FIXED POINT THEOREMS
IN METRIC SPACES

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ABSTRACT

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In this thesis we study fixed point theorems for functions in metric spaces.

In Chapter I, we consider the Banach Contraction Principle and its various offshoots. We give two new theorems which generalize some previous theorems.

The second chapter deals with contractive and nonexpansive mappings, both separately and jointly, and their fixed point.

In Chapter III, we give a brief survey of fixed point theorems for multivalued mappings, and conditions under which the common fixed point of two multivalued mappings will be unique. We also provide a localization theorem for the fixed points of a certain multivalued function. We extend some theorems for single-valued function to the multi-valued case, by defining a sequence of iterates for a multivalued mapping in a natural way.

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In a similar vein, thanks for the encouragement, G.L.!

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INTRODUCTION

I am never forgetting the day I first meet the great
Lobachevsky!
In one word he taught me secret of success in mathematics:
Plagiarize! Plagiarize!
Let no one else's work evade your eyes,
Remember, why the good Lord made your eyes,
So don't shade your eyes,
But plagiarize, plagiarize, plagiarize;
Only be sure always to call it, please, 'research'!

T. Lehrer.

In 1912, Brouwer gave the first fixed point theorem, stated as: Any continuous function from a disk in E^n into itself has a fixed point. The interest in fixed points grew since Brouwer's result, and in 1974 alone over 75 papers were published and several books, exclusively, on fixed point theorems were available. A selected bibliography (by no means complete) of papers in Fixed Point Theory for an eight-year period between 1965 and 1972, published in the Rocky Mountain Journal of Mathematics (Vol.4, No.1, 1974) lists over 390 papers and books.

Because of the vastness of the field, we concentrate on one part of it. We choose metric spaces to work in, as they are the "happy medium" between the more general topological space and the restrictive Banach spaces.

S. Banach, in 1922, among his many mathematical contributions proved the following statement: "A contraction mapping from a complete metric space into itself has a fixed point."

Due to its simplicity and usefulness, notably in Integral and Differential Equations, this theorem has been pounced upon, mutilated and improved upon by several mathematicians. In Chapter I, we give a brief survey of fixed point theorems for "contraction-type" mappings, and following some of the routine techniques, we too "plagiarize" Banach, by generalizing and extending some of the previous results. In the later portion of the second section, we extend a theorem of Wong [30] to generalized metric spaces. In the third section of the first chapter, we look at localized versions of fixed point theorems and prove a localized version of the theorem of Hardy and Rogers [17].

In Chapter II, we consider more general mappings than contractions, the so-called contractive and nonexpansive mapping. In the first section we consider the work of Singh [27] and Furi and Vignoli [15,16]. We also prove a theorem similar to that of Singh [27]. We show how contractive and nonexpansive mappings have been "mixed" to give fixed point theorems on metric spaces, which are not necessarily complete. In the end of the section, we prove a quite general result regarding the common fixed point of two mappings satisfying the Wong-type [30] condition in the lines of Cheney and Goldstein [7].

In Chapter III, we have given a brief account of some results on multivalued mappings due to Nadler [24] and Dube [12]. Two partial answers to the question posed by Dube [12] concerning the uniqueness of fixed points of multivalued mappings have been given.

In the second section of this chapter, we provide an altogether different method of proving the existence of fixed points of multivalued mappings under certain restrictions, by defining a sequence of iterates of multivalued mappings in a "natural" way, unlike the previous approaches. Using our appraoch we have extended the theorems of Belluce and Kirk [6] and that of Cheney and Goldstein [7] to multivalued mappings.

CHAPTER 1

CONTRACTIONS AND THEIR FIXED POINTS

1.1 Preliminaries.

Definition 1.1.1: Let X be a set and let $\mathbb{R}^+$ denote the nonnegative real numbers. A function $d: X \times X \to \mathbb{R}^+$ is said to be a metric if it satisfies the following:

i) $d(x,y) = 0$ iff $x = y$.

ii) $d(x,y) = d(y,x)$, for all $x, y \in X$.

iii) $d(x,z) \leq d(y,z) + d(x,y)$ for all $x, y, z \in X$ (the triangle inequality).

The set X with a metric d is called a metric space, and is denoted by (X,d).

Definition 1.1.2: A sequence of points $\{x_n\}$ in a metric space (X,d) is said to converge to a point x of X if, for every positive real number ε, there exists a natural number N such that whenever $n \geq N$, then $d(x_n,x) < \varepsilon$, i.e. if $d(x_n,x)$ tends to zero as n tends to infinity.

Definition 1.1.3: A sequence of points $\{x_n\}$ in a metric space (X,d) is called a Cauchy sequence if, for every positive real number ε, there exists a natural number N, such that $d(x_n,x_m) < \varepsilon$, for $n, m \geq N$.

Definition 1.1.4 : A metric space (X,d) is said to be complete if every Cauchy sequence in (X,d) is convergent in (X,d).

Definition 1.1.5 : Let (X,d) and (X,d_1) be two metric spaces. The metrics d and d_1 are equivalent if and only if the sequence $\{x_n\}$ converges to x in (X,d) implies the sequence also converges to x in (X,d_1) and conversely.

Definition 1.1.6 : The diameter of a nonempty subset A of a metric space (X,d) is defined as the $\sup\{d(x,y) : x, y \in A\}$ and is denoted by $\delta(A)$.

A is bounded if and only if $\delta(A) < \infty$.

Remark 1.1.1 : It should be noted that not every metric space is bounded. An example of this is the real number line, with the usual metric. However, for every metric space (X,d) we can define an equivalent metric d_1, such that (X,d_1) is a bounded metric space. One way to define d_1 is by $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$.

Definition 1.1.7 : A mapping T of a nonempty set A into itself is said to have a fixed point if there exists a point $x \in A$ such that $T(x) = x$. In other words, a point x is a fixed point of T if it remains invariant under the mapping T. The point $x \in A$ is said to be the unique fixed point of T if $Tx = x$ and if $Ty = y$

implies $x = y$.

Definition 1.1.8: A mapping T of a metric space (X,d) into itself is said to satisfy the Lipschitz condition if there exists a real number k such that

$$d(Tx, Ty) \leq kd(x,y) \quad \text{for all } x,y \in X .$$

T is said to be a contraction if $0 \leq k < 1$, and nonexpansive if $k = 1$.

Functions of the form $T(x) = x/p$, defined on the real numbers, with p a real number greater than one, are examples of contraction mapping (and hence, also examples of nonexpansive mappings).

1.2 Banach Contraction Principle and its Generalizations.

We would like to look at what conditions are needed for certain mappings to have a fixed point. One of the best known fixed point theorems is due to Banach [2], called the Banach Contraction Principle. We state this theorem, with a short outline of the proof.

Theorem 1.2.1: Let (X,d) be a complete metric space and $T : X \to X$ be a contraction, i.e.,

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all } x,y \in X$$

with $0 \leq k < 1$. Then T has a unique fixed point.

The proof of this theorem is by the method of successive approximations. By taking an arbitrary point x_0, a sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$, for $n = 0,1,2,\ldots$. This sequence turns out to be Cauchy and hence convergent, since X is complete. The limit of this sequence is the unique fixed point of T.

<u>Remark 1.2.1</u>: (i) It is interesting to note that the limit of the sequence, defined above, does not depend on the initial point x_0. So the result of successive applications of T is the "shrinking" of the space X into the unique fixed point of T. Mathematically, if u is the fixed point of T then $\lim_{n\to\infty} T^n(X) = u$.

(ii) This method also yields a way of finding an "approximate" value of the fixed point. The error of the approximations is estimated by the inequality

$$d(x_n,u) \leq \frac{k^n}{1-k}\, d(x_0,x_1) .$$

(iii) The sequence defined above is called the sequence of successive approximations.

The Banach Contraction Principle has been generalized and extended in many different ways by several mathematicians. We state some of these extensions without proof.

Chu and Diaz [8] assumed that T^n, rather than T, was a contraction for some natural number n, and showed that T still had a unique fixed point.

<u>Theorem 1.2.2</u>: Let (X,d) be a complete metric space

such that for some $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq k d(x,y) \quad \text{for all } x,y \in X$$

with $0 \leq k < 1$. Then, T has a unique fixed point.

The following example illustrates that this theorem is more general than the theorem due to Banach.

Example 1.2.1 : Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$T(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

where $\mathbb{R}$ is equipped with the usual metric. It is easy to see that this function does not satisfy the conditions of Theorem 1.2.1. However $T^2 x = 1$, for all x and thus T^2 is a contraction. Therefore, by Theorem 1.2.2, T has a unique fixed point (by inspection, $x = 1$ is the unique fixed point of T).

Rakotch [25] replaced the constant k by a real-valued function. He defined a family F of functions $\lambda(x,y)$ satisfying:

(1) $\lambda(x,y) = \lambda(d(x,y))$, i.e. λ depends only on the distance between x and y .

(2) $0 \leq \lambda(d) < 1$, for $d > 0$.

(3) $\lambda(d)$ is a monotone decreasing function of d .

He then proved the following:

Theorem 1.2.3 : If $T : X \to X$ is a mapping from a complete metric space into itself such that

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) \quad \text{for } x,y \in X \text{ and } \alpha \in F,$$

then T has a unique fixed point.

Browder [5] gave a similar result to that of Rakotch.

Theorem 1.2.4 : Let (X,d) be a complete metric space, and T be a mapping of X into itself such that

$$d(Tx,Ty) \leq f(d(x,y))$$

where f is a right continuous, nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$, with $f(t) < t$, for $t > 0$. Then T has a unique fixed point.

Luxemburg [22] modified the concept of a metric space by allowing infinite distances between points.

Definition 1.2.1 : Let X by a set and $d : X \times X \to \mathbb{R}^*$, where $\mathbb{R}^*$ is the extended nonnegative real numbers. If d satisfies the conditions of a metric (Definition 1.1.1) then (X,d) is said to be a generalized metric space.

(The extended real line or the extended complex plane with the usual metrics are easy examples of generalized metric spaces.)

With this definition, Luxemburg gave the following:

Theorem 1.2.5 : Let T be a mapping from a complete generalized metric space (X,d) into itself, satisfying:

i) There exists a constant k, with $0 \leq k < 1$, such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$ with $d(x,y) < \infty$.

ii) For every sequence of successive approximations, there exists an index $N(x_0)$ depending on the initial element $x_0 \in X$, such that $d(x_N, x_{N+\ell}) < \infty$, for all $\ell = 1,2,\ldots$.

iii) If x,y are two fixed points of T, $d(x,y)$ is finite.

Under these conditions, T has a unique fixed point, and every sequence of successive approximations with any arbitrary initial element converges to the fixed point of T.

Remark 1.2.2 : It is easily seen that if (X,d) is a complete metric space, then conditions (ii) and (iii) of the theorem are trivially satisfied and condition (i) yields the Banach Contraction Principle.

