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STUDENTS' PERCEPTIONS OF INSTITUTIONAL PRACTICES: THE CASE OF LIMITS OF FUNCTIONS IN COLLEGE LEVEL CALCULUS COURSES

Nadia Hardy

Department of Mathematics and Statistics, Concordia University

1455 de Maisonneuve Blvd. West, Montreal QC, Canada H3G 1M8

1-514-252-0388

hardy.nadia@gmail.com

Abstract. This paper presents a study of instructors' and students' perceptions of the knowledge to be learned about limits of functions in a college level Calculus course, taught in a North American college institution. I modeled these perceptions using a theoretical framework which combines elements of the Anthropological Theory of the Didactic, developed in mathematics education, with a framework for the study of institutions developed in political science. While a model of the instructors' perceptions could be formulated mostly in mathematical terms, a model of the students' perceptions included an eclectic mixture of mathematical, social, cognitive and didactic norms. I describe the models and illustrate them with examples from the empirical data on which they have been built.

Keywords. *Institution, Calculus, Limits, Anthropological Theory of the Didactic, Praxeology, Institutional Theory*

1. Introduction

This paper presents partial results of a larger research project aimed at investigating the influence of institutional practices – in the form of definitions, properties, examples and exercises appearing in textbooks and examinations – on students' perceptions of the *knowledge to be learned* about limits of functions in a large, multi-section Calculus course in a North American college institution. In such courses, a common final examination carries substantial weight in students' assessment. This paper focuses on the influence of the final examination, in abstraction from the teacher's practices in the classroom.

At college level, topics related to limits are not necessarily associated with the limit concept or its definition. On the one hand, links among intuitive ideas, the formal definition, and techniques are often hidden in college level Calculus textbooks (Lithner 2004; Raman 2004). The ε-δ definition is presented in a different section than that where intuitive ideas about limits are discussed, and that in which the algebraic calculations for finding limits are presented. Algebraic techniques for finding limits are presented in self-contained sections. On the other hand, the teaching of the formal definition and its uses is dissociated from the teaching of "finding" limits. In a discussion of the restrictions imposed by an atomized curriculum on the teacher's practice, Barbé, Bosch, Espinoza and Gascón (2005) have shown that (a) the mathematical organization of the teaching of the limit definition consists only of a theoretical block (a corresponding practical block consisting of tasks and techniques is missing); and (b) the mathematical organization of the teaching of the algebra of limits consists only of a practical block – tasks and techniques – and a corresponding theoretical block is missing. These authors have investigated the restrictions imposed by the *knowledge to be taught,* as defined in curricular documents, on the *knowledge actually taught* in the classroom. At college level, the components textbook – curriculum – exams can be taken as a reflection of what students are studying and what educational institutions are expecting students to learn. In my research, I have analyzed tasks proposed in final examinations together with the types of solutions that students were expected to present. Based on this analysis, I have built models of students' and instructors' perceptions of the *knowledge to be learned*, searching for the differences between these models and the influence that routine tasks and the absence of theoretical discourses may have on students' perceptions.

The teaching and learning of limits has been studied from many different theoretical points of view, e.g., concept image and concept definition (Tall and Vinner 1981), APOS Theory (Cottril, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic 1996), the Anthropological Theory of the Didactic (Barbé et al. 2005). Most recently, Kidron (2008) discussed the complementary roles of three different frameworks in studying the teaching and learning of limits: procept theory, instrumentation theory and the model of abstraction-in-context. In my research, I have taken an *institutional practices* perspective (Artigue, Batanero and Kent 2007), close to that of Barbé et al.'s (2005) study. In Barbé et al.'s paper, however, the *institutional* status of the studied social practices was not questioned; no distinctions were made among the different *mechanisms* that regulate institutional practices. In my research, such distinctions became important, and I had to adjust my theoretical framework accordingly. I have thus used a combination of the Anthropological Theory of the Didactic (Chevallard 1999, 2002) with a framework for institutional analysis developed in political sciences ("Institutional Analysis and Development" or IAD framework; Ostrom 2005), already proposed in Sierpinska, Bobos and Knipping (2008). The

IAD framework describes what counts as an institution and identifies its main components and mechanisms of functioning.

This paper is structured as follows. The next section briefly describes the studied educational institution. Section 3 focuses on the theoretical framework. Sections 4, 5 and 6 give an account of research results related to instructor's and students' models of knowledge to be learned about limits of functions. Each section contains research methodology information pertinent for the discussed results. The final section 7 discusses the notion of knowledge to be learned, in the light of the anthropological and institutional perspectives and the results of the research.

2. A brief description of the studied educational institution

In the educational system studied in this research, "college" refers to an educational institution situated between high school and university. The high school curriculum in mathematics does not include Calculus. A first one-variable Calculus course is taught only at the college level, in academically oriented (as opposed to vocational oriented) programs leading to studying health sciences, engineering, mathematics, computer science, etc., at the university level. The majority of students enrolled in these courses are 17-18 years-old. The course is usually a multi-section course, with the number of sections in large urban colleges often exceeding 15.

In the studied college, at the time of the research, there were nineteen sections of the first Calculus course, taught by 14 different instructors, with 25-35 students enrolled in each section. The course in the college was run collectively by committees of instructors responsible for selecting an official textbook to be used in all sections, preparing the common "course outline", and writing the common final examination. All instructors teaching the course in a given semester would be automatically members of the ad-hoc "Final Examination Committee" for that semester. The course outline would be quite detailed, so that, in a given week, all sections would often be studying the same mathematical topic and working on the same homework assignments. Students from different sections usually study together, compare notes, and prepare for the final exam together, thus forming a "community of study", which has some control over what is going on in the individual sections. Students may, for example, inform their section instructor that another instructor is more (or less) advanced in the syllabus, or doing less (more) difficult problems.

