CANADIAN THESES

THÈSES CANADIENNES

Estimation of Reliability Parameters of a Redundant System with one Standby and one Repair Facility

Neyaz A. Shaheen

A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montréal, Québec, Canada

January 1987

ISBN 0-315-35529-8

ABSTRACT

Estimation of Reliability Parameters of a Redundant System with One Standby and One Repair Facility

Neyaz A. Shaheen

This thesis deals with probabilistic and statistical analysis of N-Components Series System supported by an active standby and one repair facility. System failure is shown to be convolution of two independent exponential distributions. The failure time and repair time distribution of the unit are considered exponential and independent of each other.

Two sampling schemes are considered to obtain maximum likelihood estimates of the reliability function. A computer program is developed to solve the given set of non linear likelihood equations iteratively. Information matrix is supplied. Numerical results obtained by the two schemes are presented and a comparative study is also done.

ACKNOWLEDGEMENT

The author wishes to thank his supervisor, Dr. Zohel Khalil for his guidance and support throughout the project, Dr. T. Shrivastava and Dr. Y.P. Chaubey for their time-to-time advice and help during his stay at Concordia University.

The author also wishes to thank Nancy Hagberg for her skill and patience in typing the Thesis and Erminia Pannese for preparing the tables.

Finally, the author wishes to acknowledge the moral support given by his brother and his family during the course of this work.

In memory of my mother, Maimoona Khatoon 1923-1985

Table of Contents

CHAPTER 1

Introduction

1.1 Concept of Reliability

Reliability of a system could be defined as a measure of how well a system performs or meets its design requirements. Since the past three decades technology has taken a turn towards complex systems containing subsystems, components and parts. This goes with the development of Space Vehicles, Telecommunication and Computer Systems and with these developments the study of reliability has been of major importance. Because a knowledge of the failure behavior of a component can lead to savings in its costs of production and in many cases the preservation of human life. The recent failures in aviation and space field are obvious examples.

It has been observed that mechanical failures tend to occur quite randomly and random behavior of failure leads to the conclusion that reliability studies should be made from a probabilistic or statistical view point. Such an approach is widely and commonly used within the field of reliability theory.

The failure behavior of a system can be specified in terms of a function f(t) which is known as failure density function of the system. If it is assumed that the failure does not occur before the beginning of the use, then

$f(t)dt$ = probability of failure in the small interval $(t, t+dt)$. The corresponding to $f(t)$ its distribution function is

$$F(t) = \int_0^t f(x)\,dx. \tag{1.1}$$

$F(t)$ can be interprated as the probability of the system failure occuring before time t.

Our main interest here is the probability that the system does not fail up to time t and this is described by the reliability function, given by

$$R(t) = 1-F(t). \tag{1.2}$$

Another function which is of equal importance called the hazard rate function is defined by

$$h(t) = f(t)/R(t). \tag{1.3}$$

This gives the failure rate conditional on past survival.

The expected time until failure assuming survival to time t is given by

$$E(t) = [1/F(t)]\int_0^\infty x f(t+x)\,dx. \tag{1.4}$$

$E(t)$ is known as mean residual life. Models described in this way are called continuous failure models and the exponential life time distribution is the only distribution with a constant failure rate. [Barlow 6]

If it is assumed that the conditional survival probability decreases with the age t, then $R(x)$ is decreasing in x $(0<x<\infty)$.

The function h(t) can be interpreted as the instantaneous probability of failure conditional on survival up to time t, since,

$$h(t) = \lim_{x \to 0} [\{ R(t) - R(t+x) \}/xR(t)]. \quad (1.5)$$

If aging is beneficial such that the conditional survival probability increases with age, then h(t) is a decreasing function of age and the corresponding distribution is known as a DFR (decreasing failure rate) distribution.

The behavior of failure rate as a function of age is known as the Lambda Characteristic, Life Characteristic, Mortality Curve or Hazard function. The function is also called Bath Tub Curve because of the following typical survival characteristics of a system. (Fig. 1.1).

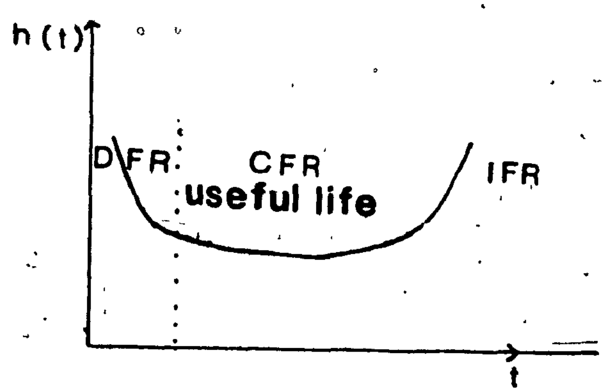
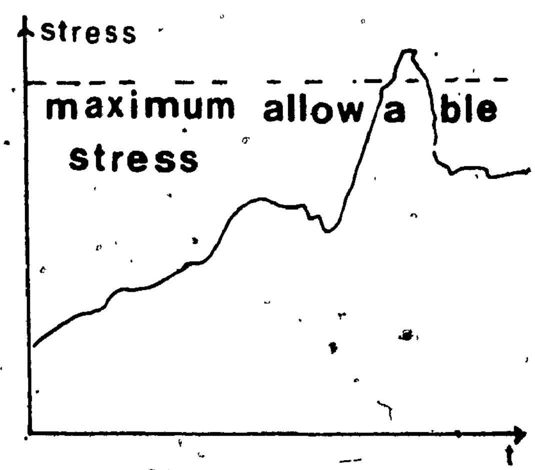


fig. 1.1

In terms of reliability, when a system is put into operation, then a failure can occur due to various reasons. They are:

i) Infant Mortality:- During the early life period the failures are of indogenious type, which are caused by inherent defects in the system attributed to faulty design, manufacturing, or assembly. During this period, the failure rate is expected to drop with age, as shown in the figure (1.1).

ii) Random Failure:- Once the system has been debugged it becomes prone to random failure due to environmental conditions under which the system is operating. In figure (1.1), the middle part is usually called useful life phase. In this phase failure rate is approximately constant and the exponential model is quite acceptable. Suppose a component is operating in an environment that subjects it to a stress which is varying in time. A failure occurs when the environmental stress exceeds its limit (maximum allowable stress). We call it peak. The "peak" stress may be assumed to follow a Poisson distribution with parameter λt (Fig 1.2) where λ is the constant rate of occurance of peak loads. Suppose in the



fig· 1· 2

interval (o,t], there are N peak stresses, then

$$P\ (N_t = n) = \exp(-\lambda t)(\lambda t)^n/n! \quad >o, \quad n = 0,1,2,\ldots$$

Now suppose X is the lifetime of the component, which corresponds to the event $\{N_t = 0\}$, then

$$R(t) = P\ (X>t)$$

$$= P\ (N_t = 0) = \exp(-\lambda t). \qquad (1.6)$$

iii) Wear Out:- When the components come to its expected life, the failure rate begins to increase. This phase of components life is called wear-out phase. This happens because of the accumulated wear and tear, abrasion, fatigue, creep, etc.

Assuming that the failure occurs randomly, the reliability of the system may be improved in the following ways:

(a) Introduction of Redundancy: This implies that more units are available for performing the system function, when fewer are actually required.

(b) Repair and Preventive maintenance: In this method when a component fails, it is repaired and put back to the operation.

In this thesis we are mainly concerned with the study of repairable redundant system and estimation of its reliability parameters which we will look into at the end of this chapter. The detail survey of the literature regarding the topic is presented in chapter 2 followed by the main results and conclusions in chapter 3 and chapter 4 respectively.

1.2 Models

In this section we will discuss the basic definition of redundancy and different models used in the study of repairable redundant system. As defined in the introduction of redundancy, suppose two or more units are available in a system in which only one unit is needed for actual operation, so that when failure occurs the other in standby replaces it and performs the task without going to idle state. An example of redundancy may be seen in an aeroplane, (with three engines) only two of them working at a time; if one fails the third engine replaces it without stop over.

Types of Redundancy

There are two types of redundancy.

1.2.1 Parallel Redundancy: A parallel redundant system with n components is one in which all the components operate simultaneously but the system requires at least one to operate (fig. 1.3). The system goes down only when all

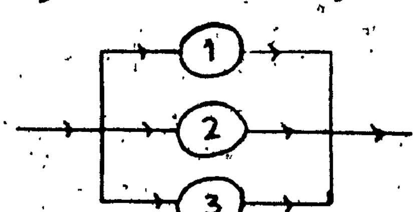


fig.1.3

the n components are in failure state. As an example, the reliability R of such a system with n=3 is given by

$$R = p_1 + p_2 + p_3 - (p_1 * p_2) - (p_1 * p_3) - (p_2 * p_3) + (p_1 * p_2 * p_3) \quad (1.7)$$

where p_i = probability of survival for the ith component.

And the MTTF(Mean time to first failure) is given by

$$= (1/\lambda_1) + (1/\lambda_2) + (1/\lambda_3) - (1/(\lambda_1+\lambda_2)) - (1/(\lambda_1+\lambda_3)) - (1/(\lambda_2+\lambda_3)) + (1/(\lambda_1+\lambda_2+\lambda_3)) \quad (1.8)$$

where λ_1, λ_2 and λ_3 are the constant failure rate of the three units respectively. There may be some parallel redundant systems where more than one unit are required to operate for the system to be functional . Suppose m units are required to operate the n unit redundant system where $m < n$, then the reliability of such a system is given by

$$R_m = \sum_{i=m}^{n} \binom{n}{i} p^i (1-p)^{n-i} \quad (1.9)$$

where $p_1 = p_2 = \ldots . p_m$ [for proof see Barlow[5]].

Gaver [13] studied 2-unit parallel redundant systems with constant state dependent hazard rate and arbitrary repair, and obtained the Laplace Transform of the reliability of the system.

1.2.2 Standby Redundant System:

A standby redundant system with n components is one in in which the components operate one at a time and when failure occurs in the operating unit, another component from the standby is switched on to work. This kind of system is also called sequentially redundant system of order n.

When a unit is in standby, it could be classified as hot, warm and cold standby (Gnedenko [14]) as follows. Suppose failure rate of a stanby unit is λ_1 and failure rate of unit in operation is λ. A standby is defined as hot when $\lambda_1 = \lambda$ and warm when $0 < \lambda_1 < \lambda$.

A unit in standby which can not fail or loose the potential is defined as cold standby.

Standby redundancy is of two kinds, one without renewal and the other with renewal. In both the redundancies all three classifications of redundancy such as hot, warm and cold are considered given below.

1.2.3 Standby Redundancy Without Renewal:

As mentioned above we divide standby redundancy without renewal in three classes.

(a) Loaded Standbys (or Hot Standbys).

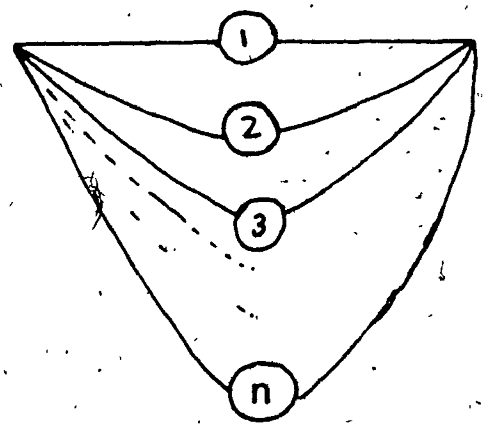
In this particular case the standby unit has the same failure behavior before and after it is put in to operation. The reliability of each unit is independent of the moment it entered into operation or in stand by state. The replacement time is negligible.

In figure 1.4 we have one basic unit and other n-1 units are in standby. Let the probability of ith individidual unit be $p_i(t)$, $i=1,2, \ldots\ldots n$. Since failure time of each unit is independently distributed, the probability of system failure (also called unreliability), is given by

$$Q_n(t) = q_1(t).q_2(t) \ldots\ldots\ldots q_n(t),$$

where $q_i(t) = 1 - p_i(t)$ and its reliability is given by

$$R_n(t) = 1 - Q_n(t) = 1 - \{1-p_1(t)\}\{1-p_2(t)\}\ldots\{1-p_n(t)\}$$



fig·1·4

From here it is also obvious that the order of standby is not important. If the standbys and basic units are identical (i.e; $p_i(t)=p(t)$ for all i) then

$$R_n(t) = 1-\{1-p(t)\}^n. \qquad (1.10)$$

If we assume that failure rate of the units are exponentially distributed with parameter λ then the meantime to failure free operation is given by

$$T_n = 1/\lambda \{\sum_{n=0}^{m} 1/n\} \qquad \text{for identical units}$$

$$T_n = 1/\lambda \left\{\ln C + C + 1/2n\right\} \qquad \text{(for large n).} \qquad (1.11)$$

where c = .57722 Euler's Constant (Gnedenko [14]).

If the failure time is distributed according to Weibull's distribution then in the case of equal reliabilities

$$P(t) = \exp(-\lambda t^{\alpha}). \qquad (1.12)$$

The MTTF is given by

$$T_n = (1/\lambda)\sum_{k=1}^{n}\{C_n^k\ (-1)^{k-1}\}/\{k^{1/\alpha}\}. \qquad (1.13)$$

If the failure time is distributed according to Power Law then

$$P\ (t) = 1/\{1 + t/t_o\}^{\alpha} \qquad (1.14)$$

where t_o = mean life of a unit, then MTTF is given by

$$T_n = 1/[(n-1/\alpha)\ \ldots.(1-1/\alpha)].\ n!\ t_o. \qquad (1.15)$$

(b) Unloaded Standby (Cold Standby)

Here the assumption is that a standby unit can not fail as long as it is in standby. The time required to replace the failed unit with one of the standby is negligible, i.e; the switch device is 100% reliable. Once again if we have one basic and n-1 standby units then the system's unreliability is given by:

$$Q_n(t) = \{(\lambda t)^n / n!\}.\{(1-\lambda t)/(n+1)\}.\exp(-\lambda t) \qquad (1.16)$$

And this is an approximation formula which involves a relative error of an amount which equals

$$(\lambda t)^2 / \{(n+1)^2 (n+2)\}. \qquad (1.17)$$

Usually $\hat{Q}_n(t) = q_1(t).q_2(t)\ldots\ldots\ldots q_n(t)/n!$ (1.18)

gives good approximation or an upper bound for $Q_n(t)$. (For proof see Gnedenko [14]).

(c) Lightly Loaded (Warm) Standbys

Lightly loaded or warm standby implies that the unit could fail even if it is in the standbys. switchtime is assumed to be negligible. An approximation formula for the unreliability of the standby group is given by

$$Q_n(t) = \Lambda_1(\Lambda_2+\lambda_2)(\Lambda_3+2\lambda_3)\ldots\ldots(\Lambda_n+\lambda_n(n-1))/n!. \qquad (1.19)$$

where Λ_k is the failure rate of the kth unit in an operating state and k denotes the failure rate of the kth unit in a non-operating state. (See Gnedenko [14].)

1.2.4 Standby Redundancy with Renewal.

In many cases renewal is applicable. A repair facility is used to increase the reliability, and consequently the time of service of a system is increased. The standby redundancy with renewal also could be classified in three categories; hot , warm and cold standbyes. A special case of standby redundancy with renewal, has been studied in chapter 2.

However, in this case the general assumptions are that a system has n+m units where n units are needed to operate and m (m ≥1) units stay in standby (hot, warm, cold). As soon as any unit fails it is sent to the repair facility. If the failed unit is on-line, then a standby unit unit is switched to operate in its place. The switch-over time is usually considered negligible. The repair of the failed unit statrts immediately, if a repair facility is free. The system goes down if more than m units are in failure state or less than n units are in working condition.

(Gnedenko [14]).

Khalil [17] considered the case of n+1 element system where one element works and n elements are in standby (unloaded) with repair faciliteis that could vary from 1 to n.

1.3 Our Model

In this thesis the model under study is a particular case of warm standby repairable redundant system. In this system N identical independent components are put into operation with one redundant element of same type in warm standby and with one repair facility. The on-line components fail according to exponential law with parameter λ and a standby unit could also fail according to the same distribution with parameter λ_1 where $\lambda > \lambda_1$. All failure times (on-line or standby) are considered to be independent random variable. Upon failure of any working unit the standby unit is switched to operate and failed unit is put into repair instantaneously (if available). The switchover time is zero. The repair time distribution is exponentially distributed with parameter μ.