Diaz and Margolis [10] extended the theorem of Luxemburg as follows:

Theorem 1.2.6 : Suppose that (X,d) is a generalized complete metric space and $T : X \to X$ satisfies

$d(Tx,Ty) \leqslant kd(x,y)$ whenever $d(x,y) < \infty$, with $0 \leqslant k < 1$.

Let $x_0 \in X$ and consider the sequence $x_0, Tx_0, T^2x_0, \ldots, T^mx_0, \ldots$. Then the following alternative holds: either

A) for every integer $m = 0,1,2,\ldots$, one has

$$d(T^mx_0, T^{m+1}x_0) = \infty, \quad \text{or}$$

B) the sequence $x_0, Tx_0, T^2x_0, \ldots$ is d-convergent to a fixed point of T.

Kannan [18] gave the following:

<u>Theorem 1.2.7</u>: Let (X,d) be a complete metric space and let $T: X \to X$ satisfy:

(1.2A) $\quad d(Tx,Ty) \leqslant \alpha\{d(x,Tx) + d(y,Ty)\}$, for all $x,y \in X$ where $0 \leqslant \alpha < \frac{1}{2}$.

Then T has a unique fixed point.

It is not too difficult to see that condition (1.2A) is different from the condition in Banach's Theorem, for the condition in the latter implies the continuity of T, whereas (1.2A) does not in general. The next example illustrates this.

<u>Example 1.2.2</u>: Let (X,d) be the interval $[0,1]$ with the usual metric. Define $T: [0,1] \to [0,1]$ by

$$Tx = \begin{cases} x/4 & \text{for } x \in [0,\tfrac{1}{2}) \\ x/5 & \text{for } x \in [\tfrac{1}{2},1] \end{cases}$$

It is clear that T is discontinuous at $x = \frac{1}{2}$, therefore the Banach Contraction Principle can not be used. However for $\alpha = 4/9$, Theorem 1.2.6 is applicable.

Kannan [18] also gave the following theorem for the fixed points of two mappings:

<u>Theorem 1.2.8</u> : let (X,d) be a complete metric space. If T_1 and T_2 are two mappings of X into itself satisfying

$$d(T_1x , T_2y) \leq \alpha\{d(x,T_1x) + d(y,T_2y)\} \text{ for all } x,y \in X , \text{ with } 0 \leq \alpha < \tfrac{1}{2} ,$$

then T_1 and T_2 have a unique common fixed point. (u is a common fixed point of T_1 and T_2 if $u = T_1u = T_2u$)

The proof of this theorem is of a similar vein as in the Banach Contraction Principle. The sequence of successive iterates is constructed by taking an arbitrary point $x_0 \in X$, and defining $x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$, ... This sequence turns out to be Cauchy and hence convergent, due to the completeness of (X,d). The limit of this sequence is the unique common fixed point of T_1 and T_2.

Dube [11] generalized the above theorem as follows:

Theorem 1.2.9 : Let T_1 and T_2 be two mappings of a complete metric space (X,d) into itself. If there exists two non-negative real numbers α and β such that $\alpha + \beta < 1$, and

$$d(T_1x, T_2y) \leq \alpha d(x, T_1x) + \beta d(y, T_2y) , \quad \text{for all } x,y \in X ,$$

then T_1 and T_2 have a unique common fixed point.

The following is an extension of Theorem 1.2.7 due to Hardy and Rogers [17] :

Theorem 1.2.10 : Let T be a mapping of a complete metric space (X,d) into itself. Suppose there exist non-negative real numbers $a_1 , a_2 , a_3 , a_4 , a_5$ such that

(1) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and

(2) $d(Tx,Ty) \leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx)$

$$+ a_5 d(x,y) , \quad \text{for all } x,y \in X ,$$

then T has a unique fixed point.

Wong [30] exhausts all the previous Kannan-type results of the Banach Contraction Principle with this result:

Theorem 1.2.11 : Let T_1 and T_2 be two mappings of a complete metric space (X,d) into itself. Suppose there exists nonnegative real numbers a_1, a_2, a_3, a_4, a_5 such that

(1) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$,

(2) $a_1 = a_2$ or $a_3 = a_4$, and

(3) $d(T_1x, T_2y) \leq a_1 d(x, T_1x) + a_2 d(y, T_2y) + a_3 d(x, T_2y)$

$+ a_4 d(y, T_1x) + a_5 d(x, y)$, for all $x, y \in X$,

then T_1 and T_2 have a common unique fixed point.

We now extend the above theorem to generalized metric spaces by the method of "alternatives" as used by Chu and Diaz [8] in extending Banach's Theorem.

<u>Theorem 1.2.12</u> : Let (X,d) be a generalized complete metric space and T_1 and T_2 be two mappings from X into itself satisfying conditions (1), (2) and (3) of Theorem 1.2.11 whenever $d(x,y) < \infty$. Let $x_0 \in X$ and consider the sequence defined as follows:

$$x_0 , x_1 = T_1x_0 , x_2 = T_2T_1x_0 , x_3 = T_1T_2T_1x_0 , \ldots$$

i.e., $x_{2n+1} = T_1(x_{2n})$ for $n = 0,1,2,\ldots$ and

$x_{2n} = T_2(x_{2n-1})$ for $n = 1,2,\ldots$.

Then the following alternative holds: either

(A) for every positive integer n , $d(x_n, x_{n+1}) = \infty$ or,

(B) the sequence $\{x_n\}$ is convergent to a common fixed point of T_1 and T_2 .

Proof: There are only two possibilities for the extended real-valued sequence $\{d(x_n, x_{n+1})\}$, either

(a) for every positive integer n, $d(x_n, x_{n+1}) = \infty$, or

(b) for some positive integer m, $d(x_m, x_{m+1}) < \infty$.

If (a) holds, we have part (A) of the conclusion of the theorem. Hence, it remains only to show that (b) implies conclusion (B) of the theorem. Now for some m, let $d(x_m, x_{m+1}) = M < \infty$. Let m be an even integer, so that $x_{m+1} = T_1 x_m$ and $x_{m+2} = T_2 x_{m+1}$, and so on.

Since $d(x_m, x_{m+1}) < \infty$, by condition (3) we have

$$d(x_{m+1}, x_{m+2}) \leq a_1 d(x_m, x_{m+1}) + a_2 d(x_{m+1}, x_{m+2}) + a_3 d(x_m, x_{m+2}) + a_4 d(x_{m+1}, x_{m+1}) + a_5 d(x_m, x_{m+1})$$

i.e., $(1 - a_2) d(x_{m+1}, x_{m+2}) \leq (a_1 + a_5) d(x_m, x_{m+1}) + a_3 d(x_m, x_{m+2})$.

Using the triangle inequality, we get

$$(1 - a_2 - a_3) d(x_{m+1}, x_{m+2}) \leq (a_1 + a_3 + a_5) d(x_m, x_{m+1}) \,, \quad \text{or}$$

$$d(x_{m+1}, x_{m+2}) \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} d(x_m, x_{m+1}) \,.$$

Thus $d(x_{m+1}, x_{m+2}) < \infty$. Again using condition (3) we have

$$d(x_{m+2}, x_{m+3}) \leq a_1 d(x_{m+2}, x_{m+3}) + a_2 d(x_{m+1}, x_{m+2}) + a_3 d(x_{m+2}, x_{m+2}) + a_4 d(x_{m+1}, x_{m+3}) + a_5 d(x_{m+1}, x_{m+2}) \quad \text{or}$$

$$d(x_{m+2}, x_{m+3}) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} d(x_{m+1}, x_{m+2}) \,.$$

Let $r = \dfrac{a_1 + a_3 + a_5}{1 - a_2 - a_3}$ and $s = \dfrac{a_2 + a_4 + a_5}{1 - a_1 - a_4}$.

Since $d(x_{m+2}, x_{m+3}) \leq s \cdot r \cdot M < \infty$, we have by (3),

$$d(x_{m+3}, x_{m+4}) \leq rd(x_{m+2}, x_{m+3})$$
$$\leq r^2 sd(x_m, x_{m+1}) .$$

Proceeding in this manner, we get, for any nonnegative integer k, $d(x_{m+2k+1}, x_{m+2k+2}) \leq rd(x_{m+2k+1}, x_{m+2k})$ and $d(x_{m+2k+3}, x_{m+2k+2}) \leq sd(x_{m+2k+2}, x_{m+2k+1})$ i.e., $d(x_{m+2k+1}, x_{m+2k+2}) \leq r(rs)^k M$ and

$$d(x_{m+2k+2}, x_{m+2k+3}) \leq (rs)^{k+1} M .$$

Therefore $\sum_{k=0}^{\infty} d(x_{m+k}, x_{m+k+1}) \leq (1+r) \sum_{k=0}^{\infty} (rs)^k M$.

Since $rs < 1$, $(1+r)M \sum_{k=0}^{\infty} (rs)^k$ converges. Thus $d(x_{m+k}, x_{m+k+1})$ tends to zero as k tends to infinity, i.e. $\{x_n\}$ is a Cauchy sequence. As X is complete, $\{x_n\}$ converges to some $u \in X$.

Since $\{x_n\}$ converges to u, there exists an N such that for $i > N$, $d(x_i, u) < M < \infty$. Choose $i > N$ such that $x_{i+1} = T_1 x_i$, and fix i. By the triangle inequality,

(i) $d(u, T_1 u) \leq d(u, x_{i+1}) + d(x_{i+1}, T_1 u)$

Now since $d(x_k,u) < \infty$, for $k > N$, it follows that

$$\text{(ii)}\quad d(x_{i+1},T_2u) \leq d(T_1x_i,T_2u) \leq a_1d(x_i,x_{i+1}) + a_2d(u,T_2u) + a_3d(x_i,T_2u) + a_4d(u,x_{i+1}) + a_5d(x_i,u) .$$

By letting i tend to infinity we have by (i) and (ii),

$$d(u,T_2u) \leq (a_2 + a_3)d(u,T_2u) .$$

As $(a_2 + a_3) < 1$, $d(u,T_2u) = 0$ or, $u = T_2u$.

Similarly, it can be shown that $u = T_1u$. Thus, if m is even, the sequence $\{x_n\}$ converges to a common fixed point of T_1 and T_2. In an analagous manner, by taking $x_{m+1} = T_2x_m$, the proof would follow for the case of m being odd. Thus the sequence $\{x_n\}$ converges to a common fixed point of T_1 and T_2.

This completes the proof.

Remark 1.2.4 : In the above theorem

(i) if (X,d) is a complete metric space, we obtain Theorem 1.2.11 as a simple corollary to our theorem.

(ii) if $T_1 = T_2 = T$ and by setting the appropriate constants to zero, we get Theorem 1.2.6.

(iii) if (X,d) is a complete metric space and $T = T_1 = T_2$ we also have Theorem 1.2.10 as a corollary.

