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Dimensions of Attractors

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Abstract

Dimension of Attractors

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This thesis studies different types of dimensions of attractors in low dimensional dissipative dynamical systems. Some of them can be calculated by looking directly at the attractor, other by looking at the system, taking or not into account a probability distribution. We give some simple examples to make the ideas clear, but the generalized baker's transformation is taken as a model for such studies. This transformation is used to illustrate some conjectures about typical chaotic attractor. This thesis may be considered as a partial report of the seminal article of Farmer *et al* [12].

Dedication

I would like to dedicate this thesis to my newborn daughter Neha and my parents.

Acknowledgements

I wish to express my great gratitude to my supervisor, Dr. Pierre Joyal for his superb guidance, encouragement and support in the preparation and composition of this thesis. I really feel fortunate to have had Dr. Pierre Joyal as my thesis supervisor.

Last, but not least, I wish to express my special thanks to my parents and my wife for their continuous encouragement and moral support during my studies.

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Chapter 1

Preliminaries

In addition to an introduction about the notion of dimension, this chapter contains definitions and results (sections 1.3 and 1.4) which are neccessary to the next chapters. In section 1.2, we present the generalized baker's transformation whose attractor is certainly the most studied one. In section 1.3, we give some conjectures about typical chaotic attractors, conjectures that have been taken from [12].

1.1 Introduction

There is no unique notion of dimension. In the familiar territory of Euclidean space, we may all be slaves to a different aspect of dimension and yet arrive at the same conclusions. But, if we venture into the realm of the bizarre, distinct notions of dimension diverge. In the domain of chaos, deterministic structure amplifies the uncertainty inherent in measurement, until only probabilistic information remains. To comprehend the strange objects that inhabit this world, we must expand our concept of dimension to encompass chance as well as certainty. Dimension of an attractor is the first kind of knowledge to characterize it. Roughly speaking, the dimension of a set indicates the amount of information necessary to specify a given location with a desired precision.

In this thesis, we present a certain number of definitions of dimension that, we think, are among the most relevant ones: in chapter 2: similarity dimension and Lyapunov dimension, in chapter 3: capacity and Hausdorff dimension, in chapter 4: information dimension and pointwise dimension. We are not maintaining the historical order of these dimensions. We are presenting them in the order of convenience, starting with those that are easier to compute or less involved. In particular, it has been possible to introduce similarity dimension in a general theoritical set up. Capacity and Hausdorff dimension are called metric dimensions, since the phase space they live in must be equiped with a metric. Information and pointwise dimensions are called probabilistic dimensions, since in their definitions we must take into account a natural probability measure defined on an attractor.

The different notions of dimension of attractors are defined for general mappings, but most of our examples of attractors will come from piecewise affine maps, essentially because they are easier to handle. We compute explicitly all the above dimensions for the attractor of the generalized baker's transformation.

As dimension theory for attractors is an active field of interest, something new is coming every year. This report can be taken as an introduction along this direction.

1.2 Generalized baker's transformation

In this section we define the generalized baker's transformation, which is, as its name says, a generalization of the following map called the baker's transformation:

$$B(x, y) = \begin{Bmatrix} (2x, y/2) & 0 \le x < 1/2, 0 \le y \le 1 \\ (2x - 1, y/2 + 1/2) & 1/2 \le x \le 1, 0 \le y \le 1 \end{Bmatrix}.$$

We shall study the attractor of the generalized baker's transformation in detail throughout the thesis (concerning the dimension properties). This transformation is general enough to be typical (according to Farmer *et al* [12]). It is also simple enough to calculate all the dimensions we are interrested in. The generalized baker's transformation is:

$$x_{n+1} = \begin{cases} \lambda_a x_n, & \text{if } y_n < \alpha \\ \lambda_b x_n + 1/2, & \text{if } y_n \geq \alpha \end{cases} \tag{1.1a}$$

$$y_{n+1} = \begin{cases} (1/\alpha) y_n, & \text{if } y_n < \alpha \\ (1/\beta)(y_n - \alpha), & \text{if } y_n \geq \alpha \end{cases} \tag{1.1b}$$

where $\beta = 1 - \alpha$, $0 \leq x_n, y_n \leq 1$ and $\alpha, \lambda_a, \lambda_b \leq 1/2$. We have illustrated in figure 1 the first two iterates of the transformation.

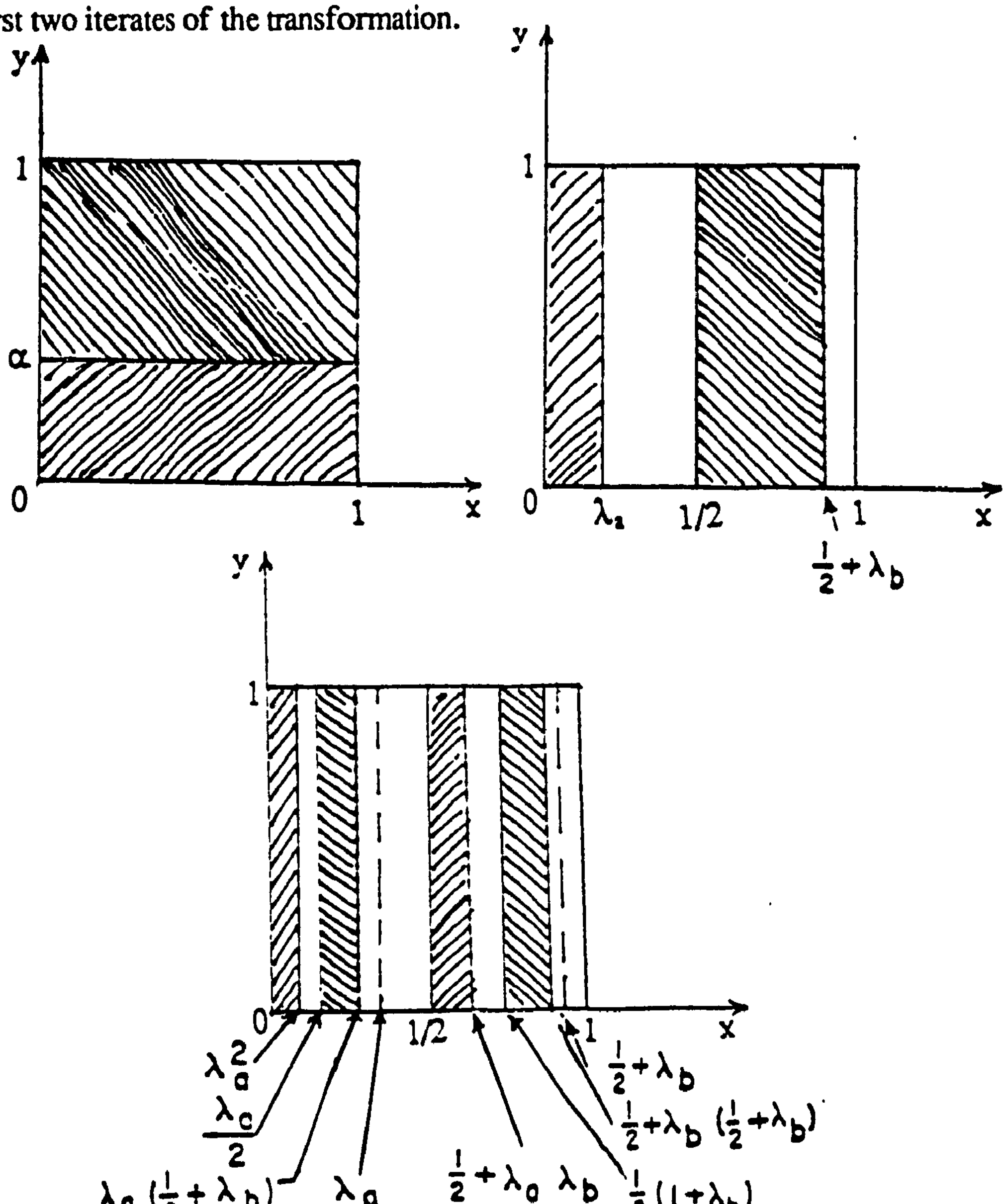


Figure 1

1.3 Conjectures

The goal of this section is to present the conjectures (about typical attractors) that we can find in [12]. We start with some definitions to introduce the notion of attractor.

Definition 1.3.1a and b: Let F: X—> X be a mapping. For any $x_1 \in X$, the set of points $\omega(x_1) = \{x_1, F(x_1), F^2(x_1), \ldots\}$ is called the *trajectory of F starting at x_1*, or the *forward orbit of F starting at x_1*. x_1 is called an *initial condition.*

Definition 1.3.2: For any trajectory $\omega(x_1)$ the set

$$L(\omega(x_1)) = \bigcap_{n=0}^{\infty} \text{closure}(F^n(\omega(x_1))$$

is called the *limit set* of $\omega(x_1)$.

Definition 1.3.3: A set A is *invariant under F* (or just invariant), if $F^{-1}(A) = A$.

Definition 1.3.4: An *attractor of F* is an invariant compact set A, with the property that there is a neighborhood of A such that for almost every initial condition x_1, $L(\omega(x_1)) = A$.

Definition 1.3.5: The *basin of attraction* of an attractor A is the closure of the set of initial conditions x_1 such that $L(\omega(x_1)) = A$.

In the following conjectures, the first two are true for the generalized baker's transformation, though the last one is satisfied for special values of λ_a, λ_b and α: $\lambda_a = \lambda_b$ and $\alpha = \beta = 1/2$. The first conjecture is about metric dimensions and the second, about probabilistic and Lyapunov dimensions (the definitions of dimensions will be given later).

Conjecture 1.3.1: For typical attractor, the capacity and the Hausdorff dimension are equal.

Conjecture 1.3.2: For typical attractor, Lyapunov dimension, information dimension and the pointwise dimension are equal.

Conjecture 1.3.3: If every (not just almost every) initial condition generates the same set of Lyapunov numbers and if the first Lyapunov number is greater than 1, then for a typical attractor, metric dimensions, Lyapunov dimension and probabilistic dimensions are equal.

1.4 Ergodicity of the map (1.1b)

We shall use in the next chapters a fundamental property of the map defined by equation (1.1b): ergodicity. The main goal of this section is to prove it. A great deal of the materials of this section comes from [25]. It should be mentioned that as for as we know our proof of ergodicity of (1.1b) is new, though inspired from an example found in [25]. In the following, we shall suppose that the reader knows what a measure space and a probability space are.

Definitions 1.4.1a and b: Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{A}, \nu)$ be measure spaces.
a) A map $T : X \longrightarrow Y$ is *measurable* if $T^{-1}(A) \in \mathfrak{B}$ for all $A \in \mathfrak{A}$.
b) A measurable map $T : X \longrightarrow Y$ is *nonsingular* if $\mu(T^{-1}(A)) = 0$ for all $A \in \mathfrak{A}$ such that $\nu(A) = 0$.

Definition 1.4.2a, b and c: Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{A}, \nu)$ be measure spaces and

$T : X \longrightarrow Y$ be a measurable map. Then we say that T is *measure–preserving*, or that μ is *invariant under T*, if $\mu(T^{-1}(A)) = \nu(A)$ for any $A \in \mathfrak{A}$. If $T: X \longrightarrow X$ is a measure–preserving transformation, then T is called an *automorphism*.

Remark 1.4.1: If T is measure–preserving, then T is nonsingular.

Definition 1.4.3: Let $(X, \mathfrak{B}, \mu)$ be a probability space and $T : X \longrightarrow X$ be a nonsingular transformation. T is *ergodic* if each invariant set is trivial in the sense of having measure either 0 or 1.

The next theorem may be considered as a corner–stone of Ergodic theory. A complete proof of this theorem can be found in [4].

Theorem 1.4.1 (Birkhoff Ergodic theorem): Let $(X, \mathfrak{B}, \mu)$ be a probability space and $T : X \longrightarrow X$ be a measure–preserving transformation. If $f: X \longrightarrow \mathbb{R}$ is integrable, then there exists an integrable function $\hat{f}: X \longrightarrow \mathbb{R}$ such that:

a) $\hat{f}(x) = \lim_{n \to \infty} 1/n \sum_{k=0}^{n-1} f(T^k x)$ a.e. (almost everywhere),

b) $\hat{f}(T(x)) = \hat{f}(x)$ a.e.($\hat{f}$ is invariant a.e.),

c) $\int_X \hat{f}\, d\mu = \int_X f\, d\mu$,

d) If T is ergodic, then $\hat{f} = \int_X f\, d\mu$ a.e..

Definition 1.4.4a and b: The function $\hat{f}$ associated to f in part a) of the above theorem is called the *orbital average of f.* When f is the characteristic function f_A of $A \in \mathfrak{B}$, the number $\hat{f}_A(x)$ is called the *average time spent by x in A* (denoted by $\tau_A(x)$).

Remark 1.4.2: The average time spent by a point in a set is a particularly meaningful application of the concept of orbital average. Observe that:

$$\tau_A(x) = \lim_{n \to \infty} (1/n)(\mathrm{card}\{0 \le j \le n-1 : T^j(x) \in A\}) \quad \text{and} \quad \int_X \tau_A d\mu = \mu(A).$$

Lemma 1.4.2: Let $(X, \mathfrak{B}, \mu)$ be a probability space and T be an automorphism of X. T is ergodic if and only if $\tau_A = \mu(A)$ almost everywhere, for every $A \in \mathfrak{B}$.

