MACHINES, SYSTEMS
AND CATEGORIES

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ABSTRACT

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We present the Arbib and Manes theory of machines and behaviors in a category. For discrete, linear machines and automata in a monoïdal category, Goguen proved that minimal realization is right adjoint to behavior, as functors between certain categories of machines and behaviors. We show that this result holds in the more general context of Arbib and Manes. As examples we give a complete study of discrete, linear, group and tree machines. We also show that decomposable systems are Arbib machines thus establishing a link between machine and system theory.

CONTENTS

INTRODUCTION

The more we progressed in the study of pure mathematics during our graduate studies, the stronger became our feeling that we were dealing with abstract concepts too far away from practical applications.

We talked out our doubts with some of our instructors in the Mathematics Department, Concordia University, Sir George Williams Campus in Montreal. One of them, Dr. COHEN, suggested to us to read the book "Discrete Mathematics" from BOBROW and ARBIB (our reference [3]).

While doing so, we discovered how so many concepts of modern algebra could be applied to that most earthly and practical branch of applied Mathematics, automata theory. Under the guidance of Professor Cohen, we undertook the fourteen months effort whose result is this thesis. Its purpose is to present how the most abstract branch of algebra, category theory, can be used to help solve problems in machine and control systems theory.

One of the most basic of these problems is, given a function (a behavior), to find a "machine" which will compute (realize) this function in the most "economical" way. This problem is called minimal realization.

The concepts of machine and behavior have been axiomatized by ARBIB and MANES. Technically these notions take form of functors which are moreover related by a condition known as adjointness. This condition allows one to shift freely between two seemingly different contests according to the nature of the particular problem under consideration.

It turns out that minimal realization is right adjoint to behaviour as functors between certain categories of machines and behaviors. GOGUEN proved this for discrete and linear machines in 1 , for automata in closed symmetric monidal categories, with countable co-products and canonical cofactorizations, in [2] .

In chapter I we show that this very important result holds in the much more general context of machines in categories as defined by ARBIB and MANES (see [3] or [7]). Thus category theory applied to automata theory gives us the following very strong results:

1. For a fairly large class of functions we call A-behaviors (A for Arbib who describes them in [3]), there exists a minimal realization;
2. The way to "build" them is fundamentally unique and we have precise "clues" how to do it;
3. These minimal realizations are uniquely characterized up to isomorphism.

In chapter II we discuss three examples of A-machines: discrete, linear and tree-automata.

In chapter III category theory allows us to "build a bridge" between control theory and automata theory with the concept of decomposable systems as defined by PADULO and ARBIB in [4]. In this book, the authors give the conditions for such a system to have a minimal realization and describe this realization. Our contribution is to show that decomposable systems with these suitable conditions are, in fact, A-machines (this is claimed but not proved in [4]).

As examples of decomposable systems we again discuss linear machines in this new context and we give a complete study of group-machines (about which only a few facts are given in [4]).

Decomposable systems not only give us a somewhat shorter way to prove that a function (or a machine) is an A-behavior (or an A-machine) and provide a link between systems and automata, but also allows to define observability in categorical terms. In chapter IV we discuss observability in the context of set theory versus the categorical one of minimality. We then expose the ARBIB and MANES theory of state-behavior machines, as described in [9], where a categorical definition of observability (which generalizes the one seen in chapter III) is given and the usual result: "A machine is minimal if and only if it is reachable and observable" is proved.

In part V we outline the adjoint machine concept and duality theory of Arbib and Manes as exposed in [9], giving only the few simple proofs not contained in that paper.

All the definitions and results of category theory used in this thesis are listed in the annex.

One word about our notation:

DN. 3 = 3rd definition in the text.

TH 4 = 4th theorem in the text.

DN a 2.5 = part 5 of the 2nd definition in the annex.

$f : S \longrightarrow S'$ = f is a map which sends

$s \longmapsto f(s)$ $\quad s \in S$ into $f(s) \in S', \forall s \in S$.

$$\begin{tikzcd}
\bullet \arrow[rr, mapsto] \arrow[dddr, bend right] & & \Lambda \arrow[dddl, bend left] \\
\{\cdot\} \arrow[r, "\eta"] \arrow[dr, "\sigma"'] & X^* \arrow[d, "\delta^+"] & \\
& S & \\
& \sigma(\cdot)=\sigma &
\end{tikzcd}$$

commutes means that $\sigma = \delta^+ \cdot \eta$ (internal diagrams) and the external diagram shows the actions of the various maps on the elements of the sets involved.

We would like to express our very deep gratitude to Concordia University (Sir George Williams University a few years back). For years it has been the only University, as far as we know, which offered all its regular courses in the evening as well as during the day. Without its pioneering work in adult and continuous education thousands of persons like ourselves could not have realized their expectations.

Finally we must thank our director of thesis, Dr. GERARD ELIE COHEN, for his trust, his encouragements, his help, his patience without which we will not have begun this work, to say nothing of completing it.

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CHAPTER I

A - MACHINES, A - BEHAVIORS AND THEIR MINIMAL REALIZATION

In this chapter we will define A - Machines, A - Behaviors, their respective categories and show that the functor minimal realization is right adjoint to external behavior.

1. MACHINE IN A CATEGORY. —

Let $\mathcal{C}$ be a category.

DN 1. Process.

A process X is a functor $X : \mathcal{C} \longrightarrow \mathcal{C}$.

DN 2. X dynamics.

$\forall s \in |\mathcal{C}|$, $\forall \delta \in \mathcal{C}(Xs, s)$, δ is called an X dynamics.

DN 3. X dynamorphisms

$\delta : Xs \longrightarrow s$, $\delta' : Xs' \longrightarrow s'$ are X-dynamics

An X - dynamorphism $f : \delta \longrightarrow \delta'$ is a morphism

$f \in \mathcal{C}(s, s') \ni \delta' . Xf = f . \delta$

i.e. $\ni$ the diagram

$$\begin{array}{ccc} Xs & \xrightarrow{\delta} & s \\ \downarrow{\scriptstyle Xf} & & \downarrow{\scriptstyle f} \\ Xs' & \xrightarrow[\delta']{} & s' \end{array}$$

commutes

DN 4. Dyn X

Dyn X is the category whose:

objects are X-dynamics,

morphisms are X-dynamorphisms.

$\because$ DN3. DYN $X(\delta, \delta') \subseteq \mathcal{C}(s, s')$

$\therefore$ We can define composition and identities in DYN X as in $\mathcal{C}$. We show: DYN X is indeed a category.

Let $\delta : Xs \longrightarrow s$, $\delta' : Xs' \longrightarrow s'$, $\delta'' : Xs'' \longrightarrow s''$ be X-dynamics and $f : \delta \longrightarrow \delta'$ and $g : \delta' \longrightarrow \delta''$ be X-dynamorphisms. By DN 3 both parts of the diagram commutes,

$\therefore$ the whole diagram commutes and

X a functor $\Longrightarrow$

$X(g \bullet f) = Xg \bullet Xf$

$$\begin{array}{ccc} Xs & \xrightarrow{\delta} & s \\ {\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle f} \\ Xs' & \xrightarrow{\delta'} & s' \\ {\scriptstyle Xg}\downarrow & & \downarrow{\scriptstyle g} \\ Xs'' & \xrightarrow{\delta''} & s'' \end{array}$$

$\therefore$ $\delta'' \bullet X(g \bullet f) = (g \bullet f) \bullet \delta$ and $g \bullet f$ is a X-dynamorphism $\delta \longrightarrow \delta''$.

Identy: let $Id_\delta = Id_s$

We have again a commuting diagram as X a functor $\Longrightarrow$

$X\,Id_s = Id_{Xs}$

$$\begin{array}{ccc} Xs & \xrightarrow{\delta} & s \\ {\scriptstyle X\,Id_s}\downarrow & & \downarrow{\scriptstyle Id_s} \\ Xs & \xrightarrow[\delta]{} & s \end{array}$$

and $\delta \bullet Id_{Xs} = \delta$, $Id_s \bullet \delta = \delta$ in $\mathcal{C}$;

$\therefore$ $Id_\delta = Id_s$ is an X-dynamorphism which inherits the usual identity properties from $\mathcal{C}$.

NOTE 1. $\delta : X s \longrightarrow s$ an X-dynamics, i.e. $\delta \in \mathcal{C}(X s, s)$,

is also a X-dynamorphism $X\delta \longrightarrow \delta$;

indeed the diagram

$$\begin{tikzcd} X(X s) \arrow[r, "X\delta"] \arrow[d, "X\delta"'] & X s \arrow[d, "\delta"] \\ X s \arrow[r, "\delta"'] & s \end{tikzcd}$$

obviously commutes

$\therefore$ δ has two different roles: $\delta \in \mathcal{C}(Xs, s)$ is an object of DYN X while the same $\delta \in$ DYN X $(X\delta, \delta)$ is a morphism of DYN X $\neq$ $\mathrm{Id}_\delta = \mathrm{Id}_s$

DN 5. Input process

$X : \mathcal{C} \to \mathcal{C}$ is an input process if the forgetful functor U: DYN X $\longrightarrow \mathcal{C}$

on objects $(\delta : Xs \to s) \longmapsto s$

on morphisms $(f : \delta \to \delta')$ — $(f : s \to s')$

has a left-adjoint.

In this case $\forall s \in \mathcal{C}, \exists\, (\delta_s^* \in /\text{DYN X}/, \eta_s : s \longrightarrow U\delta_s)$ universal from s to U;

$\therefore$ $\forall\, (\delta' : X s' \longrightarrow s', f : s \longrightarrow U\delta' = s')$ $\exists$ a unique X-dynamorphism ψ $\ni$ the two diagrams

commute:

$$\begin{tikzcd} s \arrow[r, "\eta_s"] \arrow[dr, "f"'] & U\delta_s \arrow[d, "\psi"] \\ & s' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(U\delta_s) \arrow[r, "\delta_s"] \arrow[d, "X\psi"'] & U\delta_s \arrow[d, "\psi"] \\ X s' \arrow[r, "\delta'"'] & U\delta' = s' \end{tikzcd}$$

Ψ is called the <u>unique dynamorphic extension of f</u>

δ_s is called the <u>free dynamics over s w.r.t. to $\mathcal{U}$</u>

and we will write <u>$\mathcal{U}\delta_s = X^* s$</u> where $X^* = \mathcal{U}.V : \mathcal{C} \to \mathcal{C}$, $(V \dashv \mathcal{U})$.

NOTE 2. (1) Let $V : \mathcal{C} \longrightarrow$ DYN X be the left adjoint of $\mathcal{U}$,

i.e. $V \dashv \mathcal{U}$, then V is defined by

objects: $s \longmapsto Vs := \delta_s$

morphisms: $f \; \begin{matrix} s \\ \downarrow \\ s' \end{matrix} \longmapsto \begin{matrix} \delta_s \\ \downarrow \\ \delta_{s'} \end{matrix} \; Vf$ where

$Vf : X^* s \longrightarrow X^* s'$ in $\mathcal{C}$ is the unique dynamorphic extension of $\eta_{s'} \cdot f$ i.e. $\ni$

$$\begin{tikzcd} s \arrow[r, "\eta_s"] \arrow[dr, "f"'] & X^* s \arrow[dd, "Vf"] \\ & s' \arrow[d, "\eta_{s'}"'] \\ & X^* s' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X X^* s \arrow[r, "\delta_s"] \arrow[d, "X(Vf)"'] & X^* s \arrow[d, "Vf"] \\ X X^* s' \arrow[r, "\delta_{s'}"'] & X^* s' \end{tikzcd} \quad \text{commute}$$

In short $V \dashv \mathcal{U} \Longrightarrow$

$\mathcal{C}(s, \mathcal{U}\delta' = s') \cong$ DYN X $(Vs = \delta_s, \delta')$

$$\begin{matrix} s \\ \downarrow \\ s' \end{matrix} f \longleftrightarrow \Psi \begin{matrix} \mathcal{U}\delta_s = X^*s \\ \downarrow \\ s' \end{matrix}$$

and $\mathcal{C}(s, \mathcal{U}\delta_{s'}) \cong$ DYN X $(Vs = \delta_s, \delta_{s'})$

$$\begin{matrix} s \\ \downarrow \\ \mathcal{U}\cdot\delta_{s'} \end{matrix} \eta_{s'} f \longleftrightarrow Vf \quad \begin{matrix} \mathcal{U}\delta_s = X^*s \\ \downarrow \\ \mathcal{U}\delta_{s'} = X^*s' \end{matrix}$$

(2) $X^* = \mathcal{U} \cdot V : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor as it is a composition of two functors.

DN 6. Machine in a category*

1. A machine in a category $\mathcal{C}$ is a septuple

$M = (X, s, y, i, \delta, \sigma, \lambda)$ where

X is an input process,

s = state object $\in /\mathcal{C}/$

y = output object $\in /\mathcal{C}/$

i = initial state object $\in /\mathcal{C}/$

$\delta : Xs \longrightarrow s$, an X-dynamics, is a $\mathcal{C}$ morphism

$\sigma : i \longrightarrow s$, initial state morphism, is a $\mathcal{C}$ morphism,

$\lambda : s \longrightarrow y$, output morphism, is a $\mathcal{C}$ morphism.

2. Let $(\delta_i : X(X^* i) \longrightarrow X^* i,\ \eta_i = i \longrightarrow X^* i)$ be universal from i to $\mathcal{U}$

– the object of inputs $x^* = X^* i$

– the reachability morphism $\delta^+ : x^* \longrightarrow s$

is the unique dynamorphic extension of the initial state morphism $\sigma : i \longrightarrow s$,

3. M is reachable if δ^+ is a coequalizer.

NOTE 3.(1) $V \dashv U \Rightarrow \exists \ \delta^+$ as

$$\mathcal{C}(i, U\delta = s) \cong \text{DYN } X\,(Vi = \delta_i, \delta)$$

$$\begin{tikzcd} i \arrow[d, "\sigma"'] \\ s \end{tikzcd} \longleftrightarrow \delta^+ \begin{tikzcd} U\delta_i = x^* \arrow[d] \\ U\delta = s \end{tikzcd}$$

and the two following diagrams commute:

$$\begin{tikzcd} i \arrow[r, "\eta_i"] \arrow[dr, "\sigma"'] & x^* \arrow[d, "\delta^+"] \\ & s \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Xx^* \arrow[r, "\delta_i"] \arrow[d, "X\delta^+"'] & x^* \arrow[d, "\delta^+"] \\ Xs \arrow[r, "\delta"'] & s \end{tikzcd}$$

2. We can define $\delta^* : X^* s \longrightarrow s$ as the unique dynamorphic extension of Id_s i.e. $\ni$

$$\begin{tikzcd} s \arrow[r, "\eta_s"] \arrow[dr, "Id_s"'] & X^*_s \arrow[d, "\delta^*"] \\ & s \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} XX^*_s \arrow[r, "\delta_s"] \arrow[d, "X\delta^*"'] & X^*_s \arrow[d, "\delta^*"] \\ Xs \arrow[r, "\delta"'] & s \end{tikzcd} \quad \text{commutes}$$

3. $X^*\sigma = V\sigma$ is the unique dynamorphic extension of $\eta_s \cdot \sigma$

i.e. $\ni$

$$\begin{tikzcd} i \arrow[r, "\eta_i"] \arrow[d, "\sigma"'] & x^* \arrow[dd, "X^*_\sigma"] \\ s \arrow[dr, "\eta_s"'] & \\ & X^*_s \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Xx^* \arrow[r, "\delta_i"] \arrow[d, "XX^*_\sigma"'] & x^* \arrow[d, "X^*_\sigma"] \\ XX^*_s \arrow[r, "\delta_s"'] & X^*_s \end{tikzcd} \quad \text{commutes}$$

∴ By uniqueness in the following commutative diagram

$$\begin{tikzcd}
i \arrow[rr, "\eta_i"] \arrow[r, "\sigma"] \arrow[drr, "\sigma"'] & & \mathcal{X}^* \arrow[d, "X^*_\sigma"] \arrow[dd, bend left, "\delta^+"] \\
& s \arrow[r, "\eta_s"] \arrow[dr, "Id_s"] & X^*_s \arrow[d, "\delta^*"] \\
& & s
\end{tikzcd}$$

we have $\delta^+ = \delta^* . X^*_\sigma$

x x x

2. CATEGORY OF MACHINES

DN 7. <u>Machine - Morphism</u>

Let $M = (X, s, y, i, \delta, \sigma, \lambda)$ and $M' = (X', s', y', i', \delta', \sigma', \lambda')$ be two machines in a category , $\mathcal{X}^*$ and $\mathcal{X}'^*$ their object of inputs, δ^+ and δ'^+ their reachability morphism respectively. A <u>machine morphism</u> $M \longrightarrow M'$ is a triple (a, b, c) where $a : \mathcal{X}^* \longrightarrow \mathcal{X}'^*$, $b: s \longrightarrow s'$, $c: y \longrightarrow y'$ are morphisms such that the two following diagrams commute:

$$\begin{tikzcd}
\mathcal{X}^* \arrow[r, "\delta^+"] \arrow[d, "a"'] & s \arrow[d, "b"] \\
\mathcal{X}'^* \arrow[r, "\delta'^+"'] & s'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
s \arrow[r, "\lambda"] \arrow[d, "b"'] & y \arrow[d, "c"] \\
s' \arrow[r, "\lambda'"'] & y'
\end{tikzcd}$$

We call a, b, c respectively the input, state and output components of (a, b, c) or, simply the 1st, 2nd and 3rd components.

DN 8. Let $(a, b, c) : M \longrightarrow M'$, $(a', b', c') : M' \longrightarrow M''$ be a machine morphisms, we define their composition by $(a, b, c) \cdot (a', b', c') = (a.a', b.b', c.c')$.

$\mathcal{C}$ is a category $\Longrightarrow$ $\exists$ components of $(a.a', b.b', c.c')$, they are associative and $\exists$ identities $Id_{\mathcal{X}^*}$, Id_{S}, $Id_{\mathcal{Y}}$;

$\therefore$ composition of machine-morphisms is defined, associative and $\exists$ identy $Id_M = (Id_{\mathcal{X}^*}, Id_S, Id_{\mathcal{Y}})$ with the usual properties.

Besides $\because$ DN 7 the various parts of the following diagrams commute:

$$\begin{tikzcd}
\mathcal{X}^* \arrow[r, "\delta^+"] \arrow[d, "a"'] & S \arrow[d, "b"] \\
\mathcal{X}'^* \arrow[r, "\delta'^+"] \arrow[d, "a'"'] & S' \arrow[d, "b'"] \\
\mathcal{X}''^* \arrow[r, "\delta''^+"] & S''
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\lambda"] \arrow[d, "b"'] & \mathcal{Y} \arrow[d, "c"] \\
S' \arrow[r, "\lambda'"] \arrow[d, "b'"'] & \mathcal{Y}' \arrow[d, "c"] \\
S'' \arrow[r, "\lambda''"] & \mathcal{Y}''
\end{tikzcd}$$

$$\Longrightarrow \delta''^{+}_{\bullet}(a' \cdot a) = b' \cdot \delta'^{+} \cdot a = (b' \cdot b) \cdot \delta^{+}$$

and $\lambda''_{\bullet}(b' \cdot b) = c' \cdot \lambda' \cdot b = (c' \cdot c) \cdot \lambda$

i.e. the whole diagrams commute and $(a, b, c) \cdot (a', b', c')$ is a machine morphism.

Finally

$$\begin{array}{ccc} X^* & \xrightarrow{\delta^+} & S \\ {\scriptstyle Id_{X^*}}\downarrow & & \downarrow{\scriptstyle Id_S} \\ X^* & \xrightarrow[\delta^+]{} & S \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S & \xrightarrow{\lambda} & Y \\ {\scriptstyle Id_S}\downarrow & & \downarrow{\scriptstyle Id_Y} \\ S & \xrightarrow[\lambda]{} & Y \end{array}$$

obviously commute $\Longrightarrow$ Id_M is a machine-morphism.

We are now able to define the category of machines in a category

DN 9. MAC $\mathcal{C}$ = category whose objects are machines in the category $\mathcal{C}$, morphisms are machine-morphisms.

x x x

3. CATEGORY OF BEHAVIOURS

As seen in DN 7, given $i \in |\mathcal{C}|$, $X : \mathcal{C} \longrightarrow \mathcal{C}$ an input process, $\exists$ an object of input $X^* = X^* i$

Let $Y \in |\mathcal{C}|$. A behavior is a morphism $\beta : X^* \longrightarrow Y$

DN 10. Category of behavior in category $\mathcal{C}$ = Beh $\mathcal{C}$.

Objects: behaviors $\beta : X^* \longrightarrow y$

Morphism: behavior-morphisms $(a, c) : \beta \begin{matrix} X^* \\ \downarrow \\ y \end{matrix} \longrightarrow \beta' \begin{matrix} X'^* \\ \downarrow \\ y' \end{matrix}$

where $a : X^* \longrightarrow X'^*$, $c : y \longrightarrow y'$

are $\mathcal{C}$ morphism,

$$\begin{tikzcd} X^* \arrow[r, "\beta"] \arrow[d, "a"'] & y \arrow[d, "c"] \\ X'^* \arrow[r, "\beta'"'] & y' \end{tikzcd}$$ commutes

i.e. $c.\beta = \beta'.a$

Composition: $(a, b) . (a', b') = (a.a', b.b') : \beta \longrightarrow \beta''$

with $(a', b') : \beta' \longrightarrow \beta''$.

Identity morphism: $Id_\beta = (Id_{X^*}, Id_y)$

Like MAC $\mathcal{C}$, Beh $\mathcal{C}$ is indeed a category (similar proof).

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4. CATEGORIES OF A BEHAVIORS AND A MACHINES.

DN 11. Let $\delta : X c \longrightarrow c$ be an X-dynamics.

