Inequalities Concerning Polynomials
in
Complex Domain

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ABSTRACT

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Let $P(z)$ be a polynomial of degree n in complex domain and $P'(z)$ be its derivative. The study of the inequalities concerning $|P'(z)|$ on $|z| \leq 1$ had been introduced by S. Bernstein in 1912. Since then, a number of related results and their generalizations in different directions have been obtained in this area. There are also several interesting inequalities for the polynomials having one zero prescribed, namely $P(1) = 0$. For this class of polynomials, the precise estimate of $|P'(z)|$ on $|z| \leq 1$ is still unknown.

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INTRODUCTION

Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n in complex domain; the coefficients a_ν, $\nu=1,\ldots,n$ are complex. By $P'(z)$ we denote the derivative of $P(z)$. Concerning the estimate of $|P'(z)|$ on the unit disk $|z| \le 1$, the following result is known as:

Bernstein's Theorem: Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $\max_{|z|=1} |P(z)| = 1$, then

$$|P'(z)| \le n$$

for $|z| \le 1$. The result is best possible and the equality holds if and only if $P(z) = \alpha z^n$, $|\alpha| = 1$.

In Chapter I, we deal with a discussion on Bernstein's Theorem. There we present a proof of the Theorem due to M. Reisz deduced from a result on the derivative of a trigonometric polynomial $t(\theta) = \sum_{\nu=0}^{n} (a_\nu \cos\nu\theta + b_\nu \sin\nu\theta) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ of order n. We also give a simple proof of Bernstein's Theorem. Next, we study a generalization of Bernstein's Theorem and some related results.

With regard to Chapter II, we study:

Turan's Theorem: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2}$$

The result is best possible and equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| = \frac{1}{2}$

The interest in this Theorem lies in the fact that it describes $|P'(z)|$ under the influence of zeros of $P(z)$. We discuss generalizations of this Theorem in two different directions.

The influence of zeros on the estimate of different norm of a polynomial is a study of considerable interest. During the past thirty years, a number of interesting and significant results have been obtained when only one of the zeros of the polynomial is prescribed. In Chapter III, we study some of the results inspired by the following:

<u>Callahan's Theorem</u>: *If* $P(z)$ *is a polynomial of degree* n *with* $P(1) = 0$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^2 d\theta \le \frac{n}{n+1} \max_{|z|=1} |P(z)|^2$$

The result is best possible.

Chapter IV is primarily concerned with the estimate of $|P'(z)|$ when $P(1) = 0$. The main result we study here is:

<u>Giroux and Rahman Theorem</u>: *Let* P_n *be the class of all polynomials* $\{P(z)\}$ *of degree at most* n *with* $\max_{|z|=1} |P(z)| = 1$ *and* $P(1) = 0$. *Then there exists an absolute constant* $c > 0$ *such that*

$$\max_{P(z)\varepsilon P_n} \left(\max_{|z|=1} |P'(z)|\right) \ge n - \frac{c}{n}$$

The proof of the above Theorem does not reflect how small the constant 'c' could be expected. Without much success, we have made some computer calculation to observe the $\max_{|z|=1} |P'(z)|$ when $P(1) = 0$ for polynomials of degree $n = 2,3,4$. This will be discussed in Chapter V.

The presentations in this thesis base primarily on the known results concerning the inequalities of polynomials. Whenever a result is of our own, it is marked by an asterisk *, e.g. Theorem 1.7(*).

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Chapter I

Bernstein's Theorem

1.1

Each of the following two theorems are known as Bernstein's Theorem:

Theorem 1.1: Let $P(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be a polynomial of degree n and $\max_{|z|=1} |P(z)|=1$, then

$$|P'(z)| \le n \qquad (1.1)$$

for $|z| \le 1$. The result is best possible and the equality in (1.1) holds if and only if $P(z) = \alpha z^n$, $|\alpha| = 1$.

Theorem 1.2: Let $t(\theta) = \sum_{\upsilon=0}^{n} (a_\upsilon \cos \upsilon\theta + b_\upsilon \sin \upsilon\theta) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}$ be a trigonometric polynomial of order n and $\max_\theta |t(\theta)| = 1$, then

$$|t'(\theta)| \le n \qquad (1.2)$$

for all θ. The result is best possible and the equality in (1.2) holds if $t(\theta) = \lambda \sin n(\theta - \varphi)$, $|\lambda| = 1$, φ is real. Further, if $t(\theta)$ is a real trigonometric polynomial and there is equality in (1.2), for some θ, then $t(\theta) = \lambda \sin n(\theta - \varphi)$, λ and φ are real, $|\lambda| = 1$.

The sentence "the result is best possible" means that the inequality cannot be further refined and there is a polynomial satisfying the hypothesis of the Theorem for which there is equality, such a polynomial is called an extremal polynomial. So, $P(z) = \alpha z^n$, $|\alpha| = 1$, is an extremal polynomial for the inequality (1.1) and $t(\theta) = \lambda \sin n(\theta - \varphi)$, $|\lambda| = 1$, λ and φ are real is an extremal polynomial for the inequality (1.2).

In 1912, Bernstein [4] considered the problem of estimating $|t'(\theta)|$ when the trigonometric polynomial $t(\theta)$ in its absolute value is bounded by one and proved that $|t'(\theta)| \leq 2n$; for details see [9]. He deduced this result from the following two Theorems:

Theorem 1.3: Let $P(x) = \sum_{\upsilon=0}^{n} a_\upsilon x^\upsilon$ be a polynomial of degree n and $\max_{-1 \leq x \leq 1} |P(x)| = 1$, then

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}} \tag{1.3}$$

for $-1 < x < 1$. The result is best possible and there is equality in (1.3) for Chebysev polynomial $T(x) = \cos n \arccos x$ at $x = \cos \frac{2k+1}{2n}\pi$, $k = 0, \ldots, 2n-1$.

Theorem 1.4: Let $P(x) = \sum_{\upsilon=0}^{n} a_\upsilon x^\upsilon$ be a polynomial of degree n and $\max_{-1 \leq x \leq 1} |P(x)\sqrt{1-x^2}| = 1$, then

$$|[P(x)\sqrt{1-x^2}\,]'| \leq \frac{n+1}{\sqrt{1-x^2}} \tag{1.4}$$

for $-1 < x < 1$. The result is best possible and there is equality in (1.4) for $P(x) = \frac{\sin(n+1)\arccos x}{\sqrt{1-x^2}}$ at $x = \cos\frac{k\pi}{n+1}$, $k = 0, 1, \ldots, 2n+1$.

The observation due to Bernstein attracted the attention of other mathematicians, in particular on proving the best possible result on the derivative of trigonometric polynomial. In 1914, M. Riesz [20] was the first one in establishing Theorem 1.2 and from where he also deduced Theorem 1.1 (see page 12). In the literature, these two Theorems are also referred as Riesz Theorem or Bernstein-Riesz Theorem. Around the same time, F. Riesz and de La Vallée Poussin also gave a proof of Theorem 1.2. Recently, Boas revisited a proof of Theorem 1.2

relying heavily on geometrical considerations in his expositary article [7]. The idea seems to be introduced by Erdös [11]. It is interesting to note that the method presented by Boas can be easily used in proving the following inequality due to Vander Corput and Schaake [22] which is better than (1.2) in the case when the trigonometric polynomial is real:

Theorem 1.5: Let $t(\theta) = \sum_{\upsilon=0}^{n} (a_\upsilon \cos \upsilon\theta + b_\upsilon \sin \upsilon\theta)$ be a real trigonometric polynomial of order n and $\max_\theta |t(\theta)| = 1$, then

$$\{t'(\theta)\}^2 + n^2\{t(\theta)\}^2 \le n^2 \tag{1.5}$$

The result is best possible and there is equality in (1.5) if $t(\theta) = \lambda \sin n(\theta - \varphi)$, $|\lambda| = 1$. Furthermore if there is equality in (1.5) where $t'(\theta) \neq 0$, for some θ, then $t(\theta) = \lambda \sin n(\theta - \varphi)$, $|\lambda| = 1$.

Remark 1.1: If one writes a real trigonometric polynomial $t(\theta) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}$, then $c_{-\upsilon} = \overline{c_\upsilon}$, $\upsilon = 1,\ldots,n$

In 1930, Bernstein [5] proved the following interesting generalization of Theorem 1.1:

Theorem 1.6: Let Q(z) be a polynomial of degree n having all its zeros in $|z| \le 1$ and P(z) be another polynomial of degree not exceeding n. If for $|z| = 1$

$$|P(z)| \le |Q(z)|$$

then

$$|P'(z)| \le |Q'(z)| \tag{1.6}$$

on $|z| = 1$.

In this chapter, we first discuss the proof of Theorem 1.2 and 1.1 due to M. Riesz, and also give the proof of Theorem 1.2 due to Boas, [7] in a modified form from where Theorem 1.5 is derived. We present a new proof of Theorem 1.1. Following the technique used here, we prove a new result (Theorem 1.7) which is an improvement upon Theorem 1.1; from where Theorem 1.2 is also deduced. This is interesting in view of the fact that it is not yet known whether Theorem 1.2 can be deduced from Theorem 1.1. Moreover, we also prove an inequality (1.27) which is more refined than (1.6) and discuss some further consequences.

1.2

Now, we return to present the proof of Theorem 1.2 due to M. Riesz.*

In order to prove Theorem 1.2, M. Riesz first established an interpolation formula as in the following:

Lemma 1.1: Let $t(\theta) = \sum_{\upsilon=0}^{n} (a_\upsilon \cos \upsilon\theta + b_\upsilon \sin \upsilon\theta)$ be a trigonometric polynomial of order n, then

$$t(\theta) = a_n \cos n\theta + \frac{\cos n\theta}{2n} \sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2} \qquad (1.7)$$

where $\theta_1 = \frac{\pi}{2n}$, $\theta_2 = \frac{3\pi}{2n}$, ..., $\theta_r = \frac{(2r-1)\pi}{2n}$, ..., $\theta_{2n} = \frac{(4n-1)\pi}{2n}$

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* We would like to add here that this work of M. Riesz is in German. To our request, Mr. L. Marcoux has translated this work into English for the purpose of this thesis and our presentations rely on his translation.

(i.e. $e^{i\theta_r}$ is the $2n^{th}$ root of -1).

Proof of lemma 1.1: First, we establish the following:

$$(-1)^r \cos n\theta \cot \frac{\theta-\theta r}{2} = \sin n(\theta-\theta_r)\cot\frac{\theta-\theta r}{2}$$

$$= \frac{\sin\,[(n+\frac{1}{2})(\theta-\theta_r)]}{\sin\frac{\theta-\theta r}{2}} - \cos n(\theta-\theta_r) \qquad (1.8)$$

In fact,

$$\sin n(\theta-\theta_r) = \sin n\theta\cos n\theta_r - \cos n\theta \sin n\theta_r$$

$$= \sin n\theta\cos\frac{(2r-1)\pi}{2} - \cos n\theta\sin\frac{(2r-1)\pi}{2}$$

$$= (-1)^r \cos n\theta$$

This proves

$$(-1)^r \cos n\theta\cot\frac{\theta-\theta r}{2} = \sin n(\theta-\theta_r)\cot\frac{\theta-\theta r}{2}.$$

Now, consider

$$\sin n(\theta-\theta_r)\cot\frac{\theta-\theta r}{2}$$

$$= \sin n(\theta-\theta r)\frac{\cos\frac{\theta-\theta r}{2}}{\sin\frac{\theta-\theta r}{2}}$$

$$= \frac{\frac{1}{2}\{\sin[n(\theta-\theta_r)+\frac{\theta-\theta r}{2}]+\sin[n(\theta-\theta_r)-\frac{\theta-\theta r}{2}]\}}{\sin\frac{\theta-\theta r}{2}}$$

$$= \frac{\frac{1}{2}\{\sin[(n+\frac{1}{2})(\theta-\theta r)]+\sin n(\theta-\theta r)\cos\frac{\theta-\theta r}{2} - \cos n(\theta-\theta r)\sin\frac{\theta-\theta r}{2}\}}{\sin\frac{\theta-\theta r}{2}}$$

$$= \frac{1}{2}\,\frac{\sin[(n+\frac{1}{2})(\theta-\theta_r)]}{\sin\frac{\theta-\theta_r}{2}} + \frac{1}{2}\sin n(\theta-\theta_r)\cot\frac{\theta-\theta_r}{2} - \frac{1}{2}\cos n(\theta-\theta_r)$$

Transposing $\frac{1}{2}\sin n(\theta-\theta_r)\cot\frac{\theta-\theta_r}{2}$ to the left, one gets (1.8).

By using the fact

$$\frac{\sin[(n+\frac{1}{2})(\theta-\theta_r)]}{\sin\frac{\theta-\theta_r}{2}} = 2\{\frac{1}{2} + \cos(\theta-\theta_r)+\ldots+\cos n(\theta-\theta_r)\}$$

in (1.8), we get

$$(-1)^r\cos n\theta\cot\frac{\theta-\theta_r}{2} = 2\{\frac{1}{2}+\cos(\theta-\theta_r)+\ldots+\cos n(\theta-\theta_r)\}-\cos n(\theta-\theta_r).$$

Now, we return to show (1.7). Consider

$$\frac{\cos n\theta}{2n}\sum_{r=1}^{2n} t(\theta_r)(-1)^r\cot\frac{\theta-\theta_r}{2}$$

$$= \frac{1}{2n}\sum_{r=1}^{2n} t(\theta_r)(-1)^r\cos n\theta\cot\frac{\theta-\theta_r}{2}$$

$$= \frac{1}{2n}\sum_{r=1}^{2n} t(\theta_r)\{2\,[\frac{1}{2}+\cos(\theta-\theta_r)+\ldots+\cos n(\theta-\theta_r)]-\cos n(\theta-\theta_r)\} \qquad (1.9)$$

Indeed, for $0 \le k \le n$

$$\sum_{r=1}^{2n} 2t(\theta_r)\cos k(\theta-\theta_r)$$

$$= \sum_{r=1}^{2n}\sum_{\upsilon=0}^{n} 2(a_\upsilon\cos\upsilon\theta_r + b_\upsilon\sin\upsilon\theta_r)\cos k(\theta-\theta_r)$$

$$= \sum_{r=1}^{2n}\sum_{\upsilon=0}^{n} [2a_\upsilon\cos\upsilon\theta_r\cos k(\theta-\theta_r) + 2b_\upsilon\sin\upsilon\theta_r\cos k(\theta-\theta_r)]$$

$$= \sum_{r=1}^{2n}\sum_{\upsilon=0}^{n} \{a_\upsilon\cos[(\upsilon-k)\theta_r+k\theta] + a_\upsilon\cos[(\upsilon+k)\theta_r - k\theta]$$
$$+ b_\upsilon\sin[(\upsilon-k)\theta_r+k\theta] + b_\upsilon\sin[(\upsilon+k)\theta_r - k\theta]$$

$$= \sum_{\upsilon=0}^{n} \{a_\upsilon \mathrm{Re} \sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} e^{ik\theta} + a_\upsilon \mathrm{Re} \sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r} e^{-ik\theta}$$

$$+ b_\upsilon \mathrm{Im} \sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} e^{ik\theta} + b_\upsilon \mathrm{Im} \sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r} e^{-ik\theta}\}$$

$$= \sum_{\upsilon=0}^{n} \{a_\upsilon \mathrm{Re}\, e^{ik\theta} \sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} + a_\upsilon \mathrm{Re}\, e^{-ik\theta} \sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r}$$

$$+ b_\upsilon \mathrm{Im}\, e^{ik\theta} \sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} + b_\upsilon \mathrm{Im}\, e^{-ik\theta} \sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r}\}$$

In the above $\sum_{\upsilon=0}^{n}$, we split the sum into three parts, the first part consists of $\upsilon=0$, the second one consists of $1 \le \upsilon \le n-1$ and the third one consists of $\upsilon=n$.

i) For $\upsilon=0$, consider

$$\sum_{r=1}^{2n} a_0 \cos k(\theta-\theta_r) = a_0 \mathrm{Re}\, e^{ik\theta} \sum_{r=1}^{2n} e^{-ik\theta_r}$$

If $k \neq 0$, by the fact that θ_r is $2n^{\underline{th}}$ root of -1 and r runs from 1 to 2n, we have $\sum_{r=1}^{2n} e^{-ik\theta_r} = 0$. On the other hand, if $k = 0$, we have $\sum_{r=1}^{2n} e^{-ik\theta_r} = 2n$. Hence

$$\sum_{r=1}^{2n} a_0 \cos k(\theta-\theta_r) = 0 \qquad \text{for } k \neq 0$$

$$= 2n\, a_0 \qquad \text{for } k = 0$$

ii) For $1 \le \upsilon \le n-1$, if $\upsilon \neq k$, then $(\upsilon-k)$ and $(\upsilon+k)$ are integers which are not equal to zero, so we have $\sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} = 0$ and $\sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r} = 0$. Also, we note that if $\upsilon = k$, $\sum_{r=1}^{2n} e^{i(\upsilon-k)\theta_r} = 2n$ and $\sum_{r=1}^{2n} e^{i(\upsilon+k)\theta_r} = \sum_{r=1}^{2n} e^{i2k\theta_r} = 0$. Hence, $1 \le k \le n-1$

$$\sum_{\upsilon=1}^{n-1}\sum_{r=1}^{2n}(a_\upsilon\cos\upsilon\theta_r + b_\upsilon\sin\upsilon\theta_r)\cos k(\theta-\theta_r)$$

$$= 2n(a_k\cos k\theta + b_k\sin k\theta)$$

iii) For $\upsilon = n$, consider

$$\sum_{r=1}^{2n}(a_n\cos n\theta_r + b_n\sin n\theta_r)\cos k(\theta-\theta_r)$$

$$= \frac{1}{2}\{a_n \operatorname{Re} e^{ik\theta}\sum_{r=1}^{2n} e^{i(n-k)\theta_r} + a_n \operatorname{Re} e^{-ik\theta}\sum_{r=1}^{2n} e^{i(n+k)\theta_r}$$

$$+ b_n \operatorname{Im} e^{ik\theta}\sum_{r=1}^{2n} e^{i(n-k)\theta_r} + b_n \operatorname{Im} e^{-ik\theta}\sum_{r=1}^{2n} e^{i(n+k)\theta_r}\} \quad (1.10)$$

If $k\neq n$, then $\sum_{r=1}^{2n} e^{i(n-k)\theta_r} = 0$ and $\sum_{\upsilon=1}^{2n} e^{i(n+k)\theta_r} = 0$;

hence (1.10) equals to zero. If $k = n$, we rewrite (1.10) as

$$\sum_{r=1}^{2n}[a_n\cos n\theta_r + b_n\sin n\theta_r]\cos n(\theta-\theta_r)$$

$$= \sum_{r=1}^{2n}\frac{1}{2}\{a_n[\cos n\theta + \cos(2n\theta_r - n\theta)] + b_n[\sin n\theta + \sin(2n\theta_r - n\theta)]\}$$

$$= \sum_{r=1}^{2n}\frac{1}{2}\{a_n[\cos n\theta + \cos(2r-1)\pi\cos n\theta + \sin(2r-1)\pi\sin n\theta]$$

$$+ b_n[\sin n\theta + \sin(2r-1)\pi\cos n\theta - \cos(2r-1)\pi\sin n\theta]\}$$

$$= \sum_{r=1}^{2n}\frac{1}{2}\{a_n[\cos n\theta + (-1)\cos n\theta] + b_n[\sin n\theta - (-1)\sin n\theta]\}$$

$$= 2n\,b_n\sin n\theta$$

From i), ii), iii) and (1.19), we have

$$a_n\cos n\theta + \frac{\cos n\theta}{2n}\sum_{r=1}^{2n} t(\theta_r)(-1)^r\cot\frac{\theta-\theta_r}{2}$$

$$= a_n\cos n\theta + \frac{1}{2n}\{2na_0 + 2n\sum_{\upsilon=1}^{n-1}(a_\upsilon\cos\upsilon\theta + b_\upsilon\sin\upsilon\theta) + 2n\,b_n\sin n\theta\}$$

$$= a_0 + \sum_{\upsilon=1}^{n} (a_\upsilon \cos \upsilon\theta + b_\upsilon \sin \upsilon\theta)$$

$$= t(\theta)$$

This proves lemma 1.1.

Proof of Theorem 1.2: By lemma 1.1, we have

$$t(\theta) = a_n \cos n\theta + \frac{\cos n\theta}{2n} \sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2}$$

Differentiating this formula, one gets

$$t'(\theta) = -n\, a_n \sin n\theta + \left[\sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2}\right] \frac{d}{d\theta}\left(\frac{\cos n\theta}{2n}\right) +$$

$$\left(\frac{\cos n\theta}{2n}\right)\frac{d}{d\theta}\left[\sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2}\right]$$

$$= -n\, a_n \sin n\theta + \left[\sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2}\right]\left(-\frac{n \sin n\theta}{2n}\right) +$$

$$\frac{\cos n\theta}{2n}\left[\sum_{r=1}^{2n} t(\theta_r)(-1)^r(-1)\left(\csc^2 \frac{\theta-\theta_r}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$= -n\, a_n \sin n\theta - \frac{\sin n\theta}{2} \sum_{r=1}^{2n} t(\theta_r)(-1)^r \cot \frac{\theta-\theta_r}{2}$$

$$+ \frac{\cos n\theta}{2n} \sum_{r=1}^{2n} \frac{(-1)^{r+1} t(\theta_r)}{2\sin^2 \frac{\theta-\theta_r}{2}}$$

For $\theta=0$ in the above, one has

$$t'(0) = \frac{1}{2n} \sum_{r=1}^{2n} \frac{(-1)^{r+1} t(\theta_r)}{2\sin^2 \frac{\theta_r}{2}} \qquad (1.11)$$

Setting $t(\theta)=\sin n\theta$, we have

$$t(\theta_r) = \sin n\theta_r = \sin \frac{(2r-1)\pi}{2} = (-1)^{r-1}$$

Put it back into (1.11), then

$$t'(0) = \frac{1}{2n}\sum_{r=1}^{2n}\frac{(-1)^{r+1}(-1)^{r-1}}{2\sin^2\frac{\theta_r}{2}}$$

$$= \frac{1}{2n}\sum_{r=1}^{2n}\frac{1}{2\sin^2\frac{\theta_r}{2}}$$

As $t'(\theta) = n\cos n\theta$, $t'(0) = n$. Hence, we get

$$\frac{1}{2n}\sum_{r=1}^{2n}\frac{1}{2\sin^2\frac{\theta_r}{2}} = n \qquad (1.12)$$

Now, if we let

$$T(\varphi) = a_n\cos n\varphi + \frac{\cos n\varphi}{2n}\sum_{r=1}^{2n} T(\theta_r)(-1)^r \cot\frac{\varphi-\theta_r}{2}$$

then $t(\theta+\varphi) = T(\varphi)$ is also a trigonometric polynomial and

$t'(\theta+\varphi) = T'(\varphi)$; for $\varphi = 0$, $t'(\theta) = T'(0)$. Hence

$$t'(\theta) = \frac{1}{2n}\sum_{r=1}^{2n}\frac{(-1)^{r+1}\,t(\theta+\theta_r)}{2\sin^2\frac{\theta_r}{2}} \qquad (1.13)$$

Since $\max_\theta|t(\theta)| = 1$, for fixed value of θ and for all values of θ_r, we have $|t(\theta+\theta_r)| \le 1$, then for this fixed value of θ

$$|t'(\theta)| = \left|\frac{1}{2n}\sum_{r=1}^{2n}\frac{(-1)^{r+1}\,t(\theta+\theta_r)}{2\sin^2\frac{\theta_r}{2}}\right|$$

$$\le \frac{1}{2n}\sum_{r=1}^{2n}\frac{|t(\theta+\theta_r)|}{2\sin^2\frac{\theta_r}{2}}$$

$$\le \frac{1}{2n}\sum_{r=1}^{2n}\frac{1}{2\sin^2\frac{\theta_r}{2}}$$

$$= n$$

This proves (1.2).

To show that the result is best possible, we observe that the equality holds for $t(\theta) = \lambda \sin n(\theta-\varphi)$, $|\lambda| = 1$, φ is real because $t'(\theta) = \lambda n \cos n(\theta-\varphi)$, $\max_\theta |t(\theta)| = 1$ and $\max_\theta |t'(\theta)| = n$.

So far there is no restriction that $t(\theta)$ must be a real trigonometric polynomial. In the case when $t(\theta)$ is a real trigonometric polynomial, we note that the equality holds only for $t(\theta) = \lambda \sin n(\theta-\varphi)$, for some $|\lambda| = 1$, λ and φ are real. i.e. if there is equality in (1.2), then $t(\theta) \equiv \lambda \sin n(\theta-\varphi)$, for some $|\lambda| = 1$, λ and φ are real. Now, let $t(\theta)$ be such a trigonometric polynomial satisfying $\max_\theta |t(\theta)| = 1$ and $\max_\theta |t'(\theta)| = n$. If $|t'(\theta)|$ attains its maximum at $\theta=\varphi$, i.e. $|t'(\varphi)| = n$, then from (1.13), we get

$$|t'(\varphi)| = \left| \frac{1}{2n} \sum_{r=1}^{2n} \frac{(-1)^{r+1} t(\varphi+\theta_r)}{2\sin^2 \frac{\theta_r}{2}} \right| = n$$

But in view of (1.12), it is possible only if

(i) $(-1)^{r+1} t(\varphi+\theta_r) = 1$ for $r=1,2,\ldots,2n$

or (ii) $(-1)^{r+1} t(\varphi+\theta_r) = -1$ for $r=1,2,\ldots,2n$.

In case (i), if follows that the trigonometric polynomial has the property $t(\theta) = (-1)^{r+1} = (-1)^{r-1}$ at $\theta=\varphi+\theta_r$, $r=1,\ldots,2n$. The function $\sin n(\theta-\varphi)$ has the same property at $\theta=\varphi+\theta_r$, $r=1,\ldots,2n$. Hence, the trigonometric polynomial $S(\theta) = t(\theta) - \sin n(\theta-\varphi)$ equals to zero at all points $\theta=\varphi+\theta_r$, $r=1,\ldots,2n$. Further, we note that $S'(\theta)$ also has zeros at these points. In fact, $t(\theta)$ has alternating relative maximum $+1$ or minimum -1 at all points $\theta=\varphi+\theta_r$, $r=1,\ldots,2n$. So

$$\begin{aligned} S'(\theta) &= t'(\theta) - n\cos n(\theta-\varphi) \\ &= t'(\varphi+\theta_r) - n\cos n\theta_r \\ &= 0 \end{aligned}$$

Hence $S(\theta)$ has 4n zeros. Since a trigonometric polynomial of order n cannot have more than 2n zeros. We conclude that $S(\theta)\equiv 0$ or $t(\theta) = \sin n(\theta-\varphi)$. In case (ii), we get $t(\theta) = -\sin n(\theta-\varphi)$. Hence, $t(\theta) = \lambda \sin n(\theta-\varphi)$, $|\lambda| = 1$, λ and φ are real.

Further, M. Riesz continued to deduce Theorem 1.1 as following:

Proof of Theorem 1.1: Let $P(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ be a polynomial of degree n with complex coefficients and $\max_{|z|=1} |P(z)| = 1$. For $z = e^{i\theta}$

$$P(e^{i\theta}) = \sum_{\upsilon=0}^{n} c_\upsilon e^{i\upsilon\theta} = t(\theta)$$

is a trigonometric polynomial of order n and so we have $|t'(\theta)| \le n$. Also, note that for $|z| = 1$

$$\begin{aligned} |P'(z)| &= \left|\sum_{\upsilon=0}^{n} \upsilon c_\upsilon z^{\upsilon-1}\right| \\ &= \left|ie^{i\theta}\sum_{\upsilon=0}^{n} \upsilon c_\upsilon e^{i(\upsilon-1)\theta}\right| \\ &= \left|\sum_{\upsilon=0}^{n} i\upsilon c_\upsilon e^{i\upsilon\theta}\right| \\ &= |t'(\theta)| \\ &\le n \end{aligned}$$

Hence, (1.1) is proved.

Now, for $P(z) = \alpha z^n$, $|\alpha| = 1$, there is equality in (1.1) because $\max_{|z|=1} |P'(z)| = \max_{|z|=1} |\alpha n z^{n-1}| = n$.

Further, we shall show that if there is equality in (1.1), then $P(z) = \alpha z^n$, $|\alpha|=1$. Suppose at $z = e^{i\theta^*}$, $|P'(e^{i\theta^*})| = n$. Let $\gamma = \arg P'(e^{i\theta^*})$, then we have $e^{-i\gamma}P'(e^{i\theta^*}) = n$.