3. Theoretical framework

From an epistemological point of view, according to the theory of didactic transposition (Chevallard 1985), any didactic phenomenon involves the production, teaching, learning and practice of some mathematical activities. The form of these activities depends on the process of didactic transposition, this is, the changes that a body of knowledge has to go through to become knowledge that can be taught and learned at school. Considering Chevallard's original distinctions and subsequent refinements (e.g. Barbé et al. 2005, and Bosch, Chevallard and Gascón 2005), we can analyze school mathematics into several kinds of knowledge:

- scholarly knowledge, understood as knowledge produced by professional mathematicians;
- knowledge to be taught, described in curricular documents;
- knowledge actually taught which can be gleaned from the teachers' classroom discourse and the tasks he or she prepares for the students;
- knowledge to be learned, which can be a subset of the knowledge to be taught or of the knowledge actually taught and whose minimal core can be deduced from the assessment instruments;
- and knowledge actually learned which can be accessed somewhat from students' responses to tasks, clinical interviews, observations of students' behavior in the ordinary classroom or in specially designed problem solving situations.

This framework, called the Anthropological Theory of the Didactic (ATD; Chevallard 1999, 2002) provides an epistemological model to describe mathematical knowledge as one human activity among others, as it is practiced in various institutions (research mathematics; applied mathematics; engineering; school mathematics at different educational levels; mathematics teacher training institutes, etc.). The model proposed by the ATD states that any mathematical knowledge can be described in terms of a mathematical organization, or a praxeological organization of mathematical nature also called "mathematical praxeology". Mathematical praxeology is a special case of praxeology of any activity, which is defined as a system made of four main components:

- a collection T of types of tasks which define (more or less directly) the nature and goals of the activity;
- a corresponding collection τ of techniques available to accomplish each type of tasks;
- a technology θ that justifies these techniques; and
- a theory Θ that justifies the technology.

The term “technology” is understood as the *logos* or the discourse about the techniques, which allows the practitioners to think of, about, and out the techniques. A technology can be a framework of concepts, procedures and rules for applying them. The theory Θ provides a coherent system in which concepts are defined and rules and procedures are justified. The subsystem [T, τ] corresponds to the know-how, and is called the *practical block* of the praxeology, while the *theoretical block* [θ, Θ] describes, explains and justifies the practical block. It is the theoretical block that makes it possible to preserve the activity as a practice and communicate it to others, so that they, too, can participate in it. This suggests that there is a didactic intention in any cultural practice (if there are no means to teach and therefore perpetuate an activity, it cannot become part of a practice), whence the word “didactic” in the name of the theory.

From the perspective of the ATD, the primary object of research in mathematics education is institutionalized mathematical activity (Bosch et al. 2005). This implies the need to clearly define the institutions taken into account in studying a didactic phenomenon. For example, Barbé et al. (2005), in studying the teaching of limits of functions in Spanish secondary schools, consider the following institutions: mathematical community, educational system, and classroom. To this list, in their exposition of the ATD perspective, Bosch et al. (2005) add “community of study”, whose status as an institution is perhaps less obvious. Yet, my own research made me realize how very real this institution can be (see Section 2).

In ATD, the term “institution” is treated as a “primitive term” and it is not defined (Chevallard 1992, p. 144-145). This may not be a problem in research where the institutional status of the studied social practices is not questioned. This is not necessarily the case in my research. I have, therefore, found it useful to combine ATD with elements of the framework Institutional Analysis and Development (IAD; Ostrom 2005), as has already been done in the study of students’ frustration in prerequisite mathematics courses (Sierpinska et al. 2008). From the perspective of this framework, an institution is an organization of *repetitive* interactions between individuals whose aim is to achieve certain outcomes (an IAD term) or fulfill certain tasks (an ATD term). The organization defines who are the participants, what positions they can occupy relative to the tasks and outcomes, and what *rules*, *norms* and *strategies* (IAD terms) or techniques (ATD term) will regulate and make possible the accomplishment of the tasks. These means of regulation require a specific set of discourses. The discourses can be analyzed into “technologies” and “theories” as in ATD. Rules are explicit, established by a recognized (legal) authority, and contain sanctions against those who break them. Mathematical theorems can be interpreted as ‘rules’ in this sense. Techniques to find limits follow such mathematical rules; for example, the rule “the limit of the product of a function tending to zero by a bounded function is zero” is a basis for certain techniques to find limits. Breaking a mathematical rule leads to contradictions, incorrect results, which are the above mentioned “sanctions” in this case. Norms, on the other hand, carry no formal sanctions. Norms function more like precepts for prudent or moral behavior; they are part of the generally accepted moral fabric of a community, based on habit and custom. Norms do not have to be precise or even explicit, like rules. Newcomers into a practice get to know there is a norm, when they inadvertently transgress it and experienced practitioners tell them that “that’s not how we normally do things here”. As will be discussed later, the *final examination institution* that I have studied has not – not even once in the last six years – included a problem of finding a limit involving radicals for which the rationalization technique would not apply. Thus, the sentence “to find the limit of a function involving radicals, start by multiplying and dividing by the conjugate” represents an implicitly accepted norm in this institution. Norms regulate even the mathematical activity of research mathematicians: for example, there exist unwritten norms regulating the amount of detail that a published proof must have or the style in which it should be written. The third regulatory mechanism in institutions is based on strategies. This term refers to plans of action adopted by participants in an institution for accomplishing a task or achieving a goal. For example, assessment “en masse” of students through a common final examination was the accepted evaluation strategy adopted in the studied college institution. Strategies in mathematical activity include all kinds of heuristics, such as, “study a particular case”, or “build a model of the situation”. Rules, norms and strategies are closely intertwined in the functioning of all institutions. It is no different in mathematical practices, but here the all-encompassing rule is consistently decided by means of reasoning; norms of, say, the elegance of a proof, cannot override the flaws in its mathematical correctness; a strategy in mathematical activity is only valid if it leads to mathematical truth and can be justified by mathematical rules, etc.

