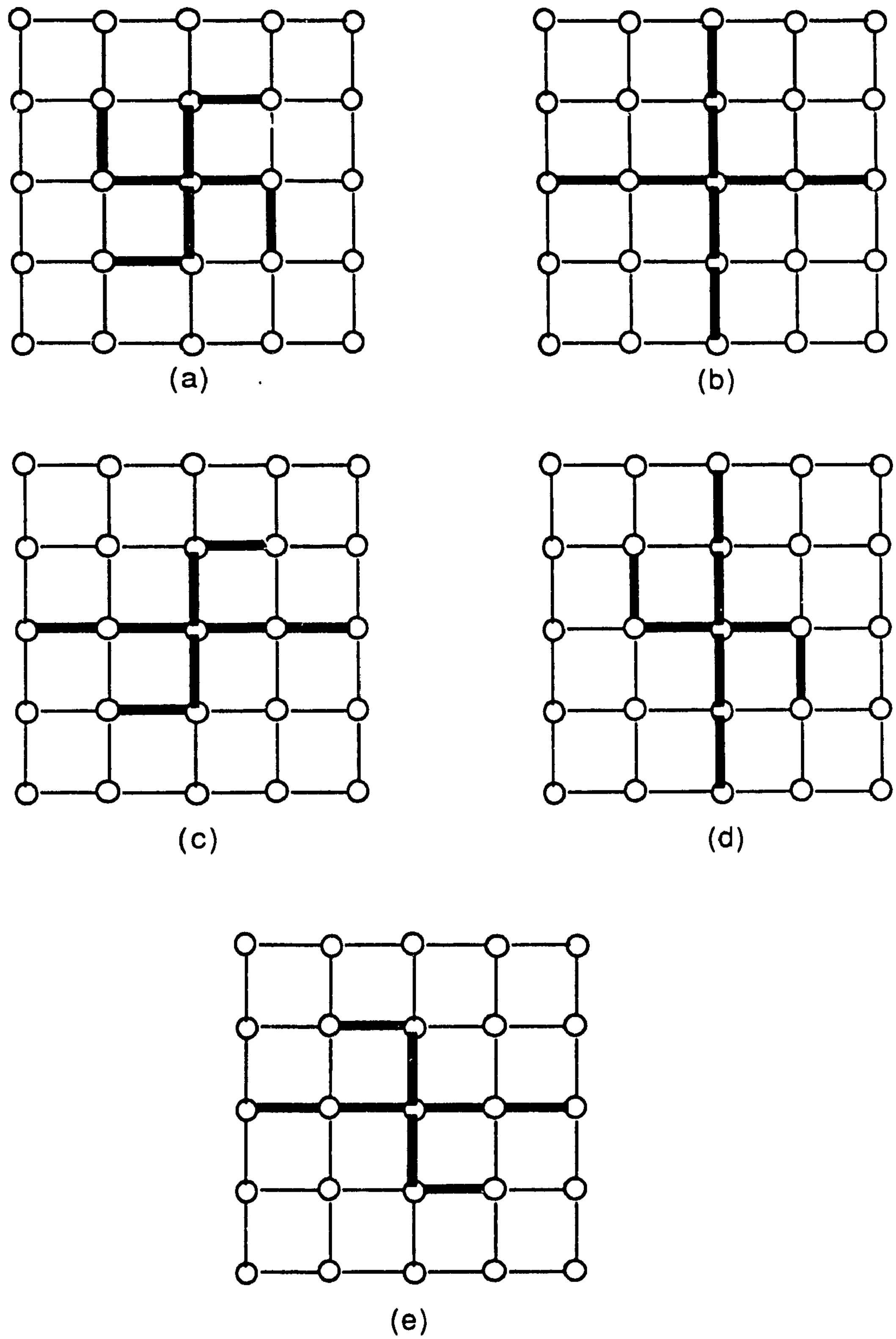
5. If $SUM > t$ then declare the unit faulty; else declare the unit fault-free.

end.

We observe that only Step 2 onwards constitutes the syndrome decoding part and each of these steps can be executed in constant time. Thus the syndrome decoding algorithm executes in constant parallel time.

We note that Fig. 5.3. just shows one possible local neighborhood of order six for a hexagonal grid structure. It is possible that a single predefined local neighborhood at a unit u may have more than t faulty units whereas another local neighborhood may not have more than t faulty units. Therefore for a given syndrome, Algorithm 5.1 executed for two different local neighborhoods at a unit u may evaluate the status of that unit differently. Thus, in order to improve the quality of diagnosis, Algorithm 5.1 may be modified to incorporate more than one local neighborhood at each unit.

As an example, we consider the rectangular grid structure to see how this modification helps. Fig. 5.4 shows five different local neighborhoods defined at unit u. It can be seen that two syndrome routing functions are sufficient for u to obtain information regarding any unit at distance two in a rectangular grid structure. Thus, the central unit u sends two bits of information to and receives



(a) (b) (c) (d) (e)

Fig. 5.4 Local neigborhoods of order 4 in a rectangular grid.

two bits of information from each neighboring unit. The information gathering process is independent of the number of local neighborhoods of order t being considered at each unit. The evaluation of the size of $F(u) \cap R_t(u)$ is then carried out for each of the specified local neighborhoods. An odd number of local neighborhoods is chosen at each unit so that majority voting can be applied. The unit u is evaluated to be faulty if and only if more than half the local neighborhoods have more than t units in $F(u) \cap R_t(u)$.

As stated in the introduction the development of the local diagnosis criterion has been motivated primarily from the viewpoint of its application to regular interconnection structures such as rectangular, hexagonal and octagonal grids with wraparound in two dimensional structures, binary n-cube connected cycles, and hypercube connections. In these architectures each processing element is connected to the same fixed number of other processing elements with perfect symmetry with respect to interconnections. Thus, it is easy to observe that each of these architectures permits the construction of local neighborhoods of order t around each unit, for t equal to the number of neighbors of any unit. The t-diagnosis algorithms permit the correct identification of all units in the system provided the total number of faulty units in the entire system does not exceed t.

For example, consider an $n * n$ rectangular grid with wraparound connections. Consider two versions of the local diagnosis algorithm, say A and B. Algorithm A uses a single local neighborhood as shown in Fig. 5.4(a); Algorithm B uses three local neighborhoods, namely the ones shown in Fig. 5.4(a), Fig. 5.4(b) and Fig. 5.4(c). There are $C_5^{n^2}$ fault patterns of size five that can occur in the system. Classical t-diagnosis considers fault patterns of size at most 4 only for this interconnection structure, and cannot correctly diagnose any of these 5-fault patterns.

In algorithm A, a local neighborhood corresponds to a 3 * 3 subsquare in a rectangular grid. Thus, Algorithm A, executed at each unit, results in the correct diagnosis of the entire system if the 5-fault pattern is not confined to a single 3 * 3 subsquare. The total number of fault patterns which may lead to incorrect diagnosis at a unit u is C_5^9 . Since the intersection of the local neighborhoods of two adjacent units can contain six different 5-fault patterns, the total number of 5-fault patterns which may lead to incorrect diagnosis is given by $n^2 * (C_5^9 - (6*4)/2) = 114n^2$.

A 5-fault pattern may lead to a wrong diagnosis for Algorithm B at a unit u if the fault pattern occurs in at least two of the three local neighborhoods considered at u. There are a total of 41 such fault patterns for each unit. None of these fault patterns can lead to an incorrect diagnosis at a neighboring unit, if n is greater than 6. Thus, in this case, the total number of 5-fault patterns which may result in incorrect diagnosis of the entire system is $41n^2$.

From the above analysis, we observe that for regular interconnected structures, the local diagnosis criterion developed in this paper permits the correct diagnosis of fault patterns which cannot be handled by classical t-diagnosability theory, even if only one neighborhood around each processor is considered. However, the question still remains as to whether or not a local neighborhood of order t can be constructed around every unit in an arbitrary t-diagnosable system so that the local diagnosis approach can be used. Unfortunately, this is not so. Fig. 5.5 presents a system whose connection assignment permits 3-diagnosability, but does not permit the construction of a local neighborhood of order 3 for the units u_i and u_j. On the other hand the converse of the above result is true.

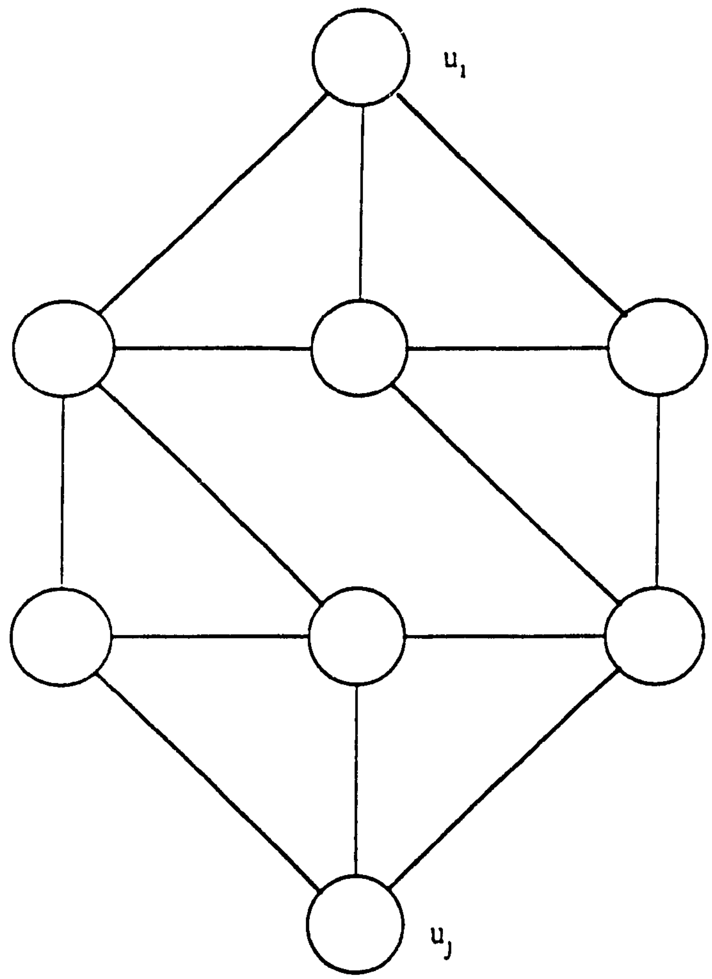


Fig. 5.5 A 3-diagnosable system which does not have local neighborhoods of order 3 for units u_i and u_j.

