Finite Element Analysis for Eigensolution Variability of Non Self-Adjoint Mechanical Systems with Stochastic Parameters

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Abstract

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This thesis concerns itself with quantifying the free dynamic response of non self-adjoint mechanical systems, the parameters of which have a stochastic distribution. A general non self-adjoint eigenproblem with stochastic coefficients, that models the vibration response of a Multi-Degree-of-Freedom (MDOF) system with stochastic parameters is considered. This MDOF mechanical system is obtained using the Finite Element Method. The Young's modulus and mass density are modelled as independent stochastic fields. The axial loadings are modelled as a random variable. Employing an asymptotic series solution, the sample functions of eigensolutions are expressed through the sample functions of stochastic system parameters and the stochastic sensitivity gradients. Based on these asymptotic solutions and the response moment calculation method, the stochastic characteristics of eigensolutions are quantified. Expressions for the first- and second-order probabilistic moments of eigenvalues and eigenvectors are derived. The relative influences on the eigensolution variability of randomness in various system parameters are quantified.

The applications of the developed solutions to the whirl speed analysis of high speed rotor-bearing systems are considered. The solutions developed for a general class of non self-adjoint systems is deployed as the basis for the determination of stochastic characteristics of whirl speeds and whirl modes. A parametric study is conducted in order to determine (i) the effects of the correlation characteristics of material property stochastic fields, (ii) the influences of bearing flexibility and (iii) the influences of the mesh size on the probabilistic moments of whirl speeds. Based on the results of this parametric study, design implications are determined and discussed.

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Nomenclature

[A], [B], [C]	Characteristic matrices of a mechanical system
$[B_r]$	Connectivity matrix corresponding to the r-th finite element
$[C^b]$	Damping matrix of the bearing
[G]	Gyroscopic matrix
$[G^d]$	Gyroscopic matrix of the disk
$[G^e]$	Gyroscopic matrix of the finite rotor element
[I]	Identity matrix
[K]	Stiffness matrix of the rotor shaft
$[K^b]$	Stiffness matrix of the bearing
$[K_A{}^e]$	Stiffness matrix due to axial load of the rotor element
$[K_B{}^e]$	Stiffness matrix due to flexural deformation of the rotor element
[M]	Mass matrix of the rotor shaft
$[M^b]$	Mass matrix of the bearing
$[M_T{}^d]$	Mass matrix of the disk corresponding to translational motion
$[M_R{}^d]$	Mass matrix of the disk corresponding to rotational motion
$[M_T{}^e]$	Mass matrix of the finite rotor element corresponding to translational displacements.
$[M_R{}^e]$	Mass matrix of the finite rotor element corresponding to rotational displacements.
$[N^e]$	A non-symmetric matrix as formed in Eq. (4.37)

[R]	Orthogonal rotation matrix
abc, a'b'c', a''b''c''	Triads attached to the cross-section of the shaft
XYZ	Stationary frame of reference of the rotor-bearing system
xyz	Rotating frame of reference of the rotor-bearing system
c^b	Damping coefficient of the bearing
c	Random variable that characterizes the fluctuations in axial load
E	Young's modulus
f	Wave frequency of the stochastic field
I	Second moment of area of the shaft cross-section
I_D, I_P	Diametral Inertia and Polar Inertia respectively
k^b	Stiffness coefficient of the bearing
m_d	Disk mass
n	Number of eigenvalues and eigenvectors
n_r	Number of degrees of freedom of the r-th finite element
P	Axial compressive load
$R_{a_{ij}}$, $R_{b_{ij}}$	Autocorrelation functions of a_{ij} and b_{ij}
$S_{a_{ij}}$, $S_{b_{ij}}$	Power Spectral Density functions of a_{ij} and b_{ij}
U	Potential energy
U_A^e	Potential energy due to axial load of the finite rotor element
U_B^e	Potential energy due to bending of the finite rotor element
s	Axial position along the rotor shaft
t	Time
a(s), b(s)	One-dimensional univariate homogeneous stochastic fields

$d_x(i,k)$	i, k = 1, 2, 3, ..., n; coefficients of the linear transformation of the k-th eigenvector to the right
$d_y(i,k)$	i, k = 1, 2, 3, ..., n; coefficients of the linear transformation of the k-th eigenvector to the left
$B(s,t)$, $\Gamma(s,t)$	Rotations of rotor cross-section about the Y and Z axes respectively in the stationary frame of reference
$S_{aa}(f)$, $S_{bb}(f)$	Power Spectral Density functions of stochastic fields a (s) and b (s)
$V(s,t)$, $W(s,t)$	Translational displacements of rotor cross-section along the Y and Z axes respectively in the stationary frame of reference
$v(s,t)$, $w(s,t)$	Translational displacements of rotor cross-section along the y and z axes respectively in the rotating frame of reference
$[\Psi]$	Matrix of interpolation functions for translational displacements; $\psi_i(s)$, i = 1, 2, 3, 4
$[\Phi]$	Matrix of interpolation functions for rotational displacements; $d\psi_i(s)/ds$, i = 1, 2, 3, 4
α	Asymptotic parameter
Λ	Eigenvalue of the rotor-bearing system in the stationary frame of reference
β, γ	Rotational displacements of rotor cross-section corresponding to $B(s,t)$ and $\Gamma(s,t)$ in the rotating frame of reference
δ	Kronecker's delta
ϕ	Spin angle about the axis a' in the triad a'b'c'
χ, κ	Non-zero scalar quantities
λ	Eigenvalue of the Non Self-Adjoint system

υ	Whirl ratio
μ	Mass per unit length of the rotor shaft
μ_{ij} , ν_{ij}	Coefficients introduced in Eq. (4.58)
η_d, ζ_d	Location of disk mass center in directions b and c of the triad abc
η (s), ζ (s)	Location of mass center of element cross-section in the directions of b and c in the triad abc
η_L, ζ_L	$\eta (s)_{s=0}$, $\zeta (s)_{s=0}$
η_R, ζ_R	$\eta (s)_{s=1}$, $\zeta (s)_{s=1}$
ξ	Lag distance of the stochastic fields
σ_a, σ_b	Standard Deviations of stochastic fields a (s) and b (s) respectively
$\sigma_{a_{ij}}$, $\sigma_{b_{ij}}$	Standard Deviations of coefficients a_{ij} and b_{ij} respectively
σ_c	Standard Deviation of random variable c
τ	Kinetic energy
τ^d	Kinetic energy of the rigid disk
τ^e	Kinetic energy of the finite rotor element
ω	Whirl speed of the rotor
ω_a, ω_b, ω_c	Angular rate components of the triad abc with respect to the stationary frame of reference XYZ
Ω	Spin speed of the rotor
R_{aa}, R_{bb}	Autocorrelation functions of stochastic fields a (s) and b (s)
$\{h\}$	State vector
$\{h_0\}$	Amplitude of the assumed solution for the state vector
$\{H\}$	Force vector in the first-order state vector form

$\{q\}$	Vector of cross-section displacements in the stationary frame of reference, XYZ
$\{p\}$	Vector of cross-section displacements in the rotating frame of reference, xyz
$\{p^s\}$	Nodal displacement vector of the rotor-bearing system in the rotating frame of reference
$\{p_0\}$	Constant solution for the system displacement vector $\{p^s\}$
$\{Q\}$, $\{P\}$	External force vectors relative to the XYZ and xyz triads
$\{x\}$	Eigenvector to the right
$\{y\}$	Eigenvector to the left

Where appropriate, the following apply:

~	*indicates the* stochastically-fluctuating component
overbar	*denotes the* mean value
overdot	*denotes* differentiation with respect to time
^	*denotes the* transformed matrix
< >	*represents the* expectation operation

Superscripts

T	*refers to* the transpose of a vector/matrix
d, e, b, s	*refer to* the disk, element, bearing and system (the entire rotor-bearing system) respectively

Subscripts

T	*refers to* translational displacement
R	*refers to* rotational displacement
B	*refers to* bending deformation
A	*refers to* deformation due to axial load

Chapter 1

Introduction

1. 1 Randomness in System Parameters

The parameters of any mechanical or structural system possess a random variation when mass products are considered. The randomness in system parameters encompasses the uncertainties involved at the design and production stages, as well as the uncertain nature of the operating conditions. At the design stage, randomness is present in the test data regarding material strength values, elastic constants, engineering constants, deterioration parameters, damage parameters, parameters that correspond to the environmental effects and the material properties pertinent to the service life. At the production stage, manufacturing errors, machining errors, measurement errors, variation in the strengths of fasteners, joint strengths etc., give rise to system parameter fluctuations over mass products. At the service stage, the environmental effects, deterioration of the structural components, degradation of structural joints and variation in joint strengths lead to the system parameter randomness. The randomness in material properties significantly affects the functioning of the mechanical component and is unavoidable even with the best quality control measures.

Tests on a single material specimen or structure yield a definite value for each material parameter such as the elastic constant, engineering constant and damage parameter. But when a number of specimens are tested, (i) the parameter values randomly fluctuate from specimen to specimen, (ii) within the same structure itself, the values of any parameter display an uncertain spatial variation, (iii) due to environmental degradation the parameters have uncertain fluctuations. The sample-to-sample variation, spatial fluctuations within the structure, structure-to-structure variations and variations due to environmental effects of strength, deterioration, deformation and damage parameters of most of the present-day engineering materials are random [Hori, 1973]. This is particularly the case with fibre reinforced composite materials. Variations in fibre size, fibre volume fraction, fibre orientation, void content, matrix properties, interfaces and thickness of lamina are always present and unavoidable. As a result, the elastic constants, engineering constants and deformation parameters of fibre reinforced composites have a random variation [Borri, 1993].

At the structural level, the geometric properties, damping constant, stress propagation parameters, joint strengths, degree of fixity, fastener strengths, boundary conditions, and damage properties always possess a random variation from structure-to-structure as well as within a single structure itself. As a result, the response and deformation parameters of the structural system or a mechanical component possess a random spatial variation.

1. 2 Sensitivity of Dynamic Response

The dynamic response of continuous mechanical and structural systems, i.e. systems with infinite degrees-of-freedom, are described by partial differential equations in space and time. A finite element formulation or any other suitable method can be employed to obtain the relevant discrete mechanical and structural systems that have finite degrees of freedom. The dynamic response of these discrete mechanical and structural systems, are described by ordinary differential equations in time. The system matrices such as the stiffness matrix and mass matrix that define the dynamic response are obtained through the finite element formulation. Depending on whether these matrices are symmetric or non-symmetric, the mechanical or structural system is accordingly called as self-adjoint or non self-adjoint system [Meirovitch, 1980].

Many important mechanical systems including rotating machinery components, aircraft structures, gyroscopic systems, actively-controlled mechanical or structural systems, coupled fluid-structure systems and non-conservatively loaded structural systems belong to the category of non self-adjoint systems. The non self-adjointness in these systems arise due to the interaction of fast moving continuous media with deformable bodies, coriolis accelerations, gyroscopic forces and active control forces.

The vibratory response and stability of a mechanical or structural system have been shown through past investigations [Rajan, Nelson and Chen, 1986; Pedersen and Seyranian, 1983; Plaut and Huseyin, 1973; Fox and Kapoor, 1968] to be highly sensitive to any small fluctuations or variations in one or more parameters of the system. In particular, the sensitivities of the response and stability characteristics of the non self-adjoint systems [Leipholz, 1988; Meirovitch, 1980] are considerably large [Nelson and Nataraj, 1986; Pedersen and Seyranian, 1983].

Significant research activity has taken place during the recent past regarding the quantification of sensitivities of mechanical and structural systems [Rajan et al, 1986; Fox and Kapoor, 1968]. However, all these works deal only with the sensitivities of system response characteristics to known deterministic variations in the parameters of the system. When the system parameter variations are known and deterministic, the sensitivities of response and stability characteristics can be calculated and they also are deterministic. On the other hand, when the system parameter variations are random, the sensitivities correspondingly become random and the quantification of the absolute values is not possible. The probabilistic viewpoint must be employed in this case. The sensitivities can then be quantified using the theory of stochastic processes and the resulting sensitivities become correlated random processes [Sankar et al, 1993; Ramu and Ganesan, 1993].

1. 3 Random Eigenvalues and Eigenvectors

The eigenvalues and eigenvectors essentially constitute the performance indicators and state descriptors of a dynamic system, since the system charcteristics, properties and hence the behaviour of the system are all embedded in them. The dynamic response as well as the stability of a dynamic system are represented in terms of its eigenvelues and eigenvectors. For instance, the natural frequencies of a dynamic system can be expressed as the eigenvalues of the system of algebraic equations of motion describing the free response. The forced response of the structural system is represented in terms of the eigenvalues and eigenvectors of the dynamic system using the concepts of modal analysis. Further, the critical loads or buckling loads of structural and mechanical systems are expressed as the eigenvalues of the corresponding algebraic equilibrium equations in terms of the flexural as well as the geometric stiffness coefficients pertinent to the static response. In the case of non self-adjoint systems, the so-called kinetic or dynamic method of stability investigation [Leipholz, 1980; Bolotin, 1963] leads to the calculation of the critical loads of the structural system. The corresponding mathematical problem is an eigenvalue problem, and the critical loads are related to the eigenvalues of the system. This way, the eigenvalue problems are central to the response and stability analysis of any mechanical or structural system.

The magnitude and distribution of eigenvalues are dictated by the inertial (mass), energy dissipation (damping) and compliance (flexibility or stiffness) characteristics of the dynamic system. The compliance and inertial characteristics are functions of both the material properties as well as geometric properties of the structural component. When these properties have a random variation, the inertial, energy dissipation and compliance properties of the mechanical system become random. Hence, when the material properties such as the Young's modulus and mass density have a random variation, the eigenvalues and eigenvectors of the mechanical system will also have a random variation. A probabilistic approach should then be employed for analyzing the mechanical systems with random material property variations. The probabilistic characteristics of eigenvalues and eigenvectors are functions of the probabilistic characteristics of system parameters which are in turn functions of material and geometric properties.

The probabilistic description of the eigenvalues and the eigenvectors has been developed in past investigations [Benaroya, 1991; Ibrahim, 1987; Soong and Cozzarelli, 1976; Boyce, 1968]. However, only the self-adjoint systems have been analyzed and even among them a few simple cases have been treated. Moreover, in most of the existing works the random material properties are modelled only as random variables. Further details in this regard are provided in the next chapter.

1. 4 Scope and Objectives of the Thesis

Non self-adjoint mechanical systems that have a random distribution of material properties have not so far, been analyzed for the probabilistic characteristics of the eigenvalues and eigenvectors. However, (i) many of the important mechanical systems such as the rotating machinery systems are non self-adjoint in nature, (ii) the sensitivities are large when the mechanical system is non self-adjoint. So, a detailed analysis of non self-adjoint systems with random parameter variations becomes only natural and essential. Such an analysis of non self-adjoint systems is considered in this thesis. The development of computational solutions that both characterize and quantify the variability in eigenvalues and eigenvectors of non self-adjoint mechanical systems is the objective of this thesis. Since the Finite Element Method (FEM) has established itself as a powerful and versatile tool for analyzing mechanical and structural systems, computational solutions that can be directly employed in a finite element formulation are developed in this thesis.

Modelling of the random parameter variations using random variables can not take into account the spatial correlation properties of the material properties, which are very important in that they considerably influence the second-order probabilistic moments of response variables such as the eigenvalues and eigenvectors. Considering this fact, the material properties are rigorously modelled using one-dimensional univariate homogeneous stochastic fields [Vanmarcke, 1983] that are characterized by the ensemble mean values, standard deviations, and autocorrelation functions or equivalent power spectral density functions in each case. The probabilistic moments up to the second order i.e. the ensemble mean values and standard deviations of both the eigenvalues and the eigenvectors of the mechanical system are determined. The solution is sought in a form suitable for finite element formulation.

Based on this solution, a probabilistic description for the whirl speeds and whirl modes of high speed rotor-bearing systems is developed. A finite element solution is developed. The finite element formulation that has already been developed for the dynamic analysis of high speed rotor-bearing systems with deterministic parameters [Nelson and McVaugh, 1976] is employed for analyzing the deterministic counterpart of the rotor-bearing system with random parameters. The random variations in the material properties of the rotor-bearing system are modelled using homogeneous stochastic fields. The sensitivities of whirl speeds and whirl modes are quantified. The probabilistic moments of the whirl speeds and whirl modes of the rotordynamic system are determined in terms of both the probabilistic moments as well as the power spectral density functions of the material property stochastic fields. The sensitivities of the rotordynamic system are employed as a basis for solution development. Associated software is fully developed.

Design implications are systematically brought out based on the developed solution. Since the field data on the random variations in the material properties of the commonly-used engineering materials have different correlation characteristics, the effects of the correlation structure of the material property variations on the second-order probabilistic moments of whirl speeds are quantified. A parametric study is carried out in this regard.

1. 5 Organization of the Thesis

The detailed literature survey regarding the random eigenvalue problems is presented in chapter 2. The non self-adjoint eigenproblem with stochastic parameters is considered in chapter 3. The stochastic parameters are modelled as multidimensional multivariate homogeneous stochastic fields. An asymptotic solution that represents a sample function of eigenvalues and eigenvectors corresponding to a sample function of material property stochastic fields is developed. Based on this asymptotic solution, the probabilistic moments of eigenvalues and eigenvectors are determined.

In chapter 4, the high speed rotor-bearing systems with a random distribution of material properties are considered. Based on the solutions developed in chapter 3, the probabilistic moments of whirl speeds and whirl modes are determined in terms of that of the material properties and their power spectral density functions. Associated software is fully developed and described in this chapter. In chapter 5, the parametric study encompassing the effects of different correlation characteristics of the material properties and the effects of the bearing flexibility on the probabilistic moments of the whirl speeds of the rotor-bearing system is presented. Also, the effects of modelling, that is, the size of the finite elements on the variability of whirl speeds are brought out.

The research results are summarized and some of the useful design implications are described in chapter 6. Suggestions for the future work are also provided therein.

Chapter 2

Literature Survey

2. 1 Random Eigenvalue Problems

2. 1. 1 Eigenvalue Problems of Mechanical Systems

Eigenvalues and eigenvectors constitute a set of system descriptors and performance indicators of a dynamic system, since information about the characteristics, behaviour, state and time evolution of the dynamic system are embedded in them. Also, the stability parameters and critical loads of the so-called self-adjoint as well as non self-adjoint mechanical systems are usually obtained through solving an associated eigenvalue problem. Whirl speeds of high speed rotor-bearing systems, critical flow velocities of fluid conveying pipes, critical active control forces and other similar stability parameters can be cited in this regard. Moreover, modal analysis is an effective way of conducting a forced vibration analysis of dynamic systems, for which the eigensolutions obtained from the homogeneous case serve as a basis.

2. 1. 2 Sensitivities of Eigensolutions

The sensitivity of the response of mechanical and structural systems to variations in their parameters is one of the basic aspects in the design of structures and machine components. The sensitivity of the stability parameters and critical loads of structural and mechanical systems is another basic aspect in the design. Since information about the behaviour of a dynamic system are embedded in its eigenvalues and eigenvectors, changes in the system behaviour due to parameter variations are reflected in its eigenvalues and eigenvectors. Hence the sensitivities of eigenvalues and eigenvectors of mechanical and structural systems should be examined when dealing with the design of systems that have stochastically varying geometric and material properties. Appropriate design modifications should be worked out based on the resulting information.

Eigenvalue and eigenvector sensitivities have been extensively used in analyzing and designing self-adjoint systems [Fox and Kapoor, 1968] and non self-adjoint systems [Plaut and Huseyin, 1973]. Lund [1979] calculated the sensitivities of the critical speeds of a rotor shaft resting on isotropic bearing supports to changes in the design parameters using a state vector-transfer matrix formulation. Palazzolo et al [1983] presented a generalized receptance method for eigensolution reanalysis of rotordynamic systems. A non-perturbation approach has been used in their work that has the advantage of accomodating original system modifications of arbitrary magnitude. Such modifications are treated simultaneously so as to eliminate error propagation which occurs when modifications are treated in sequence. The efficiency of the procedure developed increases with the total number of analyses made. Rajan et al [1986] developed equations for the sensitivities of the whirl speeds of linear rotor-bearing systems modelled using finite elements, including the bearing damping. They have shown that the (damped) critical speed sensitivity coefficients can be evaluated with a knowledge of the spin speed (whirl frequency) sensitivity coefficients as well as the design parameters.

2. 1. 3 Eigenvalue problems of Uncertain Systems

Eigenvalue problems of uncertain structural systems subjected to conservative loadings have been analyzed in the works of Boyce [1968], Shinozuka and Astill [1972] and Soong and Cozzarelli [1976]. Probabilistic modelling of system parameters such as material and geometric properties results in random eigenvalues and eigenvectors, the statistical properties of which are determined by the random coefficients associated with the inertia and stiffness terms of the equations of motion. The probabilistic description of the eigenvalues and eigenvectors has been obtained, in the literature, only for a limited and simple class of continuous and discrete systems [Benaroya, 1991; Ibrahim, 1987]. Statistical measures such as the expected values, variances, and covariances of the eigenvalues and eigenvectors have been determined. The treatment of these systems is based on the analysis of random matrices and random differential operators [Scheidt and Purkert, 1983].

2. 2 Random Eigenvalue Problems of Continuous Systems

Continuous systems involve uncertainties in the geometric properties, material properties and the boundary conditions [Boyce and Goodwin, 1964]. For these systems the random eigenvalues have been analyzed using both analytical and numerical approaches.

2. 2. 1 Numerical Solutions

The numerical solutions have been obtained using the following two methods:

1) the Monte Carlo Simulation (MCS) method, and

2) the Stochastic Finite Element Method (SFEM).

2. 2. 1. 1 Monte Carlo Simulation (MCS) method

In the Monte Carlo simulation method, a set of random samples of uncertain system parameters are generated and they are used for computing numerically the eigenvalues and eigenvectors of the system. These eigenvalues and eigenvectors constitute a sample set. A number of such sample sets are determined from which the statistics of eigensolutions are computed. This is a very computationally-expensive method since it requires a large number of numerical solutions, and hence it becomes unrealistic and impractical when dealing with medium or large size systems. The MCS methods have been used to solve problems that are related to the nonlinear random vibration of mechanical and structural systems. Different versions of MCS methods involving digital simulation of random fields have been used to analyse mechanical systems with stochastic parameters. The applicability and limitations of these methodologies have been reviewed by Shinozuka [1987], Shinozuka and Deodatis [1988] and Spanos and Mignolet [1989]. Each of these reviews have been confined to a certain simulation strategy, thereby lacking a detailed comparative study of the various simulation strategies. Further, the computing environments have since seen rapid improvements in terms of computational efficiency and accuracy. An exhaustive and comprehensive review has been put forth by Ramu et al [1996] that addresses the following issues pertinent to the various approaches used for simulation: (i) the accuracy achieved for a particular computational effort involved, (ii) the suitability of each method to cope with modern computational systems, (iii) the flexibility and expandability so as to account for more general system behaviour, and (iv) the evaluation of the merits and demerits of these approaches.

The various simulation models proposed in the literature can be classified according to the representation of the stochastic field that is being simulated. On this basis, the available works can be grouped into three categories: Spectrum models, Time Series representation, Filter models and Hybrid models.

Spectrum models

This category of simulation methods consists of methods that are based on the summation of trigonometric functions. The methods are, hence, appropriately termed as wave superposition or spectrum models. The basic representation of the stochastic field [Spanos and Mignolet, 1987] is an adoption of Rice's formula [1954], i.e. the simulation is accomplished by summing a large number of cosine functions with weighted amplitudes and random phase angles. Such a representation has first been used by Borgman [1969], wherein the simulation of ocean surface elevation as a multidimensional process has been attempted. Practical methods of simulation of stochastic fields have been further developed by Shinozuka [1971]. Through a series of papers, significant contributions to the analog and digital simulation of stochastic fields in space by spectral representation method have been made [Shinozuka and Deodatis, 1988; Yamazaki and Shinozuka, 1988; Shinozuka, 1987; Shinozuka and Lenoe, 1976; Shinozuka, 1972; Shinozuka and Jan, 1972; Shinozuka and Wen, 1972]. The first work that uses spectrum model for the generation of multivariate and multidimensional random fields in space has been reported in [Shinozuka, 1974]. In this work an efficient use of Fast Fourier Transform (FFT) techniques, the algorithms for which were just then available have been made, thereby rendering the method computationally powerful. A natural extension to simulate non-gaussian fields and stochastic waves has also been performed recently [Deodatis and Shinozuka, 1989] so that the more complex problems of natural disaster engineering can be addressed. A detailed discussion of the other works that make use of the spectrum model for Monte Carlo Simulation can be found in [Ramu et al, 1996].

Time series representation

The second category of simulation methods is based on the time series interpretation of stochastic fields. Parametric models in terms of the differential equations are developed and employed to describe the dynamic systems in this category. Unlike the FFT-based techniques, the usage of parametric models avoids the demand for large memory space because a very limited data such as the coefficient matrices of stochastic fields, are stored. A recursive relationship which depends on the model under consideration generates the output stochastic field using this limited information. Homogeneous and non-homogeneous fields are generated [Spanos and Mignolet, 1987], although, the non-gaussian nature has not been fully represented. For the purposes of digital simulation, the Auto-Regressive Moving Average (ARMA) model is a very effective tool.

Filter models

The third category constitutes the classical method of digital generation of random data. This method consists of generating a random process as the output of an appropriate analytical filter subjected to a simulated white noise. While most of the works treat nonstationary cases, those of Schueller and Bucher [1988] as well as Amman [1990] pay particular attention to non-gaussian cases.

Hybrid Methods

By suitably modifying the filter and time series models, and combining these methodologies, various new models have been devised. These models are called "Hybrid models". For instance, a slightly modified version of the linear filtering models constitutes the so-called "Pulse Train Method" which involves the generation of a pseudo white noise process [Itagaki and Ogawa, 1975]. In [Li and Kareem, 1991], another hybrid method is adopted wherein the linear filtering model and the wave summation model are combined together to yield, a non-gaussian distribution for a general n-dimensional and m-variate stochastic field. Quite recently, Li and Kareem [1993] have generated long-duration multivariate random processes using a hybrid discrete Fourier transform and a digital filtering approach.

From a study of the models described above, the following observations can be made regarding the advantages/disadvantages of these approaches [Ramu et al, 1996]:-

a) The trigonometric summation method becomes ineffective when a wide-band process is considered, due to the large number of cosine terms. However, this drawback has been overcome by Yang [1972] so as to enable the usage of the FFT technique in an efficient manner even for large number of terms.

b) It is well known that obtaining an exact solution for a set of nonlinear equations is very difficult if not impossible. Simulating this problem based on a parametric model is quite involved.

c) The model order selection scheme in the case of parametric models is rather empirical and is not well established.

d) The filter models are restricted only to one-dimensional, univariate cases and are less effective than both the time series representation and spectrum models.

e) Generation of envelope processes, which are needed in reliability studies of machines and structures, is too difficult with parametric models when compared with the wave summation models.

f) The capabilities of wave summation models such as the flexibility to incorporate methods based on "orthogonal increments" [Shinozuka, 1987] underscore their superiority over other methods, despite the computational disadvantages associated with the wave summation models in extreme cases.

2. 2. 1. 2 Stochastic Finite Element Method

The finite element method is widely used as an efficient numerical method for systems whose parameters are assumed to be of a deterministic nature. However, in practice, such parameters exhibit randomness thereby violating the above assumption. Analyses of systems with random parameters have been carried out based on statistical and non-statistical approaches [Liu et al, 1985]. Stochastic finite element method is a non-statistical approach that applies the finite element analysis in a probabilistic framework. Knowledge of the first two probabilistic moments of system parameters is sufficient for this method and there is no need for the multivariate distribution functions. A comprehensive review of the various versions of the stochastic finite element method that have been developed during recent times can be found in the work of Benaroya and Rehak [1988]. Most of these methods use as a basis the finite element method as applicable to a deterministic problem and are categorized as: (1) first-order second-moment methods [Ramu and Ganesan, 1993; Spanos and Ghanem, 1989; Yamazaki et al, 1988; Contreras, 1980] and (2) reliability methods [Liu et al, 1986].

Astill et al [1972] used a combination of the deterministic finite element method and the Monte Carlo Simulation method to analyze the problem of impact loading of structures with random geometric and material properties. Nakagiri and Hisada [1982] developed a stochastic finite element method based on a mid-point discretization scheme to estimate the statistics of eigenvalues and eigenvectors of structures with uncertain material and geometric properties. Vanmarcke and Grigoriu [1983] defined a scale of fluctuation that can be used in the finite element modelling. They developed a solution using the finite difference method for obtaining the first-order and second-order statistics of the deflection of beams whose properties vary randomly along the longitudinal axis. Weeks and Cost [1980], in their study of the reliability of a composite structure subjected to random loading, justified the use of finite elements in light of the abrupt changes in mass and stiffness in composites. Nakagiri et al [1987] used a version of the stochastic finite element method to analyse the eigenvalue problem of laminated composite plates considering the stochastic variations in the stiffness. Random variables have been used for modelling the material property variations. Zhu and Wu [1991] determined the eigensolution statistics using local averages and showed the superiority of their stochastic finite element method over that developed earlier by Nakagiri and Hisada [1982].

In the stochastic finite element methods discussed above, either a random variable modelling of material properties has been employed or the stochastic fields that model the material properties are converted into a set of random variables. Further, the calculation of the covariances between coefficients of characteristic matrices requires repeated integration of full length autocorrelation functions. This results in the finite element discretization becoming a function of the material property stochastic fields. Also, a significant loss in the accuracy with which the response moments are evaluated has been observed. Further, most of these works consider only one material property variability at a time. The Monte Carlo Simulation of stochastic fields along with the recursive solution of system equations renders them computationally inefficient and expensive. Ramu and Ganesan [1993] developed a new stochastic finite element methodology using the Galerkin weighted residual method to analyse structures in which more than one system parameters are stochastic. The superiority of their methodology over the methodologies developed earlier has also been brought out in this work.

2. 2. 2 Analytical Solutions

The analytical solutions, as detailed in the works of Boyce [1968] and Scheidt and Purkert [1983], fall into two categories depending on whether the statistical or non-statistical part of the analysis is performed first in determining the statistical moments of eigenvalues. One approach consists of first expressing the solution in terms of the system parameters, without regard to whether these parameters are random or deterministic. The statistical properties are determined after obtaining the solution. Such an approach is referred to as "honest approach" by Keller [1962, 1964] and the solution can be determined by using the perturbation method, variational method, asymptotic estimate method, or the integral equation method. Of these four methods, the perturbation method is the one that has been most widely used in the existing works [Hoshiya and Shah, 1971; Bliven and Soong, 1969; Boyce and Goodwin, 1964].

The approach is said to be "dishonest" [Boyce, 1967] if the statistics of the eigenvalues are directly determined by performing averaging analysis based on the system's partial differential equation and its associated boundary conditions. The statistics can be evaluated using either the iteration methods or the hierarchy methods. In the iteration methods the averaged integral equations of the random eigenvalue problem are solved based on assumed correlation relations. The hierarchy methods [Adomian, 1983; Haines, 1967] enable all statistical functions in question to be calculated. A detailed account of the various methods used to study the random eigenproblem of continuous systems can be found in the paper by Boyce [1968].

Purkert and Scheidt [1979a, 1979b, 1977] treated the random eigenvalue problem of ordinary differential equations with deterministic boundary conditions. They studied the effects of correlation length on the eigenvalues and eigenvectors. Boyce and Xia [1985] obtained the upper bounds for the mean values of eigenvalues through a variational characterization of the eigenvalues. Boyce [1966] and Linde [1969] considered a Sturm-Liouville boundary value problem with a stochastic non-homogeneous term. Scheidt and Purkert [1983] analyzed the probabilistic moments of the eigenvalues and mode shapes of random matrices as well as random ordinary differential operators. Day [1980] developed a number of asymptotic expansions for the random eigenvalues and eigenvectors of continuous systems. The concept of the Wiener field, obtained by replacing the time variable of a Wiener process by a space coordinate, has been adopted by Wedig [1977, 1976] as a basis to model the randomly distributed loadings or imperfections in continuous structural systems.

The statistics of random eigenvalues of elastic strings and bars have been determined by Boyce [1962] as well as Goodwin and Boyce [1964]. The statistics of the eigenvalues of elastic beams have been determined by Boyce and Goodwin [1964] using the perturbation method, the integral equation method as well as a numerical solution. The geometry of the cross-section of the beam and its support mechanism have been considered as random variables. Bliven and Soong [1969] carried out a similar study for the case of a simply supported elastic beam using the perturbation method. The beam has been considered to possess random imperfections and these imperfections have been modeled using a lumped-parameter model. Hoshiya and Shah [1971] used the perturbation method to analyze a beam-column with end axial load and spring-type boundary conditions. The n-th natural frequency of the beam-column has been determined using a probabilistic framework and the sensitivity analyses have been performed. Shinozuka and Astill [1972] also examined a beam-column supported at its ends by rotational springs for vibration as well as buckling. The spring supports and the axial force have been treated as random variables. The distribution of material and geometric properties has been considered as correlated homogeneous random functions that are generated using a Monte Carlo Simulation. The mean and variance of the eigenvalues have been determined using both the perturbation analysis and the MCS.

Vaicaitis [1974] employed a two-variable perturbation expansion procedure to determine the eigenvalues and normal modes of beams with random and/or non-uniform characteristics which do not deviate considerably from their mean properties. A Monte Carlo Simulation has been used to determine the statistical averages of the eigenvalues and mode shapes of the beam. Hart and Collins [1970], Collins et al [1971], and Hasselman and Hart [1972, 1971] developed a numerical method for computing the statistical properties of mode shapes using the component mode synthesis method. Hart [1973] developed a general algorithm for calculating the statistics of the natural frequencies and mode shapes of structures. Lin and Yang [1974] suggested that practical periodic structures always have some disorder due to imprecision in measurements, imperfect alignment of parts etc. They analysed the random eigenvalue problem of a disordered N-span periodic beam using the first-order perturbation analysis to determine the natural frequencies and their associated normal modes.

2. 3 Random Eigenvalue Problems of Discrete Systems

The statistics of random eigenvalues and eigenvectors of discrete systems have been determined using the transfer matrix method, the random perturbation method and the Monte Carlo Simulation method. Kerner [1954, 1956] developed the transfer matrix method for disordered periodic lattice systems. Soong and Bogdanoff [1963] examined the statistics of the random eigenvalues of a disordered multi-degree-of-freedom using the transfer matrix method. An N-degree-of-freedom spring-mass chain has been considered where the masses and stiffnesses have been modelled as independent random variables with equal mean values and equal standard deviations. Their method is an extension of the transfer matrix method developed originally for free vibration of deterministic discrete systems [Thomson, 1981] and utilizes a perturbational expansion of the random eigenvalues in terms of the random perturbation of the system parameters.

The random perturbation method [Scheidt and Purkert, 1983] is based on an asymptotic expansion and combines the perturbation and multivariate statistical analyses. The multivariate analysis establishes the probability distributions of random eigenvalues in terms of the coefficients of the governing equations of motion. This method is an extension of the perturbation method developed for the deterministic eigenvalue problems [Cole, 1968; Meirovitch, 1980]. Each of the random eigenvalues can be expressed, in the form of a Taylor series, as the sum of the deterministic component (mean value) and the fluctuating component representing the first and higher order terms. Similarly, each eigenvector is expressed as the sum of the deterministic component as well as the fluctuating component. The analysis is called first-order perturbation analysis if the fluctuating component consists only of first-order terms. It is referred to as the second-order perturbation analysis if in the fluctuating component terms up to second order are retained while higher-order terms are discarded. Second-order perturbation analysis is tedious and computationally-expensive and it involves multivariate statistical analysis.

As a consequence, most of the works reported in the existing literature deal only with the first-order perturbation analysis.

Collins and Thomson [1969] employed the first-order perturbation analysis and derived the eigenvalue and eigenvector statistics of a multi-degree-of-freedom system in terms of the covariance matrix of the system parameters. Assuming that the perturbations are sufficiently small, they computed the mean eigenvalues and eigenvectors as well as the variances of the eigenvalues from the mean mass and stiffness matrices. Their work considered the statistical correlation between the system influence coefficients which has not been considered by McCalley [1960] while studying the sensitivity of the eigenvalues and eigenvectors to variations in the elements of the mass and stiffness matrix. For a simple chain of springs and masses with uncorrelated random masses or with random uncorrelated stiffnesses, Collins and Thomson [1969] have shown that the standard deviations of the vibration frequencies are linearly related to the standard deviations of the mass and stiffness influence coefficients. Their results closely agree with the results of Soong and Bogdanoff [1963] and have further been confirmed by Monte Carlo Simulation. However, such a linear dependence vanishes in the presence of correlations between the mass or stiffness influence coefficients and also, the eigenvalues are not closely spaced. More recently, Pierre [1985] considered two different discrete systems, a mass-spring chain with random mass, and a chain of coupled pendula with random lengths. The statistics of their eigenvalues have been obtained by employing a first-order perturbation analysis. The results obtained in this work have been in agreement with the results obtained by Soong and Bogdanoff [1963].

2. 4 Non Self-Adjoint Systems with Random Parameters

Existing works deal only with self-adjoint systems such as structures subjected to conservative loadings and mechanical systems that do not involve coriolis forces, gyroscopic effects etc. However, many important mechanical systems including high speed rotors, actively-controlled systems, aeroelastic structures and pipes conveying high velocity fluid flow, are described by non self-adjoint differential equations. The interaction of fast moving continuous media with deformable bodies, coriolis accelerations, gyroscopic forces and active control forces give rise to the non self-adjointness of the system. Safe and reliable designs of such non self-adjoint systems have to be developed, taking into account the uncertainties in the parameters of the mechanical system through a probabilistic framework. A stochastic field description of system parameters involves considerable amount of complexity in the analysis procedures. The use of only the mean and variance through the random variable modelling of system parameters is inadequate to precisely describe the stochastic nature of parameter variation. This is so because, such a parameter variation is described through a unique autocorrelation function or scale of fluctuation. Further, when the versatile finite element modelling is employed to analyse the complex behaviour of real life mechanical systems, procedures are to be developed to handle the stochasticity in both the stiffness and mass matrices corresponding to the global dynamic response parameters.

Non self-adjoint systems with stochastic parameters have been first analyzed in the works of Sankar et al [1993], Ganesan et al [1993], Ramu et al [1992], Ramu and Ganesan [1992] and Ganesan [1996]. A closed form solution to the free vibration problems of non self-adjoint type, wherein more than one parameter behave in a stochastic manner, has been obtained by Ganesan et al [1993]. Beck's column and Leipholz's column whose Young's modulus and mass per unit length have a stochastic variation have been investigated in the work of Ramu and Ganesan [1992]. In these works, an asymptotic solution to the individual realizations of eigenvalues and eigenvectors of the non self-adjoint differential equation that describes the free-undamped oscillations of the system is obtained. Complete covariance structures of both eigenvalues and eigenvectors are obtained through computationally efficient expressions. Stochastic sensitivities of eigenvalues and eigenvectors of the non self-adjoint system to the variations in material properties are systematically brought out.

In all these works, analytical solutions have been developed for non self-adjoint eigenproblems. A numerical solution based on the finite element formulation has first been developed in the works of Ganesan et al [1993] and Sankar et al [1993]. The stochastic finite element method developed by Ramu and Ganesan [1993] has been employed to analyse the high speed rotordynamic systems [Sankar et al, 1993] and the beam column with non-conservative loadings [Ganesan et al, 1993] in which the material properties such as the elastic modulus and mass density fluctuate in a stochastic manner. This methodology has further been extended in the work of Venkatesan and Ganesan [1995] to analyse a general class of non self-adjoint mechanical systems with stochastic parameters. Eigensolution variability of non self-adjoint mechanical systems that have an uncertain distribution of material properties and loadings has been evaluated based on a stochastic modelling of uncertain system parameters [Venkatesan and Ganesan, 1995].

Chapter 3

Probabilistic Moments of Eigensolutions

3. 1 Introduction

The algebraic eigenproblem corresponding to the non self-adjoint mechanical systems such as the high speed rotor-bearing systems is considered. The mechanical system is treated to be a multi-degree-of-freedom system whose material and geometric properties as well as loadings have a stochastic distribution. Correspondingly,

(i) the coefficients of the characteristic matrices such as the stiffness and mass matrices are modelled as stochastic processes, and

(ii) the eigensolutions are modelled as random variables.

The approach to the probabilistic analysis deployed in the works of Ganesan [1996], Ramu and Ganesan [1993] and Ganesan et al [1993] is followed in this thesis. The formulation and the solutions for the probabilistic moments of eigensolutions, developed in the past investigations [Ganesan et al, 1993; Sankar et al, 1993] are deployed as the basis in this work [Venkatesan and Ganesan, 1995].

The sample realizations of the eigensolutions and the coefficients of characteristic matrices are expanded into asymptotic series. Based on these expansions and the first-order second-moment response method of probabilistic analysis, the probabilistic moments of the eigensolutions are determined. Corresponding computational algorithms are developed.

3.2 The Random Eigenvalue Problem

The vibrational and stability characteristics of a mechanical system can be obtained by solving the associated eigenproblem. The eigenproblem can be written in the following form [Meirovitch, 1986]:

$$[A]\{x\} = \lambda [B]\{x\} \tag{3.1}$$

In the above, square matrices [A] and [B] are the characteristic matrices of the mechanical system, λ is the eigenvalue and $\{x\}$ is the eigenvector. Both the matrices [A] and [B] are of order n×n where n is the number of degrees of freedom of the mechanical system. Correspondingly, n number of eigenvalues, λ_i, i = 1, 2, 3, ..., n, and n number of eigenvectors, $\{x\}_i$, i = 1, 2, 3, ..., n, can be obtained by solving the algebraic eigenproblem given by Eq. (3.1).

The coefficients of the characterisitic matrices [A] and [B] are deterministic functions of the material properties, geometric properties and, in some cases, the loadings on the mechanical system being analyzed. For instance, the coefficients of matrices [A] and [B] can respectively be the functions of the Young's modulus and the mass per unit length (or the mass per unit area/volume in the case of 2-D/3-D structures) of a mechanical system. As a result, when the material properties, geometric properties and loadings are random functions, the coefficients of matrices [A] and [B] become random functions. Due to this, both the eigenvalues λ_i, i = 1, 2, 3, ..., n, and the elements of their corresponding eigenvectors, $\{x\}_i$, i = 1, 2, 3, ..., n, become stochastic quantities. When each of the material properties, geometric properties and loadings are modelled by stochastic fields, each of the matrix coefficients a_{ij}; i, j = 1, 2, 3, ..., n and b_{ij}; i, j = 1, 2, 3, ..., n are obtained as the derived stochastic fields. In this case, λ_i, i = 1, 2, 3, ..., n, become random variables and each of the corresponding eigenvectors $\{x\}_i$, i = 1, 2, 3, ..., n, become n-dimensional random vectors, the components of which are correlated random variables. A sample realization of stochastic fields that model the material, geometric and loadings of the mechanical system leads to corresponding sample realizations of coefficients a_{ij} and b_{ij} (and thus to sample realizations of matrices [A] and [B]). Based on the sample realizations of matrices [A] and [B], the algebraic eigenvalue problem expressed by Eq. (3.1) can be solved to yield sample realizations of eigenvalues λ_i and their corresponding eigenvectors $\{x\}_i$. The algebraic eigenvalue problem that is expressed by Eq. (3.1) is called as the "Random Eigenvalue Problem".

For a non self-adjoint mechanical system, such as the high speed rotor-bearing system, both the matrices [A] and [B] are non-symmetric. The i-th solution pair $(\lambda_i, \{x\}_i)$ is obtained by solving the equation

$$[A]\{x\}_i = \lambda_i [B]\{x\}_i \qquad (3.2)$$

Because of the non-symmetric nature of the characteristic matrices, it is not possible to obtain a unique set of eigenvectors $\{x\}_i$. Based on the so-called "Adjoint System" [Meirovitch, 1980], another set of eigenvectors $\{y\}_i$ can also be obtained. The adjoint system is described by the following algebraic eigenproblem:

$$[A]^T\{y\}_i = \lambda_i [B]^T\{y\}_i \tag{3.3}$$

This equation can be re-written as

$$\{y\}_i^T[A] = \lambda_i\{y\}_i^T[B] \tag{3.4}$$

The i-th adjoint solution pair $(\lambda_i,\{y\}_i)$ is obtained from Eq. (3.4). Based on Eqs. (3.2) and (3.4), it is customary to call $\{x\}_i$ as the "Eigenvector to the right" and $\{y\}_i$ as the "Eigenvector to the left". Both the basic eigenproblem and the adjoint eigenproblem yield the same set of eigenvalues λ_i. As is the case of the eigenvector to the right $\{x\}_i$, the eigenvector to the left, $\{y\}_i$, is also an n-dimensional random vector, the components of which are correlated random variables. The eigenproblem given by Eq. (3.3) or Eq. (3.4) is called in this thesis as the "Adjoint Random Eigenvalue Problem", and correspondingly the eigenproblem given by Eq. (3.1) is called as the "Basic Random Eigenvalue Problem".

The probabilistic measures such as the ensemble mean, standard deviation and autocorrelation function (or its equivalent power spectral density function), of the stochastic fields that model the material properties, geometric properties and loadings of the mechanical system, are prescribed. From these, the stochastic characteristics of coefficients a_{ij}; i, j = 1, 2, 3, ..., n and b_{ij}; i, j = 1, 2, 3, ..., n can be determined based on the deterministic relationships between these coefficients and the material properties, geometric properties and loadings of the mechanical system. This way, each of the coefficients a_{ij} and b_{ij} have their corresponding ensemble mean values, standard deviations and autocorrelation functions (or their equivalent power spectral density functions). The ensemble mean values are denoted by $\bar{a}_{ij}$ and $\bar{b}_{ij}$, the standard deviations are denoted by $\sigma_{a_{ij}}$ and $\sigma_{b_{ij}}$, and the autocorrelation functions are denoted by $R_{a_{ij}}$ and $R_{b_{ij}}$ (the equivalent power spectral density functions are denoted by $S_{a_{ij}}$ and $S_{b_{ij}}$). The sample realizations of both the matrices [A] and [B] are used to obtain the sample realizations of eigenvalues λ_i, their corresponding eigenvectors to the right $\{x\}_i$ and eigenvectors to the left $\{y\}_i$, through solving Eqs. (3.1) and (3.3). The properties of these sample realizations of eigensolutions are now described. To this end, eqs. (3.2) and (3.4), which are in the form of the generalized eigenproblem, are written in the form of the standard eigenproblem as follows (assuming that the [B] matrix is invertible):

Generalized eigenproblem		**Standard eigenproblem**
$[A]\{x\} = \lambda[B]\{x\}$	$\longrightarrow$	$[C]\{x\} = \lambda\{x\}$
$\{y\}^T[A] = \lambda\{y\}^T[B]$	$\longrightarrow$	$\{y\}^T[C] = \lambda\{y\}^T$

The probabilistic characteristics of coefficients c_{ij}; i, j = 1, 2, 3, ..., n have to be determined from that of a_{ij} and b_{ij} through the deterministic relationships between them.

(i) If $\{x\}_i$ is an eigenvector to the right corresponding to the eigenvalue λ_i, then $\chi\{x\}_i$ is also an eigenvector to the right, where $\chi \neq 0$ is an arbitrary scalar. Similarly, if $\{y\}_i$ is an eigenvector to the left corresponding to the same eigenvalue λ_i, then $\kappa\{y\}_i$ is also an eigenvector to the left, where $\kappa \neq 0$ is an arbitrary scalar.

(ii) Both the eigenvector to the right and the eigenvector to the left can correspond to only one eigenvalue, but an eigenvalue, say λ_i can have many eigenvectors to the right, viz. $\{x\}_{i1}$, $\{x\}_{i2}$, ..., $\{x\}_{ik}$ and many eigenvectors to the left, viz. $\{y\}_{i1}$, $\{y\}_{i2}$, ..., $\{y\}_{ik}$. Indeed, any linear combination of the eigenvectors to the right $\{x\}_{i1}$, $\{x\}_{i2}$, ..., $\{x\}_{ik}$, corresponding to the eigenvalue λ_i is also an eigenvector to the right. As well, any linear combination of the eigenvectors to the left $\{y\}_{i1}$, $\{y\}_{i2}$, ..., $\{y\}_{ik}$, corresponding to the eigenvalue λ_i is also an eigenvector to the left.

(iii) If the eigenvalues of matrix [C] are distinct, then the eigenvectors to the right and eigenvectors to the left corresponding to these eigenvalues are independent, in the deterministic sense.

(iv) For distinct eigenvalues of [C], the eigenvectors to the left and eigenvectors to the right of [C] are orthonormal, that is, $\{y\}_j^T\{x\}_i = 0$ if $i \neq j$. Further, for $i = j$, the eigenvector to the right and that to the left for a given eigenvalue λ_j satisfy the relation $\{y\}_j^T\{x\}_j = 1$.

These properties will be used while deriving the probabilistic measures of eigensolutions.

3. 3 Asymptotic Expansions

In order to characterize and account for the stochastic nature of matrices [A] and [B], an asymptotic parameter α, $0 \le \alpha \le 1$, which is a deterministic parameter, is introduced into the mathematical formulation such that the sample realizations of the matrices [A] and [B] are expanded into their corresponding asymptotic series, according to

$$[A] = [\bar{A}] + \alpha[A(\sim)] + \alpha^2[A(\sim)] + ... \tag{3.5}$$

$$[B] = [\bar{B}] + \alpha[B(\sim)] + \alpha^2[B(\sim)] + ... \tag{3.6}$$

In the above, overbars denote ensemble mean values. As can be seen from the above two equations, the sample realizations of the characteristic matrices [A] and [B] are split into two components in each case: the mean component (the first term of the asymptotic series) and the zero-mean stochastically-fluctuating component (the second, third and higher order terms of the asymptotic series). A procedure for determining the mean component [Ā] from a random matrix [A] may be found in the appendix. Based on Eqs. (3.5) and (3.6), the coefficients a_{ij} and b_{ij} of the [A] and [B] matrices are expressed as:

$$a_{ij} = \overline{a_{ij}} + \alpha a_{ij}(\sim) + \alpha^2 a_{ij}(\sim) + ... \tag{3.7}$$

$$b_{ij} = \overline{b_{ij}} + \alpha b_{ij}(\sim) + \alpha^2 b_{ij}(\sim) + ... \tag{3.8}$$

The deterministic case is obtained when α=0 and the extreme case of stochasticity is represented by letting α=1. Substituting Eqs. (3.5) and (3.6) into Eqs. (3.2) and (3.3), the following pair of matrix equations is obtained:

$$\begin{aligned}([\bar{A}] - \lambda_i[\bar{B}])\{x\}_i = &- \alpha([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i \\ &- \alpha^2([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i - ....\end{aligned} \tag{3.9}$$

$$\begin{aligned}([\bar{A}]^T - \lambda_i[\bar{B}]^T)\{y\}_i = &- \alpha([A(\sim)]^T - \lambda_i[B(\sim)]^T)\{y\}_i \\ &- \alpha^2([A(\sim)]^T - \lambda_i[B(\sim)]^T)\{y\}_i - ....\end{aligned} \tag{3.10}$$

Corresponding to the asymptotic expansions of characteristic matrices, the sample realizations of eigensolutions λ_i, $\{x\}_i$ and $\{y\}_i$ are expanded into their relevant asymptotic series, given by

$$\begin{aligned}\lambda_i &= \lambda_i(a_{rs}^0,\ 0,\ 0,\ldots,\ 0,\ b_{rs}^0,\ 0,\ 0,\ldots,\ 0) \\ &\quad + \alpha\lambda_{i1}(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots) \\ &\quad + \alpha^2\lambda_{i2}(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots) + \ldots \\ &= \overline{\lambda}_i + \alpha\lambda_{i1} + \alpha^2\lambda_{i2} + \ldots\end{aligned} \tag{3.11}$$

$$\begin{aligned}\{x\}_i &= \{x(a_{rs}^0,\ 0,\ 0,\ldots,\ 0,\ b_{rs}^0,\ 0,\ 0,\ldots,\ 0)\}_i \\ &\quad + \alpha\{x(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots)\}_{i1} \\ &\quad + \alpha^2\{x(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots)\}_{i2} + \ldots \\ &= \{\overline{x}\}_i + \alpha\{x\}_{i1} + \alpha^2\{x\}_{i2} + \ldots\end{aligned} \tag{3.12}$$

$$\begin{aligned}\{y\}_i &= \{y(a_{rs}^0,\ 0,\ 0,\ldots,\ 0,\ b_{rs}^0,\ 0,\ 0,\ldots,\ 0)\}_i \\ &\quad + \alpha\{y(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots)\}_{i1} \\ &\quad + \alpha^2\{y(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots)\}_{i2} + \ldots \\ &= \{\overline{y}\}_i + \alpha\{y\}_{i1} + \alpha^2\{y\}_{i2} + \ldots\end{aligned} \tag{3.13}$$

where,

$$a_{rs}^m = \alpha^m a_{rs}(\sim)\ ;\ b_{rs}^m = \alpha^m b_{rs}(\sim)\ ; \qquad m=1,2,\ldots \tag{3.14}$$

$$a_{rs}^0 = \overline{a}_{rs}\ ;\ b_{rs}^0 = \overline{b}_{rs} \qquad r,\ s,\ = 1,2,\ldots,n \tag{3.15}$$

and α is the asymptotic parameter which is a deterministic quantity.