System goes down when an operating unit fails while another one is still under repair. In other words system is up if n-out-of-(n+1) units are working. The state of the system could be described as 0,1 or 2 according to whether system is operative with no units down, system is operative with one unit down or system is down (when more than one unit is down) respectively.

Khalil and Dharmadhikari [19] modelled the system's operating, repair and idle states as a Markov process and derived the following failure density function

$$f(t) = [ab/(a-b)][\exp(-bt)-\exp(-at)] \qquad (1.20)$$

where a>b>0 and t>0.

The main objective in this thesis is to find the estimate of a and b which are functions of λ, λ_1 and μ. To find the estimates of a and b, two sampling schemes have been considered.

In Scheme 1, m identical systems of type n-out-of-n+1 are put on test. In the interval (O,T] where T is fixed time epoch, they are kept under continuous observation. The average time to failure (when the system reaches from first state O to state 2 for the first time) t_{10}, t_{12}, t_{01} and t_{21}, n_{10}, n_{01}, n_{12} and n_{21} are observed.Let t_{ij}'s and n_{ij}'s are total amount of time the system stays in state i before entering state j and total number of transitions from i to j respectively for all n realization (see chaptor 3).

Using these we find the estimates of λ, λ_1 and μ. By replacing λ, λ_1 and μ by $\hat{\lambda}$, $\hat{\lambda}_1$ and $\hat{\mu}$ we find a and b. Using a and b in equation (1.20) we can find R(t). (see chaptor 3).

In Scheme 2, m identical systems of type n-out-of-n+1 are put on test simultaneously. The record of first failure time epoch of the m systems are observed. the first failure time epoch t_1, t_2,..........., t_m are i.i.d. ramdom variables. The maximum likelihood estimators of a and b are obtained by maximizing the likelihood function:

$$L\ (t_1,\ t_2,\ \ldots\ldots,t_m,\ a,\ b,) = \{(ab)/(a-b)\}^m \prod_{i=1}^{m}\{\exp(-bt_i)-\exp(-at_i)\}. \qquad (1.22)$$

Using the estimates of a and b an estimator of reliability is readily available.

Chapter 2

Historical Review

2.1 Literature Review of Standby Redundancy with Repair

In this chapter a historical review of the literature on standby redundancy and the estimation of reliability parameters are presented. Many authors considered the probabilistic analysis of standby systems. First attempt in the study of standby systems can be found in Epstein and Hosford [12], Barlow and Proschan [5], Gnedenko et al [15] etc.

For two units standby redundant system, Gnedenko [15], Osaki [28] and Buzacott [9] have used the renewal theoretic approach to analyze system behaviour. Sririvasan [39], Osaki [28, 30, 31], Arora [1,2], Osaki and Nakagawa [32] and Osaki and Okumoto [33] have used the Markov renewal process to analyze other redundant systems.

For multiple unit system not much progress has been made. In case of multiple unit cold standby system with a repair facility, Mine et al [21] obtained the MTSF (Mean Time to System Failure) of a 2-out-of n:F (system consists of n (>2) units and the system failes if two units fail) system with exponential failure and general repair by approximation method. Natarajan [27] studied the system with general failure and exponential repair and obtained the reliabiity and steady state availability of the system. Further, he approximated the probability density function of the life time of the unit by an Erlangian distribution. Kumagi [20] analysed the same problem as Natarajan [27] but did not use approximation. Bhat [7] obtained the reliability of an N-unit and S-spare system with exponential failure and general repair distribution using some recurrence relations. Nakagawa [25] used again Natarajan's [27] system and analysed it using Markov renewal process and discrete transforms. He also derived the expected number of visits to any state before a failure of the system occurs.

But very few advancements have been made towards the analysis of multiple unit redundant systems with multiple repair facility due to the complex natûre of the problem. Natarajan [26] has analysed cold standby redundant systems with multiple repair facility. Ramanarayana [34, 35] analysed some n-unit systems with multiple repair facility. Khalil [18] has done a comparative study for multiple standby and multiple repair facility. He obtained the mean time to system failure for special systems with different combinations of number of standbys and repair units. He further analysed the effect of adding a standby or adding a repair unit and compared them.

In the direction of statistical analysis, we find quite few statisticians who took interest. They are Naik [22, 23], Sarmah and Dharmadhikari [36, 38], Gross and Clark [16] etc. Naik [22] has obtained the MLE of transition rates in a finite state continuous time Markov process occuring in deteriorating systems. In his paper he assumed that the performance of the units may not be observed separately. Naik [23] has obtained the MLE of parameters in a 2-out-of-3:F (system fails when two units fail) system. He assumed that the stochastic process in the system could be observed as (1) Markov Chains (2) Markov Process and (3) semi-Markov Process, with the underlying failure and repair time distribution being discrete, exponential and general for the three cases 1, 2 and 3 respectively.

For estimation of parameters in case 1, the likelihood function is

$$L\ (P_{ij}) = C.\prod_{ij} P_{ij}^{n_{ij}} \text{ for } i,j = 0,1,2,3,4. \qquad (2.0)$$

The system states are defined as

0 - all components are working } System

i - component i has failed i = 1, 2, 3 } is up

4 - 2 components have failed, system is down.

In equation (2.0)

n_{ij} = total number of transitions from i to j

for all n realizations and

$\hat{P}_{ij} = n_{ij}/n_i$ is the maximum likelihood estimator of p .

where $n_i = \sum_j n_{ij}$.

For example $\hat{p}_{01} = n_{01}/n_0$ is the maximum likelihood estimator of p_{01} .

In both of his papers [22] and [23] he has shown that the estimators are assymptotically normal and $\sqrt{n_i}\,(\hat{P}_{ij} - P_{ij})$ converges in distribution to S-normal with mean 0 and variance σ^2.

$$\sigma^2 = -p_{ij}\,(1-p_{ij})\ D(1)/P_{i0}$$

where $D(1) = \left| I-Q \right|$ and P_{ij} is the coefficient of p_{ij} in the expansion of D(1).

Q is defined from the transition probability matrix

$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$ where Q, R, I, 0 are matrices,

Assuming the system as Markov process with failure and repair time distribution being exponential he obtained the estimator of transition rate matrix $A=(a_{ij})$, for that he defined the likelihood function

$$L(a_{ij}) = C.(\prod_{ij} p_{ij}^{n_{ij}}) \{\prod_{i} a_i^{n_i} \exp(-a_i t_i)\} \qquad (2.1)$$

where t_i is the total amount of time that the system was in state i. The MLE of $\hat{a}_i$, $\hat{a}_{ij}$, and $\hat{p}_{ij}$ are

$$\hat{a}_i = n_i/t_i, \quad \hat{a}_{ij} = \hat{p}_{ij}\,\hat{a}_i, \quad \hat{p}_{ij} = n_{ij}/n_i .$$

The estimate of a_{ij} is asymptotically S-normal with mean zero and variance

$$= (2\, a_{ij}^2 P_{ij} - a_i a_{ij} D(1) - 2\, D(1) a_{ij}^2 P_{ij}) / P_{io}$$

where D(1) and P_{ij} are as defined in the first case. For the third case where the failure and repair time distribution are general, the system is governed by a Semi-Markov Process. He obtained the estimator of parameter vector Q (as defined above). For that he defined the likelihood function

$$L(P_{ij}, Q) = (c \prod_{ij} P_{ij}^{n_{ij}}) \{\prod_{ij} \partial F_{ij}(Q,t)\}. \qquad (2.2)$$

P_{ij}'s are as defined above and Q on which $F_{ij}(t)$ depends. Q can be estimated by maximizing $\prod_{ij} \partial F_{ij}(Q,t)$.

Sarmah and Dharmadhikari [36, 38] have considered estimation of parameters of 1-out-of-2 : G repairable system and 1-out-of-n:G (a redundant system composed of, n(>1) units and the system functions if and only if at least 1 of the components functions) repairable system.

In their paper [36] they assumed that the failure and repair time distribution of the units are exponential with known parameters λ and μ respectively and the moment estima- tors are supplied under four sampling schemes. The estima- tors are asymptotically normally distributed in every sampling scheme. The model they considered is as follows:

a) In the system there are two identical units with one repair facility,

b) at t=o one unit starts operating and the other remains on standby,

c) Switch-over time is negligible,

d) Repair is done on first come first served basis.

e) The repair unit is as good as new.

In all four schemes λ and μ are unknown. In plan 1 or scheme 1 they assumed that the units might be observed separately contrary to the assumption made by Naik [22]. In the other three plans the assumption is similar to Naik [22, 23] that the unit performance may not be observed separately. The estimators for operating characteristics are supplied keeping in view that since they are obtained from the roots of quadratic equation, it should not be negative or imaginary. The system behavior of estimators are also kept into consideration so that the mean failure time is greater than that of mean service time.

The probabilistic analysis of the above model has been done by Birolini [8] and Dharmadhikari [11]. Sarmah and Dharmadhikari [36] assumed in plan 1 that the units might be observed separately though such plans might not be feasible. They considered, "initially one unit starts operating and upon failure it goes for repair. Upon completion of repair the unit is switched on". The switch over epochs are observed and denoted by Z_i's. $i=1,2,\ldots\ldots n$ and $Z_i = X_i + Y_i$. Further they referred Z_i's i.i.d. random variables with probability density function

$$h_1(t) = \{\lambda\mu/(\lambda-\mu)\}\ \{\exp(-\mu t) - \exp(-\lambda t)\} \qquad (2.3)$$

where $\lambda, \mu > 0 \quad \lambda \neq \mu \quad t > 0$.

They denoted such plan by R {1,O(F+K),n}.

where R implies that the system is repairable, n is the number of records available and O(F+K) denotes that the record is the sum of a life time and repair time.

The moment estimators of λ and μ are given as follows:

If $\lambda > \mu$

$$\hat{\mu} = (\bar{t}-\sqrt{A_1})/K \; , \; \hat{\lambda} = 2/(\hat{\mu})K$$

and if $\hat{\mu} > \lambda$ then

$$\hat{\mu} = (\bar{t}+\sqrt{A_1})/K \; , \; \hat{} = 2/(\hat{\mu})K$$

where $A = 2s^2 - \bar{t}^2 \qquad K = \bar{t}^2 - s^2$

and $\hat{\lambda}$ and $\hat{\mu}$ are asymptotically normal with mean O and variance which can be derived from the leading terms of Cramer [10] (p.354).

In plan 2 they considered that the available records were of switchover epochs T_i. The T_i's are time epochs at which the switch is used for ith time.

$T_0 = 0, \; t_1 = T_1 \quad t_k = T_k - T_{k-1} \quad i > 1$

$t_k = \max(X_k, Y_k)$ and t_k's are i.i.d. random variable with p.d.f.

$$h_2(t) = \lambda \exp(-\lambda t) + \mu \exp(-\mu t) - (\lambda+\mu) \exp(-(\lambda+\mu) t) \qquad (2.4)$$

$\lambda, \mu > 0$ and $t > 0$.

They denoted such plan as R {2, O(S,w,T), n} and the moment estimators of λ and μ are given by

$$\hat{\lambda} = (1 \pm \sqrt{1-A_2})/\sqrt{k} \qquad \hat{\mu} = (1 \mp \sqrt{1-A_2})/\sqrt{k}$$

where $A_2 = 4K/(2\bar{t}\sqrt{K} + K) \qquad K = \bar{t}^2 - s^2$

again $\hat{\lambda}$ and $\hat{\mu}$ are asymptotically normal with mean λ and μ respectively and variance can be obtained from Cramer [10], (p. 351).

But in Plan 3 they considered same switch-over epochs but with additional information that the system does not fail in a given interval this implies that the failure time of the unit was more than the corresponding repair time. This plan is denoted by

R {2, O(S,w;T,F>K), n}.

Here O(S,w;T,F>K) denotes that the switch-over epoch is due to failure of the on-line unit and such n observations are available.

If the time epoch is T_k as mentioned above, the $t_k = T_k - T_{k-1}$ and $\{t_k\}_{k=1}^{n}$ are i.i.d. random variable with p.d.f.

$$h_3(t) = \lambda/(\lambda+\mu) \{EXP(-\lambda t) - EXP(-(\lambda+\mu) t)\}, \quad \lambda, \mu > 0, t > 0 \qquad (2.5)$$

with the additional information that $\mu > \lambda$ the moment estimators are given by

$$\hat{\lambda} = (\bar{t} - \sqrt{A_3})/K \ , \quad \hat{\mu} = 2/(\bar{t} - \sqrt{A_3}) - \hat{\lambda}$$

where $A_3 = 2s^2 - \bar{t}^2$.

$\hat{\lambda}$ and $\hat{\mu}$ are asymptotically normally distributed with mean λ and μ respectively and variance can be obtained by Cramer [10] (p.351).

In the fourth plan they took such n systems and the life time of all n systems were recorded and denoted by T_i, $\{T_i\}_{i=1}^{n}$ are i.i.d. random variable with p.d.f.

$$h_4(t) = \{b_1 b_2/(b_1 - b_2)\} \{\exp(-b_1 t) - \exp(-b_2 t)\}, \ t>0 \qquad (2.6)$$

where $b_1 = \{(2\lambda + \mu) - \sqrt{[(2\lambda + \mu)^2 - 4\lambda^2]}\} / 2$,

$b_2 = \{(2\lambda + \mu) + \sqrt{[(2\lambda + \mu)^2 - 4\lambda^2]}\} / 2$

and the moment estimator of λ and μ are respectively given by

$$\hat{\lambda} = \sqrt{2/k} \ , \quad \hat{\mu} = (2\bar{t}/k) - (2\sqrt{2/k})$$
$$= \hat{\lambda}(\bar{t}\hat{\lambda} - 2)$$

The same model can be used to study the compartment analysis in Ecology and same model can be extended for N out of N+1 system.

Sarmah and Dharmadhikari [38] suggested three sampling plans to obtain the moment estimators of λ and μ (failure and repair rate) for a system operating with one unit, supported by N-1 inactive standbys and one repair facility. System is defined as 1-out-of-N:G repairable system. λ and μ are unknown.

Assuming failure and repair time distribution $f(t,\lambda)$ and $g(t,\mu)$ exponential, the mathematical analysis of the model has been done by Sarmah and Dharmadhikari [37] for special case taking N=2 and 3. In their analysis they obtained Laplace Transforms of the operating characteristics and ultimately obtained results for reliability, availability, expected number of repairs of the units in the system and expected number of failures of the system.

To estimate the parameters of 1-out-of N:G repairable system [38] they considered the switch over epochs T_k's where T_K denotes the time when O switch is used for the kth time and the plan is denoted by R {N,O (O-S;w:T), n}. In this plan N is total number of units in the system and O(O-S;w:T) denotes that the records are of O-switch over epochs. When an operating unit fails, switch of standby unit is used to operate if any standby unit is available and n denotes the number of records available. Where T>0, the system is in state i,

$1<i<N$, with probability p_{iN}

$\sum_{i=1}^{N-1} p_{iN} = 1$ and $P_{N-1,N} > 0$

The observation is started with switch over epoch $T_0 = 0$

$T_k > 0$ and $t_k = T_{k+1} - T_k \quad K=1,2,\ldots\ldots.n$

t_k's are i.i.d. random variable with p.d.f.

$w(t) = p\, f(t) + (1-p)\, h(t) \qquad P=\sum_{i=1}^{N-2} P_{iN}$

$h(t) = f(t)\, G(t) + g(t)\, F(t)$

They presented it as a theorem namely:

Theorem 1. The random variable $\{t_i\}_{i=1}^{n}$ defined as

$t_k = T_{k+1} - T_k$ ($T_0 = 0$, K=1,2,....,n and T are switch-over time epoch) in the sampling scheme

R {N,O (O-s,w;T), n} follow the family

$\{W(t;\lambda,\mu), \lambda, \mu > 0,\ t>0\}$

where $W(t) = Pf(t) + (1-P)\ h(t)$ $\quad P=\sum_{i=1}^{n-2} P_{in}$

$h(t) = f(t)\ G(t) + g(t)F(t)$

(The proof is given in Sarmah and Dharmadhikari [38]).

