GUIDED DISCOVERY TEACHING IN MATHEMATICS

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ABSTRACT

GUIDED DISCOVERY TEACHING IN MATHEMATICS by IVAN GOMBOS

Motivated by a desire to foster a more active learning environment I decided, approximately two years ago, to incorporate a guided discovery approach into my general teaching strategy. In Chapter I of the thesis a background of the events that led up to this decision is given, along with a brief survey of "discovery" terminology and related matters.

Guided discovery teaching proceeds simultaneously at two levels in my courses. At one level - which I partition into three stages - the students and I engage in a dialogue centering on the general themes of experimenting with mathematical patterns and relationships, forming subsequent generalizations, studying methods of proof and solving various types of problems. At the other level I select specific course topics which I feel can easily blend in with a discovery scheme. Level 1 (the General or Global Level) is discussed in Chapter II, while a representative sample of the more highly course-specific discovery work (Level 2) forms the core of Chapter III. Two Questionnaires and the taping of a lesson on Elementary Matrices have been used to gage student reaction to my teaching style. Thus, the final Chapter contains a report and an analysis of the feedback obtained from these sources.

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CHAPTER 1

INTRODUCTION

The Origins of an Idea

In 1971 I began teaching Mathematics, at Vanier C.E.G.E.P. Soon afterwards I joined the Dawson Maths. Department. Thus, it is now my 10th year at Dawson College.

As it happens quite often, I imagine, my classroom presentation in the beginning strongly reflected my own experiences with teachers in university and in high school. The first year or so of a novice teacher is spent largely on course preparation. So it was with me. I loved my job (still do!), compiled notes and exercise sets eagerly, and tried to imitate Professor George P. Styan's teaching style. (In graduate school at McGill, his lectures were meticulously organized, his presentations always clear, his patience in answering questions quite exemplary.)

Of course, I had had my share of poor, even atrocious teachers and therefore had to work very hard <u>not</u> to transmit variations of their mistakes, methods or teaching procedures to my own students. No one wants to spread an epidemic of bad teaching.

Two or three semesters went by and it was time to relax a little - to sit back and reflect upon my approach to teaching. I felt comfortable with the classical lecture or "talk-and-chalk" method but was ready to experiment with the occasional variation. For instance, I organized some group-problem sessions. In each course, I would present 2 lectures a week to the whole class and then meet with smaller groups once a week for problem-solving. This worked very well - at first. Then I noticed that attendance was dropping. Oh, the best students kept coming faithfully. But the weaker students slowly drifted away. After 2 terms I abandoned the idea. It was clearly necessary to devote some serious thinking as to what had gone wrong. (Now, in 1981, I am planning to return to the small-group-session model once again, hopefully with better results.)

In 1974 I decided to implement a system of handing out detailed set of notes to my students prior to each topic to be covered. No longer would they have to be passive note-takers. They could sit back, listen, ask questions and learn. I was prepared to answer some well-thought-out, even profound questions. After all, they had a chance to study the <u>same</u> material that I would cover in class! From <u>passive</u> note-takers my students changed to <u>passive</u> listeners. A poor accomplishment. They did <u>not</u> pepper me with a barrage of penetrating questions. They didn't - for the most part - even read the notes I had given out. And attendance was on the decline, again. A few brave souls came up to me and <u>asked me</u> to stop distributing packages of lecture notes. I was stunned and disappointed. They said they were <u>bored</u>, not having anything to do in class.

But there was something to be learned from both unsuccessful ventures. In the problem-solving groups <u>I</u> had done most of the problem-solving. By giving hand-outs I had provided a convenient security blanket: a nice set of notes to glance through just before the test. In both cases I had done little to encourage <u>active</u> learning. Naturally, <u>another teacher</u> could have perhaps taken either or both of my little projects and turned them into successful - and active - learning experiences. (In fact, one of my colleagues - Jill Britton - has been <u>consistently</u> very successful with sets of typed-up notes that she provides for her students.) And, as I've already said I intend to try the problem-solving sessions once more, after making a few strategic changes. But this will not be the central issue here.

My "failures" led me to re-think my teaching approach. My mind focused on the word "passive". Here was an adjective I really began to dislike. I resolved to start varying my sequence of presentation. Instead of always stating theorems or properties first, proving them, and then giving a set of examples, I would try, whenever feasible, to "pick away" at a general concept via preliminary examples, to then develop a proof (with the help of the class, as much as possible), and follow it by a good solid selection of examples and exercises. This, I found was quite conducive to increasing the dialogue between my students and myself. I enjoyed the ensuing dialogues and became aware of a rise in the level of classroom activity. Moreover, there were new challenges in store: How to direct discussion? How to find time for the new dialogue approach and still cover all the required material? How to know when a certain topic was appropriate for introductory experimentation? I am still working on the answers to these questions, using mostly the rather slow, learn-by-experience method. [See, for example, Chapter IV.]

Increasing student participation by generating class dialogue - this was my initial aim. While I was busy with such plans, we opened our new Math Learning Centre. It had the usual tables, reference books, Calculus film loops etc. found in any math study room, and one more thing: each teacher picked one period a week during which he'd be available to assist anyone working in the Centre. In this way I met a large cross-section of students, from those struggling through elementary algebra to the more experienced devotees of Calculus II and III. One particular type of complaint caught my attention. It usually came from shy, nervous-looking individuals but the vocal, agressive students were not excluded either. The complaint boiled down to: "I just can't do this stuff!" Furthermore, they refused to try even after I had given them some hints. They preferred to sit and wait in silence until I had outlined the complete solution. They had no confidence whatsoever in their abilities when it came to certain areas in mathematics. I suspected that this "mental block", in many cases, had taken a few years to fully develop. My suspicions were confirmed when a few students were angry enough to vent their feelings at length. Perhaps, I thought, one of my teaching objectives should be to somehow let students discover that they can indeed do such-and-such a problem. I fully believed then, and do now, that my efforts at promoting dialogue would be helpful in this regard.

Then in the fall of '78 I registered for Math 649 (Heuristics) and in the spring of '79 I enrolled in Math 624 (Mathematics Education), both given at Concordia University, Montreal. By means of these courses, my knowledge of problem-solving and of the psychology of learning mathematics increased greatly. I embarked on re-reading Polya's classical volumes and was introduced to new ways of looking at heuristics. Then came discussions of the work of Piaget, Ausubel, Gagné, Bruner, et al. in Maths. 624. For the first time I read about "discovery" and "guided discovery" as viewed by experienced psychologists and educators. I was motivated to try channelling class discussions towards helping students do their own mathematical detective work. (I had not forgotten my earlier objective - the one involving confidence-building. On the contrary, I suspect that strengthening student confidence and teaching via guided discovery are intimately connected. However, experimentally substantiating such connection is beyond the intent or the scope of this thesis.)

Thus starting from about 1978 my interest in the guided discovery approach has grown steadily. How this approach can be a part of college mathematics - and specifically, how I've incorporated it into my courses at Dawson - is at the core of the present thesis.

Some Remarks On Definitions and the Concept of "Discovery"

The concept of discovery was defined by J.S. Bruner as "a matter of re-arranging or transforming evidence in such a way that one is enabled to go beyond the evidence so re-assembled to new insights". [1] A critical person may then ask: "What are insights?" He might look up "insight" and run across other difficult words which are used to define it; difficult perhaps because human beings are attempting to describe their own cognitive processes. But we know that discussions of learning and teaching continue despite difficulties in descriptions of complex human behaviour.

The noun "discovery" is sometimes accompanied by one or more adjectives. Wittrock, for instance, talks about three types of discovery: deductive guided discovery, in which some general rules or solutions are provided by the instructor; inductive guided discovery, in which student and teacher work jointly to build up generalizations or rules from special cases; and pure discovery, in which no hints are given concerning appropriate algorithms, rules or procedures. [2] Robert Gagné speaks of guided discovery as a kind of discovery experience in which the teacher provides hints or clues to a problem in order to decrease the "search time" required to resolve the problem. [3] The paper referred to in [3] was presented at the 1965 New York Conference on Learning by Discovery. The proceedings at this conference are published in Learning by Discovery: A Critical Appraisal, edited by Lee S. Shulman and Evan R. Keislar. [4] The participants at the conference generally agree that discovery is rarely a haphazard, do-your-own-thing kind of experience, whether or not they employ the adjective "guided". (One member of a workshop even went as far as saying that pure discovery was pure nonsense! [5])

"What do I mean by guided discovery instruction in this thesis? Simply: teacher-directed manipulation of student behaviour to assist in the finding and understanding of certain processes, patterns, relationships and generalizations in Mathematics.

It is worth noting that while Learning by Discovery is the title of Keislar and Shulman's book, it contains a number of references to discovery teaching. One need only peruse Lee J. Cronbach's article, for instance. [6] Moreover, the index on page 215 of Learning of Discovery contains no less than 9 references to discovery teaching. Now, one would usually expect that learning will take place whenever there is teaching. The question is: "What is the effect of guided discovery teaching on learning?" or "How does varying a teaching method influence the learning?" A modest, and somewhat indirect, attempt to answer these questions is made in Chapter IV.

Now, one may ask: "What name is given to the opposite of discovery?" (This, of course, is a cautious way of asking: "What is the opposite of discovery?") Ausbel writes about reception learning in contrast to discovery learning and carefully notes that he does not equate this with rote learning. [7] On the teaching end of the scale, we note that Bruner refers to <u>expository</u> teaching as opposed to discovery teaching [8], while Cronbach cites <u>didactic</u> teaching as the antithesis to the discovery method. [9] In my own naive terms: "non-discovery" equates to laying all your cards on the table, wherein the teacher exposes all the elements of a particular problem or topic to his students. (Incidentally, one of the most imaginative comparisons of active "discovery" versus the more passive "reception" was made recently by one of my students. It is quoted in Chapter IV.)

For my own way of using the discovery technique, <u>guidance</u> is very important. It provides continuity in the classroom by minimizing the number of minutes of silence resulting from students having reached a cul-de-sac along their path of reasoning. (The challenge for the teacher is to learn how to distinguish dead silence because of a dead end from dead silence because of intensive mental activity inside a student's head.) Guidance clearly helps to cut down on the amount of time needed to "find an answer" or to make an observation. Without it, covering the required course material might become virtually impossible.

A Brief Guide to Some Related Literature

One possible source to use for starting an investigation into learning/teaching by discovery is the comprehensive volume, <u>Learning by Discovery</u>, mentioned earlier. This book contains a selection of opinions and observations by such prominent workers in the field as Bruner, Gagné, Cronbach, and Wittrock as well as a number of others. Each of the four major sessions at the conference (corresponding to Parts One to Four in the book) is followed by a discussion or summary of the pertinent issues that had surfaced during that particular session. Furthermore, an extensive bibliography is available on pp. 200-212 of the text.

As a mathematics teacher I turned first to Robert B. Davis' article on pages 114-128 of Shulman and Keislar [10]. In this article, examples of discovering patterns and rules in Mathematics at the elementary and junior high school level are given. The examples include graphing linear equations, solving simple quadratics and noting patterns in certain matrices. Prof. Davis feels that the discovery experience allows the student to gain some insight into how Mathematics is often developed through trials, failures, modest successes, looking at problems through many different angles. The students then can perhaps better appreciate the struggles and excitement that has been a part of much of the history of Mathematics. [11] He also relates an interesting case involving elementary arithmetic in which a teacher who did not reject a seemingly incorrect solution from one of her students actually learned something from that student - a new algorithm for the subtraction of integers. [12] (This - learning from one's students during a discovery session - is an important "fringe benefit" of discovery teaching that had not previously occurred to me.) Davis also points out that discovery experiences may be more valuable to the general process of education - as opposed to the narrower or more limited process of training for specific tasks by exposition - by being more realistic in not giving away all goals, algorithms, rules and methods at the beginning of a presentation. [13] Finally, he lists fifteen general goals that he feels one should seek after when teaching Mathematics. Among these goals are: mastery of basic techniques, improvement in students' facility to relate various parts of Mathematics to each other, to allow students to have a realistic assessment of their own abilities to discover Mathematics, to show students that Mathematics is discoverable, and to show that Mathematics can be fun, exciting and worthwhile. [14] In short, according to Dr. Davis, education is for people, not robots, and discovery teaching is a realistic way of obtaining an education in Mathematics, with its emphasis on creativity and divergent ways of thinking, as opposed to rote, mechanical training.

Another reference source is Bayne Logan's On Children's Mathematics [15]. It begins with a discussion of the theoretical rationale behind the discovery approach as envisaged by Bruner and then focuses on the theoretical expectations corresponding to this approach as postulated by Brunerian theory. Such expectations are:

(a) the ability on the part of the student to obtain greater understanding of problem solving strategies.

(b) greater "intellectual potency"; that is, an increase in the ability of the student to devise and master strategies on his own (coming from prolonged experience in discovering patterns and relationships).

(c) less difficulty in overcoming the language in which a problem may be couched and less difficulty in getting to the essence of a problem.

(d) greater flexibility in problem solving - that is, students "should be less susceptible to functional fixedness, that

(d) continued...

inability to perceive and ~~to~~ pursue viable alternative solutions".

(e) students should be able to cope with various notions in Mathematics at various stages in their cognitive development provided this Mathematics is approached at an appropriate level of sophistication. [16]

Logan then reviews some of the work of Piaget, Wilson, Moody, Peters et al. related to research in Mathematics education, particularly at the elementary school level. He observes that there seems to be evidence that:

(a) maturation is more important in how a student will learn Mathematics than in determining what he will learn.

(b) intelligence, reading ability and problem-solving ability are probably inter-related.

(c) the discovery approach tends to lead to a greater ability in general problem-solving or heuristics. [17]

To compare the expository and discovery methods~~,~~ Logan than poses a few research hypotheses. In these, he postulates that, allowing for control for variations in intelligence and reading ability, the discovery approach should yield better results in:

(H1) comprehension of number and operation concepts.

(H2) learning of personalized problem-solving strategies.

(H3) flexibility in problem-solving strategies.

(H4) ability to apply correct strategies to problems of various

(H4) continued...

levels of contextual complexity. [18]

He then proceeds to test the hypotheses via three research instruments: The Sequential Tests of Educational Progress: Mathematics Levels 4A and 4B, The Sequential Tests of Educational Progress: Reading Levels 4A and 4B, and The Longe-Thorndike Intelligence Tests. [19] The tests were administered to students in grades 4, 5 and 6 ranging in age from eight to twelve years. A complete discussion of the above tests, of general experimental design, of the selection of experimental "subjects", and so on, is available in Logan, pp. 26-50.

Briefly, the results of the investigations are as follows. Three of the four hypotheses (H1, H2, H3) were supported at the .95 level of significance. But, in the case of the fourth experimental hypothesis, H4, no significant differences were seen between groups exposed to the two "treatments" (discovery and expository methods). Both the children taught by guided discovery and the children taught by the expository technique could handle the mathematical operations involved in problems with equal ease (or difficulty) provided they understood the language and concepts associated with the problems. Although, things may portend well for the discovery procedure, Logan cautiously calls for continuing and extensive research into the two major teaching strategies discussed in his manuscript. [20]. He concludes with a quote from Kahlil Gibran's The Prophet. The last part of the quotation, referring to "the teacher" reads:

"If he is indeed wise he does not bid you enter the house of his wisdom, but rather leads you to the threshold of your own mind." [21]

I think that a more elegant description of guided discovery teaching may be very hard to find.

The bibliography in On Children's Mathematics lists contributions by Bruner as well as others to the field of learning theory and to the special area of guided discovery.

An Overview and A Look Ahead

D. Wheeler quoted in [22] recommends: "We must astonish the most anxious and insecure into some success, and in such a way that they know it is their success." This statement summarizes beautifully my basic motivation to improve my teaching methods during the past 5-6 years. Above all, I have always wanted my students to like Mathematics and to improve upon the way they were exposed to Mathematics in my classroom. Some general discussion on how I have tried to achieve these goals was given in the first section of this chapter.

In the past few semesters at Dawson, I have done a lot of work in

(a) looking at the major components or overall plan of guided discovery in elementary Mathematics, with a view of classifying and compiling examples and exercises as well as proof-types that are, in my opinion, representative of or amenable to the discovery process at the college level,

(b) re-writing my notes and problem sets in Linear Algebra, Calculus and College Mathematics with the intention of treating topics by the discovery mode whenever possible, and

(c) obtaining feed-back from students about my teaching and about their strengths, creativity and weaknesses.

Roughly speaking, Chapters II, III and IV, respectively, are representative of the outcomes of my efforts described under (a), (b) and (c) above.

The sections of Chapter II break the process of guided discovery teaching up into 3 major stages: experimentation/generalization proving, and consolidation. Although some of the examples given in Chp. II could perhaps be routinely interwoven into one or another Dawson Math Course my plan for the chapter is a more general one. What types of proofs arise frequently - and confound students just as frequently - at the college level? What types of "warm-up" exercises work rather well? What kind of follow-up problems tend to firm up the ideas presented in the inductive stages of experimentation and formalization? These are the questions I have in mind when I think of Chp. II. Of course, each of the 3 fundamental stages (which constitute the 3 sections of the chapter) are themselves quite amenable to guided discovery instruction - so we also have "embedded guided discovery". In my teaching I take advantage of every spare minute - sometimes taking an entire lecture session - to talk, in a general way, about the kinds of problems, proofs, patterns etc. exemplified in Chp. II. Briefly then, Section 1/Chp. II deals with suggested "warm-up" or pre-generalization examples [Experimentation]. Section 2 follows with a classification and overview of standard proofs in elementary Mathematics [Proving or Formalization]. Section 3 concludes with a list of problem types which I feel may be effective in consolidating the ideas introduced in Stages 1 and 2 [Consolidation]. The dialogues scattered throughout the chapter reflect my feeling as to what good discovery-dialogue sessions can be like.

Chapter III is a bread-and-butter chapter. In it I turn to specific instances or topics which I have presented as regular course material using the guided discovery technique. From three Dawson courses (Linear Algebra, College Mathematics and Calculus I) I choose representative topics for analysis; representative in the sense that I have taught these over the last 2 years via "discovery". Each section is followed by a list of "special" or "consolidation" exercises. Many of these exercises are fairly challenging in that they test the students' ability to "think hard" or to reason beyond the level required by most textbook problems. Most of the problems in these sections have been sprinkled into my exercise sets and homework assignments. I FEEL that the number of successful attempts at these exercises is directly proportional to my success at getting across Stage 1 and Stage 2 discovery notions (see Chp. II).

Finally, Chapter IV is reserved largely for a summary of student feed-back. The results of 2 questionnaires are discussed and sample responses given. An experiment involving the taping of a "typical" discovery session on Elementary Matrices is discussed. A brief critique on the possible benefits and drawbacks of guided discovery and a few words on my future plans relating to discovery teaching conclude the main body of the thesis. A list of references and the Bibliography follow Chapter IV.

CHAPTER II

A MULTI-STAGE APPROACH TO DISCOVERY TEACHING

Section 1 - Stage 1: Experimenting with Patterns and Relationships

In order to encourage student participation and to introduce the discovery process I often spend one or two lectures at the beginning of a course looking at number patterns, geometric relationships, symmetries, "black box problems" and the like. I have found this to be an enjoyable, informal way of "warming up" to Linear Algebra, Calculus or College Mathematics (Dawson mathematics courses 105, 103 and 101, respectively).

The Stage 1, or experimenting process, is followed by standard course material (where guided discovery is used once again to present certain topics) and by another "intermission" or Special Class on "generalizing and proving" during the first few weeks of the course. (see Stage 2; Section 2 of this Chapter.) In this way, guided discovery teaching proceeds at two levels: in a specific way during daily classroom work (see Chapter III) and in a more global and informal manner in which students are briefly exposed to mathematics not necessarily covered in the standard syllabus. The headings which follow reflect units of class presentation; that is, I generally handle "black box problems" or problems involving symmetries or problems involving counting patterns in any one given "special" lecture or part of a lecture.

Black Box Problems

The students are asked to imagine some mysterious machine that is fed certain data. By some process the machine converts these data into specific outputs. Their job is to discover a rule by which the input is transformed by this "black box" apparatus. (Of course, occasionally a student may discover several possible rules being used by one "machine". In such cases I like to praise the student for having made such a discovery.)

The following is an example of a typical "black box problem" I might pose in class.

Black Box A takes the triplets $I_1(3, 2, 4)$ and $I_2(1, 2, 3)$ and turns them into $O_1(5, 1, 81)$ and $O_2(3, -1, 36)$, respectively. What is a rule by which this black box operates?

After a while, if there are no answers forthcoming, I might offer a few hints:

<u>HINT 1</u>: What can you say about the last integer in O_1 and O_2?
(Right! They are perfect squares.)

<u>HINT 2</u>: In what way is the number 81 related to the numbers 3, 2, 4 in I_1?
(Correct! $(3 + 2 + 4)^2 = 81$)

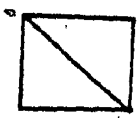
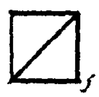
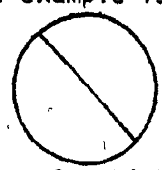
<u>HINT 3</u>: In what way are the numbers 3, 2 in I related to 5, 1 in O_1?
(Yes! $3 + 2 = 5$ and $3 - 2 = 1$)

Now compare I_2 with O_2. What is <u>a</u> possible rule?

