EXISTENCE AND PROPERTIES OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR HIGHER DIMENSIONAL CHAOTIC TRANSFORMATIONS

by

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ABSTRACT

Existence and Properties of Absolutely Continuous Invariant Measures for Higher Dimensional Chaotic Transformations

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We will study the following three problems:

(1) Let Ω be a bounded region in R^n and let $\mathcal{P}_\Omega = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of subsets having piecewise C^2 boundaries. The boundaries may contain cusps. Let $\tau : \Omega \to \Omega$ be piecewise C^2 on $\mathcal{P}_\Omega$ where, $\tau_i = \tau_{|P_i}$ is a C^2 diffeomorphism onto its image and expanding in the sense that there exists $\alpha > 1$ such that for any $i = 1, 2, \cdots, m$ $\|D\tau_i^{-1}\| < \alpha^{-1}$, where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. The main result provides a lower bound on α which guarantees the existence of an absolutely continuous invariant measure for τ.

(2) Let Ω be a region in R^n and let $\mathcal{P}_\Omega = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of closed subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau : \Omega \to \Omega$ be piecewise C^2 on $\mathcal{P}_\Omega$ where, $\tau_i = \tau_{|P_i}$ is a C^2 diffeomorphism onto its image and expanding in the sense that there exists $\alpha > 1$ such that for any $i = 1, 2, \cdots, m$ $\|D\tau_i^{-1}\| < \alpha^{-1}$, where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\|\cdot\|$ is the Euclidean matrix norm. By means of an example, we will show that the simple bound of one-dimensional dynamics cannot be generalized to higher dimensions. In fact, we

will construct a piecewise expanding C^2 transformation on a fixed partition with 10 elements but which have an arbitrarily large number of ergodic, absolutely continuous invariant measures.

(3) Let Ω be an open bounded region in R^n and let $\mathcal{P}_\Omega = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of closed subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau : \Omega \to \Omega$ be an expanding Markov transformation on $\mathcal{P}_\Omega$ where, $\tau_i = \tau_{|P_i}$ and $\tau_i^{-1} \in C^M$, $M \geq 2$. Then the τ-invariant density $h \in C^{M-2}$.

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INTRODUCTION

Absolutely continuous invariant measures (**acim**) play an important role in higher dimensional dynamical systems. Our focus in this thesis would be to establish existence results and study the properties of **acim** for higher dimensional piecewise C^2 and expanding maps τ. We will prove the existence of an **acim** for Lasota-Yorke maps [**L-Y**] in higher dimensions under general conditions. Our main tool is the Perron-Frobenius operator associated with τ. In this setting, we invoke the Ionescu Tulcea and Marinescu Theorem [**I-M**] to derive a spectral decomposition for the Perron-Frobenius operator of τ and, as a consequence obtain useful ergodic properties of τ itself. We will also study the problem of finding an upper bound for the number of ergodic **acim** for such class of maps. The property of smoothness of the invariant density of an **acim** in higher dimensions for the special class of Rényi maps [**Rén**] will also be studied.

In Chapter 1, the preliminary definitions, notations and results from ergodic theory, weakly differentiable functions and functions of bounded variation in higher dimensions relevant to this work are given.

In Chapter 2, we establish the existence of an **acim** in higher dimensions for certain class of maps using "bounded variation techniques".

This topic has an interesting history going back almost two decades.

In 1973, Lasota and Yorke [**L-Y**] used the "bounded variation techniques" to prove a general sufficient condition for existence of an **acim** for expanding, piecewise C^2 transformations on an interval. In spite of the suggestion at the end of [**L-Y**], that the " bounded variation techniques" of [**L-Y**] can be easily used to obtain analogous results in higher dimensions, the generalization of the main result of [**L-Y**] has taken much longer than expected. This was partly due to the difficulty in finding the appropriate definition of variation in higher dimensions. For smooth maps on domains without boundaries, general results for the existence of **acim** were known as early as 1969 [**K-Sz**]. For piecewise C^2 maps in $\mathbb{R}^n$, the first major attempt to prove the existence of **acim** came in 1979 [**K-S 2**]. There the authors do not use a bounded variation argument and the proof, based on a one-dimensional version [**K-S 1**], is flawed. In 1979, Keller [**Kel 1**] used a complicated definition of bounded variation to prove a partial result in dimension 2. The complicated definition given in [**Kel 1**] cannot be used beyond two dimensions.

The simplest higher-dimensional generalization of the result of [**L-Y**] came in 1983. Jabłoński [**Jab**] proved the existence of an **acim** for expanding, piecewise C^2 transformations on $[0,1]^n$. The author worked with rectangular partitions (which is a very restrictive condition) and *Tonnelli definition* of variation. The technique used was analogous to that of [**L-Y**]. This result has been generalized to larger classes of maps and various properties of the density of the **acim** have been studied by Lou [**Lou2**].

In [**Str**] a necessary and sufficient condition for the existence of an **acim** in higher dimensions is presented, but in most cases it cannot be applied.

With the publication of [**Giu**] a new tool became available. Using the definition of bounded variation of a function in $\mathbb{R}^n$ as the integral of its generalized derivative given by Giusti [**Giu**] some partial results were obtained in [**I-K**] and [**Can**]. In [**Can**] the author considers piecewise C^2 transformations on a rectangular partition satisfying a strong expansive condition (which depends on the dimension n of the space) and proves the existence of an **acim**. In 1989, Góra and Boyarsky [**G-B 1**] followed through the approach of [**Can**] in a more general setting and proved the existence of an **acim** for piecewise C^2 transformations in $\mathbb{R}^n$ for domains with piecewise C^2 boundaries. Their basic assumption was that the angle subtended by the tangents at the point of contact of the C^2 segments of the boundaries is *bounded away* from 0.

For boundaries for which the angle mentioned above may become 0 (i.e. the boundaries of partitions may contain "cusps", see Definitions 2.6 and 2.7) Keller [**Kel 2**] proved the existence of an **acim** in two dimensions giving a rather complicated sufficient condition.

Using *Giusti's definition* of bounded variation for a function in $\mathbb{R}^n$ [**Giu**] we prove several results that were proved in [**Kel 2**] in n dimensions. With the aid of these results, we prove our first theorem (Theorem 2.1) which relates the variation of a function of bounded variation to the variation of its image under Perron-Frobenius operator. The properties of the Perron-Frobenius operator are described in [**L-M**]. We then estimate the "trace" of f along a "cuspless" $(n-1)$-dimensional hypersurface, and use this result to prove our second theorem (Theorem 2.2) which yields a sufficient condition for the existence of an **acim** for piecewise C^2 transformations in $\mathbb{R}^n$ defined on domains with cuspless boundaries. We next consider domains which have at least one cusp on the boundary. By putting certain mild conditions on the boundaries, we prove several results which will provide sufficient condition for the existence of an **acim** for piecewise C^2 transformations in $\mathbb{R}^n$ defined on domains whose boundaries *contain* cusps. As a consequence, we prove the main results of [**Kel 2**] and [**Jab**] in form of examples. Finally, at the end of Chapter 2, using the Ionescu Tulcea and Marinescu Theorem, we obtain a spectral decomposition of the Perron-Frobenius operator and corresponding ergodic properties.

In Chapter 3, we investigate the problem of finding an upper bound on the number of **acim** for higher dimensional transformations. Related to the problem of the number of **acim** is the question of the support of invariant densities: knowing that the support of the density has an interior is important in this approach.

For one-dimensional transformations [**Li-Y**], $\tau : I \rightarrow I, I = [0,1]$, it is well known that the number of discontinuities of $\tau'(x)$ provides an upper bound for the number of independent **acim**. This result has been improved in [**Boy1**], [**Pia**], [**B-H**] and [**B-B**]. The key to all these bounds lies in the fact that invariant densities for piecewise C^2 expanding transformations are of bounded variation. In one dimension, a density of bounded variation is bounded and it can be proved that its support consists of a finite union of closed intervals. A simple argument then shows that at least one point of discontinuity of τ' must lie in the largest closed interval, thus this will provide the upper bound on the number of **acim**. In higher dimensions, the much more complicated geometrical setting and the complicated form of the definition of bounded variation [**Giu**] do not permit an easy generalization of the one-dimensional result. For example, in two dimensions, the variation in one direction is integrated along the other direction. It is this integration which allows a function of bounded variation in $\mathbb{R}^n$ to be unbounded and its support to be devoid of interior. In general, dynamical systems can have a large set of invariant measures. For example, higher dimensional point transformation models for cellular automata [**G-B 2**], have many **acim**.

In 1990, Góra, Boyarsky and Proppe [**G-B-P**], outlined the possibility of constructing a piecewise expanding C^2 transformation on a fixed partition with a finite number of elements but which have an arbitrarily large number of ergodic **acim**. There, it is suggested to use certain triangles having a particular geometry as the supports of ergodic **acim**. By means of a sketch (without proof, however), it is suggested that it is possible to take care of the trapezoidal regions between triangles satisfying all conditions . As it will be seen, although the conjecture turns out to be correct, the construction cannot be done in a simple manner.

We use the triangles suggested in [**G-B-P**] as supports of ergodic **acim**. For the trapezoidal regions between the triangles, we use another set of triangles which are not supports of **acim**, satisfying the following conditions:

(1) The triangles are mapped in an expanding manner similar to that of the triangles which are supports of ergodic **acim**,

(2) the intersection of images of triangles which are supports of ergodic **acim** and images of triangles which are not supports of ergodic **acim** is empty.

This reduces the trapezoidal regions to rectangular regions. We then map each such rectangular region to a "tube" in a C^2 and expanding manner in such a way that the tube does not intersect the images of the triangles that support the ergodic **acim**.

Finally, by making small perturbation to these maps near the "vertical" edges of these rectangular regions, we can obtain a map that is C^2 and expanding on all of S_1 (respectively S_{-1}) (see Figure 3.2).

In Chapter 4, we study the smoothness of invariant densities in higher dimensions for a certain class of maps. The smoothness property of their invariant densities (see Definition 4.1) for several classes of transformations of an interval has been studied by many authors. Here, we mention a few examples:

(i) Rényi [**Rén**] in 1957 proved that piecewise transformations of the unit interval onto itself, satisfying a distortion condition, admits an **acim**. Halfant [**Hal**] in 1976 proved , for the maps considered in [**Rén**], that if the transformation is of class C^M then the invariant density is of class C^{M-2}.

(ii) Lasota and Yorke [**L-Y**] in 1973 proved the existence of **acim** for C^2 expanding transformation of an interval, and Misiurewicz [**Mis**] proved the existence of an **acim** for negative Schwarzian maps without sinks and such that the set of critical points is separated from the trajectory. Szewc [**Sze**] proved that, the densities of invariant measures for Misiurewicz maps and Lasota-Yorke maps of class C^M on certain intervals are of class C^{M-1}.

(iii) For expanding maps on an interval the question of smoothness of invariant densities was answered by Sacksteder [**Sac**] (unfortunately the paper contains a mistake) and Krzyżewski [**Krz**] who proved that, if an expanding map is of class C^M then the invariant density is of class C^{M-1}.

(iv) For C^1 expanding transformation of an interval satisfying the Schmitt's [**Schm**] condition the existence of an **acim** and their smoothness of invariant densities were studied in [**Gór**].

In higher dimensions, the existence of an **acim** for expanding Markov maps (which contains the class of maps considered by Rényi [**Rén**]) was proved by Mané [**Man**]. We prove that if a transformation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is expanding Markov and of class C^M, then its invariant density is of class C^{M-2}.

Applications of ergodic theory for higher dimensional transformations can be found in [**Sch1**], [**Sch2**] and [**Yur**].

CHAPTER 1

PRELIMINARIES

Weakly Differentiable Functions and Functions of Bounded Variation

Given Ω an open bounded subset of $\mathbb{R}^n$ we denote the space of all continuously differentiable functions by $C^1(\Omega)$, and by $C_0^1(\Omega)$, the space of all $C^1(\Omega)$ functions with compact support in Ω. We denote by $\mathcal{L}^1(\Omega)$ the set of all integrable functions in Ω.

We begin with a short excursion in the theory of distributions. Let Ω be a nonempty domain (for our purpose an open bounded subset) in $\mathbb{R}^n$. The space $\mathcal{D}(\Omega)$ is the set of all ϕ in $C_0^\infty(\Omega)$ endowed with a topology so that a sequence $\{\phi\}_{k=1}^\infty$ converges to ϕ in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that:

(1) supp $\phi_k \subset K$ for all $k \in \mathbb{N}$ and

(2) $\lim_{k\to\infty} D^\alpha \phi_k = D^\alpha \phi$ uniformly on K for each multi-index α.

A mapping $< f, \cdot >: \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ is called a *distribution* if the following two conditions are satisfied:

(1) $< f, \alpha\phi + \beta\psi >= \alpha < f, \phi > +\beta < f, \psi >$ for all $\alpha, \beta \in \mathbb{R}$, $\phi, \psi \in \mathcal{D}(\Omega)$;

(2) $\lim_{k\to\infty} < f, \phi_k >= 0$ for arbitrary sequence $\{\phi_k\}_{k=1}^\infty \subset \mathcal{D}(\Omega)$ such that $\phi_k \to 0$ in $\mathcal{D}(\Omega)$.

The set of all distributions is denoted by $\mathcal{D}'(\Omega)$. An example of a distribution, relevant for our considerations is the following:

Let $f \in \mathcal{L}^1(\Omega)$ and define $< f, \cdot >$ by

$$< f, g >= \int_\Omega fgd\lambda_n, \quad g \in C_0^1(\Omega, \mathbb{R}^n), \quad \| g \|_\infty \leq 1,$$

where λ is the Lebesgue measure. It should be noted that we took $g \in C_0^1(\Omega)$ instead of the ususal $\phi \in \mathcal{D}(\Omega)$ since in our construction we only need first derivatives.

In general the first distributional derivative of $< f, \cdot > \in \mathcal{D}'(\Omega)$ is defined by the relation

$$< D_i f, \phi >= - < f, D_i \phi >, \quad \phi \in \mathcal{D}(\Omega),$$

where $D_i = \frac{\partial}{\partial x_i}$. It is easy to see that $< D_i f, \cdot >$ is a distribution. Similarly we can define the α-th distributional derivative of $< f, \cdot >$ by

$$< D^\alpha f, \phi >= (-1)^{|\alpha|} < f, D^\alpha \phi >, \quad \phi \in \mathcal{D}(\Omega),$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

A complete study of distributions is presented in [**K-J-F**] and [**Zie**].

Let $f \in \mathcal{L}^1(\Omega)$. We denote by Df the gradient of f in the distributional sense. If f is differentiable, then Df is equal to the usual gradient, where

$$\mathrm{grad} f = (D_1 f, D_2 f, \cdots, D_n f).$$

In the above example of distribution, it is easy to see that:

$$< \sum_{i=1}^n D_i f, g >= - \int_\Omega f \mathrm{div} g d\lambda_n, \quad g \in C_0^1(\Omega, \mathbb{R}^n), \quad \| g \|_\infty \leq 1.$$

We will denote, $< \sum_{i=1}^{n} D_i f, g >$ by the following more informative notation needed in our consideration $\int_\Omega < Df, g > d\lambda_n$.

We proceed with the definition of n-dimensional function of bounded variation.

Definition 1.1. Given Ω an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary of finite $(n-1)$-dimensional measure for $f \in \mathcal{L}^1(\Omega)$ we define its *variation* to be:

$$V_\Omega(f) = \sup\{ \int_\Omega < Df, g > d\lambda_n \quad , g \in C^1(\Omega, \mathbb{R}^n) \quad , \| g \|_\infty \leq 1 \}$$

and f is of *bounded variation* if $V_\Omega(f) < \infty$.

We denote by $BV(\Omega)$ the space of all functions of bounded variation on Ω. With the norm

$$\| \cdot \|_{BV(\Omega)} = \| \cdot \|_{\mathcal{L}^1(\Omega)} + V_\Omega(\cdot),$$

BV is a Banach space.

If $f \in BV(\Omega)$ and Df is the gradient of f in the sense of distributions (see[**Mor**]), then Df is a vector valued Radon measure and $V_\Omega(f)$ is the total variation of f on Ω. Thus we may extend the definition of $V_P(f)$ to include cases where $P \subset \Omega$ is not necessarily open.

$V_\Omega(f)$ satisfies the following elementary properties:

(1) $V_\Omega(f_1 + f_2) \leq V_\Omega(f_1) + V_\Omega(f_2) \;\; \forall f_1, f_2 \in \mathcal{L}^1(\Omega)$,

(2) $V_\Omega(\alpha f) = |\alpha| V_\Omega(f) \;\; \forall f \in \mathcal{L}^1(\Omega), \;\; \alpha \in \mathbb{R}$,

(3) $V_{\Omega_1}(f) + V_{\Omega_2}(f) \leq V_\Omega(f) \;\; \forall f \in \mathcal{L}^1(\Omega)$,

where $\Omega_1, \Omega_2 \subset \Omega$ with $\Omega_1 \cap \Omega_2 = \emptyset$.

Definition 1.2. For $f \in BV(\Omega)$ we define the *trace of* f as

$$\operatorname{tr}_\Omega(f(y)) = \lim_{\mu(B)\to 0} \frac{1}{\mu(B)} \int_B f(z)dz$$

for $B \subset \Omega, y \in \partial B \cap \partial\Omega$ and $\mu(B)$ the n-dimensional measure of B. It has been shown in [**Giu**, Lemma 2.4] that the above limit exists for almost every $y \in \partial \cap \partial\Omega$ with respect to $(n-1)$-dimensional Hausdorff measure.

The following remark is an immediate consequence of [**Giu** Theorem 2.11].

Remark 1.1. Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and let $f \in BV(\Omega)$. Then for $g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$:

$$\int_\Omega f \operatorname{div}(g) d\lambda_n + \int_\Omega < Df, g > d\lambda_n = \int_{\partial\Omega} \operatorname{tr}_\Omega(f) < g, \vec{n} > d\lambda_{n-1}$$

where $\vec{n}$ is the outward normal vector to $\partial\Omega$.

Remark 1.2. Let $g \in C^1, \{g_j\}_1^\infty, g_j \in C_o^1(\Omega), g_j \to g$ (pointwise) such that $\| g \|_\infty \leq 1$ and $\| g_j \|_\infty \leq 1$ for all j and $f \in \mathcal{L}^1$ then we have:

$$\int_\Omega < Df, g > d\lambda_n = \lim_{j\to\infty} \int_\Omega < Df, g_j > d\lambda_n.$$

Proof. Since $\int_\Omega < Df, \cdot > d\lambda_n$ is a measure, by bounded convergence theorem the result follows. $\square$

The next remark is [**Giu** Theorem 1.17].

Remark 1.3. Let $f \in BV(\Omega)$. Then, there exists a sequence $\{f_k\}$ in $C^\infty(\Omega)$ such that:

$$\lim_{k\to\infty} \int_\Omega |f_k - f| d\lambda_n,$$

and

$$\lim_{k\to\infty} V_\Omega(f_k) = V_\Omega(f).$$

Let $P \subset \Omega$ be a closed domain with piecewise C^2 boundary of finite $(n-1)$-dimensional measure. Then, we have the following result from [**Giu** Proposition 2.6].

Remark 1.4. Let $f \in BV(P)$ and let $\{f_k\} \subseteq BV(P)$ be a sequence converging to f in $\mathcal{L}^1(P)$ such that

$$\lim_{k\to\infty} V_P(f_k) = V_P(f).$$

Then

$$\lim_{k\to\infty} \mathrm{tr}_P(f_k) = \mathrm{tr}_P(f) \quad \text{in } \mathcal{L}^1(\partial P).$$

From [**Giu** Theorem 2.11], we have the following remark:

Remark 1.5. Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, and let $\{f_k\}, f$ be functions in $BV(\Omega)$ satisfying:

$$\lim_{k\to\infty} \int_\Omega |f_k - f| d\lambda_n = 0,$$

and

$$\lim_{k\to\infty} V_\Omega(f_k) = V_\Omega(f).$$

Then, we have:

$$\lim_{k\to\infty} \int_{\partial\Omega} |\mathrm{tr}_\Omega(f_k) - \mathrm{tr}_\Omega f| d\lambda_{n-1} = 0.$$

The following remark follows from [**Giu** Theorem 1.19]:

Remark 1.6. (Compactness) Let Ω be a bounded open set in $\mathbb{R}^n$ which is sufficiently regular (e.g, the boundary of Ω is Lipschitz continuous). Then sets of functions uniformly bounded in BV-norm are relatively compact in $\mathcal{L}^1(\Omega)$.