Participants of an institution are assigned – or assign to themselves – to different available *positions*. These positions are associated with different *actions* (“action situations”, Ostrom 2005, p. 33) to be taken to achieve certain goals. An institution is, by definition, an organization of repetitive interactions. By observing these repetitive interactions and usual outcomes, participants construct “spontaneous models” (ibid.) for *acting* in the institution. These models – constructed empirically and not always quite consciously by actual participants are not directly accessible to an external analyst who can only build theoretical models of the spontaneous models. The theoretical models of the functioning of an institution are expected to allow institutional analysts to *predict* interactions and outcomes (ibid.). In my research, I have built theoretical models of instructors' spontaneous models of the knowledge to be learned and theoretical models of students' spontaneous models of the knowledge to be learned (about limits of functions). This allowed me to see the differences between what instructors and students perceived as knowledge to be learned about limits of functions, and the influence that instructors' models had over students' models.

The ATD framework does not distinguish between rules and norms. Both these regulatory mechanisms are covered under the term "technology". The normative character of practices in the mathematical classroom has been already pointed out (e.g., Balacheff 1999; Chevallard 1999). IAD provides a language and a theoretical framework in which this normative character, in general, and the normative function of the didactic contract in particular, is highlighted against the mechanisms that regulate mathematical activity – rules and strategies. Balacheff (1999), however, highlights the often overlooked distinction between the more permanent ("institutionalized") normative mechanism of "classroom custom" and "didactic contract", which acts locally, in relation to a particular classroom activity, more as a temporary didactic strategy than as a stable institutional norm.

From the perspective of the IAD framework, it can be said that the teaching of Calculus in colleges is an institution in itself; I call it *College-Calculus*. Different sub-institutions take part in the didactic phenomenon, e.g., the classroom, the curriculum committee, the final examination committee, or the community of study. As was pointed out before, my goal is to investigate the influence of institutional practices on students' perceptions of the knowledge to be learned, in abstraction from the personal mediation of a teacher in the classroom. Thus, assuming that a core of the knowledge to be learned can be conjectured from final examinations, I focused on analyzing this particular action situation of the College-Calculus institution. "There is a common final exam" is a rule in the College-Calculus institution which is explicit and, for all intents and purposes, interpreted the same way by instructors and students. Not so with the content of the final exam. There are some unwritten norms, traditions, because the final exams do not change much over the years. Of course, some things change from one exam to the next, like the formulas of the functions whose limits are to be calculated. But some things don't, like the type of the function. And these "constants" point to the existence of norms. One can glean these norms from empirical data, for example, the texts of the past final exams, textbooks, or interviews with students. Analyzing the texts of past final exams and textbooks, I have built a theoretical model of instructors' spontaneous models of the knowledge to be learned – considering instructors as participants of the final examination committee (as participants of other sub-institutions, such as the classroom, instructors might have different spontaneous models). I formulated this model in terms of praxeologies. Then I conducted 28 "task-based interviews" (Goldin 1997) with students, using tasks that were designed so as to visually resemble the typical final examination tasks but differed from them on the conceptual level. Based on students' responses, noticing their expectations about the tasks, I have built theoretical models of their spontaneous models of the knowledge to be learned – considering students as participants of the community of study (as participants of other sub-institutions, such as the classroom, students might have different spontaneous models).

It is the existence of norms that generates the variety of the spontaneous models co-existing within an institution. The distinction between rules and norms, afforded by the IAD framework, allowed me to explain, in the particular case of the institution I was studying, the difference between the instructors' praxeologies and the students' praxeologies representing the knowledge to be learned.

4. Description of mathematical praxeologies related to limits in the College Calculus institution

The institution which decides about the contents and competencies to be taught is not to be confused with the institution that prepares the final exam, or with the classroom institution. Members of the college's mathematics department are participants in all these institutions but they do not occupy the same positions (Ostrom 2005, p. 18, 40); they abide by different rules in each of these institutions. Being a member of the curriculum committee imposes other loyalties relative to mathematical knowledge than being a member of the final examination committee or being a section instructor who has to cope daily with many students' lack of basic mathematical skills and who is interested in obtaining a good average grade for his or her students.

To characterize the mathematical praxeologies of the College-Calculus institution, I have used the following reference documents: the official textbook, topics listed in the course outline, past final examinations from the last six years, and solutions for these examinations, written by teachers and made available to students. Considering that the only control that the College-Calculus institution has over the knowledge to be learned is through the common final exams, I characterize institutional tasks related to limits of functions according to the tasks proposed in final examinations. The corresponding techniques are described following teachers' solutions and the techniques presented in the textbooks. The description of the theoretical block [θ, Θ] is based on topics listed in the outline, properties and theorems used in textbooks to justify techniques, and teachers' solutions.

From an analysis of teachers' solutions and examples in the textbook, I concluded that types of tasks and techniques belong to the domain of knowledge to be learned, while their respective technologies belong to the domain of knowledge to be taught but not of the knowledge to be learned. I divided theories into an informal justification, belonging to the knowledge to be taught, appearing in the textbook to support the corresponding technology, and a formal mathematical justification that belongs to the scholarly knowledge, appearing in a section of the textbook not listed in the outline of the course.

By analyzing the final examinations of the past 6 years (2002-2007), I identified the following three mathematical praxeologies. I give an example of task for each of the praxeologies.