Proof: If T is ergodic then:

$$\tau_A(x) = \hat{f}_A(x) = \int_X f_A d\mu = \mu(A) \text{ almost everywhere.}$$

Conversly, let $A \in \mathfrak{B}$ be T–invariant. Assume $\mu(A) > 0$. $\tau_A(x) = 1$ for any $x \in A$, since A is T–invariant. It follows that $\mu(A) = 1$.

∎

In order to show that the map (1.1b) is ergodic, we shall introduce three other important notions: mixing, Frobenius–Perron anf Koopman operators.

Definition 1.4.5: Let $(X, \mathfrak{B}, \mu)$ be a probability space and $T : X \longrightarrow X$ be a measure–preserving transformation. T is called *mixing* if $\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B)$ for all A, B belonging to $\mathfrak{B}$.

The following lemma shows that the mixing function form a subset of the ergodic function. We shall prove later that the map (1.1b) is in fact mixing.

Lemma 1.4.3: If T is mixing, then T is ergodic.

Proof: If A is T–invatiant ($T^{-n}(A) = A$ for every n) and if T is mixing:

$$\mu(A)\mu(A^c) = \lim_{n\to\infty} \mu(T^{-n}(A) \cap A^c) = \mu(A \cap A^c) = 0,$$

which implies that $\mu(A) = 0$ or $\mu(A) = 1$.

■

Notation 1.4.1: Let $(X, \mathfrak{B}, \mu)$ be a measure space. L^1 will denote (as usual) the space of functions $f: X \to \mathbb{R}$ such that $\int_X |f|\, d\mu < \infty$ and L^∞, the space of all bounded measurable functions, except possibly for a set of measure zero. The norm of f in L^1 is $\| f \| = \int_X |f|\, d\mu$ and the norm of f in L^∞ is $\| f \| = \inf\{M: \mu\{t : f(t) > M\} = 0\}$.

Definition 1.4.6: Assume that a nonsingular transformation $T: X \to X$ on a measure space $(X, \mathfrak{B}, \mu)$ is given. The *Frobenius–Perron operator* $P: L^1 \to L^1$ associated with T is defined by the unique $Pf \in L^1$ satisfying:

$$\int_A Pf(x) d\mu = \int_{T^{-1}(A)} f(x) d\mu \text{ , for any } A \in \mathfrak{B}.$$

Lemma 1.4.4: Let $P: L_1 \to L_1$ be the Frobenius–Perron operator associated with T. If A is an interval, say [a, x], then:

$$Pf(x) = d/dx \int_{T^{-1}(A)} f(t) dt.$$

Proof: Differentiating both sides of the equation given in definition 1.4.6, we get the result.

■

Definition 1.4.7: Let $(X, \mathfrak{B}, \mu)$ be a measure space and P be an operator on L^1. Any f in L^1 is called a *fixed point of P*, if $Pf = f$.

Theorem 1.4.5: Let $(X, \mathfrak{B}, \mu)$ be a measure space and $T : X \longrightarrow X$ is a nonsingular transformation, and P the Frobenius–Perron operator associated with T. Consider a nonnegative $f \in L^1$. Then a measure μ_f given by:

$$\mu_f(A) = \int_A f(x)d\mu$$

is invariant under T if and only if f is a fixed point of P. In particular, μ is invariant under T if and only if $f = 1$ is a fixed point of P.

Proof: Let μ_f be invariant, i.e.$\mu_f(A) = \mu_f(T^{-1}(A))$ for $A \in \mathfrak{B}$. This is equivalent to:

$$\int_A f(x)d\mu = \int_{T^{-1}(A)} f(x)d\mu \text{ for } A \in \mathfrak{B}.$$

Then by definition of the Frobenius–Perron operator, we get:

$$\int_A Pf(x)d\mu = \int_{T^{-1}(A)} f(x)d\mu = \int_A f(x)d\mu,$$

for any $A \in \mathfrak{B}$. This implies that $Pf = f$. The converse follows also automatically.

■

Proposition 1.4.6: The transformation $y = T(x)$ of equation (1.1b) is Lebesgue measure–preserving.

Proof: Let [0, x] is a subset of [0, 1]. By definition $P1 = d/dy \int_{T^{-1}[0,\, y]} dt$.

Since $T^{-1}y = \alpha y$, for $0 \le x < \alpha$ and $T^{-1}y = (1-\alpha)y + \alpha$ for $\alpha \le x \le 1$, then:

$$T^{-1}[0, y] = [0, \alpha y] \cup [\alpha, (1-\alpha)y + \alpha].$$

$P1 = d/dy[\int_0^{\alpha y} dt + \int_\alpha^{(1-\alpha)y+\alpha} dt\ ] = d/dy[y] = 1$. Then by theorem 1.4.5, we get the result.

■

Definition 1.4.8: Let $(X, \mathfrak{B}, \mu)$ be a measure space, $T: X \longrightarrow X$ be a nonsingular transformation, and $f \in L^\infty$. The operator $U: L^\infty \longrightarrow L^\infty$ defined by:

$$Uf(x) = f(T(x)),$$

is called the *Koopman operator* associated with T.

Remark 1.4.3: Due to the nonsigularity of T, U is well defined since $f_1(x) = f_2(x)$ a.e. implies that $f_1(T(x)) = f_2(T(x))$ a.e. .

Theorem 1.4.7: Let $(X, \mathfrak{B}, \mu)$ be a measure space, $T: X \longrightarrow X$ be a nonsingular transformation, and $f \in L^\infty$. The Koopman operator U associated with T satisfies the following property:

$$\text{For every } f \in L^1, g \in L^\infty, \int_X Pf(x)\, g(x) d\mu = \int_X f(x) Ug(x) d\mu.$$

Proof: Let us first show the property for $g = g_A$ (the characteristic function of A). The left hand side of the equation becomes:

$$\int_X Pf(x) g_A(x) d\mu = \int_A Pf(x) d\mu = \int_{T^{-1}(A)} f(x) d\mu ,$$

while the right hand side becomes:

$$\int_X f(x) Ug_A(x) d\mu = \int_X f(x) g_A(T(x)) d\mu = \int_{T^{-1}(A)} f(x) d\mu.$$

Because the equation is true for any characteristic function, it is true for any simple function. Thus it is true for any function g in L^∞.

■

Now we are in position to prove the ergodicity of the map $y = T(x)$ defined by (1.1.b). Here is the procedure that we shall use to attain our goal. Let P be the Frobenius–Perron operator associated to this map. We prove that:

1° $\{P^n f\}$ is strongly convergent to $\int_0^1 f(t)dt$ for $f \in L^1([0, 1])$.

2° If a sequence of functions $\{p_n\}$, $p_n \in L^1$, is strongly convergent to $p \in L^1$, then $\{p_n\}$ is weakly convergent to p.

3° If $\{P^n f\}$ is weakly convergent to $\int_X f\, d\mu$ for $f \in L^1$, then T is mixing.

Therefore according to lemma 1.4.3, T is ergodic.

Definition 1.4.8: A sequence of functions $\{p_n\}$, $p_n \in L^1$, is *strongly convergent* to $p \in L^1$ if $\lim_{n \to \infty} \| p_n - p \| = 0$.

Definition 1.4.9: A sequence of functions $\{p_n\}$, $p_n \in L^1$, is *weakly convergent* to $p \in L^1$ if:

$$\lim_{n \to \infty} \int_X p_n(x)\, g(x) d\mu = \int_X p(x) g(x) d\mu \quad \text{for all } g \in L^\infty.$$

Proof of 3°: Let $f = f_A$ and $g = g_B$ be the characteristic functions of $A, B \in \mathfrak{B}$ respectively. We have:

$$\lim_{n \to \infty} \mu((A) \cap T^{-n}(B)) = \lim_{n \to \infty} \int_X f_A(x) g_B(T^n(x)) d\mu ,$$

$$= \lim_{n\to\infty} \int_X f_A(x) U^n g_B(x) d\mu \text{ , (by definition of U)}$$

$$= \lim_{n\to\infty} \int_X P^n f_A(x) g_B(x) d\mu \text{ , (by theorem 1.4.7)}$$

$$= \int_X \left(\int_X f_A(x) d\mu\right) g_B(x) d\mu \text{ , (by hypothesis)}$$

$$= \int_X f_A(x) d\mu \int_X g_B(x) d\mu = \mu(A)\mu(B).$$

■

Proof of 2^o: From Cauchy–Hölder inequality, we have:

$\left| \int_X (p_n - p)(x) g(x) d\mu \right| \leq \| p_n - p \|_1 \| g \|_\infty$, $g \in L^\infty$ and thus, if $\| p_n - p \|$ converges to zero, so must $\int_X (p_n - p)(x) g(x) d\mu$. Hence strong convergence implies weak convergence.

■

Proof of 1^o: Since the set of continuous functions on [0, 1] is dense in $L^1([0, 1])$ (see [28] theorem 4.3.13), it suffices to prove the statement for any continuous function f on [0, 1]. Let $\beta = 1-\alpha$. By the argument of proposition 1.4.6, we have:

$$Pf(y) = d/dy\left[\int_0^{\alpha y} f(t)\, dt + \int_\alpha^{\beta y + \alpha} f(t)\, dt \right] = \alpha f(\alpha y) + \beta f(\beta y + \alpha),$$

$$P^2 f(y) = \alpha^2 f(\alpha^2 y) + \alpha\beta f(\alpha\beta y + \alpha^2) + \alpha\beta f(\alpha\beta y + \alpha) + \beta^2 f(\beta^2 y + \alpha + \alpha\beta).$$

Let us introduce the following non–commutative products. Letting:

$$(\alpha fp + \beta fq)^{*2} = \alpha fp(\alpha fp + \beta fq) + \beta fq(\alpha fp + \beta fq) = \alpha^2 fp^2 + \alpha\beta fpq + \alpha\beta fqp + \beta^2 fq^2.$$

$$(p \cup q)^{*2} = p(p \cup q) \cup q(p \cup q) = p^2 \cup pq \cup qp \cup q^2,$$

we define by recurrence:

$$(\alpha fp + \beta fq)^{*n}(y) = (\alpha fp + \beta fq)(\alpha fp + \beta fq)^{*(n-1)}(y),$$

$$(p \cup q)^{*n} = (p \cup q)(p \cup q)^{*(n-1)}.$$

Let $p(y) = \alpha y$ and $q(y) = \beta y+\alpha$ and $(p\cup q)(y) = p(y)\cup q(y)$. One can see that $Pf(y) = (\alpha fp + \beta fq)(y)$ and $(p\cup q)([0, 1]) = p([0, 1])\cup q([0, 1]) = [0, \alpha]\cup[\alpha, 1] = [0, 1]$. One proves by induction that $P^n f(y) = (\alpha fp + \beta fq)^{*n}(y)$ and $(p\cup q)^{*n}([0, 1]) = [0, 1]$. The latter equality is trivial. To prove the former one, it suffices to remark that by induction there are 2^{n-1} terms of the type $\alpha^r\beta^s fp^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y)$ ($a_k, b_k \geq 0$, $a_1 +\ldots + a_i = r$, $b_1 +\ldots + b_j = s$ and $r + s = n-1$) in $(\alpha fp + \beta fq)^{*(n-1)}(y)$ and that:

$$P\alpha^r\beta^s fp^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y) = \alpha^{r+1}\beta^s fp^{a_1+1}q^{b_1}\ldots p^{a_i}q^{b_j}(y) + \alpha^r\beta^{s+1} fqp^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y).$$

Moreover $(p\cup q)^{*n}([0, 1])$ gives a partition of the interval $[0, 1]$. Indeed, if we suppose by induction that $(p\cup q)^{*(n-1)}([0, 1])$ is a partition of $[0, 1]$ having two consecutive intervals with α as an endpoint, then $p(p\cup q)^{*(n-1)}([0, 1])$ sends the partition $(p\cup q)^{*(n-1)}([0, 1])$ onto $[0, \alpha]$ and $q(p\cup q)^{*(n-1)}([0, 1])$ sends the same partition onto $[\alpha, 1]$. In particular the k^{th} term in $(p\cup q)^{*n}([0, 1])$ is the k^{th} interval in the partition (from left to right).

We want to show that $\lim_{n\to\infty}(\alpha fp + \beta fq)^{*n}(y)$ is, for any $y \in [0, 1]$, the Riemann integral of $f(x)$ over $[0, 1]$. If $a_1 +\ldots + a_i = r$, $b_1 +\ldots + b_j = s$, the coefficient of y in $p^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y)$ is $\alpha^r\beta^s$. It implies that the length of the interval $p^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}([0, 1])$ of the partition $(p\cup q)^{*n}([0, 1])$ is $\alpha^r\beta^s$ (with $r + s = n$). Since $p^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y) \in p^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}([0, 1])$ (for any $y \in [0, 1]$), $\alpha^r\beta^s fp^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y)$ is the area of a rectangle with base length equal to $\alpha^r\beta^s$ and height equal to $fp^{a_1}q^{b_1}\ldots p^{a_i}q^{b_j}(y)$, that is the height of f at a point on the base of the rectangle. Then for large n, $(\alpha fp + \beta fq)^{*n}(y)$ is an approximation of the Riemann integral. Therefore:

$$\lim_{n\to\infty} P^n f(y) = \int_0^1 f(t)dt.$$

■

Chapter 2

Similarity and Lyapunov dimensions

This chapter deals with the notions of *Similarity dimension* and *Lyapunov dimension*. These dimensions have the advantage of being easily calculable for affine systems. Similarity dimension is defined only for special linear systems: the *similitudes*. The organization of this chapter is following: in the first section we present the notions of similitude and similarity dimension and compute the similarity dimension of the generalized baker's transformation. In the second section, we present the notion of *Lyapunov numbers*, Lyapunov dimension, and the computation of 1^{o} the Lyapunov numbers of the generalized baker's transformation and 2^{o} the Lyapunov dimension of the attractor of this transformation.