In the above expressions, overbars denote ensemble mean values. Further λ_{i1}, λ_{i2},..., $\{x\}_{i1}$, $\{x\}_{i2}$,... and $\{y\}_{i1}$, $\{y\}_{i2}$,... are the zero-mean stochastically-fluctuating components of eigensolutions. Now, the asymptotic expansions given by Eqs. (3.5) and (3.6) and (3.11-3.13) are substituted in the basic random eigenvalue problem as well as the adjoint random eigenvalue problem:

$$\begin{aligned}([\bar{A}] + \alpha[A(\sim)] + \alpha^2[A(\sim)] + \ldots)(\{\bar{x}\}_i + \alpha\{x\}_{i1} + \alpha^2\{x\}_{i2} + \ldots) \\ = (\bar{\lambda}_i + \alpha\lambda_{i1} + \alpha^2\lambda_{i2} + \ldots)([\bar{B}] + \alpha[B(\sim)] + \alpha^2[B(\sim)] + \ldots) \\ (\{\bar{x}\}_i + \alpha\{x\}_{i1} + \alpha^2\{x\}_{i2} + \ldots)\end{aligned} \tag{3.16}$$

$$\begin{aligned}(\{\bar{y}\}_i^T + \alpha\{y\}_{i1}^T + \alpha^2\{y\}_{i2}^T + \ldots)([\bar{A}] + \alpha[A(\sim)] + \alpha^2[A(\sim)] + \ldots) \\ = (\bar{\lambda}_i + \alpha\lambda_{i1} + \alpha^2\lambda_{i2} + \ldots)(\{\bar{y}\}_i^T + \alpha\{y\}_{i1}^T + \alpha^2\{y\}_{i2}^T + \ldots) \\ ([\bar{B}] + \alpha[B(\sim)] + \alpha^2[B(\sim)] + \ldots)\end{aligned} \tag{3.17}$$

After expanding these two equations and equating the terms of like powers in the asymptotic parameter α, the following system of equations is obtained:

$$[\bar{A}]\{\bar{x}\}_i = \bar{\lambda}_i[\bar{B}]\{\bar{x}\}_i \tag{3.18}$$

The above equation, obtained by equating the coefficients of α^0 on both sides of Eq. (3.16), can be equivalently expressed as

$$\begin{aligned}([\bar{A}] - \lambda_i\,(a_{rs}^0,\, b_{rs}^0\,;\, r,s=1,2,\ldots n)\,[\bar{B}]) \\ \{x\,(a_{rs}^0,\, b_{rs}^0\,;\, r,s=1,2,\ldots n)\}_i = 0\end{aligned} \tag{3.19}$$

Eq. (3.17) yields the following equation based on the same procedure:

$$[\bar{A}]^T\{\bar{y}\}_i = \bar{\lambda}_i\,[\bar{B}]^T\{\bar{y}\}_i \tag{3.20}$$

The above equation is equivalently written as,

$$\begin{aligned}([\bar{A}]^T - \lambda_i\,(a_{rs}^0,\, b_{rs}^0\,;\, r,s=1,2,\ldots n)\,[\bar{B}]^T) \\ \{y\,(a_{rs}^0,\, b_{rs}^0\,;\, r,s=1,2,\ldots n)\}_i = 0\end{aligned} \tag{3.21}$$

In a similar manner, comparing the coefficients of α on both sides of Eq. (3.16), one gets

$$\begin{aligned}[\bar{A}]\{x\}_{i1} + [A(\sim)]\{\bar{x}\}_i = \bar{\lambda}_i[\bar{B}]\{x\}_{i1} + \bar{\lambda}_i[B(\sim)]\{\bar{x}\}_i + \\ \lambda_{i1}[\bar{B}]\{\bar{x}\}_i\end{aligned} \tag{3.22}$$

or equivalently,

$$
\begin{aligned}
&([\bar{A}] - \lambda_i(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n) [\bar{B}]) \\
&\quad \{x(a^0_{rs}, a^1_{rs}, \dots, b^0_{rs}, b^1_{rs}, \dots)\}_{iI} = \\
&\quad - ([A(\sim)] - \lambda_i(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n) [B(\sim)]) \\
&\qquad \{x(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n)\}_i \\
&\quad + \lambda_{iI}(a^0_{rs}, a^1_{rs}, \dots, b^0_{rs}, b^1_{rs}, \dots)[\bar{B}] \\
&\qquad \{x(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n)\}_i
\end{aligned} \tag{3.23}
$$

Similarly, comparison of coefficients of α on both sides of Eq. (3.17) yields,

$$
\begin{aligned}
&[\bar{A}]^T\{y\}_{iI} + [A(\sim)]^T\{\bar{y}\}_i = \bar{\lambda}_i[\bar{B}]^T\{y\}_{iI} + \bar{\lambda}_i[B(\sim)]^T\{\bar{y}\}_i + \\
&\qquad \lambda_{iI}[\bar{B}]^T\{\bar{y}\}_i
\end{aligned} \tag{3.24}
$$

which can be written as,

$$
\begin{aligned}
&([\bar{A}]^T - \lambda_i(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n) [\bar{B}]^T) \\
&\quad \{y(a^0_{rs}, a^1_{rs}, \dots, b^0_{rs}, b^1_{rs}, \dots)\}_{iI} = \\
&\quad - ([A(\sim)]^T - \lambda_i(a_{rs}, b_{rs} ; r,s = 1, 2, \dots n) [B(\sim)]^T) \\
&\qquad \{y(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n)\}_i \\
&\quad + \lambda_{iI}(a^0_{rs}, a^1_{rs}, \dots, b^0_{rs}, b^1_{rs}, \dots)[\bar{B}]^T \\
&\qquad \{y(a^0_{rs}, b^0_{rs} ; r,s = 1, 2, \dots n)\}_i
\end{aligned} \tag{3.25}
$$

Similarly, other systems of equations that pertain to higher order solutions can be obtained.

Characteristic matrices $[\bar{A}]$ and $[\bar{B}]$ contain coefficients $\bar{a}_{ij}$ and $\bar{b}_{ij}$ respectively, which are in turn obtained from the ensemble mean values of material properties, geometric properties and loadings. This way, the algebraic eigenvalue problem expressed by Eq. (3.18) is defined in terms of the ensemble mean values of material properties, geometric properties and loadings. The ensemble mean values of λ_i and the elements of $\{x\}_i$ can be obtained by solving Eq. (3.18). In a similar manner, the ensemble mean values of the elements of $\{y\}_i$ are obtained by solving Eq. (3.20). Now, using these ensemble mean values of eigensolutions, the zero-mean stochastically-fluctuating components λ_{iI}, $\{x\}_{iI}$ and $\{y\}_{iI}$ are obtained based on Eqs. (3.22) and (3.24). To this end, Eq. (3.22) is pre-multiplied by the transpose of the ensemble mean matrix of the eigenvector to the left, $\{\bar{y}\}_i^T$, to yield the following equation:

$$\begin{aligned} \{\bar{y}\}_i^T[\bar{A}]\{x\}_{iI} + \{\bar{y}\}_i^T[A(\sim)]\{\bar{x}\}_i &= \bar{\lambda}_i\{\bar{y}\}_i^T[\bar{B}]\{x\}_{iI} \\ &+ \bar{\lambda}_i\{\bar{y}\}_i^T[B(\sim)]\{\bar{x}\}_i + \lambda_{iI}\{\bar{y}\}_i^T[\bar{B}]\{\bar{x}\}_i \end{aligned} \tag{3.26}$$

This equation can be rewritten as

$$\begin{aligned} (\{\bar{y}\}_i^T[\bar{A}] - \bar{\lambda}_i\{\bar{y}\}_i^T[\bar{B}])\{x\}_{iI} + \{\bar{y}\}_i^T[A(\sim)]\{\bar{x}\}_i \\ - \bar{\lambda}_i\{\bar{y}\}_i^T[B(\sim)]\{\bar{x}\}_i = \lambda_{iI}\{\bar{y}\}_i^T[\bar{B}]\{\bar{x}\}_i \end{aligned} \tag{3.27}$$

The adjoint eigenproblem defined by the ensemble mean values of characteristic matrices, Eq. (3.20), can be rewritten as

$$\{\bar{y}\}_i^T[\bar{A}] = \bar{\lambda}_i\{\bar{y}\}_i^T[\bar{B}] \tag{3.28}$$

After post-multiplying the above equation by the stochastic component matrix of the eigenvector to the right, $\{x\}_{iI}$ and rearranging, one gets

$$\{\bar{y}\}_i^T[\bar{A}]\{x\}_{iI} - \bar{\lambda}_i\{\bar{y}\}_i^T[\bar{B}]\{x\}_{iI} = 0 \tag{3.29}$$

Further, from the properties of the random eigenproblems that are listed in Section 3.2, it can be shown that

$$\{\bar{y}\}_i^T[\bar{B}]\{\bar{x}\}_i = 1 \tag{3.30}$$

Substitution of Eqs. (3.29) and (3.30) into Eq. (3.27) yields the expression for the stochastic component of eigenvalue, λ_{i1}, as

$$\lambda_{iI} = \{\bar{y}\}_i^T[A(\sim)]\{\bar{x}\}_i - \bar{\lambda}_i \{\bar{y}\}_i^T[B(\sim)]\{\bar{x}\}_i \tag{3.31}$$

Taking the transpose of Eq. (3.20) and imposing the resulting equation as well as the othogonality condition described by Eq. (3.30), on Eq. (3.27) yields the following equation:

$$\begin{aligned}\lambda_{iI}(a_{rs}^0, a_{rs}^1, \ldots, b_{rs}^0, b_{rs}^1, \ldots) = {} & \{y(a_{rs}^0, b_{rs}^0 ; r,s = 1, 2, \ldots n)\}_i^T [A(\sim)] \\ & \{x (a_{rs}^0, b_{rs}^0 ; r,s = 1, 2, \ldots n)\}_i \\ & - \lambda_i(a_{rs}^0, b_{rs}^0 ; r,s = 1, 2, \ldots n) \\ & \{y(a_{rs}^0, b_{rs}^0 ; r,s = 1, 2, \ldots n)\}_i^T [B(\sim)] \\ & \{x(a_{rs}^0, b_{rs}^0 ; r,s = 1, 2, \ldots n)\}_i\end{aligned} \tag{3.32}$$

It can now be observed that a sample realization of the stochastic component of the eigenvalue can be obtained as a linear function of the ensemble mean components of eigenvectors $\{x\}_i$ and $\{y\}_i$, the ensemble mean value of eigenvalue λ_i and the zero-mean components of characteristic matrices $[A(\sim)]$ and $[B(\sim)]$. Now, the sample realizations of stochastic components $\{x\}_{i1}$ and $\{y\}_{i1}$ are sought. To this end, the following linear transformations are employed based on the properties of eigensolutions listed in Section 3. 2:

$$\{x\}_{iI} = \sum_{k=1}^{n} d_x(i,k)\{x\}_k \tag{3.33}$$

or equivalently,

$$\{x(a_{rs}^0, a_{rs}^1, \ldots, b_{rs}^0, b_{rs}^1, \ldots)\}_{iI} = \sum_{k=1}^{n} d_x(i,k)\{x\}_k \tag{3.34}$$

and

$$\{y\}_{i1} = \sum_{k=1}^{n} d_y(i,k)\{y\}_k \tag{3.35}$$

or equivalently,

$$\{y(a_{rs}^0,\ a_{rs}^1,\ldots,\ b_{rs}^0,\ b_{rs}^1,\ldots)\}_{i1} = \sum_{k=1}^{n} d_y(i,k)\{y\}_k \tag{3.36}$$

These linear transformations express the sample realizations of $\{x\}_{i1}$ (or $\{y\}_{i1}$) in terms of coefficients $d_x(i,k)$ (or $d_y(i,k)$) as well as eigenvectors $\{x\}_k$ (or $\{y\}_k$). The orthogonality condition with respect to the characteristic matrix [B] is now considered in order to solve for $\{x\}_{i1}$ and $\{y\}_{i1}$.

$$\{y\}_i^T[B]\{x\}_i = \delta_{ii} = 1 \tag{3.37}$$

where δ_{ii} is the Kronecker's delta. Upon differentiation, the above orthogonality condition yields

$$d\{y\}_i^T[B]\{x\}_i + \{y\}_i^T d[B]\{x\}_i + \{y\}_i^T[B]d\{x\}_i = 0 \tag{3.38}$$

Since, for a sample realization $d[B] = [B(\sim)]$, $d\{y\}_i^T = \{y\}_{i1}^T$ and $d\{x\}_i = \{x\}_{i1}$, Eq. (3.38) is expressed as

$$\{y\}_{i1}^T[B]\{x\}_i + \{y\}_i^T[B(\sim)]\{x\}_i + \{y\}_i^T[B]\{x\}_{i1} = 0 \tag{3.39}$$

It may be noted here that, Eq. (3.39) corresponds to a single realization of the ensemble of equations obtained by differentiating Eq. (3.37). Further, in Eq. (3.39) and in the sequel $\{y\}_{i1}^T$ represents $\{y(a_{rs}^0,\ a_{rs}^1,\ a_{rs}^2,\ \ldots,\ b_{rs}^0,\ b_{rs}^1,\ b_{rs}^2,\ \ldots\ )\}^T$. A similar representation is valid for $\{x\}_{i1}$ also. The linear transformations for $\{x\}_{i1}$ and $\{y\}_{i1}$, that are given by Eqs. (3.33) and (3.35), are utilized in the above expressions and hence the following constraint equation for the coefficients $d_x(i,k)$ and $d_y(i,k)$ is derived.

$$\{y\}_{i,j}^T[B]\{x\}_i + \{y\}_i^T[B(\sim)]\{x\}_i + \{y\}_i^T[B]\{x\}_{i,j} = 0$$

$$\left\{ \sum_{k=1}^{n} d_y(i,k)\{y\}_k^T \right\} [B]\{x\}_i + \{y\}_i^T[B(\sim)]\{x\}_i + \{y\}_i^T[B] \left\{ \sum_{k=1}^{n} d_x(i,k)\{x\}_k \right\} = 0$$

$$d_y(i,i)\{y\}_i^T[B]\{x\}_i + \{y\}_i^T[B(\sim)]\{x\}_i + \{y\}_i^T[B]d_x(i,i)\{x\}_i = 0 \tag{3.40}$$

$$d_x(i,i) + d_y(i,i) + \{y\}_i^T[B(\sim)]\{x\}_i = 0$$

$$d_x(i,i) + d_y(i,i) = - \{y\}_i^T[B(\sim)]\{x\}_i$$

The following orthogonality relationships have been made use of, while deriving the above result:

$$\{y\}_k^T[B]\{x\}_k = 1 \quad ; \quad \{y\}_k^T[B]\{x\}_l = 0 \tag{3.41}$$

$$\{y\}_k^T[B]\{y\}_k = 1 \quad ; \quad \{y\}_k^T[B]\{y\}_l = 0 \tag{3.42}$$

The above relationships can also be formulated as

$$\{y\}_k^T[B]\{x\}_l = \delta_{kl} \tag{3.43}$$

$$\{y\}_k^T[B]\{y\}_l = \delta_{kl} \tag{3.44}$$

This way, the constraint equation for both the coefficients $d_x(i,i)$ and $d_y(i,i)$ is obtained as a linear function of the stochastic component of the characteristic matrix [B], eigenvector to the left $\{y\}_i$ and eigenvector to the right $\{x\}_i$.

In order to solve for the coefficients $d_x(i,k)$, $i \neq k$, first the basic eigenvalue equation is pre-multiplied by the transpose of the eigenvector to the left.

$$\{y\}_j^T([A] - \lambda_i[B])\{x\}_i = 0 \tag{3.45}$$

Differentiation of the above equation yields the following equation:

$$\{y\}_{jI}^T([A] - \lambda_i[B])\{x\}_i + \{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)] - \lambda_{iI}[B])\{x\}_i + \{y\}_j^T([A] - \lambda_i[B])\{x\}_{iI} = 0 \tag{3.46}$$

At this stage, the linear transformation for the stochastic component $\{x\}_{iI}$, given by Eq. (3.33) is substituted into the above equation.

$$\begin{aligned} &\{y\}_{jI}^T([A] - \lambda_i[B])\{x\}_i + \\ &\{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)] - \lambda_{iI}[B])\{x\}_i + \\ &\{y\}_j^T([A] - \lambda_i[B])\sum_{k=1}^{n} d_x(i,k)\{x\}_k = 0 \end{aligned} \tag{3.47}$$

The first term on the left hand side vanishes by virtue of Eq. (3.2). Substitution of the orthogonality relationships in terms of the characteristic matrix [B], that are similar to Eqs. (3.41-3.44), into the resulting equation leads to the expression for the coefficient $d_x(i,k)$ as detailed below:

$$\begin{aligned}
&\{y\}_{jI}^T([A] - \lambda_i[B])\{x\}_i + \{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)] - \lambda_{iI}[B])\{x\}_i + \\
&\qquad \{y\}_j^T([A] - \lambda_i[B])\sum_{k=1}^{n} d_x(i,k)\{x\}_k = 0 \\
&0 + \{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i - \lambda_{iI}\{y\}_j^T[B]\{x\}_i + \\
&\qquad \{y\}_j^T([A] - \lambda_i[B])\sum_{k=1}^{n} d_x(i,k)\{x\}_k = 0 \\
&\{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i - 0 + \{y\}_j^T([A] - \lambda_i[B])\, d_x(i,j)\{x\}_j = 0 \\
&\{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i + d_x(i,j)\,\{y\}_j^T[A]\{x\}_j \\
&\qquad - \lambda_i\, d_x(i,j)\,\{y\}_j^T[B]\{x\}_j = 0 \\
&\{y\}_j^T([A(\sim)] - \lambda_i[B(\sim)])\{x\}_i + \lambda_j\, d_x(i,j) - \lambda_i\, d_x(i,j) = 0 \\
&d_x(i,j) = \{y\}_j^T[A(\sim)]\{x\}_i \,/\, (\lambda_i - \lambda_j) - \\
&\qquad \lambda_i\{y\}_j^T[B(\sim)]\{x\}_i \,/\, (\lambda_i - \lambda_j) \quad , i \neq j \\
&d_x(i,k) = \{y\}_k^T[A(\sim)]\{x\}_i \,/\, (\lambda_i - \lambda_k) - \\
&\qquad \lambda_i\{y\}_k^T[B(\sim)]\{x\}_i \,/\, (\lambda_i - \lambda_k) \quad , i \neq k
\end{aligned} \tag{3.48}$$

Now the adjoint eigenproblem is considered and the corresponding equation is pre-multiplied by the transpose of the eigenvector to the right. Following an analysis which is similar to the above procedure, the expression for the coefficient $d_y(i,k)$ can be obtained in the following form:

$$d_y(i,k) = \{y\}_i^T[A(\sim)]\{x\}_k \;/\; (\lambda_i-\lambda_k) - \lambda_i\{y\}_i^T[B(\sim)]\{x\}_k \;/\; (\lambda_i-\lambda_k) \quad , \; i \neq k \tag{3.49}$$

Also based on the same analyses, the expressions for the coefficients $d_x(i,k)$ and $d_y(i,k)$, i=k, can be obtained as

$$d_x(i,i) = -\,(1/2)\{y\}_i^T[B(\sim)]\{x\}_i \tag{3.50}$$

$$d_y(i,i) = -\,(1/2)\{y\}_i^T[B(\sim)]\{x\}_i \tag{3.51}$$

These relationships satisfy the constraint equation for $d_x(i,i)$ and $d_y(i,i)$ given by Eq. (3.40). Eqs. (3.48) and (3.49) correspond to a single realization of the respective stochastic fields. Further, $d_x(i,i)$ and $d_y(i,i)$ are free from the randomness in [A], and are explicitly governed only by the randomness in [B].

3. 4 Probabilistic Moments of Eigensolutions

In the previous section, closed form solutions for the first-order terms of the asymptotic expansions given by Eqs. (3.11-3.13) have been obtained. The asymptotic expansions are now used as a basis to obtain the probabilistic moments of the eigenvalues, eigenvectors to the right and eigenvectors to the left. It has already been said that Eqs. (3.11-3.13) describe sample realizations of eigensolutions in terms of asymptotic series.

3. 4. 1 Ensemble Mean Values of Eigensolutions

The asymptotic expansion for the eigenvalue is first considered and the equations for two sample realizations are written down as

$$\lambda_i^{(1)} = \overline{\lambda}_i + \alpha\lambda_{i1}^{(1)} + \ldots. \tag{3.52}$$

$$\lambda_i^{(2)} = \overline{\lambda}_i + \alpha\lambda_{i1}^{(2)} + \ldots. \tag{3.53}$$

where the superscripts indicate the sample realizations 1 and 2. The ensemble mean value of any eigenvalue can be obtained by performing the expectation operation on both sides of any of the above two equations.

$$<\lambda_i^{(1)}> = <\overline{\lambda}_i> + <\alpha\lambda_{i1}^{(1)}> + \ldots. \tag{3.54}$$

Since the first term on the right hand side of the above equation is the eigenvalue obtained by solving Eq. (3.18) and further, the parameter α is a deterministic quantity, the above equation can be rewritten as

$$<\lambda_i^{(1)}> = \overline{\lambda}_i + \alpha<\lambda_{i1}^{(1)}> + \ldots. \tag{3.55}$$

It has already been said that the stochastically-fluctuating terms λ_{i1} are zero-mean random quantities and so, the above equation can be further reduced to

$$<\lambda_i^{(1)}> = \overline{\lambda}_i \tag{3.56}$$

In a similar way, it can be shown that the ensemble mean value of any sample realization of λ_i is equal to the eigenvalue obtained from Eq. (3.18).

$$<\lambda_i^{(2)}> = \overline{\lambda}_i \tag{3.57}$$

From asymptotic expansions given by Eqs. (3.11 - 3.13), the equivalent expression is written as

$$<\lambda_i> = \lambda_i(a_{rs}^0, 0, \ldots, 0, b_{rs}^0, 0, \ldots, 0) \tag{3.58}$$

In a similar manner, the ensemble mean values of eigenvectors are obtained as

$$<\{x\}_i> = \{x(a_{rs}^0, 0, \ldots, 0, b_{rs}^0, 0, \ldots, 0)\}_i \tag{3.59}$$

$$<\{y\}_i> = \{y(a_{rs}^0, 0, \ldots, 0, b_{rs}^0, 0, \ldots, 0)\}_i \tag{3.60}$$

since a_{rs}^m, b_{rs}^m are linear functions of zero-mean input random fields, for $m \geq 1$. Hence the mean values of eigensolution can be obtained by solving only once the so-called "Averaged Problem" or "Averaged Multi-Degree-of-Freedom (MDOF) system", given by

$$([\overline{A}] - \overline{\lambda}_i[\overline{B}])\{\overline{x}\}_i = 0 \tag{3.61}$$

$$([\overline{A}]^T - \overline{\lambda}_i[\overline{B}]^T)\{\overline{y}\}_i = 0 \tag{3.62}$$

The second-order probabilistic moments of eigensolutions are now evaluated, in terms of the second order probabilistic moments of material property stochastic fields as well as the ensemble mean values of eigensolutions.

3. 4. 2 Standard Deviations of Eigensolutions

The two sample realizations of any eigenvalue λ_i given by Eqs. (3.52) and (3.53) are multiplied after their respective ensemble mean values are deducted from them. The following equation is obtained as a result.

$$\begin{aligned}(\lambda_i^{(1)} - <\lambda_i^{(1)}>)(\lambda_i^{(2)} - <\lambda_i^{(2)}>) \\ = [(\overline{\lambda}_i + \alpha\lambda_{i/}^{(1)} + \ldots) - \overline{\lambda}_i][(\overline{\lambda}_i + \alpha\lambda_{i/}^{(2)} + \ldots) - \overline{\lambda}_i]\end{aligned} \tag{3.63}$$

At this stage, the expectation operation is performed on both the sides of the above equation according to

$$<(\lambda_i^{(1)} - \overline{\lambda}_i)(\lambda_i^{(2)} - \overline{\lambda}_i)> = \alpha^2 <\lambda_{i1}^{(1)}\lambda_{i1}^{(2)}> \tag{3.64}$$

The above equation is obtained from Eq. (3.63) after substituting Eqs. (3.56) and (3.57) in Eq. (3.63) and retaining terms upto second-order in asymptotic parameter α. The term on the left hand side of the above equation can be recognized to be the variance of the eigenvalue λ_i. In a similar manner, the first term on the right hand side can be recognized as the variance of the stochastically-fluctuating component λ_{i1}, since it is a zero-mean random variable. As a result, the variance of any eigenvalue is given as a linear function of the variance of the stochastically-fluctuating component and the squared-value of the asymptotic parameter α.

$$Var(\lambda_i) = \alpha^2 Var(\lambda_{i1}) \tag{3.65}$$

The variance of the stochastically-fluctuating component λ_{i1} is now derived from Eq. (3.31) as follows. The two sample realizations $\lambda_{i1}^{(1)}$ and $\lambda_{i1}^{(2)}$ can be expressed, based on Eq. (3.31) as

$$\begin{aligned} \lambda_{i1}^{(1)} &= \{\overline{y}\}_i^T [A^{(1)}(\sim)] \{\overline{x}\}_i \\ &\quad - \overline{\lambda}_i \{\overline{y}\}_i^T [B^{(1)}(\sim)] \{\overline{x}\}_i \end{aligned} \tag{3.66}$$

$$\begin{aligned} \lambda_{i1}^{(2)} &= \{\overline{y}\}_i^T [A^{(2)}(\sim)] \{\overline{x}\}_i \\ &\quad - \overline{\lambda}_i \{\overline{y}\}_i^T [B^{(2)}(\sim)] \{\overline{x}\}_i \end{aligned} \tag{3.67}$$

In the above, superscripts (1) and (2) associated with characteristic matrices [A($\sim$)] and [B($\sim$)] indicate their sample realizations 1 and 2. Multiplying the above two equations and performing the expectation operation on both the sides of the resulting equation, the following expression is obtained:

$$
\begin{aligned}
<\lambda_{i1}^{(1)}\lambda_{i1}^{(2)}> &= <(\{\bar{y}\}_i^T[A^{(1)}(\sim)]\{\bar{x}\}_i - \bar{\lambda}_i\{\bar{y}\}_i^T[B^{(1)}(\sim)]\{\bar{x}\}_i) \\
&\quad (\{\bar{y}\}_i^T[A^{(2)}(\sim)]\{\bar{x}\}_i - \bar{\lambda}_i\{\bar{y}\}_i^T[B^{(2)}(\sim)]\{\bar{x}\}_i)> \\
&= <\{\bar{y}\}_i^T[A^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[A^{(2)}(\sim)]\{\bar{x}\}_i \\
&\quad + (\bar{\lambda}_i)^2<\{\bar{y}\}_i^T[B^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[B^{(2)}(\sim)]\{\bar{x}\}_i> \\
&\quad - \bar{\lambda}_i<\{\bar{y}\}_i^T[B^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[A^{(2)}(\sim)]\{\bar{x}\}_i> \\
&\quad - \bar{\lambda}_i<\{\bar{y}\}_i^T[A^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[B^{(2)}(\sim)]\{\bar{x}\}_i>
\end{aligned}
\tag{3.68}
$$

When the coefficients of the characteristic matrices [A] and [B] are statistically independent, the last two terms on the right hand side of the above equation are equal to zero. Since the stochastically-fluctuating components λ_{i1} and λ_{i2} are zero-mean random quantities, the term on the left hand side of the above equation is recognized as the variance of λ_{i1}. As a result, one gets

$$
\begin{aligned}
Var(\lambda_{i1}) &= <\{\bar{y}\}_i^T[A^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[A^{(2)}(\sim)]\{\bar{x}\}_i> \\
&\quad + (\bar{\lambda}_i)^2<\{\bar{y}\}_i^T[B^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_i^T[B^{(2)}(\sim)]\{\bar{x}\}_i>
\end{aligned}
\tag{3.69}
$$

The above equation is recast into the following form such that the variances of the coefficients of characteristic matrices can be identified.

$$
\begin{aligned}
Var(\lambda_{i1}) &= <\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\, a_{rs}^{(1)}(\sim)\bar{x}_{si}\, \bar{y}_{ip}a_{pq}^{(2)}(\sim)\bar{x}_{qi}> \\
&\quad + (\bar{\lambda}_i)^2<\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\, b_{rs}^{(1)}(\sim)\bar{x}_{si}\, \bar{y}_{ip}b_{pq}^{(2)}(\sim)\bar{x}_{qi}>
\end{aligned}
\tag{3.70}
$$

In the above, y_{ir} and y_{ip} indicate the coefficient at the location (r, p) of the column vector y_i. In a similar manner x_{si} and x_{qi} indicate the coefficient at the location (s, q) of the column vector x_i. Since these coefficients are ensemble mean values, the above equation turns into

$$Var(\lambda_{i1}) = \sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\,\bar{y}_{ip}\,\bar{x}_{si}\,\bar{x}_{qi}\,Cov(a_{rs}(\sim),\,a_{pq}(\sim))$$
$$+\,(\bar{\lambda}_i)^2\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\,\bar{y}_{ip}\,\bar{x}_{si}\,\bar{x}_{qi}\,Cov(b_{rs}(\sim),\,b_{pq}(\sim)) \tag{3.71}$$

Substituting the above equation in Eq. (3.65) yields the equation for variance of any eigenvalue λ_i as

$$Var(\lambda_i) = \alpha^2\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\,\bar{y}_{ip}\,\bar{x}_{si}\,\bar{x}_{qi}\,Cov(a_{rs}(\sim),\,a_{pq}(\sim))$$
$$+\,(\alpha\bar{\lambda}_i)^2\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\,\bar{y}_{ip}\,\bar{x}_{si}\,\bar{x}_{qi}\,Cov(b_{rs}(\sim),\,b_{pq}(\sim)) \tag{3.72}$$

3. 4. 3 Covariance Functions of Eigenvalues

The sample realizations of any two eigenvalues λ_i and λ_j are multiplied after their respective ensemble mean values are deducted from them, and the following equation is obtained.

$$(\lambda_i^{(1)} - \langle\lambda_i^{(1)}\rangle)(\lambda_j^{(2)} - \langle\lambda_j^{(2)}\rangle)$$
$$= [(\bar{\lambda}_i + \alpha\lambda_{i1}^{(1)} + \,....) - \bar{\lambda}_i][(\bar{\lambda}_j + \alpha\lambda_{j1}^{(2)} + \,....) - \bar{\lambda}_j] \tag{3.73}$$
$$= \alpha^2\lambda_{i1}^{(1)}\lambda_{j1}^{(2)}$$

In the above equation, terms up to second-order in asymptotic parameter α have been retained. At this stage, the expectation operation is performed on both the sides of the above equation and makes use of Eqs. (3.56) and (3.57).

$$\langle(\lambda_i^{(1)}-\bar{\lambda}_i)(\lambda_j^{(2)}-\bar{\lambda}_j)\rangle = \alpha^2\langle\lambda_{i1}^{(1)}\lambda_{j1}^{(2)}\rangle \tag{3.74}$$

The term on the left hand side of the above equation can be recognized to be the covariance between any two eigenvalues λ_i and λ_j. In a similar manner, the first term on the right hand side can be recognized as the covariance between the two stochastically-fluctuating components λ_{i1} and λ_{j1}, since they are zero-mean random quantities. As a result, the covariance function of eigenvalues is given as a linear function of the covariance between two stochastically-fluctuating components λ_{i1} and λ_{j1} and the squared-value of the asymptotic parameter α.

$$Cov(\lambda_i,\lambda_j) = \alpha^2 Cov(\lambda_{i1},\lambda_{j1}) \tag{3.75}$$

The covariance between the two stochastically-fluctuating components λ_{i1} and λ_{j1} is now derived from Eq. (3.31) as follows. The two sample realizations $\lambda_{i1}^{(1)}$ and $\lambda_{j1}^{(2)}$ can be expressed, based on Eq. (3.31) as

$$\lambda_{i1}^{(1)} = \{y\}_i^T[A^{(1)}(\sim)]\{x\}_i - \bar{\lambda}_i\{y\}_i^T[B^{(1)}(\sim)]\{x\}_i \tag{3.76}$$

$$\lambda_{j1}^{(2)} = \{y\}_j^T[A^{(2)}(\sim)]\{x\}_j - \bar{\lambda}_j\{y\}_j^T[B^{(2)}(\sim)]\{x\}_j \tag{3.77}$$

In the above, superscripts (1) and (2) associated with characteristic matrices $[A(\sim)]$ and $[B(\sim)]$ indicate their sample realizations 1 and 2. Multiplying the above two equations and performing the expectation operation on both the sides of the resulting equation, the following expression for the covariance between any two eigenvalues λ_i and λ_j is obtained.

$$\begin{aligned}
\langle\lambda_{i1}^{(1)}\lambda_{j1}^{(2)}\rangle &= \langle(\{y\}_i^T[A^{(1)}(\sim)]\{x\}_i - \bar{\lambda}_i\{y\}_i^T[B^{(1)}(\sim)]\{x\}_i) \\
&\quad (\{y\}_j^T[A^{(2)}(\sim)]\{x\}_j - \bar{\lambda}_j\{y\}_j^T[B^{(2)}(\sim)]\{x\}_j)\rangle \\
&= \langle\{y\}_i^T[A^{(1)}(\sim)]\{x\}_i\{y\}_j^T[A^{(2)}(\sim)]\{x\}_j\rangle \\
&\quad + \bar{\lambda}_i\bar{\lambda}_j\langle\{y\}_i^T[B^{(1)}(\sim)]\{x\}_i\{y\}_j^T[B^{(2)}(\sim)]\{x\}_j\rangle \\
&\quad - \bar{\lambda}_j\langle\{y\}_i^T[A^{(1)}(\sim)]\{x\}_i\{y\}_j^T[B^{(2)}(\sim)]\{x\}_j\rangle \\
&\quad - \bar{\lambda}_i\langle\{y\}_i^T[B^{(1)}(\sim)]\{x\}_i\{y\}_j^T[A^{(2)}(\sim)]\{x\}_j\rangle
\end{aligned} \tag{3.78}$$

When the coefficients of the characteristic matrices [A] and [B] are independent, the last two terms on the right hand side of the above equation are equal to zero. Since the stochastically-fluctuating components λ_{i1} and λ_{j1} are zero-mean random variables, the term on the left hand side of the above equation is recognized to be the covariance between λ_{i1} and λ_{j1}. As a result, one gets

$$Cov(\lambda_{i1},\lambda_{j1}) = <\{\bar{y}\}_i^T[A^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_j[A^{(2)}(\sim)]\{\bar{x}\}_j> + \overline{\lambda}_i\overline{\lambda}_j<\{\bar{y}\}_i^T[B^{(1)}(\sim)]\{\bar{x}\}_i\{\bar{y}\}_j^T[B^{(2)}(\sim)]\{\bar{x}\}_j> \tag{3.79}$$

The equivalent expression for the covariance between any two eigenvalues λ_i and λ_j is given by

$$\begin{aligned} Cov(\lambda_i,\lambda_j) &= <(\lambda_i - <\lambda>_i)(\lambda_j - <\lambda>_j)> \\ &= <\{y(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_i^T [A^{(1)}(\sim)] \\ &\quad \{x(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_i \\ &\quad \{y(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_j^T [A^{(2)}(\sim)] \\ &\quad \{x(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_j > \\ &\quad + \lambda_i(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0) \\ &\quad \lambda_j(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0) \\ &\quad < \{y(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_i^T [B^{(1)}(\sim)] \\ &\quad \{x(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_i \\ &\quad \{y(a_{ij}^0, 0,\dots, 0, b_{ij}^0, 0,\dots, 0)\}_j^T [B^{(2)}(\sim)] \\ &\quad \{x(a_{ij}^0, 0,\dots, 0, b_{j}^0, 0,\dots, 0)\}_j > \end{aligned} \tag{3.80}$$

In the above, Eqs. (3.11-3.15) and Eqs. (3.58-3.60) have been made use of. The cross covariance terms vanish since matrices $[A^{(1)}(\sim)]$ and $[B^{(2)}(\sim)]$ are statistically independent.

Equation (3.79) is recast into the following form such that the covariance functions of the coefficients of characteristic matrices can be identified.

$$Cov(\lambda_{i1},\lambda_{j1}) = <\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ a_{rs}^{(1)}(\sim)\bar{x}_{sj}\ \bar{y}_{ip}a_{pq}^{(2)}(\sim)\bar{x}_{qj}>$$
$$+\ \bar{\lambda}_i\ \bar{\lambda}_j<\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ b_{rs}^{(1)}(\sim)\bar{x}_{sj}\ \bar{y}_{ip}b_{pq}^{(2)}(\sim)\bar{x}_{qj}> \tag{3.81}$$

In the above, y_{ir} and y_{ip} indicate the coefficient at the location (r, p) of the column vector y_i. In a similar manner, x_{sj} and x_{qj} indicate the coefficient at the location (s, q) of the column vector x_j. Since these coefficients are ensemble mean values, the above equation turns into

$$Cov(\lambda_{i1},\lambda_{j1}) = \sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ \bar{y}_{ip}\ \bar{x}_{sj}\ \bar{x}_{qj}\ Cov(a_{rs}(\sim),\ a_{pq}(\sim))$$
$$+\ \bar{\lambda}_i\ \bar{\lambda}_j\ \sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ \bar{y}_{ip}\ \bar{x}_{sj}\ \bar{x}_{qj}\ Cov(b_{rs}(\sim),\ b_{pq}(\sim)) \tag{3.82}$$

The product $\bar{y}_{ir}\bar{y}_{ip}$ of Eq. (3.82) is called here as the stochastic sensitivity gradient of λ_i with respect to a_{rs}, while the product $\bar{x}_{sj}\bar{x}_{qj}$ is referred to as the stochastic sensitivity gradient of λ_j with respect to a_{pq}. Similarly, the product $\bar{\lambda}_i\bar{y}_{ir}\bar{y}_{ip}$ represents the stochastic sensitivity gradient of λ_i with respect to b_{rs} while the product $\bar{\lambda}_j\bar{x}_{sj}\bar{x}_{qj}$ denotes the stochastic sensitivity gradient of λ_j with respect to b_{pq}. Substiituting the above equation in Eq. (3.75) yields the equation for covariance between any two eigenvalues λ_i and λ_j as

$$Cov(\lambda_i,\lambda_j) = \alpha^2\ \sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ \bar{y}_{ip}\ \bar{x}_{sj}\ \bar{x}_{qj}\ Cov(a_{rs}(\sim),\ a_{pq}(\sim))$$
$$+\ \alpha^2\ \bar{\lambda}_i\ \bar{\lambda}_j\ \sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\bar{y}_{ir}\ \bar{y}_{ip}\ \bar{x}_{sj}\ \bar{x}_{qj}\ Cov(b_{rs}(\sim),\ b_{pq}(\sim)) \tag{3.83}$$

It may be noted that when i = j, $Cov(\lambda_i,\ \lambda_j) = Var(\lambda_i)$. From the above equation, substituting i = j the expression for variance of any eigenvalue λ_i should be obtained. The resulting expression can be seen to be the same as the expression for Var (λ_i) that is given by Eq. (3.72).

The covariance and variance functions corresponding to coefficients a_{rs}, a_{pq}, b_{rs} and b_{pq} are determined from their respective power spectral density functions or their equivalent autocorrelation functions. For instance, one has

$$Var(a_{rs}) = \int_{-\infty}^{+\infty} S_{a_{rs}}(f)\, df = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{a_{rs}}(\xi)\, e^{if\xi}\, d\xi\, df \tag{3.84}$$

$$Var(b_{pq}) = \int_{-\infty}^{+\infty} S_{b_{pq}}(f)\, df = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{b_{pq}}(\xi)\, e^{if\xi}\, d\xi\, df \tag{3.85}$$

where f and ξ are wave frequency and lag distance, respectively, of the stochastic fields of a_{rs} and b_{pq}. The Wiener-Khinchine relationship [Vanmarcke, 1983] has been made use of in deriving Eqs. (3.84) and (3.85).

The equivalent expression for Eq. (3.83) is written as

$$Cov(\lambda_i,\lambda_j) = \alpha^2 \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \bar{y}_{ip}\, \bar{x}_{iq}\, \bar{y}_{rj}\, \bar{x}_{sj}\, Cov(a_{pq}(\sim), a_{rs}(\sim)) \; +$$

$$\alpha^2\, \bar{\lambda}_i\, \bar{\lambda}_j \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \bar{y}_{ip}\, \bar{x}_{iq}\, \bar{y}_{rj}\, \bar{x}_{sj}\, Cov(b_{pq}(\sim), b_{rs}(\sim)) \tag{3.86}$$

where,

$$\bar{y}_{mp} = y_{mp}(a_{rs}^0,\, 0 \,...\, 0,\, b_{rs}^0,\, 0 \,...\, 0) \quad ; \quad m,\, p=1,\, 2 \,...\, n \tag{3.87}$$

$$\bar{x}_{mp} = x_{mp}(a_{ij}^0,\, 0 \,...\, 0,\, b_{ij}^0,\, 0 \,...\, 0) \quad ; \quad m,\, p=1,\, 2 \,...\, n \tag{3.88}$$

$$\overline{\lambda}_i = \lambda_i(a^0_{rs}, 0 \ldots 0, b^0_{rs}, 0 \ldots 0) \quad ; \quad i=1, 2, \ldots n \tag{3.89}$$

$$\overline{\lambda}_j = \lambda_j(a^0_{rs}, 0 \ldots 0, b^0_{rs}, 0 \ldots 0) \quad ; \quad j=1, 2, \ldots n \tag{3.90}$$

y_{mp} and x_{mp} correspond to the coefficient at the location p of $\{y\}_m$ and $\{x\}_m$. When i=j, Eq. (3.86) corresponds to Var(λ_i). A similar analysis shows that, after making use of Eqs. (3.48-3.51), the equations for covariance between any two eigenvectors can be written as

$$\begin{aligned} Cov(\{x\}_i,\{x\}_j) = \sum_{k=1}^{n}\sum_{l=1}^{n} 1/\{(\overline{\lambda}_i - \overline{\lambda}_k)(\overline{\lambda}_j - \overline{\lambda}_l)\} \\ <(\{\overline{y}\}_k^T [A(\sim)]\{\overline{x}\}_i - \overline{\lambda}_i \{\overline{y}\}_l^T [B(\sim)]\{\overline{x}\}_l) \\ (\{\overline{y}\}_k^T [A(\sim)]\{\overline{x}\}_j - \overline{\lambda}_j \{\overline{y}\}_l^T [B(\sim)]\{\overline{x}\}_l)> \end{aligned} \tag{3.91}$$

$$\begin{aligned} Cov(\{y\}_i^T,\{y\}_j^T) = \sum_{k=1}^{n}\sum_{l=1}^{n} 1/\{(\overline{\lambda}_i - \overline{\lambda}_k)(\overline{\lambda}_j - \overline{\lambda}_l)\} \\ <(\{\overline{y}\}_i^T [A(\sim)]\{\overline{x}\}_k - \overline{\lambda}_i \{\overline{y}\}_i^T [B(\sim)]\{\overline{x}\}_l) \\ (\{\overline{y}\}_j^T [A(\sim)]\{\overline{x}\}_k - \overline{\lambda}_j \{\overline{y}\}_l^T [B(\sim)]\{\overline{x}\}_l)> \end{aligned} \tag{3.92}$$

where,

$$\{x\}_t = \{x(a^0_{rs}, 0 \ldots 0, b^0_{rs}, 0 \ldots 0)\}_t \quad ; \quad t=1, 2, \ldots n \tag{3.93}$$

$$\{y\}_t = \{y(a^0_{rs}, 0 \ldots 0, b^0_{rs}, 0 \ldots 0)\}_t \quad ; \quad t=1, 2, \ldots n \tag{3.94}$$

$$\lambda_t = \lambda_t(a^0_{rs}, 0, \ldots 0, b^0_{rs}, 0, \ldots 0) \quad ; \quad t=1, 2, \ldots n \tag{3.95}$$

Variances can be obtained from the above covariance equations by substituting i = j.

3. 4. 4 Complete Covariance Matrix of Eigenvalues

Based on Eq. (3.83), the matrix of covariance functions of eigenvalues is now obtained. This matrix is expressed as

$$\begin{bmatrix} C \\ \lambda \end{bmatrix} = \begin{bmatrix} Var(\lambda_1) & Cov(\lambda_1,\lambda_2) & . & . & Cov(\lambda_1,\lambda_n) \\ Cov(\lambda_2,\lambda_1) & Var(\lambda_2) & . & . & Cov(\lambda_2,\lambda_n) \\ . & . & . & . & . \\ . & . & . & . & . \\ Cov(\lambda_n,\lambda_1) & Cov(\lambda_n,\lambda_2) & . & . & Var(\lambda_n) \end{bmatrix} \tag{3.96}$$

A matrix, of order $2n^2 \times 2n^2$, consisting of the covariance functions of both stiffness and mass coefficients is formed according to

$$\begin{bmatrix} \Sigma \\ AB \end{bmatrix} = \begin{bmatrix} Var(a_1) & Cov(a_1,a_2) & . & Cov(b_1,a_1) & . & Cov(b_{n^2},a_1) \\ Cov(a_1,a_2) & Var(a_2) & . & Cov(b_1,a_2) & . & Cov(b_{n^2},a_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ Cov(b_1,a_1) & Cov(b_1,a_2) & . & Var(b_1) & . & Cov(b_{n^2},b_1) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ Cov(b_{n^2},a_1) & Cov(b_{n^2},a_2) & . & Cov(b_{n^2},b_1) & . & Var(b_{n^2}) \end{bmatrix} \tag{3.97}$$

where the square matrices [A] and [B] of size n×n are mapped onto their corresponding column vectors of size $n^2 \times 1$, so that the covariance functions of their coefficients can be determined. Accordingly, one can write

$$
\begin{aligned}
&k_{ij},\ i=1;\ j=1,\ 2,\ \dots\ n=k_j \\
&k_{ij},\ i=2;\ j=1,\ 2,\ \dots\ n=k_{(n+j)} \\
&\quad\vdots \\
&k_{ij},\ i=n;\ j=1,\ 2,\ \dots\ n=k_{((n-1)n+j)}
\end{aligned}
\qquad (3.98)
$$

Based on the elements of the eigenvectors of the averaged system, the following two matrices, one corresponding to the elements of matrix [A] and another corresponding to the elements of matrix [B] each of size nxn^2, are formed.

$$
[\lambda_{,A}] = \begin{bmatrix}
\lambda_{11,11} & \lambda_{11,12} & \lambda_{11,13} & \cdot & \cdot & \lambda_{11,nn} \\
\lambda_{21,11} & \lambda_{21,12} & \lambda_{21,13} & \cdot & \cdot & \lambda_{21,nn} \\
\lambda_{31,11} & \lambda_{31,12} & \lambda_{31,13} & \cdot & \cdot & \lambda_{31,nn} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_{nl,11} & \lambda_{nl,12} & \lambda_{nl,13} & \cdot & \cdot & \lambda_{nl,nn}
\end{bmatrix} \qquad (3.99)
$$

where,

$$
\lambda_{11,11} = \bar{y}_{11}\,\bar{x}_{11}\ ;\ \lambda_{11,12} = \bar{y}_{11}\,\bar{x}_{12}\ ; \\
\lambda_{21,11} = \bar{y}_{12}\,\bar{x}_{12}\ ;\ \dots\dots\ \lambda_{pl,rs} = \bar{y}_{rp}\,\bar{x}_{sp}\ ; \qquad (3.100)
$$

In a similar manner, matrix $[\lambda_{,B}]$ is formed according to

$$
[\lambda_{,B}] = \begin{bmatrix}
-\lambda_1(\lambda_{11,11}) & -\lambda_1(\lambda_{11,12}) & -\lambda_1(\lambda_{11,13}) & \cdot & \cdot & -\lambda_1(\lambda_{11,nn}) \\
-\lambda_2(\lambda_{21,11}) & -\lambda_2(\lambda_{21,12}) & -\lambda_2(\lambda_{21,13}) & \cdot & \cdot & -\lambda_2(\lambda_{21,nn}) \\
-\lambda_3(\lambda_{31,11}) & -\lambda_3(\lambda_{31,12}) & -\lambda_3(\lambda_{31,13}) & \cdot & \cdot & -\lambda_3(\lambda_{31,nn}) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\lambda_n(\lambda_{nl,11}) & -\lambda_n(\lambda_{nl,12}) & -\lambda_n(\lambda_{nl,13}) & \cdot & \cdot & -\lambda_n(\lambda_{nl,nn})
\end{bmatrix} \qquad (3.101)
$$

Combining Eqs. (3.96-3.101) and by virtue of Eq. (3.83), the complete covariance matrix of eigenvalues is written as

$$\begin{bmatrix} C \\ \lambda \end{bmatrix} = [[\lambda,_A] \mid [\lambda,_B]] \begin{bmatrix} \Sigma \\ AB \end{bmatrix} [[\lambda,_A] \mid [\lambda,_B]]^T \qquad (3.102)$$

Following the same procedures, the complete covariance matrix of eigenvector to the right, denoted by $[\Sigma_X]$ and of size $n^2 \times n^2$, can be constructed as follows:

$$\begin{bmatrix} \Sigma \\ x \end{bmatrix} = [[x,_A] \mid [x,_B]] \begin{bmatrix} \Sigma \\ AB \end{bmatrix} [[x,_A] \mid [x,_B]]^T \qquad (3.103)$$

where,

$$\begin{bmatrix} \Sigma \\ x \end{bmatrix} = \begin{bmatrix} Var(x_1) & Cov(x_2,x_1) & Cov(x_3,x_1) & \cdot\ \cdot & Cov(x_{n^2},x_1) \\ Cov(x_2,x_1) & Var(x_2) & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & Var(x_3) & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ Cov(x_{n^2},x_1) & \cdot & \cdot & \cdot\ \cdot & Var(x_{n^2}) \end{bmatrix} \qquad (3.104)$$

$$\begin{gathered} x_{ij},\ i=1;\ j=1,2,\ldots n = x_j \\ x_{ij},\ i=2;\ j=1,2,\ldots n = x_{(n+j)} \\ \cdot \\ \cdot \\ \cdot \\ x_{ij},\ i=n;\ j=1,2,\ldots n = x_{((n-1)n+j)} \end{gathered} \qquad (3.105)$$

$$[x_{,A}] = \begin{bmatrix} x_{1,11} & x_{1,12} & x_{1,13} & \cdot & \cdot & x_{1,nn} \\ x_{2,11} & x_{2,12} & x_{2,13} & \cdot & \cdot & x_{2,nn} \\ x_{3,11} & x_{3,12} & x_{3,13} & \cdot & \cdot & x_{3,nn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n^2,11} & x_{n^2,12} & x_{n^2,13} & \cdot & \cdot & x_{n^2,nn} \end{bmatrix} \tag{3.106}$$

$$[x_{,B}] = \begin{bmatrix} -\lambda_1(x_{1,11}) & -\lambda_1(x_{1,12}) & -\lambda_1(x_{1,13}) & \cdot & \cdot & -\lambda_1(x_{1,nn}) \\ -\lambda_2(x_{2,11}) & -\lambda_2(x_{2,12}) & -\lambda_2(x_{2,13}) & \cdot & \cdot & -\lambda_2(x_{2,nn}) \\ -\lambda_3(x_{3,11}) & -\lambda_3(x_{3,12}) & -\lambda_3(x_{3,13}) & \cdot & \cdot & -\lambda_3(x_{3,nn}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\lambda_{n^2}(x_{n^2,11}) & -\lambda_{n^2}(x_{n^2,12}) & -\lambda_{n^2}(x_{n^2,13}) & \cdot & \cdot & -\lambda_{n^2}(x_{n^2,nn}) \end{bmatrix} \tag{3.107}$$

with matrices $[x_{,A}]$ and $[x_{,B}]$ being of order $n^2 \times n^2$.

Chapter 4

Whirl Speed Analysis of Stochastic Rotor-Bearing Systems

4. 1 Introduction

The high speed rotor-bearing systems that have a stochastic distribution of material properties and axial loadings, called herein as "Stochastic Rotor-Bearing Systems", are considered. The stochastic (dynamic) analysis of such rotor-bearing systems is developed based on a consistent finite element formulation. The variation of material properties, the Young's modulus and mass density, as well as the fluctuations in axial loadings over mass products are considered to constitute one-dimensional univariate homogeneous spatial stochastic fields. The finite element formulation that had been developed earlier in the work of Nelson and McVaugh [1976] is considered. This formulation has been developed for a rotor-bearing system with deterministic material properties and loadings. This formulation is so furthered and adapted as to account for the randomness in material properties and axial loadings. The non-self-adjoint eigenproblem that quantifies the whirl speeds and whirl modes is formulated. The equations for the probabilistic moments of eigensolutions that have been developed in Chapter 3 are further modified in order to derive the probabilistic moments of whirl speeds and whirl modes. The relevent equations are then recast into a compuationally-convenient form.