$\delta^{(n)} : X^n c \longrightarrow c$ is defined inductively as follows:

$$\begin{cases} \delta^{(0)} = Id_c \\ \delta^{(n+1)} = X^{n+1}c \xrightarrow{X^n\delta} X^n c \xrightarrow{\delta^{(n)}} c \end{cases}$$

DN 12. An A behavior $\beta : \mathcal{X}^* \longrightarrow y$ is a behavior satisfying the following four postulates:

1. $\exists \; E_\beta \in |\mathcal{C}|$ and a pair of $\mathcal{C}$-morphisms $E_\beta \underset{\gamma}{\overset{\alpha}{\rightrightarrows}} \mathcal{X}^* \ni$:

   a) $\beta . \delta_j^{(n)} . (X^n\alpha) = \beta . \delta_j^{(n)}(X\gamma) \quad \forall n \in \mathbb{N}$;

   b) given $\Gamma \in |\mathcal{C}|$ and $\Gamma \underset{p'}{\overset{p}{\rightrightarrows}} \mathcal{X}^*$ a pair of $\mathcal{C}$-morphisms $\ni \beta . \delta_i^{(n)} . (X^n p) = \beta . \delta_i^{(n)}(X^n p') \quad \forall n \in \mathbb{N}$

   $\forall \varphi \ni \varphi . \alpha = \varphi . \gamma \Rightarrow \varphi . p = \varphi . p'$ (†)

The following diagrams can be helpful:

$$\begin{tikzcd} X^n E_\beta \arrow[r, shift left, "X^n\alpha"] \arrow[r, shift right, "X^n\gamma"'] & X^n \mathcal{X}^* \arrow[r, "\delta_j^{(n)}"] & \mathcal{X}^* \arrow[r, "\beta"] & y \\ & X^n \Gamma \arrow[u, shift left, "X^n p"] \arrow[u, shift right, "X^n p'"'] & & \end{tikzcd} \quad \forall n \in \mathbb{N}$$

and

$$\begin{tikzcd} E_\beta \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathcal{X}^* \arrow[r, "\forall\varphi"] & C \\ & \Gamma \arrow[u, shift left, "p"] \arrow[u, shift right, "p'"'] & \end{tikzcd}$$

2. $\exists \; s_\beta \in |\mathcal{C}|$ and a $\mathcal{C}$-morphism $\delta_\beta^+ : \mathcal{X}^* \longrightarrow s_\beta$ $\ni \delta^+ =$ coeq (α, γ)

3. $\exists$ an X-dynamics $\delta_\beta : X s_\beta \longrightarrow s_\beta \ni \delta^+$ is an X-dynamorphism i.e.

$$\begin{tikzcd} X\mathcal{X}^* \arrow[r] \arrow[d, "X\delta_\beta^+"'] & \mathcal{X}^* \arrow[d, "\delta_\beta^+"] \\ X s_\beta \arrow[r] & s_\beta \end{tikzcd}$$ commutes

4. $X \delta_\beta^+$ is epi.

(†) In [3] Arbib gives a slightly different postulate (1b):

$\exists\ \psi\ :\ n \rightarrow E_\beta \ni\ p = \alpha \cdot \psi$ and
$p' = \gamma \cdot \psi$ (see page 685).

Arbib (1b) $\Longrightarrow$ ours as

$\varphi \cdot \alpha = \varphi \cdot \gamma$

$\Longrightarrow \varphi \cdot \alpha \cdot \psi = \varphi \cdot \gamma \cdot \psi$

$\Longrightarrow \varphi \cdot p = \varphi \cdot p'$.

But I do not know if the converse is true. If such is the case our 1(b) is somewhat less restrictive. Only our condition is needed for the proof of TH 1 given in [3] page 692.

<u>NOTE 4.</u> $E_\beta \overset{\alpha}{\underset{\gamma}{\rightrightarrows}} x^*$ is called "the" <u>Nerode equivalence</u> for β. The word "the" is justified by the fact that if $E \overset{\alpha}{\underset{\gamma}{\rightrightarrows}} x^*$ and $E' \overset{\alpha'}{\underset{\gamma'}{\rightrightarrows}} x^*$ are both Nerode equivalences for β (i.e. β satisfies the four postulates for each of them) then the corresponding Nerode realizations (to be defined later) $N\beta$ and $N'\beta$ of β are isomorphic. We will discuss this point later on, after TH 1: see Note 7.

<u>DN 13.</u> A machine M <u>realizes</u> a behavior $\beta : x^* \longrightarrow y$ if $\lambda \cdot \delta^+ = \beta$; $\lambda \cdot \delta^+$ is called the <u>external behavior</u> (or <u>response</u> morphism) of M.

DN 14. An A machine M is a machine in a category $\mathcal{C}$ ∋ its external behavior $\lambda \cdot \delta^{+}$ is an A-behavior.

From now on we will consider only the two following sub-categories of MAC $\mathcal{C}$ and Beh $\mathcal{C}$:

DN 15. $\mathcal{M}$ = category of reachable A machines and machine-morphisms between them.

$\mathcal{B}$ = category of A behaviors and behavior-morphisms between them.

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5. FUNCTOR E : $\mathcal{M} \longrightarrow \mathcal{B}$

DN 16. Functor E : $\mathcal{M} \longrightarrow \mathcal{B}$

$\forall M \in |\mathcal{M}|$ and (a, b, c) : $M \longrightarrow M' \in \mathcal{M}$

$EM \doteq \lambda \cdot \delta^{+}$ the external behavior of M,

$E\,(a, b, c) = (a, c) : EM \longrightarrow EM'$.

We show E is indeed a functor.

$\lambda . \delta^{+} : x^{*} \longrightarrow y$ is an A behavior as $M \in \mathcal{M}$

E (a, b, c) is a behavior-morphism as by DN 7 the two parts of the following diagram commutes:

$$\begin{tikzcd} x^{*} \arrow[r, "\delta^{+}"] \arrow[d, "a"'] & s \arrow[r, "\lambda"] \arrow[d, "b"] & y \arrow[d, "c"] \\ x'^{*} \arrow[r, "\delta'^{+}"'] & s' \arrow[r, "\lambda'"'] & y' \end{tikzcd} \qquad (*)$$

$\Longrightarrow (\lambda' . \delta'^{+}) . a = \lambda' . b . \delta^{+} = c . (\lambda . \delta^{+})$ as required .

Let $(a, b, c) : M \longrightarrow M'$, $(a', b', c') : M' \longrightarrow M''$

$E\big((a, b, c) \cdot (a', b', c')\big) = E \quad (a.a', b.b', c.c')$

$\because$ DN of $\cdot$ in $\mathcal{M}$

$= (a.a', c.c')$

$\because$ DN of E

$= (a, c) \cdot (a', c')$

$\because$ DN of . in $\mathcal{B}$

$= E (a, b, c) \cdot E(a', b', c')$

$\because$ DN of E .

Indeed this is a behavior-morphism as the diagram commutes, because its two parts commute by (*)

$$\begin{tikzcd} x^{*} \arrow[r, "EM"] \arrow[d, "a"'] & y \arrow[d, "c"] \\ x'^{*} \arrow[r, "EM'"'] \arrow[d, "a'"'] & y' \arrow[d, "c'"] \\ x''^{*} \arrow[r, "EM''"'] & y'' \end{tikzcd}$$

Finally $E\ (Id_{x^*}, Id_S\ , Id_y) = (Id_{x^*}, Id\ y)$

i.e. $E\ (Id_M) = Id_{\lambda \cdot \delta^+} = Id_{EM}$

x x x

6. FUNCTOR $N : \mathcal{B}_X \longrightarrow \mathcal{M}_X$ -

DN 17. $\forall \beta \in \mathcal{B},\ N\beta = (X, s_\beta, y, i, \delta_\beta, \sigma_\beta, \lambda_\beta) \in \mathcal{M}$ is the Nerode realization of β ; where s_β, δ_β, δ_β^+ are defined by DN 1~~9 of~~ an A behavior, $\sigma_\beta = \delta_\beta^+ \cdot \eta_i$ (where (δ_i, η_i) is the universal arrow from i to $\mathcal{U}$)

$\lambda_\beta =$ the unique $\mathcal{C}$ -morphism $\ni$

$$\begin{tikzcd} E_\beta \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & x^* \arrow[r, "\delta_\beta^+"] \arrow[dr, "\beta"'] & s_\beta \arrow[d, "\lambda_\beta"] \\ & & y \end{tikzcd}$$

commutes,

i.e. $\underline{\beta = \lambda_\beta \cdot \delta_\beta^+}$

NOTE 5. $\exists\ \lambda_\beta$ as $\beta \cdot \alpha = \beta \cdot \gamma$ $\because$ DN 12.1(a) with n = 0 and $\delta_\beta^+ = \text{coeq}\ (\alpha, \gamma)$ $\because$ DN 12.2

$N\beta \in |\mathcal{M}|$ as:

- the two following diagrams commute

$$\begin{tikzcd} i \arrow[r, "\eta_i"] \arrow[dr, "\sigma_\beta"'] & x^* \arrow[d, "\delta_\beta^+"] \\ & s_\beta \end{tikzcd}$$

$\because$ DN of σ_β

and

$$\begin{tikzcd} X x^* \arrow[r, "\delta_i"] \arrow[d, "X\delta_\beta^+"'] & x^* \arrow[d, "\delta_\beta^+"] \\ X s_\beta \arrow[r, "\delta_\beta"'] & s_\beta \end{tikzcd}$$ $\because$ DN 12.3

- the external behavior of $N\beta$ is $\lambda_\beta \cdot \delta_\beta^+ = \beta$ $\because$ DN of λ_β, where β is an A behavior,
- δ_β^+ is a coequalizer $\because$ DN 12.2

i.e. <u>$N\beta$ is always reachable</u>.

<u>DN 18.</u> We say that M is a <u>reachable realization of β</u> if its reachability morphism δ^+ is a coequalizer and $\lambda . \delta^+ = \beta$.

<u>DN 19.</u>(i) M is a <u>minimal realization</u> of a behabior $\beta : x^* \to y$ if M is a reachable realization of β and $\forall M'$, M' a reachable realization of β, $\exists$ a unique machine morphism $(Id_{x^*}, b, Id_y) : M' \to M$.

(ii) This is equivalent to say that M is a minimal realization of β if it is a <u>terminal object</u> in the subcategory $\mathcal{M}_\beta$ of reachable realizations of β.

$\Longrightarrow M$ is unique up to isomorphism.

(iii) $b : s' \longrightarrow s$ is called a <u>simulation</u> $M' \longrightarrow M$ if it is the 2nd component of a machine morphism of the form (Id_{x^*}, b, Id_y) i.e. $\ni \delta^+ = b \cdot \delta'^+$,

$\lambda \cdot b = \lambda'$ $\because$ DN 7

and $\check{\sigma} = b \cdot \sigma'$ $\because$ DN 6 as

$$\sigma = \delta^+ \cdot \eta_i = b \cdot \delta'^+ \cdot \eta_i = b : \sigma'.$$

NOTE 6. $b : s' \longrightarrow s$ is an X-dynamorphism if $X\delta'^+$ is epi.

Indeed consider the diagram:

$$\begin{tikzcd}
X x^* \arrow[r, "\delta_1"] \arrow[d, "X\delta'^+"'] \arrow[dd, bend right=60, "X\delta^+"'] \arrow[dr, phantom, "(1)"] & x^* \arrow[d, "\delta'^+"] \arrow[dd, bend left=60, "\delta^+"] \\
X s' \arrow[r, "\delta'"] \arrow[d, "Xb"'] \arrow[dr, phantom, "(2)"] & s' \arrow[d, "b"] \\
X s \arrow[r, "\delta"'] & s
\end{tikzcd}$$

Part (1) and the whole diagram commute (see note 3)

$\Rightarrow b \cdot \delta'^+ \cdot \delta_1 = \delta \cdot Xb \cdot X\delta'^+$

$\therefore b \cdot \delta' \cdot X\delta'^+ = \delta \cdot Xb \cdot X\delta'^+$

$\Rightarrow b \cdot \delta' = \delta \cdot Xb$

if $X\delta'^+$ is epi, which is the case if β' is an A-behavior.

TH 1. NERODE REALIZATION TH.

$\beta \in \mathcal{B} \implies N\beta$ is the minimal realization of β.

Proof: see [3] page 692.

NOTE 7. Given a behavior $\beta : X^* \longrightarrow y$, assume it satisfies the four postulates for

$E \underset{\gamma}{\overset{\alpha}{\rightrightarrows}} X^* \xrightarrow{\delta^+} s$, $\delta^+ = \text{coeq}\,(\alpha, \gamma)$, $\delta : Xs \longrightarrow s$

and also for $E' \underset{\gamma'}{\overset{\alpha'}{\rightrightarrows}} X^* \xrightarrow{\delta'^+} s'$, $\delta'^+ = \text{coeq}\,(\alpha', \gamma')$

$\delta' : Xs' \longrightarrow s'$

$\because$ (1b), $\delta^+.\alpha = \delta^+.\gamma \implies \delta^+.\alpha' = \delta^+.\gamma'$ and

$\delta'^+.\alpha' = \delta'^+.\gamma' \implies \delta'^+.\alpha = \delta'^+.\gamma$

$\therefore$ $\delta^+ = \text{coeq}\,(\alpha, \gamma) \implies \exists$ a unique $b : s \longrightarrow s' \ni \delta'^+ = b.\delta^+$

and $\delta'^+ = \text{coeq}\,(\alpha', \gamma') \implies \exists$ a unique $b' : s' \longrightarrow s \ni \delta^+ = b'.\delta'^+$

$\therefore$ $\delta^+ = b'.b.\delta^+$ and $\delta'^+ = b.b'.\delta'^+$

$\implies b'.b = Id_s$ and $b.b' = Id_{s'}$

$\implies s \cong s'$

Besides $\forall f \ni f.\alpha = f.\gamma$ $\exists$ unique $\psi \ni f = \psi.\delta^+$

$(\because \delta^+ = \text{coeq}\,(\alpha, \gamma)) \implies f = \psi.b.\delta'^+$

ψ and b unique $\Longrightarrow$ $\psi \cdot b$ is unique

$\therefore$ δ'^{+} = coeq (α, γ)

Similarly δ^{+} = coeq (α', γ').

Now by TH 1, DN 19, and above, the simulation

$(Id_{\mathcal{X}^*}, b, Id_{\mathcal{Y}}) : N\beta = (\mathcal{X}, s, y, i, \delta, \sigma, \lambda)$

$\longrightarrow N\beta' = (\mathcal{X}, s', y, i, \delta', \sigma', \lambda')$

has an inverse $(Id_{\mathcal{X}^*}, b', Id\ y)$, $\therefore$ it is an isomorphism and $N\beta \cong N\beta'$.

$\therefore$ <u>the Nerode realization of β is unique up to isomorphism</u>.

Given i, y $\in /\mathcal{C}/$, X an input-process, and hence $\mathcal{X}^*$, we are basically interested in studying minimal realizations for behaviors $\beta : \mathcal{X}^* \longrightarrow y$. Therefore we can now restrict ourselves to special A machines and A behaviors.

<u>DN 20.</u> $\mathcal{M}_X$ and $\mathcal{B}_X$ are the subcategories of $\mathcal{M}$ and $\mathcal{B}$ whose objects are machines and behaviors respectively sith same initial object i and same input process X. Hence they have the same $\mathcal{X}^*$ and their morphisms are of the form $(Id_{\mathcal{X}^*}, b, c)$ and $(Id_{\mathcal{X}^*}, c)$ respectively.

Let $(Id_{X^*}, c) : \beta \longrightarrow \beta' \in \mathcal{B}_X$

$\Longrightarrow \beta' = c.\beta$ as

$$\begin{tikzcd} X^* \arrow[r, "\beta"] \arrow[d, "Id_{X^*}"'] & y \arrow[d, "c"] \\ X^* \arrow[r] & y' \end{tikzcd}$$

commutes $\because$ DN 8

Let the Nerode realization of β' and β be respectively:

$N\beta' = (X, s'_{\beta'}, y', i, \delta'_{\beta'}, \sigma'_{\beta'}, \lambda'_{\beta'})$ with $\delta'^{+}_{\beta'}$ its reachability morphism

$N\beta = (X, s_\beta, y, i, \delta_\beta, \sigma_\beta, \lambda_\beta)$ with δ^+_β its reachability morphism

$\therefore$ $M^\bullet = (X, s_\beta, y', i, \delta_\beta, \sigma_\beta, c.\lambda_\beta)$ is also a coequalizer reachable realization of β' as δ^+_β is a coequalizer $\because$ DN 12.2 and

$$(c.\lambda_\beta).\delta^+_\beta = c.(\lambda_\beta.\delta^+_\beta) = c.\beta = \beta'$$

$\therefore$ by TH 1 and DN 19, $\exists$ a unique simulation $M^\bullet \longrightarrow N\beta'$

$b' : s_\beta \longrightarrow s'_{\beta'} \ni \delta'^{+}_{\beta'} = b'.\delta^+_\beta,\ \sigma'_{\beta'} = b'.\sigma_\beta,\ c.\lambda_\beta = \lambda'_{\beta'}.b'$

$\Longrightarrow$

$$\begin{tikzcd} X^* \arrow[r, "\delta^+_\beta"] \arrow[d, "Id_{X^*}"'] & s_\beta \arrow[d, "b'"] \\ X^* \arrow[r, "\delta'^{+}_{\beta'}"'] & s'_{\beta'} \end{tikzcd}$$

and

$$\begin{tikzcd} s_\beta \arrow[r, "\lambda_\beta"] \arrow[d, "b'"'] & y \arrow[d, "c"] \\ s'_{\beta'} \arrow[r, "\lambda'_{\beta'}"'] & y' \end{tikzcd}$$

commutes.

$\therefore\ (\mathrm{Id}_{\mathcal{X}^*}, b', c) : N\beta \longrightarrow N\beta'$ is a machine-morphism in $\mathcal{M}_X$. We can now define the functor

N : $\mathcal{B}_X \longrightarrow \mathcal{M}_X$, called the Nerode realization functor.

DN 21. $N : \mathcal{B}_X \longrightarrow \mathcal{M}_X$

object $\beta \longmapsto N\beta$

morphism $(\mathrm{Id}_{\mathcal{X}^*}, c) \;\; \begin{matrix}\beta\\ \downarrow\\ \beta'\end{matrix} \longmapsto (\mathrm{Id}_{\mathcal{X}^*}, b', c) \;\; \begin{matrix}N\beta\\ \downarrow\\ N\beta'\end{matrix}$

where b' is defined as above.

LEMMA 1. $EN = \mathrm{Id}_{\mathcal{B}_X} : \mathcal{B}_X \longrightarrow \mathcal{B}_X$.

Proof: By DN 16 and 17 we have

$$\forall \beta \in /\mathcal{B}_X/,\ EN\beta = E(N\beta) = \lambda_\beta \cdot \delta_\beta^* = \beta .$$

By DN 16 and 21, $\forall\, (\mathrm{Id}_{\mathcal{X}^*}, c) : \beta \longrightarrow \beta'$ in $\mathcal{B}_X$

$EN(\mathrm{Id}_{\mathcal{X}^*}, c) = E(\mathrm{Id}_{\mathcal{X}^*}, b', c) = (\mathrm{Id}_{\mathcal{X}^*}, c)$.

x x x

7. NATURAL TRANSFORMATION $\eta : Id_{\mathcal{M}_X} \Longrightarrow NE : \mathcal{M}_X \longrightarrow \mathcal{M}_X$.

DN 22.1. $\forall M = (X, S, Y, i, \delta, \sigma, \lambda)$, $EM = \lambda \cdot \delta^+$ $\because$ DN 15

$NEM = (X, S_{EM}, Y, i, \delta_{EM}, \sigma_{EM}, \lambda_{EM})$

as defined in DN 17, is called its reduced machine .

2. Since both M and NEM are reachable realizations of EM $\therefore$ by TH 1 and DN 19 $\exists$ a unique simulation

$$b_{EM} : S \longrightarrow S_{EM} \quad \ni$$

$$\delta^+_{EM} = b_{EM} \cdot \delta^+, \qquad \sigma_{EM} = b_{EM} \cdot \sigma \text{ and } \lambda = \lambda_{EM} \cdot b_{EM}$$

$\therefore \eta_M = (Id_{X^*}, b_{EM}, Id_Y) : M \longrightarrow NEM$ is a

machine mosphism as both diagrams

$$\begin{array}{ccc} X^* & \xrightarrow{\delta^+} & S \\ {\scriptstyle Id_{X^*}}\downarrow & & \downarrow{\scriptstyle b_{EM}} \\ X^* & \xrightarrow[\delta^+_{EM}]{} & S_{EM} \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{\lambda} & Y \\ {\scriptstyle b_{EM}}\downarrow & & \downarrow{\scriptstyle Id_Y} \\ S_{EM} & \xrightarrow[\lambda_{EM}]{} & Y \end{array}$$

commute.

LEMMA 2. $\eta = \{ \eta_M : M \longrightarrow NEM \mid M \in \mathcal{M}_X \}$

is a natural transformation $Id_{\mathcal{M}_X} \Rightarrow NE : \mathcal{M}_X \longrightarrow \mathcal{M}_X$.

<u>Proof</u>: Let $(Id_{\mathcal{X}^*}, b, c) : \mathcal{M} \longrightarrow \mathcal{M}'$ be a machine morphism in $\mathcal{M}_X$

$\Longrightarrow \quad \delta'^{+} = b \cdot \delta^{+}$ $\quad \because$ DN 7

$N\,E\,(Id_{\mathcal{X}^*}, b, c) = N\,(Id_{\mathcal{X}^*}, c)$ $\quad \because$ DN 16

$\qquad\qquad = (Id_{\mathcal{X}^*}, b', c)$ $\quad \because$ DN 21

$\eta_M = (Id_{\mathcal{X}^*}, b_{EM}, Id_{\mathcal{Y}})$, $\quad \eta_{M'} = (Id_{\mathcal{X}^*}, b_{EM'}, Id_{\mathcal{Y}})$

$\Longrightarrow \quad \delta^{+}_{EM} = b_{EM} \cdot \delta^{+}, \; \delta'^{+}_{EM'} = b_{EM'} \cdot \delta'^{+}$ $\quad \because$ DN 22

and $\delta'^{+}_{EM'} = b' \cdot \delta^{+}_{EM}$ $\quad \because$ DN 21

$$\therefore \; b_{EM'} \cdot b \cdot \delta^{+} = b_{EM'} \cdot \delta'^{+} = \delta'^{+}_{EM'} = b' \cdot \delta^{+}_{EM} = b' \cdot b_{EM} \cdot \delta^{+}$$

$\Longrightarrow \quad b_{EM'} \cdot b = b' \cdot b_{EM}$ as δ^{+}, a coequalizer, is epi; i.e. the diagram

$$\begin{tikzcd} S \arrow[r, "b_{EM}"] \arrow[d, "b"'] & S_{EM} \arrow[d, "b'"] \\ S' \arrow[r, "b_{EM'}"'] & S'_{EM'} \end{tikzcd}$$

commutes

$$\Longrightarrow \quad \begin{array}{ccc} M & \xrightarrow{\eta_M = (Id_{\mathcal{X}^*},\, b_{EM},\, Id_y)} & N\,E\,M \\ {\scriptstyle (Id_{\mathcal{X}^*},\, b,c)}\downarrow & & \downarrow {\scriptstyle NE(Id_{\mathcal{X}^*},\, b,c) = (Id_{\mathcal{X}^*},\, b', c)} \\ M' & \xrightarrow[\eta_{M'} = (Id_{\mathcal{X}^*},\, b_{EM'},\, Id_y)]{} & N\,E\,M' \end{array}$$

commutes as required; indeed

$$\begin{aligned}(Id_{\mathcal{X}^*}, b_{EM'}, Id_y)\cdot(Id_{\mathcal{X}^*}, b, c) &= (Id_{\mathcal{X}^*}, b_{EM'}\cdot b, c)\\ &= (Id_{\mathcal{X}^*}, b'\cdot b_{EM}, c)\\ &= (Id_{\mathcal{X}^*}, b', c)\cdot(Id_{\mathcal{X}^*}, b_{EM}, Id\, y)\end{aligned}$$

x x x

8. <u>TH. 2.</u> $E \dashv N : \mathcal{M}_X \rightleftarrows \mathcal{B}_X$

<u>LEMMA 3.</u> $E\,\eta_M = Id_{EM} : EM \longrightarrow EM$ in $\mathcal{B}$, $\forall M \in \mathcal{M}$.