The following remark is [**Giu** Theorem 1.9].

Remark 1.7. (Semicontinuity) Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\{f_k\}$ a sequence of functions in $BV(\Omega)$ which converges in $\mathcal{L}^1(\Omega)$ to a function f. Then

$$V_\Omega(f) \leq \lim_{k\to\infty} \inf V_\Omega(f_k).$$

Proof. Let $g \in C^1(\Omega, \mathbb{R}^n)$ be such that $\|g\|_\infty \leq 1$. Then

$$\begin{aligned}
-\int_\Omega f\mathrm{div}(g)d\lambda_n &+ \int_{\partial\Omega} \mathrm{tr}_\Omega(f) < g, \vec{n} > d\lambda_{n-1} \\
&= -\int_\Omega \lim_{k\to\infty} f_k \mathrm{div}(g)d\lambda_n \\
&\quad + \int_{\partial\Omega} \mathrm{tr}_\Omega(\lim_{k\to\infty} f_k) < g, \vec{n} > d\lambda_{n-1} \\
&= \lim_{k\to\infty}\Big(-\int_\Omega f_k \mathrm{div}(g)d\lambda_n \\
&\qquad + \int_{\partial\Omega} \mathrm{tr}_\Omega(f_k) < g, \vec{n} > d\lambda_{n-1}\Big) \\
&\leq \lim_{k\to\infty} \inf V_\Omega(f_k).
\end{aligned}$$

Taking the supremum over all such g completes the proof. $\square$

Basics from Ergodic Theory

In this section we give basic definitions from ergodic theory which we will use in the sequel. For more detailed information see for example [**L-M**]. Let X be a set (for our purpose $X = \Omega$, where Ω is an open bounded subset of $\mathbb{R}^n$). Let $\mathcal{B}$ be a σ-algebra of subsets of X, i.e., (i) $X \in \mathcal{B}$; (ii) if $B \in \mathcal{B}$ then $X \backslash B \in \mathcal{B}$: (iii) if $B_n \in \mathcal{B}$ for $n \geq 1$ then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$. We then call the pair $(X, \mathcal{B})$ a *measurable space*. A *finite measure* on $(X, \mathcal{B})$ is a function $m : \mathcal{B} \rightarrow R^+$ satisfying $m(\emptyset) = 0$ and $m\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} m(B_n)$ whenever $\{B_n\}_1^{\infty}$ is a sequence of elements of $\mathcal{B}$ which are pairwise disjoint subsets of X. A finite measure space is a triple $(X, \mathcal{B}, m)$ where $(X, \mathcal{B})$ is a measurable space and m is a finite measure on $(X, \mathcal{B})$. We say $(X, \mathcal{B}, m)$ is a *probability space*, or a normalised measure space, if $m(X) = 1$.

Let $(X, \mathcal{B})$ be a measurable space and suppose μ, m are two probability measures on $(X, \mathcal{B})$. We say μ *is absolutely continuous with respect to* m (we denote it by $\mu \ll m$) if $\mu(B) = 0$ whenever $m(B) = 0$. For our purpose $m = \lambda$ the normalised Lebesgue measure.

The following remark is [**Wal** Theorem 0.10 (Radon-Nikodym)].

Remark 1.8. Let μ, m be two probability measures on the measurable space $(X, \mathcal{B})$. Then $\mu \ll m$ iff there exists $f \in \mathcal{L}_m^1(\cdot)$, with $f \geq 0$ and $\int f dm = 1$, such that $\mu(B) = \int_B f dm \; \forall B \in \mathcal{B}$. The function f is unique a.e.

For a transformation $\tau : \Omega \rightarrow \Omega$ we denote by J_τ the Jacobian matrix of τ and by $\mathcal{J}_\tau$ the absolute value of determinant of J_τ.

Definition 1.3. Let $\tau : \Omega \rightarrow \Omega$ and $\{P_1, P_2, ..., P_m\}$ be a partition of Ω. Let $\tau_i = \tau_{|P_i}$ be a diffeomorphism onto its image for $i = 1, 2, \cdots, m$. We define the Perron-Frobenius operator

$$P_\tau : \mathcal{L}^1 \to \mathcal{L}^1,$$

as follows:

$$P_\tau f(x) = \sum_{i=1}^{m} \frac{f(\tau_i^{-1}(x))}{\mathcal{J}_\tau(\tau_i^{-1}(x))}, \quad f \in \mathcal{L}^1(\Omega).$$

The Perron-Frobenius operator has the following properties:

(1) $P_\tau f \geq 0$ for $f \geq 0$;

(2) $||P_\tau f||_{\mathcal{L}^1(\Omega)} = ||f||_{\mathcal{L}^1(\Omega)}$ for $f \geq 0$.

It is well known **[L-M]** that, $h \in \mathcal{L}^1$ is a density of a τ-invariant absolutely continuous measure iff $P_\tau h = h$.

We denote the projection on the eigenspace of P_τ corresponding to eigenvalue 1 by P_{τ_1}, and the operator adjoint to P_{τ_1}, by $P_{\tau_1}^*$, i.e., operator from $\mathcal{L}^\infty(\Omega) \to \mathcal{L}^\infty(\Omega)$ such that:

$$\int_\Omega (P_{\tau_1} f) g d\lambda_n = \int_\Omega f(P_{\tau_1}^* g) d\lambda_n,$$

for any $f \in \mathcal{L}^1(\Omega)$ and any $g \in \mathcal{L}^\infty(\Omega)$.

The transformation $\tau : X \longrightarrow X$ is *measurable* if $\tau^{-1}(\mathcal{B}) \subset \mathcal{B}$, i.e. $B \in \mathcal{B} \Rightarrow \tau^{-1}(B) \in \mathcal{B}$, where $\tau^{-1} \equiv \{x \in X : \tau(x) \in B\}$. The measurable transformation $\tau : X \to X$ preserves measure μ or μ is *τ-invariant* if $\mu(\tau^{-1}(B) = \mu(B)$ for all $B \in \mathcal{B}$.

Definition 1.4. The measure preserving transformation $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is *ergodic* if for any $B \in \mathcal{B}$, such that $\tau^{-1}B = B$, $\mu(B) = 0$ or $\mu(X \backslash B) = 0$.

Definition 1.5. We say $\tau : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is *weakly mixing* if for all $A, B \in \mathcal{B}$,

$$\frac{1}{k}\sum_{i=0}^{k-1} |\mu(\tau^{-i}A \cap B) - \mu(A)\mu(B)| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

τ is *strongly mixing* if for all $A, B \in \mathcal{B}$,

$$\mu(\tau^{-k}A \cap B) \rightarrow \mu(A)\mu(B), \text{ as } n \rightarrow +\infty.$$

Definition 1.6. Let $(X, \mathcal{B}, \mu)$ be a normalised measure space and l $\tau : X \rightarrow X$ be measure preserving such that $\tau(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$. If

$$\lim_{k \rightarrow \infty} \mu(\tau^k B) = 1$$

for each $B \in \mathcal{B}, \mu(B) > 0$, then τ is called *exact*.

The following remark is a direct consequence of [**Man**, Chapter III, Theorem 1.3].

Remark 1.9. Let Ω be a bounded subset of $\mathbb{R}^n$, λ a normalised Lebesgue measure on the Borel σ-algebra of Ω and $\tau : \Omega \rightarrow \Omega$ an expanding (see Definition 4.2) and piecewise onto map of $(\Omega, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra of Ω. Then, there exists a unique probability measure μ on the Borel σ-algebra which is τ-invariant and absolutely continuous with respect to λ. Moreover, τ is exact with respect to μ.

It is well known [**L-M**] that exactness implies mixing which in turn implies ergodicity.

CHAPTER 2

ON THE EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR CHAOTIC MAPS IN HIGHER DIMENSIONS

In 1989, Góra and Boyarsky [**G-B 1**], proved the existence of **acim** for expanding piecewise C^2 transformations on $\mathbb{R}^n$ for domains with piecewise C^2 boundaries. Their basic assumption was that the angle subtended by the tangents at the point of contact of the C^2 segments of the boundaries is bounded away from 0.

For boundaries for which the angle mentioned above may become 0 (i.e., the boundaries of partitions may contain "cusps", see Definitions 2.6 and 2.7), Keller [**Kel 2**] gave a complicated sufficient condition for the existence of an **acim** in dimension two.

We use the definition of bounded variation for functions in $\mathbb{R}^n$ given in [**Giu**] and prove several results that were proved in [**Kel 2**] in dimension n. Using these results, we will prove our first theorem (Theorem 2.1), which relates the variation of a function of bounded variation to the variation of its Perron-Frobenius operator. The properties of the Perron-Frobenius operator are described in [**L-M**]. We will then estimate the "trace" of f along a "cuspless" $(n-1)$-dimensional hypersurface. By aid of this result, we will prove our second theorem (Theorem 2.2), which in turn will give a sufficient condition for proving the existence of an **acim** for piecewise C^2 transformations in $\mathbb{R}^n$ defined on domains with cuspless boundaries.

Next, we will consider domains which have at least one cusp on the boundary. By setting certain mild conditions on the boundaries, we will prove several theorems which will provide sufficient conditions for existence of an **acim** for piecewise C^2 transformations in $\mathbb{R}^n$, defined on domains whose boundaries *contain cusps*. As a consequence, we prove the main results of **[Kel 2]** and **[Jab]** in form of examples. Finally, at the end of Chapter 2, using the Ionescu Tulcea and Marinescu Theorem, we obtain a spectral decomposition of the Perron-Frobenius operator and corresponding ergodic properties.

Main Results

Let $A = (a_{ij})$ be an $n \times n$ matrix; we denote the Euclidean norm of A by $\| A \|_2 = \sup_i (\sum_{j=1}^n a_{ij}^2)^{1/2}$.

Now using Definition 1.1 for n-dimensional function of bounded variation, we generalize several results of **[Kel 2]** to dimension n.

Lemma 2.1.

Let Ω_1 and Ω_2 be open bounded subsets of $\mathbb{R}^n$ and $\tau : \Omega_1 \to \Omega_2$ a diffeomorphism of class C^2. Then

(i)

$$\int_{\Omega_2} (f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}} \, div \, g d\lambda_n$$

$$= \int_{\Omega_1} f \left[div((J_{\tau^{-1}} \cdot g) \circ \tau) - \sum_{j,k} (g_j \circ \tau) \frac{\partial}{\partial x_k} \left(\frac{\partial (\tau^{-1})_k}{\partial y_j} \circ \tau \right) \right] d\lambda_n.$$

(ii)

$$\int_{\Omega_2} < D((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}), g > d\lambda_n = \int_{\Omega_1} < Df, (J_{\tau^{-1}} \cdot g) \circ \tau > d\lambda_n$$

$$+ \int_{\Omega_1} \sum_{j,k} f \cdot (g_j \circ \tau) \frac{\partial}{\partial x_k} \left(\frac{\partial (\tau^{-1})_k}{\partial y_j} \circ \tau \right) d\lambda_n,$$

where $g \in C^1(\Omega_2)$.

(iii)

$$\int_S |tr_{\Omega_2}((f \circ \tau^{-1})\mathcal{J}_{\tau^{-1}})| d\lambda_{n-1} \leq \sup_{Z \in \Omega_2} \| (J_{\tau^{-1}})_{|Z} \|_2 \int_{\tau^{-1}S} |tr_{\Omega_1}(f)| d\lambda_{n-1},$$

for any hypersurface $S \subseteq \partial\Omega_2$.

Proof.

(i) We have:

$$(J_{\tau^{-1}} \cdot g) \circ \tau = \begin{pmatrix} \frac{\partial(\tau^{-1})_1}{\partial y_1} \circ \tau & \dots & \frac{\partial(\tau^{-1})_1}{\partial y_i} \circ \tau & \dots & \frac{\partial(\tau^{-1})_1}{\partial y_n} \circ \tau \\ \frac{\partial(\tau^{-1})_i}{\partial y_1} \circ \tau & \dots & \frac{\partial(\tau^{-1})_i}{\partial y_i} \circ \tau & \dots & \frac{\partial(\tau^{-1})_i}{\partial y_n} \circ \tau \\ \frac{\partial(\tau^{-1})_n}{\partial y_1} \circ \tau & \dots & \frac{\partial(\tau^{-1})_n}{\partial y_i} \circ \tau & \dots & \frac{\partial(\tau^{-1})_n}{\partial y_n} \circ \tau \end{pmatrix} \begin{pmatrix} g_1 \circ \tau \\ \vdots \\ g_i \circ \tau \\ \vdots \\ g_n \circ \tau \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^n (\frac{\partial(\tau^{-1})_1}{\partial y_j} \circ \tau)(g_j \circ \tau) \\ \vdots \\ \sum_{j=1}^n (\frac{\partial(\tau^{-1})_i}{\partial y_j} \circ \tau)(g_j \circ \tau) \\ \vdots \\ \sum_{j=1}^n (\frac{\partial(\tau^{-1})_n}{\partial y_j} \circ \tau)(g_j \circ \tau) \end{pmatrix}.$$

Therefore

$$\operatorname{div}\big((J_{\tau^{-1}} \cdot g) \circ \tau\big) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \Big[\big(\frac{\partial(\tau^{-1})}{\partial y_j} \circ \tau \big)(g_j \circ \tau) \Big].$$

Differentiating the right hand side yields the following:

(2.1)

$$\operatorname{div}\big((J_{\tau^{-1}}\cdot g)\circ\tau\big)=\sum_{i,j=1}^{n}\Big[\big(\frac{\partial(\tau^{-1})}{\partial y_j}\circ\tau\big)\frac{\partial}{\partial x_i}(g_j\circ\tau)+(g_j\circ\tau)\frac{\partial}{\partial x_i}\big(\frac{\partial(\tau^{-1})}{\partial y_j}\circ\tau\big)\Big],$$

and thus we have:

$$\int_{\Omega_1} f\Big[\operatorname{div}\big((J_{\tau^{-1}}\cdot g)\circ\tau\big)-\sum_{i,j=1}^{n}(g_j\circ\tau)\frac{\partial}{\partial x_i}\big(\frac{\partial(\tau^{-1})}{\partial y_j}\circ\tau\big)\Big]d\lambda_n$$

$$=\int_{\Omega_1}\sum_{i,j=1}^{n} f\cdot\big(\frac{\partial(\tau^{-1})}{\partial y_j}\circ\tau\big)\frac{\partial}{\partial x_i}(g_j\circ\tau)d\lambda_n.$$

On the other hand since τ is a diffeomorphism we have :

$$\frac{\partial g_j}{\partial y_j}=\frac{\partial(g_j\circ\tau\circ\tau^{-1})}{\partial y_j}=\sum_i\frac{\partial(g_i\circ\tau)}{\partial x_i}\circ\tau^{-1}.\frac{\partial(\tau^{-1})_i}{\partial y_j},$$

where $x_i=(\tau^{-1})_i$ and thus we get:

$$\frac{\partial g_j}{\partial y_j}\circ\tau=\sum_i\frac{\partial(g_j\circ\tau)}{\partial x_i}.\frac{\partial(\tau^{-1})_i}{\partial y_j}\circ\tau.$$

Therefore, using this in the above equation we obtain:

$$\int_{\Omega_1} f\Big[\operatorname{div}\big((J_{\tau^{-1}}\cdot g)\circ\tau\big)-\sum_{i,j=1}^{n}(g_j\circ\tau)\frac{\partial}{\partial x_i}\big(\frac{\partial(\tau^{-1})_i}{\partial y_j}\circ\tau\big)\Big]d\lambda_n$$

$$=\int_{\Omega_1} f\sum_j\big(\frac{\partial g_j}{\partial y_j}\circ\tau\big)d\lambda_n.$$

For a fixed j, by change of variables we get:

$$\int_{\Omega_1} f\big(\frac{\partial g_j}{\partial y_j}\circ\tau\big)d\lambda_n=\int_{\Omega_2}(f\circ\tau^{-1})\mathcal{J}_{\tau^{-1}}\frac{\partial g_j}{\partial y_j}d\lambda_n.$$

Therefore by summing over all j's we obtain:

$$\int_{\Omega_1} f \sum_j \left(\frac{\partial g_j}{\partial y_j} \circ \tau\right) d\lambda_n = \int_{\Omega_2} (f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}} \text{div } g d\lambda_n,$$

for $g \in C^1(\Omega)$, which completes the proof of part (i).

(ii)

For $g \in C_0^1(\Omega_2)$ we have

$$\int_{\Omega_2} (f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}} \text{div } g d\lambda_n = - \int_{\Omega_2} < D\big((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}\big), g > d\lambda_n,$$

$$\int_{\Omega_1} f \text{div}\big((J_{\tau^{-1}} \cdot g) \circ \tau\big) d\lambda_n = - \int_{\Omega_1} < Df, (J_{\tau^{-1}} \cdot g) \circ \tau > d\lambda_n.$$

Thus by part (i) we get part (ii) for $g \in C_0^1(\Omega_2)$.

Now for $g \in C^1(\Omega_2)$ we obtain the result using Remark 1.2 by approximation for $f \in BV(\Omega_1)$.

(iii)

Given $\epsilon > 0$ there exists $g \in C^1(\Omega_2)$ such that $\| g \|_\infty \leq 1$, $g(x) = 0$ $\forall x \in (\partial\Omega_2 \backslash S)$ and

$$\int_S \left| \text{tr}_{\Omega_2}\big((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}\big) \right| d\lambda_{n-1} \leq \left| \int_{\partial\Omega_2} \text{tr}_{\Omega_2}\big((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}\big) < g, \vec{n} > d\lambda_{n-1} \right| + \epsilon.$$

By Remark 1.1 we get:

$$\int_S \left| \text{tr}_{\Omega_2}\big((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}\big) \right| d\lambda_{n-1}$$

$$\leq \left| \int_{\Omega_2} (f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}} \text{div } g d\lambda_n + \int_{\Omega_2} < D\big((f \circ \tau^{-1}) \mathcal{J}_{\tau^{-1}}\big), g > d\lambda_n \right| + \epsilon.$$

By parts (i) and (ii), we get:

$$\int_S \left| \mathrm{tr}_{\Omega_2}\big((f\circ\tau^{-1})\mathcal{J}_{\tau^{-1}}\big)\right| d\lambda_{n-1}$$
$$\leq \left| \int_{\Omega_1} f \mathrm{div}\big((J_{\tau^{-1}}\cdot g)\circ\tau\big) d\lambda_n + \int_{\Omega_1} < Df, (J_{\tau^{-1}}\cdot g)\circ\tau > d\lambda_n \right| + \epsilon.$$

Thus, applying Remark 1.1 once again, we have:

$$\int_S \left| \mathrm{tr}_{\Omega_2}\big((f\circ\tau^{-1})\mathcal{J}_{\tau^{-1}}\big)\right| d\lambda_{n-1} \leq \left| \int_{\partial\Omega_1} \mathrm{tr}_{\Omega_1}(f) < (J_{\tau^{-1}}\cdot g)\circ\tau, \vec{n} > d\lambda_{n-1}\right| + \epsilon,$$

and thus:

$$\int_S \left| \mathrm{tr}_{\Omega_2}\big((f\circ\tau^{-1})\mathcal{J}_{\tau^{-1}}\big)\right| d\lambda_{n-1} \leq \sup_{\|g\|\leq 1} \| J_{\tau^{-1}}\cdot g \|_2 \int_{\tau^{-1}S} |\mathrm{tr}_{\Omega_1}(f)| d\lambda_{n-1}. \quad \square$$

Lemma 2.2.

Let Ω, Ω_1 be open bounded subsets of $\mathbb{R}^n$ with piecewise C^1 boundary, and $\Omega_2 := \Omega\backslash\bar{\Omega}_1$ and $\Gamma := \partial\Omega_1\backslash\partial\Omega = \partial\Omega_2\backslash\partial\Omega$. Then for every $f \in BV(\Omega)$ we prove the following equality for the variation of f in Ω:

$$V_\Omega(f) = V_{\Omega_1}(f) + V_{\Omega_2}(f) + \int_\Gamma \left| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f)\right| d\lambda_{n-1}.$$

Proof.

