EXTENDED M-MATRICES

by

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ABSTRACT

Extended M-Matrices

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Let K be a proper cone in R^n and let A be a real nxn matrix which is K-regular; that is A = B-aI for an a>0 and a K-nonnegative matrix B. Then if the spectral radius of B is less than a, -A is called an M-Matrix.

The purpose of the present work is to generalize certain results on M-Matrices, when the structural condition of K-regularity is replaced by the weaker condition of "K-exponential nonnegativity"; that is if $e^{tA} K \subset K$ for every $t \geq 0$. This results to the introduction of a new class of matrices, the "Extended M-Matrices".

The absence of regularity is being overcome by the use of the concept of subtangentiality. Consider the vector differential equation $\dot{x} = Ax$. Then the velocity vector Ax is "tangent or points into a positive invariant set", for each x on the boundary of that set. This geometric condition is also used here for the proof of some further results, such as a characterization of an arbitrary matrix as being an Extended M-Matrix.

C O N T E N T S

1. INTRODUCTION

Let $\mathcal{K} \subset R^n$ be a proper cone, and let A be a real $n \times n$ matrix which is $\mathcal{K}$-regular; that is, $A = B - \alpha I$ for some $\alpha > 0$ and some matrix B which is $\mathcal{K}$-nonnegative (i.e. $B\mathcal{K} \subset \mathcal{K}$). Then $-A$ is called an M-matrix with respect to the cone $\mathcal{K}$ provided that the spectral radius of B is less than α. Our general reference on M-matrices is Berman and Plemmons [1], which also contains further bibliographic information.

The purpose of the present work is to generalize certain results on M-matrices presented in [1], when the condition of $\mathcal{K}$-regularity is replaced by the weaker condition $\mathcal{K}$-exponential nonnegativity; that is, $e^{tA}\mathcal{K} \subset \mathcal{K} \quad \forall t \geq 0$. Unlike several well known results on M-matrices, in the present work, conditions on "extended" M-matrices involving spectral radii are not relevant, as regularity may not hold. Also, in view of the absence of regularity, it is not surprising that the present work involves "operator theoretic" properties (spectral conditions, types of monotonicity and seminegativity, etc.) as opposed to properties involving "internal structure" as are known in particular for $\mathcal{K} = R^n_+$ (e.g. conditions involving diagonal dominance, principal minors, etc.)

The next section contains definitions and preliminary results. Then in Section 3, the concept of an extended M-matrix is introduced. In the main result of that section, Theorem 3.5, we obtain a characterization of extended M-matrices which generalizes a result in Stern [4] that dealt with the nonsingular case. In the present work, as in [4], the lack of regularity is overcome by making use of the concept of subtangentiality, which is a certain geometric condition imposed by exponential nonnegativity. Some further results, which generalize results in [1], are given in Section 4; these also make use of subtangentiality. Examples and a conjecture are presented in Section 5.

2. DEFINITIONS AND PRELIMINARY RESULTS

A nonempty set $\mathcal{K} \subset R^n$ is said to be a cone if $a\mathcal{K} \subset \mathcal{K}$ $\forall a \geq 0$. The cone $\mathcal{K}$ is polyhedral if it is the intersection of a finite number of closed halfspaces (or equivalently if it is generated by a finte set of vectors). A cone $\mathcal{K}$ is proper if it is closed, convex, pointed (i.e. $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$) and solid (i.e. has a nonempty interior, denoted by $\text{int}\ \mathcal{K}$).

Now we introduce the terminology to be used, where A is a real matrix of order n. In brackets are the notational equivalents.

DEFINITIONS 2.1. Let $\mathcal{K} \subset R^n$ be a proper cone and let

$$\mathcal{S}_A = \bigcap_{m=0}^{\infty} \mathcal{R}(A^m) .$$

where $\mathcal{R}$ denotes range. Then A is

(2.1.1) $\mathcal{K}$-<u>nonnegative</u> if $A\mathcal{K} \subset \mathcal{K}$. [$A \in n(\mathcal{K})$]

(2.1.2) $\mathcal{K}$-<u>regular</u> if there exist $\alpha > 0$ and $B \in n(\mathcal{K})$ such that $A = B - \alpha I$. [$A \in r(\mathcal{K})$; (r)]

(2.1.3) $\mathcal{K}$-<u>exponentially nonnegative</u> if $e^{tA}\mathcal{K} \subset \mathcal{K}$, $\forall t \geq 0$. [$A \in e(\mathcal{K})$; (e)]

(2.1.4) $\mathcal{K}$-<u>negatively monotone on</u> $\mathcal{S}_A$ if

$-Ax \in \mathcal{K}$, $x \in \mathcal{S}_A \Rightarrow x \in \mathcal{K}$. [$A \in m(\mathcal{K})$; (m)]