<u>Proof:</u> $\eta_M = (Id_{\mathcal{X}^*}, b_{EM}, Id_y) : M \longrightarrow N\,E\,M$ $\because$ DN 22

$\therefore$ $E\,\eta_M = (Id_{\mathcal{X}^*}, Id_y) : EM \longrightarrow ENEM = EM$

by DN 16 and lemma 1,

i.e. $E\,\eta_M = Id_{EM}$.

<u>Lemma 3 means that reduction preserves behavior $\forall$ reachable A machines.</u>

LEMMA 4. $\eta_{N\beta} = Id_{N\beta}$ in $\mathcal{M}_X$.

Proof: $\forall \beta : x^* \longrightarrow y$ in $|\mathcal{B}_X|$

$N\beta = (X, s_\beta, y, i, \delta_\beta, \sigma_\beta, \lambda_\beta) \in |\mathcal{M}_X|$ $\because$ DN 17

$\eta_{N\beta} = (Id_{x^*}, b_{EN\beta}, Id_y) : N\beta \longrightarrow NEN\beta$ $\because$ DN 22

$EN\beta = \beta$ $\because$ lemma 1

$\because$ by DN 21 the two diagrams 1 and 2

$$\begin{tikzcd} x^* \arrow[r, "\delta_\beta^+"] \arrow[d, "Id_{x^*}"'] \arrow[dr, phantom, "1"] & s_\beta \arrow[r, "\lambda_\beta"] \arrow[d, "b_{EN\beta} = b_\beta"] \arrow[dr, phantom, "2"] & y \arrow[d, "Id_y"] \\ x^* \arrow[r, "\delta^+_{EN\beta} = \delta^+_\beta"'] & s_\beta \arrow[r, "\lambda_{EN\beta} = \lambda_\beta"'] & y \end{tikzcd}$$ commute

But
$$\begin{tikzcd} x^* \arrow[r, "\delta_\beta^+"] \arrow[d, "Id_{x^*}"'] & s_\beta \arrow[r, "\lambda_\beta"] \arrow[d, "Id_{s_\beta}"] & y \arrow[d, "Id_y"] \\ x^* \arrow[r, "\delta^+_\beta"'] & s_\beta \arrow[r, "\lambda_\beta"'] & y \end{tikzcd}$$ also commute

$\therefore$ By uniqueness property ($\because$ TH 1 and DN 19) $b_{EN\beta} = Id_{s_\beta}$

$\Longrightarrow \eta_{N\beta} = (Id_{x^*}, Id_{s_\beta}, Id_y) = Id_{N\beta}$.

Proof of TH 2

Let $\varepsilon : E\,N = Id_{\mathcal{B}_X} \Longrightarrow Id_{\mathcal{B}_X}$ be the identity i.e.

$$\varepsilon = \{ \varepsilon_\beta = Id_\beta : EN\beta = \beta \longrightarrow \beta \mid \forall \beta \in \mathcal{B}_X \}.$$

We show that the natural transformations

$\eta : Id_{\mathcal{M}_X} \Longrightarrow N\,E$ and $\varepsilon : E\,N \Longrightarrow Id_{\mathcal{B}_X}$ are respectively the unit and the co-unit of the adjunction $E \dashv N : \mathcal{B}_X \longrightarrow \mathcal{M}_X$.

$\forall M \in \mathcal{M}_X$,	$E\,N\,E\,M = E\,M$	$\because$	Lemma 1
	$(E\eta)_M = E\eta_M$	$\because$	DN a7.5
	$= Id_{EM}$	$\because$	lemma 3
	$(\varepsilon E)_M = \varepsilon_{EM}$	$\because$	DN a7.5
	$= Id_{EM}$	$\because$	DN of ε.

$\therefore$

$$\begin{tikzcd} & ENEM = EM \arrow[dr, "(\varepsilon E)_M"] & \\ EM \arrow[ur, "(E\eta)_M"] \arrow[rr, "Id_{EM}"'] & & EM \end{tikzcd}$$ commutes $\forall M$

$\Longrightarrow$ (a) :

$$\begin{tikzcd} & ENE \arrow[dr, Rightarrow] & \\ E.Id_{\mathcal{M}_X} = E \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & Id_{\mathcal{B}_X}.E = E \end{tikzcd}$$ commutes .

$$
\forall \beta \in \mathcal{B}_x, \quad
\begin{array}{rcll}
NEN\beta & = & N\beta & \because \text{ lemma 1} \\
(\eta N)_\beta & = & \eta_{N\beta} & \because \text{ DN a 7.5} \\
 & = & Id_{N\beta} & \because \text{ lemma 4} \\
(N\varepsilon)_\beta & = & N\varepsilon_\beta & \because \text{ DN a 7.5} \\
 & = & N\, Id_\beta & \because \text{ DN of } \varepsilon \\
 & = & Id_{N\beta} & \because \ N \text{ a functor .}
\end{array}
$$

$$
\therefore \quad
\begin{tikzcd}
 & NEN\beta = N\beta \arrow[dr, "(N\varepsilon)_\beta"] & \\
N\beta \arrow[ur, "(\eta N)_\beta"] \arrow[rr, "Id_{N\beta}"'] & & N\beta
\end{tikzcd}
\quad \text{commutes ,}
$$

$$
\Longrightarrow \ (b): \quad
\begin{tikzcd}
 & NEN \arrow[dr, Rightarrow, "N\varepsilon"] & \\
Id_{\mathcal{M}_x} \cdot N = N \arrow[ur, Rightarrow, "\eta N"] \arrow[rr, Rightarrow, "Id_N"'] & & N = N \cdot Id_{\mathcal{B}_x}
\end{tikzcd}
\quad \text{commutes}
$$

(a) and (b) $\Longrightarrow$ $E \dashv N$ and η and ε are the unit and the co-unit of the adjunction

$\because$ TH a 5.4 .

x x x

9. **TH 3.** $\forall$ functor $F : \mathcal{B}_X \longrightarrow \mathcal{M}_X$, F is a <u>minimal realization functor</u> (i.e. $F\beta$ is a minimal realization of β , $\forall \beta \in \mathcal{B}_X$)

$$\Longleftrightarrow \quad E \dashv F : \mathcal{M}_X \rightleftarrows \mathcal{B}_X .$$

<u>Proof:</u> By DN 19, $\forall \beta \in \mathcal{B}_X$, $N\beta$ and $F\beta$ are minimal realization of $\beta \Longleftrightarrow \exists$ a unique $\varphi : N\beta \longrightarrow F\beta \ni \varphi$ is a machine isomorphism (i.e. $\exists$ unique φ^{-1});

$E \dashv N \quad \Longleftrightarrow \mathcal{M}_X(M, N\beta) \cong \mathcal{B}_X(EM, \beta) \quad \because$ TH a 5.1

$\Longleftrightarrow \mathcal{M}_X(M, F\beta) \cong \mathcal{B}_X(EM, \beta)$ as

$\forall f : M \longrightarrow F\beta$ corresponds a unique $\varphi^{-1} \cdot f : M \longrightarrow N\beta$ i.e. a unique $E(\varphi^{-1} \cdot f) : EM \longrightarrow \beta$;

and $\forall g : EM \longrightarrow \beta$ corresponds a unique $h = Ng \cdot \eta_M : M \longrightarrow N\beta$ i.e. a unique $\varphi \cdot h : M \longrightarrow F\beta$ (η_M given by DN 21)

$\Longleftrightarrow E \dashv F \quad \because$ TH a 5.1

<u>TH 3 means that a minimal realization functor F is exactly a right adjoint to the bahavior functor E i.e. it is naturally isomorphic to N.</u>

NOTE 8. $\forall \beta \in \mathcal{B}_X$, $N\beta \cong F\beta \quad \Longleftrightarrow$ F is naturally isomorphic to N $\because$ DN a 7.2

The proof of TH 3, suitably modified, yields the following more general result of category theory:

Proposition 1: G is a right (left) adjoint of F,

H is naturally isomorphic to G

$\Longrightarrow$ H is a right (left) adjoint of F.

TH 3 can then be considered as a corollary of proposition 1 as:

$F\beta$ a minimal realization $\Longrightarrow F\beta \cong N\beta \quad \forall \beta \in \mathcal{B}_X$

$\Longrightarrow$ F naturally isomorphic to N

$\Longrightarrow E \dashv F$ as $E \dashv N$.

x x x

CHAPTER II

EXAMPLES OF A-MACHINES

We will now study three examples of A-Machines: discrete, linear and tree automata.

1 DISCRETE MACHINES.

DN 23. A discrete machine (or automaton) is a sextuple

$$= (X, S, Y, \delta, \sigma, \lambda)$$

when X = set of inputs,

S = set of states,

Y = set of outputs,

$\delta = X \times S \longrightarrow S$ is the transition function

$\lambda = S \longrightarrow Y$ is the output function,

$\sigma \in S$ is the initial state

Example: We want a decimal computer for adding integers consisting of at most n digits.

We use n counters modulo 10 which are n identical machines C_{10}^{1}, C_{10}^{2}, ... such that:

$$X = \{0, 1\} = Y, \quad S = \{0, 1\} \times \mathbb{Z}_{10}$$

$$\delta(0, (y, z)) = (0, z)$$

$$\delta(1, (y, z)) = \begin{pmatrix} (1, s(z)) \text{ if } z = 9 \\ (0, s(z)) \text{ otherwise} \end{pmatrix}$$

where $s(z) = z + 1$ (modulo 10)

$$\lambda(y, z) = y.$$

For instance let's add 3 to 397; we send in our computer

$W = 111 \in \{0, 1\}^*$; we have:

c_{10}^4		c_{10}^3		c_{10}^2		c_{10}^1		
(0,0)		(0,3)		(0,9)		(0,7)		initial state
(0,0)	$\xleftarrow{0}$	(0,3)	$\xleftarrow{0}$	(0,9)	$\xleftarrow{0}$	(0,8)	$\xleftarrow{1}$	state after 1st input
(0,0)	$\xleftarrow{0}$	(0,3)	$\xleftarrow{0}$	(0,9)	$\xleftarrow{0}$	(0,9)	$\xleftarrow{1}$	state after 2nd input
(0,0)	$\xleftarrow{0}$	(0,4)	$\xleftarrow{1}$	(1,0)	$\xleftarrow{1}$	(1,0)	$\xleftarrow{1}$	final state.

The result at each stage is given by reading in the proper order the 2nd components z of each state: 397, 398, 399 and, finally, 400.

<u>Proposition 2.</u> A discrete machine is an A-machine $M = (X\times\_\,, S, Y, I, \delta; \sigma, \lambda)$ in $\mathcal{S}$, the category of sets.

<u>Proof:</u> Let X = set of inputs

$X \times \_ : \mathcal{S} \longrightarrow \mathcal{S}$ is a functor, ∴ a process.

$$S \longmapsto X \times S$$

$$\begin{array}{ccc} S & & X \times S \\ f \downarrow & \longmapsto & \downarrow Id_X \times f \\ S' & & X \times S \end{array}$$

defined by $Id_X \times f : (x, s) \longmapsto (x, f(s))$

DYN X is the category of all dynamics i.e. all maps

$\delta : X \times S \longrightarrow S$ and dynamorphisms

$f : (\delta : X \times S \longrightarrow S) \longrightarrow (\delta' : X \times S' \longrightarrow S')$

where $f : S \longrightarrow S'$ is a map $\ni \delta' \cdot (\mathrm{Id}_X \times f) = f \cdot \delta$

X^* is the free monoid whose words are strings of elements of X, Λ = empty word, and the operation is concatenation.

$X \times \_$ is an input-process as $\forall S \in |\mathcal{S}|$

$$\left(\eta_S : S \longrightarrow X^* \times S, \; \delta_S : X \times (X^* \times S) \longrightarrow X^* \times S \right)$$
$$s \longmapsto (\Lambda, s) \qquad (x, w, s) \longmapsto (xw, s)$$

is universal from S to U, the forgetful functor

DYN $X \longrightarrow \mathcal{S}$.

Indeed $\forall f : S \longrightarrow S'$, the map

$\psi : U\delta_S = X^* \times S \longrightarrow U\delta' = S'$ uniquely defined by

$\psi(w, s) = \delta'^*(w, f(s))$ where given $\delta : X \times S \rightarrow S$

δ^* is inductively defined by:

$$\begin{cases} \delta^*(\Lambda, s) = s \\ \delta^*(xw, s) = \delta(x, \delta^*(w, s)) \end{cases} \quad \forall x \in X, w \in X^*, s \in S.$$

makes the two following diagrams commute:

$$
\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & X^* \times S \arrow[d, "\Psi"] \\
 & S'
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
X \times X^* \times S \arrow[r, "\delta_S"] \arrow[d, "Id_X \times \Psi"'] & X^* \times S \arrow[d, "\Psi"] \\
X \times S' \arrow[r, "\delta'"] & S'
\end{tikzcd}
$$

$$s \longmapsto (\Lambda, s) \longmapsto f(s) = \delta'^*(\Lambda, f(s))$$

$$(x, w, s) \longmapsto (xw, s) \longmapsto \delta'^*(xw, f(s))$$

$$(x, w, s) \longmapsto (x, \delta'^*(w, f(s))) \longmapsto \delta'(x, \delta'^*(w, f(s))) = \delta'^*(xw, f(s))$$

S = set of states

Y = set of outputs

I = $\{\cdot\}$ is the initial state object

σ : I $\longrightarrow$ S is the initial state morphism

$\cdot \longrightarrow \sigma(\cdot) = \sigma$

δ : X $\times$ S $\longrightarrow$ S is the transition function, an X dynamics

λ : S $\longrightarrow$ Y is the output morphism.

X^* is the object of inputs.

δ^+, the reachability morphism, is uniquely defined by $\delta^+ : X^* \longrightarrow S$, $w \longrightarrow \delta^*(w, \sigma)$ i.e. by
$$\begin{cases} \delta^+(\Lambda) = \sigma \\ \delta^+(x, w) = \delta(x, \delta^+(w)) \end{cases}$$

It is the unique dynamorphic extension of σ° as

$$\begin{tikzcd}
\bullet \arrow[rr, mapsto] \arrow[dddr, bend right] & & \Lambda \arrow[dddl, bend left] \\
\{\cdot\} \arrow[r, "\eta_S"] \arrow[dr, "\sigma"'] & X^* \times \{\cdot\} \cong X^* \arrow[d, "\delta^+"] & \\
& S & \\
& \sigma &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(x, w) \arrow[r, mapsto] \arrow[ddd] & xw \arrow[ddd] \\
X \times X^* \arrow[r, "\delta_I"] \arrow[d, "1d_x \times \delta^+"'] & X^* \arrow[d, "\delta^+"] \\
X \times S \arrow[r, "\delta"] & S \\
(x, \delta^+(w)) \arrow[r, mapsto] & \delta(x, \delta^+(w)) = \delta^+(xw)
\end{tikzcd}$$

commute .

$\therefore M$ is a machine in category $\mathcal{D}$ $\because$ DN

with external behavior $EM = \lambda . \delta^+ : X^* \longrightarrow Y$

Let $E_{EM} = \{ (w_1, w_2) \,/\, EM(w w_1) = EM(w w_2) \ \forall w \in X^* \}$

This is the Nerode equivalence relation

Let $\alpha, \gamma : E_{EM} \longrightarrow X^*$ be the usual projections

$$\delta_I^{(n)} : X^n \times X^* \longrightarrow X^* \qquad \forall n \in \mathbb{N}$$

$$(x_n, \dots, x_1, w) \longmapsto x_n \dots x_1 w$$

$$\forall \; (x_n \ldots x_1 , (w_1 , w_2)) \in X^n \times E_{EM}$$

$$\lambda . \delta^+ . \delta_I^{(n)} . (X \times -)^n \alpha \, (x_n \ldots x_1 , (w_1 , w_2)) = \lambda . \delta^+ . \delta_I^{(n)} (x_n \ldots x_1 \, , \, w_1)$$

$$= \lambda . \delta^+ (x_n \ldots x_1 \; w_1)$$

$$= \lambda . \delta^+ (x_n \ldots x_1 \; w_2)$$

$$(\because (w_1 , w_2) \in E_{EM})$$

$$= \lambda . \delta^+ . \delta_I^{(n)} (x_n \ldots x_1 \, , \, w_2)$$

$$= \lambda . \delta^+ . \delta_I^{(n)} . (X \times -)^n \gamma \, (x_n \ldots x_1 , (w_1 , w_2)) .$$

$\therefore$ postulate (1 a) is satisfied

Assume $\exists R$ and $p , p' : R \longrightarrow X^* \ni$

$$\lambda . \delta^+ . \delta_I^{(n)} . (X^n p) = \lambda . \delta^+ . \delta_I^{(n)} (X^n p')$$

$$\Longrightarrow \forall n \in \mathbb{N} , \forall w \in X^* \ni w = x_n \ldots x_1 \, , \, \forall r \in R$$

$$\lambda . \delta^+ (w \, p(r)) = \lambda . \delta^+ (w \, p'(r))$$

$$\Longrightarrow (p(r) , \; p'(r)) \in E_{EM}$$

$$\therefore \exists\ \varphi : \mathcal{R} \longrightarrow E_{EM}$$
$$r \longmapsto (p(r), p'(r))$$

$$\ni\ p = \alpha . \varphi \quad \text{and} \quad p' = \gamma . \varphi$$

$$\therefore \forall \psi \ni \psi . \alpha = \psi . \gamma$$
$$\psi . \alpha . \varphi = \psi . \gamma . \varphi$$
$$\psi . p = \psi . p'$$

i.e. postulates 1b is satisfied.

Let $S_{EM} = X \,/\, E_{EM}$ whose elements are equivalence classes $[w]$ of E_{EM}

$$\delta^+_{EM} : X^* \longrightarrow S_{EM} \quad \text{is onto} \iff \delta^+ = \text{coeq}\,(\alpha, \gamma)$$
$$w \longmapsto [w] \qquad \text{in } \mathcal{S}$$

$\therefore$ postulate 2 is satisfied and so is 3 with

$$\delta_{EM} : X \times S_{EM} \longrightarrow S_{EM} \quad \text{as}$$
$$(x, [w]) \longmapsto [xw]$$

$$\begin{tikzcd}
(x, w) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & xw \arrow[ddd, mapsto] \\
& X \times X^* \arrow[r, "\delta_1"] \arrow[d, "Id_X \times \delta^+_{EM}"'] & X^* \arrow[d, "\delta^+_{EM}"] & \\
& X \times S_{EM} \arrow[r, "\delta_{EM}"'] & S_{EM} & \\
(x, [w]) \arrow[rrr, mapsto] & & & [xw]
\end{tikzcd}$$

commutes.

Postulate 4 is satisfied as δ^+ onto $\Longleftrightarrow$ it is a split-epi in $\mathcal{A}$ and any functor preserves split-epi.

$\therefore$ $EM = \lambda \, . \, \delta^+$ is an A-behavior and M is an A-machine whose Nerode realization is

$$NEM = \left(X \times -, S_{EM}, Y, \{\cdot\}, \delta_{EM}, \underset{\cdot \mapsto [\Lambda]}{\sigma_{EM} : \{\cdot\} \to S_{EM}}, \underset{[w] \mapsto EM(w)}{\lambda_{EM} : S_{EM} \to Y}\right)$$

<u>NOTE</u> For proof of uniqueness of the maps ψ, δ^+, etc. see [3] , chap. 9

Given M , an informal description of an algorithm for state-merging (i.e. to find S_{EM}), which terminates in at most

$n - 2$ steps if $/S_{EM}/ \leq n$, is given in [3] page 152.

x x x

2. LINEAR MACHINES

DN 24. An R-linear machine $\mathcal{M}$ is an automaton $(X, S, Y, \sigma = 0_S, \delta, \lambda)$ where X, S, Y are R-modules,

$$\delta : X \oplus S \longrightarrow S$$

and $\lambda : S \longrightarrow Y$ are R-linear maps.

Example: Let $X = S = Y = \mathcal{M}$ an R-module, $\delta = +_M$, $\lambda = Id_M$ then our machine is simply the R-module $\mathcal{M}$.

Note 9. If R is a field then X, S, Y are vector-spaces and δ and λ are linear-transformations.

Proposition 3. $\mathcal{M}$ is the A-machine $(X \oplus - , S, Y, I, \delta, \sigma, \lambda)$ in the category Mod R of R-modules and R-linear maps as morphisms. For details

Proof: Given an R-module X

$X \oplus - :$ R-Mod $\longrightarrow$ R-Mod is a functor

$S \longmapsto X \oplus S$ $\therefore$ a process

$$\begin{array}{ccc} S & & X \oplus S \\ f \big\downarrow & \longmapsto & \big\downarrow \\ S' & & X \oplus S' \end{array} \quad Id_X \oplus f \text{ where } Id_X \oplus f : (x, s) \mapsto (x, f(s))$$

DYN X is the category of all dynamics, i.e. of all R-linear maps $\delta : X \oplus S \longrightarrow S$ and dynamorphisms

$$h : (\delta : X \oplus S \longrightarrow S) \longrightarrow (\delta' : X \oplus S' \longrightarrow S')$$

where $h : S \longrightarrow S'$ is a linear map

$$\ni \delta' \cdot (Id_X \oplus h) = h . \delta$$

<u>NOTE 10</u>. 1 $\forall (x, s) \in X \oplus S$, δ linear

$$\Longrightarrow \delta(x, s) = \delta(0_X, s) + \delta(x, 0_S)$$

Let $f : S \longrightarrow S$ and $g : X \longrightarrow S$

$$s \longrightarrow \delta(0_X, s) \qquad x \longrightarrow \delta(x, 0_S).$$

Again linear $\Longrightarrow$ f and g are; conversely given two linear maps $f : S \longrightarrow S$ and $g : X \longrightarrow S$

$$\delta : X \oplus S \longrightarrow S$$
$$(x, s) \longmapsto f(s) + g(x)$$

is certainly linear $\therefore$ a dynamics.

$\therefore \delta : X \oplus S \longrightarrow S$ is a dynamics $\Longleftrightarrow$

<u>$\delta(x, s) = f(s) + g(x)$</u> for some linear maps $f : S \longrightarrow S$ and $g : X \longrightarrow S$.