Case 1: $[\leq]$

We have:

$$V_\Omega(f) = \sup\left\{ \int_\Omega < Df, g > d\lambda_n : \quad g \in C^1(\Omega) \quad, \| g \|_\infty \leq 1 \right\}.$$

Letting $\vec{n}$ to be the field of unit normal vectors to $\partial\Omega$ pointing to the exterior of Ω, by Remark 1.1 we get:

$$V_\Omega(f) = \sup\left\{-\int_\Omega f\mathrm{div}\ g d\lambda_n + \int_{\partial\Omega} \mathrm{tr}_\Omega(f) < g, \vec{n} > d\lambda_{n-1}\right\},$$

where the supremum is taken over all $g \in C^1(\Omega)$, $\| g \|_\infty \leq 1$. We note that $\vec{n}, \vec{n}_i$ are defined almost everywhere with respect to $(n-1)$-dimensional Hausdorff measure and are piecewise continuous.

Now let $\vec{n}_i$ for $i = 1, 2$ to be the field of unit normal vectors to Γ pointing to the exterior of Ω_i. Then we get:

$$\begin{aligned} V_\Omega(f) = \sup\Big\{ & -\int_{\Omega_1} f\mathrm{div}\ g d\lambda_n - \int_{\Omega_2} f\mathrm{div}\ g d\lambda_n \\ & + \int_{\partial\Omega_1} \mathrm{tr}_{\Omega_1}(f) < g, \vec{n}_1 > d\lambda_{n-1} + \int_{\partial\Omega_2} \mathrm{tr}_{\Omega_2}(f) < g, \vec{n}_2 > d\lambda_{n-1} \\ & - \int_\Gamma \mathrm{tr}_{\Omega_1}(f) < g, \vec{n}_1 > d\lambda_{n-1} - \int_\Gamma \mathrm{tr}_{\Omega_2}(f) < g, \vec{n}_2 > d\lambda_{n-1} \\ & : g \in C^1(\Omega) , \| g \|_\infty \leq 1\Big\}. \end{aligned}$$

Now we note that $\vec{n}_1 = -\vec{n}_2$ along Γ, and thus:

$$\begin{aligned} V_\Omega(f) = \sup\Big\{ & \int_{\Omega_1} < Df, g > d\lambda_n + \int_{\Omega_2} < Df, g > d\lambda_n \\ & + \int_\Gamma (\mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f)) < g, \vec{n}_1 > d\lambda_{n-1} : g \in C^1(\Omega) , \| g \|_\infty \leq 1\Big\}. \end{aligned}$$

Taking supremum over each term we obtain:

$$V_\Omega(f) \leq \sup\left\{\int_{\Omega_1} < Df, g > d\lambda_n : \ g \in C^1(\Omega) \ , \ \| g \|_\infty \leq 1\right\}$$

$$+ \sup\left\{\int_{\Omega_2} < Df, g > d\lambda_n : \ g \in C^1(\Omega) \ , \ \| g \|_\infty \leq 1\right\}$$

$$+ \sup\left\{\int_\Gamma \left| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \right| < g, \vec{n}_1 > d\lambda_{n-1} : \ g \in C^1(\Omega) \ , \ \| g \|_\infty \leq 1\right\}.$$

Note that $\| g \|_\infty \leq 1$ and $\vec{n}_1$ is a normal unit vector thus $\|< g, \vec{n}_1 >\| \leq 1$. Therefore applying Definition 1.1 to the first two terms on the right-side we get:

$$V_\Omega(f) \leq V_{\Omega_1}(f) + V_{\Omega_2}(f) + \int_\Gamma \left| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \right| d\lambda_{n-1}.$$

Case 2: [$\geq$]

Let $U_\delta(\Gamma) := \{x \in \Omega | \operatorname{dist}(x, \Gamma) < \delta\}$. There exists $g_o = (g_{o_1}, g_{o_2}, ..., g_{o_n}) \in C_o^1(U_\delta(\Gamma))$ such that:

$$\int_\Gamma \left| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \right| d\lambda_{n-1} \leq \epsilon + \int_\Gamma \left(\mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f)\right) < g_o, \vec{n} > d\lambda_{n-1}.$$

Using Definition 1.2 on the right-side we get:

$$\int_\Gamma \left| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \right| d\lambda_{n-1}$$

$$\leq \epsilon + \int_\Gamma \left(\mathrm{tr}_{\Omega_1 \cap U_\delta(\Gamma)}(f) - \mathrm{tr}_{\Omega_2 \cap U_\delta(\Gamma)}(f)\right) < g_o, \vec{n} > d\lambda_{n-1}.$$

Since $U_\delta(\Gamma)$ is open if intersected with Ω_i we get $\partial(\Omega_i \cap U_\delta(\Gamma)) = \Gamma$ for $i = 1, 2$.

Thus, it follows that:

$$\int_{\Gamma}\left|\mathrm{tr}_{\Omega_1}(f)-\mathrm{tr}_{\Omega_2}(f)\right|d\lambda_{n-1}\leq\epsilon+\int_{\partial(\Omega_1\cap U_\delta(\Gamma))}\mathrm{tr}_{\Omega_1\cap U_\delta(\Gamma)}(f)<g_\circ,\vec{n}_1>d\lambda_{n-1}$$
$$+\int_{\partial(\Omega_2\cap U_\delta(\Gamma))}\mathrm{tr}_{\Omega_1\cap U_\delta(\Gamma)}(f)<g_\circ,\vec{n}_2>d\lambda_{n-1}.$$

Since $g_\circ$ has compact support in $U_\delta(\Gamma)$ and $\vec{n}=\vec{n}_1=-\vec{n}_2$

$$\int_{\Gamma}\left|\mathrm{tr}_{\Omega_1}(f)-\mathrm{tr}_{\Omega_2}(f)\right|d\lambda_{n-1}\leq\epsilon+\int_{\Omega_1\cap U_\delta(\Gamma)}f\mathrm{div}\ g d\lambda_n+\int_{\Omega_2\cap U_\delta(\Gamma)}f\mathrm{div}\ g d\lambda_n$$
$$+\int_{\Omega_1\cap U_\delta(\Gamma)}<Df,g_\circ>d\lambda_n$$
$$+\int_{\Omega_2\cap U_\delta(\Gamma)}<Df,g_\circ>d\lambda_n,$$

and

$$\int_{\Gamma}\left|\mathrm{tr}_{\Omega_1}(f)-\mathrm{tr}_{\Omega_2}(f)\right|d\lambda_{n-1}\leq 3\epsilon+\int_{U_\delta(\Gamma)}<Df,g_\circ>d\lambda_n. \tag{2.2}$$

Now we choose $g_i=(g_{i_1},g_{i_2},...,g_{i_n})\in C^1_\circ(\Omega_i\setminus\bar{U}_\delta(\Gamma))$ for $i=1,2$ with $\|g_i\|_\infty\leq 1$ so that:

$$V_{\Omega_i}(f)\leq\epsilon+\int_{\Omega_i\setminus\bar{U}_\delta(\Gamma)}<Df,g_i>d\lambda_n. \tag{2.3}$$

Then from Equations 2.2 and 2.3 we get:

$$V_{\Omega_1}(f)+V_{\Omega_2}(f)+\int_{\Gamma}\left|\mathrm{tr}_{\Omega_1}(f)-\mathrm{tr}_{\Omega_2}(f)\right|d\lambda_{n-1}$$
$$\leq 5\epsilon+\sum_{i=1}^{2}\int_{\Omega_i\setminus\bar{U}_\delta(\Gamma)}<Df,g_i>d\lambda_n+\int_{U_\delta(\Gamma)}<Df,g_\circ>d\lambda_n.$$

Thus:

$$V_{\Omega_1}(f) + V_{\Omega_2}(f) + \int_\Gamma \Big| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \Big| d\lambda_{n-1} \leq 5\epsilon + \int_\Omega < Df, g > d\lambda_n,$$

where $g := 1_{U_\delta(\Gamma)} \cdot g_0 + 1_{\Omega_1 \setminus \bar{U}_\delta(\Gamma)} \cdot g_1 + 1_{\Omega_2 \setminus \bar{U}_\delta(\Gamma)} \cdot g_2$.

Thus, using Definition 1.1, we have the following:

$$V_{\Omega_1}(f) + V_{\Omega_2}(f) + \int_\Gamma \Big| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \Big| d\lambda_{n-1} \leq 5\epsilon + V_\Omega(f).$$

Since ϵ is arbitrary we get:

$$V_{\Omega_1}(f) + V_{\Omega_2}(f) + \int_\Gamma \Big| \mathrm{tr}_{\Omega_1}(f) - \mathrm{tr}_{\Omega_2}(f) \Big| d\lambda_{n-1} \leq V_\Omega(f).$$

This completes the proof. □

The following corollary is a direct consequence of Lemma 2.2.

Corollary 2.2.1.

Let $\Omega_1 \subseteq \Omega$ be an open bounded subset of $\mathbb{R}^n$ with piecewise C^1 boundary. Then for every $f \in BV(\Omega_1)$

$$V_\Omega(f \cdot 1_{\Omega_1}) = V_{\Omega_1}(f) + \int_{\partial\Omega_1 \setminus \partial\Omega} |tr_{\Omega_1}(f)| d\lambda_{n-1}. \quad \square$$

Definition 2.1. Let $\tau : \Omega \to \Omega$ and $\{P_1, P_2, ..., P_m\}$ be any finite partition of Ω. Let $\tau_i \equiv \tau_{|P_i}$. We say τ is *α-expanding* if

$$\| J_{\tau_i^{-1}} \|_2 < \alpha^{-1},$$

for $i = 1, 2, ..., m$ and $\alpha > 1$.

Definition 2.2. Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary. A partition $P_\Omega = \{P_1, P_2, ..., P_m\}$, $m < \infty$ of Ω is said to be *smooth* if each P_i, $i = 1, 2, \cdots, m$ has a piecewise C^2 boundary.

Theorem 2.1.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation, then:

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + \sum_{i=1}^{m} \alpha^{-1} \cdot \int_{\partial P_i \setminus \tau_i^{-1}(\partial\Omega)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1},$$

where $K > 0$ is a constant independent of f and $P_i \in P_\Omega$, $i = 1, 2, \cdots m$.

Proof.

We start with $V_{\tau P_i}\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}}\right]$ for fixed i. First we note that, for $g \in C_0^1(\tau P_i)$ we have:

$$\int_{\partial\tau P_i} \mathrm{tr}_{\tau P_i}\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}}\right] < g, \vec{n} > d\lambda_{n-1} = 0.$$

Thus, using the previous result in Remark 1.1, we get the following:

$$V_{\tau P_i}\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}}\right] = \sup\left\{-\int_{\tau P_i} (f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}} \mathrm{div}\, g d\lambda_n\right\},$$

where the supremum is taken over all g with $\| g \|_\infty \leq 1$, $g \in C_0^1(\tau P_i, \mathbb{R}^n)$.

We have:

$$\int_{\tau P_i} (f \circ \tau_i^{-1}) \mathcal{J}_{\tau_i^{-1}} \text{div } g d\lambda_n = \sum_{j,k} \int_{P_i} f \left(\frac{\partial(\tau_i^{-1})_k}{\partial y_j} \circ \tau \right) \left(\frac{\partial}{\partial x_k} (g_j \circ \tau) \right) d\lambda_n.$$

Using Equation 2.1 in the right-side we get:

$$\int_{\tau P_i} (f \circ \tau_i^{-1}) \mathcal{J}_{\tau_i^{-1}} \text{div } g d\lambda_n$$

$$= \int_{P_i} f \left[\text{div}\big((J_{\tau_i^{-1}} \cdot g) \circ \tau\big) - \sum_{j,k} (g_j \circ \tau) \frac{\partial}{\partial x_k} \left(\frac{\partial\big(\tau_i^{-1}\big)_k}{\partial y_j} \circ \tau \right) \right] d\lambda_n.$$

Let $\phi = \alpha^{-1} g$. Then $\| J_{\tau^{-1}} \cdot \phi \|_\infty \leq 1$, and we get:

$$\int_{P_i} f \text{div}\big((J_{\tau_i^{-1}} \cdot g) \circ \tau\big) d\lambda_n = \alpha^{-1} \int_{P_i} f \text{div}\big((J_{\tau_i^{-1}} \cdot \phi) \circ \tau\big) d\lambda_n.$$

Thus, we have:

$$V_{\tau P_i} \left[(f \circ \tau_i^{-1}) \mathcal{J}_{\tau_i^{-1}} \right] = \sup \left\{ \int_{P_i} f \left[\sum_{j,k} (g_j \circ \tau) \frac{\partial}{\partial x_k} \left(\frac{\partial\big(\tau_i^{-1}\big)_k}{\partial y_j} \circ \tau \right) \right. \right.$$

$$\left. \left. - \alpha^{-1} \cdot \text{div}\big((J_{\tau_i^{-1}} \cdot \phi) \circ \tau\big) \right] d\lambda_n \right\},$$

where the supremum is taken over all g such that $\| g \|_\infty \leq 1, \quad g \in C_0^1(\tau P_i)$.

Now let $K > 0$ be the bound for $\sum_{j,k} (g_j \circ \tau) \frac{\partial}{\partial x_k} \left(\frac{\partial(\tau_i^{-1})_k}{\partial y_j} \circ \tau \right)$. Then we obtain:

$$\text{(2.4)} \qquad V_{\tau P_i} \left[(f \circ \tau_i^{-1}) \mathcal{J}_{\tau_i^{-1}} \right] \leq \alpha^{-1} \cdot V_{P_i}(f) + K \cdot \int_{P_i} |f| d\lambda_n.$$

On the other hand we have:

$$V_\Omega(P_\tau f) \le \sum_{i=1}^{m} V_\Omega\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}} \cdot 1_{\tau P_i}\right].$$

From Corollary 2.2.1 we have:

$$\sum_{i=1}^{m} V_\Omega\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}} \cdot 1_{\tau P_i}\right]$$

$$= \sum_{i=1}^{m}\left\{V_{\tau P_i}\left[(f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}}\right] + \int_{\partial(\tau P_i)\setminus\partial\Omega}\left|\mathrm{tr}_{\tau P_i}\left((f \circ \tau_i^{-1})\mathcal{J}_{\tau_i^{-1}}\right)\right|d\lambda_{n-1}\right\}.$$

Using Equation 2.4 and Lemma 2.2 in the above equation we get:

$V_\Omega(P_\tau f)$

$$\le \sum_{i=1}^{m}\left[\alpha^{-1} \cdot V_{P_i}(f) + K \cdot \int_{P_i} |f| d\lambda_n + \alpha^{-1} \cdot \int_{\partial P_i \setminus \tau_i^{-1}(\partial\Omega)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}\right]$$

$$\le \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + \sum_{i=1}^{m} \alpha^{-1} \cdot \int_{\partial P_i \setminus \tau_i^{-1}(\partial\Omega)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

Example 2.1. Under the conditions of Theorem 2.1, if $\tau(P_i) = \Omega$ for every i, then

$$\sum_{i=1}^{m} \alpha^{-1} \int_{\partial P_i \setminus \tau_i^{-1}(\partial\Omega)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = 0.$$

Thus:

$$V_\Omega(P_\tau f) \le \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \cdot \qquad \square$$

Definition 2.3. We say K is a *regular cone* in $\mathbb{R}^n$ if K is a cone whose base is a $(n-1)$-dimensional disk B and such that the central ray L joining the vertex to the center of the disk B is perpendicular to the disk.

Definition 2.4. We define the angle θ subtended at the vertex of a regular cone K to be the angle between L and any line joining the vertex to a point on the boundary of B, and we denote this angle by $\theta(K)$.

Let P be a closed bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary. We denote by $\mathcal{K}(P)_{|_p}$ the set of all regular cones with vertex at p which are locally contained in the closure of P in the vertex angle sense. Let $\partial P^* \subseteq \partial P$, and $\mathcal{S}$ be the set of singular points of ∂P^*. Put

$$\theta(\partial P^*, P)_{|_p} = \sup\{\theta(K) \ : \ K \in \mathcal{K}(P_i)_{|_p}\},$$

where $p \in \mathcal{S}$. If $\theta(\partial P^*, P)_p > \frac{\pi}{2}$ we set $\theta(\partial P^*, P)_p = \frac{\pi}{2}$ Define

$$\omega(\partial P^*, P) = \min_{p \in \partial \mathcal{S}} \theta(\partial P^*, P)_{|_p}$$

and for $\omega(\partial P^*, P) > 0$ let

$$\rho(\partial P^*, P) = \frac{1}{|\sin(\omega(\partial P^*, P))|}.$$

Now we will construct a C^1 field of segments L_p, $p \in \partial P^*$, every L_p being a central ray of a regular cone contained in P, with angle subtended at vertex p greater than or equal to $\omega(\partial P^*, P)$.

We start at points $p \in \mathcal{S}$, where the minimal angle $\omega(\partial P^*, P)$ is attained, defining L_p to be central rays of the regular cones $K \in \mathcal{K}(P)_{|_p}$ such that $\theta(K) = \theta(\partial P^*, P)_{|_p}$. Then we extend this field of segments to be the C^1 field we want, making L_p short enough to avoid overlapping. Let $\delta(p)$ be the length of L_p, $p \in \partial P^*$. By compactness of ∂P^* we have:

$$\Delta(\partial P^*, P) = \inf_{p \in \partial P^*} \delta(p) > 0.$$

Definition 2.5. Let P be a closed bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary, we say $\partial P^* \subseteq \partial P$ is *(ω, P)-regular* if $\omega(\partial P^*, P) > 0$.

Lemma 2.3.

Let $P \subset \Omega$ be a closed domain with piecewise C^2 boundary of finite $(n-1)$-dimensional measure which is (ω, P)-regular and let $f \in BV(P)$. Then:

$$\int_{\partial P} |tr_P(f)| d\lambda_{n-1} \leq \rho(\partial P, P) \cdot \left[V_P(f) + \Delta(\partial P, P) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(P)}\right].$$

Proof. For $f \in C^1(P)$, we have, $\mathrm{tr}_P(f) = f$. Thus, for $f \in C^1$ the result follows from the proof of [**G-B 1**, Lemma 3]. For a function $f \in BV(P)$, there exists a sequence $\{f_k\}$ of C^1 functions which converges to $f \in BV(P)$ in $\mathcal{L}^1(P)$ such that:

$$\lim_{k \to \infty} V_P(f_k) \to V_P(f).$$

Then by Remark 1.4 we get:

$$\begin{aligned}
\int_{\partial P} |\mathrm{tr}_P(f)| d\lambda_{n-1} &= \int_{\partial P} \lim_{k \to \infty} |\mathrm{tr}_P(f_k)| d\lambda_{n-1} \\
&= \lim_{k \to \infty} \int_{\partial P} |\mathrm{tr}_P(f_k)| d\lambda_{n-1} \\
&\leq \lim_{k \to \infty} \rho(\partial P, P) \cdot \left[V_P(f_k) + \Delta(\partial P, P) \cdot \|f_k\|_{\mathcal{L}^1_{\lambda_n(P)}}\right] \\
&= \rho(\partial P, P) \cdot \left[V_P(f) + \Delta(\partial P, P) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n(P)}}\right]. \quad \square
\end{aligned}$$

We now present an alternate proof of the main result of [**G-B 1**, Lemma 4].

Theorem 2.2 (Góra-Boyarsky).

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary. Let $P_\Omega = \{P_1, P_2, \cdots, P_m\}$ be a smooth partition of Ω such that ∂P_i is (ω, P_i)-regular for $i = 1, 2, \cdots, m$. If $\tau : \Omega \to \Omega$ is an α- expanding and piecewise C^2 transformation, then:

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot (1+\rho) \cdot V_\Omega(f) + M \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)},$$

where $M > 0$, and $\rho < \infty$ are constants independent of f.

Proof.