Next, consider the real trigonometric polynomial $t(\theta) = \operatorname{Im} e^{-i(\gamma+\theta^*)}P(e^{i\theta})$. It is of order n and $|t(\theta)| \le 1$. Since

$$t'(\theta^*) = \operatorname{Im}\{i\, e^{-i(\gamma+\theta^*)}e^{i\theta^*}P'(e^{i\theta^*})\}$$
$$= e^{-i\gamma}P'(e^{i\theta^*})$$
$$= n$$

by Theorem 1.2, we know that the equality holds only for $t(\theta) = \lambda \sin n(\theta-\varphi)$, λ and φ are real, $|\lambda| = 1$. Hence, let $\gamma'=\gamma+\theta^*$, one gets

$$\operatorname{Im}\{e^{-i\gamma'}P(e^{i\theta}) - \lambda e^{in(\theta-\varphi)}\} = 0.$$

from where

$$\operatorname{Im}\{e^{-i\gamma'}P(z) - \lambda e^{-in\varphi}z^n\} = 0$$

for $|z| = 1$. Since $\operatorname{Im}\{e^{-i\gamma'}P(z) - \lambda e^{-in\varphi}z^n\}$ is a harmonic function vanishing on $|z| = 1$, from the mean-value property, $\operatorname{Im}\{e^{-i\gamma'}P(z) - \lambda e^{-in\varphi}z^n\} \equiv 0$ for $|z| \le 1$. Thus, $e^{-i\gamma'}P(z) - \lambda e^{-in\varphi}z^n = c$ (constant) for $|z| \le 1$ and by uniqueness Theorem for all z. So, $P(z) = \alpha z^n + c e^{i\gamma'}$, where $\alpha = \lambda e^{-in\varphi}e^{i\gamma'}$. As $\max_{|z|=1}|P(z)|=1$,

c must be zero. Consequently, $P(z) = \alpha z^n$, $|\alpha|=1$.

1.3

Now, we present a new and simple proof of Theorem 1.1. We also establish an inequality by using that method employed here and from where Theorem 1.2 is deduced.

We need the following:

<u>Lemma 1.2</u>: <u>If</u> $R(z)$ <u>is a polynomial of degree</u> n <u>having all its zeros in</u> $|z| \le 1$, <u>then</u>

$$|z\, R'(z)| \geq \frac{n}{2}\, |R(z)| \tag{1.14}$$

for $|z| = 1$.

Proof of lemma 1.2: Let the zeros of $R(z)$ be $z_\upsilon = \rho_\upsilon e^{i\theta_\upsilon}$, $0 \leq \rho_\upsilon \leq 1$, $\upsilon = 1,\ldots,n$. For $|z| = 1$, we get

$$\begin{aligned}
\left|z\frac{R'(z)}{R(z)}\right| &= \left|e^{i\theta}\frac{R'(e^{i\theta})}{R(e^{i\theta})}\right| \\
&\geq \left|\mathrm{Re}\sum_{\upsilon=1}^{n}\frac{e^{i\theta}}{e^{i\theta}-\rho_\upsilon e^{i\theta_\upsilon}}\right| \\
&= \left|\mathrm{Re}\sum_{\upsilon=1}^{n}\frac{e^{i\theta}(e^{-i\theta}-\rho_\upsilon e^{-i\theta_\upsilon})}{(e^{i\theta}-\rho_\upsilon e^{i\theta_\upsilon})(e^{-i\theta}-\rho_\upsilon e^{-i\theta_\upsilon})}\right| \\
&= \left|\mathrm{Re}\sum_{\upsilon=1}^{n}\frac{1-\rho_\upsilon e^{i(\theta-\theta_\upsilon)}}{1-2\rho_\upsilon\cos(\theta-\theta_\upsilon)+\rho_\upsilon^2}\right| \\
&\geq \left|\sum_{\upsilon=1}^{n}\frac{1-\rho_\upsilon\cos(\theta-\theta_\upsilon)}{2[1-\rho_\upsilon\cos(\theta-\theta_\upsilon)]}\right| \\
&= \sum_{\upsilon=1}^{n}\frac{1}{2} \\
&= \frac{n}{2}
\end{aligned}$$

This proves (1.14).

Proof of Theorem 1.1(*): Let $P(z)$ be a polynomial of degree n and $\max_{|z|=1}|P(z)|=1$. So $\left|\frac{P(z)}{z^n}\right| \leq 1$ on $|z| = 1$. Moreover, for any $|\alpha| > 1$, we have

$$|P(z)| < |\alpha z^n|$$

on $|z| = 1$. Let $R(z) = P(z) - \alpha z^n$. By Rouche's Theorem, $R(z)$ has as many zeros in $|z| \leq 1$ as αz^n. Further, there is no zero of $P(z) - \alpha z^n$ on $|z| = 1$. Hence, $R(z)$ has all its zeros in $|z| < 1$ when $|\alpha| > 1$.

From lemma 1.2, we have

$$|zP'(z) - n\alpha z^n| \geq \frac{n}{2}|P(z) - \alpha z^n| \quad (1.15)$$

on $|z| = 1$.

Suppose, on contrary, that there exists z^*, $|z^*| = 1$ such that $|P'(z^*)| > n$. From (1.15), we have

$$|z^*P'(z^*) - \alpha n z^{*n}| \geq \frac{n}{2}|P(z^*) - \alpha z^{*n}| \quad (1.15)'$$

Note that for suitable choice of α, $|\alpha| > 1$, we have the left hand side of (1.15)' $P'(z^*) - \alpha n z^{*n-1} = 0$ and so the right hand side of (1.15)', $P(z^*) - \alpha z^{*n} = 0$. It contradicts that all the zeros of $R(z) = P(z) - \alpha z^n$ are in $|z| < 1$ and that no zeros are on $|z| = 1$. Hence $|P'(z)| \leq n$ on $|z| = 1$. This proves (1.1).

Now, we prove:

Theorem 1.7(*): If $P(z)$ is a polynomial of degree n and $\max_{|z|=1} |P(z)| = 1$, then

$$\left|\frac{zP'(z)}{n} - \frac{P(z)}{2}\right| \leq \frac{1}{2} \quad (1.16)$$

for $|z| = 1$. The result is best possible and the equality in (1.16) holds for $P(z) = \alpha z^n + \beta$, $|\alpha| + |\beta| = 1$.

Proof of Theorem 1.7: Let $P(z)$ satisfy the hypothesis of the theorem, then as in the preceding proof, one gets, for $|\alpha|>1$ and $|z|=1$

$$|zP'(z) - n\alpha z^n| \geq \frac{n}{2}|P(z) - \alpha z^n| \quad (1.15)''$$

We note that from (1.15)'', for any $|\beta| < 1$,

$$[zP'(z) - n\alpha z^n] + \frac{n}{2}\beta[P(z) - \alpha z^n] \neq 0$$

and so

$$[zP'(z) + \frac{n}{2}\beta P(z)] \neq \alpha n(1+\frac{\beta}{2})z^n \tag{1.17}$$

For an appropriate choice of the argument of α, for each z in (1.17), one gets

$$\left|zP'(z) + \frac{n}{2}\beta P(z)\right| \neq n|\alpha|\left|1+\frac{\beta}{2}\right|$$

Hence, we have either

$$\left|zP'(z) + \frac{n}{2}\beta P(z)\right| < n|\alpha|\left|1+\frac{\beta}{2}\right| \tag{1.18}$$

or

$$\left|zP'(z) + \frac{n}{2}\beta P(z)\right| > n|\alpha|\left|1+\frac{\beta}{2}\right| \tag{1.19}$$

for all $|\alpha| > 1$. But a sufficiently large value of $|\alpha|$ violates (1.19). Hence, (1.18) holds. Making $\alpha \to 1$ in (1.18), one gets

$$\left|zP'(z) + \frac{n}{2}\beta P(z)\right| \leq n\left|1+\frac{\beta}{2}\right| \tag{1.20}$$

Taking $\beta \to -1$ in (1.20), we get

$$|zP'(z) - \frac{n}{2}P(z)| \leq \frac{n}{2}$$

This proves (1.16).

If $P(z) = \alpha z^n + \beta$ with $|\alpha| + |\beta| = 1$, then

$$\left|\frac{zP'(z)}{n} - \frac{P(z)}{2}\right| = \left|\alpha z^n - \frac{\alpha z^n + \beta}{2}\right|$$

$$= \left|\frac{\alpha z^n - \beta}{2}\right|$$

$$= \frac{1}{2}$$

for $\arg z = -\frac{1}{n}\arg\left(\frac{\alpha}{-\beta}\right)$; $|z| = 1$.

As a consequence of Theorem 1.7, we mention:

<u>Remark 1.2(*)</u>: From Theorem 1.7, we deduce a proof of Theorem 1.2.

Let $t(\theta) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ and $\max_\theta |t(\theta)| = 1$. We note that

$$t(\theta) = e^{-in\theta} \sum_{\nu=0}^{2n} c_{\nu-n} e^{i\nu\theta}$$

$$= e^{-in\theta} R_{2n}(e^{i\theta})$$

where $R_{2n}(z)$ is a polynomial of degree $2n$; for $|z| = 1$, one also has

$$|t'(\theta)| = |e^{-in\theta} R_{2n}'(e^{i\theta}) ie^{i\theta} - ine^{-in\theta} R_{2n}(e^{i\theta})|$$

$$= |e^{i\theta} R_{2n}'(e^{i\theta}) - nR_{2n}(e^{i\theta})| \tag{1.21}$$

Applying Theorem 1.7 to $R_{2n}(z)$, one gets on $|z| = 1$,

$$|z R_{2n}'(z) - n R_{2n}(z)| \le n \tag{1.22}$$

From (1.21) and (1.22), we conclude

$$|t'(\theta)| \le n$$

Theorem 1.2 is proved.

Remark 1.3(*): Here we discuss the case of equality in (1.16). In fact, we claim:

Let $P(z)$ be a polynomial of even degree and $\max_{|z|=1} |P(z)| = 1$. If for some $|z^*| = 1$, there is equality in (1.16), then $P(z) = \alpha z^n + \beta$ where $|\alpha| + |\beta| = 1$.

Let $n = 2k$ and $P(z) = \sum_{\nu=0}^{2k} a_\nu z^\nu$ with $\max_{|z|=1} |P(z)| = 1$ and there is equality in (1.16) for $z = e^{i\eta}$, i.e.

$$\left| e^{i\eta} \frac{P'(e^{i\eta})}{2k} - \frac{P(e^{i\eta})}{2} \right| = \frac{1}{2}$$

Define $S(\theta) = \sum_{\nu=-k}^{k} a_{\nu+k} e^{i\nu\theta}$. So $S(\theta) = e^{-ik\theta} P(e^{i\theta})$. Let $\gamma = \arg S'(\eta)$.

Consider the real trigonometric polynomial $t(\theta) = \operatorname{Re} e^{-i\gamma} S(\theta)$. From the preceding calculation, we have

$$|t'(\eta)| = |\operatorname{Re} e^{-i\gamma} S'(\eta)|$$
$$= |S'(\eta)|$$
$$= |e^{i\eta} P'(e^{i\eta}) - kP(e^{i\eta})|$$
$$= k.$$

Thus $t(\theta) = \lambda \sin k(\theta - \varphi)$, φ is real but fixed (see page 10) which further implies that

$$\operatorname{Re} e^{-i\gamma} e^{-ik\theta} P(e^{i\theta}) = \lambda \sin k(\theta - \varphi)$$

Putting $\psi = \theta - \varphi$ and making a change of variable, we get

$$\operatorname{Re} e^{-ik\psi} Q(e^{i\psi}) = \lambda \sin k\psi \qquad (1.23)$$

where $Q(e^{i\psi}) = e^{-i\gamma} e^{-ik\varphi} P(e^{i(\psi+\varphi)})$ is a trigonometric polynomial of order $2k$ which we write as $Q(e^{i\psi}) = \sum_{\upsilon=0}^{2k} a_\upsilon^* e^{i\upsilon\psi}$. We claim $a_\upsilon^* = 0$ for $\upsilon = 1, \ldots, 2k-1$.

Let $a_\upsilon^* = b_\upsilon^* + i c_\upsilon^*$, so $b_\upsilon^* = \operatorname{Re} a_\upsilon^*$ and $c_\upsilon^* = \operatorname{Im} a_\upsilon^*$. Hence from (1.23), we get

$$\lambda \sin k\psi = \operatorname{Re} e^{-ik\psi} Q(e^{i\psi})$$
$$= \operatorname{Re} e^{-ik\psi} \sum_{\upsilon=0}^{2k} (b_\upsilon^* + i c_\upsilon^*) e^{i\upsilon\psi}$$
$$= \operatorname{Re} \sum_{\upsilon=0}^{2k} (b_\upsilon^* + i c_\upsilon^*) e^{i(\upsilon-k)\psi}$$
$$= \sum_{\upsilon=0}^{2k} [b_\upsilon^* \cos(\upsilon-k)\psi - c_\upsilon^* \sin(\upsilon-k)\psi] \qquad (1.24)$$

Since $[\sin \ell\psi, \cos \ell\psi]_{\ell=0}^{k}$ forms an orthogonal family, multiplying (1.24) by $\sin \ell\psi$, $\ell = 0, \ldots, k-1$, $\cos \ell\psi$, $\ell = 0, \ldots, k$; integrating over $[-\pi, \pi]$, we observe that $b_\upsilon^* = 0$, for all υ and $c_\upsilon^* = 0$ for $\upsilon = 1, 2, \ldots, 2k-1$. Multiplying

(1.24) by $\sin k\psi$ and integrating on $[-\pi,\pi]$, one gets $c_0^* - c_{2k}^* = \lambda$ implying that c_0^* and c_{2k}^* are not simultaneously zero. Thus, $Q(e^{i\psi}) = c_0^* - c_{2k}^* e^{i2k\theta}$. Since $P(e^{i\theta}) = e^{i(\gamma+k\varphi)}Q(e^{i(\theta-\varphi)})$, we conclude that $P(e^{i\theta})$ must be of the form $\alpha + \beta e^{i2k\theta}$. Since $\max_\theta |P(e^{i\theta})| = 1$, one also has $|\alpha| + |\beta| = 1$. The claim is verified.

Remark 1.4(*): If $P(\gamma) = 0$, for $|\gamma| = 1$, then from Theorem 1.7, one has a result due to Rahman and Mohammad [17] mentioned in Theorem 3.8.

1.4

In 1930, Bernstein [5] proved Theorem 1.6 from where the inequality (1.1) is easily deduced. In fact, he proved Theorem 1.1 by using the Gauss-Lucas Theorem:

Theorem 1.8 (Gauss-Lucas): The closed convex hull that contains all zeros of a polynomial $P(z)$ also contains all zeros of its derivative $P'(z)$.

By choosing $Q(z) = z^n$ in Theorem 1.6, we get Theorem 1.1 immediately. Now, we are going to study an extension of Theorem 1.8 [18] where the differential operator $P'(z)$ is replaced by more general operator B defined as:

Definition 1.1: Let $\mathbb{P}$ be the space of all polynomials of degree $\leq n$ and $\mathbb{P}^*$ be subset of $\mathbb{P}$ consisting of all those polynomials having all its zeros in $|z| \leq 1$. Then a linear operator defined on $\mathbb{P}$ which maps $\mathbb{P}^*$ into itself is called a B-operator.

Now, we prove:

Theorem 1.9: Let $Q(z) \in \mathbb{P}^*$ and $P(z)$ be any polynomial of degree not exceeding that of $Q(z)$. If

$$|P(z)| \le |Q(z)|$$

on $|z| = 1$, then for any B- operator

$$|B P(z)| \le |B Q(z)| \tag{1.25}$$

on $|z| = 1$.

Proof of Theorem 1.9: Let $P(z)$ and $Q(z)$ satisfy the hypothesis of Theorem 1.9, then $\frac{P(z)}{Q(z)}$ is analytic in $|z| \ge 1$ and $\max_{|z|\ge 1}\left|\frac{P(z)}{Q(z)}\right| = \max_{|z|=1}\left|\frac{P(z)}{Q(z)}\right| = 1$. Hence, for any $|\alpha| > 1$, $P(z) - \alpha Q(z)$ has all its zeros in $|z| < 1$. Thus, $P(z) - \alpha Q(z) \in \mathbb{P}^*$. For any B-operator, we have $B[P(z) - \alpha Q(z)] \in \mathbb{P}^*$, i.e. $BP(z) - \alpha BQ(z)$ has all its zeros in $|z| \le 1$. So, for any $|z| \ge 1$, for any $|\alpha| \ge 1$,

$$B P(z) \ne \alpha BQ(z)$$

From where, we have

$$|B P(z)| \le |B Q(z)|$$

We prove (1.25).

Now, we consider the following examples of B-operator:

Example 1.1(*): Let $P(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be a polynomial of degree n, then $B P(z) = a_n z^n + a_0$ is a B-operator. Note that if $P(z)$ has all its zeros in $|z| \le 1$, then $P(z) = a_n \prod_{\upsilon=1}^{n} (z - z_\upsilon)$ where $|z_\upsilon| \le 1$, $\upsilon = 1, \ldots, n$. Hence $\left|\frac{a_0}{a_n}\right| = (-1)^n z_1 \cdots z_n$ which implies $\left|\frac{a_0}{a_n}\right| \le 1$. It follows that $a_n z^n + a_0$ has all its zeros in $|z| \le 1$. Thus, for any $|P(z)| \le 1$, we have

$$|P(z)| \le |z^n|$$

on $|z| = 1$. By Theorem 1.9, we get

$$|B P(z)| \le |B z^n|$$

on $|z| = 1$, i.e.

$$|a_n z^n + a_0| \leq 1 \tag{1.26}$$

on $|z| = 1$. Choosing an appropriate argument of z in (1.26), one gets

$$|a_n| + |a_0| \leq 1$$

on $|z| = 1$ which is an inequality due to Visser [18].

Example 1.2: The differential operator is a B-operator. In fact, it follows from Gauss-Lucas Theorem. Note that the unit disk is a convex set.

Example 1.3(*): $B\,P(z) = zP'(z) + \beta\frac{n}{2}P(z)$ is a B-operator, $|\beta| < 1$. To verify this, let $P(z)$ have all its zeros in $|z| \leq 1$; then we have $\left|\frac{zP'(z)}{P(z)}\right| \geq \frac{n}{2}$ for $|z| = 1$ (see lemma 1.2). Since $P'(z)$ also has all its zeros in $|z| \leq 1$, so $\frac{zP'(z)}{P(z)} \neq 0$ in $|z| \geq 1$. By Maximum Modulus Theorem, the $\min\limits_{|z|\geq 1}\left|\frac{zP'(z)}{P(z)}\right|$ is attained on $|z| = 1$. Thus, $\min\limits_{|z|\geq 1}\left|\frac{zP'(z)}{P(z)}\right| \geq \frac{n}{2}$. Hence, for $|\beta| < 1$,

$$zP'(z) + \beta\frac{n}{2}P(z) \neq 0$$

in $|z| \geq 1$. Consequently, $zP'(z) + \beta\frac{n}{2}P(z)$ has all its zeros in $|z| \leq 1$. Hence, $B\,P(z) = zP'(z) + \beta\frac{n}{2}P(z)$ is B-operator

This immediately gives:

Theorem 1.10(*): Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P(z)$ be another polynomial of degree not exceeding n. If

$$|P(z)| \leq |Q(z)|$$

on $|z| = 1$, then for any $|\beta| < 1$

$$\left|\frac{zP'(z)}{n} + \beta\frac{P(z)}{2}\right| \leq \left|\frac{zQ'(z)}{n} + \beta\frac{Q(z)}{2}\right| \tag{1.27}$$

on $|z| = 1$.

We also deduce the following

Corollary 1.1: Under the hypothesis of Theorem 1.10

$$\left|\frac{zP'(z)}{n}\right| + \left|\frac{Q(z)}{2}\right| \le \left|\frac{zQ'(z)}{n}\right| + \left|\frac{P(z)}{2}\right| \qquad (1.28)$$

for $|z| = 1$.

Proof of Corollary 1.1: In (1.27), choose β such that

$$\left|\frac{zQ'(z)}{n} + \frac{\beta Q(z)}{2}\right| = \left|\frac{zQ'(z)}{n}\right| - \left|\frac{Q(z)}{2}\right|$$

It is obvious that

$$\left|\frac{zP'(z)}{n}\right| - \left|\frac{P(z)}{2}\right| \le \left|\frac{zP'(z)}{n} + \frac{\beta P(z)}{2}\right|$$

Hence (1.28).

Corollary 1.2: Under the hypothesis of Theorem 1.10

$$\left|\frac{zP'(z)}{n} - \frac{P(z)}{2}\right| \le \left|\frac{zQ'(z)}{n} - \frac{Q(z)}{2}\right|$$

for $|z| = 1$.

For the proof of Corollary 1.2, choose $\beta \to -1$ in (1.27).

We now give an interesting application of Corollary 1.2 in proving an equality for trigonometric polynomials. Results of this type seem to have been discussed in [18]. We present:

Theorem 1.11: Let $t(\theta)$ and $S(\theta)$ be trigonometric polynomials and the order of $t(\theta)$ not exceeding that of $S(\theta)$ which has all its zeros in $\operatorname{Im}\theta \ge 0$. If

$$|t(\theta)| \le |S(\theta)|$$

for all θ, then

$$|t'(\theta)| \le |S'(\theta)|$$

for all θ.

Proof of Theorem 1.11: Let $t(\theta) = \sum_{\upsilon=-m}^{m} c_\upsilon e^{i\upsilon\theta}$ and $S(\theta) = \sum_{\upsilon=-n}^{n} d_\upsilon e^{i\upsilon\theta}$,

$m \le n$, satisfy the hypothesis of Theorem 1.11. We can write

$$t(\theta) = e^{-im\theta} \sum_{\upsilon=0}^{2m} c_{\upsilon-m} e^{i\upsilon\theta} = e^{-im\theta} P_{2m}(e^{i\theta})$$

and $$S(\theta) = e^{-in\theta} \sum_{\upsilon=0}^{2n} c_{\upsilon-n} e^{i\upsilon\theta} = e^{-in\theta} Q_{2n}(e^{i\theta})$$

where $P_{2m}(e^{i\theta})$ and $Q_{2n}(e^{i\theta})$ are polynomials of degree 2m and 2n respectively. Since $S(\theta)$ has all its zeros in $\text{Im}\,\theta \ge 0$, $Q_{2n}(e^{i\theta})$ has all its zeros in $|z| \le 1$. Hence, P_{2n} and Q_{2n} satisfy the hypothesis of Theorem 1.10. By Corollary 1.2, we get

$$|z P'_{2m}(z) - m P_{2m}(z)| \le |z Q'_{2n}(z) - n Q_{2n}(z)|$$

for $|z|=1$. But $|t'(\theta)| = |z P'_{2m}(z) - m P_{2m}(z)|$ and $|S'(\theta)| = |z Q'_{2n}(z) - n Q_{2n}(z)|$; see (1.21). Hence

$$|t'(\theta)| \le |S'(\theta)|$$

for all θ. The proof is complete.

1.5

Now, we return to discuss the modified proof of Theorem 1.5 due to Boas.

Proof of Theorem 1.5 : Let $t(\theta)$ be a real trigonometric polynomial with $\max_\theta |t(\theta)| = 1$. Choose any $\theta^* \in [0,2\pi)$, if $t'(\theta^*) = 0$, the inequality (1.5) obviously holds. So, let $t'(\theta^*) \neq 0$. Now, we choose a number φ such that $t(\theta^*) = \sin n(\theta^* - \varphi)$ and $\operatorname{sgn} t'(\theta^*) = \operatorname{sgn} \cos n(\theta^* - \varphi)$. We claim that

$$|t'(\theta^*)| \le n|\cos n(\theta^* - \varphi)| \qquad (1.29)$$

Without any loss of generality, we may assume that $t'(\theta^*) > 0$ and that the slope of $\sin n(\theta - \varphi)$ at $\theta = \theta^*$ is also positive.

Let us suppose, on contrary, that

$$t'(\theta^*) > n\cos n(\theta^* - \varphi) \tag{1.30}$$

Also, let $\theta_r = \frac{(2r-1)\pi}{2n} + \varphi$, $r = 1,\ldots,2n$ be $2n$ points in $[0,2\pi)$, then $\sin n(\theta-\varphi) = (-1)^{r-1}$ at $\theta=\theta_r$, $r = 1,\ldots,2n$. Consider $S(\theta) = t(\theta) - \sin n(\theta-\varphi)$, since $|t(\theta)| \le 1$, for all θ, we have:

$$S(\theta_1) \le 0,\ S(\theta_2) \ge 0,\ldots,\ S(\theta_{2n-1}) \le 0,\ S(\theta_{2n}) \ge 0$$

In the case where $S(\theta_r) = 0$, for some r, we note that both of $t(\theta)$ and $\sin n(\theta-\varphi)$ attains their extrema at $\theta = \theta_r$. So, θ_r must be a double zero of $S(\theta)$. (see Figure 1.1)

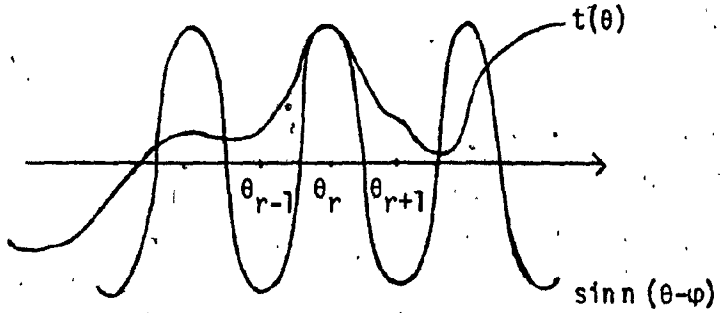

Figure 1.1

If $S(\theta_{r-1}) > 0$ and $S(\theta_r) < 0$, then $S(\theta)$ has at least one zero in the interval (θ_{r-1},θ_r). (See Figure 1.2)

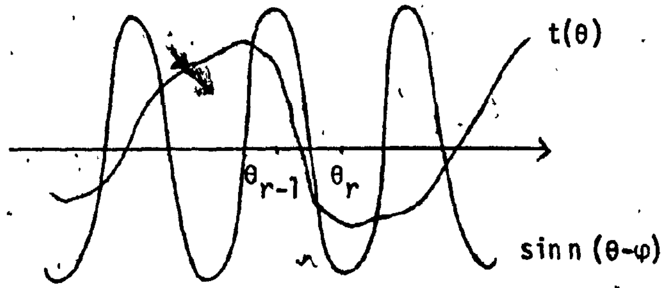

Figure 1.2

Hence, corresponding to each interval $[\theta_{r-1}, \theta_r]$, $S(\theta)$ has at least one zero. In the case where θ_r is a double zero, one of those two zeros is associated to $[\theta_{r-1}, \theta_r]$ and the other to $[\theta_r, \theta_{r+1}]$. Since there are 2n such intervals $[\theta_1,\theta_2],[\theta_2,\theta_3],\ldots,[\theta_{2n-1},\theta_{2n}]$, $[\theta_{2n},\theta_{2n+1} = \theta_1]$, following the preceding observation, we know that there are at least 2n zeros of $S(\theta)$.

Further, one of these intervals, say $[\theta_{k-1}, \theta_k]$, contains θ^*. Since $t'(\theta^*) > n \cos n(\theta^* - \varphi)$, the graph of $t(\theta)$ meets the graph of $\sin n(\theta-\varphi)$ from below to above. Hence, we conclude (geometrically) that $S(\theta)$ must have at least three zeros in $[\theta_{k-1}, \theta_k]$; see Figure 1.3.

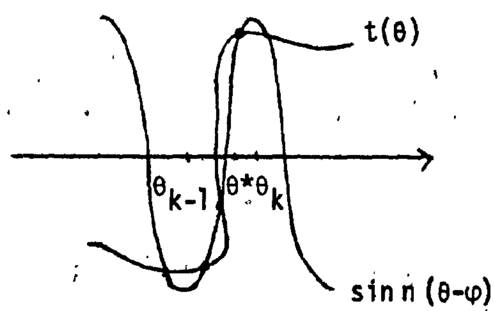

Figure 1.3

Consequently, $S(\theta)$ has at least 2n+2 zeros. Since a trigonometric polynomial of order n cannot have more than 2n zeros, we conclude $S(\theta) \equiv 0$, i.e. $t(\theta) = \sin n(\theta-\varphi)$. This contradicts (1.30) and so (1.29) is established.