For the sampling Plan 1 denoted by R {N,O(O-s, w; T), n} the density is given by

$W_1(t) = Pf(t) + (1-P)h(t)$

A similar mixture of densities has been used by Aston [4] in studying the "Distribution for gaps in Road Traffic". From the above density $W_1(t)$ the rth moment could be derived as

$$\mu_r = r!\{PQ_1^r + (1-P)(Q_1^r + Q_2^r - (Q_1 Q_2)^r/(Q_1+Q_2)^r\} \quad r \geq 1 \qquad (2.7)$$

where $Q_1 = \lambda^{-1}$ and $Q_2 = \mu^{-1}$ and the moment estimators of Q_1 and Q_2 are

$$\hat{Q}_1 = (\sqrt{k_1} - b_1)/2 \quad , \quad \hat{Q}_2 = (\sqrt{k_2} - b_2)/2a \text{ where}$$

$$K_1 = (\hat{Q}_2 + m_1')^2 - 4\hat{Q}_2(1-P),$$

$$K_2 = \{4\hat{Q}_1(m_1' - Q_1)\}^2 + (2\hat{Q}_1 - m_2')^2,$$

$$b_1 = (\hat{Q}_2 - m_1'), \qquad b_2 = \hat{Q}_1(4m_1' - m_2'\hat{Q}_1) \text{ and}$$

$$a = 2(m_1' - \hat{Q}_1)$$

for $(Q_1/Q_2) > \sqrt{P}$, $\hat{Q}_1, \hat{Q}_2$ are the consistent estimator of Q_1 and Q_2.

The solution of Q_1 and Q_2 are not possible but an iterative solution is always possible. They suggested the iterative procedure outlined by Tallis and Light (1968). Later they showed that

$\hat{Q}$ is asymptotically normally distributed, with mean Q, and variance $V(\hat{Q})$ where $V(\hat{Q})$ is given by

$$V(\hat{Q}) = U(Q)^{-1}\ V(S)\ U(Q)^{-1}$$

where

$$V(S) = \left| S_{ij} \right|_{2\times 2}, \quad U(Q) = \left\| U_{ij} \right\|_{2\times 2} \quad \text{non singular}$$

$$S_{ij} = (m'_{i+j} - m'_i m'_j)/n$$

$$U_{ij} = (d/dQ_j) \; m'_i \Big|_{\hat{Q}=Q} \qquad i,j = 1,2.$$

Elements of U(Q) are given as

$$U_{11} = 1 - (1-P)\{Q_2 /(Q_1 + Q_2)\}^2,$$

$$U_{12} = (1-P)\{1 - [(Q_1 /(Q_1 + Q_2)]^2\},$$

$$U_{21} = 4\, Q_1 \{1 - (1-P)[Q_2 / (Q_1 + Q_2)]^3\}$$

$$U_{22} = 4\, Q_2 \{(1-P)[1-(Q_1 / (Q_1+Q_2)]^3\}$$

and S_{ij} could be obtained from Eq (2.7).

In the second plan they replaced T_k O-switch-over time epoch by X_k, R-switch-over time epoch when it was used for the kth time. Otherwise it is similar to the plan denoted by R{N,O (R-s.w;x), n} and t_k is again defined as

$$t_k = X_{k-1} - X_k \qquad k = 1, \ldots\ldots, n$$

and the density of $\{t_i\}_{i=1}^{n}$ is given by

$$W_2(t) = Pg(t) + (1-P)\, h(t) \qquad t \geq 0$$

and rth moment under this plan is obtained as

$$\mu'_r = r!\{PQ_1 + (1-P)\, Q_2\, [Q_1^r/Q_3 - Q_3^r/Q_1]\}, \quad r \geq 1 \qquad (2.8)$$

where the moment estimators of Q_1, Q_2 and Q_3 are given as

$$\hat{Q}_1 = (\sqrt{k_3} - b_3)/2, \qquad \hat{Q}_2 = (-\sqrt{k_4} - b_4)/a$$

$$Q_3 = 1/(\lambda + \mu) \quad \text{where}$$

$$k_3 = \{(\hat{Q}_2(2-P) - m'_1) + 4\, m'_1 \hat{Q}_2\},$$

$$k_4 = -2\, \hat{Q}_1^4 (1-P)\, (m'_2 - 2m'_1 \hat{Q}_1),$$

$$b_3 = \hat{Q}_2(2-P) - m'_1,$$

$$b_4 = (m'_2 - 2m'_1 \hat{Q}_1)\, \hat{Q}_1, \text{ and}$$

$$a = \{m'_2 - 2m'_1 \hat{Q}_1 - (1-P)\, \hat{Q}_1^2\}.$$

Again, $\hat{Q}_1$, $\hat{Q}_2$ are consistent estimators of $\hat{Q}_1$ and $\hat{Q}_2$ under the condition that

$$Q_1 / (Q_1 + Q_2) < 1/\sqrt{2}.$$

They have shown that $\hat{\underline{Q}}$ is asymptotically normally distributed with mean $\hat{\underline{Q}}$ and variance $V(\hat{\underline{Q}})$ given by

$$V(\hat{\underline{Q}}) = U(\underline{Q})^{-1} \; S(\underline{Q}) \; U(\underline{Q})'^{-1}$$

where

$$U_{11} = P + (1-P) \{1 + [Q_1^2 / (Q_1 + Q_2)^2]\}$$

$$U_{12} = (1-P) \{1 - Q_2^2 + [2Q_1Q_2/(Q_1 + Q_2)^2]\}$$

$$U_{21} = 2 [2PQ_1 + (1-P) \{2Q_1 + Q_2 - Q_2^3 (Q_2-Q_1)/(Q_1 + Q_2)^3\}]$$

$$U_{22} = 2 [(1-P) \{1-Q_2^2 (Q_2+3Q_1)/(Q_1+Q_2)^3\}]$$

and S_{ij} could be derived from Equation (2.8).

The third plan assumes that the system does not fail in $(0, t_n]$, the plan is denoted by

R_E R {N, 0 (0-s. w; T), n} where R_E denotes that the system is reliable and the density function of t under the third plan is given by

$$W_3(t) = P f(t) + (1-P) h_1(t)$$

where

$$h_1(t) = \{(\lambda+\mu)/\mu\} f(t) G(t).$$

For explanation see Sarmah and Dharmadhikari [38].

And for a particular case where $\lambda = \mu$ the density function of r.v. t is given by

$$W_4(t) = Pf(t) + 2(1-P) f(t) F(t)$$

and the moment estimator of Q_1 is given by

$$\hat{Q}_1 = 2m_1'/(3-P)$$

when P is unknown then

$\hat{Q}_1 = \sqrt{k_5} + 3m_1' \qquad \hat{P} = 3m_1'/(2\hat{Q}_1) \qquad k_5 = \sqrt{(9m_1'^2 - 16m_2')}$

and $\hat{Q}_1$ is asymptotically normal with mean Q_1 and variance $V(\hat{Q}_1)$ where

$V(\hat{Q}_1) = \hat{Q}_1^2 \cdot (5-P^3)(3-P^3)/((3-P^3) \cdot n)$.

A similar study has been done by Gross and Clark [16] with the exception that the component after failure can not be renewed. They considered two organ system. When both organs are functioning the failure rate of each organ is constant (λ_0). After one organ fails, the failure rate of the remaining organs is $\lambda_1 > \lambda_0$ and it is constant.

There are two states, S_0 and S_1 in which an individual may be still alive (kidney, lungs), where S_0 denotes when both organs are functioning and S_1 denotes when one has failed and the individual is left with only one organ functioning. The probability functions are given for different states follows:

$P_0(t)$ = P(system is in S_0 at time t)

$= \exp(-2\lambda_0 t)$ for $t \geq 0$ and

$P_1(t)$ = P(system is in S_1 at time t)

$= 2\lambda_0/(\lambda_1 - 2\lambda_0)\{\exp(-2\lambda_0 t) - \exp(-\lambda_1 t)\}$ for $t \geq 0$.

Also,

S(t) = P (that the individual survives to time t)

$= P_0(t) + P_1(t)$

$= \exp(-2\lambda_0 t) + \{2\lambda_0/(\lambda_1 - 2\lambda_0)\}\{\exp(-2\lambda_0 t) - \exp(-\lambda_1 t)\}$ for $t \geq 0$.

F(t) = 1 - S(t) = P(the individual dies prior to t)

$f(t) = F'(t) = \lambda_1 P_1(t)$ (2.8a)

$h(t) = 2\lambda_0\lambda_1/[(\lambda_1-2\lambda_0)/\{1-\exp(-(\lambda_1-2\lambda_0)t)\}+2\lambda_0]$

for $\lambda_1 \neq 2\lambda_0$, h(t) is an increasing function

$h(0) = 0$, $h(\infty) = 2\lambda_0$ or λ_1 according as $\lambda_1 \gtrless 2\lambda_0$

rth moment can be obtained from M.g.f.

$$M(Q) = \{2\lambda_0\lambda_1/(\lambda_1 - 2\lambda_0)\}\{(Q-\lambda_1)^{-1} - (Q-2\lambda_0)^{-1}\},$$

as $\mu_r = r!(2\lambda_0\lambda_1)^{-r}\{(\lambda_1^{r+1}-(2\lambda_0)^{r+1})/(\lambda_1-2\lambda_0)\}$.

The mean and variance of f(t) in (2.8a) are

$\mu = \lambda_1^{-1} + (2\lambda_0)^{-1}$ and $\sigma^2 = 1/\lambda_1^2 + 1/(2\lambda_0)^2$.

Gross and Clark [16] used MLE to estimate the parameters of λ_1 and $2\lambda_0$ for the above model observing t_i, $i=1,2,\ldots,N$, failure time on i.i.d organ system with failure density given by f(t), and likelihood function $L(\alpha,\beta)$ for $\alpha = 2\lambda_0$ and $\beta = \lambda_1$ is

$$L = \prod_{i=1}^{N} \{\alpha\beta/(\beta-\alpha)\ (\exp(-\alpha t_i)-\exp(-\beta t_i)\}$$

$dL/d\alpha$ and $dL/d\beta$ are obtained. They are

$$N\hat{\beta}/\hat{\alpha}(\hat{\beta}-\hat{\alpha}) - \sum_{i=1}^{N}[\{t_i\exp(-\hat{\alpha}t_i)\}/\{\exp(-\hat{\alpha}t_i)-\exp(-\hat{\beta}t_i)\}] = 0.$$

$$-N\hat{\alpha}/\hat{\alpha}(\hat{\beta}-\hat{\alpha}) + \sum_{i=1}^{N}[t_i\exp(-\hat{\beta}t_i)/\{\exp(-\hat{\alpha}t_i)-\exp(-\hat{\beta}t_i)\}] = 0.$$

Solving these two equations iteratively $\hat{\alpha}$ and $\hat{\beta}$ are obtained. They did a comparative study for $\hat{\alpha}$ and $\hat{\beta}$, by Newton-Raphson method and the method of scoring.

To start the two methods, initial $\hat{\alpha}$ and $\hat{\beta}$ are obtained in the following way:

$$\hat{\beta}_0 = 2/\{\bar{t} \mp (\sqrt{2s^2-\bar{t}^2})\} \qquad \text{for } \bar{t} > \sqrt{(2s^2-\bar{t}^2)}$$
$$= 2/\bar{t} \qquad \text{otherwise}$$

where $\bar{t} = \sum_{i=1}^{N} t_i/N$ and $\hat{\alpha}_0 = 2/\{\bar{t} \pm (\sqrt{2s^2-\bar{t}^2})\}$

$$s^2 = \sum_{i=1}^{N} (t_i - \bar{t})^2/(N-1)$$

By using initially $\hat{\alpha}_0$ and $\hat{\beta}_0$, $\hat{\alpha}$ and $\hat{\beta}$ are obtained by Newton-Raphson method and by method of scoring and ultimately the $\hat{\lambda}_0$ and $\hat{\lambda}_1$. They found that the results obtained by the two methods were quite similar and $\hat{\lambda}_0$, $\hat{\lambda}_1$ (their histograms) appeared to be normal.

2.2 An example: - Derivation of Reliability equation of a standby system with a repair facility (a special case).

For this example, we choose 1-out-of-2 system. It can be explained as an active identical unit parallel system. As soon as one unit fails, the repair begins and the system is still up. The system goes down if the 2nd unit also fails before the repair of the first failed unit is finished.

The Markov state space diagram is given in the Figure 2.1. We use the following notations:

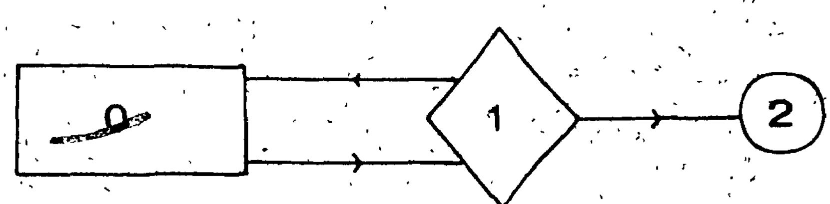


Figure 2.1

$P_i(t)$ = Probability that at time t, the system is in state i for i = 0,1,2,

λ_j = failure rate of the unit j, j = 1,2

μ = repair rate of failed unit

S = Laplace Transform variable

for $\lambda_1 = \lambda_2$ the transition rate matrix is given by

$$\underline{P} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

State 2 is an absorbing state. Corresponding to this matrix, a system of differential equations can be written:

$$P_0'(t) = -2\lambda P_0(t) + \mu P_1(t),$$

$$P_1'(t) = 2\lambda P_0(t) - (\lambda+\mu) P_1(t),$$

$$P_2'(t) = \lambda P_1(t)$$

with initial conditions

$$P_0(0) = 1,\ P_1(0) = P_2(0) = 0 .$$

The Laplace Transforms of these equations are

$$\left.\begin{aligned} &P_0(S)(S+2\lambda) - \mu P_1(S) = 1, \\ &-2\lambda P_0(S) + (S+\lambda+\mu)P_1(S) = 0, \\ &\lambda P_1(S) - SP_2(S) = 0, \end{aligned}\right\} \qquad (2.9)$$

Solving for $P_1(S)$ we get

$$P_1(S) = 2\lambda/(S^2 + (3\lambda+\mu)S + 2\lambda^2). \qquad (2.10)$$

Letting, $S_1 = \{-(3\lambda+\mu) + \sqrt{\lambda^2+\mu^2+6\lambda\mu}\}/2$ (2.11)

and $S_2 = \{-(3\lambda+\mu) - \sqrt{\lambda^2+6\lambda\mu+\mu^2}\}/2$ (2.12)

(2.10) can be written in the form

$$P_1(S) = 2\lambda/\{(S-S_1)(S-S_2)\} \qquad (2.13)$$

and by using partial fraction (Trivedi [42]) expansions

$P_1(S)$ can be written as

$$P_1(S) = \{2\lambda / (S_2 - S_1)\}\{1/(S-S_2) - 1/(S-S_1)\}. \qquad (2.14)$$

Now the inverse transform of $P_1(S)$ is given by

$$P_1(t) = \{2\lambda/(S_1 - S_2)\}\{\exp(-S_1 t) - \exp(-S_2 t)\} \qquad (2.15)$$

and $P_0(S)$ can be obtained by substituting $P_1(S)$ in (2.9) as

$$P_0(S) = (S + \mu + \lambda) / \{(S - S_1)(S - S_2)\}$$

$$= [(S_1 + \mu + \lambda) / \{(S_1 - S_2)(S - S_1)\}] - [(S_2 + \mu + \lambda) / \{(S_1 - S_2)(S - S_2)\}]$$

$$= 1/(S_1 - S_2) \{(S_1 + \mu + \lambda)/(S - S_1) - (S_2 + \mu + \lambda)/(S - S_2)\}. \qquad (2.16)$$

Inverting $P_0(S)$ we have

$$P_0(t) = \{1/(S1 - S2)\}\{(\lambda + \mu + S_1)\exp(S_1 t) - (\lambda + \mu + S_2)\exp(S_2 t)\} \qquad (2.17)$$

The reliability function R(t) is

$$R(t) = P_0(t) + P_1(t) =$$

$$= \{-S_2 \exp(S_1 t) + S_1 \exp(S_2 t)\} / (S_1 - S_2). \qquad (2.18)$$

Reliability models for standby redundancy with $\lambda_1 \neq \lambda_2$ and with a repair facility can be obtained similarly.