Theorem 5.2 : Let S be a system under the comparison model with a local neighborhood of order t defined around each unit. Then S is t-diagnosable.

Proof : If there are at most t faults in the system, then for every unit $u_i \in U$, there are at most t faults in $R_t(u_i)$. Thus, by Theorem 5.1, every unit $u_i \in U$ can be correctly identified. Hence S is t-diagnosable. //

5.2 Diagnosis of a Ring of Processors under Local Fault Constraints

In this section, we analyze the implication of imposing local fault constraints on a ring of processors. Specifically, we determine if, given a syndrome, we can uniquely determine the set of faulty processors as long as at most p out of any q consecutive processors are faulty.

Theorem 5.3 : Let S be a ring of n processors where n is even. Given that at most p processors are faulty out of any q consecutive processors, the values for p and q which admit the maximum numb of fault sets that can be uniquely diagnosed are p=2 and q=5.

Proof: Let $\{u_1, u_2, \ldots, u_n\}$ be the ring of processors.

case 1: $3 \leq$ p $\leq$ n .

We show that in this case, the set of permissible fault sets cannot be uniquely diagnosed. Consider the following syndrome: $a_{n1}=1$, $a_{12}=1$, $a_{23}=1$, $a_{34}=1$ and all other outcomes have value 0. Both $F_1= \{u_1, u_2, u_3\}$ and $F_2= \{u_1, u_3\}$ are allowable fault sets for this syndrome.

case 2: p=2, q $\geq$ 3 .

case 2.1: p=2, q=3 .

Consider the following syndrome: $a_{n1}=1$, $a_{12}=1$, $a_{23}=1$, $a_{34}=1$, $a_{45}=1$ and all other outcomes are 0. The fault sets $F_1= \{u_1, u_3, u_4\}$ and $F_2= \{u_1, u_2, u_4\}$.two permissible fault sets under the given fault constraint, are allowable faults for this syndrome.

case 2.2: p=2, q=4 .

Since n is even, let F be a fault set containing alternate processors in S. Then the fault sets F and F^c are allowable fault sets for the syndrome in which all outcomes are 1.

case 2.3: p=2, q $\geq$ 5 .

We note that if there are at most 2 faulty processors in any consecutive 5 processors then for any processor u , $L_2(u) \bigcup \{u\}$ which consists of 5 processors contains at most 2 faulty processors. Thus, given a permissible syndrome, the local diagnosis algorithm (Algorithm 5.1) developed in the previous section can be used to identify all processors correctly. Since the constraint p=2 and q=5 permits all fault sets which are valid when p=2 and q $\geq$ 5, these values for p and q admit the maximum number of fault sets which can be uniquely diagnosed. //

If a fault constraint permits a fault set F and its complement F^c to be permissible fault sets, then given a valid syndrome, the faulty processors may not be correctly identified; the fault sets F and F^c generate a common syndrome. We note that if initially one processor v is correctly determined to be fault-free or less than half the total number of processors in the system are faulty then for any subset F of U, at most one of the subsets F and F^c can be an allowable fault set for a given syndrome.

Theorem 5.4 : Let S be a ring of n processors in which one of the following conditions is satisfied:

(1) some processor is known to be fault-free and $n \geq 5$

(2) less than half processors in the system are faulty and $n \geq 7$

(3) n is odd.

Then the values for p and q which permit the maximum number of fault sets which can be uniquely diagnosed in S under the local constraint of at most p faulty processors in any q consecutive processors are $p = 2$ and $q = 4$ respectively.

Proof: It can be verified as in the proof of Theorem 5.3 that the case $p \geq 3$ and the case $p = 2$ and $q = 3$ may result in the syndromes which cannot be uniquely diagnosed.

Assume $p = 2$ and $q = 4$. We first show that if one fault-free processor v is known to be fault-free, all other processors can be identified correctly. We assume that the diagnosis procedure initiated at v proceeds clockwise. If a processor is fault-free then the adjacent processor can be correctly identified. If two consecutive processors are identified as faulty then the next processor can be correctly identified as fault-free. Thus only the situation shown in Fig. 5.6 could pose a problem.

Since there are at most 2 faulty processors in any 4 consecutive processors, there are at most 2 faulty processors in A. Hence B contains at most 2 faulty processors. The processor w has at most 2 faulty processors in its local neighborhood $L_2(w) \bigcup \{w\}$. Thus Algorithm 5.1 can be carried out with respect to processor w to determine its status. Thus if one processor is given to be fault-free or can be identified correctly to be fault-free then all other processors can be identified correctly.

We now show how one processor can be identified correctly if either (2) or (3) is true. We note that if a valid syndrome contains the sequence of consecutive outcomes 00, 011 or 110 then the processors adjacent to the 0-links are fault-free; for otherwise there is a sequence of 4 consecutive processors of which at least three are faulty.

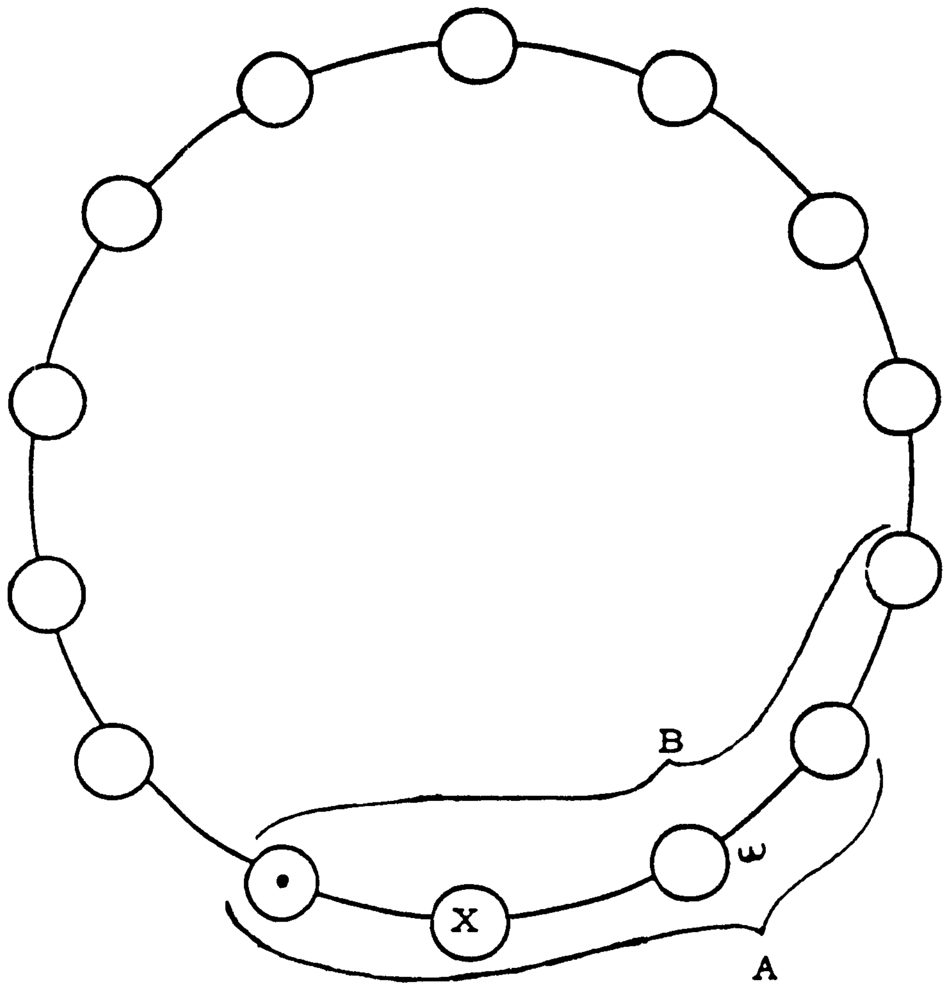


Fig. 5.6 Illustration for Local Diagnosis Algorithm
in a ring of processors.

We now claim that for a valid syndrome one of the following sequences of outcomes 00, 011 or 110 occurs. Assume the contrary. Then the following syndromes are the only syndromes which do not contain any of the sequences 00, 011 or 110: the syndrome s_1 in which all outcomes have value 1 and the syndrome s_2 in which 0 and 1 outcomes alternate.

case 1: Less than half the processors are faulty and $n \geq 7$.

In this case, S contains two consecutive fault-free processors. Hence there exists at least one 0-link and the syndrome s_1 cannot occur. Since at most 2 of any 4 consecutive processors can be faulty, the syndrome s_2 corresponds to fault sets in which two faulty processors are followed by two fault-free processors and vice versa. But this contradicts the assumption that the number of faulty processors is less than the number of fault-free processors.

case 2: n is odd.

In this case, the syndrome s_2 cannot be present. Since at most 2 out of any 4 consecutive processors can be faulty, the syndrome s_1 corresponds to fault sets in which faulty and fault-free processors alternate; this is not possible since n is odd and S is a ring of processors.