Recall that $t(\theta^*) = \sin n(\theta^* - \varphi)$, so

$$\varphi = \theta^* - \frac{1}{n} \arcsin t(\theta^*)$$

and in consequence, (1.29) gives

$$|t'(\theta^*)|^2 \le n^2|\cos^2 n(\theta^* - \varphi)|$$
$$= n^2|\{\cos n\,[\tfrac{1}{n} \text{ arc sin } t(\theta^*)]\}^2$$
$$= n^2\{1 - [\sin \text{ arc sin } t(\theta^*)]^2\}$$
$$= n^2 - n^2 t^2(\theta^*)$$

From where, we prove

$$\{t'(\theta)\}^2 + n^2\{t^2(\theta)\} \le n^2$$

For $t(\theta) = \sin n\,(\theta-\varphi)$, there is equality in (1.5), for all θ. On the other hand, if there is equality at $\theta = \theta^*$ in (1.5), where $t'(\theta^*) \neq 0$, then

$$\{t'(\theta^*)\}^2 = n^2 - n^2\{t(\theta^*)\}^2$$
$$= n^2 - n^2 \sin^2 n(\theta^* - \varphi)$$
$$= n^2 \cos^2 n(\theta^* - \varphi)$$

where $\varphi = \theta^* - \frac{1}{n}$ arc sin $t(\theta^*)$.

In this case, we note that the graph of $t(\theta)$ is tangential to the graph of $\sin n\,(\theta-\varphi)$ at θ^*; and θ^* is not the extrema of $\sin n\,(\theta-\varphi)$; see Figure 1.4

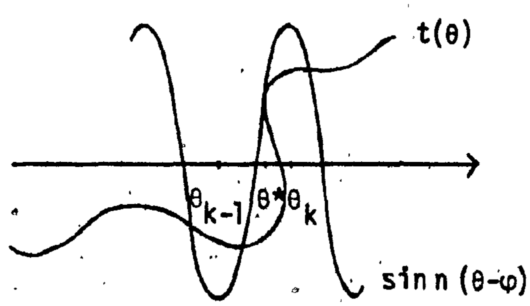

Figure 1.4

and so $S(\theta) = t(\theta) - \sin n\,(\theta-\varphi)$ has at least two zeros in $\theta_{k-1} < \theta^* < \theta_k$ implying $S(\theta)$ has $2n+1$ zeros in $[0,2\pi)$. Hence $t(\theta) \equiv \sin n\,(\theta-\varphi)$.

The proof of Theorem 1.6 is complete.

As an observation in the proof of Theorem 1.5, we note the following:

Remark 1.5: If $t(\theta)$ is a real trigonometric polynomial of order n with $\max_\theta |t(\theta)| = 1$ and $t(\theta) - \sin n(\theta-\varphi)$ has a double zero at a point $\theta = \theta^*$ where $t'(\theta^*) \neq 0$, then $t(\theta) \equiv \sin n(\theta-\varphi)$.

Further, we note that we can prove "only if" part of Theorem 1.1 by means of the above remark. Let us suppose that for some $|z^*| = 1$, we have $|P'(z^*)| = n$. Then there exists an $|\alpha| = 1$ such that $P'(z^*) - \alpha n z^{*\,n-1} = 0$. From (1.15), $P(z) - \alpha z^n$ must also have a zero at z^*, so it has a double zero at z^*. Let $z^* = e^{i\theta^*}$, $t(\theta) = \mathrm{Re}\,P(e^{i\theta})$, $S(\theta) = \mathrm{Im}\,P(e^{i\theta})$, then for $\alpha = e^{-in\varphi}$, $z = e^{i\theta}$, we have

$$\mathrm{Re}\,\alpha z^n = \cos n(\theta-\varphi)$$

and

$$\mathrm{Im}\,\alpha z^n = \sin n(\theta-\varphi)$$

Note that $[t(\theta) - \cos n(\theta-\varphi)] + i[S(\theta) - \sin n(\theta-\varphi)]$ has a double zero at $\theta = \theta^*$. Hence $t(\theta) - \cos n(\theta-\varphi)$ as well as $S(\theta) - \sin n(\theta-\varphi)$ both have a double zero at $\theta = \theta^*$. Moreover, both $t'(\theta^*)$ and $S'(\theta^*)$ cannot be simultaneously zero. If $t'(\theta^*) \neq 0$, from remark 1.5, we get $t(\theta) = \cos n(\theta-\varphi)$ and so $S(\theta) = \sin n(\theta-\varphi)$. Thus $P(z) = \alpha z^n$, $|\alpha| = 1$.

Remark 1.6: The case of equality in (1.7) has recently been discussed in [16].

1.6

Recently, Abi-Khuzam [1] has proved an interesting result related to the following inequality:

$$|P'(\rho e^{i\theta})| \le \frac{nM(\rho)}{\rho} \qquad (1.31)$$

where $M(\rho) = \max_{|z|=\rho} |P(z)|$, $\rho > 0$. In fact, this inequality (1.31) can be directly obtained from (1.1) by considering $Q(z) = P(\rho z)$. So, it is known that for any polynomial the inequality (1.31) holds. Abi-Khuzam established a converse of such a result in showing that an entire function (a function which is analytic in the whole plane) satisfying such an inequality for every $\rho > 0$ is in fact a polynomial. He proved:

Theorem 1.12: Let $P(z)$ be an entire function. Assume that there exists a constant $n(\ge 0)$ such that

$$|P'(\rho e^{i\theta})| \le \frac{nM(\rho)}{\rho} \qquad (1.31)'$$

for every $\rho > 0$; where $M(\rho) = \max_{|z|=\rho}|P(z)|$. Then $P(z)$ is a polynomial of degree $\le n$.

Proof of Theorem 1.12: Let $P(z)$ be an entire function, so we can write

$$P(z) = \sum_{\upsilon=0}^{\infty} a_\upsilon z^\upsilon \qquad (1.32)$$

Given any complex number $z = Re^{i\theta}$, $R > 0$, we have

$$P(Re^{i\theta}) = P(0) + \int_0^R P'(\rho e^{i\theta})\, e^{i\theta} d\rho \qquad (1.33)$$

For a fixed real number a, $0 < a < R$, we rewrite (1.33) as

$$\begin{aligned} P(Re^{i\theta}) &= P(0) + \int_0^a P'(\rho e^{i\theta})e^{i\theta}d\rho + \int_a^R P'(\rho e^{i\theta})e^{i\theta}d\rho \\ &= P(0) + P(a) - P(0) + \int_a^R P'(\rho e^{i\theta})e^{i\theta}d\rho \\ &= P(a) + \int_a^R P'(\rho e^{i\theta})e^{i\theta}d\rho \end{aligned}$$

Hence

$$|P(Re^{i\theta})| \le |P(a)| + |\int_a^R P'(\rho e^{i\theta})e^{i\theta}d\rho|$$

$$\le |P(a)| + \int_a^R |P'(\rho e^{i\theta})|d\rho$$

From (1.31)', we get

$$M(R) \le |P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho \qquad (1.34)$$

Dividing both sides by $|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho$, we have

$$\frac{M(R)}{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho} \le 1$$

Multiplying both sides by $\frac{n}{R}$, we get

$$\frac{\frac{n}{R}M(R)}{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho} \le \frac{n}{R} \qquad (1.35)$$

Let $F(R) = \log\{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho\}$. Since $|P(a)| > 0$, $F(R)$ is absolutely continuous on $[a,x]$, $a > 0$. Hence

$$F(x) - F(a) = \int_a^x F'(R)dR$$

$$= \int_a^x \frac{\frac{d}{dR}\left\{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho\right\}}{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho} \qquad (1.36)$$

Since $M(\rho)$ is known to be continuous, so $n\int_a^R \frac{M(\rho)}{\rho}d\rho$ is an absolutely continuous function of R whose derivative with respect to R equals to $\frac{nM(R)}{R}$. Therefore, combining (1.35) and (1.36), we have

$$F(x) - F(a) \le \int_a^x \frac{\frac{n}{R}M(R)dR}{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}d\rho}$$

$$\leq n \int_a^x \frac{dR}{R}$$

$$= n \log \left(\frac{x}{a}\right) \qquad (1.37)$$

From (1.34) and (1.37), we get

$$\log M(R) \leq \log\{|P(a)| + n\int_a^R \frac{M(\rho)}{\rho}\, d\rho\}$$

$$= F(R)$$

$$\leq F(a) + n \log \left(\frac{R}{a}\right)$$

Hence $M(R) \leq c(\frac{R}{a})^n$ where $c = e^{F(a)}$. Note that in (1.32), if $\upsilon > n$, by Cauchy's inequality, we have

$$|a_\upsilon| \leq \frac{M(R)}{R^\upsilon} \leq c\left(\frac{R}{a}\right)^n . \left(\frac{1}{R^\upsilon}\right) = \frac{c}{a^n} \cdot \frac{1}{R^{\upsilon-n}}$$

Letting $R \to \infty$, we get $|a_\upsilon| \leq 0$ which implies $a_\upsilon = 0$. It follows that $P(z)$ is a polynomial of degree $\leq n$. The proof is complete.

Chapter II

Turan's Theorem

2.1

In 1939, Turan [23] first discussed the lower estimate of $\max_{|z|=1} |P'(z)|$ on $|z| = 1$ when all the zeros of the polynomial $P(z)$ lie in $|z| \le 1$. He proved the following:

Theorem 2.1: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \tag{2.1}$$

The result is best possible and equality in (2.1) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| = \frac{1}{2}$.

The proof of Theorem 2.1 is easy. In fact, if $P(z)$ satisfies the hypothesis of Theorem, then

$$|z\, P'(z)| \ge \frac{n}{2}\, |P(z)| \tag{2.2}$$

on $|z| = 1$ (see lemma 1.2), so the inequality (2.1) is immediate if one takes the maximum of both sides in (2.2).

In this chapter, we consider two types of generalizations of Theorem 2.1. First, we discuss a recent result where $\max_{|z|=1} |P(z)|$ is replaced by $\left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q}$ in the right hand side of (2.1). The other generalization is when the zeros are in $|z| \le k$, $k \ge 1$. In this direction, we shall present the work of Aziz [3] and Malik [15].

2.2

We present the following result from [15]:

Theorem 2.2: If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$, we have

$$(A_q)^{1/q} \max_{|z|=1} |P'(z)| \geq n\left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{1/q} \tag{2.3}$$

where $A_q = \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta = 2^{q+1}\sqrt{\pi}\,\dfrac{\Gamma(\frac{1}{2}q + 2)}{\Gamma(\frac{1}{2}q + 1)}$. The result is best possible and equality in (2.3) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

We note that the idea of proving such a result is due to Saff and Sheil-Small [21]. In 1973, they proved the following conjecture proposed by Erdös:

Let $t(\theta)$ be a trigonometric polynomial of degree n having $2n$ zeros in $[0, 2\pi)$ and let $M = \max_\theta |t(\theta)| \leq 1$, then

$$\int_0^{2\pi} |t(\theta)| d\theta \leq 4\pi$$

To prove the above conjecture, Saff and Shiel-Small established the following:

Theorem 2.3: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has all its zeros on $|z| = 1$, then for each $q > 0$, we have

$$\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \leq \left(\tfrac{1}{2}\right)^q A_q \tag{2.4}$$

where $A_q = \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta = 2^{q+1}\sqrt{\pi}\,\dfrac{\Gamma(\frac{1}{2}q+2)}{\Gamma(\frac{1}{2}q+1)}$. The result is best possible and equality in (2.4) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

In order to prove Theorem 2.3, Saff and Shiel-Small, first established:

$$n\int_0^{2\pi} |P(e^{i\theta})|^q d\theta \leq \left(\max_{|z|=1} |P'(z)|\right)^q (A_q) \tag{2.5}$$

and then deduced (2.4) from (2.5) by the use of the following Erdös-

Lax Theorem [14]:

Theorem 2.4: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has all its zeros in $|z| \geq 1$, then for $|z| = 1$

$$|P'(z)| \leq \frac{n}{2} \tag{2.6}$$

The result is best possible and equality in (2.6) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| = \frac{1}{2}$.

For the details, see [9].

For the proof of Theorem 2.2, we also need the following known:

Lemma 2.1: If $Q(z)$ is a polynomial of degree n and $Q(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$

$$|Q'(z)| \leq |P'(z)|$$

where $P(z) = z^n \overline{Q(1/\overline{z})}$.

For the proof, see Theorem 1.6.

Now, we return to present:

Proof of Theorem 2.2: Let $Q(z)$ be a polynomial of degree n. Define $Q(z) = z^n \overline{P(1/\overline{z})}$. For $|z| = 1$, we have

$$|Q(z)| = |e^{in\theta} \overline{P(1/e^{-i\theta})}| = |P(e^{i\theta})| = |P(z)| \tag{2.7}$$

Moreover, note that $P(z) = z^n \overline{Q(1/\overline{z})}$, so

$$P'(z) = n z^{n-1} \overline{Q(1/\overline{z})} - z^{n-2} \overline{Q'(1/\overline{z})}$$

$$= z^{n-1}[n \overline{Q(1/\overline{z})} - (1/z) \overline{Q'(1/\overline{z})}] .$$

It follows

$$z^{n-1} \overline{P'(1/\overline{z})} = [nQ(z) - zQ'(z)] \tag{2.8}$$

Since $|z^{n-1} \overline{P'(1/\overline{z})}| = |P'(z)|$ on $|z| = 1$, from (2.8), we get

$$|P'(z)| = |nQ(z) - zQ'(z)| \tag{2.9}$$

Further,

$$n|Q(z)| = |nQ(z) - zQ'(z) + zQ'(z)|$$

$$= |nQ(z) - zQ'(z)|\left|1 + \frac{zQ'(z)}{nQ(z)-zQ'(z)}\right| \qquad (2.10)$$

Let $W(z) = \frac{zQ'(z)}{nQ(z)-zQ'(z)}$, we rewrite (2.10) as

$$n|Q(z)| = |nQ(z) - zQ'(z)||1 + W(z)| \qquad (2.11)$$

Combining (2.7), (2.9) and (2.11), one gets

$$n|P(z)| = |P'(z)||1 + W(z)| \qquad (2.12)$$

Now, we claim that if P(z) is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

(1) W(z) is analytic in $|z| \leq 1$

(2) $1 + W(z)$ is subordinate to $1 + z$ for $|z| \leq 1$.

First, we show that W(z) is analytic in $|z| \leq 1$. Since P(z) has all its zeros in $|z| \leq 1$, by Gauss-Lucas Theorem, (see Theorem 1.8), P'(z) also has all its zeros in $|z| \leq 1$. Hence, from (2.8)

$$z^{n-1}\,\overline{P'(1/\overline{z})} = nQ(z) - zQ'(z)$$

has no zeros in $|z| < 1$. So W(z) is analytic in $|z| < 1$. Moreover, if $|\alpha| = 1$ is a zero of order $k(<n)$ of $nQ(z) - zQ'(z)$, then from (2.9), α is also a zero of order k of P'(z). Note that the unit disk is strictly convex. So, we can conclude that the zeros of P'(z) on $|z| = 1$ must be the zeros of P(z) on $|z| = 1$. In fact, suppose on contrary, there exists a β with $|\beta| = 1$ such that $P'(\beta) = 0$ and $P(\beta) \neq 0$, by Gauss-Lucas Theorem, β does not belong to the convex set which is formed by joining all the zeros of P(z) on $|z| = 1$. Hence, we have contradiction. Further, if P'(z) has a zero α of order k, then α is a zero of order k+1 of P(z). In fact, suppose

that α is a zero of order $m(\le n)$ of $P(z)$, so

$$P(z) = (z-\alpha)^m \tau(z)$$

where $\tau(z)$ is a polynomial of degree $n-m$; $\tau(\alpha) \neq 0$. It follows:

$$P'(z) = (z-\alpha)^{m-1}[m\,\tau(z) + (z-\alpha)\tau'(z)].$$

Since α is a zero of order k of $p'(z)$, we conclude that $m = k+1$. From (2.7), we know that α is a zero of order $k+1$ of $Q(z)$. Hence, α is a zero of order k of $Q'(z)$. Note that

$$W(z) = \frac{zQ'(z)}{nQ(z) - zQ'(z)}$$

and both $Q'(z)$ and $nQ(z) - zQ'(z)$ have the same order of zeros on $|z| = 1$, we conclude that $W(z)$ is analytic in $|z| \le 1$.

Now, we show that $1 + W(z)$ is subordinate to $1+z$ for $|z| \le 1$. By lemma 2.1, we have

$$|Q'(z)| \le |P'(z)|$$

on $|z| = 1$. From (2.9), we have

$$|W(z)| = \left|\frac{zQ'(z)}{nQ(z)-zQ'(z)}\right| = \left|\frac{Q'(z)}{P'(z)}\right| \le 1$$

on $|z| = 1$. Furthermore, $W(0) = 0$. So $W(z)$ satisfies the condition of Schwarz lemma. Hence $|W(z)| \le |z|$. It follows that $1 + W(z)$ is subordinate to $1+z$ for $|z| \le 1$.

Thus, applying a well-known property of the principle of subordination as in [21] to (2.12), we deduce that for each $q > 0$,

$$n\left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{1/q} \le \max_{|z|=1} |P'(z)| \left(\int_0^{2\pi} |1+W(e^{i\theta})|^q d\theta\right)^{1/q}$$

$$\le \max_{|z|=1} |P'(z)| \left(\int_0^{2\pi} |1+e^{i\theta}|^q d\theta\right)^{1/q}$$

$$= (A_q)^{1/q} \max_{|z|=1} |P'(z)|$$

This completes the proof of Theorem 2.2.

Remark 2.1: We note that Theorem 2.1 can be easily deduced from Theorem 2.2 by making $q \to \infty$ in (2.3). In fact,

$$\lim_{q\to\infty} \left(\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right)^{1/q} = \max_{|z|=1} |P(z)|$$

and

$$\lim_{q\to\infty} (A_q)^{1/q} = \lim_{q\to\infty} \left(\int_0^{2\pi} |1+e^{i\theta}|^q d\theta\right)^{1/q} = 2 .$$

Hence, Theorem 2.2 is a simultaneous generalization of Theorem 2.1 and of a result due to Saff and Shiel-Small; see (2.5).

Remark 2.2: An inequality of the type (2.4) is not expected for a polynomial having all its zeros in $|z| > 1$. This is easily seen in considering the polynomial $P(z) = 1 + (kz)^n$ where $k < 1$. It is obvious that as $n \to \infty$, $\max_{|z|=1} |P'(z)|$ tends to zero whereas the left hand side in (2.4) tends to 2π.

2.3

Now, we consider the other generalization due to Aziz [3] which is in fact a refinement of the result of Govil [12]. In 1973, Govil gave an extension result of Theorem 2.1 where he considered all the zeros in $|z| \le k$, $k \ge 1$ and proved:

Theorem 2.5: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \tag{2.13}$$

The result is best possible and equality in (2.13) holds for $P(z) = \dfrac{z^n+k^n}{1+k^n}$.

In 1981, Aziz [3] refined the above result as in:

Theorem 2.6: If $P(z) = \prod_{\upsilon=1}^{n} (z-z_\upsilon)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and P(z) has all its zeros in $|z| \le k$, $k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{2}{1+k^n} \sum_{\upsilon=1}^{n} \frac{k}{k+|z_\upsilon|} \tag{2.14}$$

The result is best possible and equality in (2.14) holds for

$$P(z) = \frac{z^n + k^n}{1+k^n}.$$

It is clear that (2.13) can be easily obtained from (2.14). In fact, $|z_\upsilon| \le k$, $\forall \upsilon$, so

$$\sum_{\upsilon=1}^{n} \frac{k}{k+|z_\upsilon|} \ge \sum_{\upsilon=1}^{n} \frac{k}{2k} = \frac{n}{2}$$

Before giving the proof, we need the following lemmas, the first one is well-known and the others are due to Aziz [3].

Lemma 2.2: If P(z) is a polynomial of degree n, then on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \le n \max_{|z|=1} |P(z)|$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof of lemma 2.2: Let P(z) satisfy the hypothesis of lemma and let $M = \max_{|z|=1} |P(z)|$. Define $R(z) = P(z) - Me^{i\alpha}$, $0 < \alpha \le 2\pi$, then R(z) is a polynomial of degree n and $R(z) \ne 0$ in $|z|<1$. By lemma 2.1, for $|z| = 1$, we have

$$|R'(z)| \le |T'(z)| \tag{2.15}$$

where $T(z) = z^n \overline{R(1/\bar{z})} = Q(z) - Me^{-i\alpha}z^n$, $0 < \alpha \le 2\pi$. Further, note that $|R'(z)| = |P'(z)|$ and $|T'(z)| = |Q'(z) - nMe^{-i\alpha}z^{n-1}|$. From (2.15), one gets

$$|P'(z)| \le |Q'(z) - nMe^{-i\alpha}z^{n-1}|$$

For a suitable choice of α, we have

$$|P'(z)| \leq nM - |Q'(z)|$$

on $|z| = 1$. This proves lemma 2.2.

Lemma 2.3: If $P(z)$ is a polynomial of degree n, then for $|z| = 1$ and for all $R \geq 1$,

$$|P(Rz) - P(z)| + |Q(Rz) - Q(z)| \leq (R^n - 1) \max_{|z|=1} |P(z)|$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$

Proof of lemma 2.3: Let α be any complex number such that $|\alpha| = 1$, then

$$|P'(z) + \alpha Q'(z)| \leq |P'(z)| + |\alpha Q'(z)| = |P'(z)| + |Q'(z)| \quad (2.16)$$

Applying lemma 2.2 to (2.16), we have

$$|P'(z) + \alpha Q'(z)| \leq n \max_{|z|=1} |P(z)| \quad (2.17)$$

on $|z| = 1$. Since $P'(z) + \alpha Q'(z)$ is a polynomial of degree $n-1$, then for $t > 1$, we get

$$\max_{|z|=t>1} |P'(z) + \alpha Q'(z)| \leq t^{n-1} \max_{|z|=1} |P'(z) + \alpha Q'(z)|$$

Thus, from (2.17) and $0 \leq \theta \leq 2\pi$, one gets

$$|P'(te^{i\theta}) + \alpha Q'(te^{i\theta})| \leq n\, t^{n-1} \max_{|z|=1} |P(z)|$$

Now, for a suitable choice of the argument of α, it follows

$$|P'(te^{i\theta})| + |Q'(te^{i\theta})| = |P'(te^{i\theta}) + \alpha Q'(te^{i\theta})| \leq n\, t^{n-1} \max_{|z|=1} |P(z)| \quad (2.18)$$

Also note that for each θ, $0 \leq \theta \leq 2\pi$ and $R > 1$, we have

$$P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta})dt \quad (2.19)$$

and

$$Q(Re^{i\theta}) - Q(e^{i\theta}) = \int_1^R e^{i\theta} Q'(te^{i\theta})dt \quad (2.20)$$

Combining (2.18), (2.19) and (2.20), we get

$$|P(Re^{i\theta}) - P(e^{i\theta})| + |Q(Re^{i\theta}) - Q(e^{i\theta})|$$

$$\leq \int_1^R \{|P'(te^{i\theta})| + |Q'(te^{i\theta})|\}dt$$

$$\leq \int_1^R n\,t^{n-1} \max_{|z|=1} |P(z)|dt$$

$$= (R^n - 1) \max_{|z|=1} |P(z)|$$

which is equivalent to the desired result.

Lemma 2.4: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and has all its zeros in the disk $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} \qquad (2.21)$$

The result is best possible and equality in (2.21) holds for $P(z) = \frac{z^n + k^n}{1 + k^n}$

Proof of lemma 2.4: Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, we can write $P(z) = c \prod_{\upsilon=1}^{n} (z - \gamma_\upsilon e^{i\theta_\upsilon})$ where $\gamma_\upsilon \leq k$, $\upsilon = 1,\ldots,n$. Then, for points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, other than the zeros of $P(z)$, we have

$$\left|\frac{P(k^2 e^{i\theta})}{P(e^{i\theta})}\right| = \prod_{\upsilon=1}^{n} \left|\frac{k^2 e^{i\theta} - \gamma_\upsilon e^{i\theta_\upsilon}}{e^{i\theta} - \gamma_\upsilon e^{i\theta_\upsilon}}\right|$$

$$= \prod_{\upsilon=1}^{n} \left\{\frac{(k^2\cos\theta - \gamma_\upsilon\cos\theta_\upsilon)^2 + (k^2\sin\theta - \gamma_\upsilon\sin\theta_\upsilon)^2}{(\cos\theta - \gamma_\upsilon\cos\theta_\upsilon)^2 + (\sin\theta - \gamma_\upsilon\sin\theta_\upsilon)^2}\right\}^{1/2}$$

$$= \prod_{\upsilon=1}^{n} \left\{\frac{k^4 + \gamma_\upsilon^2 - 2k^2\gamma_\upsilon\cos(\theta - \theta_\upsilon)}{1 + \gamma_\upsilon^2 - 2\gamma_\upsilon\cos(\theta - \theta_\upsilon)}\right\}^{1/2}$$

$$= \prod_{\upsilon=1}^{n} k \left\{\frac{k^2 + (\frac{\gamma_\upsilon}{k})^2 - 2\gamma_\upsilon\cos(\theta - \theta_\upsilon)}{1 + \gamma_\upsilon^2 - 2\gamma_\upsilon\cos(\theta - \theta_\upsilon)}\right\}^{1/2} \qquad (2.22)$$

Since $k \geq \gamma_\upsilon$, $\upsilon = 1,\ldots,n$ and $k \geq 1$, so $k^2(k^2-1) \geq \gamma_\upsilon^2(k^2-1)$. Hence, $k^4 + \gamma_\upsilon^2 \geq k^2(\gamma_\upsilon^2 + 1)$ which implies $k^2 + (\frac{\gamma_\upsilon}{k})^2 \geq 1 + \gamma_\upsilon^2$

Thus, we can rewrite (2.22) as

$$\left|\frac{P(k^2e^{i\theta})}{P(e^{i\theta})}\right| \geq \prod_{\upsilon=1}^{n} k \left\{\frac{1+\gamma_\upsilon^2 - 2\gamma_\upsilon\cos(\theta-\theta_\upsilon)}{1+\gamma_\upsilon^2 - 2\gamma_\upsilon\cos(\theta-\theta_\upsilon)}\right\}^{1/2}$$

$$= \sum_{\upsilon=1}^{n} k$$

$$= k^n$$

Furthermore, if the point $e^{i\theta}$ is a zero of $P(z)$, then $|P(k^2e^{i\theta})| \geq k^n|P(e^{i\theta})| = 0$ is always true. Hence, we have

$$|P(k^2z)| \geq k^n|P(z)| \tag{2.23}$$

on $|z| = 1$.

Now, let $G(z) = P(kz)$ and $H(z) = z^n\,\overline{G(1/\overline{z})} = z^n\,\overline{P(k/\overline{z})}$. Applying lemma 2.3 to polynomial $G(z)$, for $|z| = 1$ and for all $k \geq 1$, we get

$$|G(kz) - G(z)| + |H(kz) - H(z)| \leq (k^n - 1) \max_{|z|=1} |G(z)|$$

Since $|G(kz)| - |G(z)| \leq |G(kz) - G(z)|$

$$|H(kz)| - |H(z)| \leq |H(kz) - H(z)|$$

and $\max\limits_{|z|=1} |G(z)| = \max\limits_{|z|=1} |z^n\,\overline{G(1/\overline{z})}| = \max\limits_{|z|=1} |H(z)|$, we have

$$|G(kz)| + |H(kz)| \leq (k^n-1) \max_{|z|=1} |G(z)| + |G(z)| + |H(z)|$$

$$\leq (k^n-1) \max_{|z|=1} |G(z)| + 2 \max_{|z|=1} |G(z)|$$

$$= (k^n+1) \max_{|z|=1} |G(z)| \tag{2.24}$$

Further, note that $G(kz) = P(k^2z)$ and

$H(kz) = (kz)^n\,\overline{P(k/\overline{kz})} = k^nz^n\,\overline{P(1/\overline{z})}$, then from (2.23) and for $|z| = 1$, one gets

$$|G(kz)| = |P(k^2z)| \geq k^n|P(z)|$$

and $|H(kz)| = |k^nz^n\,\overline{P(1/\overline{z})}| = k^n|P(z)|$

Hence, we rewrite (2.24) as

$$k^n|P(z)| + k^n|P(z)| \le (k^n+1)\max_{|z|=1}|G(z)|$$
$$= (k^n+1)\max_{|z|=k}|P(z)|$$

on $|z| = 1$. It follows

$$\max_{|z|=k}|P(z)| \ge \frac{2k^n}{k^n+1}\max_{|z|=1}|P(z)| = \frac{2k^n}{k^n+1}$$

The proof is completed.

