National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

NL-339 (r. 88/04) c

Canada

Existence and Approximation of Absolutely Continuous Measures Invariant Under Higher Dimensional Transformations

You-shi Lou

A Thesis
in
The Special
Individual Program

Presented in Partial Fulfillment of the requirements
for the degree of Doctor of Philosophy at
Concordia University
Montreal, Quebec, Canada

April, 1991

© You-shi Lou, 1991

National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-64757-4

Canadä

ABSTRACT

Existence and Approximation of Absolutely Continuous Measures Invariant Under Higher Dimensional Transformations

You-shi Lou, Ph.D.
Concordia University, 1991

Let $I^n = [0, 1]^n$ and L^1 be the space of all integrable functions on I^n. Let τ: $I^n \longrightarrow I^n$ be a measurable, nonsingular transformation on I^n. It is well known that f^* is the density of an absolutely continuous measure invariant under τ if and only if f^* is a fixed point of P_τ, where P_τ is the Frobenius-Perron operator associated with τ.

Using a Theorem of Rychlik's, we prove a sufficient condition for the existence of an absolutely continuous invariant measure for n-dimensional $C^{1+\varepsilon}$ Jablonski transformations. We present existence theorems for countable Jablonski transformations, random Jablonski transformations and higher dimensional Markov transformations. The existence of finite approximations to the invariant densities for Jablonski transformations and random Jablonski transformations is established. We also present a compactness theorem which is useful in approximating the invariant densities for Jablonski transformations.

ACKNOWLEDGEMENTS

The author would like to express his deepest gratitude to his thesis supervisor, Dr. Abraham Boyarsky of Concordia University, for his teaching, guidance, encouragement, valuable suggestions and support throughout the preparation of this thesis. The numerous conversations the author had with Dr. A. Boyarsky were most helpful and stimulating.

Also, the author would like to thank Dr. P. Gora of Warsaw University for his valuable assistance.

DEDICATION

To my wife

Mrs. Zhong-jing Lee Lou,

my parents

Mr. and Mrs. Er-pin Lou

and my daughters

Miss Jia-yun Lou, Jia-lan Lou and Jia-yang Lee

Table of Contents

CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

In 1973 Lasota and Yorke [7] proved a general sufficient condition for the existence of an absolutely continuous invariant measure for expanding, piecewise C^2 transformations on the interval. In spite of the suggestion at the end of [7] that the "bounded variation" technique can be used to obtain analogous results in higher dimensions, the generalization of the main result in [7] has taken much longer than expected. This was partly due to the difficulty in finding the right definition of variation in higher dimensions. For smooth maps on boundaryless domains, general results for the existence of absolutely continuous invariant measure were known as early as 1969 [21]. For piecewise C^2 maps in R^n, the first major attempt to prove an existence result came in 1979 [22]. The authors of [22] do not use a bounded variation argument but the proof, based on a one-dimensional version [5], is flawed. The first correct, but partial result, appeared in [11]. There, Keller considers expanding, piecewise analytic transformation on the unit square partitioned by smooth boundaries. A complicated definition of bounded variation is used and the method can not be extended beyond dimension 2. For boundaries which are not analytic, the sufficient condition that arises is rather complicated [23].

Working on rectangular partitions and with expanding, piecewise C^2 transformations which are very restrictive (the ith component of the transformation depends only on the ith variable), Jablonski [1] proved the existence of an absolutely continuous invariant measure using the Tonnelli definition of bounded variation. The technique in this special setting is exactly analogous to that in [7].

In this thesis we study Jablonski transformations and more general Markov transformations in higher dimensions.

In Chapter 1, we introduce Jablonski transformations and prove the denseness of these transformations in L^1. Also we review the proof of the existence of an absolutely continuous invariant measure for these transformations and prove a new result for countable Jablonski transformations. In Chapter 2, a result of Rychlik [18] is used to present a new sufficient condition for the existence of an absolutely continuous invariant measure for $C^{1+\varepsilon}$ Jablonski transformations in R^n. In Chapter 3, some new existence theorems are proved for higher dimensional Markov transformations. These results are needed in Chapter 5. In Chapter 4, we study small stochastic perturbations for n-dimensional Jablonski transformations and discuss some finite approximations to the invariant densities and the uniqueness. In Chapter 5, we discuss the compactness of invariant densities for families of Jablonski transformations. In Chapter 6, we approximate the absolutely continuous invariant measure for non-expanding Jablonski transformations. In Chapter 7, we prove a new sufficient condition for the existence of an absolutely continuous invariant measure for higher dimensional random Jablonski transformations. In Chapter 8, we study finite approximations to the invariant densities for random Jablonski transformations.

1.2 Higher Dimensional Transformation and Frobenius-Perron Operator

Let $I^n = [0, 1]^n$ and let $\mathcal{B}$ denote the Borel σ-algebra on I^n. The n-dimensional Lebesgue measure on I^n will be denoted by λ, and we write

$$\lambda(dx) = d\lambda_n = dx = dx_1 \ldots dx_n.$$

Let $L^1 = L^1(I^n, \mathcal{B}, \lambda)$ be the space of all Lebesgue integrable functions on I^n. The transformation $\tau: I^n \to I^n$ is defined by

$$\tau(x) = (\varphi_1(x), \ldots, \varphi_n(x)),$$

where $x = (x_1, \ldots, x_n)$ and $\varphi_i(x)$ is a function from I^n into $[0,1]$, $i = 1, \ldots, n$.

We say that a transformation $\tau: I^n \longrightarrow I^n$ is measurable if for any measurable subset A of I^n, $\tau^{-1}(A)$ is a measurable subset of I^n. A measurable transformation $\tau: I^n \longrightarrow I^n$ is nonsingular if $\lambda(A) = 0$ implies $\lambda(\tau^{-1}(A)) = 0$. For τ nonsingular, we define the Frobenius-Perron operator $P_\tau: L^1 \longrightarrow L^1$ by the formula

$$\int_A P_\tau f \, dx = \int_{\tau^{-1}(A)} f \, dx,$$

where $A \subseteq I^n$ is measurable. It follows that

$$P_\tau f(x) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_{\tau^{-1}\left(\prod_{i=1}^{n} [0, x_i]\right)} f(y) \, dy.$$

It is well known [2] that the operator P_τ is linear and satisfies the following conditions:

1) P_τ is positive: $f \geq 0$ implies $P_\tau f \geq 0$;

2) P_τ preserves integrals: for any $f \in L^1$, we have

$$\int_{I^n} P_\tau f \, dx = \int_{I^n} f \, dx;$$

3) $P_{\tau^k} = P_\tau^{\,k}$ (τ^k denotes the nth iterate of τ);

4) $P_\tau f = f$ if and only if the measure $d\mu = f\,dx$ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable subset A of I^n.

1.3 Jablonski Transformations and Functions of Bounded Variation in R^n

For $I^n = [0, 1]^n$, let $\mathcal{P} = \{D_1, \ldots, D_p\}$ be a partition of I^n such that $p < \infty$, (i.e., we have a finite partition) and

$$\bigcup_{j=1}^{p} D_j = I^n, \quad D_j \cap D_k = \phi \quad \text{for} \quad j \neq k.$$

A transformation τ: $I^n \longrightarrow I^n$ is called a Jablonski transformation if it is given by the formula

$$\tau(x) = (\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)), \quad x \in D_j, \quad 1 \le j \le p,$$

where $D_j = \prod_{i=1}^{n} [a_{ij}, b_{ij})$, φ_{ij}: $[a_{ij}, b_{ij}] \longrightarrow [0,1]$, and we write

$$[a_{ij}, b_{ij}) = \begin{cases} [a_{ij}, b_{ij}) & \text{if } b_{ij} < 1, \\ [a_{ij}, b_{ij}] & \text{if } b_{ij} = 1. \end{cases}$$

To define the variation of a function of n variables, we use the Tonelli definition ([1], [10]). Denote by $\prod_{i=1}^{n} A_i$ the Cartesian product of the sets A_i and by P_i the projection of R^n onto R^{n-1} defined by

$$P_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Let $A = \prod_{i=1}^{n} [a_i, b_i]$ and let g: $A \rightarrow R$. Fixing i, we define a function $\overset{A}{\underset{i}{V}} g$ of the n-1 variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$ by the formula

$$\overset{A}{\underset{i}{V}} g = \underset{i}{V} g = \sup \left\{ \sum_{k=1}^{r} \left| g(x_1, \ldots, x_i^k, \ldots, x_n) - g(x_1, \ldots, x_i^{k-1}, \ldots, x_n) \right| : \right.$$

$$\left. a_i = x_i^0 < x_i^1 < \ldots < x_i^r = b_i, \quad r \in N \right\}.$$

We now define the variation $\overset{A}{V} f$ as

$$\overset{A}{V} f = V f = \sup_{1 \le i \le n} \overset{A}{\underset{i}{V}} f,$$

where

$$\overset{A}{\underset{i}{V}} f = \inf \left\{ \int_{P_i(A)} \underset{i}{V}\, g \, d\lambda_{n-1} : \ g = f \ \text{ almost everywhere, } \ \underset{i}{V}\, g \text{ measurable} \right\}$$

and $d\lambda_{n-1} = dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$. If $\overset{A}{V} f < \infty$, then f is of bounded variation on A and its total variation is $\overset{A}{V} f$.

The existence of an absolutely continuous invariant measure for a Jablonski transformation τ was proved in [1]:

<u>Theorem 1.1</u> Consider a partition

$$\bigcup_{j=1}^{p} D_j = I^n, \quad D_j \cap D_k = \phi \quad \text{for} \quad j \neq k,$$

of I^n into sets D_j of the form $D_j = \prod_{i=1}^{n} D_{ij}$, $j = 1, 2, \ldots, p$, where

$$D_{ij} = [a_{ij}, b_{ij}) \ \text{ if } \ b_{ij} < 1 \ \text{ and } \ D_{ij} = [a_{ij}, b_{ij}] \ \text{ if } \ b_{ij} = 1.$$

Let τ: $I^n \longrightarrow I^n$ be the transformation given by the formula

$$\tau(x_1, \ldots, x_n) = (\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)), \quad (x_1, \ldots, x_n) \in D_j,$$

where φ_{ij}: $[a_{ij}, b_{ij}] \longrightarrow [0,1]$ are C^2 functions and

$$\inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 1.$$

Then for any $f \in L^1$ the sequence

$$\frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k f$$

is convergent in norm to a function $f^* \in L^1$. The limit function has the following properties:

1) $f \geq 0$ implies $f^* \geq 0$;

2) $\int_{I^n} f^* \, dx = \int_{I^n} f \, dx$;

3) $P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* dx$ is invariant under τ;

4) The function f^* is of bounded variation. Moreover, there exists a constant C independent of the choice of initial f such that the variation of the limiting f^* satisfies the inequality

$$Vf^* \le C \|f\|_1.$$

The following lemmas, proved in [1], are used in the proof of Theorem 1.1:

<u>Lemma 1.1</u> If f: A ⟶ R is a function of the n-dimensional interval $A = \prod_{i=1}^{n} [a_i, b_i]$ into R and g is given by

$$g = \int_{a_j}^{b_j} f \, dx_j,$$

then for $i \neq j$, $\underset{i}{V} g \le \underset{i}{V} f$.

<u>Lemma 1.2</u> Let S be a set of functions $f: I^n \longrightarrow R$ such that

1) $f \ge 0$;

2) $V f \le M$;

3) $\|f\| \le 1$.

Let f_i be such that

$$\int_{P_i(I^n)} \underset{i}{V} f_i \, d\lambda_{n-1} \le \underset{i}{V} f + \varepsilon \qquad (\varepsilon > 0)$$

and $f_i = f$ almost everywhere. Then, for $i=1,2,\ldots,n$

$$\lim_{k \longrightarrow \infty} \{\sup_{f \in S} \lambda_{n-1}(P_i(B_{f,k}))\} = 0,$$

where

$$B_{f,k} = \bigcup_{i=1}^{n} \{x \in I^n: f_i(x) \ge k\}.$$

__Lemma__ __1.3__ If a set S of functions $f\colon I^n \longrightarrow R$ satisfies the conditions of Lemma 1.2, then S is weakly relatively compact in L^1.

Let $f\colon \prod_{i=1}^{n} [a_i, b_j] \longrightarrow R$ and A be a subset of the interval $\prod_{i=1}^{n} [a_i, b_i]$. For this function and the set A, a function $V_{i,A}$ of the n-1 variables $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ is defined by

$$\underset{i,A}{V} f = \sup\left\{ \sum_{k=1}^{q} |f(x_1,\ldots,x_i^k,\ldots,x_n)-f(x_1,\ldots,x_i^{k-1},\ldots,x_n)| :\right.$$

$$\left. a_i \le x_i^0 < x_i^1 < \ldots < x_i^q \le b_i \quad (x_1,\ldots,x_i^k,\ldots, x_n)\notin A\right\}.$$

__Lemma__ __1.4__ Let A be a subset of the interval [a, b] and let a sequence of functions $f_\ell\colon [a,b] \longrightarrow R$ converge to a function $f\colon [a,b] \longrightarrow R$ pointwise on $[a,b]\setminus A$. Then

$$\underset{1,A}{V} f \le \liminf_{\ell \to \infty} \underset{1,A}{V} f_\ell \le \liminf_{\ell \to \infty} \underset{1}{V} f_\ell,$$

and there exists a function $\bar{f}\colon [a,b] \longrightarrow R$ such that $\bar{f} = f$ almost everwhere on $[a,b]\setminus A$ and

$$\underset{1}{V} \bar{f} = \liminf_{\ell \to \infty} \underset{1}{V} f_\ell.$$

__Lemma__ __1.5__ If a sequence of functions $f_\ell\colon [0,1]^n \longrightarrow R$ converges to a function $f\colon [0,1]^n \longrightarrow R$ in the norm of L^1, then

$$V f \le \limsup_{\ell \to \infty} V f_\ell.$$

1.4 Denseness of Jablonski Transformations in L^1

In this section, we will prove that Jablonski transformations are dense in L^1.

A transformation $\tau: I^n \longrightarrow I^n$ is called piecewise C^2 transformation if there is a partition $\mathcal{P} = \{D_1, \ldots, D_p\}$ of I^n,

$$\bigcup_{j=1}^{p} D_j = I^n \ , \quad D_j \cap D_k = \phi \text{ for } j \neq k,$$

where each D_j has piecewise C^2 boundaries and τ is given by the formula

$$\tau(x) = (\varphi_{1j}(x), \ldots, \varphi_{nj}(x)), \quad x \in D_j, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$ and $\varphi_{ij}(x)$: $\bar{D}_j \longrightarrow [0, 1]$ are C^2 functions. Note that τ need not be continuous on the boundary of D_j.

The C^2 Jablonski transformations on I^n with their rectangle boundaries and special coordinate functions are the simplest possible piecewise C^2 transformations. For a general piecewise C^2 transformation $\tau: I^n \longrightarrow I^n$ we are interested in knowing whether there exists a sequence of C^2 Jablonski transformations $\{\tau_\ell\}$ such that τ_ℓ converges to τ in L^1 as $\ell \to \infty$. With τ as in (1.1), we define its L^1 norm as follows:

$$\|\tau\|_1 = \sum_{i=1}^{n} \|\varphi_i\|_1 = \sum_{i=1}^{n} \int_{I^n} |\varphi_i| dx$$

$$= \int_{I^n} \left(\sum_{i=1}^{n} |\varphi_i| \right) dx$$

and $\tau_\ell(x) = (\varphi_1^\ell(x), \ldots, \varphi_n^\ell(x))$ converges to $\tau(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ in L^1 as $\ell \to \infty$ if and only if for any $1 \le i \le n$, $\varphi_i^\ell(x)$ converges to $\varphi_i(x)$ in L^1 as $\ell \to \infty$. i.e., $\lim_{\ell \to \infty} \|\tau - \tau_\ell\|_1 = 0$ if and only if for any $1 \le i \le n$, $\lim_{\ell \to \infty} \|\varphi_i - \varphi_i^\ell\|_1 = 0$.

We now prove this for dimension 2. (For dimension n, n > 2, we can get the same results.)

1. Case 1: Jablonski partition $\mathcal{P}$.

If τ is a piecewise C^2 transformation with respect to the Jablonski partition $\mathcal{P} = \{D_1,\dots,D_p\}$:

$$I^2 = \bigcup_{j=1}^{p} D_j = \bigcup_{j=1}^{p} [a_{1j},b_{1j}) \times [a_{2j},b_{2j}).$$

Let ℓ be an integer. For any $D_j = [a_{1j},b_{1j}) \times [a_{2j},b_{2j})$ we divide it into ℓ^2 equal subrectangles

$$E^{j\ell}_{km} = [a_{1j} + \frac{k-1}{\ell}(b_{1j}-a_{1j}),\ a_{1j} + \frac{k}{\ell}(b_{1j}-a_{1j}))$$

$$\times [a_{2j} + \frac{m-1}{\ell}(b_{2j}-a_{2j}),\ a_{2j} + \frac{m}{\ell}(b_{2j}-a_{2j}))$$

$$= E^{j\ell}_{1,k} \times E^{j\ell}_{2,m} \qquad k, m = 1, 2, \dots, \ell.$$

For any $f(x_1,x_2) \in C^2(D_j)$, let

$$Q^j_{\ell\, x_1} f(x_1,x_2) = \sum_{k=1}^{\ell} \frac{1}{\lambda(E^{j\ell}_{1,k})} \int_{E^{j\ell}_{1,k}} f(x_1,x_2)dx_1 \quad \chi_{E^{j\ell}_{1,k}}(x_1),$$

$$Q^j_{\ell\, x_2} f(x_1,x_2) = \sum_{m=1}^{\ell} \frac{1}{\lambda(E^{j\ell}_{2,m})} \int_{E^{j\ell}_{2,m}} f(x_1,x_2)dx_2 \quad \chi_{E^{j\ell}_{2,m}}(x_2).$$

Let

$$\varphi^{\ell}_{1j}(x_1,x_2) = Q^j_{\ell x_2}\varphi_{1j}(x_1,x_2) = \sum_{m=1}^{\ell} \psi^{j\ell}_{1,m}(x_1)\, \chi_{E^{j\ell}_{2,m}}(x_2),$$

$$\varphi^{\ell}_{2j}(x_1,x_2) = Q^j_{\ell x_1}\varphi_{2j}(x_1,x_2) = \sum_{k=1}^{\ell} \psi^{j\ell}_{2,k}(x_2)\, \chi_{E^{j\ell}_{1,k}}(x_1),$$

where

$$\psi^{j\ell}_{1,m}(x_1) = \frac{\ell}{b_{2j}-a_{2j}} \int_{a_{2j}+\frac{m-1}{\ell}(b_{2j}-a_{2j})}^{a_{2j}+\frac{m}{\ell}(b_{2j}-a_{2j})} \varphi_{1j}(x_1,x_2)dx_2,$$

$$\psi^{j\ell}_{2,k}(x_2) = \frac{\ell}{b_{1j}-a_{1j}} \int_{a_{1j}+\frac{k-1}{\ell}(b_{1j}-a_{1j})}^{a_{1j}+\frac{k}{\ell}(b_{1j}-a_{1j})} \varphi_{2j}(x_1,x_2)dx_1.$$

Now we define a Jablonski transformation τ_ℓ as follows: let the partition be given by $\mathcal{P}_\ell = \{E^{j\ell}_{km}\}$, $j = 1,2,\ldots,p$, $k,m = 1,2,\ldots,\ell$, and let

$$\tau_\ell(x_1,x_2) = (\varphi^\ell_1(x_1,x_2),\ \varphi^\ell_2(x_1,x_2))$$

$$= (\psi^{j\ell}_{1,m}(x_1),\ \psi^{j\ell}_{2,k}(x_2)),\quad (x_1,x_2) \in E^{j\ell}_{km}.$$

<u>Theorem</u> <u>1.2</u> $\tau_\ell(x_1,x_2)$ converges to $\tau(x_1,x_2)$ in L^1.

<u>Proof</u> For any $j = 1,2\ldots,p$, it follows from Lemma 2 of [9] that

$$\Phi^\ell_j(x_1) = \int_{a_{2j}}^{b_{2j}} |\varphi_{1j}(x_1,x_2) - \varphi^\ell_1(x_1,x_2)|dx_2 \longrightarrow 0 \text{ as } \ell \longrightarrow \infty,$$

and

$$\Psi^\ell_j(x_2) = \int_{a_{1j}}^{b_{1j}} |\varphi_{2j}(x_1,x_2) - \varphi^\ell_2(x_1,x_2)|dx_1 \longrightarrow 0 \text{ as } \ell \longrightarrow \infty.$$

Since φ_{ij} is C^2 on $\bar{D}_j$, they are bounded on D_j, it implies φ^ℓ_i is bounded on $\bar{D}_j$ and $\Phi_j(x_1)$, $\Psi_j(x_2)$ are bounded.

By the Lebesgue Convergence Theorem ([19], p. 88),

$$\iint_{D_j} |\varphi_{1j}(x_1,x_2) - \varphi^\ell_1(x_1,x_2)|dx_1\, dx_2$$

$$= \int_{a_{1j}}^{b_{1j}} \left(\int_{a_{2j}}^{b_{2j}} |\varphi_{1j}(x_1, x_2) - \varphi_1^{\ell}(x_1, x_2)| dx_2 \right) dx_1$$

$$= \int_{a_{1j}}^{b_{1j}} \Phi_j^{\ell}(x_1) dx_1 \longrightarrow 0 \quad \text{as } \ell \longrightarrow \infty,$$

and

$$\iint_{D_j} |\varphi_{2j}(x_1, x_2) - \varphi_2^{\ell}(x_1, x_2)| dx_1 \, dx_2$$

$$= \int_{a_{2j}}^{b_{2j}} \left(\int_{a_{1j}}^{b_{1j}} |\varphi_{2j}(x_1, x_2) - \varphi_2^{\ell}(x_1, x_2)| dx_1 \right) dx_2$$

$$= \int_{a_{2j}}^{b_{2j}} \Psi_j^{\ell}(x_2) dx_2 \longrightarrow 0 \quad \text{as } \ell \longrightarrow \infty.$$

Therefore,

$$\|\tau - \tau_{\ell}\|_1 = \|\varphi_1 - \varphi_1^{\ell}\|_1 + \|\varphi_2 - \varphi_2^{\ell}\|_1$$

$$= \iint_{I^2} |\varphi_1 - \varphi_1^{\ell}| dx_1 dx_2 + \iint_{I^2} |\varphi_2 - \varphi_2^{\ell}| dx_1 dx_2$$

$$= \sum_{j=1}^{p} \left(\iint_{D_j} |\varphi_{1j} - \varphi_1^{\ell}| dx_1 dx_2 + \iint_{D_j} |\varphi_{2j} - \varphi_2^{\ell}| dx_1 dx_2 \right) \to 0 \text{ as } \ell \to \infty.$$

Q.E.D.

2. Case 2: some D_j is not a rectangle.

First we prove Lemma 1.6 for completeness.

<u>Lemma 1.6</u> Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $\{f_n\}$ be a sequence in $\mathcal{X}$. If for any integer n, the sequence $\{f_{n\ell}\}$ converges to f_n as $\ell \to \infty$ and the sequence $\{f_n\}$ converges to f as $n \to \infty$, then we can take a sequence $\{f_{n\ell_n}\}$

such that it converges to f as $n \to \infty$.

Proof For any $\varepsilon > 0$ we can take an f_n such that $\|f_n - f\| < \frac{\varepsilon}{2}$. For this f_n, we can take $f_{n\ell_n}$ such that $\|f_{n\ell_n} - f_n\| < \frac{\varepsilon}{2}$. Therefore

$$\|f_{n\ell_n} - f\| \le \|f_{n\ell_n} - f_n\| + \|f_n - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

Let ℓ be an integer. We divide I^2 into ℓ^2 equal subsquars

$$E^{\ell}_{km} = [\tfrac{k-1}{\ell}, \tfrac{k}{\ell}) \times [\tfrac{m-1}{\ell}, \tfrac{m}{\ell})$$

$$= E^{\ell}_{1k} \times E^{\ell}_{2m}, \quad k,m = 1,2,\ldots,\ell.$$

For any $f(x_1,x_2) \in L^1(I^2)$, let

$$Q_{\ell x_1} f(x_1,x_2) = \sum_{k=1}^{\ell} \frac{1}{\lambda(E^{\ell}_{1k})} \int_{E^{\ell}_{1k}} f(x_1,x_2)dx_1 \, \chi_{E^{\ell}_{1k}}(x_1),$$

$$Q_{\ell x_2} f(x_1,x_2) = \sum_{m=1}^{\ell} \frac{1}{\lambda(E^{\ell}_{2m})} \int_{E^{\ell}_{2m}} f(x_1,x_2)dx_2 \, \chi_{E^{\ell}_{2m}}(x_2).$$

Let

$$\varphi^{\ell}_1 (x_1,x_2) = Q_{\ell x_2} \, \varphi_1(x_1,x_2) = \sum_{m=1}^{\ell} \psi^{\ell}_{1m}(x_1) \, \chi_{E^{\ell}_{2m}}(x_2),$$

$$\varphi^{\ell}_2 (x_1,x_2) = Q_{\ell x_1} \, \varphi_2(x_1,x_2) = \sum_{k=1}^{\ell} \psi^{\ell}_{2k}(x_2) \, \chi_{E^{\ell}_{1k}}(x_1),$$

where

$$\psi^{\ell}_{1m}(x_1) = \ell \int_{\frac{m-1}{\ell}}^{\frac{m}{\ell}} \varphi_1(x_1,x_2)dx_2,$$

$$\psi^{\ell}_{2k}(x_2) = \ell \int_{\frac{k-1}{\ell}}^{\frac{k}{\ell}} \varphi_2(x_1,x_2)dx_1.$$

We take a partition $\mathcal{P}_\ell = \{E^\ell_{km}\}$, $k,m = 1,2,\ldots,\ell$ and

$$\tau_\ell(x_1,x_2) = (\varphi^\ell_1(x_1.x_2), \varphi^\ell_2(x_1,x_2))$$

$$= (\psi^\ell_{1,m}(x_1), \psi^\ell_{2k}(x_2)). \quad (x_1,x_2) \in E^\ell_{km}.$$

As in Theorem 1.2, we have $\|\tau - \tau_\ell\|_1 \longrightarrow 0$ as $\ell \longrightarrow \infty$. But $\tau_\ell(x_1,x_2)$ is not a C^2 Jablonski transformation because we do not have $\psi^\ell_{1m}(x_1)$, $\psi^\ell_{2k}(x_2) \in C^2$, they may be discontinuous on the boundary of D_j, $j = 1,2,\ldots,p$. So we use a smoothing operator. Let

$$\alpha(x) = \begin{cases} C\, e^{\frac{1}{r^2-1}}, & r < 1, \\ 0, & r \geq 1, \end{cases}$$

where C is a constant such that $\int_{R^2} \alpha(x)dx = \int_{r<1} \alpha(x)dx = 1$ and $r = (x_1^2+x_2^2)^{1/2}$.

Let $\alpha_\varepsilon(x) = \varepsilon^{-2}\alpha(\frac{x}{\varepsilon})$. We know that $\alpha(x) \in C^\infty$, $\alpha_\varepsilon(x) \in C^\infty$ and for any $f(x) \in L^1$,

$$f_\varepsilon(x) = \int \alpha_\varepsilon(y)f(x-y)dy \in C^\infty$$

is such that $\|f - f_\varepsilon\|_1 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus, for $\tau_\ell(x_1,x_2)$, we can find $\tau_{\ell\varepsilon}(x_1,x_2) \in C^\infty$ such that $\|\tau_{\ell\varepsilon} - \tau_\ell\|_1 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. By virtue of Lemma 1.6, we have:

<u>Theorem</u> <u>1.3</u> For any piecewise C^2 transformation τ: $I^2 \longrightarrow I^2$, we can find a sequence of C^∞ Jablonski transformations τ_ℓ: $I^2 \longrightarrow I^2$ such that $\|\tau_\ell - \tau\|_1 \longrightarrow 0$ as $\ell \longrightarrow \infty$.

1.5 Countable Jablonski Transformations

In this section, we will prove the existence of the fixed point of P_τ for a countable Jablonski transformation τ.

A transformation τ: $I^n \longrightarrow I^n$ will be called a countable Jablonski transformation if there exists a partition

$$\mathcal{P} = \{D_1, D_2, \ldots\} = \{D_j\}_{j=1}^{\infty},$$

$$\bigcup_{j=1}^{\infty} D_j = I^n, \quad D_j \cap D_k = \phi \quad \text{for } j \neq k$$

of I^n into sets D_j of the form $D_j = \prod_{i=1}^{n} D_{ij}$, $j = 1, 2, \ldots$, where

$$D_{ij} = [a_{ij}, b_{ij}) \text{ if } b_{ij} < 1 \text{ and } D_{ij} = [a_{ij}, b_{ij}] \text{ if } b_{ij} = 1$$

and τ is given by the formula

$$\tau(x_1, \ldots, x_n) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)) = (\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)), \quad (x_1, \ldots, x_n) \in D_j,$$

where $\varphi_{ij} : [a_{ij}, b_{ij}] \longrightarrow [0,1]$.

We say that a countable Jablonski transformation τ is a C^2 transformation if for any $1 \leq i \leq n$ and $j = 1, 2, \ldots$, $\varphi_{ij} \in C^2$. A C^2 countable Jablonski transformation τ has finite image with respect to the partition $\{D_j\}$ if in the collection $\{\tau(D_j)\}$ there are only finitely many different rectangles.

As in [20], if τ is a C^2 countable Jablonski transformation with finite image, then it is easy to show that τ^ℓ, $\ell \geq 1$, is a C^2 countable Jablonski transformation with finite image.

Theorem 1.4 Let τ: $I^n \longrightarrow I^n$ be a C^2 countable Jablonski transformation with finite image. If

$$\text{(1)} \qquad \inf_{i,j} \left\{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \right\} = s > 1;$$

(2) there exists a constant M such that for any fixed $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$,

$$\overset{1}{\underset{0}{V}} \frac{1}{|\varphi_i'|} \le M < \infty.$$

(Where φ_i' is not defined, we define it as the left limit). Then there exists an absolutely continuous measure invariant under τ.

Proof   Take N such that $s^N \ge \frac{1}{\mu} > 6$. Set $T = \tau^N$ and let $\{E_j\}_{j=1}^{\infty}$ be the partition of I^n relative to T. T is a C^2 countable Jablonski transformation with finite image. Let

$$T = \tau^N = (\Phi_{1j}(x_1),\dots,\Phi_{nj}(x_n)), \quad (x_1,\dots,x_n) \in E_j,$$

where $E_j = \prod_{i=1}^{n} [c_{ij}, d_{ij})$. T satisfies all conditions of this theorem and $|\Phi_{ij}'| \ge \frac{1}{\mu} > 6$. We have

$$P_T f = \sum_j f(\psi_{1j}(x_1),\dots, \psi_{nj}(x_n))\, \sigma_{1j}(x_1)\dots\sigma_{nj}(x_n)\, \chi_{I_j}(x),$$

where $\psi_{ij} = \Phi_{ij}^{-1}$, $\sigma_{ij} = |\psi_{ij}'(x_i)|$ and $I_j = T(\bar{E}_j)$.

Let $f \ge 0$ be a function of bounded variation and $\int_{I^n} f(x)\,dx = 1$. We show that there exists a constant $K > 0$ independent of f such that

$$VP_T f \le 6\mu\, Vf + K.$$

Denote by $\mathcal{G}$ the set of functions of the form

$$g = \sum_{j=1}^{L} g_j\, \chi_{A_j},$$

where χ_{A_j} is the characteristic function of the set $A_j = \prod_{i=1}^{n} [\alpha_{ij}, \beta_{ij}] \subset I^n$ (we do not assume that $\alpha_{ij} < \beta_{ij}$, the interval $[\alpha_{ij}, \beta_{ij}]$ can be degenerate) and $g_j\colon I^n \longrightarrow R$ is a C^1-function on A_j. By the proof of Theorem 1 of [1], it is enough to show that for any $i = 1,2,\dots,n$ and $f \in \mathcal{E}$, $f \geq 0$ and $f_i \in \mathcal{E}$, $f_i = f$ a.e., $\int_{P_i(I^n)} \bigvee_i^{I^n} f_i \, d\lambda_{n-1} = \bigvee_i f$, we have

$$\int_{P_i(I^n)} \bigvee_i^{I^n} P_T f_i \, d\lambda_{n-1} \leq 6\mu \int_{P_i(I^n)} \bigvee_i^{I^n} f_i \, d\lambda_{n-1} + K.$$

Now we show that.

$$\bigvee_i^{I^n} P_T f_i \leq \sum_j \bigvee_i^{I_j} f_i(\psi_{1j}(x_1),\dots,\psi_{nj}(x_n))\sigma_{1j}(x_1)\dots\sigma_{nj}(x_n)\chi_{I_j}(x)$$

$$+ \sum_j \Big(|f_i(\psi_{1j}(x_1),\dots,c_{ij},\dots,\psi_{nj}(x_n))|\sigma_{ij}(\psi_{ij}(c_{ij}))$$

$$+ |f_i(\psi_{1j}(x_1),\dots,d_{ij},\dots,\psi_{nj}(x_n))|\sigma_{ij}(\psi_{ij}(d_{ij}))\Big)$$

$$\prod_{\substack{j=1 \\ j\neq i}}^{n} \sigma_{ij}(x_j) \; \chi_{I_j}(x_1,\dots,c_{ij},\dots,x_n).$$

Hence

$$\int_{P_i(I^n)} \bigvee_i^{I^n} P_T f_i \, d\lambda_{n-1}$$

$$\leq \sum_j \int_{P_i(I_j)} \bigvee_i^{I_j} f_i(\psi_{1j}(x_1),\dots,\psi_{nj}(x_n))\sigma_{1j}(x_1)\dots\sigma_{nj}(x_n) \, d\lambda_{n-1}$$

$$+ \sum_j \int_{P_i(I_j)} \Big(|f_i(\psi_{1j}(x_1),\dots,c_{ij},\dots,\psi_{nj}(x_n))|\sigma_{ij}(\Phi_{ij}(c_{ij}))$$

$$+ |f_i(\psi_{1j}(x_1),\dots,d_{ij},\dots,\psi_{nj}(x_n))|\sigma_{ij}(\Phi_{ij}(d_{ij}))\Big) \prod_{\substack{j=1 \\ j\neq i}}^{n} \sigma_{ij}(x_j) \, d\lambda_{n-1}$$

$$\le 2 \sum_j \int_{P_i(I_j)} \bigvee_i^{I_j} f_i(\psi_{1j}(x_1),\dots,\psi_{nj}(x_n))\sigma_{1j}(x_1)\dots\sigma_{nj}(x_n)\, d\lambda_{n-1}$$

$$+ \frac{2}{\delta} \int_{I^n} f_i(x)\, dx,$$

where we used

$$|g(x) + g(y)| \le \bigvee_x^y g(t) + \left|\frac{2}{y-x} \int_x^y g(t)\, dt\right|$$

and

$$\delta = \min_{i,j} |\Phi_{ij}(c_{ij}) - \Phi_{ij}(d_{ij})| > 0.$$

As the proof of Theorem 1 in [20], the first term less than

$$6\mu \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + \frac{4\mu}{\delta_i} \int_{P_i(I^n)} d\lambda_{n-1}$$

$$= 6\mu \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + \frac{4\mu}{\delta_i}$$

for some $\delta_i > 0$. The second term is equal to $\frac{2}{\delta}$. Therefore

$$\int_{P_i(I^n)} \bigvee_i^{I^n} P_T f_i\, d\lambda_{n-1}$$

$$\le 6\mu \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + \frac{4\mu}{\delta_i} + \frac{2}{\delta}$$

$$\le 6\mu \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + K,$$

where $K = \max_i \frac{4\mu}{\delta_i} + \frac{2}{\delta}$ is independent of f. Now we have $VP_T f \le 6\mu\, Vf + K$. Since $6\mu < 1$, interaction of the above inequality yields

$$\lim_{m \to \infty} \sup \mathbf{V}\, P_T^m f \le \frac{K}{1 - 6\mu}.$$

It implies that the sequence $\left\{P_T^m f\right\}_{m=1}^{\infty}$ is weakly compact in $L^1(I^n)$ and the same holds for $\left\{P_\tau^m f\right\}_{m=1}^{\infty}$. From the Kakutani-Yoshida Theorem ([8], VIII 5.3), it follows that the sequence

$$\left\{\frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k f\right\}$$

converges strongly in L^1 to, say, f^* which is a fixed point of P_τ. A standard functional analysis argument proves that f^* is of bounded variation.