This shows that one fault-free processor can be determined if either (2) or (3) is true. //

We observe that the diagnosis algorithms outlined in the proof of the above two theorems can be designed to run sequentially on a host processor or in a distributed manner on the ring of processors.

5.3. Summary

In this chapter, we have studied the problem of fault diagnosis of multiprocessor systems under local constraints. Assuming there are no more than t

faulty processors in an appropriately constructed local neighborhood around each processor, we show how to determine the faulty or fault-free nature of each processor using syndrome information available in the local neighborhood. The algorithm can be implemented on $O(1)$ parallel time. We have also presented an analysis of fault coverage for an $n * n$ rectangular grid. It has been shown that all but $114n^2$ of the possible 5-fault patterns can be diagnosed correctly. This is in contrast to the t-diagnosis algorithms which do not permit more than 4 faults in the entire $n * n$ grid. We have shown that using more than one neighborhood around each processor and using a form of majority voting, the quality of diagnosis can be improved. In fact, if we use three neighborhoods around each processor, then in the case of the $n * n$-rectangular grid no more than $41n^2$ of all possible 5-fault patterns will result in incorrect diagnosis.

We have also considered the application of Algorithm Local Diagnosis in the analysis of a ring of processors when, instead of a single global fault constraint, local fault constraints are imposed. Specifically, we determine if, given a syndrome, the set of faulty processors can be uniquely determined as long as at most p out of any q consecutive processors are faulty.

CHAPTER VI

t-in-L_1-DIAGNOSABILITY AND DIAGNOSIS

In this chapter we continue our study of the problem of diagnosing faulty processors in a multiprocessor system imposing local constraints on the number of faulty processors in the neighborhood of each processor. Specifically, we introduce in the following section the concept of t-in-L_k-diagnosability and present several basic results which lead to a sufficient condition for unique diagnosis when certain local fault constraints are satisfied. In section 6.2 we study t-in-L_k-diagnosability of certain regular interconnected systems: the closed rectangular, hexagonal and octagonal grid systems and the hypercube systems. In section 6.3 we present t-in-L_1-diagnosis algorithms for these regular systems as well as those which satisfy certain conditions. These diagnosis algorithms can be implemented in a distributed manner on the system itself. As in the previous chapter our study in this chapter is based on the comparison model (See section 2.3).

6.1. t-in-L_1-Diagnosable Systems

Given a multiprocessor system with test interconnection $G=(U,E)$, a processor u_j is said to belong to a **local domain** $L_k(u_i)$ if u_j lies within a distance k of u_i. A system S is defined to be t-in-L_k-diagnosable if, given a syndrome, all faulty processors can be correctly identified provided that there are at most t faulty processors in $L_k(u_i) \bigcup \{u_i\}$ for every processor u_i in S. In this section we study t-in-L_1-diagnosability. Recall that the distance between any two processors x and y is denoted by $d(x,y)$.

Lemma 6.1 : Given a system S and a syndrome, let F_1 and F_2 be two distinct allowable fault sets for the given syndrome such that $F_1 \cup F_2 \neq U$ and for all processors $u \in U$, $L_1(u)$ F_1 and $L_1(u)-F_2$ are both non-empty. Then there exist processors $x, y \in U$ such that

(1) $x \in U - (F_1 \cup F_2)$

(2) $y \in F_1 \oplus F_2$

(3) $2 \leq d(x,y) \leq 3$ and $d(x,y)$ is minimum among all x and y satisfying (1) and (2).

Proof : Since $F_1 \cup F_2 \neq U$ the set $U - (F_1 \cup F_2)$ is non-empty. Furthermore, since F_1 and F_2 are distinct, there exists at least one processor which belongs to one fault set and is not contained in the other. Thus there exist processors in U satisfying conditions (1) and (2). Now let x and y be processors in U satisfying conditions (1) and (2), respectively such that the distance $d(x,y)$ is minimum.

Assume $d(x,y) \geq 4$. Consider a processor w such that w is at a distance of at most $\lceil d(x,y)/2 \rceil$ from both x and y. Since $L_1(w)-F_1$ and $L_1(w)-F_2$ are both non-empty, there exists a processor $z \in L_1(w)$ satisfying condition (1) or (2). If z satisfies condition (1), then $d(z,y) \leq d(w,y) + 1 < d(x,y)$; if z satisfies condition (2), then $d(x,z) \leq d(x,w) + 1 < d(x,y)$. In either case, the minimality of $d(x,y)$ is contradicted. Hence $d(x,y) \leq 3$.

To prove that $d(x,y) \geq 2$, we show that the assumption $d(x,y) = 1$ leads to a contradiction. Assume $d(x,y) = 1$. Then the edge between x and y is a 0-link with respect to one fault set and a 1-link with respect to the other, contradicting the assumption that F_1 and F_2 share a common syndrome. //

Lemma 6.2 : Let S be a system with test interconnection graph $G=(U,E)$ in which for every processor $u \in U$, there are at most $\lfloor k/2 \rfloor + 1$ faulty processors in $L_1(u) \bigcup \{u\}$ where $k = |L_1(u)|$. Given a syndrome and two allowable fault sets F_1 and F_2 corresponding to this syndrome, the following conditions hold for every $x \in F_1 \oplus F_2$:

(1) there is at most one processor $w \in L_1(x)$ with $w \in F_1 \cap F_2$.

(2) there are at least $k-1$ processors $v \in L_1(x)$ with $v \in F_1 \oplus F_2$.

Proof : Without loss of generality assume $x \in F_1 - F_2$. Then x is faulty with respect to F_1 and fault-free with respect to F_2. Let X denote the subset of processors in $L_1(x)$ which are fault-free with respect to F_2. The processors in X are all faulty with respect to F_1 since they have 0-links with x and x is faulty with respect to F_1. $|X| \leq \lfloor k/2 \rfloor$ since there are at most $\lfloor k/2 \rfloor +1$ faulty processors in $L_1(x) \bigcup \{x\}$. Furthermore $|X| \geq \lceil k/2 \rceil -1$, since the processors in $L_1(x)-X$ are all faulty with respect to F_2 and there are at most $\lfloor k/2 \rfloor$ faulty processors in $L_1(x) \bigcup \{x\}$. Thus

$$\lceil k/2 \rceil -1 \leq |X| \leq \lfloor k/2 \rfloor$$

Now consider the processors in $L_1(x)-X$. They are all faulty with respect to F_2. Now, if more than one processor in $L_1(x)-X$ is also faulty with respect to F_1 then the number of faulty processors in $L_1(x) \bigcup \{x\}$ with respect to F_1 is greater than $\lfloor k/2 \rfloor + 1$, contradicting our assumption that F_1 is a permissible fault set. This shows that (1) is true.

Since all processors in X are contained in $F_1 - F_2$ and all processors except at most one in $L_1(x)-X$ belong to $F_2 - F_1$, there are at least $|L_1(x)| - 1$ processors in $L_1(x)$ which also belong to $F_1 \oplus F_2$. Since $|L_1(x)| = k$, it follows that (2) holds. //

Lemma 6.3 : Let S be a system with test interconnection graph $G=(U,E)$ in which the number of faulty processors is less than $|U|/2$ and for every processor $u \in U$, there are at most $\lfloor k/2 \rfloor + 1$ faulty processors in $L_1(u) \bigcup \{u\}$ where $k = |L_1(u)|$. Given a syndrome and two allowable fault sets F_1 and F_2 corresponding to this syndrome, there exist two processors $x, y \in U$ such that

(1) $x \in U - (F_1 \cup F_2)$

(2) $y \in F_1 \oplus F_2$

(3) $2 \leq d(x,y) \leq 3$ and $d(x,y)$ is minimum among all x and y satisfying (1) and (2).

(4) If w lies on any shortest path between x and y and $d(w,y)=2$ then

there is exactly one path of length 2 between w and y.

Proof : Since the system S satisfies the conditions of Lemma 6.1, it follows that there exist x and y satisfying (1), (2) and (3). Now assume (4) is not true. Then there exists a processor w lying on a shortest path between x and y with $d(w,y)=2$ and there are two or more paths of length 2 between w and y. We note that w could be the processor x itself. We also observe that the system S, the fault sets F_1 and F_2, and the processor y satisfy the conditions of Lemma 6.2. Hence there exists at most one processor in $L_1(y)$ which belongs to $F_1 \cap F_2$ and all other processors belong to $F_1 \oplus F_2$. Since there are two or more paths of length 2 between w and y, there is at least one processor in $L_1(y)$ belonging to $F_1 \oplus F_2$ which is closer to x than y. If $w=x$, this will contradict (3); if $x \neq w$, this will contradict the minimality of $d(x,y)$. //

Note: For every pair of processors x and y satisfying conditions (1)-(3) of the above lemma, condition (4) holds for every shortest path between x and y.

The following theorem follows directly from Lemma 6.3.

Theorem 6.1 : Let S be a system with test interconnection graph $G=(U,E)$ in which the number of faulty processors is less than $|U|/2$ and for every processor $u_i \in U$ there are at most $\lfloor k_i/2 \rfloor + 1$ faulty processors in $L_1(u_i) \bigcup \{u_i\}$ where $k_i = L_1(u_i)$. The system S is uniquely diagnosable if between any two processors at distance 2 from each other, there are at least two disjoint paths of length 2. //

A straight forward consequence of the above theorem is given below.