4. 2 Finite Element Formulation

The flexible rotor-bearing system that consists of discrete disks, a rotor shaft with distributed mass and stiffness, and discrete bearings at the ends is shown in Fig. (4.1). In order to model the dynamic characteristics of this rotor-bearing system the equations of motion should be developed. This is carried out in several steps.

4. 2. 1 Description of Dynamic Motion

The first step is to define the reference frames which are essential and useful for both qualifying and quantifying the dynamic motions of the system. For the rotor-bearing system under consideration, two reference frames, namely, a stationary (fixed) reference frame and a rotating reference frame, are used to describe the dynamic motions of the system. As can be seen from Fig. (4.1), the rotor motion is described with respect to two inertial frames of reference which are (i) the XYZ triad (which is the stationary frame of reference) with the X-axis coinciding with the undeformed center line of the rotor shaft, and (ii) the xyz triad (which is the rotating frame of reference) with the x-axis coinciding with the undeformed center line of the rotor shaft. This way, the X and x axes are colinear and coincident with the undeformed center line of the rotor shaft.

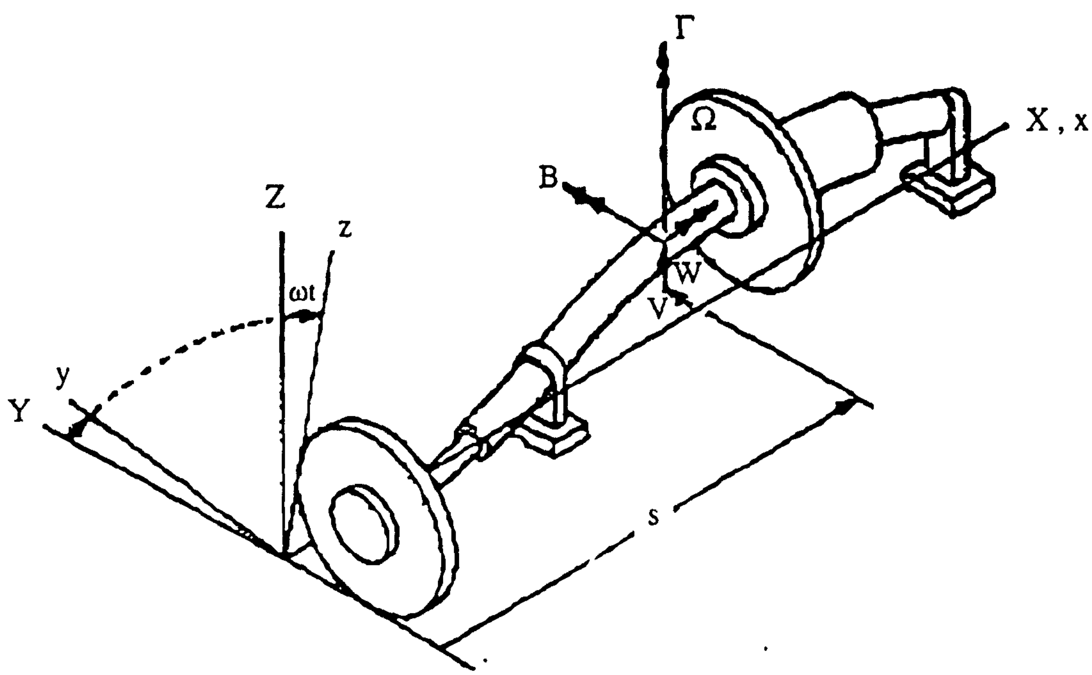

Fig. 4. 1: Typical configuration of a rotor-bearing system

(After Nelson and McVaugh, 1976 [58])

The second step is to describe the dynamic motions of the rotor-bearing system in terms of translational and rotational displacements. The dynamic response of the rotor-bearing system is defined with respect to the stationary frame of reference, by two translations V(s, t) and W(s, t) in the Y and Z directions respectively, and two small angle rotations B(s, t) and Γ(s, t) about Y and Z axes respectively. Here, s denotes the axial position along the rotor shaft and t denotes the time. This way, the elastic center line of the deformed rotor system is located using translations V(s, t) and W(s, t), and the plane of the cross-section is oriented using rotations B(s, t) and Γ(s, t). Further, the cross-section also spins normal to its face relative to the stationary frame of reference. The cross-section rotation angles are defined using another triad denoted as abc which is attached to the cross-section of the shaft such that the a-axis is normal to the cross-section. The frame abc is defined such that:

(i) Small angle rotation Γ(s, t) about Z-axis defines the triad a''b''c''

(ii) Small angle rotation B(s, t) about b'' defines the triad a'b'c'

(iii) Spin angle ϕ about a' defines abc

These cross-section rotation angles are shown in Fig. (4.2). The angular rate of the triad abc relative to the stationary frame of reference XYZ is denoted in terms of its components ω_a, ω_b, ω_c that are defined by

$$\begin{Bmatrix} \omega_a \\ \omega_b \\ \omega_c \end{Bmatrix} = \begin{bmatrix} -\sin B & 1 & 0 \\ \cos B \sin\phi & 0 & \cos\phi \\ \cos B \cos\phi & 0 & -\sin\phi \end{bmatrix} \begin{Bmatrix} \dot{\Gamma} \\ \dot{\phi} \\ \dot{B} \end{Bmatrix} \tag{4.1}$$

When only small deformations are considered, the small angle rotations B(s, t) and Γ(s,

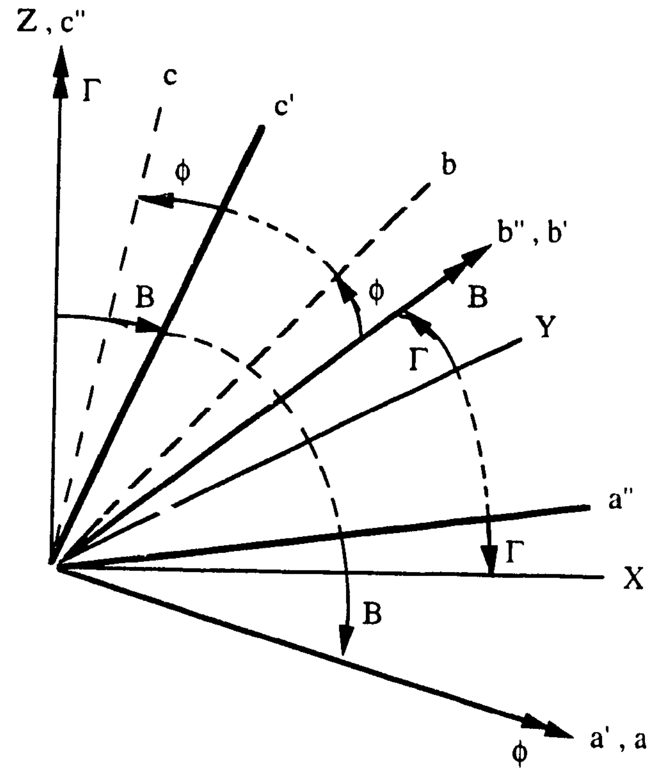

Fig. 4. 2: Cross-section rotation angles

t) are approximately colinear with the Y and Z axes (of the stationary frame of reference) respectively. Further, when the system is considered to run at a constant speed and when the torsional deformation is negligible, the spin angle ϕ is equal to Ωt, where Ω denotes the rotor spin speed.

The translational displacements V(s, t) and W(s, t), and the rotational displacements B(s, t) and Γ(s, t) of a typical cross-section (relative to the stationary frame of reference) are transformed into corresponding translational displacements v and w, and rotational displacements β and γ (that are relative to the rotating frame of reference), through the following orthogonal transformation:

$$\{q\} = [R]\{p\} \tag{4.2}$$

where,

$$\{q\} = \begin{Bmatrix} V \\ W \\ B \\ \Gamma \end{Bmatrix} \quad ; \{p\} = \begin{Bmatrix} v \\ w \\ \beta \\ \gamma \end{Bmatrix} \tag{4.3}$$

and

$$[R] = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 & 0 \\ \sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{bmatrix} \tag{4.4}$$

In Eq. (4.4), ω denotes the whirl speed.

4. 2. 2 Finite Element Modelling

The third step is to model the physical (infinite-degree-of-freedom) rotor-bearing system as a mathematical finite-degree-of-freedom system i.e. Multi-Degree-of-Freedom (MDOF) model, that consists of an interconnected set of discrete elements. This way, the entire rotor-bearing system is modelled as being comprised of a set of interconnecting components that are (i) rigid disks, (ii) rotor shaft segments which are the finite elements, and (iii) linear bearings. Essentially, the rotor-bearing system is modelled as a collection of finite elements and rigid bodies.

The fourth step is the development of equations of motion for various discrete elements of the MDOF model. In the sequel, first the equations of motion for the rigid disk are developed using a Lagrangian formulation.

4. 2. 2. 1 Rigid Disk

Turbine and compressor wheels as well as gears are usually modelled as rigid disks, and it is usually assumed that the inertial coupling between the longitudinal, torsional and lateral motions of the disk is negligible. In many engineering applications, it is also acceptable to consider the width of the disk to be negligible in comparison with the overall length of the rotating assembly. The resulting "thin-disk model" is depicted in Fig. (4.3). The kinetic energy of a typical rigid disk with its mass center coincident with the elastic center line of the shaft is given as

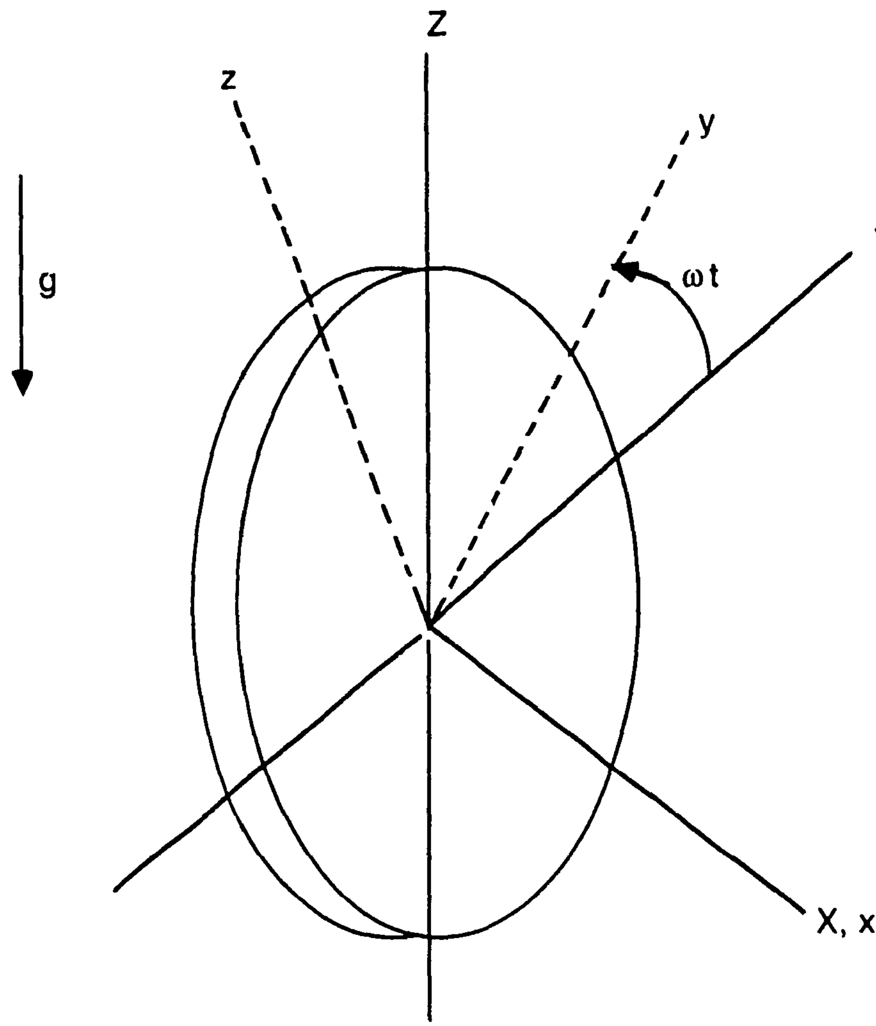

Fig. 4. 3: Thin-rigid-disk model

$$
\tau^{d} = \frac{1}{2}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix}^{T}\begin{Bmatrix}m_d & 0\\ 0 & m_d\end{Bmatrix}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}\omega_a\\ \omega_b\\ \omega_c\end{Bmatrix}^{T}\begin{Bmatrix}I_D & 0 & 0\\ 0 & I_D & 0\\ 0 & 0 & I_p\end{Bmatrix}\begin{Bmatrix}\omega_a\\ \omega_b\\ \omega_c\end{Bmatrix} \tag{4.5}
$$

Eq. (4.1) that defines the angular rate components of the triad abc (relative to the stationary frame of reference) is now substituted in the above equation. Further, only terms up to second order are retained in the resulting equation so that the kinetic energy is given by

$$
\tau^{d} = \frac{1}{2}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix}^{T}\begin{Bmatrix}m_d & 0\\ 0 & m_d\end{Bmatrix}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}\dot{B}\\ \dot{\Gamma}\end{Bmatrix}^{T}\begin{Bmatrix}I_D & 0\\ 0 & I_D\end{Bmatrix}\begin{Bmatrix}\dot{B}\\ \dot{\Gamma}\end{Bmatrix} - \dot{\phi}\dot{\Gamma}BI_p \tag{4.6}
$$

Based on the above expression, the Lagrangian equation of motion of the rigid disk with respect to the stationary frame of reference XYZ and for constant spin speed, $d\phi/dt = \Omega$, is written.

$$
([M_T^d] + [M_R^d])\{\ddot{q}^d\} - \Omega[G^d]\{\dot{q}^d\} = \{Q^d\} \tag{4.7}
$$

In the above, M_T^d and M_R^d denote the mass matrices of the disk corresponding to the translational and the rotational motions respectively and they are given as

$$
[M_T^d] = \begin{Bmatrix}m_d & 0 & 0 & 0\\ 0 & m_d & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{Bmatrix} \tag{4.8}
$$

$$[M_R^d] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_D & 0 \\ 0 & 0 & 0 & I_D \end{bmatrix} \tag{4.9}$$

The matrix G^d of Eq. (4.7) denotes the gyroscopic matrix of the disk and is given by

$$[G^d] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_p \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{4.10}$$

Further, the velocity and acceleration vectors $\{dq^d/dt\}$ and $\{d^2q^d/dt^2\}$ are, by virtue of Eqs. (4.3) and (4.4), given by

$$\{\dot{q}\} = \omega\,[S]\,\{p\} + [R]\,\{\dot{p}\} \tag{4.11}$$

$$\{\ddot{q}\} = [R]\,(\{\ddot{p}\} - \omega^2\,\{p\}) + 2\omega\,[S]\,\{\dot{p}\} \tag{4.12}$$

where overdot denotes differentiation with respect to time, and

$$[S] = \frac{1}{\omega}[\dot{R}] = \begin{bmatrix} -\sin\omega t & -\cos\omega t & 0 & 0 \\ \cos\omega t & -\sin\omega t & 0 & 0 \\ 0 & 0 & -\sin\omega t & -\cos\omega t \\ 0 & 0 & \cos\omega t & -\sin\omega t \end{bmatrix} \tag{4.13}$$

The force vector $\{Q^d\}$ relative to the stationary frame of reference is now determined. The forcing functions include the effects of mass unbalance, interconnection forces and other external forces on the disk. The force vector $\{Q^d\}$ can be formed corresponding to each case. In what follows, the proceoure of constructing the force vector due to mass unbalance is given. Considering a disk, the mass center of which is located at (η_d, ζ_d) relative to the triad abc, the unbalance force vector in the stationary frame of reference is

$$\{Q^d\} = \begin{Bmatrix} \eta_d \\ \zeta_d \\ 0 \\ 0 \end{Bmatrix} \cos\Omega t + \begin{Bmatrix} -\zeta_d \\ \eta_d \\ 0 \\ 0 \end{Bmatrix} \sin\Omega t \qquad (4.14)$$

$$= \{Q_C^d\} \cos\Omega t + \{Q_S^d\} \sin\Omega t$$

It is convenient to write the equation of motion with respect to the rotating frame of reference. In doing so, an inertial restoring force appears and further, the so-called Green's gyroscopic stiffening effect [Green, 1948] is also present. In order to write the equation of motion in the rotating frame of reference, use is made of the orthogonal transformation, given by Eqs. (4.2-4.4) as well as Eqs. (4.11-4.13). These equations are substituted in Eq. (4.7). Pre-multiplying the resulting equation by $[R]^T$ the following equation of motion is obtained.

$$([M_T^d] + [M_R^d])\{\ddot{p}^d\} + \omega(2([\hat{M}_T^d] + [\hat{M}_R^d]) - \upsilon[G^d])\{\dot{p}^d\} - \omega^2(([M_T^d] + [M_R^d]) + \upsilon[\hat{G}^d])\{p^d\} = \{P^d\} \qquad (4.15)$$

The Green's gyroscopic stiffening effect is quantified by the term

$$-\omega^2([M_T^d] + [M_R^d] + \upsilon[\hat{G}^d])\{p^d\}$$

For the case of a thin disk wherein $I_P = 2I_D$, the above equation reduces to

$$([M_T^d] + [M_R^d])\{\ddot{p}^d\} + \omega(2[\hat{M}_T^d] + (1-\upsilon)[G^d])\{\dot{p}^d\} - \omega^2([M_T^d] + (1-2\upsilon)[M_R^d])\{p^d\} = \{P^d\} \qquad (4.16)$$

In Eqs. (4.15) and (4.16), υ denotes the whirl ratio which is equal to the quotient Ω/ω.

4. 2. 2. 2 Finite Rotor Element

The distribution of mass and stiffness in a rotor shaft is generally irregular. So, the rotating assembly (rotor shaft) is divided into a number of shaft segments which are individually variable and discontinuous in cross-section. Subsequently, each of the shaft segments with distributed mass and stiffness (i.e. the case of infinite-degree-of-freedom) are reduced to a finite-degree-of-freedom element, following either of the following two approaches: (i) lumped mass model and (ii) consistent mass model. The consistent mass model approach is very useful and is described herein.

In the rotordynamic system, the material properties viz. the elastic modulus E and the mass per unit length μ, and the axial compressive load P are considered as fluctuating randomly about their respective ensemble mean values. The random fluctuations of the material properties are modelled here as independent one-dimensional univariate zero-mean stochastic fields in space and the random fluctuations of axial loading are modelled as a zero-mean random variable. This way, the elastic modulus and mass per unit length are represented by

$$E(s) = \overline{E}(1+\alpha a(s)) \tag{4.17}$$

$$\mu(s) = \overline{\mu}(1+\alpha b(s)) \tag{4.18}$$

where, overbars indicate the ensemble mean values, s represents the axial position along the rotor shaft, a(s) and b(s) are two independent one-dimensional univariate homogeneous real stochastic fields with zero mean, and α is an asymptotic parameter introduced to characterize the stochastic fields in the equations of motion. These two stochastic fields are quantified by their respective variances σ_a^2 and σ_b^2, and autocorrelation functions $R_{aa}(\xi)$ and $R_{bb}(\xi)$, where ξ is the lag vector, or their equivalent power spectral density functions $S_{aa}(f)$ and $S_{bb}(f)$, where f is the wave frequency.

The compressive axial loading is given by

$$P = \bar{P}(1+c) \tag{4.19}$$

where overbar indicates the ensemble mean value and c is a random variable quantified through its variance σ_c^2.

For a rotor shaft element, the time-dependent (translational and rotational) displacements, V(s, t), W(s, t), B(s, t) and Γ(s, t), of the cross-section are also functions of position s along the axis of the element. A consistent mass model of the shaft element with uniform cross-section is now developed by representing the translation of a typical cross-section of the element in terms of the endpoint coordinates (i.e. nodal displacements of the finite element). The interpolation using cubic Hermitian polynomials is employed for this purpose. The rotation of the cross-section can be obtained in a similar manner from the interpolated displacements. The endpoint coordinates of a finite element are denoted by q_i^e, i = 1, 2,....,8, which are the time-dependent displacements (both rotational and translational) of the finite rotor element shown in Fig. (4.4). Based on q_i^e, i = 1, 2,....,8, the translation of a typical point within the finite rotor element is given by

$$\begin{Bmatrix} V(s,t) \\ W(s,t) \end{Bmatrix} = [\Psi(s)]\{q^e(t)\} \tag{4.20}$$

where the matrix of interpolation functions, that are functions of axial position s, is given

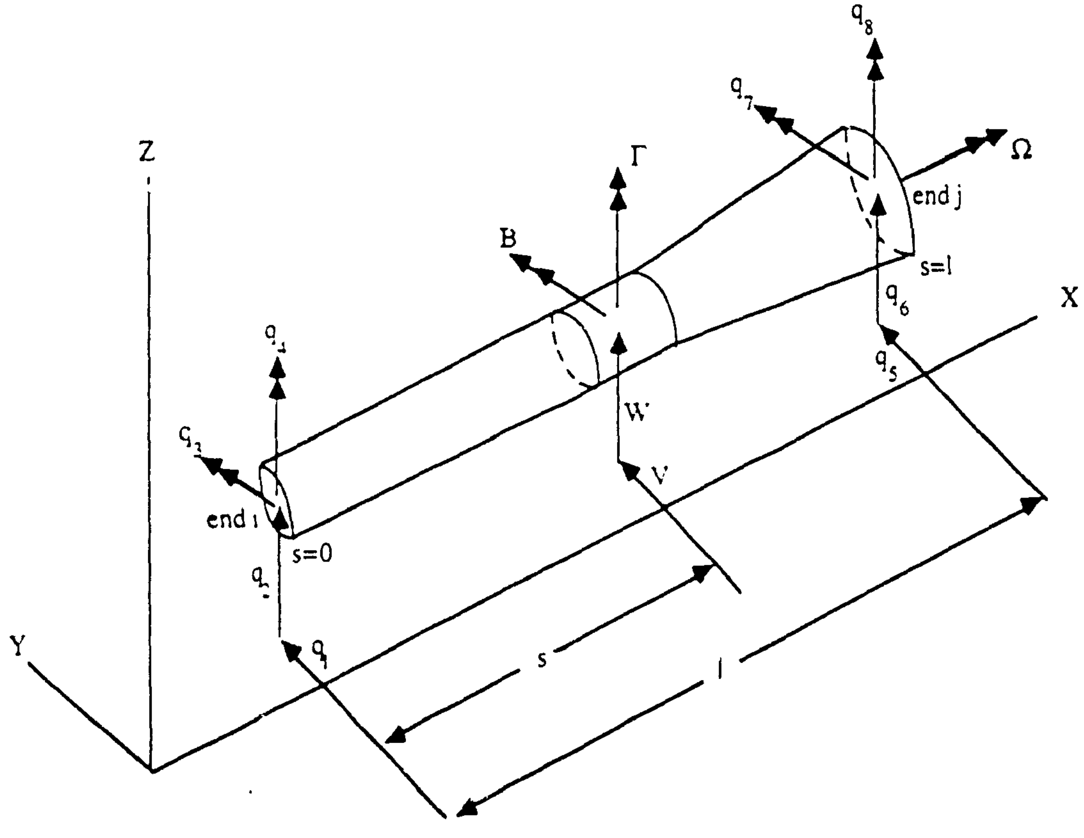

Fig. 4. 4: Finite rotor element

by

$$[\Psi] = \begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & \psi_3 & 0 & 0 & \psi_4 \\ 0 & \psi_1 & -\psi_2 & 0 & 0 & \psi_3 & -\psi_4 & 0 \end{bmatrix} \tag{4.21}$$

These interpolation functions are the static displacement modes associated with a unit displacement of one of the endpoint coordinates while all others are constrained to zero. These functions can be shown to be

$$\begin{aligned} \psi_1 &= 1-3\left(\frac{s}{l}\right)^2+2\left(\frac{s}{l}\right)^3 \\ \psi_2 &= s\left[1-2\left(\frac{s}{l}\right)+\left(\frac{s}{l}\right)^2\right] \\ \psi_3 &= 3\left(\frac{s}{l}\right)^2-2\left(\frac{s}{l}\right)^3 \\ \psi_4 &= l\left[-\left(\frac{s}{l}\right)^2+\left(\frac{s}{l}\right)^3\right] \end{aligned} \tag{4.22}$$

where l is the length of the finite element.

The rotational displacements can now be related to the translational displacements through

$$\begin{aligned} B &= -\frac{\partial W}{\partial s} \\ \Gamma &= \frac{\partial V}{\partial s} \end{aligned} \tag{4.23}$$

As a result, the rotation of a typical point within the shaft finite element is given, by combining Eqs. (4.20) and (4.23), as

$$\begin{Bmatrix} B(s,t) \\ \Gamma(s,t) \end{Bmatrix} = [\Phi(s)]\{q^{\bullet}(t)\} \tag{4.24}$$

where the matrix of interpolation functions corresponding to the rotational displacements is given as

$$[\Phi] = \begin{bmatrix} \lfloor\Phi_B\rfloor \\ \lfloor\Phi_\Gamma\rfloor \end{bmatrix} = \begin{bmatrix} 0 & -\psi_1' & \psi_2' & 0 & 0 & -\psi_3' & \psi_4' & 0 \\ \psi_1' & 0 & 0 & \psi_2' & \psi_3' & 0 & 0 & \psi_4' \end{bmatrix} \tag{4.25}$$

where primes denote differentiation with respect to the axial coordinate s.

The kinetic and potential energy functions are defined in terms of the element interpolation functions and the Lagrangian equation of motion is then stated. In order to write the element equations for the kinetic and potential energies (due to the flexural deformation as well as the axial load), these quantities are first stated for a differential element of length ds. Integration of the resulting expressions over the length of the finite element then yields the energy functions corresponding to the finite element. Physically, the shaft element is viewed as being composed of a set of disks. For one such disk that is located at an axial location s, the expressions for the kinetic energy $d\tau^e$ and the potential energy due to flexural deformation dU_B^e are

$$d\tau^e = \frac{1}{2}\begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix}^T \begin{bmatrix}\bar{\mu}(1+\alpha b(s)) & 0 \\ 0 & \bar{\mu}(1+\alpha b(s))\end{bmatrix} \begin{Bmatrix}\dot{V}\\ \dot{W}\end{Bmatrix} ds + \frac{1}{2}\dot{\phi}^2 i_p ds + \frac{1}{2}\begin{Bmatrix}\dot{B}\\ \dot{\Gamma}\end{Bmatrix}^T \begin{bmatrix}I_D & 0 \\ 0 & I_D\end{bmatrix} \begin{Bmatrix}\dot{B}\\ \dot{\Gamma}\end{Bmatrix} ds - \dot{\phi}\dot{\Gamma}B i_p ds \tag{4.26}$$

$$dU_B^e = \frac{1}{2}\begin{Bmatrix}V''\\ W''\end{Bmatrix}^T \begin{bmatrix}E(1+\alpha a(s))I & 0 \\ 0 & E(1+\alpha a(s))I\end{bmatrix} \begin{Bmatrix}V''\\ W''\end{Bmatrix} ds \tag{4.27}$$

The potential energy due to axial load dU_A^e is given by

$$dU_A^e = -\frac{1}{2}\begin{Bmatrix} V' \\ W' \end{Bmatrix}^T \begin{bmatrix} \bar{P}(1+c) & 0 \\ 0 & \bar{P}(1+c) \end{bmatrix} \begin{Bmatrix} V' \\ W' \end{Bmatrix} ds \qquad (4.28)$$

Substituting the equations for the translation and rotation of a typical point within the finite element, in the above three equations, one gets

$$\begin{aligned} d\tau^e = &\frac{1}{2}\bar{\mu}(1+\alpha b(s))\{\dot{q}^e\}^T[\Psi]^T[\Psi]\{\dot{q}^e\}ds + \frac{1}{2}\dot{\phi}^2 i_p ds \\ &+\frac{1}{2} i_D \{\dot{q}^e\}[\Phi]^T[\Phi]\{\dot{q}^e\}ds \\ &-\dot{\phi} i_p \{\dot{q}^e\}^T[\Phi_\Gamma]^T[\Phi_B]\{q^e\}ds \end{aligned} \qquad (4.29)$$

$$dU_B^e = \frac{1}{2}\bar{E}(1+\alpha a(s))I\{q^e\}^T[\Psi'']^T[\Psi'']\{q^e\}ds \qquad (4.30)$$

$$dU_A^e = -\frac{1}{2}\bar{P}(1+c)\ \{q^e\}^T[\Psi']^T[\Psi']\{q^e\}ds \qquad (4.31)$$

It can be observed that the first term in the right hand side of Eq. (4.29) accounts for the kinetic energy due to rectilinear translation, the second term accounts for the spin-axis rotation, the third for rotary inertia, and the fourth for gyroscopic coupling. Based on these three equations, the kinetic energy, potential energy due to bending, and potential energy due to axial load of the finite element are obtained, through integration over the entire element length. The mass and stiffness matrices of the shaft element are identified while performing these integrations. The total energy of the element is the sum of kinetic energy and potential energies due to both flexural deformation and axial load, and it is given by

$$U_B^e + U_A^e + \tau^e = \frac{1}{2}\{q^e\}^T([\overline{K}_B^e + K_B^e(\sim) - \overline{K}_A^e - K_A^e(\sim)])\{q^e\}$$
$$+ \frac{1}{2}\{\dot{q}^e\}^T([\overline{M}_T^e + M_T^e(\sim) + M_R^e])\{\dot{q}^e\} \tag{4.32}$$
$$+ \frac{1}{2} I_P^e \dot{\phi}^2 + \dot{\phi}\{\dot{q}^e\}^T[N^e]\{q^e\}$$

The mass matrices are given as

$$[M_T^e] = \int_0^1 \overline{\mu}(1+\alpha b(s))[\Psi]^T[\Psi]\,ds \tag{4.33}$$

$$[M_R^e] = \int_0^1 i_D[\Phi]^T[\Phi]\,ds \tag{4.34}$$

where $[M_T^e]$ and $[M_R^e]$ represent the mass matrices of the finite rotor element corresponding to translational and rotational displacements respectively.

The stiffness matrix due to flexural deformation, denoted by $[K_B^e]$, and the stiffness matrix due to axial load (the so-called geometric stiffness matrix), denoted by $[K_A^e]$, are given as

$$\begin{aligned}
[K_B^e] &= \int_0^1 \overline{E}(1+\alpha a(s)) I[\Psi'']^T[\Psi'']\,ds \\
&= \int_0^1 \overline{E}I[\Psi'']^T[\Psi'']\,ds \\
&\qquad + \int_0^1 \alpha\overline{E}a(s) I[\Psi'']^T[\Psi'']\,ds \\
&= [\overline{K}_B^e] + [K_B^e(\sim)]
\end{aligned} \tag{4.35}$$

$$
\begin{aligned}
[K_A^e] &= \int_0^1 \bar{P}(1+c)\,[\Psi']^T[\Psi']\,ds \\
&= \int_0^1 \bar{P}[\Psi']^T[\Psi']\,ds \\
&\qquad + \int_0^1 \bar{P}c\,[\Psi']^T[\Psi']\,ds \\
&= [\bar{K}_A^e] + [K_A^e(\sim)]
\end{aligned}
\tag{4.36}
$$

All of the above matrices can be seen to be symmetric. The matrix, $[N^e]$, in Eq. (4.32) is a non-symmetric matrix and is formed as

$$
[N^e] = \int_0^1 i_P[\Phi_\Gamma]^T[\Phi_B]\,ds \tag{4.37}
$$

Corresponding to the constant spin speed restriction, i.e. $d\phi/dt = \Omega$, the Lagrangian equation of motion in the stationary frame of reference, for the finite rotor element is derived, based on the expressions for element energies. It is given as

$$
\begin{aligned}
&(([\bar{M}_T^e] + [M_T^e(\sim)]) + [M_R^e])\,\{\ddot{q}^e\} - \Omega[G^e]\,\{\dot{q}^e\} \\
&\quad + (([\bar{K}_B^e] + [K_B^e(\sim)]) - ([\bar{K}_A^e] + [K_A^e(\sim)]))\,\{q^e\} = \{Q^e\}
\end{aligned}
\tag{4.38}
$$

where the gyroscopic (coupling) matrix $[G^e]$ that is skew-symmetric is given by

$$
[G^e] = ([N^e] - [N^e]^T) \tag{4.39}
$$

As in the case of rigid disk, the force vector $\{Q^e\}$ includes the effects of mass unbalance, interconnection forces and other external forces acting on the rotor element. Considering the effects of mass unbalance, Archer [1963] derived an expression for the equivalent unbalance force, based on the consistent mass matrix approach. For an element with distributed mass center eccentricity (η (s), ζ(s)) the relevant expression is

$$\{Q^{e}\}_{8\times1} = \int_{0}^{1} \overline{\mu}\,(1+b(s))\,\Omega^{2}\,\{\psi\}^{T} \left(\begin{Bmatrix}\eta(s)\\ \zeta(s)\end{Bmatrix}\cos\Omega t + \begin{Bmatrix}-\zeta(s)\\ \eta(s)\end{Bmatrix}\sin\Omega t\right) \tag{4.40}$$
$$= (\{\overline{Q}^{e}_{c}\} + \{Q^{e}_{c}(\sim)\})\cos\Omega t + (\{\overline{Q}^{e}_{s}\} + \{Q^{e}_{s}(\sim)\})\sin\Omega t$$

where the force vectors $\{\overline{Q}_c^e\}$, $\{Q_c^e(\sim)\}$, $\{\overline{Q}_s^e\}$ and $\{Q_s^e(\sim)\}$ have to be obtained considering a particular case of mass unbalance distribution. The equation (4.39) can be employed to obtain the nodal forces due to mass unbalance corresponding to a particular distribution of mass center eccentricity over the rotor element. For instance, when the case of a linear mass unbalance distribution given by

$$\begin{aligned}\eta(s) &= \eta_L\left(1-\frac{s}{l}\right) + \eta_R\left(\frac{s}{l}\right)\\ \zeta(s) &= \zeta_L\left(1-\frac{s}{l}\right) + \zeta_R\left(\frac{s}{l}\right)\end{aligned} \tag{4.41}$$

is considered, the ensemble mean components of the equivalent nodal unbalance force vector are given as

$$\{\bar{Q}_c^e\} = \bar{\mu}\Omega^2 \begin{Bmatrix} \frac{7}{20}\eta_L l + \frac{3}{20}\eta_R l \\ \frac{7}{20}\zeta_L l + \frac{3}{20}\zeta_R l \\ -\frac{1}{20}\zeta_L l^2 - \frac{1}{30}\zeta_R l^2 \\ \frac{1}{20}\eta_L l^2 + \frac{1}{30}\eta_R l^2 \\ \frac{3}{20}\eta_L l + \frac{7}{20}\eta_R l \\ \frac{3}{20}\zeta_L l + \frac{7}{20}\zeta_R l \\ \frac{1}{30}\zeta_L l^2 + \frac{1}{20}\zeta_R l^2 \\ -\frac{1}{30}\eta_L l^2 - \frac{1}{20}\eta_R l^2 \end{Bmatrix} \qquad \{\bar{Q}_s^e\} = \bar{\mu}\Omega^2 \begin{Bmatrix} -\frac{7}{20}\zeta_L l - \frac{3}{20}\zeta_R l \\ \frac{7}{20}\eta_L l + \frac{3}{20}\eta_R l \\ -\frac{1}{20}\eta_L l^2 - \frac{1}{30}\eta_R l^2 \\ -\frac{1}{20}\zeta_L l^2 - \frac{1}{30}\zeta_R l^2 \\ -\frac{3}{20}\zeta_L l - \frac{7}{20}\zeta_R l \\ \frac{3}{20}\eta_L l + \frac{7}{20}\eta_R l \\ \frac{1}{30}\eta_L l^2 + \frac{1}{20}\eta_R l^2 \\ \frac{1}{30}\zeta_L l^2 + \frac{1}{20}\zeta_R l^2 \end{Bmatrix} \tag{4.42}$$

In the above, (η_L, ζ_L) denotes the mass center eccentricity at the left end of the element, i.e. at $s = 0$, and (η_R, ζ_R) denotes the mass center eccentricity at the right end of the element, i.e. at $s = l$.

Substituting Eqs. (4.2-4.4), and (4.11-4.13), after they have been extended to include four coordinates at each end of the element, into Eq. (4.38) and pre-multiplying the resulting equation by $[R]^T$ yields the equation of motion in the rotating frame of reference as

$$\begin{aligned} &(([\bar{M}_T^e] + [M_T^e(\sim)]) + [M_R^e])\{\ddot{p}^e\}_{8\times1} \\ &\quad + \omega(2[\bar{M}_T^e + \hat{M}_T^e(\sim)] + (1-\upsilon)[G^e])\{\dot{p}^e\} \\ &\quad + ((([\bar{K}_B^e] + [K_B^e(\sim)]) - [K_A^e]) - \omega^2([(([\bar{M}_T^e] \\ &\qquad + [M_T^e(\sim)]))] + (1-2\upsilon)[M_R^e]))\{p^e\} = \{P^e\} \end{aligned} \tag{4.43}$$

where the transformed mass matrix is given by

$$[\overline{M}_T^e + \hat{M}_T^e(\sim)] = [R]^T\,([\overline{M}_T^e] + [M_T^e(\sim)])\,[S] \qquad (4.44)$$

In deriving the above equation of motion, due to the fact that the polar moment of inertia of the element cross-section I_P is equal to twice the diametral moment of inertia of the element cross-section I_D, the following identity has been made use of.

$$[R]^T\,[M_R^e]\,[S] = \frac{1}{2}[G^e] \qquad (4.45)$$

Now it can be observed that the finite element formulation described above includes the effects of the distributed rotary inertia and gyroscopic coupling within the rotor element. The formulation also accounts for the distributed mass unbalance acting along the length of the element. Further, the finite element model derived here is based on the so-called Euler-Bernoulli beam model, which does not account for shear deflections. Consideration of shear deflections, however, would require the introduction of new interpolation functions (that can be obtained from the static solutions for the so-called Timoshenko beam model) into the foregoing analysis.

4. 2. 2. 3 Bearings

The rotating assembly of a machine interacts with other rotating assemblies and the static support structure through a variety of mechanisms, the most prevalent being the rolling-element bearings, fluid-film bearings and dampers, seals, splines, couplings and aerodynamic interconnection mechanisms. In most cases, these components are non-linear in their force-displacement and force-velocity relationships. However, the analysis methods can be based on the assumption that the rotor-bearing system operates in a small neighbourhood of the static equilibrium. When the system is subjected to dynamic loadings which are superposed on the static loading, it is assumed that the corresponding interconnection forces acting on the rotating assemblies can be closely approximated by linear force-displacement and force-velocity relationships. The bearing systems considered in the present analysis are limited to those which can be represented by the governing equation of motion of the form

$$[C^b]\{\dot{q}^b\} + [K^b]\{q^b\} = \{Q^b\}_{2\times1} \tag{4.46}$$

which has been written in stationary frame of reference. In the above

$$\{q^b\} = \begin{Bmatrix} V \\ W \end{Bmatrix} \quad ; \quad [K^b] = \begin{bmatrix} k^b_{VV} & k^b_{VW} \\ k^b_{WV} & k^b_{WW} \end{bmatrix} \quad ; \quad [C^b] = \begin{bmatrix} c^b_{VV} & c^b_{VW} \\ c^b_{WV} & c^b_{WW} \end{bmatrix} \tag{4.47}$$

and $\{Q^b\}$ is the external force vector. Transformation of the equation of motion given above to the rotating frame of reference can be performed by making use of Eqs. (4.2-4.4) in Eq. (4.46) and pre-multiplying the resulting equation by $[R]^T$. As a result, one gets

$$[R]^T[C^b]\,[R]\{\dot{p}^b\} + [R]^T[K^b]\,[R]\{p^b\} = \{P^b\}_{2\times1} \tag{4.48}$$

For isotropic bearings, this equation reduces to the following form

$$c^b[I]\{\dot{p}^b\} + k^b[I]\{p^b\} = \{P^b\}_{2\times1} \tag{4.49}$$

where c^b and k^b are the damping and stiffness coefficients of the isotropic bearing.

4. 2. 3 System Equations of Motion

The fifth step is to assemble the system equations of motion based on the element equations of motion. The procedures for assembly of the system equations of motion in the stationary and rotating reference coordinates are conceptually equivalent. When the system equations of motion are assembled in the stationary frame of reference it is possible to transform the assembled system equations to the rotating frame of reference through the transformation relationship stated in section 4. 2. 1.

The system displacement vector $\{q^s\}$ is formed to be consisting of all the nodal (translational and rotational) displacements and is defined as

$$\{q^s\}^T_{n\times 1} = [\underbrace{\{q_1\}^T, \ldots, \{q_6\}^T}_{element1}, \underbrace{\{q_7\}^T, \ldots, \{q_{10}\}^T}_{element2}, \underbrace{\ldots\ldots}_{\ldots\ldots}] \tag{4.50}$$

Next, the connectivity relationships of each of the finite elements (and the element nodal displacements) with the system displacement vector given above, are defined. The set of element nodal displacements constitutes a dependent set whereas the set of system nodal displacements constitutes an independent set, and the connectivity relationships essentially represent the geometric constraint relations between the element nodal displacements and the system nodal displacements. For the r-th element with n_r degrees of freedom, the connectivity relationship is written as

$$\{q^r\}_{n_r\times 1} = [B_r]\{q^s\}_{n\times 1} \tag{4.51}$$

where $[B_r]$ is the connectivity matrix corresponding to the r-th finite element. The coefficients of this matrix are equal to unity when the element and system reference frames are the same and also both the element and nodal degrees of freedom are the same. These coefficients of the connectivity matrix essentially identify the components of the system displacement vector $\{q^s\}$ that are common to the element displacement vector $\{q^r\}$.

The equations of motion corresponding to the finite rotor elements are assembled based on the above connectivity relationship. The assembled undamped system equations of motion, consisting of the component equations of motion (4.7), (4.38) and (4.46), are of the form

$$\begin{aligned}&([\overline{M}^s] + [M^s(\sim)])\{\ddot{q}^s\} - \Omega\,[G^s]\,\{\dot{q}^s\}\\&+ ([\overline{K}^s] + [K^s(\sim)])\{q^s\} = \{\{\overline{Q}^s\} + \{Q^s(\sim)\}\}_{4n\times 1}\end{aligned} \tag{4.52}$$

in the stationary frame of reference. For computational purposes, it is convenient to rewrite the above equation of motion in the first-order state-vector form

$$\begin{bmatrix}[0] & ([\overline{M}^s] + [M^s(\sim)])\\ ([\overline{M}^s] + [M^s(\sim)]) & -\Omega\,[G^s]\end{bmatrix}\{\dot{h}\} + \begin{bmatrix}([\overline{M}^s] + [M^s(\sim)]) & [0]\\ 0 & ([\overline{K}^s] + [K^s(\sim)])\end{bmatrix}\{h\} = \{H\}_{8n\times 1} \tag{4.53}$$

where

$$\{h\} = \begin{Bmatrix}\{\dot{q}\}\\ \{q\}\end{Bmatrix}\ , \quad \{H\} = \begin{Bmatrix}\{0\}\\ \{\overline{Q}^s\} + \{Q^s(\sim)\}\end{Bmatrix} \tag{4.54}$$

4. 3 Whirl Speed Analysis

For an assumed solution of the form, $\{h\} = \{h_0\}e^{\Lambda t}$, the eigenvalue problem corresponding to the homogeneous case of equation (4.53), is

$$\begin{bmatrix} [0] & [I] \\ -\{([\overline{K}^s] + [K^s(\sim)])^{-1} & \\ \quad ([\overline{M}^s] + [M^s(\sim)])\} & \{\Omega([\overline{K}^s] + [K^s(\sim)])^{-1}[G^s]\} \end{bmatrix} \{h_0\} = \frac{1}{\Lambda}\{h_0\} \tag{4.55}$$

For orthotropic bearings (and when $k_{VW}{}^b = k_{WV}{}^b = 0$) and zero damping, the eigenvalues of equation (4.55) appear as pure imaginary conjugate pairs with the magnitude equal to the natural whirl speed. The superposition of a solution with its conjugate represents an associated elliptical precession mode. Isotropic bearings produce circular precession modes for a rotating assembly.

In rotating coordinates and for the case of isotropic bearings, Eq. (4.52) transforms into the form

$$\begin{aligned} &[\overline{M}^s] + [M^s(\sim)] \{\ddot{p}^s\} + \omega(2([\overline{M}^s] + [\hat{M}^s(\sim)]) \\ &- \upsilon[G^s])\{\dot{p}^s\} + (([\overline{K}^s] + [K^s(\sim)].) \\ &- \omega^2(([\overline{M}^s] + [M^s(\sim)]) + \upsilon[\hat{G}^s]))\{p^s\} = \{P^s\}_{4n\times 1} \end{aligned} \tag{4.56}$$

The natural (circular) whirl speeds and the associated mode shapes can be obtained from the homogeneous form of Eq. (4.56). These modes are constant relative to the rotating reference frame and, further, the two planes of motion are 90 degrees out of phase. It is necessary, therefore, to consider only one of the two planes of motion and assume a constant solution $\{p^s\}=\{p_o\}$=constant. The associated eigenvalue problem is given by

$$([\overline{K}^s] + [K^s(\sim)])\{p_0\} = \omega^2(([\overline{M}^s] + [M^s(\sim)]) + \nu[G^s])\{p_0\}_{2n\times 1} \tag{4.57}$$

The 2n number of eigenvalues are real and, further, the positive values ω_r with associated vectors $\{p_0\}^{(r)}$ represent the natural (circular) whirl speeds and the mode shapes relative to the rotating frame of reference at the specified whirl ratio ν.

4.4 Stochastic Properties of Characteristic Matrices

The stochastic properties of the coefficients of both the stiffness and mass matrices due to the randomness in material properties and axial loadings, are now determined. Each of these matrices have been split into a mean component and a stochastically-fluctuating component. The ensemble mean values of the coefficients μ_{ij}, are given by

$$\begin{aligned}\langle\mu_{ij}\rangle &= \int_0^l \langle\overline{\mu}(1+\alpha b(s))(\Psi_i\Psi_j + I\,\Phi_i\Phi_j)\rangle ds \\ &= \int_0^l \langle\overline{\mu}(1+\alpha b(s))\nu_{ij}\rangle ds \\ &= \int_0^l \overline{\mu}\nu_{ij} ds = \overline{\mu}_{ij}\end{aligned} \tag{4.58}$$

where, I is the second moment of the area of cross-section while Ψ and Φ are matrices of interpolation functions of size 2x8 as can be seen from Eqs. (4.21) and (4.25). The variance of any coefficient μ_{ij} is derived by considering two sample realizations of the coefficient and performing the expectation operation after multiplying them together.

$$\mathrm{Var}(\mu_{ij}) = \mathrm{Var}(\mu_{ij}(\sim))$$

$$= (\alpha\bar{\mu})^2 \int_0^1\int_0^1 \langle b(\xi_1)\, b(\xi_2)\rangle \nu_{ij}(\xi_1)\, \nu_{ij}(\xi_2)\, d\xi_1 d\xi_2$$

$$= (\alpha\bar{\mu})^2 \int_0^1\int_0^1 R_{bb}(\xi_1 - \xi_2)\, \nu_{ij}(\xi_1)\, \nu_{ij}(\xi_2)\, d\xi_1 d\xi_2 \tag{4.59}$$

The covariance between any two coefficients μ_{ij} and μ_{rs} is determined by considering two sample realizations of μ_{ij} and μ_{rs}, multiplying them together and performing the expectation operation.

$$Cov(\mu_{ij}, \mu_{rs}) = C^{m}_{ij,rs}$$

$$= (\alpha\bar{\mu})^2 \int_0^1\int_0^1 R_{bb}(\xi_1 - \xi_2)\, \nu_{ij}(\xi_1)\, \nu_{rs}(\xi_2)\, d\xi_1 d\xi_2 \tag{4.60}$$

The stochastic properties of the coefficients of the stiffness matrix, k_{ij}, and the coefficients of the gyroscopic matrix, g_{ij}, can be derived in a similar manner. The probabilistic moments of k_{ij} and g_{ij} are given by

$$\langle k_{ij}\rangle = \int_0^1 \langle \bar{E}(1+\alpha a(s))\, I(\Psi_i''\Psi_j'')\rangle ds = \bar{k}_{ij}$$

$$\langle g_{ij}\rangle = 2\int_0^1 \langle \bar{\mu}(1+\alpha b(s))\, I(\Phi_{2i}\Phi_{1j} - \Phi_{1i}\Phi_{2j})\rangle ds \tag{4.61}$$

$$= \int_0^1 \langle \bar{\mu}(1+\alpha b(s))\, f_{ij}\rangle ds = \bar{g}_{ij}$$

$$Var(k_{ij}) = (\alpha\overline{EI})^2\int_0^1\int_0^1 R_{aa}(\xi_1 - \xi_2)$$

$$\psi_i''(\xi_1)\psi_j''(\xi_1)\psi_i''(\xi_2)\psi_j''(\xi_2)\,d\xi_1 d\xi_2 \qquad (4.62)$$

$$+ \overline{P}^2\sigma_c^2\int_0^1\int_0^1 \psi_i'(\xi_1)\psi_j'(\xi_2)\,d\xi_1 d\xi_2$$

$$Var(g_{ij}) = (\alpha\overline{\mu})^2\int_0^1\int_0^1 R_{bb}(\xi_1 - \xi_2)\,f_{ij}(\xi_1)\,f_{ij}(\xi_2)\,d\xi_1 d\xi_2 \qquad (4.63)$$

The covariance between any two coefficients of stiffness matrices is given by

$$Cov(k_{ij}\,,\,k_{rs}) = C^k_{ij,rs} = (\alpha\overline{EI})^2\int_0^1\int_0^1 R_{aa}(\xi_1 - \xi_2)$$

$$\psi_i''(\xi_1)\psi_j''(\xi_1)\psi_r''(\xi_2)\psi_s''(\xi_2)\,d(\xi_1)\,d(\xi_2) \qquad (4.64)$$

$$+ \overline{P}^2\sigma_c^2\int_0^1\int_0^1 \psi_i'(\xi_1)\psi_j'(\xi_1)\psi_r(\xi_2)\psi_s(\xi_2)\,d\xi_1 d\xi_2$$

The cross covariances between the coefficients of mass and stiffness matrices are zero as the two stochastic fields a(s) and b(s) are independent.

4. 5 Probabilistic Moments of Whirl Speeds and Whirl Modes

Comparing Eq. (4.57) with Eq. (3.1), the probabilistic moments of whirl speeds can be obtained. Here the matrix [B] consists of two correlated parts and further, one scalar multiplier (the whirl ratio) is involved. So, the formulas for the second order probabilistic moments developed earlier in Chapter 3 are now further adapted.

The expected mean value of any whirl speed ω_i^2 is given by

$$<\omega^2_i> = \overline{\omega^2_i} \tag{4.65}$$

The covariance function of whirl speeds is now determined.

$$Cov(\omega^2_i, \omega^2_j) = <(\omega^2_i - \overline{\omega^2_i}) . (\omega^2_j - \overline{\omega^2_j})> = <\omega^2_{i,1} , \omega^2_{j,1}> \tag{4.66}$$

$$\begin{aligned}
&= <(\{ p_0^- \}_i^T [K^s(\sim)] \{ \overline{p_0} \}_i - \overline{\omega^2_i} \{ p_0^- \}_i^T ([M^s(\sim)] \\
&\quad + \mu [G^s(\sim)]) \{ \overline{p_0} \}_i) . (\{ p_0^- \}_j^T [K^s(\sim)] \{ \overline{p_0} \}_j \\
&\quad - \overline{\omega^2_j} \{ p_0^- \}_j^T ([M^s(\sim)] + \mu [G^s(\sim)]) \{ \overline{p_0} \}_j)>
\end{aligned}$$

$$\begin{aligned}
&= <\{ p_0^- \}_i^T [K^s(\sim)] \{ \overline{p_0} \}_i . \{ p_0^- \}_j^T [K^s(\sim)] \{ \overline{p_0} \}_j > \\
&\quad + \overline{\omega^2_i} . \overline{\omega^2_j} <\{ p_0^- \}_i^T ([M^s(\sim)] + \mu [G^s(\sim)]) \{ \overline{p_0} \}_i \\
&\quad \{ p_0^- \}_j^T ([M^s(\sim)] + \mu [G^s(\sim)]) \{ \overline{p_0} \}_j > \\
&\quad + \omega^2_j <\{ p_0^- \}_i^T [K^s(\sim)] \{ \overline{p_0} \}_i . \{ p_0^- \}_j^T ([M^s(\sim)] \\
&\quad + \mu [G^s(\sim)]) \{ \overline{p_0} \}_j > + \omega^2_i <\{ p_0^- \}_i^T ([M^s(\sim)] \\
&\quad + \mu [G^s(\sim)]) \{ \overline{p_0} \}_i . \{ p_0^- \}_j^T [K^s(\sim)] \{ \overline{p_0} \}_j>
\end{aligned} \tag{4.67}$$

where the symbol $\sim$ indicates the adjoint system eigenvector and the symbol $\approx$ indicates the corresponding ensemble mean value.