Remark 1.2.5 : We cannot ask for our fixed point to be unique. On the other hand, if T_1 and T_2 have two common fixed points, then the two points must necessarily be infinitely far apart; that is if x and y are such that $x \neq y$ with $T_1x = x = T_2x$ and $T_1y = y = T_2y$, then $d(x,y) = \infty$, for if $d(x,y) < \infty$, then

$$d(x,y) = d(T_1x,T_2y) \leq a_1d(x,T_1x) + a_2d(y,T_2y) + a_3d(x,T_2y) + a_4d(y,T_1x) + a_5d(x,y) ,$$

$$\leq (a_3 + a_4 + a_5)\, d(x,y) ,$$

and since $a_3 + a_4 + a_5 < 1$, $d(x,y) = 0$ which is a contradiction.

1.3 Localized Versions of Fixed Point Theorems.

We would now like to look at a less restricted type of condition for fixed point theorems. In the previous section, we considered mappings which satisfied certain conditions for the whole space. Some authors have shown that in certain cases, mappings satisfying conditions only "locally" will still have a fixed point. The most notable of these is the localized version of the Banach Contraction Principle due to Edelstein [13]. First we give the following definition:

Definition 1.3.1: Let (X,d) be a metric space and ε a positive real number. A finite sequence $x_0, x_1, x_2, \ldots, x_n$ of points of X is an ε-chain joining x_0 and x_n, if $d(x_{i-1},x_i) < \varepsilon$, for $i = 1,2,\ldots,n$. (X,d) is said to be ε-chainable if, for each pair (x,y) of its points, there exists an ε-chain joining x and y. (X,d) is called well-chained if it is ε-chainable for any ε.

The real numbers with the usual metric is an example of a well-chained metric space. The real numbers with the metric defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

is an example of a metric space which is not ε-chainable for any $0 < \varepsilon < 1$, and hence not well-chained.

On an ε-chainable metric space (X,d), one can define an equivalent metric d_ε by

$$d_\varepsilon(x,y) = \inf \sum_{i=1}^{n} d(x_{i-1},x_i),$$

where the infimum is taken over all ε-chains joining $x = x_0$ and $y = x_n$. It is readily seen that d_ε is indeed a metric on X with $d(x,y) \leq d_\varepsilon(x,y)$ and $d(x,y) = d_\varepsilon(x,y)$, if $d(x,y) < \varepsilon$. Now suppose that $\{x_n\}$ is a sequence in X that is d-convergent to x in X. Then there exists an N such that $n > N$ implies $d(x_n,x) < \varepsilon$. This in turn implies that $d_\varepsilon(x_n,x) < \varepsilon$. Therefore, $\{x_n\}$ is also d_ε-convergent to x. Similarly if $\{x_n\}$ is d_ε-convergent to x, then it is also d-convergent to x. Therefore, by definition 1.1.5, d and

d_ε are equivalent metrics on X.

With the help of this equivalent metric, Edelstein was able to prove this "localized" version of the Banach Contraction Principle:

Theorem 1.3.1: Let (X,d) be a complete ε-chainable metric space and T, a self-map on X. If there exists a $k \in [0,1)$, such that

$$d(x,y) < \varepsilon \Rightarrow d(Tx,Ty) \leq kd(x,y)\ ,\ x,y \in X\ ,$$

then T has a unique fixed point $u \in X$ and $u = \lim_{n\to\infty} T^n x_0$, for any $x_0 \in X$.

Proof: Let d_ε be the metric defined above. As (X,d) is complete and d_ε is an equivalent metric, then (X,d_ε) is also complete.

Let $x,y \in X$ and $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ be any ε-chain joining x and y. As $d(x_{i-1},x_i) < \varepsilon$ for each $i = 1,2,\ldots,n$, we have by the hypothesis of the theorem,

$$d(Tx_{i-1},Tx_i) \leq kd(x_{i-1},x_i) < \varepsilon\ ,\ \text{for}\ \ i = 1,\ldots,n$$

Hence, $Tx_0, Tx_1, \ldots, Tx_{n-1}, Tx_n$ is an ε-chain joining Tx and Ty. Therefore

$$d\,(Tx,Ty) \leq \sum_{i=1}^{n} d(Tx_{i-1},Tx_i)\ .$$

Since for any ε-chain $x_0, x_1, \ldots, x_n$ joining x and y, we have

$$\sum_{i=1}^{n} d(Tx_{i-1}, Tx_i) \leq k \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

it follows that $d_\varepsilon(Tx, Ty) \leq k d_\varepsilon(x, y)$, i.e., T is a contraction with respect to (X, d_ε). Hence, by the Banach Contraction Principle, T has a unique fixed point u, with $\lim_{n\to\infty} d_\varepsilon(T^n x_0, u) = 0$, for any $x_0 \in X$.

However, since d is an equivalent metric to d_ε, we also have $\lim_{n\to\infty} d(T^n x_0, u) = 0$, for any $x_0 \in X$, i.e., $\lim_{n\to\infty} T^n x_0 = u$.

This finishes the proof of the theorem.

Meir and Keeler [23] have given another theorem for a mapping that satisfies a condition only locally. First, they give the following definition:

Definition 1.3.2: Let $T : (X, d) \to (X, d)$. T is said to be a weakly uniformly strict contraction if, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon .$$

The theorem in [23] is stated as:

Theorem 1.3.2: If (X, d) is a complete metric space and $T : X \to X$ is a weakly uniformly strict contraction, then T has a unique fixed point.

Following the method of Diaz and Margolis [10], we give a localized version of the theorem due to Hardy and Rogers (Theorem 1.2.10) as follows:

Theorem 1.3.3: Let (X,d) be a generalized complete metric space and T, a mapping of X into itself. Suppose there exists nonnegative real numbers, a_1, a_2, a_3, a_4, a_5 and a positive real number c such that

(1) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$

(2) whenever $d(x,y) \leq c$ one has

$$d(Tx,Ty) \leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx) + a_5 d(x,y) .$$

Let $x_0 \in X$, and consider the sequence: $x_0, Tx_0, T^2x_0, \ldots, T^n x_0, \ldots$. Then the following alternative holds: either

(A) for every integer $n = 0,1,2,\ldots$, one has

$$d(T^n x_0, T^{n+1} x_0) > c , \text{ or}$$

(B) the sequence of successive approximations $x_0, Tx_0, T^2x_0, \ldots, T^n x_0, \ldots$ is d-convergent to a fixed point of T.

Proof: Suppose $d(x,y) \leq c$, then in addition to (2) of the theorem we have

(3) $$d(Ty,Tx) \leq a_1 d(y,Ty) + a_2 d(x,Tx) + a_3 d(y,Tx) + a_4 d(x,Ty) + a_5 d(x,y)$$

By adding (2) and (3) and dividing both sides of the resultant inequality by two, we have

$$(2')\quad d(Tx,Ty) \leq \frac{a_1+a_2}{2}\, d(x,Tx) + \frac{a_1+a_2}{2}\, d(y,Ty) + \frac{a_3+a_4}{2}\, d(x,Ty) + \frac{a_3+a_4}{2}\, d(y,Tx) + a_5 d(x,y)$$

or $(2')$ $d(Tx,Ty) \leq a_1' d(x,Tx) + a_2' d(y,Ty) + a_3' d(x,Ty) + a_4' d(y,Tx) + a_5 d(x,y)$.

with $a_1' + a_2' + a_3' + a_4' + a_5' < 1$, and $a_1' = a_2'$, $a_3' = a_4'$

Now suppose conclusion (A') of the theorem does not hold. Thus for some nonnegative integer m we have

$$d(T^m x_0, T^{m+1} x_0) \leq c .$$

By invoking condition (2') of the theorem, we get

$$d(T^{m+1}x_0, T^{m+2}x_0) \leq a_1' d(T^m x_0, T^{m+1}x_0) + a_2' d(T^{m+1}x_0, T^{m+2}x_0) + a_3' d(T^m x_0, T^{m+2}x_0) + a_4' d(T^{m+1}x_0, T^{m+1}x_0) + a_5 d(T^m x_0, T^{m+1}x_0)$$

or, by collecting terms:

$$(1-a_2')d(T^{m+1}x_0, T^{m+2}x_0) \leq (a_1' + a_5) d(T^m x_0, T^{m+1}x_0) + a_3' d(T^m x_0, T^{m+2}x_0)$$

Using the triangle inequality, we have

$$d(T^{m+1}x_0, T^{m+2}x_0) \leq \frac{a_1' + a_3' + a_5}{1 - a_2' - a_3'}\, d(T^m x_0, T^{m+1}x_0) ,$$

with $\dfrac{a_1' + a_3' + a_5}{1 - a_2' - a_3'} < 1$.

Therefore $d(T^{m+1}x_0, T^{m+2}x_0) < c$, and hence we can use condition (2') again to obtain,

$$d(T^{m+2}x_0, T^{m+3}x_0) \leq a_1' d(T^{m+1}x_0, T^{m+2}x_0) + a_2' d(T^{m+2}x_0, T^{m+3}x_0)$$

$$+ a_3' d(T^{m+1}x_0, T^{m+3}x_0) + a_4' d(T^{m+2}x_0, T^{m+2}x_0) + a_5 d(T^{m+1}x_0, T^{m+2}x_0)$$

or, $$d(T^{m+2}x_0, T^{m+3}x_0) \leq \frac{a_1' + a_3' + a_5}{1 - a_2' - a_3'} d(T^{m+1}x_0, T^{m+2}x_0)$$

$$\leq \left[\frac{a_1' + a_3' + a_5}{1 - a_2' - a_3'}\right]^2 d(T^m x_0, T^{m+1}x_0) .$$

Let $r = \dfrac{a_1' + a_3' + a_5'}{1 - a_2' - a_3'}$.

Proceeding successively, we get for any nonnegative integer k ,

$$d(T^{m+k}x_0, T^{m+k+1}x_0) \leq r^k d(T^m x_0, T^{m+1}x_0) .$$

Let $x_k' = T^{m+k}x_0$, then $d(x_k', x_{k+1}') \leq r^k d(x_0', x_1')$. As $0 \leq r < 1$, by letting k tend to infinity, we have that $\{x_k'\}$ is a Cauchy sequence. Since X is complete, $\{x_k'\}$ converges to an element of X , say u . Therefore, there exists a natural number N , such that

$$k \geq N \Rightarrow d(x_k', u) \leq c .$$

Assume $k \geq N$, and consider $d(x_{k+1}', Tu)$:

(i) $$d(x_{k+1}', Tu) = d(Tx_k', Tu) \leq a_1' d(x_k', x_{k+1}') + a_2' d(u, Tu)$$

$$+ a_3' d(x_k', Tu) + a_4' d(u, x_{k+1}') + a_5 d(x_k', u)$$

By letting k tend to infinity, we have that $d(x'_k, u)$, $d(x'_{k+1}, u)$ and $d(x'_k, x'_{k+1})$ all tend to zero. In fact, the first two tend to zero since $\{x'_k\}$ converges to u, and the latter, since $\{x'_k\}$ is a Cauchy sequence. Therefore, as k tends to infinity, (i) becomes

$$d(u,Tu) \leq (a'_2 + a'_3)\, d(u,Tu)$$

or $[1 - a'_2 - a'_3]\, d(u,Tu) \leq 0$ and since $1 - a'_2 - a'_3 > 0$, it follows that $d(u,Tu) = 0$, i.e., $u = Tu$.