Corollary 6.1.1 : Let S be a system with test interconnection graph $G=(U,E)$ in which the number of faulty processors is less than $|U|/2$. If between any two processors u_i and u_j at distance 2 from each other in G, there are at least two paths of distance 2, then S is t-in-L_1 diagnosable for $t = \lfloor \delta/2 \rfloor + 1$, where δ is the minimum degree of G. //

As we will see in the following section, the condition given in the above corollary for t-in-L_1 diagnosability with $t = \lfloor \delta/2 \rfloor + 1$ is not in general necessary.

6.2. t-in-L_1-Diagnosability of Regular Interconnected Systems

In this section we study the t-in-L_1 diagnosability of certain regular interconnected systems - the closed rectangular, hexagonal and octagonal grid systems and the hypercube systems. First we consider the hypercube systems.

Theorem 6.2 : Let S be a hypercube system containing 2^k processors. The system S is t-in-L_1 diagnosable for $t = \lfloor k/2 \rfloor + 1$ provided less than half the total number of processors in S are faulty.

Proof : The above result follows immediately from Corollary 6.1.1 and the observation that in a hypercube system there are two disjoint paths of length 2 between any two processors at distance 2 from each other. //

The condition of Theorem 6.1 is not satisfied by the other regular interconnected systems to be considered in this section. We now proceed to determine the maximum value of t for which these systems are t-in-L_1 diagnosable under the assumption that less than half the total number of processors in these systems are faulty. Interestingly, we will see that the value of t is equal to $\lfloor \delta/2 \rfloor + 1$ in these cases too.

Theorem 6.3 : The maximum value of t which permits a closed rectangular grid S to be t-in-L_1 diagnosable given that less than half the processors in S are faulty, is 3.

Proof : The theorem is proved by contradiction. Assume there exist two permissible fault sets F_1 and F_2 sharing a common syndrome s, such that there are at most 3 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S and F_1 and F_2 contain less than half the total number of processors in the system. Since $|L_1(u)| = 4$ for every processor u, the system S and the two fault sets F_1 and F_2 satisfy the requirements of Lemma 6.3. Thus there exist processors x and y satisfying the conditions (1),(2), (3) and (4) of this lemma. Assuming conditions (1),(2) and (4) are satisfied by x and y we arrive at a contradiction by showing that (3) is violated.

We observe that the status of all processors in $L_1(x)$ remain unchanged with respect to both F_1 and F_2 since x is fault-free in the presence of either fault set. This means that all processors which share a 1-link with x belong to $F_1 \bigcap F_2$. We also note that there cannot be a path of fault-free processors between x and y with respect to either fault set; otherwise F_1 and F_2 cannot share a common syndrome.

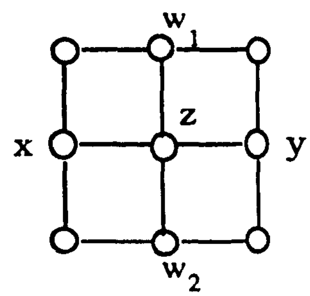
case 1 $d(x, y) = 2$.

Consider Fig. 6.1(a). All other cases with $d(x, y) = 2$ satisfying (1),(2) and (4) are symmetric to this case.

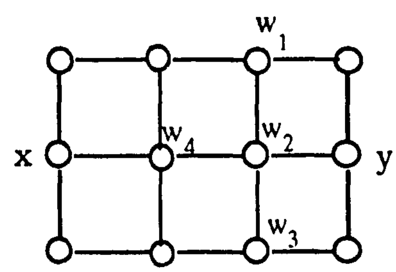
The processor z must be faulty with respect to both fault sets, otherwise there is a path of fault-free processors between x and y. Now the following subcases arise.

case 1.1 w_1 or w_2 belongs to $F_1 \oplus F_2$.

In this case, from Lemma 6.2, x is adjacent to a processor belonging to



(a)



(b)

Fig. 6.1 Illustrations for proof of Theorem 6.3

$F_1 \oplus F_2$; this contradicts the observation that all processors in $L_1(x)$ belong to $F_1 \cap F_2$ or $(F_1 \cup F_2)^c$.

case 1 2. w_1 and w_2 belong to $F_1 \cap F_2$.

If both w_1 and w_2 are faulty with respect to F_1 and F_2, then since y is faulty in the presence of one of these fault sets, $L_1(z) \cup \{z\}$ contains more than 3 faulty processors with respect to F_1 or F_2; a contradiction.

Note that if w_1 or w_2 is in $(F_1 \cup F_2)^c$, then there would be two paths of length 2 between $y \in F_1 \oplus F_2$ and w_1 (or w_2) $\in (F_1 \cup F_2)^c$, contradicting condition (4) of Lemma 6.3.

case 2: $d(x,y) = 3$.

We consider Fig. 6.1(b). All other cases with $d(x,y)=3$ satisfying (1), (2) and (4) are symmetric to to this case.

The processors w_2 and w_4 belong to $F_1 \cap F_2$; otherwise the minimality of the distance $d(x,y)$ is violated. Similarly w_1 and w_3 cannot be fault-free with respect to both F_1 and F_2 and so they both belong to $F_1 \cup F_2$. Since w_1 and w_3 are at distance 3 from x and both have two disjoint paths of length 2 to w_4, by Lemma 6.3, they cannot belong to $F_1 \oplus F_2$. Thus w_1 and w_3 are in $F_1 \cap F_2$. But then $L_1(w_2) \cup \{w_2\}$ will have more than 3 faulty processors with respect to either F_1 or F_2.

It follows that the system S is 3-in-L_1 diagnosable given that less than half the processors in S are faulty. Fig. 6.2 shows a closed rectangular grid in which two fault sets having at most 4 faulty processors in $L_1(u) \cup \{u\}$ for every processor u, share a common syndrome. In this syndrome, the comparison outcome between faulty processors is 1. This proves that the maximum value of t is 3. //

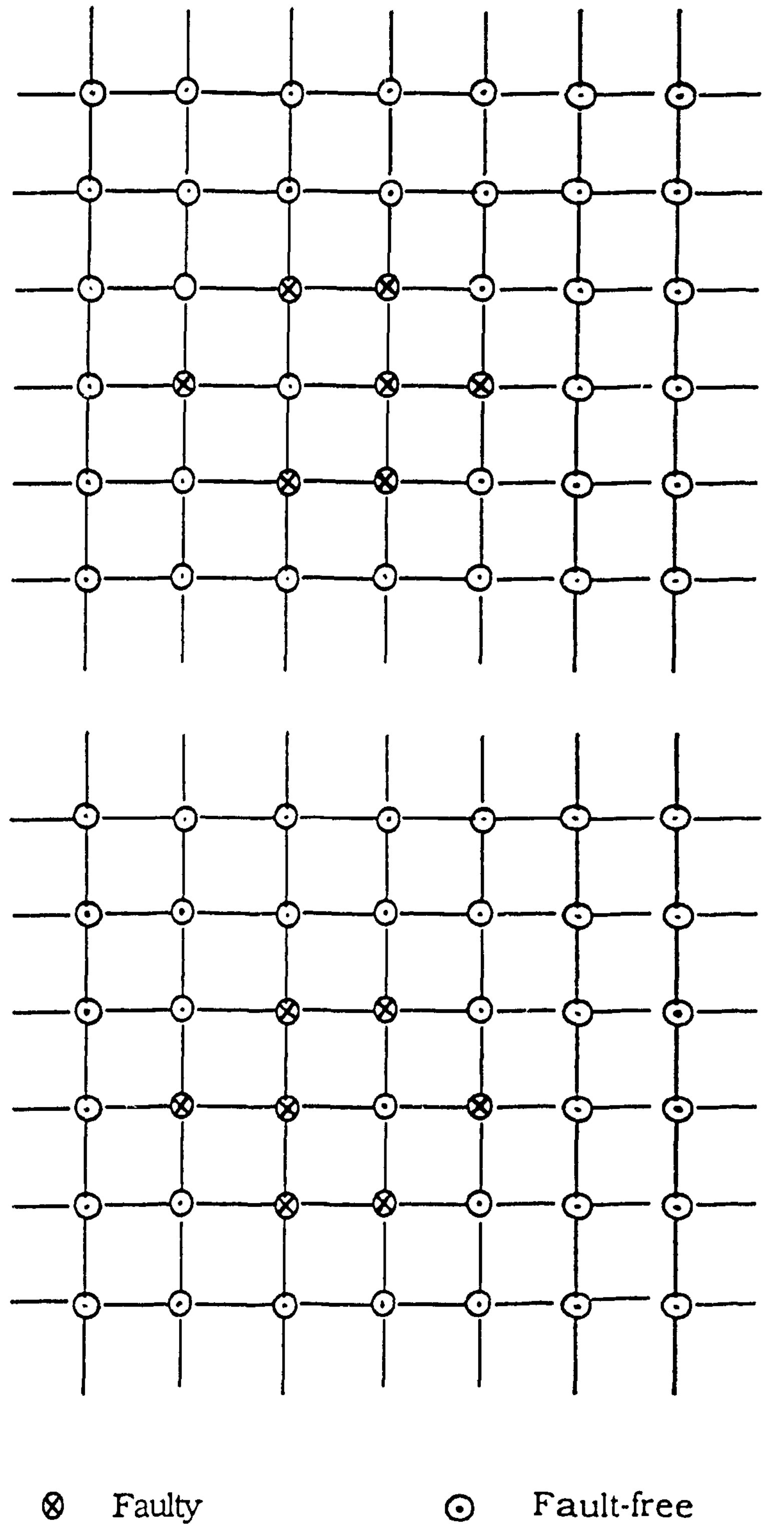


Fig 6.2 Two fault sets generating a common syndrome in a rectangular grid.

Theorem 6.4 : The maximum value of t which permits a closed hexagonal grid S to be t-in-L_1 diagnosable given that less than half the processors in S are faulty, is 4.