2.1 Similarity dimension

We shall present the notion of *(hyperbolic) iterated function systems*. M. F. Barnsley gives a good introduction about these systems in [2, 3]. We shall use this notion to compute the similarity dimension of the generalized baker's transformation.

In order to define an iterated function system, a series of definitions and lemmas is needed.

Definition 2.1.1: Let (X, d) be a metric space. A map $f: X \longrightarrow X$ is called a *contraction* if $d(f(x), f(y)) \leq r\, d(x, y)$, for all x and y in X where $0 \leq r < 1$ is called the *contractivity factor* of f.

Definitions 2.1.2 a), b) and c): Let (X, d) be a complete metric space. Let $\mathcal{H}(X)$ denote the space whose points are the compact subsets of X other than empty set.

a) Let $x \in X$ and $B \in \mathcal{H}(X)$.

$$d(x, B) = \min\{d(x, y) : y \in B\}$$

is called the *distance from the point* x *to the set* B.

b) Let $A, B \in \mathcal{H}(X)$.

$$d(A, B) = \max\{d(x, B) : x \in A\}$$

is called the *distance from the set* A *to the set* B.

c) The *Hausdorff distance between the points* A *and* B in $\mathcal{H}(X)$ is defined by:

$$h(A, B) = \max(d(A, B), d(B, A)).$$

Remark 2.1.1: The Hausdorff distance h depends on the metric d. Sometimes we shall use the notation h(d).

Example 2.1.1: See the following figure 2.

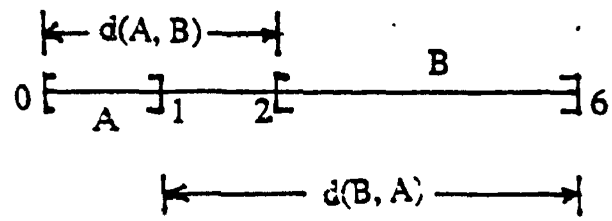


Figure 2

Proposition 2.1.1: h: $\mathcal{H}(X) \longrightarrow R^+$ is a metric.

Proof: Let A, B, C $\in \mathcal{H}(X)$. In order to show that h is a metric on $\mathcal{H}(X)$, we need to satisfie the metric axioms:

A1) $h(A, B) \geq 0$ by definition.

A2) $h(A, A) = \max(d(A, A), d(A, A)) = d(A, A) = \max\{d(a, A): a \in A\} = 0$.

A3) $h(A, B) = \max(d(A, B), d(B, A)) = \max(d(B, A), d(A, B)) = h(B, A)$.

A4) In order to show that $h(A, B) \leq h(A, C) + h(C, B)$, we shall show that $d(A, B) \leq d(A, C) + d(C, B)$. Consequently we shall obtain:

$h(A, B) = \max(d(A, B), d(B, A)) \leq \max((d(A, C) + d(C, B)), (d(B, C) + d(C, A)))$

$\leq \max(d(A, C), d(C, A)) + \max(d(C, B), (d(B, C)) = h(A, C) + h(C, B)$.

Now for any $a \in A$, we have:

$d(x, B) = \min\{d(x, y): y \in B\} \leq \min\{d(x, z) + d(z, y): y \in B\}$ for all $z \in C$, so

$\leq \min\{d(x, z) : z \in C\} + \max\{\min\{d(z, y): y \in B\} z \in C\} = d(x, C) + d(C, B)$.

Therefore, $d(A, B) \leq d(A, C) + d(C, B)$.

∎

Lemma 2.1.2: Let $f : X \longrightarrow X$ be a contraction map on the metric space (X, d). Then f is continuous.

Proof: Let $\varepsilon > 0$ be given. Then we must find a $\delta > 0$ such that:

$$\text{if } d(x, y) < \delta, \text{ then } d(f(x), f(y)) < \varepsilon$$

Since f is a contraction map, so we have $d(f(x), f(y)) \leq r\, d(x, y)$, r is the contractivity factor of the map f. Then $d(f(x), f(y)) < r\, \varepsilon/r = \varepsilon$, $(\delta = \varepsilon/r)$.

■

Lemma 2.1.3: Let $f : X \longrightarrow X$ be a contraction map on (X, d) with contractivity factor r. Then $f : \mathcal{H}(X) \longrightarrow \mathcal{H}(X)$ defined by $f(A) = \{ f(x) : x \in A\}$ for all $A \in \mathcal{H}(X)$ is well defined and is a contraction map on $(\mathcal{H}(X), h(d))$ with contractivity factor r.

Proof: Since f is a contraction, then by lemma 2.1.1 f is continuous. Since f is continuous, f(A) is compact if A is compact. Now we have to show that f is a contraction map on $(\mathcal{H}(X), h(d))$ with contractivity factor r, i.e., we want to show that:

$$h(f(A), f(B)) \leq r\, h(A, B) \text{ for all } A, B \in \mathcal{H}(X).$$

Let $A, B \in \mathcal{H}(X)$, then:

$$\begin{aligned} d(f(A), f(B)) &= \max\{\min\{d(f(x), f(y)) : y \in B\} : x \in A\} \\ &\leq \max\{\min\{r\, d(x, y) : y \in B\} : x \in A\} \\ &= r \max\{\min\{ d(x, y) : y \in B\} : x \in A\} \\ &= r\, d(A, B), \end{aligned}$$

Similarly $d(f(B), f(A)) \leq r\, d(B, A)$. Therefore

$h(f(A), f(B)) = \max(d(f(A), f(B)), d(f(B), f(A))) \leq r \max(d(A, B), d(B, A)) = r\, h(A, B)$.

■

Lemma 2.1.4: Let A, B and $C \in \mathcal{H}(X)$, and C be a subset of B. Then:

$$d(A, C) \geq d(A, B).$$

Proof: $d(A, B) = \max\{d(x, B) : x \in A\} = \max\{\min(d(x, y) : y \in B) : x \in A\}$

$$\leq \max\{\min(d(x, y) : y \in C) : x \in A\} = \max\{d(x, C) : x \in A\} = d(A, C).$$

The inequality follows from the definition and the fact that C is a subset of B.

■

Lemma 2.1.5: Let $A, B, C \in \mathcal{H}(X)$. Then:

$$d(A \cup B, C) = \max\{d(A, C), d(B, C)\}.$$

Proof: $d(A \cup B, C) = \max\{d(x, C) : x \in A \cup B\}$

$$= \max\{\max\{d(x, C) : x \in A\}, \max\{d(x, C) : x \in B\}\}$$

$$= \max\{d(A, C), d(B, C)\}.$$

■

Lemma 2.1.6: Let A, B, C and $D \in \mathcal{H}(X)$. Then:

$$d(A \cup B, C \cup D) \leq \max\{d(A, C), d(B, D)\}.$$

Proof: By lemma 2.1.5 we know that:

$$d(A \cup B, C \cup D) = \max\{d(A, C \cup D), d(B, C \cup D),$$

Since C and D are subsets of $C \cup D$, we know by lemma 2.1.4 that:

$$\max\{d(A, C \cup D), d(B, C \cup D) \leq \max\{d(A, C), d(B, D)\},$$

Now combining these two, we get:

$$d(A \cup B, C \cup D) \leq \max\{d(A, C), d(B, D)\}$$

■

Lemma 2.1.7: Let A, B, C and $D \in \mathcal{H}(X)$. Then:

$$h(A \cup B, C \cup D) \leq (h(A, C), h(B, D)).$$

Proof: By lemma 2.1.6: $d(A \cup B, C \cup D) \le \max\{d(A, C), d(B, D)\}$. Similarly we have: $d(C \cup D, A \cup B) \le \max\{d(C, A), d(D, B)\}$. Therefore:

$$\begin{aligned} h(A \cup B, C \cup D) &= \max\{d(A \cup B, C \cup D), d(C \cup D, A \cup B)\} \\ &\le \max\{\max(d(A, C), d(B, D)), \max(d(C, A), d(D, B))\} \\ &= \max\{\max(d(A, C), d(C, A)), \max(d(B, D), d(D, B))\} \\ &= \max(h(A, C), h(B, D)). \end{aligned}$$

■

Theorem 2.1.8: [2, 3] Let (X, d) be a complete metric space. Let $\{f_n\colon n = 1, \ldots k\}$ be contraction maps on $(\mathcal{H}(X), h(d))$. Let the contractivity factor of f_n be r_n. Then $F\colon \mathcal{H}(X) \longrightarrow \mathcal{H}(X)$, defined by $F(A) = \bigcup_{n=1}^{k} f_n(A)$ for all $A \in \mathcal{H}(X)$, is a contraction map on $(\mathcal{H}(X), h(d))$ with contractivity factor $r = \max\{r_n\}$. Its unique fixed point $A_0 \in \mathcal{H}(X)$ obeys $A_0 = F(A_0) = \bigcup_{n=1}^{k} f_n(A_0)$ and is given by $A_0 = \lim_{n \to \infty} F^n(B)$ for any $B \in \mathcal{H}(X)$.

Proof: Obviously F is well defined. Let us prove first that F is a contraction map. Let $k = 2$ and $A, B \in \mathcal{H}(X)$. Then:

$$\begin{aligned} h(F(A), F(B)) &= h(f_1(A) \cup f_2(A), f_1(B) \cup f_2(B)) \\ &\le \max\{\, h(f_1(A), f_1(B)), h(f_2(A), f_2(B))\} \\ &\le \max\{r_1\, h(A, B), r_2\, h(A, B)\} \le r\, h(A, B), \end{aligned}$$

where $r = \max\{r_1, r_2\}$. The first inequality follows by lemma 2.1.7 and the second by the definition of the contraction map. By an induction argument, we prove that F is a contraction map on $(\mathcal{H}(X), h(d))$ i.e.,

$$h(F(A), F(B)) \le (\max r_n) h(A, B).$$

This completes the first part. Now the uniqueness of the fixed point follows from the fixed point theorem (see Theorem 1 on page 66 of [24]).

■

Definition 2.1.3: A *(hyperbolic) iterated function system* consists of a complete metric space (X, d) together with a finite set of contraction maps:

$$f_n: X \longrightarrow X$$

with respective contractivity factors r_n, for $n = 1, \ldots, k$. The abbreviation of an iterated function system is IFS and denoted by $\{X: f_n, n = 1, \ldots k\}$ with *contractivity factor* $r = \max\{r_n: n = 1, \ldots, k\}$.

Definition 2.1.4: The *fixed point* $A_0 \in \mathcal{H}(X)$ described in the above theorem is called the *attractor* of the IFS.

Now we are in position to define similarity dimension which was originally given by B. B. Mandelbrot [27]. The advantage of the similarity dimension is that it is very easy to calculate and under certain condition it is exactly equal to the Hausdorff dimension and the capacity of an attractor (see next chapter).

Definition 2.1.5: A map $f: X \longrightarrow X$ is called a *similitude* if $d(f(x), f(y)) = r\, d(x, y)$, for all x and y in X, where $r \in \mathbb{R}$ is called the *scale (scaling) factor* of the similitude.

Definition 2.1.6: Let $\{X : f_n, n = 1, \ldots, k\}$ be an IFS with $r_n > 0$, $n = 1, \ldots, k$. Assume that f_n is a similitude. The *similarity dimension* d_S *of the IFS* is the unique positive number which satisfies the following equation:

$$\sum_{n=1}^{k} (r_n)^{d_S} = 1. \qquad (2.1)$$

Remark 2.1.2: d_S is unique , because $(r_n)^{d_S}$ ($0 < r_n < 1$) is strictly decreasing in d_S. If $r_n = r$ (for $n = 1, \ldots, k$), $d_S = \log(1/k) / \log r$. When the r_n are different, d_S can be computed using Newton's method:

Let $f(d_S) = \sum_{n=1}^{k} (r_n)^{d_S} - 1$. Since $f'(d_S) \neq 0$, the following formula is well defined:

$$d_{v+1} = d_v + f(d_v) / f'(d_v).$$

and $d_S = \lim_{v \to \infty} d_v$.

Example 2.1.2: *The classical Cantor set.* Let $\{[0, 1]: f_1, f_2\}$ be an IFS, where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$ are similitudes. Then the Cantor ternary set is the attractor of this IFS and the similarity dimension d_S of the IFS is log2/log3, since equation (2.1) becomes $2(1/3)^{d_S} = 1$.

Remark 2.1.3: The similarity dimension is associated with the IFS, not to a possible attractor of the IFS. The next example shows why.

Example 2.1.3: Let $\{[0, 1]: f_1, f_2\}$ and $\{[0, 1]: f_1, f_2, f_3\}$ be two IFS, where $f_1(x) = x/3$, $f_2(x) = x/3 + 2/3$, and $f_3(x) = x/3$. Then the Cantor ternary set is the attractor of each of the IFS and the similarity dimension d_S of the first IFS is log2/log3 and that of the second is 1, since equation (2.1) becomes $3(1/3)^{d_S} = 1$ for second IFS.