Q.E.D.

CHAPTER 2

The Existence of Invariant Measures under $C^{1+\varepsilon}$ Jablonski Transformation

In [18] Rychlik proved a sufficient condition for the existence of a fixed point of P_τ. Using his result, we present a sufficient condition for the existence of an absolutely continuous invariant measures under $C^{1+\varepsilon}$ Jablonski transformations in R^n. As a special case, when n = 1, the Lasota and Yorke's sufficient condition can be weakened from piecewise C^2 to piecewise $C^{1+\varepsilon}$ for any $0 < \varepsilon < 1$.

2.1 Rychlik's Sufficient Conditions

Let (X,Σ,λ) be a Lebesgue space with a σ-algebra Σ and a probability measure λ. Let $\tau: X \longrightarrow X$ be a measurable, nonsingular transformation and let g_τ be the absolute value of the reciprocal of the Jacobian of τ. Then the Frobenius-Perron operator can be represented in the following way:

$$P_\tau f(x) = \sum_{y\in\tau^{-1}(x)} g_\tau(y)\, f(y), \quad x \in X.$$

Let $\mathcal{P}$ be a partition of X which is a generator for τ, i.e., $\bigvee_{k=0}^{\infty} \tau^{-k}(\mathcal{P}) = \varepsilon$, where ε is a partition into points. For any positive integer ℓ let $\mathcal{P}_\ell$ be a partition of X which is a generator for τ^ℓ. Let $g = g_1 = g_\tau$ and $g_\ell = g_{\tau^\ell}$. For every $A \in \Sigma$, we define $\mathcal{P}(A) = \{B \in \mathcal{P}: \lambda(A\cap B) > 0\}$.

Condition 2.1 (Distortion condition) There exists a positive number b such that for any $\ell \geq 1$ and any $B \in \mathcal{P}_\ell$ we have

$$\sup_B g_\ell \leq b \inf_B g_\ell.$$

Condition 2.2 (Localization condition) There exist $\varepsilon > 0$ and $0 < \gamma < 1$ such that for any $\ell \geq 1$ and $B \in \mathcal{P}_\ell$ $\lambda(\tau^\ell B) < \varepsilon$ implies $\sum_{B' \in \mathcal{P}(\tau^\ell B)} \sup_{B'} g \leq \gamma$.

Condition 2.3 (Boundedness condition) $\sum_{B \in \mathcal{P}} \sup_B g < \infty$.

Under these 3 conditons, Rychlik [18] proved the existence of a fixed point for P_τ.

Theorem 2.1 Assume Condition 2.1, 2.2 and 2.3 are satisfied. Then the sequence $\left\{P_\tau^\ell 1\right\}_{\ell=1}^{\infty}$ is bounded in L^∞, and the averages $\frac{1}{\ell}\sum_{j=0} P_\tau^j 1$ converge in L^1 to some $\varphi \in L^\infty$ such that $P_\tau \varphi = \varphi$.

2.2 $C^{1+\varepsilon}$ Jablonski Transformations

Let $X = I^n$. For any $x,y \in I^n$, we define

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2},$$

where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$.

Lemma 2.1 If there exists a positive number M such that for any $\ell \geq 1$, $B \in \mathcal{P}_\ell$ and any $x, y \in B$, we have $\frac{g_\ell(x)}{g_\ell(y)} \leq M$, then Condition 1 is satisfied.

Lemma 2.2 Let $\tau: I^n \longrightarrow I^n$ be a nonsingular piecewise $C^{1+\varepsilon}$ transformation, i.e., there exists a constant $\bar{c}$ such that for any $x, y \in D_j$, $j = 1,\dots,p$, $|J(x) - J(y)| \leq \bar{c}\, d(x,y)^\varepsilon$, where J is the absolute value of the Jacobian of τ. If there exist constants $\alpha > 1$, $C > 0$ such that for $\ell \geq 1$, $B \in \mathcal{P}_\ell$ and $x,y \in B$, we have $d(\tau^\ell x, \tau^\ell y) \geq C\, \alpha^\ell d(x,y)$, then there exists a constant M such that $\frac{g_\ell(x)}{g_\ell(y)} \leq M$ and Condition 1 is satisfied.

Proof For any $x, y \in B \in \mathcal{P}$, we have $d(\tau x, \tau y) \geq C\, \alpha\, d(x, y)$. Let $\mathbb{J}$ be the Jacobi matrix of τ and $J = |\det \mathbb{J}|$.

If $s_1,\dots,s_n$ and $v_1,\dots,v_n$ are the eigenvalues and the eigenvectors of $\mathbb{J}^T\mathbb{J}$, then, for $i = 1,\dots,n$, we have

$$C^2\alpha^2|v_i|^2 \leq |\mathbb{J}v_i|^2 = (\mathbb{J}v_i, \mathbb{J}v_i) = (\mathbb{J}^T\mathbb{J}v_i, v_i) = s_i(v_i, v_i) = s_i|v_i|^2$$

and $s_i \geq C^2\alpha^2$. Therefore,

$$J^2 = (\det \mathbb{J})^2 = \det \mathbb{J}^T\mathbb{J} = s_1 s_2 \dots s_n \geq C^{2n}\alpha^{2n}$$

and $J \geq C^n\alpha^n$.

For $x,y \in B \in \mathcal{P}_\ell$, $k = 0, 1, \dots, \ell-1$, we have

$$d(\tau^k x, \tau^k y) \leq C^{-1}\, \alpha^{-(\ell-k)}\, d(\tau^\ell x, \tau^\ell y) \leq C^{-1}\, \alpha^{-(\ell-k)}\sqrt{n}.$$

Then, using the fact that τ is piecewise $C^{1+\varepsilon}$, we have

$$\frac{J(\tau^k y)}{J(\tau^k x)} = \left| 1 + \frac{J(\tau^k y) - J(\tau^k x)}{J(\tau^k x)} \right|$$

$$\leq 1 + \frac{|J(\tau^k y) - J(\tau^k x)|}{J(\tau^k x)}$$

$$\leq 1 + \frac{\bar{c}\, d(\tau^k x,\ \tau^k y)^{\varepsilon}}{C^n \alpha^n}$$

$$\leq 1 + \frac{\bar{c}\sqrt{n^{\varepsilon}}}{C^{n+\varepsilon}\ \alpha^n}\ \alpha^{-(\ell-k)\varepsilon}, \qquad k = 0,\ 1, \ldots,\ \ell-1,$$

and

$$\frac{g_\ell(x)}{g_\ell(y)} = \frac{J(\tau^{\ell-1}y)\ J(\tau^{\ell-2}y)\ldots J(\tau y)\ J(y)}{J(\tau^{\ell-1}x)\ J(\tau^{\ell-2}x)\ldots J(\tau x)\ J(x)}$$

$$\leq \prod_{k=0}^{\ell-1}\left(1 + \frac{\bar{c}\sqrt{n^{\varepsilon}}}{C^{n+\varepsilon}\ \alpha^n}\ \alpha^{-(\ell-k)\varepsilon}\right)$$

$$= \prod_{i=1}^{\ell}\left(1 + \frac{\bar{c}\sqrt{n^{\varepsilon}}}{C^{n+\varepsilon}\ \alpha^n}\ \alpha^{-\varepsilon i}\right)$$

$$\leq \prod_{i=1}^{\infty}\left(1 + \frac{\bar{c}\sqrt{n^{\varepsilon}}}{C^{n+\varepsilon}\ \alpha^n}\ \alpha^{-\varepsilon i}\right) = M.$$

Q.E.D.

Let τ: $I^n \longrightarrow I^n$ be a Jablonski transformation, i.e.,

$$\tau(x) = (\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)),\ x \in D_j,$$

where for any $1 \leq j \leq p$, D_j is a rectangle, $D_j \cap D_k = \phi$ for $j \neq k$ and $\bigcup_{j=1}^{p} D_j = I^n$. $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$. For any $\ell \geq 1$, let $\mathcal{P}_\ell = \left\{D_1^{(\ell)}, \ldots, D_{p_\ell}^{(\ell)}\right\}$.

Then every $D_j^{(\ell)}$ is a rectangle and τ^ℓ is a Jablonski transformation.

Fix $(x_1,\ldots,x_{i-1}, x_{i+1},\ldots,x_n)$, and let $L_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ be the straight line. Let $r_i^{(\ell)}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ be the number of $D_j^{(\ell)}$'s for which $D_j^{(\ell)} \cap L_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \neq \phi$. For any $1 \le i \le n$ let

$$r_i^{(\ell)} = \sup_{0\le x_k\le 1,\, 1\le k\le n,\, k\ne i} r_i^{(\ell)}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), \quad r_i = r_i^{(1)},$$

$$r^{(\ell)} = r_1^{(\ell)} r_2^{(\ell)} \cdots r_n^{(\ell)} = \prod_{i=1}^{n} r_i^{(\ell)}, \quad r = r^{(1)},$$

$$\bar{r}_i^{(\ell)} = r^{(\ell)}/r_i^{(\ell)} = r_1^{(\ell)} \cdots r_{i-1}^{(\ell)} r_{i+1}^{(\ell)} \cdots r_n^{(\ell)} = \prod_{j=1,\, j\ne i}^{n} r_j^{(\ell)}, \quad \bar{r}_i = \bar{r}_i^{(1)},$$

$$\bar{r}^{(\ell)} = \max_{1\le i\le n} \bar{r}_i^{(\ell)}, \quad \bar{r} = \bar{r}^{(1)}.$$

i.e., $r_i^{(\ell)}$ is the maximal number of parts of $\mathcal{P}_\ell$ in the x_i direction. For example, in Figure 2.1, for n = 2, we have $r_1^{(\ell)} = 4$, $r_2^{(\ell)} = 5$,

$$r^{(\ell)} = r_1^{(\ell)} r_2^{(\ell)} = 20, \quad \bar{r}_1^{(\ell)} = r^{(\ell)}/r_1^{(\ell)} = 5,$$

$$\bar{r}_2^{(\ell)} = r^{(\ell)}/r_2^{(\ell)} = 4 \quad , \quad \bar{r}^{(\ell)} = \max(\bar{r}_1^{(\ell)}, \bar{r}_2^{(\ell)}) = 5.$$

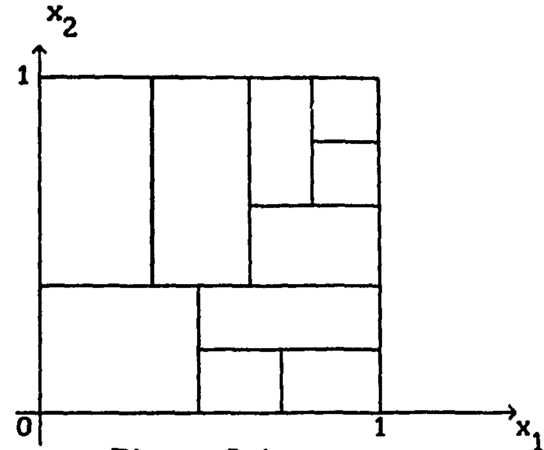

Figure 2.1

<u>Lemma 2.3</u> Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation and assume that

$s = \inf_{i,j} \inf_{i \le x_i \le 1} |\varphi'_{ij}| > 1$. If $\bar{r}\ s^{-n} < \frac{1}{2}$, then Condition 2 is satisfied.

<u>Proof</u> Take $\varepsilon > 0$ such that $\varepsilon^{1/n} < d = \min_{1 \le j \le p} d_j$, where d_j is the minimal width of D_j, and $0 < \gamma < 1$ such that $0 < 2\ \bar{r}\ s^{-n} < \gamma < 1$. Since $|\varphi'_{ij}| \ge s$, we have $g \le s^{-n}$. Since $d > \varepsilon^{1/n}$, for any $B \in \mathcal{P}_\ell$, $m(\tau^\ell B) < \varepsilon$ means at least one of the sides of $\tau^\ell B$ has length smaller than $\varepsilon^{1/n} < d$. and the number of $\mathcal{P}(\tau^\ell B)$ is at most $2\bar{r}$. Hence,

$$\sum_{B' \in \mathcal{P}(\tau^\ell B)} \sup_{B'} g < \frac{\text{the number of } \beta(\tau^\ell B)}{s^n} \le \frac{2\bar{r}}{s^n} < \gamma.$$

Q.E.D.

<u>Lemma 2.4</u> Let $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$, $p < \infty$. If for any D_j, $\sup_{D_j} g < \infty$, then Condition 3 is satisfied.

<u>Proof</u> Let $M = \max_{1 \le j \le p} \sup_{D_j} g$. Then we have $\sum_{B \in \mathcal{P}} \sup_{B} g \le p\ M < \infty$, which is Condition 3.

Q.E.D.

By Lemma 2.2, 2.3 and 2.4, and Theorem 2.1 we have:

Theorem 2.2 Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation having the finite partition $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$, and let $s = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. If for any i and j, $\varphi_{ij} \in C^{1+\varepsilon}$ with $0 < \varepsilon < 1$ and $\bar{r}s^{-n} < \frac{1}{2}$, then the sequence $\left\{P^{\ell}_{\tau} 1\right\}_{\ell=1}^{\infty}$ is bounded in L^{∞} and the averges $\frac{1}{\ell}\sum_{j=0}^{\ell-1} P^{j}_{\tau} 1$ converge in L^1 to some $\varphi \in L^{\infty}$ such that $P_{\tau}\varphi = \varphi$.

Lemma 2.5 If φ is a fixed point of $P_{\tau^t} = P^t_{\tau}$ for some positive integer t, then $\psi = \frac{1}{t}(\varphi + P_{\tau}\varphi + \ldots + P^{t-1}_{\tau}\varphi)$ is a fixed point of P_{τ}.

For a Jablonski transformation τ and any $\ell \ge 1$, we have $r_i^{(\ell)} \le r_i^{\ell}$, $r^{(\ell)} \le r^{\ell}$, $\bar{r}_i^{(\ell)} \le \bar{r}_i^{\ell}$, $\bar{r}^{(\ell)} \le \bar{r}^{\ell}$, $g_{\ell} < \alpha^{-n\ell}$. Hence, we have:

Theorem 2.3 Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation with a finite partition $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$, and assume $s = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. If, for any i and j, $\varphi_{ij} \in C^{1+\varepsilon}$ with $0 < \varepsilon < 1$ and $\bar{r}s^{-n} < 1$, then there exists a function $\psi \in L^{\infty}$ such that $P_{\tau}\psi = \psi$.

Proof Si· ·e $\bar{r}s^{-n} < 1$. We can take an integer $t \ge 1$ such that $(\bar{r}s^{-n})^t < \frac{1}{2}$. Since τ^t satisfies all conditions of Theorem 2.2, there exists a function $\varphi \in L^{\infty}$ such that $P^t_{\tau}\varphi = P_{\tau^t}\varphi = \varphi$. By Lemma 2.5, we know that

$$\psi = \frac{1}{t}\sum_{s=0}^{t-1} P^{s}_{\tau}\varphi$$

satisfies $P_{\tau}\psi = \psi$.

Q.E.D.

For $n = 1$, we have $\bar{r} = 1$. By virtue of Theorem 2.3, we have:

Theorem 2.4 Let $\tau: I \longrightarrow I$ be a piecewise expanding and piecewise $C^{1+\varepsilon}$

transformation with $0 < \varepsilon < 1$. Then there exists a function $\varphi \in L^\infty$ such that $P_\tau \varphi = \varphi$.

<u>Theorem</u> <u>2.5</u> Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation with a finite partition $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$, and $s = \inf_{i,j} \inf_{0 \le x_i \le 1} |\varphi'_{ij}| > 1$. If for any i and j, $\varphi_{ij} \in C^{1+\varepsilon}$, $0 < \varepsilon < 1$, and there exists a positive integer $t \ge 1$ such that $\bar{r}^{(t)}/s^{nt} < 1$, then there exists a function $\psi \in L^\infty$ such that $P_\tau \psi = \psi$.

<u>Proof</u> Since τ^t satisfies all the conditions of Theorem 2.3, there exists a function $\varphi \in L^\infty$ such that $P_\tau^t \varphi = P_{\tau^t} \varphi = \varphi$. By Lemma 2.5, $\psi = \frac{1}{t} \sum_{s=0}^{t-1} P_\tau^s \varphi$ satisfies $P_\tau \psi = \psi$.

Q.E.D

2.3 Generalized Jablonski Transformations

In this section, we will prove the existence of the fixed point of P_τ for a generalized Jablonski transformation τ.

A transformation $\tau: I^n \longrightarrow I^n$ is called a generalized Jablonski transformation : it is given by the formula

$$\tau(x) = (\varphi_{1j}(x_1),\ \varphi_{2j}(x_1,x_2),\ \ldots,\ \varphi_{nj}(x_1,\ldots,x_n)),\ x \in D_j,\ 1 \le j \le p < \infty.$$

The image of any hyperplane in I^n is a collection of segments of hyperplanes in I^n and the inverse image of any hyperplane in I^n is a collection of segments of hyperplanes in I^n, where D_j is the same as for a Jablonski transformation and $\varphi_{ij}(x_1,\ldots,x_i)$: $\prod_{k=1}^{i} \left[a_{kj}\ ,\ b_{kj}\right] \longrightarrow [0,1]$.

Lemma 2.6 Let $\tau: I^n \longrightarrow I^n$ be a non-singular piecewise $C^{1+\varepsilon}$ transformation. If there exists a constant $\alpha > 1$ such that for any $B \in \mathcal{P}$ and $x,\ y \in B$ we have

$$d(\tau x, \tau y) \ge \alpha\ d(x,y), \tag{2.1}$$

then Condition 1 holds.

Proof This is the special case of Lemma 2.2 when $C = 1$.

Q.E.D.

Lemma 2.7 Let $\tau: I^n \longrightarrow I^n$ be a generalized Jablonski transformation. If (2.1) is satisfied with $\alpha > 1$ and $2\bar{r}\alpha^{-n} < 1$, then Condition 2 holds.

Proof Let $\varepsilon > 0$ such that $\varepsilon^{1/n} < \frac{d}{2^{n-1}}$, where $d = \min_{1 \le j \le p} d_j$ and d_j is the minimal width of D_j, and $0 < \gamma < 1$ such that $0 < 2\bar{r}\alpha^{-n} < \gamma < 1$. Since for any $x,\ y \in B \in \mathcal{P}$, $d(\tau x,\ \tau y) \ge \alpha d(x,\ y)$, we have $g \le \alpha^{-n}$. Since $d > 2^{n-1}\varepsilon^{1/n}$, for any $B \in \mathcal{P}_\ell$, $m(\tau^\ell B) < \varepsilon$ means that at least one of the maximal lengths of $\tau^\ell B$ in the x_i direction is smaller than $\varepsilon^{1/n} < \frac{d}{2^{n-1}}$, and the number of $\mathcal{P}(\tau^\ell B)$ is at most $2\bar{r}$. Hence

$$\sum_{B' \in \mathcal{P}(\tau^{\ell} B)} \sup_{B'} g \le \frac{\text{the number of } \mathcal{P}(\tau^{\ell} B)}{\alpha^n} \le \frac{2\bar{r}}{\alpha^n} < \gamma.$$

Q.E.D.

By Lemmas 2.4 and 2.6 and 2.7, and Theorem 2.1 we obtain:

<u>Theorem 2.6</u> Let $\tau: I^n \longrightarrow I^n$ be a non-singular, piecewise $C^{1+\varepsilon}$ generalized Jablonski transformation with a finite partition $\mathcal{P} = \left\{D_1, \ldots, D_p\right\}$. If for any $x, y \in B \in \mathcal{P}$ $d(\tau x, \tau y) \ge \alpha\, d(x,y)$ for some constant $\alpha > 1$ and $2\bar{r}\alpha^{-n} < 1$, then the sequence $\left\{P_\tau^{\ell} 1\right\}_{\ell=1}^{\infty}$ is bounded in L^∞ and the averages $\frac{1}{\ell} \sum_{j=0}^{\ell-1} P_\tau^j 1$ converge in L^1 to some $\varphi \in L^\infty$ such that $P_\tau \varphi = \varphi$.

2.4 Examples

In Theorem 2.5 we need a condition $\frac{\bar{r}^{(t)}}{s^{nt}} < 1$ for some integer $t \geq 1$.

Now we show that, in general, $s > 1$ does not imply this condition.

Example 2.1 Consider the partition $\mathcal{P} = \{D_1, D_2, \ldots, D_9\}$ shown in Figure 2.2 for the unit square in 2-dimension. Let τ be a Jablonski transformation defined as follows. For $j = 1,2,4$ and $i = 1,2$, φ_{ij} is a linear function which is onto [0, 1] with $|\varphi'_{ij}| = 3$.

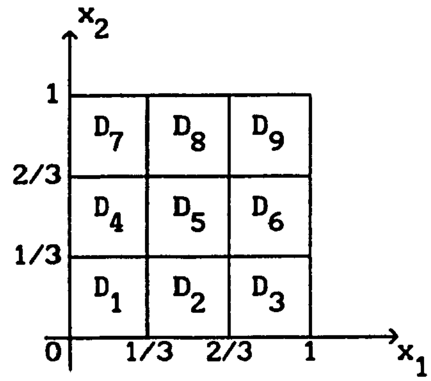

Figure 2.2

For $j = 3,5,6,7,8,9$ and $i = 1,2$, $\varphi_{ij} \in C^{1+\varepsilon}$ with $\inf_{x_i} |\varphi'_{ij}| = 1.1$. Therefore, $s = 1.1$, $s^2 = 1.21$, $\bar{r} = 3$. $\bar{r}^{(2)} \geq 2\bar{r} + 1 > 2\bar{r}$. For $\ell > 1$, $\bar{r}^{(\ell)} > 2^{\ell-1}\,\bar{r} = 3 \cdot 2^{\ell-1} = 1.5 \cdot 2^{\ell}$. This means that for any positive integer $t \geq 1$

$$\frac{\bar{r}^{(t)}}{s^{nt}} = \frac{\bar{r}^{(t)}}{s^{2t}} = \frac{\bar{r}^{(t)}}{1.21^t} > 1.5\left(\frac{2}{1.21}\right)^t > 1.$$

Example 2.2 Let $n = 2$ and the partition $\mathcal{P} = \{D_1, \ldots, D_9\}$ be the same as in Example 2.1. Assume that for every i and j, $\varphi_{ij} \in C^{1+\varepsilon}$, and that

$$s = \inf_{i,j} \inf_{x_i} |\varphi'_{ij}| > \sqrt{3}.$$

Then we have

$$\frac{\bar{r}}{s^n} < \frac{3}{(\sqrt{3})^2} = 1.$$

For example, let $\varphi(x) = x^{3/2} + 1.74x$. Then

$$\varphi'(x) = \frac{3}{2} x^{1/2} + 1.74 \geq 1.74 > \sqrt{3}$$

for $x \geq 0$ and $\varphi(x)$ maps $[0, 1/3]$ onto $[0, 0.77245]$. Let

$$\varphi_{ij}(x_i) = \begin{cases} \varphi(x_i), & \text{if } [a_{ij}, b_{ij}] = [0, \frac{1}{3}], \\ 1.2\varphi(x_i - \frac{1}{3}), & \text{if } [a_{ij}, b_{ij}] = [\frac{1}{3}, \frac{2}{3}], \\ \varphi(x_i - \frac{2}{3}) + 0.2, & \text{if } [a_{ij}, b_{ij}] = [\frac{2}{3}, 1], \end{cases}$$

and we have $\varphi_{ij}(x_i) \in C^{1+\frac{1}{2}}$, $s = 1.74 > \sqrt{3}$.

By Theorem 2.3, there exists a function $\psi \in L^\infty$ such that $P_\tau \psi = \psi$.

2.5 A New Sufficient Condition

In this section, we will present a sufficient condition for the existence of the fixed point of P_τ. In Rychlik's sufficient condition, the Jacobian of τ, which is the quotient of volumes, was used. We want to change the volume to the diameter.

Let $\tau: I^n \longrightarrow I^n$, $\mathcal{P}$, $\mathcal{P}_\ell$, g_τ and $\mathcal{P}(A)$ be the same as in 2.1 and

$$P_\tau f(x) = \sum_{y\in\tau^{-1}x} g_\tau(y)\, f(y), \quad x \in I^n.$$

For any $x,\ y \in I^n$, let

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{1/2},$$

where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$.

For any $B \in \mathcal{P}_\ell$, let

$$d(B) = \sup_{x,y\in B} d(x,y).$$

For any $\ell \ge 1$ and $x,y \in B \in \mathcal{P}_\ell$ let

$$\tilde{h}_\ell(x,y) = \frac{d(\tau^\ell x, \tau^\ell y)}{d(x,y)},$$

$$h_\ell(x) = \sup_{y\in B} \tilde{h}_\ell(x,y).$$

Condition 2.4 There exists a positive number b such that for any $\ell \ge 1$ and $B \in \mathcal{P}_\ell$ we have

$$\sup_B \frac{1}{h_\ell} \le b \inf_B \frac{1}{h_\ell}.$$

Condition 2.5 There exist $\varepsilon > 0$ and $0 < \gamma < 1$ such that for any $\ell \ge 1$ and $B \in \mathcal{P}_\ell$, $d(\tau^\ell B) < \varepsilon$ implies

$$\sum_{B'\in\mathcal{P}(\tau^\ell B)} \sup_{B'} g \le \gamma.$$

Condition 2.6 $\sum_{B\in\mathcal{P}} \sup_B g < \infty.$

Condition 2.7 There exists $C > 0$ such that for any $\ell \geq 1$ and $B \in \mathcal{P}_\ell$ we have $d(B) \leq C\,\lambda(B)$.

Theorem 2.7 Assume that Conditions 2.4, 2.5, 2.6 and 2.7 are satisfied. Then the sequence $\left\{P_\tau^\ell 1\right\}_{\ell=1}^\infty$ is bounded in L^∞, and the averages $\frac{1}{\ell}\sum_{j=0}^{\ell-1} P_\tau^j 1$ converge in L^1 to some $\varphi \in L^\infty$ such that $P_\tau\varphi = \varphi$.

Proof If the averages $\frac{1}{\ell}\sum_{j=0}^{\ell-1} P_\tau^j 1$ are bounded in L^∞ then they form a weakly sequentially compact sequence and, hence, by the Kakutani-Yoshida Theorem ([8], VIII5.3), they converge to $\varphi \in L^\infty$, where $P_\tau\varphi = \varphi$.

Therefore, we only need check that $\left\{P_\tau^\ell 1\right\}_{\ell=1}^\infty$ is bounded in L^∞.

Let $\alpha_\ell = \sum_{B\in\mathcal{P}_\ell} \sup_B g_\ell$. It is obvious that $\|P_\tau^\ell 1\|_\infty \leq \alpha_\ell$ for every $\ell \geq 1$. If we can prove that α_ℓ is bounded then the theorem follows. We will prove by induction that

$$\alpha_{\ell+1} \leq \gamma\,\alpha_\ell + C_1,$$

where $C_1 = \alpha_1 b^n C^n/\varepsilon^n$. (Condition 2.6 implies that $\alpha_1 < \infty$.)

Since $g_{\ell+1} = g_\ell(g\circ\tau^\ell)$, we have for every $B' \in \mathcal{P}_\ell$:

$$\sum_{\left\{B\in\mathcal{P}_{\ell+1}:\ B\subset B'\right\}} \sup_B g_{\ell+1} \leq \sup_{B'} g_\ell \sum_{B''\in\mathcal{P}(\tau^\ell B')} \sup_{B''} g. \qquad (2.2)$$

Let

$$\mathcal{P}' = \left\{B' \in \mathcal{P}_\ell : d(\tau^\ell B') < \varepsilon\right\},$$

$$\mathcal{P}'' = \left\{B' \in \mathcal{P}_\ell : d(\tau^\ell B') \geq \varepsilon\right\}.$$

If $B' \in \mathcal{P}'$, then the right-hand side of (2.2) is not greater than $\gamma \sup_{B} g_\ell$ by Condition 2.5.

Since for every $B \in \mathcal{P}_\ell$, $g_\ell \le \sup_B \left(\frac{1}{h_\ell}\right)^n$ and

$$d(\tau^\ell B) = \sup_{x,y\in B} d(\tau^\ell x, \tau^\ell y)$$

$$= \sup_{x,y\in B} \tilde{h}_\ell(x,y)\, d(x,y) \le \sup_B h_\ell(x)\, d(B).$$

Hence,

$$\inf_B \frac{1}{h_\ell} = (\sup_B h_\ell)^{-1} \le \frac{d(B)}{d(\tau^\ell B)} .$$

Therefore, if $B' \in \mathcal{P}''$, then

$$\sup_{B'} g_\ell \le \sup_{B'} \left(\frac{1}{h^\ell}\right)^n \le b^n \inf_{B'} \left(\frac{1}{h_\ell}\right)^n$$

$$\le b^n \left[\frac{d(B')}{d(\tau^\ell B')}\right]^n \le \frac{b^n}{\varepsilon^n} C^n\, [\lambda(B')]^n \le \frac{b^n}{\varepsilon^n} C^n\, [\lambda(B')]$$

by Conditions 2.4 and 2.7.

For every $B \in \mathcal{P}_{\ell+1}$ there is an unique $B' \in \mathcal{P}_\ell$ containing B. We have

$$\alpha_{\ell+1} = \sum_{B\in\mathcal{P}_{\ell+1}} \sup_B g_{\ell+1}$$

$$= \sum_{B'\in\mathcal{P}_\ell} \sum_{\{B\in\mathcal{P}_{\ell+1}, B\subset B'\}} \sup_B g_{\ell+1}$$

$$\le \sum_{B'\in\mathcal{P}'} \gamma \sup_{B'} g_\ell + \sum_{B'\in\mathcal{P}''} \alpha_1 \frac{b^n}{\varepsilon^n} C^n\, m(B')$$

$$\le \gamma\, \alpha_\ell + C_1.$$

Since $0 < \gamma < 1$, this implies that $\left\{\alpha_\ell\right\}_{\ell=1}^{\infty}$ is bounded.

Q.E.D.

CHAPTER 3

Markov Transformations

A Markov transformation is a special kind of transformations which map boundaries into boundaries. For these transformations some special methods can be used to prove existence theorems.

In 1972 Kosyakin and Sandler [5] proved an existence theorem of invariant measures for Markov transformations on an interval. In 1981 Friedman and Boyarsky [3] discussed piecewise linear Markov transformations on the interval and proved that it had a piecewise constants function as an invariant density. In 1987 Mane [17] got a more general result.

In this chapter we will generalize some results to higher dimensional Markov transformations and they will be compared with Mane's result.

3.1 Piecewise Linear Markov Transformations

We generalize results of [3] to higher dimensions.

Let $\mathcal{P} = \left\{ D_1, D_2, \ldots, D_p \right\}$, $p < \infty$, be a partition of I^n:

$$\bigcup_{j=1}^{p} D_j = I^n \text{ and } D_j \cap D_k = \phi \text{ for } j \neq k.$$

<u>Definition 3.1</u> $\tau(x_1, \ldots, x_n)$: $I^n \longrightarrow I^n$ is called a non-singular piecewise C^2 transformation with respect to the partition $\mathcal{P}$ if for any $j = 1, 2, \ldots, p$,

$$\tau(x_1, \ldots, x_n) = \tau_j(x_1, \ldots, x_n) \text{ on } D_j,$$

τ_j is a C^2 function on $\bar{D}_j$ and the Jacobian matrix $A_j = \frac{\partial \tau_j}{\partial x}$ satisfies $\det A_j \neq 0$.

<u>Definition 3.2</u> $\tau(x_1, \ldots, x_n)$: $I^n \longrightarrow I^n$ is called a Markov transformation with respect to the partition $\mathcal{P}$ if for any $j = 1, 2, \ldots, p$ the image of $\bar{D}_j$ is an

union of $\bar{D}_k$'s, i.e.,

$$\tau(\bar{D}_j) = \bar{D}_{j_1} \cup \dots \cup \bar{D}_{j_\ell} \quad \text{for some } D_{j_1}, \dots, D_{j_\ell}.$$

And the image of the boundary of D_j is the boundary of some D_k's.

Let S be the class of all functions which are piecewise constant on $\mathcal{P}$, that is

$$f \in S \text{ if and only if } f = \sum_{j=1}^{p} C_j \chi_{D_j}$$

for some constants $C_1, \dots, C_p$. Such an f will also be represented by the column vector $(C_1, \dots, C_p)^T$, where T denotes transpose.