By Theorem 2.1 we have:

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + \sum_{i=1}^{m} \alpha^{-1} \int_{\partial P_i \setminus \tau_i^{-1}(\partial\Omega)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + \sum_{i=1}^{m} \alpha^{-1} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

By Lemma 2.3, for every i, we get:

$$\int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \leq \rho(\partial P_i, P_i)[V_{P_i}(f) + \Delta(\partial P_i, P_i) \| f \|_{\mathcal{L}^1_{\lambda_n}(P_i)}].$$

Let $\rho = \max_i \rho(\partial P_i, P_i)$ and $\Delta = \max_i \Delta(\partial P_i, P_i)$. Then

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot V_\Omega(f) + K \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + \alpha^{-1} \cdot \rho \cdot \left[V_\Omega(f) + \Delta \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)}\right]$$

$$= \alpha^{-1} \cdot (1+\rho) \cdot V_\Omega(f) + (K + \alpha^{-1} \cdot \Delta) \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)}$$

$$= \alpha^{-1} \cdot (1+\rho) \cdot V_\Omega(f) + M \cdot \| f \|_{\mathcal{L}^1_{\lambda_n}(\Omega)},$$

where $M = (K + \alpha^{-1} \cdot \Delta)$. $\square$

Definition 2.6. We say P_i has a cusp at a point $p \in \partial P_i \cap \mathrm{int}(\Omega)$ if

$$\theta(\partial P_i, P_i)|_p = 0.$$

Definition 2.7. A hypersurface C is a *cusp with respect to* P_i if

$$\theta(\partial P_i, P_i)|_p = 0 \quad \forall p \in C.$$

Definition 2.8. We say a hypersurface C is a *simple cusp of order 1* if

(1) C is in the interior of Ω,

(2) $\lambda_n(C) = 0$; $\lambda_{n-1}(C) = 0$; $\lambda_{n-2}(C) > 0$,

(3) $\theta(\partial P_i, P_i)_{|_p} = 0$ for all $p \in C$,

(4) C is a cusp only with respect to a unique $P_i \in \mathcal{P}_\Omega$,

(5) If $C \cap \partial P_j \neq \emptyset$ for $j \neq i$, then $\omega(\partial P_j, P_j) > 0$, and

(6) there exists an open neighborhood N of C such that $N \subset \mathrm{int}(\Omega)$ and $\partial P_i \cap N$ is $\big(\omega, N \backslash (N \cap P_i)\big)$ -regular.

Remark 2.1. Let P be an element of a partition of Ω. If C is a simple cusp with respect to P, then clearly ∂P is not (ω, P)- regular in the sense of Definition 2.4, along the hypersurface C. On the other hand, we may take $C \subset \partial P^* \subset \partial P$ such that ∂P^* is $(\omega, \Omega \backslash P)$-regular. In particular if we take $N \subset \Omega$ an open neighborhood of ∂P^* then $\partial P^* = \partial P \cap N$ would be $\big(\omega, N \backslash (N \cap P)\big)$-regular.

Remark 2.2. Under the situation of Remark 2.1 when calculating ρ and Δ, we calculate them with respect to $N \backslash (N \cap P)$ as it regards to the simple cusp C.

Figure 2.1. Illustrates a very simple situation in dimension 2, in connection with Remark 2.1. In this figure C is a simple cusp with respect to P_2. If we take $\partial P_2^* = \partial P_2 \cap N$, then ∂P_2^* is $(\omega, N \backslash (N \cap P_2))$-regular.

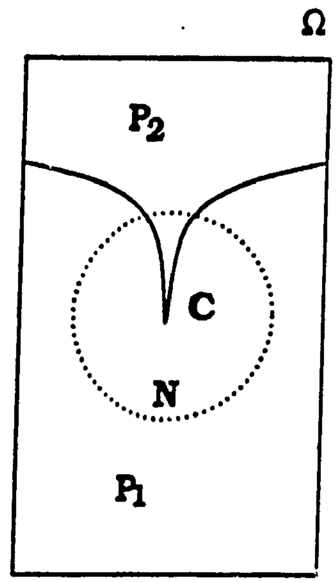

Figure 2.1

Lemma 2.4.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If a hypersurface C is a simple cusp of order 1 with respect to a $P_i \in P_\Omega$, then:

$$\int_{\partial P_i \cap N} |tr_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \rho(\partial P_i \cap N, N \backslash (N \cap P_i)) \left\{ V_N(f) + \Delta(\partial P_i \cap N, N \backslash (N \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N)} \right\},$$

where N is an open neighborhood of C such that $N \subset int(\Omega)$ and $\partial P_i \cap N$ is $(\omega, N \backslash (N \cap P_i))$ -regular.

Proof. By Definition 1.2, we have:

$$\int_{\partial P_i \cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \int_{\partial P_i \cap N} |\mathrm{tr}_{N \cap P_i}(f)| d\lambda_{n-1}$$

$$= \int_{\partial P_i \cap N} \left| \mathrm{tr}_{N\setminus(N\cap P_i)}(f) - \mathrm{tr}_{N\setminus(N\cap P_i)}(f) + \mathrm{tr}_{(N\cap P_i)}(f) \right| d\lambda_{n-1}.$$

Thus, by the triangle inequality, we get:

$$\int_{\partial P_i \cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \le \int_{\partial P_i \cap N} \left| \mathrm{tr}_{N\setminus(N\cap P_i)}(f) - \mathrm{tr}_{(N\cap P_i)}(f) \right| d\lambda_{n-1}$$

$$+ \int_{\partial P_i \cap N} |\mathrm{tr}_{N\setminus(N\cap P_i)}(f)| d\lambda_{n-1}.$$

Applying Lemma 2.3 to the second term on the right hand side of the above equation, we get:

(2.5)

$$\int_{\partial P_i \cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$\le \int_{\partial P_i \cap N} \left| \mathrm{tr}_{N\setminus(N\cap P_i)}(f) - \mathrm{tr}_{N\cap P_i}(f) \right| d\lambda_{n-1}$$

$$+ \rho(\partial P_i \cap N, N\setminus(N\cap P_i)) \times$$

$$\left\{ V_{N\setminus(N\cap P_i)}(f) + \Delta(\partial P_i \cap N, N\setminus(N\cap P_i)) \int_{N\setminus(N\cap P_i)} |f| d\lambda_n \right\}.$$

By Lemma 2.2, we have:

(2.6)

$$V_N(f) = V_{N\setminus(N\cap P_i)}(f) + V_{N\cap P_i}(f)$$

$$+ \int_{\partial(N\cap P_i)\setminus\partial N} \left| \mathrm{tr}_{N\setminus(N\cap P_i)}(f) - \mathrm{tr}_{N\cap P_i}(f) \right| d\lambda_{n-1}.$$

Therefore

$$V_N(f) = V_{N\backslash(N\cap P_i)}(f) + V_{N\cap P_i}(f)$$
$$+ \int_{\partial P_i\cap N} \left| \mathrm{tr}_{N\backslash(N\cap P_i)}(f) - \mathrm{tr}_{N\cap P_i}(f) \right| d\lambda_{n-1}.$$

Since $V_{N\cap P_i}(f) \geq 0$, we have:

$$V_N(f) \geq V_{N\backslash(N\cap P_i)}(f) + \int_{\partial P_i\cap N} \left| \mathrm{tr}_{N\backslash(N\cap P_i)}(f) - \mathrm{tr}_{N\cap P_i}(f) \right| d\lambda_{n-1}.$$

Since $\rho(\partial P_i \cap N, N\backslash(N \cap P_i)) \geq 1$, using the above equation in Equation 2.5, we get:

$$\int_{\partial P_i\cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$
$$\leq \rho(\partial P_i\cap N, N\backslash(N\cap P_i)) \left\{ V_N(f) + \Delta(\partial P_i\cap N, N\backslash(N\cap P_i)) \int_{N\backslash(N\cap P_i)} |f| d\lambda_N \right\}.$$

$$\int_{\partial P_i\cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$
$$\leq \rho(\partial P_i \cap N, N\backslash(N \cap P_i)) \left\{ V_N(f) + \Delta(\partial P_i \cap N, N\backslash(N \cap P_i)) \int_N |f| d\lambda_n \right\}.$$

$$\int_{\partial P_i\cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$
$$= \rho(\partial P_i\cap N, N\backslash(N\cap P_i)) \left\{ V_N(f) + \Delta(\partial P_i\cap N, N\backslash(N\cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N)} \right\}. \qquad \square$$

Lemma 2.5.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If a hypersurface C is a simple cusp of order 1 with respect to a $P_i \in P_\Omega$, and there is no other cusp along the boundary of P_i, then:

$$\int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \rho(\partial P_i \cap N, N \backslash (N \cap P_i)) \Big\{ V_N(f) + \Delta(\partial P_i \cap N, N \backslash (N \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N)} \Big\}$$

$$+ \rho(\partial P_i \backslash (\partial P_i \cap N), P_i) \Big[V_{P_i}(f) + \Delta(\partial P_i \backslash (\partial P_i \cap N), P_i) \|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)} \Big],$$

where N is an open neighborhood of C such that $N \subset \Omega$ and $\partial P_i \cap N$ is $(\omega, N \backslash (N \cap P_i))$ -regular.

Proof.

We have:

$$\int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \int_{\partial P_i \backslash (\partial P_i \cap N)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} + \int_{\partial P_i \cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

Applying Lemma 2.3 to the first term on the right hand side we obtain:

$$\int_{\partial P_i \backslash (\partial P_i \cap N)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \rho(\partial P_i \backslash (\partial P_i \cap N), P_i) \Big\{ V_{P_i}(f) + \Delta(\partial P_i \backslash (\partial P_i \cap N), P_i) \|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)} \Big\}.$$

Thus applying Lemma 2.4 to the second term on the right hand side completes the proof. $\square$

Theorem 2.3.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly one cusp (hypersurface) C which is a simple cusp of order 1 with respect to P_i, there exist $M > 0$ and $\rho < \infty$ such that

$$V_\Omega(P_\tau f) \leq \alpha^{-1}(1+2\rho)V_\Omega(f) + M \parallel f \parallel_{\mathcal{L}^1_{\lambda_n}(\Omega)} .$$

Proof. Let N be an open neighborhood of C such that $N \subset \Omega$ and $\partial P_i \cap N$ is $\big(\omega, N \backslash (N \cap P_i)\big)$ -regular.

By Theorem 2.1 we have:

$$\begin{aligned} V_\Omega(P_\tau f) \leq \alpha^{-1} V_\Omega(f) &+ K \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\ &+ \sum_{j=1, j\neq i}^{m} \alpha^{-1} \int_{\partial P_j} |\mathrm{tr}_{P_j}(f)| d\lambda_{n-1} \\ &+ \alpha^{-1} \int_{\partial P_i \backslash (\partial P_i \cap N)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} + \alpha^{-1} \int_{\partial P_i \cap N} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}. \end{aligned}$$

Using Lemma 2.3 and Lemma 2.5, we get:

$$
\begin{aligned}
V_\Omega(P_\tau f) \leq \alpha^{-1} V_\Omega(f) + K\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\
&+ \alpha^{-1} \sum_{j=1, j\neq i}^{m} \Big[\rho(\partial P_j, P_j)\big(V_{P_j}(f) + \Delta(\partial P_j, P_j)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_j)}\big)\Big] \\
&+ \alpha^{-1}\rho\big(\partial P_i \backslash (\partial P_i \cap N), P_i\big) \\
&\quad \times \Big[V_{P_i}(f) + \Delta\big(\partial P_i \backslash (\partial P_i \cap N), P_i\big)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\Big] \\
&+ \alpha^{-1}\rho\big(\partial P_i \cap N, N \backslash (N \cap P_i)\big) \\
&\quad \times \Big\{V_N(f) + \Delta\big(\partial P_i \cap N, N \backslash (N \cap P_i)\big)\|f\|_{\mathcal{L}^1_{\lambda_n}(N)}\Big\}.
\end{aligned}
$$

Let

$$
\rho = \max\Big\{\rho(\partial P_j, P_j)(j \neq i),\ \ \rho\big(\partial P_i \backslash (\partial P_i \cap N), P_i\big),\ \ \rho\big(\partial P_i \cap N, N \backslash (N \cap P_i)\big)\Big\},
$$

and

$$
\Delta = \max\Big\{\Delta(\partial P_j, P_j)(j \neq i),\ \Delta\big(\partial P_i \backslash (\partial P_i \cap N), P_i\big),\ \Delta\big(\partial P_i \cap N, N \backslash (N \cap P_i)\big)\Big\}.
$$

Then:

$$
\begin{aligned}
V_\Omega(P_\tau f) \leq \alpha^{-1} V_\Omega(f) + K\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\
&+ \alpha^{-1}\rho\big[V_\Omega(f) + \Delta\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}\big] \\
&+ \alpha^{-1}\rho\big[V_\Omega(f) + \Delta\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}\big].
\end{aligned}
$$

Thus:

$$
V_\Omega(P_\tau f) \leq \alpha^{-1}(1 + 2\rho) V_\Omega(f) + (K + 2\alpha^{-1}\rho\Delta)\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.
$$

Letting $M = K + 2\alpha^{-1}\rho\Delta$ completes the proof. $\square$

Definition 2.9. Simple cusps of order 1 $C_1, C_2, \cdots, C_k$ are said to be *separable* if there exist neighborhoods $N_1, N_2, \cdots, N_k$ of $C_1, C_2, \cdots, C_k$ respectively such that:

(1) $N_j \subset \Omega$, for $\quad 1 \leq j \leq k$

(2) $N_j \cap N_i = \emptyset$, if $\quad j \neq i$

(3) if C_j is cusp with respect to P_i then $\partial P_i \cap N_j$ is $(\omega, N_j \backslash (N_j \cap P_i))$ -regular.

Lemma 2.6.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If hypersurfaces $\{C_1, C_2, \cdots, C_k\}$ are pairwise separable cusps each of order 1 with respect to some $P_i \in P_\Omega$, then:

$$\sum_{i=1}^{m} \int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1} \leq 2\rho \big[V_\Omega(f) + \Delta \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \big].$$

Proof. For the cusps $\{C_1, C_2, \cdots, C_k\}$ let $N_1, N_2, \cdots, N_k$ be the neighborhoods of C_j, for $1 \leq j \leq k$ as in Definition 2.9.

Let, $I_i = \big\{ j : C_j$ is a simple cusp with respect to $P_i \big\}$ for $1 \leq i \leq m$. Then

$$\int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \int_{\partial P_i \backslash \bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \int_{\bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

Thus, summing over all i's, gives:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \sum_{i=1}^{m} \int_{\partial P_i \setminus \bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{i=1}^{m} \int_{\bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

We denote the region for which C_j is a simple cusp of order 1 with respect to by $P_{i(j)}$ then the above equation will be equivalent to the following:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \sum_{i=1}^{m} \int_{\partial P_i \setminus \bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{j=1}^{k} \int_{\partial P_{i(j)} \cap N_j} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

By Lemma 2.4, we have:

$$\int_{\partial P_{i(j)} \cap N_j} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \le \rho\big(\partial P_{i(j)} \cap N_j, N_j \setminus (N_j \cap P_{i(j)})\big) \times$$

$$\left\{ V_{N_m}(f) + \Delta\big(\partial P_{i(j)} \cap N_j, N_j \setminus (N_j \cap P_{i(j)})\big) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_m)} \right\},$$

and by Lemma 2.3, we have:

$$\int_{\partial P_i \setminus \bigcup_{j \in I_i} (\partial P_i \cap N_j)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$\le \rho(\partial P_i \setminus \bigcup_{j \in I_i} (\partial P_i \cap N_j), P_i) \left[V_{P_i}(f) + \Delta(\partial P_i \setminus \bigcup_{j \in I_i} (\partial P_i \cap N_j), P_i) \|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)} \right].$$

Thus, we obtain:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1} \le \sum_{i=1}^{m}\rho\big(\partial P_i\setminus\bigcup_{j\in I_i}(\partial P_i\cap N_j),P_i\big)\times$$

$$\Big[V_{P_i}(f)+\Delta\big(\partial P_i\setminus\bigcup_{j\in I_i}(\partial P_i\cap N_j),P_i\big)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\Big]$$

$$+\sum_{j=1}^{k}\rho\big(\partial P_{i(j)}\cap N_j,N_j\setminus(N_j\cap P_{i(j)})\big)\times$$

$$\Big\{V_{N_m}(f)+\Delta\big(\partial P_{i(j)}\cap N_j,N_j\setminus(N_j\cap P_{i(j)})\big)\|f\|_{\mathcal{L}^1_{\lambda_n}(N_m)}\Big\}.$$

Let

$$\rho=\max\Big\{\rho\big(\partial P_i\setminus\bigcup_{j\in I_i}(\partial P_i\cap N_j),P_i\big):1\le i\le m,$$

$$\rho\big(\partial P_{i(j)}\cap N_j,N_j\setminus(N_j\cap P_{i(j)})\big):1\le j\le K\Big\},$$

and

$$\Delta=\max\Big\{\Delta\big(\partial P_i\setminus\bigcup_{j\in I_i}(\partial P_i\cap N_j),P_i\big):1\le i\le m,$$

$$\Delta\big(\partial P_{i(j)}\cap N_j,N_j\setminus(N_j\cap P_{i(j)})\big):1\le j\le K\Big\}.$$

Then:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}\le 2\rho\big[V_\Omega(f)+\Delta\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}\big].\quad\square$$

Theorem 2.4.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly k cusps (hypersurfaces) $\{C_1, C_2, \cdots, C_k\}$ each of them simple of order 1 which are pairwise separable, there exist $M > 0$ and $\rho < \infty$ such that:

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot (1 + 2\rho) \cdot V_\Omega(f) + M \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.$$

Proof. Using Lemma 2.6 in Theorem 2.1 and putting $M = K + 2\alpha^{-1} \cdot \rho \cdot \Delta$ completes the proof. $\square$

Definition 2.10. We say a hypersurface C is *a cusp of order 1* if

(1) C is in the interior of Ω,

(2) $\lambda_n(C) = 0$; $\lambda_{n-1}(C) = 0$; $\lambda_{n-2}(C) > 0$,

(3) C is a cusp with respect to every P_i, $i \in I \subset \{1, 2, \ldots, m\}$ where $|I| > 1$,

(4) $\theta(\partial P_i, P_i)_{|_p} = 0$ for all $p \in C$,

(5) If $C \cap \partial P_j \neq \emptyset$ for $j \neq i$, then $\omega(\partial P_j, P_j) > 0$, and

(6) there exists an open neighborhood N of C which can be divided by hypersurfaces into open neighborhoods $N_i, i \in I$ such that $N = \bigcup_{i \in I} N_i$, where N_i's are pairwise disjoint and $\partial P_i \cap N_i$ is $(\omega, N_i \backslash (N_i \cap P_i))$- regular.

Lemma 2.7.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose Ω contains a single cusp of order 1 along the *hypersurface C with respect to P_i for $i \in I \subset \{1, 2, \cdots, m\}$. then:*

$$\sum_{i\in I} \int_{\partial P_i \cap N_i} |tr_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \sum_{i\in I} \rho(\partial P_i \cap N_i, N_i \backslash (N_i \cap P_i)) \left[V_{N_i}(f) + \Delta(\partial P_i \cap N_i, N_i \backslash (N_i \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i)} \right].$$

Proof. Straightforward consequence of Lemma 2.4. $\square$

Lemma 2.8.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose Ω contains exactly one cusp (hypersurface) C which is of order 1 with respect to P_i for $i \in I \subset \{1, 2, \cdots, m\}$. Then there exist ρ and Δ such that:

$$\sum_{i=1}^{m} \int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1} \leq 2\rho \left[V_\Omega(f) + \Delta \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \right].$$

Proof.