Now, we begin to prove Theorem 2.6.

<u>Proof of Theorem 2.6</u>: Define $G(z)=P(kz)$. Since $P(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k \ge 1$. Hence,

$$G(z) = P(kz) = \prod_{\nu=1}^{n}(kz-z_\nu) = k^n\prod_{\nu=1}^{n}(z-\frac{z_\nu}{k})$$

has all its zeros in $|z| \le 1$. For $0 \le \theta \le 2\pi$, we have

$$\left|\frac{G'(e^{i\theta})}{G(e^{i\theta})}\right| = \left|\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})}\right| \ge \operatorname{Re}\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})}$$
$$= \operatorname{Re}\sum_{\nu=1}^{n}\frac{e^{i\theta}}{e^{i\theta}-z_\nu/k}$$
$$= \sum_{\nu=1}^{n}\operatorname{Re}\frac{ke^{i\theta}}{ke^{i\theta}-z_\nu}$$
$$= \sum_{\nu=1}^{n}\operatorname{Re}\frac{k}{k-z_\nu e^{-i\theta}}$$
$$\ge \sum_{\nu=1}^{n}\operatorname{Re}\frac{k}{k+|z_\nu|}$$
$$= \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}$$

which implies

$$\max_{|z|=1}|G'(z)| \ge \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|G(z)| \qquad (2.25)$$

Since $G'(z) = kP'(kz)$, we rewrite (2.25) as

$$k \max_{|z|=1} |P'(kz)| \geq \sum_{\upsilon=1}^{n} \frac{k}{k+|z_\upsilon|} \max_{|z|=1} |P(kz)| \qquad (2.26)$$

Consider the left hand side of (2.26). We know that $P'(kz)$ is a polynomial of degree n-1. Hence

$$\max_{|z|=1} |P'(kz)| \leq k^{n-1} \max_{|z|=1} |P'(z)| \qquad (2.27)$$

Further, applying lemma 2.4 to the right hand side of (2.26), we get

$$\max_{|z|=1} |P(kz)| = \max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} \qquad (2.28)$$

From (2.26), (2.27) and (2.28) we conclude

$$k^n \max_{|z|=1} |P'(z)| \geq \sum_{\upsilon=1}^{n} \frac{k}{k+|z_\upsilon|} \frac{2k^n}{1+k^n}$$

So, (2.14) is proved.

To see the result is best possible, we note that the zeros of the polynomial $\frac{z^n+k^n}{1+k^n}$ must be on $|z| = k$. Hence, Theorem 2.6 is proved.

Chapter III

Callahan's Theorem

3.1

This chapter is mainly devoted to the study of inequality concerning $\int_0^{2\pi}|P(e^{i\theta})|^2d\theta$ and $\max_{|z|=1}\left|\frac{P(z)}{z-1}\right|$ when one of the zeros of $P(z)$ is prescribed, namely, $P(1) = 0$. The polynomial with one prescribed zero seems to be of considerable interest and a number of interesting inequalities have been obtained in the past thirty years. The inspiration of such a study originates in the work of Callahan where he obtained the estimate of $\int_0^{2\pi}|P(e^{i\theta})|^2d\theta$ in terms of $\max_{\theta}|P(e^{i\theta})|$ when $P(1) = 0$. It is obvious that for any polynomial $P(z)$ there holds

$$\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2d\theta \leq \max_{|z|=1}|P(z)| \tag{3.1}$$

Of course, there is equality in (3.1) for $P(z)$ = constant. Prescribing one of the zeros of a polynomial of degree n, Callahan [8] obtained a best possible inequality, see Theorem 3.1. In this chapter, we discuss the work of Callahan and subsequent works of Aziz, Boas, Donaldson, Mohammad and Rahman.

3.2

In 1959, Callahan proved:

<u>Theorem 3.1</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>with</u> $P(1) = 0$, <u>then</u>

$$\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2d\theta \leq \frac{n}{n+1}\max_{|z|=1}|P(z)|^2 \tag{3.2}$$

<u>The result is best possible.</u>

For the construction of extremal polynomial, see remark 3.1.

In order to prove Theorem 3.1, we need the following lemma due to Fejér and Riesz [19]:

Lemma 3.1: Let $h(z) = \sum_{\upsilon=-n}^{n} h_\upsilon z^\upsilon (\overline{h}_\upsilon = h_{-\upsilon})$. Then there exists a polynomial $P(z)$ of degree n such that $h(z) = |P(z)|^2$ for $|z| = 1$ if and only if $h(z) \geq 0$ for $|z| = 1$.

Proof of lemma 3.1: We may suppose that $h(z) > 0$ on $|z| = 1$ since the general case can be reduced to this case by adding to $h(z)$ the constant term $\varepsilon > 0$, which then is made to tend to 0. Now, let

$$\begin{aligned} g(z) &= z^n \left(\sum_{\upsilon=-n}^{n} h_\upsilon z^\upsilon\right) \\ &= \sum_{\upsilon=-n}^{n} h_\upsilon z^{\upsilon+n} \\ &= \sum_{\upsilon=0}^{2n} h_{\upsilon-n} z^\upsilon \end{aligned}$$

We claim that $g(z) = z^{2n}\, \overline{g(1/\overline{z})}$. In fact,

$$\begin{aligned} z^{2n}\, \overline{g(1/\overline{z})} &= z^{2n} \overline{[h_{-n} + h_{-n+1}(1/\overline{z}) + \ldots + h_{n-1}(1/\overline{z})^{2n-1} + h_n (1/\overline{z})^{2n}]} \\ &= z^{2n}[\overline{h}_{-n} + \overline{h}_{-n+1}(1/z) + \ldots + \overline{h}_{n-1}(1/z)^{2n-1} + \overline{h}_n (1/z)^{2n}] \\ &= \overline{h}_{-n} z^{2n} + \overline{h}_{-n+1} z^{2n-1} + \ldots + \overline{h}_{n-1} z + \overline{h}_n \\ &= h_n z^{2n} + h_{n-1} z^{2n-1} + \ldots + h_{-n+1} z + h_{-n} \\ &= g(z) \end{aligned} \tag{3.3}$$

because $\overline{h}_\upsilon = h_{-\upsilon}$. Further, from (3.3), we observe that if α is an exterior zero to the unit circle, then there must be an interior zero, $\frac{1}{\overline{\alpha}}$, of $g(z)$ and conversely. Note that α is not a zero when $|\alpha| = 1$.

Hence, we get

$$g(z) = c(z-\alpha_1).....(z-\alpha_n)(z-\frac{1}{\bar{\alpha}_1}).....(z-\frac{1}{\bar{\alpha}_n}) \qquad (3.4)$$

Secondly, we note that on $|z| = 1$ and in view of (3.4), we have

$$h(e^{i\theta}) = e^{-in\theta}g(e^{i\theta})$$
$$= e^{-in\theta}c\,(e^{i\theta}-\alpha_1).....(e^{i\theta}-\alpha_n)(e^{i\theta}-\frac{1}{\bar{\alpha}_1}).....(e^{i\theta}-\frac{1}{\bar{\alpha}_n})$$
$$= c(e^{i\theta}-\alpha_1)\,(e^{i\theta}-\alpha_2).....(e^{i\theta}-\alpha_n)\,(1-\frac{e^{-i\theta}}{\bar{\alpha}_1}).....(1-\frac{e^{-i\theta}}{\bar{\alpha}_n})$$
$$= \frac{(-1)^n c}{\bar{\alpha}_1.....\bar{\alpha}_n}\,(e^{i\theta}-\alpha_1).....(e^{i\theta}-\alpha_n)\,(e^{-i\theta}-\bar{\alpha}_1).....(e^{-i\theta}-\bar{\alpha}_n)$$
$$= \frac{(-1)^n c}{\bar{\alpha}_1.....\bar{\alpha}_n}|(e^{i\theta}-\alpha_1).....(e^{i\theta}-\alpha_n)|^2$$

Setting $\beta = \frac{(-1)^n c}{\bar{\alpha}_1.....\bar{\alpha}_n}$, we have $\beta > 0$ because $h(z) > 0$. It follows that $P(z) = \sqrt{\beta}\prod_{\upsilon=1}^{n}(z-\alpha_\upsilon)$ is the polynomial which satisfies the requirements of the lemma.

Now, we begin to present:

Proof of Theorem 3.1: Let $P(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ and $M = \max_{|z|=1}|P(z)|$, then $|P(z)|^2 \le M^2$. Hence, $M^2-|P(z)|^2 \ge 0$. For $|z| = 1$

$$M^2 - |P(e^{i\theta})|^2 = M^2 - P(e^{i\theta})\overline{P(e^{i\theta})}$$
$$= M^2 - (\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon)(\overline{\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon})$$
$$= M^2 - [c_0\bar{c}_n e^{-in\theta} + (\sum_{\upsilon=0}^{1} c_\upsilon\bar{c}_{\upsilon+(n-1)})e^{-i(n-1)\theta} + \ldots + (\sum_{\upsilon=0}^{n-1} c_\upsilon\bar{c}_{\upsilon+1})e^{-i\theta}$$
$$+ \sum_{\upsilon=0}^{n} c_\upsilon\bar{c}_\upsilon + (\sum_{\upsilon=0}^{n-1} c_{\upsilon+1}\bar{c}_\upsilon)e^{i\theta} + \ldots + (\sum_{\upsilon=0}^{1} c_{\upsilon+(n-1)}\bar{c}_\upsilon)e^{i(n-1)\theta} + \bar{c}_0 c_n e^{in\theta}]$$

We note that $M^2-|P(e^{i\theta})|^2$ satisfies the conditions of lemma 3.1, so there exists a polynomial $g(z)$ of degree n such that

$$M^2 - |P(z)|^2 = |g(z)|^2 \qquad (3.5)$$

Further, we observe that the polynomial $g(z)$ has the following properties:

1) $|g(z)| \le M$ for $|z| \le 1$

2) Since $P(1) = 0$, so $|g(1)|^2 = M^2$. Without loss of generality, we take $g(1) = M$.

3) For $z = e^{i\theta}$, we integrate both sides of (3.5) and have

$$\frac{1}{2\pi}\int_0^{2\pi}(M^2 - |P(e^{i\theta})|^2)d\theta = \frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta$$

$$\text{So} \quad M^2 - \frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta \tag{3.6}$$

In order to select the greatest values of $\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta$ from the class of polynomials satisfying (3.6), we must minimize the integral $\frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta$. Let $g(z) = \sum_{\upsilon=0}^{n} g_\upsilon z^\upsilon$ and consider

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{\upsilon=0}^{n} g_\upsilon e^{i\upsilon\theta}\Big)\Big(\sum_{\upsilon=0}^{n}\bar{g}_\upsilon e^{-i\upsilon\theta}\Big)d\theta \\ &= \sum_{\upsilon=0}^{n}|g_\upsilon|^2 \end{aligned}\tag{3.7}$$

On the other hand, recall that $g(1) = M$, we have

$$\sum_{\upsilon=0}^{n} g_\upsilon = M \tag{3.8}$$

As it turns out, we want to minimize the quantity $\sum_{\upsilon=0}^{n}|g_\upsilon|^2$ subject to the constraint (3.8). Applying Schwarz inequality to (3.8), we have

$$M = \sum_{\upsilon=0}^{n} g_\upsilon \cdot 1$$

$$\leq \left(\sum_{\upsilon=0}^{n} |g_\upsilon|^2\right)^{1/2} \left(\sum_{\upsilon=0}^{n} 1^2\right)^{1/2}$$

$$= \left(\sum_{\upsilon=0}^{n} |g_\upsilon|^2\right)^{1/2} (n+1)^{1/2} \qquad (3.9)$$

Setting $g_\upsilon = \frac{M}{n+1}$, for all υ we get the smallest value in the right hand side of (3.9) because

$$\left(\sum_{\upsilon=0}^{n} |g_\upsilon|^2\right)^{1/2} (n+1)^{1/2}$$

$$= \left(\sum_{\upsilon=0}^{n} \left|\frac{M}{n+1}\right|^2\right)^{1/2} (n+1)^{1/2}$$

$$= \left[\left(\frac{M}{n+1}\right)^2 \sum_{\upsilon=0}^{n} 1\right]^{1/2} (n+1)^{1/2}$$

$$= \left[\left(\frac{M}{n+1}\right)^2 (n+1)\right]^{1/2} (n+1)^{1/2}$$

$$= M$$

Therefore, we obtain that the corresponding polynomial is

$$g(z) = \sum_{\upsilon=0}^{n} \frac{M}{n+1} z^\upsilon$$

$$= \frac{M}{n+1} \sum_{\upsilon=0}^{n} z^\upsilon \qquad (3.10)$$

It is well known that the polynomial (3.10) is equal to M for $z = 1$ and vanishes when z is any of other $(n+1)^{\underline{st}}$ roots of unity because

$$g(z) = \frac{M}{n+1} \frac{z^{n+1} - 1}{z - 1}$$

Also, note that for $z = e^{i\theta}$, we have

$$|g(e^{i\theta})| = \left|\frac{M}{n+1} \cdot \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}\right|$$

$$= \left| \frac{M}{n+1} \cdot \frac{e^{\frac{i(n+1)\theta}{2}}\left(e^{\frac{i(n+1)\theta}{2}} - e^{\frac{-i(n+1)\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}\right)} \right|$$

$$= \left| \frac{M}{n+1} \cdot \frac{\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}} \right|$$

Recall that $|\sin n\theta| \le n\,|\sin\theta|\ \forall n$, so we have

$$|g(e^{i\theta})| = \frac{M}{n+1} \left| \frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \right|$$

$$\le \frac{M}{n+1} \cdot \frac{(n+1)\left|\sin\frac{\theta}{2}\right|}{\left|\sin\frac{\theta}{2}\right|}$$

$$= M. \tag{3.11}$$

From (3.11), it is clear that $\max_{|z|=1} |g(z)| = M$, for $|z| \le 1$. As a consequence, $M^2 - |g(z)|^2 \ge 0$. By the previous argument, we know that $M^2 - |g(z)|^2$ satisfies the conditions of lemma 3.1. Hence the associated polynomial $P(z)$ of degree n is

$$M^2 - |g(z)|^2 = |P(z)|^2 \tag{3.12}$$

and has the following properties:

(1) $|P(z)| \le M$ for $|z| \le 1$. Further, since $g(z)$ vanishes when z is any of $(n+1)^{\underline{st}}$ roots of unity except $z = 1$, so let $z^* \ne 1$ be any of $(n+1)^{\underline{st}}$ roots of unity, by (3.12), we have

$$|P(z^*)|^2 = M^2 - |g(z^*)|^2 = M^2 - 0 = M^2$$

i.e. the maximum is attained. It follows that

$$\max_{|z|=1} |P(z)| = M$$

2) Since $g(1) = M$, one has $P(1) = 0$.

3) Integrate both sides of (3.12) and apply (3.7), one gets

$$\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}(M^2 - |g(e^{i\theta})|^2)d\theta$$

$$= M^2 - \frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^2 d\theta$$

$$= M^2 - \sum_{\upsilon=0}^{n}|g_\upsilon|^2$$

Since when $g_\upsilon = \frac{M}{n+1}$, for all υ, the value of $\sum_{\upsilon=0}^{n}|g_\upsilon|^2$ is the smallest, so we get the greatest value of:

$$\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta = M^2 - \sum_{\upsilon=0}^{n}\left(\frac{M}{n+1}\right)^2$$

$$= M^2 - \left(\frac{M}{n+1}\right)^2(n+1)$$

$$= \frac{n}{n+1}M^2$$

$$= \frac{n}{n+1}\max_{|z|=1}|P(z)|^2$$

Note that this associated polynomial $P(z)$ satisfies the conditions of Theorem 3.1. We complete the proof.

Remark 3.1(*): The above method is in fact constructive. For each n, we can construct an extremal polynomial $P(z)$ for which $P(1) = 0$ and there is equality in (3.2). Callahan obtained:

for $n = 1$, $P(z) = M(\frac{z-1}{2})$

for $n = 2$, $P(z) = \frac{M}{\sqrt{18}}\left[(1+\sqrt{3})z^2 - 2z + (1-\sqrt{3})\right]$

Following a lengthy calculation, we obtain for $n = 3$, $\quad \tilde{P}(z) = \dfrac{|(1-i)+\sqrt{-2i-1}|M}{4}\Big\{z^3+[1-\sqrt{-2i-1}+\sqrt{2i-1}]\,z^2$

$$+\,[-2+\sqrt{-2i-1} - \sqrt{2i-1}+|(1-i)-\sqrt{-2i-1}|^2]z-|(1-i)-\sqrt{-2i-1}|^2\Big\}$$

We can explain the above calculation in short. Since (3.5) holds for $|z| = 1$, we observe that

$$\begin{aligned}
|P(z)|^2 &= M^2 - |g(z)|^2 \\
&= M^2\left\{1 - \left|\frac{1+z+z^2+z^3}{4}\right|^2\right\} \\
&= M^2\left\{1-\frac{1}{16}(1+z+z^2+z^3)(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3})\right\} \\
&= M^2\left\{-\frac{1}{16}[(\frac{1}{z^3}+z^3)+2(\frac{1}{z^2}+z^2)+3(z+\frac{1}{z})-12]\right\}
\end{aligned}$$

Let $y = z + \frac{1}{z}$ so $z^2 + \frac{1}{z^2} = y^2-2$ and $z^3 + \frac{1}{z^3} = y^3 - 3y$

Hence

$$\begin{aligned}
|P(z)|^2 &= M^2\left\{-\frac{1}{16}(y^3 + 2y^2 - 16)\right\} \\
&= M^2\left\{-\frac{1}{16}(y-2)[y-(-2+2i)][y-(-2-2i)]\right\} \\
&= M^2\left\{-\frac{1}{16z^3}(z^2-2z+1)[z^2+(2-2i)z+1][z^2+(2+2i)z+1]\right\} \\
&= M^2\Big\{-\frac{1}{16z^3}(z-1)^2[z+(1-i)-\sqrt{-2i-1}][z+(1-i)+\sqrt{-2i-1}] \\
&\qquad [z+(1+i)-\sqrt{2i-1}][z+(1+i)+\sqrt{2i-1}]\Big\} \qquad (3.13)
\end{aligned}$$

Since $\dfrac{1}{(1-i)-\sqrt{-2i-1}} = (1-i)+\sqrt{-2i-1}$ and

$$\frac{1}{(1+i)-\sqrt{2i-1}} = (1+i)+\sqrt{2i-1}$$

Hence, we rewrite (3.13) as

$$|P(z)|^2 = M^2 \left\{\frac{1}{16}(z-1)(\frac{1}{z}-1)\left[z+(1-i)+\sqrt{-2i-1}\right]\left[\frac{(1-i)+\sqrt{-2i-1}}{z}+1\right]\right.$$

$$\left.\left[z+(1+i)+\sqrt{2i-1}\right]\left[\frac{(1+i)-\sqrt{2i-1}}{z}+1\right]\right\}$$

$$= M^2 \left\{\frac{|(1-i)+\sqrt{-2i-1}|^2}{16}(z-1)\overline{(z-1)}\left[z+(1-i)-\sqrt{-2i-1}\right]\right.$$

$$\left.\left[\overline{z+(1-i)-\sqrt{-2i-1}}\right]\left[z+(1+i)+\sqrt{2i-1}\right]\left[\overline{z+(1+i)+\sqrt{2i-1}}\right]\right\}.$$

Note that $\frac{|(1-i)+\sqrt{-2i-1}|^2}{16}$ is positive real number.

So $P(z) = \frac{|(1-i)+\sqrt{-2i-1}|M}{4}(z-1)\left[z+(1-i)-\sqrt{-2i-1}\right]\left[z+(1+i)+\sqrt{2i-1}\right]$

$$= \frac{|(1-i)+\sqrt{-2i-1}|M}{4}\left\{z^3+\left[1-\sqrt{-2i-1}+\sqrt{2i-1}\right]z^2\right.$$

$$\left.+\left[-2+\sqrt{-2i-1}-\sqrt{2i-1}+|(1-i)-\sqrt{-2i-1}|^2\right]z-|(1-i)-\sqrt{-2i-1}|^2\right\}$$

3.3

Later, Boas [6] gave a sharpened form of Callahan's inequality by taking the maximum of $|P(z)|$ on the right hand side of (3.2) over the $(n+1)^{\underline{th}}$ roots of unity rather than on $|z|=1$. In doing so, he gave an alternate proof of Theorem 3.1. He established:

Theorem 3.2: If $P(z)$ is a polynomial of degree n with $P(1)=0$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta \le \frac{n}{n+1}\max_{1\le\nu\le n}|P(e^{\frac{2\pi\nu i}{n+1}})|^2 \qquad (3.14)$$

In fact, he deduced Theorem 3.2 from the following result concerning trigonometric polynomial:

Theorem 3.3: Let $t(\theta) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}$ be a trigonometric polynomial of order n with $t(0) = 0$, then

$$|c_0| \le \frac{n}{n+1} \max_{1\le\upsilon\le n} |t(\frac{2\pi\upsilon}{n+1})| \tag{3.15}$$

For the proof of Theorem 3.2, let $P(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be the polynomial of degree n with $P(1) = 0$. Since

$$|P(e^{i\theta})|^2 = \sum_{\upsilon=0}^{n} a_\upsilon e^{i\upsilon\theta} \sum_{\upsilon=0}^{n} \bar{a}_\upsilon e^{-i\upsilon\theta}.$$

$$= \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}.$$

$$= t(\theta).$$

where $c_0 = \sum_{\upsilon=0}^{n} |a_\upsilon|^2$ is a trigonometric polynomial with $t(0) = |P(1)|^2 = 0$, in view of (3.15) and the fact

$$\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta = \sum_{\upsilon=0}^{n} |a_\upsilon|^2 = c_0 ,$$

(3.14) is proved immediately.

Now, we return to present the proof of Theorem 3.3 due to Boas.

Proof of Theorem 3.3: First, we proved the special case of discrete form of Poisson's summation formula stated as following: if $t(\theta) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}$ and s is a positive integer, then

$$\sum_{p=0}^{s-1} e^{\frac{-2\pi ipm}{s}} t(k + \frac{2\pi p}{s}) = s \sum_{\upsilon=m(\text{mod } s)} c_\upsilon e^{i\upsilon k} \tag{3.16}$$

where k is an integer. In fact, since $t(\theta) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon\theta}$,

so

$$t(k + \frac{2\pi p}{s}) = \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon(k+ \frac{2\pi p}{s})}$$

Multiplying both sides by $\sum_{p=0}^{s-1} e^{\frac{-2\pi ipm}{s}}$, where m is an integer, we get

$$\sum_{p=0}^{s-1} e^{\frac{-2\pi ipm}{s}} t(k + \frac{2\pi p}{s})$$

$$= \sum_{p=0}^{s-1} e^{\frac{-2\pi ipm}{s}} \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon(k+ \frac{2\pi p}{s})}$$

$$= \sum_{p=0}^{s-1} \left[\sum_{\upsilon=-n}^{n} c_\upsilon e^{(i\upsilon k+ \frac{2\pi ip\upsilon}{s})} \right] e^{\frac{-2\pi ipm}{s}}$$

$$= \sum_{p=0}^{s-1} \left(\sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon k} e^{\frac{2\pi ip\upsilon}{s}} \right) e^{\frac{-2\pi ipm}{s}}$$

$$= \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon k} \left(\sum_{p=0}^{s-1} e^{\frac{2\pi ip\upsilon}{s}} e^{\frac{-2\pi ipm}{s}} \right)$$

$$= \sum_{\upsilon=-n}^{n} c_\upsilon e^{i\upsilon k} \left[\sum_{p=0}^{s-1} e^{\frac{2\pi ip(\upsilon-m)}{s}} \right] \qquad (3.17)$$

Since υ runs from $-n$ to n, let us consider the following cases in (3.17): i) $\upsilon=m$; ii) $\upsilon-m$ is divided by s and iii) $\upsilon-m$ is not divided by s. In the first case, we get $e^{\frac{2\pi ip(\upsilon-m)}{s}} = 1$, so $\sum_{p=0}^{s-1} e^{\frac{2\pi ip(\upsilon-m)}{s}} = s$. Concerning the second case, we can express υ as $m + ks$ for $k = \dots,-1,0,1,\dots$. Hence $e^{\frac{2\pi ip(\upsilon-m)}{s}} = e^{\frac{2\pi ip(m+ks-m)}{s}} = e^{2\pi ipk} = 1$, so $\sum_{p=0}^{s-1} e^{\frac{2\pi ip(\upsilon-m)}{s}} = s$ too. With regard to the last case, we know that $e^{\frac{2\pi ip(k-m)}{s}}$ is $s^{\underline{th}}$ roots of unity, so $\sum_{p=0}^{s-1} e^{\frac{2\pi ip(\upsilon-m)}{s}} = 0$. This proves (3.16).

In order to prove (3.15), we only need to take $s=n+1$, $m=0$, $k=0$ in (3.16). Using such specified values, we get

$$\sum_{p=0}^{n} t\left(\frac{2\pi p}{n+1}\right) = (n+1) \sum_{\upsilon \equiv 0 (\mathrm{mod}\ n+1)} c_{\upsilon} e^{i\upsilon k}$$

Since υ runs through $-n$ to n, the only possibility of υ is zero. Therefore, the equality

$$\sum_{p=0}^{n} t\left(\frac{2\pi p}{n+1}\right) = (n+1)\ c_0 \tag{3.18}$$

holds. Next, taking absolute value on both sides of (3.18) and use the fact $t(0) = 0$, we have

$$|c_0| = \left|\frac{\sum_{p=0}^{n} t\left(\frac{2\pi p}{n+1}\right)}{n+1}\right| = \left|\frac{\sum_{p=1}^{n} t\left(\frac{2\pi p}{n+1}\right)}{n+1}\right|$$

$$\leq \frac{n \max\limits_{1\leq p\leq n} \left|t\left(\frac{2\pi p}{n+1}\right)\right|}{|n+1|}$$

$$\leq \frac{n}{n+1} \max_{1\leq p\leq n} \left|t\left(\frac{2\pi p}{n+1}\right)\right|$$

We complete the proof of Theorem 3.3.

3.4

Further, we consider some other results in this direction. Given β an arbitrary non-negative real number and $P(z)$ be a polynomial of degree n with $P(\beta) = 0$. We ask how large can $\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta$ be if $\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta = 1$. In 1972, Donaldson and Rahman [10] answered this question as following:

Theorem 3.4: If $P(z)$ is a polynomial of degree n with $\int_0^{2\pi}|P(e^{i\theta})|^2 d\theta = 1$ and $P(\beta) = 0$, where β is an arbitrary non-negative real number, then

$$\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \le [1+\beta^2-2\beta\cos(\tfrac{\pi}{n+1})]^{-1} \tag{3.19}$$

Proof of Theorem 3.4: First, if we let

$\frac{P(z)}{z-\beta} = \alpha_{n-1} z^{n-1}+\alpha_{n-2} z^{n-2}+\ldots+\alpha_1 z+\alpha_0$, where $\alpha_{n-1}\neq 0$, then

$$\begin{aligned}P(z) &= (z-\beta)\,\frac{P(z)}{z-\beta} \\ &= (z-\beta)(\alpha_{n-1} z^{n-1}+\alpha_{n-2} z^{n-2}+\ldots+\alpha_1 z+\alpha_0) \\ &= \alpha_{n-1}z^n+\alpha_{n-2}z^{n-1}+\ldots+\alpha_1 z^2+\alpha_0 z-\beta\alpha_{n-1}z^{n-1}-\beta\alpha_{n-2}z^{n-2}-\ldots-\beta\alpha_1 z-\alpha_0\beta \\ &= \alpha_{n-1}z^n+(\alpha_{n-2}-\beta\alpha_{n-1})z^{n-1}+\ldots+(\alpha_1-\beta\alpha_2)z^2+(\alpha_0-\beta\alpha_1)z-\alpha_0\beta\end{aligned}$$

Now, let us consider

$$\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \Big/ \int_0^{2\pi}|P(e^{i\theta})|^2 d\theta$$

$$= \frac{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}{|\alpha_{n-1}|^2+\sum_{\upsilon=1}^{n-1}|\alpha_{\upsilon-1}-\beta\alpha_\upsilon|^2+\beta^2|\alpha_0|^2}$$

$$\le \frac{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}{|\alpha_{n-1}|^2+\sum_{\upsilon=1}^{n-1}(|\alpha_{\upsilon-1}|^2-2\beta|\alpha_{\upsilon-1}||\alpha_\upsilon|+\beta^2|\alpha_\upsilon|^2)+\beta^2|\alpha_0|^2}$$

$$= \frac{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}{(1+\beta^2)\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2-2\beta\sum_{\upsilon=1}^{n-1}|\alpha_{\upsilon-1}||\alpha_\upsilon|}$$

$$= \frac{1}{(1+\beta^2)-2\beta\dfrac{\sum_{\upsilon=1}^{n-1}|\alpha_{\upsilon-1}||\alpha_\upsilon|}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}} \tag{3.20}$$

In view of (3.20), we note that we can prove inequality (3.19) if the maximum of the function

$$f(|\alpha_0|,|\alpha_1|,\ldots,|\alpha_{n-1}|) = \frac{\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2} \tag{3.21}$$

is less than 1 and the maximum value is cos $(\frac{\pi}{n+1})$.