(2.1.5) <u>Nonpositive stable</u> if Re[Spectrum (A)] ≤ 0 . [$A \in a$; (a)]

(2.1.6) $\mathcal{K}$-<u>seminegative on</u> $\mathcal{S}_A$ if there exists $x \in \mathcal{K} \cap \mathcal{S}_A$ such that $-Ax \in \{\text{int } \mathcal{K}\} \cap \mathcal{S}_A$. [$A \in s(\mathcal{K})$; (s)]

(2.1.7) $\mathcal{K}$-<u>weakly seminegative on</u> $\mathcal{S}_A$ if there exists $x \in \mathcal{K} \cap \mathcal{S}_A$ such that $-Ax \in \{\mathcal{K} \cap \mathcal{S}_A\}/\{0\}$. [$A \in \bar{s}(\mathcal{K})$; $(\bar{s})$]

(2.1.8) $(\mathcal{K} \cap \mathcal{S}_A)$-<u>zeroed</u> if $\{x \in \mathcal{K} \cap \mathcal{S}_A : Ax \in \mathcal{K}\} = \{0\}$.

[$A \in z(\mathcal{K})$; (z)]

(2.1.9) $(\mathcal{K}^* \cap \mathcal{S}_{A^T})$-<u>zeroed</u> if $\{x \in \mathcal{K}^* \cap \mathcal{S}_{A^T} : A^T x \in \mathcal{K}^*\} = \{0\}$.

[$A \in z(\mathcal{K}^*)$; $(z)^*$] where

$$\mathcal{K}^* = \{y \in R^n : \langle x,y \rangle \geq 0 , \ \forall x \in \mathcal{K}\}$$

is the <u>dual cone</u> of $\mathcal{K}$. Here A^T denotes the transpose of A and $\langle .,. \rangle$ denotes the inner product on R^n.

REMARKS 2.2.

(2.2.1) If A is a nonsingular matrix, then $\mathcal{S}_A = R^n$ and (2.1.4), (2.1.5), (2.1.6), (2.1.8), (2.1.9) become respectively

(2.1.4)' $-Ax \in \mathcal{K} \Rightarrow x \in k$,

(2.1.5)' $\mathrm{Re}[\mathrm{Spectrum}\ (A)] < 0$,

(2.1.6)' $\exists\, x \in \mathcal{K}$ such that $-Ax \in \mathrm{int}\ \mathcal{K}$,

(2.1.8)' $\{x \in \mathcal{K} : Ax \in \mathcal{K}\} = \{0\}$, (2.1.9)′ $\{x \in \mathcal{K}^*: A^T x \in \mathcal{K}^*\} = \{0\}$, as in [4].

(2.2.2) If $\mathcal{K} = R^n_+$, the nonnegative orthant, then $A \in \Gamma(\mathcal{K})$ is <u>equivalent</u> to A being <u>essentially nonnegative</u>; i.e. $a_{ij} \geq 0$ for $i \neq j$.

(2.2.3) It was proven in [4] that (r) ⇒ (e), with equivalence for polyhedral $\mathcal{K}$.

Now we review some required basic definitions and results on generalized inverses. The <u>index</u> of a square matrix A is the smallest nonnegative integer k such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$. Then

$\mathcal{S}_A = \bigcap_{m=0}^{k} \mathcal{R}(A^m)$. A matrix $X \in R^{n\times n}$ which satisfies $XAX = X$,

$AX = XA$, $A^{p+1}X = A^p$ $\forall p \geq \text{index}(A)$, exists uniquely and is called the <u>Drazin inverse</u> of A, denoted by A^D. A square matrix Y is a <u>generalized left inverse</u> of A if $YAx = x$ $\forall x \in \mathcal{S}_A$. The Drazin inverse of A is easily recognized to be a generalized left inverse of A, since for $x \in \mathcal{S}_A$ and $p \geq \text{index}(A)$ there exists $y \in R^n$ such that $x = A^p y$, whence $A^D A x = A^D A^{p+1} y = A^p y = x$. We also note that if $x \in \mathcal{S}_A \cap \mathcal{N}(A)$ ($\mathcal{N}$ denoting null space), then $Ax = 0$ and at the same time there exists $y \in R^n$ such that $A^k y = A^D A^{k+1} y = 0$, where $k = \text{index}(A)$. Therefore

(2.3) $\quad \mathcal{S}_A \cap \mathcal{N}(A) = \{0\}$ (and similarly $\mathcal{S}_{A^T} \cap \mathcal{N}(A^T) = \{0\}$).