<u>ANSWER</u>: $(a, b, c) \rightarrow (a + b, a - b, (a + b + c)^2)$

(In this situation the answer is not unique - the images of two elements in R^3 do not determine a unique rule for the above non-linear transformation. In general, it may be very difficult to determine all possible rules for a given "non-unique solution" type of black box problem, and just as difficult to find the most likely rule. A relative frequency approach - tabulating and comparing the frequencies of various kinds of solutions - may be of some help.)

Students enjoy working out these types of problems and require little encouragement to make up a few "black box puzzles" to challenge their classmates and, of course, their teacher. It is also interesting to observe that many of those figure analogy questions appearing on standard I.Q. tests are simply variations on the black box problem theme. (They, however, are constructed so as to yield a unique solution, or at least, a most obvious solution. An example is:

 is to  as  is to ____?____

The examinee is after an underlying rule which, in this case, converts the large figures into the small ones. Unfortunately the discovery must be made without any guidance from anyone. (Note that, in this case,  and  may both be acceptable answers.)

Symmetries

One example which I use in class to demonstrate certain kinds of symmetry is the following classification exercise taken from page 89 of George Polya's book: Mathematics and Plausible Reasoning [23]

Exercise: Consider the following arrangement of the 26 letters of our alphabet.

GROUP 1: A M T U V W Y

GROUP 2: B C D E K

GROUP 3: N S Z

GROUP 4: H I O X

GROUP 5: F G J L P Q R

What is the rationale for placing a given letter in a certain group?

The key clue I would give to my class here is: SYMMETRY. Once they are aware of this - that symmetry is the guide to sorting - the students quickly realize that all the letters in Group 1 have one thing in common; namely, they possess vertical symmetry. Now the discussion proceeds quickly. Someone shouts out that the letters in Group 2 are symmetric about a horizontal line. Someone else notes that the members of Group 3 are characterized by point symmetry (S, N, Z). (In fact N rotates into Z and vice versa.) Horizontal, point and vertical symmetry are <u>all</u> found in the letters listed under Group 4. Those in the last group are seen to possess <u>none</u> of the above symmetries. But this is not the end - I now introduce the following related exercise which also relies on the notion of symmetry. [see "Type G" problems described in Section 3].

<u>Exercise</u>: Give equations (examples) of curves satisfying the various conditions listed below:

(a) symmetric about the Y-axis.

(b) symmetric about the line $y = 1$.

(c) symmetric about the X-axis.

(d) symmetric about the line $x = -1$.

(e) symmetric about the origin.

(f) having horizontal, vertical and point symmetry.

<u>Analogies</u>

(a) <u>Quadratics and Related Equations</u>

Most high school graduates are quite adept at solving "ordinary" quadratic equations. But solving for x in $6 \sin^2 x - \sin x - 1 = 0$ or in $\log^2 x - 4 \log x + 3 = 0$ causes many difficulties. I have found that it pays to spend a little time discussing the similarities as well as the differences between procedures for solving quadratics in the variable x, y, or z, say, and quadratics in sin x, log y, e^z and so on. We compare the methods of solution as well as the NUMBER OF ROOTS of $ax^2 + bx + c = 0$ versus the analogous pseudo-quadratics.

(b) Linear Systems and Pseudo-linear Systems

In an introductory class in Linear Algebra I might ask the students to solve:

$$\left.\begin{aligned} x - y &= 1 \\ 2x + y &= 2 \end{aligned}\right\}$$

by any method they wish to use. Immediately afterwards I'll put some set of equations such as

$$\left.\begin{aligned} x^2 - y^2 &= 3 \\ 2x^2 + 3y^2 &= 11 \end{aligned}\right\}$$

on the board and ask for solutions. Before everyone starts scribbling, I'll ask for a prediction about the number of solutions. The responses range from "none" to "two" to "infinitely many". The students discover that even though they can solve this system in a manner similar to solving bona fide linear systems, there are 4 sets of ordered pairs that satisfy both equations. We then look at the geometry of the situation. I allow a few minutes for a brief review of ellipses and hyperbolas. Can the students visualize the 4 points of intersection of the above conic sections? One or two students can then come to the board to sketch the given curves. The class is asked to help them. (NOTE: The question about the number of solutions of a system gains further importance in subsequent analyses of m equations in n unknowns.

Counting Patterns

(a) The Number of Diagonals of an n-gon

Problem: Suppose we have a polygon of n sides. How many diagonals can we draw in such a polygon?

Ideally, a discussion of this problem might proceed as follows:

Instructor: Consider a triangle. How many diagonals can you draw?

Student A: None.

Instructor: Consider a quadrilateral. Observe that it has 2 diagonals. How would you define "diagonal"?

Student A: A diagonal is a line that joins two opposite vertices.

Instructor: Any comments?

Student B: In the case of a pentagon how do I choose opposite vertices?

Instructor: Good point. Anything else?

Student C: Here's my definition. A diagonal is a line segment joining a pair of non-adjacent vertices of a polygon.

Instructor: Very well. Let's go on. Now then - how many diagonals does a pentagon have?

Student A: I count 5 of them.

Instructor: Correct. Now consider a 6-sided figure. Try to find a systematic way of counting your diagonals.

Student D: I got it. Take any vertex of the hexagon. It can be joined to 3 other (non-neighbouring) vertices. Now go around the hexagon, visiting each vertex once, and keep drawing diagonals. Near the end of this "trip" no new diagonals will be created.

Altogether I count 9 diagonals.

Instructor: Quite right! Now, can anyone predict how many diagonals a 7-sided polygon has?

Student C: Certainly. Each vertex can be joined to 4 others - it can't be joined to itself or to its immediate neighbours. As we journey around our heptagon we could get $7 \cdot 4 = 28$ diagonals, but these would not all be distinct. Every diagonal is counted twice. Therefore the actual number of diagonals is just $^{28}/_{2}$ or 14.

Instructor: Excellent! Can anyone give me a general formula for the number of diagonals of an n-gon?

Student B: Sure. The number of diagonals is clearly

$$\frac{n(n-3)}{2}$$

and n, of course, is bigger than or equal to 3. We have already verified this formula for n = 3, 4, 5, 6 and 7.

Instructor: Does anyone care to suggest a proof of our formula?

The preceding problem and dialogue, in my mind, exemplifies the guided discovery method. In some classes the interchange of ideas may proceed much more slowly; in others, despite all good intentions, the discovery method may simply fail. Generally, however, I have been able to generate lively discussions with the previous n-gon problem.

(b) The Partitioning of Space

I have recently begun to introduce one or two classical partitioning problems, via a special "interest" lecture, into my College Math course. They are:

Problem 1: Find the number of regions into which n lines will divide 2-space. Assume the most general configurations; that is, no 3 (or more) lines should be concurrent, lines should not be parallel or co-incident.

Problem 2: Find the number of regions into which n planes will partition 3-space. (Assume general configurations analogous to those in Problem 1.)

Problem 1 is ideal for class discussion in that you can get your hands dirty (literally and figuratively) by drawing pictures on the board. (Drawings may be attempted for Problem 2 as well but, of course, it is not very easy to represent 3-dimensional situations on a plane surface.) It may not be a bad idea to make an even humbler start. Begin with:

Problem 0: Find the number of segments into which n points will partition a straight line.

and tabulate your results thusly:

Number of Points	Number of Line Segments
0	1
1	2
2	3
3	4
.	.
.	.
.	.
n	n + 1

[see [24]]

Someone in the class usually spots that each time you bring in a new point you create a new segment. Good practice for the more complex Problem 1 and 2.

Similarly, students can be guided in tabulating values for Problem 1:

Number of Lines	Number of Regions
0	1
1	2
2	4
3	7
4	11
5	16
.	.
.	.
.	.
etc.	etc.

[see. [24]]

Throughout this exercise the instructor can steer the students via appropriate leading questions and challenge them in various ways. Here are some possible: "leading questions" and "challenges":

(1) "When I add a new line to the 4 I have drawn so far, how many <u>new</u> regions do you think I will obtain?"

(2) "Count the number of segments into which the $5^{\underline{th}}$ line is broken by the previous 4 lines. What is the relationship between this number and the number of new regions we create? How do you account for this relationship?"

(3) "Describe the number pattern between successive terms of the sequence 1, 2, 4, 7, ..."

(4) [CHALLENGE] "Can you <u>guess</u> what sequence we'll obtain in the right-hand column, for Problem 2?"

(5) [CHALLENGE] "What is the general, or n^{th} term of the sequence 1, 2, 4, 7...?"

(6) [CHALLENGE] "Can you spot any relationship between the combinatorial coefficients of Pascal's Triangle and the right-hand column sequence of Problem 1? Can you predict any relationship between the combinatorial coefficients of Pascal's Triangle and the right-hand column of Problem 2?"

A much more elaborate treatment of Problems 1 and 2 and related matters can be found in Polya's Mathematics and Plausible Reasoning and in the November 1978 issue of The Mathematics Teacher. [24] and [25].

Concluding Remarks

Black box problems, problems involving symmetries, pseudo-quadratics and pseudo-linear systems, diagonal counting and space partitioning problems are just a few mathematical vehicles for examining patterns and relationships. In the years to come I will be looking for other types of problems to supplement the standard course material-problems which have a potential for student discovery and, hopefully, problems which are fun to do, as well.

Of course, effective experimentation (where students are motivated to find patterns etc.) pre-supposes ample class time, availability of problems of appropriate levels of difficulty and good teacher-student rapport. In addition, a degree of showmanship on the part of the instructor, a knack for building up suspense, a talent for boosting student confidence and co-ordinating discussion are all definite assets. Thus, besides maintaining a continuous search for further problems and problem types, I will not be able to ignore the "management" and "artistic" aspects of any future discovery teaching.

Let us now examine Stage 2 of the "global" discovery process - proving various hypotheses.

Section 2 - Stage 2: Proving Generalizations and Other Hypotheses

It is perhaps debatable whether forming generalizations from specific instances should be lumped together with "experimentation" (Stage 1) or with "proving" (Stage 2). Since I try to get students to generalize from certain patterns during my "special" lectures on experimentation (see Section 1) as well as immediately after I introduce certain standard course topics, I consider "forming generalizations" usually to be the last part of the Stage 1 work described in the previous section. Thus, studying patterns and relationships and making conjectures based on these observations are often closely related (chronologically and logically) in my classroom presentations. In the next few pages, then, I will focus on proving conjectures, the second stage of the "global" discovery process. (It is in this sense that I sometimes refer to the section as the section on Generalizations or Formalization of concepts.)

I am motivated and somewhat encouraged by Dr. Polya's words:

> "What can the Mathematics teacher do? He can, first of all, acquaint his students with mathematical proofs. ...a good textbook and a good teacher using good examples should make him [the student] understand the role and the interest of strict proofs. This could be for a few students a great experience, but for all there is a good chance to enrich essentially their general culture and take a stride forward to mental maturity." [26]

Unfortunately, I have found that most students feel very uncomfortable with proofs; even the ones who have no difficulty in making conjectures. Many simply lack experience - they rarely, if ever had to carry out a formal proof in pre-college courses. Almost all have great difficulty in starting a proof. Some believe that checking a half-dozen or so numerical examples constitutes a proof. Some don't believe in proofs at all; more precisely, in the need to carry out the formal justification of a hypothesis [see, for example, responses to the First Questionnaire, in Chp. IV].

I spend several hours, in each course, speaking about proofs _in general_ (Stage 2 discovery activity). A small portion of this time is spent on what observers would call "pep talk" - bringing in instances (legal, medical and mathematical) where a proof might be considered a necessity, and trying to instil a little self-confidence. But "pep talks" are not very effective without a lot more solid support. In this instance, the "support" comes from a discussion of some elementary logic (necessity and sufficiency, proof versus disproof, negations, contradictions, and the like), general proof formats and proof strategies.

For a start, I give the students some general guidelines to follow. These are:

(a) put down all given conditions or assumptions.

(b) write down what is required.

(c) look up relevant definitions, axioms and prior theorems or lemmas.

(d) ask yourself such questions as:

- Would it be easier to begin working on the left side or on the right side of the "required-to-prove"? (This is an important consideration in simple _Cross-over Proofs_ (L.H.S. $\longleftrightarrow$ R.H.S.), the kind commonly employed in proving elementary trig. identities, for example.)
- If I start with a certain step, how will I proceed?

- What would the last step of the proof be? What about the second-to-last step?
- What TYPE of proof is called for?

The last question then leads to an examination of the various types of proofs commonly used in college math courses. This takes up most of the time I devote to my "special" lectures on "proving".

Classification of Proofs

One broad classification of proofs is suggested by Lakatos. In his dichotomy, there are proofs which "improve" through the discovery of new and unexpected aspects of a set of conjectures, and there are proofs that "do not improve" upon our present state of knowledge. [27] The first kind is generally associated with growing theories appearing usually at the graduate or research level. The second kind is associated with "mature" theories which are standard fare in high school, college and undergraduate university courses. Of course, one can sometimes return to an established theorem and sharpen or extend previously known results.

A more applicable classification, for my own teaching needs, consists of pigeon-holing theorems according to the MODE of proof, or the STRATEGY involved in the proof. Thus I tell students about Proofs by Step-Reversal, Recursive Proofs, Proofs by Contradiction, Proofs by Pattern Analysis, Cross-over Proofs, and so forth. Naturally, certain theorems may require a combination of strategies. Moreover, the list does not, of course, exhaust all possibilities. As I gain more experience in teaching and encounter the need for other problem-solving or proof strategies, more names may be added to the list. [see also Chp. III]

Some Proof Strategies (Each example is used to illustrate the strategy given under the various headings.)

Example A [A Recursive Proof]

Given that $a^2 = a + 1$ show that $a^4 = 3a + 2$.

A discussion in the guided discovery vein might (ideally) proceed as follows:

Instructor: How would you start?

Sam: Solve for 'a' from $a^2 = a + 1$ and then use this value of 'a' to check that a^4 is the same as $3a + 2$.

Instructor: Go ahead, Sam.

Sam (a short while later): I give up! What's the 4^{th} power of

$$\frac{1 + \sqrt{5}}{2}?$$

Instructor: Rather than doing some messy arithmetic, why don't we try another approach? Any suggestions?

Sandra: Well, $a^4 = (a^2)^2 = (a + 1)^2 = a^2 + 2a + 1$, but then what?

Joe: Solve for 'a'!

Sam: No way!

Linda: We know that $a^2 = a + 1$, so $a^2 + 2a + 1$ would equal $a + 1 + 2a + 1 = 3a + 2$.

Instructor: Excellent. Now express a^5 in terms of 'a'.

Sam: Aha! $a^5 = a^4 a = (3a + 2)\cdot a = 3a^2 + 2a$

$$= 3(a + 1) + 2a = 5a + 3.$$

Instructor: Now, we're rolling! Would anyone care to try a^6? ...

Example B [Decomposing a Proof into its Fundamental Parts]

Given a point $P_0(x_0, y_0)$ and a line $ax + by + c = 0$ prove that the distance from P_0 to the line is given by:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Analysis by Chief Components:

First Component: Find the equation of the line ℓ going through P_0 and perpendicular to the given line.

- Sub-goal 1: Find the slope of $ax + by + c = 0$
- Sub-goal 2: Find the slope of ℓ, using the "slope relationship" between perpendicular lines.
- Sub-goal 3: Find the point-slope equation for ℓ.
- Sub-goal 4: Write the equation for ℓ in general form.

Second Component: Find Q, the point of intersection of the ~~two~~ lines.

Third Component: Compute the distance from P_0 to Q, to obtain the required formula.

Comments: Via this example, and others like it, the student should learn to look for the chief components of long proofs. For they, somewhat like skeleton outlines in the writing of an essay, can act as fine threads that hold together the fabric of proof.

Of course, the given hypothesis can be proved in a much more elegant way using vectors and the concept of scalar projection. I have found that a comparison of the 2 methods is useful in highlighting the power of vector methods. [Such vector methods are discussed in our Linear Algebra course at Dawson College.]

But my main reason for introducing the longer, perhaps more awkward proof of the distance formula is to demonstrate to the class yet another general strategy they may find helpful. This is to survey the overall objective, to look for key "landmarks" before wading into the specific, detailed components of a proof.

Example C [Proof by Pattern Analysis and Step Reversal]

Show that $(a^2+1)(b^2+1)(c^2+1)(d^2+1) \geq 16\,abcd$, for all numbers a, b, c, d.

Simulated guided-discovery discussion

Instructor: I think of the number 4 when I look at this inequality. Why is that?

Ben: There are 4 similar terms on the left side and 4 simple factors a, b, c, d on the right side.

Instructor: Good. But what about the 16?

Marvin: Since $2^4 = 16$ we can associate four 2's with the number 16.

Instructor: Fine. Now can anyone start the proof?

DEAD SILENCE! (More hints needed.)

Instructor: Can anyone think of a really BAD way to begin - a way which would result in complicated algebraic manipulations?

Sam: Multiply everything out on the left-hand side!

Instructor: Right - er, wrong! Now that we know what not to do, what should we do? Try to solve a SIMPLER but SIMILAR problem by selecting one of the groups of four. (Inspiration from Dr. Polya.)

Susan: Well, let's take the "$a^2 + 1$", the "a" and one of the '2's. Suppose we try to prove that $a^2 + 1 \geq 2a$.

Instructor: A very promising suggestion, Susan. Proof, anyone?

Marvin: We did something like this, last year. Well, $a^2 + 1 \geq 2a$ implies $a^2 - 2a + 1 \geq 0$ or $(a - 1)^2 \geq 0$ which happens to be true for any $a \in R$. So, by REVERSING the steps we can say that since $(a - 1)^2 \geq 0$ clearly holds for all 'a', it follows that $a^2 + 1 \geq 2a$ for all 'a', as well.

Instructor: Correct. Notice that ...

Sam: Sure, $b^2 + 1 \geq 2b$, $c^2 + 1 \geq 2c$, $d^2 + 1 \geq 2d$ for the same reason!

Susan: Then it's obvious that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq 2a \cdot 2b \cdot 2c \cdot 2d \text{ or}$$

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq 16\ abcd.$$

Instructor: Almost obvious, but please note the following fine point. Since $a^2 + 1 = |a|^2 + 1 \geq 2|a|$, $b^2 + 1 \geq 2|b|$, $c^2 + 1 \geq 2|c|$, $d^2 + 1 \geq 2|d|$ and $|abcd| \geq abcd$ we can multiply corresponding terms to get:

$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq 2^4|abcd| \geq 16\ abcd$ and not have to worry about a possible reverse in the sign of the inequality. Before we leave this example can anyone make up another inequality which could be proved in the same way?

Susan: Yes. How about $(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq 32\ abcde$?

Marvin: I got a better one! Prove that $(a^2 + 4)(b^2 + 4)(c^2 + 4) \geq 64\ abc$.

Instructor: All right, Marvin. We'll assign your inequality as an exercise for tomorrow.

Example D [More Pattern Analysis]

Prove that if $1 \leq a < b < c < d < e$ for integers a, b, c, d, e then

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16}$$ where $[m, n]$ denotes the Least Common Multiple of m and n.

ANALYSIS

This problem is taken from the May 1979 Canadian Mathematics Olympiad contest paper. Like the previous inequality

(a) it may be disected into 4 key groups

and (b) it has pleasing patterns and is easily generalizable.

Of course, it calls for a new concept - the L.C.M. of 2 numbers. It is usually necessary to define "L.C.M." at the start of this problem. After I discuss the solution with my class I tell them they have just participated in solving a contest problem. Announcing this at the beginning, I feel, would tend to intimidate a fair number of students. During the process of solving the problem I usually offer guidance in the form of:

HINT NO. 1: Write $^{15}/_{16}$ as $1 - {}^{1}/_{2^4}$. Now think of Example C.

HINT NO. 2: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$ Notice the 4 terms in the sum.

HINT NO. 3: Associate $\frac{1}{2}$ with $\frac{1}{[a,b]}$, $\frac{1}{4}$ with $\frac{1}{[b,c]}$ and so on.

SUB-GOAL: Prove that $[a, b] \geq 2$. [This is easy.] Hence $\frac{1}{[a,b]} \leq \frac{1}{2}$.

At this stage, the class has little difficulty with the corresponding arguments that:

$[b,c] \geqslant 4$, $[c,d] \geqslant 8$, $[d,e] \geqslant 16$.

Thus $\frac{1}{[a,b]} \leqslant \frac{1}{2}$, $\frac{1}{[b,c]} \leqslant \frac{1}{4}$, $\frac{1}{[c,d]} \leqslant \frac{1}{8}$, $\frac{1}{[d,e]} \leqslant \frac{1}{16}$ so that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

Extending the proven result is simple:

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} + \frac{1}{[e,f]} \leqslant \frac{31}{32} \quad \text{for}$$

$1 \leqslant a < b < c < d < e < f$.

Naturally, as the number of terms on the left increases we get closer and closer to the upper bound of 1.

Example E [A Proof by Contradiction]

Prove that there are no matrices A, B such that AB - BA = I, where I = identity matrix.