<u>Theorem 3.1</u> Let $\tau(x_1, \dots, x_n)$: $I^n \longrightarrow I^n$ be a non-singular piecewise C^2 Markov transformation with respect to the partition $\mathcal{P}$ of I^n. Assume, for any $j = 1, 2, \dots, p$,

(1) τ_j is a homeomorphism from $\bar{D}_j$ onto $\bar{D}_{j_1} \cup \dots \cup \bar{D}_{j_\ell}$ and has $\tau_j^{-1}(x_1, \dots, x_n)$ as its inverse;

(2) det A_j is constant on $\bar{D}_j$, where $A_j = \frac{\partial \tau_j}{\partial x}$, then,

(1) there exists a pxp matrix M_τ such that $P_\tau f = M_\tau f$ for every $f \in S$;

(2) $M = M_\tau$ has 1 as the eigenvalue of maximum modulus;

(3) there exists a function $f \in S$ with $\|f\| = 1$ and $P_\tau f = f$, i.e., it is invariant under P_τ.

<u>Proof</u> (1) A simple computation [2] shows that the Frobenius-Perron operator for τ is given by

$$P_\tau f(x_1, \dots, x_n) = \sum_{j=1}^{p} f(\tau_j^{-1}(x_1, \dots, x_n)) \, |\det A_j^{-1}| \, \chi_{\tau_j(D_j)}.$$

Suppose first that $f = \chi_{D_k}$ for some $1 \le k \le p$ then

$$P_\tau f = P_\tau \chi_{D_k} = \sum_{j=1}^{p} \chi_{D_k}(\tau_j^{-1}(x_1,\ldots,x_n))|\det A_j^{-1}|\ \chi_{\tau_j(D_j)}.$$

Since τ_j^{-1} has range D_j, $\chi_{D_k}(\tau_j^{-1}(x_1,\ldots,x_n))$ will be zero for $j \neq k$. Thus

$$P_\tau \chi_{D_k} = |\det A_k^{-1}|\ \chi_{\tau_k(D_k)}.$$

Now let $f \in S$, i.e.,

$$f = \sum_{k=1}^{p} C_k \chi_{D_k} = (C_1,\ldots,C_p)^T.$$

Since P_τ is a linear operator we have

$$P_\tau f = \sum_{k=1}^{p} C_k P_\tau \chi_{D_k} = \sum_{k=1}^{p} C_k |\det A_k^{-1}|\ \chi_{\tau_k(D_k)}. \tag{3.1}$$

It means $P_\tau f \in S$.

Let us write $P_\tau f = (d_1,\ldots,d_p)^T$. When $(x_1,\ldots,x_n) \in D_j$, $P_\tau f = d_j$. Now the kth term on the right hand side of (3.1) equals $C_k|\det A_k^{-1}|$ if and only if $(x_1,\ldots,x_n) \in \tau_k(D_k)$, i.e., if $D_j \subset \tau_k(D_k)$.

Let

$$\Delta_{jk} = \begin{cases} 1 & \text{if } D_j \subset \tau_k(D_k) \\ 0 & \text{otherwise} \end{cases}$$

and

$$M_{jk} = \Delta_{jk}\ |\det A_k^{-1}|.$$

Define

$$M_\tau = (M_{jk}).$$

Then

$$d_j = \sum_{k=1}^{p} C_k M_{jk}$$

and

$$M_\tau \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_p \end{pmatrix} = P_\tau f.$$

(2) M_τ is called the matrix induced by τ. It is nonnegative and for each $j = 1,2,\ldots,p$ the non-zero entries in the jth column are equal to $|\det A_j^{-1}|$.

Let G_j be the volume of D_j, i.e.,

$$G_j = \int_{D_j} dx,$$

where $dx = dx_1 \ldots dx_n$.

If the image of D_j under τ is $D_{j_1} \cup \ldots \cup D_{j_\ell}$ then

$$\begin{aligned} G_{j_1} + \ldots + G_{j_\ell} &= \int_{D_{j_1} \cup \ldots \cup D_{j_\ell}} dy \\ &= \int_{D_j} |\det A_j| \, dx \qquad (y = \tau_j(x)) \\ &= |\det A_j| \int_{D_j} dx = |\det A_j| \, G_j. \end{aligned} \tag{3.2}$$

We recall that the eigenvalues of a matrix are invariant under similarity transformations and under transposition. Let us define

$$\delta = \prod_{j=1}^{p} G_j,$$

$$\delta_i = \delta / G_i = \prod_{\substack{j=1 \\ j \neq i}}^{p} G_j,$$

$$D = \begin{pmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_p \end{pmatrix}, \qquad D^{-1} = \begin{pmatrix} \delta_1^{-1} & & & \\ & \delta_2^{-1} & & \\ & & \ddots & \\ & & & \delta_p^{-1} \end{pmatrix}.$$

$$B = D^{-1} M D = (b_{rs}).$$

Then

$$b_{rs} = \delta_r^{-1} M_{rs} \delta_s.$$

We claim that B is column stochastic. Consider the column sum of the jth column for B:

$$\sum_{r=1}^{p} b_{rj} = \sum_{r=1}^{p} \delta_r^{-1} M_{rj} \delta_j = \sum_{D_r \subset \tau_j(D_j)} \delta_r^{-1} |\det A_j^{-1}| \delta_j$$

$$= \frac{|\det A_j^{-1}| \delta}{G_j} \sum_{D_r \subset \tau_j(D_j)} \delta_r^{-1}$$

$$= \frac{|\det A_j^{-1}|}{G_j} \sum_{D_r \subset \tau_j(D_j)} \frac{\delta}{\delta_r}$$

$$= \frac{|\det A_j^{-1}|}{G_j} \sum_{D_r \subset \tau_j(D_j)} G_r$$

$$= \frac{|\det A_j^{-1}|}{G_j} (G_{j_1} + \ldots + G_{j_\ell})$$

$$= \frac{|\det A_j^{-1}|}{G_j} |\det A_j| G_j = 1,$$

where we have used (3.2) and $|\det A_j^{-1}| = |\det A_j|^{-1}$.

Thus B^T is row stochastic. By Theorem 9.5.1 in [4] the matrix B^T has one as the eigenvalue of maximum modulus. So does M.

(3) By (2) the system of linear equations $M_\tau \Pi = \Pi$ always has non-trivial solutions. So there always exists some functions $f \in S$ such that $P_\tau f = f$, i.e., it is invariant under P_τ.

Since $P_\tau f = f$, for any constant a we have $P_\tau(af) = af$. This means that one of them satisfies $\|f\| = 1$.

Q.E.D.

From Theorem 3.1, we have:

<u>Theorem 3.2</u> If $\tau(x_1,\ldots,x_n)$: $I^n \longrightarrow I^n$ is a non-singular piecewise linear Markov transformation with respect to the partition $\mathcal{P}$ of I^n, then

(1) there exists a pxp matrix M such that Pf = Mf for every $f \in S$;

(2) $M = M_\tau$ has one as the eigenvalue of maximum modulus;

(3) there exists a function $f \in S$ with $\|f\| = 1$ and $P_\tau f = f$, i.e., it is invariant under P_τ.

For Theorem 3.1 and Theorem 3.2 we need if τ: $I^n \longrightarrow I^n$ to be piecewise linear transformation, then for any $1 \le j \le p$, det A_j is a constant. But the converse is not true, i.e., if det A_j is a constant, τ does not have to be piecewise linear. For example, for n = 2, let

$$\begin{cases} u = xy \\ v = \ln y \end{cases}$$

and

$$\tau(x,y) = (u,v) = (xy, \ln y).$$

It is not linear, but

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 0 & \frac{1}{y} \end{vmatrix} = 1.$$

Hence det A_j is a constant.

3.2 Quasi-Expanding Markov Transformations

For the Markov transformation τ, not necessarily piecewise linear, we set

$$D_{i_1 i_2} = D_{i_1} \cap \tau^{-1} D_{i_2}$$

$$D_{i_1 \ldots i_\ell i_{\ell+1}} = D_{i_1 \ldots i_\ell} \cap \tau^{-1} D_{i_2 \ldots i_{\ell+1}}$$

and if $D_{i_1 i_2} \neq \phi$, then we have

$$\tau\left(D_{i_1 i_2}\right) = D_{i_2}.$$

If $D_{i_1 \ldots i_{\ell+1}} \neq \phi$, then

$$\tau\left(D_{i_1 \ldots i_{\ell+1}}\right) = D_{i_2 \ldots i_{\ell+1}};$$

$$\tau^\ell\left(D_{i_1 \ldots i_{\ell+1}}\right) = D_{i_{\ell+1}};$$

i.e., τ^ℓ maps $D_{i_1 \ldots i_{\ell+1}}$ onto $D_{i_{\ell+1}}$. Let us denote by $f_{i_1 \ldots i_{\ell+1}}$ the function which gives this mapping. If it is a one to one function then its inverse function $S_{i_1 \ldots i_{\ell+1}}$ gives the mapping of $D_{i_{\ell+1}}$ onto $D_{i_1 \ldots i_{\ell+1}}$.

For any Lebesgue measurable subset A of I^n we define the diameter of A as follows:

$$d(A) = \sup_{x,y \in A} d(x,y),$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$.

<u>Theorem 3.3</u> A function $\rho(x)$ is the fixed point of P_τ, i.e., $P_\tau \rho = \rho$ or for any measurable subset A of I^n

$$\int_A \rho(x)dx = \int_{\tau^{-1}A} \rho(x)dx$$

if and only if for an arbitrary integrable function $\varphi(x)$

$$\int_{I^n} \varphi(x)\ \rho(x)dx = \int_{I^n} \varphi(\tau x)\ \rho(x)dx.$$

Proof Although the proof is straightforward, we present it for completeness.

(1) If ρ is a fixed point of P_τ, then for the simple function

$$\varphi(x) = \sum_{k=1}^{m} a_k\, \chi_{A_k}(x)$$

we have

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx$$

$$= \int_{I^n} \sum_{k=1}^{m} a_k\, \chi_{A_k}(x)\ \rho(x)\ dx$$

$$= \sum_{k=1}^{m} a_k \int_{I^n} \chi_{A_k}(x)\ \rho(x)\ dx$$

$$= \sum_{k=1}^{m} a_k \int_{A_k} \rho(x)\ dx = \sum_{k=1}^{m} a_k \int_{\tau^{-1}A_k} \rho(x)\ dx$$

$$= \sum_{k=1}^{m} a_k \int_{I^n} \chi_{\tau^{-1}A_k}(x)\ \rho(x)\ dx = \sum_{k=1}^{m} a_k \int_{I^n} \chi_{A_k}(\tau(x))\ \rho(x)\ dx$$

$$= \int_{I^n} \sum_{k=1}^{m} a_k\, \chi_{A_k}(\tau(x))\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx.$$

Since the set of all simple functions is dense in L_1, we have for any integrable function $\varphi(x)$,

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx.$$

(2) If for any integrable function $\varphi(x)$ we have

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx.$$

Letting $\varphi(x) = \chi_A(x)$ we have

$$\int_A \rho(x)dx = \int_{I^n} \chi_A(x)\ \rho(x)\ dx$$

$$= \int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx$$

$$= \int_{I^n} \chi_A(\tau x)\ \rho(x)\ dx = \int_{I^n} \chi_{\tau^{-1}A}(x)\rho(x)dx$$

$$= \int_{\tau^{-1}A} \rho(x)dx.$$

So $\rho(x)$ is a fixed point of P_τ.

Q.E.D.

<u>Definition 3.3</u> A Markov transformation $\tau: I^n \longrightarrow I^n$ with respect to the partition $\mathcal{P}$ of I^n is called a quasi-directionally expanding transformation if there exist constants $\alpha > 1$ and $0 < C \le 1$ such that for any $x, y \in D_j$, $\tau x, \tau y \in D_{j_1}, \ldots, \tau^\ell x, \tau^\ell y \in D_{j_\ell}$, we have

$$d(\tau^\ell x, \tau^\ell y) \ge C\ \alpha^\ell\ d(x,y).$$

<u>Theorem 3.4</u> Let $I^n = [0,1]^n$ and $\tau(x_1,\ldots,x_n): I^n \longrightarrow I^n$ be a non-singular piecewise $C^{1+\varepsilon}$ Markov transformation with respect to the partition $\mathcal{P}$ of I^n. If τ is quasi-directionally expanding with constants $\alpha > 1$ and $1 \ge C > 0$, then there exists a function $\rho(x_1,\ldots,x_n)$ such that $0 \le \rho(x) \le K$ for some constant K and for an arbitrary integrable function $\varphi(x_1,\ldots,x_n)$

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx$$

and $\rho(x)$ is a fixed point of P_τ.

<u>Proof</u> For an arbitrary integrable function $\varphi(x)$, we have

$$\int_{I^n} \varphi(\tau^\ell x)\ dx = \sum_{i_1, i_2, \ldots, i_{\ell+1}=1}^{p} \int_{D_{i_1 i_2 \ldots i_{\ell+1}}} \varphi(\tau^\ell x)dx$$

$$= \sum_{i_{\ell+1}=1}^{p} \sum_{i_1,\dots,i_\ell} \int_{D_{i_1 i_2 \dots i_{\ell+1}}} \varphi(\tau^\ell x)dx$$

$$= \sum_{i_{\ell+1}=1}^{p} \sum_{i_1 \dots i_\ell} \int_{D_{i_{\ell+1}}} \varphi(y) \left| \frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} \right| dy \qquad (y=\tau^\ell x)$$

$$= \sum_{i_{\ell+1}=1}^{p} \int_{D_{i_{\ell+1}}} \varphi(y)\ \rho_\ell(y)\ dy = \int \varphi(y)\ \rho_\ell(y)\ dy,$$

where $\left| \frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} \right|$ is the absolute value of the Jacobian of $S_{i_1,\dots,i_{\ell+1}}$

and

$$\rho_\ell(y) = \sum_{i_1,\dots,i_\ell} \left| \frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} \right| \geq 0,$$

and the summation is carried out according to the same indices $i_1,\dots,i_\ell$ for which $D_{i_1 \dots i_\ell} \neq \phi$.

Since τ^ℓ: $D_{i_1 \dots i_{\ell+1}} \longrightarrow D_{i_{\ell+1}}$, for x_1, $x_2 \in D_{i_1 \dots i_{\ell+1}}$, ...,

$\tau^\ell x_1$, $\tau^\ell x_2 \in D_{i_{\ell+1}}$, we have

$$d(\tau^\ell x_1,\ \tau^\ell x_2) \geq C\alpha^\ell d(x_1,x_2).$$

Therefore

$$d(D_{i_1 \dots i_{\ell+1}}) = \sup_{x_1,x_2 \in D_{i_1 \dots i_{\ell+1}}} d(x_1,x_2)$$

$$\leq C^{-1}\ \alpha^{-\ell} \sup_{x_1,x_2 \in D_{i_1 \dots i_{\ell+1}}} d(\tau^\ell x_1,\ \tau^\ell x_2)$$

$$\le C^{-1}\alpha^{-\ell}\sup_{\tau^\ell x_1,\ \tau^\ell x_2\in D_{i_{\ell+1}}} d(\tau^\ell x_1,\ \tau^\ell x_2)$$

$$= C^{-1}\alpha^{-\ell}\, d(D_{i_{\ell+1}}) \le \sqrt{n}\ C^{-1}\alpha^{-\ell}.$$

Therefore, for any $x_1, x_2 \in D_{i_1\ldots i_{\ell+1}}$, we have $\tau^k x_1,\ \tau^k x_2 \in D_{i_{k+1}\ldots i_{\ell+1}}$ and

$$d(\tau^k x_1,\ \tau^k x_2) \le C^{-1}\alpha^{-(\ell-k)} d(\tau^\ell x_1,\ \tau^\ell x_2)$$

$$\le C^{-1}\alpha^{-(\ell-k)} d\left(D_{i_{\ell+1}}\right) \le \sqrt{n}\ C^{-1}\alpha^{-(\ell-k)},\quad k = 0, 1, \ldots, \ell-1.$$

If $\bar{D}_j$ is not convex, since τ_j is $C^{1+\varepsilon}$ on D_j, it can be extended to $\bar{E}_j$ as a $C^{1+\varepsilon}$ function, where $\bar{E}_j$ is the smallest convex set containing D_j. So there exists a constant $\bar{C}$ such that for any $x_1, x_2 \in \bar{D}_j$ we have

$$\left|\frac{\partial\,\tau_j(x_1)}{\partial x} - \frac{\partial\,\tau_j(x_2)}{\partial x}\right| \le \bar{C}\ d(x_1,\ x_2)^{\varepsilon}.$$

We know

$$f_{i_1\ldots i_{\ell+1}}(x) = \tau^\ell(x) = \tau(\tau(\tau\ldots\tau(x)))\qquad (\ell \text{ times}).$$

If $y = \tau^\ell(x)$ then

$$\frac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} = \left(\frac{\partial\, f_{i_1,\ldots,i_{\ell+1}}(x)}{\partial x}\right)^{-1}$$

$$= \left(\frac{\partial\tau(\tau^{\ell-1}x)}{\partial x}\ \frac{\partial\tau(\tau^{\ell-2}x)}{\partial x}\cdots\frac{\partial\tau(x)}{\partial x}\right)^{-1}.$$

By the proof of Lemma 2.2, we know that $\left|\frac{\partial\tau(x)}{\partial x}\right| \ge C^n\alpha^n$. Therefore,

$$\left| \frac{\dfrac{\partial\tau(\tau^k x_2)}{\partial x}}{\dfrac{\partial\tau(\tau^k x_1)}{\partial x}} \right| = \left| 1 + \frac{\dfrac{\partial\tau(\tau^k x_2)}{\partial x} - \dfrac{\partial\tau(\tau^k x_1)}{\partial x}}{\dfrac{\partial\tau(\tau^k x_1)}{\partial x}} \right|$$

$$\leq 1 + \frac{\left| \dfrac{\partial\tau(\tau^k x_2)}{\partial x} - \dfrac{\partial\tau(\tau^k x_1)}{\partial x} \right|}{\left| \dfrac{\partial\ \tau(\tau^k x_1)}{\partial x} \right|}$$

$$\leq 1 + \frac{\bar{C}\ d(\tau^k x_2,\ \tau^k x_1)^\varepsilon}{C^n\ \alpha^n}$$

$$\leq 1 + \frac{\bar{C}\ \sqrt{n}^\varepsilon}{C^{n+\varepsilon}\alpha^n}\ \alpha^{-(\ell-k)\varepsilon} \quad (k = 0, 1, \ldots, \ell-1).$$

This means that for $y_1 = \tau^\ell x_1$, $y_2 = \tau^\ell x_2$, $(x_1, x_2 \in D_{i_1 \ldots i_{\ell+1}},\ y_1, y_2 \in D_{i_{\ell+1}})$

$$\left| \frac{\dfrac{\partial\ S_{i_1,\ldots,i_{\ell+1}}(y_1)}{\partial y}}{\dfrac{\partial\ S_{i_1,\ldots,i_{\ell+1}}(y_2)}{\partial y}} \right| = \left| \frac{\dfrac{\partial\ \tau(\tau^{\ell-1}x_2)}{\partial x}\ \dfrac{\partial\ \tau(\tau^{\ell-2}x_2)}{\partial x} \cdots \dfrac{\partial\ \tau(x_2)}{\partial x}}{\dfrac{\partial\ \tau(\tau^{\ell-1}x_1)}{\partial x}\ \dfrac{\partial\ \tau(\tau^{\ell-2}x_1)}{\partial x} \cdots \dfrac{\partial\ \tau(x_1)}{\partial x}} \right|$$

$$\leq \prod_{k=0}^{\ell-1} \left(1 + \frac{\bar{C}\ \sqrt{n}^\varepsilon}{C^{n+\varepsilon}\alpha^n}\ \alpha^{-(\ell-k)\varepsilon} \right)$$

$$= \prod_{i=1}^{\ell} \left(1 + \frac{\bar{C}\ \sqrt{n}^\varepsilon}{C^{n+\varepsilon}\alpha^n}\ \alpha^{-i\varepsilon} \right)$$

$$\leq \prod_{i=1}^{\infty} \left(1 + \frac{\bar{C}\sqrt{n^{\varepsilon}}}{C^{n+\varepsilon}\alpha^{n}} \alpha^{-i\varepsilon}\right) = C^{*}.$$

Since

$$\lambda(D_{i_1,\ldots,i_{\ell+1}}) = \int_{D_{i_1,\ldots,i_{\ell+1}}} dx$$

$$= \int_{D_{i_{\ell+1}}} \left| \frac{\partial S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| dy \qquad (y = \tau^{\ell} x)$$

$$= \left| \frac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y_0)}{\partial y} \right| \lambda(D_{i_{\ell+1}})$$

for some $y_0 \in D_i$, we have

$$\frac{\left| \dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| \lambda\,(D_{i_{\ell+1}})}{\lambda\,(D_{i_1,\ldots,i_{\ell+1}})}$$

$$= \frac{\left| \dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| \lambda\,(D_{i_{\ell+1}})}{\left| \dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y_0)}{\partial y} \right| \lambda\,(D_{i_{\ell+1}})}$$

$$= \frac{\left| \dfrac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} \right|}{\left| \dfrac{\partial S_{i_1,\dots,i_{\ell+1}}(y_0)}{\partial y} \right|} \le C^*,$$

and

$$\rho_\ell(y) = \sum_{i_1 \dots i_\ell} \left| \frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} \right|$$

$$\le \sum_{i_1 \dots i_\ell} \frac{C^*}{\lambda(D_{i_{\ell+1}})} \lambda(D_{i_1 \dots i_{\ell+1}})$$

$$\le \frac{C^*}{K_1} \sum_{i_1 \dots i_\ell} \lambda(D_{i_1 \dots i_{\ell+1}})$$

$$\le \frac{C^*}{K_1} = K,$$

where $K_1 = \min\limits_{1 \le j \le p} \lambda(D_j)$.

Hence $\left\{\rho_\ell\right\}_{\ell=1}^{\infty}$ is a weakly compact subset of L^1, and for any $\varphi \in L^1$

$$\int_{I^n} \varphi(\tau^\ell x)\, dx = \int_{I^n} \varphi(x)\, \rho_\ell(x)\, dx.$$

We can take $\rho_0(x) = 1$, and

$$\int_{I^n} \varphi(\tau^{\ell+1} x)\, dx = \int_{I^n} \varphi(x)\, \rho_{\ell+1}(x)\, dx.$$

Since

$$\int_{I^n} \varphi(\tau^{\ell+1} x)\, dx = \sum_{i_{\ell+1}=1}^{p} \sum_{i_1 \dots i_\ell} \int_{D_{i_1,\dots,i_{\ell+1}}} \varphi(\tau^{\ell+1} x)\, dx$$

$$= \sum_{i_{\ell+1}=1}^{p} \sum_{i_1 \ldots i_\ell} \int_{D_{i_{\ell+1}}} \varphi(\tau y) \left| \frac{\partial S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| dy \qquad (y=\tau^\ell x)$$

$$= \sum_{i_{\ell+1}=1}^{p} \int_{D_{i_{\ell+1}}} \varphi(\tau y)\, \rho_\ell(y)\, dy = \int_{I^n} \varphi(\tau y)\, \rho_\ell(y)\, dy.$$

Hence for any $\varphi \in L^1$ we have

$$\int_{I^n} \varphi(\tau y)\, \rho_\ell(y)\, dy = \int_{I^n} \varphi(y)\, \rho_{\ell+1}(y)\, dy.$$

Let $\varphi(y) = \chi_{\prod_{i=1}^{n} [0,x_i]}(y)$, we get

$$\int_{\tau^{-1}\left(\prod_{i=1}^{n} [0,x_i]\right)} \rho_\ell(y)dy = \int_0^{x_1} \cdots\cdot \int_0^{x_n} \rho_{\ell+1}(y)dy.$$

So

$$\rho_{\ell+1}(x) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_0^{x_1} \cdots\cdot \int_0^{x_n} \rho_{\ell+1}\ (y)dy$$

$$= \frac{\partial^n}{\partial x_1 \ \ldots\ \partial x_n} \int_{\tau^{-1}\left(\prod_{i=1}^{n} [0,x_i]\right)} \rho_\ell(y)\ dy = P_\tau\ \rho_\ell(x).$$

In general, $\rho_\ell(x) = P_\tau^\ell\ \rho_0(x)$. By the Mazur's Theorem ([8], p.416, V.2.6) $\left\{\frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k\ \rho_0\right\}$ is weakly compact. By the Kakutani-Yoshida's Theorem ([8] p.662, VIII 5.3), $\frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k\ \rho_0$ converges to $\rho(x)$ in L^1 for some $\rho \in L^1$. This $\rho(x)$ is a fixed point of P_τ, i.e., $P_\tau\ \rho(x) = \rho(x)$. By Theorem 3.3, for any $\varphi \in L^1$

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx.$$

Since for any ℓ we have $0 \le \rho_\ell(x) \le K$, it follows that for any m

$$0 \le \frac{1}{m} \sum_{k=0}^{m-1} \rho_k(x) = \frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k\ \rho_0(x) \le K$$

and we have $0 \le \rho(x) \le K$. Q.E.D.

3.3 Expanding Markov Transformations

Definition 3.4 A Markov transformation $\tau: I^n \longrightarrow I^n$ with respect to the partition $\mathcal{P}$ of I^n is called a volume expanding transformation if there exists a constant $\beta > 1$ such that for any $j = 1,2,\ldots,p$

$$\left|\det A_j\right| \geq \beta > 1.$$

Definition 3.5 A Markov transformation $\tau: I^n \longrightarrow I^n$ with respect to the partition $\mathcal{P}$ of I^n is called a directionally expanding transformation if there exists a constant $\alpha > 1$ such that for any $1 \leq j \leq p$ and x, $y \in D_j$, we have

$$d(\tau x, \tau y) \geq \alpha\, d(x,y),$$

where $x = (x_1, x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$ and $d(x,y) = \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{1/2}$.

It is obvious that directionally expanding with constant $\alpha > 1$ implies volume expanding with constant $\beta = \alpha^n > 1$, but the converse is not true, i.e., volume expanding does not imply directionally expanding.

If τ is volume expanding with a constant $\beta > 1$, then

$$\left|\frac{\partial f_{i_1,\ldots,i_{\ell+1}}}{\partial x}\right| \geq \beta^{\ell}$$

and the inverse function $S_{i_1,\ldots,i_{\ell+1}}$ gives the mapping of $D_{i_{\ell+1}}$ onto $D_{i_1\ldots i_{\ell+1}}$

and

$$\left|\frac{\partial S_{i_1,\ldots,i_{\ell+1}}}{\partial x}\right| \leq \beta^{-\ell}.$$

Theorem 3.5 Let $I^n = [0,1]^n$ and $\tau(x_1,\ldots,x_n)$: $I^n \longrightarrow I^n$ be a non-singular piecewise $C^{1+\varepsilon}$ Markov transformation with respect to the partition $\mathcal{P}$ of I^n. If τ is directionally expanding with a constant $\alpha > 1$, then there exists a function $\rho(x_1,\ldots,x_n)$ such that $0 \leq \rho(x) \leq K$ for some constant K and for an arbitrary integrable function $\varphi(x_1,\ldots,x_n)$

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx$$

and $\rho(x)$ is a fixed point of P_τ.

Proof This is a special case of Theorem 3.4 for C = 1.

Q.E.D.

Theorem 3.6 Let $I^n = [0,1]^n$ and $\tau: I^n \longrightarrow I^n$ be a non-singular piecewise $C^{1+\varepsilon}$ Markov transformation with respect to the partition $\mathcal{P}$ of I^n. If τ is volume expanding with a constant $\beta > 1$ and there exists a constant $C > 0$ such that for any $D_{i_1,\ldots,i_\ell}$ we have

$$d(D_{i_1,\ldots,i_\ell}) \le C\ (\lambda(D_{i_1,\ldots,i_\ell}))^{1/n}$$

then there exists a function $\rho(x_1,\ldots,x_n)$ such that $0 \le \rho(x_1,\ldots,x_n) \le K$ for some constant K and for an arbitrary integrable function $\varphi(x_1,\ldots,x_n)$

$$\int_{I^n} \varphi(x)\ \rho(x)\ dx = \int_{I^n} \varphi(\tau x)\ \rho(x)\ dx$$

and $\rho(x)$ is a fixed point of P_τ.

Proof As in the proof of Theorem 3.4, for an arbitrary integrable function $\varphi(x)$, we have

$$\int_{I^n} \varphi(\tau^\ell x)\ dx = \int_{I^n} \varphi(y)\ \rho_\ell(y)\ dy,$$

where

$$\rho_\ell(y) = \sum_{i_1,\ldots,i_\ell} \left| \frac{\partial\ S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| \ge 0,$$

and the summation is carried out according to the same indices $i_1,\ldots,i_\ell$ for which $D_{i_1,\ldots,i_{\ell+1}} \ne \phi$.

Since $S_{i_1,\ldots,i_{\ell+1}}$ is a mapping of $D_{i_{\ell+1}}$ onto $D_{i_1,\ldots,i_{\ell+1}}$ and

$$\left| \frac{\partial S_{i_1,\dots,i_{\ell+1}}}{\partial x} \right| \le \beta^{-\ell},$$

we have

$$\lambda\left(D_{i_1,\dots,i_{\ell+1}}\right) \le \beta^{-\ell}\, \lambda\left(D_{i_{\ell+1}}\right) \le \beta^{-\ell}.$$

We know

$$f_{i_1,\dots,i_{\ell+1}}(x) = \tau^{\ell}(x) = \tau(\tau(\tau \;\dots\; \tau(x)\;)) \qquad (\ell \text{ times}).$$

If $y = \tau^{\ell}(x)$ then

$$\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y} = \left(\frac{\partial f_{i_1,\dots,i_{\ell+1}}(x)}{\partial x}\right)^{-1}$$

$$= \left(\frac{\partial\tau(\tau^{\ell-1}x)}{\partial x}\,\frac{\partial\tau(\tau^{\ell-2}x)}{\partial x}\;\dots\;\frac{\partial\tau(x)}{\partial x}\right)^{-1}.$$

For any x_1, $x_2 \in D_{i_1,\dots,i_{\ell+1}}$ we have $\tau^k x_1$, $\tau^k x_2 \in D_{i_{k+1},\dots,i_{\ell+1}}$ and

$$d(\tau^k x_1\;,\tau^k x_2) \le d\left(D_{i_{k+1}\dots i_{\ell+1}}\right)$$

$$\le C\,\left(\lambda\left(D_{i_{k+1}\dots i_{\ell+1}}\right)\right)^{1/n} \le C\,\beta_1^{-(\ell-k)},$$

where $\beta_1 = \beta^{1/n} > 1$.

Therefore, if $y_1 = \tau^{\ell}(x_1)$, $y_2 = \tau^{\ell}(x_2)$, then

$$\frac{\dfrac{\partial S_{i_1,\dots,i_{\ell+1}}(y_1)}{\partial y}}{\dfrac{\partial S_{i_1,\dots,i_{\ell+1}}(y_2)}{\partial y}} = \frac{\dfrac{\partial\,\tau(\tau^{\ell-1}x_2)}{\partial x}\,\dfrac{\partial\,\tau(\tau^{\ell-2}x_2)}{\partial x}\cdots\dfrac{\partial\,\tau(x_2)}{\partial x}}{\dfrac{\partial\,\tau(\tau^{\ell-1}x_1)}{\partial x}\,\dfrac{\partial\,\tau(\tau^{\ell-2}x_1)}{\partial x}\cdots\dfrac{\partial\,\tau(x_1)}{\partial x}}$$

and since τ_j is $C^{1+\varepsilon}$ on $\bar{D}_j$, if D_j is not convex, τ_j can be extended to $\bar{E}_j$ as a $C^{1+\varepsilon}$ function, where E_j is the smallest convex set containing D_j. So there is a constant $\bar{C}$ such that for any x_1, $x_2 \in \bar{D}_j$ we have

$$\left| \frac{\partial\ \tau_j(x_1)}{\partial x} - \frac{\partial\ \tau_j(x_2)}{\partial x} \right| \le \bar{C}\ d(x_1,\ x_2)^{\varepsilon}$$

and

$$\left| \frac{\dfrac{\partial\ \tau(\tau^k x_2)}{\partial x}}{\dfrac{\partial \tau(\tau^k x_1)}{\partial x}} \right| = \left| 1 + \frac{\dfrac{\partial \tau(\tau^k x_2)}{\partial x} - \dfrac{\partial \tau(\tau^k x_1)}{\partial x}}{\dfrac{\partial\ \tau(\tau^k x_1)}{\partial x}} \right|$$

$$\le 1 + \frac{\left| \dfrac{\partial \tau(\tau^k x_2)}{\partial x} - \dfrac{\partial \tau(\tau^k x_1)}{\partial x} \right|}{\left| \dfrac{\partial\ \tau(\tau^k x_1)}{\partial x} \right|}$$

$$\le 1 + \frac{\bar{C}\ d(\tau^k x_1, \tau^k x_2)^{\varepsilon}}{\beta}$$

$$\le 1 + \frac{\bar{C}\ C^{\varepsilon}}{\beta}\ \beta_1^{-(\ell-k)\varepsilon} .$$

Therefore,

$$\left| \frac{\dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y_1)}{\partial y}}{\dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y_2)}{\partial y}} \right| \le \prod_{k=0}^{\ell-1} \left(1 + \frac{\overline{C}\, C^{\varepsilon}}{\beta}\, \beta_1^{-(\ell-k)\varepsilon}\right)$$

$$= \prod_{i=1}^{\ell} \left(1 + \frac{\overline{C}\, C^{\varepsilon}}{\beta}\, \beta_1^{-i\varepsilon}\right)$$

$$\le \prod_{i=1}^{\infty} \left(1 + \frac{\overline{C}\, C^{\varepsilon}}{\beta}\, \beta_1^{-i\varepsilon}\right) = C^*.$$

Since

$$\lambda\left(D_{i_1,\ldots,i_{\ell+1}}\right) = \int_{D_{i_1,\ldots,i_{\ell+1}}} dx$$

$$= \int_{D_{i_{\ell+1}}} \left| \frac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| dy \qquad (y = \tau^{\ell} x)$$

$$= \left| \frac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y_0)}{\partial y} \right| \lambda(D_{i_{\ell+1}})$$

for some $y_0 \in D_{i_{\ell+1}}$. We now have

$$\frac{\left| \dfrac{\partial\, S_{i_1,\ldots,i_{\ell+1}}(y)}{\partial y} \right| \lambda(D_{i_{\ell+1}})}{\lambda(D_{i_1,\ldots,i_{\ell+1}})}$$

$$= \frac{\left|\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y}\right| \lambda(D_{i_{\ell+1}})}{\left|\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y_0)}{\partial y}\right| \lambda(D_{i_{\ell+1}})}$$

$$= \frac{\left|\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y}\right|}{\left|\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y_0)}{\partial y}\right|} \le C^*$$

and

$$\rho_\ell(y) = \sum_{i_1\dots i_\ell} \left|\frac{\partial S_{i_1,\dots,i_{\ell+1}}(y)}{\partial y}\right| \le \sum_{i_1\dots i_\ell} \frac{C^*}{\lambda\left(D_{i_{\ell+1}}\right)} \lambda\left(D_{i_1\dots i_{\ell+1}}\right)$$

$$\le \frac{C^*}{K_1} \sum_{i_1\dots i_\ell} \lambda\left(D_{i_1,\dots,i_{\ell+1}}\right) \le \frac{C^*}{K_1} = K,$$

where $K_1 = \min_{1\le j\le p} m(D_j)$.