If we apply the Schwarz inequality to the numerator of (3.21), we have

$$f(|\alpha_0|,|\alpha_1|,\ldots,|\alpha_{n-1}|)$$

$$\leq \frac{\left(\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon|^2\right)^{1/2}\left(\sum_{\upsilon=1}^{n-1} |\alpha_{\upsilon-1}|^2\right)^{1/2}}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}$$

$$\leq \frac{\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)^{1/2}\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)^{1/2}}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}$$

$$= 1.$$

Moreover, we note that the maximum is attained when all of the numbers $|\alpha_\upsilon|$ are non-zero. If we suppose for some υ, $\alpha_\upsilon=0$ and j is the smallest positive integer such that $\alpha_{\upsilon-j}$, $\alpha_{\upsilon+j}$ are not both zero, then we can find out a number

$$|\alpha'_\upsilon| \leq \frac{|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|}{f(|\alpha_0|,|\alpha_1|,\ldots,|\alpha_{\upsilon-1}|,0,|\alpha_{\upsilon+1}|,\ldots,|\alpha_{n-1}|)} \tag{3.22}$$

such that

$$f(|\alpha_0|,|\alpha_1|,\ldots,|\alpha_{\upsilon-1}|,0,|\alpha_{\upsilon+1}|,\ldots,|\alpha_{n-1}|)$$
$$\leq f(|\alpha_0|,|\alpha_1|,\ldots,|\alpha_{\upsilon-1}|,|\alpha'_\upsilon|,|\alpha_{\upsilon+1}|,\ldots,|\alpha_{n-1}|). \tag{3.23}$$

In fact, let

$$A = |\alpha_1||\alpha_0| + |\alpha_2||\alpha_1| + \ldots + |\alpha_{\upsilon-j}||\alpha_{\upsilon-j-1}| + |\alpha_{\upsilon+j+1}||\alpha_{\upsilon+j}| + \ldots + |\alpha_{n-1}||\alpha_{n-2}|$$

$$B = |\alpha_0|^2 + |\alpha_1|^2 + \ldots + |\alpha_{\upsilon-j}|^2 + |\alpha_{\upsilon+j}|^2 + \ldots + |\alpha_{n-1}|^2$$

From (3.22), we have

$$|\alpha'_\upsilon| \le \frac{|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|}{\frac{A}{B}}$$

It follows that

$$f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{\upsilon-1}|, 0, |\alpha_{\upsilon+1}|, \ldots, |\alpha_{n-1}|)$$

$$= \frac{A}{B}$$

$$\le \frac{|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|}{|\alpha'_\upsilon|}$$

$$= \frac{|\alpha'_\upsilon|(|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|)}{|\alpha'_\upsilon|^2}$$

Hence

$$A|\alpha'_\upsilon|^2 \le B|\alpha'_\upsilon|(|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|)$$

$$\Rightarrow AB + A|\alpha'_\upsilon|^2 \le AB + B|\alpha_\upsilon|(|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|)$$

$$\Rightarrow \frac{A}{B} \le \frac{A + |\alpha'_\upsilon|(|\alpha_{\upsilon-j}| + |\alpha_{\upsilon+j}|)}{B + |\alpha'_\upsilon|^2}$$

$$= f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{\upsilon-1}|, |\alpha'_\upsilon|, |\alpha_{\upsilon+1}|, \ldots, |\alpha_{n-1}|)$$

This proved (3.23).

On the other hand, we also note that if one or more of the numbers $|\alpha_\upsilon|$ are allowed to be arbitrarily large, then the function (3.21) is bounded above by $\frac{n-1}{n}$. In fact, suppose

$|\alpha_0| = \ldots = |\alpha_{n-1}| \to \infty$, then

$$f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|)$$

$$= \frac{\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}$$

$$= \frac{(n-1)|\alpha_\upsilon|^2}{n|\alpha_\upsilon|^2}$$

$$\to \frac{n-1}{n}.$$

Next, we want to find out the maximum value of the function (3.21). Let us consider the partial derivatives of (3.21) with respect to the variables $|\alpha_0|$, $|\alpha_1|, \ldots, |\alpha_{n-1}|$, then we get

$$\frac{\partial f}{\partial |\alpha_0|} = \frac{\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)\frac{\partial}{\partial|\alpha_0|}\left(\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|\right) - \left(\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|\right)\frac{\partial}{\partial|\alpha_0|}\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)}{\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)^2}$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}\left[|\alpha_1| - 2\,\frac{\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}\,|\alpha_0|\right]$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2}\left[|\alpha_1| - 2.f.|\alpha_0|\right]$$

$$\frac{\partial f}{\partial |\alpha_\mu|} = \frac{\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)\frac{\partial}{\partial|\alpha_\mu|}\left(\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|\right) - \left(\sum_{\upsilon=1}^{n-1} |\alpha_\upsilon||\alpha_{\upsilon-1}|\right)\frac{\partial}{\partial|\alpha_\mu|}\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)}{\left(\sum_{\upsilon=0}^{n-1} |\alpha_\upsilon|^2\right)^2}$$

$$= \frac{\left(\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2\right)\left(|\alpha_{\mu-1}|+|\alpha_{\mu+1}|\right)-\left(\sum_{\upsilon=1}^{n-1}|\alpha_\upsilon||\alpha_{\upsilon-1}|\right).2|\alpha_\mu|}{\left(\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2\right)^2}$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}\left[|\alpha_{\mu-1}|+|\alpha_{\mu+1}|-2.\frac{\sum_{\upsilon=1}^{n-1}|\alpha_\upsilon||\alpha_{\upsilon-1}|}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}.|\alpha_\mu|\right]$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}\left[|\alpha_{\mu-1}|+|\alpha_{\mu+1}|-2.f.|\alpha_\mu|\right]$$

for $\mu = 1,\ldots, n-2$

$$\frac{\partial f}{\partial|\alpha_{n-1}|} = \frac{\left(\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2\right)\frac{\partial}{\partial|\alpha_{n-1}|}\left(\sum_{\upsilon=1}^{n-1}|\alpha_\upsilon||\alpha_{\upsilon-1}|\right)-\left(\sum_{\upsilon=1}^{n-1}|\alpha_\upsilon||\alpha_{\upsilon-1}|\right)\frac{\partial}{\partial|\alpha_{n-1}|}\left(\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2\right)}{\left(\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2\right)^2}$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}\left[|\alpha_{n-2}|-2.\frac{\sum_{\upsilon=1}^{n-1}|\alpha_\upsilon||\alpha_{\upsilon-1}|}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}.|\alpha_{n-1}|\right]$$

$$= \frac{1}{\sum_{\upsilon=0}^{n-1}|\alpha_\upsilon|^2}\left[|\alpha_{n-2}|-2f|\alpha_{n-1}|\right]$$

In order to find a local maximum, we must solve the following equations:

$$\left\{\begin{array}{ll} |\alpha_1| - 2f|\alpha_0| = 0 & \\ |\alpha_{\mu-1}|+|\alpha_{\mu+1}| - 2f|\alpha_\mu| = 0 & \mu = 1,2,\ldots, n-2 \\ |\alpha_{n-2}| - 2f|\alpha_{n-1}| = 0 & \end{array}\right. \quad (3.24)$$

Now, let us suppose that the required local maximum is λ. Since the maximum value of (3.21) is less than 1, we know that $\lambda < 1$ and write $\lambda = \cos\gamma$ $(\gamma \neq 0)$. Solving the equations of the system (3.24), we first claim

$$U_{\mu+1}(\lambda) = 2\lambda U_{\mu}(\lambda) - U_{\mu-1}(\lambda) \tag{3.25}$$

where $U_{\mu}(\lambda) = \dfrac{\sin(\mu+1)\gamma}{\sin\gamma}$ is the Chebyshev polynomial of the second kind of degree μ in λ. In fact,

$$\begin{aligned} & 2\lambda U_{\mu}(\lambda) - U_{\mu-1}(\lambda) \\ &= \frac{2\cos\gamma\sin(\mu+1)\gamma - \sin\mu\gamma}{\sin\gamma} \\ &= \frac{2.\frac{1}{2}[\sin(\mu+2)\gamma + \sin\mu\gamma] - \sin\mu\gamma}{\sin\gamma} \\ &= \frac{\sin(\mu+2)\gamma}{\sin\gamma} \\ &= U_{\mu+1}(\lambda) \end{aligned}$$

This proves (3.25)

Now, we return to solve the equations (3.24). From the first equation of (3.24), we have

$$\begin{aligned} |\alpha_1| &= 2\lambda|\alpha_0| \\ &= 2\cos\gamma\,|\alpha_0| \\ &= \frac{2\cos\gamma\sin\gamma\,|\alpha_0|}{\sin\gamma} \\ &= \frac{\sin 2\gamma}{\sin\gamma}|\alpha_0| \\ &= U_1(\lambda)|\alpha_0| \end{aligned}$$

With regard to the middle equations and (3.25) we use the fact $U_0(\lambda) = 1$ and then have

$$|\alpha_2| = 2\lambda|\alpha_1| - |\alpha_0|$$
$$= 2\lambda U_1(\lambda)|\alpha_0| - U_0(\lambda)|\alpha_0|$$
$$= [2\lambda U_1(\lambda) - U_0(\lambda)]|\alpha_0|$$
$$= U_2(\lambda)|\alpha_0|$$

Suppose $|\alpha_\mu| = U_\mu(\lambda)|\alpha_0|$ for $\mu = 3,\ldots,n-2$. By induction, we have

$$|\alpha_{n-1}| = 2\lambda|\alpha_{n-2}| - |\alpha_{n-3}|$$
$$= [2\lambda U_{n-2}(\lambda) - U_{n-3}(\lambda)]|\alpha_0|$$
$$= U_{n-1}(\lambda)|\alpha_0|$$

Hence, we obtain

$$|\alpha_\mu| = U_\mu(\lambda)|\alpha_0| \qquad \mu = 1,2,\ldots,n-1 \tag{3.26}$$

Further, consider the last equation of the system (3.24) together with (3.25) and (3.26), we get $U_{n-2}(\lambda)|\alpha_0| - 2\lambda U_{n-1}(\lambda)|\alpha_0| = 0$, which implies $U_n(\lambda)|\alpha_0| = 0$. Since the maximum is not attained when one or more of the numbers $|\alpha_\nu|$ are zero. Hence, we have $U_n(\lambda) = 0$, i.e.

$$\sin(n+1)\gamma = 0 \tag{3.27}$$

The only solution of (3.27) which is consistent with all the numbers $|\alpha_\nu|$ being non negative is $\gamma = \frac{\pi}{n+1}$. Therefore,

$$\lambda = \cos\left(\frac{\pi}{n+1}\right)$$

Since $\cos\left(\frac{\pi}{n+1}\right) \geq \frac{n-1}{n}$, the required maximum of the function $f(|\alpha_0|, |\alpha_1|, \ldots, \alpha_{n-1})$ is $\cos\left(\frac{\pi}{n+1}\right)$. This immediately leads to the inequality (3.19) if $\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta = 1$.

3.5

Recently, Aziz [2] gave a related result in terms of $\max_{1\le k\le n} |P(z_k)|$, where $z_1,\ldots, z_n$ are the zeros of z^n+1. In fact, he proved:

Theorem 3.5: If $P(z)$ is a polynomial of degree n with $P(\beta)=0$, where β is an arbitrary non-negative real number, then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \le \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{(1+\beta^n)^2}\max_{1\le k\le n}|P(z_k)|^2 \qquad (3.28)$$

where $z_1, z_2,\ldots,z_n$ are the zeros of z^n+1. The result is best possible and equality holds for $P(z)=z^n-\beta^n$.

For the proof of Theorem 3.5, we need the following:

Lemma 3.2: If $z_1,\ldots,z_n$ are the zeros of z^{n+1}, then for an arbitrary non-negative real number β, we have

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-\beta|^2}=\frac{1+\beta^2+\ldots+\beta^{2(n-2)}+\beta^{2(n-1)}}{(1+\beta^n)^2} \qquad (3.29)$$

Proof of Lemma 3.2: First, we consider the case $\beta=0$. Since $z_1,\ldots, z_n$ are zeros of z^n+1, so

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k|^2}=\frac{1}{n}\sum_{k=1}^{n}1=1$$

It follows (3.29).

Secondly, consider the case $\beta\ne 0$. If $P(z)$ is a polynomial of degree n such that $P(\beta)=0$, then $\frac{P(z)}{z-\beta}$ is a polynomial of degree $n-1$. Using Lagrange's interpolation formula with $z_1,\ldots,z_n$ as the basic points of interpolation, we can write

$$\frac{P(z)}{z-\beta}=\sum_{k=1}^{n}\left[\frac{P(z_k)}{z_k-\beta}\right]\left[\frac{z^n+1}{(z-z_k)n\, z_k^{\,n-1}}\right] \qquad (3.30)$$

Since $z_k^n = -1$, so $z_k^{n-1} = \frac{-1}{z_k}$ for $k=1,\ldots,n$. Hence, we rewrite (3.30) as

$$\frac{P(z)}{z-\beta} = \sum_{k=1}^{n}\left[\frac{P(z_k)}{z_k-\beta}\right]\left[\frac{z^n+1}{n(-\frac{1}{z_k})(z-z_k)}\right]$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{P(z_k)z_k(z^n+1)}{(z_k-\beta)(z_k-z)}\,.$$

Taking in particular $P(z) = z^n-\beta^n$, we obtain

$$\frac{z^n-\beta^n}{z-\beta} = \frac{1}{n}\sum_{k=1}^{n}\frac{(z_k^n-\beta^n)z_k(z^n+1)}{(z_k-\beta)(z_k-z)} \tag{3.31}$$

With regard to the left hand side of (3.31), we get $z^{n-1}+\beta z^{n-2}+\ldots+\beta^{n-2}z+\beta^{n-1}$. Applying the fact $z_k^n = -1$ in the right hand side of (3.31), we have

$$z^{n-1}+\beta z^{n-2}+\ldots+\beta^{n-2}z+\beta^{n-1} = \frac{1}{n}\sum_{k=1}^{n}\frac{(1+\beta^n)z_k(z^n+1)}{(z_k-\beta)(z-z_k)}$$

$$= \frac{(1+\beta^n)}{n}\sum_{k=1}^{n}\frac{z_k(z^n+1)}{(z_k-\beta)(z-z_k)} \tag{3.32}$$

Putting $z = \frac{1}{\beta}$ in (3.32), we note that the left hand side is

$$(\tfrac{1}{\beta})^{n-1} + \beta(\tfrac{1}{\beta})^{n-2}+\ldots+\beta^{n-2}(\tfrac{1}{\beta}) + \beta^{n-1}$$

$$= \frac{1+\beta^2+\ldots+\beta^{2(n-2)}+\beta^{2(n-1)}}{\beta^{n-1}}$$

and the right hand side is

$$\frac{(1+\beta^n)}{n}\sum_{k=1}^{n}\frac{z_k[(\frac{1}{\beta})^n+1]}{(z_k-\beta)(\frac{1}{\beta}-z_k)}$$

$$= \frac{(1+\beta^n)}{n}\sum_{k=1}^{n}\frac{z_k[\frac{1+\beta^n}{\beta^n}]}{(z_k-\beta)(\frac{1-\beta z_k}{\beta})}$$

$$= \frac{(1+\beta^n)^2}{n\beta^{n-1}} \sum_{k=1}^{n} \frac{z_k \overline{z_k}}{\overline{z_k}\,(z_k-\beta)(1-\beta z_k)}$$

$$= \frac{(1+\beta^n)^2}{n\beta^{n-1}} \sum_{k=1}^{n} \frac{|z_k|^2}{(z_k-\beta)(\overline{z_k}-\beta|z_k|^2)}$$

$$= \frac{(1+\beta^n)^2}{n\beta^{n-1}} \sum_{k=1}^{n} \frac{1}{(z_k-\beta)(\overline{z_k}-\beta)}$$

$$= \frac{(1+\beta^n)^2}{n\beta^{n-1}} \sum_{k=1}^{n} \frac{1}{|z_k-\beta|^2}$$

Since $\beta \neq 0$, we cancel β^{n-1} in both sides and transpose $(1+\beta^n)^2$ from the right hand side to the left hand side. Hence (3.29) follows.

Now, we apply lemma 3.2 to prove Theorem 3.5.

Proof of Theorem 3.5: Since $P(\beta)=0$, then $\frac{P(e^{i\theta})}{e^{i\theta}-\beta}$ is a polynomial of degree n-1. Hence $t(\theta) = \left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 = \sum_{\upsilon=-(n-1)}^{n-1} c_\upsilon e^{i\upsilon\theta}$ is a trigonometric polynomial of order n-1. Thus, we also have (3.16). Taking s=n, m=0 and $k=\frac{\pi}{n}$ in (3.16), we have

$$\sum_{p=0}^{n-1} e^{\frac{-2\pi i p(0)}{n}} \left|\frac{P[e^{i(\frac{\pi}{n}+\frac{2\pi p}{n})}]}{e^{i(\frac{\pi}{n}+\frac{2\pi p}{n})}-\beta}\right|^2 = n \sum_{\upsilon\equiv 0 \pmod n} c_\upsilon e^{i\upsilon(\frac{\pi}{n})} \qquad (3.33)$$

Simplifying the left hand side of (3.33), we have

$$\sum_{p=0}^{n-1} \left|\frac{P[e^{i(1+2p)\frac{\pi}{n}}]}{e^{i(1+2p)\frac{\pi}{n}}-\beta}\right|^2 = \sum_{k=1}^{n} \left|\frac{P(z_k)}{z_k-\beta}\right|^2$$

Concerning the right hand side of (3.33), we have $n c_0$ since υ runs from -(n-1) to (n-1). But, we know that $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta$, so we established

$$\frac{n}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta = \sum_{k=1}^{n}\left|\frac{P(z_k)}{z_k-\beta}\right|^2$$

Transpose n from left hand side to right hand side and take the maximum over $P(z_1),\ldots,P(z_n)$ to get

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \le \left\{\frac{1}{n}\sum_{k=1}^{n}\left|\frac{1}{z_k-\beta}\right|^2\right\}\cdot \max_{1\le k\le n}|P(z_k)|^2 \tag{3.34}$$

Applying lemma 3.2 to the left hand side of (3.34). (3.28) is immediately proved.

For $P(z) = z^n - \beta^n$, the left hand side of (3.28) becomes $1 + \beta^2 + \ldots + \beta^{2(n-2)} + \beta^{2(n-1)}$. On the other hand, $\max_{1\le k\le n}|P(z_k)|^2=(1+\beta^n)^2$ in the right hand side, so the equality holds.

3.6

Now, we discuss the results concerning the estimate of $\max_{|z|=1}\left|\frac{P(z)}{z-1}\right|$ where $P(z)$ is a polynomial of degree n with $\max_{|z|=1}|P(z)|=1$ and $P(1)=0$. The first result is due to Rahman and Mohammad [17]. They proved

Theorem 3.6: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1}|P(z)|=1$ and $P(1)=0$, then for $|z|\le 1$

$$\left|\frac{P(z)}{z-1}\right| \le \frac{n}{2} \tag{3.35}$$

The equality holds if $P(z) = \frac{z^n-1}{2}$.

Proof of Theorem 3.6: Define $\tilde{P}(z)=\frac{P(z)}{z-1}$ and let the maximum of $|\tilde{P}(e^{i\theta})|$ for $-\pi\le\theta\le\pi$ occur when $\theta=2\theta_0$. Now, we may suppose that $|e^{2i\theta_0}-1| < \frac{2}{n}$, otherwise, there is nothing to prove. Note that $P(1) = 0$, so $\tilde{P}(z)$ is a polynomial of degree n-1. Now, let $\tilde{P}(z) = \sum_{\upsilon=0}^{n-1} a_\upsilon z^\upsilon$ and consider

$$t(\theta) = e^{-i(n-1)\theta}\tilde{P}(e^{i2\theta})$$

$$= e^{-i(n-1)\theta}\left(\sum_{\upsilon=0}^{n-1} a_\upsilon e^{i2\upsilon\theta}\right)$$

$$= \sum_{\upsilon=0}^{n-1} a_\upsilon e^{i[2\upsilon-(n-1)]\theta}$$

$$= \sum_{\upsilon=-(n-1)}^{(n-1)} a_\upsilon^* e^{i\upsilon\theta}$$

then we have that $t(\theta)$ is a trigonometric polynomial of order n-1 and

$$|t(\theta_0)| = |e^{-i(n-1)\theta_0}\tilde{P}(e^{i2\theta_0})| = |\tilde{P}(e^{i2\theta_0})|$$

is the maximum on $[-\pi,\pi]$

Moreover, we choose γ such that $e^{i\gamma}t(\theta_0)$ is real and consider the real trigonometric polynomial $T(\theta) = \mathrm{Re}\{e^{i\gamma}t(\theta)\}$. Since $|T(\theta)| = |e^{i\gamma}t(\theta)| = |\tilde{P}(e^{i2\theta})|$ and $\tilde{P}(e^{i2\theta})$ has its maximum at θ_0, so does $T(\theta)$. Hence, we conclude that

$$T'(\theta_0) = 0 \tag{3.36}$$

Next, let us consider the real trigonometric polynomial $2\sin\theta T(\theta)$. We note that such a trigonometric polynomial is of order n and for $\theta \in [-\pi, \pi]$,

$$|2\sin\theta\; T(\theta)| = \left|2\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) T(\theta)\right|$$

$$= |(e^{i\theta}- e^{-i\theta})T(\theta)|$$

$$= |e^{-i\theta}(e^{i2\theta} - 1)T(\theta)|$$

$$\le |(e^{i2\theta} - 1)\tilde{P}(e^{i2\theta})|$$

$$= |P(e^{i2\theta})|$$
$$\leq 1$$

Apply Theorem 1.5 to $2\sin\theta\, T(\theta)$ and get

$$[\sin\theta\, T'(\theta) + \cos\theta\, T(\theta)]^2 + n^2\sin^2\theta T^2(\theta) \leq \frac{n^2}{4}$$

Setting $\theta=\theta_0$ and using (3.36), we get

$$\cos^2\theta_0 T^2(\theta_0) + n^2\sin^2\theta_0 T^2(\theta_0) \leq \frac{n^2}{4} \tag{3.37}$$

Simplifying the left hand side of (3.37), we obtain

$$[\cos^2\theta_0 + \sin^2\theta_0 - \sin^2\theta_0 + n^2\sin^2\theta_0]\, T^2(\theta_0)$$
$$= [1 + (n^2 - 1)\sin^2\theta_0]\, T^2(\theta_0)$$
$$\geq T^2(\theta_0)$$

Hence $T^2(\theta_0) \leq \frac{n^2}{4}$ and $|T(\theta_0)| \leq \frac{n}{2}$. It follows $|\tilde{P}(e^{i2\theta_0})| \leq \frac{n}{2}$. Since the maximum of $\tilde{P}(e^{i\theta})$ occurs at $2\theta_0$, so $|\tilde{P}(e^{i\theta})| \leq \frac{n}{2}$ which implies (3.35).

Further, for $P(z) = \frac{z^n \pm 1}{2}$, we have

$$\frac{1}{2} \max_{|z|=1} \left|1 + z + \ldots + z^{n-2} + z^{n-1}\right| = \frac{n}{2}$$

Thus, we complete the proof of Theorem 3.6.

In (3.35), making $z \to 1$, they also obtained:

<u>Theorem 3.7</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>with</u> $\max_{|z|=1} |P(z)|=1$ <u>on</u> $|z| \leq 1$ <u>and</u> $P(1) = 0$, <u>then</u>

$$|P'(1)| \leq \frac{n}{2} \tag{3.38}$$

<u>The result is best possible and there is equality in</u> (3.38) <u>for</u> $P(z) = \frac{z^n-1}{2}$.

Now, we present a generalization of Theorem 3.7 by using the technique we discuss in Chapter I (see Theorem 1.7). In fact, we prove.

<u>Theorem 3.8</u> (*): <u>If</u> P(z) <u>is a polynomial of degree</u> n <u>with</u> $\max_{|z|=k} |P(z)|=k^n$ <u>on</u> $|z|\le k$, $k \le 1$ <u>and</u> $P(\gamma)=0, |\gamma| =1$, <u>then</u>

$$P'(\gamma) = \frac{nk}{1+k} \tag{3.39}$$

<u>Proof of Theorem 3.8:</u> Since P(z) is a polynomial of degree n with $\max_{|z|=k} |P(z)|=k^n$, then for $|\alpha| > 1$, $P(z) - \alpha z^n$ has all its zeros in $|z|\le k$, $k < 1$. Let $R(z) = P(z) - \alpha z^n$ and the zeros of R(z) be $z_\upsilon = \rho_\upsilon e^{i\theta\upsilon}$, $\rho_\upsilon \le k$, $\forall\upsilon$, then on $|z|=1$, we have

$$\left|\frac{R'(e^{i\theta})}{R(e^{i\theta})}\right| = \left|\frac{e^{i\theta}R'(e^{i\theta})}{R(e^{i\theta})}\right|$$

$$= \sum_{\upsilon=1}^{n} \frac{e^{i\theta}}{e^{i\theta}-\rho_\upsilon e^{i\theta\upsilon}}$$

$$\ge \sum_{\upsilon=1}^{n} \frac{1}{1+\rho_\upsilon}$$

$$\ge \sum_{\upsilon=1}^{n} \frac{1}{1+k}$$

$$= \frac{n}{1+k}$$

Hence, one gets

$$|P'(z) - \alpha n z^{n-1}| \ge \frac{n}{1+k} |P(z) - \alpha z^n|$$

on $|z|=1$. For any $|\beta| < 1$,

$$[z P'(z) - \alpha n z^n] + \frac{\beta n}{1+k} [P(z) - \alpha z^n] \ne 0$$

and so

$$[z P'(z) + \frac{\beta n}{1+k} P(z)] \ne \alpha n(1 + \frac{\beta}{1+k}) z^n \tag{3.40}$$

For an appropriate choice of the argument of α in (3.40), one gets

$$|z P'(z) + \frac{\beta n}{1+k} P(z)| \ne n |\alpha| \left|\frac{\beta}{1+k}\right|$$

for all $|\alpha| > 1$. We get

$$|z P'(z) + \frac{\beta n}{1+k} P(z)| < n|\alpha| \left|1 + \frac{\beta}{1+k}\right|$$

Making $|\alpha| \to 1$, we have

$$|z P'(z) + \frac{\beta n}{1+k} P(z)| \le n \left|1 + \frac{\beta}{1+k}\right| \tag{3.41}$$

Taking $\beta \to -1$ in (3.41), we get

$$|z P'(z) - \frac{n}{1+k} P(z)| \le n \left|\frac{k}{1+k}\right|$$

Since $P(\gamma) = 0$ and $|\gamma| = 1$, we have (3.39). Hence, the proof is complete.

3.7

Later, Donaldson and Rahman [10] gave a generalization by considering the case when for a non-negative real number β, $P(\beta) = 0$. They first constructed a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and $P(\beta) = 0$ to show that in this case $\max_{|z|=1} \left|\frac{P(z)}{z-\beta}\right| > \frac{n}{2}$ if β is arbitrary. Now, we present:

Example 3.1: For $n > 1$, consider the polynomial

$$P(z) = \frac{n}{2\sqrt{n^2-1}} (1 + z + z^2 + \ldots + z^{n-1})(z - 1 + \frac{2}{n^2}). \tag{3.42}$$

We claim that $\max_{|z|=1} |P(z)| = 1$ and if we take $\beta = 1 - \frac{2}{n^2}$, we have $\max_{|z|=1} \left|\frac{P(z)}{z-\beta}\right| > \frac{n}{2}$. First, if $z = e^{i\theta}$, we consider two cases 1) $\cos\theta \le 1 - \frac{2}{n^2}$ and 2) $\cos\theta \ge 1 - \frac{2}{n^2}$.