2. Let k be a dynamorphism $\delta \longrightarrow \delta'$

$$\Longrightarrow k \cdot \delta(x, s) = \delta' \cdot (Id_X \oplus k)(x, s)$$

$$= \delta'(x, k(s))$$

i.e. $k(f(s) + g(x)) = f' \cdot k(s) + g'(x)$

Putting $x = 0_X$ we have $k \cdot f = f' \cdot k$

and $s = 0_S$ we have $k \cdot g = g'$

$\therefore$ $k : S \longrightarrow S'$, with $(\delta : X \oplus S \longrightarrow S) = f + g$

and $(\delta' : X \oplus S' \longrightarrow S') = f' + g'$, is a dynamorphism

$$\Longleftrightarrow \underline{k \cdot f = f' \cdot k \quad \text{and} \quad k \cdot g = g'}$$

$X^+ = \coprod_{n=0}^{\infty} X_n =$ countable copower of $X = X_n \ \forall n \in \mathbb{N}$

i.e. $X^+ = \{(x_0, \ldots, x_n, 0, \ldots) \mid n \in \mathbb{N}\} =$ set of infinite tuples

$\ni x_i \in X$ and $x_i = 0$ for all but a finite number of x_i's.

It is an R-module with addition and scalar multiplication defined component wise by the addition and scalar multiplication in X .

In the string $w = x_0 \ldots x_n\, 0\, 0 \ldots\ldots$, x_n is the first non zero input, x_0 the last input activating the machine.

$X \oplus -$ is an input process as $\forall\, S \in /\,\mathrm{Mod}\ R\,/$:

$$\left(\ \eta_S : S \longrightarrow X^+ \oplus S^+\right.$$

$$s \longmapsto (00\cdots, s00\ldots)$$

$$\delta_S : X \oplus (X^+ \oplus S^+) \longrightarrow X^+ \oplus S^+$$

$$(x,\ x_0 \ldots x_n 0 \ldots, s_0 \ldots s_m 0 \ldots) \longmapsto (x x_0 \ldots x_n 0 \ldots, 0 s_0 \ldots s_m 0 \ldots)$$

is universal from S to U , the forgetful functor DYN X $\longrightarrow$ Mod R .

Indeed $\forall\ \varphi : S \longrightarrow S'$ linear

$$\Psi(x_0 \ldots x_n 0 \ldots, s_0 \ldots s_m 0 \ldots) = \Sigma f'^j \varphi(s_j) + \Sigma f'^i g(x_i)$$

is the unique linear map $\ni$ the following diagrams commute:

$$s \longmapsto (0 \ldots, s0 \ldots)$$

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "\varphi"'] & X^+ \oplus S^+ \arrow[d, "\Psi"] \\ & S' \end{tikzcd}$$

$$s \longmapsto \varphi(s) = f'^0 \varphi(s) + 0$$

$$(0 \ldots, s0 \ldots) \longmapsto \varphi(s) = f'^0 \varphi(s) + 0$$

$$
\begin{tikzcd}
(x, x_0 \ldots x_n 0 \ldots, s_0 \ldots s_m 0 \ldots) \arrow[r, mapsto] \arrow[ddd] & (x x_0 \ldots x_n 0 \ldots, 0 s_0 \ldots s_m 0 \ldots) \arrow[ddd] \\
X \oplus (X^+ \oplus S^+) \arrow[r, "\delta_S"] \arrow[d, "Id_X \oplus \psi"'] & X^+ \oplus S^+ \arrow[d, "\psi"] \\
X \oplus S' \arrow[r, "\delta'"] & S' \\
(x, \Sigma f'^j \cdot \varphi(s_j) + \Sigma f'^i \cdot g(x_i)) \arrow[r, mapsto] & \Sigma f'^{j+1} \cdot \varphi(s_j) + \Sigma f'^{i+1} \cdot g(x_i) + g(x)
\end{tikzcd}
$$

S = R-module of states

Y = R-module of outputs

I = $\{0\}$ is the initial state R-module

$\sigma : \{0\} \longrightarrow S$ is the initial state linear morphism

$0 \longmapsto 0_S$

X^+ = the R-module of inputs

$\delta^+ : X^+ \longrightarrow S$, the reachability morphism

$(x_0 \ldots x_n 0 \ldots) \longmapsto \Sigma f^i \cdot g(x_i)$

is the unique dynamorphique extension of σ as

$$
\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[ddd, bend right] & & (00 \ldots\ldots) \arrow[ddd] \\
\{0\} \arrow[r, "\eta_S"] \arrow[drr, "\sigma"'] & X^+ \times \{0\} \cong X^+ \arrow[d, "\delta^+"] & \\
 & S & \\
 & 0_S &
\end{tikzcd}
$$

and

$$\begin{tikzcd}
(x, w = x_0 \ldots x_n 0 \ldots) \arrow[rrr, mapsto] \arrow[ddd] & & & (x x_0 \ldots x_n 0 \ldots) \arrow[ddd, "\text{commute}"] \\
& X \oplus X^+ \arrow[r, "\delta_i"] \arrow[d, "Id_X \oplus \delta^+"'] & X^+ \arrow[d, "\delta^+"] & \\
& X \oplus S \arrow[r, "\delta"] & S & \\
(x, \Sigma F^i g(x_i)) \arrow[rrr, mapsto] & & & \Sigma F^{i+1} g(x_i) + g(x)
\end{tikzcd}$$

$\therefore$ M is a machine in Mod R $\because$ DN 6

with external behavior $\underline{E\,M(w) = \lambda \cdot \delta^+(w) = \Sigma h . F^i g(x_i)}$

if we put $\lambda = h : S \longrightarrow Y$.

Let $\overline{EM} : X^+ \longrightarrow Y^{IN}$

$$(w = x_0 \ldots x_n 0 \ldots) \mapsto (EM(w), EM(0w), \ldots, EM(0^n w), \ldots)$$

with $0^n w = \underbrace{0 \ldots 0}_{n \text{ zeros}} x_0 \ldots x_n 0 \ldots$

i.e. $\overline{EM}(w) = (\Sigma h . F^i g(x_i), \Sigma h . F^{i+1} g(x_i), \ldots, \Sigma h . F^{i+n} g(x_i), \ldots)$

Then the Nerode equivalence relation for E M is

$$E_{EM} = \{ (w_1, w_2) \mid w_1, w_2 \in X^+ \text{ and } \overline{EM}(w_1) = \overline{EM}(w_2) \}$$

as $\forall \varphi : X^+ \longrightarrow Y$ linear

$$\varphi(w\, w_1) = \varphi(w w_2)\ \forall w \in X^*$$

$$\Longleftrightarrow \varphi(w w_1) = \varphi(w + 0^{|w|} w_1) = \varphi(w) + \varphi(0^{|w|} w_1) = \varphi(w) + \varphi(0^{|w|} w_2)$$

$$\Longleftrightarrow \varphi(0^{|w|} w_1) = \varphi(0^{|w|} w_2),$$ where $|w|$ = length of w

$$\Longleftrightarrow \varphi(0^n w_1) = \varphi(0^n w_2) \quad \forall n \in \mathbb{N}.$$

$$S_{EM} = X^+ / E_{EM}$$

$$\begin{aligned} \delta^+_{EM} : X^+ &\longrightarrow S_{EM} \\ w &\longmapsto [w] \end{aligned}$$

$$\begin{aligned} \delta_{EM} : X \times S_{EM} &\longrightarrow S_{EM} \\ (x, [w]) &\longmapsto [xw] \end{aligned}$$

E M satisfies the four postulates of DN 12

(see [3] p. 696 to 701)

$\therefore$ M is an A-machine and its Nerode realization is:

$$N E M = \left(X \oplus -, S_{EM}, Y, \{0\}, \delta_{EM}, \tau_{EM} : \underset{0 \mapsto [0]}{\{0\} \to S_{EM}}, \lambda_{EM} : \underset{[w] \to EM(w)}{S_{EM} \to Y}\right)$$

NOTE. Again for proof of uniqueness of ψ , δ^* , etc. see [3] , chap. 9

The realization algorithm for linear machines is given in [3] page 582 to 585 and a partial realization algorithm is given in [3] page 587.

x x x

3. TREE AUTOMATA.

DN 2.5.1 A multigraded set is a set Ω together with a function $\nu : \Omega \longrightarrow 2^{\mathbb{N}}$ which assigns to each $\omega \in \Omega$ a finite set of arities $\nu(\omega) \subset \mathbb{N}$

We put $\Omega_n = \nu^{-1}(n) \subset \Omega$. We will use in particular $\Omega_0 = \{\omega \in \Omega \mid 0 \in \nu(\omega)\}$

2 An Ω -tree is a tree $T \subset \mathbb{N}^*$ together with a function $t : T \longrightarrow \Omega$ such that if $w \in T$ has n successors, then $n \in \nu(t(w))$.

3 An Ω -algebra (S, δ) is a set S (the carrier) together with a map $\delta : \omega \longmapsto (\delta^n_\omega : S^n \longrightarrow S)$ $\forall \omega \in \Omega$ and $\forall n \in \nu(\omega)$

4 <u>An Ω -algebra homomorphism</u> is a map $h : S \longrightarrow S'$ $\ni$:

- (S, δ) and (S', δ') are Ω -algebras,

- $h\,(\delta^n_\omega\,(s_1,\dots,s_n)) = \delta'^n_\omega(\,h\,(s_1),\dots,h(s_n)\,)$ $\forall \omega \in \Omega,\ \forall n \in \mathbb{N}$

i.e. the diagram

$$\begin{tikzcd} S^n \arrow[r, "\delta^n_\omega"] \arrow[d, "h^{(n)}"'] & S \arrow[d, "h"] \\ S'^n \arrow[r] & S' \end{tikzcd}$$

commutes

where $h^{(n)} :\ S \longrightarrow S'$

$(s_1,\dots,s_n) \longmapsto (h\,(s_1),\dots,\ h\,(s_n)\,)$

DN 26.1 <u>A tree automaton</u> is a quintuple

$M = (S, \Omega, Y, \delta, \lambda)$ $\ni$:

S = set of states

Ω = multigraded set of inputs, with arity function ν

Y = set of outputs

(S, δ) is an Ω -algebra, δ is the transition function,

$\lambda :\ S \longrightarrow Y$ is a map called the output function.

2 <u>An input-structure</u> for M is an Ω -tree $t : T \longrightarrow \Omega_S$,

where $\Omega_S = S \cup \Omega$ with arity function defined by :

$$\begin{cases} \nu(\omega_S) = \{0\} & \text{if } \omega_S \in S \\ \nu(\omega_S) = \nu(\omega_S) & \text{if } \omega_S \in \Omega \end{cases}$$

3 <u>The R U N of M on t</u> is the tree $\bar{t} : T \longrightarrow S$ $\ni$:

- if w is a terminal node of T then -

$$\begin{cases} \bar{t}\,(w) = t\,(w) & \text{if } t\,(w) \in S \\ \bar{t}\,(w) = \delta^0_{t(w)}(\Lambda) \in S & \text{if } t\,(w) \in \Omega_0 \end{cases}$$

- if w has successors $w0, \ldots\ldots, w(n-1)$ then.

$$\bar{t}(w) = \delta^{n}_{t(w)}(\bar{t}(w0), \ldots\ldots, \bar{t}(w(n-1)))$$

4 <u>The evaluation of t by M</u> is the output $\lambda(\bar{t}(\Lambda))$ of the state obtained at the Λ node.

<u>Example:</u> We want a tree automaton and an input-structure to compute $(1 + 3) + (4 \times 2) + 3 = 15$

Let $S = Y = \mathbb{N}$, $\Omega = \{+, \times\}$ $\Rightarrow \Omega_S = \mathbb{N} \cup \{+, \times\}$

with $\gamma(n) = 0 \quad \forall n \in \mathbb{N}$ and $\gamma(+) = \gamma(\times) = 2$

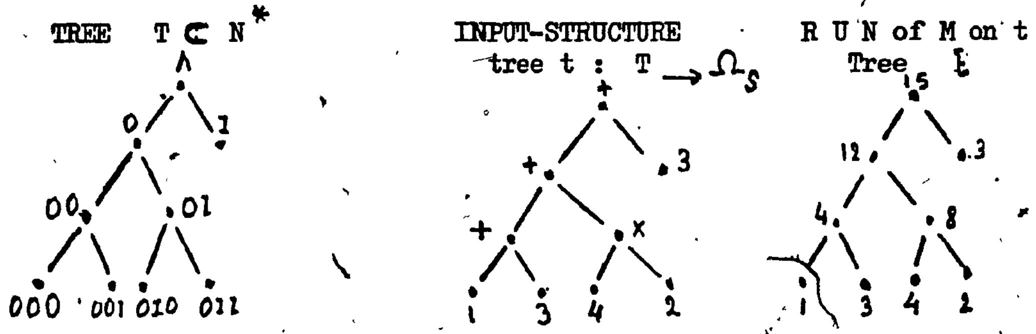
(+ and x are the usual addition and multiplication of integers).

We have

$$\delta = \begin{cases} \delta^{0}_{n} : \mathbb{N}^{0} \longrightarrow \mathbb{N} \\ \qquad \Lambda \longmapsto n . \\ \delta^{2}_{+} : \mathbb{N}^{2} \longrightarrow \mathbb{N} \\ \qquad (n, m) \longmapsto n + m \\ \delta^{2}_{\times} : \mathbb{N}^{2} \longrightarrow \mathbb{N} \\ \qquad (n, m) \longmapsto n \times m \end{cases}$$

$$\lambda = Id_{\mathbb{N}}$$

We have then:



The evaluation of t by M is: $Id_{\mathbb{N}}(15) = 15$

INPUT PROCESS $X : \mathcal{S} \longrightarrow \mathcal{S}$. We define X as follows:

$\forall S \in |\mathcal{S}|$, $XS = \coprod_{n \in v(\omega)} S^n \times \{\omega\}$

$= \{(s_1, \ldots, s_n, \omega) / s_1, \ldots, s_n \in S, \omega \in \Omega, n \in v(\omega)\}$

$\forall$ map $h : S \longrightarrow S'$

$Xh : XS \longrightarrow XS'$

$(s_1, \ldots, s_n, \omega) \longrightarrow (h(s_1), \ldots, h(s_n), \omega)$

h a map $\Longrightarrow$ Xh is well defined.

Assume we have $S \xrightarrow{h} S' \xrightarrow{h'} S''$:

$X(h'.h)(s_1, \ldots, s_n, \omega) = (h'.h(s_1), \ldots, h'.h(s_n), \omega)$ ∵ DN of X

$= Xh'(h(s_1), \ldots, h(s_n), \omega)$ ∵ DN of . and X

$= (Xh' . Xh)(s_1, \ldots, s_n, \omega)$ ∵ DN of X and .

Besides $X(Id_S)(s_1, \ldots, s_n, \omega) = (s_1, \ldots, s_n, \omega) = Id_{XS}(s_1, \ldots, s_n, \omega)$

∴ X is a functor.

<u>An X-dynamics</u> is a map $\delta : \coprod_{n\in\nu(\omega)} S \times \{\omega\} \longrightarrow S$ $\forall \omega \in \Omega$ and $\forall n \in \nu(\omega)$. To specify δ is just to specify $\forall \omega \in \Omega$ and $\forall n \in \nu(\omega)$ a map

$$\delta^n_\omega : S^n \longrightarrow S .$$

But this is the definition of an Ω-algebra (S, δ)

$\therefore$ <u>the X-dynamics are precisely the Ω-algebras</u>.

An map $h : S \longrightarrow S'$ is an <u>X-dynamorphism</u> $h : (\delta : XS \rightarrow S) \longrightarrow (\delta' : XS' \longrightarrow S')$ i.e. a map $h : (S, \delta) \longrightarrow (S', \delta')$ of Ω-algebras if the diagram

$$\begin{tikzcd}
(s_1,\ldots,s_n,\omega) \arrow[rrr, mapsto] \arrow[ddd] & & & \delta^n_\omega(s_1,\ldots,s_n) \arrow[ddd] \\
& XS \arrow[r, "\delta"] \arrow[d, "Xh"] & S \arrow[d, "h"] & \\
& XS' \arrow[r, "\delta'"'] & S' & \\
(h(s_1),\ldots,h(s_n),\omega) \arrow[rrr, mapsto] & & & \delta'^n_\omega(h(s_1),\ldots,h(s_n)) = h(\delta^n_\omega(s_1,\ldots,s_n))
\end{tikzcd}$$

commutes

But this is the definition of an Ω-algebra homomorphism.

$\therefore$ <u>the X-dynamorphisms are precisely the Ω-algebra homomorphisms</u>.

<u>Now we show that X is an input-process</u> i.e. that $\forall S \in |\mathcal{A}|\ \exists\ X^* S \in |\mathcal{A}|$, an X-dynamics $\delta_S : X(X^*S) \longrightarrow X^*S$ and a map $\eta_S : S \longrightarrow X^*S$ $\ni$ (δ_S, η_S) is universal over S with respect to U, the forgetful functor DYN $X \longrightarrow \mathcal{A}$.

Let $\Omega_S = \Omega \cup S$ with arity function $\nu_S : \Omega_S \longrightarrow 2^{\mathbb{N}}$
defined by $\nu_S(\omega_S) = \begin{cases} \nu(\omega_S) & \text{if } \omega_S \in \Omega \\ \{0\} & \text{if } \omega_S \in S \end{cases}$

<u>Let X^*S = set of Ω_S - trees</u>

$\Longrightarrow \quad X(X^*S) = \{(t_1,\dots,t_n,\omega)/t_1,\dots,t_n$ are Ω_S-trees and $n \in \nu(\omega)\}$ $\therefore$ DN of X

Let $\delta_S : X(X^*S) \longrightarrow X^*S$
$(t_1,\dots,t_n,\omega) \longmapsto$  $= (\delta_S)^n_\omega(t_1,\dots,t_n)$

δ_S is an X-dynamics as (X^*S, δ_S) is an Ω-algebra.

Let $\eta_S : \quad S \longrightarrow X^*S$
$\qquad\qquad s \longmapsto \cdot s$ (one node tree)

We show by induction that:
$\forall \ (\delta' : XS' \longrightarrow S') \in |DYN\ X|$ and $\forall f: \ S \longrightarrow S'$
$\exists$ a unique map $\psi : X^*S \longrightarrow S'$ $\ni$ the two diagrams

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & X^*S \arrow[d, "\psi"] \\ & S' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(X^*S) \arrow[r, "\delta_S"] \arrow[d, "X\psi"'] & X^*S \arrow[d, "\psi"] \\ XS' \arrow[r, "\delta'"'] & S' \end{tikzcd} \quad \text{commute}$$

(so that ψ is an X-dynamorphism $\delta_S \longrightarrow \delta'$)

1. Let t be a one node tree. Two cases are possible:
   a) $t(\Lambda) = \cdot s \in S$
   We want that

$$\begin{tikzcd} s \arrow[rr, mapsto] \arrow[drr, bend right, mapsto] & & \cdot s = t(\Lambda) \arrow[d, mapsto] \\ & S \arrow[r, "\eta_S"] \arrow[d, "f"'] & X^*S \arrow[d, "\psi"] \\ & & S' \\ & & f(s) \end{tikzcd} \quad \text{commutes}$$

We must have $\underline{\psi(t) = f(s)}$ (1)

b) $t(\Lambda) = \omega \in \Omega_0$ (i.e. $0 \in \forall(\omega)$)

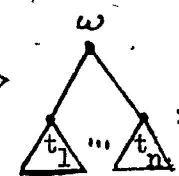
we want that $(\ , \omega) \longmapsto \omega = t(\Lambda)$

$$\begin{tikzcd} X(X^*S) \arrow[r, "\delta_S"] \arrow[d, "X\Psi"'] & X^*S \arrow[d, "\Psi"] \\ X S' \arrow[r, "\delta'"] & S' \end{tikzcd} \quad \text{commutes}$$

$(\ , \omega) \longmapsto \delta_\omega^{j_0}(\Lambda)$

We must have $\underline{\Psi(t) = \delta_\omega^{j_0}(\Lambda) \in S'}$ (2)

2. Let $t = \delta_S(t_1, \ldots, t_n, \omega)$

We want that $(t_1, \ldots, t_n, \omega) \longmapsto$  $= t$

$$\begin{tikzcd} X(X^*S) \arrow[r, "\delta_S"] \arrow[d, "X\Psi"'] & X^*S \arrow[d, "\Psi"] \\ X S' \arrow[r, "\delta'"] & S' \end{tikzcd} \quad \text{commutes}$$

$(\psi(t_1), \ldots, \psi(t_n), \omega) \longmapsto \delta_\omega^{j_n}(\psi(t_1), \ldots, \psi(t_n))$

We must have $\underline{\psi(t) = \delta_\omega^{j_n}(\psi(t_1), \ldots, \psi(t_n))}$ (3)

$\therefore$ (1), (2) and (3) define inductively a unique dynamorphism ψ which uses $f: S \longrightarrow S'$ to relabel terminal nodes to form, from an Ω_S-tree, a corresponding $\Omega_{S'}$-tree and run the X-dynamics δ' on this tree to read out the result from the Λ node.

<u>We show that a tree automaton is a machine in the category $\mathcal{S}$.</u>

Consider the machine in $\mathcal{S}$

$M = (X, S, \Omega, I, Y, \sigma, \delta, \lambda)$ where

X = the input-process just defined

S = set of states

Ω = multigraded set of inputs

I = set of initial states

Y = set of outputs.

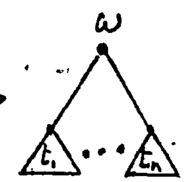
(Ω, δ) is an Ω-algebra

$\sigma : I \longrightarrow S$ is the initial map

$\lambda : S \longrightarrow Y$ is the output map.

By previous discussion the 2 diagrams

$$\begin{tikzcd}
i \arrow[rr, mapsto] \arrow[ddd, bend right, mapsto] & & \cdot\, i \arrow[ddd, mapsto] \\
I \arrow[r, "\eta_I"] \arrow[dr, "\sigma"'] & X^*I = X^* \arrow[d, "\delta"] & \\
& S & \\
& & \sigma(i)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(t_1, \ldots, t_n, \omega) \arrow[rr, mapsto] \arrow[ddd, mapsto] & & \text{tree} \arrow[ddd, mapsto] \\
X(X^*I) \arrow[r, "\delta_I"] \arrow[d, "X\delta^+"'] & X^*I \arrow[d, "\delta^+"] & \\
XS \arrow[r, "\delta"] & S & \\
(\delta^+(t_1), \ldots, \delta^+(t_n), \omega) \arrow[rr, mapsto] & & \delta^n_\omega(\delta^+(t_1), \ldots, \delta^+(t_n))
\end{tikzcd}$$



commute: with δ^+ defined inductively by:

a) $t(\Lambda) = i \in I \implies \delta^+(t) = \sigma(i)$

$\quad t(\Lambda) \in \Omega_0 \implies \delta^+(t) = \delta^0_\omega \in S$

b) $t = \delta_I(t_1, \ldots, t_n, \omega) \implies \delta^+(t) = \delta^n_\omega(\delta^+(t_1), \ldots, \delta^+(t_n))$

i.e. if t' is the Ω_S tree obtained by relabelling any terminal node labelled $i \in I$ of an Ω_I-tree t with the state $\sigma(i) \in S$, $\delta^+(t) = t'(\Lambda)$, <u>the run of M on t'</u>; furthermore the external behavior of M

$EM(t) = \lambda . \delta^+(t) = \lambda(t'(\Lambda))$ is the <u>evaluation of t' by M</u>.