So $\left\{\rho_\ell\right\}_{\ell=1}^{\infty}$ is a weakly compact subset of L^1 and for any $\varphi \in L^1$ we have

$$\int_{I^n} \varphi(\tau^\ell x)\, dx = \int_{I^n} \varphi(x)\, \rho_\ell(x)\, dx.$$

As in the proof of Theorem 3.4, we take $\rho_0(x) = 1$ and we have

$$\rho_\ell(x) = P_\tau\, \rho_{\ell-1}(x) = P_\tau^\ell\, \rho_0(x).$$

Therefore $\left\{\frac{1}{m} \sum_{k=0}^{m-1} P_\tau^k \rho_0\right\}$ is weakly compact and it converges to some $\rho(x)$ in L^1.

This $\rho(x)$ is a fixed point of P_τ, i.e., $P_\tau\rho(x) = \rho(x)$, $0 \le \rho(x) \le K$ and for any $\varphi \in L^1$ we have $\int_{I^n} \varphi(x)\, \rho(x)\, dx = \int_{I^n} \varphi(\tau x)\, \rho(x)\, dx$. Q.E.D.

3.4 Comparison with Mane's Results

In [17] R. Mane gave the following more general definition of an expanding transformation:

Definition 3.6 Let $(X, \mathcal{A}, \mu)$ be a probability space, where X is a separable metric space and $\mathcal{A}$ is its Borel σ-algebra. We say that a transformation $f: X \longrightarrow X$ is expanding if there exists a sequence of partitions $(\mathcal{P}_i)_{i \geq 0}$ such that

a) $\bigcup_{P \in \mathcal{P}_0} P = X$

b) For every $n \geq 0$ and $P \in \mathcal{P}_{n+1}$, $f(P)$ is an union (mod 0) of atoms of $\mathcal{P}_n$, and $f|P$ is injective.

c) There exist $0 < \lambda < 1$ and $K > 0$ such that, denoting by $\mathcal{P}_n(x)$ the atom of $\mathcal{P}_n$ which contains x, we have

$$d(x, y) \leq K \lambda^n d(f^n(x), f^n(y))$$

for every $n \geq 0$, $x \in X$, $y \in \mathcal{P}_n(x)$.

d) There exists $m > 0$ such that, for every pair of atoms $P, Q \in \mathcal{P}_0$, we have $\mu(f^{-m}(P) \cap Q) \neq 0$.

e) There exist $J: X \longrightarrow R^+$, $0 < \gamma < 1$ and $C > 0$ such that, for every $n \geq 0$ and every Borel set A contained in an atom of $\mathcal{P}_0$, we have

$$\mu(f(A)) = \int_A J \, d\mu$$

and for every x, y contained in the same atom of $\mathcal{P}_n$ we have

$$\left| \frac{J(y)}{J(x)} - 1 \right| \leq C \, d(f(x), f(y))^{\gamma}.$$

For this expanding transformation Mane proved

Theorem 3.7 Let X be a bounded metric space, λ a probability measure on the Borel σ-algebra of X and $f: X \longrightarrow X$ an expanding transformation of $(X, \mathcal{A}, \lambda)$, where $\mathcal{A}$ is the Borel σ-albegra of X. Then there exists a unique probability measure μ on the Borel σ-algebra of X which is f-invariant and absolutely continuous with respect to λ. This measure satisfies the following properties:

a) $\frac{d\mu}{d\lambda}$ is Hölder continuous;

b) $\inf_{x\in P} \left(\frac{d\mu}{d\lambda}\right)(x) > 0$ and $\sup_{x\in P} \left(\frac{d\mu}{d\lambda}\right)(x) < \infty$ for every $P \in \mathcal{P}$;

c) f is exact with respect to μ;

d) $\lim_{n\to\infty} \lambda(f^{-n}(A)) = \mu(A)$ for every $A \in \mathcal{A}$.

When $X = I^n$ and λ is the Lebesgue measure on I^n, let $f = \tau$ and J be the absolute value of the Jacobian of τ, Mane's expanding transformation is our quasi-directional expanding Markov transformation but with the additional condition d).

Theorem 3.4 establishes existence while Mane's Theorem 3.7 proves uniqueness and properties a), b), c), d).

The methods of the proof for Theorem 3.4 and Theorem 3.7 are different.

For example, let $n = 1$, $\mathcal{P}_0 = \{D_k\}_{k=1}^3$, $D_k = [\frac{k-1}{3}, \frac{k}{3})$, $k = 1,2,3$ and $\tau: I \to I$ be defined by $\tau = 2x - \frac{2}{3}(k-1)$, $x \in D_k$ (figure 3.1).

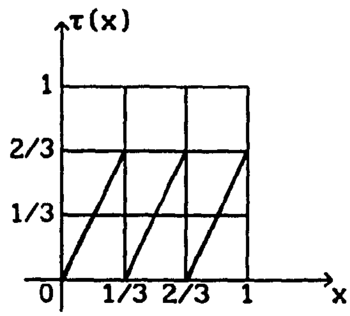

Figure 3.1

τ satisfies all the conditions of Theorem 3.1, 3.2, 3.4 (with $\alpha = 2$ and $C = 1$), 3.5 (with $\alpha = 2$) and 3.6 (with $\beta = 2$ and $C = 1$), but it does not satisfy the condition d) of Theorem 3.7. Since when we take $P = D_3$, for any m, we have $\tau^{-m}(P) = \phi$ and $\lambda(\tau^{-m}(P) \cap Q) = 0$ for any $Q \in \mathcal{P}_0$, where λ is the Lebesgue measure on I.

CHAPTER 4

Approximating the Invariant Densities of Jablonski Transformation

In 1960 Ulam [27, p.75] conjectured that it was possible to construct finite dimensional operators which approximate P_τ and whose fixed points approximate the fixed point of P_τ. In 1976 this conjecture was proved by Li [9] for a class of one dimensional piecewise expanding transformations. In 1984 Gora [24] approximated the invariant density using small perturbations.

The aim of this chapter is to prove Ulam's conjecture and to generalize Gora's result to higher dimensional Jablonski transformations.

4.1 Approximation the Invariant Density by Piecewise Constant Functions

Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation. For each $\ell = 1,2,\dots,$ we divide I^n into ℓ^n equal subsets $I_1, I_2, \dots, I_{\ell^n}$ with

$$I_k = \left[\frac{r_1}{\ell}, \frac{r_1+1}{\ell}\right) \times \left[\frac{r_2}{\ell}, \frac{r_2+1}{\ell}\right) \times \dots \times \left[\frac{r_n}{\ell}, \frac{r_n+1}{\ell}\right)$$

for $r_1, r_2, \dots, r_n = 0, 1, \dots, \ell-1$ and

$$\lambda(I_k) = \frac{1}{\ell^n}, \quad k = 1,2,\dots,\ell^n.$$

Define P_{st} to be the fraction of I_s which is mapped into I_t by τ , i.e.,

$$P_{st} = \lambda(I_s \cap \tau^{-1}(I_t))/\lambda(I_s).$$

Let Δ_ℓ be the ℓ^n dimensional linear subspace of L^1 which is the finite element space generated by $\{\chi_k\}_{k=1}^{\ell^n}$, where χ_k denotes the characteristic function of I_k, i.e. $f \in \Delta_\ell$ if and only if $f = \sum_{k=1}^{\ell^n} a_k \chi_k$ for some constants

$a_1, a_2, \ldots, a_{\ell^n}$.

Define a linear operator $P_\ell = P_\ell(\tau)\colon \Delta_\ell \longrightarrow \Delta_\ell$ by

$$P_\ell(\tau)\chi_k = \sum_{t=1}^{\ell^n} P_{kt}\, \chi_t.$$

<u>Lemma 4.1</u> Let $\Delta_\ell^1 = \left\{ \sum_{k=1}^{\ell^n} a_k\chi_k\colon\ a_k \geq 0 \text{ and } \sum_{k=1}^{\ell^n} a_k = 1 \right\}$. Then P_ℓ maps Δ_ℓ^1 to a subset of Δ_ℓ^1.

<u>Proof</u> Let $f \ \sum_{k=1}^{\ell^n} P_{kt}\, \chi_t$, $a_k \geq 0$ and $\sum_{k=1}^{\ell^n} a_k = 1$. Then

$$P_\ell f = P_\ell\left(\sum_{k=1}^{\ell^n} a_k\, \chi_k\right) = \sum_{k=1}^{\ell^n} a_k\left(P_\ell\, \chi_k\right)$$

$$= \sum_{k=1}^{\ell^n} a_k\left(\sum_{t=1}^{\ell^n} P_{kt}\, \chi_t\right) = \sum_{t=1}^{\ell^n}\left(\sum_{k=1}^{\ell^n} a_k\, P_{kt}\right)\chi_t.$$

Now, for all $k = 1, 2, \ldots, \ell^n$,

$$\sum_{t=1}^{\ell^n} P_{kt} = \sum_{t=1}^{\ell^n} \frac{m(I_k \cap \tau^{-1}(I_t))}{m(I_k)} = 1.$$

Hence, for any $t = 1, 2, \ldots, \ell^n$, $\sum_{k=1}^{\ell^n} a_k\, P_{kt} \geq 0$ and

$$\sum_{t=1}^{\ell^n}\left(\sum_{k=1}^{\ell^n} a_k\, P_{kt}\right) = \sum_{k=1}^{\ell^n} a_k \left(\sum_{t=1}^{\ell^n} P_{kt}\right) = \sum_{k=1}^{\ell^n} a_k = 1.$$

Therefore $P_\ell f \in \Delta_\ell^1$.

Q.E.D.

<u>Definition 4.1</u> For $f \in L^1$ and for any positive integer ℓ we define Q_ℓ: $L^1 \longrightarrow \Delta_\ell$ by $Q_\ell f = \sum_{k=1}^{\ell^n} C_k\, \chi_k$, where

$$C_k = \frac{1}{\lambda(I_k)} \int_{I_k} f(x)\, dx.$$

__Lemma 4.2__ If $f \in L^1$ then the sequence $Q_\ell f$ converges in L^1 to f as $\ell \longrightarrow \infty$.

__Proof__ Since $f \in L^1$, for any $\varepsilon > 0$ there exists a continuous function g such that $\|g-f\| < \frac{\varepsilon}{3}$. Since g is continuous on I^n, g is uniformly continuous. We may choose N large enough such that for $\ell > N$, we have $|g(x)-g(y)| < \frac{\varepsilon}{3}$ for all x, $y \in I_k$, $k = 1,2,\ldots,\ell^n$. It follows that

$$\int_{I_k} |(Q_\ell g(x) - g(x)|\, dx = \int_{I_k} \left|\frac{1}{\lambda(I_k)} \int_{I_k} g(x')dx' - g(x)\right|dx$$

$$= \int_{I_k} \left|\frac{1}{\lambda(I_k)} \int_{I_k} (g(x') - g(x))dx'\right|dx \le \int_{I_k} \left(\frac{1}{\lambda(I_k)} \int_{I_k} |g(x') - g(x)|dx'\right)dx$$

$$\le \lambda(I_k)\, \frac{\varepsilon}{3}.$$

Hence,

$$\|Q_\ell g - g\| = \int_{I^n} |Q_\ell g-g|dx = \sum_{k=1}^{\ell^n} \int_{I_k} |Q_\ell g-g|\, dx \le \sum_{k=1}^{\ell^n} \lambda(I_k)\, \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

On the other hand, for $f \ge 0$

$$\int_{I^n} Q_\ell f\, dx = \int_{I^n} \sum_{k=1}^{\ell^n} \left(\frac{1}{\lambda(I_k)} \int_{I_k} f(x')dx'\right) \chi_k(x)\, dx$$

$$= \sum_{k=1}^{\ell^n} \int_{I_k} f(x')\, dx' = \int_{I^n} f(x)\, dx.$$

Therefore, $\|Q_\ell\| = 1$ and

$$\|Q_\ell f - f\| \le \|Q_\ell f - Q_\ell g\| + \|Q_\ell g - g\| + \|g - f\|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Q.E.D.

Lemma 4.3 If $f \in \Delta_\ell$ then $P_\ell f = Q_\ell P_\tau f$.

Proof We only need to show $P_\ell \chi_k = Q_\ell P_\tau \chi_k$ for $1 \le k \le \ell^n$. Since

$$(P_\tau f)(x) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} \int_{\tau^{-1}\left(\prod_{i=1}^{n} [0,x_i]\right)} f(x')dx',$$

we have

$$Q_\ell P_\tau \chi_k = \sum_{j=1}^{\ell^n} \left[\frac{1}{\lambda(I_j)} \int_{I_j} \left(\frac{\partial^n}{\partial x_1 \dots \partial x_n} \int_{\tau^{-1}\left(\prod_{i=1}^{n} [0,x_i]\right)} \chi_k(x')dx'\right)dx\right] \chi_j$$

$$= \sum_{j=1}^{\ell^n} \left[\frac{1}{\lambda(I_j)} \int_{\tau^{-1}(I_j)} \chi_k(x')dx'\right]\chi_j = \sum_{j=1}^{\ell^n} \frac{\lambda(I_k \cap \tau^{-1}(I_j))}{\lambda(I_j)} \chi_j$$

$$= \sum_{j=1}^{\ell^n} \frac{\lambda(I_k \cap \tau^{-1}(I_j))}{\lambda(I_k)} \chi_j = \sum_{j=1}^{\ell^n} P_{kj} \chi_j = P_\ell \chi_j.$$

Q.E.D.

Lemma 4.4 If $f \in \Delta_\ell$ then the sequence $P_\ell f$ converges to $P_\tau f$ in L^1 as $\ell \longrightarrow \infty$.

Proof By Lemma 4.3 $P_\ell f = Q_\ell P_\tau f$ and by Lemma 4.2 it converges to $P_\tau f$ in L^1 as $\ell \longrightarrow \infty$.

Q.E.D.

Lemma 4.5 For any integer ℓ there exists $f_\ell \in \Delta_\ell$ such that $P_\ell f_\ell = f_\ell$ and $\|f_\ell\| = 1$. i.e. P_ℓ has a fixed point of norm 1.

Proof By Lemma 4.1 $P_\ell(\Delta_\ell^1) \subset \Delta_\ell^1$ is a compact convex set. So by the Brouwer fixed point theorem there exists a point $g_\ell \in \Delta_\ell^1$ for which $P_\ell g_\ell = g_\ell$. Let $f_\ell = \ell^n g_\ell$, we have $f_\ell \in \Delta_\ell$, $P_\ell f_\ell = f_\ell$ and $\|f_\ell\| = 1$.

Q.E.D.

From [1] we have

Lemma 4.6 Let τ be a Jablonski transformation

$$\tau(x) = (\varphi_{1j}(x_1), \dots, \varphi_{nj}(x_n)) \quad x \in D_j,$$

with

$$s = \inf_{i,j}\left\{ \inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}| \right\} > 2.$$

Then, for $f \in L^1$

$$\overset{I^n}{V} P_\tau f \le K_\tau \|f\| + \alpha \overset{I^n}{V} f,$$

where K_τ is a constant depending on τ and $\alpha = 2s^{-1} < 1$.

__Lemma 4.7__ If $f \in L^1$, then $\overset{I^n}{V} Q_\ell f \le \overset{I^n}{V} f$.

__Proof__ Let $I_k = \prod_{i=1}^{n} \left[\frac{r_i}{\ell}, \frac{r_i+1}{\ell}\right) = \prod_{i=1}^{n} J_{r_i}$ for some $r_i = 0,1,\ldots,\ell-1$, $k = 1,2,\ldots,\ell^n$ and $\lambda(I_k) = \prod_{i=1}^{n} \lambda\left(J_{r_i}\right)$.

Let

$$Q_{\ell_i} f(x) = \sum_{r_i=0}^{\ell-1} \left(\frac{1}{\lambda(J_{r_i})} \int_{J_{r_i}} f(x)\, dx_i\right) \chi_{J_{r_i}}(x_i).$$

Then

$$Q_\ell f(x) = Q_{\ell_1} Q_{\ell_2} \cdots Q_{\ell_n} f(x) = \left(\prod_{i=1}^{n} Q_{\ell_i}\right) f(x).$$

By Lemma 2.6 of [9] we have

$$\overset{I^n}{\underset{i}{V}} Q_\ell f = \overset{I^n}{\underset{i}{V}} \left(\prod_{j=1}^{n} Q_{\ell_j}\right) f = \overset{I^n}{\underset{i}{V}} Q_{\ell_i} \left(\prod_{j=1,\, j\ne i}^{n} Q_{\ell_j}\right) f \le \overset{I^n}{\underset{i}{V}} \left(\prod_{\substack{j=1\\ j\ne i}}^{n} Q_{\ell_j}\right) f.$$

If we could show

$$\int_{I^{n-1}} \overset{I^n}{\underset{i}{V}} \left(\prod_{\substack{j=1\\ j\ne i}}^{n} Q_{\ell_j}\right) f \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right) \le \int_{I^{n-1}} \overset{I^n}{\underset{i}{V}} f \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right), \tag{4.1}$$

then we would have

$$\overset{I^n}{\underset{i}{V}} Q_\ell f = \inf \left\{ \int_{I^{n-1}} \overset{I^n}{\underset{i}{V}} h \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right),\ h = Q_\ell f \text{ a.e.},\ \overset{I^n}{\underset{i}{V}} h \text{ measurable} \right\}$$

$$\le \inf\left\{ \int_{I^{n-1}} \bigvee_i^{I^n} Q_\ell g \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right),\ g = f \text{ a.e.},\ \bigvee_i^{I^n} Q_\ell g \text{ measurable} \right\}$$

$$\le \inf\left\{ \int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{\substack{j=1\\ j\ne i}}^{n} Q_{\ell_j}\right) g \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right),\ g = f \text{ a.e.},\ \bigvee_i^{I^n} g \text{ measurable} \right\}$$

$$\le \inf\left\{ \int_{I^{n-1}} \bigvee_i^{I^n} g \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right),\ g = f \text{ a.e.},\ \bigvee_i^{I^n} g \text{ measurable} \right\} = \bigvee_i^{I^n} f.$$

In the foregoing argument we have used the fact that for any positive integer ℓ and $f, g \in L^1$, $f = g$ a.e. implies $Q_\ell f = Q_\ell g$ a.e. and $\bigvee_i^{I^n} g$ measurable implies $\bigvee_i^{I^n} Q_\ell g$ measurable. Therefore,

$$\bigvee^{I^n} Q_\ell f = \max_i \bigvee_i^{I^n} Q_\ell f \le \max_i \bigvee_i^{I^n} f = \bigvee^{I^n} f.$$

We now prove (4.1). For any $0 = x_i^0 < x_i^1 < \dots < x_i^{r-1} < x_i^r = 1$,

$$\sum_{k=1}^{r} \left| \prod_{\substack{j=1\\ j\ne i}}^{n} Q_{\ell_j} f\left(x_1,\dots,x_i^{k-1},\dots,x_n\right) - \prod_{\substack{j=1\\ j\ne i}}^{n} Q_{\ell_j} f\left(x_1,\dots,x_i^{k},\dots,x_n\right) \right|$$

$$= \sum_{k=1}^{r} \left| \prod_{\substack{j=1\\ j\ne i}}^{n} \sum_{r_j=0}^{\ell-1} \frac{1}{\prod_{\substack{j=1\\ j\ne i}}^{n} \lambda(J_{r_j})} \int_{\prod_{\substack{j=1\\ j\ne i}}^{n} J_{r_j}} \left(f(x_1,\dots,x_i^{k-1},\dots,x_n) - f(x_1,\dots,x_i^{k},\dots,x_n)\right) \right.$$

$$\left. \left(\prod_{\substack{j=1\\ j\ne i}}^{n} dx_j\right) \left(\prod_{\substack{j=1\\ j\ne i}}^{n} \chi_{J_{r_j}}(x_j)\right) \right|$$

$$\leq \sum_{k=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \sum_{r_j=0}^{\ell-1} \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \lambda(J_{r_j})} \int_{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} J_{r_j}} |f(x_1, \ldots, x_i^{k-1}, \ldots, x_n) - f(x_1, \ldots, x_i^{k}, \ldots, x_n)| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \chi_{J_{r_j}}(x_j) \right) \right);$$

Now,

$$\int_{I^{n-1}} \sum_{k=1}^{r} \left| \prod_{\substack{j=1 \\ j \neq i}}^{n} Q_{\ell_j} f(x_1, \ldots, x_i^{k-1}, \ldots, x_n) - \prod_{\substack{j=1 \\ j \neq i}}^{n} Q_{\ell_j} f(x_1, \ldots, x_i^{k}, \ldots, x_n) \right| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right)$$

$$\leq \int_{I^{n-1}} \sum_{k=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \sum_{r_j=0}^{\ell-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \lambda(J_{r_j}) \right)^{-1} \int_{\prod\limits_{j=1, j \neq i}^{n} J_{r_j}} |f(\ldots x_i^{k-1} \ldots) - f(\ldots x_i^{k} \ldots)| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \chi_{J_{r_j}}(x_j) \right) \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right)$$

$$= \sum_{k=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \sum_{r_j=0}^{\ell-1} \int_{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} J_{r_j}} |f(x_1, \ldots, x_i^{k-1}, \ldots, x_n) - f(x_1, \ldots, x_i^{k}, \ldots, x_n)| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right) \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \lambda(J_{r_j})} \int_{I^{n-1}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \chi_{J_{r_j}}(x_j) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right) \right)$$

$$= \sum_{k=1}^{r} \int_{I^{n-1}} |f(x_1, \ldots, x_i^{k-1}, \ldots, x_n) - f(x_1, \ldots, x_i^{k}, \ldots, x_n)| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right)$$

$$= \int_{I^{n-1}} \sum_{k=1}^{r} |f(x_1, \ldots, x_i^{k-1}, \ldots, x_n) - f(x_1, \ldots, x_i^{k}, \ldots, x_n)| \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right).$$

Therefore,

$$\int_{I^{n-1}} \overset{I^n}{\underset{i}{V}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} Q_{\ell_j} f \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right) \le \int_{I^{n-1}} \overset{I^n}{\underset{i}{V}} f \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} dx_j \right).$$

Q.E.D.

<u>Lemma 4.8</u> Let τ be a Jablonski transformation

$$\tau(x) = (\varphi_{1j}(x_1), \dots, \varphi_{nj}(x_n)) \quad x \in D_j$$

and let $f_\ell \in \Delta_\ell$ be the fixed point of $P_\ell(\tau)$ with $\|f_\ell\| = 1$. If

$$s = \inf_{i,j} \left\{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \right\} > 2,$$

then the sequence $\left\{ \overset{I^n}{V} f_\ell \right\}_{\ell=1}^{\infty}$ is bounded.

<u>Proof</u> By Lemma 4.3, $f_\ell = P_\ell f_\ell = Q_\ell P_\tau f_\ell$ for all ℓ. Hence by Lemma 4.6 and Lemma 4.7,

$$\overset{I^n}{V} f_\ell = \overset{I^n}{V} Q_\ell P_\tau f_\ell \le \overset{I^n}{V} P_\tau f_\ell \le K_\tau \|f_\ell\| + \alpha \overset{I^n}{V} f_\ell$$

$$= K_\tau + \alpha \overset{I^n}{V} f_\ell,$$

where $K_\tau > 0$ and $0 < \alpha < 1$. Since $\overset{I^n}{V} f_\ell < \infty$, we have

$$\overset{I^n}{V} f_\ell \le \frac{K_\tau}{1-\alpha}.$$

Q.E.D.

<u>Lemma 4.9</u> For any $f \in L^1$, $\ell = 1, 2, \dots$ and measurable subset A of I^n

$$\int_{I^n} \chi_A Q_\ell f \, dx = \int_{I^n} f \, Q_\ell \chi_A \, dx.$$

<u>Proof</u> $$\int_{I^n} \chi_A(x) \, Q_\ell f(x) \, dx = \int_{I^n} \chi_A(x) \left(\sum_{k=1}^{\ell^n} \frac{1}{\lambda(I_k)} \int_{I_k} f(y) dy \, \chi_k(x) \right) dx$$

$$= \sum_{k=1}^{\ell^n} \frac{1}{\lambda(I_k)} \int_{I_k} f(y) dy \int_{I^n} \chi_A(x) \, \chi_k(x) \, dx$$

$$= \sum_{k=1}^{\ell^n} \frac{\lambda(A\cap I_k)}{\lambda(I_k)} \int_{I_k} f(y)dy = \sum_{k=1}^{\ell^n} \frac{\lambda(A\cap I_k)}{\lambda(I_k)} \int_{I_k} f(x)dx$$

$$= \int_{I^n} f(x) \left(\sum_{k=1}^{\ell^n} \frac{1}{\lambda(I_k)} \int_{I_k} \chi_A(y)dy \, \chi_k(x)\right) dx$$

$$= \int_{I^n} f(x) \, Q_\ell \chi_A(x) \, dx.$$

Q. E. D.

<u>Lemma</u> <u>4. 10</u> If $f_\ell \in L^1$, $\ell = 1,2,\dots$, with $f_\ell \geq 0$, $\|f_\ell\| \leq M$ for some constant M and a subsequence f_{ℓ_j} converges weakly to f as $j \longrightarrow \infty$, then $Q_{\ell_j} f_{\ell_j}$ converges weakly to f as $j \longrightarrow \infty$.

<u>Proof</u> It is enough to show that for any measurable subset A of I^n we have

$$\lim_{j\to\infty} \int_{I^n} \chi_A \, Q_{\ell_j} f_{\ell_j} \, dx = \int_{I^n} \chi_A \, f \, dx.$$

By Corollary IV.8.11 of [8, p.294],

$$\int_E f_{\ell_j}(x) \, dx \to 0 \text{ as } \lambda(E) \longrightarrow 0 \text{ uniformly in } j.$$

Since $\|f_{\ell_j}\| \leq M$, $f_{\ell_j} \geq 0$ and f_{ℓ_j} converges weakly to f as $j \longrightarrow \infty$, Theorem 7.5.3 of [26, p.296] implies that $\{f_{\ell_j}\}$ are uniformly integrable, i.e.,

$$\int_{\{|f_{\ell_j}|\geq C\}} |f_{\ell_j}| \, dx \longrightarrow 0 \text{ as } C \longrightarrow \infty \text{ uniformly in } j.$$

Therefore, for any $\varepsilon > 0$, there exists $C > 0$ such that for all ℓ_j

$$2 \int_{\{|f_{\ell_j}|\geq C\}} |f_{\ell_j}| \, dx < \varepsilon.$$

Hence,

$$\int_{I^n} f_{\ell_j} (Q_{\ell_j}\chi_A - \chi_A) \, dx \leq \int_{I^n} |f_{\ell_j}| \, |Q_{\ell_j}\chi_A - \chi_A| \, dx$$

$$= \int_{\{|f_{\ell_j}|\geq C\}} |f_{\ell_j}| \, |Q_{\ell_j}\chi_A - \chi_A| \, dx + \int_{\{|f_{\ell_j}|<C\}} |f_{\ell_j}| \, |Q_{\ell_j}\chi_A - \chi_A| \, dx$$

$$\leq 2 \int_{\{|f_{\ell_j}|\geq C\}} |f_{\ell_j}| \, dx + C \int_{\{|f_{\ell_j}|<C\}} |Q_{\ell_j}\chi_A - \chi_A| \, dx$$

$$\le 2 \int_{\{|f_{\ell_j}| \ge C\}} |f_{\ell_j}| \, dx + C \int_{I^n} |Q_{\ell_j}\chi_A - \chi_A| \, dx.$$

The first term is less than ε and the second term approaches 0 as $j \longrightarrow \infty$ by Lemma 4.2. i.e.,

$$\lim_{j\to\infty} \int_{I^n} f_{\ell_j} (Q_{\ell_j}\chi_A - \chi_A) \, dx = 0.$$

Thus, by Lemma 4.9,

$$\lim_{j\to\infty} \int_{I^n} \chi_A \, Q_{\ell_j} f_{\ell_j} \, dx = \lim_{j\to\infty} \int_{I^n} f_{\ell_j} \, Q_{\ell_j}\chi_A \, dx$$

$$= \lim_{j\to\infty} \int_{I^n} f_{\ell_j} (Q_{\ell_j}\chi_A - \chi_A) \, dx + \lim_{j\to\infty} \int_{I^n} f_{\ell_j} \, \chi_A \, dx$$

$$= \int_{I^n} f \, \chi_A \, dx.$$

Q.E.D.

By [2, p.43] we have:

Lemma 4.11 The Frobenius-Perron operator is weakly continuous, i.e., if $\{f_\ell\}_{\ell=1}^{\infty} \subset L^1$ and $f_\ell \longrightarrow f$ weakly as $\ell \longrightarrow \infty$, then $P_\tau f_\ell \longrightarrow P_\tau f$ weakly as $\ell \longrightarrow \infty$.

Theorem 4.1 Let τ be a non-singular Jablonski transformation with respect to the partition $\left\{D_1, \ldots, D_p\right\}$ and $s = \inf_{i,j} \left\{ \inf_{[a_{ij}, b_{ij}]} \left|\varphi'_{ij}\right| \right\} > 2$. Suppose P_τ has a unique fixed point. Then for any positive integer ℓ, $P_\ell(\tau)$ has a fixed point f_ℓ in Δ_ℓ with $\|f_\ell\| = 1$ and the sequence $\left\{f_\ell\right\}$ converges weakly to the fixed point of P_τ as $\ell \longrightarrow \infty$.

Proof It follows from Lemma 1.3 and Lemma 4.8 that the set $\left\{f_\ell\right\}_{\ell=1}^{\infty}$ is weakly relatively compact in L^1. Let $\left\{f_{\ell_j}\right\}$ be any weakly convergent subsequence of $\left\{f_\ell\right\}_{\ell=1}^{\infty}$ and let $f = \lim_{j\longrightarrow\infty} f_{\ell_j}$ weakly. Then for any continuous function g

$$\left|\int_{I^n} g\,(f - P_\tau f)dx\right| \le \left|\int_{I^n} g\,(f - f_{\ell_j})dx\right| + \left|\int_{I^n} g\left(f_{\ell_j} - Q_{\ell_j} P_\tau f_{\ell_j}\right)dx\right|$$

$$+ \left|\int_{I^n} g\left(Q_{\ell_j} P_\tau f_{\ell_j} - P_\tau f\right)dx\right|.$$

The first term approaches 0 since f_{ℓ_j} converges weakly to f as $j \longrightarrow \infty$. By Lemma 4.3, $Q_{\ell_j} P_\tau f_{\ell_j} = P_{\ell_j} f_{\ell_j} = f_{\ell_j}$, the second term is identically 0. By Lemma 4.11, $P_\tau f_{\ell_j}$ converges weakly to $P_\tau f$ as $j \to \infty$. The last term approaches 0 by Lemma 4.10.

We have, therefore, established that for any continuous function g

$$\int_{I^n} g(x)\ (f(x) - P_\tau f(x))\ dx = 0.$$

This means $P_\tau f(x) = f(x)$ almost everywhere. If f^* is the unique fixed point of P_τ then $f = f^*$ a.e. and $f_{\ell_j} \xrightarrow{W} f^*$ as $j \longrightarrow \infty$. We have, therefore, shown that any weakly convergent subsequence of $\{f_\ell\}$ converges weakly to f. Hence $f_\ell \xrightarrow{W} f$ as $\ell \longrightarrow \infty$.

Q.E.D.

<u>Corollary 4.1</u> If the fixed point of P_τ is not unique in Theorem 4.1 then any weak limit point of $\left\{f_\ell\right\}_{\ell=1}^{\infty}$ is a fixed point of P_τ.

<u>Theorem 4.2</u> Let τ be a nonsingular Jablonski transformation with

$$s = \inf_{i,j} \left\{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \right\} > 1.$$

Suppose P_τ has a unique fixed point. Let k be an integer such that $s^k > 2$. Let $\phi = \tau^k$ and f_ℓ be a fixed point of $P_\ell(\phi)$. Let

$$g_\ell = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_\ell.$$

Then $\{g_\ell\}$ converges weakly to the fixed point of P_τ.

Proof Notice that $P_{\tau^j} = (P_\tau)^j$. By Lemma 4.11 and Theorem 4.1

$$g_\ell \longrightarrow g = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f \text{ weakly as } \ell \longrightarrow \infty.$$

Therefore,

$$P_\tau g = \frac{1}{k} \sum_{j=1}^{k} P_{\tau^j} f = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f = g,$$

where f is the fixed point of $P_\phi = P_{\tau^k}$, i.e.

$$P_{\tau^k} f = f.$$

Q.E.D.

Corollary 4.2 If the fixed point of P_τ is not unique in Theorem 4.2 then any weak limit point f of $\{f_\ell\}_{\ell=1}^{\infty}$ is a fixed point of P_ϕ and $g = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{\ell_i} \longrightarrow f$ weakly as $i \longrightarrow \infty$ then $g_{\ell_i} = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_{\ell_i} \rightarrow g$ weakly as $i \rightarrow \infty$.

4.2 Uniqueness

We say f is an invariant density under τ if f is a fixed point of P_τ. Let F be the set $f \in L^1$ which is invariant under τ. F is a subspace of L^1.

Definition 4.2 We write "$A \subset B$ a.e" if A, $B \subset I^n$ and $x \in B$ for almost all x in A. We write "A = B a.e." if both $A \subset B$ a.e. and $B \subset A$ a.e. are satistied. We say a set A is invariant (under τ) if A is measurable subset of I^n and $\tau(A) = A$ a.e.

Lemma 4.12 — 4.19 can be found in [14], we include the proofs for completeness.

Lemma 4.12 If A and B are invariant sets then

1) $\lambda(\tau(A)) = \lambda(A)$;

2) $\tau(A) \subset A$ a.e.;

3) $\tau(A \cup B) = A \cup B$ a.e.

Proof The first one and second one are obvious. For the third one we have $\tau(A \cup B) = \tau(A) \cup \tau(B) = A \cup B$ a.e.

Q.E.D.

Lemma 4.13 Let A be a measurable set satisfying $\tau(A) \subset A$ a.e., then, for any invariant function f, we have

$$\int_{\tau^{-1}(A)-A} f = 0.$$

Proof Since $\tau(A) \subset A$ a.e. So $A \subset \tau^{-1}(A)$ a.e. Then

$$\int_{\tau^{-1}(A)-A} f = \int_{\tau^{-1}(A)} f - \int_A f$$

$$= \int_A P_\tau f - \int_A f = \int_A f - \int_A f = 0.$$

Q.E.D.