DEFINITION 2.4. Let A be an $(n\times n)$ matrix and let $\mathcal{K} \subset R^n$ be a proper cone. Then a generalized left inverse of A, say Y, is said to be $\mathcal{K}$-nonnegative on $\mathcal{S}_A$ if $Y(\mathcal{K} \cap \mathcal{S}_A) \subset \mathcal{K}$.

The following result parallels Theorem 5.4.24 in [1], where $\mathcal{K} = R^n_+$. We include the proof for completeness.

THEOREM 2.5. For an $(n\times n)$ matrix A and proper cone $\mathcal{K} \subset R^n$, the following statements are equivalent:

(i) A has a generalized left inverse Y such that $-Y$ is $\mathcal{K}$-nonnegative on $\mathcal{S}_A$.

(ii) For every generalized left inverse of A, say L, $-L$ is $\mathcal{K}$-nonnegative on $\mathcal{S}_A$. In particular $-A^D(\mathcal{K} \cap \mathcal{S}_A) \subset \mathcal{K}$.

(iii) A is $\mathcal{K}$-negatively monotone on $\mathcal{S}_A$.

PROOF:

<u>(i) ⇒ (ii)</u>: Let L be a generalized left inverse of A, while (i) holds. Let $x \in \mathcal{K} \cap \mathcal{S}_A$. Then there exists $z \in R^n$ such that $x = A^{k+1}z$, $k = \text{index}(A)$, for which

$$-A^k z = -YA^{k+1}z = -Yx \in \mathcal{K}.$$

Then $-Lx = -LA^{k+1}z = -A^k z \in \mathcal{K}$, and therefore (ii) holds.

<u>(ii) ⇒ (iii)</u>: Assume that (ii) holds and $x \in \mathcal{S}_A$, $-Ax \in \mathcal{K}$. Since $-Ax \in \mathcal{K} \cap \mathcal{S}_A$, we have $x = LAx = -L(-Ax) \in \mathcal{K}$. Hence (iii) holds.

<u>(iii) ⇒ (i)</u>: Let $-Ax \in \mathcal{K}$, $x \in \mathcal{S}_A$ imply that $x \in \mathcal{K}$, and let $w \in \mathcal{K} \cap \mathcal{S}_A$. It is enough to show that $-A(-A^D w) \in \mathcal{K}$ and $-A^D w \in \mathcal{S}_A$. But this is true since A^D and A commute.

□

COROLLARY 2.6. <u>If it is assumed that</u> A <u>is nonsingular, then</u> A <u>is</u> $\mathcal{K}$<u>-negatively monotone</u>, (i.e. $-Ax \in k \Rightarrow x \in k$) <u>if and only if</u> $-A$

is K-inverse positive (i.e. $-A^{-1}(K/\{0\}) \subset \text{int } K$).

PROOF: This follows immediately from Theorem 2.5 and the fact that if A is nonsingular, then $A^D = A^{-1}$ and $S_A = R^n$.

□

Following now is a summary of required results on positive invariance.

DEFINITION 2.7. For an $(n \times n)$ real matrix A, consider the linear autonomous differential equation

$$\text{(2.7.1)} \qquad \dot{x}(t) = Ax(t) .$$

A set $\Gamma \subset R^n$ is said to be positively invariant with respect to (2.7.1) if $x(0) \in \Gamma$ implies that $x(t) = e^{tA}x(0) \in \Gamma \ \ \forall t \geq 0$; that is, if $e^{tA}\Gamma \subset \Gamma \ \ \forall t \geq 0$.

If $\Gamma \subset R^n$ is closed and convex, we define the set of nonzero outward unit normal vectors to Γ at a point $x \in \partial\Gamma$, (the boundary) as

$$N_\Gamma(x) = \{\nu \in R^n : \langle \nu, y-x \rangle \le 0 \quad \forall y \in \Gamma, \|\nu\| = 1\}$$

where $\|.\|$ denotes the euclidean norm.

DEFINITION 2.8. For a closed convex set $\Gamma \subset R^n$, a vector $z \in R^n$ is <u>subtangential</u> to Γ at $x \in \partial\Gamma$ if $\langle z, \nu \rangle \le 0 \quad \forall \nu \in N_\Gamma(x)$.

The following theorem characterizes positive invariance of a closed convex set with respect to (2.7.1) as equivalent to the "velocity" vector, Ax, being "tangent to or pointing into the set" for each point x on the boundary of the set.

THEOREM 2.9. *A closed convex set* $\Gamma \subset R^n$ *is positively invariant if and only if* Ax *is subtangential to* Γ *for every* $x \in \partial\Gamma$.