DISCUSSION:

The proof is a difficult one calling for a rather inspired start. It underscores the importance of guidance in guided discovery teaching. I have not as yet had any student who could begin the proof until I had given the key word: TRACE. The proof then proceeds like this:

Assume there is a set of matrices A, B such that AB - BA = I. (For conformability purposes we take A, B, I to be of dimension n by n.) Then taking the TRACE of each side we get:

$$\text{tr}(AB - BA) = \text{tr } I$$

But tr I = n while tr(AB - BA) = tr AB - tr BA = 0 - an obvious contradiction. Hence there are no matrices A, B such that AB - BA = I.

It is important to point out to the class that proofs requiring such "clever" starts do not occur too often, in standard exercise sets. But they should know about the basic strategy; namely, <u>Proofs by Contradiction</u>. In CONTRAST to the above example, most classroom proof strategies will seem, hopefully, quite mundane.

It is interesting to note that after we had spent some time discussing Proofs by Contradicition, Robert Broca - a student in my Linear Algebra course - easily gave the following proof of "$AB = 0$ means at least one of the matrices A, B must be singular":

<u>His Proof</u>: If A^{-1}, B^{-1} were to exist then

$$A^{-1}ABB^{-1} = A^{-1}0B^{-1} \text{ which means } I = 0!$$

Therefore, A, B can't <u>both</u> be non-singular. It took Robert no more than 2 minutes to come up with this proof.

Other examples that I have used in class to illustrate Proofs by Contradiction include Euclid's proof of the infinity of prime numbers and the proof of the irrationality of $\sqrt{2}$.

<u>In Summary</u>

There are two central ideas being promoted in this section:

<u>One</u>: That by classifying or naming different proof types during the "special" lectures (which are actually intended to be general - as opposed to course/topic-specific - in nature) and analysing the corresponding strategies some progress can be made in helping students in what, for them, may be one of the most difficult areas in Mathematics.

Two: That by presenting the various proofs through the method of guided discovery the students can become actively involved in "proving" - a process which pervades all of Mathematics as well as all subjects which are founded on rigorous logical bases.

It is too early to declare total success in teaching proof-strategies. After all, the problems I have outlined have only been classroom tested during the past 3 or 4 semesters. (Nor have I had any recent feed-back from my students about how they fared with proofs in their subsequent university courses.)

But I can say that there are at least some encouraging signs. On the whole, my students seem to cope a little better with proofs when they take a second (or third) course from me. (The mere fact that many of them register for second and third courses with me is, perhaps, a good sign.) Moreover, I am beginning to notice increasingly better response to my "special consolidation exercises" both in the classroom and in homework assignments. This brings us to the last Stage of the "global" guided discovery theme: solving and proposing exercises which may help to consolidate learning.

Section 3 - Stage 3: Consolidation Through Problem Solving

Since the Fall of 1979 I have developed an interest in certain kinds of mathematical exercises. These problems all have one or more of the following characteristics:

(a) they call for a solution strategy not commonly found in text-book exercises.

(b) they bring together elements from two or more areas in Mathematics.

(c) they require "translation from English to Mathematics".

(d) they provide practice in areas which the majority of college students find difficult.

(e) they have a potential for demonstrating student creativity.

At first I simply combed through various textbooks, old lecture notes and exams and started putting together a file of Miscellaneous Exercises. Then I decided to attach names to the various problem types. (Interestingly, at the same time, I thought it would be nice to make up some of these exercises rather than to rely always on somebody else's efforts. Furthermore, perhaps I could encourage some of my students to make up a few problems, I thought. After all, this was what characteristic (e) was all about.)

The names I finally chose were as follows:

Type A: Reconstruction Exercises [characteristics (a) and (d)].

Type B: Reverse Procedure Exercises [characteristics (a) and (d)].

Type C: Student-generated Problems [characteristic (e) and possibly others].

Type D: Linkage Exercises [any of characteristics (a) to (e)].

Type E: Story Problems [characteristics (b), (c) or (d)].

Type F: Proofs and Disproofs [chiefly characteristic (d)].

Type G: Example Construction [characteristics (a), (d) and/or (e)].

Type H: Common Feature Exercises [chiefly characteristics (a) and (e)].

Now a "Story Problem" may be generated by a student and call for a "proof" or "reverse procedure". I would file copies of this problem under each of the 3 or 4 chief headings [Type E, F, C and B]. I am rather pleased that the above eight categories are not mutually exclusive. For, I believe that coming up with a "multiple-feature" problem is a challenge to one's imagination or creativity. Moreover, such a problem might prove to be exceptionally good for testing understanding of several mathematical concepts or strategies.

Just as with the earlier problem-solving processes - going from specific patterns to general hypotheses and then discussing ways and means of proving various conjectures - these eight problem types provide a source by which one can study solution strategies. I have found that Story Problems, Reverse Procedure Exercises, etc. are not only amenable to a guided discovery approach but that they are quite enjoyable to solve together with groups of students, as well.

Many of these special exercises, especially those of Type A, B, F and G, can be designed to test an understanding of proofs, patterns and relationships, as will be evident from examples coming up in this section as well as in Chapter III. Hence I have chosen the umbrella term "Consolidation Exercises" to describe the set of all eight types of problems.

Now some Consolidation Exercises can be incorporated into problem sets which follow up guided discovery teaching sessions covering specific course topics. Selections of such problems are included in the "problems of special interest" sections of Chapter III. Other Consolidation Exercises can be used to round out general Stage 1 and Stage 2 class discussions, or to test student comprehension of standard course topics which are not necessarily presented via guided disovery. The next few pages focus on Consolidation Exercises of tnis more general variety.

Reconstruction Exercises

Many people love jigsaw puzzles, crossword puzzles or putting clues together to solve a crime in a mystery novel. Why not carry this enjoyment of "reconstruction" over to mathematical problem solving? I like to involve my classes in piecing together a graph, a function, a matrix or an equation from given fragments of data. For instance, consider the following:

Example 1: Find a quadratic equation with integer coefficients which will have $x = \frac{1}{2}$ and $x = \frac{1}{3}$ as roots.

Comment: This little problem always results in a kind of initial reaction best typified by the word "Huh?". Even though many textbooks may carry similar exercises, it seems to jolt students into non-mechanical problem solving - they can't simply "grind out" answers by plugging into the Quadratic Formula.

Example 2: Find a non-trivial set of 3 equations in 3 unknowns whose ONLY solution is $x = 1$, $y = 2$ and $z = 3$.

<u>Comments</u>: (a) One must be careful not to select a system having an infinite number of solutions, among which (1, 2, 3) is just one particular solution. This caution always surprises some of my students, particularly those who have harbored a belief that 3 equations in 3 unknowns always have a unique solution.

(b) This problem puts the student in the role of the teacher who might be looking for a question on linear systems to put on a test. A number of Type A exercises possess a teacher-student role reversal property.

<u>Example 3</u>: Suppose f is a function with the properties:

(1) $f(x)$ is defined for every natural number x.

(2) $f(x) \in N$, for all $x \in N$.

(3) $f(2) = 2$.

(4) $f(x) > f(y)$ for all $x, y \in N$ such that $x > y$.

(5) $f(xy) = f(x) \cdot f(y)$ for all $x, y \in N$. Find f.

<u>Hint to Students</u>: You should be able to IDENTIFY this function f.

<u>Comment</u>: Example 3 has been perceived by my students as more difficult than either Example 1 or 2. It is very rare for a junior college student, I believe, to have had any previous experience solving such problems. More examples will be provided in the next Chapter of Reconstruction Exercises in specific course settings.

<u>Reverse Procedure Exercises</u>

I am not too surprised that Reverse Procedure Exercises pose a considerable challenge to many students. For it is possible that a great many learn theorems, rules or other mathematical concepts with certain fixed <u>purposes</u> in mind. In fact, I have been <u>told</u> by several students that this was so in their case. (For example, the Binomial Theorem "is good for expanding things". Or, certain trigonometric identities serve to transform sums to products or reduce expressions to simpler forms.) "Reverse" exercises force students to steer away from the usual or conventional application of a rule or formula and to ask themselves: "What <u>else</u> might this rule or proposition say that may have direct bearing on my problem?"

<u>Example 1</u>: Simplify $(x - 1)^4 + 4(x - 1)^3 + 6(x - 1)^2 + 4(x - 1) + 1$.

<u>Comment</u>: Almost every student to whom I've taught the Binomial Theorem responded by expanding $(x - 1)^4$, $(x - 1)^3$, $(x - 1)^2$ and collecting terms. I can recall at most TWO students who instantly recognized that the above expression was merely the expansion of $\left[(x - 1) + 1\right]^4$, and hence equal to x^4. The instruction "Expand $(x + y)^n$" elicits the correct response quite quickly possibly because the Binomial Theorem is learned "left-to-right". But when I ask my class to find a closed form expression for (i.e. simplify) a certain collection of terms <u>resulting from</u> some expansion, the responses are for from instantaneous.

Now consider the following example:

<u>Example 2</u>: Compute $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 6 \\ 0 & 2 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 6 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 6 \\ -1 & -3 & 1 \end{vmatrix}$

<u>Comment</u>: In Math 105 (Linear Algebra) we study the properties of Determinants, among which we include the Disection Property:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g+r & h+s & i+t \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ r & s & t \end{vmatrix}$$

Yet, once again, only a very small number of students recognize that they should re-combine the 3 given determinants into a single determinant using the Disection Property (twice) <u>from right to left</u>.

<u>Example 3</u>: A fraction $\frac{a}{b}$ is said to be expressed in Egyptian Form if $\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}$ for some distinct natural numbers $n_1, n_2, \ldots, n_k$. Write $5/6$ in Egyptian Form.

<u>Comment</u>: One way to do this is:

$$\frac{5}{6} = \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{1}{2} + \frac{1}{3}.$$

After a little more prodding, someone in the class will come up with an alternative; namely,

$$\frac{5}{6} = \underbrace{\frac{1}{12} + \frac{1}{12} + \ldots + \frac{1}{12}}_{\text{to 10 terms}} = \frac{6}{12} + \frac{3}{12} + \frac{1}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{12},$$

and so on. Egyptian Form questions are a nice way to supplement the usual "simplify the sum" exercises on fractions. Some students might even conjecture that if a fraction can be written in Egyptian Form one way then it can be written in this form in an unlimited number of ways. This is actually a well-known theorem. For a proof, as well as a more complete treatment of Egyptian fractions see Beck's <u>Excursions into Mathematics</u>. [28]

Example 3 can also be viewed as a forerunner to partial fraction decomposition needed to evaluate certain integrals in Calculus. The next example may baffle, temporarily, at least, some students who thought they knew all there was to know about the Quadratic Formula.

Example 4: Given $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, write down the value of

$ax^2 + bx + c$. You have 10 SECONDS.

Clearly there is no limit to the number of Type A or Type B exercises one can manufacture. (The Russian researcher V.A. Krutetskii, for instance, illustrates his discussion on "reversibility of thought" with several such examples. He feels that such examples can be important indicators of mathematical ability. [29])

But the ultimate - and somewhat whimsical - reverse exercise can be found in the December 1978 issue of Crux Mathematicorum:

Exercise: Find a question whose answer is $\frac{22}{7} - \pi$.

Comment: Not many can top the response sent in by one Kenneth S. Williams of Carleton University, Ottawa. His "question" was:
"What is the value of the integral $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$?"
Remarkable!

The little poem by Piet Hein, also found in this issue of Crux Mathematicorum is very appropriate:

"Solutions to problems
are easy to find:
the problem's a great
contribution.

What is truly an art
is to wring from your mind
a problem to fit
a solution." [30]

Student-generated Problems

Once in a while the student should have the opportunity to play the role of teacher [see also Example 2 of the Reconstruction Exercise section]. He should be asked to make up exercises on the various topics discussed in class. In some of my sections at Dawson College I have told the students to generate a few problems. Naturally I expect each person to be able to solve his or her <u>own</u> problem. (Copying out some super-hard miscellaneous exercise from a library book is of little educational value!) For a creative (and completely solved) problem the student earns a few bonus marks. His problem, with due credit of course, may even be used in subsequent problem sets presented to future generations of students. Specific examples of student-generated problems will be given in Chapter IV.

Linkage Exercises

Problems which link together concepts from several topics or even several branches of Mathematics are, I believe, exceptionally useful learning tools. They usually possess several of the desirable features listed on the first page of this section.

I make an effort to include a few Type D problems in my general interest or problem-solving sessions. Some illustrations follow.

<u>Example 1</u>: Find the value of $k \in R$ such that the lines $x - y = 0$, $x + y = 2$ and $3x - 2y = k$ will enclose a triangle of zero area.

<u>Example 2</u>: Let S be the set of all systems of equations of the form

$$\left.\begin{aligned} ax + by &= 10 \\ x + 2y &= 15 \end{aligned}\right\}$$

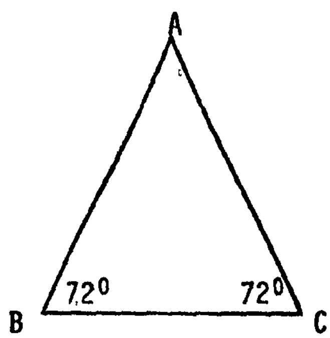
where a, b are integers with $1 \leq a \leq 20$ and $1 \leq b \leq 20$.

Choose one of these systems of equations at random from S.

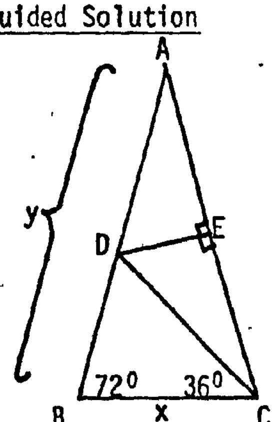
Find the PROBABILITY that this system has a solution. [31]

Example 3:

Given isosceles triangle ABC, with $\angle ABC = \angle ACB = 72^0$. With the help of a construction determine the EXACT VALUE of $\cos 36^0$.



Guided Solution



Hint 1: Let $BC = x$, $AB = y$. Bisect $\angle ACB$ by line segment CD and drop the perpendicular DE to side AC. Now show that Δ's ADE and DEC are CONGRUENT. Find EC in terms of x or y.

Desired Response: $\angle ECD = \angle DAE = 36^0$

side DC = side DA

side DE is common.

Clearly, $\Delta ADE \cong \Delta DEC$. Hence $AE = EC = y/2$.

Hint 2: Write $\cos \angle ECD = \cos 36^0$ in terms of x and y.

Desired Response: $\cos 36^0 = \frac{EC}{DC} = \frac{y/2}{x} = \frac{y}{2x}$, noting that $DC = BC = x$

since $\angle BDC = \angle DBC$

Hint 3: Use similar triangles ABC and DBC to obtain a relationship between y and x.

Desired Response: $\frac{AB}{BC} = \frac{DC}{DB}$ or $\frac{y}{x} = \frac{x}{y - x}$

$$\therefore \quad y^2 - xy - x^2 = 0$$

Hint 4: Express y as a function of x.

Desired Response: $y = \frac{x \pm \sqrt{x^2 + 4x^2}}{2} = \frac{x \pm \sqrt{5x^2}}{2} = \frac{x(1 \pm \sqrt{5})}{2}$

But y has to be POSITIVE. Thus $y = \frac{x(1 + \sqrt{5})}{2}$

Hint 5: Now find cos 36^0.

Desired Response: $\cos 36^0 = \frac{y}{2x} = \frac{x(1 + \sqrt{5})}{2} \cdot \frac{1}{2x} = \frac{1 + \sqrt{5}}{4}$.

Comment: "Whew!" is a common student reaction, at the end of this problem, although after offering the above hints, many are able to proceed to the next stage on their own. (I think everyone would agree that without some hints to serve as stepping-stones, there is little chance that a student picked at random could give a complete solution.) More than most problems one encounters, Example (3) underscores the importance of guidance in certain teaching situations. Solving at least one or two such lengthy problems (preferably in a spare hour set aside for general problem-solving) by means of a step-by-step analysis of its chief components is, I firmly believe, valuable training for encounters with "tough" or "substantial" problems that some students will have in later years.

Story Problems

One of the 2 major hurdles in undergraduate mathematics is the so-called "story" or "word" problems. (The other is, I believe, the notion of "proof".) But of the (small percentage of) students who will one day use Mathematics in their work, the majority will have to deal with mathematizing complex situations - in short, cope with "word" problems. Thus, Type E Problems are important. And, they are everywhere. Linear Algebra has its Linear Programming, Calculus has its Related Rates and Optimization Exercises, Finite Math has its Probability and Markov Chains, High School Algebra has its Mixture, Work, Distance-Rate-Time Problems. All these are available in standard school texts. However, contest problems, mathematical game-and-puzzle books, math journals and special problem collections (such as those of Martin Gardner, Charles Salkind, and William Moser) are excellent sources for those who seek that extra challenge.

I have observed that there is a certain type of story problem which students are generally eager to solve. (And providing problems which they are at least interested in solving is a major step towards helping those who have trouble with "word problems" - the desire to find a solution can be a strong counter-force to the traditional fear that they may evoke.) Such a problem either couches a standard mathematical process in "interesting surroundings" or presents a situation close to the heart of people involved in the process of education. Consider, for instance, the next 3 examples.

Example 1: Abe and Alan have the habit of re-copying examination questions before solving them. On a certain exam Question 1 involves the solution of a quadratic equation. Abe makes a mistake in copying down the CONSTANT TERM and gets the roots 4 and 1. Alan makes a mistake in copying the COEFFICIENT OF THE FIRST DEGREE TERM and gets the roots -3 and 2. What is the actual equation given on the exam?

Comment: One might justifiably classify this also as a Type A story problem.

Example 2: In a certain class there are more than 20 and fewer than 40 students. On a recent test the average failing mark was 48. [Sixty-five is the established pass mark.] The CLASS AVERAGE was 66. The teacher then raised every grade by 5 points. (Assume that the highest score had been 95, originally, and that the average passing mark had been 75.) As a result of the grade adjustments, the average passing mark was raised to 77.5 while the average failing mark became 45. How many students had ~~their~~ grades changed from failing to passing? [32]

Example 3: A classical archimedean weighing problem, such as the one found in Christy [33] can also arouse student interest, I have found (while teaching Algebraic Functions).

Incidentally I am convinced that most students will not respond enthusiastically to problems that begin with: "A works twice as fast as B" or "two trains leave a station at the same time" or even "a ball is dropped vertically from a tall building". They have seen too many of these, and repetition can easily dull any enthusiasm.

Proofs and Disproofs

We have looked at proofs and proving in Section 2 of this Chapter.

Every student in my class learns to EXPECT problems to prove on class tests and assignments. He also has to be ready to DISPROVE a statement by means of counter-examples. Let me present a few suggested exercises, appropriate for various courses.

Example 1: [Disproving statements of the type: $f(a+b) = f(a) + f(b)$.]

DISPROVE each of the following:

(a) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, for all positive real numbers 'a' and 'b'.

(b) $\sqrt[3]{a+b} = \sqrt[3]{a} + \sqrt[3]{b}$, for all real numbers 'a' and 'b'.

(c) $(a+b)^3 = a^3 + b^3$, for all real numbers 'a' and 'b'.

(d) $P(A \cup B) = P(A) + P(B)$, for all events A and B (where P is the probability function).

(e) $\det(A+B) = \det A + \det B$, for all square matrices A and B.

(f) $(A+B)^{-1} = A^{-1} + B^{-1}$, for all square matrices A and B.

(g) $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$, for all vectors $\vec{v}$, $\vec{w}$ in 2-space (where $\|\vec{v}\|$ represents the norm of $\vec{v}$).

Comments: (1) (a), (b), (c) are appropriate for high school algebra; (d) is extracted from College Math 101 (Dawson College); (e), (f), (g) are appropriate in a Vector-and-Linear-Algebra course.

(2) Prior to assigning Type F problems the distinction between the GENERAL NATURE of a "proof" and the "DISPROOF" BY COUNTER-EXAMPLES is discussed in class.

Example 2: [Experimenting with Number Theory]

Observe that for the PERFECT NUMBER 6,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$$

and that for the perfect number 28,

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{14}+\frac{1}{28}=2$$

Make a conjecture about the sum of the reciprocals of the positive divisors of any perfect number n. Prove your conjecture.

Example 3: PROVE or DISPROVE each of the following:

(a) $aX = bX \Rightarrow a = b$ for all $a, b \in R$ and matrix X.

(b) $x^y = x^z \Rightarrow y = z$ for all $x, y, z \in R^+$

(c) For any 2 distinct real numbers a, b the quantity

$$\ell = \frac{a + b + |a - b|}{2}$$

always equals the LARGER of the two.

(d) det C = 0 for any square matrix C whose elements are consecutive natural numbers, beginning with $C_{11} = 1$.

(e) det S = 0 for any k x k skew-symmetrix matrix.

(f) $x > y \Rightarrow x \log_{10}a > y \log_{10}a$ for all $x, y \in R^+$, $a > 0$.