By Lemma 4.13 we have

Lemma 4.14 If A is invariant and f is an invariant function, then

$$\int_{\tau^{-1}(A)-A} f = 0,$$

i.e., for any invariant measure μ with $d\mu = fdm$ we have

$$\mu(\tau^{-1}(A) - A) = 0.$$

Lemma 4.15 If A is an invariant set, then for any invariant function f, $f\chi_A$ is invariant.

Proof For any measurable set S we have

$$\int_S f\,\chi_A = \int_{S\cap A} f = \int_{\tau^{-1}(S\cap A)} f = \int_{\tau^{-1}(S)\cap\,\tau^{-1}(A)} f$$

$$= \int_{\tau^{-1}(S)\cap(\tau^{-1}(A)-A)} f + \int_{\tau^{-1}(S)\cap A} f$$

$$= \int_{\tau^{-1}(S)\cap A} f = \int_{\tau^{-1}(S)} f\chi_A.$$

It means that $f\chi_A$ is invariant, where we use Lemma 4.14, from $\int_{\tau^{-1}(A)-A} f = 0$, we get $\int_{\tau^{-1}(S)\cap(\tau^{-1}(A)-A)} f = 0.$

Q.E.D.

For any function f: $I^n \longrightarrow R$ we write

$P(f) = \{x \in I^n \mid f(x) > 0\},$

$N(f) = \{x \in I^n \mid f(x) < 0\}$

and

$Z(f) = \{x \in I^n \mid f(x) = 0\}.$

Lemma 4.16 If f is invariant, then the sets P(f) and N(f) are invariant.

Proof Since

$$\int_{P(f)} f = \int_{\tau^{-1}(P(f))} f$$

$$= \int_{\tau^{-1}(P(f))\cap N(f)} f + \int_{\tau^{-1}(P(f))\cap P(f)} f + \int_{\tau^{-1}(p(f))\cap Z(f)} f$$

$$= \int_{\tau^{-1}(P(f))\cap N(f)} f + \int_{\tau^{-1}(P(f))\cap P(f)} f$$

$$\leq \int_{\tau^{-1}(P(f))\cap P(f)} f \leq \int_{P(f)} f.$$

Where the equal sign holds only if

$$\lambda(\tau^{-1}(P(f))\cap P(f)) = \lambda(P(f))$$

and

$$\lambda(\tau^{-1}(P(f))\cap N(f)) = 0.$$

That is

$$\tau^{-1}(P(f)) \supset P(f) \text{ a.e.}$$

and

$$\tau^{-1}(P(f))\cap N(f) = \phi \text{ a.e.}$$

Hence $\tau(P(f)) \subset P(f)$ a.e.

Let $A = \tau(P(f))$ then $\tau(A) \subset A$ a.e. By Lemma 4.13,

$$\int_{\tau^{-1}(P(f))-P(f)} f = 0$$

and

$$\int_{\tau^{-1}(A)-A} f = 0.$$

But

$$P(f) \subset \tau^{-1}(A) \subset \tau^{-1}(P(f)) \text{ a.e.}$$

It follows that

$$\int_{\tau^{-1}(A)-A} f = \int_{P(f)-\tau(P(f))} f + \int_{\tau^{-1}(A)-P(f)} f$$

$$= \int_{P(f)-\tau(P(f))} f = 0.$$

Since $f > 0$ on $P(f) - \tau(P(f))$, hence $m(P(f) - \tau(P(f))) = 0$. Therefore $\tau(P(f)) = P(f)$ a.e. Same as above we have $\tau(N(f)) = N(f)$ a.e.

Q.E.D.

By Lemma 4.15 and Lemma 4.16 we have

<u>Lemma 4.17</u> If f is invariant then $f\chi_{P(f)}$ and $f\chi_{N(f)}$ are invariant.

Definition 4.3 Let f be a function defined on I^n. We call the set, on which the function f is nonzero, the support of f and denote it by spt f.

Lemma 4.18 Let f be invariant and S = spt f. Then S is invariant.

Proof Since spt f = P(f) ∪ N(f). By Lemma 4.16 P(f) and N(f) are invariant. By Lemma 4.12 (3) S is invariant.

Q.E.D.

Lemma 4.19 If f_1 and f_2 are linearly independent functions in F with $\|f_1\| = \|f_2\| = 1$, then there exist f_1^* and f_2^* such that

(1) $f_1^* \geq 0$, $f_2^* \geq 0$, $\|f_1^*\| = \|f_2^*\| = 1$ and $f_1^* \in F$, $f_2^* \in F$;

(2) spt f_1^* and spt f_2^* are disjoint;

(3) for each i = 1, 2, spt f_i^* is an union of disjoint regions contained in spt f_1 ∪ spt f_2.

Proof If for i = 1 or i = 2 we have $m(P(f_i)) > 0$ and $m(N(f_i)) > 0$ then we may let

$$f_1^* = f_i \chi_{P(f_i)} / \|f_i \chi_{P(f_i)}\|,$$

$$f_2^* = -f_i \chi_{N(f_i)} / \|f_i \chi_{N(f_i)}\|.$$

If $f_i \geq 0$ a.e. or $f_i \leq 0$ a.e. for each i, we assume $f_i \geq 0$ a.e. for each i, replacing f_i by $- f_i$ if necessary. In this case, neither $f_1 \geq f_2$ a.e. nor $f_2 \geq f_1$ a.e. is true, otherwise since $\|f_1\| = \|f_2\|$, $f_1 = f_2$ a.e. So $m(P(f_1-f_2)) > 0$ and $m(N(f_1 - f_2)) > 0$. We can take

$$f_1^* = (f_1 - f_2) \chi_{P(f_1 - f_2)} / \|(f_1 - f_2)\chi_{P(f_1 - f_2)}\|,$$

$$f_2^* = -(f_1 - f_2) \chi_{N(f_1 - f_2)} / \|(f_1 - f_2)\chi_{N(f_1 - f_2)}\|.$$

Since F is a subspace of L , by Lemma 4.15 f_1^* and f_2^* are invariant, i.e. $f_1^* \in F$, $f_2^* \in F$.

Q.E.D.

Let τ: $I^n \longrightarrow I^n$ be a Jablonski transformation. Without loss of generality we shall assume that there exist

$$0 = a_{i,0} < a_{i,1} < \dots < a_{i\, r_i} = 1\ ,\ i = 1,2,\dots,n,$$

for some positive integers r_1, r_2 ,..., r_n such that the partition $\mathcal{P}$ is

$$D_{s_1,\dots,s_n} = \prod_{i=1}^{n} D_{s_i},$$

where $D_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$, $s_i = 1,2,\dots,r_i-1$ and $D_{r_i} = [a_{i,r_i-1}, a_{i,r_i}]$ and τ is given by the formula

$$\tau(x) = (\varphi_{1,s_1,\dots,s_n}(x_1),\dots,\varphi_{n,s_1,\dots,s_n}(x_n))\ ,\ x \in D_{s_1,\dots,s_n},$$

where $\varphi_{i,s_1,\dots,s_n}: \bar{D}_{s_i} \longrightarrow [0,1]$ are C^2 functions.

Definition 4.4 We say that the partition $\mathcal{P}$ has the communication property under the transformation $\tau: I^n \longrightarrow I^n$ if for any parts $D'_{s_1,\dots,s_n}$ and $D''_{s_1,\dots,s_n}$ of $\mathcal{P}$ there exist integers u and v such that $D'_{s_1,\dots,s_n} \subset \tau^u(D''_{s_1,\dots,s_n})$ and $D''_{s_1,\dots,s_n} \subset \tau^v(D'_{s_1,\dots,s_n})$.

Definition 4.5 A Jablonski transformation $\tau: I^n \longrightarrow I^n$ is in class $\mathcal{C}$ if it satisfies following conditions for the fixed partition $\mathcal{P}$:

(1) $\inf |\varphi'_i| > 0$ and $|(\varphi_i^w)'| > 1$ for some integer w.

(2) τ is piecewise C^2;

(3) the partition $\mathcal{P}$ has the communication property under τ.

We associate with each $D_{s_1,\dots,s_n}$ a symbol such as α, β, γ,... and code the orbit by a sequence $\langle x\rangle = .\alpha\,\beta\,\gamma \dots$ if $x \in D(\alpha)$, $\tau(x) \in D(\beta)$, $\tau^2(x) \in D(\gamma)$, ..., where $D(\alpha)$ is some $D_{s_1,\dots,s_n}$ whose symbol is α.

Lemma 4.20 Let $\tau: I^n \longrightarrow I^n$ be a C^2 Jablonski transformation and satisfy condition (1) defining class $\mathcal{C}$. Then $\langle x\rangle = \langle y\rangle$ implies $x = y$.

Proof We write

$$\tau(x) = (\tau_1(x_1),\dots,\tau_n(x_n)).$$

Assume $x \neq y$ but $\langle x\rangle = \langle y\rangle$, then $x_i \neq y_i$ for some $1 \le i \le n$. By condition (1) there exists an integer w such that

$$\left|\frac{d\,\tau_i^w}{dx_i}\right| \geq d > 1.$$

Now $\langle x\rangle = \langle y\rangle$ implies that $\tau^{\ell w+j}(x)$ and $\tau^{\ell w+j}(y)$ belong to the same $D_{s_1,\ldots,s_n}$ for each ℓ and j, $0 \leq j \leq w-1$, but (1) implies that if $x_i \neq y_i$ then

$$|\tau_i^{\ell w+j}(x_i) - \tau_i^{\ell w+j}(y_i)| \geq d^\ell s^j |x_i - y_i| \longrightarrow \infty \text{ as } \ell \longrightarrow \infty,$$

where $s = \inf |\tau_i'| > 0$. This is a contradiction.

Q.E.D.

Lemma 4.21 Let τ be same as Lemma 4.20. If $\sigma = .\alpha_1\alpha_2\ldots$ is a sequence with the property that $\tau(D(\alpha_k)) \supset D(\alpha_{k+1})$, $k = 1,2,\ldots$, then there exists a unique $x \in I^n$ such that $\langle x\rangle = \sigma$.

Proof Let $D(\alpha_k) = \prod_{i=1}^{n} I_i(\alpha_k)$ and $\tau(x) = (\tau_1(x_1),\ldots, \tau_n(x_n))$. We have $\tau_i(I_i(\alpha_k)) \supset I_i(\alpha_{k+1})$. By Lemma 2 of [13], there exists a unique $x \in I^n$ such that $\langle x\rangle = \sigma$.

Q.E.D.

Lemma 4.22 Let τ be same as Lemma 4.20 and $\xi \subset P$ be a collection regions satisfying the communication property: for any D_1, $D_2 \in \xi$ there exist integers u and v such that $D_1 \subset \tau^u(D_2)$ and $D_2 \subset \tau^v(D_1)$. Assume that ξ contains at least two D_i^s and $V = \bigcup_{D\in\xi} D$. Then there exists an $x \in V$ such that $\left\{\tau^\ell(x)\right\}_{\ell=1}^{\infty}$ is dense in V.

Proof Consider the set of all possible finite sequences $.\alpha_1\,\alpha_2\ldots\alpha_k$, where $D(\alpha_j) \in \xi$, $j = 1,2,\ldots,k$ and $\tau(D(\alpha_j)) \supset D(\alpha_{j+1})$, $1 \leq j \leq k-1$, $k = 1,2,\ldots$. This set is countable. Let $S_1, S_2, S_3,\ldots$ be an enumeration, and form the sequence

$$\langle x\rangle = .S_1\, T_1\, S_2\, T_2\, S_3\, T_3\, \ldots,$$

where the T_i are finite sequences joining the last symbol of S_i to the first symbol of S_{i+1}. This can be done because of the assumption of the communication property. Thus, by Lemma 4.21, a point x exists corresponding to the coding <x>.

Now, given $y \in V$ and $\varepsilon > 0$, we claim there exists ℓ such that

$$\left(\sum_{i=1}^{n} (\tau_i^{\ell w}(x_i) - y_i)^2\right)^{1/2} < \varepsilon.$$

To see this, note that for any m, the symbol S corresponding to $y \in D(\alpha)$, $\tau(y) \in D(\beta), \ldots, \tau^m(y) \in D(\gamma)$ occurs in the coding of x. This implies that for some ℓ $\tau^{\ell w+i}(x)$ and $\tau^i(y)$ belong to the same D, $i = 0, 1, \ldots, mw$. Now

$$\left(\sum_{i=1}^{n} \left(\tau_i^{\ell w}(x_i) - y_i\right)^2\right)^{1/2} \le \frac{1}{d}\left(\sum_{i=1}^{n}\left(\tau_i^{(\ell+1)w}(x_i) - \tau_i^{w}(y_i)\right)^2\right)^{1/2}$$

$$\le \frac{1}{d^m}\left(\sum_{i=1}^{n}\left(\tau_i^{(\ell+m)w}(x_i) - \tau_i^{mw}(y_i)\right)^2\right)^{1/2} \le \frac{2\sqrt{n}\,M}{d^m} < \varepsilon$$

for m sufficiently large, where $M = \sup_{x \in I^n, 1 \le i \le n} |\tau_i(x_i)|$. Thus the orbit of x is dense in V.

Q.E.D.

By Lemma 4.22 we have

<u>Lemma</u> <u>4.23</u> If τ is the same as in Lemma 4.20 and satisfies condition (3) defining class $\mathcal{C}$ then there exists a dense orbit in all of I^n.

<u>Theorem</u> <u>4.3</u> If $\tau \in \mathcal{C}$, then there exists exactly one absolutely continuous measure invariant under τ.

<u>Proof</u> Assume there exist two such measure with densities f_1 and f_2. By Lemma 4.19, there are two invariant densities $f_1^* \ge 0$, $f_2^* \ge 0$, $\|f_1^*\| = \|f_2^*\| = 1$ such that $S_1 = \text{spt } f_1^*$ and $S_2 = \text{spt } f_2^*$ are disjoint and S_i is an union of disjoint regions, $i = 1,2$. From [25] we know that each S_i has interior.

Now let $x \in I^n$ be a point which has a dense orbit in I^n. By Lemma 4.23 such point exists. The denseness of the orbit $\left\{\tau^{\ell}(x)\right\}_{\ell=1}^{\infty}$ implies there exist points $u = \tau^{\ell_1}(x)$ and $v = \tau^{\ell_2}(u)$ such that $u \in \text{int } S_1$ and $v \in \text{int } S_2$, where int denotes interior. By the piecewise continuity of τ there exists an open ball O_1 centered at u and in S_1 such that for $u \in O_1$, $v = \tau^{\ell_2}(u) \in \text{int } S_2$. But by Lemma 4.18, S_1 and S_2 are invariant sets, i.e. $\tau(S_i) = S_i$ a.e. $i = 1,2$. It is a contradiction. Hence there exists only one absolutely continuous invariant measure under τ.

Q.E.D.

__Theorem__ __4.4__ Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation with respect to the partition $\mathcal{P} = \left\{D_{s_1,\ldots,s_n}\right\}_{s_i=1,2,\ldots,r_i,\, i=1,2,\ldots,n}$ given by the formula

$$\tau(x) = (\varphi_{1,s_1,\ldots,s_n}(x_1),\ldots,\varphi_{n,s_1,\ldots,s_n}(x_n)), \quad x \in D_{s_1,\ldots,s_n}$$

such that

(1) for any $s_1,\ldots,s_n$ and i $\varphi_{i,s_1,\ldots,s_n}(x_i) \in C^2$ and

$$|\varphi'_{i,s_1,\ldots,s_n}(x_i)| \geq s > 1;$$

(2) the partition $\mathcal{P}$ has the communication property under τ.

Then P_τ has a fixed point f with $\|f\| = 1$ and it is unique.

__Proof__ τ satisfies all the conditions of Theorem 4.3.

Q.E.D.

4.3 Examples

Example 4.1 If for any part $D_{s_1,\dots,s_n}$ of P $\varphi_{i,s_1,\dots,s_n}$ is a C^2 bijective of closed interval $\bar{D}_{s_i}$ onto [0,1], the restriction $\tau_{s_1,\dots,s_n}$ of τ on $D_{s_1,\dots,s_n}$ is a C^2 bijective transformation of $\bar{D}_{s_1,\dots,s_n}$ onto I^n and

$$s = \inf |\varphi'_{i,s_1,\dots,s_n}| > 1,$$

then $\tau \in \mathscr{E}$ and by Theorem 4.3 the absolutely continuous invariant measure under τ is unique.

If $s > 2$ then by Theorem 4.1 we have a sequence of piecewise constant functions f_ℓ with $\|f_\ell\| = 1$ which converges weakly to the fixed point of P_τ.

If τ is piecewise linear and Markovian, then the fixed point of P_τ is a piecewise constant function.

So it is easy to get an example of τ which absolutely continuous invariant measure is unique and it is a weak limit of piecewise constant functions.

Example 4.2 We present an example where $\tau(\bar{D}_{s_1,\dots,s_n}) \neq I^n$ for some $D_{s_1,\dots,s_n}$. Let for n = 2 and

$$I_1 = J_1 = [0, \tfrac{1}{4}),\ I_2 = J_2 = [\tfrac{1}{4}, \tfrac{1}{2}),$$

$$I_3 = J_3 = [\tfrac{1}{2}, \tfrac{3}{4}),\ I_4 = J_4 = [\tfrac{3}{4}, 1]$$

and $D_{kj} = I_k \times J_j$, $k,j = 1,2,3,4$ (Figure 4.1).

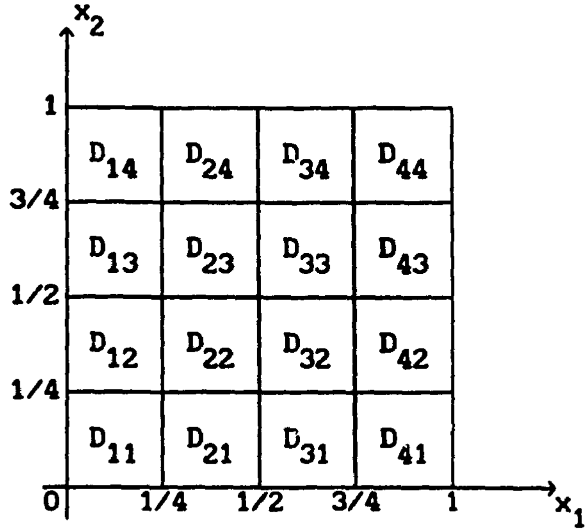

Figure 4.1

$$f_1(x) = 2.4(x^2+x); \quad f_2(x) = f_1(x - \tfrac{1}{4});$$

$$f_3(x) = f_1(x - \tfrac{1}{2}); \quad f_4(x) = f_1(x - \tfrac{3}{4});$$

$$g_i(y) = f_i(y), \qquad i = 1,2,3,4;$$

$$f(x) = 4x; \quad g(y) = 4y$$

and

$$\tau(x,y) = \begin{cases} (f_k(x), g_j(y)) & (x,y) \in D_{kj},\ D_{kj} \neq D_{11}, \\ (f(x),\ g(y)) & (x,y) \in D_{11}. \end{cases}$$

Since

$$\tau\left(\bar{D}_{kj}\right) = [0, \tfrac{3}{4}] \times [0, \tfrac{3}{4}] \quad (D_{kj} \neq D_{11});$$

$$\tau(\bar{D}_{11}) = I^2.$$

By [1], we know that P_τ has a fixed point, and by Theorem 4.4, it is unique. Also by Theorem 4.1, we have $f_\ell \in \Delta_\ell$ with $\|f_\ell\| = 1$ and $\left\{f_\ell\right\}$ converges weakly to the fixed point of P_τ as $\ell \longrightarrow \infty$.

4.4 Small Stochastic Perturbations

For any positive integer ℓ we consider a family of probability densities $q^{\ell}(x,\cdot)$, $x \in I^n$, with respect to the measure λ. The densities of q^{ℓ} are bounded and measurable as functions of 2n variables. The family of transition densities $p^{\ell}(x,\cdot) = q^{\ell}(\tau(x),\cdot)$, $\ell = 1,2,\ldots$, with respect to λ is call a stochastic perturbation of the mapping τ. It is called small if for any $r > 0$ we have

$$\inf_{x\in I^n} \int_{O(x,r)} q^{\ell}(x,y)\, dy \longrightarrow 1 \quad \text{as } \ell \longrightarrow \infty,$$

where $O(x,r) = \{y \mid d(x,y) < r\}$. Perturbations considered in the sequel are small as they are local, i.e., for $\ell = 1,2,\ldots$, there exists $r_{\ell} > 0$ such that $q^{\ell}(x,y) = 0$ for $d(x,y) > r_{\ell}$, and $r_{\ell} \longrightarrow 0$ as $\ell \longrightarrow \infty$.

The Frobenius-Perron operator $P_{\tau}\colon L^1 \longrightarrow L^1$ is

$$(P_{\tau}f)\ (y) = \sum_{k=1}^{q} \frac{f(\tau_k^{-1}y)}{|J_k(\tau_k^{-1}y)|}\ \chi_{\tau_k(D_k)}\ (y),$$

where J_k is the Jacobian of τ on D_k and

$$\tau(x) = \tau_k(x),\ x \in D_k,\ \bigcup_{k=1}^{p} D_k = I^n,\ D_j \cap D_k = \phi \text{ for } j \neq k.$$

We define operators Q_{ℓ} and P_{ℓ}, $\ell = 1,2,\ldots$, from L^1 to L^1 as follows:

$$(Q_{\ell}f)(y) = \int_{I^n} q^{\ell}\ (x,y)\ f(x)\ dx, \quad y \in I^n;$$

$$(P_{\ell}f)(y) = \int_{I^n} p^{\ell}\ (x,y)\ f(x)\ dx, \quad y \in I^n.$$

We have $P_{\ell} = Q_{\ell}\circ P_{\tau}$. Since

$$[(Q_{\ell}\circ P_{\tau})f](y) = (Q_{\ell}(P_{\tau}f))(y) = \int_{I^n} q^{\ell}(x,y)\ P_{\tau}f(x)dx$$

$$= \sum_{k=1}^{p} \int_{I^n} q^{\ell}(x,y)\ \frac{f(\tau_k^{-1}\ x)}{|J_k(\tau_k^{-1}x)|}\ \chi_{\tau_k(D_k)}\ (x)dx$$

$$= \sum_{k=1}^{p} \int_{\tau_k(D_k)} q^{\ell}(x,y) \frac{f(\tau_k^{-1} x)}{|J_k(\tau_k^{-1} x)|} dx$$

$$= \sum_{k=1}^{p} \int_{D_k} q^{\ell}(\tau_k(z),y) \; f(z)dz \qquad (z = \tau_k^{-1}x, \quad x = \tau_k z)$$

$$= \int_{I^n} q^{\ell}(\tau(z),y) \; f(z)dz = \int_{I^n} p^{\ell}(z,y) \; f(y)dz$$

$$= (P_{\ell}f)(y).$$

The transition density p^{ℓ}, $\ell = 1,2,\dots$, has at least one invariant probability measure μ_{ℓ}, i.e.

$$\mu_{\ell}(A) = \int_{I^n} \left(\int_A p^{\ell}(x,y)dy\right) d\mu_{\ell}(x)$$

for any Borel subset A of I^n. The measure μ_{ℓ} is of the form $\mu_{\ell} = f_{\ell}\lambda$, where $f_{\ell} \in L^1$, $f_{\ell} \geq 0$ and $P_{\ell}f_{\ell} = f_{\ell}$.

Let τ: $I^n \longrightarrow I^n$ be a C^2 Jablonski transformation with respect to a finite partition $\mathcal{P} = \{D_1,\dots,D_p\}$ and $s = \inf_{i,j} \inf_{0 \leq x_i \leq 1} |\varphi'_{ij}| > 2$, i.e.,

$$\tau(x) = (\varphi_{1j}(x_1),\dots,\varphi_{nj}(x_n)), \quad x \in D_j, \quad j = 1,2,\dots,p,$$

and for any $1 \leq i \leq n$, $1 \leq j \leq p$, $\varphi_{ij} \in C^2$ and $\inf |\varphi'_{ij}| > 2$.

Let $\Pi_{\ell} = \left\{D_{\ell,1},\dots,D_{\ell,m(\ell)}\right\}$ be a partition of I^n into rectangles such that

$$\max_{1 \leq j \leq m(\ell)} d(D_{\ell,j}) \longrightarrow 0 \text{ as } \ell \longrightarrow \infty.$$

Let us define

$$q^{\ell}(x,y) = q(\Pi_{\ell})(x,y) = \begin{cases} [\lambda(D_{\ell,j})]^{-1} & \text{for } x,y \in D_{\ell j}; \\ 0 & \text{otherwise} \end{cases}$$

$$p(\Pi_{\ell}) = p^{\ell}; \quad Q(\Pi_{\ell}) = Q_{\ell}; \quad P(\Pi_{\ell}) = P_{\ell}$$

for $\ell = 1,2,\dots$.

Lemma 4.24 For any positive integer ℓ and for any $f \in L^1$ we have

$$\overset{I^n}{V} (Q(\Pi_\ell)f) \le \overset{I^n}{V} f.$$

Lemma 4.25 For any $f \in L^1$ we have $Q(\Pi_\ell)f \longrightarrow f$ as $\ell \longrightarrow \infty$ in the L^1 norm.

Lemma 4.26 For any $f \in L^1$, $\ell = 1,2,\ldots$ and measurable subset A of I^n

$$\int_{I^n} \chi_A \, Q(\Pi_\ell)f \, dx = \int_{I^n} f \, Q(\Pi_\ell)\chi_A \, dx.$$

Lemma 4.27 If $f_\ell \in L^1$, $\ell = 1,2,\ldots,$ with $f_\ell \ge 0$, $\|f_\ell\| \le M$ for some constant M and its a subsequence f_{ℓ_j} converges weakly to f as $j \longrightarrow \infty$, then $Q(\Pi_{\ell_j})f_{\ell_j}$ converges weakly to f as $j \longrightarrow \infty$.

The proofs of Lemma 4.24, Lemma 4.25, Lemma 26 and Lemma 27 are analogous to the proofs of Lemma 4.7, Lemma 4.2, Lemma 4.9 and Lemma 4.10.

Theorem 4.5 Let τ: $I^n \longrightarrow I^n$ be a nonsingular C^2 Jablonski transformation with respect to a finite partition $\mathcal{P} = \{D_1,\ldots,D_p\}$ and

$$s = \inf_{i,j} \left\{ \inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}| \right\} > 2.$$

If, for any positive integer ℓ, $f_\ell \in L^1$, $f_\ell \ge 0$, $\|f_\ell\|_1 = 1$ is a fixed point of P_ℓ, then the set $\{f_\ell: \ell = 1,2,\ldots\}$ is weakly compact in L^1 and its weak limit points are fixed points of P_τ.

Proof Since $P(\Pi_\ell) = P_\ell = Q_\ell \circ P_\tau = Q(\Pi_\ell) \circ P_\tau$, Lemma 4.6 and Lemma 4.24 imply that

$$\overset{I^n}{V} f_\ell = \overset{I^n}{V} P(\Pi_\ell) f_\ell = \overset{I^n}{V} Q(\Pi_\ell) (P_\tau f_\ell)$$

$$\le \overset{I^n}{V} P_\tau f_\ell \le K_\tau \|f_\ell\| + \alpha \overset{I^n}{V} f_\ell = K_\tau + \alpha \overset{I^n}{V} f_\ell.$$

Hence

$$\overset{I^n}{V} f_\ell \le \frac{K_\tau}{1 - \alpha}$$

and the set $\left\{f_\ell : \ell = 1,2,\dots\right\}$ is weakly compact in L^1.

Let $\left\{f_{\ell_k}\right\}_{k=1}^{\infty}$ be a subsequence of $\left\{f_\ell\right\}$ which converges weakly to a function f in L^1 as $k \to \infty$. Then for any continuous function g

$$|\int_{I^n} g\,(f - P_\tau f)dx|$$

$$\le |\int_{I^n} g(f - f_{\ell_k})dx| + |\int_{I^n} g(f_{\ell_k} - Q(\Pi_{\ell_k})\, P_\tau f_{\ell_k})dx|$$

$$+ |\int_{I^n} g(Q(\Pi_{\ell_k})\, P_\tau f_{\ell_k} - P_\tau f)dx|.$$

The first term approaches 0 as $k \longrightarrow \infty$ since f_{ℓ_k} converges to f weakly. Since $(Q(\Pi_{\ell_k})\, P_\tau f_{\ell_k} = P(\Pi_{\ell_k})f_{\ell_k} = P_{\ell_k} f_{\ell_k} = f_{\ell_k}$ the second term is identically 0. By Lemma 4.11 $P_\tau f_{\ell_k}$ converges to $P_\tau f$ weakly as $k \to \infty$. The last term approches 0 by Lemma 4.27.

We have established that for any continuous function g,

$$\int_{I^n} g(x)\ (f(x) - P_\tau f(x))\ dx = 0.$$

This means $P_\tau f(x) = f(x)$ a.e., i.e. f(x) is a fixed point of P_τ.

Q.E.D.

<u>Theorem</u> <u>4.6</u> Suppose P_τ has a unique fixed point $f^*(x)$. Then $f_\ell(x)$ converges to $f^*(x)$ weakly in L^1.

<u>Proof</u> Let f(x) be a weak limit point of $\{f_\ell\}$. By Theorem 4.5, f(x) is a fixed point of P_τ. But P_τ has only one fixed point $f^*(x)$. So $f(x) = f^*(x)$ a.e. We have, therefore, shown that any weakly convergent subsequence of $\{f_\ell\}$ converges weakly to f^*. Hence $f_\ell \xrightarrow{w} f^*$ as $\ell \longrightarrow \infty$.

Q.E.D.

In fact Theorem 4.1 and Corollary 4.1 are the special cases of Theorem 4.6 and Theorem 4.5.

CHAPTER 5

Compactness of Invariant Densities for Families of Piecewise Expanding Jablonski Transformations

In 1988 Gora and Boyarsky [15] proved a compactness result for the invariant densities for families of expanding, piecewise monotone transformations on an interval.

In this chapter we will prove an analogous result for invariant densities for families of piecewise expanding higher dimensional Jablonski transformations.

5.1 Compactness

Let $\tau: I^n \longrightarrow I^n$ be a Jablonski transformation. As in Chapter 4, we assume that for any $i = 1,2,\dots,n$, there exists a partition J_i of I:

$$0 = a_{i,0} < a_{i,1} < \dots < a_{i,r_i} = 1$$

for some integers $r_1,\dots,r_n$ such that the partition $\mathcal{P}$ of I^n is $\left\{D_{s_1,\dots,s_n}\right\}$ and

$$D_{s_1,\dots,s_n} = \prod_{i=1}^{n} D_{s_i},$$

where $D_{s_i} = \left[a_{i,s_i-1},\ a_{i,s_i}\right)$, $s_i = 1,2,\dots,r_i$, $i = 1,2,\dots,n$ and τ is given by the formula

$$\tau(x) = (\varphi_{1,s_1,\dots,s_n}(x_1),\dots,\ \varphi_{n,s_1,\dots,s_n}(x_n)), \quad x \in D_{s_1,\dots,s_n},$$

where for any $1 \le i \le n$, $s_i = 1,\dots,r_i$, $\varphi_{i,s_1,\dots,s_n}(x_i)$: $\left[a_{i,s_i-1}, a_{i,s_i}\right] \longrightarrow [0,1]$.

Let $\varphi_{i,s_1,\dots,s_n}(x_i)$ on (a_{i,s_i-1}, a_{i,s_i}) be a C^1-function and the limits

$\varphi'_{i,s_1,\dots,s_n}\left(a^+_{i,s_i-1}\right)$ and $\varphi'_{i,s_1,\dots,s_n}\left(a^-_{i,s_i}\right)$ exist.

A Jablonski transformation $\tau: I^n \longrightarrow I^n$ is called piecewise expanding if

(1) $$\inf_{i,s_1,\ldots,s_n} \left\{ \inf_{(a_{i,s_i-1},a_{i,s_i})} \left| \varphi'_{i,s_1,\ldots,s_n} \right| \right\} = s > 1.$$

(2) for any $s_1,\ldots,s_n$ and i, $\left|\dfrac{1}{\varphi'_{i,s_1,\ldots,s_n}}\right|$ is a function of bounded variation on $\left[a_{i,s_i-1},\ a_{i,s_i}\right]$.

Theorem 5.1 Let $\{\tau_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of piecewise expanding Jablonski transformations with respect to the partitions $\{\mathcal{P}_\alpha\}_{\alpha\in\mathcal{A}}$ and satisfying the following conditions:

(1) There exists a constant $s > 1$ such that

$$\left|\varphi'_{\alpha,i,s_1,\ldots,s_n}\right| \geq s$$

whenever the derivative exists for any $\alpha \in \mathcal{A}$ and $i = 1,2,\ldots,n$, $s_i = 1,2,\ldots,r_i$.

(2) There exists a constant $W > 0$ such that for any $\alpha \in \mathcal{A}$, $i = 1,2,\ldots,n$, and fixed $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$,

$$\bigvee_0^1 \left|\frac{1}{\varphi'_{\alpha,i}(x_i)}\right| \leq W.$$

(3) There exists a constant $\delta > 0$ such that for any $\alpha \in \mathcal{A}$ there exists a finite partition $K_\alpha = \left\{D_{\alpha,s_1,\ldots,s_n}\right\}_{s_i=1,2,\ldots,r_i,\ i=1,2,\ldots,n}$,

where

$$D_{\alpha,s_1,\ldots,s_n} = \prod_{i=1}^{n} \left[a_{\alpha,i,s_i-1},\ a_{\alpha,i,s_i}\right)$$

such that τ_α is a Jablonski transformation on K_α and for any $D_{\alpha,s_1,\ldots,s_n}$, τ_α on $D_{\alpha,s_1,\ldots,s_n}$ is an injection, $\tau_\alpha(D_{\alpha,s_1,\ldots,s_n})$ is a rectangle and

$$L(D_{\alpha,s_1,\dots,s_n}) = \min_{1\le i\le n} (a_{\alpha,i,s_i} - a_{\alpha,i,s_i-1}) > \delta.$$

(4) For any $\ell \ge 1$ there exists $\delta_\ell > 0$ such that if

$$K_\alpha^{(\ell)} = \bigvee_{j=0}^{\ell-1} \tau_\alpha^{-j} (K_\alpha),$$

then

$$\min_\alpha \min_{D\in K_\alpha^{(\ell)}} L(D) \ge \delta_\ell > 0.$$

Then, for any density f of bounded variation, there exists a constant V such that any $\alpha\in\mathcal{A}$ and any $k = 1,2,\dots$

$$\bigvee P_{\tau_\alpha}^k f \le V.$$

This implies that any τ_α, $\alpha \in \mathcal{A}$, admits an invariant density f_α with $\bigvee f_\alpha < V$ and by the Lemma 1.3 the set $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is weakly precompact in L^1.