Therefore the sequence $\{T^n x_0\}$ converges to a fixed point of T, and the theorem is proved.

Remark 1.3.1 : The uniqueness of the fixed point in our theorem is not assured. However, we note that if u and v are two distinct fixed points of T, then we must have $d(u,v) > c$. For if $d(u,v) \leq c$, then by condition (2) of the theorem, $d(u,v) = d(Tu,Tv) \leq a_1 d(u,Tu) + a_2 d(v,Tv)$ $+ a_3 d(v,Tv) + a_4 d(v,Tv) + a_5 d(u,v)$, or $d(u,v) \leq (a_3 + a_4 + a_5)\, d(u,v)$. Since $a_3 + a_4 + a_5 < 1$, $d(u,v) = 0$ and thus $u = v$. Therefore, if u and v are two distinct fixed points of T, then $d(u,v) > c$. It follows therefore, that F, the set of all fixed points of T, is a scattered set. Hence by Kuratowski [21, p.252], F is at most a countable set if X is separable.

CHAPTER II

FIXED POINTS OF CONTRACTIVE AND NONEXPANSIVE MAPPINGS

In this chapter, we will study contractive and nonexpansive mappings which are more general than contractions. Although, seperately neither one of these mappings on complete metric spaces necessitates the existence of fixed points, the second section will show how these two mappings have been "intertwined" to insure the existence of fixed points.

2.1 Contractive Mappings:

In section 1.3, we saw that a mapping T from a complete metric space X into itself satisfying

(2.1.1) $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $\varepsilon < d(x,y) < \varepsilon + \delta$
implies $d(Tx,Ty) < \varepsilon$ for all $x,y \in X$, $x \neq y$

has a unique fixed point.

We notice that (2.1.1) clearly implies

(2.1.2) $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$, $x \neq y$.

Definition 2.1.1 : Let (X,d) be a metric space. A mapping $T : X \to X$ is called contractive if (2.1.2) holds.

Remark 2.1.1 : We note that a contractive mapping on a complete metric space does not necessarily imply the existence of a fixed point. The following example illustrates this.

Example 2.1.1 : Let $T:[1,\infty) \to [1,\infty)$ be defined by $T(x) = x + 1/x$, where $[1,\infty)$ is equipped with the usual metric.

Thus
$$d(Tx,Ty) = |x + \frac{1}{x} - y - \frac{1}{y}|$$
$$= |(x-y) + \frac{x-y}{xy}|$$
$$= |x-y| \, |\frac{xy-1}{xy}| .$$

Since $|\frac{xy-1}{xy}| < 1$, for any $x,y \in [1,\infty)$, it follows that T is a contractive mapping. However, suppose that there exists a fixed point $x_0 \in [1,\infty)$ of T . Then $x_0 = Tx_0 = x_0 + \frac{1}{x_0}$, i.e., $\frac{1}{x_0} = 0$, which is impossible.

Although T being contractive does not necessarily imply the existence of a fixed point, it does imply the uniqueness of the fixed point. Suppose x and y are two distinct fixed points of T , then

$$d(x,y) = d(Tx,Ty) < d(x,y)$$

which is impossible.

In [14], Edelstein showed that the compactness of the space does insure the existence of a fixed point for a contractive mapping. In fact, he relaxed the condition even further in the following:

Theorem 2.1.1 : Let T be a contractive self-map of a metric space (X,d) . If the sequence $\{T^n x\}$, for some

$x \in X$, has a convergent subsequence, then the limit of this subsequence is the unique fixed point of T.

As every sequence in a compact space has a convergent subsequence, the following is an obvious corollary.

Corollary 2.1.1 : A contractive mapping on a compact metric space into itself has a unique fixed point.

Bailey [1] gave the following extension of the above result.

Theorem 2.1.2 : Let (X,d) be a compact metric space and T be a continuous self-map on X. If there exists an $n = n(x,y)$ such that T^n is a contractive mapping, then T has a unique fixed point.

Furi and Vignoli [15] relaxed the condition of T being contractive with this definition:

Definition 2.1.2 : Let (X,d) be a metric space and F a real-valued lower semicontinuous function on $X \times X$, with $F(x,y) \leq 0$ implying $x = y$. Suppose T is a self-map on X. T is said to be weakly F-contractive if

$$F(Tx,Ty) < F(x,y)$$

for all $x,y \in X$, $x \neq y$.

Note that a contractive map is weakly contractive but a weakly contractive map need not be contractive.

Before we give the results of Furi and Vignoli, we need a measure of noncompactness given by Kuratowski [21] as follows:

Let A be a bounded set of a metric space X, and let $\alpha(A)$ denote the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε.

<u>Definition 2.1.3</u> : A continuous mapping T of a metric space X into itself is called densifying, if for every bounded subset A of X, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

<u>Remark 2.1.2</u> : $\alpha(A)$ satisfies the following properties:

(i) $0 \leq \alpha(A) \leq \delta(A)$.

(ii) $\alpha(A) = 0$ iff A is totally bounded.

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(iv) $\alpha(\bar{A}) = 0$ iff $\alpha(A) = 0$, where $\bar{A}$ is the closure of A.

(v) $A \subset B$ implies $\alpha(A) < \alpha(B)$.

We are now in a position to give the results of Furi and Vignoli [15].

<u>Theorem 2.1.3</u> : Let (X,d) be a complete metric space and $T: X \to X$ be a densifying weakly F-contractive map. If $\{T^n x_0\}$ is bounded for some $x_0 \in X$, then T has a unique

fixed point.

In another paper [16], the same authors gave the following theorem:

Theorem 2.1.4 : Suppose T is a densifying map from a complete subset D, of a metric space (X,d) into X. Then any bounded sequence $\{x_n\}$ of X, such that $d(x_n, Tx_n)$ tends to zero as n tends to infinity, is compact and all the limit points of $\{x_n\}$ are fixed points of T.

Singh [27] replaced the condition $F(Tx,Ty) < F(x,y)$, $x \neq y$ by

$$(2.1.3) \quad F(Tx,Ty) < \frac{1}{3}[F(x,Tx) + F(x,y) + F(y,Ty)]$$

when $Tx = x = y = Ty$ does not hold, and gave the following:

Theorem 2.1.5 : Let X be a complete metric space and T be a densifying map of X into itself satisfying (2.1.3) for some F as in Definition 2.1.2. If for some $x_0 \in X$ the sequence $\{T^n x_0\}$ is bounded, then T has a unique fixed point.

A short sketch of the proof of the above theorem is as follows. Let $A = \bigcup_{n=0}^{\infty} T^n x_0$, where x_0 is such that $\{T^n x_0\}$ is bounded. As T is densifying, it follows that $\bar{A}$ is compact. The semicontinuous function $f(x) = F(x,Tx)$ defined on $\bar{A}$, has a minimum point which is the fixed point of T.

Now we prove the following theorem similar to the above theorem of Singh [27] by replacing (2.1.3) with a different condition.

<u>Theorem 2.1.6</u>: Let (X,d) be a complete metric space and T, a densifying map of X into itself, such that for a real-valued lower semi-continuous function F on $X \times X$ with $F(x,y) = 0$ implying $x = y$, we have

$$(2.1.4)\quad F(Tx,Ty) < \frac{1}{4}[F(x,Tx) + F(x,y) + F(y,Ty) + F(Tx,y)],$$

when $Tx = x = y = Ty$ does not hold.

Then T has a unique fixed point, if for some $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ is bounded.

<u>Proof</u>. The uniqueness of the fixed point follows easily from (2.1.4). Let x and y be two distinct fixed points of T, then $Tx = x = y = Ty$ does not hold, so by (2.1.4)

$$F(x,y) = F(Tx,Ty) < \frac{1}{4}[F(x,Tx) + F(x,y) + F(y,Ty) + F(Tx,y)].$$

Since $F(x,Tx) = F(x,x) = 0 = F(y,Ty) = F(y,y)$,

$$F(x,y) < \frac{1}{2} F(x,y),$$ which is impossible.

Hence $x \in y$.

Now, let $A = \bigcup_{n=0}^{\infty} x_n$, where $x_n = T^n x_0$, and x_0 is an arbitrary point in X (it is understood that $T^0 x_0 = x_0$). Then $T(A) \subset A$. Since T is continuous (by assumption), therefore,

$$(2.1.5) \qquad T(\bar{A}) \subseteq \overline{T(A)} \subseteq \bar{A} .$$

Suppose $\alpha(A) > 0$. As T is densifying, $\alpha(T(A)) < \alpha(A)$. However, since $A = T(A) \cup \{x_0\}$, it follows that

$$\alpha(A) = \max\{\alpha(T(A)), \alpha(\{x_0\})\} = \alpha(T(A) ,$$

which is a contradiction. Therefore $\alpha(A) = 0$, and hence $\alpha(\bar{A}) = 0$. Thus, as $\bar{A}$ is totally bounded and closed, $\bar{A}$ is compact.

Now define the function $f : \bar{A} \to \mathbb{R}$, by $f(x) = F(x,T(x))$. Since f is the composition of a lower semi-continuous function and a continuous function, then f is lower semi-continuous. As $\bar{A}$ is compact, f attains its minimum value at a point $y \in \bar{A}$. We claim that y is the fixed point of T.

Suppose $y \neq Ty$. Then we have by (2.1.4),

$$f(Ty) = F(Ty,T^2y) < \frac{1}{4}[F(y,Ty) + F(y,Ty) + F(Ty,T^2y) + F(Ty,Ty)]$$

or $F(Ty,T^2y) < \frac{2}{3} F(y,Ty)$, i.e., $f(Ty) < f(y)$, which contradicts the minimality of y, since $Ty \in \bar{A}$. Therefore y is the unique fixed point of T.

It is clear that metric d, satisfies all of the conditions of the function F. Hence we have the following:

Corollary 2.1.2 : If T is a densifying map from a complete bounded metric space (X,d) satisfying:

$$d(Tx,Ty) < \frac{1}{4}[d(x,Tx) + d(y,Ty) + d(Tx,y) + d(x,y)] ,$$

then T has a unique fixed point.

Remark 2.1.3 : It is natural to ask whether one can replace the condition (2.1.4) in the above theorem by the Hardy and Rogers [17] type condition

$$(2.1.6) \quad F(Tx,Ty) < \frac{1}{5}[F(x,Tx) + F(y,Ty) + F(y,Tx) + F(x,y) + F(Ty,x)]$$

to give the same result. The answer is yes, provided the function F satisfies the inequality

$$F(T^2y,y) \leq F(Ty,T^2y) + F(y,Ty) .$$

In fact, $\bar{A}$ will still be compact, and the function f in the theorem will attain a minimum value in $\bar{A}$, say y, and

$$f(Ty) = F(Ty,T^2y) < \frac{1}{5}[F(y,Ty) + F(Ty,T^2y) + F(T^2y,y) + F(Ty,Ty) + F(y,Ty)] ,$$

or $$\frac{4}{5} F(Ty,T^2y) < \frac{2}{5} F(y,Ty) + \frac{1}{5} F(T^2y,y)$$

$$< \frac{3}{5} F(y,Ty) + \frac{1}{5} F(Ty,T^2y) .$$

or $f(Ty) < f(y)$, which gives the same contradiction.