Comment: Such problems require DECISION-MAKING, based on experimentation, trial-and-error, and knowledge of propositions or properties covered in class. Even false assertions provide beautiful opportunities for class participation. Janet's Hypothesis can be disproved by Nick's Counterexample, and so on. I GUIDE the class towards the correct solutions, letting the students take full credit for all proofs or disproofs. Names of students are appended to all hypotheses, proofs and counter-examples. I have found that such personal involvement tends to heighten student interest in class work.

<u>Example 4</u>: [Special Treat for the Advanced Student]

Prove that if $\frac{x^2 - 5x + 6}{x^2 - 11x + 30} < 0$ then $\sin 2x < 0$, where x is in radians.

<u>Comment</u>: This problem is taken from an examination given to prospective high school Mathematics teachers applying for entrance to the Moscow Levin State Pedagogical Institute. It can also be classified as a Type D problem, combining knowledge of simple trigonometry with the elementary laws of inequalities. For a solution to this rather novel exercise, see [34].

Clearly, some Type F exercises are especially suited for introducing unusual and/or difficult problems to the gifted math student. After they easily dispense with the "routine" proofs there is no reason why they should remain idle, waiting for the rest of the class to catch up. I like to refer them, for instance, to Ross Honsberger's <u>Ingenuity in Mathematics</u>. Here they will find an ingenious proof that there exist powers of 2 beginning with any given sequence of digits. They will learn why a rectangle of incommensurable dimensions cannot be tiled with squares. They will read about Sam Beatty's Theorem, The Theorem of Barbier and other non-standard mathematical tidbits [35]. Moreover, their ability to carry out proofs will be truly tested.

<u>Example Construction</u>:

Many class exercises begin with "compute" or "evaluate" or "prove that". For a welcome change of pace it is good to interject a few problems that start with "give an example of". It's also an alternative for texting definitions, necessary and sufficient conditions and other basic concepts. Some sample examples follow.

Example 1: Find a point which is at a distance of 1 unit from the line $y = 2x - 1$. [This is also Type B.]

Example 2: Give an example of a quadratic equation in sin x which has NO REAL SOLUTIONS.

Example 3: Give an example of a straight line that does NOT go through any point (x, y) where x, y are both integers.

Comment: This is a difficult Type G problem. One possible answer is $y = \sqrt{7}\,x + \sqrt{2}$.

Example 4: Give an angle θ for which $\cos\theta = \sin\theta$.

Example 5: Give an example of a function which is neither ODD nor EVEN.

Example 6: Give a 2x2 matrix A for which det A = 1 but none of whose entries equals 1.

Example 7: Write down the equation of a circle which lies entirely in Quadrant I and which is not tangent to either of the co-ordinate axes.

Example 8: Give a 2nd degree equation $(ax^2 + bxy + cy^2 + dx + ey + f = 0)$ that corresponds to:

(a) a circle (b) an ellipse (c) a single point in the plane (d) two intersecting straight lines.

Comment: Type G problems may take up little writing space but they often call for much more than a little thought. Moreover, they are perfect catalysts to class debate, and hence a natural medium for the discovery process.

Common Feature Exercises

Type H problems test the ability to spot patterns and relationships and are easily created by both teachers and STUDENTS. Moreover, they are FUN to solve. Here are a few illustrations.

WHAT IS THE COMMON FEATURE AMONG EACH OF THE FOLLOWING GROUPS?

(1) (a) 3, 4, 5 (b) 5, 12, 13 (c) 7, 24, 25 (d) 9, 12, 15

(2) (a) $\sin 162^0$ (b) $\cos 312^0$ (c) $2^{-\frac{1}{2}}$ (d) $\log_{10}77$.

(3) (a) $\sin 189^0$ (b) $\cos 162^0$ (c) -1782 (d) $\log_{10}(1/15)$.

(4) (a) $x^2 + y^2 + 1 = 0$ (b) $2 \sin x = 5$

(c) $\log_{10}(\log_{10}\sin x) = 1$ (d) $\dfrac{1}{2^x + 2^{-x}} = -5.$

(5) (a) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$

Some Concluding Remarks

In this Chapter I outlined a basic three-part structure upon which I build my global guided discovery teaching strategy. Examples were drawn from various areas of Mathematics to illustrate the kind of problem solving work that takes place at each of the three Stages. In the Chapter to follow I will describe how I have employed discovery teaching in three course topics at Dawson College. The topics under consideration will be:

(1) Curve Sketching, as taught in Mathematics 103 (Introduction to Calculus).

(2) Elementary Matrices and The Determinant Product Theorem, as taught in Mathematics 105 (Linear Algebra).

(3) Conditional Probability and Bayes' Theorem, as taught in Mathematics 101 (College Mathematics).

CHAPTER III

GUIDED DISCOVERY TEACHING IN COLLEGE MATHEMATICS

Section 1: Curve Sketching

Curve Sketching Without the Derivative

Prior to studying Calculus, every math student learns how to sketch polynomial, trigonometric, exponential and logarithmic curves, by the point-plot method and/or by analysing the given function's periodicity, symmetry, domain and so on. For this reason, perhaps, no one seems worried when I start sketching the graph of $y = \frac{x-1}{x}$ in Mathematics 103 (Introductory Calculus). At first, all I want is a very rough sketch. The students find that (1, 0) is an X-intercept and note that there will be no Y-intercept. We then examine the behaviour of the given function around $x = 0$ and values for x approaching $\pm\infty$. I draw something like FIGURE 1 on the board.

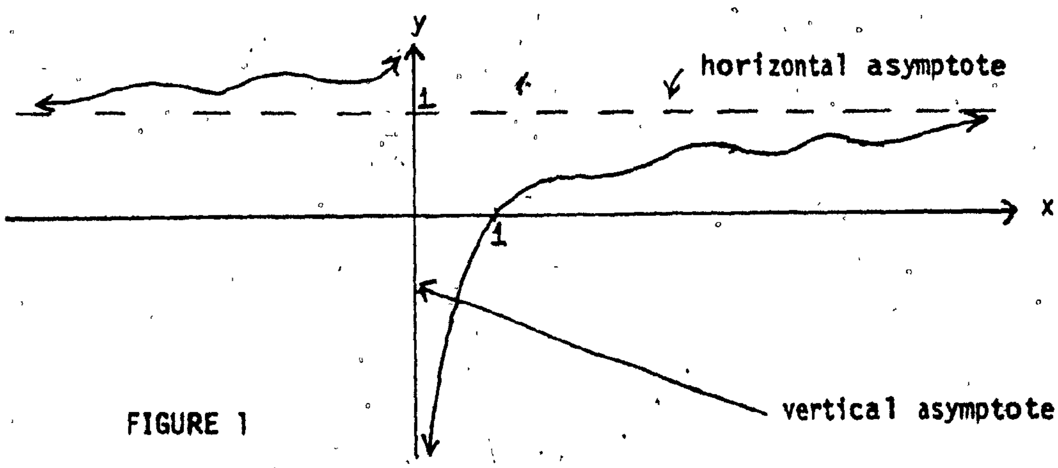


FIGURE 1

The class laughs at my crude FIRST APPROXIMATION. When the laughter dies down I ask them to sketch $y = x^2 - 5x + 6$ using only 2 plotted points. "That's easy! This is a parabola, which ...", someone might say. "Parabola? Never heard of it!", I reply, and using the points (0, 6) and (2, 0), sketch:

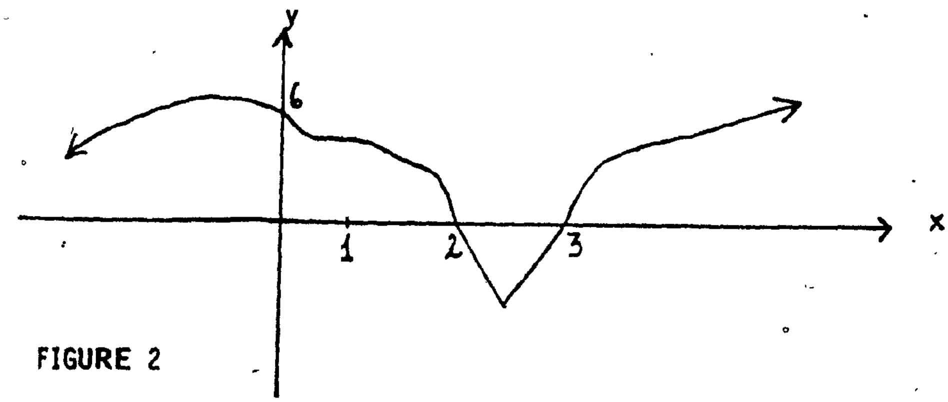


FIGURE 2

Someone else suggests that, to be fair, I should plot a few more points. I do so, and draw a new curve, distorting it as much as possible. After a short debate and the usual accusations of unfair tactics we make the observation:

- The point-plot method can sometimes leave us guessing about the way in which we should connect pairs of points, ESPECIALLY IF THE FUNCTION IS UNFAMILIAR or COMPLICATED. -

Next I ask for a few points on the graph of $y = 3x^4 - x^3 + 56$. There's a frantic scramble for pocket calculators. I cut off the search for points after about a minute and wait for comments. It's not long before most students are convinced that:

- For some curves it may be difficult to find more than one or two points. Sketching graphs on such limited basis can be a very shaky operation. -

Of course, in certain cases we may just want to know the shape of the graph in some fixed interval, or just the maximum and minimum points. In these cases especially, Calculus techniques will pay off.

First Leading Question: How can we know more about the graph of $y = \frac{x - 1}{x}$? OR How can we iron out some of the wrinkles in FIGURE 1?

Answer (to be elicited from the class): By finding out where the curve is INCREASING and where it is DECREASING.

Curve Sketching and The First Derivative

Exploratory Stage: Typical Dialogue

Instructor: Consider the curve

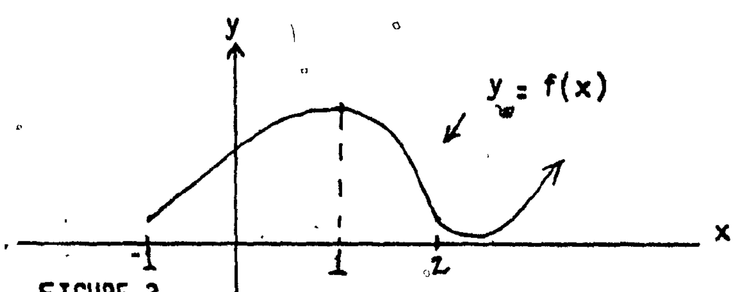


FIGURE 3

What is the SIGN of the slope at $(-\frac{1}{2}, f(-\frac{1}{2}))$? At $(0, f(0))$? At $(1, f(1))$? At $(2, f(2))$?

Student A: Positive, positive, ZERO, and negative.

Instructor: Good. So the sign of the slope changes from positive to negative about the turning point $(1, f(1))$, and the graph changes from INCREASING to DECREASING. What would be a good name for this "turning point"?

Student B: A maximum point.

Instructor: That's fine. What is the DERIVATIVE at $(1, f(1))$?

Student B: Zero, of course. The tangent is horizontal!

Instructor: O.K., but now can someone come up to the board and draw a picture of a curve having a maximum at (1, f(1)) but with no derivative there?

Student X: But you just did that!

Instructor: I said "NO derivative", not "zero derivative"!

Several students are coaxed into going up to the board. They draw graphs such as:

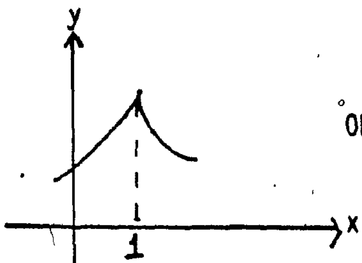


FIGURE 4

OR

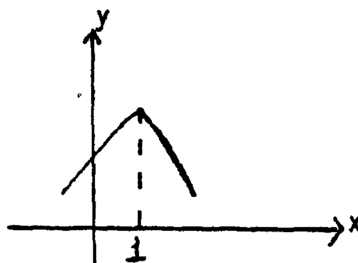


FIGURE 5

OR

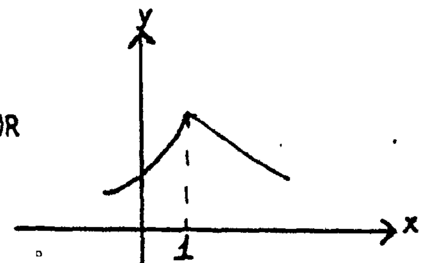


FIGURE 6

Instructor: Some graphs have maxima at places where the derivative is zero, some at places where the derivative is undefined, while some maxima occur at end-points. Let's call these maxima Type I, Type II and Type III, respectively. Now I'd like you to draw one large curve having each of these 3 types of maxima on it.

Student A: What are end-points?

Instructor explains.

Student A: Here's my answer then.

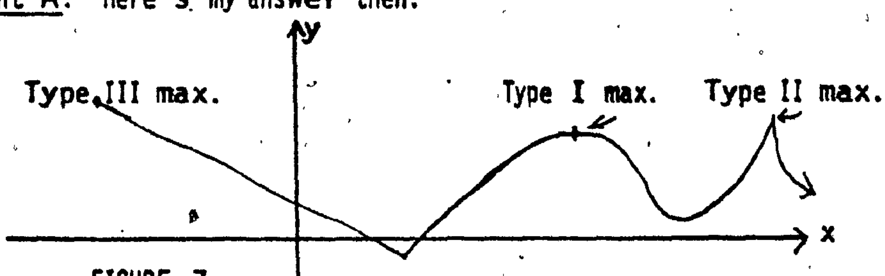


FIGURE 7

<u>Instructor</u>: Which of these maxima is the "highest maximum" on your graph?

<u>Student A</u>: The first one on the left, obviously.

<u>Instructor</u>: Right. Now let's put together some definitions.

The class and I go about defining "critical points" and "local versus global maxima". We also look at some <u>minima</u>, both relative and absolute. Along the way, I ask for, <u>and generally get</u>, examples of critical points which are neither maxima nor minima. We end up with criteria for classifying critical points as "maxes", "mins", "global maxes", "global mins", "local maxes", "local mins" or "none of the above". Now, one can open any standard Calculus text and find all the necessary definitions and criteria. But I prefer to have the <u>students</u> do the work, <u>BEFORE</u> they start flipping pages in a book. They are GUIDED in DISCOVERING things for themselves. As I listen to them <u>explain</u> concepts in curve sketching, I know that the guided discovery method is having some benefit - the students are <u>actively</u> participating in learning. They learn from me as well as from each other.

<u>Generalizations</u>

One important outcome of "the exploratory stage" is the formulation of <u>generalizations</u>. We get, for instance:

"If f is a function such that $f'(x) > 0$ for every $x \in (a, b)$ then f is INCREASING in (a, b). If f is a function such that $f'(x) < 0$ for every $x \in (c, d)$ then f is DECREASING in (c, d)."

The formal proof of this is not given in our general Calculus course at Dawson College. I give what Polya would call "a plausible argument";

namely, that a POSITIVE DERIVATIVE at a point implies a tangent line with an inclination between 0^0 and 90^0 (i.e. POSITIVE SLOPE) which, in turn, implies an INCREASING FUNCTION while, on the other hand a NEGATIVE DERIVATIVE is associated with a tangent line having inclination between 90^0 and 180^0 (i.e. NEGATIVE SLOPE) which gives rise to a DECREASING FUNCTION.

We also record (without proof) the Theorem:

"If a function $y = f(x)$ has a relative maximum (or minimum) at the point $P(x, y)$ and $f'(x)$ EXISTS then $f'(x) = 0$",

carefully noting that $f'(a) = 0$ or $f'(a)$ undefined merely implies that $(a, f(a))$ is a POSSIBLE relative max. or min.. Finally, so that we can routinely separate "maxes" from "mins", the First Derivative Test is "discovered" after some teacher-student dialogue and included in the students' notes.

<u>Consolidation</u> or <u>Problem Solving</u>

One of my favourite exercises, following the preceding class work, is the <u>Type G problem</u>:

"Give several examples of functions which are increasing for all values of x in the domain."

Early responses are: $y = 3x$ or $y = 7x + 1$ or some other linear function. But with a little more reflection and prodding I get answers like: $y = 2^x$ or $y = x^3$ or, on a particularly lucky occasion, even $y = \text{Arctan } x$. But, to use an analogy:

SPECIFIC FUNCTION is to GENERAL CLASS of FUNCTIONS as EXPLORATION is to GENERALIZATION. Thus, at the next level, I ask for whole <u>classes of functions</u>

satisfying the given condition. Usually there is no immediate response. I then say, "all linear functions $y = mx + b$ where ..." and someone completes the statement with "where m is greater than 0." Or, I might say, "all power functions $y = x^n$ where n equals" and guide the students towards the completion "1, or 3 or 5 or any other odd number." The need for guidance is very strong at all stages during this process of discovering properties of certain curves, I have found.

Exercise: We know that $f'(x) > 0$ for all $x \in (a, b)$ implies f is INCREASING on (a, b). Give an example to show that f INCREASING on (a, b) merely implies that $f'(x) \geq 0$ for all $x \in (a, b)$.

I usually get: "Look at $y = x^3$ and the point (0, 0)", since this was one of the functions which we had said was increasing for all x, in the previous exercise. Right at this point, I ask for another function, "related" to x^3 which is increasing for all x but for which $f'(0)$ is UNDEFINED. (Of course, I'm after the inverse of x^3, or $\sqrt[3]{x}$.) So far I haven't had much success with this question. Inverse functions do not strike a familiar chord with ~~most first~~-year college students. Possibly, also, I am not asking the right kinds of leading questions.

We return to our function, $y = \frac{x - 1}{x}$ and try to REFINE our sketch, using the new information we have accumulated. Our graph now looks like this:

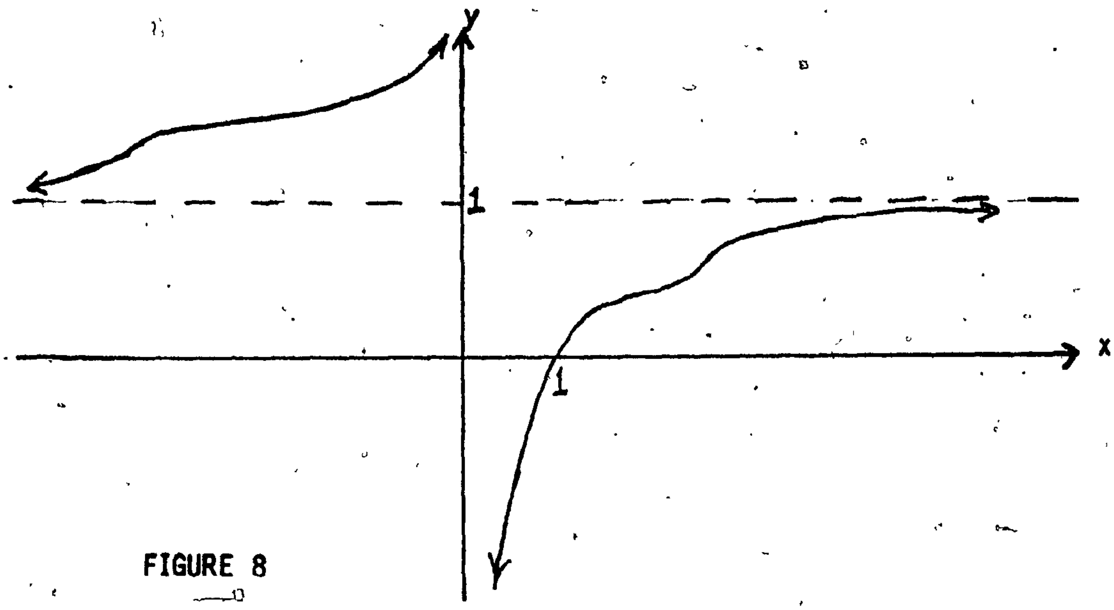


FIGURE 8

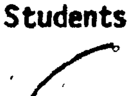
In analysing $f(x) = \frac{x-1}{x}$ we discover that f is always increasing - except, of course, at $x = 0$ - and consequently has no maxima or minima. "But there is still something wrong with the picture!" the class protests. This brings us to the next step in the discovery process.

<u>Second Leading Question</u>: How can we learn <u>more</u> about this graph? <u>OR</u> How can we smooth out the wrinkles remaining in FIGURE 8?

<u>Answer</u> (to be elicited from the class); By finding out WHICH WAY the curve is increasing.

<u>Curve Sketching and The Second Derivative</u>

<u>Exploratory Stage</u>

Students quickly realize that a curve may increase yet steadily level off:  or, it may increase in a more dramatic fashion, thus:  . Can this difference be important, they will ask? Yes! Demographers, for instance, would be especially interested in <u>how</u>

a graph is increasing if that graph happens to represent population growth! (In a situation closer to home, teachers and prospective teachers hope that the "student population DECLINE curve" looks at least like  and not like  !) I then draw a picture like the one shown in Figure 9, and give the CUE WORDS: "slope", "increasing", "decreasing", "tangent lines," and ask for comments. Once again, if there is dead silence I start the ball rolling with: "On the extreme left the slope is ? (fill in 'SIGN') and is steadily ? (fill in "increasing" or "decreasing")."