The variance function for any whirl speed is obtained from Eq. (4.67) by substituting i = j.

$$Var(\omega_i^2) = <\{p_0^-\}_i^T[K^s(\sim)]\{\overline{p_0}\}_i \;.\; \{p_0^-\}_i^T[K^s(\sim)]\{\overline{p_0}\}_i >$$
$$+ (\overline{\omega_i^2})^2) \;.\; <\{p_0^-\}_i^T([M^s(\sim)] + \mu[G^s(\sim)])$$
$$\{\overline{p_0}\}_i)\;\{p_0^-\}_i^T([M^s(\sim)] + \mu[G^s(\sim)])\;\{\overline{p_0}\}_i) >$$
$$+ \omega^2_i\;(\;<\{p_0^-\}_i^T[K^s(\sim)]\{\overline{p_0}\}_i \;.\{p_0^-\}_i^T([M^s(\sim)]$$
$$+ \mu[G^s(\sim)])\;\{\overline{p_0}\}_i> + <\{p_0^-\}_i^T([M^s(\sim)]$$
$$+ \mu[G^s(\sim)])\{\overline{p_0}\}_i \;.\{p_0^-\}_i^T[K^s(\sim)]\;\{\overline{p_0}\}_i>\;) \tag{4.68}$$

The equations for covariance function, Eq. (4.67), is recast into the following computationally-convenient form:

$$Cov(\omega^2_i\;,\;\omega^2_j) = \sum_1^n\sum_1^n\sum_1^n\sum_1^n <k^s_{pq}(\sim)\;\;k^s_{rs}(\sim)>$$
$$(\{p_0^-\}^T)_{pi}\;\;(\{\overline{p_0}\})_{qi}\;\;(\{p_0^-\}^T)_{rj}\;\;(\{\overline{p_0}\})_{sj}$$
$$+ \omega^2_i\;.\;\omega^2_j\sum_1^n\sum_1^n\sum_1^n\sum_1^n <m^*_{pq}(\sim)\;\;m^*_{rs}(\sim)> \tag{4.69}$$
$$(\{p_0^-\}^T)_{pi}\;\;(\{\overline{p_0}\})_{qi}\;\;(\{p_0^-\}^T)_{rj}\;\;(\{\overline{p_0}\})_{sj}$$

That the cross correlations between stiffness and mass matrices are zero in the above equation has already been taken into account. In the above equation, $m^*_{pq} = m_{pq} + \lambda\, g_{pq}$ and $(\{p^-_0\}^T)_{rs}$ denotes the r-th element in the array $\{p^-_0\}^T_s$ of the solution pair $(\omega^2_s\,,\,\{\overline{p}_0\}_s)$.

From Eq. (4.69) the variance function for any whirl speed ω^2_i can be obtained by substituting i = j. The complete covariance matrix of whirl speeds can be generated using the expressions developed. Following the same procedure, the probabilistic moments of whirl modes are determined as follows:

$$<\{p_0\}_i> = \{\overline{p}_0\}_i$$

$$Cov(\{p_0\}_i , \{p_0\}_j) = <(\{p_0\}_i - \{\overline{p}_0\}_i) . (\{p_0\}_j - \{\overline{p}_0\}_j)>$$

$$= <\{p_0\}_{i,1} . \{p_0\}_{i,1}>$$

$$= <\sum_1^n d_x(i,k) \{\overline{p}_0\}_k \sum_1^n d_x(j,k) \{\overline{p}_0\}_k > \quad (4.70)$$

$$Cov(\{p_0\}_i^T, \{p_0\}_j^T) = <\sum_1^n d_y(i,k) \{\overline{p}_0\}_i^T \sum_1^n d_y(i,k) \{\overline{p}_0\}_j^T> \quad (4.71)$$

i.e.,

$$Cov(\{p_0\}_i , \{p_0\}_j) = \sum_1^n \sum_1^n 1 / (\overline{\omega^2_i} - \overline{\omega^2_k}) . 1 / (\overline{\omega^2_j} - \overline{\omega^2_s})$$

$$< (\{p_0^-\}_k^T [K^s(\sim)] \{\overline{p_0}\}_i)$$

$$- \omega^2_i\{p_0^-\}_s^T [M^s*(\sim)] \{\overline{p_0}\}_i) . (\{p_0^-\}_k^T [K^s(\sim)] \{\overline{p_0}\}_j$$

$$- \omega^2_j\{p_0^-\}_s^T [M^s*(\sim)] \{\overline{p_0}\}_j) > \quad (4.72)$$

and

$$Cov(\{p_0\}_i^T , \{p_0\}_j^T) = \sum_1^n \sum_1^n 1 / (\overline{\omega^2_i} - \overline{\omega^2_k}) . 1 / (\overline{\omega^2_j} - \overline{\omega^2_s})$$

$$< (\{p_0^-\}_i^T [K^s(\sim)] \{\overline{p_0}\}_k)$$

$$- \omega^2_i\{p_0^-\}_i [M^s*(\sim)] \{\overline{p_0}\}_s) . (\{p_0^-\}_j^T [K^s(\sim)] \{\overline{p_0}\}_k$$

$$- \omega^2_j\{p_0^-\}_j [M^s*(\sim)] \{\overline{p_0}\}_s)> \quad (4.73)$$

where,

$$[M^s*(\sim)] = [M^s(\sim)] + \mu [G^s(\sim)]$$

Chapter 5

Variability of Whirl Speeds of Rotor-Bearing Systems

5. 1 Introduction

Based on the theoretical developments described in Chapter 4, the variability of whirl speeds of a high speed rotor-bearing system is studied. The effects of the stochastic distribution of Young's modulus and mass density of the material, on the whirl speeds of the rotordynamic system are quantified. The variation of the probabilistic moments of whirl speeds with the correlation properties of stochastic fields that model the uncertain material properties is analyzed in detail. Design implications are discussed.

5. 2 Description of the Rotor-Bearing System

The rotor-bearing system that has previously been employed for numerical study in the work of Lalanne and Ferraris [1990] is considered. This rotor-bearing system is shown in Fig. (5.1) and it consists of a flexible horizontal shaft and 3 disks. The shaft is mounted at its ends either on fixed supports or on flexible bearings.

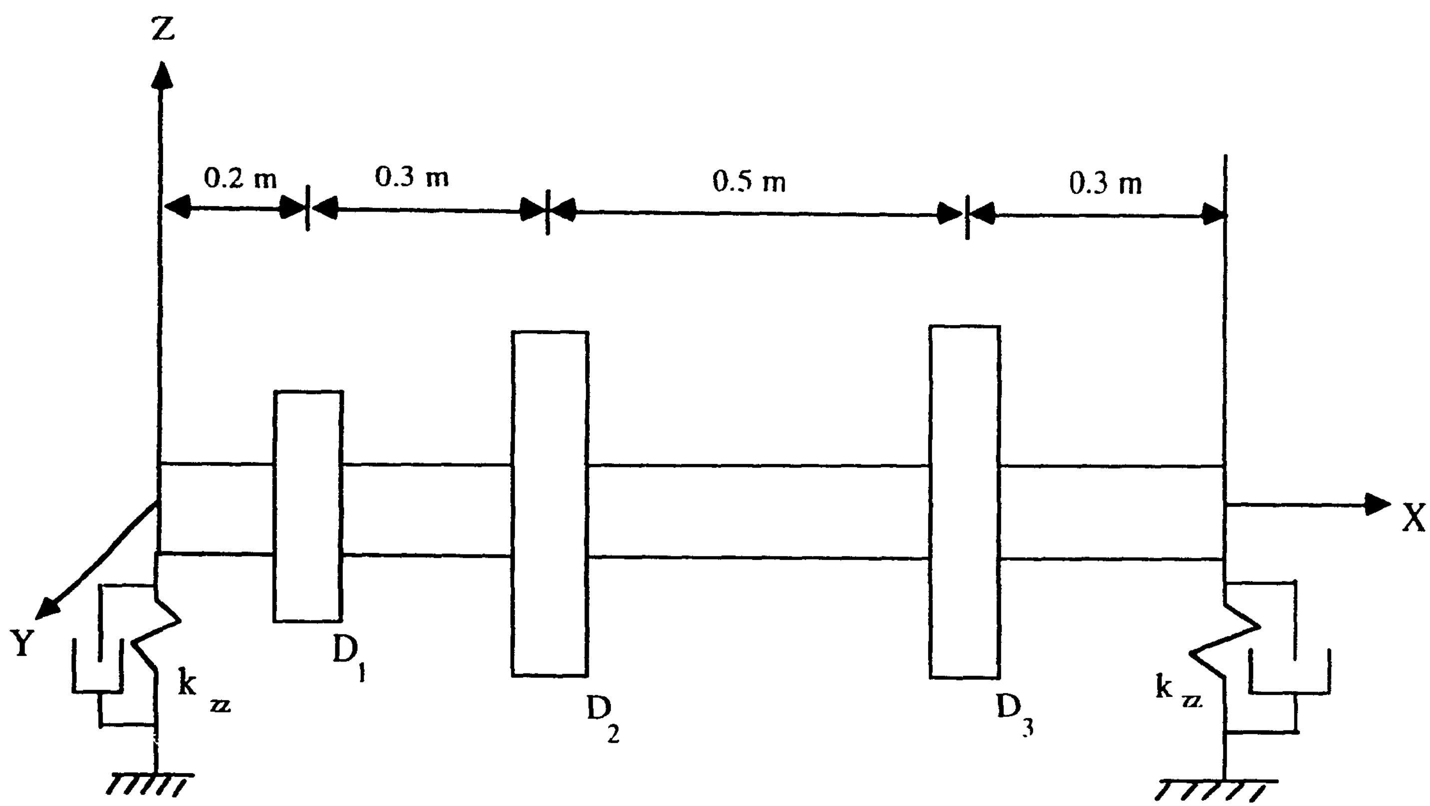

Fig. 5. 1: The rotor-bearing system employed in the parametric study

5. 2. 1 Geometric Properties

The length and cross-sectional radius of the shaft are 1.3m and 0.05m respectively. The dimensions of the three disks, denoted in Fig. (5.1) as D_1, D_2 and D_3, are given below in Table (5.1). The disks D_1, D_2 and D_3 are located along the longitudinal axis of the shaft at a distance of 0.2m, 0.5m and 1.0m respectively. The two flexible bearings at the ends of the shaft are assumed to be identical and they are characterized by their stiffnesses that are given as follows: $k_{yy} = 5\times10^7$ N/m ; $k_{zz} = 7\times10^7$ N/m ; $k_{yz} = k_{zy} = 0$.

Table 5.1: Dimensions of the disks in the rotor-bearing system

DISK	D_1	D_2	D_3
Thickness (m)	0.05	0.05	0.06
Inner Radius (m)	0.05	0.05	0.05
Outer Radius (m)	0.12	0.2	0.2

5. 2. 2 Finite Element Model

The shaft is modelled using n number of uniform finite elements. The lengths of the finite elements may or may not be equal. Further, in the parametric study, the shaft is modelled using 4, 6, 8 and 10 finite elements. The lengths of the finite elements as well as the location of the disks in each of the above cases are detailed below.

Case (i) : 4 elements

Lengths of elements 0.2m, 0.3m, 0.5m, 0.3m

Disks located at nodes 2, 3, 4

The finite element model of this configuration is shown in Fig. (5.2).

Case (ii) : 6 elements

Lengths of elements 0.2m, 0.2m, 0.1m, 0.2m, 0.3m, 0.3m

Disks located at nodes 2, 4, 6

Fig. (5.3) depicts the finite element model of the shaft comprising of six finite elements.

Case (iii) : 8 elements

Lengths of elements 0.2m, 0.2m, 0.1m, 0.2m, 0.2m, 0.1m, 0.2m, 0.1m

Disks located at nodes 2, 4, 7

The finite element model is shown in Fig. (5.4).

Case (iv) : 10 elements

Lengths of elements 0.2m, 0.1m, 0.1m, 0.1m, 0.2m, 0.2m, 0.1m, 0.1m, 0.1m, 0.1m

Disks located at nodes 2, 5, 8

The finite element model of the rotor system with three disks and a shaft consisting of ten finite elements is shown in Fig. (5.5).

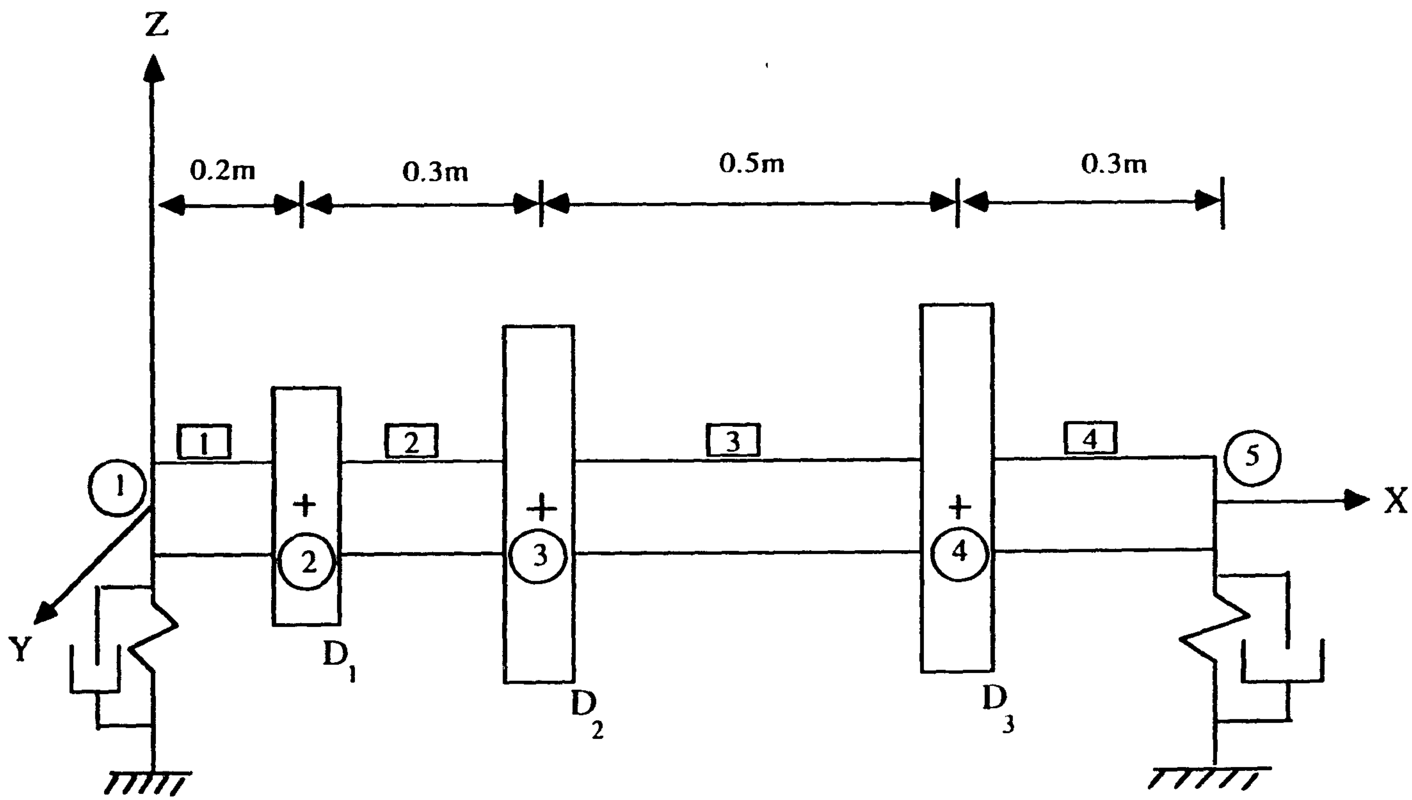

Fig. 5. 2: Finite element model of the rotor-bearing system using 4 elements

Finite elements -- [1], [2], [3], [4]

Nodes -- (1), (2), (3), (4), (5)

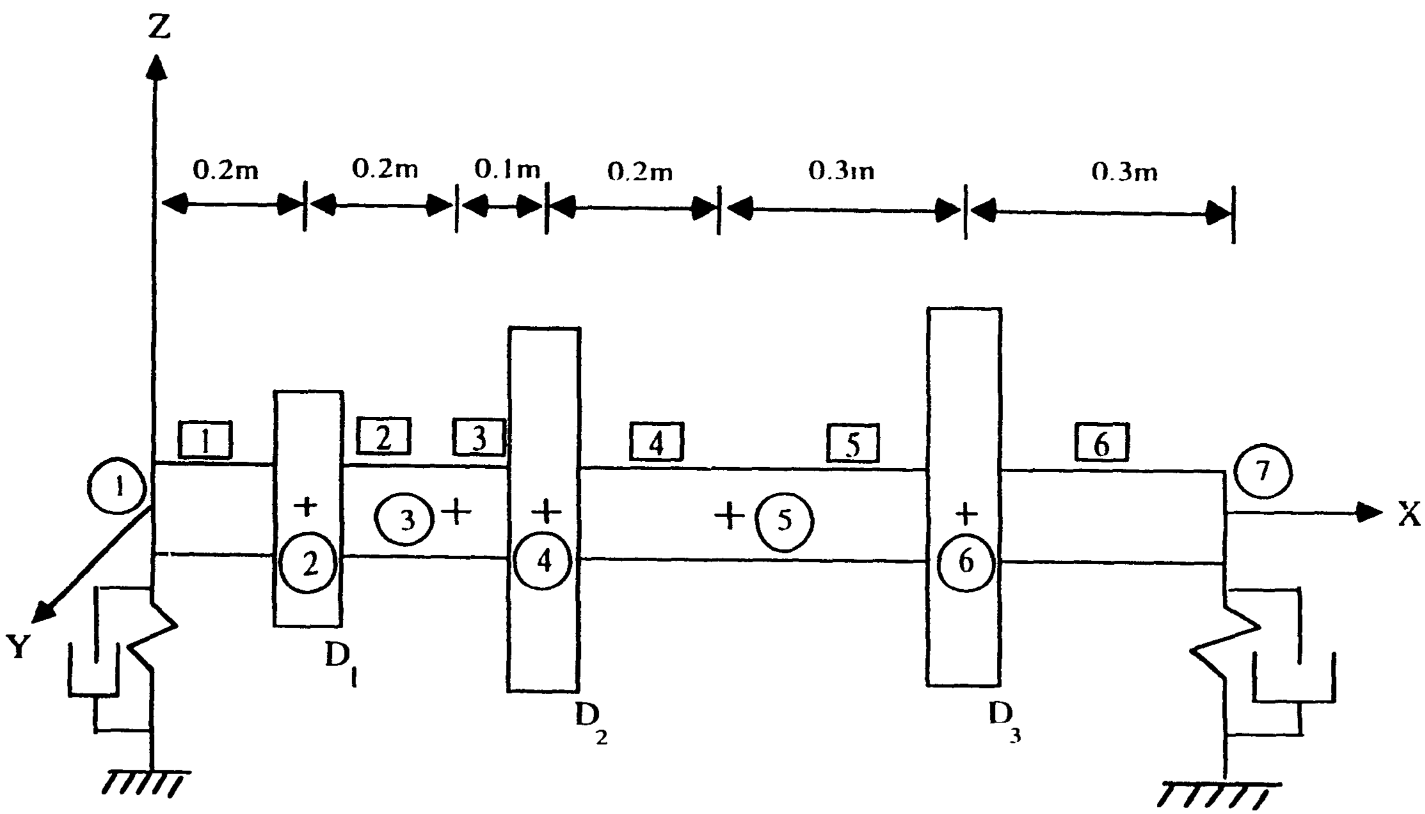

Fig. 5. 3: Finite element model of the rotor-bearing system using 6 elements

Finite elements -- 1, 2, 3, 4, 5, 6

Nodes -- 1, 2, 3, 4, 5, 6, 7

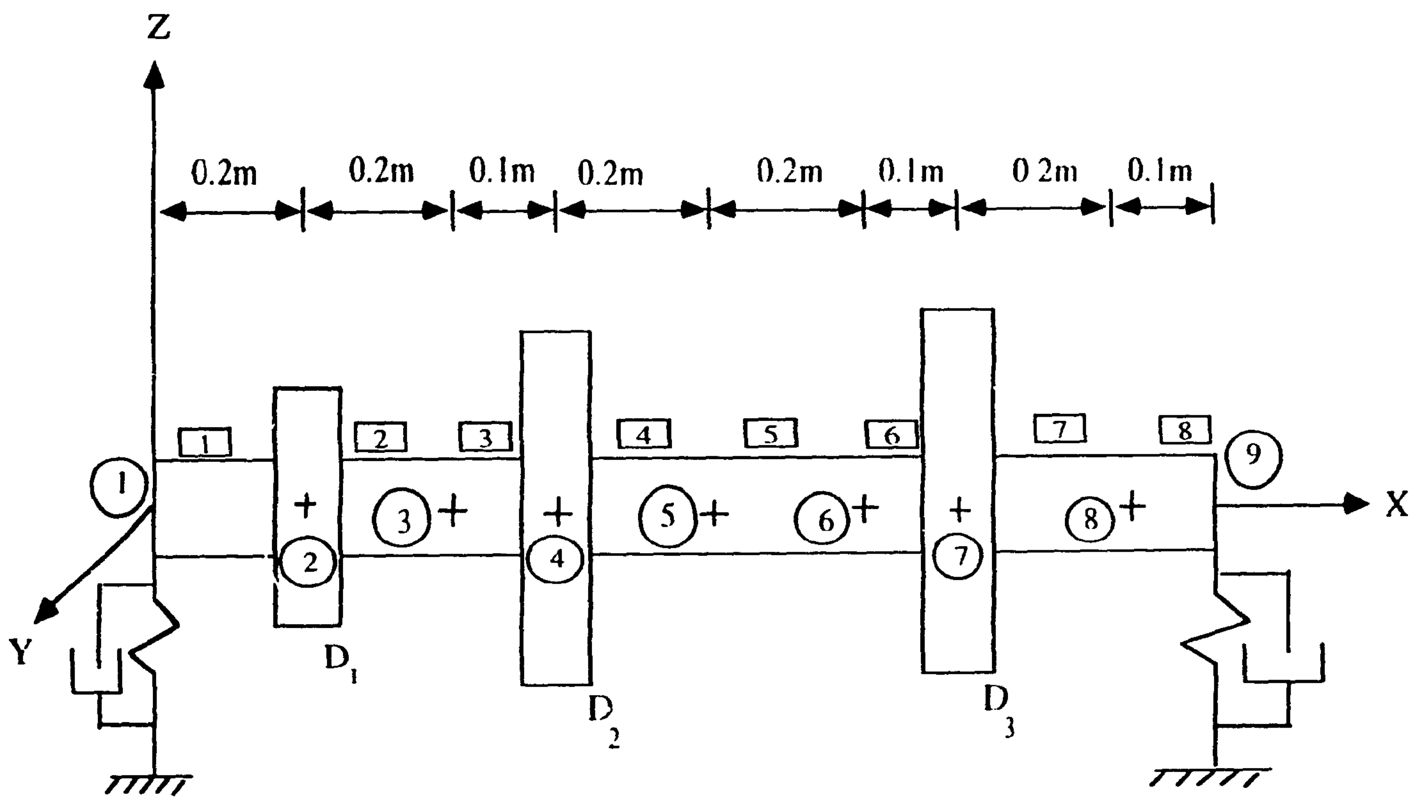

Fig. 5. 4: Finite element model of the rotor-bearing system using 8 elements

Finite elements -- 1, 2, 3, 4, 5, 6, 7, 8

Nodes -- 1, 2, 3, 4, 5, 6, 7, 8, 9

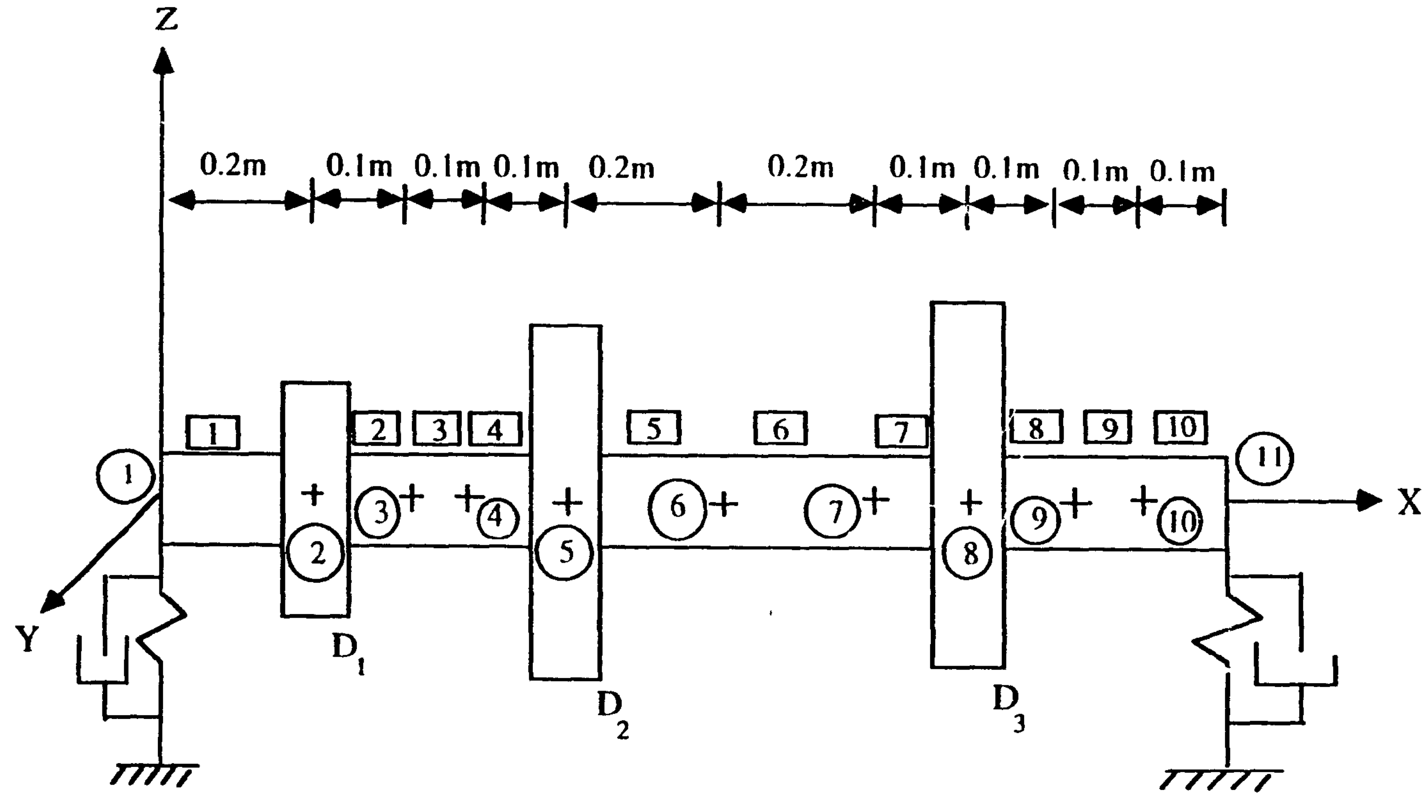

Fig. 5. 5: Finite element model of the rotor-bearing system using 10 elements

Finite elements -- 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Nodes -- 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

5. 2. 3 Stochastic Field Modelling of Material Properties

The disks and shaft used in the rotor-bearing system are made of steel. The mean value of the Young's modulus and mass density of steel are taken to be 2×10^{11} N/m^2 and 7800 kg/m^3 respectively. The Young's modulus and mass density are modelled as one-dimensional univariate homogeneous stochastic fields. The correlation properties of the stochastic fields representing the fluctuating components of the Young's modulus and mass density are represented using five different correlation models. The choice of these models in this work is due to their wide use in the literature [Vanmarcke, 1983]. These correlation models that characterize the second order stochastic properties of both Young's modulus and mass density are:-

1. Triangular Correlation Model
2. Markov Model (or First-order Autoregressive Model)
3. Second-order Autoregressive Model
4. Gaussian Correlation Model
5. Finite Power White Noise Model

Triangular Correlation Model

The triangular correlation function decreases linearly from 1 to 0 as the lag distance increases from 0 to a, i.e.

$$\varsigma(\xi_1-\xi_2) = \begin{cases} 1 - \dfrac{|\xi_1-\xi_2|}{a} & for|\xi_1-\xi_2|<a, \\ 0 & for|\xi_1-\xi_2|\geq a \end{cases}$$

where a is a constant. $(\xi_1-\xi_2)$ is the lag distance and $\varsigma(\xi_1-\xi_2)$ is the correlation function. For the purpose of illustration, the correlation values of the triangular correlation model are plotted as a function of the lag distance in Fig. (5.6). The effects of constant a on the correlation properties are brought out for three values of a, viz. 10, 30 and 50.

Markov Model

The first-order autoregressive correlation model (commonly known as AR(1) of Box-Jenkins models) is given by

$$\varsigma(\xi_1-\xi_2) = e^{-|\xi_1-\xi_2|/b} \quad , \quad b = constant = f(\varepsilon)$$

where $(\xi_1-\xi_2)$ is the lag distance and $\varsigma(\xi_1-\xi_2)$ is the correlation function. It may be noted that this model can be interpreted as the Markov process representation of the random fields. The constant 1/b governs the shape of the correlation function and is related to the correlation length ε. Figure (5.7) shows a sketch of the first-order autoregressive model as a function of the lag distance for three values of the constant b viz., 10, 30 and 50. The correlation length is determined by the condition that the correlation is negligible for $|\xi_1 - \xi_2| > \varepsilon$. For instance, if the correlation at ε is reduced to 5% of that at $|\xi_1 - \xi_2| = 0$, the correlation function can be written as

$$\varsigma(\ |\xi_1 - \xi_2| = \varepsilon) = \varsigma(\varepsilon) = 0.05$$

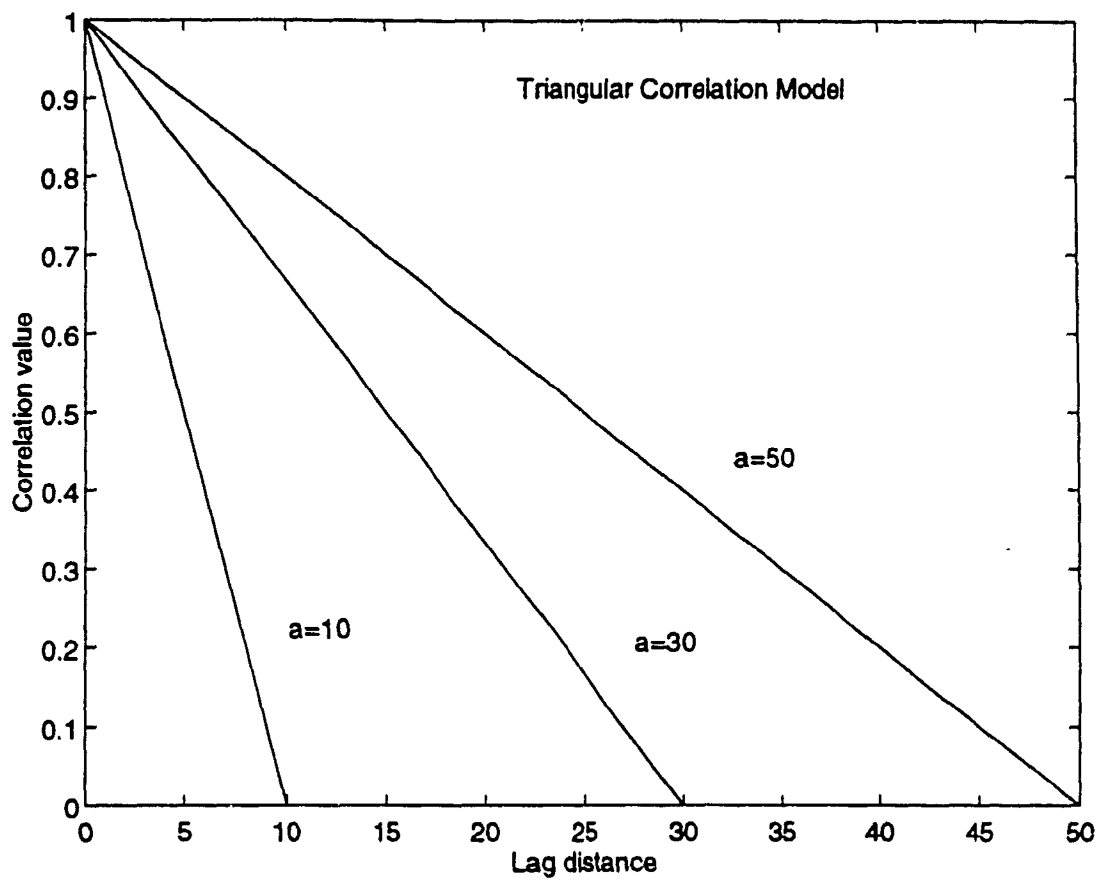

Fig. 5. 6: **The triangular correlation function for a = 10, 30 and 50.**

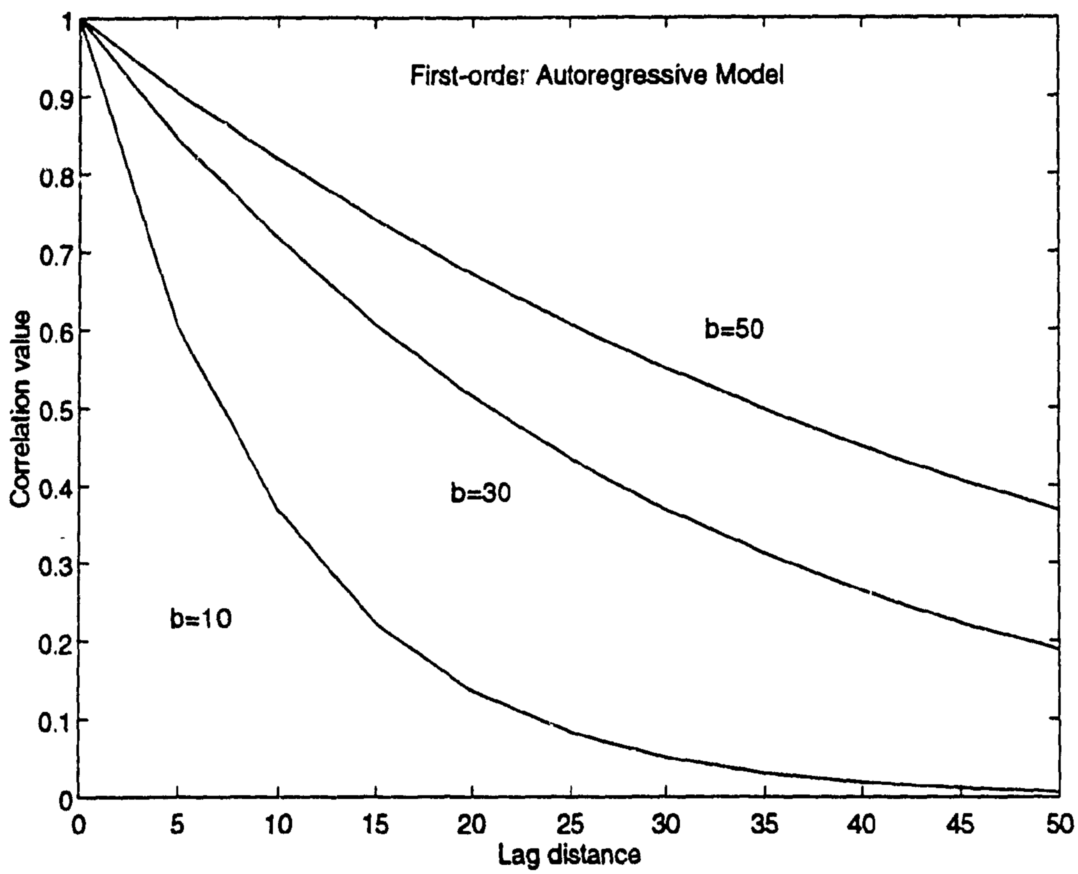

Fig. 5. 7: The first-order autoregressive function for b = 10, 30 and 50.

in which case we have $b = 0.3338\varepsilon$. This type of exponential correlation is widely used in practical applications.

Second-order Autoregressive Model

The correlation function $\varsigma(\xi_1-\xi_2)$ in the case of the second-order autoregressive correlation model is given by

$$\varsigma(\xi_1-\xi_2) = \left(1 + \frac{|\xi_1-\xi_2|}{c}\right) e^{-|\xi_1-\xi_2|/c} \quad , \quad c = constant$$

where $(\xi_1-\xi_2)$ is the lag distance. The correlation values of this model are shown in Fig. (5.8) as a function of the lag distance. The influence of the constant c on these correlation values is presented for c = 10, 30 and 50.

Gaussian Correlation Model

The squared exponential correlation model which is commonly known as the Gaussian correlation model is given by

$$\varsigma(\xi_1-\xi_2) = e^{-(|\xi_1-\xi_2|/d)^2} \quad , \quad d = constant$$

where $(\xi_1-\xi_2)$ is the lag distance and $\varsigma(\xi_1-\xi_2)$ is the correlation function. This model is illustrated in Fig. (5.9) for three values of the constant d, viz. 10, 30 and 50.

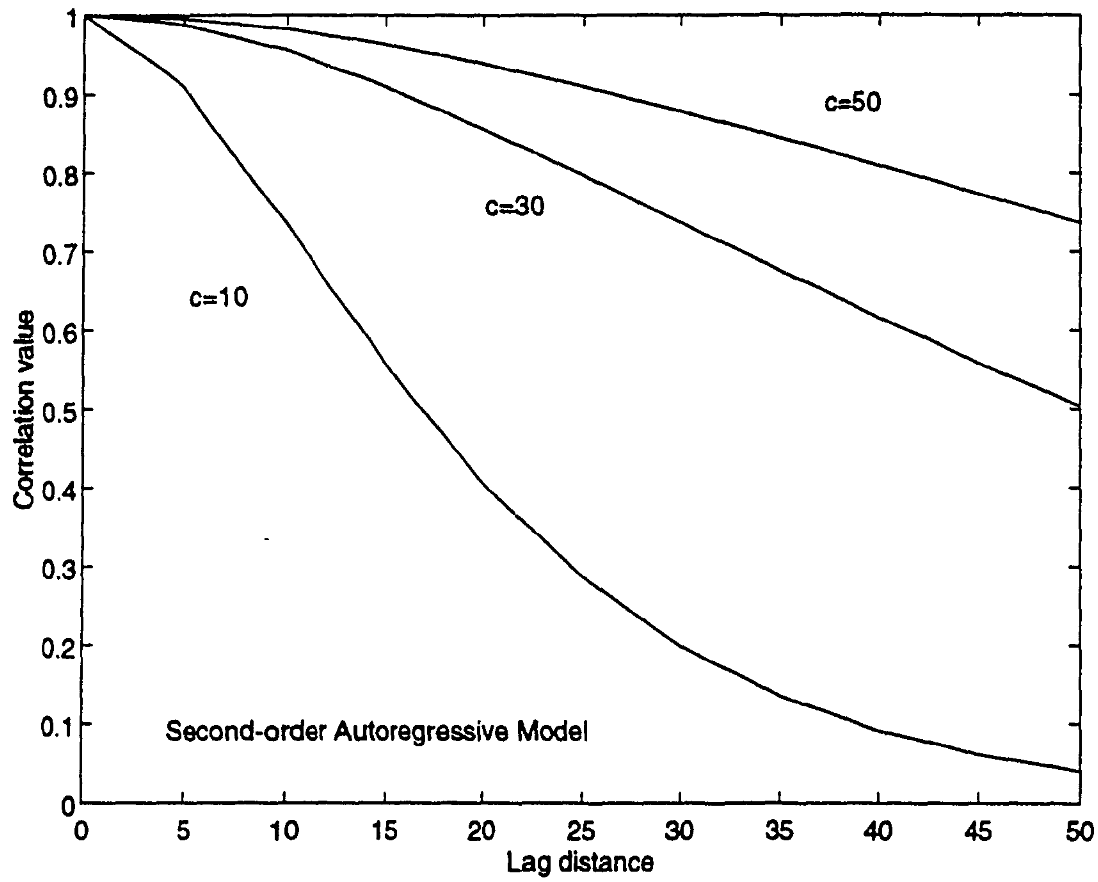

Fig. 5. 8: The second-order autoregressive function for c = 10, 30 and 50.

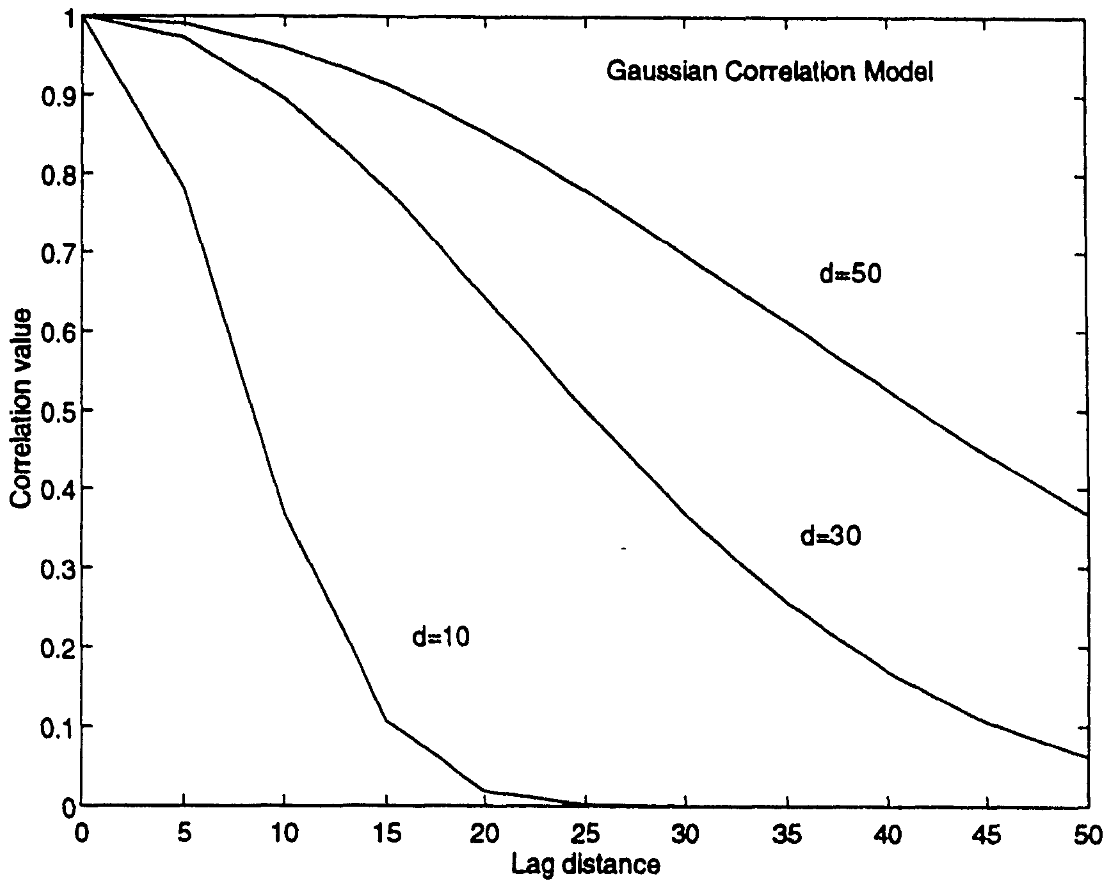

Fig. 5. 9: The Gaussian correlation function for d = 10, 30 and 50.

Finite Power White Noise Model

The sine function correlation model which represents the finite power white noise field is given by

$$\varsigma(\xi_1-\xi_2) = 2S_0\frac{\sin[f_u(\xi_1-\xi_2)]}{\xi_1-\xi_2}$$

where $(\xi_1-\xi_2)$ is the lag distance, $\varsigma(\xi_1-\xi_2)$ is the correlation function, S_0 is the strength of the white noise and f_u is the upper cut-off frequency of the power spectral density function which in turn is given by

$$S(f) = \begin{cases} S_0 = \dfrac{\sigma^2}{2\,f_u} & for|f|<f_u, \\ 0 & for|f|\geq f_u \end{cases}$$

In the above equation, σ^2 is the variance of the stochastic field. For the purpose of illustration, the finite power white noise model is sketched in Fig. (5.10) for $S_0 = 0.001$, 0.003 and 0.005.

5. 3 Software Development

The development of the algorithms for computing the covariance matrices of the eigenvalues (whirl speeds) has been carried out in several stages. As a first step, the flexural vibrations in the XZ plane of a beam comprising of N number of finite elements with translational and rotational degrees of freedom was considered. The mass and stiffness matrices for each of the finite elements were determined and assembled to yield the structural mass and stiffness matrices. The natural frequencies were determined based

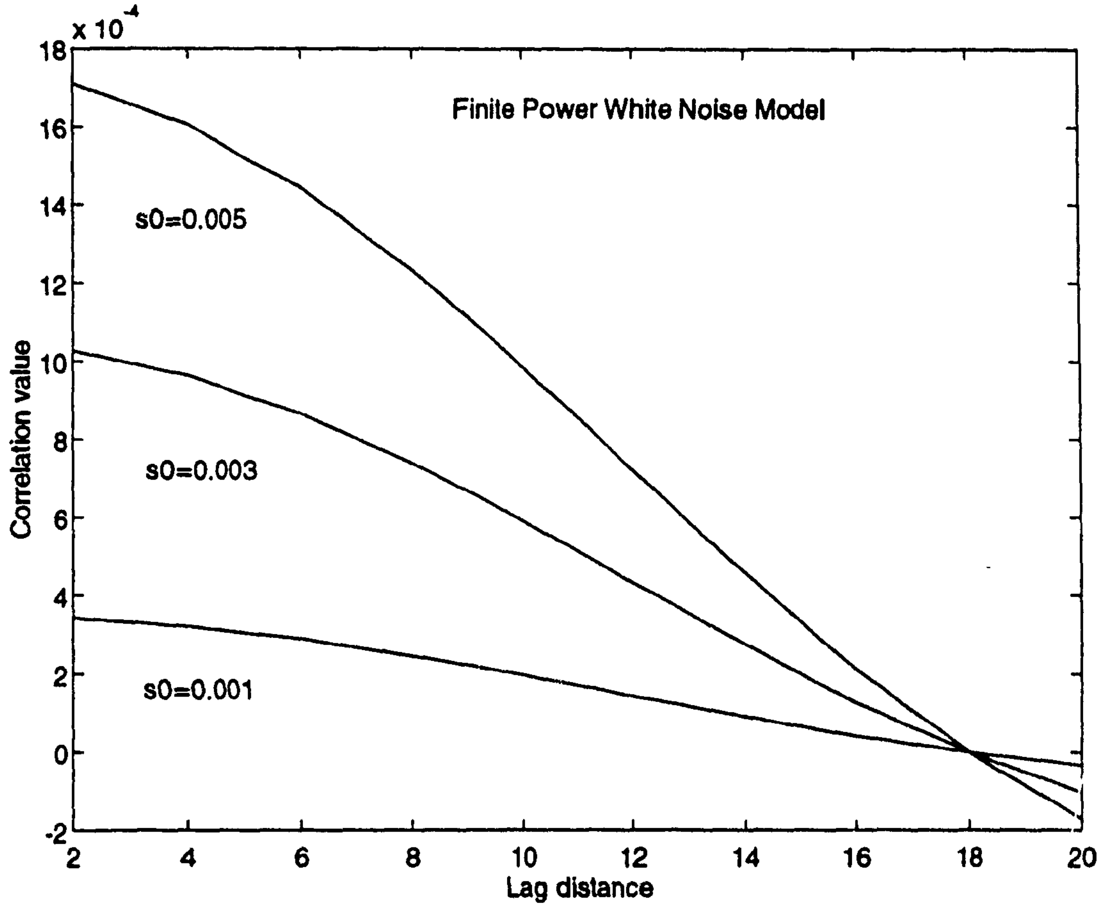

Fig. 5. 10: **The finite power white noise correlation function for S_0= 0.001, 0.003 and 0.005 ; f_s = 10.**

on this finite element model through an eigenvalue analysis. The eigenvalues and eigenvectors of the beam problem were obtained using the eigenvalue routine, that is available in MATLAB, using the structural mass and stiffness matrices. The eigenvalue routine available in MATLAB was later found to be not very robust for non self-adjoint eigenproblems and was replaced by an algorithm developed to handle any general matrix [Hatter, 1973]. The beam was modelled as 3, 4, 5 and 6 finite elements and was analyzed in each of these cases for its eigenvalues.

Since the rotordynamic system involves a coupled vibratory motion in the XYZ inertial frame, the beam considered above was extended so as to include the XY plane thus yielding a shaft. The translational and rotational degrees of freedom corresponding to both XY and YZ planes are included in the finite element model. Correspondingly, the mass and stiffness matrices were altered. The coupled motion of the rotordynamic system involves the gyroscopic matrices. These gyroscopic matrices of the shaft elements were developed [Nelson and McVaugh, 1976] and assembled to obtain the structural gyroscopic matrix. The gyroscopic matrices corresponding to the disks are formed and incorporated into the structural gyroscopic matrix. The non self-adjoint eigenproblem represented by Eq. (4.57) is then formulated. The whirl ratio was chosen as 1 in this computation.

The stochasticity in Young's Modulus and/or mass density causes variations in the whirl speeds which are quantified by the covariance matrix of whirl speeds. The relevant algorithm has been developed as follows:

$$[Cov(\lambda_i,\lambda_j)] = \begin{bmatrix}\Sigma\\ \lambda\end{bmatrix} = [\delta\lambda]\begin{bmatrix}\Sigma\\ km\end{bmatrix}[\delta\lambda]^T = [\delta\lambda]\{[\delta km]\begin{bmatrix}\Sigma\\ E\rho\end{bmatrix}[\delta km]^T\}[\delta\lambda]^T$$

$$= [\delta\lambda]\{[\delta km]\,([\delta E\rho]\begin{bmatrix}\Sigma\\ E\rho\end{bmatrix}[\delta E\rho]^T)\,[\delta km]^T\}[\delta\lambda]^T$$

$$[\delta\lambda] = \begin{bmatrix}\dfrac{\delta\lambda}{\delta k} & \Big| & \dfrac{\delta\lambda}{\delta m}\end{bmatrix}_{(n\times 2n^2)}$$

$$[\delta km] = \begin{bmatrix}\begin{bmatrix}\dfrac{\delta k}{\delta E}\end{bmatrix} & \begin{bmatrix}\dfrac{\delta k}{\delta\rho}\end{bmatrix}\\ \begin{bmatrix}\dfrac{\delta m}{\delta E}\end{bmatrix} & \begin{bmatrix}\dfrac{\delta m}{\delta\rho}\end{bmatrix}\end{bmatrix}_{(2n^2\times 2N)}$$

$$[\delta E\rho] = \begin{bmatrix}[SD_E] & [0]\\ [0] & [SD_\rho]\end{bmatrix}_{(2N\times 2N)}$$

SD ----→ Standard Deviation

$$\begin{bmatrix}\Sigma\\ E\rho\end{bmatrix} = \begin{bmatrix}[R_E] & [0]\\ [0] & [R_\rho]\end{bmatrix}_{(2N\times 2N)}$$

R ----→ Correlation Function

where n is the number of degrees of freedom of the shaft and N is the total number of finite elements that the shaft is divided into.

Using the structural stiffness matrix [K] and structural mass matrix [M] of the rotor system, the eigenvectors to the left {y} and that to the right {x} are computed. After matrices [K] and [M] are transformed into column vectors {k} and {m}, for the purposes of stochastic analysis, the sensitivity matrix [δλ] is computed using the following algorithm:

$$\frac{\delta\lambda_i}{\delta k_j}=\frac{\delta\lambda_i}{\delta K_{rs}}=\frac{x_{ri}y_{si}}{\{y\}_i^T[M]\{x\}_i}$$

$$\frac{\delta\lambda_i}{\delta m_j}=\frac{\delta\lambda_i}{\delta M_{rs}}=-\lambda_i\frac{\delta\lambda_i}{\delta K_{rs}}$$

$$i=r=s=1..n \;\; ; \;\; j=1..n^2$$

The matrix [δkm] can be assembled from {k} and {m}, i. e. the stiffness and mass matrices in the column vector form. The submatrices [δk/δρ] and [δm/δE] of [δkm] vanish since the stiffness matrix is not a function of mass density and the mass matrix is not a function of Young's modulus. The matrix [δEρ] consists of two submatrices that characterize the standard deviation of E and/or ρ corresponding to each of the finite elements.