Hence y is again the unique fixed point of T.

Thus we may have the following corollary to the above remark.

Corollary 2.1.3 : If T is a densifying map from a complete bounded metric space into itself satisfying:

$$d(Tx,Ty) < \frac{1}{5}[d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx) + d(x,y)]$$

Then T has a unique fixed point.

2.2 Nonexpansive Mappings.

Let us recall the definition of a nonexpansive mapping (Definition 1.1.8). A mapping T from a metric space X into itself is said to be nonexpansive if

$$d(Tx,Ty) \leq d(x,y) \ , \quad \text{for all} \quad x,y \in X \ .$$

Contractions and isometries are easy examples of nonexpansive mappings.

Remark 2.2.1 : Just as in the contractive case, a nonexpansive mapping on a complete metric space need not have a fixed point. To see this, one can take any function T, from the real line into itself defined by $T(x) = x + p$ for any $p \neq 0$. It is clear that such a mapping cannot have a fixed point for, if it did then p would have to equal zero.

Belluce and Kirk [3] have introduced the notion of "diminishing orbital diameters" of a mapping T. To define this term, we need another term "limiting orbital diameters" of T described below.

Let T be a mapping of a metric space X into itself. For each x, let $O(T^n x)$ denote the sequence of iterates $T^n(x)$, $O(T^n x) = \bigcup_{i=n}^{\infty} T^i x$, $n = 0,1,2,\ldots$, where $T^0 x = x$. The diameters $\delta(O(T^n x))$ of the sets $O(T^n x)$, when finite, form a nonincreasing real-valued sequence. The limit $r(x) = \lim_{n\to\infty} \delta(O(T^n x))$ is a nonnegative real number and is called the limiting orbital diameter of T at the point x.

<u>Definition 2.2.1</u>: Let $T : X \to X$ be a mapping of a metric space X into itself. If for each $x \in X$ the limiting orbital diameter $r(x)$ of T at x is less than $\delta(O(x))$ when $\delta(O(x)) > 0$, then T is said to have diminishing orbital diameters.

Contractions and contractive mappings are examples of mappings having diminishing orbital diameters.

With this definition, Belluce and Kirk [3] have given the following theorem for nonexpansive mappings.

<u>Theorem 2.2.1</u>: Let X be a complete metric space and let T be a nonexpansive mapping of X into itself which has diminishing orbital diameters. Suppose for some $x \in X$ a subsequence of the sequence $\{T^n x\}$ of iterates of T on x has limit z. Then $T^n x$ also has limit z, and z is a fixed point of T.

A simple corollary to this theorem is as follows:

Corollary 2.2.1 : If X is a compact metric space and T is a nonexpansive mapping of X into itself which has diminishing orbital diameters, then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Cheney and Goldstein [7] showed that if a nonexpansive mapping satisfies a "quasi-contractive" condition, (condition (ii) of the next theorem), then the existence of a fixed point is assured.

Theorem 2.2.2 : Let (X,d) be a metric space and let T be a mapping from X into itself such that

(i) $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in X$.

(ii) if $x \neq Tx$, then $d(T^2x,Tx) < d(x,Tx)$, and

(iii) the sequence of successive approximations has a cluster point for some initial point $x_0 \in X$.

Then T admits at least one fixed point.

The proof of the above theorem will be omitted. However, we point out that the crucial point of the proof of the existence of the fixed point is that T is continuous (which follows from (i)). The nonexpansiveness of the mapping is required to show that the sequence of successive approximations tends to the fixed point.

Singh [28] replaced the nonexpansive condition by a kind of "Kannan" condition in the following:

<u>Theorem 2.2.3</u> : Let T be a continuous mapping of a metric space (X,d) into itself such that

(i) $d(Tx,Ty) \leq \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}$, for all $x,y \in X$,

(ii) if $x \neq Tx$, then $d(T^2x,Tx) < d(Tx,x)$ and

(iii) for some $x \in X$, the sequence $\{T^n x\}$ has a convergent subsequence that converges to $z \in X$.

Then $\{T^n x\}$ converges to z , and z is the unique fixed point of T .

This type of "quasi-contractive" condition used by Cheney and Goldstein [7] and Singh [28] also lends itself to proving the existence of common fixed points of two mappings. We extend the above theorem due to Singh [28] by replacing the condition

$$d(Tx,Ty) \leq \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}$$

by the Wong [30] type condition, in the following:

<u>Theorem 2.2.4</u> : Let (X,d) be a metric space and T_1 , T_2 two continuous mappings of X into itself satisfying:

(i) $d(T_1x,T_2y) \leq \frac{1}{5}\{d(x,T_1x) + d(y,T_2y) + d(x,T_2y) + d(y,T_1x) + d(x,y)\}$

(ii) if $x \neq T_1x$ then $d(T_2T_1x,T_1x) < d(x,T_1x)$ and $d(T_1T_2x,T_2x) < d(x,T_2x)$.

(iii) the sequence of successive approximations with some initial element x_0, has a cluster point $u \in X$. Then u is the unique common fixed point of T_1 and T_2.

<u>Proof</u>; Suppose $T_1 v = v$, then by (i)

$$d(v,T_2v) = d(T_1v,T_2v) \leq \frac{2}{5} d(v,T_2v) \text{ , i.e. } d(v,T_2v) = 0 .$$

Hence, if v is a fixed point of T_1, it is necessarily a common fixed point of T_1 and T_2.

Now, suppose there are two common fixed points of T_1 and T_2, say, a and b. Then again by (i),

$$d(a,b) = d(T_1a,T_2b) \leq \frac{1}{5}\{d(a,T_1a) + d(b,T_2b) + d(a,T_2b) + d(b,T_1a) + d(a,b)\} ,$$

or $d(a,b) \leq \frac{3}{5} d(a,b)$ which implies that $d(a,b) = 0$.

Thus if there exists a common fixed point of T_1 and T_2, then it necessarily is unique.

Assume $u \neq T_1u$. Now, by condition (iii), the sequence x_n defined by

$$\begin{cases} x_{2n+1} = T_1x_{2n} & \text{for } n = 0,1,2,\ldots \\ x_{2n} = T_2x_{2n-1} & \text{for } n = 1,2,\ldots \end{cases}$$

has a cluster point u, so let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that x_{n_k} tends to u as $k \to \infty$.

Therefore

$$d(u,T_1u) = \lim_{k\to\infty} d(x_{n_k},T_1x_{n_k})$$

$$= \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1})$$

$$= \lim_{n\to\infty} d(x_n, x_{n+1})$$

$$= \lim_{n\to\infty} d(x_{n+1}, x_{n+2})$$

$$= \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2})$$

$$= d(T_1u, T_2T_1u)$$

which contradicts condition (ii), hence $u = T_1u$ and u is the unique common fixed point of T_1 and T_2.

In the proof, we used the assumption that $x_{n_k+1} = T_1x_{n_k}$. If, however, $x_{n_k+1} = T_2x_{n_k}$, then the same result would follow for we would have

$$d(u, T_2u) = d(T_2u, T_1T_2u)$$

which would still contradict (ii).

Remark 2.2.2 : (i) We note in the above theorem, both $\lim_{n\to\infty} d(x_n, x_{n+1})$ and $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1})$ do exist, since both are decreasing real-valued sequences (by condition (i) of the theorem) and both are bounded below by zero.

(ii) The uniqueness of the fixed point is assured, unlike the case of Cheney and Goldstein [7]

(iii) If the sequence $\{x_n\}$ was Cauchy, then it would also tend to the common fixed point.

CHAPTER III

MULTIVALUED MAPPINGS AND THEIR FIXED POINTS

In this chapter, multivalued mappings and their fixed points will be studied. Recently, attempts have been made by some mathematicians to extend some of the fixed point theorems for single-valued mappings to multivalued mappings. In fixed point theorems for multivalued mappings, however, only the existence of fixed points has been established. In the first section, we have given two partial answers to the question posed by Dube in [12] concerning uniqueness of fixed points for multivalued mappings. The second section deals with a "natural" definition of a sequence of iterates for multivalued mappings. Using this apparently new definition, we have been able to give extensions of the results of Cheney and Goldstein [7] and Belluce and Kirk [3] to multi-valued mappings.

3.1 Multivalued Mappings and Their Fixed Points.

We begin with the following definition:

Definition 3.1.1 : Let (X,d) be a metric space, with $A, B \subseteq X$.

(i) $2^X = \{A \subset X \mid A \neq \phi, A \text{ is closed in } X\}$.

(ii) $CB(X) = \{A \subseteq X \mid A \neq \phi, A \text{ is closed and bounded in } X\}$.

(iii) $\delta(x,B) = \inf\{d(x,y) \mid y \in B\}$.

(iv) $N(\varepsilon,A) = \{x \in X \mid d(x,a) < \varepsilon$, for some $a \in A$ with $\varepsilon > 0$ and $A \in CB(X)\}$.

(v) $H(A,B) = \inf\{\varepsilon \mid A \subseteq N(\varepsilon,B)$ and $B \subseteq N(\varepsilon,A)$, with $\varepsilon > 0$ and $A, B \in CB(X)\}$.

It can be easily shown that H satisfies all the conditions of being a metric on $CB(X)$ (in fact $(2^X,H)$ is a generalized metric space). H is called the Hausdorff metric on $CB(X)$. It is clear that the metric H depends on the metric d . However, two equivalent metrics on X do not yield two equivalent Hausdorff metrics on $CB(X)$ [19,p.131].

Definition 3.1.2 : Let (X,d_1) and (Y,d_2) be two metric spaces.

(i) F is a multivalued function on X into Y if, to every point of X , F assigns one and only one element of $CB(Y)$.

(ii) A multivalued function $F : (X,d_1) \to (CB(Y),H_2)$, where H_2 is the Hausdorff metric induced by d_2 , is said to be continuous at a point $x_0 \in X$ iff every sequence $\{x_n\}$ converging to x_0 implies the convergence of $\{F(x_n)\}$ to $F(x_0)$ in $(CB(Y),H_2)$; F is continuous in X if it is continuous at every point of X .

The function $F : \mathbb{R} \to CB(\mathbb{R})$, defined by $F(x) = [x, x+1]$, is an example of a multivalued function.

Definition 3.1.3 : A multivalued function $F : X \to CB(X)$ is said to have a fixed point if there exists $x \in X$ such that $x \in F(x)$.

It is obvious that for the multivalued function given as an example above, every point $x \in \mathbb{R}$ is a fixed point of F.