Consider the former case, we want to show

$$\frac{n}{\sqrt{n^2-1}} |z - 1 + \frac{2}{n^2}| \le |z - 1| \tag{3.43}$$

for $z = e^{i\theta}$. In fact,

$$\frac{n}{\sqrt{n^2-1}}\left|\cos\theta + i\sin\theta - 1 + \frac{2}{n^2}\right| \le \left|\cos\theta + i\sin\theta - 1\right|$$

$$\leftrightarrow \frac{n^2}{n^2-1}\left[(\cos\theta - 1 + \frac{2}{n^2})^2 + \sin^2\theta\right] \le (\cos\theta - 1)^2 + \sin^2\theta$$

$$\leftrightarrow \frac{n^2}{n^2-1}\left[\cos^2\theta + 2\left(\frac{2}{n^2} - 1\right)\cos\theta + \left(\frac{2}{n^2} - 1\right)^2 + \sin^2\theta\right]$$

$$\le \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$

$$\leftrightarrow \frac{n^2}{n^2-1}\left[1 + 2\left(\frac{2}{n^2} - 1\right)\cos\theta + \left(\frac{2}{n^2} - 1\right)^2\right] \le 2 - 2\cos\theta$$

$$\leftrightarrow n^2 + 2n^2\left(\frac{2}{n^2} - 1\right)\cos\theta + n^2\left(\frac{2}{n^2} - 1\right)^2 \le 2(n^2-1) - 2(n^2-1)\cos\theta$$

$$\leftrightarrow 2\cos\theta\left[n^2\left(\frac{2}{n^2} - 1\right) + (n^2-1)\right] \le 2(n^2-1) - n^2 - n^2\left(\frac{2}{n^2} - 1\right)^2$$

$$\leftrightarrow 2\cos\theta \le 2 - \frac{4}{n^2}$$

Hence, we have (3.43). Now, we apply (3.43) in (3.42) and get

$$|P(z)| = \frac{n}{2\sqrt{n^2-1}}\left|(z - 1 + \frac{2}{n^2})(1 + z + \ldots + z^{n-1})\right|$$

$$\le \frac{1}{2}\left|(z-1)(1 + z + \ldots + z^{n-1})\right|$$

$$\le \frac{1}{2}|z^n - 1|$$

$$\le 1 \qquad \text{on } |z| \le 1$$

Concerning the latter case, we first note that

$$|P(z)| = \frac{n}{2\sqrt{n^2-1}}\left|(1 + z + \ldots + z^{n-1})(z - 1 + \frac{2}{n^2})\right|$$

$$\le \frac{n}{2\sqrt{n^2-1}}\left|(1 + |z| + \ldots + |z|^{n-1})(z - 1 + \frac{2}{n^2})\right|$$

$$\le \frac{n^2}{2\sqrt{n^2-1}}\left|z - 1 + \frac{2}{n^2}\right| \tag{3.44}$$

for $z = e^{i\theta}$. Next, since $\cos\theta \geq 1 - \frac{2}{n^2}$, we have

$$|e^{i\theta} - 1 + \frac{2}{n^2}| = \sqrt{(\cos\theta - 1 + \frac{2}{n^2})^2 + \sin^2\theta}$$

$$= \sqrt{1 - 2(1 - \frac{2}{n^2})\cos\theta + (1 - \frac{2}{n^2})^2}$$

$$\leq \sqrt{1 - 2(1 - \frac{2}{n^2})^2 + (1 - \frac{2}{n^2})^2}$$

$$= \sqrt{1 - (1 - \frac{2}{n^2})^2}$$

$$= \frac{2\sqrt{n^2 - 1}}{n^2}$$

Hence from (3.44), one has

$$|P(z)| \leq \frac{n^2}{2\sqrt{n^2-1}} \; \frac{2\sqrt{n^2-1}}{n^2} = 1 \qquad \text{on } |z| \leq 1$$

Therefore, we conclude that $\max_{|z|=1} |P(z)| = 1$. Further if we take $\beta = 1 - \frac{2}{n^2}$, we get

$$\max_{|z|=1} \left|\frac{P(z)}{z - 1 + \frac{2}{n^2}}\right| = \frac{n}{2\sqrt{n^2 - 1}} \max_{|z|=1} |1 + z + \ldots + z^{n-1}|$$

$$= \frac{n^2}{2\sqrt{n^2 - 1}}$$

$$> \frac{n^2}{2n}$$

$$= \frac{n}{2}$$

In view of the above, the following result they proved is interesting:

Theorem 3.9: Let $P(z)$ be a polynomial of degree n with $\max_{|z|=1} |P(z)|=1$ such that $P(\beta)=0$, where β is a non-negative real number, then for $|z| \le 1$

$$\left|\frac{P(z)}{z-\beta}\right| \le \frac{n+1}{2} \tag{3.45}$$

Proof of Theorem 3.9: Let $P(z)=(z-\beta)q(z)$ and define $\tilde{P}(z)=(z-1)q(z)$. First of all, we claim

$$\max_{|z|=1} |\tilde{P}(z)| \le \frac{2}{1+\beta} \max_{|z|=1} |P(z)| \tag{3.46}$$

In order to establish (3.46), we consider the following: Let 0 be the origin of the complex plane, draw a circle which has the center at 0 with radius one and intersect the positive, negative axis at point C, A respectively. Now, suppose β at point B and $z=e^{i\theta}$ at point E. Then we join the line $\overline{BE}$; draw one straight line from A passing through E; the other passing C parallel to $\overline{BE}$ and intersect $\overline{AE}$ at D. Finally, join the line $\overline{CE}$ (see figure 3:1).

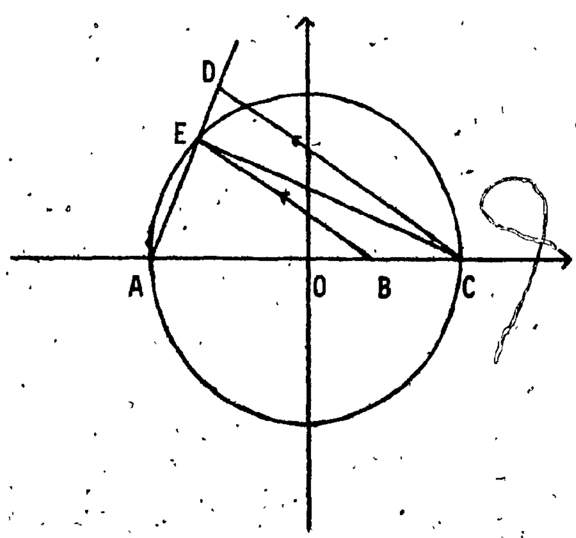

Figure 3.1

Thus, we obtain

1) $\angle DEC = \angle rt$, so $EC \le DC$

2) $EB \parallel CD$, so $\Delta ADC \sim \Delta AEB$. Hence

$$\frac{DC}{EB} = \frac{AC}{AB}$$

Now, from 1) and 2), we have

$$\left|\frac{\tilde{P}(e^{i\theta})}{P(e^{i\theta})}\right| = \left|\frac{(e^{i\theta}-1)q(e^{i\theta})}{(e^{i\theta}-\beta)q(e^{i\theta})}\right|$$

$$= \left|\frac{e^{i\theta}-1}{e^{i\theta}-\beta}\right|$$

$$= \frac{EC}{EB} \le \frac{DC}{EB} = \frac{AC}{AB} = \frac{2}{1+\beta}$$

Hence, (3.46) holds.

Further if $\max\limits_{|z|=1} |P(z)| = M$, then $\max\limits_{|z|=1} \left|\frac{\tilde{P}(z)}{M}\right| = 1$. Since

$\tilde{P}(z) = (z-1)q(z)$, we apply Theorem 3.6 to $\tilde{P}(z)$ and have

$$\left|\frac{q(z)}{M}\right| = \left|\frac{(z-1)q(z)}{M(z-1)}\right| = \left|\frac{\tilde{P}(z)}{M(z-1)}\right| \le \frac{n}{2}$$

for $|z| \le 1$. It follows

$$\max_{|z|=1} |q(z)| \le \frac{n}{2} \max_{|z|=1} |\tilde{P}(z)|. \qquad (3.47)$$

In view of inequality (3.46) and (3.47), one gets

$$\frac{2}{n} \max_{|z|=1} |q(z)| \le \frac{2}{1+\beta} \max_{|z|=1} |P(z)|$$

and so

$$\max_{|z|=1} |q(z)| \le \frac{n}{1+\beta} \max_{|z|=1} |P(z)| . \qquad (3.48)$$

Now, if $\beta \ge \frac{n-1}{n+1}$, then

$$1+\beta \ge 1+\frac{n-1}{n+1}; \quad \frac{1}{1+\beta} \le \frac{n+1}{2n} \qquad (3.49)$$

Applying (3.49) to (3.48), we get (3.45) immediately.

Next if $\beta \le \frac{n-1}{n+1}$ then we have

$$\frac{1}{1-\beta} \le \frac{1}{1-\frac{n-1}{n+1}} = \frac{n+1}{2}$$

Hence

$$\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right| \le \left|\frac{P(e^{i\theta})}{1-\beta}\right| \le \frac{n+1}{2}\,|P(e^{i\theta})|.$$

This leads to (3.45) and Theorem 3.9 is proved.

Chapter IV

Giroux and Rahman Theorem

4.1

In Chapter I, we have discussed how large $|P'(z)|$ could be on $|z| \leq 1$ if $|P(z)| \leq 1$ on $|z| \leq 1$. From Theorem 1.1, we have $|P'(z)| \leq n$ and there is equality if and only if $P(z) = \alpha z^n$, $|\alpha| = 1$. One observes that the extremal polynomial $P(z) = \alpha z^n$, in this case, has all its zeros concentrated at the origin. Hence, if we restrict ourselves to the class of polynomials which excludes αz^n, the estimate of $|P'(z)|$ must be smaller than n. This observation led Erdös to propose to study the estimate of $|P'(z)|$ where there is a restriction on the zeros of P(z). He was particularly interested in the case when all the zeros of P(z) are outside the disk $|z| < 1$. The conjecture, he proposed, was proved by Lax [14] and the result is now known as Erdös-Lax Theorem (see Theorem 2.4). Later, Boas suggested to investigate the case when a number of zeros (let k) of P(z) are in $|z| < 1$ and the remaining are in $|z| \geq 1$. It is interesting to note that if half of the zeros are inside $|z| \leq 1$ and other half are outside, then the estimate of $|P'(z)|$ can be sufficiently large, in fact, even greater than n-1. This is shown in the following (not yet published):

Example 4.1: (Rahman-Ruschewayeh): Consider

$$P(z) = \frac{1}{4}\{(1-iz)^2 + z^{n-2}(z-i)^2\}$$

This polynomial is of degree n and on $|z| = 1$

$$|P(z)| \leq \frac{1}{4}\{|1-iz|^2 + |z-i|^2\}$$

$$= \frac{1}{4}\{|z+i|^2 + |z-i|^2\}$$

$$= \frac{1}{4}\{\text{the diameter of the unit disk}\}^2$$

$$= 1$$

Since

$$|P(1)| = \frac{1}{4}\,|(1-i)^2 + (1-i)^2| = \frac{|-4i|}{4} = 1\ ,$$

so we have $\max\limits_{|z|=1} |P(z)| = 1$. Moreover, we note that

$$z^n\, P(\tfrac{1}{z}) = P(z) \tag{4.1}$$

because

$$z^n P(\tfrac{1}{z}) = \frac{z^n}{4}\{(1-\tfrac{i}{z})^2 + (\tfrac{1}{z})^{n-2}(\tfrac{1}{z}-i)^2\}$$

$$= \frac{z^n}{4}\left\{\frac{(z-i)^2}{z^2} + \frac{(1-iz)^2}{z^n}\right\}$$

$$= \frac{1}{4}\{z^{n-2}(z-i)^2 + (1-iz)^2\}$$

Hence (4.1). From here we know that if z_0 is a zero of $P(z)$, then $\frac{1}{z_0}$ is also a zero of $P(z)$; so there are as many zeros inside as there are outside. Now, consider

$$P'(z) = \frac{1}{4}\{2(1-iz)(-i) + (n-2)z^{n-3}(z-i)^2 + 2z^{n-2}(z-i)\}$$

and at $z = -i$

$$|P'(-i)| = \frac{1}{4}|\{2(1+i^2)(-i) + (n-2)(-i)^{n-3}(-i-i)^2 + 2(-i)^{n-2}(-i-i)\}|$$

$$= \frac{1}{4}|4(n-2)(-i)^{n-1} + 4(-i)^{n-1}|$$

$$= n-1\ .$$

Consequently, $\max\limits_{|z|=1} |P'(z)| \geq n-1$

The problem proposed by Boas seems to be very difficult. Up to now, there seems to be no significant result obtained in this direction. However, a number of interesting and significant results concerning the inequalities of polynomials are known if only one of the zeros of a polynomial $P(z)$ is prescribed, namely, $P(1) = 0$. Some results we have discussed in Chapter III. In this chapter, we shall present another remarkable result due to Giroux and Rahman [13].

First of all, let us introduce the notation P_n - which represents the class of all polynomials $\{P(z)\}$ of degree at most n with $\max_{|z|=1} |P(z)| = 1$ and $P(1) = 0$. In 1974, Giroux and Rahman established the following result concerning the derivative of $P(z)$ for $P(z) \varepsilon P_n$. They proved:

<u>Theorem 4.1</u>: <u>There exists an absolute constant</u> $c > 0$ <u>such that</u>

$$\max_{P(z)\varepsilon P_n} \left(\max_{|z|=1} |P'(z)| \right) \geq n - \frac{c}{n} \tag{4.2}$$

In [13], they also proved an inequality (4.9) for $|P'(z)|$ when $P(1) = a$. For the case when $a = 0$, they obtained a better inequality (4.3) from the interpolation formula due to M.Riesz (see (1.13)) together with a result due to Boas. We first study the later two inequalities and finally give a proof of Theorem 4.1

4.2

We present

<u>Theorem 4.2</u>: <u>If</u> $P(z)\varepsilon P_n$, <u>then for</u> $|z| \leq 1$

$$\max_{|z|=1} |P'(z)| \leq n - \frac{2-\sqrt{2}}{4n} \tag{4.3}$$

In order to prove Theorem 4.2, we need the following:

<u>Lemma 4.1</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>such that</u>

$\max_{0\le k\le n-1} |P[e^{i(\frac{2k+1}{n})\pi}]| \le 1$ and $P(1) = 0$, then for $|\theta| \le \frac{\pi}{n}$

$$|P(e^{i\theta})| \le |\sin \frac{n\theta}{2}| \tag{4.4}$$

Proof of lemma 4.1: Let us consider the real trigonometric polynomial $t(\theta)$ of order n with $t(0) = 0$ and $|t[\frac{(2k+1)\pi}{2n}]| \le 1$. We claim

$$|t(\theta)| \le |\sin n\theta| \tag{4.5}$$

for $|\theta| \le \frac{\pi}{2n}$.

Suppose, on contrary, that for some $\theta_0 (0 < \theta_0 < \frac{\pi}{2n})$

$t(\theta_0) > \sin n\theta_0$

This implies that in the interval $(-\frac{\pi}{2n} \le \theta \le \frac{\pi}{2n})$, the graph of $t(\theta)$ meets the graph of $\sin n\theta$ at least three times (see Figure 4.1); whereas each of the other arcs of $\sin n\theta$ rising from -1 to 1 meets

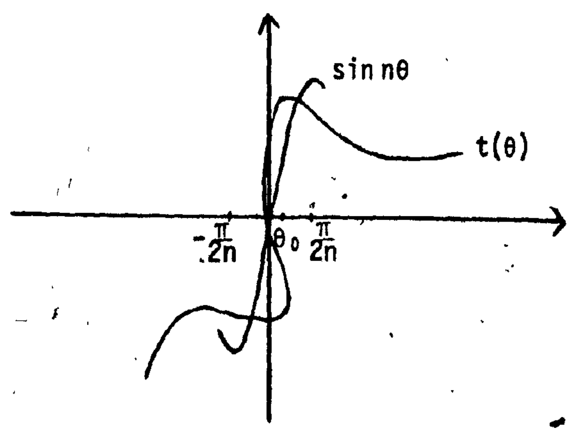

Figure 4.1

the graph of $t(\theta)$ at least once; see the proof of Theorem 1.5 on page 25. Hence the trigonometric polynomial $t(\theta) - \sin n\theta$ of order n has at least $2n+2$ zeros implying $t(\theta) \equiv \sin n\theta$. Contradiction. Hence (4.5) is established.

Moreover, the inequality (4.5) can be easily seen to hold for complex trigonometric polynomial as for any θ_0, we may consider $\mathrm{Re}\, e^{-i\gamma} t(\theta_0)$ where $\gamma = \arg t(\theta_0)$.

Now, let $P(z)$ satisfy the hypothesis of the lemma and consider $t(\theta) = e^{-in\theta} P(e^{2i\theta})$; since $t(0) = P(1) = 0$, we have

$$|P(e^{i2\theta})| \le |\sin n\theta|$$

for $|\theta| \le \frac{\pi}{2n}$. Putting $\phi = 2\theta$, we get (4.4) and the lemma is proved.

Remark 4.1: The origin of such result (see [7]) lies in the work of Schur where he proved:

If $P(x)$ is a polynomial of degree n such that $|P(x)| \le 1$ on $-1 \le x \le 1$ and $P(0) = 0$, then

$$|P(x)| \le m|x|$$

where $m = n$ or $n-1$ according as n is odd or even with equality possible only at $x = 0$ for odd n.

Now, we return to prove Theorem 4.2:

Proof of Theorem 4.2: Since $P(e^{i\theta})$ is a trigonometric polynomial of order n, recall (1.13), we have

$$P'(e^{i\theta}) = \frac{1}{2n} \sum_{\nu=1}^{2n} \frac{(-1)^{\nu+1} P[e^{i(\theta + \frac{2\nu-1}{2n}\pi)}]}{2\sin^2 \frac{(2\nu-1)\pi}{4n}}$$

$$= \frac{1}{2n} \sum_{\nu=1}^{2n} \frac{(-1)^{\nu+1}}{2\,\frac{1-\cos\frac{(2\nu-1)\pi}{2n}}{2}} P[e^{i(\theta + \frac{2\nu-1}{2n}\pi)}]$$

$$= \frac{1}{2n} \sum_{\upsilon=1}^{2n} \frac{(-1)^{\upsilon}}{1-\cos(\frac{2\upsilon+1}{2n})\pi} P[e^{i(\theta+\frac{2\upsilon+1}{2n}\pi)}] \tag{4.6}$$

Furhtermore, we have

$$\frac{1}{2n} \sum_{\upsilon=1}^{2n} \frac{1}{1-\cos\,(\frac{2\upsilon+1}{2n})\pi} = n \tag{4.7}$$

(see (1.12)). Now, consider the points at $e^{i(\theta+\frac{2\upsilon+1}{2n}\pi)}$, $1 \le \upsilon \le 2n$. First, if θ is an odd multiple of $\frac{\pi}{2n}$, say, $\theta = \frac{2m+1}{2n}\pi$, then we have

$$\upsilon = 1 \qquad \frac{2m+1}{2n}\pi + \frac{3\pi}{2n} = \frac{m+2}{n}\pi$$

$$\upsilon = 2 \qquad \frac{2m+1}{2n}\pi + \frac{5\pi}{2n} = \frac{m+3}{n}\pi$$

$$\vdots$$

$$\upsilon = 2n \qquad \frac{2m+1}{2n}\pi + \frac{4n+1}{2n}\pi = \frac{m+2n+1}{n}\pi .$$

For $m \le n$ there exists an integer k such that $m+k=n$. Hence, with regard to $\frac{m+2}{n}\pi, \frac{m+3}{n}\pi, \ldots, \frac{m+2n+1}{n}\pi$, we must have $m+k+n=2n$. In other words, for such k, $\frac{m+k+n}{n}\pi = 2\pi$. So we know that one of those points $\upsilon=j$, $e^{i[\theta+\frac{(2j+1)}{2n}\pi]}$ is 1. Since $\cos\frac{\pi}{2n} \ge -\cos\frac{2j+1}{2n}\pi$ and $P(1)=0$, from (4.6) and (4.7), we note for such value of θ, we have

$$|P'(e^{i\theta})| \le n - \frac{1}{2n} \frac{1}{1-\cos\frac{2j+1}{2n}\pi}$$

$$\le n - \frac{1}{2n} \frac{1}{1+\cos\frac{\pi}{2n}}$$

$$\le n - \frac{1}{2n}\,\frac{1}{2}$$

$$= n - \frac{1}{4n}$$

Now, if θ is not an odd multiple of $\frac{\pi}{2n}$, there are precisely two of points $\theta + \frac{2\upsilon+1}{2n}$, say $\theta + \frac{2j_1+1}{2n}$, $\theta + \frac{2j_2+1}{2n}$, lie in the arc $(-\frac{\pi}{n}, \frac{\pi}{n})$. From (4.6), we have

$$|P'(e^{i\theta})| \le \frac{1}{2n}\left\{ \sum_{\substack{\upsilon=1 \\ \upsilon\ne j_1, j_2}}^{2n} \frac{1}{1-\cos\frac{2\upsilon+1}{2n}\pi} + \frac{|P[e^{i(\theta+\frac{2j_1+1}{2n}\pi)}]|}{1-\cos\frac{2j_1+1}{2n}\pi} + \frac{|P[e^{i(\theta+\frac{2j_2+1}{2n}\pi)}]|}{1-\cos\frac{2j_2+1}{2n}\pi}\right\}$$

Applying lemma 4.1, we get

$$|P'(e^{i\theta})| \le \frac{1}{2n}\left\{ \sum_{\substack{\upsilon=1 \\ \upsilon\ne j, j}}^{2n} \frac{1}{1-\cos\frac{2\upsilon+1}{2n}\pi} + \frac{\left|\sin\frac{n(\theta+\frac{2j_1+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_1+1}{2n}\pi} + \frac{\left|\sin\frac{n(\theta+\frac{2j_2+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_2+1}{2n}\pi}\right\}$$

$$= \frac{1}{2n}\left\{ \sum_{\upsilon=1}^{2n} \frac{1}{1-\cos\frac{2\upsilon+1}{2n}\pi} - \frac{1}{1-\cos\frac{2j_1+1}{2n}\pi} - \frac{1}{1-\cos\frac{2j_2+1}{2n}\pi} + \frac{\left|\sin\frac{n(\theta+\frac{2j_1+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_1+1}{2n}\pi} + \frac{\left|\sin\frac{n(\theta+\frac{2j_2+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_2+1}{2n}\pi}\right\}$$

Using (4.7), we obtain

$$|P'(e^{i\theta})| \le n - \frac{1}{2n}\left\{ \frac{1-\left|\sin\frac{n(\theta+\frac{2j_1+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_1+1}{2n}\pi} + \frac{1-\left|\sin\frac{n(\theta+\frac{2j_2+1}{2n}\pi)}{2}\right|}{1-\cos\frac{2j_2+1}{2n}\pi}\right\} \quad (4.8)$$

Since $j_2 = j_1 + 1$, so

$$\sin \frac{n(\theta + \frac{2j_2+1}{2n}\pi)}{2} = \sin \frac{n(\theta + \frac{2j_1+2+1}{2n}\pi)}{2}$$

$$= \sin\left[\frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2} + \frac{\pi}{2} \right]$$

$$= \cos \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}$$

Hence, we rewrite (4.8) as

$$|P'(e^{i\theta})| \le n - \frac{1}{2n} \left\{ \frac{1-\left|\sin \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right|}{1-\cos \frac{2j_1+1}{2n}\pi} + \frac{1-\left|\cos \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right|}{1-\cos \frac{2j_2+1}{2n}\pi} \right\}$$

$$\le n - \frac{1}{2n} \left\{ \frac{1-\left|\sin \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right|}{1+\cos \frac{\pi}{2n}} + \frac{1-\left|\cos \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right|}{1+\cos \frac{\pi}{2n}} \right\}$$

$$\le n - \frac{1}{4n} \left\{ 2 - \left|\sin \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right| - \left|\cos \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}\right| \right\}$$

Let $\alpha = \frac{n(\theta + \frac{2j_1+1}{2n}\pi)}{2}$. Since $|\sin 2\alpha| \le 1$, so $2|\sin\alpha||\cos\alpha| \le 1$.

Hence $\sin^2\alpha + \cos^2\alpha + 2|\sin\alpha||\cos\alpha| \le 2$ which implies $|\sin\alpha| + |\cos\alpha| \le \sqrt{2}$.

Therefore, we have

$$|P'(e^{i\theta})| \le n - \frac{1}{4n}\{2 - \sqrt{2}\}$$

Note that $n - \frac{1}{4n} < n - \frac{2-\sqrt{2}}{4n}$. We prove (4.3).

4.3

It is interesting that the inequality like (4.3) can be established when $P(1) = a$. In fact, we have [13]:

<u>Theorem 4.3</u>: <u>Let</u> $P(z)$ <u>be a polynomial of degree</u> n. <u>If</u> $\max_{|z|=1} |P(z)|=1$ <u>and</u> $|P(1)|=a$, <u>then</u> for $|z|\le 1$

$$|P'(z)| \le n - \frac{(1-a)}{4\pi n}\{(1-a) - \sin(1-a)\} \tag{4.9}$$

We first prove :

<u>Lemma 4.2</u>: <u>Let</u> $P(z)$ <u>be a polynomial of degree</u> n. <u>If</u> $\max_{|z|=1} |P(z)|=1$ <u>and</u> $|P(1)|=a$, <u>where</u> $0 \le a \le 1$, <u>then for</u> $|\theta| < \frac{1-a}{n}$,

$$|P(e^{i\theta})| \le \frac{1+a}{2} \tag{4.10}$$

<u>Proof of lemma 4.2</u>: Let $z_1, \ldots, z_k$ be the zeros of $P(z)$ in $|z|<1$. First, we construct a polynomial

$$\tilde{P}(z) = P(z) \prod_{\upsilon=1}^{k} \frac{1-\bar{z}_\upsilon z}{z - z_\upsilon}.$$

We claim that the polynomial $\tilde{P}(z)$ is of degree n and does not vanish in $|z|<1$. In fact

$$\tilde{P}(z) = (z-z_1)\cdots\cdots(z-z_k)\cdots\cdots(z-z_n)\cdot\frac{1-\bar{z}_1 z}{z-z_1}\cdots\cdots\frac{1-\bar{z}_k z}{z-z_k}$$

$$= (-1)^k \bar{z}_1 \cdots\cdots \bar{z}_k \cdot (z-z_{k+1})\cdots\cdots(z-z_n)(z-\frac{1}{\bar{z}_1})\cdots\cdots(z-\frac{1}{\bar{z}_k})$$

Since $z_\upsilon = \rho_\upsilon e^{i\theta_\upsilon}$, $\upsilon = 1, \ldots, k$ are in $|z|<1$, so $\frac{1}{\bar{z}_\upsilon} = \frac{1}{\rho_\upsilon e^{-i\theta_\upsilon}} = \frac{1}{\rho_\upsilon} e^{i\theta_\upsilon}$ are outside $|z|\le 1$.

Moreover, we note that $|\tilde{P}(e^{i\theta})| = |P(e^{i\theta})|$ for $0 \le \theta \le 2\pi$ because $\left|\prod_{\upsilon=1}^{k} \frac{1-\bar{z}_\upsilon z}{z-z_\upsilon}\right| = \prod_{\upsilon=1}^{k} |z| \left|\frac{\bar{z}-\bar{z}_\upsilon}{z-z_\upsilon}\right| = 1$ on $|z|=1$.

Under the above observation, we know that without loss of generality one may suppose that $P(z) \ne 0$ in $|z|<1$. In order to prove (4.10), we note that

$$P(e^{i\theta}) = P(1) + \int_1^{e^{i\theta}} P'(z)dz$$

so we have

$$|P(e^{i\theta})| \le |P(1)| + \left|\int_1^{e^{i\theta}} P'(z)dz\right|$$

$$\le |P(1)| + |e^{i\theta} - 1| \max_{|z|=1} |P'(z)|$$

Applying Erdös-Lax Theorem (see Theorem 2.4), we get

$$|P(e^{i\theta})| \le |P(1)| + |e^{i\theta}-1| \cdot \frac{n}{2}$$

$$= a + n \left|\frac{e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}}{2}\right|$$

$$= a + n\ |\sin \frac{\theta}{2}|$$

$$\le a + n\ |\frac{\theta}{2}| \tag{4.11}$$

Now, if $|\theta| \le \frac{1-a}{n}$ in (4.11), we have

$$|P(e^{i\theta})| \le a + \frac{n}{2}\ \frac{1-a}{n}$$

$$= \frac{1+a}{2}$$

Hence we prove the result of lemma 4.2.