5. 4 Parametric Study

A parametric study has been conducted based on the rotor-bearing system described in section 5. 2, with the following objectives:

(i) to quantify in the amplitude domain the effects of stochastic distributions of Young's modulus and mass density of the shaft material, on the stochastic properties of whirl speeds of the rotor-bearing system.

(ii) to distinguish between the effects of stochastic distribution of mass density and that of stochastic distribution of Young's modulus.

(iii) to quantify the effects of frequency-domain stochastic properties of material property stochastic fields on the amplitude-domain stochastic properties of whirl speeds of the rotor-bearing system.

(iv) to quantify the effects of elastic bearing supports on the stochastic properties of the whirl speeds of the rotor-bearing system.

(v) influence of finite element modelling on the calculated stochastic properties of whirl speeds.

5. 4. 1 Mean Values and Covariances of Whirl Speeds

In order to demonstrate the variability in whirl speeds, the mean values and complete covariance matrices have been determined using the finite element model shown in Fig. (5.2) when,

(i) the Young's modulus has a stochastic distribution,

(ii) the stochastic field representing the fluctuations of Young's modulus has a triangular correlation structure,

(iii) the rotor system is supported on rigid bearings.

The mean values of the whirl speeds are given in Table (5.2) and plotted in Fig. (5.11) for the various whirl modes of the rotor system. The complete covariance matrix of the whirl speeds is given in Table (5.3).

Table 5.2: Mean values of whirl speeds of the rotor-bearing system
E is random ; Rigid bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0444*10^4$
2	$0.0629*10^4$
3	$0.1609*10^4$
4	$0.2275*10^4$
5	$0.3707*10^4$
6	$0.5246*10^4$
7	$0.5438*10^4$
8	$0.7485*10^4$
9	$0.7691*10^4$
10	$1.0588*10^4$
11	$1.8013*10^4$
12	$2.5548*10^4$
13	$2.8861*10^4$
14	$4.0754*10^4$
15	$6.5102*10^4$
16	$9.1323*10^4$

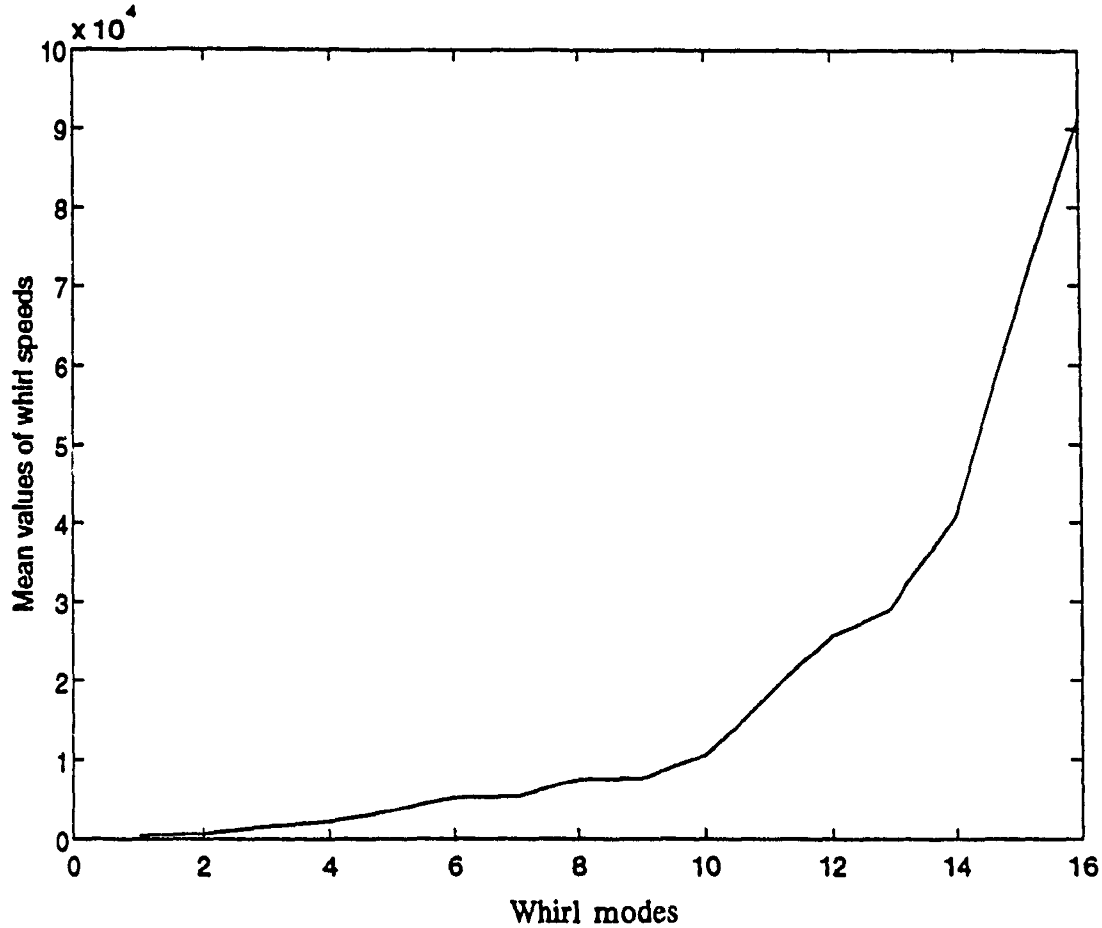

Fig. 5. 11: **Mean values of whirl speeds of the rotor system on rigid bearings; E is random and has a triangular correlation structure; Number of finite elements = 4.**

Table 5.3: Covariance matrix of whirl speeds when the rotor system is mounted on rigid bearings and E is random

$[Cov(\lambda_i , \lambda_j)] = 10^6 *$

2.1359	1.0662	1.0333	0.9711	0.8546	0.1956	1.7220	0.9290
1.0662	0.5322	0.5158	0.4847	0.4266	0.0976	0.8596	0.4637
1.0333	0.5158	0.4999	0.4698	0.4134	0.0946	0.8330	0.4494
0.9711	0.4847	0.4698	0.4415	0.3885	0.0889	0.7828	0.4223
0.8546	0.4266	0.4134	0.3885	0.3420	0.0783	0.6890	0.3718
0.1956	0.0976	0.0946	0.0889	0.0783	0.0183	0.1583	0.0859
1.7220	0.8596	0.8330	0.7828	0.6890	0.1583	1.3892	0.7503
0.9290	0.4637	0.4494	0.4223	0.3718	0.0859	0.7503	0.4059
0.9081	0.4533	0.4393	0.4128	0.3634	0.0835	0.7326	0.3957
0.4684	0.2338	0.2266	0.2129	0.1874	0.0433	0.3783	0.2046
0.8828	0.4407	0.4271	0.4013	0.3533	0.0817	0.7132	0.3860
0.9332	0.4658	0.4515	0.4243	0.3734	0.0857	0.7528	0.4065
0.2482	0.1239	0.1201	0.1128	0.0994	0.0235	0.2014	0.1097
0.0991	0.0494	0.0479	0.0450	0.0396	0.0093	0.0803	0.0436
0.1428	0.0713	0.0691	0.0649	0.0572	0.0135	0.1159	0.0631
0.1004	0.0501	0.0486	0.0456	0.0402	0.0094	0.0812	0.0441

0.9081	0.4684	0.8828	0.9332	0.2482	0.0991	0.1428	0.1004
0.4533	0.2338	0.4407	0.4658	0.1239	0.0494	0.0713	0.0501
0.4393	0.2266	0.4271	0.4515	0.1201	0.0479	0.0691	0.0486
0.4128	0.2129	0.4013	0.4243	0.1128	0.0450	0.0649	0.0456
0.3634	0.1874	0.3533	0.3734	0.0994	0.0396	0.0572	0.0402
0.0835	0.0433	0.0817	0.0857	0.0235	0.0093	0.0135	0.0094
0.7326	0.3783	0.7132	0.7528	0.2014	0.0803	0.1159	0.0812
0.3957	0.2046	0.3860	0.4065	0.1097	0.0436	0.0631	0.0441
0.3864	0.1995	0.3761	0.3970	0.1062	0.0423	0.0611	0.0428
0.1995	0.1032	0.1946	0.2050	0.0553	0.0220	0.0318	0.0222
0.3761	0.1946	0.3673	0.3864	0.1048	0.0416	0.0602	0.0419
0.3970	0.2050	0.3864	0.4080	0.1090	0.0435	0.0627	0.0440
0.1062	0.0553	0.1048	0.1090	0.0309	0.0121	0.0177	0.0120
0.0423	0.0220	0.0416	0.0435	0.0121	0.0048	0.0069	0.0048
0.0611	0.0318	0.0602	0.0627	0.0177	0.0069	0.0101	0.0069
0.0428	0.0222	0.0419	0.0440	0.0120	0.0048	0.0069	0.0048

Now, the mass density is considered to have a stochastic distribution and the mean values as well as the complete covariance matrix of whirl speeds are given in Tables (5.4) and (5.5) respectively. As can be seen from these tables,

(i) the mean values of whirl speeds are not changed, and

(ii) the values of covariances of whirl speeds are considerably larger than the covariances of whirl speeds when the Young's modulus is random.

Both the Young's modulus and mass density are now considered to have a stochastic variation and an analysis similar to the above analyses is conducted. The mean values as well as the covariances of the whirl speeds are given in Tables (5.6) and (5.7) respectively. The values of covariances of whirl speeds are seen to be closer to the values of covariances when the mass density is random while the mean values of the whirl speeds remain unchanged. The elements of the covariance matrices in Tables (5.3), (5.5) and (5.7) are real and positive. The matrices exhibit symmetricity and their diagonal elements, representing the variances of the eigenvalues, show a decreasing trend. It may thus be noted that the variance of the eigenvalue decreases with increasing eigenvalues.

5. 4. 2 Effects of Bearing Flexibility on the Mean Values and Covariances of Whirl Speeds

The two ends of the rotor-bearing system are now considered as being mounted on isotropic flexible bearings. The influence of isotropic bearings on the covariance matrix of whirl speeds is observed. The mean values and covariances of the whirl speeds are determined when

Table 5.4: Mean values of whirl speeds of the rotor-bearing system
m is random ; Rigid bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0444*10^4$
2	$0.0629*10^4$
3	$0.1609*10^4$
4	$0.2275*10^4$
5	$0.3707*10^4$
6	$0.5246*10^4$
7	$0.5438*10^4$
8	$0.7485*10^4$
9	$0.7691*10^4$
10	$1.0588*10^4$
11	$1.8013*10^4$
12	$2.5548*10^4$
13	$2.8861*10^4$
14	$4.0754*10^4$
15	$6.5102*10^4$
16	$9.1323*10^4$

Table 5.5: Covariance matrix of whirl speeds when the rotor system is mounted on rigid bearings and m is random

$[Cov(\lambda_i, \lambda_j)] = 10^9 *$

6.8693	3.4909	1.3678	0.6856	0.5359	0.2621	0.0916	0.0480
3.4909	1.7740	0.6951	0.3484	0.2723	0.1332	0.0465	0.0244
1.3678	0.6951	0.2724	0.1365	0.1067	0.0522	0.0182	0.0096
0.6856	0.3484	0.1365	0.0684	0.0535	0.0262	0.0091	0.0048
0.5359	0.2723	0.1067	0.0535	0.0418	0.0206	0.0072	0.0038
0.2621	0.1332	0.0522	0.0262	0.0206	0.0103	0.0035	0.0019
0.0916	0.0465	0.0182	0.0091	0.0072	0.0035	0.0012	0.0006
0.0480	0.0244	0.0096	0.0048	0.0038	0.0019	0.0006	0.0003
0.0458	0.0233	0.0091	0.0046	0.0036	0.0018	0.0006	0.0003
0.0240	0.0122	0.0048	0.0024	0.0019	0.0009	0.0003	0.0002
0.0219	0.0111	0.0044	0.0022	0.0017	0.0009	0.0003	0.0002
0.0112	0.0057	0.0022	0.0011	0.0009	0.0004	0.0002	0.0001
0.0041	0.0021	0.0008	0.0004	0.0003	0.0002	0.0001	0.0000
0.0021	0.0011	0.0004	0.0002	0.0002	0.0001	0.0000	0.0000
0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

0.0458	0.0240	0.0219	0.0112	0.0041	0.0021	0.0003	0.0002
0.0233	0.0122	0.0111	0.0057	0.0021	0.0011	0.0002	0.0001
0.0091	0.0048	0.0044	0.0022	0.0008	0.0004	0.0001	0.0000
0.0046	0.0024	0.0022	0.0011	0.0004	0.0002	0.0000	0.0000
0.0036	0.0019	0.0017	0.0009	0.0003	0.0002	0.0000	0.0000
0.0018	0.0009	0.0009	0.0004	0.0002	0.0001	0.0000	0.0000
0.0006	0.0003	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
0.0003	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.6: Mean values of whirl speeds of the rotor-bearing system
E and m are random ; Rigid bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0444*10^4$
2	$0.0629*10^4$
3	$0.1609*10^4$
4	$0.2275*10^4$
5	$0.3707*10^4$
6	$0.5246*10^4$
7	$0.5438*10^4$
8	$0.7485*10^4$
9	$0.7691*10^4$
10	$1.0588*10^4$
11	$1.8013*10^4$
12	$2.5548*10^4$
13	$2.8861*10^4$
14	$4.0754*10^4$
15	$6.5102*10^4$
16	$9.1323*10^4$

Table 5.7: Covariance matrix of whirl speeds when the rotor system is mounted on rigid bearings and both E and m are random

$$[\mathrm{Cov}\,(\lambda_i\,,\,\lambda_j)] = 10^9 *$$

6.8714	3.4919	1.3688	0.6866	0.5368	0.2623	0.0933	0.0489
3.4919	1.7746	0.6956	0.3489	0.2728	0.1333	0.0474	0.0248
1.3688	0.6956	0.2729	0.1370	0.1071	0.0523	0.0191	0.0100
0.6866	0.3489	0.1370	0.0689	0.0539	0.0263	0.0099	0.0052
0.5368	0.2728	0.1071	0.0539	0.0422	0.0206	0.0078	0.0041
0.2623	0.1333	0.0523	0.0263	0.0206	0.0103	0.0037	0.0020
0.0933	0.0474	0.0191	0.0099	0.0078	0.0037	0.0026	0.0014
0.0489	0.0248	0.0100	0.0052	0.0041	0.0020	0.0014	0.0007
0.0467	0.0237	0.0096	0.0050	0.0039	0.0019	0.0013	0.0007
0.0244	0.0124	0.0050	0.0026	0.0021	0.0010	0.0007	0.0004
0.0228	0.0116	0.0048	0.0026	0.0021	0.0009	0.0010	0.0005
0.0122	0.0062	0.0027	0.0015	0.0013	0.0005	0.0009	0.0005
0.0043	0.0022	0.0009	0.0005	0.0004	0.0002	0.0003	0.0001
0.0022	0.0011	0.0005	0.0003	0.0002	0.0001	0.0001	0.0001
0.0005	0.0002	0.0001	0.0001	0.0001	0.0000	0.0001	0.0001
0.0003	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001	0.0000

0.0467	0.0244	0.0228	0.0122	0.0043	0.0022	0.0005	0.0003
0.0237	0.0124	0.0116	0.0062	0.0022	0.0011	0.0002	0.0001
0.0096	0.0050	0.0048	0.0027	0.0009	0.0005	0.0001	0.0001
0.0050	0.0026	0.0026	0.0015	0.0005	0.0003	0.0001	0.0001
0.0039	0.0021	0.0021	0.0013	0.0004	0.0002	0.0001	0.0001
0.0019	0.0010	0.0009	0.0005	0.0002	0.0001	0.0000	0.0000
0.0013	0.0007	0.0010	0.0009	0.0003	0.0001	0.0001	0.0001
0.0007	0.0004	0.0005	0.0005	0.0001	0.0001	0.0001	0.0000
0.0007	0.0004	0.0005	0.0005	0.0001	0.0001	0.0001	0.0000
0.0004	0.0002	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000
0.0005	0.0003	0.0004	0.0004	0.0001	0.0000	0.0001	0.0000
0.0005	0.0002	0.0004	0.0004	0.0001	0.0000	0.0001	0.0000
0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(i) the Young's modulus has a stochastic distribution and

(ii) the stochastic field representing the fluctuations of Young's modulus has a triangular correlation structure

Table (5.8) shows the mean values of the whirlspeeds and Table (5.9) contains the covariance matrix of the whirl speeds. Figure (5.12) shows the mean values of the whirl speeds corresponding to different degrees of freedom of the rotor system supported on isotropic bearings. A comparison of Tables (5.8) and (5.9) with Tables (5.2) and (5.3) indicates that the mean values of whirl speeds and their covariances are higher when the rotor system is mounted on elastic supports.

A similar study is carried out with the mass density considered to have a stochastic distribution. The mean values of the whirl speeds are given in Table (5.10) and, as can be seen, they are same as the mean values of the whirl speeds in Table (5.8), corresponding to the case when the Young's modulus has a stochastic distribution. The covariances of the whirl speeds presented in Table (5.11) are, however, significantly higher than the covariances of the whirl speeds when the Young's modulus is random.

When the Young's modulus and the mass density are both random, the mean values and covariances of the whirl speeds are computed and presented in Tables (5.12) and (5.13) respectively. The mean values of the whirl speeds are not affected, as can be seen by comparing Table (5.12) with Tables (5.8) and (5.10). Table (5.13) indicates that the covariances of the whirl speeds are fairly close to the covariances given in Table (5.11) for the case when the mass density alone is random and are much larger than the

Table 5.8: Mean values of whirl speeds of the rotor-bearing system
E is random ; Isotropic bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0067*10^5$
2	$0.0076*10^5$
3	$0.0084*10^5$
4	$0.0089*10^5$
5	$0.0148*10^5$
6	$0.0182*10^5$
7	$0.0290*10^5$
8	$0.0407*10^5$
9	$0.0516*10^5$
10	$0.0729*10^5$
11	$0.0731*10^5$
12	$0.0980*10^5$
13	$0.1034*10^5$
14	$0.1378*10^5$
15	$0.2187*10^5$
16	$0.3112*10^5$
17	$0.4254*10^5$
18	$0.6005*10^5$
19	$0.8539*10^5$
20	$1.1957*10^5$

Table 5.9: Covariance matrix of whirl speeds when the rotor system is mounted on isotropic bearings and E is random

$[Cov(\lambda_i, \lambda_j)] = 10^7 *$

2.0658	1.2299	3.6683	3.4508	1.1891	0.9014	0.7620	0.2574	0.3883	0.2035
1.2299	0.7323	2.1845	2.0549	0.7080	0.5366	0.4535	0.1531	0.2311	0.1211
3.6683	2.1845	6.5216	6.1342	2.1128	1.6002	1.3509	0.4549	0.6883	0.3602
3.4508	2.0549	6.1342	5.7698	1.9874	1.5054	1.2710	0.4281	0.6476	0.3389
1.1891	0.7080	2.1128	1.9874	0.6847	0.5188	0.4382	0.1478	0.2233	0.1170
0.9014	0.5366	1.6002	1.5054	0.5188	0.3934	0.3327	0.1128	0.1695	0.0891
0.7620	0.4535	1.3509	1.2710	0.4382	0.3327	0.2817	0.0958	0.1435	0.0756
0.2574	0.1531	0.4549	0.4281	0.1478	0.1128	0.0958	0.0346	0.0488	0.0268
0.3883	0.2311	0.6883	0.6476	0.2233	0.1695	0.1435	0.0488	0.0731	0.0385
0.2035	0.1211	0.3602	0.3389	0.1170	0.0891	0.0756	0.0268	0.0385	0.0209
0.2056	0.1223	0.3638	0.3423	0.1181	0.0900	0.0764	0.0270	0.0389	0.0211
0.1868	0.1111	0.3301	0.3107	0.1073	0.0817	0.0694	0.0244	0.0354	0.0190
0.1146	0.0681	0.2023	0.1904	0.0658	0.0502	0.0427	0.0154	0.0218	0.0120
0.2181	0.1297	0.3858	0.3631	0.1253	0.0953	0.0809	0.0281	0.0412	0.0220
0.1490	0.0886	0.2633	0.2478	0.0855	0.0651	0.0553	0.0192	0.0282	0.0151
0.1082	0.0644	0.1912	0.1799	0.0621	0.0473	0.0402	0.0141	0.0205	0.0110
0.0616	0.0367	0.1089	0.1025	0.0354	0.0270	0.0229	0.0081	0.0117	0.0063
0.0640	0.0381	0.1131	0.1064	0.0367	0.0280	0.0238	0.0084	0.0121	0.0065
0.0539	0.0321	0.0953	0.0897	0.0310	0.0236	0.0201	0.0071	0.0102	0.0055
0.0466	0.0277	0.0822	0.0774	0.0267	0.0204	0.0174	0.0063	0.0089	0.0049

0.2056	0.1868	0.1146	0.2181	0.1490	0.1082	0.0616	0.0640	0.0539	0.0466
0.1223	0.1111	0.0681	0.1297	0.0886	0.0644	0.0367	0.0381	0.0321	0.0277
0.3638	0.3301	0.2023	0.3858	0.2633	0.1912	0.1089	0.1131	0.0953	0.0822
0.3423	0.3107	0.1904	0.3631	0.2478	0.1799	0.1025	0.1064	0.0897	0.0774
0.1181	0.1073	0.0658	0.1253	0.0855	0.0621	0.0354	0.0367	0.0310	0.0267
0.0900	0.0817	0.0502	0.0953	0.0651	0.0473	0.0270	0.0280	0.0236	0.0204
0.0764	0.0694	0.0427	0.0809	0.0553	0.0402	0.0229	0.0238	0.0201	0.0174
0.0270	0.0244	0.0154	0.0281	0.0192	0.0141	0.0081	0.0084	0.0071	0.0063
0.0389	0.0354	0.0218	0.0412	0.0282	0.0205	0.0117	0.0121	0.0102	0.0089
0.0211	0.0190	0.0120	0.0220	0.0151	0.0110	0.0063	0.0065	0.0055	0.0049
0.0213	0.0192	0.0121	0.0222	0.0152	0.0111	0.0064	0.0066	0.0056	0.0049
0.0192	0.0174	0.0109	0.0202	0.0138	0.0101	0.0058	0.0060	0.0051	0.0045
0.0121	0.0109	0.0069	0.0126	0.0086	0.0063	0.0036	0.0037	0.0032	0.0028
0.0222	0.0202	0.0126	0.0234	0.0160	0.0117	0.0067	0.0069	0.0058	0.0051
0.0152	0.0138	0.0086	0.0160	0.0110	0.0080	0.0046	0.0047	0.0040	0.0035
0.0111	0.0101	0.0063	0.0117	0.0080	0.0059	0.0033	0.0035	0.0029	0.0026
0.0064	0.0058	0.0036	0.0067	0.0046	0.0033	0.0019	0.0020	0.0017	0.0015
0.0066	0.0060	0.0037	0.0069	0.0047	0.0035	0.0020	0.0021	0.0017	0.0015
0.0056	0.0051	0.0032	0.0058	0.0040	0.0029	0.0017	0.0017	0.0015	0.0013
0.0049	0.0045	0.0028	0.0051	0.0035	0.0026	0.0015	0.0015	0.0013	0.0012

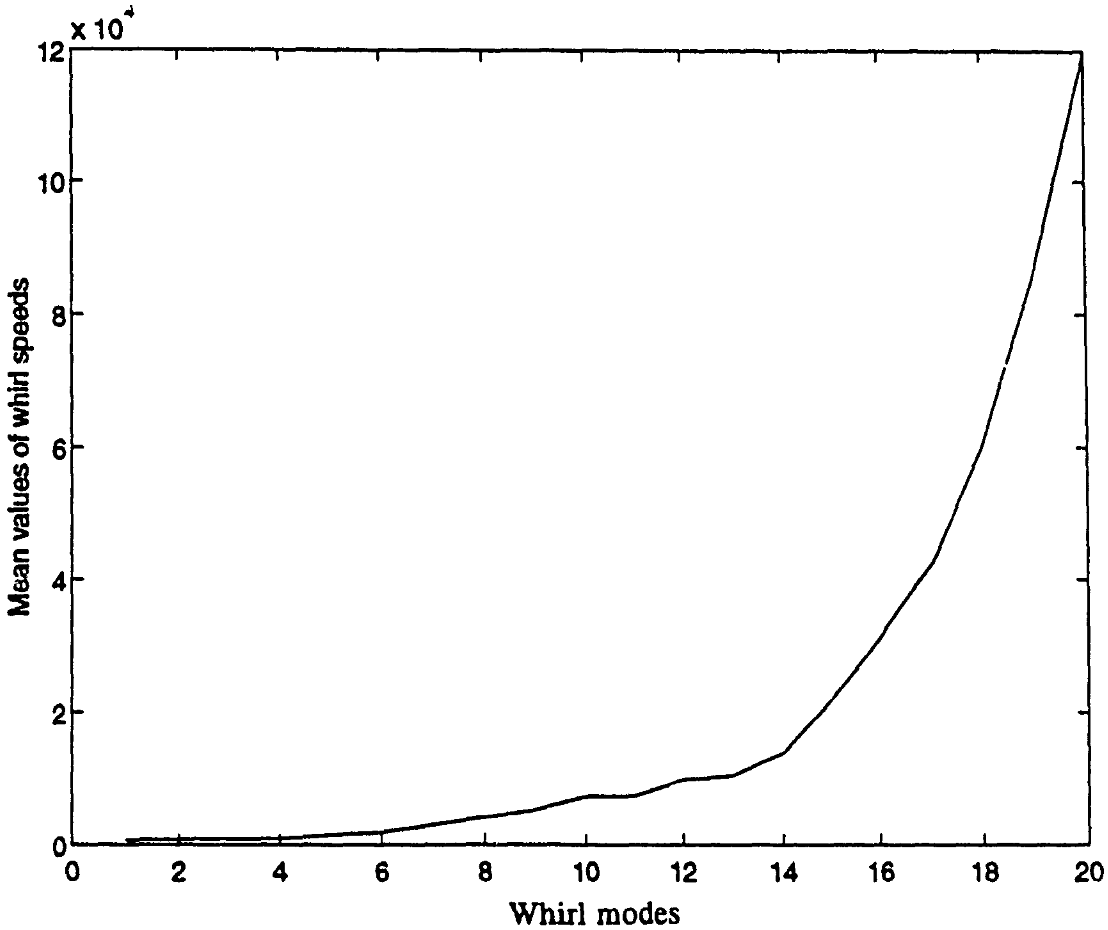

Fig. 5. 12: **Mean values of whirl speeds of the rotor system mounted on isotropic bearings; E is random and has a triangular correlation structure; Number of finite elements = 4.**

Table 5.10: Mean values of whirl speeds of the rotor-bearing system
m is random ; Isotropic bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0067*10^5$
2	$0.0076*10^5$
3	$0.0084*10^5$
4	$0.0089*10^5$
5	$0.0148*10^5$
6	$0.0182*10^5$
7	$0.0290*10^5$
8	$0.0407*10^5$
9	$0.0516*10^5$
10	$0.0729*10^5$
11	$0.0731*10^5$
12	$0.0980*10^5$
13	$0.1034*10^5$
14	$0.1378*10^5$
15	$0.2187*10^5$
16	$0.3112*10^5$
17	$0.4254*10^5$
18	$0.6005*10^5$
19	$0.8539*10^5$
20	$1.1957*10^5$

Table 5.11: Covariance matrix of whirl speeds when the rotor system is mounted on isotropic bearings and m is random

$[Cov(\lambda_i, \lambda_j)] = 10^{10} *$

2.0190	1.0297	0.5096	0.2556	0.1350	0.0665	0.0266	0.0144	0.0134	0.0072
1.0297	0.5252	0.2600	0.1304	0.0688	0.0339	0.0135	0.0073	0.0068	0.0036
0.5096	0.2600	0.1293	0.0648	0.0339	0.0167	0.0067	0.0036	0.0034	0.0018
0.2556	0.1304	0.0648	0.0324	0.0170	0.0084	0.0034	0.0018	0.0017	0.0009
0.1350	0.0688	0.0339	0.0170	0.0091	0.0045	0.0018	0.0010	0.0009	0.0005
0.0665	0.0339	0.0167	0.0084	0.0045	0.0022	0.0009	0.0005	0.0004	0.0002
0.0266	0.0135	0.0067	0.0034	0.0018	0.0009	0.0004	0.0002	0.0002	0.0001
0.0144	0.0073	0.0036	0.0018	0.0010	0.0005	0.0002	0.0001	0.0001	0.0001
0.0134	0.0068	0.0034	0.0017	0.0009	0.0004	0.0002	0.0001	0.0001	0.0000
0.0072	0.0036	0.0018	0.0009	0.0005	0.0002	0.0001	0.0001	0.0000	0.0000
0.0071	0.0036	0.0018	0.0009	0.0005	0.0002	0.0001	0.0001	0.0000	0.0000
0.0036	0.0019	0.0009	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
0.0022	0.0011	0.0006	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
0.0011	0.0006	0.0003	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0005	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

0.0071	0.0036	0.0022	0.0011	0.0005	0.0003	0.0001	0.0001	0.0001	0.0001
0.0036	0.0019	0.0011	0.0006	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000
0.0018	0.0009	0.0006	0.0003	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0009	0.0005	0.0003	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0005	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.12: Mean values of whirl speeds of the rotor-bearing system
E and m are random ; Isotropic bearings ; Number of finite elements = 4

Whirl Modes	Mean values of whirl speeds (rad. / s)
1	$0.0067*10^5$
2	$0.0076*10^5$
3	$0.0084*10^5$
4	$0.0089*10^5$
5	$0.0148*10^5$
6	$0.0182*10^5$
7	$0.0290*10^5$
8	$0.0407*10^5$
9	$0.0516*10^5$
10	$0.0729*10^5$
11	$0.0731*10^5$
12	$0.0980*10^5$
13	$0.1034*10^5$
14	$0.1378*10^5$
15	$0.2187*10^5$
16	$0.3112*10^5$
17	$0.4254*10^5$
18	$0.6005*10^5$
19	$0.8539*10^5$
20	$1.1957*10^5$

Table 5.13: Covariance matrix of whirl speeds when the rotor system is mounted on isotropic bearings and both E and m are random

$[\mathrm{Cov}(\lambda_i, \lambda_j)] = 10^{10} *$

2.0210	1.0309	0.5133	0.2590	0.1362	0.0674	0.0273	0.0146	0.0138	0.0074
1.0309	0.5259	0.2622	0.1324	0.0695	0.0344	0.0140	0.0075	0.0071	0.0038
0.5133	0.2622	0.1358	0.0709	0.0360	0.0183	0.0080	0.0041	0.0041	0.0021
0.2590	0.1324	0.0709	0.0382	0.0190	0.0099	0.0046	0.0022	0.0023	0.0012
0.1362	0.0695	0.0360	0.0190	0.0098	0.0050	0.0022	0.0011	0.0011	0.0006
0.0674	0.0344	0.0183	0.0099	0.0050	0.0026	0.0012	0.0006	0.0006	0.0003
0.0273	0.0140	0.0080	0.0046	0.0022	0.0012	0.0006	0.0003	0.0003	0.0002
0.0146	0.0075	0.0041	0.0022	0.0011	0.0006	0.0003	0.0001	0.0001	0.0001
0.0138	0.0071	0.0041	0.0023	0.0011	0.0006	0.0003	0.0001	0.0002	0.0001
0.0074	0.0038	0.0021	0.0012	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
0.0073	0.0038	0.0021	0.0012	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
0.0038	0.0020	0.0012	0.0008	0.0004	0.0002	0.0001	0.0001	0.0001	0.0000
0.0024	0.0012	0.0008	0.0005	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
0.0014	0.0007	0.0007	0.0005	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
0.0006	0.0003	0.0004	0.0003	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
0.0004	0.0002	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

0.0073	0.0038	0.0024	0.0014	0.0006	0.0004	0.0002	0.0002	0.0001	0.0001
0.0038	0.0020	0.0012	0.0007	0.0003	0.0002	0.0001	0.0001	0.0001	0.0001
0.0021	0.0012	0.0008	0.0007	0.0004	0.0003	0.0001	0.0001	0.0001	0.0001
0.0012	0.0008	0.0005	0.0005	0.0003	0.0002	0.0001	0.0001	0.0001	0.0001
0.0006	0.0004	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0003	0.0002	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.0002	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

covariances given in Table (5.9) corresponding to the case when only the Young's modulus has a stochastic distribution. The elements of the covariance matrices in Tables (5.9), (5.11) and (5.13) exhibit symmetricity and their elements are real and positive as in the case of Tables (5.3), (5.5) and (5.7). The diagonal elements of the matrices in Tables (5.11) and (5.13) show a decreasing trend, while the diagonal elements of the matrix in Table (5.9), corresponding to random Young's modulus, do not exhibit such a decreasing trend.

5. 4. 3 Effects of the Correlation Structure

The following case studies are performed so as to compute the effects of the correlation structure of material properties on the covariance matrix of whirl speeds for the rotordynamic system described in section 5. 2:-

Case 1. Rotor-Bearing System on Rigid Bearings

Case 2. Influence of Bearing Flexibility

Case 3. Influence of Finite Element Modelling

In each of these cases, after computing the covariance matrix of the whirl speeds, the sensitivity of the variance of the first whirl speed ω_1 to the stochastic fluctuations in the material properties is analyzed in detail. The effects of the various correlation models, discussed in Section 5. 2. 3, on the variance of the first whirl speed is also studied. A more standardized measure used in the literature is the coefficient of variation and is adopted in this work. The coefficient of variation of a whirl speed is defined as the ratio of the square root of the variance of that whirl speed to the mean value of the whirl speed. The coefficient of variation of the first eigenvalue, which is essentially the coefficient of variation of ω_1^2 is computed throughout this study. The coefficient of variation is normalized and the effects of the correlation structure of the material properties on this coefficient are presented graphically in each of the case studies.

Case 1. Rotor-Bearing System on Rigid Bearings

The coefficient of variation of ω_1^2 for the case of the rotor mounted on rigid bearings is determined when the Young's modulus is random and the number of finite elements is fixed. The standard deviation of the stochastic field that models the fluctuations about the mean of the Young's modulus is varied. For each value of the standard deviation, the five correlation models detailed in Section 5. 2. 3 are considered. Within each correlation structure the effect of varying the lag distance on the coefficient of variation of the first whirl speed is studied. The change in the coefficient of variation of ω_1^2 with a corresponding change in the lag distance, when the stochastic variations in Young's modulus are considered, for 5 different values of the standard deviation is shown in Figs. (5.13-5.17) corresponding to the 5 correlation models. In Figs. (5.13) and (5.14), corresponding to the cases of the Triangular and First-order Autoregressive correlation models, the coefficient of variation of ω_1^2 varies in a non-linear manner with increasing values of the lag distances. Figs. (5.15), (5.16) and (5.17) show that the sensitivity of the first whirl speed to the lag distances of the Second-order Autoregressive, Gaussian and Finite Power White Noise correlation models is not significant.

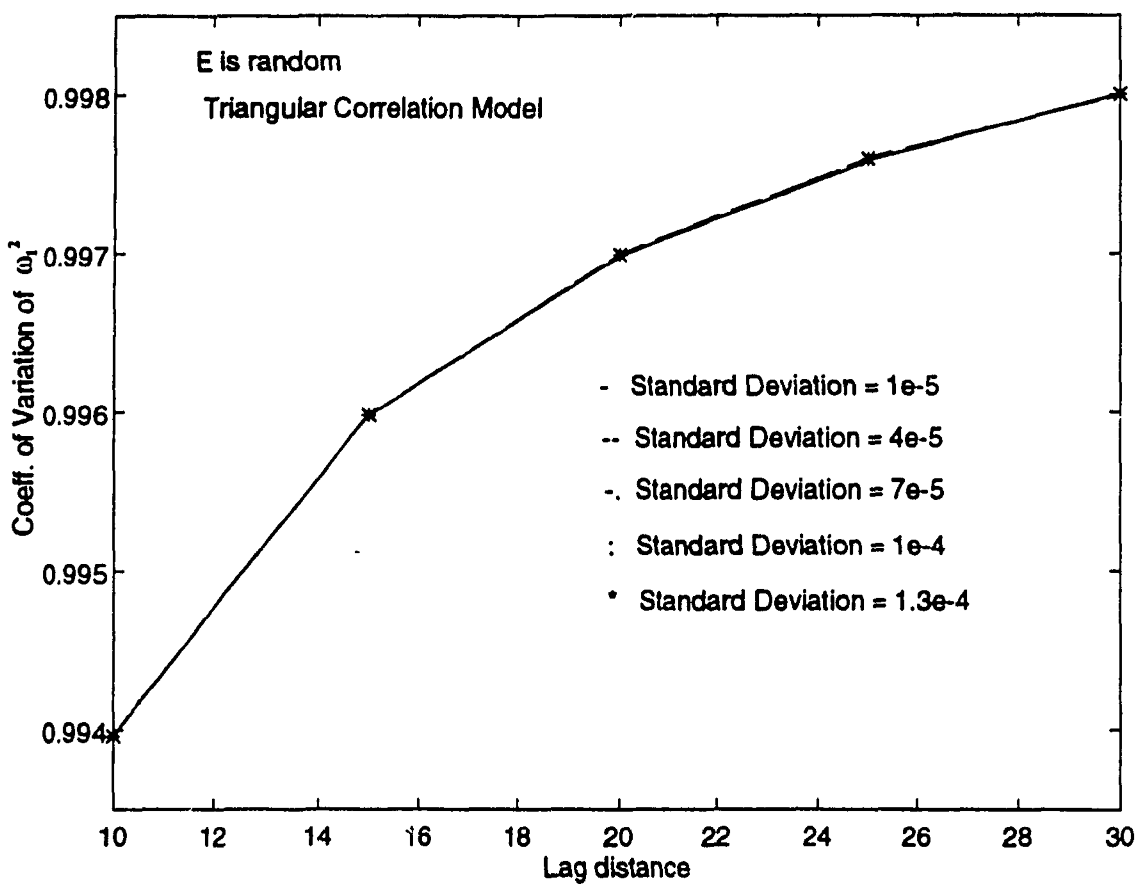

Fig. 5. 13: Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random and has a triangular correlation structure.

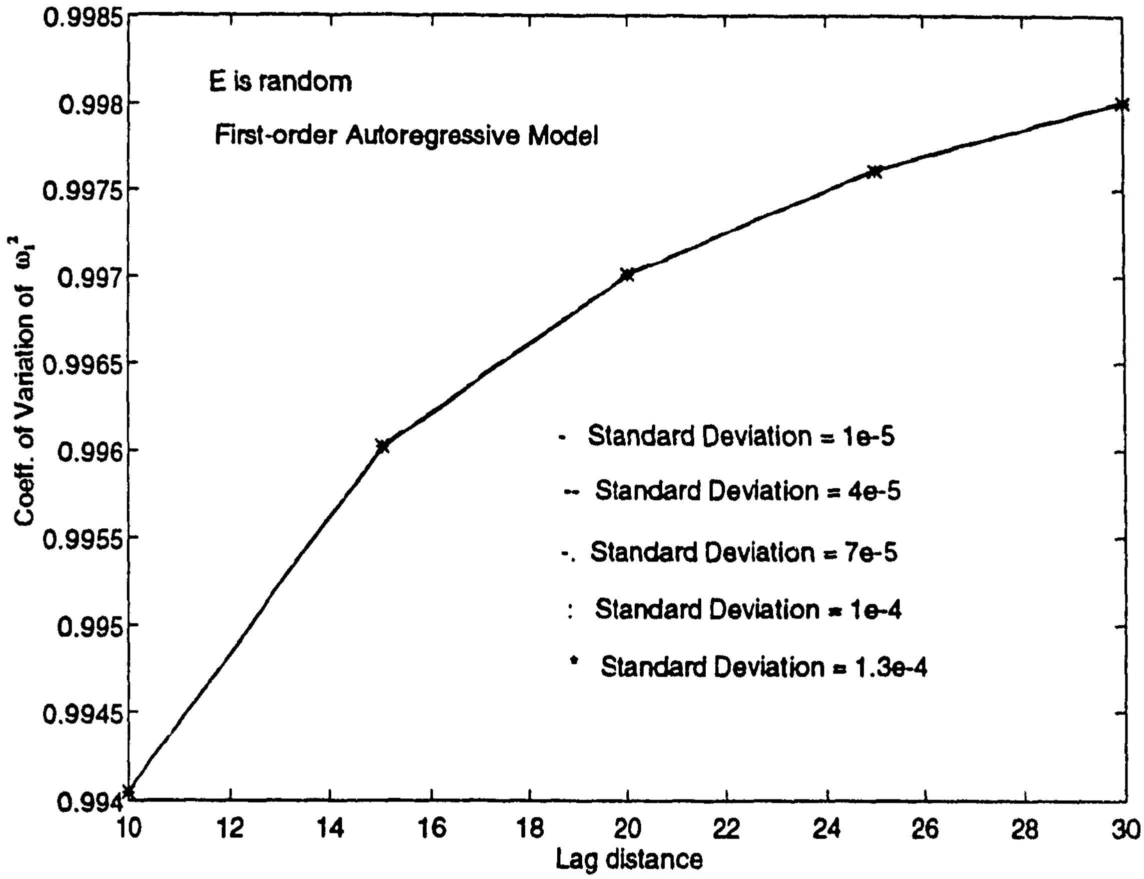

Fig. 5. 14: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random and has a first-order autoregressive correlation structure.**

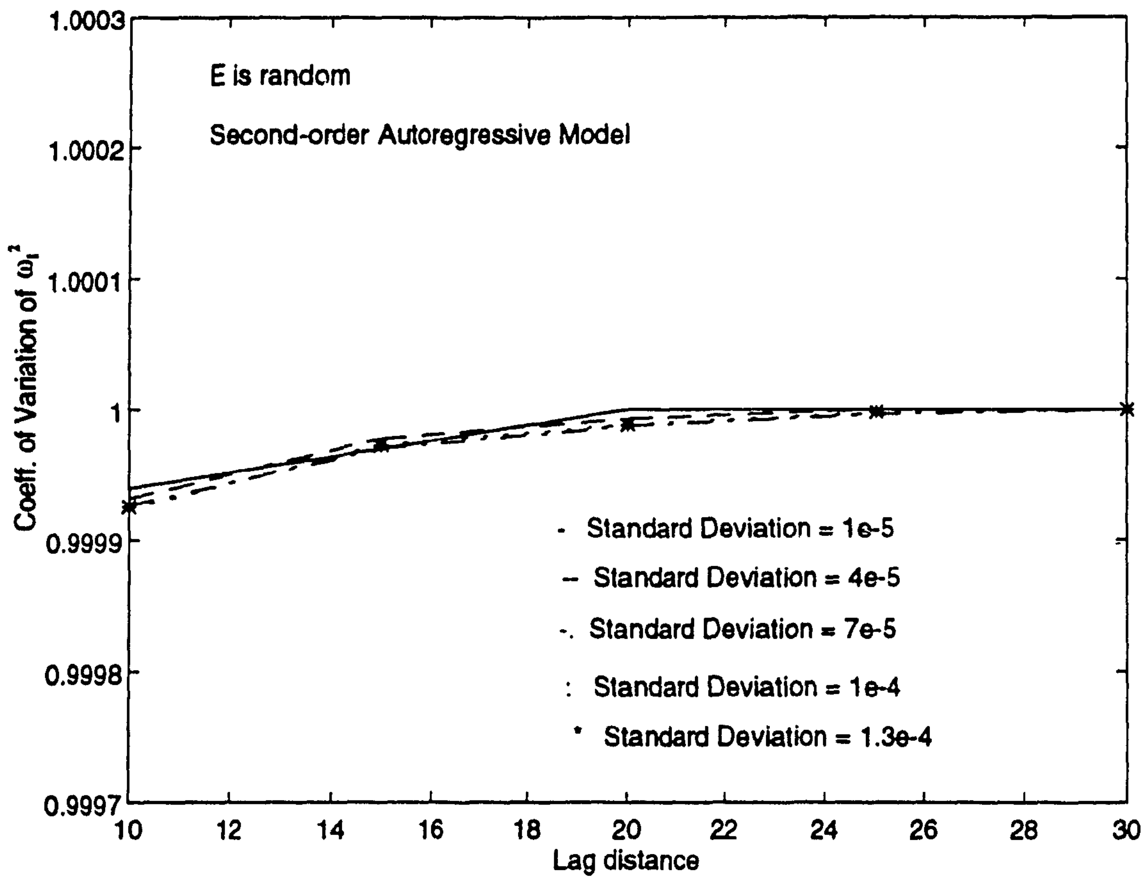

Fig. 5. 15: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random and has a second-order autoregressive correlation structure.**

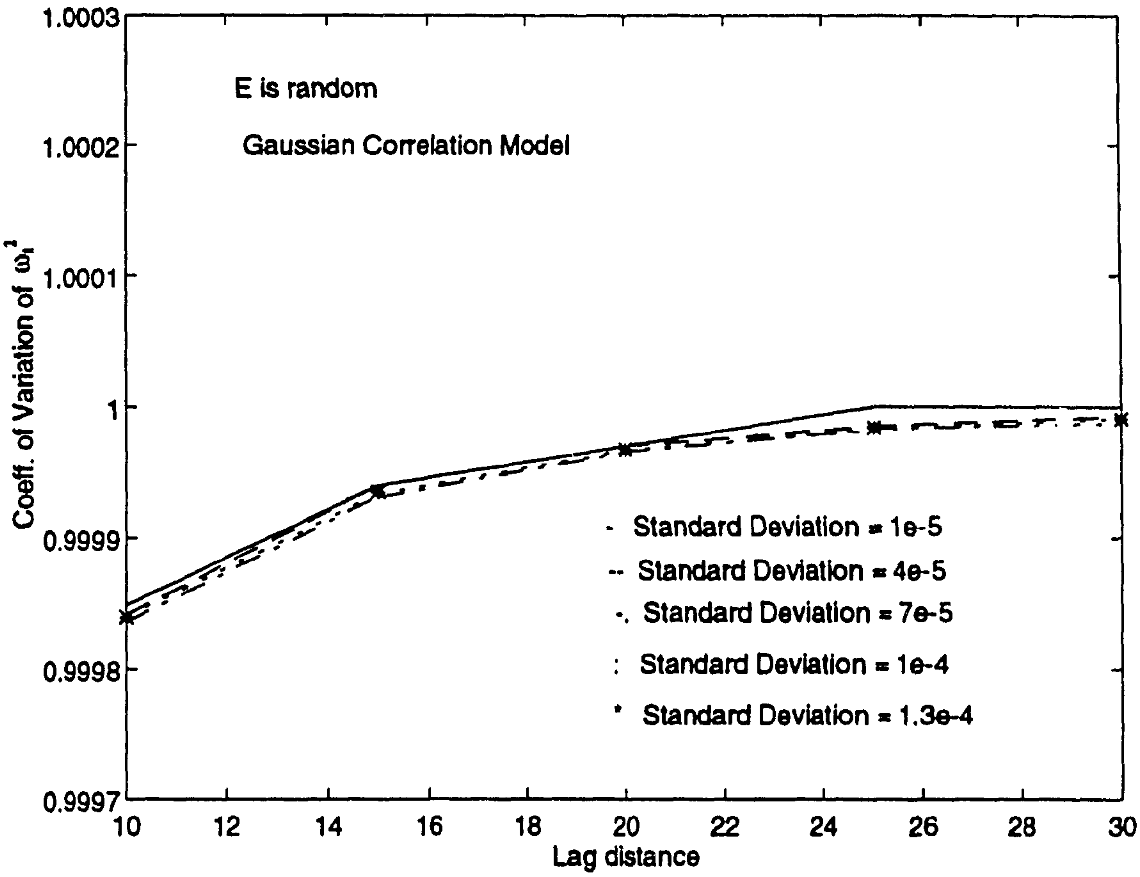

Fig. 5. 16: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random and has a Gaussian correlation structure.**

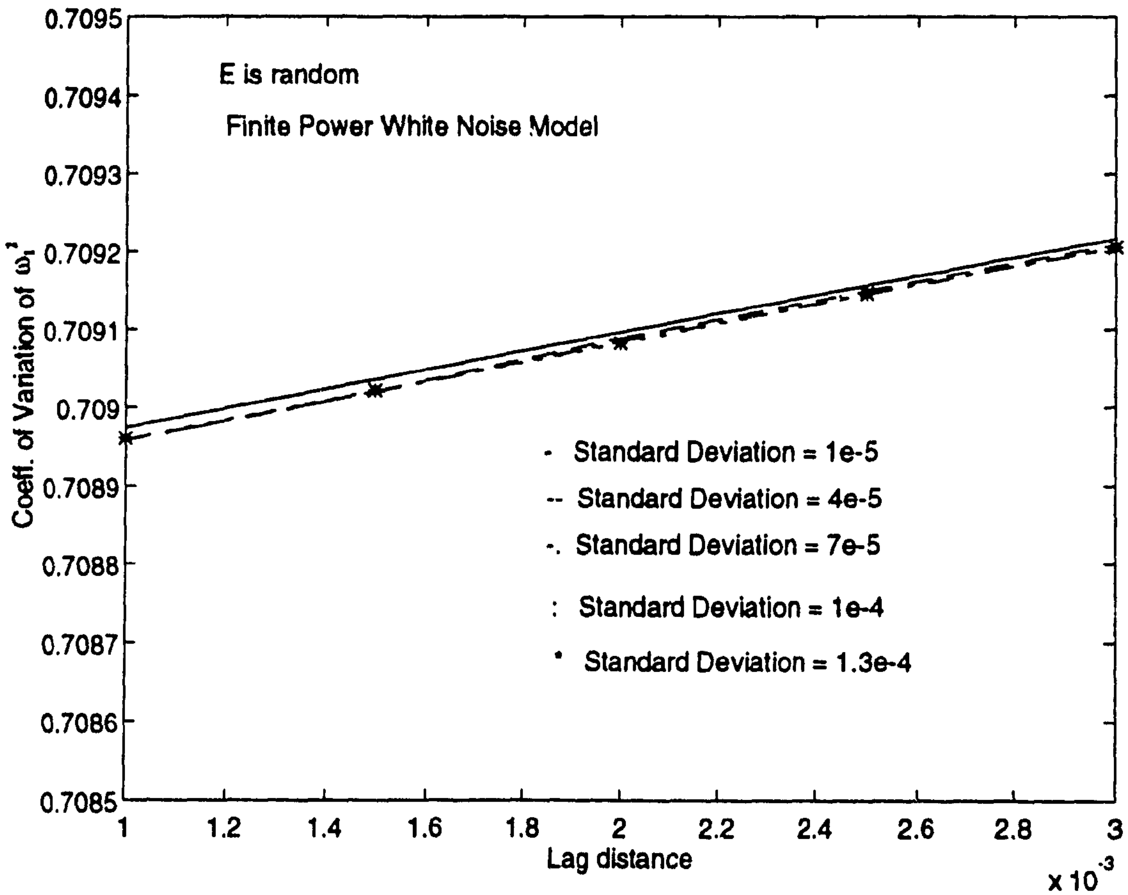

Fig. 5. 17: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random and has a finite power white noise correlation structure.**

Using the same finite element model, the effect of random mass density on the coefficient of variation of ω_1^2 is studied. The standard deviation of the stochastic field that models the fluctuations about the mean of the mass density is varied for each of the 5 correlation models. For each correlation model, the coefficient of variation of ω_1^2 is analyzed for 5 values of the standard deviation and are shown in Figs. (5.18-5.22). The results obtained in this case are found to be qualitatively similar to the results obtained when the Young's modulus is random. A similar study is carried out with a stochastic variation in both the Young's modulus and the mass density. The coefficient of variation of ω_1^2 is plotted in Figs. (5.23-5.27) as a function of the lag distances of the 5 correlation models for 5 different values of the standard deviation. The results obtained in this case, viz. random Young's modulus and mass density, are very close to the results obtained when the mass density alone is random. A comparison of the coefficient of variation of ω_1^2 prior to being normalized indicates that the stochastic variation in the mass density has a significantly larger influence on these values of the coefficient of variation of ω_1^2 than does the stochastic variation in the Young's modulus as highlighted in Fig. (5.28).

Case 2. Influence of Bearing Flexibility

The stiffnesses of the bearings are incorporated into the eigenvalue problem to study the rotor-bearing system mounted on flexible bearings. First, the Young's modulus is considered to have a stochastic distribution and the number of finite elements is fixed. For the same finite model, each and all of the five correlation models have been considered. For each correlation model, the standard deviation of the stochastic field that models the fluctuations about the mean of the Young's modulus is varied.