We have:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \sum_{i=1, i\notin I}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{i\in I} \int_{\partial P_i \backslash (\partial P_i \cap N_i)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{i\in I} \int_{\partial P_i \cap N_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

Applying Lemma 2.3 to the first two sums on the right-side and Lemma 2.7 to the third sum, we obtain the following:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1} \le \sum_{i=1,i\notin I}^{m} \rho(\partial P_i, P_i)\big(V_{P_i}(f)+\Delta(\partial P_i, P_i)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\big)$$

$$+\sum_{i\in I}\rho(\partial P_i\backslash(\partial P_i\cap N_i),P_i)\Big[V_{P_i}(f)+\Delta\big(\partial P_j\backslash(\partial P_i\cap N_i),P_i\big)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\Big]$$

$$\sum_{i\in I}\rho(\partial P_i\cap N_i, N_i\backslash(N_i\cap P_i))\Big[V_{N_i}(f)+\Delta\big(\partial P_i\cap N_i, N_i\backslash(N_i\cap P_i)\big)\cdot\|f\|_{\mathcal{L}^1_{\lambda_n}(N_i)}\Big].$$

We observe that the overlap of the regions in calculating the variation is only caused by the last term (since the regions in the first two terms are disjoint). Now, let

$$\rho=\max\Big\{\rho(\partial P_i,P_i):\ i\notin I,\ \ \rho\big(\partial P_i\backslash(\partial P_i\cap N_i),P_i\big):\ i\in I,$$

$$\rho\big(\partial P_i\cap N_i, N_i\backslash(N_i\cap P_i)\big):\ i\in I\Big\}$$

and

$$\Delta=\max\Big\{\Delta(\partial P_i,P_i):\ i\notin I,\ \ \Delta\big(\partial P_i\backslash(\partial P_i\cap N_i),P_i\big):\ i\in I,$$

$$\Delta\big(\partial P_i\cap N_i, N_i\backslash(N_i\cap P_i)\big):\ i\in I\Big\}.$$

This together with N_i's being disjoint completes the proof. □

Theorem 2.5.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau:\Omega\longrightarrow\Omega$ is an α-expanding and piecewise

C^2 transformation. Then, if Ω contains exactly one cusp (hypersurface) C, which is of order 1 with respect to $P_i \in P_\Omega$ for some i, there exist $M > 0$ and $\rho < \infty$ such that:

$$V_\Omega(P_\tau f) \leq \alpha^{-1}(1 + 2\rho)V_\Omega(f) + M\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.$$

Proof. Applying Lemma 2.8 to Theorem 2.1 and putting $M = K + 2\alpha^{-1}\rho\Delta$ completes the proof. $\square$

Definition 2.11. Cusps $C_1, C_2, \cdots, C_k$ of order 1 are said to be *pairwise separable* if there exists open neighborhoods $N(1), N(2) \cdots, N(k)$ of $C_1, C_2, \cdots, C_k$ respectively such that for every j we have:

(1) $N(j) \subset \Omega$,

(2) $N(j) \cap N(i) = \emptyset$, for $i \neq j$,

(3) $N(j) = \bigcup_{i \in I(j)} N_i(j)$, $N_i(j)$'s are pairwise disjoint and $\partial P_i \cap N_i(j)$ is $(\omega, N_i(j) \backslash (N_i(j) \cap P_i))$-regular,

where $I(j) = \{i : C_j \text{ is a cusp with respect to } P_i\}$.

Let $\mathcal{J}(P_i) = \left\{ j : C_j \text{ is a cusp with respect to } P_i \right\}$.

Lemma 2.9.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. if Ω contains k cusps $\{C_1, C_2, \cdots, C_k\}$ of order 1 which *are pairwise separable, then:*

$$\sum_{j=1}^{k}\sum_{i\in I(j)}\int_{\partial P_i\cap N_i(j)}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}$$

$$\leq \sum_{j=1}^{k}\sum_{i\in I(j)}\rho(\partial P_i\cap N_i(j),N_i(j)\backslash(N_i(j)\cap P_i))\times$$

$$\left\{V_{N_i(j)}(f)+\Delta(\partial P_i\cap N_i(j),N_i(j)\backslash(N_i(j)\cap P_i))\|f\|_{\mathcal{L}^1_{\lambda_n}(N_i(j))}\right\}.$$

Proof. Straightforward consequence of Lemma (2.4). □

Lemma 2.10.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose Ω contains k cusps $\{C_1,C_2,\cdots,C_k\}$ of order 1 which are pairwise separable, then:

$$\sum_{i=1}^{m}\int_{\partial P_i}|tr_{P_i}(f)|d\lambda_{n-1}\leq 2\rho\cdot[V_\Omega(f)+\Delta\cdot\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}].$$

Proof. Let $\mathcal{I}(j)=\bigcup_{i\in I(j)}P_i \quad \mathcal{I}=\bigcup_{j=1}^{k}\mathcal{I}(j)$ and $\mathcal{N}=\bigcup_{j=1}^{k}\mathcal{N}(j)$. Then, we can write:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}=\sum_{i\notin\mathcal{I}}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}$$

$$+\sum_{i\in\mathcal{I}}\int_{\partial P_i\backslash(\partial P_i\cap\mathcal{N})}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}$$

$$+\sum_{j=1}^{k}\sum_{i\in I(j)}\int_{\partial P_i\cap N_i(j)}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1}.$$

Applying Lemma 2.3 to the first two terms on the right-side and Lemma 2.9 to the third one we obtain the following:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1} \leq \sum_{i\notin \mathcal{I}}\rho(\partial P_i, P_i)\left[V_{P_i}(f)+\Delta(\partial P_i, P_i)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\right]$$

$$+\sum_{i\in\mathcal{I}}\rho(\partial P_i\backslash(\partial P_i\cap\mathcal{N}), P_i)\left\{V_{P_i}(f)+\Delta(\partial P_i\backslash(\partial P_i\cap\mathcal{N}), P_i)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\right\}$$

$$+\sum_{j=1}^{k}\sum_{i\in I(j)}\rho(\partial P_i\cap N_i(j), N_i(j)\backslash(N_i(j)\cap P_i))$$

$$\left\{V_{N_i(j)}(f)+\Delta(\partial P_i\cap N_i(j), N_i(j)\backslash(N_i(j)\cap P_i))\|f|_{\mathcal{L}^1_{\lambda_n}(N_i(j))}\right\}.$$

We observe that the overlap of the regions in calculating the variation is only caused by the last term (since the regions in the first two terms are disjoint). This together with $N_i(j)$'s being disjoint completes the proof if we let ρ and Δ be the maximum of all ρ's and Δ's in the above equation respectively. □

Theorem 2.6.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau:\Omega\to\Omega$ is an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly k cusps (hypersurfaces) $\{C_1, C_2, \cdots, C_k\}$ each of them of order 1 which are pairwise separable, there exist $M>0$ and $\rho<\infty$ such that

$$V_\Omega(P_\tau f)\leq \alpha^{-1}(1+2\rho)V_\Omega(f)+M\|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.$$

Proof. Applying Lemma 2.10 to Theorem 2.1 and putting $M=K+2\alpha^{-1}\rho\Delta$ completes the proof. □

Definition 2.12. We say a hypersurface C is a *simple cusp of order* K if:

(1) C is in the interior of Ω,

(2) $\lambda_n(C) = 0$; $\lambda_{n-1}(C) = 0$; $\lambda_{n-2}(C) > 0$,

(3) C is a cusp only with respect to the regions $P_i \in P_\Omega$,

for $i \in I_C \subset \{1, 2, \cdots, m\}$ where $|I_C| = K$, and

(4)

$$\theta\Big(\bigcup_{i\in I_C} \partial P_i, \bigcup_{i\in I_C} P_i\Big)_{|_p} = 0 \quad \text{for all } p \in C, \text{and}$$

(5) If $C \cap \partial P_j \neq \emptyset$ for $j \notin I_C$, then $\omega(\partial P_j, P_j) > 0$, and

(6) there exist an open neighborhood N of C such that $N = \bigcup_{i\in I_C} N_i$ where for every $i \in I_C$, N_i is an open neighborhood of C such that $\partial P_i \cap N_i$ is $(\omega, N_i \backslash (N_i \cap P_i))$-regular.

Lemma 2.11.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose that C is a simple cusp of order K in Ω. Then:

$$\sum_{i\in I_C} \int_{\partial P_i \cap N_i} |tr_{P_i}(f)| d\lambda_{n-1}$$

$$\leq \sum_{i\in I_C} \rho(\partial P_i \cap N_i, N_i \backslash (N_i \cap P_i)) \Big\{ V_{N_i}(f) + \Delta(\partial P_i \cap N_i, N_i \backslash (N_i \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i)} \Big\}.$$

Proof. Straightforward consequence of Lemma 2.4. $\square$

Lemma 2.12.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose that C is a simple cusp of order K in Ω.

Then:

$$\sum_{i=1}^{m} \int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1} \leq (1+K) \cdot \rho \cdot [V_\Omega(f) + \Delta \cdot \|f\|_{\mathcal{L}_{\lambda_n^1}(\Omega)}].$$

Proof. We have:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} = \sum_{i \notin I_C} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{i \in I_C} \int_{\partial P_i \setminus (\partial P_i \cap N_i)} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{i \in I_C} \int_{\partial P_i \cap N_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

Applying Lemma 2.3 to the first two terms on the right-side and Lemma 2.11 to the third one we obtain the following:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \leq \sum_{i \notin I_c} \rho(\partial P_i, P_i) \left[V_{P_i}(f) + \Delta(\partial P_i, P_i) \|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)} \right]$$

$$+ \sum_{i \in I_C} \rho(\partial P_i \setminus \partial P_i \cap N_i, P_i) \left\{ V_{P_i}(f) + \Delta(\partial P_i \setminus \partial P_i \cap N_i, P_i) \|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)} \right\}$$

$$+ \sum_{i \in I_C} \rho(\partial P_i \cap N_i, N_i \setminus (N_i \cap P_i)) \left\{ V_{N_i}(f) + \Delta(\partial P_i \cap N_i, N_i \setminus (N_i \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i)} \right\}.$$

We first observe that it is only the last term where the regions over which we calculate the variation overlap. Furthermore, there can only be a maximum number of k overlaps. Therefore, letting ρ and Δ be the maximum of all ρ's and Δ's in the above equation respectively, completes the proof. □

Theorem 2.7.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly one cusp (hypersurface) C which is a simple of order K, there exist $M > 0$ and $\rho < \infty$ such that:

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot \big(1 + (1 + K) \cdot \rho\big) \cdot V_\Omega(f) + M \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.$$

Proof. Applying Lemma 2.12 in Theorem 2.1 and putting $M = K + 2\alpha^{-1} \cdot \rho \cdot \Delta$ completes the proof. □

Consider a collection $G \subset P_\Omega$, $|G| = R$, and a hypersurface C such that C is a cusp with respect to every element of G. Assume G can be decomposed to (at least two) subcollections G_i's, $i \in I_C(G)$ where

$$I_C(G) = \{i : G_i \subset G \ \wedge \ \nexists(j < i \wedge P_j \in G_i)\},$$

and

$$G = \bigcup_{i \in I_C(G)} G_i, \quad \bigcap_{i \in I_C(G)} G_i = \emptyset.$$

Furthermore, there exist an integer K such that

$$K = \max_{i \in I_C(G)} |G_i|.$$

Definition 2.13. Let Ω be any open bounded subset of $\mathbb{R}^n$. We say that a subcollection of the regions $G_i \subset G \subset P_\Omega$ is *inseparable with respect to a hypersurface* C if,

$$\theta\Big(\bigcup_{P_i \in G_i} \partial P_i, \bigcup_{P_i \in G_i} P_i \Big)\Big|_p = 0 \quad \text{for all } p \in C, \text{and}$$

Definition 2.14. We say a hypersurface C is a cusp of order K if:

(1) C is in the interior of Ω,

(2) $\lambda_n(C) = 0$; $\lambda_{n-1}(C) = 0$; $\lambda_{n-2}(C) > 0$,

(3) C is a cusp with respect to R regions where $R > K$,

(4)

$$K = \max_{i \in I_C(G)} |G_i|,$$

where each G_i is an inseparable subcollection, and

(5) there exists a neighborhood N of C contained in Ω such that:

(a) $N = \bigcup N_i$ over all $i \in I_C(G)$, where N_i is an open neighborhood of C with respect to G_i for all $i \in I_C(G)$, and

(b) $N_i \cap N_j = \emptyset$ if $i \neq j$, and

(c) if P_j has a cusp with respect to C and if $P_j \in G_i$, then there exists an open neighborhood $N_i(j) \subset N_i$ of C such that $\partial P_j \cap N_i(j)$ is $\big(\omega, N_i(j) \backslash (N_i(j) \cap P_j)\big)$-regular.

Lemma 2.13.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If hypersurface C is a cusp of order K with respect to a collection $G \subset P_\Omega$, then:

$$\sum_{i \in I_C(G)} \sum_{P_j \in G_i} \int_{\partial P_j \cap N_i(j)} |tr_{P_i}(f)| d\lambda_{n-1} \leq$$

$$\sum_{i \in I_C(G)} \sum_{P_j \in G_i} \rho(\partial P_j \cap N_i(j), N_i(j) \backslash (N_i(j) \cap P_j))$$

$$\left[V_{N_i(j)} + \Delta(\partial P_j \cap N_i(j), N_i(j) \backslash (N_i(j) \cap P_j)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i(j))} \right].$$

Proof. Straightforward consequence of Lemma 2.4. $\square$

Lemma 2.14.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. If hypersurface C is a cusp of order K with respect to a collection $G \subset P_\Omega$, and Ω contains no other cusps, then:

$$\sum_{i=1}^{m} \int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1} \leq (1+K) \cdot \rho \cdot [V_\Omega(f) + \Delta \cdot \|f\|_{\mathcal{L}_{\lambda_n^1}(\Omega)}].$$

Proof. We have:

$$\sum_{i=1}^{m} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \leq \sum_{P_i \notin \bigcup_{j \in I_C(G)} G_j} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}$$
$$+ \sum_{P_i \in \bigcup_{j \in I_C(G)} G_j} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1}.$$

We note that:

$$\sum_{P_i \in \bigcup_{j \in I_C(G)} G_j} \int_{\partial P_i} |\mathrm{tr}_{P_i}(f)| d\lambda_{n-1} \leq \sum_{i \in I_C(G)} \sum_{P_j \in G_i} \int_{\partial P_j \setminus (\partial P_j \cap N_i(j))} |f| d\lambda_{n-1}$$
$$+ \sum_{i \in I_C(G)} \sum_{P_j \in G_i} \int_{\partial P_j \cap N_i(j)} |f| d\lambda_{n-1}.$$

Thus, we have:

$$\sum_{i=1}^{m}\int_{\partial P_i}|\mathrm{tr}_{P_i}(f)|d\lambda_{n-1} \leq$$

$$\sum_{P_i\notin\bigcup_{j\in I_C(G)} G_j} \rho(\partial P_i, P_i)\left[V_{P_i}(f) + \Delta(\partial P_i, P_i)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_i)}\right]$$

$$+ \sum_{i\in I_C(G)}\sum_{P_j\in G_i} \rho(\partial P_j\backslash(\partial P_j \cap N_i(j)), P_i)\times$$

$$\left\{V_{P_j}(f) + \Delta\left(\partial P_j\backslash(\partial P_j \cap N_i(j)), P_i\right)\|f\|_{\mathcal{L}^1_{\lambda_n}(P_j)}\right\}$$

$$+ \sum_{i\in I_C(G)}\sum_{P_j\in G_i} \rho(\partial P_j \cap N_i(j), N_i(j)\backslash(N_i(j)\cap P_j))\times$$

$$\left\{V_{N_i(j)}(f) + \Delta\left(\partial P_j \cap N_i(j), N_i(j)\backslash(N_i(j)\cap P_j)\right)\|f\|_{\mathcal{L}^1_{\lambda_n}(N_i(j))}\right\}.$$

We first observe that it is only the last term where the regions over which we calculate the variation overlap. Furthermore, there can only be a maximum number of k overlaps. Therefore, letting ρ and Δ be the maximum of all ρ's and Δ's in the above equation respectively, completes the proof. □

Theorem 2.8.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau : \Omega \to \Omega$ an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly one cusp (hypersurface) C which is of order K, there exist $M > 0$ and $\rho < \infty$ such that:

$$V_\Omega(P_\tau f) \leq \alpha^{-1}\cdot(1+(1+K)\cdot\rho)\cdot V_\Omega(f) + M\cdot\|f\|_{\mathcal{L}_{\lambda^1_n}(\Omega)}.$$

Proof. Applying Lemma 2.14 in Theorem 2.1 and putting $M = K + 2\alpha^{-1}\rho\Delta$ completes the proof. $\square$

Definition 2.15. If Ω contains s cusps of *possiblely different* order (some of which may be simple), we say $C_1, C_2, \cdots, C_s$ are *pairwise separable* if there exists neighborhoods $N^1, N^2, \cdots, N^s$ such that for every z we have:

(1) $N^z \subset \text{int}\Omega$,

(2) $N^y \cap N^z = \emptyset$, for $y \neq z$,

(3) if hypersurface C^z is a simple cusp of order K then, N^z satisfies Definition 2.12 with respect to C_z, and

(4) if hypersurface C^z is a cusp of order K then, N^z satisfies Definition 2.14 with respect to C_z.

We denote by $\mathcal{C}^-$ the set of all cusps in the above definition for which its respective neighborhood satisfies Definition 2.12 and by $\mathcal{C}^+$ the set of all cusps in the above definition for which its respective neighborhood satisfies Definition 2.14. Let

$$\mathcal{I}_{\mathcal{C}}^{-} = \left\{ i : i \in \bigcup_{C_z \in \mathcal{C}^-} I_{C_z} \right\}$$

and

$$\mathcal{I}_{\mathcal{C}}^{+} = \left\{ i : P_i \in \bigcup_{C_z \in \mathcal{C}^+} \bigcup_{j \in I_{C_z}} G_j \right\}.$$

Lemma 2.15.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose Ω contains s separable cusps and where K

is the order of the highest order cusp among them, then:

$$\sum_{C_z \in \mathcal{C}^-} \sum_{i \in I_{C_z}} \int_{\partial P_i \cap N_i^z} |tr_{P_i}(f)| d\lambda_{n-1}$$

$$+ \sum_{C_z \in \mathcal{C}^+} \sum_{i \in I_{C_z}(G)} \sum_{P_j \in G_i} \int_{\partial P_j \cap N_i^z(j)} |tr_{P_j}(f)| d\lambda_{n-1}$$

$$\leq \sum_{C_z \in \mathcal{C}^-} \sum_{i \in I_{C_z}} \rho(\partial P_i \cap N_i^z, N_i^z \backslash (N_i^z \cap P_i)) \times$$

$$\left\{ V_{N_i^z}(f) + \Delta(\partial P_i \cap N_i^z, N_i^z \backslash (N_i^z \cap P_i)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i^z)} \right\}$$

$$+ \sum_{C_z \in \mathcal{C}^+} \sum_{i \in I_{C_z}(G)} \sum_{P_j \in G_i} \rho(\partial P_j \cap N_i^z(j), N_i^z(j) \backslash (N_i^z(j) \cap P_j))$$

$$\left[V_{N_i^z(j)} + \Delta(\partial P_j \cap N_i^z(j), N_i^z(j) \backslash (N_i^z(j) \cap P_j)) \|f\|_{\mathcal{L}^1_{\lambda_n}(N_i^z(j))} \right].$$

Proof. Straightforward consequence of Lemma 2.4. $\square$

Lemma 2.16.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose Ω contains s separable cusps and where K is the order of the highest order cusp among them, then:

$$\sum_{i=1}^{m} \int_{\partial P_i} |tr_{P_i}(f)| d\lambda_{n-1} \leq (1+K) \cdot \rho \cdot [V_\Omega(f) + \Delta \cdot \|f\|_{\mathcal{L}_{\lambda_n^1}(\Omega)}].$$

Proof. Let $C_1, C_2, \cdots, C_s$ be the s separable cusps and $N^1, N^2, \cdots, N^s$ be the respective neighborhoods as in Definition 2.15. For every $C_i \in \mathcal{C}^-$ we can use the method of Lemma 2.12 and for every cusp $C_i \in \mathcal{C}^+$ we can use the method of Lemma 2.14. Since the cusp of highest order is of order K, thus the overlap of regions caused in calculating the variation along the boundaries of P_j's contained in any of N^i's will be less than or equal to K (N_i's being disjoint). This completes the proof. $\square$

Theorem 2.9.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary and a smooth partition P_Ω. Suppose $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation. Then, if Ω contains exactly s separable cusps and K is the order of the highest order cusp among them.

$$V_\Omega(P_\tau f) \leq \alpha^{-1} \cdot (1 + (1 + K) \cdot \rho) \cdot V_\Omega(f) + M \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)},$$

where $M > 0$.

Proof. Applying Lemma 2.16 to Theorem 2.1 and putting $M = K + 2\alpha^{-1} \cdot \rho \cdot \Delta$ completes the proof. $\square$

For Theorems 2.2-2.9 let β be the coefficient of $V_\Omega(f)$ in the statement of theorem. We then obtain the following result corresponding to those theorems.