Now, we present the proof of Theorem 4.3.

<u>Proof of Theorem 4.3</u>: Let $P(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be a polynomial of degree n with $\max_{|z|=1} |P(z)|=1$ and $|P(1)|=a$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\phi})e^{-ik\phi}d\phi \qquad (0 \le k \le n)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sum_{\upsilon=0}^{n} a_\upsilon e^{i(\upsilon-k)\phi}]d\phi$$

$$= a_k$$

Hence

$$e^{i\theta}P'(e^{i\theta}) = \sum_{\upsilon=1}^{n} \upsilon a_\upsilon e^{i\upsilon\theta}$$

$$= \sum_{\upsilon=1}^{n} \upsilon\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\phi})e^{-i\upsilon\phi}d\phi\right]e^{i\upsilon\theta}$$

$$= \sum_{\upsilon=1}^{n} \frac{\upsilon}{2\pi}\int_{-\pi}^{\pi} P(e^{i\phi})e^{i\upsilon(\theta-\phi)}d\phi$$

Let $t = \phi - \theta$, we have

$$e^{i\theta} P'(e^{i\theta}) = \sum_{\upsilon=1}^{n} \frac{\upsilon}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-i\upsilon t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]\sum_{\upsilon=1}^{n} \upsilon e^{-i\upsilon t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\sum_{\upsilon=1}^{n} \upsilon e^{-i(\upsilon-1)t}dt \qquad (4.12)$$

Since $P[e^{i(\theta+t)}]e^{-it} = \sum_{\upsilon=0}^{n} a_\upsilon e^{i\upsilon(\theta+t)-it} = \sum_{\upsilon=0}^{n} a_\upsilon e^{i\upsilon\theta}e^{i(\upsilon-1)t}$, we may add to sum $\sum_{\upsilon=1}^{n} \upsilon e^{-i(\upsilon-1)t}$ terms in $e^{-i\upsilon t}$ with $\upsilon \geq n$ without changing the value of the integral provided $n>1$. Moreover, note that

$$\left(\sum_{\upsilon=1}^{n} z^{\upsilon-1}\right)^2 = 1 + 2z + 3z^2 + \ldots + nz^{n-1} + (n-1)z^n + \ldots + z^{2(n-1)}$$

$$= \sum_{\upsilon=1}^{n} \upsilon z^{\upsilon-1} + (n-1)z^n + \ldots + z^{2(n-1)}$$

So we can write (4.12) (also if n=1) as

$$e^{i\theta}P'(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\left[\sum_{\upsilon=1}^{n} \upsilon e^{-i(\upsilon-1)t} + (n-1)e^{-int} + \ldots + e^{-i2(n-1)t}\right]dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\left[\sum_{\upsilon=1}^{n} e^{-i(\upsilon-1)t}\right]^2 dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\left(\frac{1-e^{-int}}{1-e^{-it}}\right)^2 dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\left[\frac{e^{\frac{-int}{2}}(e^{\frac{int}{2}} - e^{\frac{-int}{2}})}{e^{\frac{-it}{2}}(e^{\frac{it}{2}} - e^{\frac{-it}{2}})}\right]^2 dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P[e^{i(\theta+t)}]e^{-it}\,\frac{e^{\frac{-int}{2}}}{e^{\frac{-it}{2}}}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

Consequently

$$|P'(e^{i\theta})| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|P[e^{i(\theta+t)}]|\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$= \frac{1}{2\pi}\left(\int_{\frac{1-a}{n}\le|\theta+t|\le\pi} + \int_{|\theta+t|\le\frac{1-a}{n}}\right)|P[e^{i(\theta+t)}]|\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$= \frac{1}{2\pi}\left[\left(\int_{-\pi}^{\pi} - \int_{|\theta+t|\le\frac{1-a}{n}}\right) - \int_{|\theta+t|\le\frac{1-a}{n}}\right]|P[e^{i(\theta+t)}]|\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt \qquad (4.13)$$

Since $|P(z)| \le 1$ for $|z| \le 1$, the first two integral in (4.13) can be replaced by:

$$\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} - \int_{|\theta+t|\le\frac{1-a}{n}}\right)|P[e^{i(\theta+t)}]|\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$\le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} - \int_{|\theta+t|\le\frac{1-a}{n}}\right)\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt - \frac{1}{2\pi}\int_{|\theta+t|\le\frac{1-a}{n}}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

Since $\int_{-\pi}^{\pi}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt = 2\pi n$, in fact for $z = e^{it}$

$$\int_{-\pi}^{\pi}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt = \int_{|z|=1}\left(\frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}}\right)^2 \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{z^n - 2 + z^{-n}}{z - 2 + z^{-1}} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_{|z|=1} \frac{(z^n - 1)^2}{z^n(z-1)^2} dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{\left(\sum_{\upsilon=0}^{n-1} z^{\upsilon}\right)^2}{z^n} dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{\sum_{\upsilon=0}^{n-1} (\upsilon+1)\, z^{\upsilon} + (n-1)z^n + \ldots + z^{2(n-1)}}{z^n} dz$$

$$= \frac{1}{i}\int_{|z|=1}\left[\sum_{\upsilon=0}^{n-1} (\upsilon+1)z^{\upsilon-n} + (n-1) + \ldots + z^{(n-2)}\right] dz$$

$= \frac{1}{i}\, 2\pi i\, a_{-1}$ where a_{-1} is the coefficient of z^{-1}

$= 2\pi n$

Hence, we rewrite (4.13) as

$$|P'(e^{i\theta})| \le n - \frac{1}{2\pi}\int_{|\theta+t|\le\frac{1-a}{n}}\left\{1 - |P[e^{i(\theta+t)}]|\right\}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt.$$

Applying lemma 4.2, we get

$$|P'(e^{i\theta})| \le n - \frac{1}{2\pi}\int_{|\theta+t|\le\frac{1-a}{n}}\left(\frac{1-a}{2}\right)\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$= n - \frac{(1-a)}{4\pi}\int_{|\theta+t|\le\frac{1-a}{n}}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 dt$$

$$\leq n - \frac{(1-a)}{4\pi}\int_{-\theta-\frac{1-a}{n}}^{-\theta+\frac{1-a}{n}} \sin^2 \frac{nt}{2}\, dt$$

$$= n - \frac{(1-a)}{4\pi}\left(\frac{t}{2} - \frac{\sin nt}{2n}\right)_{-\theta-\frac{1-a}{n}}^{-\theta+\frac{1-a}{n}}$$

$$= n - \frac{(1-a)}{4\pi}\left\{\frac{1-a}{n} - \frac{-1}{2n}\,[\sin(-n\theta+1-a)+\sin(n\theta+1-a)]\right\}$$

$$= n - \frac{(1-a)}{4\pi n}\,[1 - a - \sin(1-a)\cos(-n\theta)]$$

$$\leq n - \frac{(1-a)}{4\pi n}\,[(1-a) - \sin(1-a)]$$

Hence, (4.9) is established.

Remark 4.2: For $a = 0$ in (4.9), we have

$$|P'(z)| \leq n - \frac{1 - \sin 1}{4\pi n} \tag{4.14}$$

on $|z|=1$. Hence, the inequality (4.3) is better than (4.14) because $\frac{1-\sin 1}{4\pi} = 0.012615$ is smaller than $\frac{2-\sqrt{2}}{4} = 0.146446$.

4.4

We now return to prove Theorem 4.1 mentioned in the beginning of this chapter. For the proof, one needs the following:

Lemma 4.3: The integral

$$I = \int_0^{\frac{\pi}{2}} -n \log\left\{1 - \frac{1}{(n+1)^2}\,\frac{\sin^2(n+1)\theta}{\sin^2\theta}\right\} d\theta \tag{4.15}$$

remains bounded as n tends to infinity over the positive integers.

Proof of lemma 4.3: First, we break the integral (4.15) into two parts, namely, $[0, \frac{\pi}{2n}]$ and $[\frac{\pi}{2n}, \frac{\pi}{2}]$. Let the integral over $[0,\frac{\pi}{2n}]$ be denoted by I_1, and that over $[\frac{\pi}{2n}, \frac{\pi}{2}]$ by I_2.

Next, we consider the integral I_2. Note that if $0 \le t \le \frac{\pi}{2}$, we have

$$\sin t \ge \frac{2}{\pi} t \tag{4.16}$$

So for $\frac{\pi}{2n} \le t \le \frac{\pi}{2}$

$$0 < \frac{|\sin(n+1)t|}{(n+1)\sin t} \le \frac{1}{(n+1)\sin t} \le \frac{1}{(n+1)\frac{2}{\pi} t} = \frac{\pi}{2(n+1)t} < 1$$

because if $t = \frac{\pi}{2n}$, then $\frac{\pi}{2(n+1)t} = \frac{n}{n+1} < 1$ and if $t = \frac{\pi}{2}$, then $\frac{\pi}{2(n+1)t} = \frac{1}{n+1} < 1$. Hence

$$I_2 = -n \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \log \left\{1 - \left[\frac{1}{n+1} \frac{\sin(n+1)t}{\sin t}\right]^2\right\} dt$$

$$\le -n \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \log \left\{1 - \left[\frac{\pi}{2(n+1)t}\right]^2\right\} dt$$

Using the Taylor expansion for $\log \left\{1 - \left[\frac{\pi}{2(n+1)t}\right]^2\right\}^{-1}$, we have

$$I_2 \le n \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} t^{-2k}\right\} dt$$

$$= n \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} t^{-2k}\, dt$$

$$= n \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} \left(\frac{1}{1-2k}\right)\left[\left(\frac{\pi}{2}\right)^{-2k+1} - \left(\frac{\pi}{2n}\right)^{-2k+1}\right]$$

$$= n \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\pi}{2}\right)^{2k}\left(\frac{1}{n+1}\right)^{2k}\left(\frac{1}{1-2k}\right) \left(\frac{\pi}{2}\right)^{-2k+1}\left[1 - \left(\frac{1}{n}\right)^{-2k+1}\right]$$

$$= n \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\pi}{2}\right) \left(\frac{1}{n+1}\right)^{2k}\left(\frac{1}{1-2k}\right) (1 - n^{2k-1})$$

$$= n\frac{\pi}{2}\sum_{k=1}^{\infty}\left(\frac{1}{n+1}\right)^{2k}\frac{1}{k(2k-1)}(n^{2k-1}-1)$$

$$= \frac{\pi}{2}\sum_{k=1}^{\infty}\left(\frac{1}{n+1}\right)^{2k}\frac{1}{k(2k-1)}(n^{2k}-n)$$

$$< \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k(2k-1)}\left(\frac{n}{n+1}\right)^{2k}$$

Since $\frac{n}{n+1} < 1$ for all positive n, we have $|I_2| < \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k(2k-1)}$.

Hence $|I_2|$ is bounded because $\sum_{k=1}^{\infty}\frac{1}{k(2k-1)}$ converges.

Concerning the integral I_1, we note that

$$\frac{\sin(n+1)t}{\sin t} = e^{int} + e^{i(n-2)t} + \ldots + e^{-i(n-2)t} + e^{-int} \qquad (4.17)$$

Now, if n is even, then (4.17) becomes

$$\frac{\sin(n+1)t}{\sin t} = 1 + 2\left[\frac{e^{int}+e^{-int}}{2} + \frac{e^{i(n-2)t}+e^{-i(n-2)t}}{2} + \cdots + \frac{e^{i2t}+e^{-i2t}}{2}\right]$$

$$= 1 + 2\sum_{k=1}^{n/2}\cos 2kt \qquad (4.18)$$

Since $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \qquad (4.19)$

So for $0 \le x \le 1$, $\cos x = 1 - \frac{11x^2}{24} - \left(\frac{x^2}{24} - \frac{x^4}{24}\right) + \ldots \le 1 - \frac{11x^2}{24}$.

Hence, for $0 \le t \le \frac{\pi}{2n}$ (i.e. $0 \le k \le j \le \frac{n}{\pi}$, j is an integer), we get

$$\cos 2kt \le 1 - \frac{11\cdot 4\,k^2\,t^2}{24} = 1 - \frac{11\,k^2t^2}{6} \qquad (4.20)$$

Thus, for n is even and $j < \frac{n}{\pi}$, from (4.18) and (4.20), we have

$$\frac{\sin(n+1)t}{\sin t} \le 1 + 2\sum_{k=1}^{j}\left(1 - \frac{11\,k^2t^2}{6}\right) + 2\left(\frac{n}{2} - j\right).$$

$$= 1 + 2\left(j - \frac{11\,t^2}{6}\sum_{k=1}^{j}k^2\right) + n - 2j$$

$$= (n+1) - \frac{11\cdot t^2}{3}\cdot\frac{1}{6}\,j(j+1)(2j+1)$$

$$= (n+1) - \frac{11\,t^2}{18}\, j(j+1)(2j+1) \tag{4.21}$$

If we choose $j = \left[\frac{n}{\pi}\right] > \frac{n}{\pi} - 1$, then from (4.21) we conclude that

$$0 < \frac{\sin(n+1)t}{(n+1)\sin t}$$

$$\leq \frac{1}{n+1}\left\{(n+1) - \frac{11\,t^2}{18}\left(\frac{n}{\pi} - 1\right)\left(\frac{n}{\pi} - 1 + 1\right)\left[2\left(\frac{n}{\pi} - 1\right) + 1\right]\right\}$$

$$= \frac{1}{n+1}\left\{(n+1) - \frac{11\,t^2}{9}\left(\frac{n-\pi}{\pi}\right)\left(\frac{n}{\pi}\right)\left(\frac{n - \frac{\pi}{2}}{\pi}\right)\right\}$$

$$= 1 - \frac{11\,t^2}{9}\;\frac{(n-\pi)n(n-\frac{\pi}{2})}{(n+1)\pi^3} < 1 \quad \text{if } n \geq 4$$

Hence, for even $n \geq 4$, we obtain

$$0 < I_1 \leq -n\int_0^{\frac{\pi}{2n}} \log\left\{1 - \left[1 - \frac{11\,t^2}{9}\,\frac{(n-\pi)n(n-\frac{\pi}{2})}{(n+1)\pi^3}\right]^2\right\} dt$$

Let $s = tn$, we get

$$I_1 \leq -n\int_0^{\frac{\pi}{2}} \log\left\{1 - \left[1 - \frac{11}{9}\left(\frac{S}{n}\right)^2 \frac{(n-\pi)n(n-\frac{\pi}{2})}{(n+1)\pi^3}\right]^2\right\}\frac{1}{n}\, ds$$

$$= -\int_0^{\frac{\pi}{2}} \log\left\{1 - \left[1 - \frac{11}{9}\, S^2\, \frac{(n-\pi)(n-\frac{\pi}{2})}{n(n+1)\pi^3}\right]^2\right\} dS$$

Let $\alpha = \frac{11}{9}\;\frac{(n-\pi)(n-\frac{\pi}{2})}{n(n+1)\pi^3}$, we have

$$I_1 \leq -\int_0^{\frac{\pi}{2}} \log\left[1 - (1 - \alpha S^2)^2\right] dS$$

$$= -\int_0^{\frac{\pi}{2}} \log\,(2\alpha S^2 - \alpha^2 S^4)\,ds \tag{4.22}$$

Applying integration by part in (4.22) we get

$$I_1 \le -\left\{S \log (2\alpha S^2 - \alpha^2 S^4)\right\}_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} S \frac{d}{ds} \log(2\alpha S^2 - \alpha^2 S^4)$$

$$= -\frac{\pi}{2} \log \left[2\alpha \left(\frac{\pi}{2}\right)^2 - \alpha^2 \left(\frac{\pi}{2}\right)^4\right] + \lim_{S\to 0} S \log (2\alpha S^2 - \alpha^2 S^4)$$

$$+ \int_0^{\frac{\pi}{2}} \frac{4\alpha S^2 - 4\alpha^2 S^4}{2\alpha S^2 - \alpha^2 S^4} dS$$

Note that

1) $\lim_{S\to 0} S \cdot \log (2\alpha S^2 - \alpha^2 S^4)$

$= \lim_{S\to 0} S \log [S^2(2\alpha - \alpha^2 S^2)]$

$= \lim_{S\to 0} S [\log S^2 + \log(2\alpha - \alpha^2 S^2)]$

$= \lim_{S\to 0} [2S \log S + S \log (2\alpha - \alpha^2 S^2)] = 0$

2) Since $2\alpha S^2 > \alpha S^2$, so $2\alpha S^2 - \alpha^2 S^4 > \alpha S^2 - \alpha^2 S^4$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{4(\alpha S^2 - \alpha^2 S^4)}{2\alpha S^2 - \alpha^2 S^4} dS \le \int_0^{\frac{\pi}{2}} \frac{4(\alpha S^2 - \alpha^2 S^4)}{\alpha S^2 - \alpha^2 S^4} dS = \int_0^{\frac{\pi}{2}} 4\, dS = 2\pi$$

From 1) and 2), we get

$$0 < I_1 \le -\frac{\pi}{2} \log \left[2\alpha \left(\frac{\pi}{2}\right)^2 - \alpha^2 \left(\frac{\pi}{2}\right)^4\right] + 2\pi < 10.198684 \quad \text{as}$$

$$0.039418 = \frac{11}{9\pi^3} \ge \alpha = \frac{11}{9}\frac{(n-\pi)(n-\frac{\pi}{2})}{n(n+1)\pi^3} \ge \frac{11(1-\frac{\pi}{4})(1-\frac{\pi}{8})}{18\pi^3} = 0.002568$$

and so

$$0.003216 \le 2\alpha \left(\frac{\pi}{2}\right)^2 - \alpha^2 \left(\frac{\pi}{2}\right)^4 \le 0.194482$$

Further, if n is odd, then (4.17) becomes

$$\frac{\sin(n+1)t}{\sin t} = 2\left[\frac{e^{int} + e^{-int}}{2} + \frac{e^{i(n-2)t} + e^{-i(n-2)t}}{2} + \ldots + \frac{e^{it} + e^{-it}}{2}\right]$$

$$= 2 \sum_{k=0}^{\frac{n-1}{2}} \cos(2k+1)t \qquad (4.23)$$

Recall (4.19), for $0 \le t \le \frac{\pi}{2n}$ (i.e. $0 \le k \le j < \frac{2n-\pi}{2\pi}$, j is an integer), we get

$$\cos(2k+1)t \le 1 - \frac{11(2k+1)^2t^2}{24} \qquad (4.24)$$

Thus, for odd n and $j < \frac{2n-\pi}{2\pi}$, from (4.23) and (4.24), we have

$$\begin{aligned}
\frac{\sin(n+1)t}{\sin t} &\le 2 \sum_{k=0}^{j} \left(1 - \frac{11(2k+1)^2t^2}{24}\right) + 2\left(\frac{n-1}{2} - j\right) \\
&= 2\left[j + 1 - \frac{11t^2}{24} \sum_{k=0}^{j} (2k+1)^2\right] + 2\left(\frac{n-1}{2} - j\right) \\
&= (n+1) - \frac{11t^2}{12} \sum_{k=0}^{j} (2k+1)^2 \\
&= (n+1) - \frac{11t^2}{12}\left[1 + \sum_{k=1}^{j} (4k^2 + 4k + 1)\right] \\
&= (n+1) - \frac{11t^2}{12}\left[1 + 4\sum_{k=1}^{j} k^2 + 4\sum_{k=1}^{j} k + \sum_{k=1}^{j} 1\right] \\
&= (n+1) - \frac{11t^2}{12}\left[1 + \frac{2}{3} j(j+1)(2j+1) + 2j(j+1) + j\right] \\
&= (n+1) - \frac{11t^2}{12}(j+1)(2j+1)\left(1 + \frac{2}{3}j\right) \qquad (4.25)
\end{aligned}$$

If we choose $j = \left[\frac{n-\pi}{\pi}\right] > \frac{n-\pi}{\pi} - 1 = \frac{n-2\pi}{\pi}$, then from (4.25), we conclude that

$$\begin{aligned}
0 &< \frac{\sin(n+1)t}{(n+1)\sin t} \\
&\le \frac{1}{n+1}\left\{(n+1) - \frac{11t^2}{12}\left[\frac{2}{3}\left(\frac{n-2\pi}{\pi}\right) + 1\right]\left[2\left(\frac{n-2\pi}{\pi}\right) + 1\right]\left[\frac{n-2\pi}{\pi} + 1\right]\right\} \\
&= \frac{1}{n+1}\left\{(n+1) - \frac{11t^2}{36\pi^3}(2n-\pi)(2n-3\pi)(n-\pi)\right\} \\
&= 1 - \frac{11t^2}{36}\,\frac{(2n-\pi)(2n-3\pi)(n-\pi)}{(n+1)\pi^3} < 1 \qquad \text{if } n \ge 5
\end{aligned}$$

Hence, for odd $n \geq 5$, we obtain

$$0 \leq I_1 \leq -n \int_0^{\frac{\pi}{2n}} \log \left\{1-\left[1 - \frac{11t^2}{36} \frac{(2n-\pi)(2n-3\pi)(n-\pi)}{(n+1)\pi^3}\right]^2 \right\} dt$$

Let $S = tn$, we get

$$I_1 \leq -n \int_0^{\frac{\pi}{2}} \log \left\{1 - \left[1 - \frac{11}{36}\left(\frac{s}{n}\right)^2 \frac{(2n-\pi)(2n-3\pi)(n-\pi)}{(n+1)\pi^3}\right]^2 \right\} \frac{1}{n}\, ds$$

$$= - \int_0^{\frac{\pi}{2}} \log \left\{1 - \left[1 - \frac{11}{36} s^2 \frac{(2n-\pi)(2n-3\pi)(n-\pi)}{n^2(n+1)\pi^3}\right]^2 \right\} ds$$

Let $\alpha = \frac{11}{36} \frac{(2n-\pi)(2n-3\pi)(n-\pi)}{n^2(n+1)\pi^3}$, we have

$$I_1 \leq - \int_0^{\frac{\pi}{2}} \log\left[1 - (1 - \alpha s^2)^2\right] ds$$

Using the same method as before, we get I_1 is bounded. Hence, we prove lemma 4.3.

Lemma 4.4: The integral

$$-n \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\left[1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right] \frac{dt}{1+\cos 2t} \qquad (4.26)$$

remains bounded as n tends to infinity over the odd positive integers.

Proof of lemma 4.4: We break the integral (4.26) into two parts, namely, $[\frac{\pi}{4}, \frac{\pi}{2} - \frac{\pi}{2n}]$ and $[\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2}]$. Let the integral over $[\frac{\pi}{4}, \frac{\pi}{2} - \frac{\pi}{2n}]$ be denoted by I_1, and that over $[\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2}]$ by I_2.

First, we consider I_2. Note that for odd n and $t = \frac{\pi}{2} - s$, one has

$$\begin{cases} \sin^2(n+1)t = \sin^2(n+1)s \\ \sin^2 t = \cos^2 s \\ 1 + \cos 2t = 1 - \cos 2s = 2\sin^2 s \end{cases} \tag{4.27}$$

Hence by change of variable and (4.27), one gets

$$0 \le I_2 = \int_{\frac{\pi}{2}-\frac{\pi}{2n}}^{\frac{\pi}{2}} - n \log \left\{1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\sin^2 t}\right\}\frac{dt}{1+\cos 2t}$$

$$= \int_0^{\frac{\pi}{2n}} - n \log \left[1 - \frac{\sin^2(n+1)s}{(n+1)^2\cos^2 s}\right]\frac{ds}{2\sin^2 s} \tag{4.28}$$

Using $|\sin(n+1)s| \le (n+1)|\sin s| \le (n+1)s$ and (4.16) we rewrite (4.28) as

$$I_2 \le \int_0^{\frac{\pi}{2n}} - n \log \left[1 - \frac{1}{(n+1)^2}\frac{(n+1)^2 s^2}{\cos^2 s}\right]\frac{ds}{2\sin^2 s}$$

$$\le \int_0^{\frac{\pi}{2n}} - n \log \left[1 - \frac{s^2}{\cos^2 s}\right]\frac{1}{2}\left(\frac{\pi^2}{4s^2}\right)ds$$

$$= \frac{\pi^2}{8} n \int_0^{\frac{\pi}{2n}} \log\left[1 - \frac{s^2}{\cos^2 s}\right]^{-1}\frac{ds}{s^2}$$

$$= \frac{n\pi^2}{8}\int_0^{\frac{\pi}{2n}}\left[\sum_{k=1}^{\infty}\frac{1}{k}\frac{s^{2k}}{\cos^{2k}s}\right]\frac{ds}{s^2}$$

$$\le \frac{n\pi^2}{8}\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2n}}\frac{1}{k}\frac{s^{2(k-1)}}{\cos^{2k}(\frac{\pi}{2n})}\,ds$$

$$= \frac{n\pi^2}{8}\sum_{k=1}^{\infty}\frac{1}{k(2k-1)}\left(\frac{\pi}{2n}\right)^{2k-1}\frac{1}{\cos^{2k}(\frac{\pi}{2n})}$$

$$= \frac{n\pi^2}{8}\left(\frac{\pi}{2n}\right)\frac{1}{\cos^2(\frac{\pi}{2n})}\sum_{k=1}^{\infty}\frac{1}{k(2k-1)}\left(\frac{\pi}{2n}\right)^{2k-2}\frac{1}{\cos^{2k-2}(\frac{\pi}{2n})}$$

$$\leq \frac{\pi^2}{16} \frac{\pi}{\cos^2(\frac{\pi}{2n})} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)} \left(\frac{\frac{\pi}{2n}}{\cos\frac{\pi}{2n}}\right)^{2k-2}$$

Note that $\left(\frac{\pi}{2n}\right) \leq \cos\left(\frac{\pi}{2n}\right)$ for $n \geq 3$, we know that I_2 is bounded.

Now, we consider the first integral I_1. As the argument we use before, we know that

$$I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{2n}} -n\log\left[1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\sin^2 t}\right]\frac{dt}{1+\cos 2t} \geq 0 \qquad (4.29)$$

Since $|\sin(n+1)t| \leq 1$ and from (4.16), we rewrite (4.29) as

$$I_1 \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{2n}} -n\log\left[1 - \frac{1}{(n+1)^2\sin^2 t}\right]\frac{dt}{2\cos^2 t}$$

$$\leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{2n}} -n\log\left[1 - \frac{\pi^2}{(n+1)^2 4t^2}\right]\frac{dt}{2\cos^2 t}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{2n}} -n\log\left[1 - \frac{\pi^2}{(n+1)^2 4t^2}\right]\frac{dt}{2\sin^2(t-\frac{\pi}{2})}$$

$$\leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{2n}} -n\log\left[1 - \frac{\pi^2}{(n+1)^2 4t^2}\right]\frac{\pi^2}{8}\frac{1}{(t-\frac{\pi}{2})^2}\,dt$$

Let $s = t - \frac{\pi}{2}$, we have

$$I_1 \leq \frac{n\pi^2}{8}\int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} -\log\left[1 - \frac{\pi^2}{(n+1)^2 4(s+\frac{\pi}{2})^2}\right]\frac{ds}{s^2}$$

$$= \frac{n\pi^2}{8}\int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \log\left[1 - \frac{\pi^2}{(n+1)^2 4(s+\frac{\pi}{2})^2}\right]^{-1}\frac{ds}{s^2}$$

$$= \frac{n\pi^2}{8} \int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi^2}{(n+1)^2 4(s+\frac{\pi}{2})^2}\right]^k \frac{ds}{s^2}$$

$$= \frac{n\pi^2}{8} \int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} \frac{1}{s^2(s+\frac{\pi}{2})^{2k}}\, ds$$

$$= \frac{n\pi^2}{8} \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} \int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \frac{ds}{s^2(s+\frac{\pi}{2})^{2k}}$$

Since $-\frac{\pi}{4} \le s \le -\frac{\pi}{2n}$ for $n \ge 2$, we get $s + \frac{\pi}{2} \ge \frac{\pi}{4}$. Hence,

$$I_1 \le \frac{n\pi^2}{8} \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\pi}{2(n+1)}\right]^{2k} \int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \frac{ds}{s^2(\frac{\pi}{4})^{2k}}$$

$$= \frac{n\pi^2}{8} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2k} \int_{-\frac{\pi}{4}}^{-\frac{\pi}{2n}} \frac{ds}{s^2}$$

$$= \frac{n\pi^2}{8} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2k} \left(\frac{2n}{\pi} - \frac{4}{\pi}\right)$$

$$= \frac{n^2\pi}{4} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2k} \left(1 - \frac{2}{n}\right)$$

$$< \frac{n^2\pi}{4} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2k}$$

$$= \frac{n^2\pi}{4} \left(\frac{2}{n+1}\right)^2 \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2k-2}$$

$$= \frac{n^2\pi}{(n+1)^2} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{2}{n+1}\right)^{2(k-1)}$$

$$= \frac{n^2\pi}{(n+1)^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{2}{n+1}\right)^{2k} \text{ is bounded.}$$

This proves lemma 4.4.