Chapter 3

3.1 Introduction

In the previous Chapter, few repairable systems have been discussed in which different schemes are used to estimate the reliability parameters.

In this chapter, the estimators of parameters of N-out-of-N+1 active standby system with one repair facility is obtained. Khalil and Dharmadhikari [19] considered this system with the following assumptions:

(1) The system consists of N+1 statistically independent identical unit, N operating and one standby unit with a repair facility.

(2) The system is up when N units are on-line.

(3) A unit is also subject to failure in standby position.

(4) Upon failure of a unit, it is sent to repair if the repair facility is free, otherwise, system goes down.

(5) The repaired unit is like a new unit.

(6) Switch over is instantaneous and absolutely reliable.

(7) The components on-line fail according to an exponential distribution with parameter λ and the component in standby may also fail according to an exponential distribution with parameter λ_1 and $\lambda > \lambda_1 > 0$.

The repair time distribution is exponential with parameter $\mu > 0$ and repair times of the unit are statistically independent of the life times of the unit.

Using a renewal argument, Laplace Transform of the system failure time distribution is derived. Upon inversion of this transform we consider the problem of statistical estimation of the parameter.

Operating characteristics of 1-out-of-2 repairable system (λ, λ_1 and μ known) have been obtained by Nakagawa and Osaki [25]. Our main objective is to obtain the estimators for the above operating characteristics when λ, λ_1 and μ are unknown.

3.2 System Analysis

Let X(t) denote the number of non-operative units at time t, then $\{X(t), t \geq o\}$ is a continuous time Markov process with state space $\Omega = \{0,1,2\}$. Let $Q_{ij}(t)$ denote the probability of transition from state i to j in (o,t] then

$$Q_{01}(t) = \int_0^t \{(N\lambda+\lambda_1)\exp(-N\lambda+\lambda_1)u\}\,du \qquad (3.1)$$

$$Q_{10}(t) = \int_0^t (\mu)\exp\{-(N\lambda+\mu)u\}\,du \qquad (3.2)$$

$$Q_{12}(t) = \int_0^t N\lambda\exp\{-(N\lambda+\mu)u\}\,du \qquad (3.3)$$

The matrix of transition rates $\|\lambda_{ij}\|$ is given by

$$\begin{pmatrix} -N\lambda-\lambda_1 & N\lambda+\lambda_1 & 0 \\ \mu & -N\lambda-\mu & N\lambda \\ 0 & \mu & -\mu \end{pmatrix}, \qquad (3.4)$$

$\|\lambda_{ij}\|$ specifies the Markov process completely. The general interest is to derive the cummulative distribution function of system failure time. For that the following notations are needed.

Let $H_{ij}(t)$ denote the probability distribution that the process visits state j for the first time in (0,t] given that at time t=0 the system was in state i, i=0,1 j=1,2. Suppose at t=0 all units are operative, then the cummulative distribution function $H_{02}(t)$ is given by the following integral equations:

$$H_{02}(t) = Q_{01}(t) * H_{12}(t) \tag{3.5}$$

$$H_{12}(t) = Q_{10}(t) * H_{02}(t) + Q_{12}(t) \tag{3.6}$$

where * denotes the convolution operation.

Let $h_{ij}(s)$ and $q_{ij}(s)$ be the Laplace transform of $H_{ij}(t)$ and $Q_{ij}(t)$ for all i and j.

Applying Laplace transform to equations (3.1)-(3.6) and solving the resulting algebraic equations the explicit form of $h_{02}(s)$ is: $h_{02}(s) = [N\lambda(N\lambda + \lambda 1)/\{(N\lambda + \lambda 1 + S)$

$$(N\lambda + \mu + S) - N\lambda(N\lambda - \lambda_1)\}] \tag{3.7}$$

The more general form of equation (3.7) was obtained by Gnedenko [15].

Using the partial fraction, [Trivedi 42], $h_{02}(s)$ could be simplified and upon inversion of Laplace transform of (3.7) it can be shown that the probability density function

$$f(t) = \{ab / (a-b)\}\{\exp(-bt) - \exp(-at)\} \quad t \geq 0 \quad a,b>0 \tag{3.8}$$

where

$$a = (1/2)\ (A_1 + \sqrt{A_2}) \tag{3.8a}$$

$$b = (1/2)\ (A_1 - \sqrt{A_2}) \tag{3.8b}$$

$$A_1 = 2N\lambda + \lambda 1 + \mu$$

$$A = (3N\lambda + \lambda 1 + \mu) + \lambda_1^3 \tag{3.8c}$$

If any of these three parameters λ, λ_1 and μ is unknown, two sampling schemes have been considered [19] to obtain the maximum likelihood estimators of these parameters. Equation (3.5) can be evaluated after substituting the estimator of unknown parameter in equation (3.4). The two sampling schemes are discussed in the following section and the numerical results based on these schemes are presented in Section 3.4.

3.3 Sampling Procedure

Two sampling schemes are considered to obtain the maximum likelihood estimators of λ, λ_1 and μ.

3.3.1 Sampling Scheme 1

Under this scheme it is assumed that there are M statistically identical systems of type N-out-of-N+1 put on test. For a fixed time epoch T, they are under continuous observation. The time epochs of transitions as well as states of transitions are recorded. The realization of such records are denoted by

$$(X,T) = \{(x_0,t_0),(x_1,t_1),(x_2,t_2),\ldots,(x_k,t_k)\} \qquad (3.10)$$

where $x_0 = 0$ $\quad 0 = t_0 < t_1 < t_2 < \ldots < t_k = T$ where the pair (x_i, t_i) indicates that the system enters in the state i at time epoch t_i as the effect of ith jump. The density function of such observation is given by

$$f(X,T) = \prod_{i=1}^{k-1} P_{x_i,x_{i+1}} \exp(-q_{x_i,x_{i+1}} \cdot t_{i+1}) \qquad (3.11)$$

where $\| P_{ij} \|$ is corresponding transition probability matrix and

$$t_{i+1} = T_{i+1} - T_i .$$

The likelihood function of k realization is given by

$$L(X,T) = \prod_{i=1}^{M} f_i(x,t).$$

For such realizations Albert [1] has established the maximum likelihood estimates of p_{ij}'s and q_{ij}'s. They are as follows:

Let n_{ijk} = number of transitions of the type $i \to j$ in the kth realization.define,

$t_{ik} = \sum_{j \neq i} t_{ijk}$ = Total length of time the system was in state i in kth realization,

$n_{ij} = \sum_{k=1}^{M} n_{ijk}$ = Number of transitions of the type $i \to j$,

$t_i = \sum_{k=1}^{M} t_{ik}$ = Total length of time the system spends in state i, and

$n_i = \sum_{j \neq i} n_{ij}$ = Number of transitions to state i.

Then $\hat{p}_{ij} = n_{ij}/n_i$ $\quad -(\hat{q}_i) = n_i/t_i$ and $\hat{q}_{ij} = (-\hat{q}_i)\hat{p}_{ij}$

$\hat{p}_{ij}$, $-\hat{q}_i$, $\hat{q}_{ij}$ are the maximum likelihood of estimators of p_{ij}, $-q_i$ and q_{ij} respectively.

By using the equations (3.4) and (3.11) the following is obtained:

$$\hat{\mu} = n_{10}/t \;, \quad \hat{\lambda} = (1/N)(n_{12}/t), \quad \hat{\lambda}_1 = n_{01}/t_0 - N\hat{\lambda} \quad (3.11a).$$

Where $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\lambda}_1$ are the maximum likelihood estimators of μ, λ and λ_1 respectively, and by replacing them in equations (3.8) we can obtain the reliability of the system. The simulation results are presented in the next section.

3.3.2 Sampling Scheme 2

Under this scheme it is assumed that M statistically identical system of type N-out-of-N+1:G are put to test simultaneously. The records of first failure time epochs $t_1, t_2, \ldots, t_M$ are observed for the M systems. $\{t_i\}_{i=1}^{M}$ are i.i.d. random variables from the distribution

$$F(t) = 1-\{(b \exp(-at)-a \exp(-bt)\} / (b-a) \quad (3.12)$$

and density

$$f(t) = \{ab/(a-b)\}\{\exp(-bt)-\exp(-at)\} \quad (3.12a)$$

Let the realized values be $t_1, t_2, \ldots, t_M$. The likelihood function is given by

$$L(t_1, t_2, \ldots, t_M, a, b) = \{ab/(a-b)\}^M \prod_{i=1}^{M}\{\exp(-at_i)-\exp(-bt_i)\}. \quad (3.13)$$

To maximize equation (3.13) with respect to a and b we take the log function of (3.13) and find its first deriva- tive with respect to a and b respectively. These are:

$$\ln L(t_1, t_2, \ldots, t_M, a, b) = M\ln\{ab/(a-b)\} + \sum_{i=1}^{M}\ln\{\exp(-at_i)-\exp(-bt_i)\}$$

$$\partial[\ln L(t,a,b)]/\partial a = (M/a) - \{M/(a-b)\} + \sum_{i=1}^{M} t_i \{\exp(-at_i)\} + \exp(-at_i)-\exp(-bt_i)\} \quad (3.14)$$

$$\partial[\ln L(t,a,b)]/\partial b = (M/b) + \{M/(a-b)\} - \sum_{i=1}^{M} t_i\{\exp(-bt_i)\} / \{\exp(-a\cdot t_i) - \exp(-bt_i) = 0 \quad (3.15)$$

Equations (3.14) and (3.15) can be written as

$$(M/a)-\{M/(a-b)\}+\sum_{i=1}^{M}\{t_i \exp(-at_i)/B(t_i)\} = 0 \quad (3.14a)$$

$$(M/b)+\{M/(a-b)\}-\sum_{i=1}^{M}\{t_i \exp(-bt_i)/B(t_i)\} = 0 \quad (3.15a)$$

Where $B(t_i) = \exp(-at_i)-\exp(-bt_i)$.

To obtain the maximum likelihood estimators we solve equations (3.14a) and (3.15a) iteratively. The numerical result is given in the next section.

The corresponding information matrix is given by $\| I_{ij} \|_{2\times 2}$ where

$$I_{11} = M\,[(1/a^2) - \{1/(a-b)^2\} + \Phi(a,b)], \qquad (3.16)$$

$$I_{12} = I_{21} = -M\,\{1/(a-b)^2 + \Phi(a,b)\}, \qquad (3.17)$$

$$\text{and } I_{22} = M\,[1/b^2)-\{1/(a-b)^2\} - \Phi(a,b)], \qquad (3.18)$$

$$\text{where } \Phi(a,b)=2ab/(b-a)\sum_{n=0}^{\infty} n\{((n+1)a-nb)^{-3} - (na-(n-1)b)^{-3}\}. \qquad (3.19)$$

To evaluate $\Phi(a,b)$ we need the following lemma

Lemma 1: $\{\exp(-bt)-\exp(-at)\}^{-1} = [\exp(-bt)\{1-\exp(-t(a-b))\}]^{-1}$

$$= \exp(bt)\sum_{n=0}^{\infty}\exp[-n(a-b)t] \qquad a,b>0$$

(Geometric Series)

From there we can deduce

$\{\exp(-bt)-\exp(-at)\}^{-2} = \exp(2bt)\,\{1-\exp(-t(a-b)\}^{-2}$

and $[1-\exp\{-t(a-b)\}]^{-1} = \sum_{n=0}^{\infty}\exp\{-n(a-b)t\}]$, we have

$$[1-\exp\{-t(a-b)\}]^{-2} = [-\exp\{t(a-b)\}] / (a-b)$$

$$\times (d/dt)[1-\exp\{-t(a-b)\}]^{-1}$$

$$= [-\exp\{t(a-b)\}] / (a-b) \quad [\sum_{n=1}^{\infty} -n(a-b)\cdot\exp\{-n(a-b)t\}].$$

$$\left\{\exp(-bt)-\exp(-at)\right\}^{-2} = \exp\,\{t(a+b)\}\sum_{n=1}^{\infty} n\exp\{-n(a-b)t\}.$$

Lemma 2: $E[t^2 \exp\{-t(a+b)\} \{\exp(-at)-\exp(-bt)\}^2] =$

$$2ab/(a-b)\sum_{n=1}^{\infty} n[\{(n+1)a-nb\}^{-3}-\{(na)-(n-1)b\}^{-3}]$$

Proof:- $t^2 \exp\{-t(a+b)\} \{\exp(-at)-\exp(-bt)\}^{-2}$

$$= t^2 \exp\{-t(a+b)\} \quad \exp\{t(a+b)\}\sum_{n=1}^{\infty} n \exp\{-n(a\ b)t\}$$

$$= t^2 \sum_{n=1}^{\infty} n\exp\{-n(a-b)t\}$$

$$E[t^2 \sum_{n=1}^{\infty} n\exp\{-n(a-b)t\}] = ab/(b-a) \sum_{n=1}^{\infty} n\int_0^{\infty} t^2 \exp\{-n(a-b)t\}$$

$$\{\exp(-at)-\exp(-bt)\}\, dt$$

$$= ab/(b-a)\sum_{n=1}^{\infty} n\left\{\int_0^{\infty} t^2 \exp(-t\{(n+a)a-nb\}\, dt - \int_0^{\infty} t^2 \exp(-t\{na-(n-1)b\}\, dt\right\}$$

$$= ab/(b-a)\sum_{n=1}^{\infty} 2n[\{(n+1)a-nb\}^{-3} - \{na-(n-1)b\}^{-3}],$$

In this scheme the estimators based on method of moments on M observations are used as estimators of a and b where

$$\hat{a}_o = 2/\{\bar{t} \pm \sqrt{(2s^2 - \bar{t}^2)}\} \qquad (3.20)$$

$$\hat{b}_o = 2/\{\bar{t} \mp \sqrt{(2s^2 - \bar{t}^2)}\} \qquad (3.21)$$

where $\hat{t} = 1/M \sum_{i=1}^{M} t_i$ and $s^2 = 1/(M-1) \sum_{i=1}^{M} (t_i-\bar{t})^2$

If $\bar{t} > \sqrt{2s^2 - \bar{t}^2}$ we use equations (3.20) and (3.21) to find $\hat{a}_o$ and $\hat{b}_o$.

If $t < \sqrt{2s^2 - \bar{t}^2}$ we use

$$\hat{a}_o = 2/\bar{t} \quad , \quad \hat{b}_o = 2/\{\bar{t} + \sqrt{2s^2 - \bar{t}^2}\} \qquad (3.22)$$

We obtain such pairs of $\hat{a}_o$ and $\hat{b}_o$ for a number of time (let it be N) and finally we find $\hat{a}$ and $\hat{b}$ as

$$\hat{a} = \sum_{i=1}^{N} \hat{a}_{oi} \quad \text{and} \quad \hat{b} = \sum_{i=1}^{N} \hat{b}_{oi}$$

These maximum likelihood estimators of a and b are used in equation (3.8) to obtain the estimator of reliability. Here we see that one does not require to know the explicit estimators of λ, λ_1 and μ. However, the estimators of λ, λ_1 and μ can be obtained by solving the equations (3.8 a, b).

Similar maximum likelihood estimators of a and b have been considered by Gross, Clark and Liu [16] to estimate the survival parameters when one of two organs (lungs and kidneys) must function for survival.

3.4 Numerical result for Scheme 1 and Scheme 2

For both the schemes it had been assumed that the component failure time and component repair time are exponentially distributed. Hence, to initialize the experiment, the following steps have been taken:

(1) Random numbers X,Y,Z were drawn between 0 and 1. This was done for the sample size 1000, 500 and 1000 respectively for Scheme 1 and 50, 50, 50 respectively for Scheme 2.