Lemma 2.17.

$$\|P_\tau f\|_{BV(\Omega)} \leq \beta \cdot \|f\|_{BV(\Omega)} + (M + 1 - \beta) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}.$$

Proof.

$$\begin{aligned}
\|P_\tau f\|_{BV(\Omega)} &= V_\Omega(P_\tau f) + \|P_\tau f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\
&= V_\Omega(P_\tau f) + \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\
&\leq \beta \cdot V_\Omega(f) + (M + 1) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} \\
&= \beta \cdot \|f\|_{BV(\Omega)} - \beta \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)} + (M + 1) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)}. \quad \square
\end{aligned}$$

Theorem 2.10. *Let $\tau : \Omega \to \Omega, \Omega \subset \mathbb{R}^n$, be a piecewise C^2, expanding transformation. If α (in the Theorems 2.2-2.9) is large enough so that $\beta < 1$, then τ admits an* **acim**.

Proof. It follows from Lemma 2.17, that the set $\left\{\|P_\tau^i(1)\|_{BV(\Omega)}\right\}_{i\geq 1}$ is uniformly bounded. Hence, the set $\left\{P_\tau^i(1)\right\}_{i\geq 1}$ is weakly compact in $\mathcal{L}^1_{\lambda_n}(\Omega)$ (actually it is strongly compact), and it follows from Kakutani-Yosida Theorem [**D-S**]that P_τ has a nontrivial fixed point f^* which is density of an **acim**. □

An immediate consequence of Theorem 2.10 is

Corollary 2.10.1. *Let $\tau : \Omega \to \Omega, \Omega \subset \mathbb{R}^n$, be a piecewise C^2 and expanding such that some iterate τ^k satisfies $\beta < 1$. Then τ admits an* **acim**. □

As examples we now look at two theorems which were proved in [**Kel2**] and [**Jab**].

Example (Keller's Theorem).

*Let Ω be an open bounded subset of R^2 with piecewise C^2 boundary, $\tau : \Omega \to \Omega$ an α-expanding and piecewise C^2 transformation satisfying $C(\omega)$ condition (for definition see [**Kel2**]) be given. Then:*

$$V_\Omega(P_\tau f) \leq \beta \cdot V_\Omega(f) + C \cdot \| f \|_{\mathcal{L}^1_\lambda(\Omega)},$$

where β and C are constants independent of f, $0 < \beta < 1$, $C > 0$.

Proof. This situation allows multiple cusps having different orders. We thus obtain the result by a simple application of Theorem 2.9. □

Example (Jabłoński's Theorem).

Let $\Omega = \cup_{i=1}^{m} P_i$ be an n-dimensional rectangle in $\mathbb{R}^n$ such that $P_i \cap P_k = \emptyset$ for $i \neq k$, where each P_i is also an n-dimensional rectangle in $\mathbb{R}^n$, and $T(x_1, ..., x_n) = (\Phi_{1i}(x_1), ..., \Phi_{ni}(x_n))$ for $(x_1, ..., x_n) \in P_i$ where each $\Phi_{ij}(x_i)$ is a C^2-function and α-expanding with $\alpha > 1$. Then for any $f \in \mathcal{L}^1$ the sequence $\frac{1}{n}\sum_{i=0}^{n-1} P_T^i f$ is convergent in norm to a function $f_* \in \mathcal{L}^1$.

Proof. This is a situation where the boundaries contain no cusps. An application of Theorem 2.2 and then Theorem 2.10 gives the result. □

Spectral Decomposition of P_τ

Theorem 2.11 (Ionescu-Tulcea and Marinescu). *Let $P_\tau : \mathcal{L}^1(\Omega) \to \mathcal{L}^1(\Omega)$ and let it satisfy the following properties:*

(1) *$P_\tau \geq 0$, $\int_\Omega P_\tau(f)d\lambda_n = \int_\Omega f d\lambda_n$, for $f \in \mathcal{L}^1(\Omega$, which implies that $\|P_\tau\|_{\mathcal{L}^1(\Omega)} = 1$.*

(2) *there exist constants $0 < \beta < 1$, $M > 0$ such that*

$$\|P_\tau(f)\|_{BV(\Omega)} \leq \beta\|f\|_{BV(\Omega)} + M\|f\|_{\mathcal{L}^1(\Omega)},$$

for $f \in BV(\Omega)$.

(3) *the image of any bounded subset of $BV(\Omega)$ under P_τ is relatively compact in $\mathcal{L}^1(\Omega)$.*

Then, P_τ is quasi-compact operator on $(BV(\Omega), \|\cdot\|_{BV(\Omega)}$. Thus, P_τ has only finitely many eigenvalues $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of modulus 1. The corresponding eigenspaces E_i are finite-dimensional subspaces of $BV(\Omega)$. Furthermore, P_τ

has the following representation:

$$P_\tau = \sum_{i=1}^{k} \alpha_i P_{\tau_i} + T,$$

where $P_{\tau_i} : BV(\Omega) \to BV(\Omega)$ are linear projections with finite dimensional range onto the E_i's, and $T : BV(\Omega) \to BV(\Omega)$ is a continuous linear operator. For $1 \le i, j \le \xi$, we have: $\int_\Omega \phi_i \psi_j d\lambda_n = \delta_{ij}$, where $\phi_i \in BV(\Omega)$ and $\psi \in \mathcal{L}^\infty(\Omega)$.

In this section, we will use the Theorem 2.11 **[I-M]** to obtain a spectral decomposition of P_τ.

Definition 2.16. Given $f_k, f \in \mathcal{L}^1(\Omega)$ $\Big($respectively $\mathcal{L}^\infty(\Omega)\Big)$, for $k = 1, 2, \ldots,$ we say $f_k \to f$ in $\sigma\big(\mathcal{L}^1(\Omega), BV(\Omega)\big)$–topology $\Big($respectively $\sigma\big(\mathcal{L}^\infty(\Omega), BV(\Omega)\big)$– topology$\Big)$ if and only if for any $\phi \in BV(\Omega)$, we have:

$$\int_\Omega f_k \phi \to \int_\Omega f \phi.$$

Remark 2.3. Consider the space $\big(BV(\Omega), \|\cdot\|_{BV(\Omega)}\big)$ as included in $\big(\mathcal{L}^1(\Omega), \|\cdot\|_{\mathcal{L}^1(\Omega)}\big)$. First we show that the assumptions of the Ionescu Tulcea and Marinescu Theorem are satisfied:

(1) By Remark 1.7, if $\{f_k\} \in BV(\Omega)$ (i.e., $\|f_k\|_{BV(\Omega)} \le B < \infty$) for $n = 1, 2, \cdots$, and $f_k \to f$ in $\mathcal{L}^1(\Omega)$, then $f \in BV$ (i.e., $\|f\|_{BV(\Omega)} \le B < \infty$).

(2) The operator norm of the Perron-Frobenius operator P_τ is 1.

(3) If α (in the Theorems 2.2-2.9) is large enough then, there exist constants $0 < \beta < 1$, $M > 0$ such that

$$\|P_\tau f\|_{BV(\Omega)} \leq \beta \cdot \|f\|_{BV(\Omega)} + (M + 1 - \beta) \cdot \|f\|_{\mathcal{L}^1_{\lambda_n}(\Omega)},$$

for $f \in BV(\Omega)$. (See Lemma 2.17.)

(4) The image of any bounded subset of $BV(\Omega)$ under the Perron- Frobenius operator is relatively compact in $\mathcal{L}^1(\Omega)$, by Remark 1.6.

The Ionescu Tulcea and Marinescu Theorem implies the following result:

Theorem 2.12. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with piecewise C^2 boundary. Suppose $\tau : \Omega \to \Omega$ is an α-expanding and piecewise C^2 transformation with $\beta < 1$. Then:*

(1) *P_τ (as an operator from $BV(\Omega)$ into $BV(\Omega)$) has a finite number of eigenvalues of modulus $1 : \alpha_1, \alpha_2, \cdots, \alpha_k$. They are roots of unity and*

$$P_\tau = \sum_{i=1}^{k} \alpha_i P_{\tau_i} + T,$$

where $P_{\tau_i} : BV(\Omega) \to BV(\Omega)$ are linear projections with finite dimensional range, and $T : BV(\Omega) \to BV(\Omega)$ is a continuous linear operator;

(2) $P_i^2 = P_i$, $P_i P_j = 0$ $(i \neq j)$, $P_i T = T P_i = 0$, $1 \leq i, j \leq k$;

(3) $\|T^m\| \leq \frac{\gamma}{(1+\delta)^m}$, $m = 1, 2, \cdots$ *for some* $\gamma, \delta > 0$.

Remark 2.4. (See [**Ryc**]) Operators P_{τ_i}, $i = 1, 2, \cdots, k$ and T have unique extensions onto $\mathcal{L}^1(\Omega)$. Moreover $P_{\tau_i}(\mathcal{L}^1(\Omega)) \subset BV(\Omega)$, $\|P_{\tau_i}\|_{\mathcal{L}^1(\Omega)} \leq 1$ and $\sup_m \|T^m\| < \infty$. For any $f \in \mathcal{L}^1(\Omega)$, $T^m f \to 0$ in $\mathcal{L}^1(\Omega)$, as $m \to \infty$.

The following theorem and corollaries are consequences of the representation of the Perron-Frobenius operator obtained in Theorem 2.11.

Theorem 2.13. (See [Ryc]). *Assume that* 1 *is the only eigenvalue of* P *with modulus* 1 *(we can consider* P^{η}*, where* η *is the smallest common multiplier of orders of* $\alpha_1, \alpha_2, \cdots, \alpha_k$*). Let* $U_{\varphi} = \varphi \circ \tau$*, for* $\varphi \in \mathcal{L}^{\infty}(\Omega)$*. Then there exist nonnegative functions* $\phi_1, \phi_2, \cdots \phi_{\xi} \in BV(\Omega)$ *and* $\psi_1, \psi_2, \cdots, \psi_{\xi} \in \mathcal{L}^{\infty}(\Omega)$ *such that:*

(1) *For any* $f \in \mathcal{L}^1(\Omega)$,

$$P\tau_1 f = \sum_{i=1}^{\xi} \left(\int_{\Omega} f \psi_i d\lambda_n \right) \phi_i,$$

(2) $P_{\tau} \phi_i = \phi_i, \quad U\psi_i = \psi_i, \quad i = 1, 2, \cdots, \xi.$

(3) *For* $1 \leq i, j \leq \xi$, *we have:*

$\int_{\Omega} \phi_i \psi_j d\lambda_n = \delta_{ij}$,

$\inf\{\phi_i, \phi_j\} = 0 = \inf\{\psi_i, \psi_j\}$, as $i \neq j$, *and*

$\int_{\Omega} \phi_i d\lambda_n = 1.$

(4) *There exist measurable sets* $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_{\xi} \subset \Omega$ *such that* $\psi_i = \chi_{\mathcal{M}_i}$ a.e., *for* $i = 1, 2, \cdots, \xi$ *and* $\Omega = \bigcup_{i=1}^{\xi} \mathcal{M}_i$ a.e.

(5) $\bigcap_{m=1}^{\infty} U^m(\mathcal{L}^1(\Omega)) = \bigcap_{m=1}^{\infty} U^m(\mathcal{L}^{\infty}(\Omega)) = Span\{\psi_1, \psi_2, \cdots, \psi_{\xi}\}.$

(6) *For any* $f \in \mathcal{L}^1(\Omega)$, $U^m f \to P\tau_1^* f$ *in* $\sigma(\mathcal{L}^1(\Omega), BV(\Omega))$*-topology; for any* $f \in \mathcal{L}^{\infty}(\Omega)$, $U^m f \to P_{\tau_1}^* f$ *in* $\sigma(\mathcal{L}^{\infty}(\Omega), \mathcal{L}^1(\Omega))$*- topology;*

$$P_{\tau_1}^* f = \sum_{i=1}^{\xi} \left(\int_{\Omega} f \phi_i d\lambda_n \right) \psi_i.$$

Corollary 2.12.1. (See [**Ryc**]). *For any* $1 \le i \le \xi$, $\tau^k_{\mathcal{M}_i}$ *is an exact transformation.*

Corollary 2.12.2. (See [**Ryc**]). *If we assume that* τ *is mixing (or even weakly mixing, which is equivalent in this situation), and* μ *is its unique* **acim**, *then* τ *has the property of exponential decay of correlations:*

Let $f \in BV(\Omega)$, $g \in \mathcal{L}^\infty(\Omega)$ *and* $\mu(f) = \int_\Omega f d\mu$, $\mu(g) = \int_\Omega g d\mu$. *Then*

$$\int_\Omega (fg(\tau^i) - \mu(f)\mu(g))d\mu \le \beta^i \mu(f)V(f)\|g\|_{\mathcal{L}^\infty(\Omega)}, \quad i = 1, 2, \cdots,$$

where $0 < \beta < 1$ *is the constant of Remark 2.3 condition (3).*

Corollary 2.12.3. (See [**Ryc**]). *If we assume that* τ *is mixing, then defining partition* $\{P_i\}_{i=1}^m$ *is weakly Bernoulli for* τ, *which implies that the natural extension of the dynamical system* (τ, μ) *is isomorphic to a Bernoulli shift (*μ *is a the* τ*- invariant absolutely continuous measure).*

Corollary 2.12.4. (See [**H-K**]). *Assume that* (τ, μ) *is weakly mixing (it is equivalent to being mixing or exact in our situation) . Let* $f \in BV(\Omega)$ *and* $\mu(f) = \int_\Omega f d\mu = 0$. *Define*

$$S(t) = \sum_{i=1}^{t-1} f \circ \tau^i,$$

which is a stochastic process on (S, μ). *Then the series*

$$\sigma^2 = \int_\Omega f^2 d\mu + 2\sum_{k=1}^{\infty} \int_\Omega f(f \circ \tau^k)d\mu,$$

converges absolutely, $\int_\Omega S(t)^2 d\mu = t\sigma^2 + \mathcal{O}(1)$ *and, if* $\sigma^2 \neq 0$, *the following holds:*

(1)

$$\sup_{\eta\in\mathbb{R}}\left|\mu\big((\sigma^2 t)^{\frac{-1}{2}} S(t) \leq \eta\big) - (2\pi)^{\frac{-1}{2}} \int_{-\infty}^{\eta} \exp\Big(\frac{-x^2}{2}\Big)dx\right| = \mathcal{O}(t^{-\nu}),$$

for some $\nu > 0$.

(2) *Without changing its distribution, one can define the process* $(S(t))_{t\geq 0}$ *on a richer probability space together with the standard Brownian motion* $(B(t))_{t\geq 0}$ *such that*

$$\left|\sigma^{-1}S(t) - B(t)\right| = \mathcal{O}(t^{\frac{1}{2}-\epsilon}), \quad \mu - \text{a.e.},$$

for some $0 < \epsilon < \frac{1}{2}$.

(3) *The process* $(S(t))_{t\geq 0}$ *satisfies the iterated log law and other properties of Brownian motion.*

CHAPTER 3

ON THE NUMBER OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES

For one-dimensional transformations [**Li-Y**], $\tau : I \to I, I = [0,1]$, it is well known that the number of discontinuities of $\tau'(x)$ provides an upper bound for the number of independent **acim**. This result has been improved in [**Boy1**], [**Pia**], [**B-H**] and [**B-B**]. The key to all these bounds lies in the fact that invariant densities for piecewise C^2 expanding transformations are of bounded variation. In one dimension, a density of bounded variation is bounded and it can be proved that its support consists of a finite union of closed intervals. A simple argument then shows that at least one point of discontinuity of τ' must lie in the largest closed interval, which will provide an upper bound on the number of **acim**. In higher dimensions, the much more complicated geometrical setting and the complex form of the definition of bounded variation [**Giu**] do not permit an easy generalization of the one-dimensional result. For example, in two dimensions, the variation in one direction is integrated along the other direction. It is this integration which allows a function of bounded variation in $\mathbb{R}^n$ to be unbounded and its support to be devoid of interior.

In general, dynamical systems can have a large set of invariant measures. For example, higher dimensional point transformation models for cellular automata [**G-B 2**], can have many **acim**.

In 1990, Góra, Boyarsky and Proppe [**G-B-P**], outlined the possibility of constructing a piecewise expanding C^2 transformation on a fixed partition with a finite number of elements which have an arbitrarily large number of ergodic **acim**. There the use of certain triangles having a particular geometry as the supports of ergodic **acim** is suggested. By means of a sketch, it is outlined (without proof, however) that it is possible to take care of the trapezoidal regions between triangles satisfying all conditions. Although the conjecture turns out to be correct, we will see that the construction cannot be done in a simple manner.

We use the triangles suggested in [**G-B-P**] as supports of ergodic **acim**. For the trapezoidal regions between the triangles, we use another set of triangles which are not supports of **acim**, satisfying the following conditions:

(1) The triangles are mapped in an expanding manner similar to that of the triangles which are supports of ergodic **acim**, and

(2) the intersection of images of triangles which are supports of ergodic **acim** and images of triangles which are not supports of ergodic **acim** is empty.

This reduces the trapezoidal regions to rectangular regions. We will then, by the aid of Lemma 3.1, map each such rectangular region to a "tube" in a C^2 and expanding manner in such a way that the tube does not intersect the images of the triangles that support the ergodic **acim**.

Finally, by making small perturbation to these maps near the "vertical" edges of these rectangular regions, we can obtain a map that is C^2 and expanding on all of S_1 (respectively S_{-1}) (see Figure 3.2).

Main Results

In this chapter, we will construct a piecewise expanding C^2 transformation on a fixed partition with 10 elements which has an arbitrarily large number of ergodic **acim**.

Lemma 3.1. *For $L > 0$ large enough, there exists an expanding C^2 diffeomorphism of a rectangle R with sides L and 1 into a tube T (similar to the* *one shown in Figure 3.1).*

Figure 3.1

Proof. Consider the straight line joining the point $(x_0, 0)$ to the point $(L + x_0, 0)$. Let P be the curve such that length of P is equal to αL where $\alpha > 1$,

$$(3.1) \qquad P(x_0) = 0\ ,\quad P(L + x_0) = -1\ ,\quad P'(x_0) = 1\ ,\quad P'(L + x_0) = 1.$$

Now we parametrize P by arc length $s = \alpha x$:

$$x \longrightarrow \big(u(\alpha x), v(\alpha x)\big),$$

where

$$(3.2) \qquad v(\alpha x) = P(u(\alpha x))$$

and

$$(3.3) \qquad u'^2(s) + v'^2(s) = 1.$$

If we shift the curve P along the u-axis by 1 unit i.e $\big(u(\alpha x), v(\alpha x)\big) \to \big(1 + u(\alpha x), v(\alpha x)\big)$ we can construct the tube $\mathcal{T}$ by moving P along its upward normal vector a distance of $\sqrt{2}$.

Now we would like to construct the mapping of the rectangle R to the tube $\mathcal{T}$.

Let $\vec{n}$ be the upward normal vector to the curve P, i.e,

$$\vec{n} = (n_1, n_2) = (-v', u'). \tag{3.4}$$

To derive an expression for $\vec{n}$ in terms of P, first differentiate (3.2) with respect to arc length s, which implies that

$$v' = P'u'. \tag{3.5}$$

Using (3.5) in (3.3), we get:

$$u' = \frac{1}{\sqrt{1 + P'^2}} \qquad v' = \frac{P'}{\sqrt{1 + P'^2}}. \tag{3.6}$$

Now using (3.6) in (3.4), we get:

$$\vec{n} = (n_1, n_2) = \Big(\frac{-P'}{\sqrt{1 + P'^2}}, \frac{1}{\sqrt{1 + P'^2}}\Big). \tag{3.7}$$

Now note that the width of the tube in the direction of $\vec{n}$ is $\sqrt{2}$, thus as y changes from 0 to 1 the length of the tube in the direction of the normal vector $\vec{n}$ changes by the factor of $\sqrt{2}y$.

Therefore the two-dimensional transformation τ from the rectangle to the tube is given by:

$$\tau(x, y) = \big(1 + u(\alpha x) + \sqrt{2}yn_1, v(\alpha x) + \sqrt{2}yn_2\big).$$

Thus, the Jacobian matrix of τ is given by:

$$J_\tau = \begin{pmatrix} \alpha u' + \sqrt{2} y \frac{\partial n_1}{\partial x} & \sqrt{2} n_1 \\ \alpha v' + \sqrt{2} y \frac{\partial n_2}{\partial x} & \sqrt{2} n_2 \end{pmatrix}. \tag{3.8}$$

Calculating $\frac{\partial n_1}{\partial x}$ and $\frac{\partial n_2}{\partial x}$ in terms of P and its derivatives, using (3.7), we obtain:

$$\frac{\partial n_1}{\partial x} = \frac{-\alpha P''}{(1+P'^2)^{\frac{3}{2}}} \quad \frac{\partial n_2}{\partial x} = \frac{-\alpha P' P''}{(1+P'^2)^{\frac{3}{2}}}. \tag{3.9}$$

For the curve P which is the set of points $\big(u(s), v(s)\big)$ the unit tangent vector $\vec{T}$ is given by: $\vec{T} = (u', v') = \big(u'(s), P'(u)u'(s)\big)$.