<u>Lemma 5.1</u> If the family $\{\tau_\alpha\}_{\alpha\in\mathcal{A}}$ satisfies the conditions of Theorem 5.1, then for any $\ell = 1,2,\dots,$ the family $\{\tau_\alpha^\ell\}_{\alpha\in\mathcal{A}}$ satisfies analogous conditions. Moreover, for the new partitions

$$K_\alpha^{(\ell)} = \bigvee_{j=0}^{\ell-1} \tau_\alpha^{-j}(K_\alpha),$$

we have

$$\max_{D\in K_\alpha^{(\ell)}} \bigvee_D \left| \frac{1}{(\varphi_{\alpha,i}^\ell)'} \right| \le \ell \left(\frac{1}{s}\right)^{\ell-1} W.$$

<u>Proof</u> It is obvious that conditions (1) and (2) will be satisfied for different constants. The condition (3) is satisfied by assumption (4). The proof of the inequality is the same as the proof of Lemma 1 in [15].

Q.E.D.

<u>Lemma 5.2</u> Let τ be a transformation satisfying the assumptions of

Theorem 5.1. Let

$$\eta = \max_{D\in K} \bigvee_D \left|\frac{1}{\varphi_i'}\right|.$$

Then for any density f of bounded variation

$$VP_\tau f \le \frac{1}{s^{n-1}}\left(\frac{3}{s} + \eta\right) Vf + \frac{2+\eta s}{\delta s^n} \|f\|_1.$$

Proof For any $D \in K$, $D = \prod_{i=1}^{n} [a_i, b_i]$ we define

$$(\tau|_D)^{-1} = (\phi_{D,1}(x_1), \dots, \phi_{D,n}(x_n))$$

and $A_D = \tau(D)$. Let f be a density of bounded variation. We will estimate the variation of $P_\tau f$.

For any $1 \le i \le n$ and fixed $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, let

$$0 = t_{i,0} < t_{i,1} < \dots < t_{i,r_i} = 1.$$

Then, we have:

$$\sum_{j=1}^{r_i} |P_\tau f(x_1, \dots, x_{i-1}, t_{i,j}, x_{i+1}, \dots, x_n) - P_\tau f(\dots t_{i,j-1} \dots)|$$

$$= \sum_{j=1}^{r_i} \left| \sum_{D\in K} f(\dots, \phi_{D,i}(t_j), \dots)\ |\phi_{D,1}' \dots \phi_{D,i}'(t_j) \dots \phi_{D,n}'|\ \chi_{A_D}(\dots t_j \dots)\right.$$

$$\left. - \sum_{D\in K} f(\dots, \phi_{D,i}(t_{j-1}), \dots) |\phi_{D,1}' \dots \phi_{D,i}'(t_{j-1}) \dots \phi_{D,n}'| \chi_{A_D}(\dots t_{j-1} \dots)\right|$$

$$\le \frac{1}{s^{n-1}} \sum_{j=1}^{r_i}{}' \sum_{D\in K}{}' |f(\dots \phi_{D_i}(t_j) \dots)|\phi_{D,i}'(t_j)| - f(\dots \phi_{D_i}(t_{j-1}) \dots)|\phi_{D_i}'(t_{j-1})||$$

$$+ \frac{1}{s^{n-1}} \sum_{j=1}^{r_i}{}'' \sum_{D\in K}{}'' |f(\dots \phi_{D_i}(t_j) \dots) \phi_{D_i}'(t_j)|$$

$$+ \frac{1}{s^{n-1}} \sum_{j=1}^{r_i}{}''' \sum_{D\in K}{}''' |f(\dots \phi_{D_i}(t_{j-1}) \dots) \phi_{D_i}'(t_{j-1})|,$$

where Σ' is taken over $1 \le j \le r_1$ and $D \in K$ such that $(\dots t_j \dots)$, $(\dots t_{j-1} \dots) \in A_D$;

Σ'' is taken over $1 \le j \le r_1$ and $D \in K$ such that $(\dots t_j \dots) \in A_D$, $(\dots t_{j-1} \dots) \notin A_D$;

Σ''' is taken over $1 \le j \le r_1$ and $D \in K$ such that $(\dots t_j \dots) \notin A_D$, $(\dots t_j \dots) \in A_D$.

Since τ is a Jablonski transformation with partition K, the first sum can be estimated by

$$\frac{1}{s^{n-1}} \sum_{j=1}^{r_1}{}' \sum_{D \in K}{}' \left| \left[f(\dots \phi_{D_1}(t_j) \dots) - f(\dots \phi_{D_1}(t_{j-1}) \dots) \right] \phi'_{D_1}(t_j) \right| +$$

$$+ \frac{1}{s^{n-1}} \sum_{j=1}^{r_1}{}' \sum_{D \in K}{}' |f(\dots \phi_{D_1}(t_{j-1}) \dots)| \, |\phi'_{D_1}(t_j) - \phi'_{D_1}(t_{j-1})|$$

$$\le \frac{1}{s^n} \bigvee_1 f + \frac{1}{s^{n-1}} \left(\bigvee_1 f + \frac{1}{\delta} \int_0^1 |f| \, dx_1 \right) \eta.$$

We have used the inequalities:

$$|f(\dots \phi_{D_1}(t_{j-1}) \dots)| \le \inf_{a_1 \le x_1 \le b_1} |f| + \bigvee_{1[a_1, b_1]} f$$

$$\le \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} |f| dx_1 + \bigvee_{1,[a_1, b_1]} f$$

and

$$\sum_{j=1}^{r_1}{}' |\phi'_{D_1}(t_j) - \phi'_{D_1}(t_{j-1})| \le \bigvee_1 |\phi'_{D_1}| \le \eta.$$

Let $j(D)$ be the smallest j such that $(\dots, t_j, \dots) \in D$ and $j'(D)$ be the biggest j such that $(\dots, t_j, \dots) \in D$. The remaining two sums can be estimated by

$$\frac{1}{s^n} \sum_{D \in K} \left(|f(\dots, \phi_{D_1}(t_{j(D)}), \dots)| + |f(\dots, \phi_{D_1}(t_{j'(D)}), \dots)| \right.$$

$$\le \frac{1}{s^n} \left(2 \bigvee_1 f + \frac{2}{\delta} \int_0^1 |f| dx_1 \right).$$

We have used the fact that if $(\ldots,x,\ldots)$ $(\ldots,y,\ldots) \in D$, then

$$|f(\ldots,x,\ldots)|+|f(\ldots,y,\ldots)| \le 2 \underset{i[a_i,b_i]}{V} f + 2\cdot \inf_{a_i \le x_i \le b_i} |f|.$$

Consequently, we have

$$\underset{i}{V} P_\tau f \le \left(\frac{3}{s^n} + \frac{\eta}{s^{n-1}}\right) \underset{i}{V} f + \left(\frac{\eta}{\delta s^{n-1}} + \frac{2}{\delta s^n}\right) \int_0^1 |f| dx_i$$

$$= \frac{1}{s^{n-1}} \left[(\frac{3}{s} + \eta) \underset{i}{V} f + \left(\frac{s\eta+2}{\delta s}\right)\int_0^1 |f| dx_i\right],$$

and

$$\underset{i}{V} P_\tau f = \inf\left\{\int_{I^{n-1}} \underset{i}{V} h \, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n,\ h = P_\tau f \text{ a.e., } \underset{i}{V} h \text{ measurable}\right\}$$

$$\le \inf\left\{\int_{I^{n-1}} \underset{i}{V} P_\tau g \, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n,\ f = g \text{ a.e., } \underset{i}{V} g \text{ measurable}\right\}$$

$$\le \frac{1}{s^{n-1}} \left[\left(\frac{3}{s} + \eta\right) \inf\left\{\int_{I^{n-1}} \underset{i}{V} g \, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n,\ f = g \text{ a.e., } \underset{i}{V} g \text{ measurable}\right\} + \frac{s\eta+2}{\delta s} \int_{I^n} |f| dx\right]$$

$$= \frac{1}{s^{n-1}} \left[\left(\frac{3}{s} + \eta\right) \underset{i}{V} f + \frac{2 + \eta s}{\delta s} \|f\|_1\right].$$

By $Vf = \max_i \underset{i}{V} f$, we have

$$V P_\tau f \le \frac{1}{s^{n-1}} \left[\left(\frac{3}{s} + \eta\right) V f + \frac{2+\eta s}{\delta s} \|f\|_1\right]$$

$$= \frac{1}{s^{n-1}} \left(\frac{3}{s} + \eta\right) Vf + \frac{2+\eta s}{\delta s^n} \|f\|_1.$$

Q. E. D.

<u>Lemma 5.3</u> If the conclusion of Lemma 5.2 is true for the family $\left\{\tau_p^\ell\right\}_{p\ge1}$, where ℓ is a fixed positive integer, then it is true for the family $\{\tau_p\}_{p\ge1}$

itself.

Proof Let $\tau \in \left\{\tau_p\right\}_{p\geq 1}$. It is enough to prove that if

$$V\, P^m_{\tau^\ell} f \leq V_1$$

for any m and for any f of bounded variation, then

$$V\, P^m_{\tau} f \leq V_2$$

for any m and some V_2. Let $m = k\ell + j$, $0 \leq j \leq \ell-1$. We have

$$V\, P^m_\tau f = V\, P^j_\tau P^k_{\tau^\ell} f$$

$$\leq \frac{1}{s^{n-1}} \left(\frac{3}{s} + \eta\right) V \left(P^{j-1}_\tau\, P^k_{\tau^\ell} f\right) + \frac{2+\eta s}{\delta s^n} \|f\|_1$$

$$\leq \frac{1}{s^{j(n-1)}} \left(\frac{3}{s} + \eta\right)^j V_1 + \frac{1}{s^{(j-1)(n-1)}} \left(\frac{3}{s} + \eta\right)^{j-1} \frac{2+\eta s}{\delta s^n} \|f\|_1$$

$$+ \ldots + \frac{2+\eta s}{\delta s^n} \|f\|_1.$$

It is easy to see that we can find an appropriate V_2.

Proof of Theorem 5.1 Let us fix a positive integer ℓ such that

$$\frac{1}{s^{n-1}} \left(\frac{3}{s^\ell} + \ell\, \frac{1}{s^{\ell-1}} W\right) < 1.$$

The family $\left\{\tau^\ell_\alpha\right\}_{\alpha\in\mathcal{A}}$ satisfies the conditions of Theorem 5.1 with

$$\bar{s} \leq \frac{1}{s^\ell}, \quad \bar{\eta} \leq \ell\, \frac{1}{s^{\ell-1}} W, \quad \text{and } \bar{\delta} \geq \delta_\ell$$

(Lemma 5.1). By Lemma 5.2, we have that for any $\alpha \in \mathcal{A}$ and any density of bounded variation:

$$V\, P_{\tau^\ell_\alpha} f \leq r\, V f + D,$$

where

$$r = \frac{1}{s^{n-1}} \left(\frac{1}{s^\ell} + \ell\, \frac{1}{s^{\ell-1}} W\right) < 1 \text{ and } D = \frac{2+s(\ell s^{-\ell+1} W)}{\delta_\ell}.$$

This implies that there exists a constant V_1 such that:

$$\vee P^k_{\tau^\ell_\alpha} f \le V_1$$

for any $k = 1,2,\ldots,$ and any $\alpha \in \mathcal{A}$, and hence the conclusion of Theorem 5.1 for the family $\left\{\tau^\ell_\alpha\right\}_{\alpha\in\mathcal{A}}$. Applying Lemma 5.3 completes the proof.

Q.E.D.

For any $i = 1,2,\ldots,n$, let $Q_i = \{a_{i,0}, a_{i,1},\ldots, a_{i,r_i}\}$. Let $\mathcal{P} = \{J_1 \times J_2 \times \ldots \times J_n\} = \{D_{s_1,\ldots,s_n}\}_{s_i=1,\ldots,r_i, i=1,2,\ldots,n}$. The piecewise expanding Jablonski transformation τ is called a Markov transformation with respect to the partition $\mathcal{P}$ if for any $i = 1,\ldots,n$, $\varphi_i(x_i)$ transforms the set Q_i of endpoints of intervals of J_i into itself, i.e. $\varphi_i(Q_i) \subset Q_i$.

As in [15], for any $i = 1,..,n$ and fixed $x_1,\ldots,x_{i-1}, x_{i+1},\ldots,x_n$ we have a Markov transformation $\varphi_i^{(k)}(x_i)$ (with respect to $J_i^{(k)}$ $k = 1,2,\ldots$) associated with $\varphi_i(x_i)$. For different $x_1,\ldots,x_{i-1},x_{i-1},\ldots,x_n$, there are only a finite number of different $\varphi_i^{(k)}(x_i)$. So we have a Markov transformation

$$\tau^{(k)}(x_1,\ldots,x_n) = (\varphi_1^{(k)}(x_1),\ldots,\varphi_n^{(k)}(x_n))$$

with respect to $\mathcal{P}^{(k)}=\{J_1^{(k)}\times\ldots\times J_n^{(k)}\}$. $\tau^{(k)}$ converges to τ uniformly on the set $I^n \setminus \bigcup_{i=1}^{n} \bigcup_{k\ge0} Q_i^{(k)}$ as $k \longrightarrow \infty$, where

$$Q_i^{(k)} = \{x_i | x_i \text{ is an endpoint of some interval of } J_i^{(k)}\}.$$

For any $i = 1,2,..,n$ and fixed $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$, $\varphi_i^{(k)\prime} \longrightarrow \varphi_i'$ in L^1 as $k \longrightarrow \infty$.

As in [15], we have:

<u>Theorem 5.2</u> Let τ be a piecewise expanding Jablonski transformation and

$\{\tau^{(k)}\}_{k\geq 1}$ a family of Markov transformations associated with τ. Then any $\tau^{(k)}$, $k = 1,2,\ldots$, admits a invariant density f_k and the set $\{f_k\}_{k\geq 1}$ is weakly compact in L^1.

Lemma 5.4 Let τ_k: $I^n \longrightarrow I^n$. If $\tau_k \longrightarrow \tau$ uniformly as $k \longrightarrow \infty$, $P_{\tau_k} f_k = f_k$ with $\|f_k\|_1 = 1$ and $f_k \longrightarrow f$ weakly as $k \longrightarrow \infty$ in L^1, then $P_\tau f = f$.

Proof It is enough to show that for any $g \in C^1(I^n)$

$$\int_{I^n} g\,(f - P_\tau f)dx = 0.$$

We have

$$\left|\int_{I^n} g(f-P_\tau f)dx\right| \leq \left|\int_{I^n} g(f-f_k)dx\right| + \left|\int_{I^n} g(f_k - P_k f_k)dx\right|$$

$$+ \left|\int_{I^n} g(P_k f_k - P_\tau f_k)dx\right| + \left|\int_{I^n} g(P_\tau f_k - P_\tau f)dx\right|,$$

where $P_k = P_{\tau_k}$. The first summand tends to 0 since $f_k \longrightarrow f$ weakly as $k \longrightarrow \infty$. Because $P_k f_k = f_k$, the second summand is equal to 0. The fourth summand is equal to $\left|\int_{I^n} (g\circ\tau)(f_k-f)dx\right|$ and goes to 0 since $f_k \longrightarrow f$ weakly as $k \longrightarrow \infty$ (if g is continuous then $g\circ\tau$ is bounded). Because $g \in C^1(I^n)$, there is a constant M_g such that for any $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n) \in I^n$ we have

$$|g(x) - g(y)| \leq M_g\, d(x,y),$$

where $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$. Hence the third summand is equal to

$$\left|\int_{I^n} (g\circ\tau_k - g\circ\tau)f_k dx\right| \leq \sup_x |g\circ\tau_k(x) - g\circ\tau(x)| \int_{I^n} |f_k| dx$$

$$\leq M_g \sup_x d(\tau_k(x),\ \tau(x)) \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Therefore

$$\int_{I^n} g(f - P_\tau f)dx = 0.$$

Q.E.D.

Now we will show that any weak limit point of the family of $\tau^{(k)}$-invariant densities $\{f_k\}$ is an invariant density for τ.

Theorem 5.3 Let τ be a piecewise expanding Jablonski transformation and $\{\tau^{(k)}\}_{k\geq 1}$ a family of Markov transformations associated with τ. Then any weak limit point of the family of $\tau^{(k)}$-invariant densities $\{f_k\}$ with $\|f_k\| = 1$ is an invariant density for τ.

Proof By Theorem 5.2, any $\tau^{(k)}$, $k = 1,2,\ldots$, admits an invariant density f_k and the set $\{f_k\}_{k\geq 1}$ is weakly precompact in L^1. By Lemma 5.4, any weak limit point of $\{f_k\}$ is an invariant density for τ.

Q.E.D.

As in [15], for any $k = 1,2,..$ and $i = 1,2,\ldots,n$, we can take $\varphi_i^{(k)}(x_i)$ to be a piecewise linear Markov transformation, i.e. $\tau^{(k)}$ is a piecewise linear (Jablonski) Markov transformation with respect to $\mathcal{P}^{(k)}$. By Theorem 3.2 there exists a piecewise constant function f_k with respect to $\mathcal{P}^{(k)}$ which is an invariant density under $\tau^{(k)}$ and $\|f_k\| = 1$. By Theorem 5.3, any weak limit of $\{f_k\}$ is an invariant density under τ.

5.2 Examples

Example 5.1 Let $n = 2$ and $D_{11} = [0,0.5) \times [0,0.5)$, $D_{12} = [0,0.5) \times [0.5,1]$, $D_{21} = [0,5,1] \times [0,0.5)$, $D_{22} = [0.5,1] \times [0.5,1]$ (Figure 5.1).

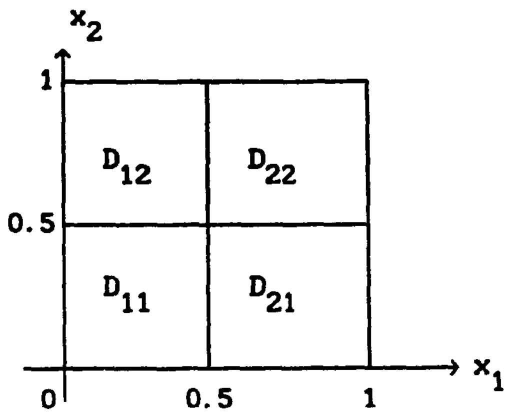

Figure 5.1

For $0 \le x_2 \le 0.5$, define (Figure 5.2)

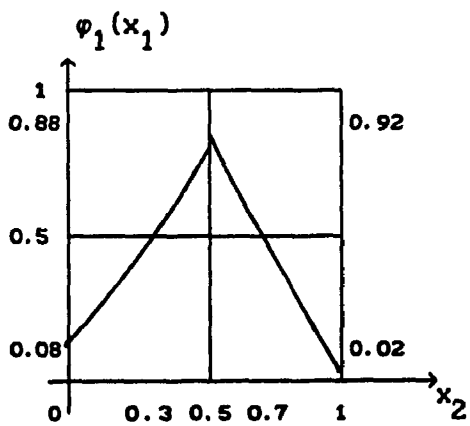

Figure 5.2

$$\varphi_{1,11}(x_1) = x_1^2 + 1.1x_1 + 0.08, \quad 0 \le x_1 < 0.5;$$

$$\varphi_{1,21}(x_1) = x_1^2 - 3.3x_1 + 2.32, \quad 0.5 \le x_1 \le 1.$$

For $0.5 \le x_2 \le 1$, define (Figure 5.3)

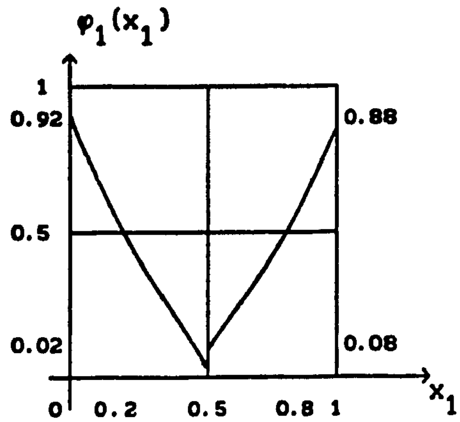

Figure 5.3

$$\varphi_{1,12}(x_1) = x_1^2 - 2.3\, x_1 + 0.92, \quad 0 \le x_1 < 0.5;$$

$$\varphi_{1,22}(x_1) = x_1^2 + 0.1\, x_1 - 0.22, \quad 0.5 \le x_1 \le 1.$$

For $0 \le x \le 0.5$, define (Figure 5.4)

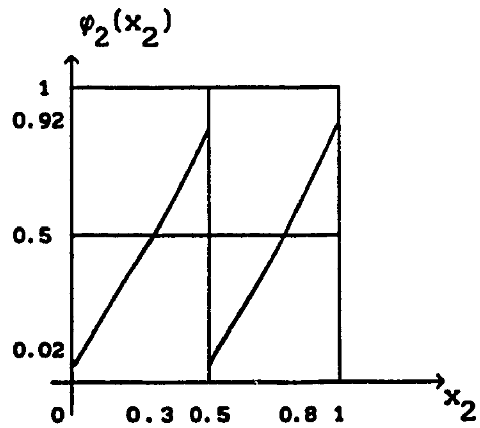

Figure 5.4

$$\varphi_{2,11}(x_2) = x_2^2 + 1.3\, x_2 + 0.02, \quad 0 \le x_2 < 0.5;$$

$$\varphi_{2,12}(x_2) = x_2^2 + 0.3\, x_2 - 0.38, \quad 0.5 \le x_2 \le 1.$$

For $0.5 \le x_1 \le 1$, define (Figure 5.5)

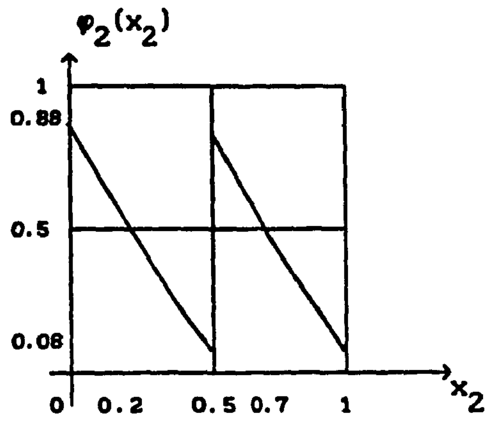

Figure 5.5

$$\varphi_{2,21}(x_2) = x_2^2 - 2.1\ x_2 + 0.88,\ 0 \le x_2 < 0.5;$$

$$\varphi_{2,22}(x_2) = x_2^2 - 3.1\ x_2 + 2.18,\ 0.5 \le x_2 \le 1.$$

Since we have

$$\varphi_{1,11}(0.3) = 0.5, \quad 1.1 \le \varphi'_{1,11}(x_1) \le 2.1;$$

$$\varphi_{1,21}(0.7) = 0.5, \quad -2.3 \le \varphi'_{1,21}(x_1) \le -1.3;$$

$$\varphi_{1,12}(0.2) = 0.5, \quad -2.3 \le \varphi'_{1,12}(x_1) \le -1.3;$$

$$\varphi_{1,22}(0.8) = 0.5, \quad 1.1 \le \varphi'_{1,12}(x_1) \le 2.1;$$

$$\varphi_{2,11}(0.3) = 0.5, \quad 1.3 \le \varphi'_{2,11}(x_2) \le 2.3;$$

$$\varphi_{2,12}(0.8) = 0.5, \quad 1.3 \le \varphi'_{2,12}(x_2) \le 2.3;$$

$$\varphi_{2,21}(0.2) = 0.5, \quad -2.1 \le \varphi'_{2,21}(x_2) \le -1.1;$$

$$\varphi_{2,22}(0.7) = 0.5, \quad -2.1 \le \varphi'_{2,22}(x_2) \le -1.1.$$

So we can take $\tau^{(1)}(x_1, x_2) = (\varphi_1^{(1)}(x_1), \varphi_2^{(1)}(x_2))$ as follows:

For $0 \le x_2 \le 0.5$, define (Figure 5.6)

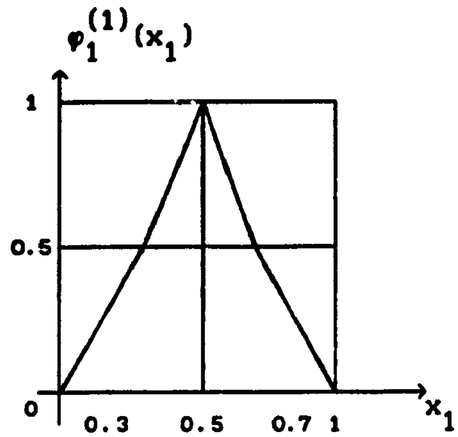

Figure 5.6

$$\varphi_1^{(1)}(x_1) = \begin{cases} \frac{5}{3} x_1, & 0 \le x_1 < 0.3, \\ 2.5\, x_1 - 0.25, & 0.3 \le x_1 < 0.5, \\ -2.5\, x_1 + 2.25, & 0.5 \le x_1 < 0.7, \\ -\frac{5}{3} x_1 + \frac{5}{3}, & 0.7 \le x_1 \le 1. \end{cases}$$

For $0.5 \le x_2 \le 1$, define (Figure 5.7)

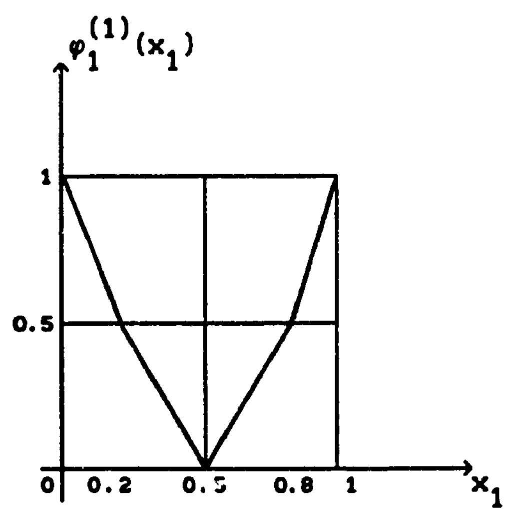

Figure 5.7

$$\varphi_1^{(1)}(x_1) = \begin{cases} -2.5\, x_1 + 1\,, & 0 \le x_1 < 0.2, \\ -\frac{5}{3}\, x_1 + \frac{5}{6}\,, & 0.2 \le x_1 < 0.5, \\ \frac{5}{3}\, x_1 - \frac{5}{6}\,, & 0.5 \le x_1 < 0.8, \\ 2.5\, x_1 - 1.5, & 0.8 \le x_1 < 1. \end{cases}$$

For $0 \le x_1 \le 0.5$, define (Figure 5.8)

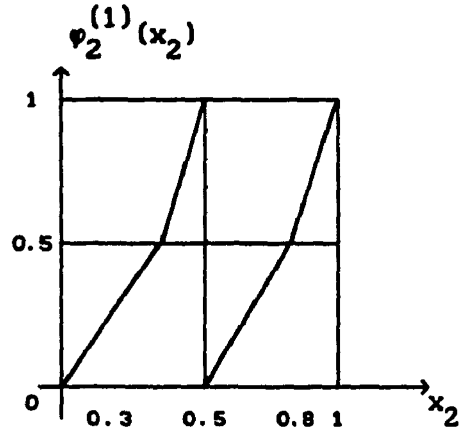

Figure 5.8

$$\varphi_2^{(1)}(x_2) = \begin{cases} \frac{5}{3}\, x_2, & 0 \le x_2 < 0.3, \\ 2.5\, x_2 - 0.25, & 0.3 \le x_2 < 0.5, \\ \frac{5}{3}\, x_2 - \frac{5}{6}\,, & 0.5 \le x_2 < 0.8, \\ 2.5\, x_2 - 1.5, & 0.8 \le x_2 \le 1. \end{cases}$$

For $0.5 \le x_1 \le 1$, define (Figure 5.9)

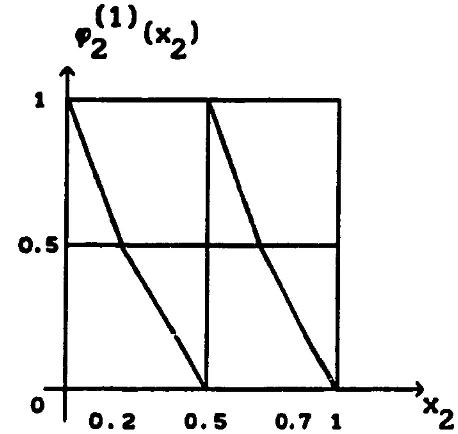

Figure 5.9

$$\varphi_2^{(1)}(x_2) = \begin{cases} -2.5\, x_2 + 1, & 0 \le x_2 < 0.2, \\ -\frac{5}{3}\, x_2 + \frac{5}{6}, & 0.2 \le x_2 < 0.5, \\ -2.5\, x_2 + 2.25, & 0.5 \le x_2 < 0.7, \\ -\frac{5}{3}\, x_2 + \frac{5}{3}, & 0.7 \le x_2 \le 1. \end{cases}$$

$\tau^{(1)}$ is a piecewise linear Markov transformation with respect a partition $\mathcal{P}^{(1)}$ (Figure 5.10). By Theorem 3.2, there exists a piecewise constant function f_1 with respect to $\mathcal{P}^{(1)}$ which is an invariant density under $\tau^{(1)}$ and $\|f_1\| = 1$.

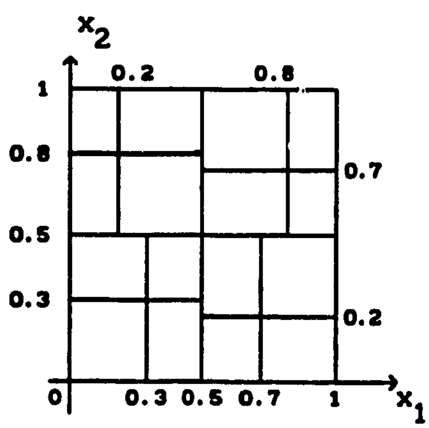

Figure 5.10

Continuing in this way, for any k we can take $\tau^{(k)}$ as a piecewise linear Markov transformation with respect to $\mathcal{P}^{(k)}$ and we have a piecewise constant function f_k with respect to $\mathcal{P}^{(k)}$, which is an invariant density under $\tau^{(k)}$ and $\|f_k\|_1 = 1$. Any weak limit of $\{f_k\}$ is an invariant density under τ.

<u>Example 5.2</u>: Let $n = 2$ and let $0 < b < 0.5$. Define $D_{11} = [0,b) \times [0,b)$, $D_{12} = [0,b) \times [b,1]$, $D_{21} = [b,1] \times [0,b)$, $D_{22} = [b,1] \times [b,1]$ (figure 5.11) and

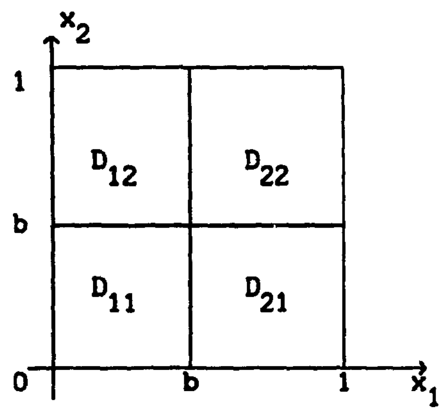

Figure 5.11

$$\varphi_{\alpha,1,11}(x_1) = \frac{1-b}{b-\alpha}x_1 + \frac{b^2-\alpha}{b-\alpha},$$

$$\varphi_{\alpha,2,11}(x_2) = \frac{1-b}{b-\alpha}x_2 + \frac{b^2-\alpha}{b-\alpha},$$

$$\varphi_{\alpha,1,12}(x_1) = \frac{1-b}{b-\alpha}x_1 + \frac{b^2-\alpha}{b-\alpha},$$

$$\varphi_{\alpha,2,12}(x_2) = \frac{x_2-b}{1-b},$$

$$\varphi_{\alpha,1,21}(x_1) = \frac{x_1-b}{1-b},$$

$$\varphi_{\alpha,2,21}(x_2) = \frac{1-b}{b-\alpha}x_2 + \frac{b^2-\alpha}{b-\alpha},$$

$$\varphi_{\alpha,1,22}(x_1) = \frac{x_1 - b}{1 - b},$$

$$\varphi_{\alpha,2,22}(x_2) = \frac{x_2 - b}{1 - b},$$

and

$$\tau_\alpha(x_1,x_2) = (\varphi_{\alpha,1,ij}(x_1), \varphi_{\alpha,2,ij}(x_2)), \ (x_1,x_2) \in D_{ij} \text{ for } \alpha \in \mathcal{A} = (0,b^2).$$

It is easy to see that $\{\tau_\alpha\}_{\alpha\in\mathcal{A}}$ satisfies assumptions (1), (2), (3) of Theorem 5.1. However, it does not satisfy assumption (4). Since

$$\min_{D\in K_\alpha^{(2)}} L(D) = \alpha$$

and therefore tends to 0 as α goes to 0. This example, therefore, shows that our proof of Theorem 5.1 does not work without assumption (4).

CHAPTER 6

Constructive Approximations to Densities Invariant under Non-Expanding Jablonski Transformations

In 1988 Gora, Boyarsky and Proppe [16] contructed some approximations to densities invariant under non-expanding maps on an interval. In this chapter we will obtain similar results for higher dimensional Jablonski transformations.

Let τ and T be two transformations from I^n into I^n. We say that τ is topologically conjugate to T if there exists a homeomorphism h from I^n onto I^n such that $h \circ \tau = T \circ h$ i.e. $\tau = h^{-1} \circ T \circ h$ or $\tau \circ h^{-1} = h^{-1} \circ T$.

<u>Lemma 6.1</u> If $\tau : I^n \longrightarrow I^n$ is topologically conjugate to T with a homeomorphism h from I^n onto I^n and f is an invariant density under T, then $g = f(h(x))\left|\frac{\partial h}{\partial x}\right|$ is an invariant density under τ, where $\frac{\partial h}{\partial x}$ is the Jacobian of h.

<u>Proof</u>

$$\begin{aligned}
\int_{\tau^{-1}A} g(x)dx &= \int_{\tau^{-1}A} f(h(x))\left|\frac{\partial h}{\partial x}\right| dx \\
&= \int_{h(\tau^{-1}A)} f(y)dy \qquad (y = h(x)) \\
&= \int_{T^{-1}(hA)} f(y)dy \qquad (h \circ \tau^{-1} = T^{-1} \circ h) \\
&= \int_{hA} f(y)\,dy \qquad (f \text{ is invariant density for } T) \\
&= \int_{A} f(h(x))\,\left|\frac{\partial h}{\partial x}\right| dx \qquad (y = h(x)) \\
&= \int_{A} g(x)\,dx.
\end{aligned}$$

Q.E.D.