(2) The random numbers were converted to random variates having exponential distribution by computing separately for two component failures and one repair.

$$X1 = \log\{(1-F_1(t)\}/-2\lambda$$

$$Y1 = \log\{(1-F_2(t)\}/-\lambda_1$$

$$Z1 = \log\{(1-F_3(t)\}/-\mu$$

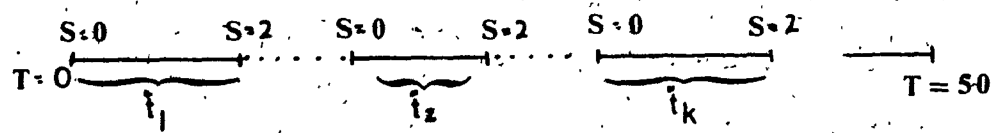
where $F_1(t)$; $F_2(t)$ and $F_3(t)$ are the three random numbers. We chose λ, λ_1 and μ such that $\mu >> \lambda > \lambda_1$, and we take $\lambda_1 = 2.5$ then $\lambda = 5$ and $\mu = 13$.

3.4.1

For Scheme 1 we took the following steps:

(1) For the life time T=50 the first failure time epoch was observed. Let it be denoted as TTF. For 150 such observations, Mean Time to First Failure (MTTF) is determined.

Again on T=50, $\dot{t}_1$, $\dot{t}_2$, ..., $\dot{t}_k$ are observed



$\dot{t}_1$ = first state 0 to first state 2

$\dot{t}_2$ = first state 0 after first state 2 to next state 2

and so on.

On the basis of $\dot{t}_1$, $\dot{t}_2$, ..., $\dot{t}_k$ mean time to system failure is determined and denoted by

$$\hat{\tau}_1 = \sum_{i}^{k} \dot{t}_i / N_1$$

where N_1 denotes the number of times the system visited state 2 starting from state 0 or, N_1 times starting from state 0 the system went down.

(2) t_i the number of times the system was in state i, n the number of transitions to state i and n_{ij} the number of transitions of the type i to j were recorded for i = 0,1,2.

(3) On the basis of such 150 records we find $\hat{\lambda}$, $\hat{\lambda}_1$ and $\hat{\mu}$ using the equations (3.11a). In finding $\hat{\lambda}$, $\hat{\lambda}_1$ and $\hat{\mu}$, $\hat{\mu} >> \hat{\lambda} > \hat{\lambda}_1$, was required. If the condition was not fulfilled then a check is needed. In this result the condition was achie[illegible]d. Using these parameters in equation (3.8 a,[illegible]). The a and b were obtained.

(4) Using $\hat{a}$ and $\hat{b}$ in equation (3.8) the reliability of the system was calculated for t = .1 t=.2,....,.9.

(5) On the basis of 150 observations from number (1) to (2) MTTF system mean time to first failure was obtained using the limit theorem $P(t) = e^{-t/MTTF}$ the reliability of the system is measured.

(6) A comparison between the reliability of the system obtained in number (4) and number (5) is presented.

3.4.2

For the second Scheme (2a) the following steps were taken:

(1) For one hundred systems (similar) the first failure time epochs were obtained say $\ddot{t}_1, \ddot{t}_2, \ldots, \ddot{t}_{100}$ on the basis of these observations. $\ddot{t}$ and s were obtained where $\ddot{t}$ and s are mean and standard deviations for the first R failures, where R = 100.

(2) Using the equations from (3.20) to (3.22) $\hat{a}$ and $\hat{b}$ are obtained.

(3) Such, 45 observations were recorded and on the basis of these 45 observations a and b were obtained where

$$\hat{\hat{a}} = \sum_{i=1}^{45} \hat{a}_i/45 \qquad \hat{\hat{b}} = \sum_{i=1}^{45} \hat{b}_i/45$$

These $\hat{\hat{a}}$ and $\hat{\hat{b}}$ could be used to initialize Newton-Raphson method. Khalil and Dharmadhikari [19] have suggested to initialize the Newton-Raphson method by such $\hat{\hat{a}}$ and $\hat{\hat{b}}$.

(4) Using these $\hat{\hat{a}}$ and $\hat{\hat{b}}$ in (3.8 a,b,c) $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\lambda}_1$ were obtained and replacing in equation (3.12) the reliability of the system for different time was obtained.

(5) On the basis of 45 observations using $\bar{\bar{t}}_i$ for $i = 1,2,....45$ the grand mean $\bar{\bar{\bar{t}}}$ was obtained. Using $\bar{\bar{\bar{t}}}$ as MTTF, the reliability of the system by limit theorem was obtained and compared with the reliability found by equation (3.12).

3.4.3

In the second scheme itself, another attempt was made to solve the likelihood equations (3.14a) and (3.15a) and to find the maximum likelihood estimators of a and b. For that, the following steps were taken. We call it Scheme 2b.

(1) On the basis of 40 observations the first average failure time is obtained say t_i and such 200 t_i's are observed.

(2) In equations (3.14a) and (3.15a) for M=200 t_i for $i=1,2,...,200$ were replaced and for different a and b the equations are solved iteratively. The pair $(\hat{a},\hat{b})$ were found for which (3.14a) and (3.15a) were maximized.

(3) Using these $\hat{a}$ and $\hat{b}$ in (3.8 a,b,c) $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\lambda}_1$ were obtained and replacing $\hat{a}$ and $\hat{b}$ in equation (3.12) the reliability of the system was obtained. Since ti's are the same as in Scheme 2a, the MTTF is also the same.

Comparing Scheme 1, Scheme 2a and Scheme 2b we find that the estimated value of $\hat{a}$ and $\hat{b}$ are very similar in all three schemes. $\hat{a}$ and $\hat{b}$ in Scheme 2a and 2b are almost equal so are the $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\lambda}_1$.

In Scheme 1, $\hat{a}$ and $\hat{b}$ are approximately equal to the feeded value with standard deviations 1.145 and .6292 respectively. $\hat{\lambda}$, $\hat{\lambda}_1$, $\hat{\mu}$ are also very close to the feeded value with standard deviations .12089, .6727 and .895 respectively.

In Scheme 2a, $\hat{a}$ and $\hat{b}$ are quite consistent but their standard deviations are not as good as in Scheme 1. The reason for this is the two conditions put to find $\hat{a}$ (3.20 to 3.22). The standard deviation of $\hat{b}$ is very reasonable with .1893.

Scheme 2b gives almost the same results as Scheme 2a for $\hat{a}$ and $\hat{b}$.

3.4.4 Tables

Sample Scheme 1

Sample Size = 150 (150 Systems)

In the span T=50 time units, we observe the first time, system failure for each realization.

Mean Time to first Failure (MTTF) = .309499

Mean Time to Failure on (t=50) = .23904.

$\hat{a}$ = 30.2034 $\hat{b}$ = 5.406 $\hat{\mu}$ = 13.054 $\hat{\lambda}$ = 5.35 $\hat{\lambda}_1$ = 2.7554

$\sigma^2_{\hat{a}}$ = 1.312 $\sigma^2_{\hat{b}}$ = .396 $\sigma^2_{\hat{\mu}}$ = .6727 $\sigma^2_{\hat{\lambda}}$ = .12089 $\sigma^2_{\hat{\lambda}_1}$ = .895

Feeded value of λ = 5 μ = 13 λ_1 = 2.5 a = 30.0699

b = 5.73

Comparison of reliability of the system obtained by two equations:

(1) $P(t) = [a\exp(-bt) - b\exp(-at)] / (a-b)$

(2) $P(t) = \exp(-t / MTTF)$

	t=.1	t=.2	t=.3	t=.4	t=.5
P(T>t)	.7007	.41273	.2404	.1401	.09032
$e^{-t/MTTF}$	.65813	.43314	.2850	.187	.1234

For sampling Scheme 1 out of 150 realizations, 10 are given in the following table:

TABLE 1

	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\lambda}_1$	Mean Time to System Failure	First System Failure in t=50 Unit Span	A1	A2	a	b
1	13.572	5.046	2.277	.231	.574	36.034	637.841	30.645	5.389
2	13.943	6.169	.58	.223	.200	39.200	718.793	33.005	6.195
3	13.614	5.663	1.645	.204	.239	37.911	674.765	31.943	5.967
4	13.229	4.573	3.413	.264	.029	34.933	622.385	29.945	4.988
5	13.474	5.191	3.366	.223	.007	37.603	684.663	31.884	5.718
6	12.840	4.866	2.897	.243	.399	35.200	601.255	29.86	5.340
7	13.394	5.001	1.888	.232	.116	35.287	613.349	30.027	5.261
8	12.354	5.331	2.904	.199	.147	36.582	608.137	30.621	5.961
9	13.847	5.116	2.46	.268	.574	36.772	665.768	31.287	5.485
10	12.150	5.005	2.902	.239	.097	35.071	572.18	29.496	5.575

Using Reliability Equation $P(t) = \frac{ae^{-bt} - be^{-at}}{a-b}$

	P(T>.1)	P(T>.2)	P(T>.3)	P(T>.4)	P(T>.5)
1	.697905	.412491	.240889	.140542	.081968
2	.654062	.356298	.191923	.103302	.055588
3	.667683	.372434	.205264	.113029	.062220
4	.718632	.44198	.268682	.163177	.099069
5	.678841	.387914	.219168	.123726	.069827
6	.702947	.418009	.245365	.143865	.084317
7	.705897	.422839	.250148	.147832	.087332
8	.672[illegible]	.376427	.20767	.114435	.063027
9	.691[illegible]	.404458	.233927	.13518	.078094
10	.693883	.403681	.231488	.132572	.075881

<u>**Sample Scheme 2a**</u>

Sample Size = 45 (4500 realization)

1 sample = t had been obtained from putting 100 system in operation and finding the first failure time epoch of the system and $\hat{t}_i$ = ti/100

grand mean $\hat{\bar{t}} = \hat{t}_i/45 = .26335$.

$\hat{a} = 32.66$ $\hat{b} = 3.7284$ $\sigma_{\hat{b}} = .1893$ $\sigma_{\hat{t}} = .02628$

$\sigma_{\hat{a}}$ is higher due to the condition put for finding $\hat{a}$.

Feeded value of a = 30.0699 b = 5.430046

Comparison of reliability of the system obtained by two equations:

(1) $P(t) = \{a\exp(-bt)-b\exp(-at)\} / (a-b)$

(2) $P(t) = \exp(-t/MTTF)$, MTTF = .26335

	t=.1	t=.2	t=.3	t=.4	t=.5
P(T>t)	.7726	.5363	.36887	.254	.1749
$e^{-t/MTTF}$	.6839	.4677	.3199	.2188	.14966

Sample scheme 2b

Sample Size=45 (45 x 200 realization)

1 sample = mean time to first system failure on the basis of 200 realization.

$\hat{a} = 32.82 \quad \hat{b} = 3.751$

The reliability comparison by the two equations is the same as in Scheme 2a, because a and b are quite similar and T is also similar.

In this scheme a pair of a and b obtained after (45 x 200) realization which maximizes the likelihood equations (3.14a) and (3.15a).

Chapter 4

Summary

In this thesis mainly three things have been discussed.

(a) The definition and classification of a redundant system.

(b) A historical review of repairable redundant systems.

(c) The estimation of reliability parameters of a repairable redundant system.

Basically a standby redundant system is a system where more than the required units are available to operate the system. A standby redundant system could be divided into three classes (i) cold standby (ii) warm standby, and (iii) hot standby.

Reliability of a standby redundant system can be improved in many ways. One of them is to make available a repair facility. A special case of a repairable redundant system is "a redundant system with one warm standby and one repair facility". In this system it is assumed that the standby unit and on-line unit fail with the rate λ_1 and λ respectively and a failed unit is repaired with rate μ. The failure time distribution and repair time distribution of the units are considered exponential and independent. If a unit fails in the system, it is sent to be repaired and the other unit performs the operation. After the repair is complete, the unit is put as standby. A failure to the system occurs if the unit fails while the other unit

is still under repair.

In this case, if λ, λ_1 and μ are known, the reliability of the system in an interval (o,t] can be obtained by

$$R(t) = 1 - \int_0^t f(t)\,dt$$

where f(t) is the probability density function of the system which is a convolution of two independent exponential distributions. If any of λ, λ_1 and μ is not known, then we estimate the unknown parameter.

In chapter three, estimation of reliability parameters have been done by two sampling methods. In method one, Albert's [1] notations have been used to estimate $\hat{\lambda}$, $\hat{\lambda}_1$ and $\hat{\mu}$ and ultimately $\hat{a}$ and $\hat{b}$ the reliability parameters. $\hat{\lambda}$, $\hat{\lambda}_1$ and $\hat{\mu}$ obtained in this scheme are distributed with very small standard deviations and so are $\hat{a}$ and $\hat{b}$.

Method two has been done in two ways called 2a and 2b. In Scheme 2a method of moments have been used to estimate $\hat{a}$ and $\hat{b}$. Here, due to the conditions put to obtain $\hat{a}$ and $\hat{b}$, we find variation in the result of $\hat{a}$ but the mean value comes as desired and $\hat{b}$ shows very reasonable variation. In this Scheme we see that to find the reliability parameters one does not require to find the estimators of λ, λ_1 and μ.

In Scheme 2b the attempt has been made to solve the non linear likelihood equations iteratively for the same system observed in Scheme 2a. We came to the result that values of $\hat{a}$ and $\hat{b}$ obtained in Scheme 2b, which maximize the likelihood equations, are quite similar to the values of $\hat{a}$

and b obtained in Scheme 2a. The results obtained in both schemes have been simulated using the computer programs (flowchart presented in the appendix).

Comparing Scheme 1 and Scheme 2 (a and b), the results obtained in Scheme 1 are very close to the feeded values and have very small and highly acceptable deviations. In Scheme 2a and 2b, the similar results have been obtained with a little difference from Scheme 1, however, the results in Scheme 2a and 2b are very consistent and the histograms of system failure time distribution tend to be normal with very small standard deviations.

REFERENCES

[1]Albert, A. Estimating the Infinitesimal Generator Of A Continuous Time Finite State Markov Process, Ann. Math. Stat. 32-727-753 (1962).

[2]Arora, J.R. Availability Of A 2-Unit Priority Standby Redundant System With Preventive Maintenance, IEEE Trans. Rel. R-25, p. 205-207 (1976a).

[3]Arora, J.R. Reliability of a 2-Unit Standby Redundant System with Constrained Repair Time, IEEE Trans. Rel. R-25, p. 203-205 (1976b).

[4]Aston, W.D. Distribution for Gaps in Road Traffic. J. Inst. Maths and Its Applics., 7, p. 37-46 (1971).

[5]Barlow, R.E., and Proschan, F. Mathematical Theory of Reliability, New York: John Wihey (1965).

[6]Barlow, R.E., and Proschan, F. Statistical Theory of Reliability and Life Testing: Probability Models, Holt, Rinehart and Winston, New York (1975).

[7]Bhat, U.N. Reliability of an Independent N-Component S-Spare System with Exponential and General Repair Times, Techno., 15, p. 529-539 (1973).

[8]Birolini, N. Comments on Renewal Theoretic Aspects of Two Unit Redundant System. IEEE Trans on Rel. Vol. 12, p. 121-123 (1975).

[9]Buzacott, J.A. Availability of Priority Standby Redundant System. IEEE Trans. Rel. R-20, p. 60-63 (1971).

[10]Cramer, H. Mathematical Methods of Statistics. Asia Pub. House (1945).

[11]Dharmadhikari, A.D. Stochastic Analysis of a Few Repairable System. Ph.D. Thesis submitted to IIT, Bombay (1980)

[12]Epstein, B., and Hosford, J. Reliability of Some 2-Unit Redundant systems, Sixth Nat. Symp. Rel. Qualt. Contr. p. 469-476 (1960).

[13]Gaver, D.P. Time to Failure and Availability of Parallel Systems with Repair, IEEE Trans. Rel. R-12, p. 30-38 (1963).

[14]Gnedenko, B.V. Some Theorems on Standbys, Proc. Fifth Berkeley Symp. Math. Stat. 3, p. 285-291 (1967).

[15]Gnedenko, B.V. Mathematical Methods for Reliability Theor, Academic Press, New York (1969).

[16]Gross, A.J., Virginia A. Clark and Vinnie Liu. Estimation of Survival Parameters when One of Two Organs Must Function for Survival; Biometrics 27, p. 369-377 (1971).