We also shall require the following lemma.

LEMMA 2.10. *Let* $K \subset R^n$ *be a proper cone. Then*

(2.10.1) $\quad \langle \nu, x \rangle = 0 \quad \forall x \in \partial K, \quad \forall \nu \in N_K(x)$.

(2.10.2) If $A \in e(\mathcal{K})$ and $x \in R^n$ is such that $Ax \in \mathcal{K}$, then the shifted set $\{x + \mathcal{K}\}$ is positively invariant.

The proofs of the above results can be found in [4]. Some further definitions are required.

DEFINITION 2.11. A proper cone $\mathcal{K} \subset R^n$ is strictly positively invariant (with respect to (2.7.1) if $e^{tA}[\mathcal{K}/\{0\}] \subset \text{int}\,\mathcal{K}\ \forall t > 0$.

DEFINITION 2.12. Let $\mathcal{K} \subset R^n$ be a proper cone and let A be a real (n×n) matrix. Then A is $\mathcal{K}$-irreducible if A has no eigenvector in $\partial\mathcal{K}$.

THEOREM 2.13 (Elsner [2], Schneider and Vidyasagar [3]). Let $\mathcal{K} \subset R^n$ be a proper cone.

(2.13.1) If $\mathcal{K}$ is positively invariant, then

$$\lambda_A := \max\{\text{Re}\,\lambda : \lambda \in \text{Spectrum}(A)\}$$

is an eigenvalue of A and has an associated eigenvector in $\mathcal{K}$.

(2.13.2) K <u>is strictly positively invariant if and only if</u> K <u>is positively invariant and</u> A <u>is</u> K-<u>irreducible.</u>

(2.13.3) <u>If</u> K <u>is strictly positively invariant, then</u> λ_A <u>is a simple eigenvalue of</u> A <u>and an associated eigenvector lies in the interior of</u> K. <u>Furthermore,</u> A <u>has only one eigenvector (up to scalar multiples) in</u> K.

We conclude this section with a needed lemma. The proof (which can be found in [4]) follows readily from the fact that $K^{**} = K$.

LEMMA 2.13. <u>Let</u> $K \subset R^n$ <u>be a proper cone. Then</u>

$$A \in e(K) \Longleftrightarrow A^T \in e(K^*) .$$

3. EXTENDED M-MATRICES

In the following theorem we will use the fact that if Re[Spectrum(A)] < 0 , then the origin is a <u>stable equilibrium</u> of the differential equation (2.7.1); that is, $e^{tA}x \to 0$ as $t \to \infty$ for every $x \in R^n$.

THEOREM 3.1. _Let_ $\mathcal{K} \subset R^n$ _be a proper cone and let_ $A \in e(\mathcal{K})$. _Then_

$$(z) \Longleftrightarrow (a) \Longleftrightarrow (m) \Longleftrightarrow (z)^* .$$

PROOF:

_(z) ⇒ (a):_ Let (z) hold and suppose (a) does not hold. Then there exists $\lambda \in \text{Spectrum}(A)$ such that $\text{Re}\,\lambda > 0$, whence $\lambda_A > 0$. Then (2.13.1) implies that there exists $0 \neq x \in \mathcal{K}$ such that $Ax = \lambda_A x$. Now note that $A^m x = \lambda^m x \;\; \forall m = 0,1,2,\ldots$. Hence $Ax \in \mathcal{S}_A$ and therefore $Ax \in \mathcal{K} \cap \mathcal{S}_A$, $x \neq 0$, violating (z).

_(a) ⇒ (m):_ If $\mathcal{K} \cap \mathcal{S}_A = \{0\}$ then (m) holds trivially. Hence we assume $\mathcal{K} \cap \mathcal{S}_A \neq \{0\}$ and that (a) holds. Suppose that (m) does not hold. Then there exists $x \in R^n$ such that $-Ax \in \mathcal{K}$, $x \in \mathcal{S}_A$ and $x \notin \mathcal{K}$. Let $\bar{x} = -x$. Then by Lemma 2.10, $\{\bar{x} + \mathcal{K}\}$ is positively invariant, while $0 \notin \{\bar{x} + \mathcal{K}\}$ due to the pointness of $\mathcal{K}$. It follows that $e^{tA}\bar{x} \not\to 0$ as $t \longrightarrow \infty$. Therefore there exists an eigenvalue of A such that $\mathrm{Re}\,\lambda \geq 0$. This implies that $\lambda_A \geq 0$. Now, viewing $\mathcal{K} \cap \mathcal{S}_A$ as a proper cone in the A-invariant subspace $\mathcal{S}_A$, and letting $\bar{A}$ be the restriction of A to $\mathcal{S}_A$, (2.13.1) plies that there exists an eigenvector $0 \neq \hat{x} \in \mathcal{K} \cap \mathcal{S}_A$ such that $A\hat{x} = \lambda_A \hat{x}$. If $\lambda_A = 0$, then $\hat{x} \in \mathcal{S}_A \cap \mathcal{N}(A) = \{0\}$ (by (2.3)). Hence $\lambda_A > 0$, contradicting (a).