A Jablonski transformation $\tau : I^n \longrightarrow I^n$ is called non-expanding if for any $s_i = 1,\dots,r_i$, $i = 1,\dots,n$. $\varphi_{i,s_1,\dots,s_n}(x_i)$ is monotonic, but the condition $|\varphi'_{i,s_1,\dots,s_n}| \geq \lambda > 1$ is not satisfied.

Lemma 6.2 Let $\tau : I^n \longrightarrow I^n$ be a non-expanding Jablonski transformation topologically conjugate to T which is a piecewise expanding Jablonski transformation and admits a unique absolutely continuous invariant measure μ with density f. Then there exists a sequence of Markov Jablonski transformation $\{\tau^k\}$ such that $\tau^{(k)} \longrightarrow \tau$ uniformly as $k \longrightarrow \infty$, and such that the set of densities $\{g_k\}$ corresponding to $\{\tau^{(k)}\}$ is weakly compact in L^1.

Proof Let $\{T^{(k)}\}$ be a family of Markov transformations associated with T. Define $\tau^{(k)}$ by

$$\tau^{(k)} \circ h^{-1} = h^{-1} \circ T^{(k)}$$

i.e., for any $i = 1,\dots, n$

$$\varphi_i^{(k)} \circ h_i^{-1} = h_i^{-1} \circ T_i^{(k)},$$

where

$$\tau^{(k)}(x) = (\varphi_1^{(k)}(x_1), \dots, \varphi_n^{(k)}(x_n)),$$

$$h(x) = (h_1(x_1), \dots, h_n(x_n))$$

and

$$T^{(k)}(x) = (T_1^{(k)}(x_1), \dots, T_n^{(k)}(x_n)).$$

As in [16], for any $i = 1,\dots,n$. $\varphi_i^{(k)} \longrightarrow \varphi_i$ uniformly as $k \longrightarrow \infty$. So $\tau^{(k)} \longrightarrow \tau$ uniformly as $k \longrightarrow \infty$.

Let f_k be the density corresponding to $\tau^{(k)}$. By Theorem 5.2, $\{f_k\}$ is weakly compact in L^1. By Lemma 6.1,

$$g_k(x) = f_k\ (h(x))\left|\frac{\partial h}{\partial x}\right|$$

and

$$\int_A g_k(x)\ dx = \int_A f_k(h(x))\left|\frac{\partial h}{\partial x}\right|\ dx = \int_{h(A)} f_k(y)\ dy.$$

Since $\{f_k\}$ is weakly compact in L^1, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $\lambda(h(A)) < \delta_1$ implies

$$\int_{h(A)} f_k(y)dy < \varepsilon$$

for all k. But h is a homeomorphism. Thus, given $\delta_1 > 0$, there exists $\delta > 0$, such that $\lambda(A) < \delta$ implies $\lambda(h(A)) < \delta_1$. Hence, we have

$$\int_A g_k(x)\ dx < \varepsilon$$

for all k if $\lambda(A) < \delta$. Thus, $\{g_k\}$ is weakly compact in L^1.

Q.E.D.

<u>Theorem 6.1</u> $g_k \longrightarrow g^*$ weakly, where g^* is the invariant density of τ.

<u>Proof</u> Since $\tau^{(k)} \longrightarrow \tau$ uniformly and

$$\|g_k\|_1 = \int_{I^n} |g_k(x)|\ dx$$

$$= \int_{I^n} |f_k(h(x)|\ \left|\frac{\partial h}{\partial x}\right|dx$$

$$= \int_{h(I^n)} |f_k(y)|dy \qquad (y=h(x))$$

$$= \int_{I^n} |f_k(y)|dy$$

$$= \|f_k\|_1 = 1.$$

Hence Lemma 5.4 yields the desired result.

Q.E.D.

CHAPTER 7

Higher Dimensional Random Maps

A random map is a discrete time dynamical system in which one of a number of transformations is selected randomly and applied. The main result of this chapter provides a sufficient condition for the existence of an absolutely continuous invariant measure for random maps on R^n. Such random maps have application to the fractals that arise from iterated function systems.

7.1 Introduction

Let $\tau_1,\dots,\tau_\ell$ be transformations on the unit cube in R^n and define the random map τ by choosing τ_i with probability p_i, $p_i > 0$, $\sum_{i=1}^{\ell} p_i = 1$. A measure μ is called invariant under τ if $\mu(A) = \sum_{i=1}^{\ell} p_i\mu(\tau_i^{-1}A)$ for each measurable set A. The one-dimensional case was studied in [28,29] for piecewise monotonic transformations. A sufficient condition is given, namely $\sum_{i=1}^{\ell} \frac{p_i}{|\tau_i'(x)|} \le \gamma < 1$ for some constant γ, which guarantees that the random map τ has an invariant measure which is absolutely continuous with respect to Lebesgue measure on [0,1].

In this chapter we extend the foregoing result to higher dimensions. This is not an obvious extension because the bounded variation techniques used in one dimension do not go over easily to higher dimensions [31].

The importance of studying higher dimensional random maps is inspired by fractals which are fixed points of iterated function systems [30], which can be viewed as random maps, where the individual transformations are contractions.

7.2 Random Maps on I^n

Let $\tau_k(x)$, $k = 1,\ldots,\ell$, be a C^2 Jablonski transformation on the rectangular partition $\mathcal{P}_k = \left\{D_{k1},\ldots,D_{kq_k}\right\}$, $k = 1,\ldots,\ell$, and q_k is a positive integer which depends on k, i.e.,

$$\tau_k(x) = (\tau_{k,1}(x),\ldots,\tau_{k,n}(x)),$$

and for any $i = 1,2,\ldots,n$ and $k = 1,2,\ldots,\ell$

$$\tau_{k,i}(x) = \psi_{k,i}(x_i)$$

$$= \varphi_{k,ij}(x_i), \quad x \in D_{kj}.$$

The main result of this chapter is

Theorem 7.1 Let $\tau(x) = \{\tau_k(x)\}_{k=1}^{\ell}$ be a random map with probabilities $p_1,\ldots,p_\ell$, $p_k > 0$, $\sum_{k=1}^{\ell} p_k = 1$, where each τ_k is a Jablonski (not necessarily expanding) transformation of I^n into I^n. Assume $\tau_{k,i}(x) = \varphi_{k,ij}(x_i)$ is C^2 and monotonic for $x \in \bar{D}_{kj}$. If, for $i = 1,2,\ldots,n$,

$$\sum_{k=1}^{\ell} \sup_j \frac{p_k}{|\varphi'_{k,ij}(x_i)|} \le \gamma < 1$$

for some constant γ, then for all $f \in L^1(I^n)$,

1) $\lim_{M\to\infty} \frac{1}{M} \sum_{t=0}^{M-1} P_\tau^t f = f^* \in L^1(I^n)$, where $P_\tau = \sum_{k=1}^{\ell} p_k P_{\tau_k}$;

2) $P_\tau f^* = f^*$;

3) $\bigvee_{I^n} f^* \le C\|f\|_1$ for some constant $C > 0$ which is independent of f.

Proof Our goal is to show that there are constants $0 < \alpha < 1$ and $K > 0$ such that $\bigvee_{I^n} P_\tau^N(f) \le \alpha \bigvee_{I^n} f + K\|f\|_1$ for some positive integer N. This will suffice, as in [1], to show that $P_\tau^* = \lim_{M\to\infty} \frac{1}{M} \sum_{t=0}^{M-1} P_\tau^t$ exists and is a non-zero projection onto the eigenspace E_1 of eigenvectors of P_τ with eigenvalue 1.

Take N such that $\gamma^N < \frac{1}{3}$, and consider the random map τ^N defined by Jablonski transformations: $\left\{ \tau_{j_N} \circ \tau_{j_{N-1}} \circ \dots \circ \tau_{j_1}(x) \right\}$ with probability $\prod_{s=1}^{N} p_{j_s}$.

The maps defining τ^N may be indexed by $\{1, 2, \dots, \ell\}^N$. Set $\tau_{j_N} \circ \dots \circ \tau_{j_1}(x) = \tau_w(x)$, where $w = (j_1, \dots, j_N)$ and $p_w = \prod_{s=1}^{N} p_{j_s}$. Let the partition of τ_w be $\left\{D_{wj}\right\}$ and

$$\tau_w(x) = (\psi_{w1}(x_1), \dots, \psi_{wn}(x_n))$$

$$= (\varphi_{w,1j}(x_1), \dots, \varphi_{w,nj}(x_n)), \ x \in D_{wj}.$$

We then have that, for any i,

$$\sum_{w \in (1,2,\dots,\ell)^N} \frac{p_w}{|\psi'_{wi}(x_i)|} = \sum_{\bar{w} \in (1,2,\dots,\ell)^{N-1}} \sum_{k=1}^{\ell} \frac{p_k \, p_{\bar{w}}}{|\psi'_{ki}(\psi_{\bar{w}i}(x_i))| \, |\psi'_{\bar{w}i}(x_i)|}$$

$$= \sum_{\bar{w}} \frac{p_{\bar{w}}}{|\psi'_{\bar{w}i}(x_i)|} \sum_{k=1}^{\ell} \frac{p_k}{|\psi'_{ki}(\psi_{\bar{w}i}(x_i))|} \le \gamma \sum_{\bar{w}} \frac{p_{\bar{w}}}{|\psi'_{\bar{w}i}(x_i)|}$$

$$\le \dots \le \gamma^N < \frac{1}{3}.$$

Thus, to simplify notation we may assume that $\gamma < \frac{1}{3}$ and show that $\overset{I^n}{V} P_\tau(f) \le \alpha \overset{I^n}{V} f + K \|f\|_1$ for $\alpha \in (0,1)$ and $K > 0$.

Let $\mathcal{P}$ be the partition of I^n into maximal rectangles on which all the maps $\left\{\tau_k\right\}_{k=1}^{\ell}$ are one to one and C^2. Without loss of generality, we assume that there exist

$$0 = a_{i,0} < a_{i,1} < \dots < a_{i,r_i} = 1, \ i = 1, 2, \dots, n,$$

for some positive integers $r_1, r_2, \dots, r_n$ such that

$$\mathcal{P} = \left\{D_{s_1,\ldots,s_n}\right\}_{s_i = 1,2,\ldots,r_i,\ i = 1,2,\ldots,n},$$

where $D_{s_1,\ldots,s_n} = \prod_{i=1}^{n} D_{s_i}$, $D_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$, $s_i = 1,2,\ldots,r_i-1$ and $D_{r_i} = [a_{i,r_i-1}, a_{i,r_i}]$, and for any $k = 1,2,\ldots,\ell$, τ_k is given by the formula

$$\tau_k(x) = (\varphi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,\varphi_{k,n,s_1,\ldots,s_n}(x_n)),\ x \in D_{s_1,\ldots,s_n},$$

where $\varphi_{k,i,s_1,\ldots,s_n}: \bar{D}_{s_i} \longrightarrow [0,1]$ are C^2 functions.

We define

$$\Psi_{k,i,s_1,\ldots,s_n} = (\varphi_{k,i,s_1,\ldots,s_n})^{-1},$$

$$\sigma_{k,i,s_1,\ldots,s_n} = |\Psi'_{k,i,s_1,\ldots,s_n}|,$$

$$I_{k,s_1,\ldots,s_n} = \prod_{i=1}^{n} \varphi_{k,i,s_1,\ldots,s_n}(\bar{D}_{s_i}).$$

Then we have

$$P_\tau f = \sum_{k=1}^{\ell} p_k P_{\tau_k} f$$
$$= \sum_{k=1}^{\ell} p_k \left(\prod_{j=1}^{n} \sum_{s_j=1}^{r_j}\right) f\left(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n)\right)$$
$$\left(\prod_{j=1}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right)\chi_{I_{k,s_1,\ldots,s_n}}(x).$$

Denote by $\mathcal{E}$ the set of functions of the form

$$g = \sum_{j=1}^{M} g_j \chi_{A_j},$$

where χ_{A_j} is the characteristic function of the set $A_j = \prod_{i=1}^{n} [\alpha_{ij}, \beta_{ij}] \subset I^n$ (we do not assume that $\alpha_{ij} < \beta_{ij}$, the interval $[\alpha_{ij}, \beta_{ij}]$ can be degenerate) and $g_j: I^n \longrightarrow R$ is a C^1-function on A_j. By the proof of Theorem 1 in [1], it is enough to show that for any $i = 1,2,\ldots,n$ and $f \in \mathcal{E}$, $f \geq 0$ and $f_i \in \mathcal{E}$,

$f_i = f$ a.e., $\int_{P_i(I^n)} \overset{I^n}{\underset{i}{V}} f_i \, d\lambda_{n-1} = \underset{i}{V} f$, we have

$$\int_{P_i(I^n)} \overset{I^n}{\underset{i}{V}} P_\tau f_i \, d\lambda_{n-1} \le \alpha \int_{P_i(I^n)} \overset{I^n}{\underset{i}{V}} f_i \, d\lambda_{n-1} + K \|f\|_1,$$

where $0 < \alpha < 1$ and $K > 0$.

Now we show that.

$$\overset{I^n}{\underset{i}{V}} P_\tau f_i \le \sum_{k=1}^{\ell} p_k \left(\prod_{j=1}^{n} \sum_{s_j=1}^{r_j}\right) \overset{I_{k,s_1,\ldots,s_n}}{\underset{i}{V}} f_i\left(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n)\right)$$

$$\left(\prod_{j=1}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right)\chi_{I_{k,s_1,\ldots,s_n}}(x)$$

$$+\sum_{k=1}^{\ell} p_k \left(\prod_{j=1}^{n} \sum_{s_j=1}^{r_j}\right)\Big(\left|f_i(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,a_{i,s_i},\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n))\right|$$

$$\sigma_{k,i,s_1,\ldots,s_n}(\varphi_{k,i,s_1,\ldots,s_n}(a_{i,s_i}))$$

$$+\left|f_i(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,a_{i,s_{i-1}},\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n))\right|$$

$$\sigma_{k,i,s_1,\ldots,s_n}(\varphi_{k,i,s_1,\ldots,s_n}(a_{i,s_{i-1}}))\Big)$$

$$\left(\prod_{\substack{j=1\\ j\ne i}}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right)\chi_{I_{k,s_1,\ldots,s_n}}(x_1,\ldots,a_{i,s_i},\ldots,x_n).$$

Hence,

$$\int_{P_i(I^n)} \overset{I^n}{\underset{i}{V}} P_\tau f_i \, d\lambda_{n-1}$$

$$\le \sum_{k=1}^{\ell} p_k \left(\prod_{j=1}^{n} \sum_{s_j=1}^{r_j}\right) \int_{P_i(I_{k,s_1,\ldots,s_n})} \overset{I_{k,s_1,\ldots,s_n}}{\underset{i}{V}} f_i\Big(\Psi_{k,1,s_1,\ldots,s_n}(x_1),$$

$$\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n)\Big) \left(\prod_{j=1}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right) d\lambda_{n-1}$$

$$+ \sum_{k=1}^{\ell} p_k \left(\prod_{j=1}^{n} \sum_{s_j=1}^{r_j}\right) \int_{P_i(I_{k,s_1,\ldots,s_n})} \Big(|f_i(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,a_{i,s_i},\ldots,$$

$$\Psi_{k,n,s_1,\ldots,s_n}(x_n))|\sigma_{k,i,s_1,\ldots,s_n}(\varphi_{k,i,s_1,\ldots,s_n}(a_{i,s_i}))$$

$$+ |f(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,a_{i,s_i-1},\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n))|$$

$$\sigma_{k,i,s_1,\ldots,s_n}(\varphi_{k,i,s_1,\ldots,s_n}(a_{i,s_i-1}))\Big) \left(\prod_{\substack{j=1\\ j\ne i}}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right) d\lambda_{n-1}.$$

Using equation (7) in [28], the first sum is less than

$$2\gamma_i \left(\prod_{\substack{j=1\\ j\ne i}}^{n} \sum_{s_j=1}^{r_j}\right) \int_{P_i(I_{k,s_1,\ldots,s_n})} \overset{P_i(I_{k,s_1,\ldots,s_n})\times I}{\underset{i}{V}} f_i\Big(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,$$

$$x_i,\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n)\Big) \left(\prod_{\substack{j=1\\ j\ne i}}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\right) d\lambda_{n-1}$$

$$+ K_i \left(\prod_{\substack{j=1\\ j\ne i}}^{n} \sum_{s_j=1}^{r_j}\right) \int_{P_i(I_{k,s_1,\ldots,s_n})} \Big(\int_0^1 f_i(\Psi_{k,1,s_1,\ldots,s_n}(x_1),\ldots,x_i,\ldots,$$

$$\Psi_{k,n,s_1,\ldots,s_n}(x_n))dx_i\Big)\Big(\prod_{\substack{j=1\\j\neq i}}^{n} \sigma_{k,j,s_1,\ldots,s_n}(x_j)\Big)d\lambda_{n-1}$$

$$= 2\gamma_i\Big(\prod_{\substack{j=1\\j\neq i}}^{n} \sum_{s_j=1}^{r_j}\Big)\int_{P_i(D_{s_1,\ldots,s_n})} \bigvee_i^{P_i(D_{s_1,\ldots,s_n})\times I} f_i(x_1,\ldots,x_n)d\lambda_{n-1}$$

$$+ K_i\Big(\prod_{\substack{j=1\\j\neq i}}^{n} \sum_{s_j=1}^{r_j}\Big)\int_{P_i(D_{s_1,\ldots,s_n})}\Big(\int_0^1 f_i(x_1,\ldots,x_n)dx_i\Big)d\lambda_{n-1}$$

$$= 2\gamma_i\int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + K_i\int_{P_i(I^n)}\Big(\int_0^1 f_i\, dx_i\Big)d\lambda_{n-1}$$

$$= 2\gamma_i\int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, d\lambda_{n-1} + K_i\,\|f\|_1$$

for some constants $0 < \gamma_i < \frac{1}{3}$ and $K_i > 0$.

Using equation (5) in [28], the second sum is less than

$$\sum_{k=1}^{\ell} p_k\Big(\sup_{s_1,\ldots,s_n}\sigma_{k,i,s_1,\ldots,s_n}\Big)\Big(\prod_{j=1}^{n}\sum_{s_j=1}^{r_j}\Big)\int_{P_i(I_{k,s_1,\ldots,s_n})}\Big(|f_i(\Psi_{k,1,s_1,\ldots,s_n}(x_1),$$

$$\ldots,a_{i,s_i},\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n))|$$

$$+|f_i(\Psi_{k,1,s_1}\ldots,_{s_n}(x_1),\ldots,a_{i,s_i-1},\ldots,\Psi_{k,n,s_1,\ldots,s_n}(x_n))|\Big)$$

$$\Big(\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_{k,j,s_1,\ldots,s_n}(x_j)\Big)d\lambda_{n-1}$$

$$\leq \sum_{k=1}^{\ell} p_k\Big(\sup_{s_1,\ldots,s_n}\sigma_{k,i,s_1,\ldots,s_n}\Big)\Big(\prod_{\substack{j=1\\j\neq i}}^{n}\sum_{s_j=1}^{r_j}\Big)\int_{P_i(I_{k,s_1,\ldots,s_n})}$$

$$\left(\bigvee_i^{P_i(I_{k,s_1,\dots,s_n}) \times I} f_i(\Psi_{k,1,s_1,\dots,s_n}(x_1),\dots,x_i,\dots,\Psi_{k,n,s_1,\dots,s_n}(x_n)) \right.$$

$$\left. + 2\int_0^1 f_i\left(\Psi_{k,1,s_1,\dots,s_n}(x_1),\dots,x_i,\dots,\Psi_{k,n,s_1,\dots,s_n}(x_n)\right) dx_i \right)$$

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{n} \sigma_{k,j,s_1,\dots,s_n}(x_j)\right) d\lambda_{n-1}$$

$$\leq \sum_{k=1}^{\ell} p_k \left(\sup_{s_1,\dots,s_n} \sigma_{k,i,s_1,\dots,s_n}\right) \left(\prod_{\substack{j=1\\ j\neq i}}^{n} \sum_{s_j=1}^{r_j}\right) \int_{P_i(D_{s_1,\dots,s_n})}$$

$$\left(\bigvee_i^{P_i(D_{s_1,\dots,s_n}) \times I} f_i(x_1,\dots,x_n) + 2\int_0^1 f_i(x_1,\dots,x_n) dx_i \right) d\lambda_{n-1}$$

$$\leq \gamma \int_{P_i(I^n)} \bigvee_i^{I^n} f_i \; dm_{n-1} + 2\gamma \int_{P_i(I^n)} \left(\int_0^1 f_i \; dx_i \right) d\lambda_{n-1}$$

$$= \gamma \int_{P_i(I^n)} \bigvee_i^{I^n} f_i \; d\lambda_{n-1} + 2\gamma \, \|f\|_1 .$$

Therefore, we have

$$\int_{P_i(I^n)} \bigvee_i^{I^n} P_\tau f_i \; d\lambda_{n-1} \leq (\gamma + 2\gamma_i) \int_{P_i(I^n)} \bigvee_i^{I^n} f_i \; d\lambda_{n-1} + (K_i + 2\gamma) \, \|f\|_1$$

$$\leq \alpha \int_{P_i(I^n)} \bigvee_i^{I^n} f_i \; d\lambda_{n-1} + K \, \|f\|_1 ,$$

where $0 < \alpha = \max_i (\gamma + 2\gamma_i) < 1$ and $K = \max_i (K_i + 2\gamma) > 0$. This α and K are independent of f. The assertions of the theorem follow immediately, as in [1].

Q.E.D.

<u>Theorem 7.2</u> Let $\tau(x)$ be the random map as in Theorem 7.1 and let

$$s_k = \inf_{i,j} \inf_{[a_{k,ij}\, b_{k,ij}]} |\varphi'_{k,ij}(x_i)|, \quad k = 1,\ldots,\ell.$$

If

$$\sum_{k=1}^{\ell} \frac{p_k}{s_k} \le \gamma < 1 \qquad (7.1)$$

for some constant γ, then all the conclusions of Theorem 7.1 hold.

Notice that we do not need every τ_k to be expanding to satisfy the inequality (7.1). When $\ell = 1$, i.e., $p_1 = 1$, Theorem 7.2 is the same as the main result of [1].

7.3 Example

Let $n = 2$ and $\ell = 5$. Let

$$\tau_1(x_1, x_2) = (\tfrac{1}{3}x_1, \tfrac{1}{3}x_2), \qquad \tau_2(x_1, x_2) = (\tfrac{1}{3}x_1 + \tfrac{2}{3}, \tfrac{1}{3}x_2),$$

$$\tau_3(x_1, x_2) = (\tfrac{1}{3}x_1, \tfrac{1}{3}x_2 + \tfrac{2}{3}), \qquad \tau_4(x_1, x_2) = (\tfrac{1}{3}x_1 + \tfrac{2}{3}, \tfrac{1}{3}x_2 + \tfrac{2}{3}),$$

and $\tau_5(x_1, x_2) = (\varphi_{1j}(x_1), \varphi_{2j}(x_2))$, $x \in D_j \in \mathcal{P}$, where $\mathcal{P}$ is a Jablonski partition for τ_5. We have $s_1 = s_2 = s_3 = s_4 = \frac{1}{3}$. The inequality (7.1) becomes

$$3\left(\sum_{k=1}^{4} p_k\right) + \frac{1}{s_5}p_5 < 1 \tag{7.2}$$

Since $\sum_{k=1}^{5} p_k = 1$, the left hand side of (7.2) is $\left(3 - \frac{1}{s_5}\right)q + \frac{1}{s_5} < 1$, where $q = \sum_{i=1}^{4} p_k$. Thus $s_5 > \frac{1-q}{1-3q}$ is sufficient to guarantee the existence of an absolutely continuous invariant measure for the random map $\tau_1, \ldots, \tau_5$.

Let $p_1 = p_2 = p_3 = p_4 = 0.05$ and $q = 0.2$. We have $s_5 > 2$ and we can take $s_5 = 3$.

Example 7.1 Let $\mathcal{P} = \{D_1, D_2, \ldots, D_9\}$ (figure 7.1).

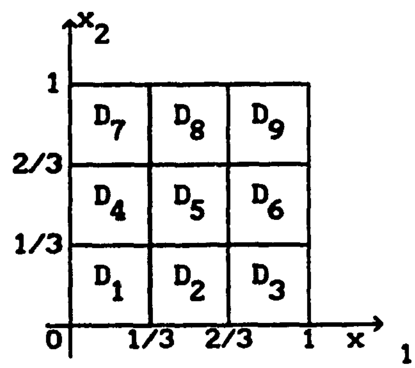

Figure 7.1

For $i = 1,2$ and $j = 1,2,\ldots,9$ we take

$$\varphi_{1j}(x) = \begin{cases} 3x, & 0 \le x < 1/3; \\ 3x-1, & 1/3 \le x < 2/3; \\ 3x-2, & 2/3 \le x \le 1. \end{cases}$$

i.e., τ_5 maps D_j , $j = 1,2,\ldots,9$, onto I^2 and $s_5 = 3$.

Example 7.2 Let $\mathcal{P} = \{D_{jk}\}_{j,k=1}^{8}$, $D_{jk} = D_j x D_k$ and $D_1 = [0,1/16)$, $D_2 = [1/16,1/8)$, $D_3 = [1/8,3/16)$, $D_4 = [3/16,1/4)$, $D_5 = [1/4,3/8)$, $D_6 = [3/8,1/2)$, $D_7 = [1/2,3/4)$, $D_8 = [3/4,1]$ (figure 7.2).

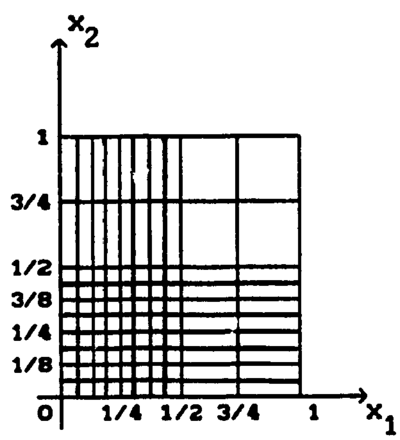

Figure 7.2

Let

$f_1(x) = g_1(x) = 4x,\qquad 0 \le x < 1/16;$

$f_2(x) = g_2(x) = -4x + 1/2,\qquad 1/16 \le x < 1/8;$

$f_3(x) = g_3(x) = 4x - 1/2,\qquad 1/8 \le x < 3/16;$

$f_4(x) = g_4(x) = -4x + 1,\qquad 3/16 \le x < 1/4;$

$f_5(x) = g_5(x) = 3x - 3/4,\qquad 1/4 \le x < 3/8;$

$f_6(x) = g_6(x) = -3x + 3/2,\qquad 3/8 \le x < 1/2;$

$f_7(x) = g_7(x) = 3x - 3/2,\qquad 1/2 \le x < 3/4;$

$f_8(x) = g_8(x) = -3x + 3,\qquad 3/4 \le x < 1.$

$\tau_5(x_1, x_2) = (f_j(x_1), g_k(x_2)),\quad (x_1,x_2) \in D_{jk}$ (figure 7.3) and $s_5 = 3$. The support of the invariant density is contained on $[0,1/4] \times [1,1/4]$.

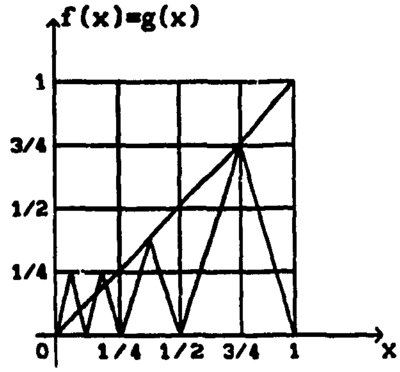

Figure 7.3

Example 7.3 For any integer $N > 2$, we take $\mathcal{P}_N = \left\{D_{jk}\right\}_{j,k=1}^{N+2} = D_j \times D_k$. $D_1 = [0, \frac{1}{2N})$, $D_2 = [\frac{1}{2N}, \frac{1}{N})$, $D_i = [\frac{i-2}{N}, \frac{i-1}{N})$, $i = 3,4,\dots,N$, $D_{N+1} = [\frac{N-1}{N}, \frac{2N-1}{2N})$ and $D_{N+2} = [\frac{2N-1}{2N}, 1]$. Define

$$f_1(x) = g_1(x) = 3x, \qquad 0 \le x < 1/2N;$$

$$f_2(x) = g_2(x) = 4x - 2/N, \qquad 1/2N \le x < 1/N;$$

$$f_i(x) = g_i(x) = 3x - (2i-3)/N, \qquad (i-2)/N \le x < (i-1)/N, \ 3 \le i \le N;$$

$$f_{N+1}(x) = g_{N+1}(x) = 4x - (3N-2)/N, \qquad (N-1)/N \le x < (2N-1)/2N;$$

$$f_{N+2}(x) = g_{N+2}(x) = 3x - 2, \qquad (2N-1)/2N \le x \le 1.$$

Now let

$$\tau_5(x_1, x_2) = (f_j(x_1), g_k(x_2)), \qquad (x_1, x_2) \in D_{jk}.$$

Since for any $i = 1, 2, \dots, N+2$

$$x - 1/N \le f_i(x) = g_i(x) \le x + 1/N,$$

when $N \to \infty$, $f_i(x) = g_i(x) \to x$. Figure 7.4 shows $\tau_5(x_1,x_2)$ for $N = 9$.

Notice that $\{\tau_1,\tau_2,\tau_3,\tau_4\}$ comprise an iterated function system [30]. It is easy to show that the attractor for this system is a two dimensional Cantor set of Lebesque measure 0. If we choose $\tau_5(x_1,x_2) = (\varphi_1(x_1), \varphi_2(x_2))$ such that for $i = 1,2$, φ_i is a piecewise monotonic map and $\tau_5(x_1,x_2)$ is close to the identify transformation $I(x_1,x_2) = (x_1,x_2)$ but has slopes sufficiently large to satisfy (7.2), we obtain a perturbation of the initial iterated function system which has an absolutely continuous invariant measure whose support is close to the attractor of the initial system. Figure 7.5 shows the two dimensional Cantor set that is the attractor for $\{\tau_1,\tau_2,\tau_3,\tau_4\}$ and Figure 7.6 shows the support of the absolutely continuous invariant measure for the random map $\{\tau_1,\tau_2,\tau_3,\tau_4,\tau_5;.05,.05,.05,.05,.8\}$ where τ_5, with N = 100, is the piecewise expanding transformation on I^2 defined above.