[17]Khalil, Z. A Limit Distribution for Failure-Free Operation Time of a Standby System with Random Switchover Time. Dirasat (A Research Journal published by the University of Jordan) Vol. 1, p. 17-21 (1974).

[18]Khalil, Z. A Comparison Between the Number of Standby Units and the Number of Repair Units in a Standby system with Repair. INFOR. Vol. 19, no. 3, p. 264-267 (1981).

[19]Khalil, Z., and Dharmadhikari, A.D. (*) Reliability Estimation in Redundant Repairable Series System. A paper under preparation.

[20]Kumagi, M. Reliability Analysis for Systems with Repair. J. Oper. Res. Soc. Japan, 14, p. 53-71 (1971).

[21]Mine, H., Osaki, S. and Asakura, T. Some Considerations for Multiple Unit Redundant Systems with Generalised Repair Time Distribution, IEEE Trans. R-17, p. 170-174 (1968).

[22]Naik, D.N. A Method for Estimating the Parameters of Reliability Models. Technical Report, I.S.I. Banglore Centre, 31 Church Street, Banglore 560001 (1980).

[23]Naik, D.N. On Estimating the Parameters of 2-Out-Of-3:F Systems. Technical Report, I.S.I. Banglore Centre, 31 Church Street, Banglore 560001 (1981).

[24]Nakagawa, T. The Expected Number of Visits to State k Before a Total System Failure of Complex System With Repair Maintenance, oper. Res. 22, p. 108-116 (1974).

[25]Nakagawa, T., and Osaki, S. Stochastic Behavior of a 2-Unit Priority Standby Redundant System with Repair, Micro. Rel. 14. p. 309-313 (1975).

[26]Natarajan, R. A Reliability Problem with Spares and Multiple Repair Facilities, Oper. Res. 16, p. 1041-1057 (1968).

[27]Natarajan, R. Reliability of a Single Unit System with Spares and Single Repair Facility, Adv. Frontiers in Oper. Res. Hinduston Publishing Company, Delhi (India), p. 227-278 (1969).

[28]Osaki, S. System Reliability Analysis by Markov Renewal Process, J. Oper. Res. Soc. Japan, 12, p. 127-188 (1970a).

[29]Osaki, S. Renewal Theoretic Aspects of Two-Unit Standby Redundant Systems, IEEE Trans. Rel, R-21, p. 105-110 (1970b).

[30]Osaki, S. On a Two-Unit Standby Redundant System with Imperfect Switchover, IEEE Trans. Rel. R-21, p. 20-24 (1972a).

[31]Osaki, S. Reliability Analysis of a Two-Unit Standby Redundant System with Preventive Maintenance. IEEE Trans. Rel. R-21, p. 24-29 (1972b).

[32]Osaki, S. and Nakagawa, T. Bibliography for Reliability and Availability of Stochastic Systems, IEEE Trans. Rel. R-25, p. 284-287 (1976).

[33]Osaki, S. and Okumoto, K. Repair Limit Suspension Policies for a Two-Unit Standby Redundant System with Two Phase Repairs, Micro. Rel. 16, p. 41-45 (1977).

[34]Ramanarayanan, R. Availability of Two-Out-Of-N:F System, IEEE Trans. Rel. R-25, p. 43 (1976).

[35]Ramanarayanan, R. Reliability and Availability of Two General Multi-Unit Systems, IEEE Trans. Rel. R-27, p. 70-72 (1978).

[36]Sarmah, P. and Dharmadhikari, A.D. Estimation of Parameters of 1-Out-Of-2:G Repairable System. Communications in Statistics Theory and Methods, 12 (14), p. 1609-1618 (1983a).

[37]Sarmah, P. and Dharmadhikari, A.D. Analysis of 1-Out-Of-N:G Adjustable Operating System. Micro. Rel., Vol. 23, No.3, p. 477-480 (1983b).

[38]Sarmah, P. and Dharmadhikari, A.D. Estimation of Parameters of 1-Out-Of-N:G Repairable System, Commun. Statist.-Theory, Meth., 13 (11), p. 1365-1374 (1984).

[39]Sririvasan, V.S. A Standby Redundant Model with Instantaneous Switchover. IEEE Trans. Rel. R-17, p. 175-178 (1968).

[40]Subramanian, R., and Ravichandran, N. Interval Reliability of a Two-Unit System, IEEE Trans. Rel. R-28, p. 84 (1979).

[41]Tallis, G.M., and Light, R. The Use of Fractional Moments for Estimating the Parameters of a Mixed Exponential Distribution. Technometrics, 10, p. 161-175 (1968).

[42]Trivedi, K.S. Probability and Statistics with Reliability, Queuing and Computer Science Applications, Prentice Hall Inc., Englewood Cliffs, N.J. (1982).

APPENDIX A

Scheme 1 Flow Chart

A: The variables used are initialized and declared.

B: X, Y, Z, uniform random variables are generated from (0,1) and then changed into exponential X_1, Y_1, Z_1 random variates. X_1, Y_1 are used as the failure times and Z_1 is used as the repair time in the program.

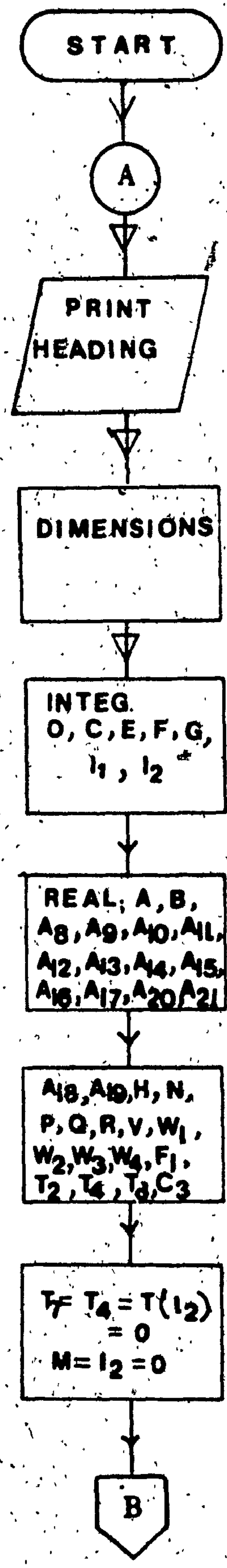
C: X_1 and Y_1 are retrieved. Here it is decided that either the active component fails or the standby. In any case, the system goes to State 1, S(I)=1.

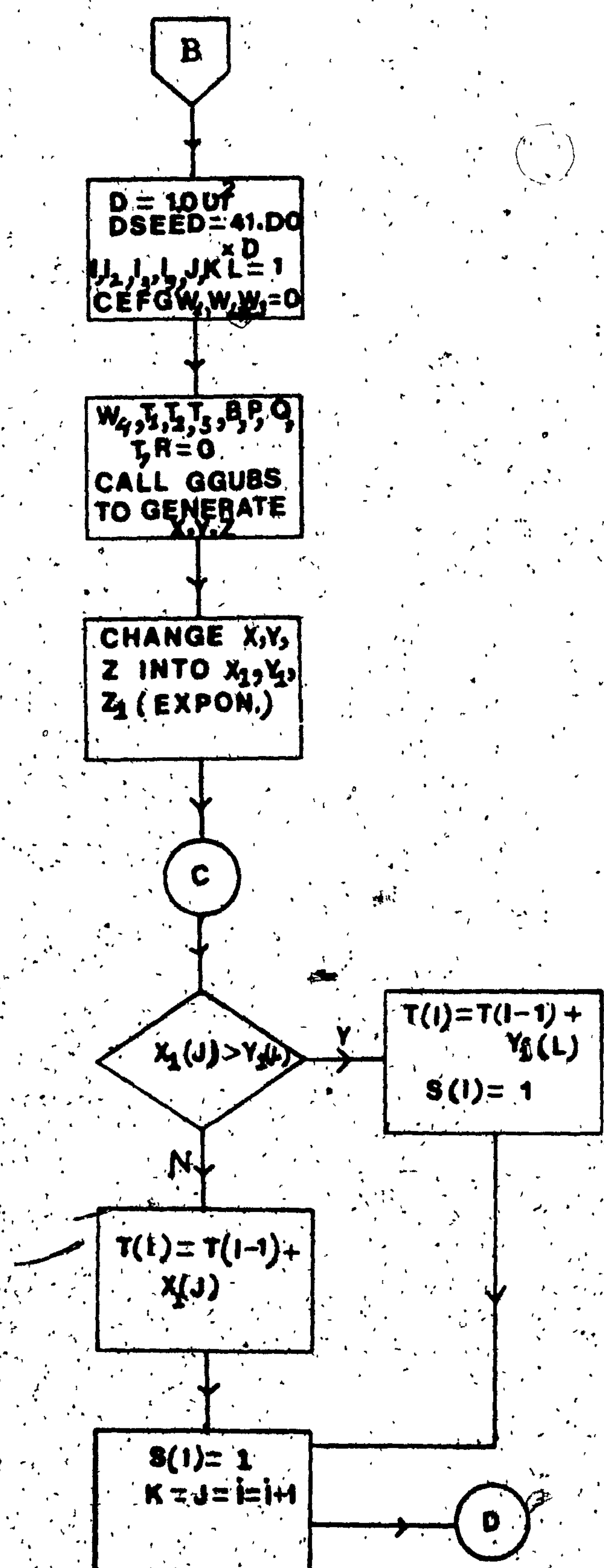
D: X_1 and Z_1 are retrieved randomly in this routine, and if $X_1 > Z_1$ then the system goes to state 0, otherwise, goes in state 2 and the system is down. After repair the system goes in state 1. We repeat this subroutine until T=50.

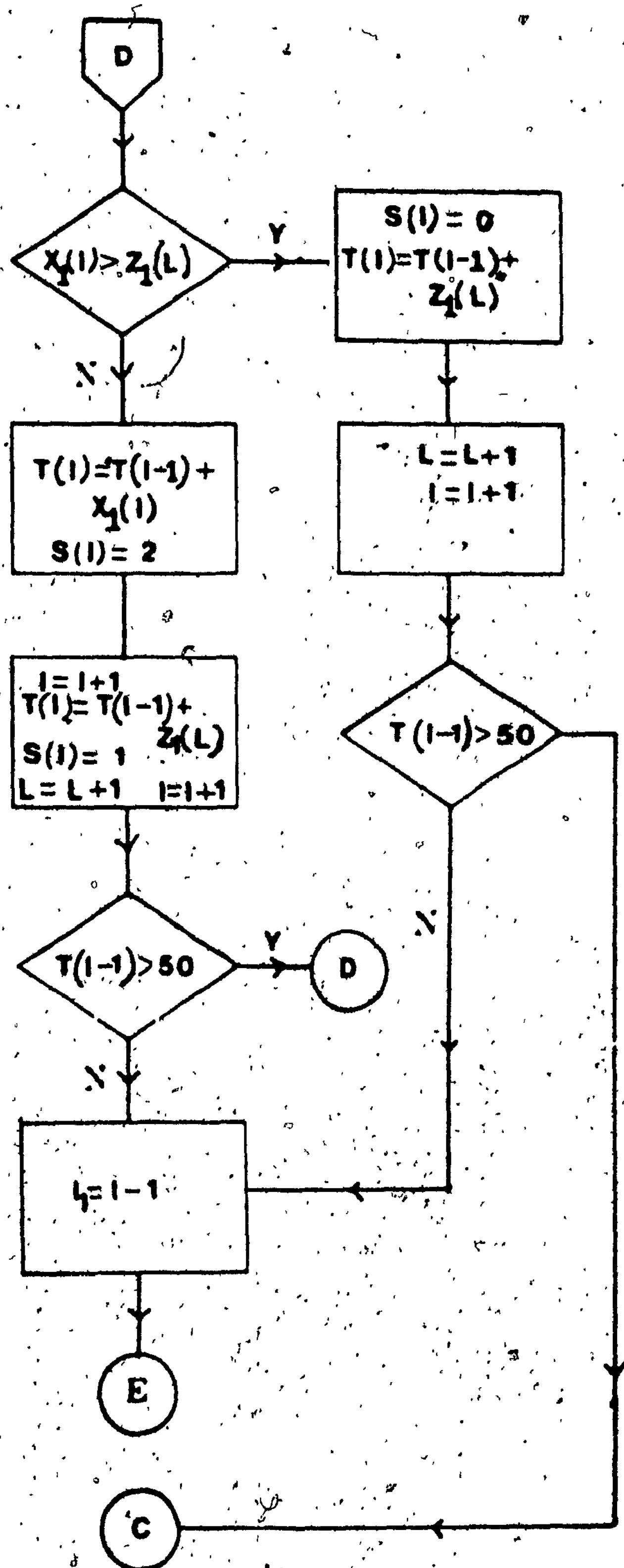
E,F: In this subroutine λ, λ_1, μ, A_1, A_2, A_3="a", and B_3="b", T_{10}, T_{01}, T_{12} are obtained. Further the probability of survival is determined for a different time period.

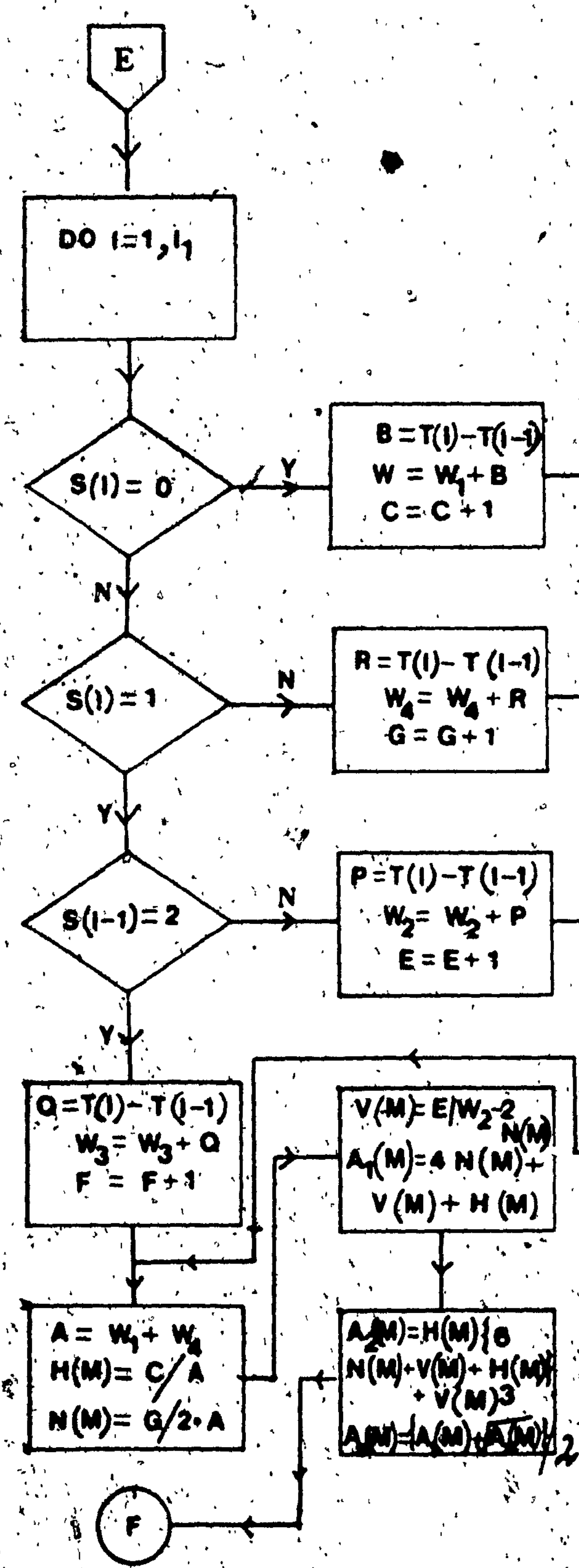
G,H,I: In these three subprograms the mean time to first failure is determined and in the end of "I" the Reliability by Limit Theorem is determined for the same system.