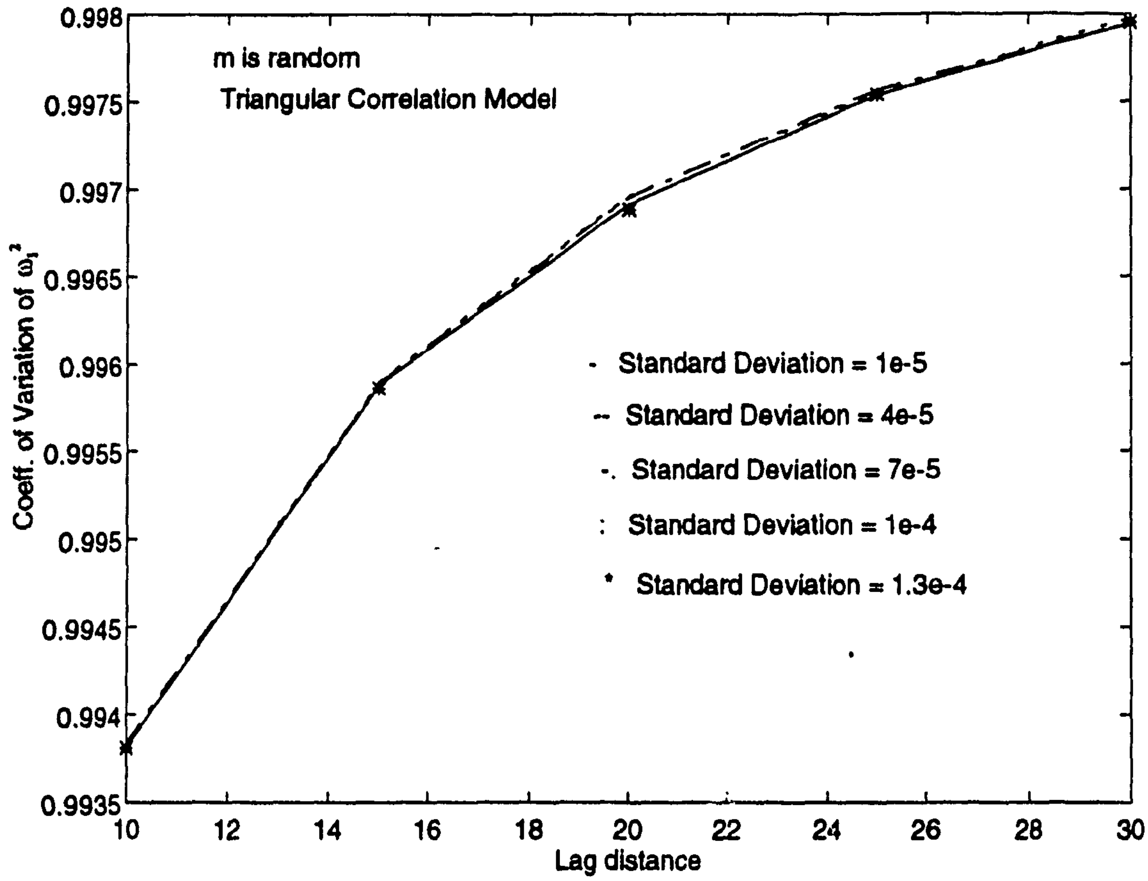

Fig. 5. 18: Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; m is random and has a triangular correlation structure.

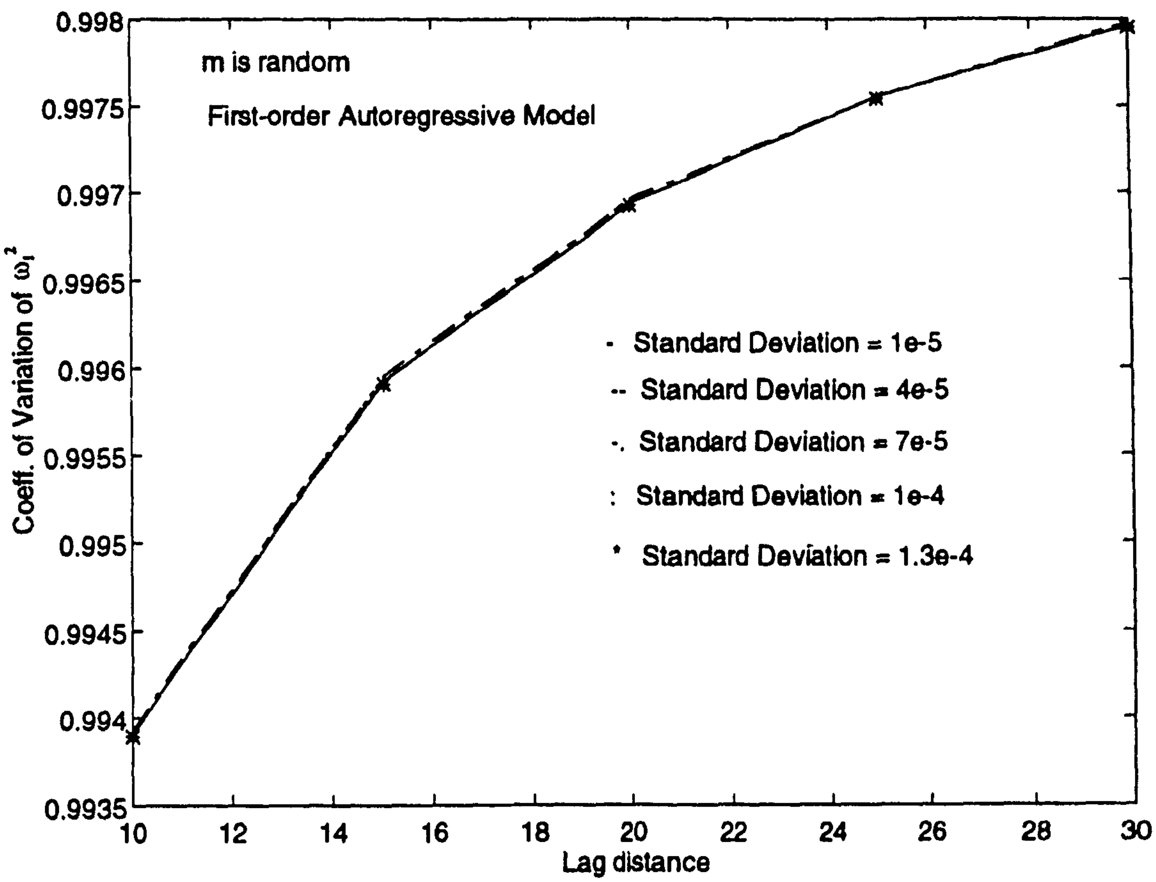

Fig. 5. 19: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; m is random and has a first-order autoregressive correlation structure.**

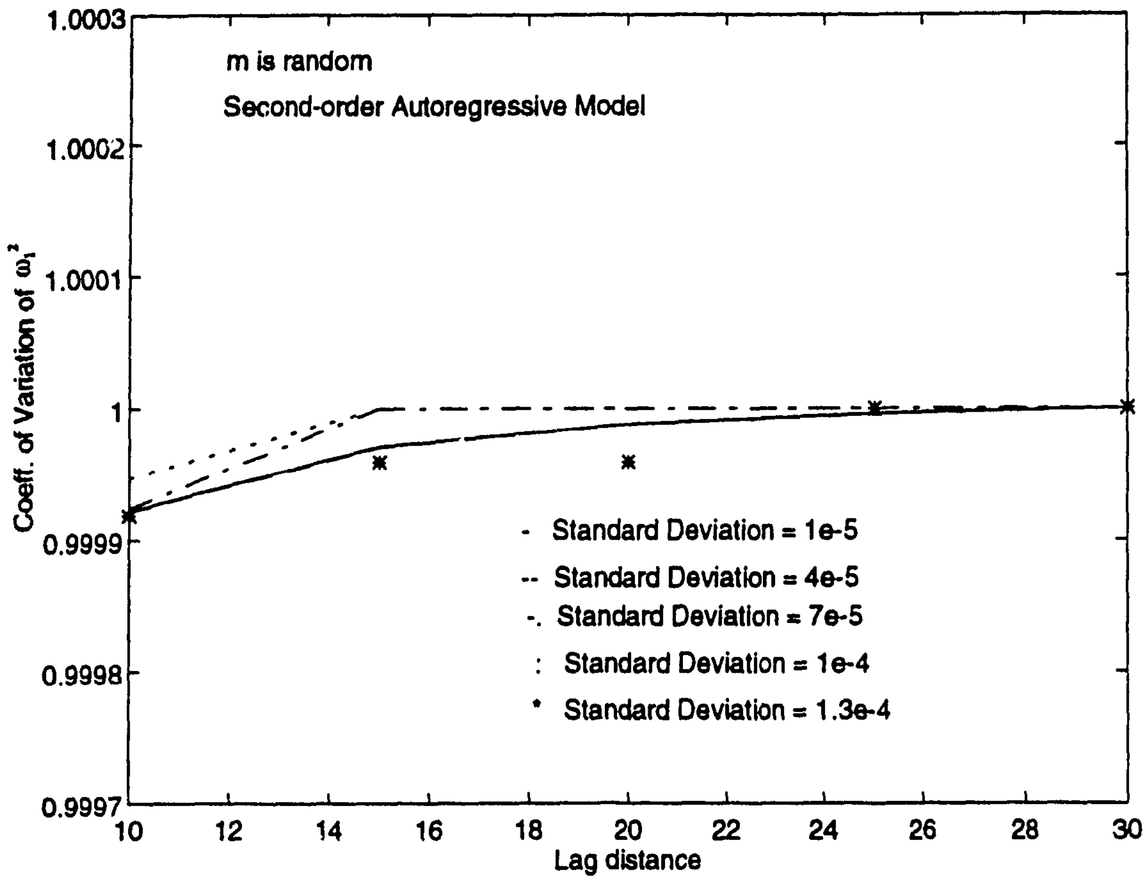

Fig. 5. 20: Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; m is random and has a second-order autoregressive correlation structure.

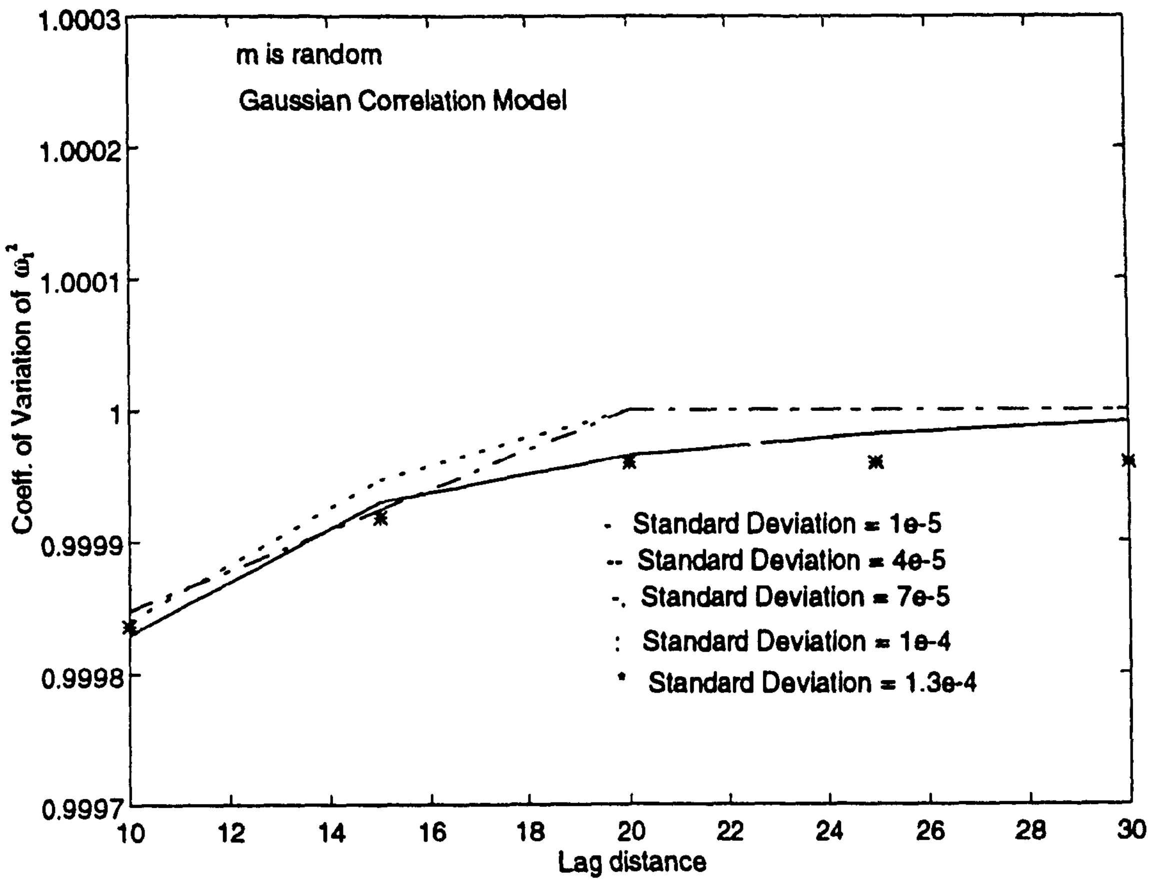

Fig. 5. 21: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; m is random and has a Gaussian correlation structure.**

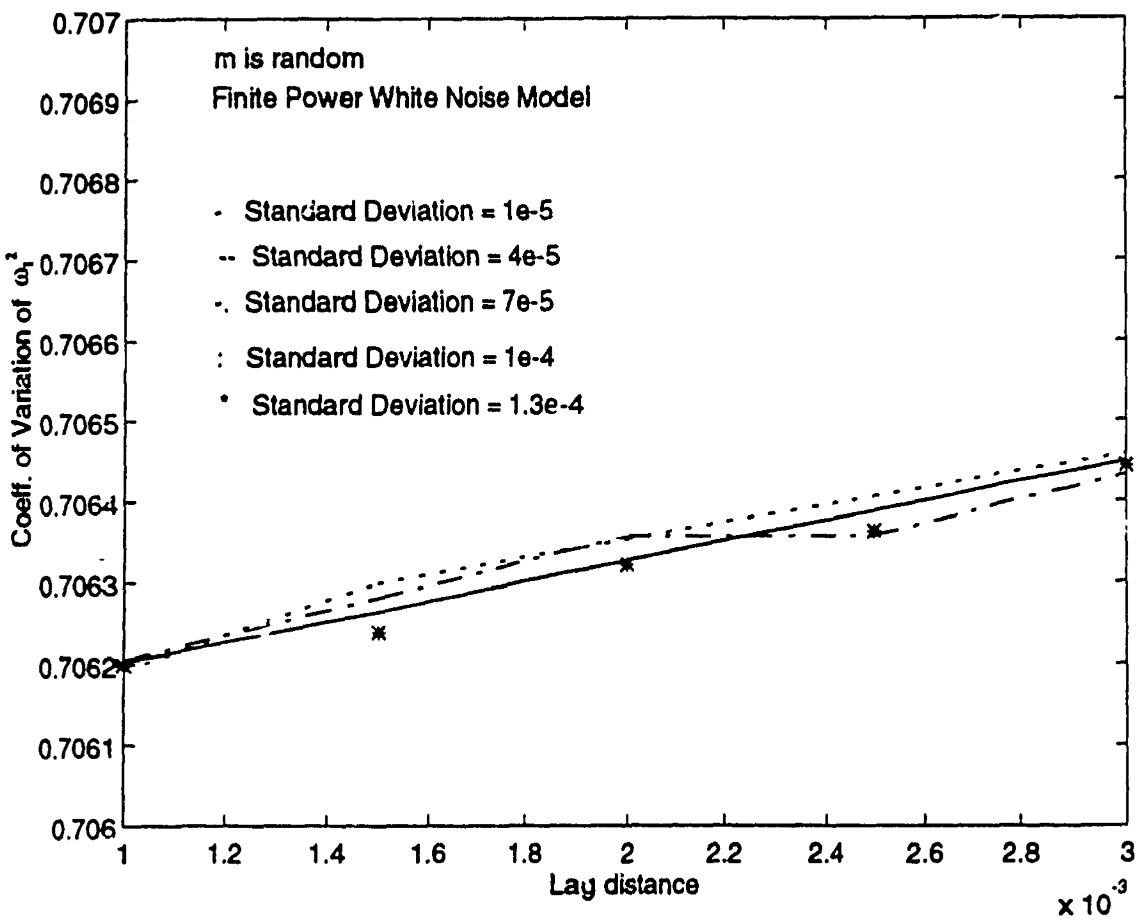

Fig. 5. 22: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; m is random and has a finite power white noise correlation structure.**

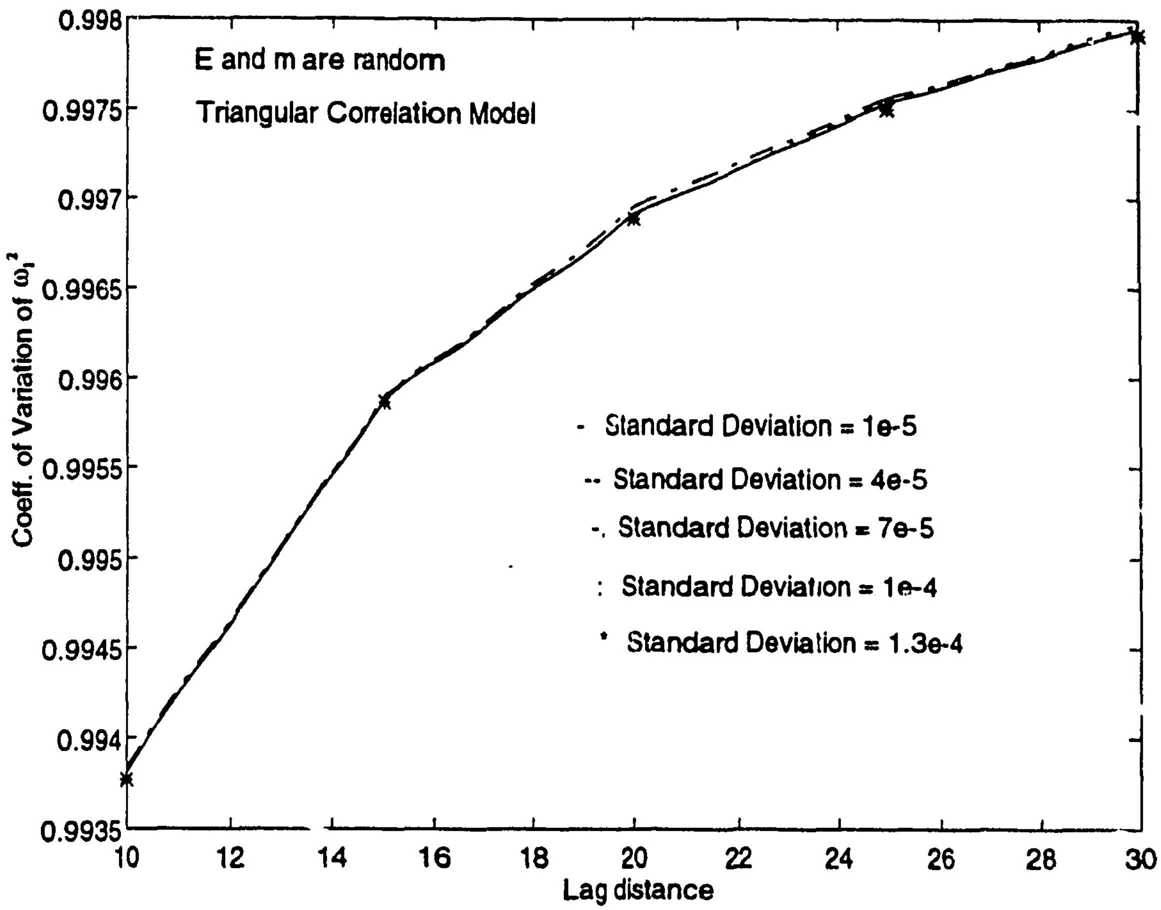

Fig. 5. 23: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; Both E and m are random and have a triangular correlation structure.**

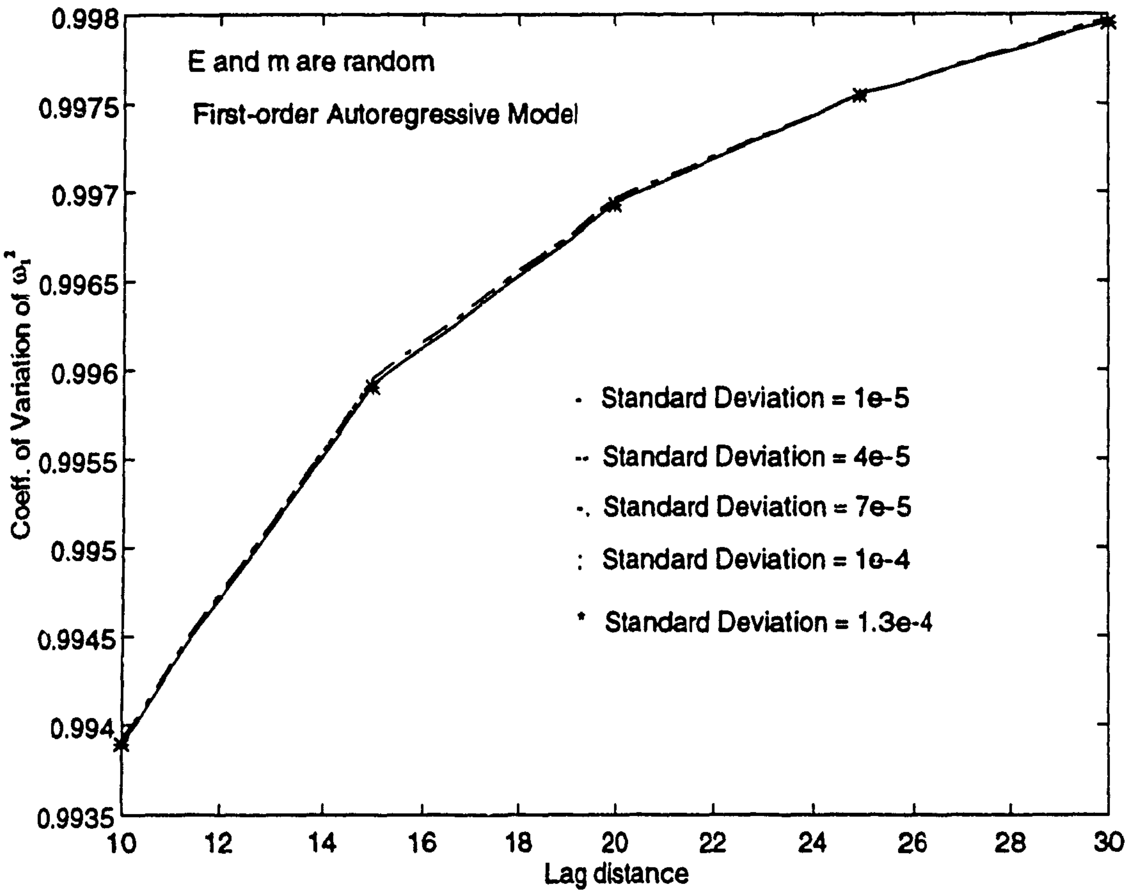

Fig. 5. 24: Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; Both E and m are random and have a first-order autoregressive correlation structure.

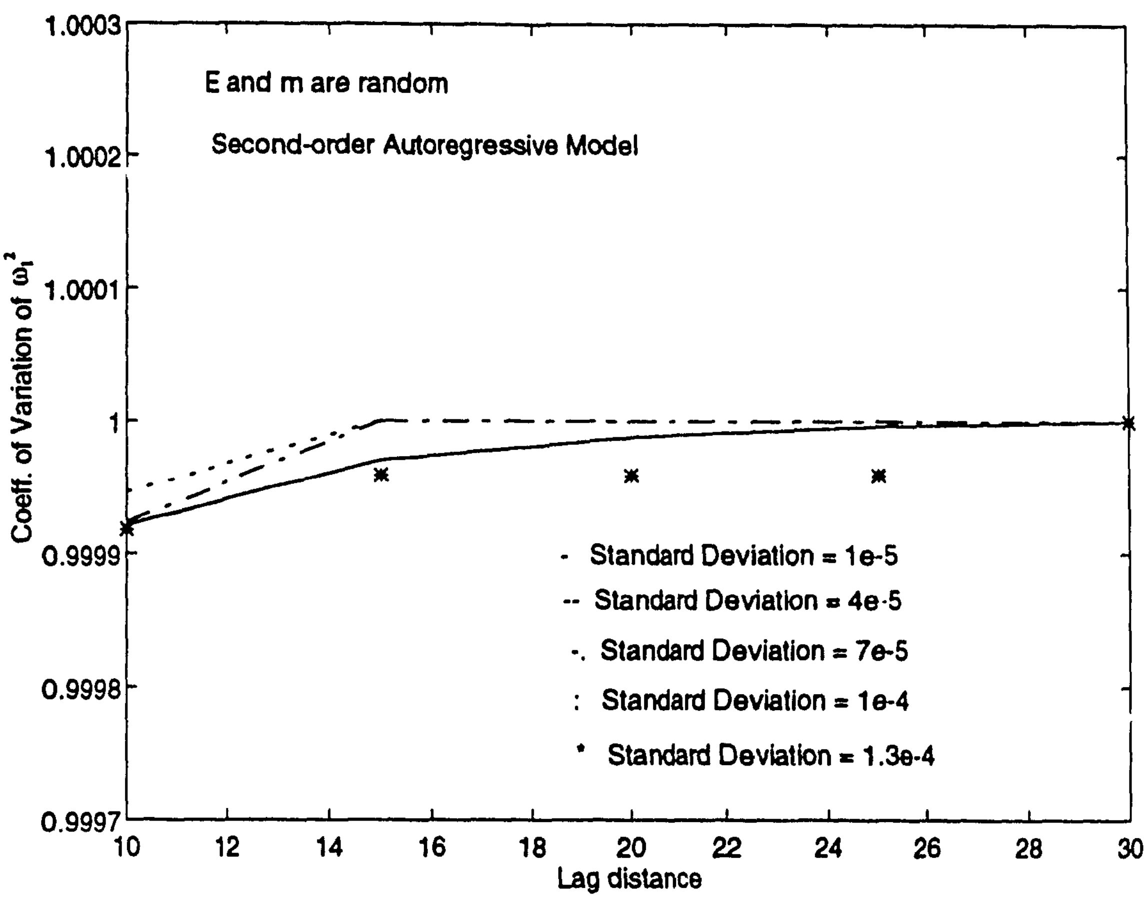

Fig. 5. 25: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; Both E and m are random and have a second-order autoregressive correlation structure.**

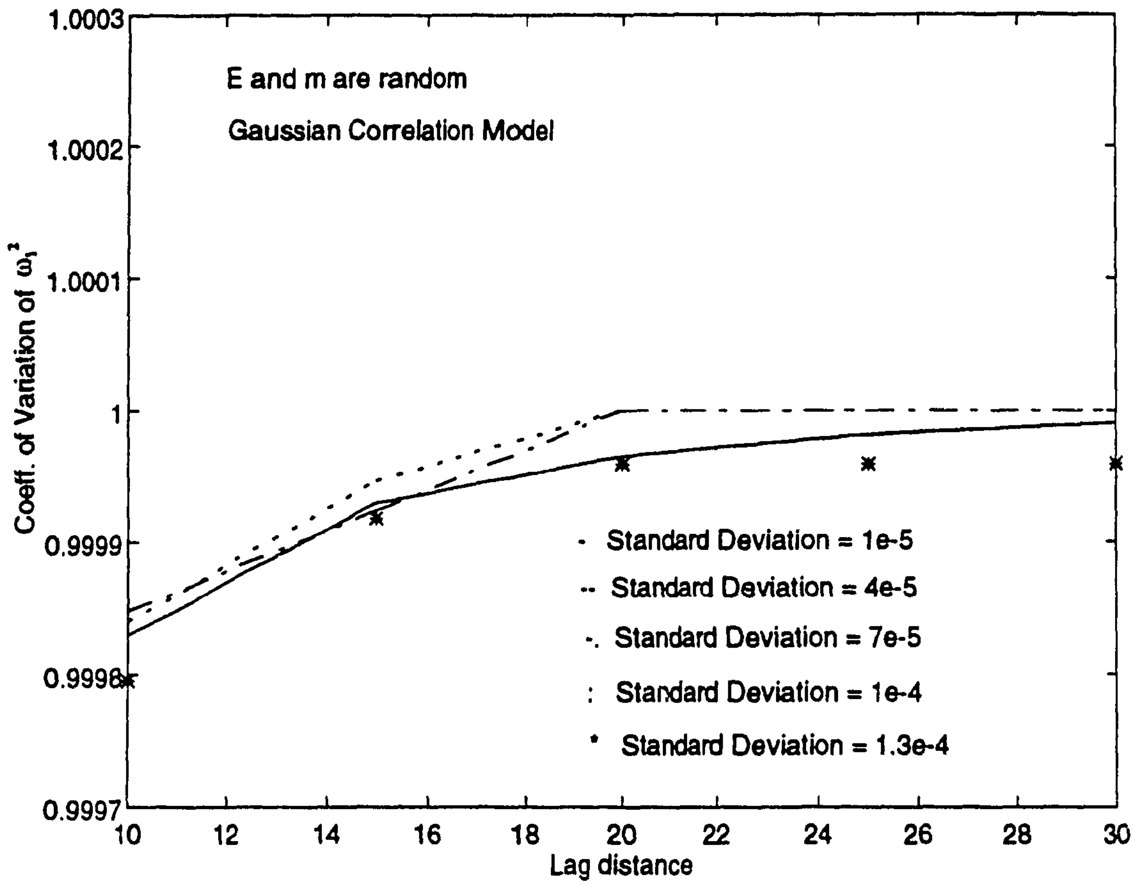

Fig. 5. 26: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; Both E and m are random and have a Gaussian correlation structure.**

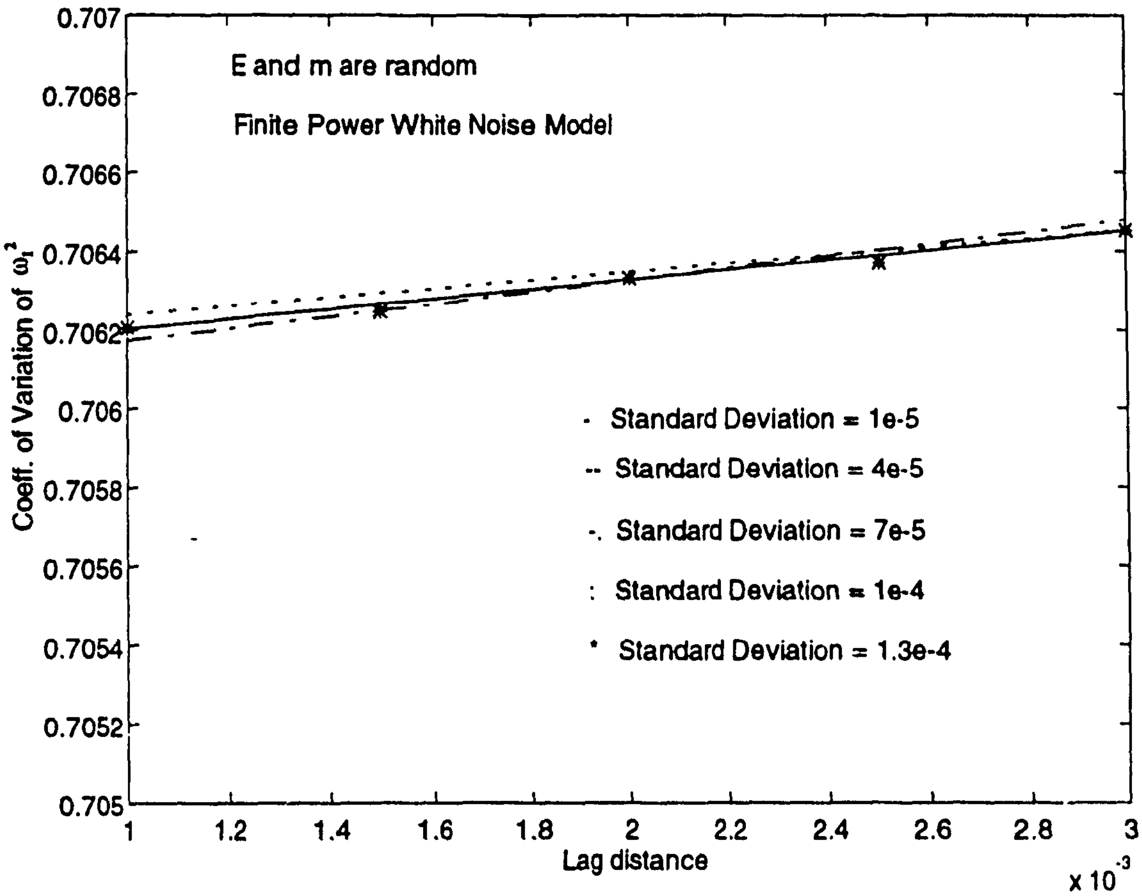

Fig. 5. 27: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; Both E and m are random and have a finite power white noise correlation structure.**

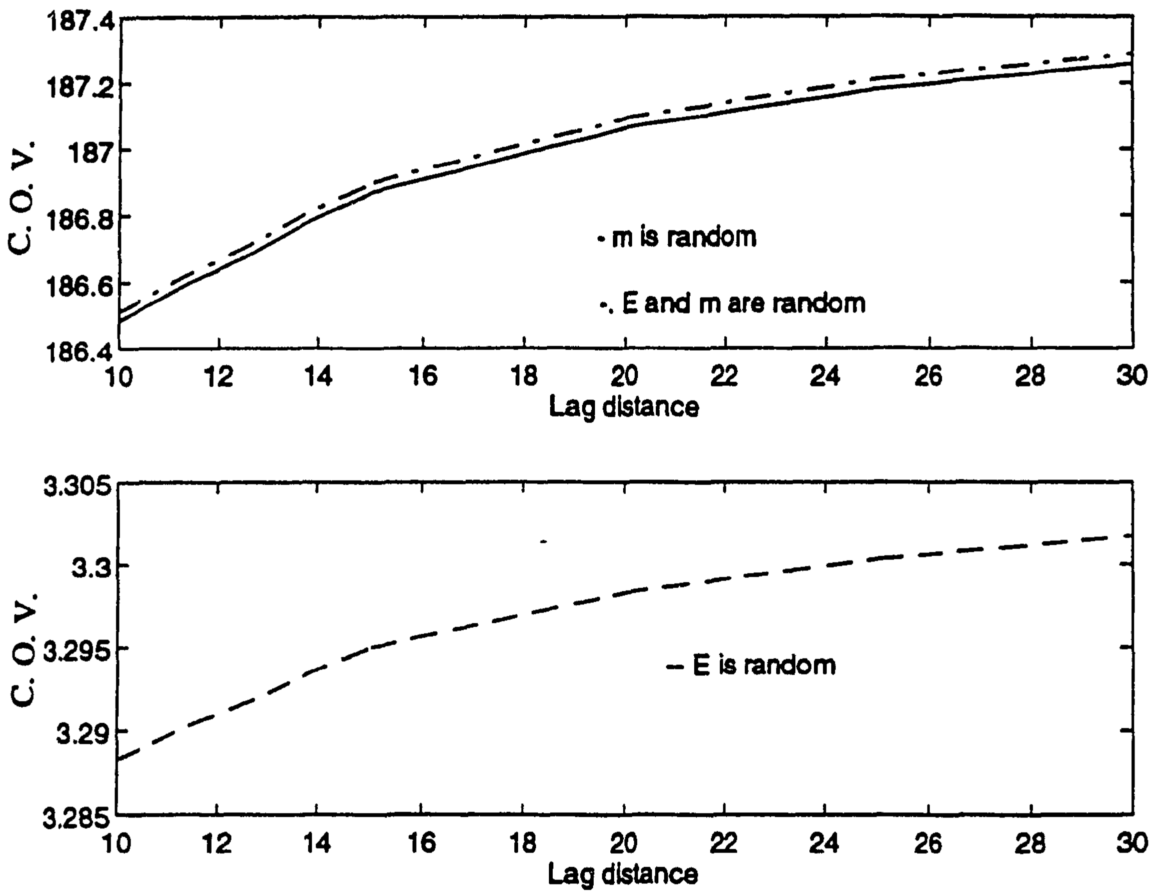

Fig. 5. 28: **Coefficient of variation of ω_1^2 prior to normalization (C. O. V.) of the rotor system on rigid bearings. Stochastic distributions in E, m and both E and m are considered to study their influence on C. O. V.**

The effect of the various correlation structures, the variation of the lag distance within each correlation structure and the different standard deviations, on the coefficient of variation of ω_1^2 is analyzed to study the influence of the bearings. The values of the coefficient of variation of ω_1^2 are found to be slightly higher than that in the case of the rotor on rigid bearings. Again, the stochastic variation in the Young's modulus is found to have not much of an influence on the coefficient of variation of ω_1^2 when compared with the stochastic variation in the mass density and this fact has been brought out in Fig. (5.29). When the Young's modulus is considered to have a stochastic variation, the change in the coefficient of variation of ω_1^2 as a function of the lag distances in each of the 5 correlation models is shown in Figs. (5.30-5.34). These figures correspond to the 5 correlation models each of which are studied for 5 different standard deviations. In a similar manner, the coefficient of variation of ω_1^2 obtained due to a stochastic variation in mass density is presented in Figs. (5.35-5.39) as a function of the lag distances of the various correlation models, for 5 different values of the standard deviation. The effect of stochastic variations in the Young's modulus and mass density on the coefficient of variation of ω_1^2 is captured in Figs. (5.40-5.44). The graphs obtained in the case of the rotor system with isotropic bearings show a trend similar to the results obtained when the rotor is mounted on rigid bearings. Although they have a qualitative resemblance, quantitatively there is a slight difference. There is a non-linear variation in the coefficient of variation of ω_1^2 for the Triangular and First-order Autoregressive correlation models, while there is a linear variation in the cases of the Second-order Autoregressive, Gaussian and Finite Power White Noise correlation models. Also, in the latter three cases, the coefficient of variation of ω_1^2 does not exhibit significant sensitivity.

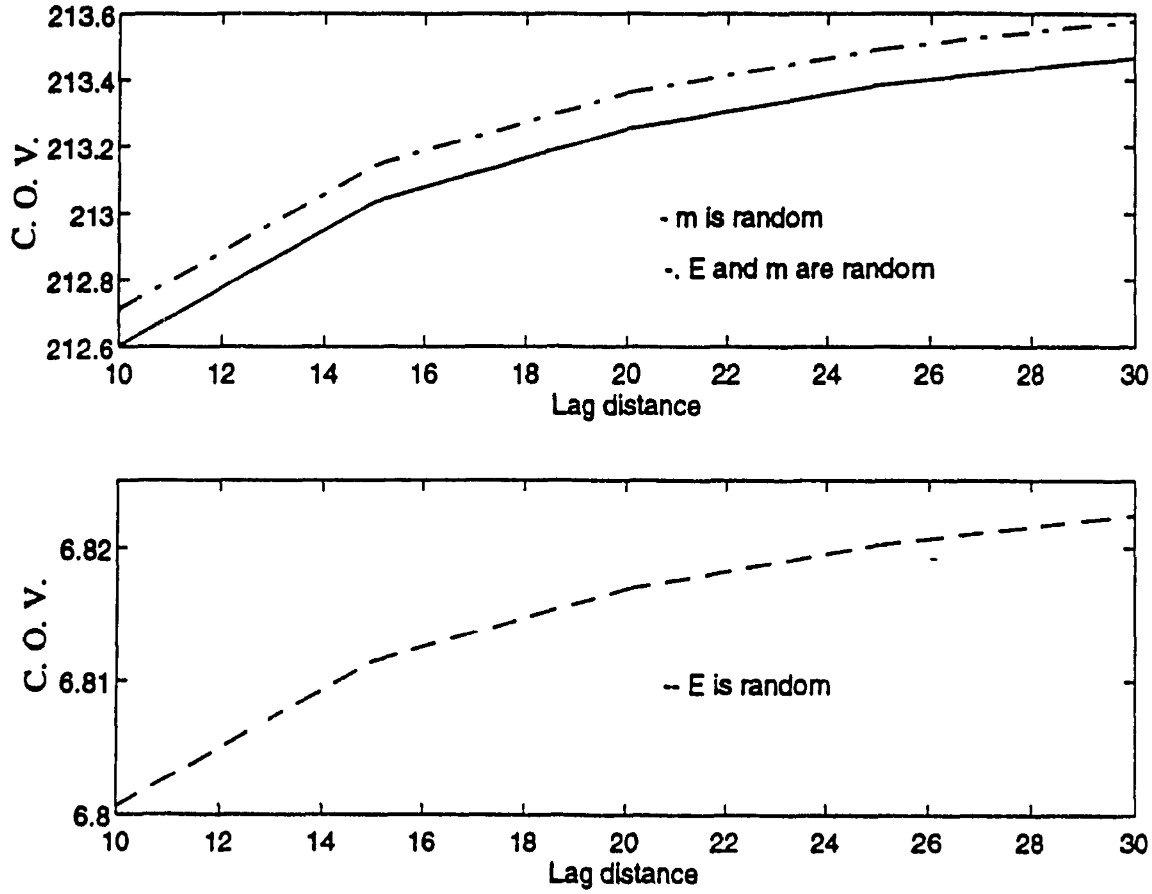

Fig. 5. 29: **Coefficient of variation of ω_1^2 prior to normalization (C. O. V.) of the rotor system on isotropic bearings. Stochastic distributions in E, m and both E and m are considered to study their influence on C. O. V.**

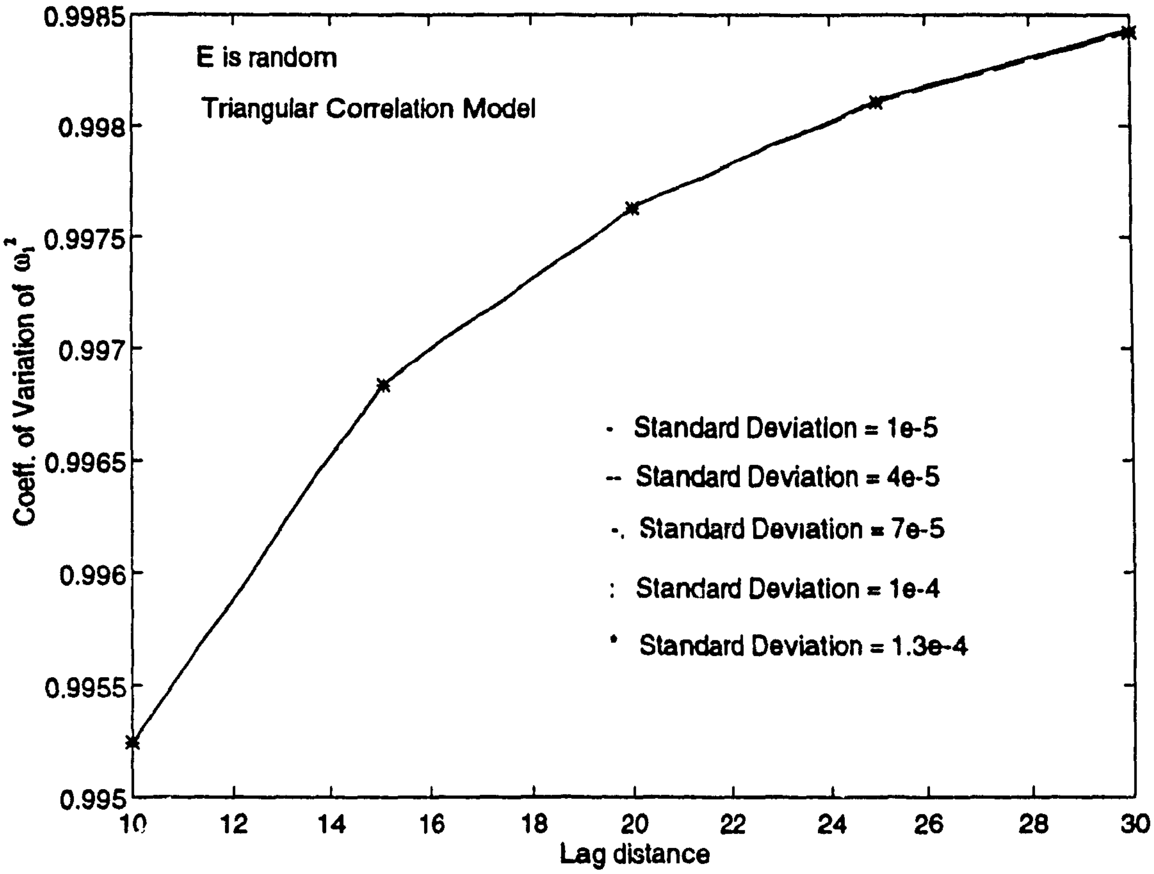

Fig. 5. 30: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random and has a triangular correlation structure.**

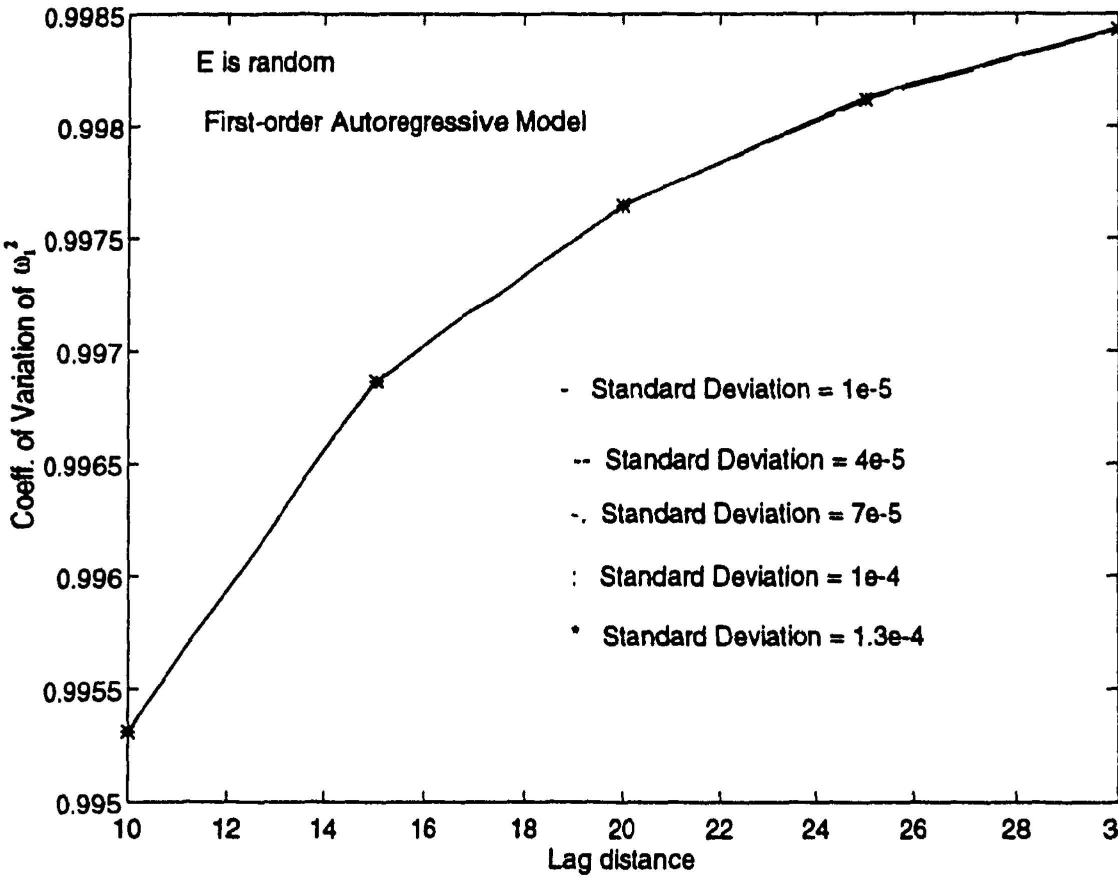

Fig. 5. 31: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random and has a first-order autoregressive correlation structure.**

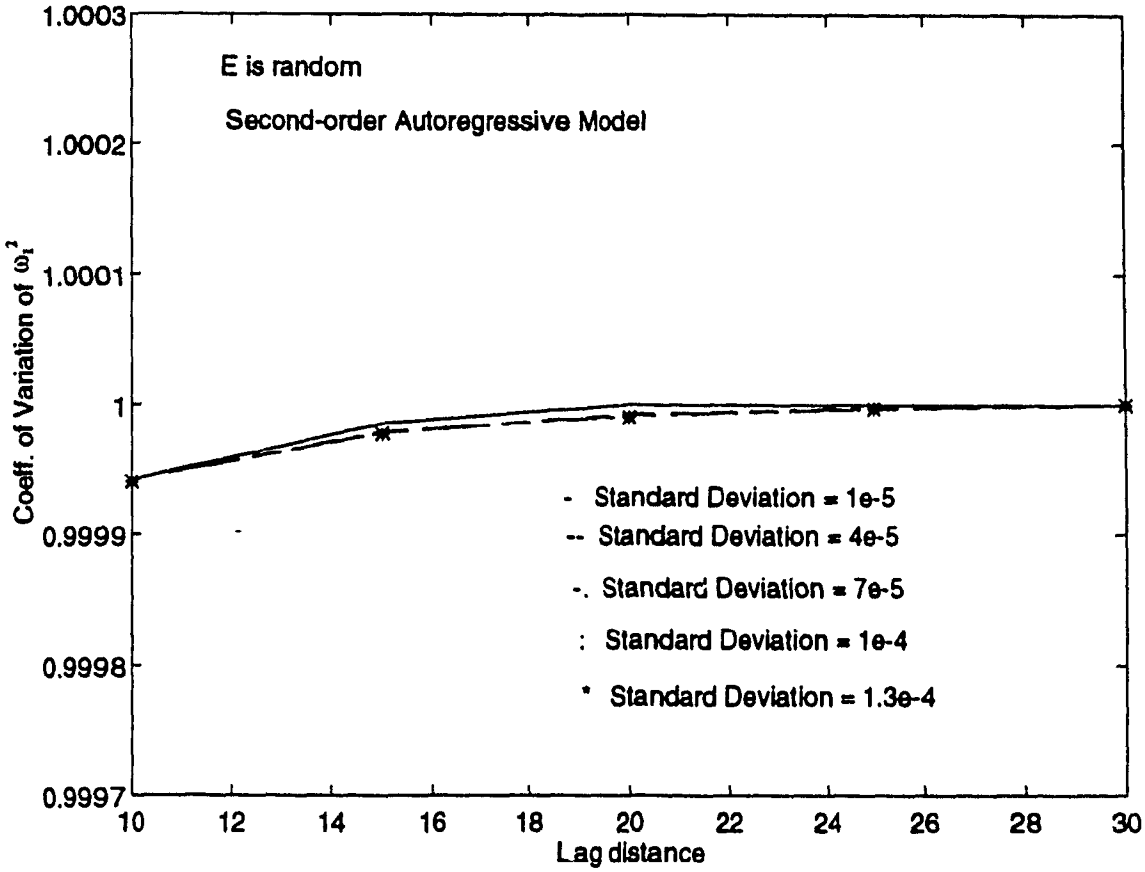

Fig. 5. 32: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random and has a second-order autoregressive correlation structure.**

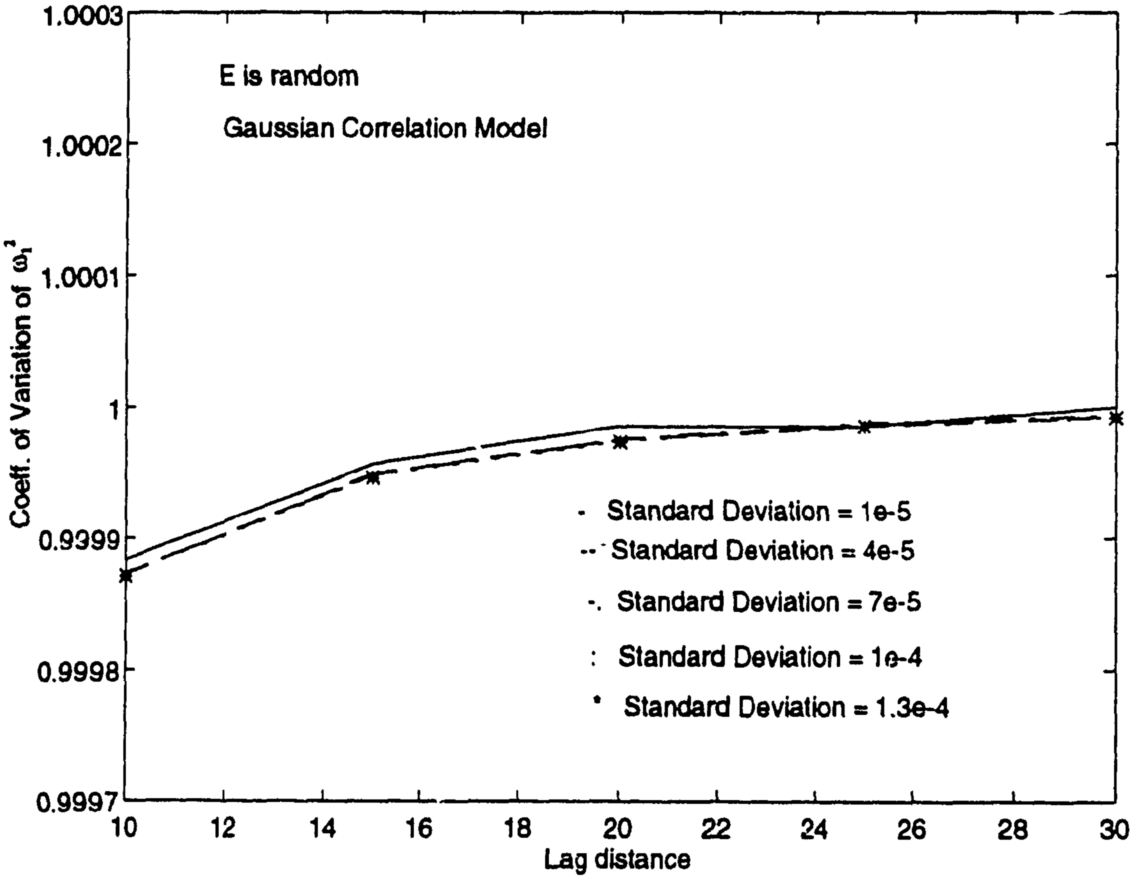

Fig. 5. 33: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random and has a Gaussian correlation structure.**

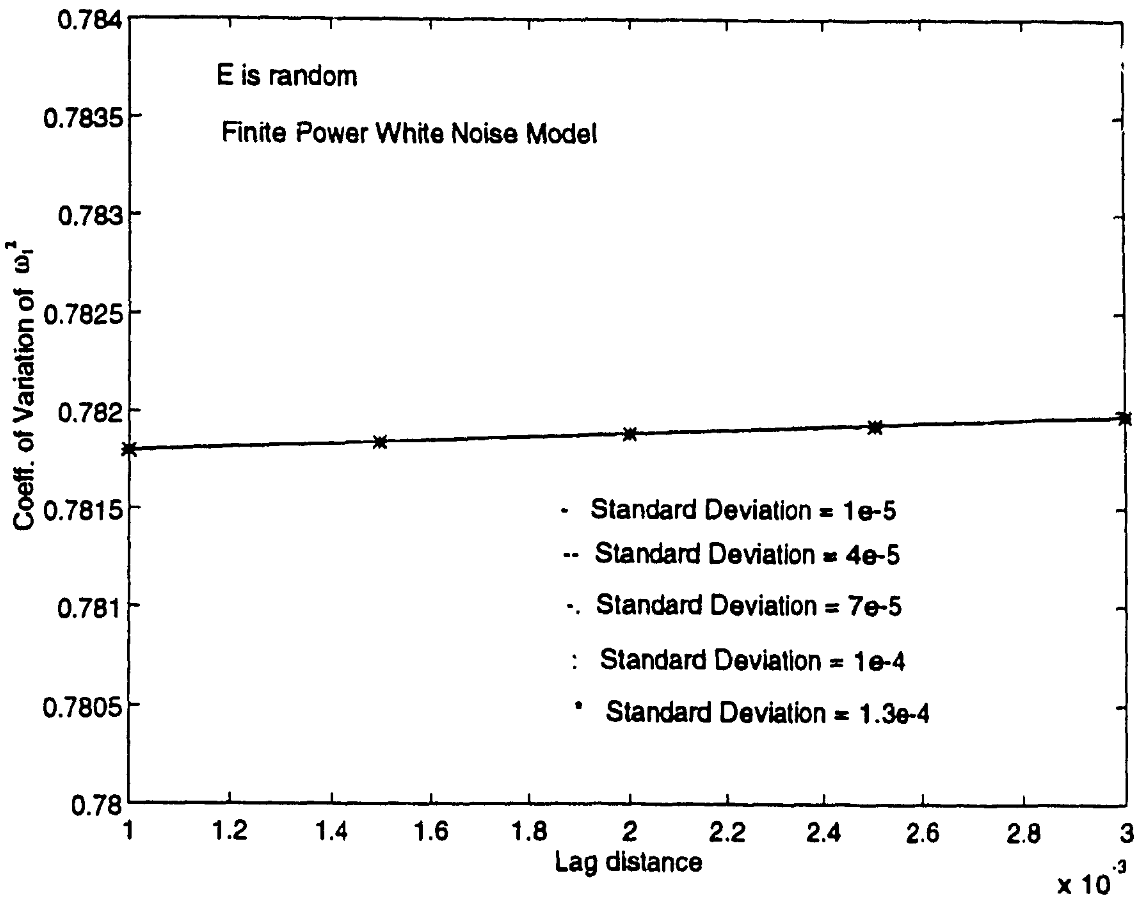

Fig 5. 34: Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random and has a finite power white noise correlation structure.