Recall that the curvature κ (which we define to be non-negative) is given by:

$\kappa^2 = |\frac{d\vec{T}}{ds}|^2$ where $\frac{d\vec{T}}{ds} = (u'', P' \cdot u'' + u' \cdot P'')$.

Thus, we have:

$$\kappa^2 = |\frac{d\vec{T}}{ds}|^2 = u''^2 + (P' \cdot u'' + u' \cdot P'')^2. \tag{3.10}$$

Using (3.6), we get:

$$u'' = \frac{-P' \cdot P''}{(1+P'^2)^{\frac{3}{2}}}. \tag{3.11}$$

Thus:

$$\begin{aligned}
\kappa^2 &= u''^2 + P'^2 u''^2 + u'^2 P''^2 + 2u'u''P'P'' \\
&= u''^2(1+P'^2) + u'^2 P''^2 + 2u'u''P'P'' \\
&= \frac{P'^2 P''^2}{(1+P'^2)^3}(1+P'^2) + \frac{1}{(1+P'^2)} P''^2 + 2\frac{1}{(1+P'^2)^{\frac{1}{2}}} \frac{P'P''}{(1+P'^2)^{\frac{3}{2}}} \\
&= \frac{P''^2}{(1+P'^2)^2},
\end{aligned}$$

and we thus obtain:

$$\kappa = \frac{|P''|}{1+P'^2}. \tag{3.12}$$

We express $\frac{\partial n_1}{\partial x}$ and $\frac{\partial n_2}{\partial x}$ in terms of κ in the case $P'' > 0$ (respectively $P'' < 0$):

$$\frac{\partial n_1}{\partial x} = \mp \kappa \cdot n_2 \quad \pm \frac{\partial n_2}{\partial x} = \kappa \cdot n_1. \tag{3.13}$$

Using (3.13) in (3.8), the Jacobian matrix of τ becomes:

$$J_\tau = \begin{pmatrix} \alpha u' \mp \sqrt{2} y \kappa n_2 & \sqrt{2} n_1 \\ \alpha v' \pm \sqrt{2} y \kappa n_1 & \sqrt{2} n_2 \end{pmatrix}.$$

Using (3.4) in the above equation we get:

$$J_\tau = \begin{pmatrix} \alpha u' \mp \sqrt{2} y \kappa u' & -\sqrt{2} v' \\ \alpha v' \mp \sqrt{2} y \kappa v' & \sqrt{2} u' \end{pmatrix}.$$

Using (3.5) in the above equation, we get:

$$J_\tau = \begin{pmatrix} \alpha u' \mp \sqrt{2} y \kappa u' & -\sqrt{2} u' P' \\ \alpha u' P' \mp \sqrt{2} y \kappa u' \cdot P' & \sqrt{2} u' \end{pmatrix}.$$

Factoring out u', we get:

$$J_\tau = u' \begin{pmatrix} \alpha \mp \sqrt{2} y \kappa & \mp\sqrt{2} P' \\ \alpha P' \mp \sqrt{2} y \kappa P' & \sqrt{2} \end{pmatrix}.$$

Thus, we have:

$$J_\tau = u' \begin{pmatrix} \beta & -\sqrt{2}P' \\ \beta P' & \sqrt{2} \end{pmatrix},$$

where $\beta = \alpha \mp \sqrt{2}y\kappa$ for $P'' > 0$ (respectively $P'' < 0$). Therefore, the eigenvalues of the Jacobian matrix will be:

$$\lambda = \frac{1}{2\sqrt{1+P'^2}}\left((\beta+\sqrt{2}) \pm \sqrt{(\beta-\sqrt{2})^2 - 4\sqrt{2}\beta P'^2}\right).$$

Now we note that if $(\beta-\sqrt{2})^2 < 4\sqrt{2}\beta P'^2$, then the eigenvalues are complex and

$$|\lambda_1| = |\lambda_2| = \frac{1}{2\sqrt{1+P'^2}}\sqrt{(\beta+\sqrt{2})^2 + 4\sqrt{2}\beta P'^2 - (\beta-\sqrt{2})^2},$$

and if $(\beta-\sqrt{2})^2 \geq 4\sqrt{2}\beta P'^2$, then the eigenvalues are real and

$$|\lambda_{\min}| = \frac{1}{2\sqrt{1+P'^2}}\left((\beta+\sqrt{2}) - \sqrt{(\beta-\sqrt{2})^2 - 4\sqrt{2}\beta P'^2}\right).$$

For $1.001 < \beta < 1.45$ and $0 \leq P' \leq 1.05$ both eigenvalues are strictly greater than 1 in absolute value, i.e. the mapping of the rectangle to the tube is expanding.

Let R^{-1} (respectively $\mathcal{T}^{-1}$) denote the reflection of R (respectively $\mathcal{T}$) about the x-axis. Let $\mathcal{T}^v$ = reflection of $\mathcal{T}$ about the vertical line $x = \frac{L}{2} + x_0$ and $\mathcal{T}^{-v} = (\mathcal{T}^v)^{-1} = (\mathcal{T}^{-1})^v$.

Note that the construction of $\mathcal{T}^{-1}$ from R^{-1} is isometric to the construction of $\mathcal{T}$ from R.

Now we will give an example of a function P which satisfies the criteria mentioned in Lemma 3.1.

Example 3.1. Let

$$P(t) = \frac{L}{2\pi}\sin\frac{2\pi}{L}(t-1) + \frac{1}{2}\cos\frac{\pi}{L}(t-1) - \frac{1}{2}$$

where L is to be determined later, and $1 \leq t \leq L+1$. Then:

$$P'(t) = \cos\frac{2\pi}{L}(t-1) - \frac{\pi}{2L}\sin\frac{\pi}{L}(t-1)$$

We note that $P(1) = 0$ $P(L+1) = -1$ and $P'(1) = P'(L+1) = 1$ (i.e Equation 3.1 is satisfied).

Also note that,

$$P''(t) = \frac{-2\pi}{L}\sin\frac{2\pi}{L}(t-1) - \frac{\pi^2}{2L^2}\cos\frac{\pi}{L}(t-1).$$

We have the following estimates for κ:

$$|P'(t)| < 1 + \frac{\pi}{2L},$$

$$\kappa \leq |P''(t)| \leq \frac{2\pi}{L} + \frac{\pi^2}{2L^2} = \frac{4\pi L + \pi^2}{2L^2}.$$

The amount of expansion applied to the line to produce the curve P is given by:

$$\alpha = \frac{1}{L}\int_1^{L+1} \sqrt{1 + P'^2(t)}dt.$$

It is easy to see that for $L > 100$, $\beta \in (1.1, 1.33)$ and $|P'| < 1.039$; therefore, if we choose $L > 100$ then both eigenvalues will be larger than 1 in absolute value.

By Δpqr we denote the triangle with vertices at the points p, q and r, and by $\Box pqrs$ we denote the rectangle with vertices at the points p, q, r and s.

We are now ready to prove the main result of this chapter, which is the construction of arbitrarily large number of ergodic **acim** for a piecewise expanding C^2 transformation in $\mathbb{R}^2$ on a fixed partitition. Proppe's **[G-B-P]** idea of using triangles as ergodic sets of **acim** is very essential in this construction.

Theorem 3.1. *For any number k there exists a two dimensional piecewise C^2 expanding transformation with 10 elements which has at least k ergodic* **acim**.

Proof. We prove the theorem by the means of a construction. Consider the following 10 elements partition of Ω: where $\ell(k) = 2k + 1 + 2kL$ and L is to be determined later, where z is large enough so that for every $(x, y) \in \mathcal{T}$ defined in Lemma 3.1 we have $y \leq z$.

Let $P^1 = \Box(1,1)(\ell(k),1)((\ell(k),z)(1,z)$ and P^{-1} be its reflection about the x-axis. Each P^j is subdivided into 4 rectangles P_i^j for $1 \leq i \leq 4$ and $j = -1, 1$ as shown in figure 3.2. The exact manner of subdivision is irrelevant. Let define τ as follows:

$$\forall 1 \leq i \leq 4, j \in \{-1, 1\} \quad \tau(P_i^j) = \bigcup_{i=1}^{4} P_i^j.$$

Thus, it remains to define τ on S_1 and S_{-1} (see figure 3.2 for definition of S_1 and S_{-1}). Now we define the sets E_i for $1 \leq i \leq 2k + 1$ on $S_1 \cup S_{-1}$

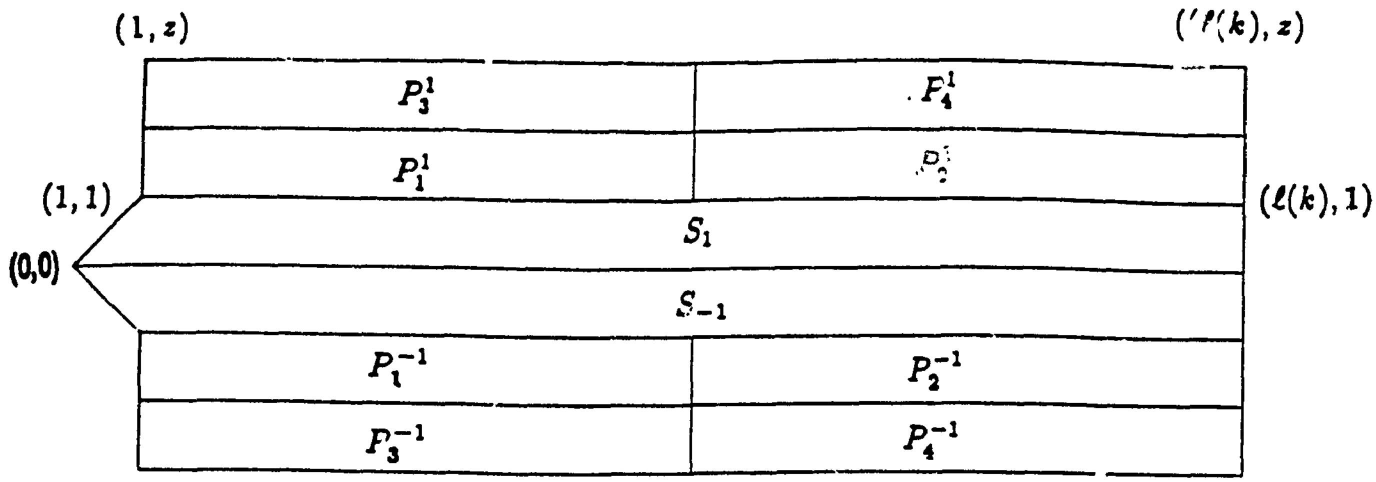

Figure 3.2

which are the sets that will produce supports for the $k+1$ ergodic **acim**. Let $E_i = \triangle a_i^0 b_i^1 b_i^{-1}$ where, $a_i = (i-1)L+i-1$, $a_i^j = (a_i, j)$ and $b_i^j = (a_i+1, j)$ for $j \in \{-1, 0, 1\}$ and $1 \le i \le 2k+1$, we also define $c_i^0 = (a_i + 2, 0)$ $\; 1 \le i \le 2k$, see Figure 3.4.

We define the triangles which are the supports of ergodic **acim** as follows:

$$T_i^j = E_i \cap S_j \quad 1 \le i \le 2k+1,\ j \in \{1, -1\},$$

see Figure 3.3.

Let $R_i = \Box a_i^{-1} a_i^1 b_i^1 b_i^{-1}$. We define the triangles which are not supports of ergodic **acim** as follows:

$$\tilde{T}_i^j = (R_i \cap S_j) \backslash E_i \quad 2 \le i \le 2k+1,\ j \in \{-1, 1\},$$

see Figure 3.3.

For $1 \le i \le 2k$ let $\tilde{E}_i = \triangle b_i^{-1} b_i^1 c_i^0$, see Figure 3.4. Let

$$v_j = \begin{pmatrix} \ell(k) \\ j(-iL + L - i) \end{pmatrix}, \quad M_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then for a point $p \in R_i \cap S_j$ we define:

$$\tau(p) = v_j + M_j \cdot p.$$

Then

$$\tau(T_i^j) = E_{2k-i+2} \text{ and } \tau(\tilde{T}_i^j) = \tilde{E}_{2k-i+2}$$

and

$$\tau(b_i^0) = a_{2k-i+2}^0 \quad \tau(a_i^0) = b_{2k-i+2}^{-j} \quad \tau(b_i^j) = b_{2k-i+2}^j \quad \tau(a_i^j) = c_{2k-i+2}^0,$$

see Figures 3.3 and 3.4.

Note that in this construction τ is continuous across the boundaries between the triangles, since τ is an affine map on $R_i \cap S_j = T_i^j \cup \tilde{T}_i^j$.

Remark 3.1. In what will follow $\mathcal{T}_i^j$ and $\mathcal{R}_i^j$ correspond to $\mathcal{T}$ and R in Lemma 3.1.

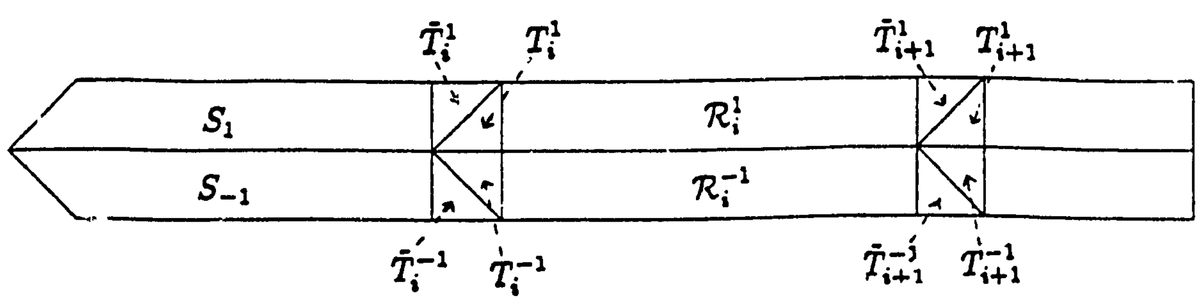

Figure 3.3

Let $\mathcal{R}_i^j = \Box b_i^0 a_{i+1}^0 a_{i+1}^j b_i^j$. and define $\tau(\mathcal{R}_i^j) = \mathcal{T}_{2k-i+2}^{jv}$ (see Lemma 3.1 and figure 3.5).

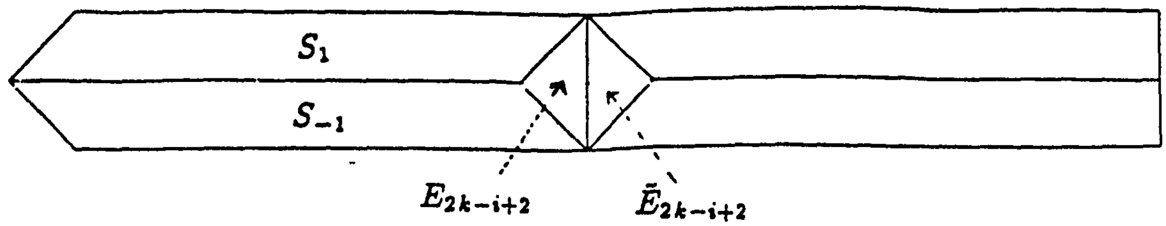

Figure 3.4

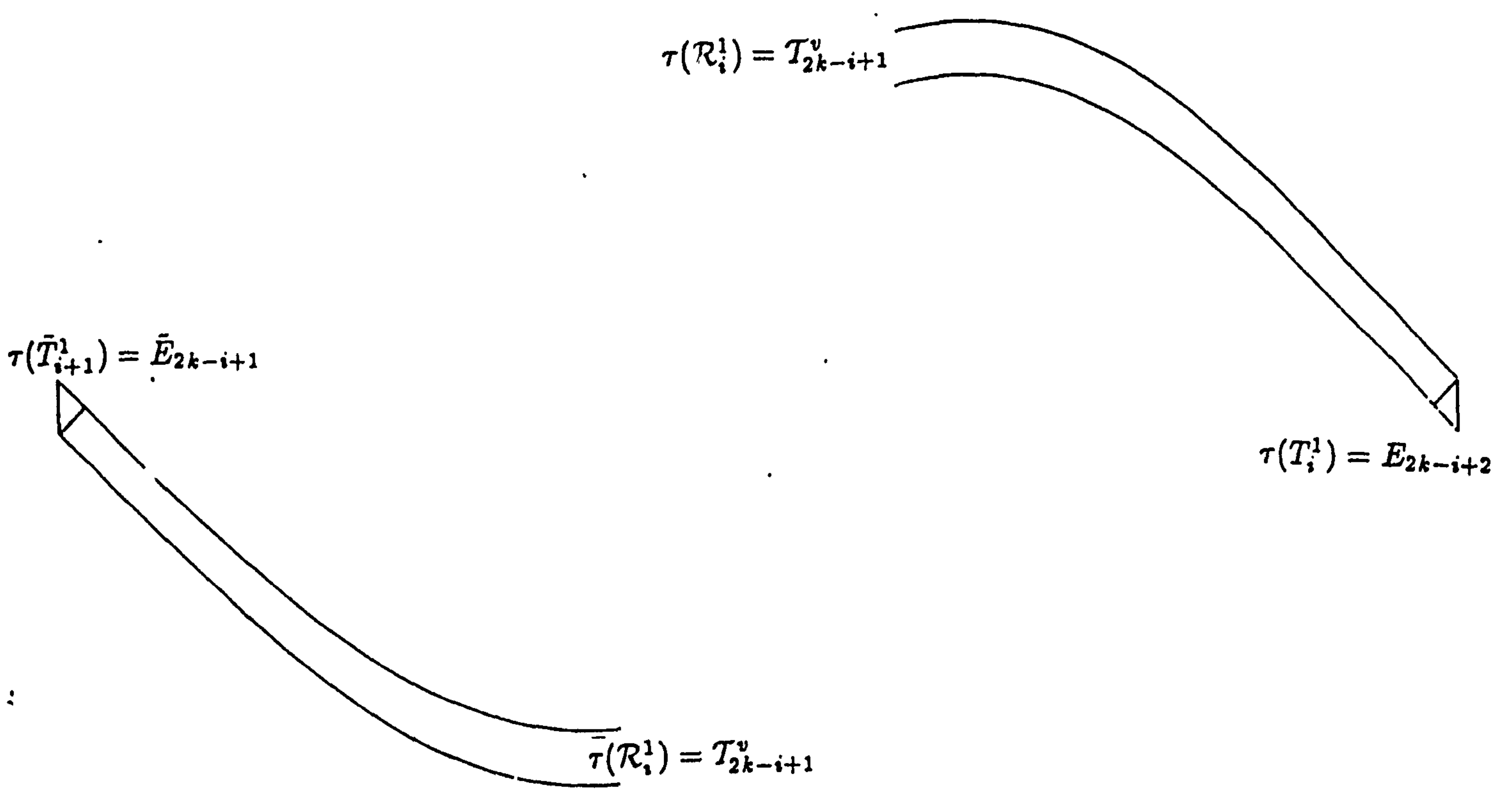

Figure 3.5

Note that τ is C^2 and expanding on each subregion of S_1 (respectively S_{-1}) and C^0 on all of S_1 (respectively S_{-1}) for the following reasons:

(1) $\tau_{|_{R_i \cap S_j}}$, $\tau_{|_{\mathcal{R}_i^j}}$ and $\tau_{|_{R_{i+1} \cap S_j}}$ are C^2 on their closed domains ($R_i \cap S_j$, $\mathcal{R}_i^j$ and $R_{i+1} \cap S_j$ respectively) and coincide on $(R_i \cap S_j) \cap \mathcal{R}_i^j$ and $(R_{i+1} \cap S_j) \cap \mathcal{R}_i^j$.

(2) $\tau|_{R_i \cap S_j}$ is an affine map with expansion constant equal to $\sqrt{2}$ and $\tau|_{\mathcal{R}_i^j}$ is expanding by Lemma 3.1 for $L \geq 100$.