Definition 3.1.4 : Let (X,d_1) and (Y,d_2) be two metric spaces and $F : (X,d_1) \to (CB(Y),H_2)$. The function F is said to be a multivalued contraction if

$$H_2(F(x),F(y)) \leq \alpha d(x,y) \quad \text{for all } x,y \in X, \text{ with } 0 \leq \alpha < 1 .$$

It should be noted that for any metric space X, the isometry $i : X \to CB(X)$ defined by $i(x) = \{x\}$ is a multivalued function. Thus fixed point theorems for multivalued functions are generalizations of their single-valued analogues.

Nadler [24] extended the Banach Contraction Principle to multivalued functions as follows:

Theorem 3.1.1 : Let (X,d) be a complete metric space. If $F : X \to 2^X$ is a multivalued contraction, then F has a fixed point.

In the same paper, the author extended the notion of an $(\varepsilon-\lambda)$-uniformly locally contractive mapping to multivalued mappings.

<u>Definition 3.1.5</u> : $F : X \to 2^X$ is an $(\varepsilon-\lambda)$-uniformly locally contractive multivalued mapping (where $\varepsilon > 0$ and $0 \leqslant \lambda < 1$) if

$$H(F(x),F(y)) \leqslant \lambda d(x,y) \text{ , for all } x,y \in X \text{ such that } d(x,y) < \varepsilon \text{ .}$$

With this definition he [24] gave the following:

<u>Theorem 3.1.2</u> : Let (X,d) be a complete ε-chainable metric space. If $F : X \to 2^X$ is an $(\varepsilon-\lambda)$-uniformly locally contractive multivalued mapping, then F has a fixed point.

Recently, Dube [12] extended one of the most general results due to Wong [31] to multivalued mapping with this theorem:

<u>Theorem 3.1.3</u> : Let (X,d) be a complete metric space. Let F_1 and F_2 be two multivalued mappings of X into $(CB(X),H)$ satisfying:

(1) $$H(F_1(x),F_2(y)) \leqslant a_1\delta(x,F_1(x)) + a_2\delta(y,F_2(y)) + a_3\delta(x,F_2(y)) + a_4\delta(y,F_1(x)) + a_5 d(x,y)$$

for all $x,y \in X$ where a_1,a_2,a_3,a_4,a_5 are nonnegative real numbers such that

(2) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and

(3) $a_1 = a_2$ or $a_3 = a_4$.

Then F_1 and F_2 have a common fixed point in X .

In order to prove this theorem, the following lemma is needed:

Lemma 3.1.1 : Let F_1 and F_2 be two multivalued mappings of X into $CB(X)$. Let $x_0, x_1 \in X$. Then for each $y \in F_2(x_1)$

$$\delta(y, F_1(x_0)) \leq H(F_2(x_1), F_1(x_0)) .$$

Proof: For any $\varepsilon > 0$ such that $y \in F_2(x_1) \subset N(\varepsilon, F_1(x_0))$, there exists $z \in F_1(x_0)$ such that $d(y,z) \leq \varepsilon$. But

$$\delta(y, F_1(x_0)) \leq d(y,z) \leq \varepsilon .$$

Hence, by definition of the metric H

$$\delta(y, F_1(x_0)) \leq H(F_2(x_1), F_1(x_0)) .$$

A similar result with F_1 and F_2 interchanged in the lemma also holds true.

Proof of Theorem 3.1.3 : Let p_0 be a point in X . If $p_0 \in F_1(p_0)$, then p_0 is a fixed point of F_1 . In fact,

$$\delta(p_0, F_2(p_0)) \leq H(F_1(p_0), F_2(p_0)) \leq$$

$$a_1\delta(p_0, F_1(p_0)) + a_2\delta(p_0, F_2(p_0)) + a_3\delta(p_0, F_2(p_0))$$

$$+ a_4\delta(p_0, F_1(p_0)) + a_5 d(p_0, p_0) , \text{ by } (1) .$$

Hence $(1-a_2-a_3)\delta(p_0,F_2(p_0)) \leq 0$. Since $1-a_2-a_3 > 0$, $\delta(p_0,F_2(p_0)) = 0$, i.e. $p_0 \in F_2(p_0)$.

Therefore, $p_0 \in F_1(p_0)$ implies that p_0 is a common fixed point of F_1 and F_2 .

Suppose $p_0 \notin F_1(p_0)$. Since $F_1(p_0)$ is non-empty, let $p_1 \in F_1(p_0)$. As $F_1(p_0), F_2(p_1) \in CB(X)$ and $p_1 \in F_1(p_0)$, for any $\varepsilon_1 > 0$, there exists a $p_2 \in F_2(p_1)$ such that

$$\begin{aligned} d(p_1,p_2) &\leq H(F_1(p_0),F_2(p_1)) + \varepsilon_1 \\ &\leq a_1\delta(p_0,F_1(p_0)) + a_2\delta(p_1,F_2(p_1)) + a_3\delta(p_0,F_2(p_1)) \\ &\qquad + a_4\delta(p_1,F_1(p_0)) + a_5 d(p_0,p_1) + \varepsilon_1 \\ &\leq a_1 d(p_0,p_1) + a_2 d(p_1,p_2) + a_3 d(p_0,p_2) + a_4 d(p_1,p_1) \\ &\qquad + a_5 d(p_0,p_1) + \varepsilon_1 \\ &\leq (a_1+a_3+a_5)d(p_0,p_1) + (a_2+a_3)d(p_0,p_1) + \varepsilon_1 \end{aligned}$$

therefore

$$(4) \qquad d(p_1,p_2) \leq \frac{a_1+a_3+a_5}{1-a_2-a_3}\, d(p_0,p_1) + \frac{\varepsilon_1}{1-a_2-a_3}$$

Also, as $F_2(p_1), F_1(p_2) \in CB(X)$ and $p_2 \in F_2(p_1)$, for and $\varepsilon_2 > 0$, there exists a $p_3 \in F_1(p_2)$ such that

$$\begin{aligned} d(p_2,p_3) &\leq H(F_1(p_2),F_2(p_1)) + \varepsilon_2 \\ &\leq a_1\delta(p_2,F_1(p_2)) + a_2\delta(p_1,F_2(p_1)) + a_3\delta(p_2,F_2(p_1)) \\ &\qquad + a_4\delta(p_1,F_1(p_2)) + a_5 d(p_2,p_1) + \varepsilon_2 \end{aligned}$$

$$\leq a_1 d(p_2,p_3) + a_2 d(p_1,p_2) + a_3 d(p_2,p_2) + a_4 d(p_1,p_3) + a_5 d(p_1,p_2) + \varepsilon_2$$

Using (4) we get

$$d(p_2,p_3) \leq \frac{a_1 + a_4 + a_5}{1 - a_1 - a_4} \cdot \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}\, d(p_0,p_1) + \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} \cdot \frac{\varepsilon_1}{1 - a_2 - a_3} + \frac{\varepsilon_2}{1 - a_1 - a_4}$$

Proceeding in this manner and choosing

$$\varepsilon_n = [(a_1 + a_3 + a_5)(a_2 + a_4 + a_5)]^n \ , \quad n = 1,2,\ldots$$

and letting $\dfrac{a_1 + a_3 + a_5}{1 - a_2 - a_3} = r$, $\dfrac{a_2 + a_4 + a_5}{1 - a_1 - a_4} = s$,

we have that p_n is a sequence in X where $p_{2n-1} \in F_1(p_{2n-2})$ and $p_{2n} \in F_2(p_{2n-1})$, $n = 1,2,\ldots$, such that

(5) $d(p_{2n-1},p_{2n}) \leq r(rs)^{n-1} d(p_0,p_1) + (rs)^n + r(rs)^n + (rs)^{n+1} + \ldots + r(rs)^{2n-2} + rs^{2n-1}$, and

(6) $d(p_{2n},p_{2n+1}) \leq (rs)^n d(p_0,p_1) + s(rs)^n + (rs)^{n+1} + s(rs)^{n+1} + (rs)^{n+2} + \ldots + s(rs)^{2n-1} + (rs)^{2n}$.

By conditions (2) and (3), $0 \leq rs < 1$. It follows therefore that p_n is a Cauchy sequence. As (X,d) is complete, p_n converges to a point of X ; say x_0 .

Now, choosing $n+1$ to be even and fixing n, then $p_{n+1} \in F_2(p_n)$. Consider $H(F_1(x_0), F_2(p_n))$.

By (1) $H(F_1(x_0), F_2(p_n)) \leq a_1\delta(x_0, F_1(x_0)) + a_2\delta(p_n, F_2(p_n))$

$$+ a_3\delta(x_0, F_2(p_n)) + a_4\delta(p_n, F_1(x_0)) + a_5 d(x_0, p_n) .$$

By Lemma 3.1.1, as $p_{n+1} \in F_2(p_n)$,

(7) $\delta(p_{n+1}, F_1(x_0)) \leq H(F_1(x_0), F_2(p_n)) < a_1\delta(x_0, F_1(x_0)) +$

$$a_2 d(p_n, p_{n+1}) + a_3 d(x_0, p_{n+1}) + a_4\delta(p_n, F_1(x_0)) +$$

$$a_5 d(x_0, p_n)$$

Now, by Kuratowski [21, p. 209], for a fixed A, $\delta(x,A)$ is a continuous function of x. Therefore, as p_n converges to x_0, we have by (7),

$$\delta(x_0, F_1(x_0)) \leq (a_1 + a_4)\delta(x_0, F_1(x_0)) .$$

Hence, since $(a_1 + a_4) < 1$, $\delta(x_0, F_1(x_0)) = 0$ and, $x_0 \in F_1(x_0)$.

Now, by (1), $H(F_1(x_0), F_2(x_0)) \leq (a_2 + a_4)\delta(x_0, F_2(x_0))$. And, by Lemma 3.1.1, $\delta(x_0, F_2(x_0)) \leq (a_2 + a_4)\delta(x_0, F_2(x_0))$, i.e., $\delta(x_0, F_2(x_0)) = 0$.

Therefore, x_0 is a common fixed point of F_1 and F_2.

The most conspicuous difference between the fixed point theorems for single-valued functions and for multi-valued mappings is the absence of the guarantee of uniqueness of fixed points in the multivalued case. Dube, [12], in his paper asked whether it was possible to alter the conditions of theorem 3.1.3 to insure the uniqueness of the fixed point. We give two partial answers to this.

If, in Theorem 3.1.3, we assume that the image of a fixed point is a singleton, then the fixed point is the unique common fixed point. We have the following:

<u>Theorem 3.1.4</u>: Let (X,d), F_1, F_2 be as in Theorem 3.1.3. If x_0 is a common fixed point of F_1 and F_2 such that $F_1(x_0) = \{x_0\}$, or $F_2(x_0) = \{x_0\}$, then x_0 is the unique common fixed point of F_1 and F_2.

<u>Proof</u>: Suppose that $F_1(x_0) = \{x_0\}$ and y_0 is any other common fixed point. Since $F_1(x_0)$ and $F_2(y_0)$ are both closed with $x_0 \in F_1(x_0)$ and $y_0 \in F_2(y_0)$, we have that $\delta(x_0,F_1(x_0)) = 0 = \delta(y_0,F_2(y_0))$.