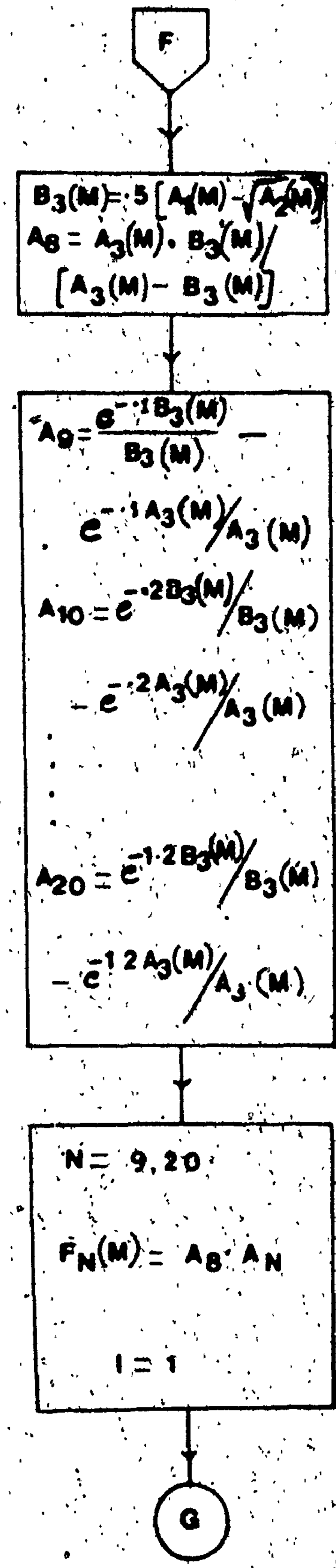
In the end $\hat{\lambda}$, $\hat{\lambda}_1$, $\hat{\mu}$, A_1, A_2, a, b, P[T] of the system and probability by approximation are printed.

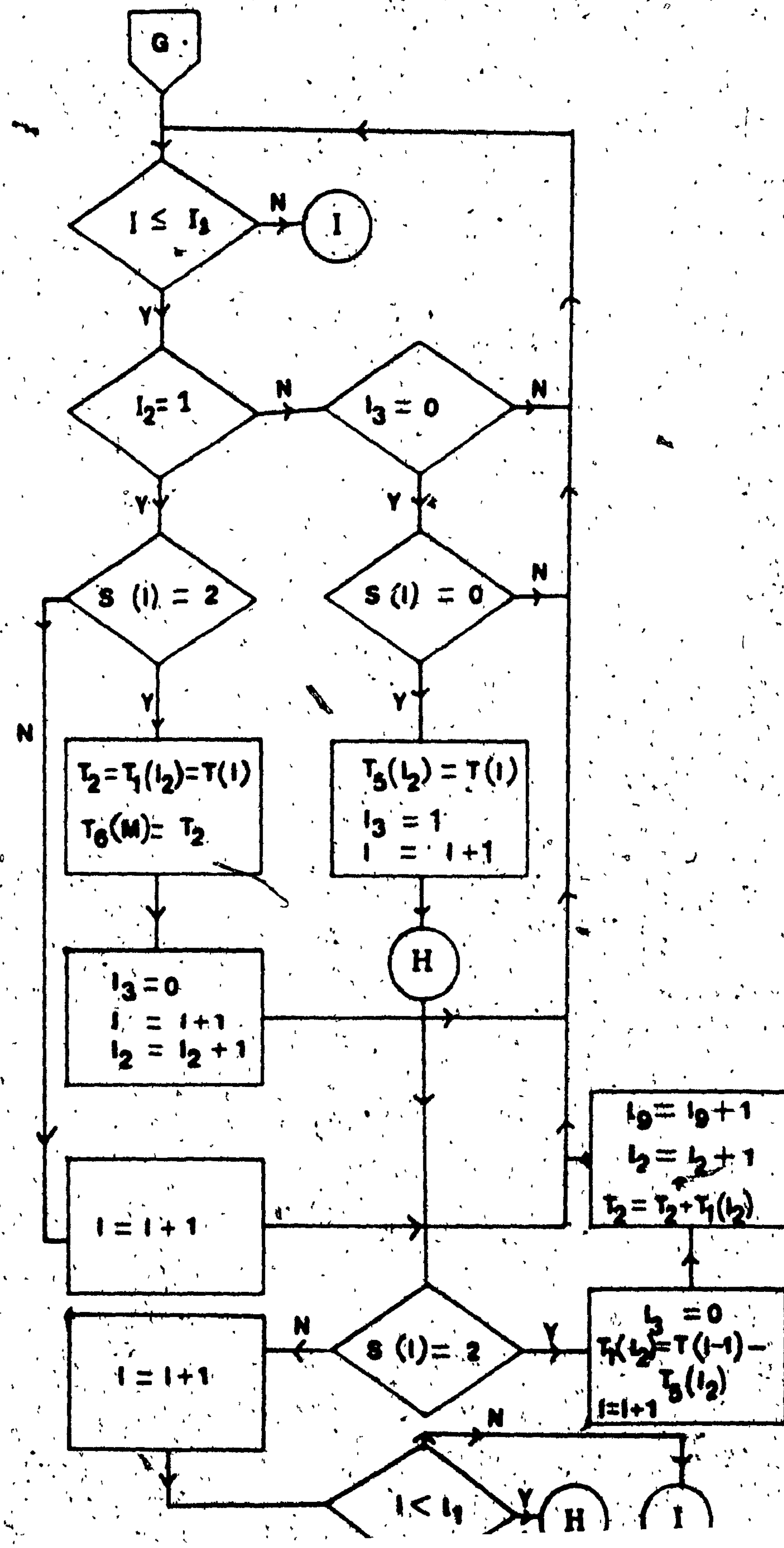

START
A
PRINT
HEADING
DIMENSIONS
INTEG.
O, C, E, F, G,
I1 , I2
REAL; A, B,
A8, A9, A10, A11,
A12, A13, A14, A15,
A16, A17, A20, A21
A18, A19, H, N,
P, Q, R, V, W1,
W2, W3, W4, F1,
T2, T4, T3, C3
T7 = T4 = T(I2)
= 0
M = I2 = 0
B

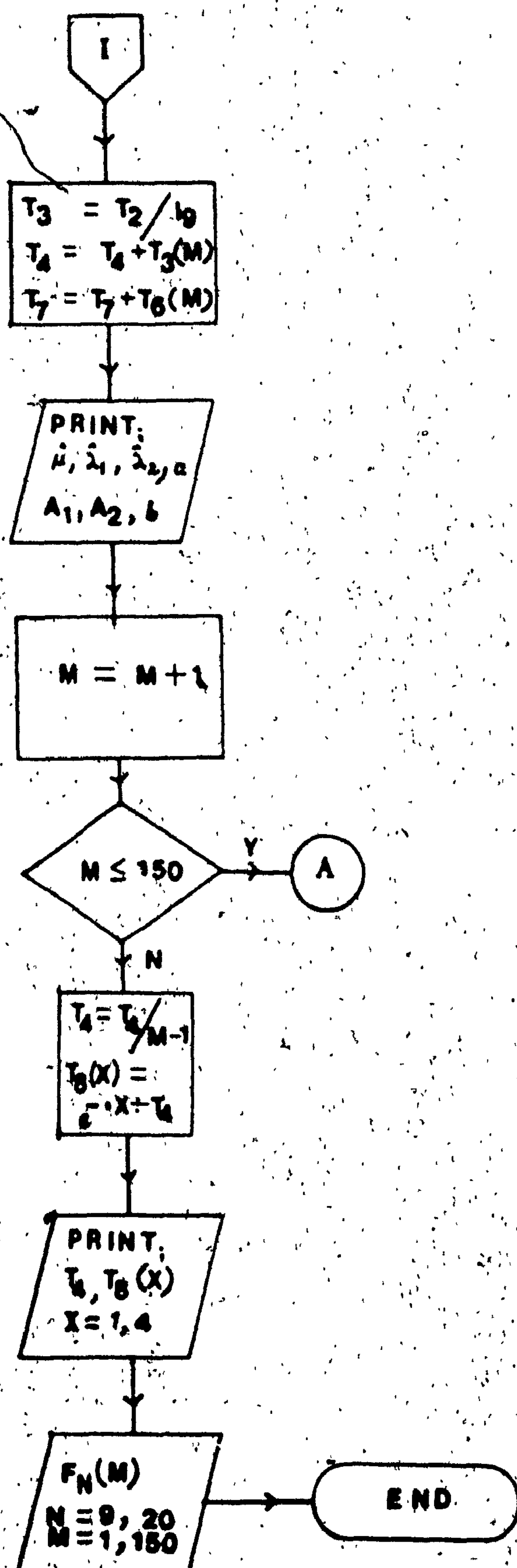

B
D = 1.0 UT
DSEED = 41.DO
× D
CEFGW, W, W = 0
T, R = 0
CALL GGUBS
TO GENERATE
X, Y, Z
CHANGE X, Y,
Z INTO X1, Y1,
Z1 (EXPON.)
C
Y
N
T(I) = T(I-1) +
Y1(L)
S(I) = 1
T(I) = T(I-1) +
X1(J)
S(I) = 1
K = J = I = I+1
D


D
X1(I) > Z1(L)
Y
N
S(I) = 0
T(I) = T(I-1) + Z1(L)
T(I) = T(I-1) + X1(I)
S(I) = 2
L = L+1
I = I+1
I = I+1
T(I) = T(I-1) + Z1(L)
S(I) = 1
L = L+1 I=I+1
T(I-1) > 50
Y
N
D
T(I-1) > 50
N
Y
I1 = I - 1
E
C


E
DO I=1, I1
S(I)= 0
Y
B=T(I)-T(I-1)
W = W1+B
C=C+1
N
S(I)=1
N
R=T(I)- T (I-1)
W4= W4 + R
G=G+1
Y
S(I-1)=2
N
P=T(I)-T(I-1)
W2= W2+P
E=E+1
Y
Q=T(I)- T(I-1)
W3= W3+ Q
F = F+1
A= W1+ W4
H(M)= C/A
N(M)= G/2·A
V(M)= E/W2-2 N(M)
A1(M)=4 N(M)+ V(M)+ H (M)
A2(M)=H(M){6 N(M)+V(M)+ H(M) + V(M)3
F


F
B3(M) = .5 [A1(M) - √A2(M)]
A8 = A3(M) · B3(M) / [A3(M) - B3(M)]
A9 = e^(-.1 B3(M)) / B3(M) - e^(-.1 A3(M)) / A3(M)
A10 = e^(-.2 B3(M)) / B3(M) - e^(-.2 A3(M)) / A3(M)
A20 = e^(-1.2 B3(M)) / B3(M) - e^(-1.2 A3(M)) / A3(M)
N = 9, 20
FN(M) = A8 · AN
I = 1
G


G
I ≤ I1
N
I
Y
I2 = 1
N
I3 = 0
N
Y
Y
S (I) = 2
S (I) = 0
N
Y
Y
N
T2 = T1(I2) = T(I)
T6(M) = T2
T5(I2) = T(I)
I3 = 1
I = I + 1
H
I3 = 0
I = I + 1
I2 = I2 + 1
I9 = I9 + 1
I2 = I2 + 1
T2 = T2 + T1(I2)
I = I + 1
N
S (I) = 2
Y
I3 = 0
T1(I2) = T(I-1) - T5(I2)
I = I + 1
I = I + 1
N
I < I1
Y
H
I


I
T3 = T2 / 19
T4 = T4 + T3(M)
T7 = T7 + T6(M)
PRINT;
A1, A2, b
M = M + 1
M ≤ 150
Y
A
N
T4 = T4 / M-1
T8(X) =
PRINT;
T4, T8 (X)
X = 1, 4
FN(M)
N = 9, 20
M = 1, 150
END

APPENDIX B

Scheme 2 Flow Chart

For both 2a and 2b initial and A subroutines are the same.
Variables are declared and initialized in the beginning.

A: X, Y, Z uniform (0,1) random variables are drawn and then changed in to X_1, Y_1, Z_1 exponential random variates. X_1, Y_1 are used as failure times of active and standby unit, and Z_1 as repair time of the unit.

Scheme 2a

B: In this subroutine for two hundred systems, average failure time is observed. This has been done 40 times.

C,D: The likelihood equation is maximized on the basis of average failure time for different "a" and "b". A simulated result is obtained in these subroutines.

Scheme 2b

B: Using the observations in A, in this subroutine, the standard deviation of system failure time is observed, and using the equation

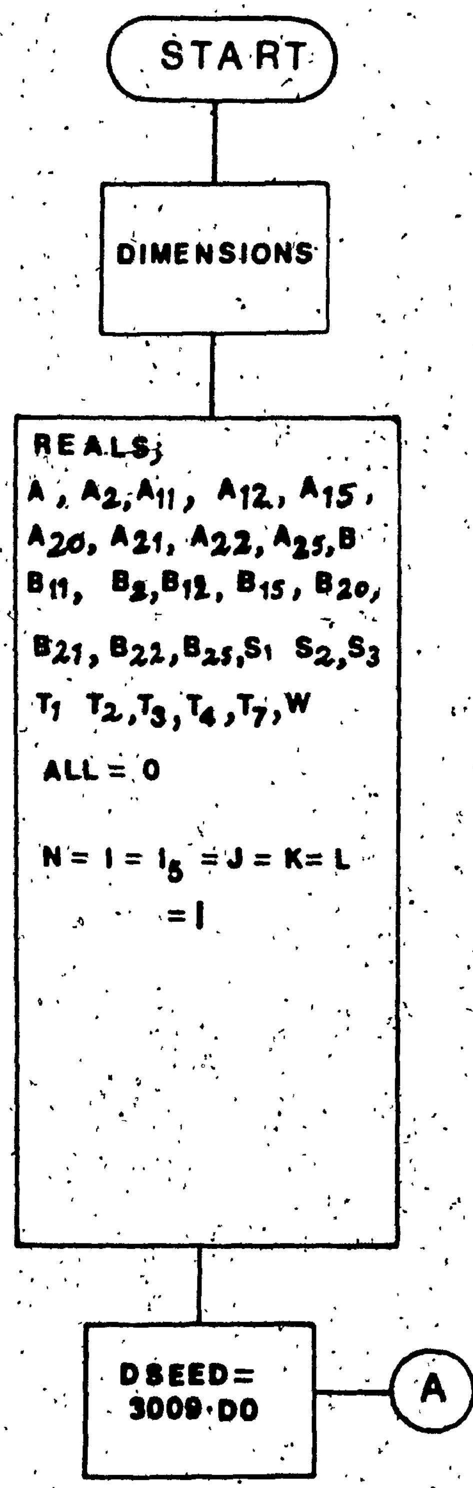
$b = 2/(\bar{t} + \sqrt{2S^2 - \bar{t}^2})$