Now, we begin to prove Theorem 4.1.

Proof of Theorem 4.1: First, we consider the following trigonometric polynomial of order n

$$1 - \frac{1}{(n+1)^2}\left(\sum_{k=0}^{n} e^{ik\theta}\right)\left(\sum_{\ell=0}^{n} e^{-i\ell\theta}\right)$$

$$= 1 - \frac{1}{(n+1)^2}\left(e^{\frac{n\theta}{2}i} \frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)\left(e^{\frac{-n\theta}{2}i} \frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)$$

$$= 1 - \frac{1}{(n+1)^2} \frac{\sin^2\frac{n+1}{2}\theta}{\sin^2\frac{\theta}{2}} \geq 0$$

By lemma 3.1, there exists a polynomial $P_n(z)$ of degree n such that

$$|P_n(e^{i\theta})|^2 = 1 - \frac{1}{(n+1)^2} \frac{\sin^2\frac{n+1}{2}\theta}{\sin^2\frac{\theta}{2}}$$

i.e. $$1 - |P_n(e^{i\theta})|^2 = \frac{1}{(n+1)^2} \frac{\sin^2\frac{n+1}{2}\theta}{\sin^2\frac{\theta}{2}} \qquad (4.30).$$

From (4.30), we know that

1) $|P_n(e^{i\theta})| \leq 1$

2) Since $\lim_{\theta\to 0} \frac{1}{(n+1)^2} \frac{\sin^2\frac{n+1}{2}\theta}{\sin^2\frac{\theta}{2}} = 1$, we have $P_n(1) = 0$.

Hence $P_n(z) \in \mathcal{P}_n$.

Now, consider $P_n^+(z)$ be the one satisfying (4.30) which does not vanish in $|z| < 1$ and assumes a positive value at the origin. In fact, for any z_υ in $|z| < 1$, the factor $(z - z_\upsilon)$ in $P_n(z) = c(z - z_1)\ldots\ldots(z - z_\upsilon)\ldots\ldots(z - z_n)$ can be replaced by $\bar{z}_\upsilon(z - {}^1/\bar{z}_\upsilon)$ to get

$$Q_n(z) = c\,\bar{z}_\upsilon(z - z_1)\ldots\ldots(z - {}^1/\bar{z}_\upsilon)\ldots\ldots(z - z_n)$$

and we easily see that $|P_n(e^{i\theta})| = |Q_n(e^{i\theta})|$. Also note that $e^{-i\gamma}P_n(0) = |P_n(0)| > 0$ where $\gamma = \arg P_n(0)$. Now, if we let $P_n^+(z) = e^{-i\gamma}Q_n(z)$, then the polynomial $P_n^+(z)$ is the required one.

Next, for $|z| < 1$, we claim that

$$\log P_n^+(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,\log|P_n^+(e^{it})|\,dt \tag{4.31}$$

In fact, given $z_0(|z_0|<1)$, by Poisson-Schwarz formula for $|z_0| < \rho < 1$, we have

$$\log P_n^+(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho e^{it}+z_0}{\rho e^{it}-z_0}\,\log|P_n^+(\rho e^{it})|\,dt \tag{4.32}$$

(see Complex analysis by Alhfors, page 167).

Since $\quad |\rho e^{it}-1|^2$

$$= \rho\left(\sqrt{\rho}\,e^{\frac{it}{2}} - \frac{1}{\sqrt{\rho}}\,e^{\frac{-it}{2}}\right)\left(\sqrt{\rho}\,e^{\frac{-it}{2}} - \frac{1}{\sqrt{\rho}}\,e^{\frac{it}{2}}\right)$$

$$= \rho\left(\rho + \frac{1}{\rho} - e^{it} - e^{-it}\right)$$

$$= \rho\left(\rho + \frac{1}{\rho} - 2\cos t\right)$$

$$= \rho\left(\rho + \frac{1}{\rho} - 2 + 4\sin^2\frac{t}{2}\right)$$

Note that $(\sqrt{\rho} - \frac{1}{\sqrt{\rho}})^2 \geq 0$, one has $|\rho e^{it}-1|^2 \geq \rho\, 4\sin^2\frac{t}{2}$. So for $\rho \geq \frac{1}{2}$ and $|\rho e^{it}-1| \leq 1$, we have

$$1 \geq |\rho e^{it} - 1| \geq 2\sqrt{\rho}\,|\sin \tfrac{t}{2}| \geq 2\,\frac{1}{\sqrt{2}}\,\frac{|t|}{\pi} = \sqrt{2}\,\frac{|t|}{\pi}$$

(recall 4.16). Hence

$$\left|\log|\rho e^{it} - 1|\right| \leq \log \frac{\pi}{\sqrt{2}} + \left|\log|t|\right| \tag{4.33}$$

For other values of ρ and t, it is obvious that $\left|\log|\rho e^{it} - 1|\right|$ is bounded. Recall that $P_n^+(z)$ has the property $P_n^+(1) = 0$, so we have

$$P_n^+(\rho e^{it}) = (\rho e^{it} - 1)g(\rho e^{it}).$$

Hence

$$\log|P_n^+(\rho e^{it})| = \log|\rho e^{it} - 1| + \log|g(\rho e^{it})|$$

$$\leq \text{constant} + \left|\log|t|\right| + \log|g(\rho e^{it})|$$

Applying Lebesque convergence Theorem and letting $\rho \to 1$ in (4.32), we get (4.31). Besides, the function $|P_n^+(e^{it})|$ is integrable over $[0, 2\pi]$, we can differentiate (4.31) under the integral sign (w.r.t z) and get

$$\frac{P_n^{+'}(z)}{P_n^+(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(e^{it}-z)(1)-(e^{it}+z)(-1)}{(e^{it}-z)^2} \log|P_n^+(e^{it})|dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2e^{it}}{(e^{it}-z)^2} \log|P_n^+(e^{it})|dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2} \log|P_n^+(e^{it})|^2\, dt \tag{4.34}$$

for $|z| < 1$. We are interested in $P_n^{+'}(-1)$ and ~~like (4.34) to hold for~~ $z = -1$ as well. Now, for odd n, we note that the function

$$\log\,[P_n^+(e^{it})]^2 = \log\left\{1 - \frac{1}{(n+1)^2}\,\frac{\sin^2 \frac{(n+1)}{2}t}{\sin^2 \frac{t}{2}}\right\} \tag{4.35}$$

is analytic in $|t-\pi| < \rho_0$, for some $\rho_0 > 0$ and has a double zero at $t = \pi$ since $(n+1)$ is even, so (4.35) becomes $\log\{P_n^+(e^{i\pi})\}^2 = \log 1 = 0$. On the other hand, $\dfrac{P_n^{+\prime}(e^{i\pi})}{P_n^+(e^{i\pi})} = 0$ which implies $(\log[P_n^+(e^{it})]^2)' = 0$ at $t = \pi$. Hence, if we let $\delta = \frac{1}{2}\min\{\rho_0, 1\}$, then for $|t-\pi| < \delta$, we have

$$\log|P_n^+(e^{it})|^2 = (t-\pi)^2\alpha(t)$$

where $\alpha(t)$ is continuous on $|t-\pi| < \delta$. So if

$$f_\rho(t) = \frac{e^{it}}{(e^{it}+\rho)^2}\log\left\{1 - \frac{1}{(n+1)^2}\,\frac{\sin^2\frac{(n+1)}{2}t}{\sin^2\frac{t}{2}}\right\}, \tag{4.36}$$

then for $0 < \rho < 1$, $|t-\pi| \le \delta$, we get

$$\begin{aligned} |f_\rho(t)| &= \frac{1}{|e^{it}+\rho|^2}(t-\pi)^2\alpha(t) \\ &= \frac{1}{|1+\rho e^{-it}|^2}(t-\pi)^2\alpha(t) \\ &= \frac{1}{|\rho e^{-i(t-\pi)}-1|^2}(t-\pi)^2\alpha(t) \end{aligned} \tag{4.37}$$

Since $|\rho e^{it}-1| \ge 2\sqrt{\rho}\,\sin\frac{t}{2}$, we rewrite (4.37) as

$$|f_\rho(t)| \le \frac{1}{\rho}\,\frac{(t-\pi)^2\alpha(t)}{4\sin^2\frac{(t-\pi)}{2}}$$

Let $\beta(t) = \dfrac{(t-\pi)^2\alpha(t)}{4\sin^2\frac{t-\pi}{2}}$, then $\beta(t)$ is integrable over $|t-\pi| \le \delta$.

Moreover, for $0 < \rho < 1$, $\delta < |t-\pi| \le \pi$, we have

$$\left|\frac{1}{(e^{it}+\rho)^2}\right| = \frac{1}{1+\rho^2+2\rho\cos t}.$$ Since $\delta \le |t-\pi|$, so $\cos t \ge \cos(\pi-\delta)$.

Hence $\frac{1}{|e^{it}+\rho|^2} \le \frac{1}{1+2\rho\cos(\pi-\delta)+\rho^2}$. Also note that

$$\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2\frac{(n+1)}{2}t}{\sin^2\frac{t}{2}}\right\} < 0,$$ we get (4.36) as

$$|f_\rho(t)| \le \frac{-1}{1+2\rho\cos(\pi-\delta)+\rho^2}\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2\frac{n+1}{2}t}{\sin^2\frac{t}{2}}\right\}$$

Applying Lebesque's convergence theorem again we have

$$\lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi} f_\rho(t)dt = \frac{1}{2\pi}\int_0^{2\pi} f_1(t)dt.$$

Since, $\log\{P_n^+(e^{it})\}^2 = 0$ at $t=\pi$, so $P_n^+(-1) = 1$. Consequently, from (4.34) and (4.35), we have

$$P_n^{+\prime}(-1) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}+1)^2}\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2\frac{n+1}{2}t}{\sin^2\frac{t}{2}}\right\}dt$$

Note that $P_n^{+\prime}(-1) < 0$, so

$$\begin{aligned}|P_n^{+\prime}(-1)| &= -P_n^{+\prime}(-1)\\ &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}+1)^2}\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2\frac{n+1}{2}t}{\sin^2\frac{t}{2}}\right\}dt\end{aligned}$$

Let $\varphi = \frac{t}{2}$, we get

$$\begin{aligned}|P_n^{+\prime}(-1)| &= -\frac{1}{\pi}\int_0^{\pi}\frac{e^{i2\varphi}}{(e^{i2\varphi}+1)^2}\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2(n+1)\varphi}{\sin^2\varphi}\right\}d\varphi\\ &= -\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\frac{e^{i2\varphi}}{(e^{i2\varphi}+1)^2}\log\left\{1-\frac{1}{(n+1)^2}\frac{\sin^2(n+1)\varphi}{\sin^2\varphi}\right\}d\varphi\right.\end{aligned}$$

$$+ \int_{\frac{\pi}{2}}^{\pi} \frac{e^{i2\varphi}}{(e^{i2\varphi}+1)^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\varphi}{\sin^2\varphi}\right\} d\varphi \Big] \qquad (4.38)$$

Consider the second integral in (4.38) and let $\psi = \pi - \varphi$, we have

$$- \int_{\frac{\pi}{2}}^{0} \frac{e^{2(\pi-\psi)i}}{[e^{i2(\pi-\psi)}+1]^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)(\pi-\psi)}{\sin^2(\pi-\psi)}\right\} d\psi$$

$$= - \int_{\frac{\pi}{2}}^{0} \frac{e^{2\pi i-2\psi i}}{[e^{2\pi i-2\psi i}+1]^2} \log \left[1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\psi}{\sin^2\psi}\right] d\psi$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{e^{-2\psi i}}{(e^{-2\psi i}+1)^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\psi}{\sin^2\psi}\right\} d\psi$$

Hence, we rewrite (4.38) as

$$- \frac{1}{\pi} \Big[\int_{0}^{\frac{\pi}{2}} \frac{e^{2it}}{(e^{i2t}+1)^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt +$$

$$\int_{0}^{\frac{\pi}{2}} \frac{e^{-2it}}{(e^{-2it}+1)^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt$$

$$= - \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \left\{\frac{e^{2it}}{(e^{2it}+1)^2} + \frac{e^{-2it}}{(e^{-2it}+1)^2}\right\} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt$$

$$= - \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{2}{(e^{it}+e^{-it})^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt$$

$$= - \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{1}{2\cos^2 t} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt$$

$$= - \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} \frac{dt}{1+\cos 2t}$$

$$\le -\frac{1}{\pi}\left[\frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left\{1-\frac{1}{(n+1)^2}\ \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt\right.$$

$$\left. + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\left\{1-\frac{1}{(n+1)^2}\ \frac{\sin^2(n+1)t}{\sin^2 t}\right\} \frac{dt}{1+\cos 2t}\right]$$

$$\le \frac{1}{\pi}\left[\frac{1}{2}\int_0^{\frac{\pi}{4}} -n\log\left\{1-\frac{1}{(n+1)^2}\ \frac{\sin^2(n+1)t}{\sin^2 t}\right\} dt\right.$$

$$\left. + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -n\log\left\{1-\frac{1}{(n+1)^2}\ \frac{\sin^2(n+1)t}{\sin^2 t}\right\} \frac{dt}{1+\cos 2t}\right]$$

Applying lemma 4.3 and lemma 4.4, we know that there exists a positive constant such that

$$|P_n^{+\prime}(-1)| \le \frac{c}{n}$$

Next, consider the case that n is even. First, if $m = \frac{n}{2}$ is odd, consider $g_n(z) = [P_m^+(z)]^2$ is a polynomial of degree n such that

1) $g_n(1) = [P_m^+(1)]^2 = 0$

2) $\max_{|z|=1} |g_n(z)| = g_n(-1) = [P_m^+(-1)]^2 = 1$

3) $|g_n'(-1)| = 2|P_m^{+\prime}(-1)| \le 2\frac{c}{m} = 2\frac{c}{\frac{n}{2}} = \frac{c'}{n}$

where $c' = 4c$.

Secondly, if $m = \frac{n}{2}$ is also even, consider the polynomial $b_n(z) = P_{m-1}^+(z)\, P_{m+1}^+(z)$ is of degree n such that

1) $b_n(1) = P_{m-1}^+(1)\, P_{m+1}^+(1) = 0$

2) $\max_{|z|=1} |b_n(-1)| = b_n(-1) = P_{m-1}^+(-1)\, P_{m+1}^+(-1) = 1$

3) $b_n'(z) = P^+_{m-1}(z)\ P^{+'}_{m+1}(z) + P^+_{m+1}(z)\ P^{+'}_{m-1}(z)$

Hence

$$b_n'(-1) = P^+_{m-1}(-1)\ P^{+'}_{m+1}(-1) + P^+_{m+1}(-1)\ P^{+'}_{m-1}(-1) = P^{+'}_{m+1}(-1) + P^{+'}_{m-1}(-1)$$

Hence

$$|b_n'(-1)| = -b_n'(-1)$$

$$\leq \frac{c}{m+1} + \frac{c}{m-1}$$

$$= \frac{c(m-1)+c(m+1)}{m^2-1}$$

$$= \frac{\frac{c(n-2)}{2} + \frac{c(n+2)}{2}}{\frac{n^2-4}{4}}$$

$$= \frac{4cn}{n^2-4} \leq \frac{c'}{n}$$

where $8c \leq c'$ and $n > 2$

Thus, there exists a positive constant c_1 and, for each n, a polynomial $P_n(z) \in \mathcal{P}_n$ with $P_n(-1) = 1$, $P_n'(-1) < 0$ and $|P_n'(-1)| \leq \frac{c_1}{n}$.

Now, let $G_n(z) = (-1)^n z^n P_n(\frac{1}{z})$ is a polynomial of degree n such that

1) $G_n(1) = (-1)^n (1)^n P_n(1) = 0$

2) $G_n(-1) = (-1)^n (-1)^n P_n(-1) = 1$

3) Since $G_n'(z) = (-1)^n \left\{-z^{n-2} P_n'(\frac{1}{z}) + n\ z^{n-1} P_n(\frac{1}{z})\right\}$

so $G_n'(-1) = (-1)^n \left\{-(-1)^{n-2} P_n'(-1) + n(-1)^{n-1} P_n(-1)\right\}$

$$= -(-1)^{2n-2} P_n'(-1) + n(-1)^{2n-1}$$

$$= -P_n'(-1) - n$$

Hence $|G_n'(-1)| = -G_n'(-1)$

$$= n + P_n'(-1)$$

$$= n - |P_n'(-1)|$$

$$\geq n - \frac{c_1}{n}$$

Hence, there exists a polynomial $G_n(z) \varepsilon \mathcal{P}_n$ for which

$$|G_n'(-1)| \geq n - \frac{c}{n}$$

Consequently, $\max_{P(z)\varepsilon\mathcal{P}_n} \left[\max_{|z|=1} |P'(z)|\right] \geq n - \frac{c}{n}$ where c is absolute constant. Thus, we complete the proof of Theorem 4.1.

Chapter V

Computer Calculation

5.1

In this chapter, we present some computer calculation concerning the estimate of $\max_{|z|=1} |P'(z)|$ when $\max_{|z|=1} |P(z)| = 1$. We know that from Theorem 1.1, $\max_{|z|=1} |P'(z)| \le n$ whereas if one of the zeros is prescribed on $|z| = 1$, namely $P(1) = 0$, from Theorem 4.1, $\max_{|z|=1} |P'(z)|$ is never equal but 'very near' to n. The proof of Theorem 4.1 does not reflect on what the constant c in (4.2) should be expected. So, it would be interesting to consider a polynomial with one zero at $z = 1$, i.e. $P(1) = 0$ and compute the concerning quantities, in particular related to c in (4.2). We make calculation for the polynomial of degree $n = 2$, 3 and 4. In selecting a polynomial, note that after one zero is prescribed at $z = 1$, all the other zeros can be taken inside $|z| < 1$ to get a larger value of $\max_{|z|=1} |P'(z)|$. This is suggested from Theorem 1.6. In fact, let $P(z)$ be a polynomial of degree n with $P(1) = 0$, $k(\le n-1)$ of its zero, $z_2,\dots,z_{k+1}$ are in $|z| > 1$ and the rest $z_{k+2},\dots,z_n$ are in $|z| \le 1$. i.e.

$$P(z) = (z-1) \prod_{\nu=2}^{k+1} (z - z_\nu) \prod_{\mu=k+2}^{n} (z - z_\mu)$$

Consider $Q(z) = \bar{z}_2 \dots \bar{z}_{k+1} (z-1) \prod_{\nu=2}^{k+1} (z - \frac{1}{\bar{z}_\nu}) \prod_{\mu=k+2}^{n} (z - z_\mu)$, then

$$|P(z)| = |Q(z)|$$

on $|z| = 1$. Since $Q(z)$ has all its zeros in $|z| \le 1$, from Theorem 1.6, we have

$$|P'(z)| \le |Q'(z)|$$

on $|z|=1$. Thus, for our purpose, we can take all the zeros inside $|z| \le 1$.

For each n, we calculate

$$B = \max_{|z|=1} |P'(z)| \Big/ \max_{|z|=1} |P(z)| \qquad (5.1)$$

and

$$C = n^2 - Bn \qquad (5.2)$$

Our study is neither comprehensive nor leading to any definite conclusion. However, it is interesting to observe a surprising coincidence. For $n=2$, the polynomial which gives max B is "approximately" the same as used by Callahan in (3.5) and Giroux-Rahman in (4.30) having all its zeros in $|z| \le 1$; we shall refer such polynomials as Callahan polynomial. This observation is repeated for degree $n=3$ and 4.

5.2

We present the above observation here:

For $n=2$, let $z_1=1$ and the other zero be $z_2=r_2e^{i\theta_2}$. We vary r_2 from 0 to 1 with increasement 0.001 and θ_2 from 0° to 180° with step 1° to generate the polynomials of degree 2 with one zero at $z=1$. For each such polynomial, we calculate B and C; and observe

$$\max B = 1.760173 \qquad (5.3)$$

$$\min C = 0.479653 \qquad (5.4)$$

The polynomial having the above B and C is

$$P(z) = (z-1)(z-0.255\, e^{i180^\circ}) \tag{5.55}$$

(see Figure 5.1)

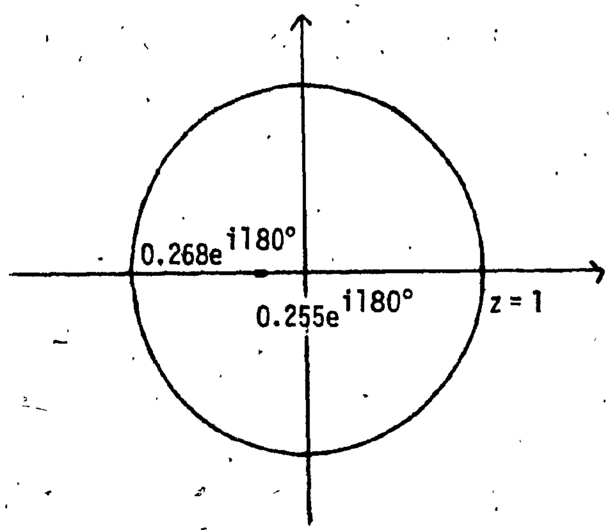

Figure 5.1

We note here that B and C in Callahan polynomial of degree 2 (see remark 3.1)

$$P(z) = \frac{(\sqrt{2}+\sqrt{6})}{6}(z-1)(z-0.268\, e^{i180^\circ}) \tag{5.6}$$

are

$$B = 1.759307 \tag{5.7}$$

$$C = 0.481386 \tag{5.8}$$

For $n = 3$, let $z_1 = 1$ and the other zeros be $z_\upsilon = r_\upsilon e^{i\theta_\upsilon}$, $\upsilon = 2,3$. First, we vary r_2, r_3 between 0 and 1 with increasement 0.1; θ_2 = 90°, 100°, 110°, 120°, 130°, 135°, 140°, 150°, 160°, 170°, 180°; θ_3 between 180° and 270° with increasement 30°. We find that the polynomial having max B and min C is

$$P(z) = (z-1)(z-0.4\, e^{i135^\circ})(z-0.3\, e^{i240^\circ}).$$

Later, we refine this result by varying r_2, r_3 with increasement 0.01; θ_2, θ_3 with increasement 1°. Finally, we observe

$$\max B = 2.797842 \tag{5.9}$$

$$\min C = 0.606475 \tag{5.10}$$

The polynomial having the above B and C is

$$P(z) = (z-1)(z-0.28\, e^{i131^\circ})(z-0.28\, e^{i228.8^\circ}) \tag{5.11}$$

(see Figure 5.2)

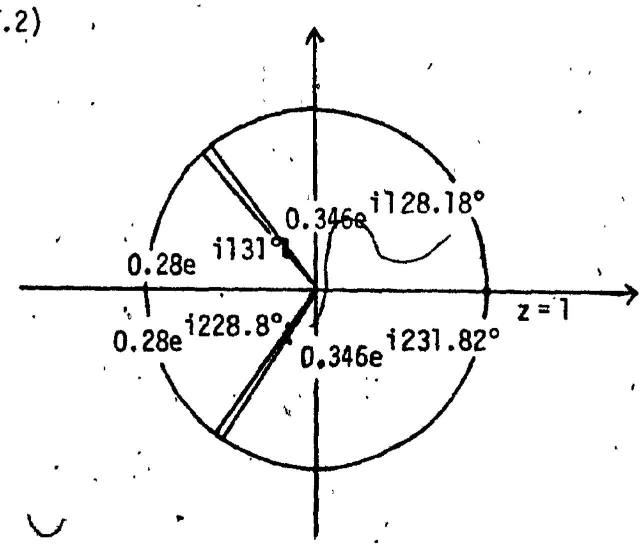

Figure 5.2

The corresponding Callahan polynomial

$$P(z) = \frac{|(1-i)+\sqrt{-2i-1}|}{4}(z-1)(z-0.346\, e^{i128.18^\circ})(z-0.346\, e^{i231.82^\circ}) \tag{5.12}$$

(see remark 3.1) has the following:

$$B = 2.781619 \tag{5.13}$$

$$C = 0.655143 \tag{5.14}$$

(see Figure 5.2).

For the case $n = 4$, let z_1 be one and the other zeros be $z_\upsilon = r_\upsilon e^{i\theta_\upsilon}$, $\upsilon = 2,3,4$. First, we vary r_2, r_3, r_4 between 0 and 0.5 with increasement 0.1; $\theta_2 = 90°$, $\theta_3 = 180°$, θ_4 between 0° and 360° with increasement 30°, we observe that the polynomial having max B and min C is

$$P(z) = (z-1)(z-0.32\, e^{i90°})(z-0.33\, e^{i180°})(z-0.32\, e^{i270°}).$$

Then we fix $\theta_3 = 180°$; vary θ_2 from 90° to 180° with increasement 10°; vary θ_3 from 180° to 270° with increasement 10°. We get the polynomial

$$P(z) = (z-1)(z-0.5\, e^{i100°})(z-0.4\, e^{i180°})(z-0.5\, e^{i260°})$$

Finally, we refine this result by varying r_υ, $\upsilon = 2,3,4$ with increasement 0.01; θ_2, θ_4 with increasement 1°, then we get

$$\max B = 3.854964 \tag{5.15}$$

$$\min C = 0.580143 \tag{5.16}$$

and the corresponding polynomial is

$$P(z) = (z-1)(z-0.38\, e^{i103°})(z-0.33\, e^{i180°})(z-0.38\, e^{i257°}). \tag{5.17}$$

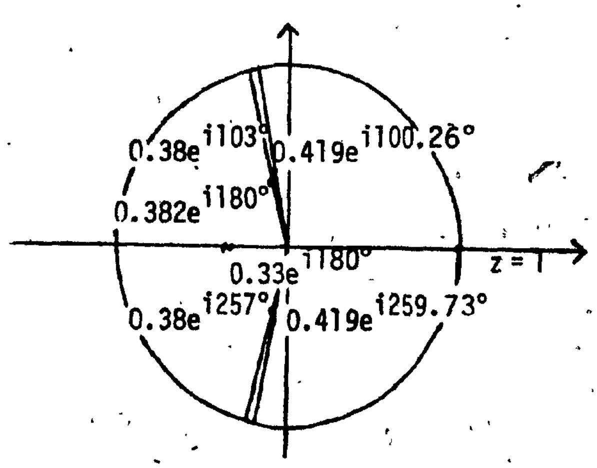

Figure 5.3

Further, following the technique used in remark 3.1, we get the Callahan polynomial of degree 4 with $\max_{|z|=1} |P(z)| = 1$:

$$P(z) = \frac{\sqrt{3+\sqrt{5}}\ |(-1+\sqrt{15}\ i) - \sqrt{-30-2\sqrt{15}\ i}\ |}{20\sqrt{2}}(z-1)(z+\frac{3-\sqrt{5}}{2})$$

$$(z - \frac{-1+\sqrt{15}\ i + \sqrt{-30-2\sqrt{15}\ i}}{4})(z - \frac{-1-\sqrt{15}\ i + \sqrt{-30+2\sqrt{15}\ i}}{4})$$

i.e.

$$P(z) = \frac{\sqrt{3+\sqrt{5}}\ |(-1+\sqrt{15}\ i) - \sqrt{-30-2\sqrt{15}\ i}\ |}{20\sqrt{2}}(z-1)(z-0.419\ e^{i100.26^\circ})$$

$$(z-0.382\ e^{i180^\circ})(z-0.419\ e^{i259.73^\circ}) \qquad (5.18)$$

(see Figure 5.3). We also calculate the B anc C for this polynomial and get:

$$B = 3.842233 \qquad (5.19)$$

$$C = 0.631069. \qquad (5.20)$$

We cannot conclude with so little evidence that the Callahan polynomial is the extremal polynomial for Theorem 4.1, but there may be such a possibility.

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