<u>(m) ⟹ (z)</u>: If $\mathcal{K} \cap \mathcal{S}_A = \{0\}$, then (z) holds trivially. Therefore assume that $\mathcal{K} \cap \mathcal{S}_A \neq \{0\}$ while (m) holds. Suppose that (z) does not hold. Then there exists $0 \neq x \in \mathcal{K} \cap \mathcal{S}_A$ such that $Ax \in \mathcal{K}$. Now consider $\bar{x} = -x$. Then $\bar{x} \in S$, $-A\bar{x} \in \mathcal{K}$, and therefore $\bar{x} \in \mathcal{K}$. In other words $x \in \mathcal{K} \cap (-\mathcal{K}) = \{0\}$, a contradiction to $x \neq 0$.

In view of Lemma 2.13 and the fact that $A \in a \Longleftrightarrow A^T \in a$, the proof is completed.

□

DEFINITION 3.2. Let $\mathcal{K} \subset R^n$ be a proper cone. If $A \in e(\mathcal{K})$ satisfies any of the equivalent conditions in Theorem 3.1, then $-A$ is called an <u>extended M-matrix with respect to</u> $\mathcal{K}$. (If A also satisfies the stronger condition $A \in r(\mathcal{K})$, then $-A$ is simply an <u>M-matrix with respect to</u> $\mathcal{K}$.)

REMARKS 3.3.

(3.3.1) Theorem 3.1 is trivially valid in case $\mathcal{K} \cap \mathcal{S}_A = \{0\}$. It will be seen in Theorem 3.5 below that when this intersection is nonzero, then the list of equivalences in Theorem 3.1 can be extended to include weak seminegativity.

(3.3.2) Theorem 3.1 is a known result on M-matrices for $A \in r(\mathcal{K})$; see e.g. [1]. Hence, in view of Remark 2.2.3, the theorem is of most interest for (non-polyhedral) proper cones $\mathcal{K}$ such that $A \in e(\mathcal{K})$ but $A \notin r(\mathcal{K})$. Our examples will illustrate such cases.

THEOREM 3.4. <u>Let</u> $\mathcal{K} \subset R^n$ <u>be a proper cone and let</u> $A \in e(k)$. <u>Assume further that</u>

(3.4.1) $\mathcal{K} \cap \mathcal{S}_A \neq \{0\}$.

<u>Then the following implications hold:</u>

$$(m) \Rightarrow (\bar{s}) \Rightarrow (z)^* .$$

PROOF:

<u>$(m) \Rightarrow (\bar{s})$</u>: Let (m) hold. Then by Theorem 2.5, $-A^D$ is $\mathcal{K}$-nonnegative on $\mathcal{S}_A$. Let $0 \neq d \in \mathcal{K} \cap \mathcal{S}_A$. Then

$$x = -A^D d \in \mathcal{K} \cap \mathcal{S}_A \Rightarrow -Ax = A^D A d = d .$$

Therefore there exists an $x \in \mathcal{K} \cap \mathcal{S}_A$ such that $-Ax \in \{\mathcal{K} \cap \mathcal{S}_A\}/\{0\}$; that is $(\bar{s})$ holds.

<u>$(\bar{s}) \Rightarrow (z)^*$</u>: Let $x \in \mathcal{K} \cap \mathcal{S}_A$ be such that $-Ax \in \{\mathcal{K} \cap \mathcal{S}_A\}/\{0\}$. If $(z)^*$ is not true, then there exists $0 \neq u \in \mathcal{K}^* \cap \mathcal{S}_{A^T}$ such that $A^T u \in \mathcal{K}$. But then $\langle Ax, u \rangle < 0$ since $-Ax \in \mathcal{K}$. Also, $u \notin \mathcal{N}(A^T)$

by (2.3). Furthermore, $\langle Ax, u\rangle = \langle x, A^T u\rangle \geq 0$ since $x \in K$ and $A^T u \in K$, which is a contradiction.

□

Putting together Theorems 3.1 and 3.4, we state the extension of Theorem 3.1 which results when (3.4.1) holds.