By condition (1) of Theorem 3.1.3,

$$H(F_1(x_0),F_2(y_0)) \leq a_3\delta(x_0,F_2(y_0)) + a_4\delta(y_0,F_1(x_0)) + a_5 d(x_0,y_0).$$

However, since $y_0 \in F_2(y_0)$ and $x_0 \in F_1(x_0)$ it follows that

$$\delta(x_0,F_2(y_0)) \leq d(x_0,y_0) \quad \text{and} \quad \delta(y_0,F_1(x_0)) \leq d(x_0,y_0).$$

Therefore,

$$H(F_1(x_0),F_2(y_0)) \leq (a_3+a_4+a_5)d(x_0,y_0) .$$

By Lemma 3.1.1,

$$\delta(y_0,F_1(x_0)) \leq H(F_1(x_0),F_2(y_0)) .$$

Thus, $\delta(y_0,F_1(x_0)) \leq (a_3+a_4+a_5)d(x_0,y_0)$.

But since $F_1(x_0) = \{x_0\}$; $\delta(y_0,F_1(x_0)) = d(x_0,y_0)$

and, $(1-a_3-a_4-a_5)d(x_0,y_0) \leq 0$.

with $a_3+a_4+a_5 < 1$, then $d(x_0,y_0) = 0$.

Similarly, $F_2(x_0) = \{x_0\}$ also yields $d(x_0,y_0) = 0$.

Therefore, x_0 is the unique common fixed point of F_1 and F_2 .

Next, as a second answer to the question of uniqueness for multivalued mappings, we give a kind of "localization" of the fixed points of one multivalued mapping satisfying the conditions of Theorem 3.1.3.

First, we prove the following lemma:

<u>Lemma 3.1.2</u>: Let (X,d) be a metric space and $A \in CB(X)$. If $x_0 \in X-A$, then there exists a sequence $\{y_n\} \subsetneq A$ such that

$$\lim_{n\to\infty} d(x_0,y_n) = \delta(x_0,A)$$

Proof: Since $A \in CB(X)$, $\delta(x_0, A)$ clearly exists. Let $\delta(x_0, A) = a$. By the definition of infimum, for every $\varepsilon > 0$, there exists $b \in A$ such that

$$a \leq d(x_0, b) < a + \varepsilon$$

By taking $\varepsilon = 1/n$, there exists $y_n \in A$ such that

$$a \leq d(x_0, y_n) < a + 1/n ,$$

and by letting n tend to infinity,

$$\lim_{n \to \infty} d(x_0, y_n) = a .$$

We now give the following theorem, which localizes the fixed points of a multivalued mapping in a closed and bounded subset of X.

Theorem 3.1.5: Let (X,d) be a complete metric space, and $F : (X,d) \to (CB(X),H)$, a multivalued mapping satisfying the conditions of Theorem 3.1.3, with $F = F_1 = F_2$. If there exists a fixed point y_0 of F such that for any $z \in F(y_0)$, $z \in F(z) \subseteq F(y_0)$, then for every fixed point x_0 of F, $x_0 \in F(y_0)$.

Proof: Let x_0 be any fixed point other than y_0. Because $F(y_0)$ is closed, we need only to show that $\delta(x_0, F(y_0)) = 0$ to prove that $x_0 \in F(y_0)$.

For any fixed point z of F, we have by condition (1) of Theorem 3.1.3,

$$H(F(x_0),F(z)) \leq a_3\delta(x_0,F(z)) + a_4\delta(z,F(x_0)) + a_5d(x_0,z)$$

By Lemma 3.1.1, as $z \in F(z)$,

$$\delta(x_0,F(z)) \leq H(F(x_0),F(z))$$

Hence $\delta(x_0,F(z)) \leq a_3\delta(x_0,F(z)) + a_4\delta(z,F(x_0)) + a_5d(x_0,z)$

and as $x_0 \in F(x_0)$ and $z \in F(z)$, we have

$$\delta(x_0,F(z)) \leq (a_3+a_4+a_5)d(x_0,z) .$$

Thus, since $F(z) \subseteq F(y_0)$,

(3.1.1) $\delta(x_0,F(y_0)) \leq \delta(x_0,F(z)) \leq (a_3+a_4+a_5)d(x_0,z)$

for any $z \in F(y_0)$.

Now, by Lemma 3.1.2, there is a sequence $\{y_n\} \subseteq F(y_0)$ such that

$$\lim_{n\to\infty} d(x_0,y_n) = \delta(x_0,F(y_0))$$

By substituting $z = y_n$, in (3.1.1), we get

$$\delta(x_0,F(y_0)) \leq \delta(x_0,F(y_n)) \leq (a_3+a_4+a_5)d(x_0,y_n)$$

Hence, by taking the limit as $n \to \infty$, we get

$$\delta(x_0,F(y_0)) \leq (a_3+a_4+a_5)\delta(x_0,F(y_0)) .$$

Therefore, since $(a_3+a_4+a_5) < 1$, $\delta(x_0,F(y_0)) = 0$, and the theorem is proved.

Remark 3.1.1: If we assumed that $a_5 = 0$ in condition (1) of Theorem 3.1.3, then we can rephrase the hypothesis of the above theorem to read: If y_0 is a fixed point of F such that for any $y \in F(y_0)$, we have $y \in F(y)$, then $F(y_0)$ contains all of the fixed points of F. For, if $a_5 = 0$ and $y \in F(y_0)$ we have for any $x \in F(y)$,

$$\delta(x,F(y_0)) \leq H(F(y),F(y_0)) \leq a_3\delta(y,F(y_0)) + a_4(y_0,F(y))$$

or since $y \in F(y_0)$,

$$(3.1.2) \quad \delta(x,F(y_0)) \leq H(F(y_0),F(y)) \leq a_4\delta(y_0,F(y)).$$

Also, since $y_0 \in F(y_0)$,

$$\delta(y_0,F(y)) \leq H(F(y_0),F(y)) \leq a_4\delta(y_0,F(y)).$$

Hence, since $a_4 < 1$, $\delta(y_0,F(y)) = 0$ and therefore, $\delta(x,F(y_0)) = 0$. Thus for every $y \in F(y_0)$, we have, for any $x \in F(y)$, $x \in F(y_0)$. Hence, $y \in F(y) \subseteq F(y_0)$. Thus the conditions of the preceding theorem are satisfied, and then $F(y_0)$ contains all the fixed points of F.

We can also see in the above remark that if $a_5 = 0$, then $H(F(y_0),F(y)) = 0$. Therefore, as a corollary we have the following which gives conditions for the uniqueness of the fixed point:

Corollary 3.1.1: If $a_5 = 0$ in the above theorem, and F is a one-to-one function, then y_0 is the unique fixed point of F.

Proof: If $a_5 = 0$, then we have

$$(3.1) \qquad H(F(y_0),F(y)) \leq a_4\delta(y_0,F(y)) .$$

However, as proved in the preceding remark, $\delta(y_0,F(y)) = 0$. Therefore, $H(F(y_0),F(y)) = 0$, and hence $F(y_0) = F(y)$, for any $y \in F(y_0)$. By the above remark, for any fixed point x_0, of F, $x_0 \in F(y_0)$. Thus, for any fixed point x_0 of F, we have $F(x_0) = F(y_0)$. Since F is one-to-one, $x_0 = y_0$ and therefore y_0 is the unique fixed point of F.

Remark 3.1.2: Let W be the set of all fixed points of a multivalued mapping F. H-M Ko [20] defined W to be a "singleton in the generalized sense", if there exists $y \in W$, such that $W \subset F(y)$. She noted that even when considering such a restricted space as a closed convex subset K of a Banach Space X, and F a nonexpansive multivalued mapping, $F : K \to 2^K$, W was not a singleton in the generalized sense. We observe that in our Theorem 3.1, W is a singleton in the generalized sense. In fact, $W \subseteq F(y_0)$, and $y_0 \in W$.

We also observe that the type of "localization" used in Theorem 3.1.4, can also be used in Theorem 3.1.1.

We have the following:

Theorem 3.1.6 : Let (X,d) be a complete metric space and $F : X \to CB(X)$, be a multivalued contraction. Then F has a fixed point, and if there exists a fixed point y_0 of F, such that for any $z \in F(y_0)$, $z \in F(z) \subseteq F(y_0)$, we have that $F(y_0)$ contains all the fixed points of F.

The proof of this theorem is along the same lines of Theorem 3.1.5. If x_0 is any fixed point of F and $z \in F(y_0)$, we have

$$\delta(x_0,F(y_0)) \leq \delta(x_0,F(z)) \leq kd(x_0,z) .$$

Again, by taking a sequence $\{y_n\} \subseteq F(y_0)$ such that $\lim_{n\to\infty} d(x_0,y_n) = \delta(x_0,F(y_0))$ we have that

$$\delta(x_0,F(y_0)) \leq \delta(x_0,F(y_n)) \leq kd(x_0,y_n) .$$

By taking the limit as $n \to \infty$, we get

$$\delta(x_0,F(y_0)) \leq k\delta(x_0,F(y_0)) .$$

Since $k < 1$, therefore $\delta(x_0,F(y_0)) = 0$, and $x_0 \in F(y_0)$.

3.2 Another Approach to Multivalued Fixed Point Theorems.

We have seen in Chapters I and II that the sequences of successive approximations played an important role in proving the existence of fixed points of single-valued functions. As can be seen in the proof of Theorem 3.1.3, the construction of a specific sequence was also crucial in proving the existence of a common fixed point for two multivalued mappings. However, we note that this sequence was constructed and not defined "naturally" as in the previous chapters.

We define the sequence of iterates for a multivalued mapping $F : X \to CB(X)$ by picking x_0 to be any point in X, and arbitrarily taking $x_{n+1} \in F(x_n)$ for $n = 1,2,\ldots$. As $F(x) \in CB(X)$, and $F(x)$ is nonempty for each x in X, the existence of such a sequence is obvious.

Remark 3.2.1: We note that as in the single-valued case, if the sequence of iterates $\{x_n\}$, as defined above, converges to $x \in X$, and F is continuous, then x is a fixed point of F. For, by Lemma 3.1.1, we have that

$$\delta(x_n, F(x)) \leq H(F(x_{n+1}, F(x)) .$$

As F is continuous, the right-hand side, and hence $\delta(x_n, F(x))$, tends to zero as $n \to \infty$. Hence as $\delta(y, F(a))$ is a continuous function of y, for fixed a, it follows that

$$\lim_{n\to\infty} \delta(x_n, F(x)) = \delta(x, F(x)) = 0$$

Thus x is a fixed point of F.

The sequence of iterates, defined above, could not be used in proving Theorem 3.1.3, due to its arbitrariness. However, we can prove the theorem, if $a_5 = 0$ and $F = F_1 = F_2$. First we prove the following lemma.

Lemma 3.2.1: If $F : (X,d) \to (CB(X),H)$ is a continuous multi-valued map, then $\delta(y, F(y))$ is a continuous function of $y \in X$.