Mathematical praxeology 1 (MP1)

TASK TYPE T1: Evaluate the following limit: $\lim_{x \to c} \frac{P(x)}{Q(x)}$.

Description: c is a fixed constant; $P(x)$ and $Q(x)$ are polynomials such that the factor $x - c$ occurs in both $P(x)$ and $Q(x)$.

TECHNIQUE τ1: Substitute c for x and recognize the indetermination *0/0*[1]. Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

EXAMPLE 1:

Task: Evaluate the following limit $\lim_{x \to 1} \frac{x^3 - 6x + 5}{x^2 - 6x + 5}$.

[1] The first step in τ1 appears in the textbooks when strategies of calculating limits are described in general. However, this step is omitted in most worked out examples in the textbooks and in solutions written by teachers and made available to students. The same is true for Mathematical Praxeology MP2.

Expected solution: (Substitution of 1 for *x* in the expression to check whether the indetermination 0/0 is the case here is not expected in students' written solutions. Students are not penalized if there are no traces on paper of this verification).

$$\lim_{x\to 1}\frac{x^3-6x+5}{x^2-6x+5}=\lim_{x\to 1}\frac{(x-1)(x^2+x-5)}{(x-1)(x-5)}=\lim_{x\to 1}\frac{x^2+x-5}{x-5}=$$
$$=\frac{1^2+1-5}{1-5}=\frac{-3}{-4}=\frac{3}{4}$$

TECHNOLOGY θ1: If two functions *f* and *g* agree in all but one value *c* then $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)$. If *r(x)* is a rational function and *c* is a real number such that *r(c)* exists, then $\lim_{x\to c} r(x)=r(c)$.

THEORY Θ1: In the analyzed textbook a graph supports the fact that two functions agreeing in all but one point have the same limit behavior (knowledge to be taught). An *ε*-*δ* proof is presented in an appendix. However this appendix and the *ε*-*δ* definition of limits are not listed in the course outline (scholarly knowledge).

Mathematical praxeology 2 (MP2)

TASK TYPE T2: Evaluate the following limit: $\lim_{x\to c}\frac{\sqrt{P(x)}-Q(x)}{R(x)}$.

Description: *P(x), Q(x)* and *R(x)* are polynomials such that $\sqrt{P(c)}-Q(c)=0$ and the factor $P(x)-[Q(x)]^2$ has degree one in *R(x)*.

TECHNIQUE τ2: Substitute *c* in *x* and recognize the indetermination *0/0*. Multiply and divide by the conjugate of $\sqrt{P(c)}-Q(c)$. Factor out $P(x)-[Q(x)]^2$ from *R(x)*. Simplify and substitute *c* for *x*. The obtained value is the limit.

EXAMPLE 2:

Task: Evaluate the following limit $\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}$.

Expected solution: (Substitute 4 for x and recognize the indetermination *0/0*; see footnote 1.)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}=\lim_{x\to 4}\frac{(\sqrt{x}-2)}{(x^2-16)}\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}=\lim_{x\to 4}\frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)}=$$
$$=\lim_{x\to 4}\frac{1}{(x+4)(\sqrt{x}+2)}=\frac{1}{(4+4)(\sqrt{4}+2)}=\frac{1}{32}$$

TECHNOLOGY θ2: If two functions *f* and *g* agree in all but one value *c* then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$. If *n* is a positive integer and *c* is a real number, then $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ for all *c* if *n* is odd and for all non-negative *c* if *n* is even.

THEORY Θ2: In the analyzed textbooks a graph supports the fact that two functions agreeing in all but one point have the same limit behavior (knowledge to be taught). *ε*-*δ* proofs of both statements in θ2 are presented in an appendix that is not listed in the course outline (scholarly knowledge).

Mathematical praxeology 3 (MP3)

TASK TYPE T3: Evaluate the following limit: $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$.

Description: *P(x)* and *Q(x)* are polynomials such that *m*, the degree of *P(x)*, is less or equal to *n*, the degree of *Q(x)*.

TECHNIQUE τ3a: Factor x^m from *P(x)* and x^n from *Q(x)*, and simplify $\frac{x^m}{x^n}$ to $\frac{1}{x^{n-m}}$. Use the fact that the limit of a constant over a positive power of *x*, as $x \to \infty$, is *0*.

τ3b: Divide every term by the highest power of *x* appearing in the rational expression, simplify and use the fact that the limit of a constant over a positive power of *x*, as $x \to \infty$, is *0*.

EXAMPLE 3:

Task: Evaluate the following limit: $\lim_{x\to\infty} \frac{5x^4 - 2x + 1}{6x - 2x^4}$.

Expected solution:

$$\lim_{x\to\infty} \frac{5x^4 - 2x + 1}{6x - 2x^4} = \lim_{x\to\infty} \frac{x^4\left(5 - \frac{2}{x^3} + \frac{1}{x^4}\right)}{x^4\left(\frac{6}{x^3} - 2\right)} = \lim_{x\to\infty} \frac{\left(5 - \frac{2}{x^3} + \frac{1}{x^4}\right)}{\left(\frac{6}{x^3} - 2\right)} = \frac{5}{-2}$$

TECHNOLOGY θ3: If *r* is a positive rational number and *c* is any real number, then $\lim_{x\to+\infty} \frac{c}{x^r} = 0$; if x^r is defined for *x* < *0*, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

THEORY Θ3: An *ε*-*δ* proof is presented in an appendix, however, this appendix is not listed in the course outline (scholarly knowledge).