THEOREM 3.5. Let $K \subset R^n$ be a proper cone, and let $A \in e(K)$. Assume that (3.4.1) holds. Then

$$(z) \Longleftrightarrow (a) \Longleftrightarrow (m) \Longleftrightarrow (\bar{s}) \Longleftrightarrow (z)^*.$$

REMARK 3.6. As we can see from Theorem 3.5, if $K \cap S_A \neq \{0\}$, then $-A$ can be characterized as an extended M-matrix by K-exponentially nonnegativity in conjunction with weak seminegativity. Note that if $(\text{int } K) \cap S_A \neq \emptyset$, then by choosing $\theta \in (\text{int } K) \cap S_A$ in the proof of Theorem 3.4, $(\bar{s})$ can be replaced by (s) ("true" seminegativity on S_A) in characterizing an extended M-matrix.

REMARK 3.7. In view of Remark (2.2.1), Theorem 3.5 yields Theorem 1.7 of [4] as a special case.

4. FURTHER RESULTS EMPLOYING SUBTANGENTIALITY

In the theorem to follow we give a necessary and sufficient condition which characterizes an arbitrary real matrix as being an extended M-matrix with respect to a proper cone $\mathcal{K}$, without assuming positive invariance. First we require the following.

LEMMA 4.1. $-A$ <u>is an extended</u> M-<u>matrix with respect to the proper cone</u> $\mathcal{K} \subset R^n$ <u>if and only if</u> $(-A + \varepsilon I)$ <u>is a nonsingular extended nonsingular</u> M-<u>matrix</u> $\forall \varepsilon > 0$.

PROOF:

<u>Only if</u>: Let $A \in e(\mathcal{K})$ and $A \in a$; that is, Re[Spectrum(A)] ≤ 0. Then $(A - \varepsilon I) \in e(\mathcal{K})$ $\forall \varepsilon > 0$, while Re[Spectrum$(A - \varepsilon I)$] < 0 $\forall \varepsilon > 0$. Therefore $(-A + \varepsilon I)$ is an extended nonsingular M-matrix $\forall \varepsilon > 0$.

<u>If</u>: Letting ε approch zero we have that Re[Spectrum(A)] ≤ 0, and since $(A - \varepsilon I) \in e(\mathcal{K})$ implies $A \in e(\mathcal{K})$, we conclude that $-A$ is an extended M-matrix.

□

The following result generalizes part of Theorem 6.4.? in [1], where it was assumed that $\mathcal{K} = R^n_+$, and regularity was explicitly employed. We circumvent the lack of regularity by using the concept of subtangentiality.

THEOREM 4.2. Let $A \in R^{n\times n}$ and let $\mathcal{K} \subset R^n$ be a proper cone. Then $-A$ is an extended M-matrix with respect to $\mathcal{K}$ if and only if $(-A + \varepsilon I)$ is $\mathcal{K}$-inverse positive $\forall \varepsilon > 0$.

PROOF:

Only if: Let $-A$ be an extended M-matrix with respect to $\mathcal{K}$. Then according to Lemma 4.1, $(-A + \varepsilon I)$ is a nonsingular extended M-matrix $\forall \varepsilon > 0$, and consequently $A - \varepsilon I$ is $\mathcal{K}$-negatively monotone. Then by Corollary 2.6 $(-A + \varepsilon I)$ is $\mathcal{K}$-inverse positive $\forall \varepsilon > 0$.

If: Let $B := -A$ and assume that $(B + \varepsilon I)$ is $\mathcal{K}$-inverse positive for every $\varepsilon > 0$. First we will prove that $-B \in e(\mathcal{K})$. Suppose not. Then by Theorem 2.9 there exists $g \in \partial\mathcal{K}$ such that $\langle \nu, -Bg \rangle > 0$ for a normal $\nu \in N_{\mathcal{K}}(g)$. For sufficiently small $\varepsilon > 0$, $(I + \varepsilon B)$ is nonsingular and the second term dominates the series expansion

$$(I + \varepsilon B)^{-1} = I - \varepsilon B + (\varepsilon B)^2 - (\varepsilon B)^3 + \dots .$$

But then $\nu^T(B + I/\varepsilon)^{-1}g = \varepsilon\nu^T(I + \varepsilon B)^{-1}g =$ $\varepsilon[r^Tg - r^T \varepsilon Bg + r^T\varepsilon^2B^2g - \dots] > 0$, since $\nu^Tg = 0$ (by Lemma 2.10.1) and so $\varepsilon\nu^TBg > 0$. Therefore $\langle r, (B + I/\varepsilon)^{-1}g\rangle > 0$, and so $(B + I/\varepsilon)^{-1}g \notin K$, a contradiction. Hence $A = -B \in e(K)$. Now it follows that $(A - \varepsilon I) \in e(K)$, and since $(A - \varepsilon I)$ is K-negatively montone $\forall \varepsilon > 0$ (by Lemma 4.1), we have that $-A$ is an extended M-matrix with respect to K.