Finally, we can perturb τ slightly in $\mathcal{R}_i^j$ near its "vertical" edges so that the resulting map is C^2 and expanding on all of S_j, $j = -1, 1$. Using Theorem 2.5 of **[Hir]**, for each $\epsilon > 0$ there are (relatively) open sets $N_i^j(\epsilon)$ of the form $(0,\epsilon) \times [0,h]$ and $(L-\epsilon, L) \times [0,h]$ in each rectangle $\mathcal{R}_i^j = [0,L] \times [0,h]$ (in local coordinates), and perturbations of τ on these open sets, so that the perturbed map $\tilde{\tau}$ is C^2 on all of S_j, $j = -1, 1$, and agrees with τ outside of these open sets. It is clear that $\tilde{\tau}$ is expanding on these open sets for suitably small ϵ.

Let

$$\mathcal{E}_i = E_i \cup E_{2k-i+2} \quad \text{for} 1 \leq i \leq k+1.$$

Then, we have, $\tau(\mathcal{E}_i) = \mathcal{E}_i = \tau^{-1}(\mathcal{E}_i)$, for $1 \leq i \leq k+1$, so each $\mathcal{E}_i$ is an invariant set of positive Lebesgue measure. Since on each $\mathcal{E}_i$, τ is piecewise expanding and onto, by Remark 1.9 each $\mathcal{E}_i$ supports exactly one ergodic **acim**. Since there are $k+1$ distinct $\mathcal{E}_i$, the proof is complete. $\square$

CHAPTER 4

SMOOTHNESS OF INVARIANT DENSITY FOR EXPANDING MARKOV TRANSFORMATIONS IN HIGHER DIMENSIONS

Rényi [**Rén**], proved that piecewise transformation of unit interval onto itself, satisfying the distortion condition admits an **acim**. Halfant [**Hal**], proved that for the maps considered in [**Rén**], if the transformation is of class C^M then the invariant density is of class C^{M-2}.

In higher dimensions, the existence of **acim** for expanding Markov (which is a class of maps that is a superset of maps considered by Rényi [**Rén**]) maps was proved by Mané [**Man**]. In this chapter, we prove that, if a transformation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is expanding Markov and of class C^M then its invariant density is of class C^{M-2}.

Main Results

We denote by λ the Lebesgue measure. For a $n \times n \times \cdots \times n$ (k-times) array M_k we define its norm

$$\|M_k\| = \max_{\mathcal{N}_k} \left|(m)_{i_1 \cdots i_k}\right|,$$

where

$$\mathcal{N}_k = \{i_1 i_2 \cdots i_k : 1 \le i_j \le n \ \text{ for } \ 1 \le j \le k\}.$$

For a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $\mathbf{D}f$ its derivative, and by $\mathbf{D}^{(M)}f$ the M-th derivative of f. If $f(x) = f(x_1, x_2, \cdots, x_n)$ then $(\mathbf{D}f)_x$ is a linear map : $\mathbb{R}^n \to \mathbb{R}$ and $(\mathbf{D}f)_x(\vec{v}) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}$.

Definition 4.1. For an invariant measure μ, absolutely continuous with respect to λ, the invariant density h is uniquely given by

$$\mu = \int h d\lambda$$

We denote by J_τ the Jacobian matrix of τ and by $\mathcal{J}_\tau$ the absolute value of the determinant of J_τ.

We also denote by $\mathcal{P}_n$ the partition of Ω under τ^{-n} and we set

$$\mathcal{I}_n = \{i : P_i \in \mathcal{P}_n\}.$$

Definition 4.2. Let $(\Omega, \mathcal{A}, \nu)$ be a probability space, where Ω is a separable metric space and $\mathcal{A}$ its Borel $\sigma-$algebra. We say that a map $\tau : \Omega \to \Omega$ is *expanding Markov* if there exists a sequence of partition $(\mathcal{P}_i)_{i \geq 0}$ such that:

(a) $\bigcup_{P \in \mathcal{P}_0} = \Omega$

(b) For every $n \geq 0$ and $P \in \mathcal{P}_{n+1}, \tau(P)$ is a union (mod 0) of atoms of $\mathcal{P}_n$, and $\tau_{|P}$ is injective.

(c) There exists $0 < \eta < 1$ and $K > 0$ such that,

$$d(x, y) \leq K\eta^n d((\tau^n(x), \tau^n(y))$$

for every $n \geq 0, x, y \in P_i$ where $i \in \mathcal{I}_n$.

(d) There exists $k > 0$ such that, for every pair of atoms $P, Q \in \mathcal{P}_0$, we have $\mu(\tau^{-k}(P) \cap Q) \neq 0$.

(e) There exist $\mathcal{J} : \Omega \to \mathbf{R}^+, 0 < \zeta < 1$ and $C > 0$ such that, for every $n \geq 0$ and every Borel set A contained in an atom of $\mathcal{P}_0$, we have

$$\nu(\tau(A)) = \int_A \mathcal{J} d\mu,$$

and for every x, y contained in the same atom P_i of $\mathcal{P}_n$ we have

$$|\frac{\mathcal{J}_{\tau_i}(y)}{\mathcal{J}_{\tau_i}(x)} - 1| \leq Cd(\tau_i(x), \tau_i(y))^{\zeta}.$$

Condition (c) of Definition 4.2 implies that:

$$d(\tau_i^{-n}(x), \tau_i^{-n}(y)) \leq K\eta^n d(x,y), \quad \text{where } \tau_i^{-n}(x), \tau_i^{-n}(y) \in P_i \in \mathcal{P}_n$$

for every $n \geq 0$. This in turn implies that, for N large enough we have:

$$(4.1) \qquad \frac{d(\tau_i^{-N}(x), \tau_i^{-N}(y))}{d(x,y)} < 1.$$

Set

$$\tau_i^{-N} = (\phi_1, \phi_2, \cdots, \phi_n).$$

Equation 4.1 is true for any $x, y \in P_i \in \mathcal{P}_N$. Thus for a fixed x, if y approaches x in the direction of any of the n coordinate axis, (4.1) will hold true. We then have:

$$|\frac{\partial \phi_j(x)}{\partial x_k}| < 1 \quad \text{where } 1 \leq j, k \leq n.$$

Therefore, there exists an N such that for all $i \in \mathcal{I}_N$ we have:

$$(4.2) \qquad \|\mathbf{D}\tau_i^{-N}(x)\| < 1 \quad \text{for all } x \in P_i.$$

Definition 4.3. We define the measures λ_n by:

$$\lambda_n(A) = \lambda(\tau^{-n}(A)).$$

Theorem 4.1 (Mané). *If τ is expanding Markov then it admits an invariant probability measure μ, absolutely continuous with respect to Lebesgue measure and*

$$\lim_{n\to\infty} \lambda_n(A) = \mu(A) \quad \forall A \in \mathcal{B},$$

where $\mathcal{B}$ is the Borel σ-algebra of Ω.

Definition 4.4. Perron-Frobenius operator $P_\tau : \mathcal{L}^1 \to \mathcal{L}^1$ is defined as follows:

$$P_\tau f(x) = \sum_{i=1}^{m} \frac{f(\tau_i^{-1}(x))}{\mathcal{J}_\tau(\tau_i^{-1}(x))}, \quad f \in \mathcal{L}^1 \text{ and } \#(\mathcal{P}_0) = m.$$

Definition 4.5. We define the iterated densities $S_n(x)$ for almost all $x \in \Omega$:

$$S_n(x) = P_\tau^n \mathbf{1}(x) = \sum_{i\in\mathcal{I}_n} \mathcal{J}_{\tau_i^{-n}}(x). \tag{4.3}$$

We note that,

$$\int_A (P_\tau^n \mathbf{1}) d\lambda = \int_{\tau^{-n}(A)} \mathbf{1} d\lambda = \int_A d\lambda_n.$$

Thus, we get:

$$S_n = \frac{d\lambda_n}{d\lambda}. \tag{4.4}$$

Lemma 4.1. *There exists, constant $B^{(0)} > 0$ such that $S_n(x) \le B^{(0)}$ for almost every $x \in \Omega$ and $n \ge 1$.*

Proof. See [Man].

Theorem 4.2. *Let $\tau : \Omega \to \Omega$ be an expanding Markov map and $\tau_i^{-1} \in C^M$, $M \geq 2$. Then the sequence*

$$\{\sup_{x \in \Omega} \|\mathbf{D}^{(M-1)} S_n(x)\|\}$$

is uniformly bounded.

Proof. We prove the theorem by induction. First we note that:

$$S_{K+N}(x) = \sum_{i \in \mathcal{I}_N} S_K(\tau_i^{-N}(x)) \mathcal{J}_{\tau_i^{-N}}(x)$$

We take N so that Equation 4.2 is satisfied. Let $M = 2$. By differentiation we obtain:

$$\mathbf{D}S_{K+N}(x) = \sum_{i \in \mathcal{I}_N} \{\mathbf{D}S_K(\tau_i^{-N}(x))\mathbf{D}\tau_i^{-N}(x)\mathcal{J}_{\tau_i^{-N}}(x) + S_K(\tau_i^{-N}(x))\mathbf{D}\mathcal{J}_{\tau_i^{-N}}(x)\}.$$

For N large enough we have:

$$\gamma = \sup_{x \in \Omega} \|\mathbf{D}\tau_i^{-N}(x)\| < \frac{1}{B^{(0)}} < 1.$$

Set

$$\beta_1^{(2)} = \sup_{x \in \Omega} \sum_{i \in \mathcal{I}_N} \|\mathbf{D}\mathcal{J}_{\tau_i^{-N}}(x)\| < \infty,$$

and

$$B_n = \sup_{x \in \Omega} \|\mathbf{D}S_n(x)\|.$$

Then using Lemma 4.1 and Equation 4.3 we have:

$$B_{K+N} \leq B_K \gamma B^{(0)} + B^{(0)} \beta_1^{(2)}.$$

This implies that the sequence

$$B_K, B_{K+N}, B_{K+2N}, \cdots$$

is uniformly bounded by some number $\hat{B}_K$ and hence entire sequence $\{B_n\}$ is bounded by $B^{(1)} = \max\{\hat{B}_0, \hat{B}_1, \cdots, \hat{B}_{N-1}\}$.

Now we assume the theorem is true for M and prove it for $M+1$ (i.e we assume that $\tau_i^{-1} \in C^{M+1}$ and prove that $\{\sup_{x\in\Omega} \mathbf{D}^{(M)} S_n(x)\}$ is uniformly bounded).

We note that:

$$\mathbf{D}^{(M)} S_{K+N}(x) = \sum_{i\in\mathcal{I}_N} \mathbf{D}^{(M)} S_K(\tau_i^{-N}(x))\left(\mathbf{D}\tau_i^{-N}(x)\right)^M \mathcal{J}_{\tau_i^{-N}}(x)$$

$$+\sum_{j=0}^{M-1}\left(\mathbf{D}^{(j)} S_K(\tau_i^{-N}(x))\sum_{r=1}^{r(j)} c_r \prod_{t=1}^{t(r)}\left(\mathbf{D}^{(a(t))}\tau_i^{-N}(x)\right)^{(b(t))}\left(\mathbf{D}^{(c(t))}\mathcal{J}_{\tau_i^{-N}}(x)\right)^{d(t)}\right)$$

where each c_r is an integer, $a(t), b(t), c(t)$ and $d(t)$ are integers less than M. It is clear that differentiating S_{N+K} will not produce any term different from the above form. Now note that by induction we have for $j = 0, 1, 2, \cdots, M-1$ constants $B^{(j)}$ which are bounds for the sequences $\{\sup_{x\in\Omega} \mathbf{D}^{(j)} S_n(x)\}$ respectively.

We also have:

$$\sup_{x\in\Omega} \|\left(\mathbf{D}^{(a(t))}\tau_i^{-N}(x)\right)^{(b(t))}\| < \infty$$

and

$$\sup_{x\in\Omega} \|\left(\mathbf{D}^{(c(t))}\mathcal{J}_{\tau_i^{-N}}(x)\right)^{d(t)}\| < \infty.$$

For each r

$$\sup_{x\in\Omega} c_r \prod_{t=1}^{t(r)} \left(\|(\mathbf{D}^{(a(t))}\tau_i^{-N}(x))^{(b(t))}\|\|(\mathbf{D}^{(c(t))}\mathcal{J}_{\tau_{|_i}^{-N}}(x))^{d(t)}\| \right) < \infty.$$

Thus for each j there exists a constant $\beta_j^{(M)}$ such that:

$$\beta_j^{(M)} = \sup_{x\in\Omega} \sum_{r=1}^{r(j)} \left(c_r \prod_{t=1}^{t(r)} \left(\|(\mathbf{D}^{(a(t))}\tau_i^{-N}(x))^{(b(t))}\|\|(\mathbf{D}^{(c(t))}\mathcal{J}_{\tau_{|_i}^{-N}}(x))^{d(t)}\| \right)\right) < \infty.$$

Setting

$$B_n^{(M)} = \sup_{x\in\Omega} \|\mathbf{D}^{(M)} S_n\|(x),$$

we get:

$$B_{K+N}^{(M)} \leq B_K^{(M)} \gamma^M B^{(0)} + \sum_{j=0}^{M-1} B^{(j)} \beta_j^{(M)}.$$

Thus

$$B_K^{(M)}, B_{K+N}^{(M)}, B_{K+2N}^{(M)}, \cdots$$

is uniformly bounded by some number $\hat{B}_K^{(M)}$. Therefore, the entire sequence $\{B_n^{(M)}\}$ is bounded by $B^{(M)} = \max\{\hat{B}_0^{(M)}, \hat{B}_1^{(M)}, \cdots \hat{B}_{N-1}^{(M)}\}$. This completes the proof. □

Lemma 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\{S_k\}$ be a sequence of functions such that $S_k : \mathbb{R}^n \to \mathbb{R}$ and $S_k \to f$. If $\mathbf{D}S_k \rightrightarrows g$ (uniformly) then $\mathbf{D}f = g$.*

Proof. Keep every coordinate fixed except the i-th coordinate. Let $f(x_1, x_2, \cdots, x_n) = F(x_i)$ and $S_k(x_1, x_2, \cdots, x_n) = T_k(x_i)$. Then we have: $T_k(x_i) \to$

$F(x_i)$. Now put $g = (g_1, g_2, \cdots, g_n)$ and $\mathbf{D}f = (f_1', f_2', \cdots, f_n')$. As $\mathbf{D}S_k \rightrightarrows g$ we get: $T_k(x_i) \rightrightarrows g_i$. By [**Kos**, Theorem 8.3.4] we have: $F'(x_i) = g_i$. Since $f_i' = F'(x_i)$, have $f_i' = g_i$. i being arbitrary, the proof is complete. □

Theorem 4.3. *Let $\tau : \Omega \to \Omega$ be expanding Markov and $\tau_i^{-1} \in C^M$, $M \geq 2$. Then the invariant density $h(x) \in C^{M-2}$.*

Proof. Using Theorem 4.2, we conclude that the sequence $\{\mathbf{D}_{S_n}^{(K)}\}$ is uniformly bounded and equicontinuous for $0 \leq K \leq M-2$. By the Ascoli-Arzela Theorem, there must exist a subsequence $\{\mathbf{D}^{(K)} S_{n_k}\}$ with a continuous limit.

Put

$$\lim_{k \to \infty} \mathbf{D}^{(K)} S_{n_k}(x) = f_K(x)$$

In particular we have:

$$\lim_{k \to \infty} S_{n_k}(x) = f(x).$$

(Therefore, by Lemma 4.2 $f_K(x) = f^{(K)}(x)$ for $K = 0, 1, 2, \cdots, M-2$). Thus integrating the above equation we obtain:

$$\lim_{k \to \infty} \int S_{n_k} d\lambda = \int f d\lambda.$$

By Equation 4.4 we note that $\{S_{n_k}\lambda = \lambda_{n_k}\}$ is a subsequence of $\{\lambda_n\}$, and thus from Theorem 4.1 we have:

$$\mu(A) = \int_A f d\lambda.$$

Thus f must be equal to the invariant density h. Hence, h is $M-2$ times differentiable and we have $f^{(j)} = h^{(j)}$, for $1 \leq j \leq M-2$. □

Corollary 4.3.1. *Let $\tau : \Omega \to \Omega$ be expanding Markov and $\tau_i^{-1} \in C^M$, $M \geq 2$. Then for $0 \leq j \leq M-2$, $\{\mathbf{D}^{(j)} S_n\}$ converges uniformly to $h^{(j)}$, as $n \to \infty$.*

Proof. Fix $j \in \{0, 1, \cdots M-2\}$, and assume the contrary. Then, it is possible to find an $\epsilon > 0$ and a subsequence $\{\mathbf{D}^{(j)} S_{n_k}\}$ of $\mathbf{D}^{(j)} S_n$ such that

$$\sup_{x \in \Omega} \|\mathbf{D}^{(j)} S_{n_k}(x) - h^{(j)}(x)\| \geq \epsilon \quad \forall k. \tag{4.5}$$

The sequence $\{\mathbf{D}^{(j)} S_{n_k}\}$ is itself uniformly bounded and equicontinuous, and posseses a uniformly convergent subsequence with limit $f^{(j)}$. By Theorem 4.3 we find that $f^{(j)} = h^{(j)}$, which is incompatible with Equation 4.5. $\square$

CONCLUSION

In the present thesis, we established existence results and studied the properties of **acim** for higher dimensional piecewise C^2 and expanding maps τ. We proved the existence of an **acim** for Lasota-Yorke maps in higher dimensions under general conditions. We obtained a spectral decomposition for the Perron-Frobenius operator of τ which yielded certain ergodic properties of τ itself. We also studied the problem of finding an upper bound for the number of ergodic **acim** for such class of maps. The property of smoothness of the invariant density of an **acim** in higher dimensions for the special class of Rényi maps was also studied.

We now briefly discuss some improvements of the results obtained in this thesis. The main results of Chapter 2, could be improved in several directions. We assumed that the partition of Ω was *smooth*. It seems possible to weaken this condition to a "Lipschitz continuous partition". To do that it might be necessary to assume some "regularity" along the boundary of elements of the partition, in the vicinity of any cusps. We also assumed that the underlying transformation was C^2. It is now known [**G-S**] that C^1 is not sufficient for the existence of an **acim**, even in one-dimension. If one assumes that the transformation is $C^{1+\varepsilon}$ with an additional condition on the derivative of the transformation (such as summable oscillations [**Gór**]) existence of an **acim** has been established in one-dimension. In higher dimensions analogous have been obtained [**Lou1**] only for a very restrictive class of Jabłoński transformations not using "bounded variation techniques". It would be of interest to obtain such results using "bounded variation techniques" for more general class of maps.

It is important to know which properties the invariant density, if one exists, inherits from its underlying transformation. Even though we proved the existence of an **acim** for Lasota-Yorke maps in higher dimensions, we were able to establish the smoothness property of the invariant density only for the restricted class of Rényi maps. Our results could be improved in two directions. One problem to consider is to establish the smoothness property of invariant densities for the Lasota-Yorke maps in higher dimensions. Another problem would be to increase the degree of smoothness from C^{M-2} to C^{M-1}. In one-dimension, such degree of smoothness of invariant densities for the Lasota-Yorke maps was proved by **[Sze]**.

In conclusion, we would like to mention some applications of our results. Even though we have the existence of an **acim**, obtaining an explicit form of the invariant density is often difficult. In **[Li]** and **[G-B 3]** results for approximation of invariant densities were obtained in one-dimension. The result of **[G-B 3]** has been generalized to higher dimensions **[B-L 1]** for Jabłoński transformations. Now, that the existence of an **acim** has been established for more general class of maps, the approximation problem could be considered for future research.

The dynamics of many physical systems are often governed by a randomly changing environment and can thus be described by a random map whose evolution is represented by choosing a transformation from a given set of transformations and applying it with a given probability. For random maps composed of Lasota-Yorke maps, existence of an **acim** was established in **[Pel]** and for random maps composed of Jabłoński transformations, existence of an **acim** was shown in **[B-L 2]**. The properties associated with the invariant densities of maps considered in **[B-L 2]** were studied in **[K-M]**. It seems possible to establish the existence of an **acim** in our setting for random maps which are expanding on average. Once this is established, one may attempt to generalize the results of **[K-M]** to random maps composed of general Lasota-

Yorke maps in higher dimensions. It also seems possible to establish certain smoothness properties of invariant densities of random maps composed of Rényi maps in higher dimensions.

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