Instructors' spontaneous models of the knowledge to be learned (instructors as participants of the final examination committee) are made only of the practical blocks. The occurrence of these blocks in final examinations does not obey rules. On the one hand, the institution College-Calculus doesn't have explicit rules stating that these types of tasks *have to* appear in final examinations. Their occurrence is based on tradition and the shared idea that this is the minimum knowledge that students should learn. These types of tasks are usually referred by teachers as "the least common denominator of what is taught in our courses". Thus, the occurrence of these tasks is the result of a practice regulated by *norms*, not by rules. On the other hand, although the committee preparing the final exam also prepares a grading scheme, there are no sanctions for not following it to the letter. It is only a suggestion and the final decisions about the grades are left to the discretion of the instructors. This implies that the institution preparing the final exam considers the techniques as norms, not as rules.

5. Interviews with students

To analyze students' interpretations of the institutional praxeologies I conducted 28 interviews. Subjects were recruited from Calculus II course sections in the winter semester of 2008. All subjects had successfully completed a Calculus I course in the previous semester, in the fall of 2007. Subjects were selected to represent a vast spectrum of the teachers teaching Calculus I in the fall of 2007. In that semester there were 19 sections taught by 14 different teachers; the sample of interviewed subjects covers at least 12 of these 14 teachers. Table 1 shows the number of interviewed students corresponding to each teacher. Five students could not remember the name of their Calculus I teacher; those were accounted for in the last column of the table.

Teacher	T1	T2	T3	T4	T5	T8	T9	T10	T11	T12	T13	T14	?
Number of students	1	1	2	2	4	2	1	1	3	2	2	2	5

Table 1. Number of students per teacher.

The interviews were based on many tasks, including classifying limits, finding limits, and graphing functions. In this paper, I will focus only on the task in which students were asked to find four limits (see below) that resembled those appearing in final examinations but differed from them on the conceptual level. Students were asked to think aloud while working on them. When they were finished, I asked them questions about their calculations, failed attempts, answers, etc. It was by analyzing students' responses to these particular tasks that I realized that students' spontaneous models of the knowledge to be learned can be quite different from the mathematical praxeologies representing the members' of the final examination committee spontaneous models of the knowledge to be learned.

1. $\lim_{x\to 1}\frac{x-1}{x^2+x}$ 2. $\lim_{x\to 2}\frac{x+3}{x^2-9}$ 3. $\lim_{x\to 5}\frac{x^2-4}{x^2-25}$ 4. $\lim_{x\to 1}\frac{x^3+4x^2+9}{x^2+2}$

The tasks corresponding to MP1 that we can find in textbooks involve rational expressions that are easily factorable using algebraic techniques such as "difference of squares", "taking common factors", "factoring by grouping", or simple cases of "undoing the distribution property". Hence, the polynomials in the rational expressions are usually of degree 2 or 3, and, on rare occasions, of degree 4. On the other hand, tasks in MP3 always involve polynomials that cannot be easily factored, or are not factorable at all. Problems 1, 2 and 4 above, can all be solved by direct substitution. Problem 3 cannot be solved algebraically, but by inspection or by making a table of values. I chose these four problems for the interview because the rational expressions in problems 1, 2 and 3 are (in the case of problem 2), or seem to be (in the case of problems 1 and 3), instances of rational expressions in type of task T1, and problem 4 belongs neither to MP1 nor to MP3, as the involved polynomials are not factorable and the limit is taken at a constant. The idea behind this choice was, in the case of problems 1, 2 and 3, to deceive students to engage into the factoring techniques typically used in tasks in MP1, and in the case of problem 4, to present students with a rational expression that is not factorable to contrast their approach to problem 2. My expectation was that in problems 1 and 3 students would get frustrated and show difficulties in providing an answer, because of the lack of common factors, while in problem 2, they would be comfortable with cancelling the common factors and then using substitution to arrive at a correct answer, without noticing that the factoring was not necessary. My expectation for students' approach to problem 4 was that they would recognize right away that the technique to find the limit is direct substitution, because the involved polynomials are not factorable. Although my hypothesis was confirmed, the reasons for this behavior were not as purely mathematical as I expected, but were generated by a mixture of *mathematical, social, cognitive* and *didactic norms.*

In the following table I show the frequencies of correct and incorrect answers and no answer at all.

N=28	Correct answer	Incorrect answer	No answer
Problem 1	82.1 (23)	10.8 (3)	7.1 (2)
Problem 2	96.4 (27)	0 (0)	3.6 (1)
Problem 3	42.9 (12)	14.3 (4)	42.9 (12)
Problem 4	92.9 (26)	7.1 (2)	0 (0)

Table 2. Frequency of correct, incorrect, and lack of answer in the four problems given to students in the interview.

Both incorrect answers in problem 4 were due to miscalculations. Prior to any intervention on my part, only by observing students' notes and listening to students' spontaneous talk, I noticed that in problems 1 to 3 most students would factor the numerator and the denominator trying to find common factors, even in cases where they tried direct substitution first. When cancellation was not possible (problems 1 and 3) some students could not produce a final answer (two students in problem 1, ten in problem 3). Because students showed a similar behavior when approaching problems 1 and 3, I discuss them first. Then I present a discussion for problems 2 and 4.

Discussion of problem 1: $\lim_{x \to 1} \frac{x-1}{x^2+x}$

Of the 28 interviewed students, twenty factored the numerator and the denominator in problem 1. These students can be divided into two groups: those who tried direct substitution first (seven) and those who factored first (thirteen). From the spontaneous talk, my first interpretation was that students who used direct substitution as a first approach in problem 1 and then proceeded with factoring did so because they were not sure about the value of 0 divided by 2. I was thus interpreting their behavior as deficiency in their algebraic knowledge. For example, student S1 said:

> S1: Ok. The first thing I do when I see limits is to put in the number it goes to, to see what it gives. So in this case I do zero over two, right? […] Then […] what I would do is factor. Now I don't remember, if I factor a negative one, can I cross them out? [Student S1 took common factor x in the denominator but then got stuck at the fact that the other factor in the denominator was $x + 1$ and not $x - 1$ as he expected. He was then trying to factor out a -1 so as to have the same factors and cancel them out; something that of course cannot be done in this case.]