In order to attach the similarity dimension to the attractor we introduce the following:

Definition 2.1.7: Let $\{X: f_n, n = 1, \ldots, k\}$ be an IFS. We say that the *open set condition* is satisfied if there exists a bounded open set V such that:

(a) $\bigcup_{n=1}^{k} f_n(V) \subset V$ and

(b) $f_n(V) \cap f_m(V) = \emptyset$ for $n \neq m$.

Remark 2.1.4: This definition is motivated by Theorem 3.3.3 which says that under the open set condition, the Hausdorff dimension of the attractor of an IFS is equal to the similarity dimension.

In example 2.1.2, there is nothing special about the factor 1/3. If we replace 1/3 by $0 < k < 1/2$ and proceed in the similar way, we shall get the following:

Example 2.1.4: *A Cantor set.* Let $\{[0, 1]: f_1, f_2\}$ be an IFS where $f_1(x) = kx$ and $f_2(x) = kx + (1-k)$ for some fixed k, $0 < k < 1/2$. The attractor of this IFS is a Cantor set obtained by removing the segment $(k, 1-k)$. It is clear that f_1 and f_2 are similitudes with scaling factors k, and the open set condition is satisfied. Since we can take $V = (0, 1)$. Therefore the similarity dimension of the attractor is: $\log 2/\log(1/k)$, because $2(k)^{d_S} = 1$.

In a similar way we can calculate the similarity dimension of a generalized Cantor set in p–dimension:

Example 2.1.5: *A generalized Cantor set (in p dimensions).*
Let $\{[0, 1]^p: f_{i_1 \dots i_p}, i = 1, 2, 3, \dots, 2^p\}$ where:

$$f_{i_1 \dots i_p}(x) = k(x_1, \dots, x_p) + (1-k)(i_1, i_2, \dots, i_p),$$

where $0 < k < 1/2$, $0 \le x_j \le 1$, $i_j = 0$ or 1 for $1 \le j \le p$ and $(i_1, i_2, \dots, i_p)$ is one of 2^p corners of the hypercube $[0, 1]^p$. $f_{i_1 \dots i_p}(x)$ is sending the hypercube $[0, 1]^p$ onto the hypercube with length k, situated at the corner $(i_1, i_2, \dots, i_p)$. So at the n^{th} iterate, the length of $(f_{i_1 \dots i_p})^n([0, 1]^p)$ is k^n. Since we can take $V = (0, 1)^p$ we can say that the open set condition is satisfied and equation (2.1) yields $2^p(k)^d = 1$, which implies that the similarity dimension of the attractor is $d = p[\log 2 / \log(1/k)]$. See figure 3.

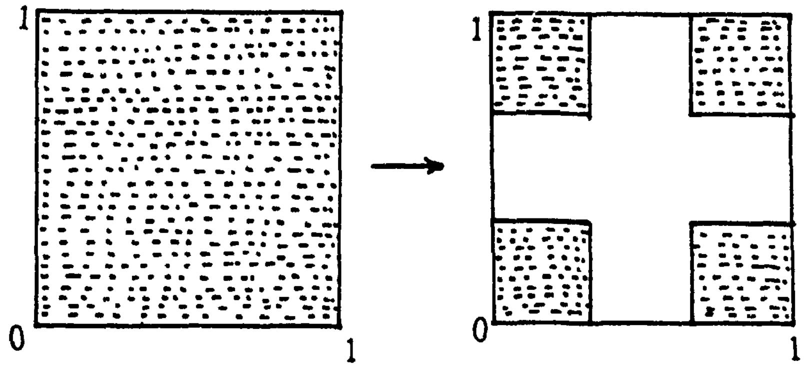


Figure 3 (for p = 2)

Now we turn to the computation of the *similarity dimension* of the attractor *along the x–axis* of the generalized bakers transformation. From the figure 1, it is clear that the attractor is a product of the interval [0, 1] along the y–axis and a (assymetric) Cantor set along the x–axis. Thus we shall compute the dimension along the x–axis and use the fact that any dimension d of the attractor is equal to $1 + \bar{d}$, where $\bar{d}$ is the dimension along the x–axis.

Computation of the similarity dimension along the x–axis of the generalized baker's transformation:

Let $\bar{d}_S$ be the similarity dimension of the following IFS: $\{[0, 1] : f_1, f_2\}$, where $f_1(x) = \lambda_a x$ and $f_2(x) = \lambda_b x + 1/2$, $0 < \lambda_a, \lambda_b \leq 1/2$. Clearly both are similitudes. The open set condition holds, since we can take V = (0, 1) and immidiately both conditions of the definition 2.1.7 are satisfied. Therefore the similarity dimension of the attractor can be calculated using this equation:

$$\lambda_a^{\bar{d}_S} + \lambda_b^{\bar{d}_S} = 1 \tag{2.2}$$

Since both terms on the left side of the equation (2.2) are monotonically decreasing, $\bar{d}_S$ obtained by solving this equation is unique. Thus the *similarity dimension of the attractor of the generalized baker's transformation is:* $1 + \bar{d}_S$, where $\bar{d}_S$ is the solution of equation (2.2).

2.2 Lyapunov dimension

The Lyapunov numbers were originally defined by A. M. Lyapunov [26] in 1950, for the study of the general problem of stability of motion. Some authors refer to Lyapunov exponents rather than Lyapunov numbers. The Lyapunov exponents are the logarithms of Lyapunov numbers.

Definition 2.2.1: Let $F : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ be a piecewise C^1–mapping and $J_n(x_1) = J(x_n)J(x_{n-1})\ldots J(x_1)$, where $J(x)$ is the Jacobian matrix of the map $x_{n+1} = F(x_n)$, i.e. $J(x) = \partial F/\partial x$ when it is defined at x. Let $j_1(n) \geq j_2(n) \geq \ldots \geq j_p(n)$ be the magnitudes of the eigenvalues of $J_n(x_1)$. The *Lyapunov numbers* of F at x_1 are:

$$\lambda_i = \lim_{n \to \infty} [j_i(n)]^{1/n}, \tag{2.3}$$

where the positive real n^{th} root is taken.

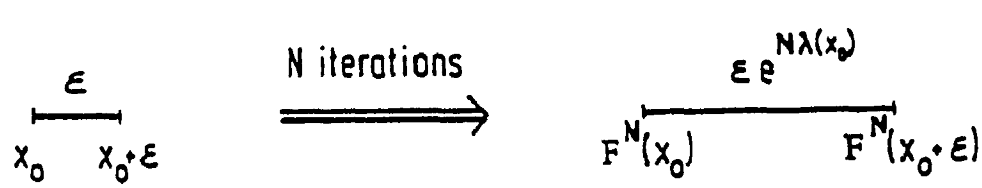


Figure 4

The Lyapunov exponent measures the exponential separation as shown in the preceding figure.

Remark 2.2.1: It can be seen that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. The Lyapunov numbers generally depend on the choice of the initial condition x_1, but not for an affine system. We shall look at examples where the Lyapunov numbers are constant. For a p–dimensional map, λ_i is the average principal stretching factor of an infinitesimal p–spherical volume in the direction of the i^{th} axis. See figure 5.

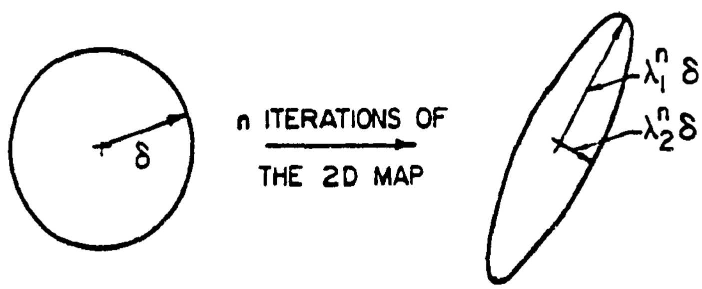


Figure 5 (for p = 2)

The Lyapunov dimension was originally defined by Kaplan and Yorke [22] in 1978 for the problem of determining the dimension of an attractor.

Definition 2.2.2: Suppose that $F : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ has Lyapunov numbers not depending on the initial condition x_1. If F has an attractor A and if k is the largest value for which $\lambda_1 \ldots \lambda_k \geq 1$ ($1 \leq k \leq p$), then the *Lyapunov dimension of A* denoted by d_L is defined by:

$$d_L = 0, \text{ if } \lambda_1 < 1,$$

$$d_L = k + \log(\lambda_1 \ldots \lambda_k)/\log(1/\lambda_{k+1}), \text{ if } 1 \leq k < p, \tag{2.4}$$

$$d_L = p, \text{ if } k = p.$$

Remark 2.2.2: Kaplan and Yorke [22] use the following conjecture to justify the integer k in the above formula: *For nearly every (in the generic sence) F satisfying the hypotheses of definition 2.2.2, the attractor A has Hausdorff dimension greater than or equal to the integer k.*

Proposition 2.2.1: For $1 \leq k < p$,, we have $0 \leq \log(\lambda_1 ... \lambda_k)/\log(1/\lambda_{k+1}) < 1$.

Proof: From the definition of Lyapunov dimension $\lambda_1 ... \lambda_k \lambda_{k+1} < 1$. Therefore $1 \leq \lambda_1 ... \lambda_k < (1/\lambda_{k+1})$. Hence the result follows.

■

Example 2.2.1: [After 30]. *The Bernoulli shift* $\sigma : [0, 1] \to [0, 1]$:

$$x_{n+1} = \sigma(x_n) = 2x_n (\text{mod } 1)$$

has the Lyapunov number 2, i.e., the Lyapunov exponent log2. The Lyapunov dimension of the Bernoulli shift is $d_L = 1$. Indeed $\lambda_1 = 2 > 1$ and by the above definition $d_L = p$, where $p = 1$.

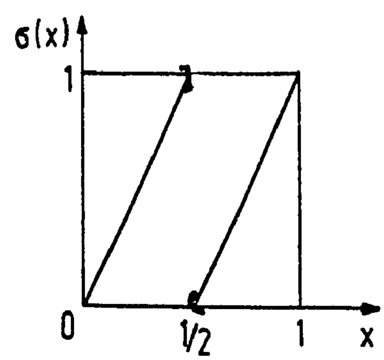


Figure 6

Example 2.2.2: [After 22]. *The Kaplan–Yorke map*:

$$x_{n+1} = 2x_n \ (\text{mod } 1)$$

$$y_{n+1} = ay_n + \cos 4\pi x_n.$$

has the Lyapunov numbers 2 and a; therefore the Lyapunov dimension is 1 + log2/log(1/a) if a < 1/2, and 2 if a ≥ 1/2. See figure 7.

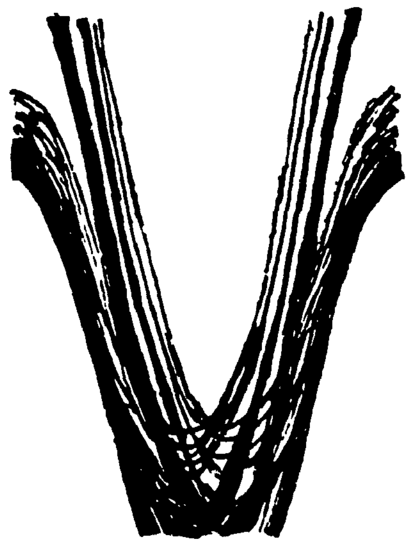


Figure 7

(for a = 1/2; taken from [22])

For more examples, see [14]. Now we turn to the computation of the Lyapunov numbers of the generalized baker's transformation and the Lyapunov dimension of it's attractor.