Proof : We prove by contradiction that S is 4-in-L_1 diagnosable. Assume the contrary. Then there exist two permissible fault sets F_1 and F_2 with at most 4 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S such that F_1 and F_2 share a common syndrome s. Then there exist two processors x and y satisfying the conditions of Lemma 6.3. Assuming conditions (1), (2), and (4) are true, we arrive at a contradiction by showing that condition (3) is violated.

case 1. $d(x,y) = 2$.

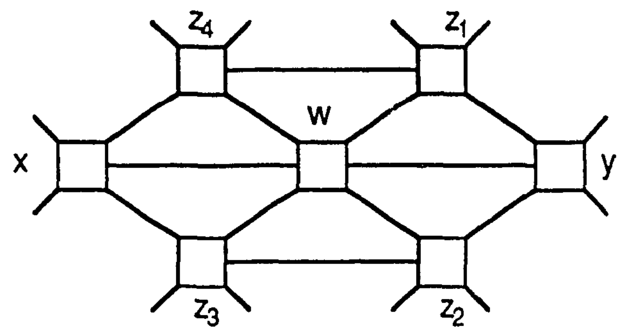
Consider Fig. 6.3(a). All other instances of x and y such that $d(x,y) = 2$ and satisfying conditions (1),(2) and (4) of Lemma 6.3 are symmetric to this case.

Since $y \in F_1 \oplus F_2$ and $w \in F_1 \bigcap F_2$, from Lemma 6.2, z_1 and z_2 belong to $F_1 \oplus F_2$. This in turn implies that z_3 and z_4 belong to $F_1 \oplus F_2$. But this contradicts the fact that both z_3 and z_4 belong to $F_1 \bigcap F_2$ or $(F_1 \bigcup F_2)^c$ as they both belong to $L_1(x)$.

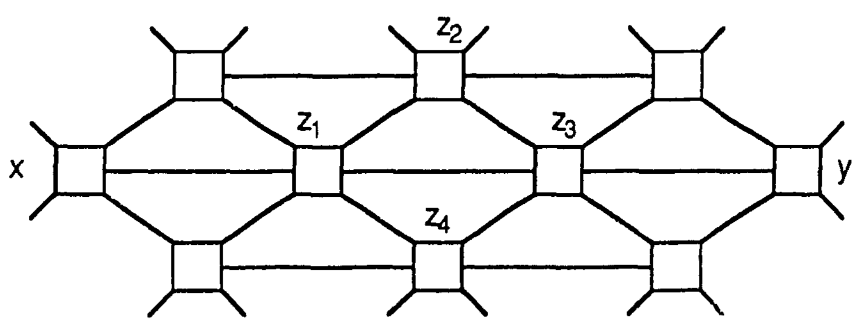
case 2. $d(x,y) = 3$

Consider Fig. 6.3(b). All other instances for x and y with $d(x,y) = 3$ and satisfying conditions (1),(2) and (4) are symmetric to this case. The processors z_1, z_2, z_3 and z_4 are all faulty with respect to both fault sets, otherwise $d(x,y)$ is not minimum. Since y is faulty in the presence of F_1 or F_2, $L_1(z_3)$ contains more than four faulty processors with respect to either F_1 or F_2, a contradiction. Thus $d(x,y) \neq 3$.

Thus the system S is 4-in-L_1 diagnosable given that less than half the processors in S are faulty. Fig. 6.4 gives an example of two fault sets in a hexagonal grid, having at most 5 faulty processors in $L_1(u)$ for every processor u such that



(a)



(b)

Fig. 6.3 Illustrations for Theorem 6.4.

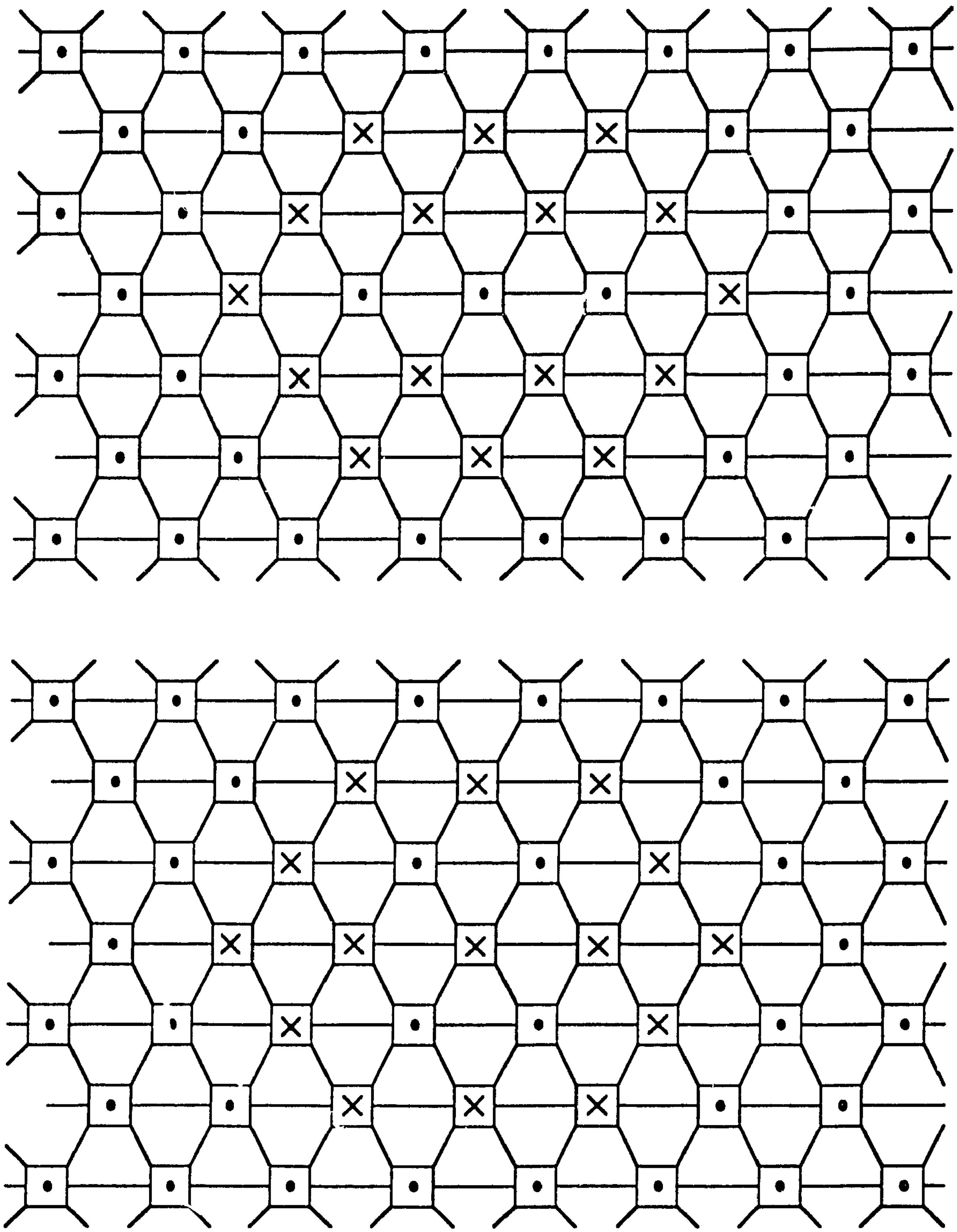


Fig. 6.4 Two fault sets generating a common syndrome in a hexagonal grid.

they both share a common syndrome. In this syndrome the comparison outcome between two faulty processors is 1. //

Theorem 6.5 : The maximum value of t which permits a closed octagonal grid S to be t-in-L_1 diagnosable given that less than half the processors in S are faulty, is 5.

Proof : We show that S is 5-in-L_1 diagnosable given that less than half the processors in S are faulty. Assume the contrary. Then there exist two permissible fault sets F_1 and F_2, with at most 5 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S, such that they share a common syndrome s. Thus there exist two processors x and y satisfying the conditions of Lemma 6.3. To arrive at a contradiction we show that condition (3) of Lemma 6.3 is violated if we assume that x and y satisfy conditions (1),(2) and (4).

case 1: $d(x,y) = 2$.

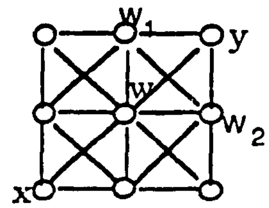
Consider Fig. 6.5(a). All other instances of x and y with $d(x,y)=2$ and satisfying conditions (1),(2) and (4) of Lemma 6.3 are symmetric to this case.

The processor w is faulty with respect to both F_1 and F_2. Otherwise $d(x,y)$ is not minimum. Since $y \in F_1 \oplus F_2$, from Lemma 6.2, both w_1 and w_2 belong to $F_1 \oplus F_2$. There are two disjoint paths of length 2 between w_1 and x, a contradiction of 4.

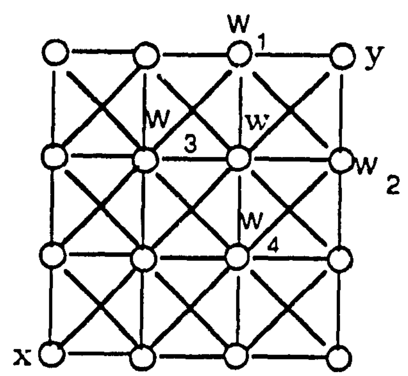
case 2: $d(x,y) = 3$.

Consider Fig. 6.5(b). All other instances of x and y with $d(x,y)=3$ and satisfying conditions (1),(2) and (4) of Lemma 6.3 are symmetric to this case.