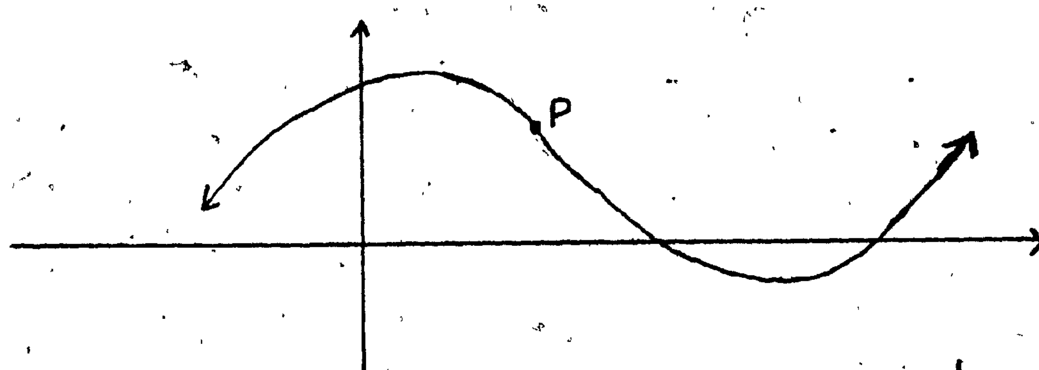


FIGURE 9

We also note how the curve always falls below its tangent lines to the left of point P. In comparison, the tangent lines "support" the curve everywhere on the right side of P. It is then useful to look at a few specific curves and see how they fare with respect to tangent lines drawn at selected points. The functions

(1) $y = x^3$ for $-1 \leqslant x \leqslant 1$ (concave down, then concave up)

(2) $y = x^{1/3}$ for $-1 \leqslant x \leqslant 1$ (concave up, then concave down)

(3) $y = x^2$ for $-2 \leqslant x \leqslant 2$ (always concave up)

(4) $y = -x^2$ for $-2 \leqslant x \leqslant 2$ (always concave down)

and (5) $y = 3x - 1$ for $-5 \leqslant x \leqslant 5$ (having no concavity).

do nicely to illustrate various possibilities. Also, I sometimes say to the students: "I want a certain function which falls below its tangent lines to the left of (0, 0) and which is above the tangent lines to the right of (0, 0). Can you name one? "I give hints when necessary. (I can then change the conditions to obtain curves having other desired characteristics.)

Generalizations and A New Look at the Parabola

The preliminary, informal work is followed by recording the rough analogies: $y = f(x)$ increasing is to $f'(x) > 0$ as $f'(x)$ increasing is to $f''(x) > 0$ AND $y = f(x)$ decreasing is to $f'(x) < 0$ as $f'(x)$ decreasing is to $f''(x) < 0$. The terms "concave up", "concave down" and "inflection point" are defined in terms of the second derivative. If student participation in the "preliminaries" is high - and this, after all, is the essence of the discovery technique - little or no "brute force" is required to have students accept the usual definitions, or the criteria given in the Second Derivative Test for Extrema. The students then get a summary sheet detailing the steps one should take in analysing a function prior to sketching its graph. The items on the sheet have all been discussed thoroughly in class - many have had a chance to ANTICIPATE the main results and definitions.

We then return to our function $y = \frac{x - 1}{x}$ for the third and last time, and sketch its graph using our previous analyses together with any information we can obtain from the 2nd derivative. And, as the students have expected, all the wrinkles disappear. Several polynomial and rational functions are then slowly analysed and their curves sketched. The following problem can also prove quite instructive, since it confirms a previously known fact (from high school algebra) with the aid of Calculus.

Exercise: Use Calculus to show that the parabola $y = ax^2 + bx + c$ has an absolute minimum at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ if $a > 0$; where a, b, c are fixed real numbers.

An Interesting Experiment

I challenge the class to give me any function they would like to see sketched. There is usually a student who wants me to work on something like $y = \dfrac{x^3 - 4x}{(-3x^2 + 5)^2}$. I politely decline saying: "Please, be reasonable!" It's not long before someone relents and requests that I sketch something simple like $y = 4x^3 - x^2 - x - 5$.

I find that I'm forced to deal with clumsy inequalities and messy critical points. Everyone realizes how difficult calculations can get without judicious prior planning. So then I pose the real challenge:

"Find a cubic or quartic polynomial which will have nice integer - or, at least, rational - critical and inflection points, along with, of course, factorable derivatives."

This is a non-trivial reverse problem, with a special bonus: with proper teacher guidance, students can start experimenting with simple anti-differentiation.

Instructor: Find $f''(x)$ so that $f''(x)$ will give an inflection point $(1, f(1))$.

Student A: $f''(x) = x - 1$

Instructor: Now find $f'(x)$.

Student B: $f'(x) = \frac{x^2}{2} - x$

Instructor: This is not the only possible answer. Find some others.

Student C: $f'(x) = \frac{x^2}{2} - x + 1$

Instructor: What is the most general answer?

Student A: $f'(x) = \frac{x^2}{2} - x + c$ ('c' any constant)

Instructor: Now find an $f'(x)$ so that $(2, f(2))$ will be a critical point.

and the dialogue continues in this fashion.

On Providing Thought-Provoking Exercises

Curve sketching affords excellent opportunities for formulating and solving non-trivial, even interesting, problems. Now, scores of Calculus texts are available with good standard problems of varying degrees of difficulty. Most class time will be taken up with solving such problems. But if one wishes to add a little more variety to problem sets or homework assignments the search for "good" problems may take a bit longer. Here then, is a sample of what I call SPECIAL PROBLEMS, related to the subject of Curve Sketching, many of which have become part of my Calculus 103 course material since the time I had gotten interested in "Consolidation Exercises". [see Section 3/Chp. II]

Some Special Problems

(1) Give an EXAMPLE of:

(a) a function which has neither an X nor a Y intercept.

(b) a function which passes through a horizontal asymptote somewhere in the interval $[-2, 2]$.

(c) a function which has an infinite number of vertical asymptotes.

(d) a function which has 2 horizontal asymptotes.

(e) a function $y = f(x)$, other than $f(x) =$ a constant, for which $f''(x) = 0$ for all values of $x \in R$.

(f) a function, other than a quadratic polynomial, which is CONCAVE DOWN at all values in its domain.

(2) What is the COMMON FEATURE among:

(a) $y = 3x^5 + 1$ (b) $2y - 3x + 5 = 0$

(c) $y = 7^x$ (d) $y = x^{1/3}$?

(3) Sketch a continuous curve $y = f(x)$, having the following properties:

(i) $f(1) = 3$, $f(-1) = -5$

(ii) $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ otherwise

(iii) $f'(1) = f'(-1) = 0$

(iv) $f''(x) < 0$ for $x > 0$ and $f''(x) > 0$ for $x < 0$.

(4) Make up a "Reconstruction Exercise" like Number 3, above.

(5) Show that the graph of $f(x) = \dfrac{x + 1}{x^2 + 1}$ has 3 inflection points which are COLLINEAR.

(6) Explain why $(0, \frac{3}{4})$ is not a point of inflection of $f(x) = \dfrac{x^4}{4} + \dfrac{3}{4}$ even though $f''(0) = 0$.

(7) Sketch the curve(s): $(xy - 3)(y - 5x) = 0$.

(8) (a) How does the quantity $b^2 - 3ac$ determine the NUMBER OF CRITICAL POINTS of the general cubic polynomial $ax^3 + bx^2 + cx + d$? ($a \neq 0$).

(b) How many critical points does $y = x^3 - x^2 + x + 1$ have?

(c) Give an example of a cubic polynomial that has NO critical points.

(d) Give an example of a cubic polynomial that has ONE critical point.

(e) Give an example of a cubic polynomial that has TWO critical points.

(f) Show that there can be no cubic polynomial with 3 or more critical points. Can you generalize this?

(10) (a) Consider $f(x) = x - \sin x$, for $\underline{x \geq 0}$. Show that $f(x)$ is INCREASING for all such x. What is the minimum point on the graph of f?

(b) Show that $g(x) = \cos x + \frac{x^2}{2} - 1$ is INCREASING for all $x \geq 0$.

(c) What is the relationship between $f(x)$ and $g(x)$?

(d) Find a function $h(x)$ such that $h'(x) = g(x)$ and so that $h(0) = 0$. Prove that this $h(x)$ is also INCREASING for $x \geq 0$.

(e) Now show that $-\cos x + \frac{x^4}{24} - \frac{x^2}{2} + 1 \geq 0$ for all $x \geq 0$.

(f) Prove that $-\sin x + \frac{x^5}{120} - \frac{x^3}{6} + x \geq 0$ for all $x \geq 0$.

(g) Prove that $x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}$ for $x \geq 0$.

(h) Verify that $.8333 < \sin 1 < .8375$. Find sin 1 using trig. tables and compare the 2 results.

This last exercise is a particularly nice example of a Type D problem, bringing together the notion of "increasing function", "anti-differentiation, "upper and lower bounds" and the idea of constructing mathemtical tables. Moreover, steps (a) to (h) clearly suggest a sequence of progressive stages or a mode of <u>guidance</u> that the instructor and students may follow. Each step is a link between the initial notion of an increasing curve and the final goal of approximating the value of sin 1.]

Section 2: Elementary Matrices and The Determinant Product Theorem.

Many matrix patterns, easy-to-spot relationships between certain matrices and determinants, proofs within reach of most students, potential for posing good, thought-provoking exercises all contribute to make Elementary matrices and The Determinant-Product Theorem amenable to guided discovery teaching.

It all begins with the basic definition:

"An Elementary Matrix is a matrix that is obtained from the identity I by one of the 3 types of Elementary Row Operations." (By this time, properties and operations on matrices, properties of determinants, as well as the row-reduction method for solving "m" linear equations in "n" unknowns have all been covered in class.) There are, of course, 3 classes or types of elementary matrices, corresponding to the 3 categories of elementary row operations. Each type is carefully analysed in the following manner.

Definition: A Type I elementary matrix is formed by interchanging a pair of rows of the Identity. E_{ij} denotes the elementary matrix which is obtained by interchanging rows i and j of the matrix I.

I then pick an arbitrary matrix, say $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

and let someone in class choose an E_{ij} suitable for pre-multiplication. Suppose that $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is selected.

We then verify that E_{12} A yields $\begin{bmatrix} c & d \\ a & b \\ e & f \end{bmatrix}$ = B, say. This is followed by a series of key questions:

(Q1) What is the effect of pre-multiplying A by E_{12}?

(Q2) What is det E_{12}?

(Q3) What is E_{12} E_{12}?

(Q4) What is E_{12}^{-1}?

Students who volunteer answers must also try to explain <u>how</u> they have arrived at these answers.

One or more such "case studies" (observing the properties and effects of E_{ij}) is carried out - the number of examples depending largely upon the strength of student response. All this preliminary activity eventually leads to the more general questions:

(Q1) What is the effect of pre-multiplying any $n \times r$ matrix A by an $n \times n$ matrix E_{ij}?

(Q2) What is det E_{ij}?

(Q3) What is E_{ij} E_{ij}?

(Q4) What is E_{ij}^{-1}?

In this way we - the class and I - put together <u>Generalizations I</u>. These are:

(1) (E_{ij})A results in a matrix B with the property that row i of A = row j of B and vice versa. All other rows of B are the same as those of A.

(2) det E_{ij} = -1, by a previous property of determinants.

(3) Since E_{ij} E_{ij} = I (by (1)) it follows that $E_{ij}^{-1} = E_{ij}$.

In every class of Linear Algebra that I've taught over the past few years there have always been a fair number of students who could discover these general rules and state them (in their own words) BEFORE I wrote the above statements on the board.

Similarly we study $E_i(k)$ [formed from I by multiplying row i by the scalar k] and $E_{ij}(k)$ [formed from I by adding k times row j to row i] and summarize our results as follows:

Generalizations II:

(1) $(E_i(k))A$ results in a matrix B which is identical to A in all but one respect: the $i^{\underline{th}}$ row of B = k times the $i^{\underline{th}}$ row of A.

(2) $\det E_i(k) = k$, by a previous property of determinants.

(3) Since $E_i(k)\, E_i\left(\frac{1}{k}\right) = I$ (by (1)) we have: $\left[E_i(k)\right]^{-1} = E_i\left(\frac{1}{k}\right)$.

Generalizations III:

(1) $(E_{ij}(k))A$ results in a matrix B which is identical to A in all but one respect: the $i^{\underline{th}}$ row of B = k times row j of A added to row i of A.

(2) $\det E_{ij}(k) = 1$, by a previous property of determinants.

(3) Since $E_{ij}(k)\, E_{ij}(-k) = I$ (by (1)) we have: $\left[E_{ij}(k)\right]^{-1} = E_{ij}(-k)$.

This last set of generalizations is usually obtained more slowly than the first 2 sets. A possible tactic is to develop Generalizations I and II in one class, define $E_{ij}(k)$, give some examples of Type III matrices, and let the students try to work out the parts of Generalizations III for homework. They can use the previous 6 results as a rough guide. I then find out how they fared with this task in the next lecture and fill in the remaining gaps. I also encourage Students X, Y and Z to fill in the gaps left by the remarks of their classmates.

I recall an interesting incident that occurred during one such "wrap-up" session about a year ago. One observant student noted: "Sir, I found an easy way to remember what the matrix $E_{ij}(k)$ looks like. Just put a "k" in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of the identity matrix!" The proud way in which he reported this small discovery told me that some positive things have happened. At least one student had begun to look for patterns and to experience the pleasure of finding something on his own - something that required more than "plugging into a formula", using a standard technique or solving a simple equation. (The same student, in a subsequent course in Finite Mathematics, independently tried to discover a divisibility property of integers and set out to prove his result. With a little help from me he was eventually successful.)

Returning to our exposition of the theory of elementary matrices, once the 3 groups of Generalizations have been established I summarize the chief points:

(A) Any elementary row operation on a given matrix A can be performed by pre-multiplying A by the appropriate **elementary matrix.**

(B) An elementary matrix is invertible and its inverse is an elementary matrix OF THE SAME TYPE.

Next I display a matrix such as $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. Working with the class, we reduce A to I by successive pre-multiplication by $E_1 = E_{13}$, $E_2 = E_3(\frac{1}{2})$, $E_3 = E_{13}(-2)$ and $E_4 = E_{12}(1)$. [E_1, E_2, E_3, ... is used when we wish to list a number of elementary matrices appearing in a certain order.]

Thus, $E_4E_3E_2E_1 A = I$. I then ask: "Can all square matrices be eventually reduced to I this way?" My question is usually greeted with silence. I will then ask Rosemary, say: "Could you reduce $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ to I, for instance?" She (hopefully) says "NO!" and even if she doesn't someone else usually volunteers the right answer. Now I can ask: "What do you think the necessary condition is for a square matrix to be reducible to I?" For those who still need some guidance, I try a gradual staging:

(Q1) How do you verify that matrix Y is the inverse of some matrix X?

(Q2) Since $(E_4E_3E_2E_1)A = I$ what can you say about the matrix product $E_4E_3E_2E_1$ in relation to A?

(Q3) So the matrix A that we worked with was...? Therefore, if A is not invertible we cannot...?

By the end of this question / answer session the whole class is generally ready for what I refer to as Generalizations IV.

Generalizations IV: (1) $A \sim I$ (A equivalent to I) means that there are elementary matrices $E_1, E_2, \ldots, E_k$, say, such that $E_k E_{k-1} \ldots E_3E_2E_1 A = I$. Moreover, $A^{-1} = E_k E_{k-1} \ldots E_3E_2E_1$ or equivalently, $A^{-1} = E_k E_{k-1} \ldots E_3E_2E_1 I$, since $E_1 I = E_1$. It is important to state the last result in words: "The SAME sequence of elementary row operations that converts A into I will change I into A^{-1}." BONUS: We have another way to find A^{-1} for a given non-singular A. [The "old" way was the adjoint-determinant method.]

(2) Since $E_1, E_2, \ldots, E_k$ are, of course, all invertible we have that $(A^{-1})^{-1} = A = (E_k E_{k-1} \ldots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1}$

and since all these k individual inverses are themselves elementary matrices we readily conclude that: ANY INVERTIBLE MATRIX "A" CAN BE FACTORED INTO A PRODUCT OF ELEMENTARY MATRICES.

(3) If A is any m x n matrix and $A \sim B$ via k elementary row operations, then there exist elementary matrices $E_1, E_2, \ldots, E_k$, say, such that

$$E_k E_{k-1} \ldots E_2 E_1 A = B \text{ or equivalently,}$$

$$A = E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1} B \text{ where, once again,}$$

$E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$ are all elementary matrices.

A few exercises are essential to consolidate this barrage of new ideas, before proceeding to the Main Theorem. Some typical exercises that I use are:

Exercise 1: Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (a) Find A^{-1} by making use of Generalizations IV, part (1).

(b) Write A as a product of elementary matrices.

(c) Write A^{-1} as a product of elementary matrices.

Exercise 2: Prove that the following 4 statements are equivalent:

(a) A has a LEFT INVERSE.

(b) A is INVERTIBLE.

(c) A is a product of elementary matrices.

(d) the system $A\underline{x} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$.

Exercise 3: Try to write $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as a product of elementary matrices. What happened? Why?

<u>Exercise 4</u>: <u>How many</u> solutions does the system have?

$$\left.\begin{aligned} 3x + 2y + z &= 0 \\ 2x + 2y + z &= 0 \\ x + y + z &= 0 \end{aligned}\right\}$$

What about the system $\left.\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}\right\}$?

<u>The Determinant Product Theorem</u>

<u>Objective</u>: To prove the Main Theorem: det AB = det A det B for any square A, B of the <u>same</u> dimensions.

This is too hard to tackle right away, so we first prove the "weaker" Lemma: det EA = det E det A where A is any n x n matrix and E is any elementary matrix of the same dimensions. (I briefly explain to the class how the terms "weak hypothesis" and "strong hypothesis" are related. We try to come up with other examples of weak versus strong hypotheses.) The proof of det EA = det E det A provides a good example of "<u>Proofs by Case Distinction</u>" [for other proof types see Section 2/ Chp. II].

For, if $E = E_{ij}$ then det EA = -det A = (-1) det A = det E det A. Similarly, one can easily establish the factorization for $E = E_i(k)$ as well as for $E = E_{ij}(k)$. We make use of Generalizations I, II and III to prove our Lemma. It can then be shown by an inductive argument that det $E_1E_2 \ldots E_k$ A = det E_1 det $E_2 \ldots$ det E_k det A for any k elementary matrices each having the same dimensions as A.

We are now prepared to prove The Main Theorem. A somewhat idealized discussion centering on the proof might be:

Instructor: Suppose A is invertible. What else can you say about A?

Student A: Well, det $A \neq 0$.

Student B: $A \sim I$.

Instructor: Good. What can you conclude from $A \sim I$?

SILENCE.

Instructor: Can you write an equation that connects A and I?

Student C: $EA = I$, maybe?

Instructor: Just one E?

Student B: We don't know how many elementary matrices are needed.

Student A: That's it! So $E_n E_{n-1} \dots E_2 E_1 A = I$, say. This is like what we did earlier.

Instructor: True - very good. But also, of course,

$A = E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1} E_n^{-1} I$. Right? Can you tell me what det A is now?

Student A: $\det A = \det E_1^{-1} \det E_2^{-1} \dots \det E_{n-1}^{-1} \det E_n^{-1} \det I$ from what we did after the last Lemma...

Instructor: Great, I...

Student A: Please ... I'm not finished yet! So,

$\det A = \det E_1^{-1} \det E_2^{-1} \dots \det E_n^{-1}$, since $\det I = 1$.

Now what?

Instructor: What is AB?

Student D: $AB = E_1^{-1} E_2^{-1} \dots E_n^{-1} IB = E_1^{-1} E_2^{-1} \dots E_n^{-1} B$.

Student A: Hey, now it's easy!

$\det AB = \det E_1^{-1} \det E_2^{-1} \dots \det E_n^{-1} \det B = \det A \det B$.

And thus the dialogue continues - we show that $\det AB = \det A \det B$, even when A is singular. (The argument hinges on the fact that if the last row of a matrix R, say, is a zero row, then the last row of RB is also a zero row.) Nor does the work end here. The Determinant-Product Theorem has many consequences, providing ample material for exercise sets. I was surprised how few <u>interesting</u> consolidation exercises are found in standard textbooks on elementary matrices and related topics, however. Here was an opportunity for me to devise a few good problems. The next set of Special Problems includes some of the results of such efforts.

More Special Problems

(1) Find matrix A given: $E_{13}(3)\ E_{13}A\ E_{12}E_2(2) = \begin{bmatrix} 18 & 6 & 15 \\ 4 & 4 & 6 \\ 3 & 2 & 5 \end{bmatrix}$

(2) Make up an exercise similar to (1).

(3) (a) Suppose $A = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & b/a \\ 1/b & a/b & 1 \end{bmatrix}$ Show that $A^2 = 3A$.

(b) Using the equation $A^2 = 3A$, solve for det A. Are 2 solutions possible? Explain.

(c) Using the "patterned matrix" A as a model find an analogous 4 x 4 matrix B so that $B^2 = 4B$. What is det B?