Computation of Lyapunov dimension:

The Jacobian of the generalized baker's transformation is diagonal and depends only on y.

$$J = \begin{bmatrix} L_2(y) & 0 \\ 0 & L_1(y) \end{bmatrix},$$

$$\text{where: } L_1(y) = \begin{cases} 1/\alpha & \text{if } y < \alpha \\ 1/(1-\alpha) & \text{if } y > \alpha \end{cases} \text{ and } L_2(y) = \begin{cases} \lambda_a & \text{if } y < \alpha \\ \lambda_b & \text{if } y > \alpha \end{cases}$$

and where $\alpha, \lambda_a, \lambda_b \leq 1/2$. $L_1(y_n)\ldots L_1(y_1) = (1/\alpha)^{n_\alpha}(1/\beta)^{n_\beta}$, where n_α is the number of times the orbit has been in the set $y < \alpha$ and n_β is the number of times the orbit has been in the set $y > \alpha$. Then $\lambda_1 = \lim_{n\to\infty}[L_1(y_n)...L_1(y_1)]^{1/n}$ which implies that:

$$\log\lambda_1 = \lim_{n\to\infty}[(n_\alpha/n)\log(1/\alpha) + (n_\beta/n)\log(1/\beta)] .$$

We know from section 1.4 that the map defined by (1.1b) is ergodic. Therefore by lemma 1.4.2, we have $\tau_{[0,\,\alpha]} = \lim_{n\to\infty} n_\alpha/n = \mu([0, \alpha]) = \alpha$ and similarly $\lim_{n\to\infty} n_\beta/n = \beta$. Thus:

$$\log\lambda_1 = \alpha\log(1/\alpha) + \beta\log(1/\beta) \tag{2.5}$$

Similarly:

$$\log\lambda_2 = \alpha\log\lambda_a + \beta\log\lambda_b \tag{2.6}$$

Then $\lambda_1 = (1/\alpha)^\alpha(1/\beta)^\beta$ and $\lambda_2 = (\lambda_a)^\alpha(\lambda_b)^\beta$. $\lambda_1 > 1$, since $1/\alpha > 1$ and $1/\beta > 1$. Let us prove that $\lambda_1\lambda_2 \le 1$, where the equality holds only if $\lambda_a = \lambda_b = 1/2$. We shall show that $\lambda_1 \le 2 \le 1/\lambda_2$ i.e., $(1/\alpha)^\alpha(1/\beta)^\beta \le 2 \le (1/\lambda_a)^\alpha(1/\lambda_b)^\beta$, where the last equalitiy holds only if $\lambda_a = \lambda_b = 1/2$. Since $1/\lambda_a \ge 1/2$ and $1/\lambda_b \ge 1/2$, then $2 \le 1/\lambda_2$, where the inequality holds only if $\lambda_a = \lambda_b = 1/2$. From elementary calculus, $\max(\lambda_1) = \max[(1/\alpha)^\alpha(1/(1-\alpha))^{1-\alpha}] = 2$ and is attained at $\alpha = 1/2$. Indeed:

$$\lambda_1' = (1/\alpha)^\alpha(1/(1-\alpha))^{1-\alpha}\ln[(1-\alpha)/\alpha] = \begin{cases} < 0 \text{ if } \alpha < 1/2 \\ = 0 \text{ if } \alpha = 1/2 \\ > 0 \text{ if } \alpha > 1/2 \end{cases}.$$

Then equality in $\lambda_1\lambda_2 \le 1$ implies that $d_L = 2$, by definition also if we put the values of λ_1 and λ_2 when the equality holds we see that $\log\lambda_1/\log(1/\lambda_2) = 1$, more than that when $\lambda_a = \lambda_b = 1/2$ the attractor is unit square, so that its dimension is 2. Therefore, the Lyapunov dimension of the attractor of the generalized baker's transformation is:

$$\begin{aligned} d_L &= 1 + \log\lambda_1/\log(1/\lambda_2) \\ &= 1 + H(\alpha)/(\alpha\log(1/\lambda_a) + \beta\log(1/\lambda_b)), \end{aligned} \tag{2.7}$$

where $H(\alpha) = \alpha\log(1/\alpha) + \beta\log(1/\beta)$ is called *the binary entropy function.*

Chapter 3

Metric Dimensions

In this chapter we present the notions of *capacity* (d_C) and *Hausdorff dimension* (d_H). Because we need a metric space to define them, we refer to them as *metric dimensions*. In sections 3.1 and 3.2 respectively, we present the notions of capacity and Hausdorff dimension. In section 3.3 we shall present relations between d_C, d_H and d_S and prove that under the open set condition (see definition 2.1.7) these three dimensions are equal when the attractor is obtained from an IFS. In the last part of section 3.3 we apply this result to compute the capacity and the Hausdorff dimension of the attractor of the generalized baker's transformation. To compute these dimensions, we look at the attractor itself instead of looking at the corresponding map, as we did in chapter 2 to compute the similarity and Lyapunov dimensions. In fact these dimensions are defined for any set.

3.1 Capacity

The capacity was originally defined by Kolmogorov [23] in 1958.

Definition 3.1.1: Let A be a bounded subset of a p–dimensional euclidean space $\mathbb{R}^p$, and $N(\varepsilon)$ be the minimum number of closed balls with radius ε needed to cover the set A.

Then the quantity $d_C(A)$ (sometimes writen as d_C) given by:

$$d_C(A) = \lim_{\varepsilon \to 0} \log N(\varepsilon)/\log(1/\varepsilon) \tag{3.1}$$

is called the *capacity* of the set A, provided the limit exists.

The following proposition shows that d_C is not greater than the dimension of the euclidean space.

Proposition 3.1.1: Let A and B be bounded subsets of $\mathbb{R}^p$ (with euclidean metric) such that $A \subset B$. Then $d_C(A) \leq d_C(B)$, provided both quantities exist. In particular $0 \leq d_C(A) \leq p$.

Proof: Let $N_1(\varepsilon)$ be the minimum number of closed balls needed to cover the set A and $N_2(\varepsilon)$ be the analogous quantity for the set B. Since the set A is contained in the set B, this implies that:

$$N_1(\varepsilon) \leq N_2(\varepsilon),\ \forall\ \varepsilon \text{ such that } 0 < \varepsilon < 1,$$

Which implies that:

$$0 \leq \log N_1(\varepsilon) / \log(1/\varepsilon) \leq \log N_2(\varepsilon) / \log(1/\varepsilon),\ \forall\ \varepsilon \text{ such that } 0 < \varepsilon < 1.$$

Now taking the limit of both sides, we get:

$$\lim_{\varepsilon \to 0} \log N_1(\varepsilon) / \log(1/\varepsilon) \leq \lim_{\varepsilon \to 0} \log N_2(\varepsilon) / \log(1/\varepsilon),$$

which implies that $d_C(A) \leq d_C(B)$. If we replace B by a unit hypercube (p–dimensional) (see example 3.1.1) we see that $d_C(A) \leq p$.

■

Definition 3.1.2: We shall say that a set A *intersects strongly* a set B if and only if there exists $x \in \mathrm{int}(A) \cap B$.

Remark 3.1.1: The set A of definition 3.1.2 must have a non–void interior to intersect strongly a set B. The definition is not symmetric. B does not necessarily intersect strongly the set A.

Theorem 3.1.2: [The Box Counting theorem] [After 2]. Let A be a bounded subset of $\mathbb{R}^p$, with euclidean metric. Cover $\mathbb{R}^p$ with closed just touching square boxes of side length $(1/2)^n$. Let $N_n(A)$ denote the number of boxes of side length $(1/2)^n$ which intersect strongly the set A. If the following limit exists:

$$\lim_{n\to\infty} \log N_n(A) / \log(2^n) \tag{3.2}$$

then the capacity of A is equal to that limit.

Proof: See the following figure 8. Let $N_n(A, 1/2^n)$ be the minimum number of closed balls with radius $1/2^n$ to cover A. We observe that $2^{-p}N_{n-1} \leq N_n(A, 1/2^n) \leq N_{k(n)}$ for all $p = 1, 2, \ldots$ and $n = 1, 2, \ldots$, where $k(n)$ is the smallest integer $k \geq n - 1 + (1/2)\log_2 p$. The first inequality will follow from lemma 3.1.3. The second inequality follows because a box of side length δ can fit inside a ball of radius ε provided $\varepsilon \geq p^{1/2}(\delta/2)$. Indeed by Pythagoras theorem half of the diameter of a box of side δ is $(p)^{1/2}(\delta/2)$. Therefore δ is less than or equal to the $2\varepsilon p^{-1/2}$. Let $\delta = (1/2)^k$ and $\varepsilon = (1/2)^n$. The unknown k must satisfy the following inequality: $2^{-k} \leq (1/2)^{n-1}\, p^{-1/2}$, which implies that $k \geq n - 1 + (1/2)\log_2 p$. On one hand:

$$\begin{aligned}\lim_{n\to\infty}\left\{\log(N_{k(n)} / \log(2^n)\right\} &= \lim_{n\to\infty} \log 2^{k(n)} \log N_{k(n)} / \log 2^n \log 2^{k(n)} \\ &= \lim_{n\to\infty} \{k(n)/n\}[\log N_{k(n)}/\log 2^{k(n)}] \\ &= \lim_{n\to\infty} \log N_{k(n)}/\log 2^{k(n)}, \text{ since } \lim_{n\to\infty} k(n)/n = 1.\end{aligned}$$

On the other hand:

$$\lim_{n\to\infty}\{\log 2^{-p}N_{n-1} / \log 2^n\} = \lim_{n\to\infty}\{\log 2^{n-1}\log(2^{-p}N_{n-1}) / \log 2^n \log 2^{n-1}\}$$

$$= \lim_{n\to\infty} \{(n-1)\log 2 \log N_{n-1}/n\log 2 \log 2^{n-1}\}$$
$$- \lim_{n\to\infty} \{p\log 2 \log 2^{n-1}/n\log 2 \log 2^{n-1}\}$$
$$= \lim_{n\to\infty} [\{(n-1)/n\}\log N_{n-1}/\log 2^{n-1}] - \lim_{n\to\infty} p/n$$
$$= \lim_{n\to\infty} \log N_{n-1}/\log 2^{n-1},$$

since $\lim_{n\to\infty} (n-1)/n = 1$ and $\lim_{n\to\infty} p/n = 0$. ∎

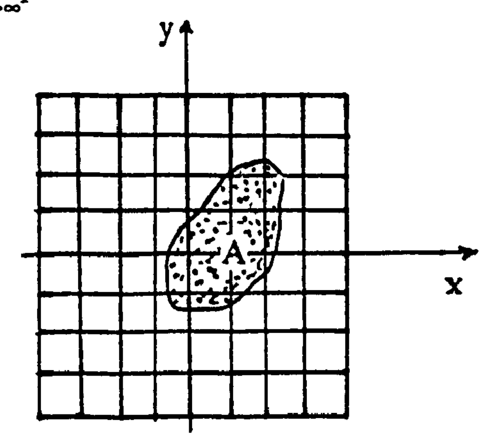


Figure 8

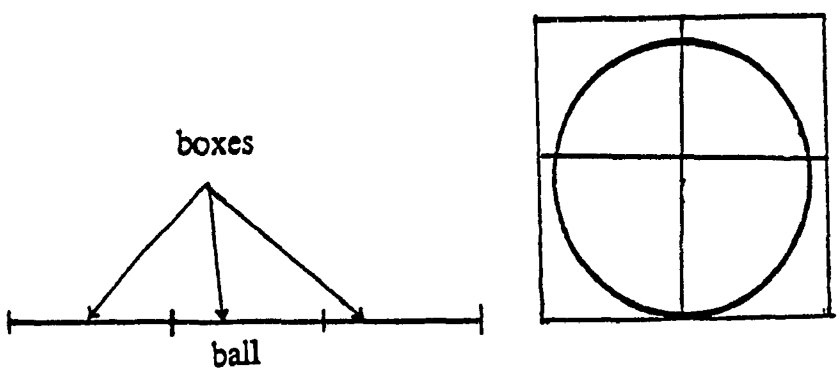
Remark 3.1.2: This theorem proves that the definition of capacity does not depend on balls. In fact we can use any simple geometric object (triangle for instance) to cover the attractor. Moreover there is nothing special about the number $(1/2)^n$, we can take any ε^n $(0 < \varepsilon < 1)$.

Lemma 3.1.3: A closed ball of radius $1/2^n$ can intersect strongly at most 2^p closed boxes of side length $1/2^{n-1}$.

Proof: The proof will be done by induction on the dimension of the space. Let $p = 1$. It is obvious that a closed ball of radius $1/2^n$ intersects strongly at most 2 closed boxes of side length $1/2^{n-1}$ (see figure 9a). Suppose that the statement is true for $1 \le p \le k-1$. Let x_i be the i^{th} coordinate in $\mathbb{R}^k$. Let $x_k = c$ be the hyperplane passing through the center of a closed ball B_k of radius $1/2^n$. The union of the closed just touching square boxes in R^p

makes a k–dimensional grid. If we project this grid on the hyperplane, we obtain a (k–1)–dimensional grid (see figure 9b). By induction hypothesis, the (k–1)–dimensional ball $B_{k-1} = B_k \cap \{x_i = c\}$ of radius $1/2^n$ intersects strongly at most 2^{k-1} closed (k–1)–dimensional boxes of side length $1/2^{n-1}$. Each such box can be seen as the projection of the base of a k–dimensional box. Since the height of two k–dimensional boxes cover the diameter of B_k (still by induction hypothesis), then there exists at most $2(2^{k-1})$ closed boxes intersected strongly by B_k.

■



Figures 9 (a and b)

Remark 3.1.3: In [2] Barnsley uses the word intersection instead of strong intersection in theorem 3.1.2. But his proof is wrong according to his statement. A closed ball in $\mathbb{R}$ with radius 1/2 can intersect 3 closed boxes (intervals) as it is shown in figure 9a.

The first of the following examples shows that the notion of capacity fits with the standard notion of dimension of a regular geometric object.

Example 3.1.1: *A hypercube.* The capacity of a p–dimensional unit hypercube is p. Indeed we can cover a unit hypercube with 2^p hypercubes each with side length (1/2) at the first stage. So we can see that $N_n = 2^{pn}$, with $\varepsilon^n = (1/2)^n$. Equation (3.2) yields that the capacity of the unit hypercube is p.

Example 3.1.2: *A Cantor set.* Let us take the attractor of the IFS of example 2.1.4. Let $\varepsilon = k$ $(0 < k < 1/2)$, then $N_n = 2^n$. Equation (3.2) yields $d_C = \log 2 / \log(1/k)$.

Example 3.1.3: [After 8]. *The Kieβwetter's curve* According to Theorem 2.1.8, there is a unique attractor for the following IFS : $\{ \mathbb{R}^2 : f_1, f_2, f_3, f_4 \}$ where:

$$f_1(x, y) = \begin{bmatrix} 1/4 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad f_2(x, y) = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix},$$

$$f_3(x, y) = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \qquad f_4(x, y) = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/2 \end{bmatrix}.$$

The attractor is a continuous curve called the Kieβwetter's curve (see figure 10). Since at the n^{th} stage there will be 4^n rectangles each with side lengths 4^{-n} and 2^{-n+1} and each of the rectangles of size 4^{-n} and 2^{-n+1} is the union of 2^{n+1} squares of side lengths 4^{-n}. In other words the set can be covered by $(4^n\, 2^{n+1})$ squares each with side length 4^{-n}. Then equation (3.2) yields $d_C = \lim_{n \to \infty} \log(4^n\, 2^{n+1}) / \log(4^n) = \lim_{n \to \infty} (3n + 1)\log 2 / 2n\log 2 = 3/2$. It should be mensioned that f_n is not a similitude, so the similarity dimension is not defined.