However, an analysis of students' explanations following my intervention showed that what was triggering the factoring was a routine sequence of techniques of which, for these students, substitution was the first in line. Although S1 said he did substitution "to see what it gives", he disregarded the result and tried to factor. In the next minutes of the interview, there was this exchange:

> I: The first thing you did was to put in the one...

S1: Yeah. The next step is to factor.

Thus, it seems that students were doing substitution not to find the limit, or to characterize an indetermination, but because that's "what you do first". They were not paying attention to the outcome of the substitution. Rather, they were engaging in a kind of *normal* behavior developed for the context of finding limits of rational functions: to find the limit of a rational function, some algebraic technique has to be applied (see discussion of problem 4). For example, realizing that there are no common factors in the expression in problem 1, student S2 tried another algebraic approach:

> S2: What I think might work is if I separate them into two different parts. If I have *x* over *x* squared plus *x* minus one over *x* squared plus *x*. Which follows the rules of how you are allowed to solve for limits. Now I can easily put in the *x* values, so one over two minus one over two, so in this case the limit does equal to zero.
> I: Ok, and why do you think you have to go through this step?
> S2: Well, I guess I could just put in the one here [in the initial expression], but I am used to have something divided by zero.

The last sentence of S2 makes explicit his expectations about the types of tasks he can be given in the context of limits of rational functions. Such expectations could also be found in other students' talk.

When questioned, many students realized that the algebraic approach was not necessary; they explained their behavior as the result of following perceived *norms*. For example, student S18 explains her behavior (she says factorable to mean that there are common factors in the numerator and denominator):

> S18: Basically I look at a problem and the first thing I see... and I always assume it is factorable, I mean, they never gave me a problem that wasn't factorable, so I wouldn't even ask whether it's factorable. I'd say, ok, where can I factor it. And I'd say ok let's look at the different categories. If I see a trinomial or a difference of squares and the method to factor them, and so long and so forth, but if it wasn't factorable... I never came across a problem that wasn't factorable.
> I: And do you remember ever coming across something like this [problem 1]?
> S18: Yes, it is one of the tricky ones. You have to think of a special... I don't know, I am not saying this would work, but you can multiply by negative one to inverse the signs of your equation and then you'd be able to simplify and it would work out.

Even in the interview, which is an event outside of the institution, student S18 thinks she will not be given a problem that isn't "factorable"; for her, problem 1 is a problem to be solved by factoring; it's just that she doesn't know how to factor it: a tricky limit.

Student S3 seemed to hold the same assumptions as S18; he factored because he thought it was a zero over zero type of problem:

> I: But why did you factor here?
> S3: Because I try the zero over zero thing not realizing...
> I: When did you realize the denominator was not zero?

S3: I was substituting one on the bottom.
I: Did you do the substitution here or here [the initial form or the factored form]?
S3: No, here [the factored form].

Student S17 made a strong assertion to defend his factoring approach. When asked, "Was it necessary to factor?", he said:

S17: Was it necessary? No. But I was taught, if you can factor, factor.

In the same vein, student S15 answered this question by saying:

S15: Oh, no, no, I do that in every problem: I see if something would cancel. Even if nothing cancels I do it anyway, in case I miss something.

Discussion of problem 3: $\lim_{x\to 5} \frac{x^2-4}{x^2-25}$

Students displayed a very similar behavior in their approaches to problem 3. Of the 28 interviewed students, sixteen factored before trying direct substitution, seven factored after trying direct substitution. Their reasons for factoring were very similar to those expressed for problem 1: either their strategy is always to factor first or they expected the problem to be an indetermination of the type zero over zero. Again, their explanations refer to *norms*. For example, student S7 factored the numerator and the denominator right away, and when asked why, she said:

S7: Well... I don't know... for me... because in most of the exercises that we were given, every time that you'd replace it'd give you zero over zero, so it's kind of a reflex.

Student S11 was frustrated by the fact that factoring wouldn't lead to simplification, but he was convinced that there was something to be done:

S11: If I open them, there's is nothing I can cancel... there must be something else. I cannot bring them up either, then I can't divide. I cannot pull any *x*s out. I'll open them [he means to factor] and I see after, maybe, but I don't think here anything will work.

The interview with student S7 also showed that she believed that some algebraic technique should be applied; factoring is the only technique she thinks she remembers.

I: What happens with that one? [She wrote that the limit equals 21/0.] Did you try anything in your mind?
S7: No, nothing works out.
I: Why do you say that nothing works out?
S7: Ok, well. Because if I put it in I get nothing, well I get twenty one over zero. Then […] if I factor it out it doesn't give me anything different, like you can't cross anything out. But then… I am trying to remember, back to Cal I, all the different steps you could do. […] There was always first you try to factor and cross out anything that you can. Then…

Discussion of problem 2: $\lim_{x \to 2} \frac{x+3}{x^2-9}$

In problem 2, seventeen students were able to produce a correct answer by doing substitution after factoring and cancelling out common terms. Student S6 gave an insight into the state of his mind, which suggests a psychological explanation why students were factoring and cancelling common terms before checking if this was necessary to find the limit:

> S6: [S6 tried substitution first in every problem except problem 2.] I think because I saw the top factors out... I think every time I see *x* square minus nine I get mentally excited and I want to factor out and cancel. And I knew I would be able to cancel so I was confident.