□

Next we will employ subtangentiality in order to obtain generalizations of certain results of Varga [5] (see also [1]) which were obtained for $K = R^n_+$ (in which case (r) $\Longleftrightarrow$ (e)) by using properties of completely monotonic functions.

THEOREM 4.3. Let A be a nonsingular real $(n \times n)$ matrix and let $K \subset R^n$ be a proper cone. Then

(4.3.1) $\quad A^{-1}[e^{tA} - I]K \subset K \quad \forall t \geq 0 \Longleftrightarrow K$ is positively invariant.

(4.3.2) $A^{-1}[e^{tA}-I](\mathcal{K}/\{0\}) \subseteq \text{int}\, k \quad \forall t > 0 \Longleftrightarrow \mathcal{K}$ is strictly positively invariant.

PROOF OF (4.3.1):

($\Leftarrow$): For $g \in R^n$ and $t \geq 0$ we define

$$x(t;g) = A^{-1}[e^{tA}-I]g .$$

We must show that $x(t;g) \in \mathcal{K} \quad \forall g \in \mathcal{K}, \quad \forall t \geq 0$. Note that $\dot{x}(t;g) = e^{tA}g$ and $x(0,g) = 0$. Therefore $x(t;g) = \int_0^t e^{sA}g\, ds$. Now we write

$$\mathcal{K} = \{x \in R^n : \langle \nu, x \rangle \leq 0 , \quad \nu \in N\}$$

where N is the set of outward normals to hyperplanes which support $\mathcal{K}$. Since $e^{tA}\mathcal{K} \subset \mathcal{K} \quad \forall t \geq 0$, it follows that $\langle \nu, e^{ta}g \rangle \leq 0$ $\forall g \in \mathcal{K}, \quad \forall t \geq 0, \quad \forall \nu \in N$. Hence $\langle \nu, \int_0^t e^{sA}g\, ds \rangle \leq 0 \quad \forall g \in \mathcal{K}$, $\forall t \geq 0, \quad \forall \nu \in N$, and consequently $x(t;g) \in \mathcal{K} \quad \forall t \geq 0, \quad \forall g \in \mathcal{K}$.

($\Rightarrow$): Suppose that $\mathcal{K}$ is not positively invariant. Then by Theorem 2.9 there exists $g \in \partial k^*$ such that Ag is not subtangential to $\mathcal{K}$ at

g. This means that there exists ν, an outer normal to a hyperplane which supports $\mathcal{K}$ at g, such that $\langle \nu, Ag \rangle > 0$. Then

$$v_g(t) := \langle \nu, e^{tA} g \rangle = \langle \nu, \sum_{j=0}^{\infty} \frac{(ta)^j}{j!} g \rangle = \langle \nu, \sum_{j=1}^{\infty} \frac{(ta)^j}{j!} g \rangle ,$$

since $\langle \nu, g \rangle = 0$ by (2.10.1). Therefore $(1/t) v_g(t) =$ $\langle \nu, Ag \rangle + \langle \nu, (e^{tA} g - tAg - g)/t \rangle$, and $\langle \nu, (e^{tA} g - tAg - g)/t \rangle =$ $\langle \nu, (e^{tA} g - tAg)/t \rangle \longrightarrow 0$ as $t \to 0$. Hence $v_g(t) > 0 \ \ \forall t \in (0, \delta)$ for some $\delta > 0$. But $\langle \nu, x(t; g) \rangle = \int_0^t v_g(s) ds > 0 \ \ \forall t \in (0, \delta)$, whence $A^{-1}[e^{tA} - I] g \notin \mathcal{K} \ \forall t \in (0, \delta)$. Then the left-hand-side of (4.3.1) does not hold, which is a contradiction.

PROOF OF (4.3.2):

$(\Rightarrow)$: From (4.3.1) we know that $e^{tA} \mathcal{K} \subset \mathcal{K} \ \ \forall t \geq 0$. Suppose A is not $\mathcal{K}$-irreducible. Then there exist $\lambda \in R$ and $0 \neq g \in \partial \mathcal{K}$ such that $Ag = \lambda g$. But then $x(t; g) = \int_0^t e^{sA} g \, ds = \int_0^t e^{s\lambda} g \, ds$. Therefore either $x(t; g) = (1/\lambda) e^{t\lambda} g$ in case $(\lambda \neq 0)$ or $x(t; g) = t g$ in case $(\lambda = 0)$, and in both cases $x(t; g) \in \partial \mathcal{K} \ \ \forall t \geq 0$. But then the left-hand-side of (4.3.2) does not hold.