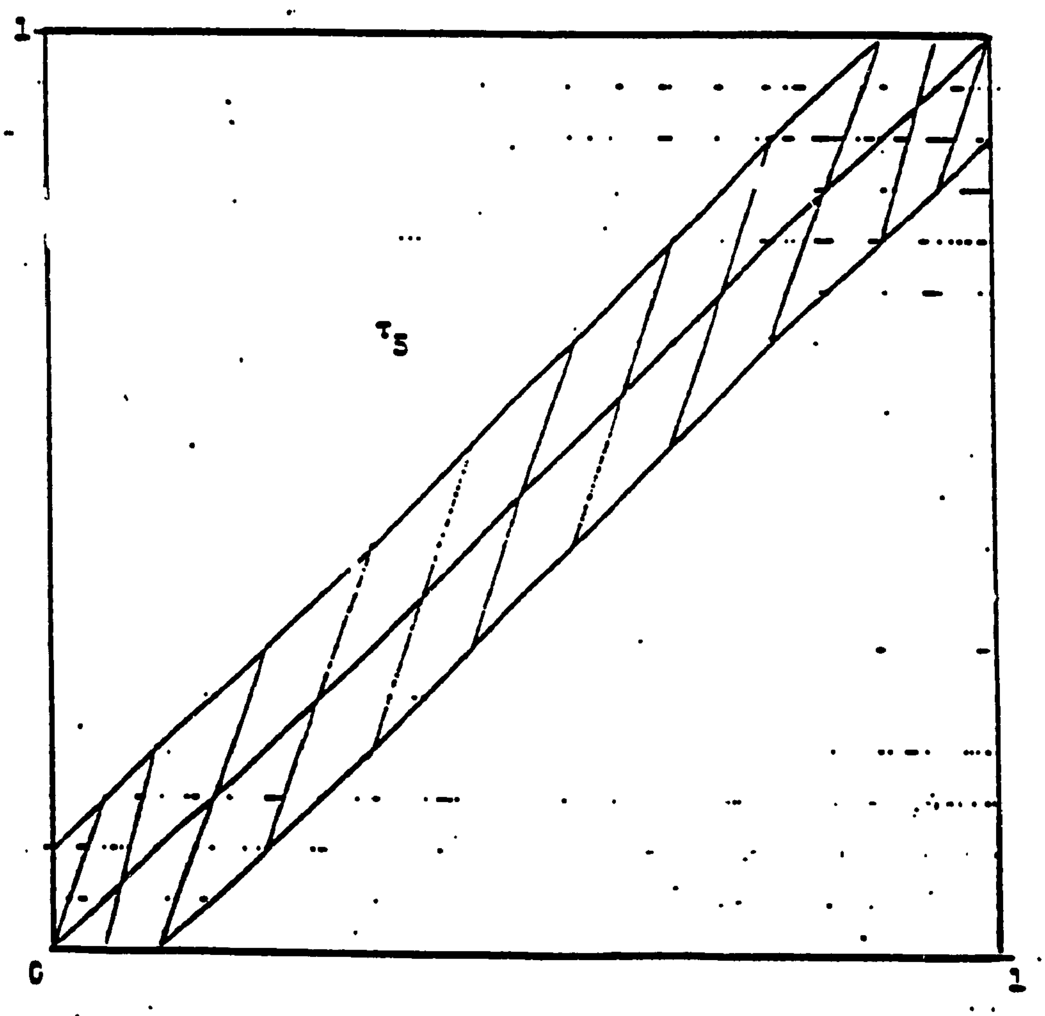

Figure 7.4

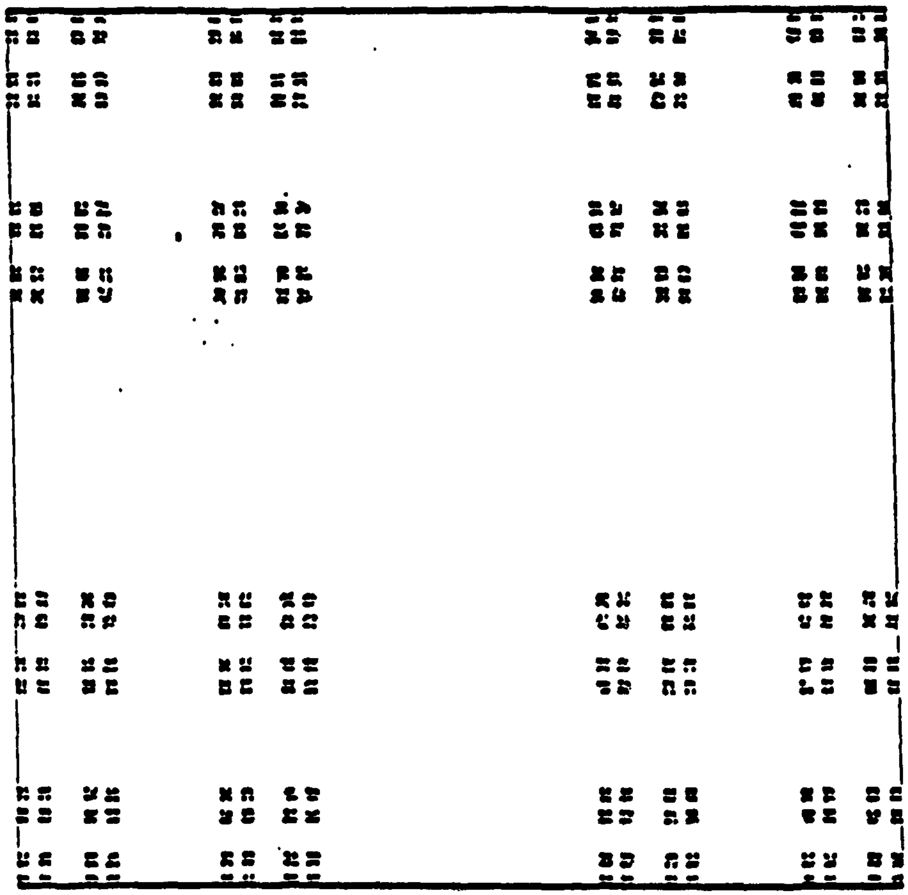

Figure 7.5

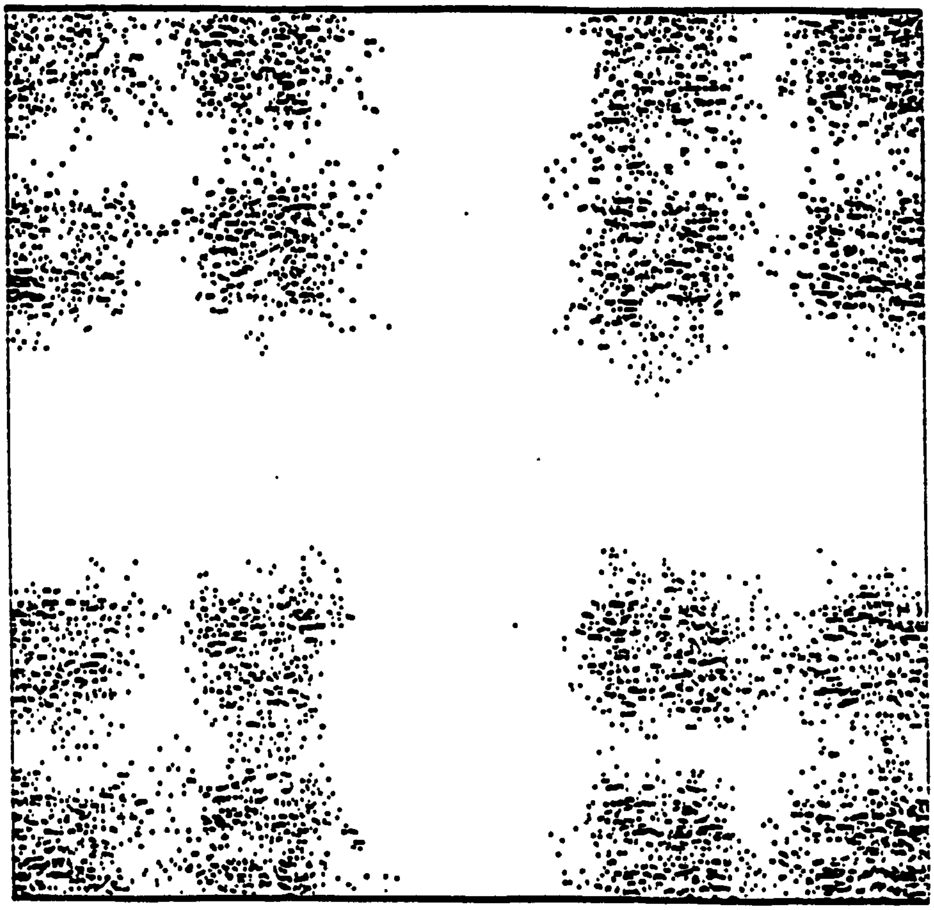

Figure 7.6

CHAPTER 8

Approximating the Invariant Densities of Random Jablonski Transformations

In Chapter 4, using piecewise constant functions, we approximated the invariant densities of Jablonski transformations. In this chapter we will use the same method to approximate the invariant densities of random Jablonski transformations.

8.1 Approximation by Piecewise Constant Function

Let $\tau(x) = \{\tau_k(x)\}_{k=1}^{\ell}$ be a random map with probabilities $p_1,\dots,p_\ell$, $p_k > 0$, $\sum_{k=1}^{\ell} p_k = 1$, where each τ_k is a Jablonski transformation of I^n into I^n on the rectangular partition $\mathcal{P}_k = \left\{D_{k1},\dots,D_{kq_k}\right\}$, $k = 1,2,\dots,\ell$, and q_k is a positive integer which depends on k, i.e.,

$$\tau_k(x) = \left(\tau_{k,1}(x),\dots,\ \tau_{k,n}(x)\right),$$

and for any $i = 1,2,\dots,n$ and $k = 1,2,\dots,\ell$

$$\tau_{k,i}(x) = \psi_{k,i}(x_i) = \varphi_{k,ij}(x_i), \quad x \in D_{kj}.$$

For any integer m, let I^n be divided into m^n equal subsets $I_1, I_2,\dots,I_{m^n}$ with

$$I_j = \left[\frac{r_1}{m}, \frac{r_1+1}{m}\right) \times \left[\frac{r_2}{m}, \frac{r_2+1}{m}\right) \times \dots \times \left[\frac{r_n}{m}, \frac{r_n+1}{m}\right)$$

for some $r_1, r_2,\dots,r_n = 0,1,\dots,m-1$ and $\lambda\left(I_k\right) = \frac{1}{m^n}$, $k = 1,2,\dots,m^n$.

Let P_{st}^k denote the fraction of I_s which is mapped into I_t by τ_k, i.e.,

$$P_{st}^{k} = \frac{\lambda\left(I_s \cap \tau_k^{-1} I_t\right)}{\lambda\left(I_s\right)},$$

and $P_{st} = \sum_{k=1}^{\ell} p_k P_{st}^{k}$.

Let Δ_m be the m^n-dimensional linear subspace of L^1 which is the finite space generated by $\left\{\chi_j\right\}_{j=1}^{m^n}$, where χ_j denotes the characteristic function of I_j, i.e., $f \in \Delta_m$ if and only if $f = \sum_{j=1}^{m^n} a_j \chi_j$ for some constants $a_1, a_2, \ldots, a_{m^n}$.

Define a linear operator $P_m(\tau) = P_m\colon \Delta_m \to \Delta_m$ by

$$P_m(\tau)\, \chi_r = P_m\, \chi_r = \sum_{s=1}^{m^n} P_{rs}\, \chi_s = \sum_{k=1}^{\ell} p_k\, P_m^k\, \chi_r,$$

where

$$P_m^k\, \chi_r = \sum_{s=1}^{m^n} P_{rs}^k\, \chi_s.$$

<u>Theorem 8.1</u> Let $\tau(x) = \{\tau_k(x)\}_{k=1}^{\ell}$ be a random map from I^n into I^n with probabilities $p_1, \ldots, p_\ell$, $p_k > 0$, $\sum_{k=1}^{\ell} p_k = 1$, where each τ_k is a C^2 Jablonski transformation. Suppose $\tau_{k,i}(x) = \varphi_{k,ij}(x_i)$ is monotonic for $x \in \bar{D}_{kj}$ and for $i = 1, 2, \ldots, n$,

$$\sum_{k=1}^{\ell} \sup_j \frac{p_k}{\left|\varphi'_{k,ij}(x_i)\right|} \le \gamma < \frac{1}{3}$$

for some constant γ and $P_\tau = \sum_{k=1}^{\ell} p_k P_{\tau_k}$ has a unique fixed point. Then for any positive integer m, P_m has a fixed point f_m in Δ_m with $\|f_m\| = 1$ and f_m converges weakly to the fixed point of P_τ as $m \to \infty$.

Before proving Theorem 8.1, we need a number of lemmas.

<u>Lemma 8.1</u> Let $\Delta_m^1 = \left\{ \sum_{r=1}^{m^n} a_r\chi_r : a_r \geq 0 \text{ and } \sum_{r=1}^{m^n} a_r = 1 \right\}$. Then P_m maps Δ_m^1 to a subset of Δ_m^1.

<u>Proof</u> Let $f = \sum_{r=1}^{m^n} a_r \chi_r \in \Delta_m^1$, then

$$P_m f = P_m \left(\sum_{r=1}^{m^n} a_r \chi_r \right) = \sum_{r=1}^{m^n} a_r \left(P_m \chi_r \right) = \sum_{r=1}^{m^n} a_r \left(\sum_{s=1}^{m^n} P_{rs} \chi_s \right) = \sum_{s=1}^{m^n} \left(\sum_{r=1}^{m^n} a_r P_{rs} \right) \chi_s .$$

But for any r = 1,2 ,..., m^n

$$\sum_{s=1}^{m^n} P_{rs} = \sum_{s=1}^{m^n} \left(\sum_{k=1}^{\ell} p_k P_{rs}^k \right) = \sum_{k=1}^{\ell} p_k \left(\sum_{s=1}^{m^n} P_{rs}^k \right) = \sum_{k=1}^{\ell} p_k \left(\sum_{s=1}^{m^n} \frac{\lambda\left(I_r \cap \tau_k^{-1} I_s\right)}{\lambda\left(I_r\right)} \right) = \sum_{k=1}^{\ell} p_k = 1.$$

Hence,

$$\sum_{s=1}^{m^n} \left(\sum_{r=1}^{m^n} a_r P_{rs} \right) = \sum_{r=1}^{m^n} a_r \left(\sum_{s=1}^{m^n} P_{rs} \right) = \sum_{r=1}^{m^n} a_r = 1.$$

Therefore, $P_m f \in \Delta_m^1$.

Q.E.D.

Since $P_m \left(\Delta_m^1 \right) \subset \Delta_m^1$ is a compact convex set, the Brouwer fixed point theorem implies that there exists a function $g_m \in \Delta_m^1$ such that $P_m g_m = g_m$. Let $f_m = m^n g_m$. Then $f_m \in \Delta_m$ and $\|f_m\| = 1$ for all m.

<u>Lemma 8.2</u> For $f \in \Delta_m$ we have $P_m f = Q_m P_\tau f$, where the operator Q_m is defined

in Definition 4.1.

Proof: By Lemma 4.3 for $k = 1,\ldots,\ell$ and $f \in \Delta_m$

$$P_m^k f = Q_m P_{\tau_k} f.$$

Hence,

$$P_m f = \sum_{k=1}^{\ell} p_k P_m^k f = \sum_{k=1}^{\ell} p_k Q_m P_{\tau_k} f$$

$$= Q_m\left(\sum_{k=1}^{\ell} p_k P_{\tau_k} f\right) = Q_m P_\tau f.$$

Q.E.D.

By Lemma 4.2 and 8.2 we have:

Lemma 8.3 For $f \in \Delta_m$, the sequence $P_m f$ converges to $P_\tau f$ in L^1 as $m \to \infty$.

By Theorem 7.1 we have

Lemma 8.4 If τ is same as in Theorem 8.1 then for $f \in L^1$

$$\overset{I^n}{\mathrm{V}} P_\tau f \le \alpha \overset{I^n}{\mathrm{V}} f + K \|f\|$$

for some constants $0 < \alpha < 1$ and $K > 0$.

Lemma 8.5 The sequence $\left\{\overset{I^n}{\mathrm{V}} f_m\right\}$ is bounded.

Proof By Lemma 8.2 $f_m = P_m f_m = Q_m P_\tau f_m$ for all m. Hence, by Lemma 8.4 and 4.7

$$\overset{I^n}{\mathrm{V}} f_m = \overset{I^n}{\mathrm{V}} Q_m P_\tau f_m \le \overset{I^n}{\mathrm{V}} P_\tau f_m$$

$$\le \alpha \overset{I^n}{\mathrm{V}} f_m + K \|f_m\| = \alpha \overset{I^n}{\mathrm{V}} f_m + K.$$

Since $\overset{I^n}{\mathrm{V}} f_m < \infty$, we have $\overset{I^n}{\mathrm{V}} f_m \le \frac{K}{1-\alpha}$.

Q.E.D.

<u>Proof of Theorem 8.1</u> By Lemma 8.5 and Lemma 8.3, we know that the set $\left\{f_m\right\}_{m=1}^{\infty}$ is weakly relatively compact in L^1. Let $\left\{f_{m_j}\right\}$ be any weakly convergent subsequence of $\left\{f_m\right\}_{m=1}^{\infty}$ and $f = \lim_{j\to\infty} f_{m_j}$ weakly. Then for any bounded and measurable function g,

$$\left|\int_{I^n} g\left(f - P_\tau f\right)dx\right|$$

$$\le \left|\int_{I^n} g\left(f - f_{m_j}\right)dx\right| + \left|\int_{I^n} g\left(f_{m_j} - Q_{m_j}P_\tau f_{m_j}\right)dx\right|$$

$$+ \left|\int_{I^n} g\left(Q_{m_j}P_\tau f_{m_j} - P_\tau f\right)dx\right|.$$

The first term approaches 0 since f_{m_j} converges weakly to f as $j \longrightarrow \infty$. By Lemma 8.2, $Q_{m_j}P_\tau f_{m_j} = P_{m_j}f_{m_j} = f_{m_j}$. The second term is identically 0. Now we consider the last term. f_{m_j} converges weakly to f as $j \longrightarrow \infty$, by Lemma 4.11, for any $k = 1,2,\dots,\ell$, $P_{\tau_k}f_{m_j}$ converges weakly to $P_{\tau_k}f$ as $j \longrightarrow \infty$. Therefore $P_\tau f_{m_j} = \sum_{k=1}^{\ell} p_k P_{\tau_k} f_{m_j}$ converges weakly to $P_\tau f = \sum_{k=1}^{\ell} p_k P_{\tau_k} f$ as $j \longrightarrow \infty$. The last term approaches 0 by Lemma 4.10.

We have established that for any bounded and measurable function g

$$\int_{I^n} g(x)\left(f(x) - P_\tau f(x)\right)dx = 0.$$

This means that $P_\tau f(x) = f(x)$ almost everywhere. Therefore, any weakly convergent subsequence of $\left\{f_m\right\}$ converges weakly to a fixed point of P_τ. By assumption, P_τ has a unique fixed point. Hence, $f_m \longrightarrow f$ weakly as $m \longrightarrow \infty$.

Q.E.D.

Corollary 8.1 If the fixed point of P_τ is not unique in Theorem 8.1, then any weak limit point of $\left\{f_m\right\}_{m=1}^{\infty}$ is a fixed point of P_τ.

Theorem 8.2 Let τ be the same as in Theorem 8.1 and assume that P_τ has a unique fixed point and

$$\sum_{k=1}^{\ell} \sup_j \frac{p_k}{|\varphi'_{k,ij}(x_i)|} \le \gamma < 1.$$

Let t be an integer such that $\gamma^t < \frac{1}{3}$. Let $\phi = \tau^t$ and f_m be a fixed point of $P_m(\phi)$. Let

$$g_m = \frac{1}{t} \sum_{j=0}^{t-1} P_{\tau^j} f_m.$$

Then g_m converges weakly to the fixed point of P_τ as $m \to \infty$.

Proof Since P_{τ^j} is continuous for all j, by Theorem 8.1

$$g_m \to g = \frac{1}{t} \sum_{j=0}^{t-1} P_{\tau^j} f$$

weakly as $m \to \infty$. Therefore,

$$P_\tau g = \frac{1}{t} \sum_{j=1}^{t} P_{\tau^j} f = \frac{1}{t} \sum_{j=0}^{t-1} P_{\tau^j} f = g,$$

where f is the fixed point of $P_\phi = P_{\tau^t}$, i.e. $P_{\tau^t} f = f$.

Q.E.D.

Corollary 8.2 If the fixed point of P_τ is not unique in Theorem 8.2, then any weak limit point f of $\left\{f_m\right\}_{m=1}^{\infty}$ is a fixed point of $P_\phi = P_{\tau^t}$ and $g = \frac{1}{t} \sum_{j=0}^{t-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{m_i} \to f$ weakly as $i \to \infty$, then

$$g_{m_i} = \frac{1}{t}\sum_{j=0}^{t-1} P_{\tau^j} f_{m_i} \rightarrow g$$

weakly as $i \rightarrow \infty$.

We know that in Theorem 8.1 if $n = 1$ (one-dimension), then $\left\{f_m\right\}_{m=1}^{\infty}$ is compact in L^1. Therefor we have

<u>Theorem 8.3</u> Let $\tau(x)$ be the same as in Theorem 8.1 with $n = 1$. Then for any positive integer m, P_m has a fixed point f_m in Δ_m with $\|f_m\| = 1$ and f_m converges to the fixed point of P_τ in L^1 as $m \rightarrow \infty$.

<u>Corollary 8.3</u> If the fixed point of P_τ is not unique in Theorem 8.3, then any limit point of $\left\{f_m\right\}_{m=1}^{\infty}$ is a fixed point of P_τ.

<u>Thorem 8.4</u> Let $\tau(x)$ be the same as in Theorem 8.2 with $n = 1$. Then g_m converges to the fixed point of P_τ in L^1 as $m \rightarrow \infty$.

<u>Corollary 8.4</u> If the fixed point of P_τ is not unique in Theorem 8.4, then any limit point f of $\left\{f_m\right\}_{m=1}^{\infty}$ is a fixed point of $P_\phi = P_{\tau^t}$ and $g = \frac{1}{t}\sum_{j=0}^{t-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{m_i} \rightarrow f$ in L^1 as $i \rightarrow \infty$, then

$$g_{m_i} = \frac{1}{t}\sum_{j=0}^{t-1} P_{\tau^j} f_{m_i} \rightarrow g$$

in L^1 as $i \rightarrow \infty$.

8.2 Uniqueness

Theorem 8.5 Let $\tau(x) = \tau_k(x)$ (prob. p_k), $k = 1,2,\ldots,\ell$, be the same as in Theorem 8.2. If for some k

$$\tau_k(x) = (\varphi_{k,1j}(x_1),\ldots,\varphi_{k,nj}(x_n)),\ x \in D_{kj},$$

satisfies the conditions $\inf|\varphi'_{k,1j}(x_1)| > 1$ and the fixed point of P_{τ_k} is unique, then the fixed point of $P_\tau = \sum_{k=1}^{\ell} p_k P_{\tau_k}$ is unique.

Proof Without loss of generality we let $k = 1$. The proof is similar to that in [32]. Since P_τ and P_{τ_1} are constrictive Markov operators, for any $f \in L^1$

$$P_\tau f = \sum_{i=1}^{\bar{\ell}} a_i(f)\, g_i(x) + Q f(x);$$

$$P_{\tau_1} f = \sum_{i=1}^{\ell_1} b_i(f)\, h_i(x) + R f(x)$$

for some $\bar{\ell}$, ℓ_1, g_i, h_i, a_i, b_i, Q and R.

Letting $P_1 = P_{\tau_1}$ and $P_2 = \frac{1}{1-p_1} \sum_{k=2}^{\ell} p_k P_{\tau_k}$, we have $P_\tau = p_1 P_1 + (1-p_1)P_2$.

Let $\bar{n}$, n_1 be the numbers of independent fixed points of P_τ and P_{τ_1} respectively. From Theorems 5 and 6 of [32], we obtain $\bar{n} \le \bar{\ell} \le \ell_1 = n_1 = 1$.

Hence P_τ has a unique fixed point.

Q.E.D.

8.3 Example

In this section we present an example and give an algorithm for computing the fractal measure.

Let $n = 2$ (so we are working on the unit square), $\ell = 5$ (i.e., 5 transformations) with $p_1 = p_2 = p_3 = p_4 = 0.0125$, $p_5 = 0.95$. To simplify notation, let $x = x_1$, $y = x_2$. We define:

$$\tau_1(x, y) = (0.5x, 0.5y),$$

$$\tau_2(x, y) = (0.5x + 0.5, 0.5y),$$

$$\tau_3(x, y) = (0.5x, 0.5y + 0.5),$$

$$\tau_4(x, y) = (0.5x + 0.5, 0.5y + 0.5),$$

and

$$\tau_5(x, y) = (f_a(x), f_b(y)), \quad (x, y) \in D_{a,b} = \left[\frac{a-1}{8}, \frac{a}{8}\right) \times \left[\frac{b-1}{8}, \frac{b}{8}\right),$$

where a, b = 1,2,...,8 and $f_1(x) = 5x$, $f_2(x) = 5x - 5/8$, $f_3(x) = 5x - 10/8$, $f_4(x) = 5x - 14/8$, $f_5(x) = 5x - 18/8$, $f_6(x) = 5x - 22/8$, $f_7(x) = 5x - 27/8$ and $f_8(x) = 5x - 4$. The one-dimensional function $f = (f_1, \ldots, f_2)$ is shown in Figure 8.1. Notice that the four transformations $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ constitute a hyperbolic iterated function system, while τ_5 is a piecewise expanding transformation on [0,1].

We have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$, $\lambda_5 = 5$ and

$$\sum_{k=1}^{5} \frac{p_k}{\lambda_k} = 2 \cdot 0.05 + 0.95/5 = 0.29 < 1/3.$$

By Theorem 1 of [1] and Theorem 4.4, P_{τ_5} has a unique fixed point. By Theorem 7.1, $P_\tau = \sum_{k=1}^{5} p_k P_{\tau_k}$ has a fixed point f^* and by Theorem 8.5 it is unique. By Theorem 8.1, there exists a sequence of piecewise constant functions and its weak limit is f^*.

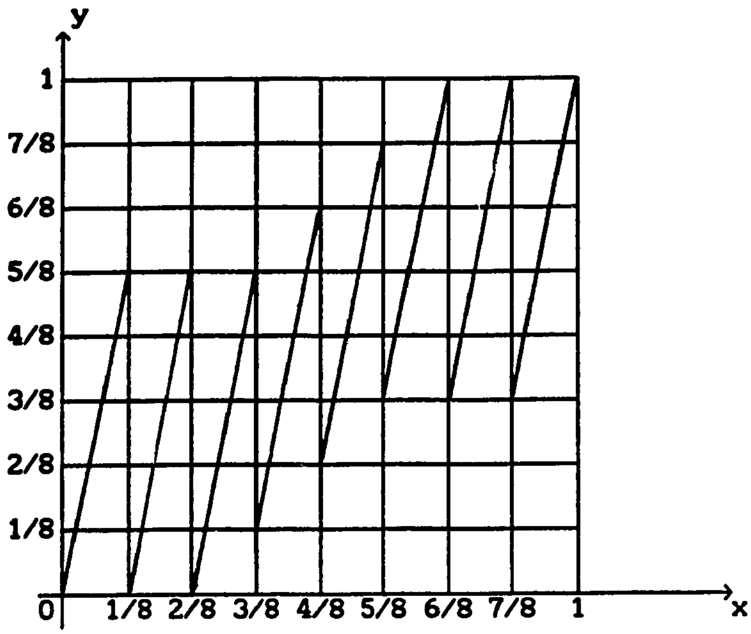

Figure 8.1

We now present an algorithm for approximating f^*. Let $m = 2^t$ for some integer $t \geq 3$. For τ_1, τ_2, τ_3 and τ_4, we have

(1) $P^1_{ij} = 1$ for $i = (r-1)2^t + s$, $j = \left[\frac{r-1}{2}\right]2^t + \left[\frac{s+1}{2}\right]$,

$P^1_{ij} = 0$ otherwise;

(2) $P^2_{ij} = 1$ for $i = (r-1)2^t + s$, $j = \left[\frac{r-1}{2}\right]2^t + \left[\frac{s+1}{2}\right] + 2^{t-1}$,

$P^2_{ij} = 0$ otherwise;

(3) $P^3_{ij} = 1$ for $i = (r-1)2^t + s$, $j = \left[\frac{r-1}{2}\right]2^t + 2^{2t-1} + \left[\frac{s+1}{2}\right]$,

$P^3_{ij} = 0$ otherwise;

(4) $P^4_{ij} = 1$ for $i = (r-1)2^t + s$, $j = \left[\frac{r-1}{2}\right]2^t + 2^{2t-1} + \left[\frac{s+1}{2}\right] + 2^{t-1}$,

$P^4_{ij} = 0$ otherwise;

where $[x]$ is the largest integer less than or equal to x, and r and s run through all integers from 1 to 2^t.

Consider τ_5, whose associated matrix P^5_{ij} have values either 0 or 0.04. All the non-zero values are determined as follows: let r and s run through all integers from 1 to 2^{t-3} and let p and q run through all integers from 1 to 5. Then

(1) $i = (r-1)2^t + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1)2^t + 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1)2^t + 2\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t + 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t + 2\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t + 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t + 2\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 5(s-1) + q;$

(2) $i = (r-1)2^t + 3\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t + 3\cdot 2^{t-3} +s,$ $j = (5(r-1)+p-1)2^t + 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t +3\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 2^{t-3} + 5(s-1) + q;$

(3) $i = (r-1)2^t + 4\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 2\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t+4\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 2\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 2\cdot 2^{t-3} + 5(s-1) + q;$

(4) $i = (r-1)2^t + 5\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t +5\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1)2^t + 6\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t + 6\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1)2^t + 7\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2^{t-3})2^t+7\cdot 2^{t-3} + s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

$i = (r-1+2\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s,$ $j = (5(r-1)+p-1)2^t + 3\cdot 2^{t-3} + 5(s-1) + q;$

(5) $i = (r-1+3\cdot 2^{t-3})2^t + s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t + 5(s-1) + q;$

$i = (r-1+3\cdot 2^{t-3})2^t + 2^{t-3} + s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t + 5(s-1) + q;$

$i = (r-1+3\cdot 2^{t-3})2^t + 2\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t + 5(s-1) + q;$

(6) $i = (r-1+3\cdot 2^{t-3})2^t+3\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t+2^{t-3}+5(s-1)+q;$

(7) $i = (r-1+3\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t+2\cdot 2^{t-3}+5(s-1)+q;$

(8) $i = (r-1+3\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

$i = (r-1+3\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

$i = (r-1+3\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s,\quad j = (5(r-1)+2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

(9) $i = (r-1+4\cdot 2^{t-3})2^t+s,\quad j = (5(r-1)+2\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+4\cdot 2^{t-3})2^t+2^{t-3}+s,\quad j = (5(r-1)+2\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+4\cdot 2^{t-3})2^t+2\cdot 2^{t-3}+s,\quad j = (5(r-1)+2\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

(10) $i = (r-1+4\cdot 2^{t-3})2^t+3\cdot 2^{t-3}+s,\quad j = (5(r-1)+2\cdot 2^{t-3}+p-1)2^t+2^{t-3}+5(s-1)+q;$

(11) $i = (r-1+4\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s,\quad j=(5(r-1)+2\cdot 2^{t-3}+p-1)2^t+2\cdot 2^{t-3}+5(s-1)+q;$

(12) $i = (r-1+4\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s,\quad j=(5(r-1)+2\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

$i = (r-1+4\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s,\quad j=(5(r-1)+2\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

$i = (r-1+4\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s,\quad j=(5(r-1)+2\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q;$

(13) $i = (r-1+5\cdot 2^{t-3})2^t + s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+5\cdot 2^{t-3})2^t + 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+5\cdot 2^{t-3})2^t+2\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+6\cdot 2^{t-3})2^t + s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+6\cdot 2^{t-3})2^t + 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+6\cdot 2^{t-3})2^t+2\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+7\cdot 2^{t-3})2^t + s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+7\cdot 2^{t-3})2^t + 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

$i = (r-1+7\cdot 2^{t-3})2^t+2\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+5(s-1)+q;$

(14) $i = (r-1+5\cdot 2^{t-3})2^t+3\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2^{t-3}+5(s-1)+q;$

$i = (r-1+6\cdot 2^{t-3})2^t+3\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2^{t-3}+5(s-1)+q;$

$i = (r-1+7\cdot 2^{t-3})2^t+3\cdot 2^{t-3}+s,\quad j = (5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2^{t-3}+5(s-1)+q;$

$$(15)\quad i = (r-1+5\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2\cdot 2^{t-3}+5(s-1)+q;$$

$$i = (r-1+6\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2\cdot 2^{t-3}+5(s-1)+q;$$

$$i = (r-1+7\cdot 2^{t-3})2^t+4\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+2\cdot 2^{t-3}+5(s-1)+q;$$

$$(16)\quad i = (r-1+5\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+5\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+5\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+6\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+6\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+6\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+7\cdot 2^{t-3})2^t+5\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+7\cdot 2^{t-3})2^t+6\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

$$i = (r-1+7\cdot 2^{t-3})2^t+7\cdot 2^{t-3}+s, \quad j=(5(r-1)+3\cdot 2^{t-3}+p-1)2^t+3\cdot 2^{t-3}+5(s-1)+q.$$

Notice that for each i there are 25 j values. The above relationships yield a $4^t x 4^t$ matrix, whose left eigenvector (associated with the eigenvalue 1) is an approximation to the fractal measure. The actual program for computing the eigenvector is left for future work.

REFERENCES

1. M. Jablonski, On invariant measures for piecewise C^2-transformations of the n-dimensional cube. Annales Polonici Mathematici XLIII (1983) 185-195.

2. A. Lasota and M. Mackey, Probabilistic properties of Deterministic Systems. Cambridge University Press, (1985).

3. N. Friedman and A. Boyarsky, Matrices and eigenfunctions induced by Markov maps. Linear Algebra and its Applications. 38 (1981) 141-147.

4. P. Lancaster, Theory of matrices, Academic Press, (1969).

5. A.A. Kosyakin and E.A. Sandler, Ergodic properties of a class of piecewise smooth transformations of an interval. (Ergodicheskie svoistva odnogo klassa kusochno-gladkikh Preobazovanii otrezko). Izvestiya VUZ Matematika 1972(3) [118] 32-40.
(English translation from the British Library, Translation Service).

6. Sinai Ya G. "Dopolnenie k knige G. M. Zaslavskogo. Statisticheskaya neobratimost" v nelinenykh sistemakh. (Addition to the book by G. M. Zaslavskii. Statistical irreversibility in non-linear systems). Nauka (1970) 124-143.

7. A. Lasota and J.A. Yorke. On the existence of invariant measures for piecewise monotonic transformation. Trans. Amer. Math. Soc. 186 (1973) 481-488.

8. N. Dunford and J.T. Schwartz. Linear operators. I. General theory. Pure and Applied Math. Vol. 7. Interscience. New York, (1958). MR22 #8302.

9. Tien-Yien Li, Finite Approximation for Frobenius-Perron Operator. A solution to Ulam's Conjecture. J. Approx. Theory 17 (1976) 173-186.

10. James A. Clarkson and C. Raymond Adams, On definitions of bounded variation for functions of two variables. Transactions, Vol. 35 (1933) 824-854.

11. Gerhard Keller, Probabilites - Ergodicite et mesures invariantes pour les transformations dilatantes par morceaux d'une region bornee du plan. C.R. Acad. Sc. Paris, t 289, Serie A. (1979) 625-627.

12. Abraham Boyarsky, Approximating the absolutely continuous measures invariant under general maps of the interval. Proc. Amer. Math. Soc. 87 (1983) 475-480.

13. Abraham Boyarsky and Manny Scarowsky. On a class of transformations which have unique absolutely continuous invariant measures. Trans. Amer. Math. Soc. 255 (1979) 243-262.

14. Tien-Yien Li and J.A. Yorke. Ergodic transformations from an interval into itself. Trans. Amer. Math. Soc. 235 (1978) 183-192.

15. D. Gora and A. Boyarsky. Compactness of invariant densities for families of expanding, piecewise monotone transformations. Can. J. Math. Vol. XLI, No.5 (1989) 855-869.

16. P. Gora, A. Boyarsky and H. Proppe. Constructive approximations to densities invariant under non-expanding maps. J. Stat. Phy. 51 (1988) 179-194.

17. Ricardo Mane. Ergodic theory and differentiable dynamics. (1987).

18. Marek Ryszard Rychlik. Invariant measures and variational principle for Lozi mappings. Preprint.

19. H.L. Royden, Real Analysis. (2nd edition). (1968).

20. Giulio Pianigiani. First return map and invariant measures. Israel Journal of Mathematics. Vol. 35 Nos. 1-2 (1980) 32-48.

21. K. Krzyzewski and W. Szlenk, On invariant measures for expanding differentiable mappings, Studia Math. 33 (1969) 82-92.

22. A.A. Kosyakin and E.A. Sandler, Stochasticity of a certain class of discrete system, translated from Automatika and Telemekhanika No. 9 (1972). 87-94.

23. Gerhard Keller, Proprietes ergodiques des endomorphismes dilatants, C^2 par morceaux, des regions bornees du plan, thesis, Universite de Rennes, (1979).

24. P. Gora, On small stochastic perturbations of mappings on the unit interval. Colloquium. Mathematicum, Vol. XLIX, Fasc. 1 (1984) 73-85.

25. P. Gora and A. Boyarsky, On the number of absolutely continuous measures invariant under higher dimensional transformations, Jour. of Stat. Physics, in Press.

26. Robert B. Ash, Real analysis and probability. Academic Press, (1972).

27. S. Ulam, Problems in Mathematics. Interscience, N.Y. (1960).

28. S. Pelikan, Invariant densities for random maps of the interval. Trans. Amer. Math. Soc. Vol. 281 No. 2 (1984). 813-825.

29. T.Morita, Random iteration of one-dimensional transformations, Osaka J. Math. 22 (1985) 489-518.

30. M. Barnsley, Fractals Everywhere, Academic press, (1988).

31. P.Gora and A. Boyarsky, On functions of bounded variation in higher dimensions, Amer. Math. Monthly, in press.

32. A. Boyarsky and R. Levesque, Spectral decomposition for combinations of Markov operators, J Math. Analy. and Appli. V.132, No.1 (1988) 251-263.