Discussion of problem 4: $\lim_{x \to 1} \frac{x^3+4x^2+9}{x^2+2}$

With respect to problem 4, two students "believed" the question was to find the limit as $x \to \infty$, while three tried long division or factoring. Twenty students solved problem 4 right away by direct substitution. Students who substituted right away observed that the problem was "too easy", but eight of these students did not do direct substitution in problem 2. I surmise that these eight students identified problem 2 as belonging to type of tasks T1, but they could immediately see that problem 4 did not belong to T1. The two students who "believed" that $x \to \infty$, considered problem 4 as an instance of type of tasks T3. The three students who tried some algebraic technique to simplify the expression, focused on the fact that $x \to 1$ and classified problem 4 as belonging to T1 and then tried techniques characteristic of *0/0* indeterminations. This suggest that students' attention is often focused on the algebraic form of the function, or at least, not on the limit expression as a whole.

The responses of student S12 provided a deep insight into her (and possibly also other students') thinking about limit problems. When given problem 4, she said:

> S12: […] Isn't this one [the limit] just infinity too? I don't really remember if we are allowed, when you can like… let's say you divide [the numerator and the denominator] by *x* cube… Can I do it, even if I have just one *x* cube? Or is the rule like… because if I do it then I'll get one over zero which is infinity… **Oh, it's one, right?** [referring to the "1" in "$x \to 1$".] […] Then it's just fourteen over three.
> I: Why did you think it was infinity?
> [...]
> S12: Well it's just because it's like you can't factor this. Can you? No, I don't think you can. So the only thing they could ask us is divide by *x*.

Her explanation was key in my understanding of what students believe to be the knowledge to be learned, and thus in the construction of a theoretical model of students' spontaneous models of the knowledge to be learned. I present this model in the section below.

6. A model of students' spontaneous models of the knowledge to be learned

Student 12 classified limits of rational functions into two types: those where the polynomials are "factorable" and those where they are not. More generally, we could say that students distinguish two types of limits. I describe them below:

Type 1. When x tends to a constant, expressions are *normally* indeterminations of the type zero over zero, and they involve binomials or trinomials. The polynomials in these expressions can be easily factored by the standard algebraic techniques that students had learned in high school: difference of squares, "undoing" the distribution property, factoring by grouping. Limits involving expressions that contain polynomials in the "high school factoring categories" cannot be found only by direct substitution, something else must be done.

Type 2. If a rational expression involves polynomials that are not binomials or trinomials easily recognized as belonging to one of the "high school factoring categories" it must be a limit at infinity, or a limit that can be found by substituting a constant for x.

These types are based on norms that are not purely mathematical but a mixture of mathematical, cognitive, social and didactic norms. For example in Type 1, the sentence "When x tends to a constant, expressions are *normally* indeterminations of the type zero over zero, and they involve binomials or trinomials" describes a *social norm*, or, using Voigt's term, a *socio-mathematical norm*, since it refers to a social convention regarding the mathematical objects that will be dealt with in the tasks given to students (Voigt 1995). Implied in students' idea of types of tasks is the norm, "If the polynomial can be easily factored, then factor it". This norm is *cognitive* in the sense that, upon recognizing such a polynomial, the student feels internally compelled to factor it. One of the students was saying that he gets *"mentally excited"* when he sees something like $x^2 - 9$ and he simply *must* factor it. It is also a *didactic norm*, however, because, in the routine tasks given to students, factorization of such polynomials is usually a useful strategy: there is a didactic intention in it, and the student is decoding it correctly by factoring the polynomial. It is a *social norm*, as well, because it can only function as a norm by being repetitively effective in similar social situations. Thus, the notion that these polynomials can be *easily* factored and the standard techniques typically used for this factoring (difference of squares, "undoing" the distribution property, factoring by grouping) are an entanglement of *social*, *didactic* and *cognitive norms*.

Students appear to classify limits of rational expressions into different types of tasks according to their algebraic appearance, instead of using some Calculus criteria such as types of indeterminations, type of technique to be applied, convergence or divergence, etc. The technique to be used to accomplish a task is chosen based on the algebraic form of the expression. In problem 4, students applied direct substitution without hesitation; they took it for granted that the denominator was not zero because it did not look like the trinomials or binomials they were used to be given. Many of these same students did not check that the expression in problem 2 was not an indetermination; they thought it was because those polynomials do fall into the "high school factoring categories".

The technology, i.e. the discourse supporting the technique, seems to be that of *norm*: "we do this because that's what we usually do under the circumstances". This is an extrapolation of what the students actually said in the interviews:

> S2: … I am used to having something divided by zero.
> S18: ... they never gave me a problem that wasn't factorable.
> S7: Most exercises that we were given... it'd give you zero over zero.
> S12: I don't think [you can factor this]. So the only thing they could ask us is to divide by *x*.

I surmise that, in students' praxeology, the role of a theory justifying this technology is played by the students' trust in the authority of the teachers, the textbooks, the solutions to past examinations. At the college level, students position themselves as subjects of a school institution who have to abide by its rules and norms (Sierpinska et al. 2008). It is the school institution, embodied in persons in the position of power over the knowledge to be taught and learned and in the official documents and texts, not the students, who is responsible for the validity of this knowledge. Theory in the mathematical sense is not under the students' jurisdiction or responsibility (Chevallard 1985, p. 75).