The processor w is faulty with respect to both F_1 and F_2. Since $y \in F_1 \oplus F_2$, and $w \in F_1 \bigcap F_2$, by Lemma 6.2, both w_1 and w_2 belong to $F_1 \oplus F_2$. For the same reason, both w_3 and w_4 belong to $F_1 \oplus F_2$. This means that there is a processor (w_3 and w_4) in $F_1 \oplus F_2$ at a distance 2 from x, contradicting the



(a)



(b)

Fig. 6.5 Illustrations for Theorem 6.5

minimality of $d(x,y)$.

It follows that S is 5-in-L_1 diagnosable provided that less than half the processors in S are faulty. Fig. 6.6 gives an example of two permissible fault sets in an octagonal grid, with at most 6 faulty processors in $L_1(u) \bigcup \{u\}$ for every u, which share a common syndrome. In this syndrome the comparison outcome between two faulty processors is always 1. //

6.3. t-in-L_1 Diagnosis

In this section we present t-in-L_1 diagnosis algorithms. To begin with, we assume that one processor is known to be fault-free and show how diagnosis can be done in the cases of all the regular interconnected systems studied in the previous section as well as when the system satisfies the conditions of Theorem 6.1. We then present a unified version of these algorithms. We next show how this basic algorithm can be used to determine a fault-free processor in linear time in all the cases considered. Finally we present the complete diagnosis algorithm.

We need the following lemma which forms the basis of all the diagnosis algorithms of this section.

Lemma 6.4 : Given a system S and a syndrome, let u be a processor in S such that $|L_1(u)| = k$, $L_1(u) \bigcup \{u\}$ has at most $\lfloor k/2 \rfloor +1$ faulty processors, and at least 2 processors in $L_1(u)$ have been correctly identified. Then u can be correctly identified.

Proof : Let X denote the set of processors in $L_1(u)$ which have been correctly identified. If any member of X is fault-free then the status of u can be determined correctly. We now consider the case when all processors in X have been identified to be faulty. Let X_0 and X_1 represent the set of processors in $L_1(u) - X$ which have 0-links and 1-links respectively with u. If

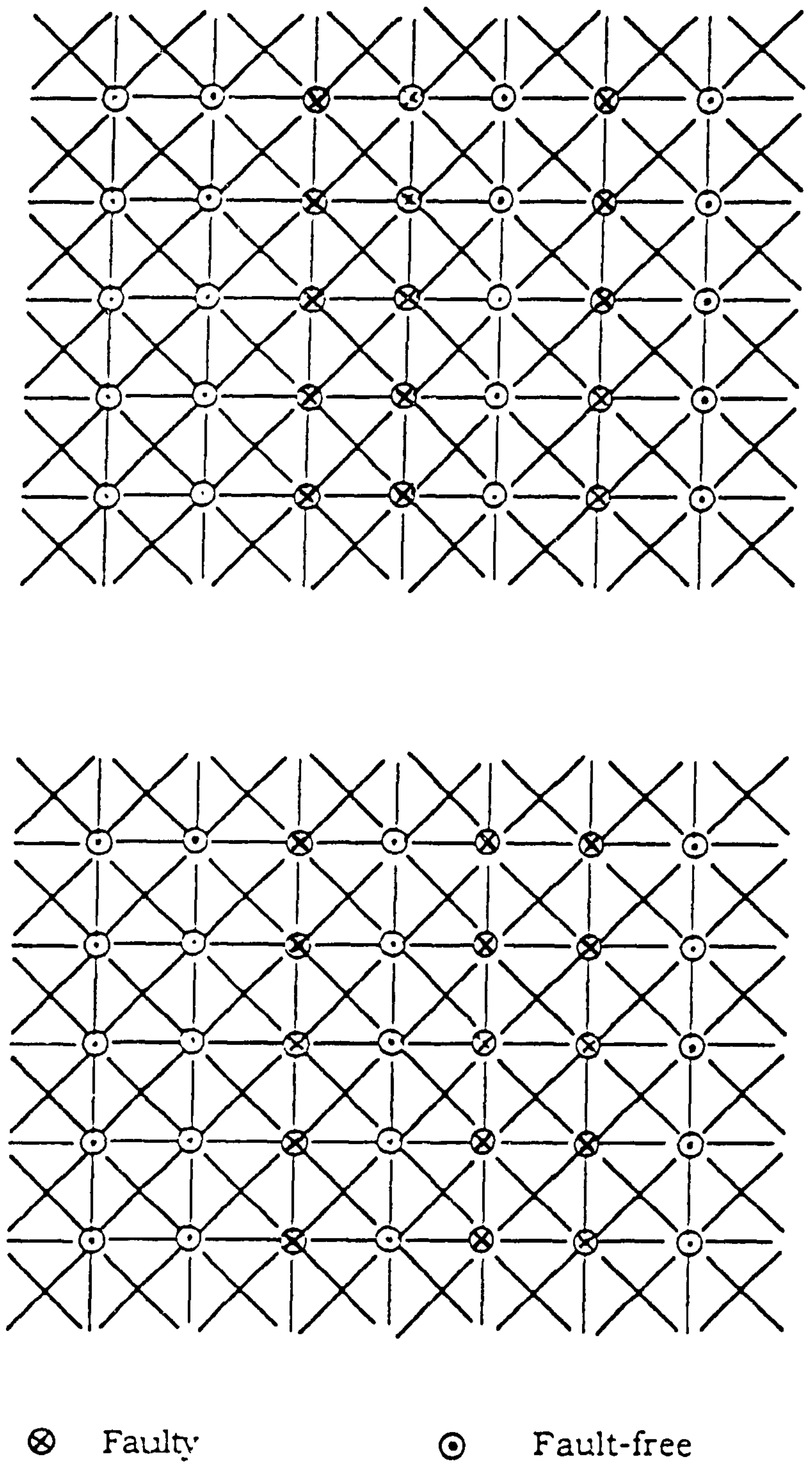


Fig. 6.6 Two fault sets wich generate a common syndrome in an octagonal grid.

$$|X| + |X_1| > \lfloor k/2 \rfloor + 1 \qquad (6.1)$$

then u can be declared faulty; u can be declared fault-free if

$$|X| + |X_0| + 1 > \lfloor k/2 \rfloor + 1 \qquad (6.2)$$

Both (6.1) and (6.2) cannot be satisfied simultaneously; for otherwise the assumption that there are at most $\lfloor k/2 \rfloor + 1$ faults in $L_1(u) \bigcup \{u\}$ is violated or the processors in F have been identified incorrectly. At least one of the conditions (6.1) and (6.2) is satisfied if we ensure that

$$|X| + \max\{X_0 + 1, X_1\} > \lfloor k/2 \rfloor + 1 \qquad (6.3)$$

Since $X_0+X_1=k-|X|$, $\max\{X_0 + 1, X_1\} \geq \lfloor (k - |F|)/2 \rfloor + 1$. But

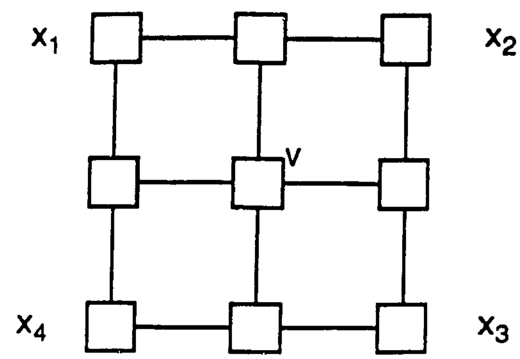
$$|X| + \lfloor (k - |X|)/2 \rfloor + 1 \geq \lfloor k/2 \rfloor + 2$$

if $|X| \geq 2$. Hence condition (6.3) is satisfied if $|X| \geq 2$. //

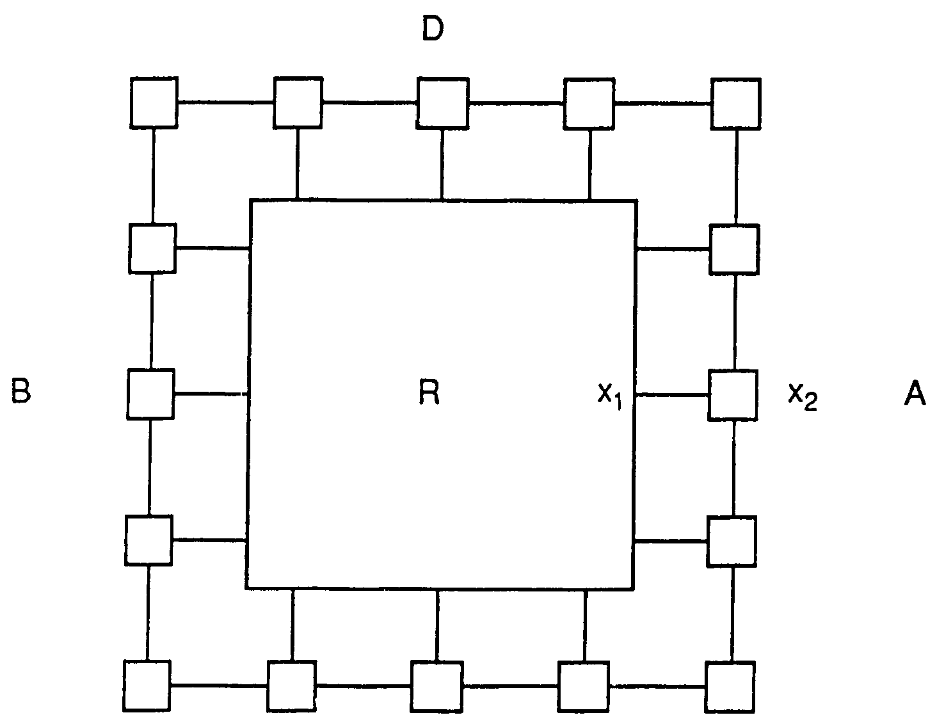
Theorem 6.6 : Let S be a closed rectangular grid in which there are at most 3 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S. Given a syndrome and a fault-free processor v in S, all processors in S can be correctly identified.