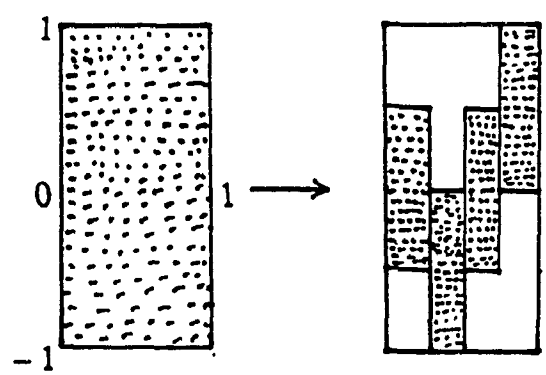


figure 10

3.2 Hausdorff dimension

The Hausdorff dimension was originaly defined by F. Hausdorff [19] in 1919. It marked the beginning of the study of geometric measure theory [13, 29]. We give the original definition of the Hausdorff dimension which is different from that of Farmer *et al* [12].

Definition 3.2.1a and b: Let A be a non–empty subset of $\mathbb{R}^p$. We define the *diameter of A* as $\mathrm{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. If $A \subset \bigcup_{n=1}^{k} A_n$ and $\mathrm{diam}(A_n) \leq \varepsilon$ for each n, we say that $\{A_n\}$ is an *ε–cover of A*

Definitions 3.2.2a and b: Let A be a subset of $\mathbb{R}^p$ and d be a non-negative number. Consider an ε–cover $\{A_n\}$ of A. For $\mathrm{diam}(A_n) \leq \varepsilon$ define the *quantity* $L_d^{\varepsilon}(A)$ (some times written as L_d^{ε}) by:

$$L_d^{\varepsilon}(A) = \inf \sum (\mathrm{diam}(A_n))^d, \tag{3.3a}$$

where the infimum extends over all countable ε–covers of A. The following limit is called the *Hausdorff d–dimensional outer measure*:

$$L_d(A) = \lim_{\varepsilon \to 0} L_d^{\varepsilon}(A). \tag{3.3b}$$

$L_d(A)$ is sometimes written as L_d.

Remark 3.2.1: The preceding limit exists but may be infinite.

Remark 3.2.2: Farmer *et al* [12] are using ε–covers of cubes only but a different measure is obtained. See Besicovitch ([5] chapter 3) who compares Hausdorff measure with Hausdorff measure obtained with ε–covers of balls only.

Remark 3.2.3: Note that an equivalent definition of Hausdorff measure is obtained if the infimum in (3.3a) is taken over ε–covers of A by convex sets rather than by arbitrary sets since any set lies in a convex set of the same diameter.

Example 3.2.1: Any countable set of points has Hausdorff d–dimensional measure zero. Indeed we cover it by the set of its points, each point having a diameter equal to zero.

Lemma 3.2.1: Let $A \subset \mathbb{R}^p$. If $L_d(A) < \infty$, then $L_{d'}(A) = 0$ for any $d' > d$.

Proof: We may assume that $d > 0$ and $L_d(A) < \infty$. Let $a > L_d(A)$. Therefore:

$$a + \delta(\varepsilon) \geq L_d^{\varepsilon}(A)$$

for small ε where $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. This implies that:

$$a + \delta(\varepsilon) \geq \inf \sum (\mathrm{diam}(A_n))^d$$

$$\varepsilon^{d'-d}(a + \delta(\varepsilon)) \geq \inf \sum (\mathrm{diam}(A_n))^d \varepsilon^{d'-d}$$

$$\geq \inf \sum (\mathrm{diam}(A_n))^d (\mathrm{diam}(A_n))^{d'-d}$$

$$\geq \inf \sum (\mathrm{diam}(A_n))^{d'}$$

Taking the limit, we get:

$$\lim_{\varepsilon \to 0} (\varepsilon^{d'-d}(a + \delta(\varepsilon))) \geq \lim_{\varepsilon \to 0} \inf \sum (\mathrm{diam}(A_n))^{d'}$$

$$0 \geq \lim_{\varepsilon \to 0} \inf \sum (\mathrm{diam}(A_n))^{d'} = L_{d'}(A).$$

This implies that $L_{d'}(A) = 0$.

■

Corollary 3.2.2: There exists a critical point d_H of L_d such that L_d is zero for $d > d_H$ and infinite for $d < d_H$.

Definition 3.2.3: The critical point $d_H(A)$ (sometimes written as d_H) of the above corollary is called the *Haudorff dimension of the set A*.It is defined as:

$$d_H(A) = \inf\{d : L_d(A) < \infty\} = \sup\{d : L_d(A) > 0\}.$$

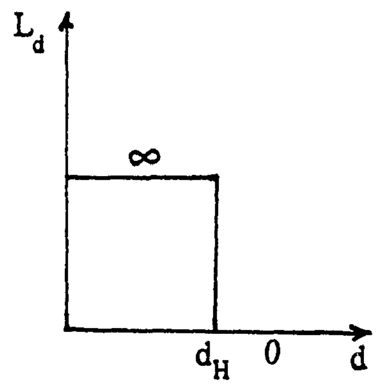


Figure 11

Proposition 3.2.2: Let A and B be any subsets of $\mathbb{R}^p$ (with euclidean metric) such that $A \subset B$. Then $d_H(A) \leq d_H(B)$. In particular $0 \leq d_H(A) \leq p$.

Proof: First of all we shall show that the Hausdorff dimension of a hypercube is p. Let C be a hypercube in $\mathbb{R}^p$ with side length r ($r \in \mathbb{N}$). Divide C into k^p ($k \in \mathbb{N}$) subcubes of side length r/k in the obvious way. In the equation (3.3a) if we take $\varepsilon = (r/k)$, then:

$$L_p^{\varepsilon}(C) \leq k^p((r/k))^p \leq r^p.$$

Now take the limit of the above expression:

$$\lim_{\varepsilon \to 0} L_p^{\varepsilon}(C) \leq r^p < \infty.$$

Thus if $s > p$, then by corollary 3.2.2 $L_s(C) = 0$ and so $d_H(C) = p$. Consequently $d_H(\mathbb{R}^p) = p$, since $d_H(C) = p$ for any $r \in \mathbb{N}$. Secondly we shall prove that $d_H(A) \leq d_H(B)$. If $A \subset B$, it is obvious that $L_d^{\varepsilon}(A) \leq L_d^{\varepsilon}(B)$ and so $L_p(A) \leq L_p(B)$. Then by definition $d_H(A) \leq d_H(B)$. Therefore for any $A \subset \mathbb{R}^p$, $d_H(A) \leq d_H(\mathbb{R}^p) = p$.

■

Example 3.2.2: *The classical Cantor set.* We shall prove that its Hausdorff dimension is log2 / log3. We shall use the last remarks to compute the Hausdorff dimension. Only intervals are convex sets on the real line.

Let $\{A_n\}$ be an ε–cover of A. Without loss of generality, we can consider an ε–cover $\{A_n\}$ such that $\mathrm{diam}(A_n) > 0$. Suppose for a while that the $3^{-n} \le \mathrm{diam}(A_n) \le 3^{-m}$ $(n > m)$. Let us show that under this restriction (about the lower bound of $\mathrm{diam}(A_n)$) $\inf\sum(\mathrm{diam}(A_n))^d = 2^n 3^{-nd}$, for any $d \ge 0$. If two intervals in an ε–cover $\{B_n\}$ of A has a nonvoid intersection, then:

$$\sum(\mathrm{diam}(B_n))^d \ge \sum(\mathrm{diam}(A_n))^d ,$$

where $\{A_n\}$ is an ε–cover with disjoint intervals. Moreover, if $\{B_n\}$ is an ε–cover with disjoint intervals such that $\mathrm{diam}(B_j) > 3^{-n}$ for one j then:

$$\sum(\mathrm{diam}(B_n))^d \ge \sum(\mathrm{diam}(A_n))^d = 2^n 3^{-nd} ,$$

where $\mathrm{diam}(A_n) = 3^{-n}$. So we get:

$$\inf\sum(\mathrm{diam}(A_n))^d = 2^n 3^{-nd},$$

where $3^{-n} \le \mathrm{diam}(A_n) \le 3^{-m}$ $(n > m)$. If $n \longrightarrow \infty$, $3^{-n} \longrightarrow 0$ and $L_d^{\varepsilon}(A) = \lim_{n \to \infty} 2^n 3^{-nd}$ for all $\varepsilon > 0$. Then $L_d = \lim_{n\to\infty} 2^n 3^{-nd}$. If $d = \log2/\log3$, one can check easily that $L_d = 1$.

Example 3.2.3: The Hausdorff dimension of the Kiesswetter's curve is 3/2.

1° Upper bound: According to the example 3.1.3 at the n^{th} stage the attractor has 4^n rectangles each with side lengths 4^{-n} and 2^{-n+1} and each of these ectangles is the union of 2^{n+1} squares of side lengths 4^{-n}. So for $\varepsilon = 2^{1/2} 4^{-n}$, we have from equation(3.3a) that $L^{\varepsilon}_{d\ (d=3/2)} \le 4^n 2^{n+1}(2^{1/2}.4^{-n})^{3/2}$. Therefore from equation (3.3b) we have $L_{3/2} \le 2^{7/4} < \infty$. Hence the Hausdorff dimension of the Kiesswetter's curve is less than or equal to 3/2.

2° Lower bound: For the lower bound see [8]. Complicated and too long.

3.3 Relations between d_C, d_H and d_S

In this section we shall present some relations between capacity, Hausdorff dimension and similarity dimension. We shall give the computation of the metric dimensions of the attractor of the generalized baker's transformation in a different way than that given in [12].

Proposition 3.3.1: The capacity of a set is greater than or equal to it's Hausdorff dimension i.e., $d_C \geq d_H$.

Proof: Consider an ε-cover consisting of balls of equal diameter ε i.e., $\mathrm{diam}(A_n) = \varepsilon$. Then due to the infimum in equation (3.4), we see that:

$$\bar{L}_d^{\varepsilon} = \sum_{1}^{N(\varepsilon)} (\mathrm{diam}(A_n))^d = N(\varepsilon)(\mathrm{diam}(A_n))^d \text{ satisfies } \bar{L}_d^{\varepsilon} \geq L_d^{\varepsilon}.$$

Let us assume that $d = d_C$ for instance, then $\bar{L}_{d_C}^{\varepsilon} \geq L_{d_C}^{\varepsilon}$ and:

$$\lim_{\varepsilon \to 0} \bar{L}_{d_C}^{\varepsilon} \geq \lim_{\varepsilon \to 0} L_{d_C}^{\varepsilon}$$

$$\lim_{\varepsilon \to 0} N(\varepsilon)\varepsilon^{d_C} \geq \lim_{\varepsilon \to 0} L_{d_C}^{\varepsilon}.$$

It is an immediate consequence of (3.1) that:

$$\lim_{\varepsilon \to 0} N(\varepsilon)\varepsilon^{d_C} = 1$$

when ε represents the radius of a ball. But by lemma 3.3.2, we have the same limit when ε represents the diameter of the ball. This implies that $\lim_{\varepsilon \to 0} L_{d_C}^{\varepsilon}$ is zero if $d_C > d_H$ or finite if $d_C = d_H$.

∎

Lemma 3.3.2: Let $N_D(\varepsilon)$ be the minimum number of closed balls with diameter ε to cover a set A and $N(\varepsilon/2)$ be the corresponding number of closed balls with radius $\varepsilon/2$.

Then:

$$\lim_{\varepsilon\to 0}\log N_D(\varepsilon)/\log(1/\varepsilon) = \lim_{\varepsilon\to 0}\log N(\varepsilon/2)/\log(2/\varepsilon),$$

provided one of the limit exists.

Proof: Note that $N_D(\varepsilon) = N(\varepsilon/2)$. Then:

$$\begin{aligned}\lim_{\varepsilon\to 0}\log N(\varepsilon/2)/\log(2/\varepsilon) &= \lim_{\varepsilon\to 0}[\log N(\varepsilon/2)/\{\log 2 + \log(1/\varepsilon)\}] \\ &= \lim_{\varepsilon\to 0}[\log N(\varepsilon/2)/\log(1/\varepsilon)] \\ &= \lim_{\varepsilon\to 0}\log N_D(\varepsilon)/\log(1/\varepsilon),\end{aligned}$$

since log2 does not make any difference in the limit.

■

Farmer *et al* [12] believe that for attractors the capacity and the Hausdorff dimension are generally equal (conjecture 1.3.1), while it is still possible to construct simple examples of sets where the Hausdorff dimension and the capacity are not equal.

Example 3.3.1: [After 12]. For the set of numbers $S = \{1, 1/2, 1/3, 1/4, \ldots\}$, the Hausdorff dimension is 0 while its capacity is 1/2.