(d) Can you generalize the patterns to any analogous n x n matrix X? What is det X?

(4) Find four 3 x 3 matrices A, B, C, D (none of them equal to I) such that det ABCD = 24.

(5) We are looking for a certain 3 x 3 matrix A. We know that

(i) $\text{ent}_{3,3}[E_3(3)\ A] = 15$ [Recall that $\text{ent}_{i,j}$ X is the i, j^{th} entry of matrix X.]

(ii) $E_{12}\ A = \begin{bmatrix} 0 & 2 & 3 \\ 7 & 5 & 9 \\ ? & ? & ? \end{bmatrix}$ and (iii) $E_{31}\ (2)\ A = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ 15 & 12 & 23 \end{bmatrix}$

Find matrix A. [Another Reconstruction Exercise.]

(6) Prove or disprove each of the following:

(a) E^n is an elementary matrix for every $E = E_{ij}$ and any $n \in N$.

(b) E^n is an elementary matrix for every $E = E_i(k)$ and any $n \in N$.

(c) E^n is an elementary matrix for every $E = E_{ij}(k)$ and any $n \in N$.

(d) The product of any 2 elementary matrices yields an elementary matrix.

(e) The transpose of any elementary matrix is an elementary matrix.

(7) Prove the following consequences of the Determinant-Product Theorem:

(a) $\det AB = \det BA$ (c) $\det (A_1A_2 \ldots A_n) = \det A_1 \det A_2 \ldots \det A_n$

(b) $\det A^{-1} = {}^1/\det A$ (d) $\det A^r B^s = (\det A)^r (\det B)^s$; $r, s \in N$.

(e) $\det (\text{adj } A) = (\det A)^{n-1}$ where A is an n x n matrix.

(8) If $\det X = 10$, find $\det (C^{-1} B^{-1} A^{-1} X ABC)$, where A, B, C are any 3 invertible matrices of the same dimensions as X.

(9) If P is ORTHOGONAL prove that $\det P = \pm 1$. <u>Give an example</u> of an orthogonal matrix whose determinant is 1. Give an example of an orthogonal matrix whose determinant is -1.

(10) If $X = E_1E_2 \ldots E_{99}E_{100}$ where <u>half</u> of the elementary matrices are Type I and <u>half</u> are Type III, what is det X? What is det X if exactly 33 matrices are Type I and the remaining 67 are Type III?

(11) <u>Make up an exercise</u> similar to (10). Why is it inconvenient to use Type II matrices in such a problem?

(12) A student once argued as follows:

(a) I know that $\det (AB)^{-1}$ exists iff $\det AB \neq 0$ since $\det X^{-1} = {}^1/\det X$.

(b) This means that <u>neither</u> det A <u>nor</u> det B can be equal to 0, since $\det AB = \det A \det B$ by the Determinant-Product Theorem.

(c) Therefore A, B are both invertible.

(d) Hence $(AB)^{-1}$ exists $\Longrightarrow A^{-1}$, B^{-1} must both exist.

But we know from previous class discussion that (d) is FALSE.

What's wrong with the student's argument?

Section 3: Conditional Probability and Bayes' Theorem

The topic of probability is greeted with mixed emotions by most classes in College Mathematics. On the one hand "it sounds interesting" in these days of the super lotto, mini-lotto, inter-lotto and all the other lottos. It is the subject everyone who is interested in gambling wants to know about. On the other hand students "have heard that probability problems can be tough." Almost all however, <u>expect</u> to get something useful out of their encounter with probability. Thus, little "selling" (pep talks on importance, future applications, etc.) is required in comparison to Math Induction, Complex Numbers, The Binomial Theorem and other items on the College Math Syllabus.

I generally begin with a brief historical introduction - the days of Pascal, Fermat and de Méré - and follow by definitions of "sample space", "elementary outcomes" and "P(A)", the probability that an event A occurs. The 3 Kolmogorov Axioms and their consequences come next. During this introductory part, the usual coin, die and card experiments are examined.

After the few preliminary lectures I introduce the notion of Conditional Probability. Now, if you open any standard textbook at random, the probability is .9 that the <u>guided discovery mode is used</u> to develop the formula:

$$P(A|B) = \text{probability event A occurs given that B has occurred}$$

$$= \frac{P(A \cap B)}{P(B)} .$$

Typically, the author will define two events A and B - say, A = event that 3 dots appear on a die toss; B = event that an odd number of dots occur - and then ask the student to compute the quantities $P(A)$, $P(B)$ and $P(A \cap B)$.

The reader is then encouraged to guess at the value of $P(A|B)$ and eventually to try to formulate a connection between $P(A|B)$, $P(A \cap B)$ and $P(B)$. (I take the same approach in class.) It's good to see such support for the guided discovery technique even if it is often limited to only one or two specific areas.

The formula for $P(A|B)$ immediately gives rise to the <u>Multiplication Rule</u>: $P(A \cap B) = P(B)\,P(A|B)$. One can also derive: $P(A \cap B) = P(A)\,P(B|A)$. Consider the following example:

<u>Example</u>: The probability that Ten-Pin Tina gets a strike in the first frame of her first game is .2. The probability that she gets a strike in the second frame given that she got a strike in frame #1 is .45. Find the probability that she rolls a "double" in the first 2 frames.

This, of course, is a rather straight-forward application of the Multiplication Rule. Nevertheless, the student must first <u>define</u> events A and B before "plugging numbers into a formula." I INSIST on this - for such "translation" exercises [English to Mathematical Symbolism] prove essential practice for subsequent, more complex problems. Next, I ask someone to <u>set up</u> a similar exercise based on 3 FRAMES of bowling. He usually has no trouble with the "question" to be asked but is uncertain about the necessary "givens".

<u>Pattern Analysis</u> [Stage I]

<u>Instructor</u>: How many "pieces of information" did we need in order to compute $P(A|B)$ in the previous problem?

<u>Student A</u>: Two.

<u>Instructor</u>: How many pieces of information will we need in the 3-frame problem? And - oh yes. What is the question?

<u>Student A</u>: We'll need to know 3 probabilities to find $P(A \cap B \cap C)$ = probability of bowling a triple.

<u>Instructor</u>: Yes. What are the first two?

<u>Student B</u>: $P(A)$, $P(B|A)$ I think.

<u>Instructor</u>: Correct. Now the third - anyone?

SILENCE

<u>Instructor</u>: Event C - the 3rd strike - is to occur after ...

<u>Student B</u>: I get it! We'll need $P(C|B|A)$!

<u>Instructor</u>: Your answer has some logic to it based on a pattern we've been following. However, there is an ambiguity - do you mean C|B GIVEN A or C GIVEN B|A? In either case, it's difficult to ...

<u>Student B</u>: I MEAN EVENT C given that there were strikes in <u>both</u> frame 1 <u>and</u> frame 2!

<u>Instructor</u>: Ah - that's quite clear. Now what symbol is generally associated with "<u>both</u> ... <u>and</u>"?

<u>Student B</u>: The "intersection" $\cap$. So, our 3rd probability will be $P(C|A \cap B)$!

<u>Student A</u>: The formula is: $P(A \cap B \cap C) = P(A)\, P(B|A)\, P(C|A \cap B)$.

<u>Instructor</u>: Looks good. Let's see how it <u>reads</u> ... It even sounds reasonable. Now we have a <u>plausible conjecture</u>, as a famous Hungarian problem strategist would say! Let's prove it!

<u>Proof</u> [Stage 2]

<u>Instructor</u>: How would you start?

Student X: I really don't know.

Instructor: Come on - think hard! Use the Multiplication Rule we've talked about.

Student X: But there are 3 events, not 2!

Instructor: Brackets are useful.

Student Y: O.K. $P(A \cap B \cap C) = P[A \cap (B \cap C)]$. Now what?

Student X: I see now - $P[A \cap (B \cap C)] = P(A)\,P(B \cap C|A)$. Now ... I'm stuck!

Instructor: Well, your statement is correct. You have found another way to compute $P(A \cap B \cap C)$! In order to apply your formula, though, one would have to know the values $P(A)$ and $P(B \cap C|A)$. In some problems, this is exactly what would be given. But right now our goal is to obtain the right-hand side: $P(A)\,P(B|A)\,P(C|A \cap B)$. These 3 numbers constitute the "givens" in our bowling problem. Remember: in doing a proof always keep one eye on WHAT YOU ARE REQUIRED TO SHOW. Maybe if you just put your brackets ...

Student X: That's it!

$$P(A \cap B \cap C) = P[(A \cap B) \cap C]$$
$$= P(A \cap B)\,P(C|A \cap B)$$
$$= P(A)\,P(B|A)\,P(C|A \cap B)$$

Instructor: Good work, X! Guess what my next question is.

Student Y: What happens for 4 events?

And on it goes, until we arrive at the "ultimate" statement:

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Thus the recurring pattern enfolds.

The next step is to talk about mathematical (or stochastic) independence. Specifically, the events A and B are said to be mathematically independent if and only if $P(A|B) = P(A)$ or equivalently, $P(B|A) = P(B)$, or $P(A \cap B) = P(A)\,P(B)$. This last factorization rule is the criterion most commonly used. Unfortunately, events that may appear to be intuitively independent (i.e. not to have any effect upon one another) are often stochastically dependent. Moreover, events which are intuitively dependent may be mathematically independent. B.H. Bissinger, for example, uses a 3-coin experiment to demonstrate that the one kind of independence does not necessarily imply the other. [36] Intuition doesn't always fail us, however. When A, B are (mathematically) independent we would expect that

(1) A and $\bar{B}$ are independent

(2) $\bar{A}$ and B are independent

(3) $\bar{A}$ and $\bar{B}$ are independent.

Here, $\bar{A}$ ("A complement") consists of all sample points in the sample space which are NOT in A. The proofs are relatively simple. For instance:

$P(A \cap \bar{B}) = P(A)\,P(\bar{B}|A)$, by Multiplication rule.

$= P(A)\,[1 - P(B|A)]$, from a proposition proved earlier in class.

$= P(A)\,[1 - P(B)]$, since A, B are independent.

$= P(A)\,P(\bar{B})$

This shows that (1) holds. The proof of (2) is very similar. Statement (3) can be verified as follows:

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}), \text{ by De Morgan's Law,}$$
$$= 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)], \text{ by the Addition Axiom of Kolmogorov.}$$
$$= 1 - P(A) - P(B) + P(A)\,P(B), \text{ since } A, B \text{ are independent.}$$
$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$
$$= [1 - P(A)]\,[1 - P(B)]$$
$$= P(\overline{A})\,P(\overline{B}).$$

At this point, I introduce the notions of <u>pairwise</u>, <u>triple-wise</u>, ... <u>n-wise independence</u>, as well as that of <u>complete independence</u>.

<u>Definitions</u>:

(1) The events $A_1, A_2, \ldots, A_n$ are PAIRWISE INDEPENDENT if $P(A_i \cap A_j) = P(A_i)\,P(A_j)$ for $i \neq j$ with $i, j = 1, 2, \ldots, n$.

(2) The events $A_1, A_2, \ldots, A_n$ are TRIPLE-WISE INDEPENDENT if $P(A_i \cap A_j \cap A_k) = P(A_i)\,P(A_j)\,P(A_k)$ for $i \neq j \neq k$ with $i, j, k = 1, 2, \ldots, n$.

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

(n - 1) The events $A_1, A_2, \ldots, A_n$ are n-WISE INDEPENDENT if $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$.

(n) The events $A_1, A_2, \ldots A_n$ are COMPLETELY INDEPENDENT if conditions (1), (2), ... (n - 1) <u>all</u> hold. (Obviously, this is a very "strong" kind of independence!)

Most students readily agree that pairwise independence does <u>not</u> necessarily imply triple-wise independence. It is easy to demonstrate this:

<u>Example</u>: Two dice are tossed.

Let A = event that an odd number of dots occur on die # 1.

Let B = event that an odd number of dots occur on die # 2.

Let C = event that the sum of the dots appearing on the 2 dice is odd.

Then by studying the 36 elementary outcomes of this 2-dice experiment we observe that

$P(A \cap B) = P(A)\,P(B)$, $P(A \cap C) = P(A)\,P(C)$, $P(B \cap C) = P(B)\,P(C)$ but $P(A \cap B \cap C) \neq P(A)\,P(B)\,P(C)$.

However intuition rebels against the mathematical reality that 3 events may be triple-wise independent and yet fail to be pair-wise independent. (A little controversy adds spice to the course material.) The challenge is to come up with an example to illustrate this property. I have found the following one to be quite appropriate:

Example: Eight billiard balls numbered 444, 454, 455, 455, 544, 545, 545, 554 are placed in a bag. One ball is drawn at random.

Let A = event that the 1st digit on the ball drawn is a '4'.

Let B = event that the 2nd digit on the ball drawn is a '4'.

Let C = event that the 3rd digit on the ball drawn is a '4'.

Then $P(A \cap B \cap C) = \frac{1}{8} = P(A)\,P(B)\,P(C)$.

But $P(A \cap B) = \frac{1}{8} \neq P(A)\,P(B)$ and the same goes for $P(A \cap C)$ as well as for $P(B \cap C)$.

The work on conditional probability prepares the class for Bayes' Theorem. When I first taught this topic, I began by stating the Theorem in general, followed by the proof and then the applications to specific examples. It was a disaster! There were red faces everywhere. Those of the students were red from effort while mine was red from embarrassment. I wasn't getting anywhere. Since then I have found that the guided discovery approach in this instance is not just a nice alternative but a necessary part of achieving the required level of understanding. Thus I now introduce Bayes' Theorem via a "typical" problem.

Example: A wealthy Roman senator wishes to choose a gladiator to bet on in the upcoming contest at the Circus Maximus. Matches are to be made by selecting gladiators from 2 groups. Group I consists of 6 Thracians and 4 Libyans. Group II consists of 3 Libyans and 7 Thracians. The senator tosses a denarius to decide which group to choose from. Given that he ends up betting on a Libyan what is the probability that his Libyan came from Group I?

The first time I used this simple problem, one student started scribbling away. After a few seconds' work he announced: "The answer is 4/7!" I looked at his work and saw something like this:

	Libyans	Thracians
Group I	4	6
Group II	3	7

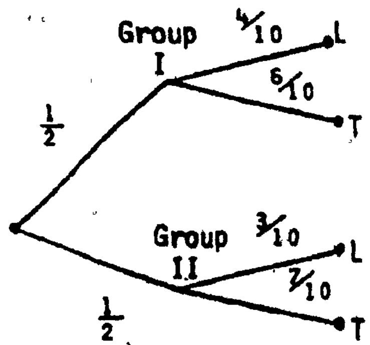
$\underbrace{\qquad}_{\text{Total } = \ 7}$ (Libyans)

probability of Libyan from Group I is 4/7.

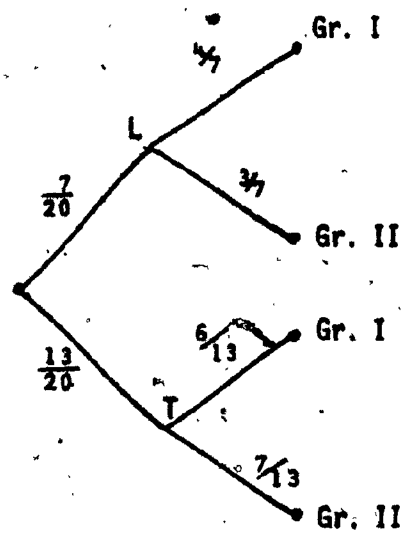
One notes that John W. Dawson at Pennsylvania State University has encouraged his students to employ similar box diagrams in solving such Bayesian problems. [37] As above, they learn to segregate various groups into different cells, to write down the corresponding cell probabilities, and to then use the table to compute the required "a posteriori" probability.

Other instructors prefer to use "tree diagrams" as suggested by

John Kemeny [38]. Using this mode of analysis, the diagrams for the gladiator problem would take on the following appearance:



FORWARD PATH



REVERSE PATH

The "path weight" along the top branches of the "forward tree" is $\frac{1}{2} \cdot \frac{4}{10} = \frac{4}{20}$ = probability of a Libyan from Group I being selected. This must correspond to the total path weight along the top branches of the "reverse tree". But, the probability of picking a Libyan out of the 20 gladiators is $\frac{7}{20}$. Hence the probability that this Libyan is coming from

Group I must be $4/7$. It is also a simple matter to compute the 3 remaining Bayesian probabilities. These are:

Probability a Libyan is chosen from Group II is $3/7$

Probability a Thracian is chosen from Group I is $6/13$

Probability a Thracian is chosen from Group II is $7/13$.

The gladiator problem lends itself to either a box or tree diagram type of approach, as a first look at a typical "reverse conditional probability" situation. But I feel that it is also important that the student learn how to solve this kind of problem by defining the events of interest and using the appropriate laws of probability. Using the basic concepts as "building blocks" and the formal operations as the "glue" that holds these units together the student can be guided to invent his own formula (otherwise known as Bayes' Theorem). This is an enjoyable and an important part of learning - putting together ideas to form a cohesive whole. The student-teacher interchange might go like this:

<u>Instructor</u>: There are 4 basic events that we should define. What are they?

<u>Student X</u>: Losing, winning, Thracians, Libyans!

<u>Instructor</u>: Not quite. I know you'd be interested in losing or winning, especially if you were the senator or the gladiator, but does the question asked have anything to do with the actual outcome of the gladiatorial contest?

<u>Student Y</u>: No. We should talk about Thracians, Libyans, Group I and Group II.

<u>Instructor</u>: Right. So let

G_1 = event that a gladiator is chosen from Group I.

G_2 = event that a gladiator is chosen from Group II.

T = event that the chosen gladiator is a Thracian.

L = event that the chosen gladiator is a Libyan.

Now tell me. What is $P(G_1)$ and $P(G_2)$?

<u>Student Z</u>: $P(G_1) = \frac{1}{2} = P(G_2)$ since the senator tosses a coin which has a 50/50 chance of falling heads or tails.

<u>Instructor</u>: Yes. Now are we looking for $P(G_1|L)$ or $P(L|G_1)$?

<u>Student Z</u>: Well, you know that a Libyan has been picked, so it must be the second one.

<u>Instructor</u>: Wrong, I'm afraid.

<u>Student Z</u>: I knew that - it's $P(G_1|L)$, of course!

<u>Instructor</u>: That's better! What is $P(L|G_1)$, incidentally?

<u>Student X</u>: $P(L|G_1) = 4/10$, since 4 out of the 10 gladiators in Group I are Libyans.

<u>Instructor</u>: Fine. What is $P(L|G_2)$?

<u>Student X</u>: 3/10.

<u>Instructor</u>: We now know the values of $P(G_1)$, $P(G_2)$, $P(L|G_1)$ and $P(L|G_2)$. We are going to use these to find our $P(G_1|L)$.

Of course, we all know that $P(G_1|L)$ can also be written as...?

<u>Student Y</u>: $P(G_1|L) = \dfrac{P(G_1 \cap L)}{P(L)}$. But I don't see any of the 4 probabilities you were talking about.

<u>Instructor</u>: Remember the Multiplication Rule!

<u>Student Y</u>: $P(G_1 \cap L) = P(G_1)\,P(L|G_1)$. Oh, I see. But what about the $P(L)$?

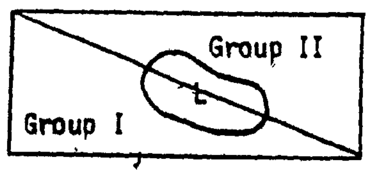
Instructor: A Libyan may come from either Group I or II. Therefore ...

SILENCE

Instructor: Therefore $L = (G_1 \cap L) \cup \ldots$? (Remember, the $\cap$ is roughly equivalent to "and" and the $\cup$ is roughly equivalent to "or".)

Student X: Aha! $L = (G_1 \cap L) \cup (G_2 \cap L)$ which accounts for both groups.

Student Y: So $P(G_1|L) = \dfrac{P(G_1)\, P(L|G_1)}{P\{(G_1 \cap L) \cup (G_2 \cap L)\}}$. The bottom of this fraction looks complicated.

Instructor: 

Look at this diagram. What can you say about the events $G_1 \cap L$ and $G_2 \cap L$?

Student Y: They're separate.

Instructor: Separate? You mean ...?

Student Y: Mutually exclusive.

Instructor: And therefore?

Student Y: Well, $P\{(G_1 \cap L) \cup (G_2 \cap L)\} = P(G_1 \cap L) + P(G_2 \cap L)$ by the Addition Axiom.

Instructor: Good. What is the next step? Think of the original 4 probability values that we had.

Student X: Easy! $P(G_1 \cap L) + P(G_2 \cap L) = P(G_1)\, P(L|G_1) + P(G_2)\, P(L|G_2)$.

Instructor: Excellent. Now let's put it all together.

Student Z: $$P(G_1|L) = \frac{P(G_1)\, P(L|G_1)}{P(G_1)\, P(L|G_1) + P(G_2)\, P(L|G_2)}$$

$$= \frac{1/2 \cdot 4/10}{1/2 \cdot 4/10 + 1/2 \cdot 3/10} = \frac{4/20}{7/20} = \frac{4}{7}.$$

Hey, part of the bottom appears on top.