Proof : Given a fault-free processor v, all processors in $L_1(v)$ can be correctly identified. Consider the processors x_1, x_2, x_3, and x_4 in Fig. 6.7(a). Since each of these is adjacent to 2 processors in $L_1(u) \bigcup \{u\}$ and there are at most 3 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S, the processors x_1, x_2, x_3, and x_4 can be correctly identified. Now assuming all processors within a rectangle with sides containing at least 3 processors have been correctly identified, we show that the processors in the enclosing rectangle can be correctly identified.

Let R be the rectangle in which all processors have been correctly identified (See Fig. 6.7(b)). Let x_1 be a processor in R which is adjacent to the unidentified processor x_2 on A such that x_1 does not lie on a corner of R. If x_1



(a)



(b)

Fig. 6.7. Propagation of fault-identification in a rectangular grid.

or one of its adjacent processors which lie on the same column as x_1 is fault-free then one of the processors on A can be identified correctly. If all three are faulty then x_2 can be identified as fault-free since there are already three faulty processors in $L_1(x_1)$. Once one processor on A has been correctly identified, the status of an adjacent processor on A can be determined since two of its neighboring processors have been correctly identified: one of the identified neighboring processors is in R and the other lies on A. Applying this technique repeatedly, all processors lying on A can be identified. The processors lying on B, C, and D can be identified similarly. The status of the processors lying on the corners of the enclosing rectangle can be determined correctly since each of these is adjacent to two processors which have been identified correctly.

By induction, all processors in the system can be correctly identified. //

Theorem 6.7 : Let S be a closed hexagonal grid in which there are at most 4 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S. Given a syndrome and a fault-free processor v in S, all processors in S can be correctly identified.

Proof : Given a fault-free processor v, all processors in $L_1(v)$ can be correctly identified. We now show that if R is a hexagonal region in S and all processors within R have been correctly identified then all processors in the enclosing hexagonal region can be correctly identified. Consider Fig. 6.8.

All processors lying on the enclosing hexagonal region except the processors $z_1, \ldots, z_6$ are adjacent to at least two processors in R. Thus from Lemma 6.4, these processors can be correctly identified. Now each of the corner processors $z_1, \ldots, z_6$ can be correctly identified as each of them is adjacent to two correctly identified processors.

By induction, all processors in S can be correctly identified. //

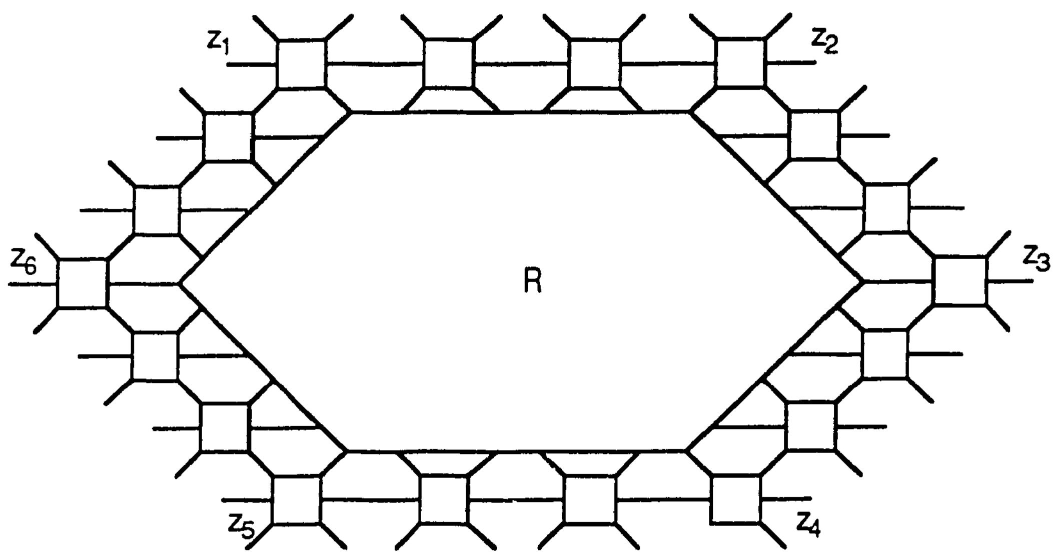


Fig. 6.8. Propagation of fault-identification in a hexagonal grid.

Theorem 6.8 : Let S be a closed octagonal grid in which there are at most 5 faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S. Given a syndrome and a fault-free processor v in S, all processors in S can be correctly identified.

Proof : Given a fault-free processor v, all processors in $L_1(v)$ can be correctly identified. We now show that if R is a rectangular region in S and all processors within R have been correctly identified then all processors in the enclosing rectangular region can be correctly identified. Consider Fig. 6.9.

All processors lying on the enclosing rectangular region except the processors $z_1, \ldots, z_4$ are adjacent to at least two processors in R. Thus from Lemma 6.4, these processors can be correctly identified. Now each of the corner processors $z_1, \ldots, z_4$ can be correctly identified as each of them is adjacent to two correctly identified processors.

By induction, all processors in S can be correctly identified. //

Theorem 6.9 : Let S be a hypercube system with 2^k processors in which there are at most $\lfloor k/2 \rfloor + 1$ faulty processors in $L_1(u) \bigcup \{u\}$ for every processor u in S. Given a syndrome and a fault-free processor v in S, all processors in S can be correctly identified.

Proof : We observe that in a hypercube system, if a processor u is at a distance $r+1$ from a processor v where $r \geq 1$, then it is adjacent to at least two processors at a distance r from v. Thus if all processors within distance r from v have been correctly identified then, by Lemma 6.4, all processors within distance $r+1$ from v can be correctly identified. If v is initially given to be fault-free, all processors within distance 1 from v can be correctly identified.

By induction, all processors in S can be correctly identified. //

Note that the procedure outlined in the above proof will work correctly even

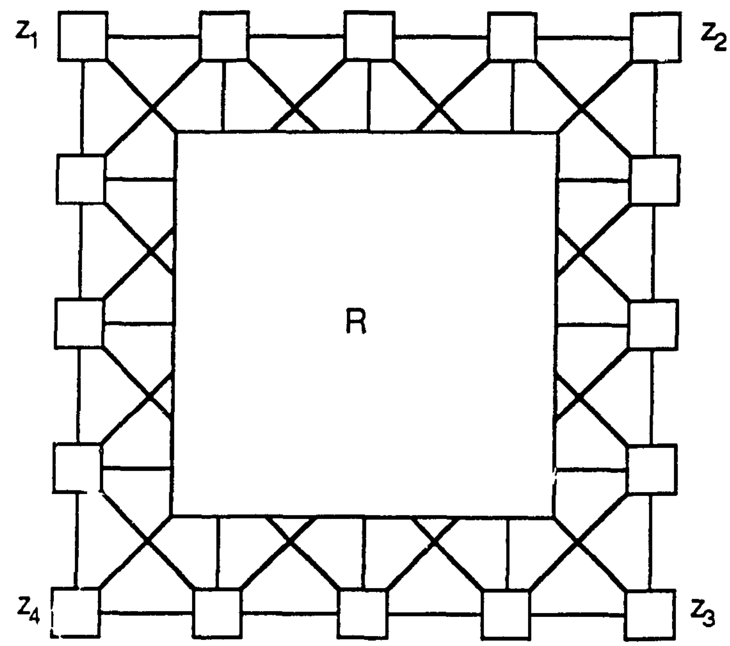


Fig. 6.9. Propagation of fault-identification in an octagonal grid.

in the case of general systems which satisfy the conditions of Theorem 6.1. Thus we have the following.

Theorem 6.10 : Let S be a system with test interconnection graph G = (U,E) satisfying the following conditions:

(1) For all $u_i \in U$, there are at most $\lfloor k_i/2 \rfloor + 1$ faulty processors in $L_1(u_i) \bigcup \{u\}$ where k_i is the degree of u_i.

(2) There are at least two distinct paths of length 2 between any two processors at distance 2 from each other in S.

Given a valid syndrome for S, all processors can be correctly identified provided one processor is initially given to be fault-free. //

The procedures discussed above to identify all processors given a fault-free processor can be used to develop diagnosis algorithms for all the cases considered when instead of being given a fault-free processor we have the constraint that less than half the processors in the system are faulty. In these diagnosis algorithms, the following procedure is called at most two times to determine at least one fault-free processor.

procedure LABEL (v : vertex)

S.1. Label vertex v fault-free. Let A := {v}.

S.2. (a) Let x be a vertex which is adjacent to a vertex in A and x is not a member of A.

(i) **if** x is adjacent to a fault-free vertex y in A **then** label x as fault-free if x and y share a 0-link; label x as faulty, otherwise.

(ii) **elseif** x is adjacent to two faulty vertices in A **then** determine the label of x using Lemma 6.4.

(iii) **elseif** x is adjacent to a faulty vertex y in A which already has $\lfloor deg(y)/2 \rfloor$ vertices labeled faulty in $L_1(y)$ **then** label x as fault-free.

(b) **if** vertex x is labeled **then** add x to the set A.