($\Leftarrow$): Let $0 \neq g \in \mathcal{K}$. Then for every $\nu \in N$ we have $\langle \nu, x(t;g)\rangle = \int_0^t \langle \nu, e^{sA}g\rangle ds < 0$, since $e^{sA}g \in \text{int}\,\mathcal{K}$ $\forall s > 0$, implies $\langle \nu, e^{sA}g\rangle < 0$ $\forall s > 0$, $\forall \nu \in N$. Therfore $x(t;g) \in \text{int}\,\mathcal{K}$ $\forall t > 0$; that is, $A^{-1}[e^{tA}-I](\mathcal{K}/\{0\}) \subset \text{int}\,\mathcal{K}$ $\forall t > 0$.

□

REMARK 4.4. If A is singular, then the fact that $A^D A g = g$ $\forall g \in \mathcal{S}_A$ enables us to work with $\mathcal{K} \cap \mathcal{S}_A$ as a proper cone in $\mathcal{S}_A$. Then, referring to the relative interior $\text{int}_R(\mathcal{K} \cap \mathcal{S}_A)$, Theorem 4.3 can immediately be extended in an obvious way:

$$\text{(4.3.1)'} \quad A^D[e^{tA}-I](\mathcal{K} \cap \mathcal{S}_A) \subset \mathcal{K}\ \forall t \geq 0 \Longleftrightarrow e^{tA}(\mathcal{K} \cap \mathcal{S}_A) \subset \mathcal{K}\ \forall t \geq 0.$$

$$\text{(4.3.2)'} \quad A^D[e^{tA}-I](\mathcal{K} \cap \mathcal{S}_A)/\{0\}) \subset \text{int}_R(\mathcal{K} \cap \mathcal{S}_A) \Longleftrightarrow e^{tA}(\mathcal{K} \cap \mathcal{S}_A) \subset \mathcal{K} \text{ and } A \text{ is } (\mathcal{K} \cap \mathcal{S}_A)\text{-irreducible.}$$

Finally, we present a result on $\mathcal{K}$-irreducible singular extended M-matrices.

THEOREM 4.5. _Let_ A _be a singular_ $\mathcal{K}$_-irreducible extended M-Matrix with respect to a proper cone_ $\mathcal{K} \subset R^n$. _Then_ $-A$ _is almost monotonic with respect to_ $\mathcal{K}$ _; that is,_

$$-Ax \in \mathcal{K} \Rightarrow Ax = 0 \, .$$

PROOF: The hypotheses imply that $-\mathcal{K}$ is strictly positively invariant with respect to (2.7.1). Then $\mathcal{K}^*$ is easily seen to be strictly positively invariant with respect to the dual differential equation $\dot{x}(t) = A^T x(t)$. Since A^T is also a singular extended M-matrix, we conclude that $\lambda_{A^T} = 0$ and that there exists $0 \neq y \in \text{int}\, \mathcal{K}^*$ such that $A^T y = 0$. Now let $x \in R^n$ be such that $-Ax \in \mathcal{K}$. If $Ax \neq 0$, then $y^T Ax \neq 0$ (since $y \in \text{int}\, \mathcal{K}^*$), a contradiction.

□

5. EXAMPLES

EXAMPLE 5.1. Let $\mathcal{K} = \{x \in R^3 : x_1^2 + x_2^2 \leq x_3^2 , \; x_3 \geq 0\}$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is readily noted that $A \in e(\mathcal{K})$, either by checking subtangentiality or by working directly with the power series e^{tA}. Since $\lambda_A = 0$, $-A$ is a singular extended M-matrix. Furthermore, $A \notin r(\mathcal{K})$; that is, $-A$ is not an M-matrix. Note also that in this example $\mathcal{K} \cap \mathcal{S}_A = \{0\}$.

EXAMPLE 5.2. Now let $\mathcal{K} = \{x \in R^4 : x_1^2 + x_2^2 \le x_3^2, x_3 \ge 0, x_4 \ge 0\}$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then again $-A$ is a singular extended M-matrix which is not an M-matrix, but now $\mathcal{K} \cap \mathcal{S}_A \neq \{0\}$.

REMARK 5.3. It seems that examples with all the features of Example 5.2 do not exist in three dimensions. (Note that the three-dimensional Example 5.1 did not satisfy (3.4.1).) It is conjectured that if A is a singular extended M-matrix with respect to a proper cone $\mathcal{K} \subset R^n$ such that $A \notin r(\mathcal{K})$, and such that (3.4.1) holds, then $n > 3$.

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