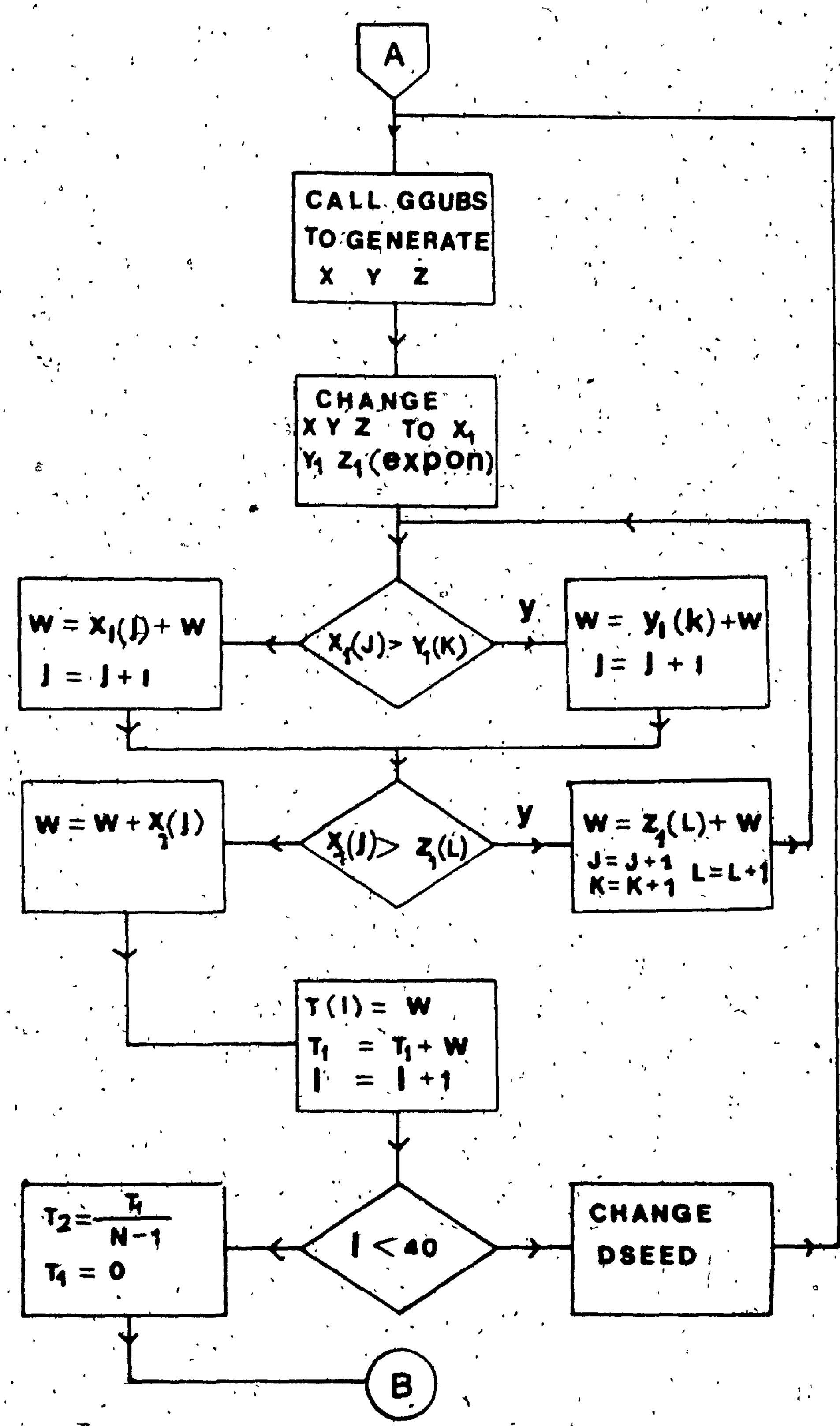
and $a = 2/\bar{t}$ for $\bar{t} < \sqrt{2S^2 - \bar{t}^2}$

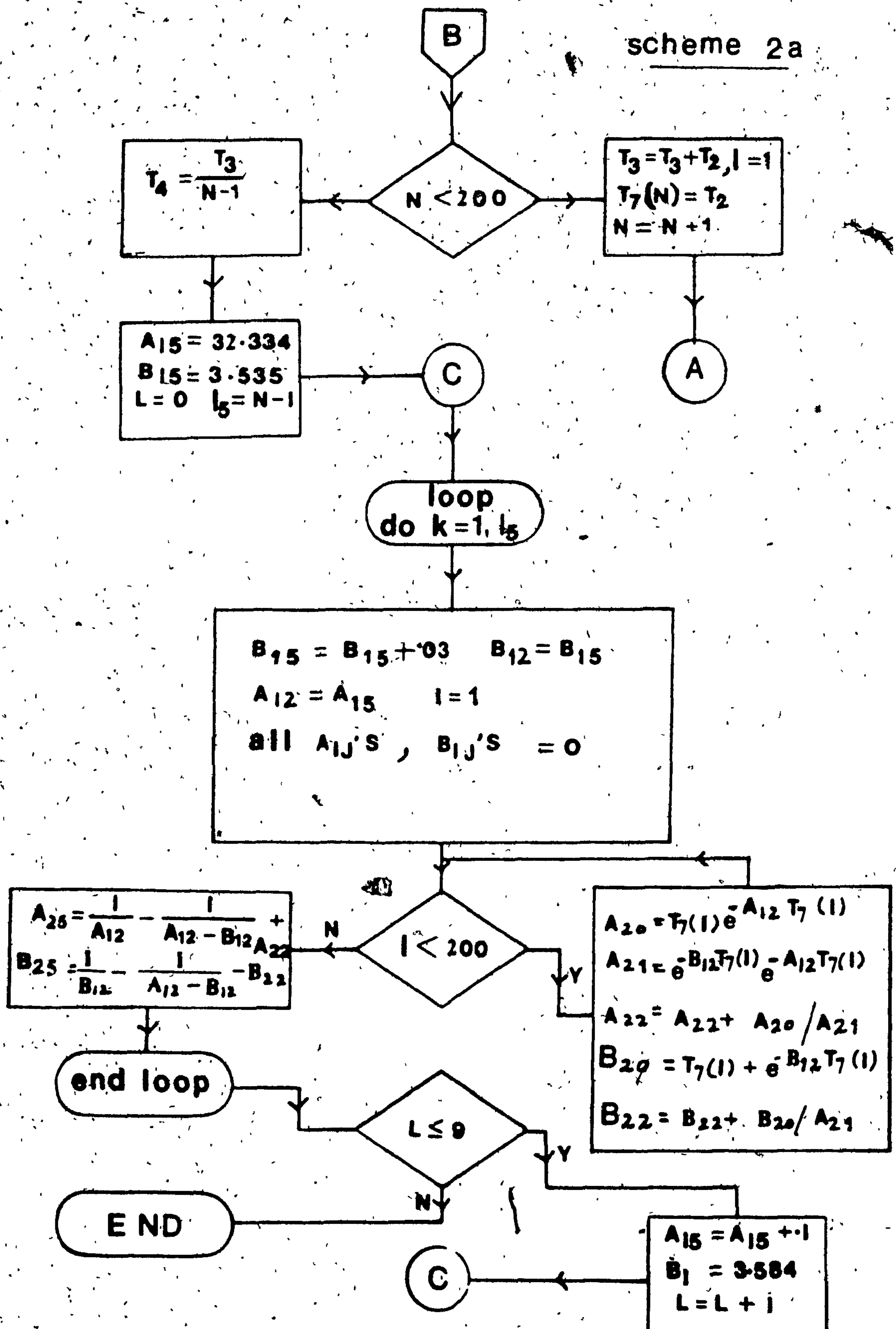
otherwise $a = 2/(\bar{t} - \sqrt{2S^2 - \bar{t}^2})$

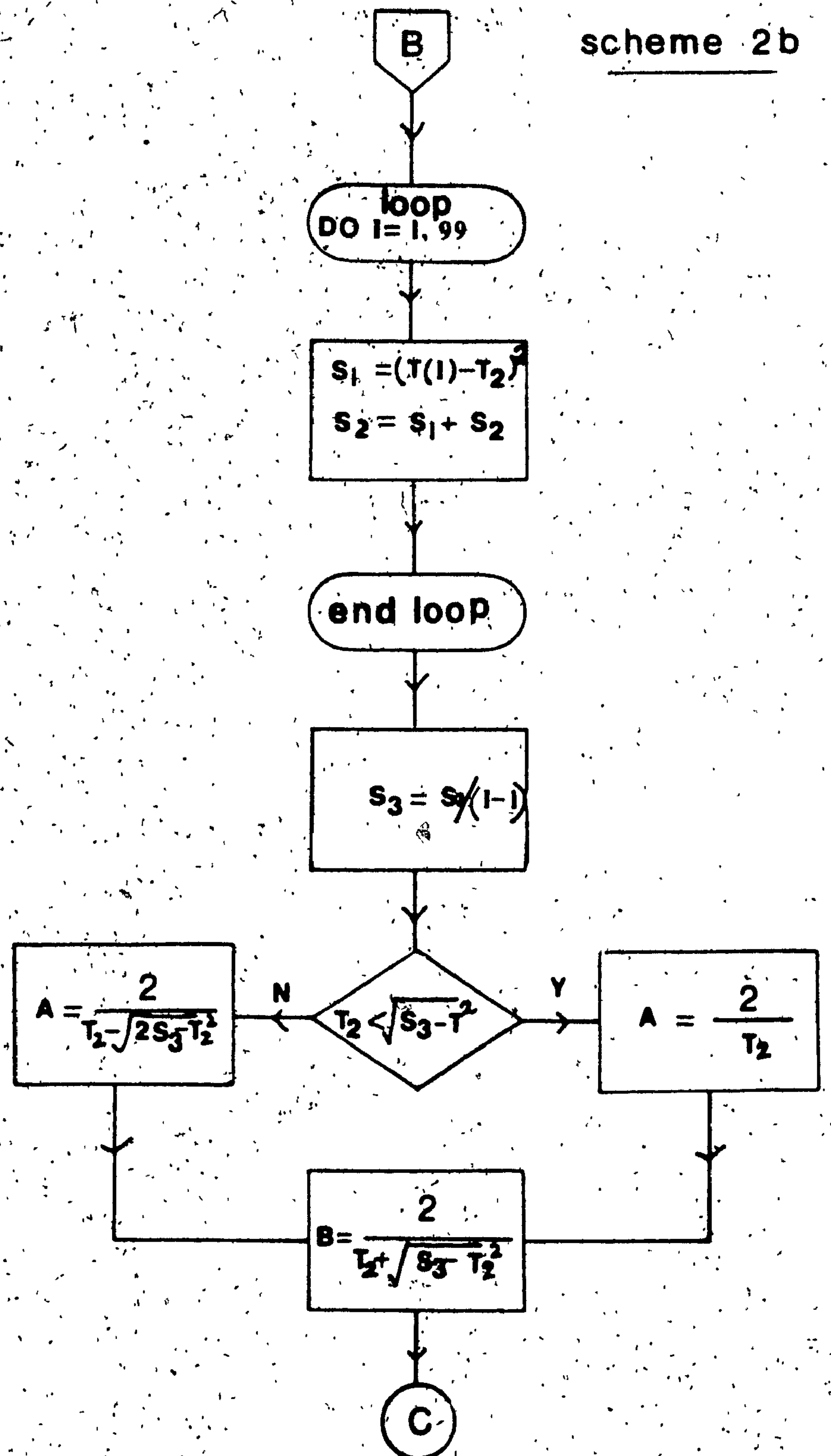
are obtained.

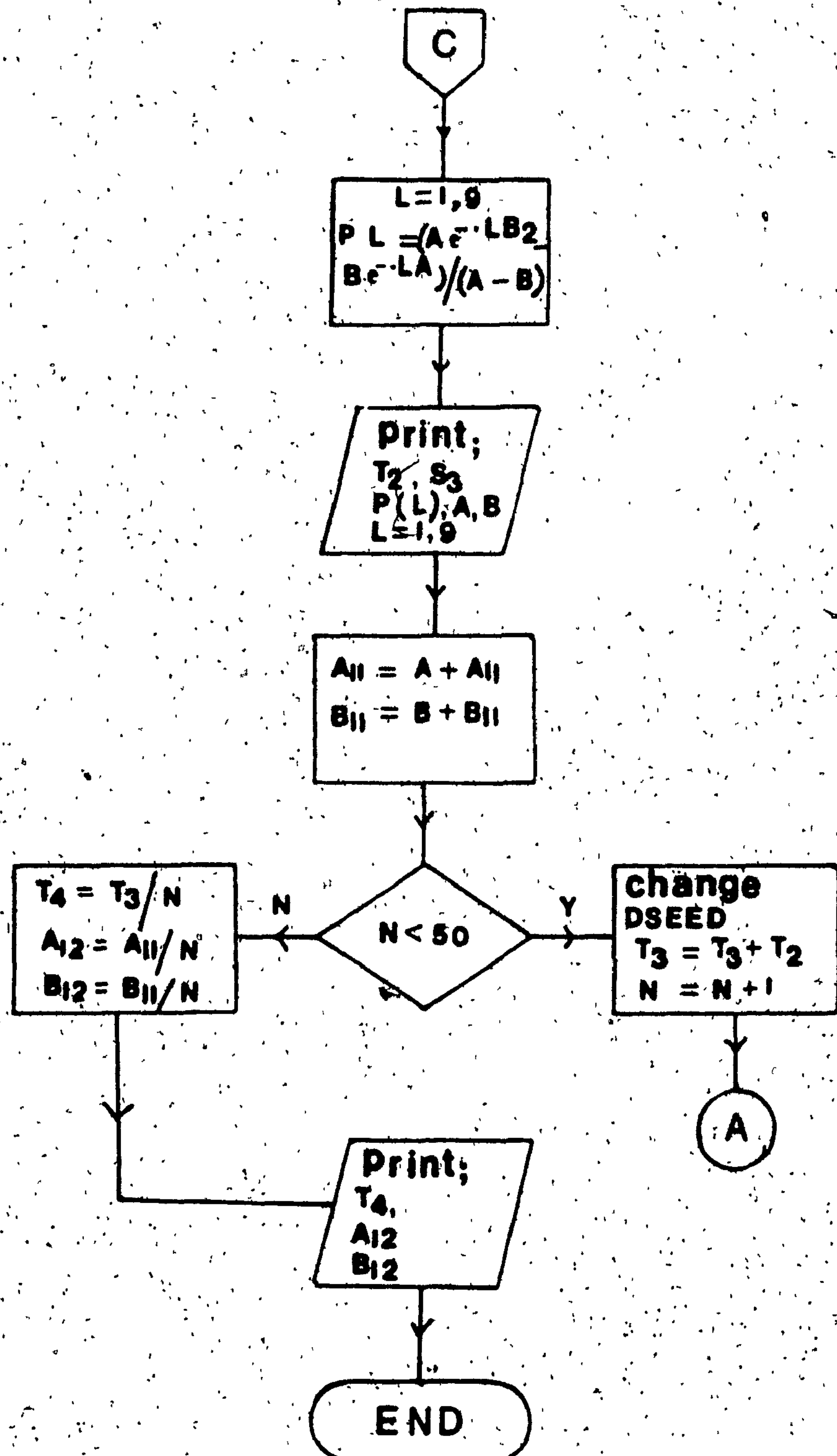
C: In this subroutine on the basis of 50 observations the mean of a and b are obtained, and printed.


START
DIMENSIONS
REALS;
A, A2, A11, A12, A15,
A20, A21, A22, A25, B
B11, B2, B12, B15, B20,
B21, B22, B25, S1 S2, S3
T1 T2, T3, T4, T7, W
ALL = 0
N = I = I5 = J = K = L
= 1
DSEED=
3009.D0
A


A
CALL GGUBS TO GENERATE X Y Z
CHANGE X Y Z TO X1 Y1 Z1 (expon)
W = X1(J) + W
J = J + 1
X1(J) > Y1(K)
y
W = Y1(k) + W
J = J + 1
W = W + X1(J)
X1(J) > Z1(L)
y
W = Z1(L) + W
J = J + 1
K = K + 1
L = L + 1
T(I) = W
T1 = T1 + W
I = I + 1
T2 = T1 / N-1
T1 = 0
I < 40
CHANGE DSEED
B


scheme 2a
B
N < 200
T4 = T3/(N-1)
T3 = T3 + T2, I = 1
T7(N) = T2
N = N + 1
A
A15 = 32·334
B15 = 3·535
L = 0 I5 = N-1
C
loop
do k = 1, I5
B15 = B15 + ·03   B12 = B15
A12 = A15   I = 1
all AIJ'S , BIJ'S = 0
I < 200
N
Y
A25 = 1/A12 − 1/(A12 − B12) + A22
B25 = 1/B12 − 1/(A12 − B12) − B22
A20 = T7(I) e^−A12 T7(I)
A21 = e^−B12 T7(I) e^−A12 T7(I)
A22 = A22 + A20/A21
B20 = T7(I) + e^−B12 T7(I)
B22 = B22 + B20/A21
end loop
L ≤ 9
END
A15 = A15 + ·1
B1 = 3·584
L = L + 1
C


B
scheme 2b
loop
DO I= 1, 99
S1 =(T(I)-T2)²
S2 = S1 + S2
end loop
S3 = S2/(I-1)
T2 < √(S3-T²)
N
Y
A = 2 / (T2-√(2S3-T2²))
A = 2 / T2
B= 2 / (T2+√(S3-T2²))
C



C
L=1,9
P L =(Ae^-LB2_ Be^-LA)/(A-B)
Print;
T2, S3
P(L), A, B
L=1,9
A11 = A + A11
B11 = B + B11
N < 50
N
Y
T4 = T3/N
A12 = A11/N
B12 = B11/N
change
DSEED
T3 = T3 + T2
N = N + 1
A
Print;
T4,
A12
B12
END