<u>Instructor</u>: I wonder if that will always happen?

At this point I assign a second exercise similar in nature to the gladiator problem. The students work mostly on their own this time. After 10 or 15 minutes I ask someone to put the solution on the board. If necessary, the student at the board is helped by his classmates or by myself. Finally, I state the Theorem of Bayes-Laplace:

"If the events $A_1, A_2, \ldots, A_k$ are such that

(1) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = S$ (S = sample space)

and (2) $A_i \cap A_j = \emptyset$ for all $i \neq j$,

then for any event $B \subset S$ we have that

$$P(A_j|B) = \frac{P(A_j)\,P(B|A_j)}{P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + \ldots \quad P(A_k)\,P(B|A_k)}$$

By this point the students are fairly easily guided into establishing the above relationship. <u>They base their arguments on the steps taken to solve the 2 preliminary exercises</u>.

<u>A Few Final Remarks on Probability</u>

The notions of independence, conditional probability, Bayes' Theorem and the "story problems" arising thereof can challenge even those students who had previously passed other college courses fairly easily. But the guided discovery technique, I have found, helps to make the subject more accessible for the student primarily because of the greater personal involvement. It also makes the teaching much more enjoyable. I have taught elementary probability many times but still look forward to the next occasion.

More Problems

(1) Prove or disprove:

(a) $P(A \cup B|C) = P(A|C) + P(B|C)$ for $A \cap B = \emptyset$ and with $P(C) \neq 0$.

(b) $P(C|A \cup B) = P(C|A) + P(C|B)$ for $P(A) \neq 0$, $P(B) \neq 0$, $P(A \cup B) \neq 0$.

[Applications of Probability Laws plus some Set Theory.]

(2) Give an example of:

(a) 2 events which are mutually exclusive.

(b) 2 events which are independent.

(c) 3 events which are pair-wise independent but not triple-wise independent.

(3) The Game of Craps

You roll 2 dice. If a "7" or an "11" turns up you win the stake. If a "2", "3" or "12" turns up, you lose. Otherwise, if "4", "5", "6", "8", "9", or "10" is rolled you keep on playing until one of two things happens: either you "make your point" by repeating the sum you rolled initially or you "crap out" by coming up with a "7". "Making your point" results in winning the stake while "crapping out" results in the loss of your money. Calculate the probability of winning according to the rules of this game.

[NOTE: This problem has been analysed in various textbooks and journals. It links the notions of "mutually exclusive events", "independent events" and "geometric series" quite beautifully.]

(4) A small box contains 2 rubies, 3 emeralds, and 5 worthless pebbles. A second box of identical size contains 3 rubies, 6 emeralds and 7 pebbles. Four coins are tossed. If an even number of heads occurs,

box #1 is selected. Otherwise, box #2 will be selected. From the box thus chosen one item will be drawn at random. Suppose this item turns out to be a pebble. What is the probability the pebble came from box #1? [Application of Bayes' Theorem].

(5) The Three-Cornered Duel

A, B and C are to fight a three-cornered pistol duel. All know that A's chance of hitting his target is .3, C's is .5 and B never misses. They are to fire at their choice of target is succession in the order A, B, C cyclically, until only one man is left unhit. (A hit man loses further turns and is no longer shot at.) What should A's strategy be? [39]

[NOTE: This is an excellent problem for class discussion. All kinds of suggestions are generated by the nature of the question - some of them are even serious ones. The ideas of elementary probability theory are once again, linked with the concept of a geometric series.]

(6) The Cautious Counterfeiter

The king's minter boxes his coins 100 to a box. In each box he puts one false coin. The king suspects the minter and from each of 100 boxes draws a coin at random and has it tested. What is the chance that the minter's counterfeiting goes undetected? What if both 100's are replaced by any natural number $n > 1$? [40].

[NOTE: The second question is a very interesting one. It gives rise to the series expansion for $(1 - 1/n)^n$; namely

$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \dots \approx e^{-1}$ for large n. Moreover, the students can generalize to 2, 3 or even 'm' false coins and thus eventually obtain the expression $(1 - m/n)^n$ which tends to e^{-m} when n gets large.]

(7) "Star Wars" Revisited

Princess Leia is captured by the evil Darth Vader. She is blindfolded and placed in confinement on an Empire star cruiser. Vader wants to transport her to either the planet Ragnar or to a dismal world, Organia. The probability that she will be taken to these places is .4 and .6, respectively. It is well known that if you randomly select a native of Ragnar, the probability that he will not be carrying a neuron disrupter is .9. The probability that an Organian carries such a weapon is .75.

After several weeks of unpleasant confinement, Leia senses the cruiser deccelerating in preparation for a landing. Shortly thereafter she's dragged out onto the planet surface and her blindfold is removed. The muzzle of a rather large neuron disrupter is pointed at her head. What is the probability that Leia has just landed on Organia? [Application of Bayes' Theorem.]

(8) In the grand old days of ancient Rome, chariot races were very popular. The Blue Team had a record of winning a race with probability $5/8$, losing with probability $2/8$ and tying with probability $1/8$. Senator Quintus Fabius Obnoxious bet 1000 sesterces that in a 14-race schedule, the Blue Team will win 9 races, lose 3 and tie 2. What is the probability that Obnoxious will win his bet? [An example of a multinomial probability distribution. Binomial probability distributions are

discussed in class. Hence, in this problem students are required to extend a notion involving a binomial dichotomy (success, failure) to one involving the trichotomy - win, loss and tie.]

(9) Suppose n coins are tossed ($n \geq 3$). Find the probability that exactly 1 head or exactly 1 tail will occur. This situation is termed "getting an odd man out". If 6 people toss 6 coins (1 coin per person) what is the probability that 4 plays or trials will be required to produce an "odd man out"? [Application of the notion of "mutually exclusive events", "independent events" and Binomial probabilities].

(10) For 2 events A and B we know that $P(A) = .6$, $P(A \cup B) = .9$, $P(A|B) = .5$. Find $P(B)$. [Application of the Laws and Axioms of Probability.]

(11) Given that $P(A) = .7$, $P(B) = .3$ and $P(A \cap \bar{B}) = .4$, find $P(B|A \cup \bar{B})$. [Application of Probability Laws.]

(12) What is the necessary condition on A, B, C so that

$$P(A \cap B|C) = P(A|C)\, P(B|C)?$$

(HINT: Independence is the key.)

CHAPTER IV: STUDENT FEED-BACK AND CONCLUDING REMARKS

Section 1: Student Feed-Back

Like many of my colleagues I have always been interested in how students perceived my work in the classroom. (Self-evaluation is, of course, an integral part of "quality control" in the teaching profession.) Course/teacher evaluations prepared by Dawson College and by Students' Associations have served to provide some general feed-back. But during the past two years my interest in certain teaching methods, in why some individuals approach Mathematics with great trepidation and especially in whether my use of the guided discovery method was having any effect have led me to seek feed-back more specific to these concerns. I wanted to try to find out why students found proofs difficult, and how they felt about "proving" in general. I wanted to know what mathematical topics they might have had trouble with. I was also curious how they would respond to being placed in the role of "teacher". Thus, in early 1979, with these points in mind, I drafted a questionnaire.

The First Questionnaire

After one or two preliminary drafts I decided to include the following six questions or directives in what I now refer to as "The First Questionnaire":

(1) What was the most difficult topic you ever studied in Mathematics? Why do you think you had so much trouble with it?

(2) A student once said that it is a waste of time to PROVE anything formally in this course, or in any course, for that matter. Do you agree or disagree? Explain.

(3) Many times a student will ask: "Will there be proofs on the exam?" Do you find proofs difficult? Explain.

(4) Many people claim that they "just can't do Mathematics". Some even admit that they would be afraid to take a math course or that they hate mathematics. What do you think causes these attitudes?

(5) Imagine that you are tutoring someone in Mathematics. Your topic for the day is: FILL IN TOPIC OF YOUR CHOICE. Briefly explain how you would go about teaching this topic.

- What kind of examples would you use?
- What, if anything, would you prove in general?
- How would you know that the person you are tutoring understood the lesson?

(6) Make up an exercise based on topics we have covered in this course. "Tough" problems copied out of textbooks will not be of special merit. Also, you should be able to SOLVE your proposed problem.

These questions were distributed to my classes during the course of two semesters. I must admit that, with the rare exception, the responses were disappointing, for the following reasons:

(a) Only a small percentage of my students - around 5-10% or about 25 individuals - returned a completed questionnaire, responding to the questions being voluntary. I generally distributed the questionnaire in the middle of our 15-week term, thus giving everyone about 7 weeks to respond. Despite frequent reminders most people simply forgot about the whole thing. Possibly their course work prevented them from attending to such an optional matter.

(b) Many students wrote long, rambling essays, especially when answering questions (2), (3) or (4). I admit that these questions may be too general and thus encourage the "padding" or vagueness prevalent in most responses. But such hindsight does little to ease my disappointment.

(c) A number of respondents jotted down one or two sentences for each of the first 5 questions and concocted a few trivial examples to satisfy the requirement for the last question/directive. They may have been satisfied that they had "done their duty". Clearly, I wasn't. Any statements I had made in class in an attempt to prevent such slap-dash, half-hearted work had not made much of an impression on these students.

What follows now is a summary and analysis of the responses given to the 6 parts of the questionnaire.

Responses to (1)

There was no general consensus when it came to selecting the "most difficult topic". Answers ranged from "Mathematical Induction" to "Probability" to "Word Problems in Calculus" to "Lines and Planes in Space". Most students didn't seem to blame anyone but themselves for any difficulties with a topic. This came as a pleasant surprise. Unfortunately though, they were not very specific about <u>what</u> aspect of a particular concept might have puzzled them, or <u>why</u> it bothered them. One of my long-standing suspicions, however, was somewhat confirmed from the answers given by about a dozen individuals. Thomas Gagnon, for example, in writing about his difficulties with Probability complained:

"There seemed to be no <u>standard techniques</u> for solving probability problems ... The teacher should have shown us some ways of working the various types of problems."

~~I have long felt that~~ not having systematic cut-and-dried procedures for solving all kinds of problems frustrated a lot of students. In Introductory Calculus they are taught specific differentiation techniques which correspond to certain kinds of functions - a highly structured situation. In many cases the wording of the problems [for example: "Use the Chain Rule to differentiate the following."] leaves no doubt as to what method to use. In College Mathematics students are much more likely to have to make their own decisions and/or more difficult decisions: "Does this problem involve permutations or combinations?", "Are repetitions (of symbols in an arrangement) allowed?", "Are we dealing with mutually exclusive or independent events?", and so on. The <u>contrast</u> between the levels of decision-making in various courses is probably a significant contributor to the amount of frustration. Now, clearly all problems either <u>cannot</u> be or <u>should not</u> be conveniently tagged to a specific method by the teacher. After all, in "real-life" or industrial or business situations the problem solver is often called upon to make decisions about <u>optimal</u> ways and <u>possible</u> ways to tackle certain problems. Thus, I contend that <u>in addition</u> to the time spent looking for "canned problem solving procedures" a teacher should devote some time to grappling with the difficult task of <u>reducing the contrast</u> between the kinds of problem solving required in various courses. In a few years I hope to be able to categorically state that guided discovery teaching is an important aid in this struggle. (Right now, I simply suspect that it may be.)

Responses to (2)

I expected most of the students to support the "disagree" side; that is, to affirm the necessity of mathematical proofs. I now realize that the question may be loaded in such a way as to elicit this "desired response". And indeed, the vast majority (¾ or more) of those who expressed their opinions were in favour of proofs. Their reasons generally were: "to understand the logic of a theorem", "to improve the ability to think", "to set things straight in your mind", and the like. One of my pupils, Allan Short, expressed his views this way:

> "To prove something formally is to see the true beauty of Mathematics. Manipulation of numbers is, in the long run, trivial. Anybody can crunch numbers on a calculator. Therefore, in the broadest sense, proving something is not a waste of time. It enables an individual to see the logic behind mathematical operations and it gives one the satisfaction that the procedures in use are not random, lucky ones, but that there are valid reasons for doing something in a certain way."

Knowing Allan fairly well - he was my student for 2 semesters and we had, during that time, many hours of discussion in my office - I feel that his answer is quite honest and represents his true feelings about the matter. But not all responses were so favourable or so well formulated. A lot of what was handed to me was, unfortunately, sheer nonsense, such as:

> "Problems involving numbers are much easier than doing a proof even if they mean the same thing."

I had the impression that some students still did not know the difference between a general proof and a mere illustration of a hypothesis via a numerical example.

A few individuals bravely went against the strong current of popular opinion and candidly let me know that they felt that proofs in class <u>were</u> a waste of time. Among these outspoken individuals was Keith Cranston who asserted:

> "Proofs in this course [Linear Algebra] are generally a waste of time. The general terms have really no concrete meaning so it makes it difficult to relate the proof to anything."

Arthur Roach said:

> "I agree with that student - that proofs are a waste of time - because in a course you might learn to prove something one way and then in the next course you'll learn a totally different way."

Evidently it will take more work on my part to convince at least <u>some</u> of my students that proofs are an important part of Mathematics. (In The Second Questionnaire I pose another question concerning proofs. We will look at the results obtained for this question, later on in this chapter.)

<u>Responses to (3)</u>

The main reason given for difficulties encountered in doing proofs was: NOT KNOWING HOW TO GET STARTED. During class discussions, as well, this appears to be a major concern. Therefore I spend several hours each term talking to my students about the various kinds of proofs [see Section 2, Chp. II], and helping them with proofs, particularly with "opening strategies".

There is no doubt, however, that a little <u>extra encouragement and guidance</u> should accompany the proof-making task in my classes. For, as Allan Short says:

> "I sometimes find proofs difficult because I reach a point where I get stuck. Many times it turns out that you have to be quite clever to be able to continue. At other times it's so easy to take a wrong track. I discover that the method I had started to use is simply not going to work. This is <u>discouraging</u>, to say the least!"

<u>Responses to (4) and (5).</u>

The 2 questions which received the greatest attention were (4) and (5) [not every student chose to answer every question]. However, while the number of responses may have been comparatively large, the quality and variety of answers left something to be desired.

In (4), most students attributed the negative feelings about Mathematics to no personal motivation, lack of patience, poor preparation or insufficient time spent studying. Two individuals felt that some people shied away from math because it is a subject "where there is ONLY ONE RIGHT ANSWER". A few took a somewhat Freudian attitude, ascribing difficulties to unpleasant incidents that may have taken place in early childhood in the first few years of elementary school. However, not one person offered any suggestions as to what I might do to help students overcome their fear of mathematics. This was mostly <u>my</u> fault. I had neglected to ask for such suggestions in the Questionnaire.

Personally, I was most disappointed with the responses I got for (5). I had been confident that the way in which the question was worded would make it clear to everyone that I wanted detailed descriptions of tutoring situations based on a specific topic. But the work submitted was, in many cases, even more vague and uninformative than it had been for the previous four questions. Several students simply ignored my instructions to select a topic and went about writing long essays on teaching methods. The gist of such essays was: always give a lot of examples, state the major theorems, make the person being tutored do a lot of exercises and give them little quizzes as frequently as possible.

Now I knew that a number of students had actually done some tutoring but not one elected to write a personal account of how these tutoring sessions had progressed! Nor did anyone speculate about what examples they might use or what properties they would bother proving in a hypothetical tutoring situation. Perhaps I had expected too much or perhaps I should have provided a much more detailed explanation as to what I expected from my students.

Among the many pages of disappointing reading I did run across at least this one gem of a remark by Arnie Di Loreto:

"The students who(m) you tutor must be able to ask you questions. The only way that this will happen is if THEY ARE TREATED AS PERSONS AND NOT AS STUDENTS."

Responses to (6)

Despite repeated requests not to hand in routine exercises copied from textbooks many students seemed to have done just that. Among the few (less then about 2 dozen) relatively interesting or creative problems that were submitted certain patterns became evident.

- About 40 percent of the problems were based on patterns or extensions of patterns discussed in class. For instance, shortly after I had done some matrix algebra, Bernie Zemaitaitis proposed the exercise:

"If A, B, C commute in pairs [that is, $AB = BA$, $AC = CA$ and $BC = CB$] prove that $A^2 B^2 C^2 = C^2 B^2 \cdot A^2$."

Matrix patterns were, in fact, quite popular. Michael Sideris suggested the following problem:

"Given $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ show that

(a) $A^2 = 4I$ (b) $A^3 = 4A$ (c) $A^{2n} = 2^{2n}I$ for all $n \in N$."

This problem might possibly appear in some textbook on Linear Algebra but Michael assured me that he had discovered the pattern by himself. I do not doubt his word.

Another (small) group of students generated some exercises whose solutions would turn out to very simple provided one started by SIMPLIFYING the given expressions. Mario Filippone, for example, contributed:

"Analyse and sketch the function $y = \frac{x^3 - x^2 - 4x + 4}{x - 1}$."

Chee Yin Lu submitted:

"Given $y = 2 \tan^2\frac{x}{4} - 2 \sec^2\frac{x}{4} + (1 - \cos 2x + 2 \cos^2 x)^3$ find $\frac{dy}{dx}$."

Imitation is a form of flattery. I constantly tell my classes to see if they can simplify various expressions before plunging into a problem.

Upon some reflection I have come to the conclusion that the quality (and the quantity) of the exercises may have been higher had I

(a) been more explicit about what I meant by "special" or "creative" problems,

(b) perhaps encouraged the students to work in teams,

and (c) provided some special incentives for working on question (6). [for example, small prizes or bonus marks.]

The greatest value, for me, of The First Questionnaire was that I learned something about realistic expectations in relation to responses to such a set of questions. I also learned that designing good questions was not as simple a matter as I had once believed. I had the feeling that a second questionnaire compiled after this experience would produce better (more informative) results. But before this took place I would attempt to use a visual medium to obtain further feed-back on my teaching style.

The Filming Session

On October 23rd, 1980 Kevin O'Connor from the Dawson Counselling Centre and Bob O'Meara together with Robert Deans from the Media Resources Department of Dawson College came into my Linear Algebra class to tape and observe a session on Elementary Matrices. This was an opportunity for me to "stand back" and study the dynamics of the guided discovery process on film. While the class was going on Mr. O'Connor sat at the back of the room observing and making notes. (The film has been transferred to standard ½ - inch tape and can be viewed on V.T.R. units found in most university resource centres or libraries. A copy of the film is in my possesion and may be borrowed at any time by contacting me at the Mathematics Department, Richelieu Campus, Dawson College.)

After class I met with Kevin O'Connor to get his views on my teaching, classroom management and the students' reactions. He raised the following positive points:

(1) A serious learning environment had been evident during the Elementary Matrix session.

(2) There was good progression throughout the lesson; as time passed teacher and students became progressively more active and involved.

(3) My speaking style and tone were good; hands and other

bodily gestures were used effectively.

(4) There was a definite improvement over last year, particularly with regard to the added emphasis being placed on student involvement. [I had participated in a similar taping/filming session supervised by Mr. O'Connor in the Fall of '79, which took place in a class on College Mathematics.]

But, of course, other points were brought up when we looked at areas that might need improvement. In particular, Mr. O'Connor noted that:

(1) The teaching pace was sometimes quite rapid. Through the use of a few more, longer pauses to allow for reflection and comments by the students, the lesson might have been more effective.

(2) Many times the students' replies were given in a low, unsure tone. Possibly more support was needed here, or a greater effort to encourage the more reserved, less vocal members of the class to participate in the discussion.

(3) I sometimes had a tendency to answer my own questions, rather than using the questions to elicit responses from the class.

I have had the opportunity to view the film on two occasions. Unfortunately, I must admit that the element of student participation or discovery wasn't as prominent as I would have hoped. I still have no doubt that the use of guided discovery teaching to "point the way" towards theorems, general properties and other mathematical concepts can be a valuable pedagogical tool. But clearly there is ample room for refining or polishing my classroom presentation so that I may use this tool more effectively. For instance I would like to find a way in which more time could be spent <u>exploring incorrect responses</u> by students without having to drastically curtail the course syllabus. I will return to this and other issues under the heading "Future Plans", later in the Chapter.

<u>The Second Questionnaire</u>

About 2 weeks after the filming experiment had taken place I decided to distribute a brief questionnaire to the same group of students. I designed 3 very specific questions (having learned something, hopefully, from my earlier efforts) to which Kevin O'Connor added 3 excellent questions of his own. These 6 questions were to comprise The Second Questionnaire.

<u>My Questions</u>

(1) In presenting a topic such as Elementary Matrices it is possible to

(a) simply list the theorems and properties and then give a set of illustrative examples, OR

(b) help students formulate many of the rules themselves through observing patterns and relationships.

Did you find that my use of method (b) actually helped you to understand the fundamentals of Elementary Matrices? Which method, (a) or (b), would <u>you generally prefer</u> and why?

(2) Has your feeling about proofs <u>changed</u> since the beginning of the course? Explain.

(3) <u>In your own words</u>, briefly describe what prior information about elementary matrices, row equivalence, determinants, etc. is required

in order to prove the ~~Determinant-Product~~ Theorem [i.e. det AB = det A det B]. DO NOT REPRODUCE THE PROOF.