7. Discussion

Instructors' spontaneous models of the knowledge to be learned could be described using criteria characteristic of college level Calculus. In the interviews, however, students revealed that their spontaneous models of the institutional praxeologies are not built using Calculus criteria. Their approaches to finding limits of rational expressions show that their models are grounded in their previously acquired knowledge, especially knowledge corresponding to high school algebra, and in a type of strategic knowledge associated with succeeding on the final examination. The institutional practices that make the tasks T1, T2, T3 and the corresponding techniques routine, are such that they have conditioned students to expect only these tasks. This does not mean that students are doomed to fail when dealing with non-routine problems (see Table 2). Nevertheless, while dealing with the proposed problems, students have shown that their thinking is not mathematical thinking. Students justify their choice of a technique to tackle a problem by stating their beliefs and convictions that the technique indeed applies (their expectations about the tasks that the institution asks them to do). These beliefs and convictions are themselves based on communicated elements of tradition (explicitly communicated in the solutions written by teachers and made available to the students). Furthermore, students' use of techniques is an algorithmic use; it is based on a recall of a set of "instructions" or "steps" given by the textbook or the instructor. The absence of a theoretical block in instructors' expectations of student's solutions to final examinations may result in these steps forming an *arbitrary* list. A simple consequence of this arbitrariness is that students have difficulties in remembering the order. Thus, for example, they hesitate if they should do direct substitution first or factoring first.

The "normative" character of instructors' models of the knowledge to be learned emphasizes learning on the plane of tradition rather than on the scientific, mathematical plane. It is as if the implicit institutional discourse was, "this technique is used to solve this problem because this is how things are usually done here" instead of, for example, "this technique is used to solve this problem because it is one of the (many) mathematical strategies to find the answer and because of this or that mathematical feature of this problem, it is an efficient strategy, better than...". This may have the effect that students end up learning how to behave *normally* rather than how to behave *mathematically*.

The four tasks presented to the students in the interview belong to what can be identified as the knowledge to be taught: these are topics covered by the textbook in sections listed in the outline of the course. Tasks of these types, however, did not appear in the final examinations of the last 6 years. The first three problems presented to the students in the interview do not look like the ones in the textbook but resemble the routine tasks in the sense that the polynomials involved can be easily factored; students approached these problems by way of algebraic techniques. It is problem 4 that they easily recognized as a non-routine but still familiar task; this task does resemble the ones that, occasionally, they had to deal with in the textbook (and probably in the classroom). For a mathematician or for a Calculus teacher at the college level, problems 1, 2 and 4 are all of the same type. But the interviews revealed that students would rather treat problems 1, 2 and 3 as belonging to the same type, while problem 4 would be in a separate category. The reasons for these models are based on norms; norms built on routine tasks. Of course, these routine tasks might be chosen so as not to trick the students into using algebraic techniques when they are not needed. The analysis of the interviews shows, however, that they have a negative impact on students' generalizations: where a mathematician sees a limit that can be found by direct substitution (e.g. task 2), students see a limit that has to be found by an algebraic technique. In the routine problems presented in textbooks and final examinations, these students have identified patterns on which they have built their spontaneous models for practices related to limits of functions. These patterns, however, are not mathematical. Yet students' models are valid or, perhaps, more accurately, viable. They are viable because the College-Calculus institution to which they belong does not propose enough tasks that would challenge them. Furthermore, students' models are emphasized and validated by the tasks proposed by the institution. They are rooted in the void left by the absence of a theoretical component in the knowledge to be learned as defined by some sub-institutions of the College-Calculus institution. Participants of these institutions fill this void with cognitive, didactic and social norms.

The different notions of knowledge described in the process of didactic transposition highlight "the institutional relativity of knowledge and situates didactic problems at an institutional level, beyond individual characteristics of the institutions' subjects" (Bosch et al. 2005). In the present work, I have tried to highlight the possible relativity of these notions when considered from an anthropological point of view. From a strictly epistemological perspective, notions such as *knowledge to be taught* or *knowledge to be learned* might be quite well-defined objects. From the anthropological point of view, however, its unity breaks down into distinct praxeologies, different for students and for teachers; they become relative to the institution that is describing them. Even this relativity is not subtle enough, however. The distinction between practices regulated by rules, and practices regulated by norms, afforded by the IAD framework, emphasizes the normative character of the theoretical blocks in students' praxeologies and highlights the different *nature* of these norms. The integrated perspective ATD/IAD allows me to zoom into the mechanisms that *link* and *regulate* the different parts of a (mathematical) praxeology, relative to the models of knowledge developed by participants in different positions with respect to the institution. I believe that all mathematical activity is regulated not only by rules and strategies but also by norms. The results presented here, however, show that the norms that regulate students' theoretical blocks interfere with their ability to think in mathematical terms. In this paper, I have tackled the question of the mechanisms that regulate students' practices as participants of the community of study. Research grounded in the ATD/IAD framework may help answer related questions such as what are the mechanisms that regulate students' and instructors' practices as participants of the different institutions that take part in didactic phenomena. Furthermore, assuming that *norms* pervade all human practices, from the combined ATD/IAD perspective, we may be able to investigate, how a College-Calculus institution can develop *norms of appropriate behavior* relative to limit tasks without losing the mathematical character of this behaviour, as has unfortunately been happening within the studied institution. The question is, where is the threshold past which institutional norms start diverting students and instructors from behaving *mathematically*.

To achieve these goals, it may be, however, necessary to refine the framework, especially as far as the concept of norm is concerned. This research has suggested that there are several kinds of norms at play in the way instructors and students view knowledge to be taught and learned. I have tried to capture differences between them using terms such as "mathematical", "social", "didactic", and "cognitive". These terms, however, did not refer to distinct kinds of norms, but only to the kinds of actions that they seemed to constrain, and each norm appeared to constrain several kinds of actions. If we want to classify norms relative to their effect on the quality of students' mathematical thinking, then there is more theoretical work to be done. There is also more empirical work to be done, since, obviously, this research was based on a single institution of College-Calculus teaching, and a small number of students. It would be interesting to conduct this type of studies in other College-Calculus institutions and compare the results.

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