The Hausdorff dimension of the set S is 0. Indeed we can cover the set by a countable set of points, so the Hausdorff dimension is 0. The computation of the capacity is more involved. We shall use the box counting theorem (Theorem 3.1.2). Let $\varepsilon_n = (1/2)^{2n}$, then we shall show that $2^n \le N_n \le 2^{n+1}$, where N_n is the number of boxes to cover S. Assuming these inequalities, we have:

$$1/2 = \log 2^n/\log 2^{2n} \le \log N_n/\log 2^{2n} \le \log 2^{n+1}/\log 2^{2n},$$

$$1/2 \le \lim_{n\to\infty}\log N_n/\log 2^{2n} \le \lim_{n\to\infty}\log 2^{n+1}/\log 2^{2n} = \lim_{n\to\infty}[\{(n+1)/2n\}\log 2/\log 2^{2n}] = 1/2.$$

Now we have to show that:

$$2^n \le N_n \le 2^{n+1}.$$

Let us start with the upper bound. Let us count the number of intervals with length $1/2^{2n}$ needed to cover $S \cap [1/2^{2n}, 1]$, one being needed to cover $S \cap [0, 1/2^{2n}]$. Since $1/2^n - 1/(2^n-1) = 1/(2^{2n}-2^n) > 1/2^{2n}$ the distance between two consequetive points of $S \cap [1/2^n, 1]$ is greater than $1/2^{2n}$. Therefore we need 2^n intervals to cover the points in $S \cap [1/2^n, 1]$. Let k be the minimum number of intervals needed to cover $[1/2^{2n}, 1/2^n[$. Then $k/2^{2n} = 1/2^n - 1/2^{2n} = (2^n-1)/2^{2n}$ and $k = 2^n-1$, so we need at most k intervals to cover $S \cap [1/2^{2n}, 1/2^n[$. Thus $N_n = 1 + 2^n + (2^n - 1) = 2^{n+1}$. For the lower bound: since we need at least 2^n intervals to cover the points in $S \cap [1/2^n, 1]$, then $2^n \leq N_n$.

Now we shall present a theorem originally given by Hutchinson [21] in 1981. It makes the computation of the capacity and the Hausdorff dimension of an attractor very easy.

Theorem 3.3.3: Let $\{ X: f_n, n = 1, 2, ..., k\}$ be an IFS, and all f_n's be similitudes and A be the attractor of the IFS. If the open set condition holds, then $d_H(A) = d_C(A) = d_S(A)$, i.e. the Hausdorff dimension, the capacity and the similarity dimension are equal.

Proof of $d_H(A) = d_S(A)$: 1^o Upper bound: Let us show that $d_H \leq d_S$, i.e., $L_{d_S}(A) < \infty$. Let $f_{n_1...n_p} = f_{n_1} o...o f_{n_p}$ and $f_{n_1...n_p}(A) = A_{n_1...n_p}$. By theorem 2.1.8, we know that A is invariant with respect to f_n, n = 1, ..., k. Observe the following:

$$A = \bigcup_{n=1}^{k} f_n(A) = \bigcup_{n,m} f_n(f_m(A)) = \bigcup_{n,m} f_{nm}(A) = \bigcup_{n,m} A_{nm} = ... = \bigcup_{n_1,...,n_p} A_{n_1...n_p}.$$

Since $\operatorname{diam}(A_{n_1...n_p}) \leq \max(r_n)^p \operatorname{diam}(A) \longrightarrow 0$ as $p \longrightarrow \infty$, $\{A_{n_1...n_p}\}$ is an ε-cover of A for sufficiently large p. By equation (3.3a), we have:

$$L^{\varepsilon}_{d_S}(A) = \inf \sum_{n_1...n_p} (\operatorname{diam} A_{n_1...n_p})^{d_S} = \inf \sum_{n_1...n_p} r_{n_1}^{d_S} ... r_{n_p}^{d_S} (\operatorname{diam} A)^{d_S} = (\operatorname{diam} A)^{d_S},$$

since $\sum_{n_1 \ldots n_p} r_{n_1}^{d_S} \ldots r_{n_p}^{d_S} = (\sum r_n^{d_S})^p = 1$, by the definition of similarity dimension. By equation (3.3b) we have $L_{d_S}(A) < \infty$.

2° Lower bound: see either [9] or [21]. ■

Example 3.3.2: According to the example 2.1.4 we know that the open set condition holds for a Cantor set and according to the example 2.1.4 we know that the similarity dimension of the corresponding IFS is log2 / log(1/k). Now according to the above theorem $d_H = d_C = d_S = \log 2 / \log(1/k)$.

Now we shall compute the capacity and the Hausdorff dimension of the attractor of the generalized baker's transformation.

Computation of capacity and Hausdorff dimension: Since the attractor is a product of the interval [0, 1] along the y–axis and a (assymetric) Cantor set along the x–axis, we shall compute the capacity $\bar{d}_C$ and the Hausdorff dimension $\bar{d}_H$ of the attractor along the x–axis to obtain $d_C = 1 + \bar{d}_C$ and $d_H = 1 + \bar{d}_H$. Using the attractor of the IFS of page 20 (Computation of the similarity dimension along the x–axis), we obtain, according to the above theorem, $\bar{d}_H = \bar{d}_C = \bar{d}_S$.

Chapter 4

Probabilistic dimensions

In this chapter we shall present the notions of *information* and *pointwise dimensions*. These dimensions require both a metric and a probability measure for their definition, and hence we shall refer to them as *probabilistic dimensions*. As for similarity and Lyapunov dimensions, information and pointwise dimensions are related to a mapping. The organization of this chapter is following: in section 4.1 we shall present the notion of the natural measure. In section 4.2 and 4.3 respectivly, we present the notion of information dimension and pointwise dimension and compute the information dimension and pointwise dimension of the attractor of the generalized baker's transformation.

4.1 Natural measure of an attractor

To define the information and pointwise dimensions of an attractor A, we shall take into account a natural measure on A. In computing d_C from equation (3.1), all cubes used in covering the attractor are equally important even though the frequencies with which an orbit on the attractor visits these cubes may be different. In order to compute information and pointwise dimensions, we need to consider not only the attractor itself, but the relative frequency with which a typical orbit visits different regions of the attractor as well. We can say that some regions of the attractor are more probable than others, or alternatively we may speak of a probabily measure on the attractor. We define the natural measure of an attractor as follows:

Definition 4.1.1a and b: Let A be the attractor of a mapping F: $\mathbb{R}^p \longrightarrow \mathbb{R}^p$. For a subset S of $\mathbb{R}^p$, an initial condition x_1 in the basin of attraction of A and an ε–neighborhood $V(\varepsilon)$ of S, let $\mu_\varepsilon(x_1, S) = \tau_{V(\varepsilon)}(x_1)$ (see definition 1.4.4b and remark 1.4.2) and let:

$$\mu(x_1,S) = \lim_{\varepsilon \to 0} \mu_\varepsilon(x_1, S) \tag{4.1}$$

If $\mu(x_1, S)$ is the same for almost every x_1 in the basin of attraction, then we denote this value by $\mu(S)$ and call it the *natural measure of the attractor A*.

Remark 4.1.1: When S is a ball or a cube, $V(\varepsilon)$ is not needed in the definition of $\mu(S)$, since $\mu(x_1,S)$ may be rather defined as the fraction of time the trajectory originating at x_1 spends in S; in both cases we obtain the same value for $\mu(S)$. But with this definition, a difficulty occur in special sets S. For example, if the attractor has zero phase–space volume and we let the set S be the attractor itself, then for almost every x_1 in the basin of attraction, $\mu(x_1,S)$ is zero (for finite length trajectories, the orbit approaches S but is not on S). A proper definition should give $\mu(S) = 1$ for this set S.

Notation 4.1.1: Let F be the generalized baker's transformation, $T_1 = [0, 1]\times[0, \alpha]$ and $T_2 = [0, 1]\times[\alpha, 1]$. The different 2^n strips obtained from T_1 and T_2 (see figure 1) after the n^{th} iterate of F will be denoted by $S_n(m)$ according to the following rule:

$S_1(1) = F(T_1)$ and $S_1(2) = F(T_2)$ and:

$S_n(m) = F(S_{n-1}(m)\cap T_1)$, for $1 \le m \le 2^{n-1}$ or $= F(S_{n-1}(m)\cap T_2)$, for $2^{n-1}+ 1 \le m \le 2^n$.

Example 4.1.1: *The natural measure of a strip obtained from the n^{th} iterate of the generalized baker's transformation* (see figure 1). After one iterate of the map we get two strips, the first strip $S_1(1)$ with width λ_a, the second strip $S_1(2)$ with width λ_b. Let us recall that n_α and n_β are respectively the numbers of times the orbit has been in the set $y < \alpha$ and $y > \alpha$ (see computation of Lyapunov dimension of the attractor of the generalized baker's transformation in chapter 2). Because the transformation is ergodic, we have $\lim_{n\to\infty} n_\alpha/n = \alpha$ and $\lim_{n\to\infty} n_\beta/n = \beta$ (see lemma 1.4.2). Then $\mu(S_1(1)) = \alpha$ and $\mu(S_1(2)) = \beta$. After the second iterate, we get four strips $S_2(m)$, where $\mu(S_2(m))$ is the m^{th} number in the "non–commutative" product $(\alpha + \beta)^{*2}$, that is the m^{th} number among α^2, $\alpha\beta$, $\beta\alpha$ and β^2 in this order. In general, the n^{th} iterate gives birth to 2^n strips $S_n(m)$, where $\mu(S_n(m))$ is the m^{th} term in the "non–commutative" product $(\alpha + \beta)^{*n}$, $m = 1,\ldots, 2^n$.

4.2 Information dimension

The information dimension was originaly defined by Balatoni and Renyi [1] in 1956. The *information dimension is a generalization of the capacity.*

Notation 4.2.1: Let A be an attractor of a mapping F, $N(\varepsilon)$ be the minimum number of hypercubes with side length ε needed to cover the set A and let $p_i = \mu(C_i)$, where C_i is the i^{th} cube in the covering of A.

Definition 4.2.1a and b: The *entropy of A* (denoted by $I(\varepsilon)$) is defined as:

$$I(\varepsilon) = \sum_{1}^{N(\varepsilon)} p_i \log(1/p_i)$$

The *information dimension of A* (denoted by $d_I(A)$ or just d_I) is defined as:

$$d_I(A) = \lim_{\varepsilon \to 0} I(\varepsilon) / \log(1/\varepsilon) \tag{4.2}$$

The quantity $I(\varepsilon)$ is the amount of information necessary to specify the state of a system to within an accuracy ε, or equivalently, it is the information obtained in making a measurement that is uncertain by an amount ε. Since for small ε, $I(\varepsilon) \approx d_I \log(1/\varepsilon)$, d_I may be viewed as telling how fast the information necessary to specify a point on the attractor increases as ε decreases. For more details see [10, 11].

Propositon 4.2.1: $d_C(A) \geq d_I(A)$. In particular:

a) If all cubes have equal measure i. e., $p_i = p$ (say), then $d_C(A) = d_I(A)$.

b) If condition a) is not satisfied, then $d_C(A) > d_I(A)$.

Proof:

a) Since $p_i = p$ for any i and:

$$\sum_{1}^{N(\varepsilon)} p_i = 1,$$

then $p = 1/N(\varepsilon)$. Therefore:

$$I(\varepsilon) = N(\varepsilon) p \log(1/p) = \log N(\varepsilon).$$

From equation (4.2) and the definition of capacity (equation (3.1)), it follows that $d_I = d_C$.

b) In order to show that $d_I < d_C$, it is sufficient to show that $I(\varepsilon) < \log N(\varepsilon)$. We have:

$$I(\varepsilon) = \sum_{1}^{N(\varepsilon)} p_i \log(1/p_i) = \sum_{1}^{N(\varepsilon)} \log(1/p_i)^{p_i}$$

$$= \log \prod_{i=1}^{N(\varepsilon)} (1/p_i)^{p_i} < \log \sum_{1}^{N(\varepsilon)} p_i(1/p_i) = \log N(\varepsilon).$$

The inequality follows from inequality (2.5.2) in Hardy *et al* [18], that is:

$$\prod_{i=1}^{N}(a_i)^{q_i} < \sum_{i=1}^{N} q_i a_i ,$$

where $a_i > 0$, $q_i \geq 0$ and $\sum_{i=1}^{N} q_i = 1$.

■

Example 4.2.1: *The classical Cantor set.* Let us take the IFS of example 2.1.2. Let $\varepsilon = (1/3)^n$. When $n = 1$, $N(\varepsilon) = 2$ and $p_1 = p_2 = 1/2$, since each interval has equal length. In general, when $n = n_0$, $N(\varepsilon) = 2^{n_0}$ and $p_i = 1/2^{n_0}$ for any i. Using (4.2), we get $d_I = \log 2/\log 3$.

Now we turn to the computation of the information dimension of the attractor of the generalized baker's transformation.