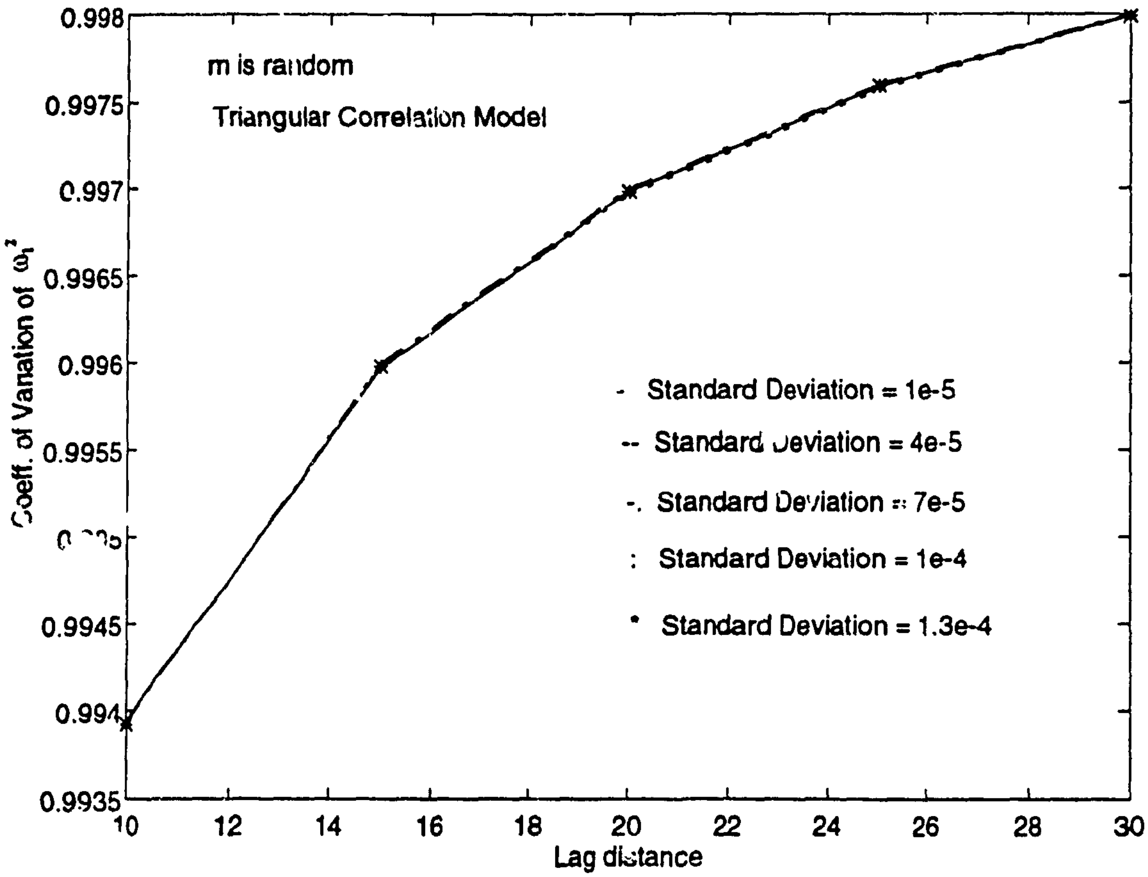

Fig. 5. 35: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; m is random and has a triangular correlation structure.**

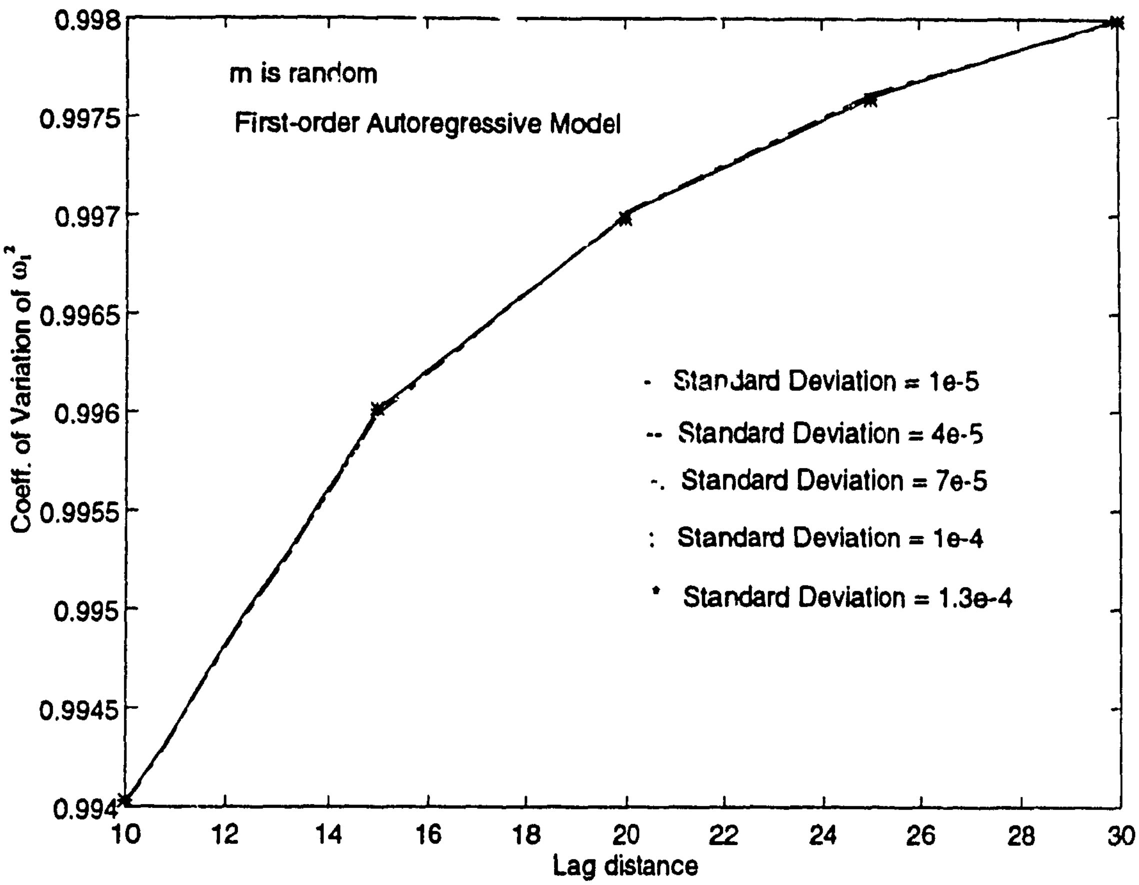

Fig. 5. 36: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; m is random and has a first-order autoregressive correlation structure.**

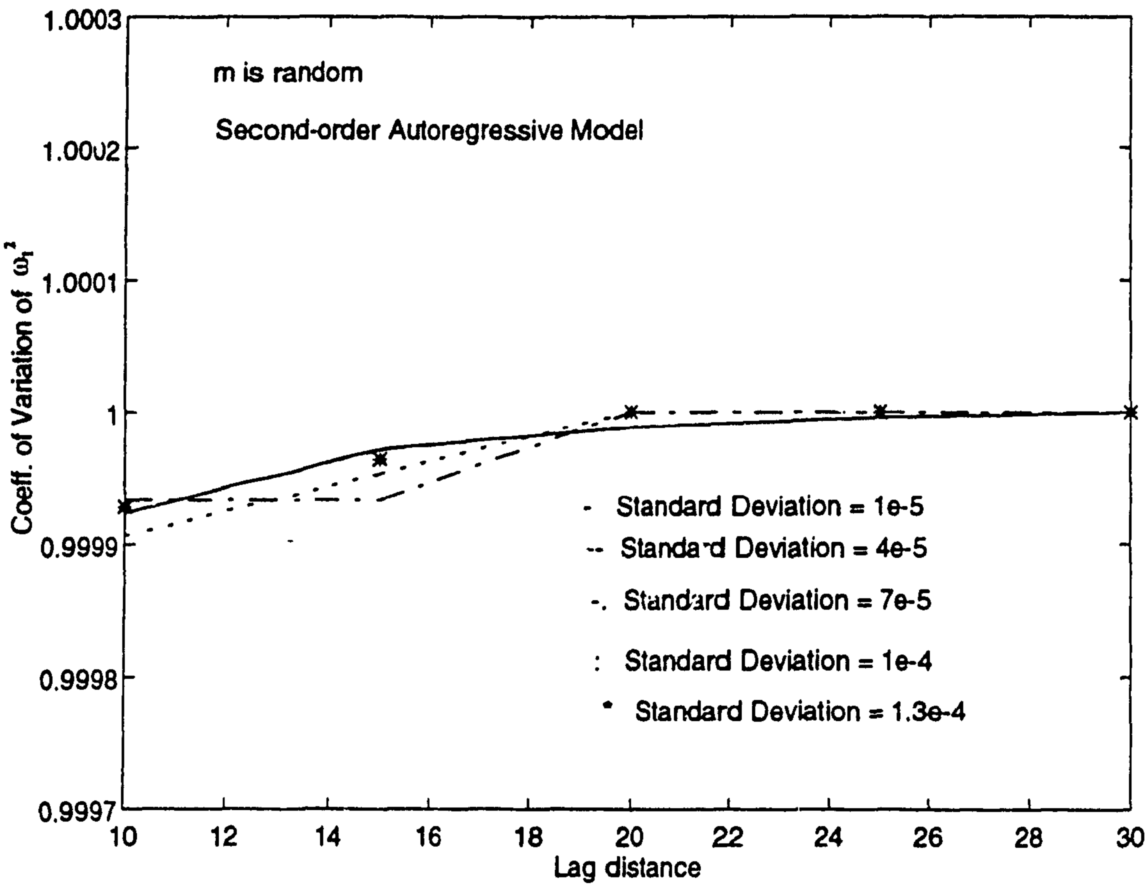

Fig. 5. 37: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; m is random and has a second-order autoregressive correlation structure.**

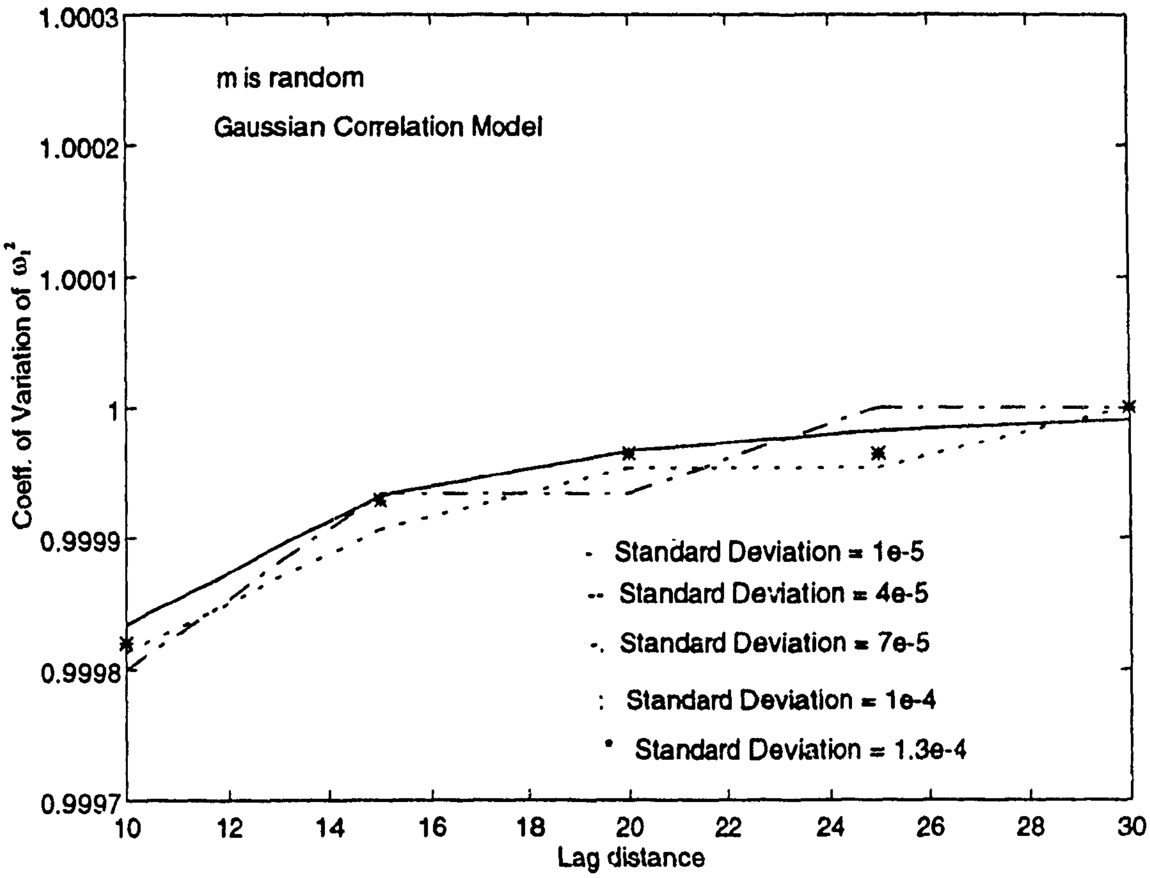

Fig. 5. 38: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; m is random and has a Gaussian correlation structure.**

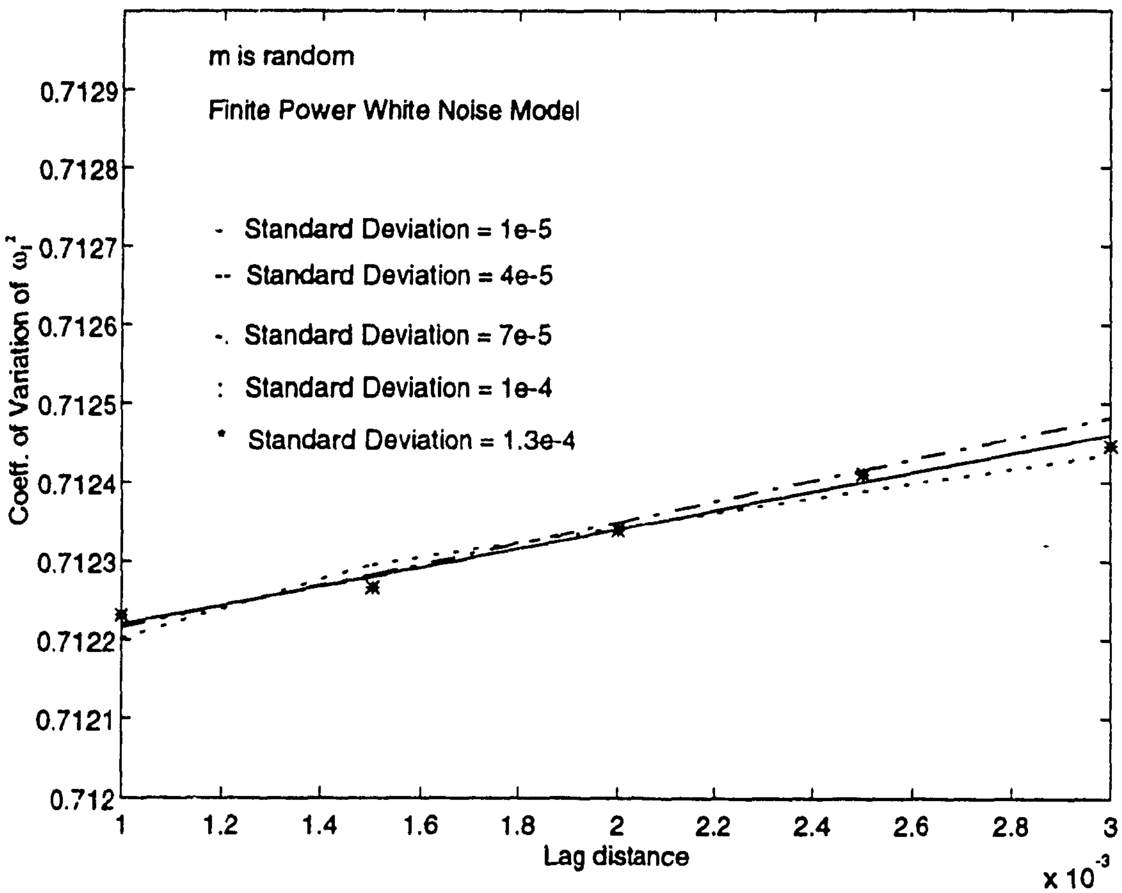

Fig. 5. 39: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; m is random and has a finite power white noise correlation structure.**

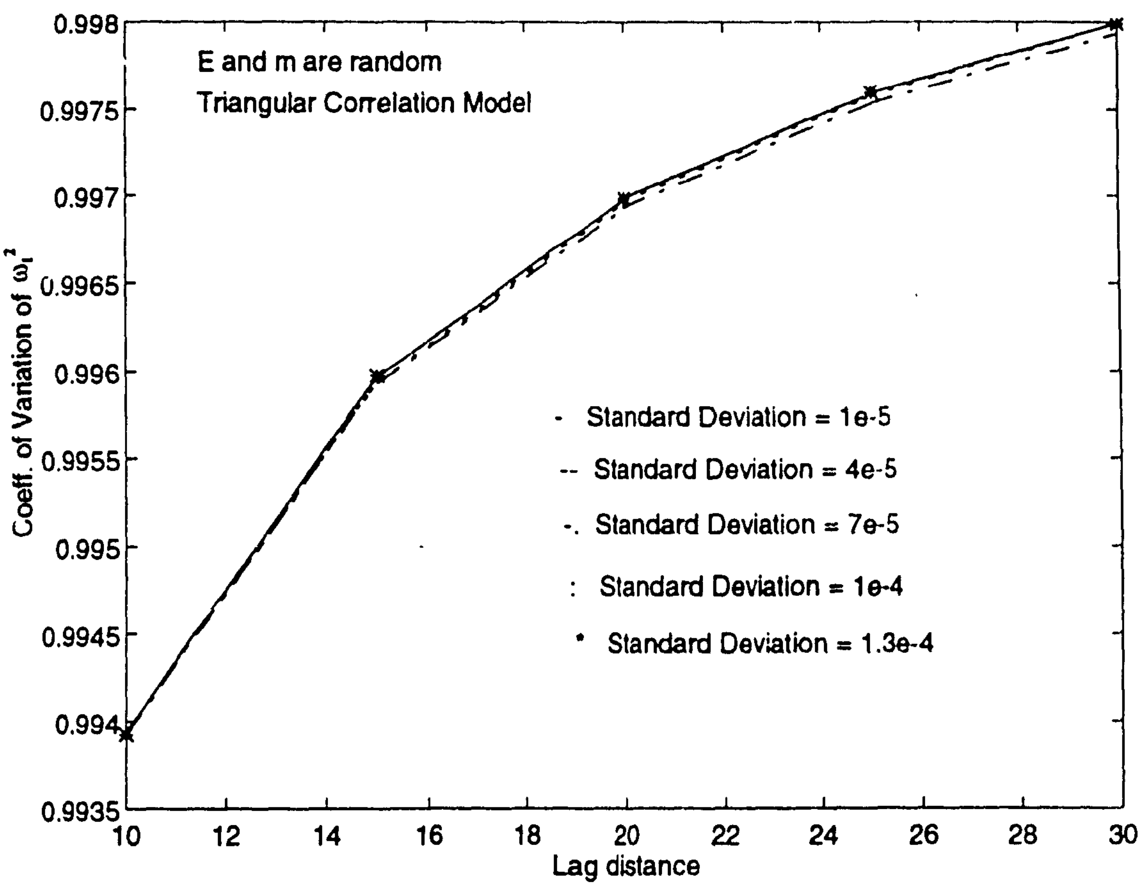

Fig. 5. 40: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; Both E and m are random and have a triangular correlation structure.**

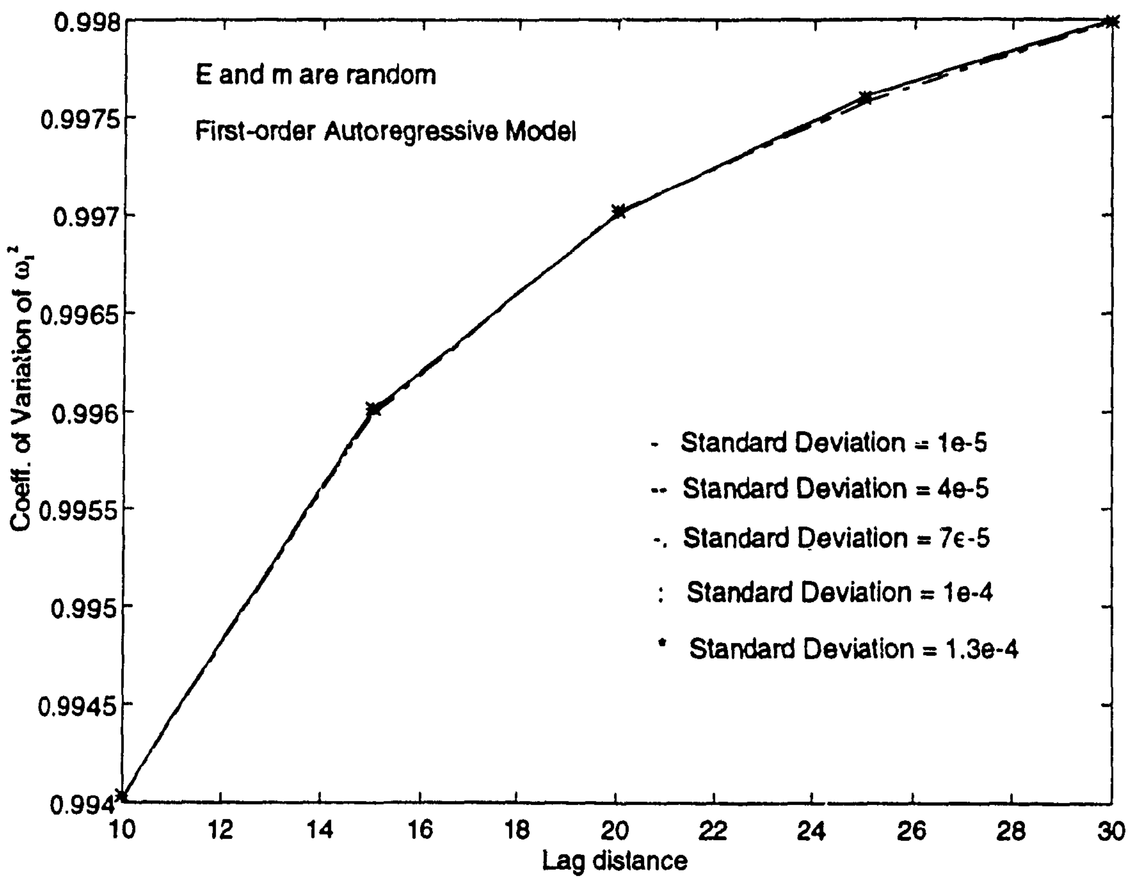

Fig. 5. 41: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; Both E and m are random and have a first-order autoregressive correlation structure.**

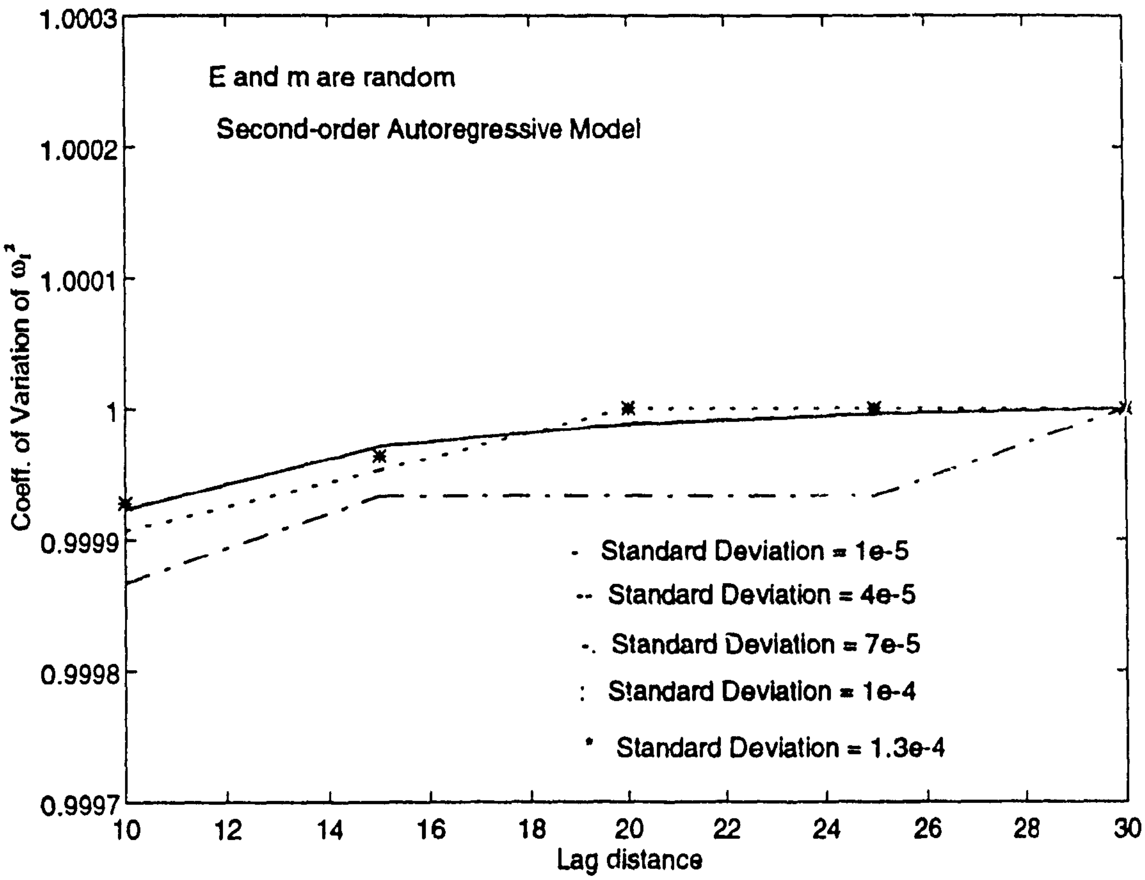

Fig. 5. 42: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; Both E and m are random and have a second-order autoregressive correlation structure.**

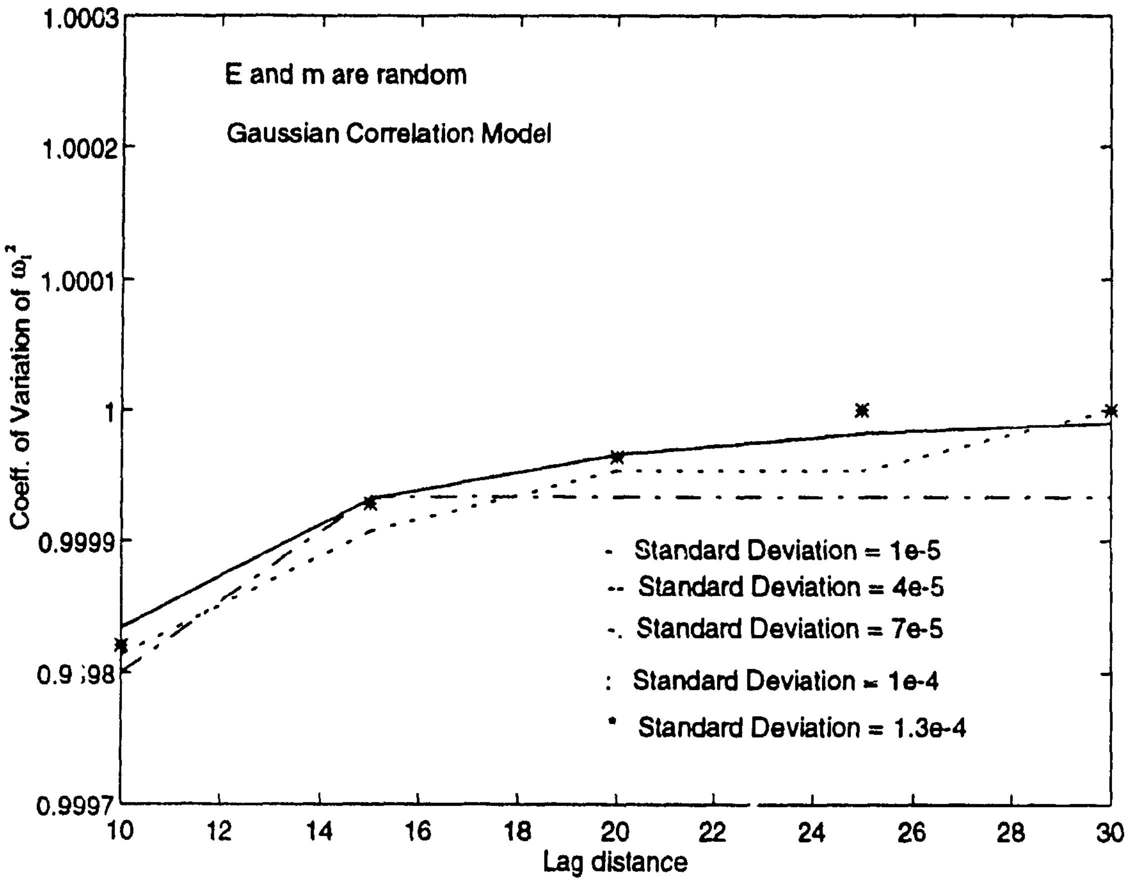

Fig. 5. 43: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; Both E and m are random and have a Gaussian correlation structure.**

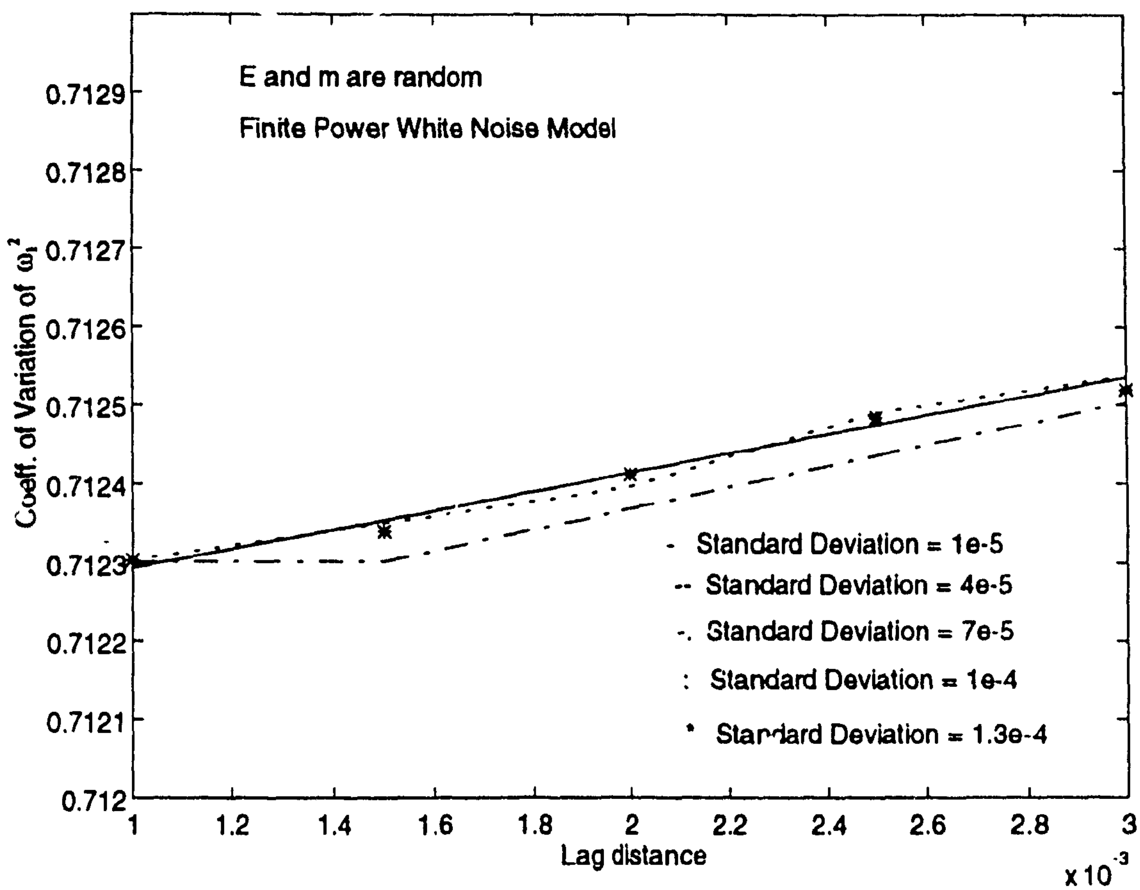

Fig. 5. 44: **Effect of the lag distance on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; Both E and m are random and have a finite power white noise correlation structure.**

Case 3. Influence of Finite Element Modelling

In this study, for a particular mesh, the rotor-bearing system with rigid bearings is considered and the effect on the coefficient of variation of ω_1^2 of the stochastic variations in the Young's modulus is quantified. The analysis is repeated so as to study the effect of changes in the mesh on the coefficient of variation of ω_1^2. The above procedure is then followed for the rotordynamic system with flexible bearing supports.

The stochastic fluctuations in Young's modulus are considered to have a triangular correlation structure. The shaft is mounted on rigid bearings and is modelled using 4, 6, 8 and 10 finite elements. In each of these cases, the coefficient of variation of ω_1^2 is computed for a given value of standard deviation of the Young's modulus The analysis is repeated for different values of the constant a of the triangular correlation model. The effects of the change in the number of finite elements on the coefficient of variation of ω_1^2 are shown for 5 values of a, viz. 10, 15, 20, 25 and 30, in Fig. (5.45). The mean values of the whirl speeds corresponding to the various degrees of freedom of the rotor system mounted on rigid bearings is shown in Fig. (5.46), corresponding to the cases when the shaft is modelled as 4, 6, 8 and 10 finite elements.

In order to study the effects of bearing flexibility on the influence of modelling, the shaft is cosidered as being mounted on elastic supports and is modelled using 4, 6, 8 and 10 elements. For a stochastically fluctuating Young's modulus with a triangular correlation structure, the influence of bearing flexibility on the coefficient of variation of ω_1^2 is studied for a fixed value of the standard deviation of the Young's modulus. The

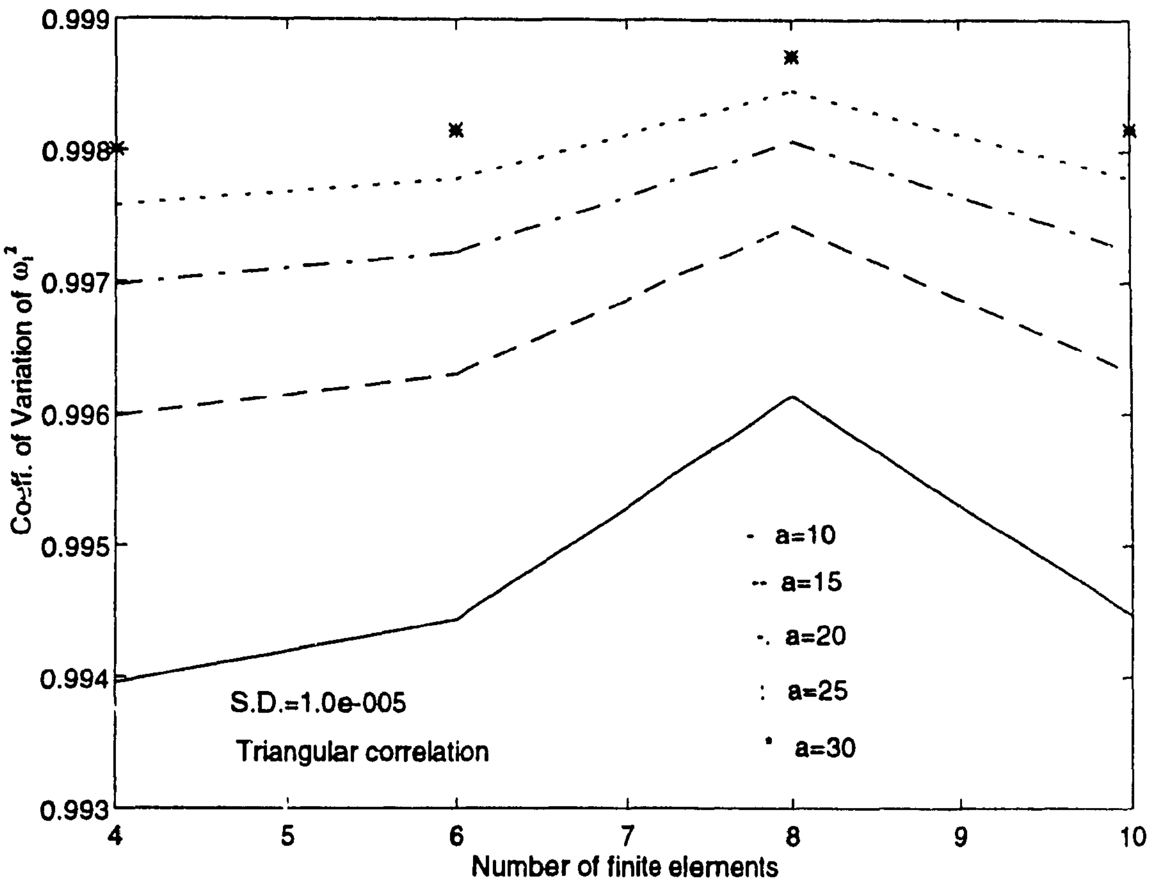

Fig. 5. 45: **Influence of finite element modelling on the coefficient of variation of ω_1^2 of the rotor system on rigid bearings; E is random with a fixed standard deviation (S. D.); The triangular correlation model is used with 5 different values of the constant a.**

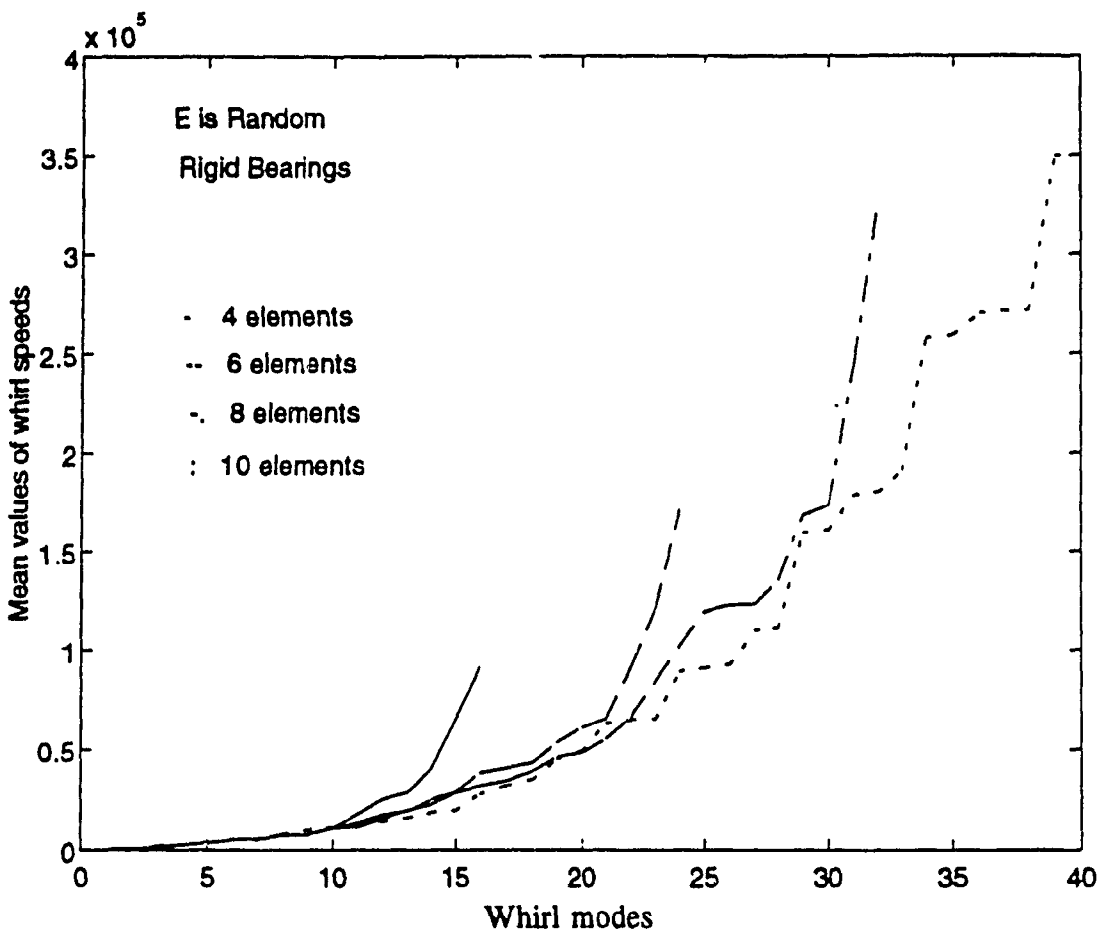

Fig. 5. 46: Influence of finite element modelling on the mean values of the whirl speeds of the rotor system on rigid bearings when E has a stochastic distribution.

change in the coefficient of variation of ω_1^2 as a function of the number of finite elements is presented in Fig. (5.47) for different values of the constant a. The mean values of the whirl speeds corresponding to various degrees of freedom of the rotordynamic system mounted on elastic supports are shown in Fig. (5.48), when the shaft is modelled using 4, 6, 8 and 10 finite elements.

5. 5 Design Aspects

The physical properties of engineering materials exhibit considerable spatial variations. The extent to which these variations occur depend on a number of factors including the chemical composition of these materials. The random nature of these variations as well as the uncertainty in the loading conditions necessitates the use of the so-called minimum strength and the safety factor during the design phase. Shinozuka and Lenoe [1976] have studied the probabilistic characteristics of the material properties so as to construct a probabilistic model that can be used for digital-analytical simulation of these material properties. When the correlation structures of the material properties cannot be precisely modelled, the probabilistic design of a rotor-support system can be based on the bounds on the second-order probabilistic moments of whirl speeds and whirl modes. For the case of a rotor-bearing system mounted on rigid bearings, the bounds on the coefficient of variation of ω_1^2 when the Young's modulus is random with a standard deviation of $\sigma_a = 1\times10^{-5}$ have been obtained for each of the five correlation structures and are given below:

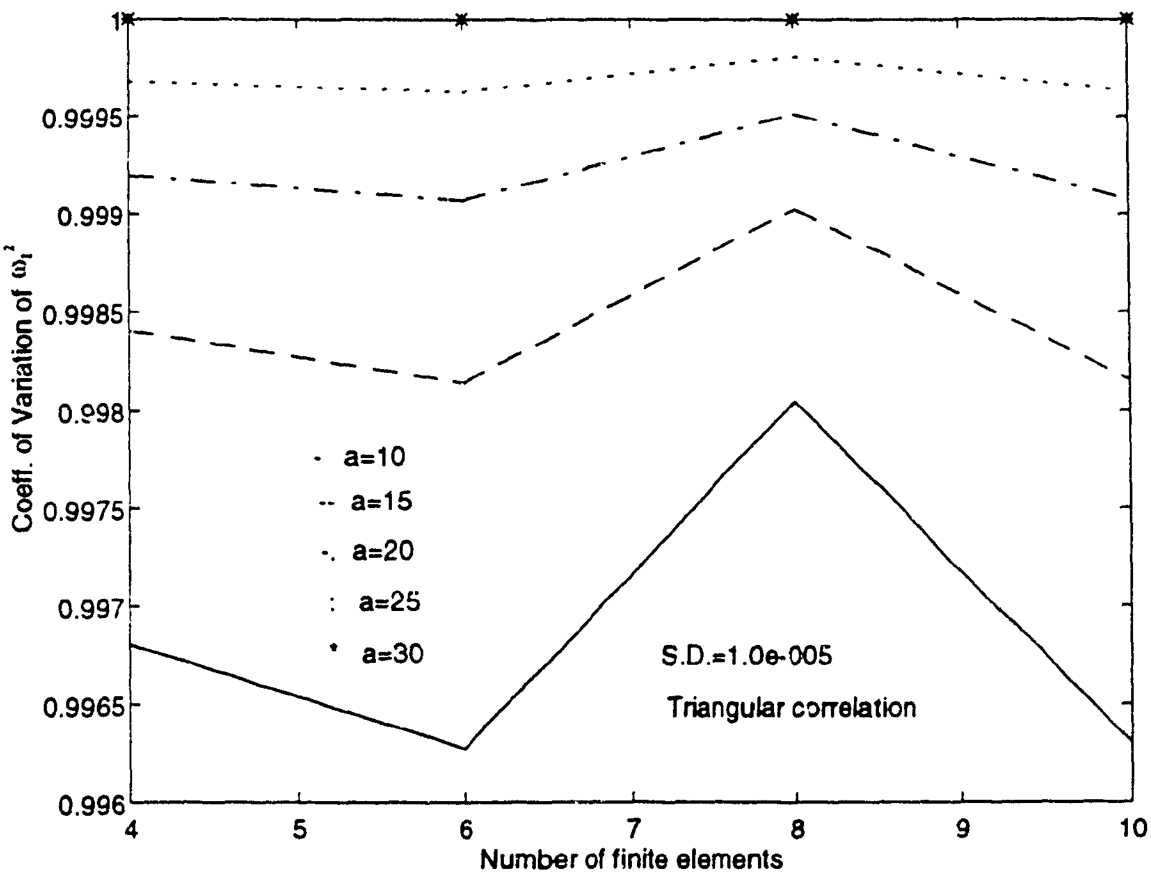

Fig. 5. 47: **Influence of finite element modelling on the coefficient of variation of ω_1^2 of the rotor system on isotropic bearings; E is random with a fixed standard deviation (S. D.); The triangular correlation model is used with 5 different values of the constant a.**

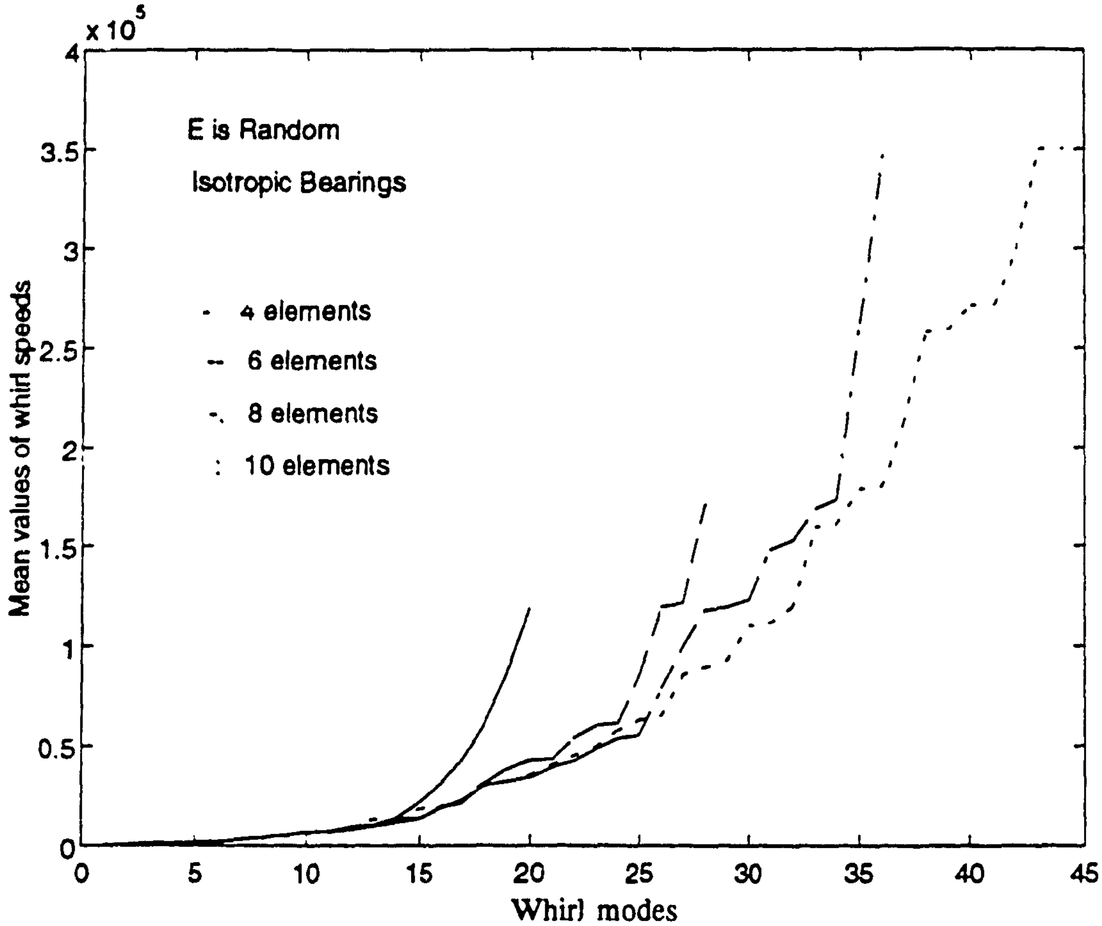

Fig. 5. 48: Influence of finite element modelling on the mean values of the whirl speeds of the rotor system on isotropic bearings when E has a stochastic distribution.