Kevin O'Connor's Questions

(4) Did you find that the rate or pace of the lessons is too fast, just right or too slow?

(5) How do you feel about answering questions in class?

(6) What recommendations can you make to help improve the class?

The Second Questionnaire was distributed by Kevin O'Connor. Unlike in the case of the First Questionnaire all students were asked to respond and to respond immediately after the Questionnaire had been given out.

Mr. O'Connor carefully explained to my students that the purpose of the questions was to obtain feed-back about my particular style of teaching. It was made clear that I was involved in a personal and completely voluntary project and that in no way were their responses to be used as a teacher evaluation by the Mathematics Department or Dawson College. This was done to put the students at ease and to try to solicit honest opinions and constructive criticism. All replies, furthermore, were to be anonymous. I returned to the classroom only after the responses had been collected by Mr. O'Connor.

Responses to Question (1)

Of the 32 students present 3 did not answer this question. Twenty-one of the remaining twenty-nine expressed a preference for the discovery method or method (b). Most of these 21 claimed that method (b) helped them get more involved in the lecture, and thus to understand the subject matter better. One rather exuberant student even wrote: "I only wish other teachers used your method!"* But the prize for originality surely goes to the individual who composed:

> "METHOD B IS LIKE BUILDING A HOUSE FROM ITS FOUNDATIONS WHILE METHOD A IS MORE LIKE BUILDING A CASTLE IN THE AIR."

I believe that in transcribing this inspired support of guided discovery teaching I'm justified in using capital letters!

The following quote represents quite well the views of those students who favoured the expository method of presentation:

> "It would be less confusing for me to understand given facts than to try to make up rules by myself."

Several people suggested that method (a) would use up less class time leaving more time for the presentation of examples.

I was obviously pleased that the discovery method was well received by the majority of the class. At the same time, my own view that guided discovery may <u>not</u> be the best method for <u>all</u> persons and under <u>all</u> occasions also received some confirmation. Some recommendations were made to keep using a <u>combination</u> of methods (a) and (b). This raises a challenge for future course planning: to find the optimal combination given certain class profiles (student attitudes and backgrounds) and the usual time constraints.

<u>Response to Question (2)</u>

Although eleven respondents felt that they were starting to feel a bit more comfortable with proofs my overall impression from studying all the responses was that the questions on "proofs" on assignments and

and especially on tests still generated a fair amount of anxiety. Over and over I read: "I need more time to get used to proofs." What I found interesting though, was that at least 6 students admitted that, as difficult as proofs may be, they were an important part of learning Mathematics. (This corresponds rather well to the results I obtained for Question (2) of the First Questionnaire.) About 5 people reported that their biggest problem had always been how to start proofs but that they had detected an improvement in their abilities to overcome this hurdle during the course of the term.

For some, of course, negative attitudes towards proofs had not changed. (There were approximately 8 in this group.) As one spokesman for this group aptly put it:

"My feelings about proofs haven't changed for years. I still hate them!"

I had not expected to help everyone overcome their malaise about proofs. All in all I'm now pretty certain that the active dialogue/discovery approach may be an important tool in bringing mathematics students and mathematical proofs closer together.

Responses to Question (3)

Only 6 students were able to describe the prerequisite knowledge one requires to prove the Determinant Product Theorem. The others left the question blank, in most cases. Three or four students made an unsuccessful attempt putting down one or two ideas, and then giving up. (One of them wrote: "If there was a test tomorrow I'd know what to do!") The class was clearly unprepared for this question. Was it because guided discovery teaching did not make it any easier for them to assimilate a long sequence of ideas? Or, was it because most students make no effort to review their notes unless they are threatened with a test? I really don't know.

Responses to Question (4)

Seven students felt that the class pace was too slow, three believed it was too fast, and twenty-two were satisfied with the pacing. A few of those who rated the lessons "too slow" added: "I would like more time to think about the things discussed in class." This lends further support to Kevin O'Connor's statement that I should allow a few extra moments for reflection, questions etc. However, on the whole, the responses indicate that my present pacing is not too far off the mark.

Responses to Question (5)

The majority of the class felt that answering questions was an important part of learning. Moreover, about 10 individuals declared that they liked answering questions and listening to the answers of others, for this way they could learn from their mistakes and the mistakes of others "right on the spot". Some reported that they felt more confident after having answered a few questions correctly.

Seven students were somewhat less enthusiastic about speaking up in class, saying that they "didn't mind provided they were sure that they had the right answer". Finally, 3 people had the courage to admit that they disliked answering questions, one of them explaining that he (she?) was just too shy. (Having viewed the "Elementary Matrix Lesson" film earlier and observed that only about 5 or 6 students regularly participated in the discussion, I'm fairly certain that these 3 spoke for a number of others who were too shy to say they were shy!)

Overall, I was definitely pleased with the responses. Comments such as:

"Answering questions in class is a good method to improve a student's confidence in himself in relation to his math abilities. Of course, no one should be forced to answer questions but rather should be encouraged to answer them. The emphasis on encouragement is handled well in class."

and

"The class has an open and friendly atmosphere so I don't mind answering questions. This type of atmosphere makes me feel comfortable among people I don't know."

were very encouraging.

Responses to Question (6)

In summary the following recommendations were made by the students (the number in brackets indicating the number of respondents):

(a) spend more time on solving problems in exercise sets. [6]

(b) find a good textbook. [6]

(c) put more emphasis on the applications of Linear Algebra. [8]

(d) put some bonus questions on tests. [5]

(e) give more people ("not just the usual smart group") a chance to answer questions. [9]

(f) have class reviews before tests. [3]

(g) allow extra time to think about harder problems. [3]

Points (a), (f), (g) all have to do with time or pacing the course. This issue had been brought up earlier and clearly will require serious consideration both on my part and the part of various course committees. Also, the Linear Algebra course committee will have to sit down and discuss the text situation since at least 6 students, in my class alone, had negative things to say about our present text. The questionnaire and the film has made me very aware of recommendation (e). Without a doubt I will have to concentrate on involving more students in my classes in dialogue or general discussion. Points (c) and (d) [on "applications" and "bonus questions"] should also be matters for the Linear Algebra course committee, since I believe that there should be a general policy discussion about these issues with the hope of achieving consistency among the various sections of the course.

At least a dozen students either had no suggestions to make or left the question blank.

Summary:

The kinds of questions posed and the way in which The Second Questionnaire was administered seemed to have made a difference. I now have a much better idea about the strengths and weaknesses in my teaching style. In particular, responses to such questions as (2), (5) and (6) have given me some direction or indication as to what aspects of my guided discovery teaching have been effective and what aspects can use some further polishing or improvement.

Section 2: Concluding Remarks

Triumphs and Pitfalls

Perhaps my use of the present tense in this thesis has tended to lend an aura of universal applicability to the technique of guided discovery teaching. However, by no means do I consider this mode of teaching to be a panacea for all pedagogical problems. For one thing, I have noted that the lack of sufficient class time can seriously effect one's teaching strategy; specifically, optimal use of the guided discovery method may require a great deal of extra time and one has to balance "method" with the overall requirements of a syllabus vis-a-vis "content". On the basis of some of the questionnaire results and the filming experiment in Linear Algebra it is very tempting to expand on the guided discovery feature, to involve more students and to re-work certain topics. But how much would "content" suffer? How many topics would have to be shortened or even eliminated? A truly frustrating dilemma!

Another possible drawback of the dialogue-discovery method is that some students may actually be intimidated by their very vocal or active classmates. Just when the exchange of ideas between the instructor and certain members of the class may be flowing beautifully, a few of the more reticent or slower students may be discovering that others are better at discovery! Moreover, the teacher will, most probably, not even be aware of such a counter-productive affect. (It takes considerable control on the part of a teacher to occasionally hold the agressive, vocal and "clever" types in check, in order to make it possible for those who don't usually participate to make their contribution. I know that I am often too eager to hear the right answer and thereby may inadvertently promote an excessive amount of showmanship by a few "smart" individuals.)

Sometimes what I may see as an excellent opportunity to generate dialogue and open the way to discovery may appear to some students as unnecessary stalling. In fact I have met a few individuals who showed considerable annoyance at my attempts to have them work towards the desired theorem or property. I have had great difficulty trying to convince these skeptics that there may be some advantage to withholding and building up to general results once in a while.

But in the final analysis, I feel that guided discovery sessions can be very successful. In the short time (approximately 2 years) that I had experimented with it I believe that I have garnered sufficient evidence to support this claim. The bulk of the responses to the Questionnaires, especially to the Second Questionnaire form part of this evidence. The decrease in the number of course withdrawals I've had (around 5 to 10 per cent less than 3 or 4 years ago) and the high class averages (in the order of 70-75 per cent, taking the 3 courses: Linear Algebra, Calculus, and College Mathematics, into account) are further evidence, I believe. The number of students coming into my office at the end of each term (at least 10-15 per term over the last couple of years) expressing their satisfaction with my teaching is surely another positive indicator.

I know that I have not developed the use of the guided discovery process to <u>my</u> ultimate satisfaction. All the better! I can look forward to continued experimentation with this exciting learning/teaching method.

<u>Future Plans</u>

As I had just stated there are areas directly and indirectly connected to guided discovery teaching which I wish to explore in the future. Some of these tentative projects are:

(a) to look for ways to get the shy or non-participating students involved in mathematical dialogue and discovery.

(b) to expand and refine my problem sets, incorporating more challenging and varied "Consolidation Exercises".

(c) to spend more time - probably in the unhurried and more personal environment of my office - following up unsuccessful student problem-solving strategies, finding out why students make certain types of unjustified assumptions, algebraic mistakes and so on.

(d) to find better ways of teaching "proofs".

(e) to investigate the role of "readiness" or "mathematical maturity" in learning proofs, abstract concepts and in devising problem-solving strategies.

(f) to develop better questionnaires and look for other sources of feed-back to gage student reaction to guided discovery teaching.

(g) to look for other topics which may be amenable to the guided discovery approach.

Prologue

Perhaps one day one of my former students will drop into my office and inform me that he has been teaching mathematics at a local educational institution. He may then relate how he has been trying to get his students involved in dialogue, in discovering some of the rules of Mathematics for themselves. That would be, for me, the ultimate form of career satisfaction.

REFERENCES

[1] Bruner, J.S. — On Knowing: Essays For The Left Hand Cambridge, Belknap, 1964, p. 82.

[2] Shulman, L.S. — Psychology and Mathematics Education., Nat. Soc. for the Study of Educ., Yearbook '69 (Part 1), Univ. of Chicago Press, 1970.

[3] Gagné, R.M. — Varieties of Learning and the Concept of Discovery., in Learning by Discovery; A Critical Appraisal., L.S. Shulman and E.R. Keislar (editors), Rand McNally and Co., 1966, pp. 146-150.

[4] Shulman, L.S. and Keislar, E.R. (editors) — Learning by Discovery: A Critical Appraisal., Rand McNally and Co., 1966.

[5] IBID., p. 130.

[6] Cronbach, L.J. — The Logic of Experiments on Discovery., in Learning by Discovery: A Critical Appraisal., pp. 76-92.

[7] Ausubel, D.P. — Learning by Discovery: rationale and mystique., Bull. nat. Assoc. Sec. Sch. Princ., Vol. 45 (1961 b), pp. 18-58.

[8] Bruner, J.S. — The Process of Education., New York, Vintage, Harvard Univ. Press, 1960.

[9] Cronbach, L.J. — The Logic of Experiments on Discovery, Op. Cit., p. 76.

[10] Davis, R.B. — Discovery in the Teaching of Mathematics., in Learning by Discovery: A Critical Appraisal. Op. Cit. pp. 114-128.

[11] IBID., p. 118-119.

[12] IBID., pp. 120-121.

[13] IBID., pp. 125-126.

[14] IBID., pp. 126-127.

[15] Logan, B. — On Children's Mathematics., Univ. of Ottawa Press, 1975.

[16] IBID., p. 12.

[17] IBID., pp. 13-25.

[18] IBID., p. 25.

[19] IBID., p. 26.

[20] IBID., pp. 55-65.

[21] IBID., p. 73.

[22] Hillel, J. Report of Working Group A. Proceedings of 1978 Conference, Canadian Math. Educ. Study Group, Kingston, Ontario, 1978.

[23] Polya, G. Mathematics and Plausible Reasoning., Princeton Univ. Press, Vol. I, 1973, p. 89.

[24] IBID., pp. 43-52.

[25] Crouse, R. Binomial Coefficients and the Partitioning of N-dimensional Space., The Mathematics Teacher, N.C.T.M. Publ., Nov. 1978, pp. 698-701.

[26] Polya, G. More on Guessing and Proving., Two-Year Coll. Math. Jnl., M.A.A. Publ., Sept. 1979, p. 257.

[27] Lakatos, I. Proofs and Refutations., Cambridge Univ. Press, 1976, p. 42.

[28] Beck, A. et al. Excursions into Mathematics., Worth Publ., 1972, pp. 421-434.

[29] Krutetskii, V.A. The Psychology of Mathematical Abilities in Schoolchildren. Univ. of Chicago Press, 1976, pp. 85, 88, 143-146, 187-189.

[30] Sauvé, L. et al. Crux Mathematicorum, Algonquin College Publ., Dec. 1978, pp. 293-295.

[31] Butts, T. Problem Solving In Mathematics., Scott, Foresman and Co., 1973, p.125.

[32] Hill, T.J. Mathematical Challenges II, Plus Six., N.C.T.M. Publ., 1974, p.11.

[33] Christy, D.T. Essentials of Pre-Calculus Mathematics., Harper and Row Publ., 1977, p. 311.

[34] Goldberg, J. and Swetz, F. Mathematics Examinations in the Soviet Union., The Mathematics Teacher, N.C.T.M. Publ., March, 1977, p. 213.

[35] Honsberger, R. — Ingenuity in Mathematics, M.A.A. Publ, Vol. 23 of the New Mathematical Library, 1977.

[36] Bissinger, B.H. — Stochastic Independence Versus Intuitive Independence., Two-Year College Math. Jnl., M.A.A. Publ., March, 1980, pp. 122-123.

[37] Dawson, J.W. — Wavefronts, Box Diagrams and The Product Rule: A Discovery Approach., Two-Year College Math. Jnl., M.A.A. Publ., March 1980, p. 106.

[38] Kemeny, J.G. et al. — Introduction to Finite Mathematics., Prentice-Hall, 1966, pp. 158-163.

[39] Mosteller, F. — Fifty Challenging Problems in Probability., Addison-Wesley, 1965, pp. 35-36.

[40] IBID., pp. 41-42.

BIBLIOGRAPHY

Ausubel, D.P. - Learning by Discovery: Rationale and Mystique., Bull. Nat. Assoc. Sec. Sch. Princ., 1961 b., pp. 18-58.

Beck, A. et al. Excursions into Mathematics., Worth Publ., 1972.

Bissinger, B.H. Stochastic Independence Versus Intuitive Independence., Two-Year Coll. Math. Jml., M.A.A. Publ., March 1980, p. 106.

Bruner, J.S. On Knowing: Essays For The Left Hand., Athenum Press, 1968.

Bruner, J.S. The Process of Education., Harvard Univ. Press, 1960.

Bruner, J.S. Some Elements of Discovery, in L.S. Shulman and E.R. Keislar (eds.), Learning by Discovery: A Critical Appraisal., Rand McNally & Co., 1966, pp. 101-113.

Butts, T. Problem Solving in Mathematics., Scott, Foresman and Co., 1973.

Charosh, M. Mathematical Challenges, N.C.T.M. Publ., 1965.

Craig, R.C. Recent Research on Discovery., Educational Leadership, Vol. 26, 1969, pp. 501-508.

Cronbach, L.J. The Logic of Experiments on Discovery., in L.S. Shulman and E.R. Keislar (eds.) Learning by Discovery: A Critical Appraisal, Rand McNally & Co., 1966, pp. 76-92.

Crouse, R. Binomial Coefficients and the Partitioning of N-dimensional Space., The Mathematics Teacher, N.C.T.M. Publ., Nov. 1978, pp. 698-701.

Cummins, R. A Student Experience-Discovery Approach to the Teaching of Calculus, in Readings from The Mathematics Teacher, N.C.T.M. Publ., 1977, pp. 31-39.

Davis, P.J. The Mathematics of Matrices., Xerox Publ., 1973.

Davis, R.B. Discovery in the Teaching of Mathematics, in L.S. Shulman and E.R. Keislar (eds.) Learning by Discovery: A Critical Appraisal, Rand McNally & Co., 1966, pp. 114-128.

Dawson, J.W. Wavefronts, Box Diagrams and the Product Rule: A Discovery Approach., Two-Year Coll. Maths. Jnl., M.A.A. Publ., March 1980, pp. 102-106.

Gagné, R.M. The Conditions of Learning., Holt, Rinehart & Winston, 1965.

Gagné, R.M. Varieties of Learning and The Concept of Discovery., in L.S. Shulman and E.R. Keislar (eds.), Learning by Discovery: A Critical Appraisal., Rand McNally and Co., 1966, pp. 135-150.

Guthrie, J.T. Expository Instruction Versus a Discovery Method., Jnl. of Educ. Psych., Vol. 63, 1967, pp. 45-59.

Hendrix, G. Learning by Discovery., The Mathematics Teacher., N.C.T.M. Publ., October 1961, pp. 290-299.

Hill, T.J. Mathematical Challenges II, Plus Six., N.C.T.M. Publ., 1974.

Hillel, J. Report of Working Group A., Proceedings of 1978 Conference, Can. Math. Educ. Study Group, Kingston, Ontario, 1978.

Honsberger, R. Ingenuity in Mathematics., Math. Assoc. of America, 1977.

Kane, R.B. On the Proof-Making Task., The Mathematics Teacher, N.C.T.M. Publ., Feb. 1975, pp. 89-94.

Kersh, B.Y. and Wittrock, M.C. Learning by Discovery: An Interpretation of Recent Research., Jnl. of Teacher Educ., Vol. 13, 1962, pp. 461-468.

Krutetskii, V.A. The Psychology of Mathematical Abilities in Schoolchildren., Univ. of Chicago Press, 1976.

Lakatos, I. Proofs and Refutations., Cambridge Univ. Press, 1976.

Logan, B. On Children's Mathematics., Univ. of Ottawa Press, 1975.

Mayer, R.E. Thinking and Problem Solving., Scott, Foresman & Co., 1977.

Mosteller, F. Fifty Challenging Problems in Probability., Addison-Wesley, 1965.

Nicolai, M.B. A Discovery in Linear Algebra., The Mathematics Teacher, N.C.T.M. Publ., May, 1974, pp. 403-404.

Olson, M.	Using Calculators to Stimulate Conjectures and Algebraic Proofs., The Mathematics Teacher, N.C.T.M. Publ., April 1979, pp. 288-289.
Partee, S. and Halsey, E.	Discovery Method Algebra at the University of Washington., Two-Year Coll. Maths. Jnl., M.A.A. Publ., Jan. 1977, pp. 27-29.
Polya, G.	How To Solve It., Doubleday Publ., 1957.
Polya, G.	Mathematical Discovery., Vols. 1 and 2, John Wiley & Sons, 1962 (Vol. 1), 1965 (Vol. 2).
Polya, G.	Mathematics and Plausible Reasoning., Vols. 1 and 2, Princeton Univ. Press, 1954.
Polya, G.	More on Guessing and Proving., Two-Year Coll. Math. Jnl., M.A.A. Publ., Sept. 1979, p. 257.
Posamentier, A.S. and Salkind, C.T.	Challenging Problems in Algebra 2., Macmillan Publ., 1970.
Roughead, W.G. and Scandura, J.M.	What is Learned in Mathematical Discovery., Jnl. of Educ. Psych., 1968, pp. 283-289.
Sawyer, W.W.	The Search for Pattern., Penguin Books, 1970.
Schaaf, O.	Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize., Doctoral dissertation, Ohio State Univ., 1954.
Shulman, L.S.	Psychology and Mathematics Education., Nat. Soc. for the Study of Educ., Yearbook 1969, Univ. of Chicago Press, 1970, pp. 23-71.
Shulman, L.S. & Keislar, E.R. (eds.)	Learning by Discovery: A Critical Appraisal., Rand McNally & Co., 1966.
Sloyer, C.W.	Algebra and Its Applications - A Problem Solving Approach., Addison-Wesley, 1970.
Stewart, I. & Tall, D.	The Foundations of Mathematics., Oxford Univ. Press, 1977.
Suchman, J.R.	Inquiry Training: Building Skills for Autonomous Discovery., Merrill-Palmer Quarterly of Behaviour Development, Vol. 7, 1961, pp. 147-169.
Wellen, D.	More Discoveries in Linear Algebra, The Mathematics Teacher, N.C.T.M. Publ., Jan. 1976, pp. 56-58.

Wickelgren, W.A. — How to Solve Problems., W.H. Freeman and Co., 1974.

Wittrock, M.C. — Verbal Stimuli in Concept Formation: Learning by Discovery., Jnl. of Educ. Psych. Vol. 54, 1963, pp. 183-190.

Worthen, B.R. — Discovery and Expository Task, Presentation in Elementary Mathematics., Jnl. of Educ. Psychology, Vol. 59, No. 1, 1968, pp. 1-13.