If we put $I = S$, $\varphi = Id_S$ our machine M is the tree automaton $(S, \Omega, Y, \delta, \lambda)$ of DN 25.

<u>Proposition 5.</u> A tree automaton is an A-machine.

<u>Proof:</u> By above it is a machine in $\mathcal{S}$. We have to show that its external behavior EM satisfies the four postulates of DN 12.

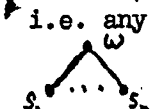
If $\delta : XS \longrightarrow S$ is an X-dynamics, by DN 9 $\delta^{(n)} : X^n S \longrightarrow S$ is defined inductively by:

$$\begin{cases} \delta^{(0)} = Id_S \\ \delta^{(n+1)} = X^{n+1} S \xrightarrow{X^n \delta} X^n S \xrightarrow{\delta^{(n)}} S \end{cases}$$

In the case at hand we have $\forall\ S \in \mathcal{S}$

$XS = \{ (s_1, \ldots, s_n, \omega) \,/\, s_1, \ldots, s_n \in S$ and $n \in \nu(\omega) \}$

i.e. any element of XS may be represented in the form



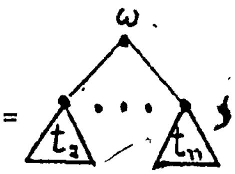
$= (\delta_S)^n_\omega \ (s_1, \ldots, s_n)$ or $\omega = (\delta_S)^0_\omega$ i.e. by elements of $X^* S$.

By induction on n, we show that any element of $X^n S$ is a tree of height at most n, where any path of length n terminates at a node labelled with an element of S, while any path of length less than n terminates at a node labelled with an element of Ω_0.

For $n = 1$ this is true by our above convention. Assume it is true for n.

$X^{n+1}S = X(X^nS) = \{(t_1,\dots,t_k,\omega)\}$ where $t_1,\dots,t_k \in X^nS$, i.e. are trees of length at most n with the required property concerning the labelling of terminal nodes,

$\omega \in \Omega_k$

$\therefore \quad (t_1,\dots,t_k,\omega) =$  is obviously

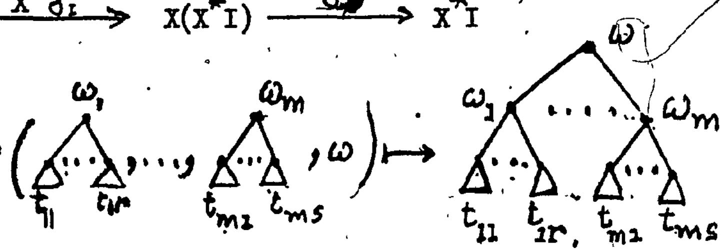
a tree of length at most $n+1$ with the required property.

<u>DN 27.</u> We may represent any such tree (whose terminal nodes are labelled by $s_1,\dots,s_m \in S$ for a path of length n for some $n \in \mathbb{N}$, and by $\omega \in \Omega_0$ for any path of length $k<n$) by $\Delta(s_1,\dots,s_m)$. If Δ is a tree obtained by replacing k of the labels s_i by s_i', $1 \leq k \leq m$, we call Δ a <u>k - ary derived operator;</u> in particular Δ is <u>unary</u> is $k = 1$.

$\therefore$ any element of $X^n(X^*I)$ is of the form $\Delta(t_1, \quad , t_n)$ where the t_i's are Ω_I-trees and $\delta_I^{(n)} : X^n(X^*I) \longrightarrow X^*I$ simply maps an Ω_{X^*I}-tree into the Ω_I-tree obtained by "unfurling" the Ω_I-trees comprising the terminal nodes of the Ω_{X^*I}-trees.

<u>Example:</u> if $n = 2$ we have:

$$\delta_I^{(2)} : X^2(X^*I) \xrightarrow{X\delta_I} X(X^*I) \xrightarrow{\delta_I} X^*I$$

$((t_{11},\dots,t_{1r},\omega_1),\dots,(t_{m1},\dots,t_{ms},\omega_m),\omega) \mapsto$ 

$r \in \nu(\omega_1) \quad s \in \nu(\omega_m) \quad m \in \nu(\omega)$

∴ There is little risk of ambiguity in using the same notation for an Ω_{X^*I}-tree and the Ω_I-trees obtained by applying $\delta_I^{(n)}$, to it.

A behavior β is a map $\beta : X I^* \longrightarrow Y$

Let $E_\beta = \{ (t, t') \in X^*I \times X^*I \mid \beta(\Delta(t)) = \beta(\Delta(t'))$

$\forall$ unary derived operators Δ $\}$

<u>E_β is an equivalent relation</u> as $\cong$ is

Let α and $\gamma : E_\beta \longrightarrow X^*I$ be the usual projections;

$$\therefore \quad X^n\alpha : X^n E_\beta \longrightarrow X^n (X^*I)$$

$$\Delta((t_1, t_1'), \quad , (t_n, t_m')) \longmapsto \Delta(t_1, \ldots, t_m)$$

$$X^n\gamma : X^n E_\beta \longrightarrow X^n (X^*I)$$

$$\Delta((t_1, t_1'), \ldots, (t_n, t_m')) \longmapsto \Delta(t' , \ldots, t_m')$$

$\delta_I^{(n)}$ applied to the trees so obtained "unfurls" them and reads out the corresponding output; ∴ to show that

$$\beta \cdot \delta_I^{(n)}(X^n\alpha) = \beta \cdot \delta_I^{(n)}(X^n\gamma),$$

We prove that $\forall i = 1, \ldots, m, \ (t_i, t_i') \in E_\beta$,

i.e. $\beta(\Delta(t_i)) = \beta(\Delta(t_i'))$ $\forall$ unary derived operators ,

$$\Rightarrow \beta(\Delta'(t_1, \ldots, t_m)) = \beta(\Delta'(t_1', \ldots, t_m'))$$

$\forall$ m - ary derived operators Δ' (*)

$$\beta(\Delta(t_1, \ldots t_{i-1}, t_i, t_{i+1}, \ldots t_m)) = \beta(\Delta(t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots t_m))$$

as this elementary translation is in fact a unary derived operator Δ_i and $\beta(\Delta_i(t_i)) = \beta(\Delta_i(t_i'))$.

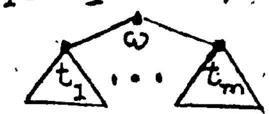
By using m suitable Δ_i's we have the desired result.

∴ β <u>satisfies postulate 1(a)</u>.

In particular we have that

$(t_i, t_i') \in E_\beta \quad \forall\, i = 1, \ldots, m$

$\Longrightarrow \big((\delta_I)^m_\omega (t_1, \ldots, t_m), (\delta_I)^m_\omega (t_1', \ldots, t_m') \big) \in E_\beta \quad \forall \omega \in \Omega$

where $(\delta_I)^m_\omega (t_1, \ldots, t_m) =$ 

as (*) valid $\forall$ m-ary derived operators

$\Longrightarrow$ it is valid for the operator $(\delta_I)^m_\omega$

$\therefore$ E_β is a congruence.

Assume $\exists$ a set R and maps $R \overset{p}{\underset{p'}{\rightrightarrows}} X^* I$

$\ni \beta . \delta_I^{(n)} (X^n p) = \beta . \delta_I^{(n)} (X^n p')$.

We have just seen that this condition means that $\forall$ m-ary derived operator and $\forall (r_1, \ldots, r_m)$

$\beta(\Delta(p(r_1), \ldots, p(r_m))) = \beta(\Delta(p'(r_1), \ldots, p'(r_m)))$

$\Longrightarrow \beta(\Delta(p(r))) = \beta(\Delta(p'(r)))$ if $m = 1$

$\Longrightarrow (p(r), p'(r)) \in E_\beta \quad \forall r \in R$.

Let $\varphi : R \longrightarrow E_\beta$
$\qquad r \longmapsto (p(r), p'(r))$

We have $p = \alpha . \varphi$ and $p' = \gamma . \varphi$

$\therefore \forall \psi \ni \psi . \alpha = \psi . \gamma$

$\psi . \alpha . \varphi = \psi . \gamma . \varphi$

i.e. $\psi . p = \psi . p'$

$\therefore$ β satisfies postulate 1 (b)

If $S_\beta = X^* I / E_\beta$ = set of equivalence classes $[t]$ with respect to the congruence E_β, the usual coeq (α, γ) (,) in $\mathcal{S}$ is :

$$\begin{aligned} \delta_\beta^+ : X^* I &\longrightarrow S_\beta \\ t &\longmapsto [t] \end{aligned}$$

$\therefore$ <u>β satisfies postulate 2.</u>

As E_β is a congruence we can define δ_β by

$$\begin{aligned} (\delta_\beta)^n_\omega : S^n &\longrightarrow S \quad , \quad n \in \nu(\omega) . \\ ([t_1], \ldots, [t_n]) &\longmapsto [(\delta_I)^n_\omega (t_1, \ldots, t_n)] \end{aligned}$$

This makes (S_β , δ_β) an Ω-algebra, i.e. δ_β is an X-dynamics, and δ_β^+ an X-dynamorphism as

$$(t_1, \ldots, t_n, \omega) \longmapsto (\delta_I)^n_\omega (t_1, \ldots, t_n) =$$

$$n \in \nu(\omega) \qquad \begin{array}{ccc} X(X^* I) & \xrightarrow{\delta_I} & X^* I \\ {\scriptstyle X\delta_\beta^+}\downarrow & & \downarrow{\scriptstyle \delta_\beta^+} \\ X S_\beta & \xrightarrow{\delta_\beta} & S_\beta \end{array} \quad \text{commutes} .$$

$$([t_1], \ldots, [t_n], \omega) \longmapsto [(\delta_I)^n_\omega (t_1, \ldots, t_n)]$$

$\therefore$ <u>β satisfies postulate 3.</u>

In $\mathcal{S}$ any coequalizer is a split-epimorphism and any functor $X : \mathcal{S} \longrightarrow \mathcal{S}$ preserves split-epimorphism as

$f \cdot f' = Id_B$

$\Longrightarrow \quad (X f) \cdot (X f') = Id_{XB}$

$\therefore$ <u>β satisfies postulate 4.</u>

$\therefore$ <u>β is an A-behavior</u> .

Putting $\beta = E M = \lambda \cdot \delta^+$ we have that our <u>tree automaton M is an A-machine.</u>

x x x

CHAPTER III

DECOMPOSABLE SYSTEMS

In this chapter we will define a "system" in categorical terms and show that it is a particular case of machines in a category, thus establishing a link between automata and system theories. We give two examples of such systems: linear and group machines.

<u>DN 28.</u> <u>A decomposable system in a category</u> $\mathcal{C}$ is a sextuple (u, x, y, f, g, h) where x, u and $y \in /\mathcal{C}/$ and
$f : x \longrightarrow x$ is an $Id_{\mathcal{C}}$-dynamics $\in \mathcal{C}$
$g : u \longrightarrow x$ is the initial state morphism $\in \mathcal{C}$
$h : x \longrightarrow y$ is the output morphism $\in \mathcal{C}$

We denote the system (f, g, h) for short.

<u>TH 4.</u> $\mathcal{C}$ a category $\ni \exists\ c^+$ the countable copower of $c\ \forall c \in /\mathcal{C}/$
$\Longrightarrow Id_{\mathcal{C}}$ is an input-process.

<u>Proof</u>: $Id_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor i.e. a process
an $Id_{\mathcal{C}}$-dynamics is any morphism $f \in \mathcal{C}(x, x)\ \ \forall x \in \mathcal{C}$.
an $Id_{\mathcal{C}}$-dynamorphism g: $(f : x \to x) \longrightarrow (f' : x' \to x')$
is an $\mathcal{C}$-morphism $g : x \longrightarrow x' \ni g.f = f'.g$

i.e. $\ni$
$$\begin{array}{ccccc} Id_{\mathcal{C}}x & = & x & \xrightarrow{f} & x \\ Id_{\mathcal{C}}g & = & g\downarrow & & \downarrow g \\ Id_{\mathcal{C}}x' & = & x' & \xrightarrow{f'} & x' \end{array}$$
commutes

We have to show that the forgetful functor
$U : DYN\ Id_{\mathcal{C}} \longrightarrow \mathcal{C}$ has a left-adjoint.
$(f : x \to x) \longmapsto x$
$(k : f \longrightarrow f') \longmapsto k : x \longrightarrow x'$

By lemma 4.1, $\forall u \in /\mathcal{C}/, \forall$ pair $(f : x \to x,\ g : u \to x)$
$f \in /DYN\ Id_{\mathcal{C}}/,\ g \in \mathcal{C}(u, x), \exists\ \delta^+$ uniquely defined by
$\delta^+.in_n = f^n.g\ \ \forall n \in \mathbb{N}$

$$\ni \quad \begin{tikzcd} u \arrow[r, "in_0"] \arrow[dr, "g"'] & u^+ \arrow[d, "\delta^+"] \\ & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} u^+ \arrow[r, "z"] \arrow[d, "\delta^+"'] & u^+ \arrow[d, "\delta^+"] \\ x \arrow[r, "f"] & x \end{tikzcd} \quad \text{commute}$$

where z is uniquely defined by $in_n . z = in_{n+1} \quad \forall n \in \mathbb{N}$ (see DN a 13.1).

$\therefore \ \delta^+ \in \text{Dyn } Id_{\mathcal{C}} \ (z: u^+ \longrightarrow u^+, \ f : x \longrightarrow x)$

and $(z : u^+ \longrightarrow u^+, \ in_0 : u \longrightarrow Uz = u^+)$

is a universal arrow from u to U.

$\Longrightarrow$ U has a left-adjoint (by TH a 5.2).

<u>Proposition 6.</u> <u>A decomposable system</u> (f, g, h) in a category $\mathcal{C}$

$\ni \ \exists u^+ \ \forall u \in /\mathcal{C}/$, is a machine

$S = (Id_{\mathcal{C}}, x, u, y, f, g, h)$.

<u>Proof:</u> S is a machine in a category $\because$ DN 6 with:

$I\partial_{\mathcal{C}}$ = input process $\because$ TH 4

u = initial state object

x = state object

y = output object

$f : x \longrightarrow x$ is an $I\partial_{\mathcal{C}}$-dynamics

$g : u \longrightarrow x$ is the initial state morphism

$h : x \longrightarrow y$ is the output morphism

$(\delta_u = z : u^+ \longrightarrow u^+; \ \eta_u = in_0 : u \longrightarrow u^+)$

is the universal arrow from u to U, z defined by

$in_n . z = in_{n+1} \ \forall n \in \mathbb{N}$; the object of inputs is $u^+ = U\delta_u$,

the reachability morphism is $\delta^+ : u^+ \longrightarrow x$ defined by

$\underline{\delta^+ . in_n = f^n . g} \quad \forall n \in \mathbb{N}$;

the external behavior is $E\,S = h.\delta^+$

i.e. the unique morphism $u^+ \longrightarrow y$ defined by

$\underline{E\,S . in_n = h.f^n.g} \quad \forall n \in \mathbb{N}.$

DN. In the following a suitable category $\mathcal{C}$ is a category which has countable powers, copowers and coeq. mono factorization $\forall$ morphisms.

DN 29. Observability map.

1. S is a decomposable system in a suitable category $\mathcal{C}$. The observability map of S is the morphism $\omega : x \longrightarrow y^x$ uniquely defined by $p_n . \omega = h . f^n$ i.e. $\ni$

$$\begin{tikzcd} y & y^x \arrow[l, "p_n"'] \\ & x \arrow[u, "\omega"'] \arrow[ul, "h.f^n"] \end{tikzcd}$$

commutes $\forall n \in IN$, where (y^x, p_n) is the countable power of y.

2. S is observable if ω is monic.

NOTE 11. By lemma 4.2, $h = p_0 . \omega$ and ω is an $Id_{\mathcal{C}}$ dynamorphism : $(f : x \to x) \longrightarrow (z' : y^x \longrightarrow y^x)$

where z' is uniquely defined by $z' . p_n = p_{n+1}$ $\forall$ $n \in IN$.

DN 30. The total external behavior of a decomposable system S in a suitable category C is the morphism
$\overline{ES} = \omega . \delta^+ : U^+ \longrightarrow y^x$.

NOTE 12.1 δ^+ and ω are $Id_{\mathcal{C}}$ dynamorphism $\Longrightarrow$ $\overline{ES}$ is a $Id_{\mathcal{C}}$ dynamorphism: $(z : u^+ \longrightarrow u^+) \longrightarrow (z' : y^x \longrightarrow y^x)$

The diagram

$$\begin{tikzcd} u^+ \arrow[r, "\delta^+"] \arrow[d, "z"'] & x \arrow[r, "\omega"] \arrow[d, "f"] & y \arrow[d, "z'"] \\ u^+ \arrow[r, "\delta^+"'] & x \arrow[r, "\omega"'] & y^+ \end{tikzcd}$$

commutes.

as δ^+ a $Id_{\mathcal{C}}$ dynamorphism $(z: u^+ \longrightarrow u^+) \longrightarrow (f: x \longrightarrow x)$
$\Longrightarrow$ the left square commutes
and ω a $Id_{\mathcal{C}}$ dynamorphism $(f: x \longrightarrow x) \longrightarrow (z' : y^x \longrightarrow y^x)$
$\Longrightarrow$ the right square commutes.

2. $ES = h.\delta^+ = p_o.\omega.\delta^+ = p_o . \overline{ES}$.

<u>DN 31.</u> Given a behavior $\beta : u^+ \longrightarrow y$ in a suitable category we uniquely define the corresponding total behavior $\bar\beta : u^+ \longrightarrow y^x$ by $p_n.\bar\beta = \beta . z^n$
i.e. $\ni$

$$\begin{tikzcd} y & & y^x \arrow[ll, "p_n"'] \\ & u^+ \arrow[ul, "\beta"] & \\ & & u^+ \arrow[ul, "z^n"] \arrow[uu, "\bar\beta"'] \end{tikzcd}$$

commutes $\forall n \in IN$.

Again we have $\beta = p_o . \bar\beta$ for $n = 0$.

<u>LEMMA 5.</u> $f_1 : x_1 \longrightarrow x_1$, $f_2 : x_2 \longrightarrow x_2$ and $f_3 : x_3 \longrightarrow x_3$ are $Id_{\mathcal{C}}$ dynamics; $\exists\ \varphi : x_1 \longrightarrow x_2$, $\psi : x_2 \longrightarrow x_3$
$\ni \varphi$ is an $Id_{\mathcal{C}}$ dynamorphism : $f_1 \longrightarrow f_2$ and is epi,
$\psi . \varphi$ is an $Id_{\mathcal{C}}$ dynamorphism: $f_1 \longrightarrow f_3 \Longrightarrow \psi$ is an $Id_{\mathcal{C}}$ dynamorphism: $f_2 \longrightarrow f_3$.

<u>Proof:</u> We have $\varphi . f_1 = f_2 . \varphi$ $\because \varphi$ a $Id_{\mathcal{C}}$ dynamorphism,

$$\psi . \varphi . f_1 = f_3 . \psi . \varphi \quad \because \psi . \varphi \text{ a } Id_{\mathcal{C}} \text{ dynamorphism,}$$

$\therefore \psi . \varphi . f_1 = \psi . f_2 . \varphi = f_3 . \psi . \varphi$
$\Longrightarrow \psi . f_2 = f_3 . \psi$ as φ is epi
i.e. ψ is a $Id_{\mathcal{C}}$ dynamorphism.

TH 5. $\beta : u^{+} \longrightarrow y$ is a behavior in a suitable category $\mathcal{C} \ni$ its total behavior $\bar{\beta}$ is an $Id_{\mathcal{C}}$ dynamorphism $(z: u^{+} \longrightarrow u^{+}) \longrightarrow (z' : y^{x} \longrightarrow y^{x})$ with a coequalizer-mono factorization

$$\bar{\beta} = u^{+} \xrightarrow{\delta_\beta^{+}} x_\beta \xrightarrow{\omega_\beta} y^{x}$$

$\Longrightarrow \beta$ is an A-behavior.

Proof: 1.a) By DN 11 and proof of proposition 6,

$$\delta_u = z : u^{+} \longrightarrow u^{+},$$

$$\delta_u^{(0)} = Id_{u^{+}} = z^{0}$$

$$\delta_u^{(n+1)} = (Id_{\mathcal{C}})^{n+1} u^{+} \xrightarrow{(Id_{\mathcal{C}})^{n} z} (Id_{\mathcal{C}})^{n} u^{+} \xrightarrow{\delta_u^{(n)}} u^{+}$$

$$= u^{+} \xrightarrow{z} u^{+} \xrightarrow{\delta_u^{(n)}} u^{+} = \delta_u^{(n)} . z$$

$\therefore \delta_u^{(n)} = z^{n} \quad \forall n \in \mathbb{N}$, by induction.

Let $\delta_\beta^{+} = \text{coeq}(\alpha, \gamma)$

$$
\begin{aligned}
\therefore \quad \beta . \delta_u^{(n)} . ((Id_{\mathcal{C}})^{n} \alpha) &= \beta . z^{n} . \alpha && \because \text{above result} \\
&= p_n . \bar{\beta} . \alpha && \because \text{DN 31} \\
&= p_n . \omega_\beta . \delta_\beta^{+} . \alpha && \because \text{hypothesis} \\
&= p_n . \omega_\beta . \delta_\beta^{+} . \gamma && \because \delta_\beta^{+} = \text{coeq}(\alpha, \gamma) \\
&= p_n . \bar{\beta} . \gamma && \because \text{hypothesis} \\
&= \beta . z^{n} . \gamma && \because \text{DN 31} \\
&= \beta . \delta_u^{(n)} . ((Id_{\mathcal{C}})^{n} \alpha) && \forall n \in \mathbb{N}.
\end{aligned}
$$

$\therefore \beta$ satisfies postulate 1a.

b) Let $r \in /\mathcal{C}/$ and $r \underset{p'}{\overset{p}{\rightrightarrows}} u^+ \ni$

$$\beta.\delta_u^{(n)}((\mathfrak{a}_{\mathcal{C}})^n p) = \beta.\delta_u^{(n)}((\mathfrak{a}_{\mathcal{C}})^n p')$$

i.e. $\beta.z^n.p = \beta.z^n.p'$

$\Longrightarrow p_n.\bar{\beta}.p = p_n.\bar{\beta}.p'$ $\quad\because$ DN 31

$\Longrightarrow \bar{\beta}.p = \bar{\beta}.p'$ by uniqueness in diagram:

$$\begin{tikzcd} y & y^x \arrow[l, "p_n"'] \\ & r \arrow[u, "\bar{\beta}.p = \bar{\beta}.p'"'] \arrow[ul, "p_n.\bar{\beta}.p = p_n.\bar{\beta}.p'"] \end{tikzcd}$$

i.e. $\omega_\beta.\delta_\beta^+.p = \omega_\beta.\delta_\beta^+.p'$ $\quad\because$ hypothesis

$\Longrightarrow \delta_\beta^+.p = \delta_\beta^+.p'$ $\quad\because \omega_\beta$ monic

$\therefore \forall \psi \ni \psi.\alpha = \psi.\gamma$

$\Longrightarrow \exists$ unique $\psi' \ni \psi = \psi'.\delta_\beta^+$ $\quad\because$ coeq $(\alpha, \gamma) = \delta_\beta^+$

$$\begin{aligned} \Longrightarrow \quad \psi.p &= \psi'.\delta_\beta^+.p \\ &= \psi'.\delta_\beta^+.p' \\ &= \psi.p' \end{aligned}$$

and postulate 1 b is satisfied.