Proof: Let $\{y_n\}$ be a sequence in X converging to y. Now,

$$|\delta(y_n, F(y_n)) - \delta(y, F(y))| \leq |\delta(y_n, F(y)) - \delta(y_n, F(y_n))|$$
$$+ |\delta(y_n, F(y)) - \delta(y, F(y))| .$$

From Kuratowski [21, p.210] we have that

$$|\delta(y_n, F(y)) - \delta(y, F(y))| < d(y_n, y) .$$

Hence, to show that $\delta(y, F(y))$ is a continuous function of y, it suffices to show that $|\delta(y_n, F(y)) - \delta(y_n, F(y_n))|$ tends to zero as n tends to infinity.

For every $\varepsilon > 0$, there is $a_n \in F(y_n)$ such that

$$d(y_n, a_n) \leq \delta(y_n, F(y_n)) + \varepsilon .$$

Since $\delta(y_n, F(y)) \leq d(y_n, a_n) + \delta(a_n, F(y))$, it follows that

$$\delta(y_n, F(y)) \leq \delta(y_n, F(y_n)) + \delta(a_n, F(y)) + \varepsilon .$$

However, as n tends to infinity, $\lim_{n\to\infty} \delta(a_n, F(y)) = 0$, since $a_n \in F(y_n)$ and $\lim_{n\to\infty} H(F(y_n),F(y)) = 0$.

Therefore, as n tends to infinity,

(3.2.1) $\delta(y_n,F(y)) - \delta(y_n,F(y_n)) \leq \varepsilon$.

Similarly, for the same $\varepsilon > 0$, as above, there is a $b \in F(y)$ such that

$$d(y_n,b) \leq \delta(y_n,F(y)) + \varepsilon .$$

And hence, $\delta(y_n,F(y_n)) - \delta(y_n,F(y)) \leq \delta(b,F(y_n)) + \varepsilon$.

Again, as $b \in F(y)$ and $\lim_{n\to\infty} H(F(y_n),F(y)) = 0$, $\delta(b,F(y_n))$ tends to zero as n tends to infinity.

Therefore, as n tends to infintiy,

(3.2.2) $\delta(y_n,F(y_n)) - \delta(y_n,F(y)) \leq \varepsilon$.

Combining (3.2.1) and (3.2.2) and, as ε is arbitrary, we have that

$$\lim_{n\to\infty} |\delta(y_n,F(y_n)) - \delta(y_n,F(y))| = 0 .$$

Thus, $\delta(y,F(y))$ is a continuous function of y .

We now prove the following theorem.

<u>Theorem 3.2.1</u> : Let (X,d) be a metric space and $F : X \to CB(X)$ be continuous. Suppose F satisfies the conditions of Theorem 3.1.3, with $F = F_1 = F_2$ and $a_5 = 0$. Then every cluster point of the sequence of iterates,

defined by picking any $x_0 \in X$ and arbitrarily picking $x_{n+1} \in F(x_n)$, $n = 1, 2, \ldots$, is a fixed point of F.

<u>Proof</u>: Suppose x is a cluster point of the sequence of iterates of F. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to x.

Since $a_5 = 0$, and $F_1 = F = F_2$, it is easily shown that

$$H(F(x_n), F(x_{n-1})) \leq k^{n-1} H(F(x_0), F(x_1)),$$

with $k = \dfrac{a_1 + a_3}{1 - a_2 - a_3} < 1$, $n > 1$.

Therefore, by Lemma 3.1.1, as $x_n \in F(x_{n-1})$,

$$\delta(x_n, F(x_n)) \leq k^{n-1} H(F(x_0), F(x_1)).$$

Thus, $$\lim_{n\to\infty} \delta(x_n, F(x_n)) = 0.$$

As $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then

$$\lim_{k\to\infty} \delta(x_{n_k}, F(x_{n_k})) = 0$$

However, by Lemma 3.2.1, since F is continuous

$$\delta(x, F(x)) = \lim_{k\to\infty} \delta(x_{n_k}, F(x_{n_k})) = 0,$$

i.e., $x \in F(x)$. Hence the theorem.

Next, we extend the result of Belluce and Kirk [3] to multivalued mappings. First we need the following definition, due to Browder and Petryshyn [6], for single-valued functions.

Definition 3.2.1: A mapping $T : X \to X$, where (X,d) is a metric space is said to be asymptotically regular on X, if $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$, for each $x \in X$.

We recall that $T^n x$ is the n^{th} term of the sequence of successive approximations. It is natural, therefore, to define asymptotic regularity for a multi-valued mapping F, as follows:

Definition 3.2.2: Let $F : (X,d) \to (CB(X),H)$ be a multi-valued mapping. F is said to be asymptotically regular if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, where x_n is the n^{th} term of the sequence of iterates of F, for every initial element $x_0 \in X$, as defined before.

Although asymptotic regularity is a "strong" condition on the sequence of successive approximations, it alone does not guarantee the existence of a fixed point for the single-valued function. Belluce and Kirk [3], however, have given this result:

Theorem 3.2.2: Let (X,d) be a compact metric space and let T be a continuous mapping of X into itself which is asymptotically regular on X. Then every sequence $\{T^n x\}$ contains a subsequence which converges to a fixed point of T.

Now, we extend the above theorem to multivalued mappings. Note that we are assuming a weaker condition than compactness of the whole space, unlike the preceding theorem.

<u>Theorem 3.2.2</u>: Let (X,d) be a metric space and F, a continuous multi-valued mapping of X into $(CB(X),H)$.

If (i) F is asymptotically regular, and

(ii) there exists a convergent subsequence $\{x_{n_k}\}$ of the sequence of iterates $\{x_n\}$,

then the subsequence $\{x_{n_k}\}$ converges to a fixed point of F.

<u>Proof</u>: Suppose that $\{x_{n_k}\}$ converges to $z \in X$. By Lemma 3.1.1, $\delta(x_{n_k+1},F(z)) \leq H(F(x_{n_k}),F(z))$.

Hence, since F is continuous, we have that

$$\lim_{k\to\infty} \delta(x_{n_k+1},F(z)) = 0 .$$

However, by the triangle inequality, we have

$$(3.2.3) \quad d(x_{n_k+1},z) \leq d(x_{n_k},z) + d(x_{n_k},x_{n_k+1})$$

Now since F is asymptotically regular, we have that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$ and hence for any subsequence $\{x_{n_k}\}$, $\lim_{k\to\infty} d(x_{n_k},x_{n_k+1}) = 0$. Therefore, by (3.2.3),

$$\lim_{k\to\infty} d(x_{n_k+1},z) = 0 .$$

Thus, $\delta(z,F(z)) = \lim_{k\to\infty} \delta(x_{n_k+1},F(z)) = 0$, and therefore,

$z \in F(z)$, i.e., z is a fixed point of F.

As a corollary to this theorem, we have the analogue of the Belluce and Kirk theorem:

<u>Corollary 3.2.1</u>: Let (X,d) be a compact metric space and $F : X \to CB(X)$ be a continuous mapping which is asymptotically regular on X. Then every sequence of iterates contains a subsequence which converges to a fixed point of F.

<u>Proof</u>: As (X,d) is compact, it is sequentially compact and hence every sequence $\{x_n\}$ of iterates of F has a convergent subsequence $\{x_{n_k}\}$. We now have the conditions of the theorem. Thus $\{x_{n_k}\}$ converges to a fixed point of F.

With our definition of the sequence of iterates for a multivalued mapping, we have also been able to extend Theorem 2.2.2, due to Cheney and Goldstein [7]. We give the following result for nonexpansive multivalued mappings.

<u>Theorem 3.2.3</u>: Let (X,d) be a compact metric space and $F : (X,d) \to (CB(X),H)$ be a multivalued nonexpansive mapping such that

(3.2.1) if $x \notin F(x)$, then $H(F(x),F(y)) < \delta(x,F(x))$,

for every $y \in F(x)$.

Then every cluster point of the sequence of iterates $\{x_n\}$ for F, as defined in Theorem 3.2.1, is a fixed point of F.

Proof: Without loss of generality, we may assume that $x_n \notin F(x_n)$, otherwise there is nothing to prove. As $x_{n+1} \in F(x_n)$ for $n = 0,1,2,\ldots$ we have by Lemma 3.1.1,

$$\delta(x_{n+1}, F(x_{n+1})) \leq H(F(x_n), F(x_{n+1}))$$

and by (3.2.1), $H(F(x_n), F(x_{n+1})) \leq \delta(x_n, F(x_n))$.

Hence $\delta(x_{n+1}, F(x_{n+1})) < \delta(x_n, F(x_n))$ for $n = 0,1,2,\ldots$

i.e., $\delta(x_n, F(x_n))$ is a nonincreasing sequence of non-negative real numbers. Therefore the sequence $\{\delta(x_n, F(x_n))\}$ has a limit. Since X is compact, and hence sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let x_{n_k} converge to u. Now,

$$\begin{aligned} \delta(u, F(u)) &= \lim_{k\to\infty} \delta(x_{n_k}, F(x_{n_k})) \\ &= \lim_{n\to\infty} \delta(x_n, F(x_n)) \\ &= \lim_{n\to\infty} \delta(x_{n+1}, F(x_{n+1})) \\ &= \lim_{k\to\infty} \delta(x_{n_k+1}, F(x_{n_k+1})) . \end{aligned}$$

Again, since X is compact, the sequence $\{x_{n_k+1}\}$ has a convergent subsequence, say $\{y_{n_k+1}\}$.

Since F is nonexpansive, it follows, by Lemma 3.1.1 that

$$\delta(x_{n_k+1},F(u)) \leq H(F(x_{n_k}),F(u)) \leq d(x_{n_k},u)$$

Therefore, $\lim_{k\to\infty} \delta(x_{n_k+1},F(u)) = 0$.

And as such, $\lim_{k\to\infty} \delta(y_{n_k+1},F(u)) = 0$.

If $\{y_{n_k+1}\}$ converges to u , then $\delta(u,F(u)) = \lim_{k\to\infty} \delta(y_{n_k+1},F(u)) = 0$, i.e., u is a fixed point of F .

Suppose $\lim_{k\to\infty} y_{n_k+1} = v \neq u$. Therefore $\lim_{k\to\infty} \delta(y_{n_k+1},F(u)) = \delta(v,F(u)) = 0$, and $v \in F(u)$.

However,

$$\begin{aligned} \delta(u,F(u)) &= \lim_{k\to\infty} \delta(x_{n_k+1},F(x_{n_k+1})) \\ &= \lim_{k\to\infty} \delta(y_{n_k+1},F(y_{n_k+1})) \\ &= \delta(v,F(v)) . \end{aligned}$$

By Lemma 3.1.1,

$$\delta(u,F(u)) = \delta(v,F(v)) \leq H(F(u),F(v)) ,$$

which contradicts (3.2.1) unless $u \in F(u)$. Thus the subsequence $\{x_{n_k}\}$ converges to a fixed point of F . Hence the theorem.

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