Computation of the information dimension: We shall use the old technique, that is, we let $d_I = 1 + \bar{d}_I$, where $\bar{d}_I$ is the information dimension of the attractor along the x–axis. Let $I(\varepsilon) = I_a(\varepsilon) + I_b(\varepsilon)$, where $I_a(\varepsilon)$ is the entropy for the strip $[0, \lambda_a]$ and $I_b(\varepsilon)$ is the analogous quantity for the strip $[1/2, 1/2 + \lambda_b]$. From the scaling property of the transformation, covering the strip $[0, \lambda_a]$ at resolution $\varepsilon\lambda_a$ requires $N(\varepsilon)$ strips of width ε. Since $\alpha = \lim_{n \to \infty} n_\alpha/n$, where n_α is the number of times the orbit has been in the set $y < \alpha$, the probability of finding a point of the attractor in the i^{th} strip of width $\varepsilon\lambda_a$ becomes αp_i, where p_i is the probability of finding a point of the attractor in the i^{th} strip of width ε. Then:

$$I_a(\varepsilon\lambda_a) = \sum_{1}^{N(\varepsilon)} \alpha p_i \log[1/(\alpha p_i)] = \alpha \sum_{1}^{N(\varepsilon)} p_i[\log(1/\alpha) + \log(1/p_i)]$$

$$= \alpha\sum_{1}^{N(\varepsilon)} p_i \log(1/\alpha) + \alpha\sum_{1}^{N(\varepsilon)} p_i \ \log(1/p_i) = \alpha\log(1/\alpha) + \alpha\sum_{1}^{N(\varepsilon)} p_i \ \log(1/p_i)$$

$$= \alpha[\log(1/\alpha) + \sum_{1}^{N(\varepsilon)} p_i \ \log(1/p_i)] = \alpha[\log(1/\alpha) + I(\varepsilon)].$$

Therefore $I_a(\varepsilon) = \alpha[\log(1/\alpha) + I(\varepsilon/\lambda_a)]$. Similarly $I_b(\varepsilon) = \beta[\log(1/\beta) + I(\varepsilon/\lambda_b)]$, where $\beta = 1-\alpha$. Thus:

$$\begin{aligned} I(\varepsilon) &= I_a(\varepsilon) + I_b(\varepsilon) \\ &= \alpha[\log(1/\alpha) + I(\varepsilon/\lambda_a)] + \beta[\log(1/\beta) + I(\varepsilon/\lambda_b)] \\ &= \alpha\log(1/\alpha) + \beta\log(1/\beta) + \alpha I(\varepsilon/\lambda_a) + \beta I(\varepsilon/\lambda_b) \\ &= H(\alpha) + \alpha I(\varepsilon/\lambda_a) + \beta I(\varepsilon/\lambda_b), \end{aligned}$$

where $H(\alpha)$ is the binary entropy function. To simplify the notations, but without loss of rigour, we assume that $I(\varepsilon) = \bar{d}_I \log(1/\varepsilon)$, since $I(\varepsilon) \approx \bar{d}_I \log(1/\varepsilon)$ for very small ε. Therefore $I(\varepsilon/\lambda_a) = \bar{d}_I \log(\lambda_a/\varepsilon)$ and $I(\varepsilon/\lambda_b) = \bar{d}_I \log(\lambda_b/\varepsilon$. Putting these values in the above equation, we get:

$$\begin{aligned} \bar{d}_I \log(1/\varepsilon) &= H(\alpha) + \alpha\bar{d}_I \log(\lambda_a/\varepsilon) + \beta\bar{d}_I \log(\lambda_b/\varepsilon) \\ H(\alpha) &= \bar{d}_I \log(1/\varepsilon) - [\alpha\bar{d}_I \log(\lambda_a/\varepsilon) + \beta\bar{d}_I \log(\lambda_b/\varepsilon)\,] \\ &= \bar{d}_I\,[\log(1/\varepsilon) - \log(\lambda_a/\varepsilon)^{\alpha} - \log(\lambda_b/\varepsilon)^{\beta}] \\ &= \bar{d}_I\,[\log(1/\varepsilon) - \log(1/\varepsilon)^{\alpha} - \log(1/\varepsilon)^{\beta} - \log(\lambda_a)^{\alpha} - \log(\lambda_b)^{\beta}] \\ &= \bar{d}_I\,[\log(1/\varepsilon) - \log(1/\varepsilon)^{\alpha+\beta} - \log(\lambda_a)^{\alpha} - \log(\lambda_b)^{\beta}] \\ &= \bar{d}_I\,[\log(1/\varepsilon) - \log(1/\varepsilon) - \log(\lambda_a)^{\alpha} - \log(\lambda_b)^{\beta}] \\ &= \bar{d}_I\,[-\log(\lambda_a)^{\alpha} - \log(\lambda_b)^{\beta}] \\ &= \bar{d}_I\,[\log(1/\lambda_a)^{\alpha} + \log(1/\lambda_b)^{\beta}]. \end{aligned}$$

Thus $\bar{d}_I = H(\alpha) / [\log(1/\lambda_a)^{\alpha} + \log(1/\lambda_b)^{\beta}]$, which is exactly $d_L - 1$.

4.3 Pointwise dimension

The pointwise dimension was originally defined by Farmer *et al* [12] in 1983. Similar quantities can be seen in Billingsley [4].

Definition 4.3.1a and b: Let $B(\varepsilon, x)$ denote a p–dimensional ball of radius ε centered at a point x on an attractor A embeded in the p–dimensional phase space of the dynamical system being considered. Then the *pointwise dimension at the point x* of the attractor is defined as:

$$d_P(x) = \lim_{\varepsilon \to 0} \log\mu(B(\varepsilon, x))/\log\varepsilon \,, \tag{4.3}$$

where μ is the natural measure of A. If $d_P(x)$ is the same for almost every x on the attractor, then we denote this value by d_P and call it the *pointwise dimension of the attractor A.*

Remark 4.3.1: One can show that $d_P(x)$ is the same for almost every x on the attractor, if the mapping F is smooth except at a finite set of points (see [2])

Example 4.3.1: *The classical Cantor set.* Its pointwise dimension is log2/log3. If we choose $\varepsilon = (1/3)^n$, then $\mu(B(\varepsilon, x)) = (1/2)^n$. Equation (4.3) yields:

$$d_P = \log(1/2)^n/\log(1/3)^n = \log2/\log3.$$

The next lemma will be needed to compute the pointwise dimension of the attractor of the generalized baker's transformation. We shall compute d_P for a special case ($\lambda_a = \lambda_b$) of the transformation, as it is done in [12]. But it should be mentioned that the result is still valid for more general λ_a and λ_b.

Lemma 4.3.1: Let $\lambda_a = \lambda_b$ in equation (1.1). Then $\mu(B(\varepsilon, x)) \leq M\mu(S_n(m))$, where $M > 0$ is defined in the proof, where $\mu(B(\varepsilon, x))$ is a ball of radius $\varepsilon = \lambda_a{}^n$ centered at $x \in S_n(m)$ (see notation 4.1.1).

Proof: Let x be in the strip $S_n(m)$. The ball $B(\varepsilon, x)$ cannot intersect more than two consecutive strips ($S_n(m)$ and $S_n(m+1)$ or $S_n(m-1)$ and $S_n(m)$), because their width is $\lambda_a{}^n$ which is the radius of the ball. According to example 4.1.1, $\mu(S_n(m)) = \alpha^{n-j}\beta^j$ where this number is the m^{th} term in the "non-commutative" product $(\alpha + \beta)^{*n}$, $m = 1,\dots, 2^n$. Therefore $\mu(B(\varepsilon, x)) \leq \mu(S_n(m)) + \mu(S_n(m+1))$ (or $\leq \mu(S_n(m-1)) + \mu(S_n(m))$), that is:

$$\begin{aligned}
\mu(B(\varepsilon, x)) &\leq \alpha^{n-j}\beta^j + \alpha^{n-j}\beta^n = 2\alpha^{n-j}\beta^j, \\
\text{or: } &\leq \alpha^{n-j}\beta^j + \alpha^{n-j-1}\beta^{j+1} = \alpha^{n-j}\beta^j(1 + \alpha^{-1}\beta), \\
\text{or: } &\leq \alpha^{n-j}\beta^j + \alpha^{n-j+1}\beta^{j-1} = \alpha^{n-j}\beta^j(1 + \alpha\beta^{-1}).
\end{aligned}$$

Let M be the $\max\{2, (1 + \alpha^{-1}\beta), (1 + \alpha\beta^{-1})\}$. Thus $\mu(B(\varepsilon, x)) \leq M\mu(S_n(m))$.

■

Now we turn to the computation of the pointwise dimension of the attractor of the generalized baker's transformation.

Computation of the pointwise dimension along the x–axis: In this computation we assume that $\lambda_a = \lambda_b$. By equation (2.6), we get: $\log\lambda_2 = \log\lambda_a{}^{\alpha+\beta} = \log\lambda_a$; then $\lambda_a = \lambda_2$, where λ_2 is the second Lyapunov number. After example 4.1.1, we know that after n iterates we get 2^n strips.

Suppose first that $\lambda_a < 1/4$. Let $x \in S_n(m)$. The width of $S_n(m)$ is $\varepsilon = (\lambda_a)^n$. Because $\lambda_a < 1/4$, the gaps between strips are bigger than the strips. Since $B(\varepsilon, x)$ covers the base of $S_n(m)$, but does not intersect more than one strip, then $\mu(B(\varepsilon, x) = \mu(S_n(m)) = \alpha^{n-j}\beta^j$. Therefore the pointwise dimension of the attractor along the x–axis is:

$$\bar{d}_p = \lim_{\varepsilon\to 0} \log\mu(B(\varepsilon, x))/\log\varepsilon = \lim_{\varepsilon\to 0} \log\mu(S_n(m))/\log\varepsilon$$

$$= \lim_{n\to\infty} \log(\alpha^{n-j}\beta^j) / \log\lambda_a{}^n$$

$$= \lim_{n\to\infty} [\ (n-j)\log\alpha + j\log\beta\ ] / n\log\lambda_a$$

$$= (1/\log\lambda_a)\lim_{n\to\infty} ((n-j)/n)\log\alpha + (j/n)\log\beta.$$

By definition of $\mu(S_n(m) = \alpha^{n-j}\beta^j$, $n-j$ is the number of times the orbit has been in the set $y < \alpha$ and j is the number of times the orbit has been in the set $y > \alpha$ (see notation 4.1.1 and example 4.1.1). By the Birkhoff Ergodic therorem (see remark 1.4.2), we have for almost every x (that is, for almost every strip $S_n(m)$):

$$\lim_{n\to\infty} (n-j)/n = \alpha \text{ and } \lim_{n\to\infty} j/n = \beta.$$

Then:

$$\bar{d}_p = (1/\log\lambda_a)[\alpha\log\alpha + \beta\log\beta]$$

$$= H(1/\alpha) / \log\lambda_a$$

$$= H(\alpha) / \log(1/\lambda_a) = \bar{d}_I.$$

Suppose now that $1/4 \le \lambda_a < 1/2$. According to lemma 4.3.1, $\mu(B(\varepsilon, x)) \le M\mu(S_n)$. Since $S_n \subset B(\varepsilon, x)$, we have also $\mu(S_n) \le \mu(B(\varepsilon, x))$. Thus $\mu(S_n) \le \mu(B(\varepsilon, x)) \le M\mu(S_n)$ and we get the same value of the pointwise dimension.

Conclusion

There are several different dimensions that can be used to describe dynamical systems and their attractors. In this report we have presented few notions of dimensions of attractors. There exists many other notions of dimension that we did not look at, for example, correlation dimension, I–capacity dimension, I–Hausdorff dimension, Hausdorff dimension of the core... But we think that we have considered some of the most important ones, at least to underline the importance of the notion of dimension of an attractor and to illustrate the different conjectures of Farmer *et al* [12].

Looking back to the preceding chapters, we conclude that the attractor of the generalized baker's transformation satisfies:

1° The conjecture 1.3.1, by the sections 2.1 and 3.3. We have obtained:

$$d_C = d_H = 1 + \bar{d}_S,$$

where $\bar{d}_S$ is the solution of $\lambda_a^{\bar{d}_S} + \lambda_b^{\bar{d}_S} = 1$.

2° The conjecture 1.3.2, by the sections 2.2, 4.2 and 4.3. We have obtained:

$$d_L = d_I = d_P = 1 + \log\lambda_1 / \log(1/\lambda_2) = 1 + H(\alpha)/(\alpha\log(1/\lambda_a) + \beta\log(1/\lambda_b)),$$

where $H(\alpha) = \alpha\log(1/\alpha) + \beta\log(1/\beta)$ is the binary entropy function.

3° The conjecture 1.3.3, if we let $\lambda_a = \lambda_b$ and $\alpha = \beta = 1/2$. Indeed in this case:

$$\bar{d}_S = \log(1/2) / \log\lambda_a,\ H(\alpha) = \log(1/\alpha) = \log 2 \text{ and } \alpha\log(1/\lambda_a) + \beta\log(1/\lambda_b) = \log(1/\lambda_a),$$

which implies that:

$$d_C = d_H = d_L = d_I = d_P.$$

Among all the notions of dimension, probabilistic dimensions are probably the most interresting ones, since as we know, capacity (a metric dimension often equal to the Hausdorff dimension, according to conjecture 1.3.1) is a limiting case of probability dimension: see introduction of section 4.1. Moreover recent works related to dimensions of attractors have introduced generalizations of probabilistic dimensions (see [6], [14] – [18] and [20]). The generalized dimension d_q (Grassberger [15]), the periodic point dimension $\hat{d}_q$ (Celso *et al* [6]), the partition function dimension $\tilde{d}_q$ (Hencel *et al* [20]) are all in fact functions of a variable q. For certain values of q, one finds again standard known notions of dimension. For example, d_0 is the capacity and d_1 is the information dimension. Conjectures about these generalized dimensions (similar to 1.3.1–1.3.3) have been also stated: for typical attractors, $d_q = \hat{d}_q = \tilde{d}_q$. The realm of dimensions is still far from being completely explored.

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