2) By hypothesis $\exists x_\beta \in /\mathcal{C}/$ and a C morphism

$\delta_\beta^+ : u^+ \longrightarrow x_\beta \ni \delta_\beta^+ = $ coeq (α, γ)

$\therefore$ postulate 2 is satisfied.

3) $\bar{\beta}$ is an $Id_{\mathcal{C}}$ dynamorphism $(z: u^+ \longrightarrow u^+) \longrightarrow (z': y^x \longrightarrow y^x)$

$\Longrightarrow \omega_\beta.\delta_\beta^+.z = \bar{\beta}.z = z'.\bar{\beta} = z'.\omega_\beta.\delta_\beta^+$

$\delta_\beta^+ = $ coeq $(\alpha, \gamma) \Longrightarrow \delta_\beta^+.\alpha = \delta_\beta^+.\gamma$

$$\begin{aligned} \therefore \omega_\beta.\delta_\beta^+.z.\alpha &= z'.\omega_\beta.\delta_\beta^+.\alpha \\ &= z'.\omega_\beta.\delta_\beta^+.\gamma \\ &= \omega_\beta.\delta_\beta^+.z.\gamma \end{aligned}$$

$\Longrightarrow \delta_\beta^+ . z. \alpha = \delta_\beta^+ . z. \gamma$ as ω_β is monic

$\therefore \exists$ unique $f_\beta : X_\beta \longrightarrow X_\beta$, a $Id_{\mathcal{C}}$ dynamics $\ni$

$$\begin{tikzcd} E_\beta \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & u^+ \arrow[r, "\delta_\beta^+"] \arrow[dr, "\delta_\beta^x \cdot z"'] & X_\beta \arrow[d, "f_\beta"] \\ & & X_\beta \end{tikzcd}$$ commutes

i.e. $\delta_\beta^+ . z = f_\beta . \delta_\beta^+$

$\Longrightarrow \delta_\beta^+$ is a Id_C dynamorphism $(z: u^+ \longrightarrow u^+) \longrightarrow (f_\beta : x_\beta \longrightarrow x_\beta)$

$\therefore$ postulate 3 is satisfied .

Besides $\omega_\beta . \delta_\beta^+$ and δ_β^+ are $Id_{\mathcal{C}}$ dynamorphisms,
δ_β^+ epi (as a coequalizer) $\Longrightarrow \omega_\beta$ is a $Id_{\mathcal{C}}$ dynamorphism
$(f_\beta : X_\beta \longrightarrow X_\beta) \longrightarrow (z' : y^x \longrightarrow y^x)$ $\because$ lemma 5 .

4) Postulate 4 is satisfied as $Id_{\mathcal{C}}\, \delta_\beta^+ = \delta_\beta^+$.
From proposition 6, TH 1, note 12(1), DN 22 and TH 5,
it follows immediately:

TH 6.1 A decomposable system $S = (f, g, h)$ in a suitable category $\mathcal{C}$, $\ni$ its total external behavior $\overline{ES}$ has a coequalizer mono factorization $\overline{ES} = u^+ \xrightarrow{\delta_{ES}^+} x_{ES} \xrightarrow{\omega_{ES}} y^x$, is an A-machine.

2. Its Nerode realization is $NES = (x_{ES}, u, y, f_{ES}, g_{ES}, h_{ES})$

where $f_{ES} : x_{ES} \longrightarrow x_{ES}$ is the unique morphism

$\ni$ $$\begin{tikzcd} E_{ES} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & u^+ \arrow[r, "\delta_{ES}^+"] \arrow[dr, "\delta_{ES}^x \cdot z"'] & X_{ES} \arrow[d, "f_{ES}"] \\ & & X_{ES} \end{tikzcd}$$ commutes

$$g_{ES} = \delta^{+}_{ES} \cdot \eta_u = \delta^{+}_{ES} \cdot in_0$$

$h_{ES} : x_{ES} \longrightarrow y$ is the unique morphism

$$\ni \quad \begin{tikzcd} E_{ES} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & u^{+} \arrow[r, "\delta^{+}_{ES}"] \arrow[dr, "ES"'] & x_{ES} \arrow[d, "h_{ES}"] \\ & & y \end{tikzcd} \quad \text{commutes}$$

i.e. $\qquad ES = h_{ES} \cdot \delta^{+}_{ES}$

$$p_0 \cdot \omega_{ES} \cdot \delta^{+}_{ES} = h_{ES} \cdot \delta^{+}_{ES} \qquad \text{as } ES = p_0 \cdot \overline{ES}$$

$$\Longrightarrow h_{ES} = p_0 \cdot \omega_{ES} \qquad \text{as } \delta^{+}_{ES} \text{ is epi .}$$

3. NES is the minimal realization of ES.

NOTE 13. Similarly if we are given $\beta : u^{+} \longrightarrow y \ni$ β is a Id_{θ} dynamorphism $z \longrightarrow z'$ and has a coequalizer-mono factorization $u^{+} \xrightarrow[\delta^{+}_{\beta}]{} x_{\beta} \xrightarrow[\omega_{\beta}]{} y^{X}$

its Nerode realization is $N\beta = (x_{\beta}, u, y, f_{\beta}, g_{\beta}, h_{\beta})$.

x x x

We will now discuss two examples of decomposable systems: linear systems and group machines.

2. LINEAR SYSTEMS

We have already shown, in the general context of part II, that a linear machine $M = (X, S, Y, 0_S, \delta, \lambda)$ as defined by DN 24 is an A-machine. Now we give the outline of the proof of this fact in the framework of decomposable systems.

Recall that $\delta : X \times S \longrightarrow S$ is linear
$\Longleftrightarrow \delta(x, s) = f(s) + g(x) \quad \forall x \in X$ and $s \in S$, f and g
are linear-maps.

Putting $u = X$, $x = S$, $y = Y$, $f = S \longrightarrow S$
$s \longmapsto \delta(0_X, s)$
$g : X \longrightarrow X$, $h = \lambda$, M becomes
$x \longmapsto \delta(x, 0_S)$
a decomposable system $\Sigma = (f, g, h)$ in R-Mod.

R-Mod is a suitable category: $\forall S \in$/R-Mod/
$\exists \; S^+ = \coprod_0^\infty S_n$, the countable copower of $S = S_n$, $\forall n \in \mathbb{N}$,
$S^x = \prod_0^\infty S_n$, the countable power of S, i.e. the countable direct sum of the S_n's, $S_n = S \; \forall n \in \mathbb{N}$; $\forall$ R-Module homomorphism $t : S \longrightarrow S'$ the usual coequalizer mono factorization
$t = S \xrightarrow{t'} \mathrm{Im}\, t \cong S/\ker t \xrightarrow{t''} S'$ where t' is onto
$s \longmapsto t(s) \longmapsto t(s)$
(therefore a coequalizer $\because$ TH a 3.5) and t'' is 1-1 (i.e. monic $\because$ TH a 2.2).

We define the required R-Module homomorphism z, in_n, p_n, δ^+, ω, $E\Sigma$, $\overline{E\Sigma}$ as follows:

$$z : X^+ \longrightarrow X^+$$
$$(x_0, \ldots, x_n 0 \ldots) \longmapsto (0 x_0, \ldots, x_n 0 \ldots)$$
$$in_n : X \longrightarrow X^+$$
$$x \longmapsto (0, \ldots, 0 x 0, \ldots)$$ where x is in the n+1st position,

$$p_n : Y^x \longrightarrow Y$$
$$(y_0, \ldots, y_n, \ldots) \longmapsto y_n$$
$$\delta^+ : X^+ \longrightarrow S$$
$$(x_0, \ldots, x_n 0 \ldots) \longmapsto \sum_{i=0}^{n} f^i . g(x_i)$$

$$\omega : S \longrightarrow Y^{x}$$
$$s \longmapsto (h s, h.f(s), \ldots, h.f^{n}(s), \ldots)$$
$$\overline{E\Sigma} = X^{+} \xrightarrow{\delta^{+}} X \xrightarrow{\omega} Y^{x}$$
$$(x_0 \ldots x_n 0 \ldots) \longmapsto \sum_i f^{i}.g(x_i) \longmapsto (\sum_i h.f^{i}.g(x_i), \sum_i h.f^{i+1}.g(x_i), \ldots)$$
$$E\Sigma = p_0.\overline{E\Sigma} = h.\delta^{+} : X^{+} \longrightarrow Y$$
$$(x_0 \ldots x_n 0 \ldots) \longmapsto \sum_i h.f^{i}.g(x_i)$$

Let $S_{E\Sigma} = X^{+}/Ker\ E\Sigma$ i.e. $\forall w = x_0 \ldots x_n 0 \ldots \in X^{+}$

$$[w] = w + \mathrm{Ker}\ E$$
$$= \{ w' \in X^{+} \ / \ \overline{E\Sigma}(w') = \overline{E\Sigma}(w)$$
$$\Longleftrightarrow E\Sigma(0^{n} w') = E\Sigma(0^{n} w) \ \forall n \in \mathbb{N} \}$$

Then $$\overline{E\Sigma} = X^{+} \xrightarrow{\delta^{+}_{E\Sigma}} S_{E\Sigma} \xrightarrow{\omega_{E\Sigma}} Y^{x}$$
$$w \longmapsto [w] \longmapsto (E\Sigma(w), E\Sigma(0w), E\Sigma(00w), \ldots)$$
$$= (\Sigma h.f^{i}.g(x_i), \Sigma h.f^{i+1}.g(x_i), \Sigma h.f^{i+2}.g(x\ ), \ldots)$$

is the required coequalizer mono factorization for $E\Sigma$ to be an A-behavior ∵ TH 5.

The external behavior $E\Sigma$ of Σ is identical to EM of M and its Nerode realization $NE\Sigma = (f_{E\Sigma}, g_{E\Sigma}, h_{E\Sigma})$ same as N E M (as defined in chapter II).

Indeed we have:

$$f_{E\Sigma} : S_{E\Sigma} \longrightarrow S_{E\Sigma}$$
$$[w] \longmapsto [0w]$$

$$g_{E\Sigma} = X \xrightarrow{in_0} X^{+} \xrightarrow{\delta^{+}_{E\Sigma}} S_{E\Sigma}$$
$$x \longmapsto (x0 \ldots) \longmapsto [x0 \ldots]$$

$$h_{E\Sigma} = S_{E\Sigma} \longrightarrow Y$$
$$w \longmapsto E\Sigma(w)$$

$$S_{EM} = S_{E\Sigma}$$
$$\delta_{EM} : X \times S_{EM} \longrightarrow S_{EM} = f_{E\Sigma} + g_{E\Sigma}$$
$$(x, [w]) \longmapsto [x\,w]$$
$$\delta^{+}_{EM} = \delta^{+}_{E\Sigma} \text{ and } \lambda_{EM} = h_{E\Sigma}.$$

x x x

3. GROUP MACHINES.

<u>DN 32.</u> $M = (X, S, Y, \delta, \lambda)$ is a group machine if X, S, Y are groups and $\delta : X \times S \longrightarrow S$ with $X \times S$ = direct product of X and S and $\lambda : S \longrightarrow Y$ are group-homomorphisms.

<u>Example:</u> Let $X = S = Y = G$ a group
$\delta = \square$ the operation of G, $\lambda = Id_G$
then our machine M is simply the group G.

<u>Proposition 7.</u> <u>M is an A-machine.</u>

<u>Proof:</u> We have to show that:
- the category Gr of groups and group homomorphisms is a suitable category;
- M is a decomposable system in Gr with a coequalizer mono factorization.

1. $\forall\, G \in Gr$, $G^{+} = \coprod_{i \in \mathbb{N}} G_i$, $G_i = G \quad \forall\, i \in \mathbb{N}$, is called the free product and is defined in the following way:

Its elements are strings of the form

$(g_{i_0}, i_0)(g_{i_1}, i_1) \ldots (g_{i_n}, i_n)$, $g_{i_j} \in G$, $i_j \in \mathbb{N}$

with the empty string denoted by Λ, and subject to the restrictions:

(i) no $g_{i_j} = 1_G$ the identity;

(ii) $\forall\, j$, $0 \leq j \leq n$, $i_j \neq i_{j+1}$

The product in G^{+} is concatenation, save that if the string so formed does not satisfy conditions (i) and (ii) we apply the following:

(a) replace consecutive elements of the form $(g, n)\ (g', n)$ by (gg', n);

(b) delete elements of the form $(1_G, n)$ until a string meeting conditions (i) and (ii) is obtained.

The injections are $\forall\, j \in \mathbb{N}$

$$
\begin{aligned}
in_j : G &\longrightarrow G^{+} \\
g \neq 1_G &\longmapsto (g, j) \\
1_G &\longmapsto \Lambda
\end{aligned}
$$

$G^{x} = \prod_{i \in \mathbb{N}} G_i$, $G_i = G \quad \forall\, i \in \mathbb{N}$ is the group of countable tuples $(g, \ldots, g_j, \ldots)$, $g \in G\ \forall\, j \in \mathbb{N}$, with the product defined component wise by the product in G. The projections are

$$p_j(g, \ldots, g_j, \ldots) = g_j$$

G^{x} is called the direct product.

$\forall\ h : G \longrightarrow G'$, a group homomorphism
$\operatorname{Im} h \cong \dfrac{G}{\operatorname{Ker} h}$ is a subgroup of G',

$\therefore$ We have the usual factorization:

$$\begin{array}{ccccc} G & \xrightarrow{h'} & \operatorname{Im} h & \xrightarrow{in} & G' \\ g & \longmapsto & h(g) & \longmapsto & h(g) \end{array}$$

where both maps are group homomorphisms, h' is onto, therefore a coequalizer ($\because$ TH a 3.5) and in 1 - 1, therefore monic ($\because$ TH a 2.2).

$\therefore$ Gr is a suitable category.

2. $\delta : X \times S \longrightarrow S$ is a group homomorphism

$$\therefore \delta(x, s) = \delta((1_X, s)(x, 1_S))$$
$$= \delta(1_X, s)\, \delta(x, 1_S) = f(s)\, g(x)$$

where $f : S \longrightarrow S$, $g : X \longrightarrow S$.
$s \longmapsto \delta(1_X, s)$ $\quad x \longmapsto \delta(x, 1_S)$

f and g are group homomorphisms as δ is. Conversely if $f : S \longrightarrow S$ and $g : X \longrightarrow S$ are group homomorphisms then

$\delta : X \times S \longrightarrow S$ is certainly one.
$(x, s) \longmapsto f(s)\, g(x)$

Let's consider our group machine as the decomposable system $(S, X, Y, f, g, h = \lambda)$, i.e. the machine $(1_{Gr}, S, X, Y, f, g, h)$ in Gr.

$z : X^+ \longrightarrow X^+$
$(x_{i_0}, i_0) \dots (x_{i_n}, i_n) \longmapsto (x_{i_0}, i_0 + 1) \dots (x_{i_n}, i_{n+1})$
is such that $\forall n \in \mathbb{N}$, $\forall x \in X$

$z \cdot in_n(x) = z(x, n) = (x, n+1) = in_{n+1}(x)$ as required.

The reachability morphism is

$$\begin{aligned} \delta^+ : X^+ &\longrightarrow S \\ \Lambda &\longmapsto 1_S \\ (x_{i_0}, i_0)\dots(x_{i_n}, i_n) &\longmapsto \prod_{j=0}^{n} f^{i_j} \cdot g(x_{i_j}) \end{aligned}$$

where $\prod$ is the product in S.

NOTE. That . is always used in this paper to indicate map composition, never product in the various groups with which we are dealing. If $\square$ is the operation of G, then $g \square g'$ is simply written gg'.

As required $\forall n \in \mathbb{N}$

$$\delta^+ . in_n(x) = \delta^+(x, n) = f^n . g(x) \quad \forall x \in X$$

i.e. $\delta^+ . in_n = f^n . g$

and both following diagram commutes:

$$\begin{tikzcd} x \arrow[r, mapsto] & (x, 0) \arrow[d, mapsto] \\ X \arrow[r, "in_0"] \arrow[dr, "g"'] & X^+ \arrow[d, "\delta^+"] \\ & S \\ & g(x) = f^0 . g(x) \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} (x_{i_0}, i_0)\dots(x_{i_n}, i_n) \arrow[r, mapsto] \arrow[d, mapsto] & (x_{i_0}, i_0+1)\dots(x_{i_n}, i_n+1) \arrow[d, mapsto] \\ X^+ \arrow[r, "z"] \arrow[d, "\delta^+"'] & X^+ \arrow[d, "\delta^+"] \\ S \arrow[r, "f"] & S \\ \prod_{j=0}^{n} f^{i_j} g(x_{i_j}) \arrow[r, mapsto] & \prod_{j=0}^{n} f^{i_j+1} g(x_{i_j}) \end{tikzcd}$$

$$\begin{aligned} \text{Let } \omega : S &\longrightarrow Y^X \\ s &\longmapsto (h(s), \dots, h.f^n(s), \dots) \end{aligned}$$

We have again as required:

$$p_n \cdot \omega(s) = p_n\ (h(s),\dots,h.f^n(s),\dots)$$
$$= h.f^n(s) \qquad \forall s \in S,$$

i.e. $p_n.\omega = h.f^n$,

$$\overline{ES} : X^+ \xrightarrow{\delta^+} S \xrightarrow{\omega} Y$$
$$(x_{i_0}, i_0)\dots(x_{i_n}, i_n) \longmapsto \prod f^{i_j} g(x_{i_j})$$
$$\longmapsto (\prod hf^{i_j} g(x\ ), \prod hf^{i_j+1} g(x_{i_j}),\dots)$$

and $ES = p_0 . \overline{ES} = \prod_{j=1}^{n} hf^{i_j} g = \lambda . \delta^+$ as

$\lambda = h$ is a group homomorphism.

<u>NOTE 14.1</u> Recall that, in classical automata theory, δ^+ is defined inductively by

$$\begin{cases} \delta^+(\Lambda) = 1_S \\ \delta^+(x,w) = \delta(x, \delta^+(w)) \quad \forall x \in X, \forall w \in X^* \end{cases}$$

We show by induction the length of w that this IN yields the same result as the one we have just obtained.

$\forall\ w = x_0 \dots\dots x_n$ can be written $(x_0,0)\dots(x_n, n)$ making X^* a subset of X^+, but not a subgroup as X^* is not a group. Indeed $X^* \cong (X^+)^* \subset X^+$ where

$$(X^+) = \{\Lambda\} \cup \{(x_{i_0}, i_0)\dots(x_{i_n}, i_n) \mid i_0 < i_1 < \dots < i_n\} \subset X^+$$

$$\theta : X^* \longrightarrow (X^+)$$ is 1 - 1 as it has an

$$\Lambda \longmapsto \Lambda$$
$$(x_0 \dots x_n) \longmapsto (x_0,0)\dots(x_n,n)$$

Inverse $\theta^{-1}: (X^+) \longrightarrow X^*$

$\Lambda \longrightarrow \Lambda$

$(x_{i_0}, i_0)\ldots(x_{i_n}, i_n) \longmapsto x_0\ldots x_j\ldots x_{i_n}$

where $\forall\ j = 0, \ldots, i_{n-1}$

$x_j = x_{i_k}$ if $j = i_k$ for some i_k

$x_j = 1_x$ otherwise

Example: $\theta^{-1}(\ (x_0,0)\ (x_2, 2)\ (x_5, 5)) = x_0\ 1_x\ x_2\ 1_x\ 1_x\ x_5 \in X^*$

For $/w/ = 0$, i.e. $w = \Lambda$ this is true by DN.

Assume it is true for $/w/ = n$; $\forall\ w \ni /w/ = n + 1$ may be written as $w = xw'$ for some $x \in X$ and $w' = x_0\ldots x_{n-1} \in X^*$

$\therefore\ \delta^+(w) = \delta(x, \delta^+(w'))$ $\because$ DN

$= \delta(x, \prod_{i=0}^{n-1} f^i.g(x_i))$

$= f(\prod f^i.g(x_i))\ \ g(x)$ as $\delta(x, s) = f(s)\ g(x)$

$= \prod_{i=0}^{n-1}(f^{i+1}.g(x_i))\ \ g(x)$ as f is a group homomorphism.

$= \prod_{i=0}^{n} f^{i'}.g(x'_{i'})$ if we put

$x = x'_0,\quad x_0 = x'_1, \ldots, x_{n-1} = x'_n.$

2. We show that δ^+ is a group homomorphism.

Case 1. Let $W_1 = (x_{j_0}, j_0)\ldots(x_{j_n}, j_n)$ and

$W_2 = (x_{k_0}, k_0)\ldots(x_{k_m}, k_m) \ni j_n \neq k_0$

$\therefore\ W_1W_2 = (x_{j_0}, j_0)\ldots(x_{j_n}, j_n)\ (x_{k_0}, k_0)\ldots(x_{k_m}, k_m)$

$= (x_{i_0}, i_0)\ldots(x_{i_n}, i_n).(x_{i_{n+1}}, i_{n+1})\ldots(x_{i_{n+m}}, i_{n+m})$

If we relabbelled $x_{j_\ell} = x_{i_\ell}$ for $\ell = 0,\ldots,n$

$x_{k_\ell} = x_{i_{n+\ell+1}}$ for $\ell = 0,\ldots,m$.

$$\therefore \quad \delta^+(W_1, W_2) = \prod_{\ell=0}^{m+n+1} f^{i_\ell}.g(x_{i_\ell})$$

$$= \prod_{\ell=0}^{n} f^{j_\ell}.g(x_{j_\ell}) \prod_{\ell=0}^{m} f^{k_\ell}.g(x_{k_\ell})$$

$$= \delta^+(W_1) \quad \delta^+(W_2) .$$

Case 2. Let $W_1 = (x, n)$, $W_2 = (x', n)$, $x' \neq x^{-1}$

$$\therefore \quad W_1W_2 = (xx', n)$$

$$\delta^+(W_1, W_2) = f^n.g(x\,x')$$

$$= f^n.\, g(x)g(x') \quad \text{as } g \text{ is a homomorphism}$$

$$= f^n\,(g(x))\; f^n(g(x')) \quad \text{as } f \text{ is a homomorphism}$$

$$= \delta^+(W_1) \quad \delta^+(W_2) .$$

Case 3, Let $W_1 = (x, n)$, $W_2 = (x^{-1}, n)$

$$\delta^+(W_1\, W_2) = \delta^+(1_S, n)$$

$$= \delta^+(\wedge) = 1_S \quad \text{as } (1_S, n) \text{ must be deleted.}$$

$$\delta^+(W_1)\; \delta^+(W_2) = (f^n.g(x)) \quad (f^n.g(x^{-1}))$$

$$= f^n\,(g(x) \quad (g(x))^{-1}) \quad \text{as } f \text{ and } g \text{ are homomorphism}$$

$$= f^n(1_S) = 1_S .$$

Again we have $\delta^+(W_1\, W_2) = \delta^+(W_1)\, \delta^+(W_2)$.