S.3. Repeat S.2 until A = U.

end procedure

From the proofs of Theorems 6.6-6.10, it can be verified that the procedure LABEL(v) will terminate after labeling all the vertices assuming that v is fault-free. If this labeling is consistent with the fault constraints and the given syndrome then v is fault-free; otherwise it will be faulty. Using breadth-first search [62] an $O(n + |E|)$ implementation of the procedure can be designed. We now present our diagnosis algorithm which first uses the above procedure to determine a fault-free processor, and then completes the diagnosis. A straight forward approach to determine a fault-free processor is to apply procedure LABEL on at most $\lfloor deg(u)/2 \rfloor$ processors adjacent to some processor u. However the complexity of such an algorithm will not be linear. We show in the following that we need to use procedure LABEL at most two times to determine a fault-free processor.

First we pick a vertex q with at least $\lfloor deg(q)/2 \rfloor - 1$ 0-links. Such a vertex exists since each fault-free processor has this property. In fact a vertex with this property can be found in the neighborhood $L_1(u) \bigcup \{u\}$ for every processor u in the system. If the degree of $q \geq 3$, then let w and z be two vertices sharing 1-links with q; if q does not have two 1-links, then q must be fault-free. If the degree of q is two, w and z will be the two vertices adjacent to q.

Having selected w and z as above, we apply procedure LABEL on these two vertices. If either one of them determines a consistent labeling, then it is fault-free and we are through. If both are faulty, then q must be fault-free, for otherwise $L_1(q) \bigcup \{q\}$ will have more than $\lfloor deg(q)/2 \rfloor + 1$ faulty processors, contradicting the local fault constraints.

Thus, we need to use procedure LABEL at most two times to determine a fault-free processor. One more application of this procedure on the fault-free processor will complete the diagnosis.

A formal presentation of the above algorithm is given below.

Algorithm 6.1: t-in-L_1-Diagnosis

S.1. Select any vertex q with at least $\lfloor deg(q)/2 \rfloor - 1$ 0-links.

/* Such a vertex always exists. */

S.2. **if** there are more than $\lfloor deg(q)/2 \rfloor$ 0-links incident on q **then** call LABEL(q). **Stop**

/* q must be fault-free. The diagnosis is complete. */

S.3. **if** q has degree two **then** let w and z be the two vertices adjacent to q.

else let w and z be two vertices sharing 1-links with q.

/* w and z exist */

S.4. Call LABEL(w).

if labeling is consistent with the fault constraints and the given syndrome **then** the current labeling determines the fault set. **Stop**

S.5 Label w permanently faulty. Call LABEL(z).

if labeling is consistent with the fault constraints and the given syndrome **then** the current labeling determines the fault set. **Stop**

S.6. Label z permanently faulty. Call LABEL(q). **Stop**

/* q must be fault-free. The diagnosis is complete */

The complexity of the above algorithm is dominated by the complexity of procedure LABEL which is called at most 3 times. So the overall complexity of this algorithm is also $O(n + |E|)$. This algorithm can also be implemented in a distributed manner using standard techniques of distributed algorithm design [61]. A synchronous distributed implementation with time complexity $O(n)$ and message complexity $O(n + |E|)$ can easily be constructed.

Proof of correctness of the above diagnosis algorithm follows from the correctness of procedure LABEL and the unique t-in-L_1 diagnosability of the regular interconnected systems we have considered as well as those which satisfy the conditions of Theorem 6.1.

The above results lead to the following.

Theorem 6.11 : Let S be a closed rectangular grid, a closed hexagonal grid, a closed octagonal grid system, or a hypercube system (with size 2^p) in which for every $u_i \in U$ there are at most $\lfloor k/2 \rfloor + 1$ faulty processors in $L_1(u_i) \bigcup \{u_i\}$ where $k = 4,6,8$ and p respectively. Given a syndrome, all processors in S can be correctly identified provided less than half the total number of processors in S are faulty. //

Theorem 6.12 : Let S be a system with test interconnection graph $G = (U, E)$ satisfying the following conditions:

(1) For all $u_i \in U$, there are at most $\lfloor k_i/2 \rfloor + 1$ faulty processors in $L_1(u_i) \bigcup \{u\}$ where k_i is the degree of u_i.

(2) There are at least two distinct paths of length 2 between any two processors at distance 2 from each other in S.

Given a valid syndrome for S, all processors can be correctly identified provided less than half the processors in S are faulty. //

Note that in the above theorem the number of faulty processors permitted in the domain $L_1(u) \bigcup \{u\}$ of each processor u depends on the degree of u. Thus, in this theorem, we do not assume that the system is regular.

6.4. Summary

In this chapter we have introduced the t-in-L_1 diagnosability theory. Assuming that less than half the processors in a system are faulty, we presented a sufficient condition for unique diagnosis of a general system when certain fault constraints in a local domain of each processor are specified. This leads to a sufficient condition for a general system to be t-in-L_1 diagnosable for $t = \lfloor \delta/2 \rfloor + 1$, where δ is the minimum degree of the interconnection graph. This condition holds in the case of a hypercube. Interestingly, we have shown that certain regular interconnected systems which do not satisfy this condition are also t-in-L_1 diagnosable for $t = \lfloor \delta/2 \rfloor + 1$. We have also presented t-in-L_1 diagnosable algorithms for all the cases considered. These algorithms are of linear complexity with respect to system size.

In most useful multiprocessor systems, each processor has direct connections to a small number of processors. If only processors with direct connections are allowed to test one another, then for most practical systems which are sparsely connected, the classical diagnosability theory will allow only a small number of faulty processors. The t-in-L_1 diagnosability theory overcomes this shortcoming of the classical diagnosis approach. Our diagnosis algorithms can be implemented in a totally distributed manner on the system itself requiring no global syndrome analysis. Synchronous implementations of these diagnosis algorithms with linear message and time complexities (with respect to system size) can easily be designed.

CHAPTER VII

SUMMARY AND FUTURE WORK

7.1 Summary

Motivated by the inadequacy of the classical t-diagnosability theory of Preparata, Metze and Chien, when applied to large sparsely interconnected multiprocessor systems, we have carried out in this thesis, a detailed study of t/s-diagnosable systems. We have also developed a theory of local diagnosis (that is, diagnosis based on local fault constraints) which allows the presence of a large number of faulty processors even when the system is sparse and yet permits correct diagnosis of all the faulty processors. The theory also admits simple diagnosis algorithms which are amenable for distributed implementation on the multiprocessor system itself. A brief summary of the main results of this thesis now follows.

In Chapter III, we presented characterizations of t/s-diagnosable systems which generalize those given earlier for t/t-diagnosable systems. We have shown how the t/s-characterization for the PMC model based on Kohda's t-characterization theorem can be easily modified to arrive at a t/s-characterization for the BGM model as well as characterizations for the sequentially t-diagnosable systems. We also presented, in this chapter, certain structural properties of general t/s-diagnosable systems which generalize some of the earlier results. These properties have led to a new $t/t+1$-characterization.

In Chapter IV, we developed an $O(n^{3.5})$ algorithm for diagnosis of a $t/t+1$-diagnosable system. We also presented a $t/t+k$ diagnosis algorithm which runs in polynomial time for each fixed positive integer k. We have shown how these algorithms can be modified to construct algorithms for sequential t-diagnosis.

In Chapter V, we presented a basic approach for local diagnosis and applied it to regular interconnected multiprocessor systems with very small connectivity. Here a local neighborhood is defined around each processor which consists of its t immediate neighbors and t subsequent neighbors. The faulty or fault-free nature of each processor is then determined as long as no more than t processors are faulty in its corresponding neighborhood. Based on this result, we also presented a simple $O(1)$ distributed diagnosis algorithm. We have studied the application of local fault constraints on a ring of processors. Specifically, we have determined if unique diagnosis is possible if p out of any q consecutive processors in the ring are faulty.

In Chapter VI, we introduced the concept of a t-in-L_k diagnosable system. We first presented certain basic results which lead to a sufficient condition for unique diagnosis of a system when certain fault constraints are satisfied in the local domain $L_1(u_i)$ of each processor u_i. In this chapter, we also studied the t-in-L_1 diagnosability of certain regular interconnected systems: the closed rectangular, the hexagonal and octagonal grid systems, and the hypercube systems. We presented t-in-L_1 diagnosis algorithms for these regular systems as well as those which satisfy certain other conditions. These diagnosis algorithms can be executed in a distributed manner on the multiprocessor system itself.

7.2 Future Work

While the results of this thesis make important contributions to the area of system-level diagnosis, they also suggest certain new problems for future research.

The properties of t/s-diagnosable systems presented in Chapter III provide much insight into the structure of these systems. They have helped us in developing efficient t/s-diagnosis algorithms. They can also be of considerable value in studying the problem of designing t/s-diagnosable systems which have minimum number of tests and which admit simple diagnosis algorithms.

The results of Chapter III and Chapter IV bring out clearly the relationship between t/s- and sequentially t-diagnosable systems. Combining the properties of these systems with Sullivan's [33] proof of the co-NP-Completeness of t/s-diagnosability, one may be able to simplify considerably Raghavan's [12] proof of the co-NP-Completeness of sequential t-diagnosability. It is worthwhile to present within a unified framework all the results relating to t/s- and sequentially t-diagnosable systems. We wish to note that Somani, Avis and Agarwal have presented such a unified theory in a different context.

Further research is needed to develop simple characterizations of t-in-L_1 diagnosable systems. Although the sufficient condition of Theorem 6.1 is not satisfied by some of the regular interconnected systems considered in Chapter VI, this condition is satisfied for most pairs of vertices in these systems. This suggests the possibility of developing a characterization which is a generalized version of Theorem 6.1. Such a generalized result will help in arriving at a unified proof of the t-in-L_1 diagnosability of all the systems considered in Chapter VI. Another interesting problem will be to develop the general theory for t-in-L_k diagnosable systems extending the results of Chapter VI for $k > 1$.

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