Triangular Correlation function ---- 3.2883 to 3.3017

AR (1) correlation function ---- 3.2886 to 3.3017

AR (2) correlation function ---- 3.3081 to 3.3083

Gaussian correlation function ---- 3.3078 to 3.3083

Finite Power White Noise

Correlation function ---- 2.3455 to 2.3463

The bounds for the coefficient of variation of the first whirl speed when the mass density is random, with a standard deviation of $\sigma_b = 1x10^{-5}$, are:

Triangular Correlation function ---- 186.4824 to 187.2589

AR (1) correlation function ---- 186.4992 to 187.2608

AR (2) correlation function ---- 187.6294 to 187.6441

Gaussian correlation function ---- 187.6121 to 187.6423

Finite Power White Noise

Correlation function ---- 132.5147 to 132.5612

Similar bounds have been established for the coefficient of variation of ω_1^2 for other values of the standard deviation corresponding to each of the above two cases as well as when both Young's modulus and mass density are random.

The influence of bearing flexibility on the bounds of the second-order probabilistic moments of whirl speeds is studied next. For a standard deviation of $\sigma_a = 1x10^{-5}$ for the Young's modulus, the bounds on the coefficient of variation of ω_1^2 for each of the five correlation structures are given below:

Triangular Correlation function	----	6.8006 to 6.8224
AR (1) correlation function	----	6.8011 to 6.8224
AR (2) correlation function	----	6.8327 to 6.8331
Gaussian correlation function	----	6.8323 to 6.8331
Finite Power White Noise Correlation function	----	5.3421 to 5.3433

When the mass density is random, the bounds on the coefficient of variation of ω_1^2, for a standard deviation of $\sigma_b = 1\times10^{-5}$, are given below:

Triangular Correlation function	----	212.6019 to 213.4688
AR (1) correlation function	----	212.6207 to 213.4709
AR (2) correlation function	----	213.8825 to 213.8988
Gaussian correlation function	----	213.8633 to 213.8967
Finite Power White Noise Correlation function	----	152.3429 to 152.3945

Similar bounds have been established for the coefficient of variation of ω_1^2 for other values of the standard deviation in each of the above two cases as well as when both Young's modulus and mass density are random.

5.5.1 Sensitivity of the Whirl Speeds to Young's Modulus and Mass Density

The stochastic variation in the mass density has been found, from Figs. (5.28) and (5.29), to have a much greater influence on the coefficient of variation of ω_1^2 than does the stochastic variation in the Young's modulus. The effect of fluctuations in the Young's modulus and mass density on the whirl speed of the rotor-bearing system is now studied. The Young's modulus and mass density have been represented by Eqs. (4.17) and (4.18) while the eigenvalue has correspondingly been expanded into an asymptotic series as shown in Eq. (3.11). The variance of the stochastic component of the eigenvalue can be shown to be the ratio of the variance of the corresponding eigenvalue (obtained from the covariance matrix, Cov (λ_i, λ_j)) to the square of the mean eigenvalue.

By varying the standard deviation of the Young's modulus (i. e. the value of a(s) in Eq. (4.17)) the standard deviation of the stochastic component of the desired eigenvalue and hence that of the corresponding whirl speed has been determined. The standard deviation of the Young's modulus has been varied from 1% of Young's modulus (a(s)=0.01) to 50% of Young's modulus (a(s)=0.5) and the corresponding standard deviation of the whirl speed can be seen in Figs. (5.49-5.51). The shaft has been modelled using 4 finite elements and the Triangular correlation model has been used. In a similar manner, the standard deviation of the mass density (i. e. the value of b(s) in Eq. (4.18)) is varied to study the change in the standard deviation of the stochastic component of the desired eigenvalue and hence that of the corresponding whirl speed. Figs. (5.52-5.54) show the change in the whirl speed when the mass density is varied from 1% of mass density (b(s)=0.01) to 50% of mass density (b(s)=0.5). It may be seen that the whirl speed is very sensitive to fluctuations in the mass density. For a 50% increase in mass density, the whirl speed increases by about 325%. On the contrary, when the Young's modulus is increased by 50%, the whirl speed increases by just over 40%.

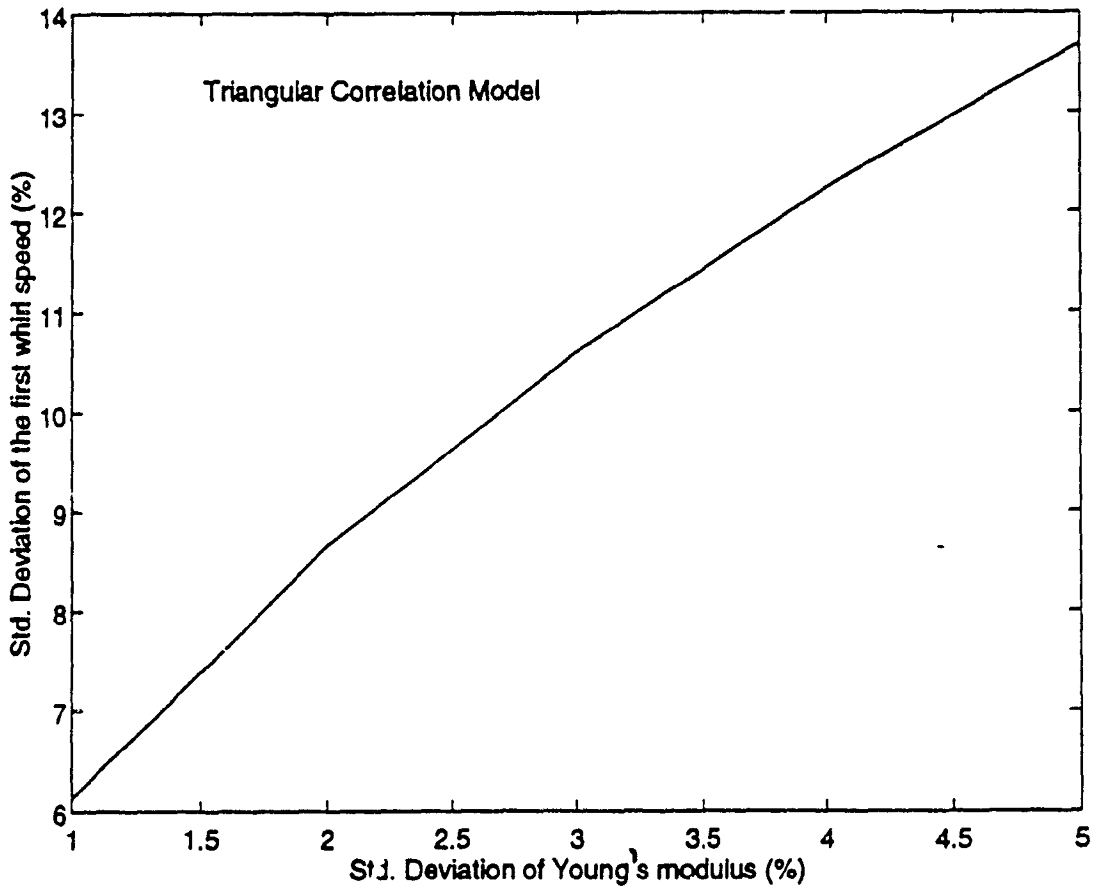

Fig. 5. 49: Variation of the standard deviation of the first whirl speed with the standard deviation of random Young's modulus -- I.

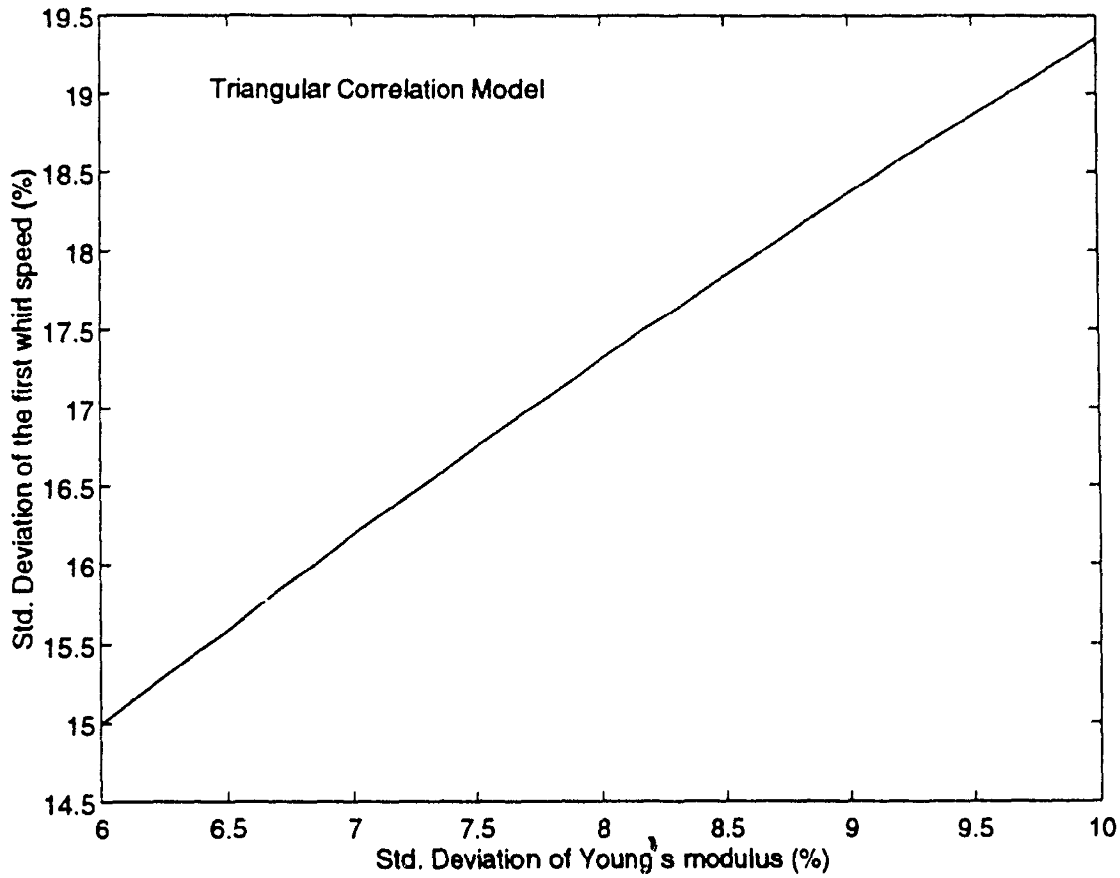

Fig. 5. 50: Variation of the standard deviation of the first whirl speed with the standard deviation of random Young's modulus -- II.

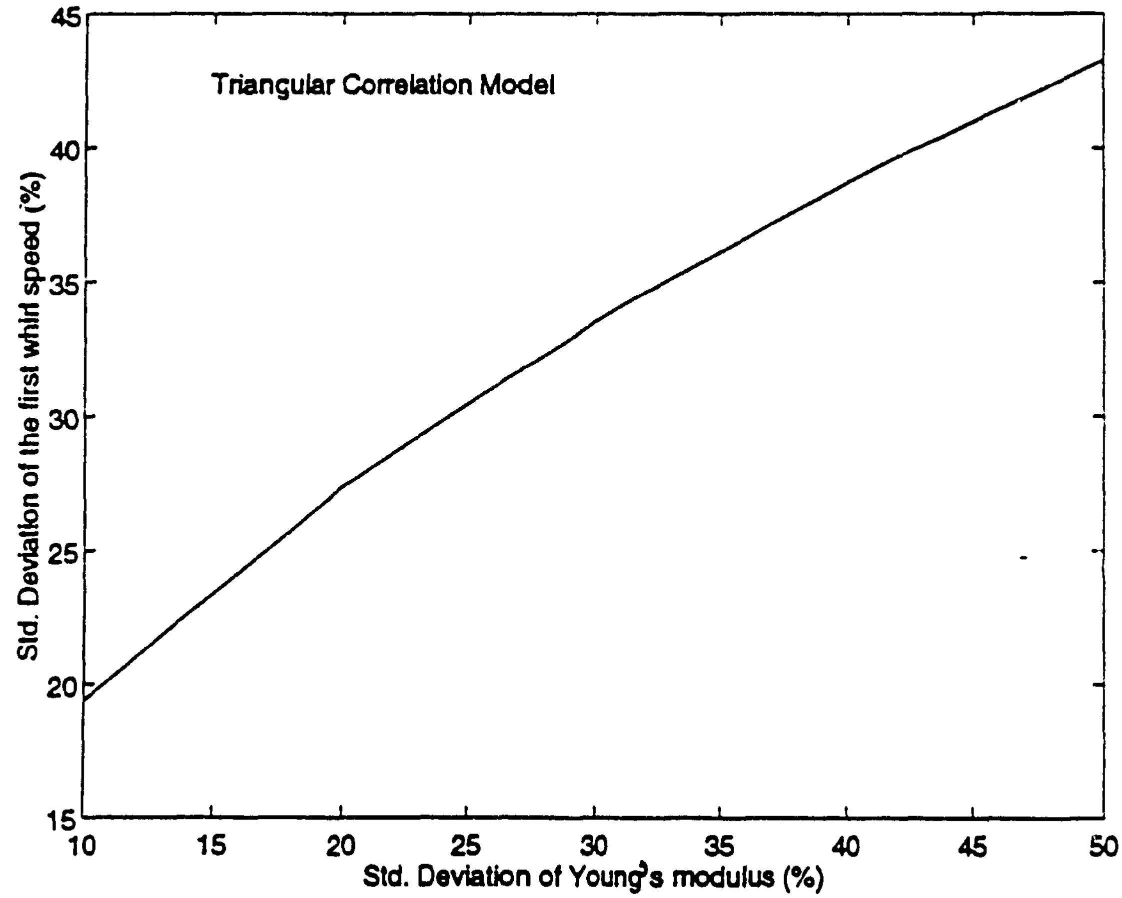

Fig. 5. 51: **Variation of the standard deviation of the first whirl speed with the standard deviation of random Young's modulus -- III.**

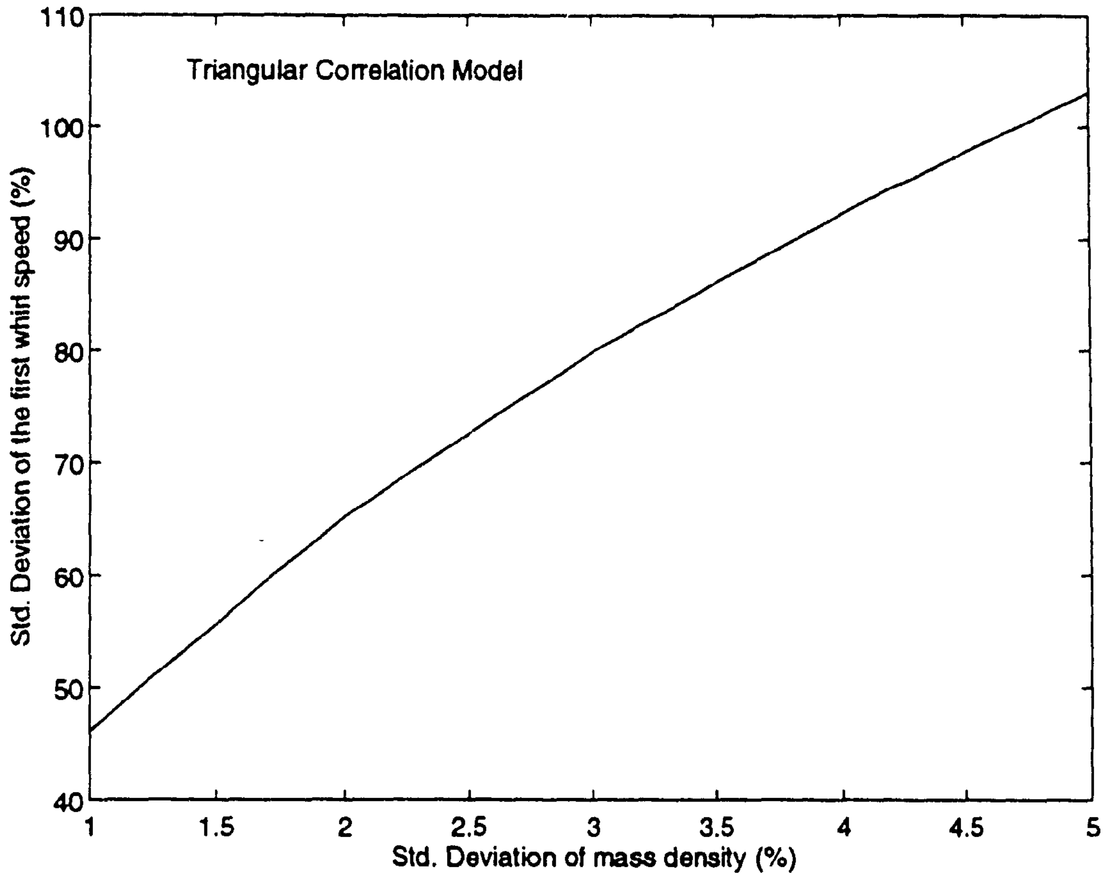

Fig. 5. 52: Variation of the standard deviation of the first whirl speed with the standard deviation of random mass density -- I.

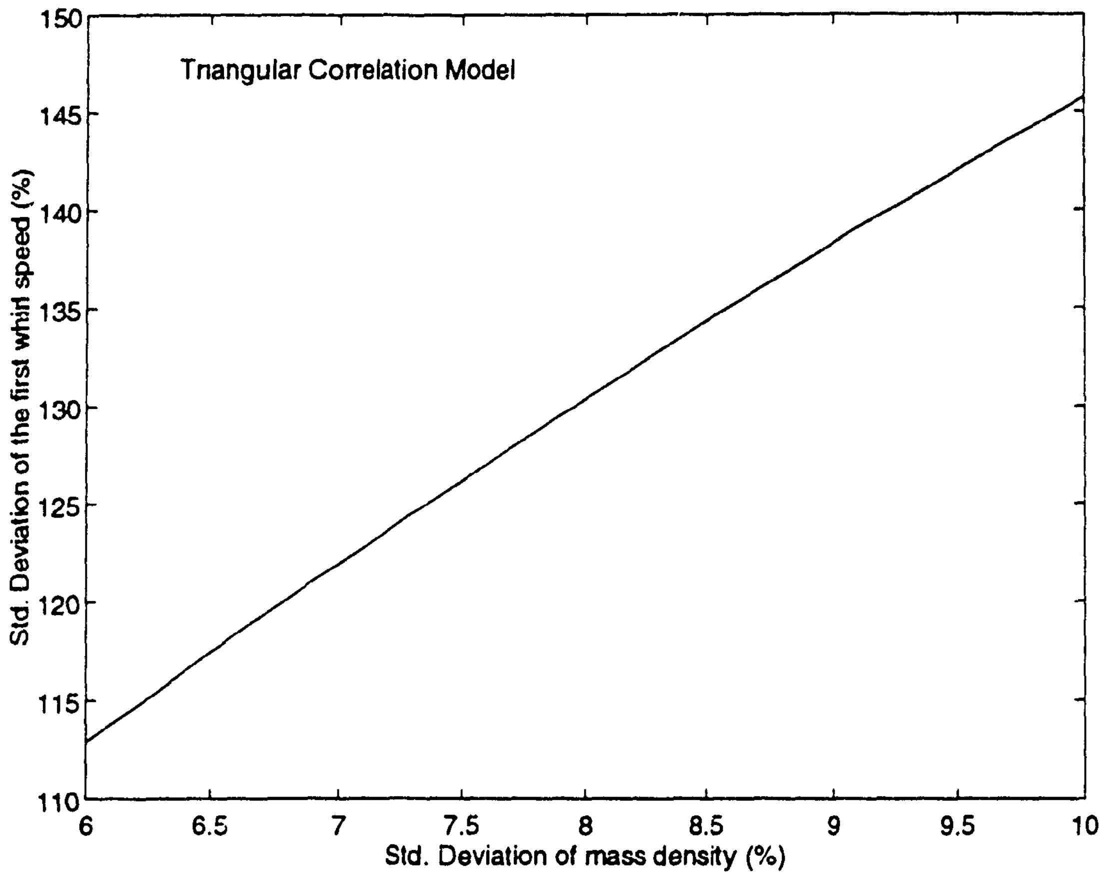

Fig. 5. 53: **Variation of the standard deviation of the first whirl speed with the standard deviation of random mass density -- II.**

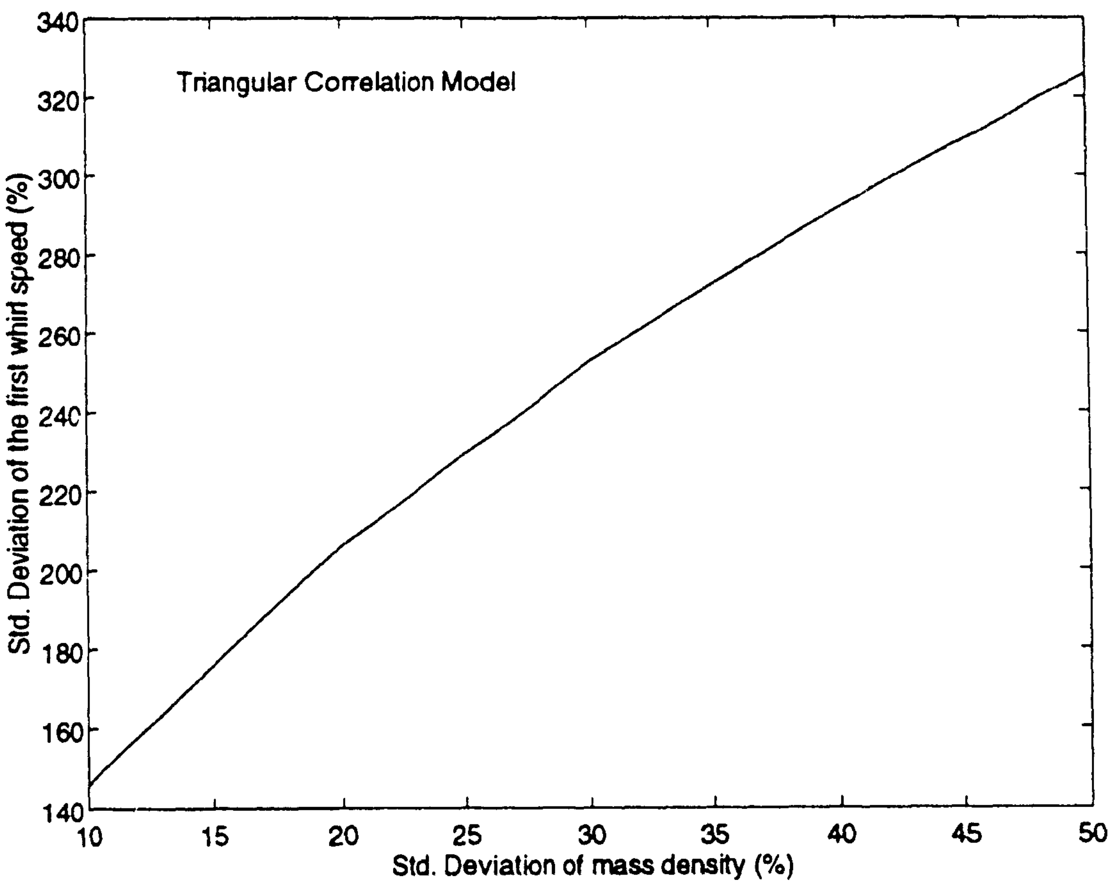

Fig. 5. 54: **Variation of the standard deviation of the first whirl speed with the standard deviation of random mass density -- III.**

Chapter 6

Discussions and Conclusions

Non self-adjoint eigenproblems in which the coefficients of the characteristic matrices are stochastic functions have been considered in this thesis. The objective has been to both qualify and quantify the effects on the variability of eigenvalues and eigenvectors of the randomness in these coefficients. For a mechanical or structural system, the coefficients of the characteristics matrices stem from the material properties and external loadings of the system. Hence, the randomness in the coefficients of the characteristic matrices arises due to the randomness in the material properties and external loadings of the mechanical or structural system. For a distributed-parameter or continuous mechanical or structural system, a corresponding MDOF system model can be obtained based on the finite element formulation. The FEM can be employed to determine the relationships between the parameters such as the stiffness coefficients and the mass coefficients of the MDOF system, and the material properties.

In the present work, the stochastic functions of MDOF system parameters are derived in terms of the stochastic functions that model the variations in the material properties and external loadings. The material properties, the Young's modulus and mass density, are modelled using independent one-dimensional univariate homogeneous stochastic fields. The external loading such as the end axial thrust is modelled using a random variable. Each of the stochastic fields are characterized by their ensemble mean values and autocorrelation functions or their equivalent power spectral density functions. The random variable is characterized by its mean value and standard deviation. The stochastic properties such as the ensemble mean values and autocorrelation functions of MDOF system parameters are derived in terms of that of the material property stochastic fields and the probabilistic moments of the random variable that models the axial loading.

The probabilistic moments of eigensolutions are determined in Chapter 3 using the asymptotic expansion and the first-order second-moment response method of probabilistic analysis. Each eigenvalue is expanded into an asymptotic series in terms of each and all of the MDOF system parameters. The randomness in the MDOF system parameters is characterized by introducing an asymptotic parameter into the asymptotic series. This asymptotic parameter takes on any value between 0 and 1 depending on the degree of randomness in the material properties and axial loading. The value of 1 represents the extreme case of randomness, i.e. a white noise stochastic field. The value of 0 represents the other extreme case of randomness, i.e. perfectly correlated stochastic field. For a particular eigenvalue, its corresponding asymptotic series is interpreted to be constituting a sample realization in the ensemble of that particular eigenvalue. This way, equations for the sample realizations of each and all of the eigenvalues of the MDOF system are derived. In a similar manner, the sample realizations of eigenvectors are determined. The sample realization of each MDOF system parameter is expressed in terms of the sample realizations of material properties and axial loading. The sample realization of each eigenvalue and eigenvector is, thus, expressed in terms of the sample realizations of material properties and axial loading.

Using the asymptotic series for the sample realization of each eigenvalue, and based on the method of response moment calculation, the probabilistic moments of each eigenvalue are determined. The relationship between the the sample realization of each eigenvalue and the sample realizations of material properties and axial loading (that has been derived previously), is central to this evaluation. The probabilistic moments of each eigenvalue are determined in terms of (i) the stochastic properties of material properties, (ii) the probabilistic moments of axial loading, (iii) the asymptotic parameter characterizing the randomness in material properties and axial loading, and (iv) the so-called "stochastic sensitivity gradients". The stochastic sensitivity gradients are expressed in terms of the eigensolutions of the so-called "averaged MDOF system".

It is observed that the first-order probabilistic moments can be obtained by solving the non self-adjoint eigenproblem with averaged coefficients, only once. From the equations for the second-order probabilistic moments of eigenvalues, it is observed that (i) for a given set of values of the coefficients of variation of the material properties and axial loading, the second-order probabilistic moments of eigenvalues vary with the correlation properties of the stochastic fields, (ii) the second-order probabilistic moments of each eigenvalue are linearly related to the autocorrelation functions of the material property stochastic fields as well as the variance of the random variable modelling the axial loading, (iii) the second-order probabilistic moments of eigenvalues are relatively more sensitive to the randomness in mass density than the randomness in the Young's modulus, and (iv) the difference between the effects of the randomness in the mass density on the second-order probabilistic moments of eigenvalues and that of the Young's modulus is of the order of the squared value of the eigenvalue under consideration.

Whirl speed analysis of high speed rotor-bearing systems has been considered in chapter 4. The effects on the variability of the whirl speeds and whirl modes, of randomness in material properties and external loadings are quantified. Using a rotating-coordinate formulation and finite element method, the whirl speed analysis is formulated into a random non self-adjoint eigenproblem. The equations developed for a general non self-adjoint system are considered and suitably modified in order to derive the equations for the first- and second-order probabilistic moments of whirl speeds and whirl modes. From the equations developed, it can be observed that (i) the variability in whirl speeds and whirl modes is severely effected by the randomness in the mass density of the rotor shaft, (ii) the second-order probabilistic moments of each whirl speed are linearly related to the ensemble mean value of that eigenvalue, and to the autocorrelation functions of the Young's modulus and mass density of the shaft material.

A parametric study is conducted, in chapter 5, in order to characterize the effects on the variability of whirl speeds and whirl modes, of power spectral density functions of material property stochastic fields. That is, the effects of the correlation properties of the material property stochastic fields on the variability of whirl speeds are considered. It is well known that in practical applications, it is seldom possible to determine precisely with a prescribed confidence level the exact autocorrelation functions or the power spectral density functions of the material properties. But, it has already been pointed out that the correlation structures or the power spectral density functions of material property stochastic fields significantly affect the second-order moments of the whirl speeds. A probabilistic design or a reliability-based design of a rotor-support system requires, as a central input information, the second-order probabilistic moments of whirl speeds and whirl modes. If the correlation structures of the material properties are not precisely modelled, the proper design configuration cannot be achieved.

However, in these circumstances, the design can be based on the bounds on the second-order probabilistic moments of whirl speeds and whirl modes. In the present thesis, these bounds are obtained in a systematic manner, by considering the most-commonly observed autocorrelation functions. Five different correlation functions viz. the triangular correlation function, the Markov process or AR (1) correlation function, second-order autoregressive correlation function i.e. AR (2) correlation function, Gaussian correlation function, and finite power white noise correlation function are considered. An extensive numerical study has been conducted and the following observations have been made:

1. A study of the absolute values of the second-order probabilistic moments of whirl speeds indicates that there is a significant increase in each and all of the coefficients of the covariance matrices of whirl speeds for increasing values of the standard deviations of the Young's modulus and mass density of the shaft material used. This has been observed for five different values of the standard deviation (σ_a in the case of Young's modulus and σ_b in the case of mass density) viz., $1x10^{-5}$, $4x10^{-5}$, $7x10^{-5}$, $1x10^{-4}$ and $1.3x10^{-4}$. For each value of the standard deviation the parametric study has been carried out for 5 different correlation functions i.e., triangular correlation function, the Markov

process, second-order autoregressive correlation function, Gaussian correlation function, and finite power white noise correlation function. The above finding has been consistent in each of these cases.

2. For a given set of values of the standard deviations of the material property random fields, the triangular correlation function, AR (1) correlation function, AR (2) correlation function and the Gaussian correlation function each yield coefficients of the covariance matrix of whirl speeds whose absolute values are very close to each other. The absolute values of these coefficients for the finite power white noise correlation function, however, are much less than that of the coefficients that correspond to the above 4 correlation functions. This was observed to be the case for each of the standard deviations considered in this study.

3. Bounds for the second-order probabilistic moments of the fundamental whirl speed of the rotor-bearing system have been obtained considering three different cases viz., when the Young's modulus is random, when the mass density is random and when both the Young's modulus and the mass density are random. In each of these cases, the bounds have been obtained for the five correlation functions corresponding to each of the five different values of the standard deviation viz , $1x10^{-5}$, $4x10^{-5}$, $7x10^{-5}$, $1x10^{-4}$ and $1.3x10^{-4}$.

4. A study of the influence of the lag distance in each of the correlation functions on the absolute values of the coefficient of variation of ω_1^2 has been performed. The coefficient of variation of ω_1^2 has been found to be very sensitive to the lag distance for the Triangular and AR(1) correlation functions. The coefficient of variation of ω_1^2 vary in

a non-linear manner with increasing values of the lag distances for these two correlation functions. The sensitivity of the coefficient of variation of ω_1^2 to the lag distances of the Second-Order Autoregressive, Gaussian and Finite Power White Noise correlation functions are found to be relatively less significant. This behaviour has been observed when the Young's modulus is random, when the mass density is random and when both Young's modulus as well as mass density are random.

5. The randomness in the mass density of the material has a considerable impact on the second-order probabilistic moments of whirl speeds when compared with that due to the randomness in the Young's modulus of the material. Since the randomness in the Young's modulus of the material has to be controlled at the micromechanics level, more attention should be paid, during the design stage, to handle randomness in the mass distribution of the mechanical component.

A comparative numerical study regarding the relative influences of randomness in Young's modulus and randomness in mass density, on the percentage variation of fundamental whirl speed is very helpful for reliability-based design applications. Such a comparative study has also been performed and the results plotted in Chapter 5.

The parametric study has been extended to the case of a rotor-bearing system mounted on isotropic bearings. As in the case when the rotor-bearing system is mounted on rigid bearings, the second-order probabilistic moments of whirl speeds are determined for 5 different values of the standard deviation of the Young's modulus and mass density. In each case, the numerical study is carried out for the 5 correlation functions so as to quantify the influence of the bearing flexibility on the second-order probabilistic moments of whirl speeds. The following observations were made:

6. The coefficient of variation of ω_1^2 due to the randomness in Young's modulus and when the rotor is mounted on flexible (and isotropic) bearings, has been determined. This value is almost twice that of the coefficient of variation of ω_1^2 due to the randomness in Young's modulus when the rotor is mounted on rigid bearings. Consistent observations have been made for the five correlation models and different values of standard deviations of material property random fields.

7. When the mass density is random, the coefficient of variation of ω_1^2 in the case of the rotor-bearing system resting on isotropic bearings is higher than the coefficient of variation of ω_1^2 when the rotor rests on rigid bearings. The proportional increase in the coefficient of variation of ω_1^2 is not as high as that when the Young's modulus is random.

8. Bounds for the second-order probabilistic moments of the fundamental whirl speed of the rotor-bearing system have been obtained for three different cases viz., when the Young's modulus is random, when the mass density is random and when both the Young's modulus and the mass density are random. In each of these cases, the bounds have been obtained for the five correlation functions corresponding to each of the five different values of the standard deviation viz., 1×10^{-5}, 4×10^{-5}, 7×10^{-5}, 1×10^{-4} and 1.3×10^{-4}.

9. Observations 1, 2, 4 and 5 made in the case of the rotor-bearing system resting on rigid bearings have also been noticed when the rotor is mounted on isotropic bearings.

The influences on the second-order probabilistic moments of whirl speeds of the mesh size in the finite element model of the actual rotor-bearing system have been considered next. A numerical study is conducted by modelling the shaft as 4, 6, 8 and 10 finite elements. The coefficient of variation of ω_1^2 is determined when the Young's modulus is random and has a standard deviation of $\sigma_a = 1 \times 10^{-5}$ and when the shaft is modelled as 4, 6, 8 and 10 finite elements. The study has been carried out for the 5 correlation functions viz. Triangular, AR(1), AR(2), Gaussian and Finite Power White Noise. The second-order probabilistic moments of whirl speeds are found to increase gradually as the number of finite elements are increased from 4 to 6. When the shaft is modelled as 8 finite elements, the coefficient of variation of ω_1^2 increases sharply. This is followed by a sharp decrease in the coefficient of variation of ω_1^2 when the shaft is modelled as 10 finite elements. The influence of finite element modelling on the second-order moments of whirl speeds, when the rotor is mounted on isotropic bearings, is also studied. The second-order probabilistic moments decrease slightly when the number of finite elements is increased from 4 to 6. When the shaft is modelled as 8 finite elements, there is a sudden increase in the coefficient of variation of ω_1^2 followed by a sharp decrease when the shaft is modelled using 10 finite elements. This sharp decrease is on account of the fact that the lag distance $(\xi_1-\xi_2)$ exceeds the triangulation correlation constant 'a', for the case when the modelling of the shaft exceeds 8 finite elements.

The values of the standard deviation in the Young's modulus and mass density, used in this thesis for the purpose of carrying out a parametric study, were arbitrarily chosen (viz., 1×10^{-5}, 4×10^{-5}, 7×10^{-5}, 1×10^{-4} and 1.3×10^{-4}) to bring out the relative effects of the various correlation structures on the coefficient of variation of the square of the first whirl speed, ω_1^2. A similar parametric study with practical values of standard deviation in Young's modulus and mass density, say 1-5 % of the mean value of the Young's modulus and mass density respectively, has been set aside for future work in the interest of time. Particular attention should be paid to the algorithm that computes the eigenvalues and eigenvectors so as to make it robust. Also, care should be taken to ensure that the resulting eigenvectors are orthonormal as this is a requirement for designing the software that computes the coefficient of variation of ω_1^2.

The probabilistic moments of whirl speeds and whirl modes have been obtained in this thesis. For normal distribution function, the first two moments completely define the probability structure of whirl speeds and whirl modes. In addition, the so-called "level crossing probabilities", i.e. the determination of the probability for a particular whirl speed to exceed a design speed, and the so-called "envelope statistics", i.e. the probabilistic structure of bounding values of whirl speeds can be determined using the probabilistic moments of whirl speeds that have been determined in the present work.

The results developed in chapter 3 encompass the cases when the moment of inertia and the cross-sectional area of rotor shafts are also random. The misalignment problems and the randomness inherent in such cases can also be treated using the results of chapter 3. The specific applications to the cases when the Young's modulus and mass density are stochastic have been considered in chapter 4. The formulations are exactly the same for the othei cases mentioned above.

The independent variable x in E(x) and m(x) is a parameter representative of the attribute of the material property randomness. It can be a metallurgical process variable, time, degree of damage to the material etc. This way, the variability in dynamic response is analytically related to a measure of the attributing factor to the randomness in the dynamic system. The results of micromechanics can thus be made use of in the design of mechanical components for reliability. Equivalently, if it is desired to reduce the variability in dynamic response of the mechanical component, the information as to what is the corresponding change that must be made at the material level, can be obtained using the probabilistic analysis presented in this thesis.

In practical applications, it is seldom possible to precisely determine the correlation function for the test data on material property fluctuations. But the type of the correlation structure has considerable influence, for the same standard deviation of the material property fluctuations, on the probabilistic moments of whirl speeds and whirl modes. It is very useful in design applications if a quantitative information as to the relative influences of various correlation structures is available. Such an information has been obtained in the present work.

Moreover, in these situations, the reliability-based design can be achieved by considering the bounds on the probabilistic moments of whirl speeds and whirl modes, through the so-called "extreme value hypothesis" of probabilistic design. It is demonstrated that these bounds can be obtained from the results of present work.

While performing a finite element analysis for design, the relevant mesh size is determined based on the type of the structural system, problem characteristic difficulties such as singularities, steep stress gradients etc. and the nature of the loadings. It has been shown in the present work that the mesh size has a direct bearing on the probabilistic moments of the whirl speeds, whirl modes and hence the dynamic response. It is shown that an increase in the number of finite elements in the finite element model of the structural system may also result in the underestimation of the variability in dynamic response, for a given standard deviation of the material properties. The design then becomes unsafe and the reliability is reduced. For a particular configuration of the structural system, the optimum mesh size from the probabilistic point of view can be determined through a parametric study. The results of the present work can be used in this regard. The relevant procedure has been demonstrated in chapter 5.

Appendix

This appendix is related to Section 3. 3 of the thesis and describes a procedure that serves to illustrate the method used to generate sample realizations of a random matrix [A] and obtain the averaged matrix $[\tilde{A}]$ of these sample realizations. Each of the elements of the various sample realizations of the random matrix [A] is characterized by a certain mean and standard deviation. Thus, the (i,j)th element of the various sample realizations have the same mean and standard deviation. Sample realizations of a random matrix [B] are generated and the averaged matrix $[\bar{B}]$ obtained in a manner identical to that for matrix [A]. Each of the elements of the various sample realizations of the random matrix [B] are characterized by a certain mean and standard deviation.

The eigenproblem is studied, using the first sample realization pair $[A_1]$ and $[B_1]$ of size m×m as the characteristic matrices, for the eigenvalues (λ_{11}, λ_{21}, ..., λ_{m1}), the eigenvectors to the left ($\{y\}_{11}$, $\{y\}_{21}$, ..., $\{y\}_{m1}$) and the eigenvectors to the right ($\{x\}_{11}$, $\{x\}_{21}$, ..., $\{x\}_{m1}$). This study is repeated with the subsequent sample realization pairs of [A] and [B] ($[A_2]$ and $[B_2]$, $[A_3]$ and $[B_3]$, ..., $[A_n]$ and $[B_n]$) and the eigenvalues (λ_{12}, λ_{22}, ..., λ_{m2} ; λ_{13}, λ_{23}, ..., λ_{m3} ; .... ; λ_{1n}, λ_{2n}, ..., λ_{mn}), the eigenvectors to the left ($\{y\}_{12}$, $\{y\}_{22}$, ..., $\{y\}_{m2}$; $\{y\}_{13}$, $\{y\}_{23}$, ..., $\{y\}_{m3}$; .... ; $\{y\}_{1n}$, $\{y\}_{2n}$, ..., $\{y\}_{mn}$) as well as the eigenvectors to the right ($\{x\}_{12}$, $\{x\}_{22}$, ..., $\{x\}_{m2}$; $\{x\}_{13}$, $\{x\}_{23}$, ..., $\{x\}_{m3}$; .... ; $\{x\}_{1n}$, $\{x\}_{2n}$, ..., $\{x\}_{mn}$) are computed in each of these cases. Finally, the eigenproblem is studied with the averaged matrices $[\bar{A}]$ and $[\bar{B}]$ and the corresponding averaged eigenvalues $(\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_m)$, the averaged eigenvectors to the left $(\{\bar{y}\}_1, \{\bar{y}\}_2, ..., \{\bar{y}\}_m)$ and the averaged eigenvectors to the right $(\{\bar{x}\}_1, \{\bar{x}\}_2, ..., \{\bar{x}\}_m)$ are determined. A comparison is done between the averaged eigenvalue $(\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_m)$ and the mean $(\lambda_1, \lambda_2, ..., \lambda_m)$ of the eigenvalues $(\lambda_{11}, \lambda_{21}, ..., \lambda_{m1} ; \lambda_{12}, \lambda_{22}, ..., \lambda_{m2} ; .... ; \lambda_{1n}, \lambda_{2n}, ..., \lambda_{mn})$. Similarly, the averaged eigenvector to the right $(\{\bar{x}\}_1, \{\bar{x}\}_2, ..., \{\bar{x}\}_m)$ is compared with the mean $(\{x\}_1, \{x\}_2, ..., \{x\}_m)$ of the eigenvectors to the right $(\{x\}_{11}, \{x\}_{21}, ..., \{x\}_{m1} ; \{x\}_{12}, \{x\}_{22}, ..., \{x\}_{m2} ; .... ; \{x\}_{1n}, \{x\}_{2n}, ..., \{x\}_{mn})$, while the averaged eigenvector to the left $(\{\bar{y}\}_1, \{\bar{y}\}_2, ..., \{\bar{y}\}_m)$ is compared with the mean $(\{y\}_1, \{y\}_2, ..., \{y\}_m)$ of the eigenvectors to the left $(\{y\}_{11}, \{y\}_{21}, ..., \{y\}_{m1} ; \{y\}_{12}, \{y\}_{22}, ..., \{y\}_{m2} ; .... ; \{y\}_{1n}, \{y\}_{2n}, ..., \{y\}_{mn})$.

Random numbers within the range of 0 to 1 are generated using the formulation presented in Gottfried [1984] that have a normal distribution. A normal random variate X is generated as:

$$X=\mu+\sigma*Z \qquad ; \qquad Z = \frac{\sum_{k=1}^{N} (U_k - N/2)}{\sqrt{N/12}}$$

where, μ and σ are the mean and standard deviation of the variate X, while Z is the standard normal and U_k represents independent uniform random numbers. In this exercise, Z is computed by choosing N=100 thus requiring U_k's to be generated 100 times by invoking the "rand" function each time. The normal random variate X is then computed using this value of Z and a knowledge of the mean μ and and standard deviation σ of X. The value of X thus computed forms the (i,j)th element of a sample realization $[A_1]$, with mean μ_{ij} and standard deviation σ_{ij}, of a random matrix [A]. The other elements of the sample realization $[A_1]$ are similarly determined with a knowledge of the mean and standard deviation of each of these elements. After constructing the sample realization $[A_1]$, subsequent sample realizations of [A] are constructed. For purposes of illustration, matrices of size 3x3 are constructed and 50 sample realizations of a matrix are generated. The different sample realizations of the random matrix [B] are generated in a manner similar to that for the random matrix [A]. For simplicity, the mean μ_{ij} and standard deviation σ_{ij} for the (i,j)th element of the various sample realizations of [B] is chosen to be the same as that for the (i,j)th element of the various sample realizations of [A]. The averaged matrix $[\bar{A}]$ is obtained by computing . . mean of the individual matrix elements of [A] over the various sample realizations. The averaged matrix $[\bar{B}]$ is formed similarly. During the generation of the various sample realizations of the random matrices [A] and [B] care is taken to discard singular matrices that results in division by zero.

The software developed to generate the various sample realizations of a random matrix and compute the averaged matrix is presented at the end of this appendix. The mean and standard deviation of the matrix elements of [A] and [B] are chosen as follows:-

$\mu_A(1,1) = \mu_B(1,1) = 0.5$; $\sigma_A(1,1) = \sigma_B(1,1) = 0.01$;

$\mu_A(1,2) = \mu_B(1,2) = 0.3$; $\sigma_A(1,2) = \sigma_B(1,2) = 0.01$;

$\mu_A(1,3) = \mu_B(1,3) = 0.7$; $\sigma_A(1,3) = \sigma_B(1,3) = 0.01$;

$\mu_A(2,1) = \mu_B(2,1) = 0.3;\ \sigma_A(2,1) = \sigma_B(2,1) = 0.1;$

$\mu_A(2,2) = \mu_B(2,2) = 0.4;\ \sigma_A(2,2) = \sigma_B(2,2) = 0.1;$

$\mu_A(2,3) = \mu_B(2,3) = 0.6;\ \sigma_A(2,3) = \sigma_B(2,3) = 0.1;$

$\mu_A(3,1) = \mu_B(3,1) = 0.2;\ \sigma_A(3,1) = \sigma_B(3,1) = 0.05;$

$\mu_A(3,2) = \mu_B(3,2) = 0.6;\ \sigma_A(3,2) = \sigma_B(3,2) = 0.05;$

$\mu_A(3,3) = \mu_B(3,3) = 0.8;\ \sigma_A(3,3) = \sigma_B(3,3) = 0.05;$

The algorithm also solves the eigenvalue problem and is used with each of the 50 sample realizations of the 3x3 random matrices [A] and [B] so as to compute the eigenvalues ($\lambda_{11}, \lambda_{21}, \lambda_{31}$; $\lambda_{12}, \lambda_{22}, \lambda_{32}$; .... ; $\lambda_{150}, \lambda_{250}, \lambda_{350}$), the eigenvectors to the left ($\{y\}_{11}, \{y\}_{21}, \{y\}_{31}$; $\{y\}_{12}, \{y\}_{22}, \{y\}_{32}$; .... ; $\{y\}_{150}, \{y\}_{250}, \{y\}_{350}$) and the eigenvectors to the right ($\{x\}_{11}, \{x\}_{21}, \{x\}_{31}$; $\{x\}_{12}, \{x\}_{22}, \{x\}_{32}$; .... ; $\{x\}_{150}, \{x\}_{250}, \{x\}_{350}$). The mean of the eigenvalues computed above are determined and denoted as ($\lambda_1, \lambda_2, \lambda_3$) Similarly, the mean of the eigenvectors to the left and the mean of the eigenvectors to the right are computed and denoted respectively as ($\{y\}_1, \{y\}_2, \{y\}_3$) and ($\{x\}_1, \{x\}_2, \{x\}_3$). Using the averaged matrices $[\bar{A}]$ and $[\bar{B}]$, the eigenvalue routine determines the averaged eigenvalues ($\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$), the averaged eigenvectors to the left ($\{\bar{y}\}_1, \{\bar{y}\}_2, \{\bar{y}\}_3$) and the averaged eigenvectors to the right ($\{\bar{x}\}_1, \{\bar{x}\}_2, \{\bar{x}\}_3$). It is noted that due to the relatively small number of sample realizations chosen, for illustrating the objective set forth in the beginning, the averaged eigenvalues ($\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$) do not compare well with the mean ($\lambda_1, \lambda_2, \lambda_3$) of the eigenvalues. Similar observations are made with regards to the eigenvectors to the left as well as the eigenvectors to the right.

ALGORITHM

Step 1 Input the mean μ_A and μ_B for each of the elements of matrices [A] and [B]:

$$\mu_A(1,1)\,,\ \mu_A(1,2),\ \ldots,\ \mu_A(3,3)\ ;\ \mu_B(1,1)\,,\ \mu_B(1,2),\ \ldots,\ \mu_B(3,3)$$

Input the standard deviation σ_A and σ_B of the [A] and [B] matrix elements:

$$\sigma_A(1,1)\,,\ \sigma_A(1,2),\ \ldots,\ \sigma_A(3,3)\ ;\ \sigma_B(1,1)\,,\ \sigma_B(1,2),\ \ldots,\ \sigma_B(3,3)$$

Step 2 Repeat steps 3 to 8 so as to construct 50 sample realizations of non-singular random matrices [A] and [B], discarding those matrices that are singular.

Step 3 Construct a sample realization of [A] by computing A(i,j) for i, j = 1, 2, 3 using steps 4 and 5.

Step 4 Generate a random number U_k $(0<U_k<1)$, k = 1, 2, ..., 100

Step 5 A(i,j) is computed using the mean $\mu_A(i,j)$, the standard deviation $\sigma_A(i,j)$ as well as the random numbers U_k, generated in step 4, as follows:

$$A(i,j) = \mu_A(i,j) + [\sigma_A(i,j) * (\frac{\sum_{k=1}^{100} (U_k - 100/2)}{\sqrt{100/12}})]$$

Step 6 Construct a sample realization of [B] by computing B(i,j) for i, j = 1, 2, 3 using steps 7 and 8.

Step 7 Generate a random number U_k ($0<U_k<1$), k = 1, 2, ..., 100

Step 8 B(i,j) is computed using the mean $\mu_B(i,j)$, the standard deviation $\sigma_B(i,j)$ as well as the random numbers U_k, generated in step 7, as follows:

$$B(i,j) = \mu_B(i,j) + [\sigma_B(i,j) * (\frac{\sum_{k=1}^{100} (U_k - 100/2)}{\sqrt{100/12}})]$$

Step 9 Using an eigenvalue routine, for each of the sample realizations $[A_i]$ and $[B_i]$, determine the eigenvalues (λ_{i1}, λ_{i2}, λ_{i3}), the eigenvectors to the left ($\{y\}_{i1}$, $\{y\}_{i2}$, $\{y\}_{i3}$) and the eigenvectors to the right ($\{x\}_{i1}$, $\{x\}_{i2}$, $\{x\}_{i3}$), i = 1, 2, ..., 50.

Step 10 Compute the mean of the eigenvalues as:

$$\lambda_1 = \frac{\sum_{i=1}^{50} \lambda_{i1}}{50} \quad ; \quad \lambda_2 = \frac{\sum_{i=1}^{50} \lambda_{i2}}{50} \quad ; \quad \lambda_3 = \frac{\sum_{i=1}^{50} \lambda_{i3}}{50}$$

Similarly, the mean ($\{y\}_1$, $\{y\}_2$, $\{y\}_3$) of the eigenvectors to the left as well as the mean ($\{x\}_1$, $\{x\}_2$, $\{x\}_3$) of the eigenvectors to the right are computed.

Step 11 The mean of the various sample realizations of [A] and [B], viz. $[\bar{A}]$ and $[\bar{B}]$, are constructed from the computation of the individual matrix elements $\bar{A}(i,j)$ and $\bar{B}(i,j)$ using the following relation, for i, j = 1, 2, 3.

$$\bar{A}(i,j) = \sum_{k=1}^{50} A_k(i,j) \quad ; \quad \bar{B}(i,j) = \sum_{k=1}^{50} B_k(i,j)$$

Step 12 Using the eigenvalue routine for the averaged matrices $[\bar{A}]$ and $[\bar{B}]$ determine the averaged eigenvalues $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$, the averaged eigenvectors to the left $(\{\bar{y}\}_1, \{\bar{y}\}_2, \{\bar{y}\}_3)$ and the averaged eigenvectors to the right $(\{\bar{x}\}_1, \{\bar{x}\}_2, \{\bar{x}\}_3)$.

Step 13 Compare the eigenvalues $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$ with $(\lambda_1, \lambda_2, \lambda_3)$; the eigenvectors to the left $(\{\bar{y}\}_1, \{\bar{y}\}_2, \{\bar{y}\}_3)$ with $(\{y\}_1, \{y\}_2, \{y\}_3)$; and the eigenvectors to the right $(\{\bar{x}\}_1, \{\bar{x}\}_2, \{\bar{x}\}_3)$ with $(\{x\}_1, \{x\}_2, \{x\}_3)$.

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