Along similar lines we can show that ω is a group homomorphism.

$\forall$ behavior $\beta : X^+ \longrightarrow Y$, i.e. β is a group homomorphism,

let $E_\beta = \{ (W_1, W_2) / W_1, W_2 \in X^+, \beta . z^n(W_1) = \beta . z^n(W_2) \ \forall n \in \mathbb{N} \}$

E_β is an equivalence relation as $=$ is and β and z^n are well defined. β and z, therefore z^n, are group homomorphisms $\Longrightarrow \beta . z^n$ is one

$\Longrightarrow E_\beta$ is a congruence.

Let $\alpha, \gamma : E_\beta \longrightarrow X^+$ be the usual projections, $S_\beta = X^+/E_\beta$, whose elements are E_β equivalence classes $[W]$, is a group with usual product $[W_1][W_2] = [W_1 W_2]$ which is well defined as E_β is a congruence.

$$\Longrightarrow \delta_\beta^+ : \begin{array}{ccc} X^+ & \longrightarrow & S_\beta \\ W & \longmapsto & [W] \end{array} \quad \text{is a group homomorphism.}$$

Besides $\delta_\beta^+ = \text{coeq}(\alpha, \gamma)$ as $\delta_\beta^+ . \alpha = \delta_\beta^+ . \gamma$ and $\forall \varphi : X^+ \longrightarrow S \ni \varphi . \alpha = \varphi . \gamma \quad \exists \psi : \begin{array}{ccc} S_\beta & \longrightarrow & S \\ [W] & \longmapsto & \varphi(W) \end{array}$

uniquely defined $\varphi = \psi . \delta_\beta^+$

$$\omega_\beta : \begin{array}{ccc} S_\beta & \longrightarrow & Y^+ \\ [W] & \longmapsto & (\beta(W), \beta . z(W), \beta . z^2(W), \ldots) \end{array}$$

is a group homomorphism as $\beta . z^n$ is one $\forall n \in \mathbb{N}$ and 1 – 1, i.e. monic by DN of S_β, ($[W] = [W'] \Longleftrightarrow \beta . z^n(W) = \beta . z^n(W')$).

Now $\beta = p_0 \cdot \omega_\beta \cdot \delta_\beta^+ = p_0 \cdot \bar{\beta}$, where $\bar{\beta} = \omega_\beta \cdot \delta_\beta^+$ is a coequalizer mono factorization.

The following diagram commute

$$
\begin{tikzcd}
w \arrow[rrr, mapsto] \arrow[dd] & & & z(w) \arrow[dd] \\
& X^+ \arrow[r, "z"] \arrow[d, "\delta_\beta^+"'] & X^+ \arrow[d, "\delta_\beta^+"] & \\
{[w]} \arrow[dd] & S \arrow[d, "\omega_\beta"'] & S \arrow[d, "\omega_\beta"] & {[z(w)]} \arrow[dd] \\
& Y^X \arrow[r, "z'"] & Y^X & \\
(\beta(w), \beta.z(w), \beta.z^2(w), \ldots) \arrow[rrr, mapsto] & & & (\beta.z(w), \beta.z^2(w), \ldots)
\end{tikzcd}
$$

where
$z' : Y^X \longrightarrow Y^X$
$(y_0, y_1, y_2, \ldots) \longmapsto (y_1, y_2, \ldots)$
is such that $p_n \cdot z' = p_{n+1}$ $\forall n \in \mathbb{N}$.

$\therefore\ \beta = \omega_\beta \cdot \delta_\beta^+$ is a Id_{Gr} dynamorphism $z \longrightarrow z'$

$\therefore\ \beta$ is an A-behavior $\because$ TH 5.

$\therefore$ M is an A-machine with $\beta = EM = \lambda \cdot \delta^+$ a group homomorphism as δ^+ and λ are.

$\therefore$ By TH 6, NEM = $(X_{EM}, X, Y, f_{EM}, g_{EM}, h_{EM})$

where $f_{EM} : X_{EM} \longrightarrow X_{EM} \ni \delta^+_{EM} \cdot z = f_{EM} \cdot \delta^+_{EM}$
$[w] \longmapsto [z(w)]$

$g_{EM} : X \longrightarrow X_{EM} \ni g_{EM} = \delta^+_{EM} \cdot in_0$
$x \longmapsto [(x, 0)]$

$h_{EM} : X_{EM} \longrightarrow Y \ni h_{EM} \cdot \delta^+_{EM} = EM$
$[w] \longmapsto EM(w)$ and $h_{EM} = p_0 \cdot \omega_{EM}$

x x x

IV

OBSERVABILITY AND MINIMALITY

We now return to the general concept of machines. In the context of set theory, observability is defined as follows:

DN 33. $M = (X, S, Y, \delta, \lambda)$ is an automaton

$\delta^* : X \times S \longrightarrow S$ is defined inductively by

$\delta^* (\Lambda, s) = s \quad \forall s \in S$

$\delta^* (wx, s) = (x, \delta^*(w, s)) \quad \forall s \in S, \forall w \in X^*$ the free monoid of words from S.

M is <u>observable</u> (or <u>reduced</u>) if

$$\omega : S \longrightarrow Y^{X^*} \quad \text{is } 1-1$$
$$s \longmapsto \lambda . \delta^* (-, s)$$

i.e. $\lambda . \delta^*(w, s) = \lambda . \delta^*(w, s') \quad \forall w \in X^* \Longleftrightarrow s = s'$

As already seen if we choose $\sigma \in S$ as initial element, M is an A-machine in $\mathcal{J}$ with

$$i = \{ . \} \quad \text{and} \quad \sigma : i \longrightarrow S$$
$$. \longmapsto \sigma(.) = \sigma$$

<u>Recall DN 19</u>. M is a minimal realization of a behavior β if M is a terminal object in the subcategory $\mathcal{M}_\beta$ of reachable realizations of β .

<u>Proposition 8.</u> $\dfrac{\text{M a reachable automaton}}{\text{M is observable} \iff \text{M is minimal}}$,

<u>Proof:</u> M reachable $\Longrightarrow \delta^+ : X^* \longrightarrow S$ is onto ($\iff$ a coequalizer in $\mathcal{S}$) $\Longrightarrow \forall s \in S \ \exists \ v \in X^* \ni s = \delta^+(v)$

$$\Longrightarrow \delta^*(w, s) = \delta^+(w\,v) \qquad (*)$$

Indeed, by induction on the length /w/ of w, we have:

$$
\begin{aligned}
/w/ = 0 \Rightarrow w = \Lambda \Rightarrow \delta^*(\Lambda, s) &= s && \because \text{DN of } \delta^* \\
&= \delta^+(v) && \because \text{above} \\
&= \delta^+(\Lambda v) && \because \text{DN of } \Lambda .
\end{aligned}
$$

Assume this result is true for n = /w/

$/w'/ = n + 1 \Rightarrow w' = xw$ for some $x \in X$

$$
\begin{aligned}
\delta^*(w', s) &= \delta^*(xw, s) && \because w' = xw \\
&= \delta(x, \delta^*(w,s)) && \because \text{DN of } \delta^* \\
&= \delta(x, \delta^+(wv)) && \because \text{induction assumption} \\
&= \delta^+(xwv) && \because \text{DN of } \delta^+ \\
&= \delta^+(w'v) && \because w' = xw .
\end{aligned}
$$

($\Leftarrow$) Let $b : S \longrightarrow S_{EM} = X^*/E_{EM}$ (recall $EM = \lambda . \delta^+$)

$$s = \delta^+(v) \longmapsto [v]$$

This map is well defined and 1 - 1 as:

M observable then by DN 33

$$\forall w \in X^*, \lambda.\delta^*(w, s) = \lambda.\delta^*(w, s') \iff s = s'$$

$$\text{i.e. } \lambda.\delta^+(w\,v) = \lambda.\delta^+(wv') \iff s = s'$$

where $s = \delta^+(v)$, $s' = \delta^+(v')$

$\therefore \quad [v] = [v'] \iff s \equiv s'$

b is onto as $\forall v \in S_{EM} \ \exists \ \delta^+(v) \in S$

$\therefore$ b is an isomorphism,

$\therefore \ (Id_{X^*}, b, Id_Y) : M \longrightarrow NEM$ is a machine isomorphism, i.e. $M \cong NEM$

$\implies$ M is terminal in $\mathcal{M}_\beta$ as NEM is,

$\implies$ M is minimal $\because$ DN 19.

$(\Longleftarrow)$ M is minimal $\iff$ M is terminal in $\mathcal{M}_\beta \because$ DN 19

$\implies \exists$ a machine isomorphism $(Id_{X^*}, b', Id_Y) : M \longrightarrow NEM$

i.e. $\ni b'. \delta^+ = \delta^+_{EM}$

and $b' : S \longrightarrow S_{EM}$ is an isomorphism. δ^+ is onto as $M \in /\mathcal{M}_\beta/ \implies$ M reachable $\implies \delta^+$ a coequalizer ($\iff$ onto in $\mathcal{S}$).

Again let $b : S \longrightarrow S_{EM}$

$$s = \delta^+(v) \longmapsto [v]$$

as already seen b is well-defined as

$$\begin{aligned} s = s' &\implies \forall w \in X^*, \ \lambda . \delta^*(w, s) = \lambda . \delta^*(w, s') \\ &\implies \lambda . \delta^+(w\,v) = \lambda . \delta^+(w\,v') \quad \because (*) \\ &\implies [v] = [v'] \\ &\implies b(s) = b(s'). \end{aligned}$$

$\therefore \forall v \in X^*, \ b'. \delta^+(v) = \delta^+_{EM}(v) = [v] = b(\delta^+(v))$

$\implies b'. \delta^+ = b. \delta^+$

$\implies b' = b$ as is onto, i.e. epi.

$$\therefore \forall s \in S, \quad s = \delta^{+}(v) \text{ for some } v \text{ as } \delta^{+} \text{ onto}$$
$$= b^{-1}([v]) \text{ as } b \text{ is an isomorphism.}$$

$$\therefore \forall w \in X^{*}, \quad \lambda.\delta^{*}(w, s) = \lambda.\delta^{*}(w, s')$$
$$\Leftrightarrow \quad \lambda.\delta^{+}(w\,v) = \lambda.\delta^{+}(w\,v') \quad \because (*)$$
$$\Leftrightarrow \quad [v] = [v']$$
$$\Leftrightarrow \quad b^{-1}([v]) = b^{-1}([v'])$$
$$\Leftrightarrow \quad s = s'$$
$$\therefore \quad M \text{ is observable } \because \text{ DN 33 .}$$

In the case of <u>decomposable systems</u> DN 29.2 gives us a definition of observability in categorical terms.

We first verify that for linear and group machines DN 29 and DN 33 are indeed equivalent.

Let $M = (X, S, Y, \delta, \lambda)$ be a linear machine. Recall from previous discussion that this is a linear system (X, S, Y, f, g, h) with $\delta(x, s) = f(s) + g(x)$ and $\lambda = h$

where $\omega : S \longrightarrow Y^{+}$

$$s \longmapsto (h(s), h.f(s), \ldots, h.f^{i}(s), \ldots).$$

We prove by induction on $/w/$ that $w = x_0 \ldots x_n$

$$\Longrightarrow \delta^{*}(w, s) = f^{n+1}(s) + \sum_{i=0}^{n} f^{i}.g(x_i)$$

if $/w/ = 1$ i.e. $w = x_0$ we have

$$\delta^{*}(x_0, s) = \delta(x_0, \delta^{*}(\Lambda, s)) \quad \because \text{D.N. of } \delta^{*}$$
$$= \delta(x_0, s) \quad \text{as } \delta^{*}(\Lambda, s) = s = f^{0}(s)$$
$$= f(s) + g(x_0) = f^{1}(s) + f^{0}.g(x_0) \text{ as required.}$$

Assume our result true for words of length n.

Let $w = x_0 \dots x_n$ i.e. $/w/ = n + 1$

$$\delta^*(w, s) = \delta(x_0, \delta^*(x_1 \dots x_n, s)) \quad \because \text{ DN of } \delta^*$$

$$= f(\delta^*(x_1 \dots x_n, s)) + g(x_0) \quad \because \delta = f + g$$

$$= f(f^n(s) + \sum_{i=1}^{n} f^i.g(x_i)) + g(x_0) \quad \because /x_1 \dots x_n/ = n$$

$$= f^{n+1}(s) + \sum_{i=0}^{n} f^i.g(x_i) \quad \because \text{ f ans g linear.}$$

Let M be observable in the sense of DN 33

$\iff [\lambda . \delta^*(w, s) = \lambda . \delta^*(w, s') \ \forall w \in X^* \iff s = s']$

$\iff [h.f^{n+1}(s) + \Sigma h.f^i.g(x_i) = h.f^{n+1}(s') + \Sigma h.f^i.g(x_i) \ \forall n \in \mathbb{N}$

and $\lambda . \delta^*(\Lambda, s) = \lambda(s) = h(s) = h(s') \iff s = s']$

$\iff [h.f^n(s) = h.f^n(s') \ \forall n \in \mathbb{N} \iff s = s']$

$\iff [\omega(s) = (h(s), h.f(s), \dots) = \omega(s') \iff s = s']$

$\iff \omega$ is 1 - 1

$\iff \omega$ is monic in $\mathcal{S}$

$\iff$ M is observable in the sense of DN 29.

For group machines, the proof is similar with $\delta^*(w, s) =$ $f^{n+1}(s) \prod_{j=0}^{n} f^{i_j}.g(x_{i_j})$ ..

The following corollary of TH 6 gives us the same result, as for general automata.

<u>Corollary</u>: $S = (f, g, h)$ is the minimal realization of its behavior ES $\Longleftrightarrow$ S is reachable and observable.

<u>Proof</u>: ($\Longleftarrow$) Assume S is reachable and observable i.e. $\overline{ES} = \omega \cdot \delta^{+} \ni \delta^{+}$ is a coequalizer and ω monic. By TH 6 putting $x_{ES} = x$, $f_{ES} = f$, $g_{ES} = g$, $h_{ES} = h$ we have $S = NES$ i.e. it is a minimal realization of ES, its behavior.

($\Longrightarrow$) S is the minimal realization of $ES = p_o \cdot \omega \cdot \delta^{+}$
$\Longrightarrow$ $\because$ DN 19 S is the terminal object in the category of reachable realizations of ES
$\Rightarrow$ δ^{+} is a coequalizer
and $\exists (Id_{u+}, b, Id_y) : S \longrightarrow NES \ni b : x \longrightarrow x_{ES}$
is an isomorphism and $\omega = \omega_{ES} \cdot b$

$\therefore \forall f, g \ni \omega . f = \omega . g$

$$\Longrightarrow \omega_{ES} \cdot b \cdot f = \omega_{ES} \cdot b \cdot g$$

$$\Longrightarrow b.f = b.g \quad \because \omega_{ES} \text{ is monic}$$

$$\Longrightarrow f = g \quad \because b \text{ is an isomorphism.}$$

$\therefore$ ω is monic and S is reachable and observable.

$\therefore$ In all cases at hand we have that:
$\forall$ <u>reachable machines M</u>, <u>minimality is equivalent to observability</u> (as defined in DN 19 and DN 33 respectively).

x x x

The following detailed study of the Arbib and Manes theory of observability for categorical machines as exposed in [9] will show that this result is more generally true.

DN 34. A functor $X : \mathcal{C} \longrightarrow \mathcal{C}$ is :

(1) An INPUT PROCESS if the forgetful functor $U : \mathrm{DYN}\ X \longrightarrow \mathcal{C}$ has a LEFT-ADJOINT (recall of DN 5).

(2) An OUTPUT PROCESS if U has a RIGHT ADJOINT.

(3) A STATE BEHAVIOR PROCESS if it is both an input and an output process.

NOTE 15. X an output process $\Longrightarrow$ by TH a 5.3

$\forall c \in \mathcal{C}\ \exists\ (\Delta_c : X(X_* c) \longrightarrow X_* c, \Lambda_c : X_* c \longrightarrow c)$

universal from U to c i.e. such that

$\forall\ (\delta' : Xc' \longrightarrow c'$, and $g : c \text{ —— } c')\ \exists$ a unique

X dynamorphism $\varphi : \delta' \longrightarrow \Delta_c \ni \Lambda_c \cdot \varphi = g$

i.e. $\ni$ the two following diagrams commute:

$$\begin{tikzcd} c & X_* c' = U\Delta_c \arrow[l, "\Lambda_c"'] \\ g & c' = U\delta' \arrow[u, "\varphi"] \arrow[ul] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X(X_* c) \arrow[r, "\Delta_c"] & X_* c \\ Xc' \arrow[u, "X\varphi"] \arrow[r, "\delta'"'] & c' \arrow[u, "\varphi"'] \end{tikzcd}$$

$X_* = U.V'$ (where $U \dashv V' : \mathrm{Dyn}\ X \rightleftarrows \mathcal{C}$)

is a functor as it is a composition of 2 functors.

If $c = s$, a state object, $g = Id_s$ then φ is called the state behavior morphism $\varphi_s : s \longrightarrow X_* s$ is the unique morphism $\ni \Lambda_s . \varphi_s = Id_s$.

DN 35. $X : \mathfrak{V} \longrightarrow \mathfrak{V}$ is an output process,
$M = (X, s, y, \delta : Xs \longrightarrow s, \lambda : s \longrightarrow y)$ has:
(1) response object $= X_* y$
(2) observability map $\omega : s \longrightarrow X_* y$ is the unique X dynamorphism $\ni \lambda = \Lambda_y . \omega$ i.e. $\ni$ the two following diagrams commute

$$
\begin{tikzcd}
y & X_* y \arrow[l, "\Lambda_y"'] \\
 & s \arrow[u, "\omega"'] \arrow[ul, "\lambda"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X(X_* y) \arrow[r, "\Delta_y"] & X_* y \\
Xs \arrow[u, "X\omega"] \arrow[r, "\delta"'] & s \arrow[u, "\omega"']
\end{tikzcd}
$$

DN 36. $X' : \mathfrak{V} \longrightarrow \mathfrak{V}$ is a state behavior process,
$M = (X, s, i, y, \delta, \sigma, \lambda)$ is a machine in $\mathfrak{V}$.
(1) The TOTAL EXTERNAL BEHAVIOR of M' is
$\overline{EM} = \omega . \delta^+ : X^* i = x^* \longrightarrow X_* y \in \text{DYN } X$.
(2) Conversely M is a realization of a given
$\bar{\beta} : x^* \longrightarrow X_* y \in \text{DYN } X$ if $\overline{EM} = \bar{\beta}$
(3) Given $\bar{\beta}$, $\beta = \Lambda_y . \bar{\beta} : x^* \longrightarrow y$ is the corresponding behavior and, conversely given a behavior $\beta : x^* \longrightarrow y$ $\exists$ a unique total behavior $\bar{\beta} \ni \Lambda_y . \bar{\beta} = \beta$ and $\bar{\beta} \in \text{Dyn } X$.
(4) M is a minimal realization of $\bar{\beta}$ if M is a reachable realization of $\bar{\beta}$ and $\forall M'$, a reachable realization of $\bar{\beta}$, $\exists$ a unique simulation $(Id_{x^*}, b, Id_y) : M' \longrightarrow M$.

We will, from now on, assume that $\mathcal{C}$ is an $(\mathcal{E},\mathcal{M})$ category (see DN a 13 and TH a 9).

LEMMA 6. M realizes $\beta \Longleftrightarrow$ M realizes $\bar{\beta}$.

Proof: $(\Longrightarrow)$ M realizes $\beta \Rightarrow \beta = EM = \lambda . \delta^+$
$= \Lambda_y . \omega . \delta^+$.

But $\bar{\beta}$ is the unique dynamorphism $\ni \beta = \Lambda_y . \bar{\beta}$

$\therefore \bar{\beta} = \omega . \delta^+ = \overline{EM}$.

$(\Longleftarrow)$ M realizes $\bar{\beta} \Longrightarrow \bar{\beta} = \omega . \delta^+$

$\therefore \beta = \Lambda_y . \bar{\beta} = \Lambda_y . \omega . \delta^+ = \lambda . \delta^+ = EM$.

LEMMA 7. M is a minimal realization of $\beta \Longleftrightarrow$ it is a minimal realization of $\bar{\beta}$.

Proof: Follows immediately from DN 19.1, DN 36.4 and lemma 6.

Starting from $\bar{\beta}$, instead of β, we now construct a minimal realization of $\bar{\beta}$, i.e. of β. As minimal realization is unique up to isomorphism, we will use for it the same symbols as for the Nerode realization of β, see TH 1, $M_\beta = (X, s_\beta, i, y, \delta_\beta, \sigma_\beta, \lambda_\beta)$. Besides TH 3 tells us that these 2 ways of obtaining the minimal realization of are basically the same (they are "naturally isomorphic").

DN 37. $\mathcal{C}$ is an $(\mathcal{E},\mathcal{M})$ category, X a state-behavior process, a machine M in $\mathcal{C}$ is reachable if $\delta^+ \in \mathcal{E}$, observable if $\omega \in \mathcal{M}$.

NOTE 16. Note that, in DN 37, the definition of reachability is slightly different of the one given in DN 6.3 where δ^+ was a coequalizer. It amounts to the same thing if $\mathcal{C}$ is a coequalizer mono category (i.e. $\exists$ a coequalizer mono factorization $\forall$ morphisms in $\mathcal{C}$) as $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category where $\mathcal{E}$ is the class of all coequalizers and then, $\mathcal{M}$ the class of all monics in $\mathcal{C}$. $\mathcal{S}$, Gr, R-Mod, among many other usual categories, have that property and so has the "suitable" category used in our study of decomposable systems.

LEMMA 8. X an input-process $\mathcal{C} \longrightarrow \mathcal{C}$
i.e. $\forall c \in \mathcal{C} \,/\, \exists\ (\delta_c : X.X^*c \longrightarrow X^*c,\ \eta_c : c \longrightarrow X^*c)$
universal from c to U the forgetful functor Dyn X $\longrightarrow \mathcal{C}$
$\Longrightarrow \eta = \{\eta_c \,/\, c \in \mathcal{C}\}$ and $d = \{\delta_c \,/\, c \in C\}$ are natural transformations.

Proof: (a) X an input-process $\Longrightarrow U$ has a left-adjoint $V \Longleftrightarrow \eta : Id_c \Longrightarrow U.V = X^* : \mathcal{C} \longrightarrow \mathcal{C}$ is natural transformation $\because$ TH a 5.4.

(b) $d : X.X^* \Longrightarrow X^* : \mathcal{C} \longrightarrow \mathcal{C}$ is a natural transformation $\because$ DN a 7.1 as $\forall c \in C$
$\exists\, \delta_c : XX^*c \longrightarrow X^*c \ni$ the diagram

$$\begin{tikzcd} XX^*c \arrow[r, "\delta_c"] \arrow[d, "XX^*f"'] & X^*c \arrow[d, "X^*f"] \\ XX^*c' \arrow[r, "\delta_{c'}"'] & X^*c' \end{tikzcd}$$

commute $\forall f : c \longrightarrow c'$ $\because$ note 2.1 and 2.2.

<u>LEMMA 9.</u> C is an $(\mathcal{E},\mathcal{M})$ category,

$f : (\delta : Xs \longrightarrow s) \longrightarrow (\delta' : Xs' \longrightarrow s') \in \mathrm{Dyn}\, X$

$\ni f = s \xrightarrow{e} \mathrm{Im} f \xrightarrow{m} s'$ is its $(\mathcal{E},\mathcal{M})$

factorization and either X or X^* preserves $\mathcal{E}$

$\Longrightarrow \exists$ a unique $\delta'' : X\,\mathrm{Im}\,f \longrightarrow \mathrm{Im} f \ni$

e and m an X-dynamorphisms $\delta \longrightarrow \delta''$ and

$\delta'' \longrightarrow \delta'$ respectively.

<u>Proof:</u> (a) X preserves (i.e. $e \in \mathcal{E} \Longrightarrow Xe \in \mathcal{E}$)

then our result is immediate

by the $(\mathcal{E},\mathcal{M})$ diagonalization property

as f a dynamorphism

$$\begin{tikzcd}
Xs \arrow[r, "Xe"] \arrow[d, "\delta"] & X\,\mathrm{Im}\,f \arrow[d, "Xm"] \arrow[ddl, dashed, "\delta''"] \\
s \arrow[d, "e"] & Xs' \arrow[d, "\delta'"] \\
\mathrm{Im} f \arrow[r, "m"] & s'
\end{tikzcd}$$

$\Longrightarrow f.\,\delta = \delta'.\, Xf$

i.e. $m.e.\,\delta = \delta'.\,X(m.e)$

$= \delta'.\,Xm\,.\,Xe$ $\because$ X a functor ;

(b) X^* preserves $\mathcal{E}$.

$\forall c \in \mathcal{C}$ let $\theta_c : Xc \longrightarrow X c$ be defined by

$$\theta_c = Xc \xrightarrow{X\eta_c} X(X^* c) \xrightarrow{\delta_c} X c$$

$$\eta = \{ \eta_c / c \in \mathcal{C} \} \quad , \quad d = \{ \delta_c / c \in \mathcal{C} \}$$

are natural transformations ($\because$ lemma 8)

$\Longrightarrow \theta = \{ \theta_c / \ c \in \mathcal{C} \}$ is a natural trans-formation (1).

Recall (note 3.2) that $\delta^* : X^* s \longrightarrow s$ is the unique dynamorphic extension of Id_s, i.e. $\delta^* . \eta_s = Id_s$. It is called the "<u>run morphism</u>".

The diagram

$$\begin{tikzcd} Xs \arrow[r, "X\eta_s"] \arrow[dr, "X Id_s = Id_{Xs}"'] \arrow[r, phantom, shift right=3, "1"] & X(X^*s) \arrow[r, "\delta_s"] \arrow[d, "X\delta^*"] \arrow[dr, phantom, "2"] & X^*s \arrow[d, "\delta^*"] \\ & Xs \arrow[r, "\delta"'] & s \end{tikzcd}$$

commutes

as triangle 1 and square 2 commute by DN of δ^*.

$\therefore$ $\underline{\delta = \delta . Id_{Xs} = \delta^* . \delta_s . X\eta_s = \delta^* . \theta_s}$ (2)

Recall that, by note 2, $X^*f = UVf$ is the unique dynamorphic extension $\forall$ f of

$\eta_{s'} . f \Longrightarrow X^*f . \eta_s = \eta_{s'} . f$

$f = f . Id_s = Id_{s'} . f$

$f = f . \delta^* . \eta_s = \delta'^* . \eta_{s'} . f = \delta'^* . X^*f . \eta_s$

$\therefore$ $f . \delta^* = \delta'^* . X^*f$

by uniqueness in

$$\begin{tikzcd} s \arrow[r, "\eta_s"] \arrow[dr, "f"'] & X^*s \arrow[d, shift right, "f.\delta^*"'] \arrow[d, shift left, "\delta'^*.X^*f"] \\ & s' \end{tikzcd}$$

$\Longrightarrow$ m.e. $\delta^* = \delta'^* . X^*m . X^*e$ $\because$ $f = m . e$

$\Longrightarrow \exists$ k : $X^*Imf \longrightarrow Imf$ $\ni$ the diagram

$$\begin{tikzcd} X^*s \arrow[r, "X^*e"] \arrow[d, "\delta^*"'] & X^*Imf \arrow[d, "X^*m"] \arrow[ddl, dashed, "k"] \\ s \arrow[d, "e"'] & Xs' \arrow[d, "\delta'^*"] \\ Imf \arrow[r, "m"'] & s' \end{tikzcd}$$

commutes (3)

$\because$ $(\mathcal{E}, \mathcal{M})$ diagonalization property as $X^*e \in \mathcal{E}$.

In the following diagram

(1) $\Longrightarrow$ the 2 parts (1) commute

(2) $\Longrightarrow$ the 2 parts (2) commute

(3) $\Longrightarrow$ part (3) commutes

$\therefore$ the whole diagram commutes .

and the desired dynamics is $\delta'' = k \cdot \theta_{Imf}$.

$$\begin{tikzcd}
Xs \arrow[rrrr, "Xe"] \arrow[dd, "\delta"'] \arrow[dr, "\theta_s"] & & \circled{1} & & X\,Imf \arrow[dl, "\theta_{Imf}"'] \arrow[ddd, "Xm"] \\
\circled{2} & X^*_s \arrow[dl, "\delta^*"] \arrow[rr, "X^*e"'] & & X^*Imf \arrow[ddd, "X^*m"] \arrow[dddlll, dashed, "k"] & \circled{1} \\
s \arrow[ddd, "e"'] & & & & \\
& & \circled{3} & & Xs' \arrow[dl, "\theta_{s'}"'] \arrow[dd, "\delta'"] \\
& & & X^*s' \arrow[dr, "\delta'^*"'] & \circled{2} \\
Imf \arrow[rrrr, "m"'] & & & & s'
\end{tikzcd}$$

δ'' is unique as if $\exists$ δ''' with the same properties

we have $m. \delta'' = \delta' .Xm = m. \delta'''$

$\Longrightarrow$ $\delta'' = \delta'''$ as m is monic.

$\delta''.Xe = e . \delta$ $\Longrightarrow$ e is a dynamorphism $\delta \longrightarrow \delta''$

$\delta'.Xm = m. \delta''$ $\Longrightarrow$ m is a dynamorphism $\delta'' \longrightarrow \delta'$

<u>LEMMA 10.</u> $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category,

$e : (\delta : Xs \longrightarrow s) \longrightarrow (\delta' : Xs' \longrightarrow s') \in DYN\,\mathcal{X}$ and

$\delta'' : Xs'' \longrightarrow s'' \in /Dyn\ X/$,

$f : s' \longrightarrow s'' \in \mathcal{C} \ni f.e$ is a dynamorphism $\delta \longrightarrow \delta''$, Either X or X^* preserves $\mathcal{E}$

$\Longrightarrow$ f is a dynamorphism $\delta' \longrightarrow \delta''$.

Proof: (a) X preserves $\mathcal{E}$, consider

$$\begin{tikzcd}
Xs \arrow[r, "Xe"] \arrow[d, "\delta"'] \arrow[dr, phantom, "1"] & Xs' \arrow[r, "Xf"] \arrow[d, "\delta'"] & Xs'' \arrow[d, "\delta''"] \\
s \arrow[r, "e"'] & s' \arrow[r, "f"'] & s''
\end{tikzcd}$$

The perimeter and square (1) commute as e and $f.e \in \mathrm{Dyn}\, X$

$\Longrightarrow$ $f.\delta'.Xe = \delta''.Xf.Xe$

$\Longrightarrow$ $f.\delta' = \delta''.Xf$ as $Xe \in \mathcal{E}$ is epi

$\therefore$ $f \in \mathrm{Dyn}\, X$

(b) X^* preserves $\mathcal{E} \Longrightarrow X^*e \in \mathcal{E}$ (as $e \in \mathcal{E}$)

$\forall\, g : (\delta : Xs \longrightarrow s) \longrightarrow (\delta' : Xs' \longrightarrow s') \in \mathrm{Dyn}\, X$

In the following diagram

$$\begin{tikzcd}
s \arrow[rrr, "g"] \arrow[dr, "\eta_s"] \arrow[ddr, "Id_s"'] & & & s' \arrow[dl, "\eta_{s'}"'] \arrow[ddl, "Id_{s'}"] \\
& X^*s \arrow[r, "Xg"] \arrow[d, "\delta^*"] & X^*s' \arrow[d, "\delta'^*"'] & \\
& s \arrow[r, "g"'] & s' &
\end{tikzcd}$$

(1) commutes as η is a natural transformation,

(2) and (3) commutes $\because$ IN of δ^*

$\Longrightarrow$ $g = g.Id_s = Id_{s'}.g$ $\because$ IN of Id

$= g.\delta^*.\eta_s = \delta'^*.\eta_{s'}.g$ $\because$ (2) and (3) commute

$= g.\delta^*.\eta_s = \delta'^*.X^*g.\eta_s$ $\because$ (1) commutes

$\Longrightarrow g.\delta^{*} = \delta'^{*}.Xg$ by uniqueness of the dynamorphic extension of g in

$$\begin{tikzcd} s \arrow[r, "\eta_s"] \arrow[dr, "g"'] & Xs \arrow[d, shift right, "g.\delta^{*}"'] \arrow[d, shift left, "\delta'^{*}.Xg"] \\ & s' \end{tikzcd}$$

∴ The left-hand side and the perimeter commute in the diagram

$$\begin{tikzcd} X^{*}s \arrow[r, "X^{*}e"] \arrow[d, "\delta^{*}"'] & X^{*}s' \arrow[r, "X^{*}f"] \arrow[d, "\delta'^{*}"] & X^{*}s'' \arrow[d, "\delta''^{*}"] \\ s \arrow[r, "e"'] & s' \arrow[r, "f"'] & s'' \end{tikzcd}$$

as e and $f.e \in \mathrm{Dyn}\ X$

$$\therefore\ f.e.\delta^{*} = \delta''^{*}.X^{*}f.X^{*}e$$
$$f.\delta'^{*}.X^{*}e = \delta''^{*}.X^{*}f.X^{*}e$$
$$\Longrightarrow f.\delta'^{*} = \delta''^{*}.X^{*}f \quad \text{as } X^{*}e \in \mathcal{E} \text{ is epi},$$

Now in

$$\begin{tikzcd} Xs' \arrow[rrr, "Xf"] \arrow[dr, "\theta_{s'}"] \arrow[dd, "\delta'"'] & & & Xs'' \arrow[dl, "\theta_{s''}"'] \arrow[dd, "\delta''"] \\ & X^{*}s' \arrow[r, "X^{*}f"] \arrow[dl, "\delta'^{*}"] \arrow[u, phantom, "(1)"] & X^{*}s'' \arrow[dr, "\delta''^{*}"'] & \\ s' \arrow[rrr, "f"'] & & & s'' \end{tikzcd}$$

(1) commutes by (1) in proof of lemma 9,

(2) and (3) commute by (2) in proof of lemma 9,

(4) commutes by above result

$\Longrightarrow$ the perimeter commutes and

$f \in \mathrm{dyn}\ X$.

LEMMA 11. $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category, X a state-behavior process,
$\bar{\beta} : X^{\#} \longrightarrow X_X y \in$ Dyn X

M' is a reachable realization of $\bar{\beta}$, M an observable realization of $\bar{\beta}$, either X or $X^{\#}$ preserves $\mathcal{E}$
$\Longrightarrow \exists$ unique simulation $(Id_{X^{\#}}, b, Id_y) : M' \longrightarrow M$.
Proof.
Define unique b : s' $\longrightarrow$ s by $(\mathcal{E}, \mathcal{M})$ diagonalization property in

$$\begin{tikzcd} X^{\#} \arrow[r, "\delta'^{+}"] \arrow[d, "\delta^{+}"'] & s' \arrow[d, "\omega'"] \arrow[dl, dashed, "b"'] \\ s \arrow[r, "\omega"'] & X_y \end{tikzcd}$$

as $\delta'^{+} \in \mathcal{E}$ (M' reachable)
$\omega \in \mathcal{M}$ (M observable)
$\bar{\beta} = \omega . \delta^{+} = \omega' . \delta'^{+}$ (M and M' realize $\bar{\beta}$).

$\therefore$ - $\delta'^{+} \in \mathcal{E}$ and δ'^{+} is a dynamorphism $\delta_i \longrightarrow \delta'$,
- $\delta_i : XX^{\#} \longrightarrow X^{\#}$, $\delta' : Xs' \longrightarrow s'$, $\delta : Xs \longrightarrow s$ $\in$ /Dyn X/,
- b $\ni$ b. δ'^{+} is a dynamorphism $\delta_i \longrightarrow \delta$,
- X or $X^{\#}$ preserves $\mathcal{E}$

$\Longrightarrow$ b is a dynamorphism $\delta' \longrightarrow \delta$ ($\because$ lemma 10)
i.e. $(Id_{X^{\#}}, b, Id_y) : M' \longrightarrow M$ is the unique (as b is unique) simulation required.

TH 7. MINIMAL REALIZATION TH. FOR STATE BEHAVIOR PROCESSES:
$\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category,
X is a state-behavior process $\ni$ X or $X^{\#}$ preserve $\mathcal{E}$, i and $y \in /\mathcal{C}/$; given a behavior
$\beta : X^{\#} = X^{\#} i \longrightarrow y$, $\bar{\beta} : X^{\#} \longrightarrow X_y$ is its corresponding dynamorphism $\delta_i \longrightarrow \Delta_y \ni \beta = \Lambda_y . \bar{\beta}$

$\Longrightarrow$ (1) $\exists$ a minimal realization of β whose state object is $\mathrm{Im}\,\bar{\beta}$,

(2) M is a minimal realization of β
$\Longleftrightarrow$ M is reachable and observable.

Proof: (1) $\mathcal{C}$ is an $(\mathcal{E},\mathcal{M})$ category $\Longrightarrow$
$\bar{\beta}$ has a unique $(\mathcal{E},\mathcal{M})$ factorization (up to isomorphism)
say $\bar{\beta} = X^{\#} \xrightarrow{\delta_\beta} \mathrm{Im}\bar{\beta} \xrightarrow{\omega_\beta} X_{\#}y$. Let $s_\beta = \mathrm{Im}\bar{\beta}$
$\because$ lemma 9 $\exists$ a unique $\delta_\beta : Xs_\beta \longrightarrow s_\beta$
$\ni \delta_\beta^{+}$ and $\omega_\beta \in$ Dyn X.
Put $\sigma_\beta = \delta_\beta^{+}.\eta_i$ and $\lambda_\beta = \Lambda_y.\omega_\beta$
$\Longrightarrow$ M $= (X, s_\beta, i, y, \delta_\beta, \sigma_\beta, \lambda_\beta)$ is a realization of
$\bar{\beta} \Longleftrightarrow$ it is a realization of β $\because$ lemma 7.

It is minimal $\because$ lemma 11
(2) ($\Longrightarrow$) M is a minimal realization of β
$\Longrightarrow$ M $\cong M_\beta$ $\because$ DN 19.2
$\Longrightarrow$ M is reachable and observable as M_β is.
($\Longleftarrow$) M is reachable and observable
$\Longrightarrow$ $\exists$ an $(\mathcal{E},\mathcal{M})$ factorization
$\bar{M} = \bar{\beta} = \omega . \delta^{+}$ unique up to isomorphism
$\Longrightarrow$ M $\cong M_\beta$, i.e. it is minimal.

Examples: 1. Decomposable systems of chapter 3 fit the above description as "suitable categories" have coequalizer mono factorization, i.e. an $(\mathcal{E},\mathcal{M})$ categories. Put X = $\mathrm{Id}_{\mathcal{C}}$
and $\forall c \in \mathcal{C}$, $X^{\#}c = c^{+}$ the countable copower of c
$X_{\#}c = c^{x}$ the countable power of c.

$\eta_c = 1\eta_c : c \longrightarrow c^+$, $\delta_c = z \times u^+ \longrightarrow u^+$,

$\Lambda_c = R_\lambda : c^X \longrightarrow c$, $\Delta_c = z' : c^X \longrightarrow c^X$

$\Longrightarrow$ $Id_{\mathcal{C}}$ is a state-behavior process, which preserves $\mathcal{E}$, $i = u$, $\sigma = g$, $s = x$, $\delta = f$, $\lambda = h$ and the system (f, g, h) is a state-behavior machine.

2. <u>Discrete machines</u> can also be put in that context.

$\mathcal{S}$ is an $(\mathcal{E}, \mathcal{M})$ category

$X = X \times - : \mathcal{S} \longrightarrow \mathcal{S}$

$\forall S \in |\mathcal{S}|$, $X_* S = S^{X^*} = \{ f / f : X^* \longrightarrow S \text{ is a map} \}$

where X^* is the free monoïd whose words are strings of elements of X;

$$\Delta_C : X \times C^{X^*} \longrightarrow C^{X^*}$$
$$(x, r) \longmapsto r \cdot R_x : X^* \longrightarrow C, \quad w \longmapsto r(wx)$$

with $R_x : X^* \longrightarrow X^*$, $w \longmapsto wx$, $\forall x \in X$

$$\Lambda_C : C^{X^*} \longrightarrow C, \quad r \longmapsto r(\Lambda)$$

Given $\delta' : X \times C' \longrightarrow C'$, $g : C' \longrightarrow C$

$\varphi : C' \longrightarrow C^{X^*}$, $c' \longmapsto g \cdot \delta'^*(-, c')$ makes the two diagrams commute:

$$\begin{tikzcd}
g(c') = g(\delta'^*(\Lambda, c')) & & g \cdot \delta'^*(-, c') \arrow[ll, mapsto] \\
C & C^{X^*} \arrow[l, "\Lambda_C"'] & \\
& C' \arrow[u, "\varphi"] \arrow[ul, "g"] & \\
& c' \arrow[uuul, mapsto, bend left] \arrow[uuur, mapsto, bend right] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, g \cdot \delta'^*(-, c')) \arrow[rr, mapsto] & & g \cdot \delta'^*(-x, c') = g \cdot \delta'^*(-, \delta'(x, c')) \\
X \times C^{X^*} \arrow[r, "\Delta_C"] & C^{X^*} & \\
X \times C' \arrow[u, "Id_X \times \varphi"] \arrow[r, "\delta'"] & C' \arrow[u, "\varphi"'] & \\
(x, c') \arrow[uuu, mapsto, bend left] \arrow[rr, mapsto] & & \delta'(x, c') \arrow[uuu, mapsto]
\end{tikzcd}$$

Recall that if $\delta : X \times S \longrightarrow S$ is a map then

$\delta^* : X^* \times S \longrightarrow S$ is defined by

$$\begin{cases} \delta^*(\Lambda, s) = s \\ \delta^*(xw, s) = \delta(x, \delta^*(w, s)) \end{cases}$$

$\therefore$ X_X - is a state behavior process.

Recall $\omega : S \longrightarrow Y^{X^*}$ we have

$$s \longmapsto \lambda.\, \delta^*(-, s)$$

$$\overline{EM} = X^* \xrightarrow{\delta^+} S \xrightarrow{\omega} Y^{X^*}$$
$$w \longmapsto \delta^+(w) \longmapsto \lambda.\, \delta^*(-, \delta^+(w)) = \lambda.\, \delta^+(-w)$$

as $\delta^+ = \delta^*(-, \sigma)$ where $\sigma \in S$ is the initial state.

$\therefore$ $EM = \Lambda_Y \cdot \overline{EM} = \overline{EM}(\Lambda) = \lambda.\, \delta^+(-\Lambda) = \lambda.\, \delta^+$

as required.

Finally Im $\overline{EM} \cong X/E_{EM} = S_{EM}$

as $E_{EM} = \{ (w_1, w_2) \,/\, EM(ww_1) = EM(ww_2) \; \forall w \; X^*$

$$\Longleftrightarrow \lambda.\, \delta^+(ww_1) = \lambda.\, \delta^+(ww_2)$$

$$\Longleftrightarrow \overline{EM}(w_1) = \overline{EM}(w_2) \}.$$

M is reachable if δ^+ is onto i.e. $\delta^+ \in \mathcal{E}$.

M is observable if ω is 1 - 1 i.e. $\omega \in \mathcal{M}$.

As foreseen by TH 3, this way of obtaining a minimal realization is basically the same as the Nerode realization discussed in chapter 2.

* * *

V

ADJOINT MACHINES AND DUALITY

For the sake of completeness we will expose in this chapter the Arbib and Manes theory of adjoint machines and dual of a machine. For missing details and examples see [9].

1. Adjoint machines.

DN 38. $X : \mathcal{C} \longrightarrow \mathcal{C}$ is an ADJOINT PROCESS if X has a right adjoint.

Proposition 9. $\mathcal{C}$ has countable coproducts which are preserved by $X : \mathcal{C} \longrightarrow \mathcal{C}$, a functor

$\Longrightarrow$ X is an input process and

$$X^{\#} = \coprod_{n \in \mathbb{N}} X^n : \mathcal{C} \longrightarrow \mathcal{C}$$

Proof: See [9] page 319.

NOTE. $\coprod_{n \in \mathbb{N}} X^n : \mathcal{C} \longrightarrow \mathcal{C}$ is defined

on object $c \longmapsto \coprod_{n \in \mathbb{N}} X^n c$

and on morphism $f \; \begin{smallmatrix} c \\ \downarrow \\ c' \end{smallmatrix} \longmapsto \bigsqcup X^n f$ the unique morphism $\ni$ the following square commutes:

$$\begin{array}{ccc} X^n c & \xrightarrow{\;in_n\;} & \bigsqcup X^n c \\ {\scriptstyle X^n f}\downarrow & & \downarrow{\scriptstyle \exists \bigsqcup X^n f} \\ X^n c' & \xrightarrow[\;in'_n\;]{} & \bigsqcup X^n c' \end{array} \qquad \forall n \in \mathbb{N}.$$

Corollary. $\mathscr{C}$ has countable co-products, X has a right-adjoint

$\Longrightarrow$ (1) X is an input process,

(2) $X^* = \bigsqcup X^n$.

Proof: $\because$ TH a 8 and above proposition 9.

LEMMA 12. $X \dashv X^\bullet : \mathscr{C} \longrightarrow \mathscr{C}$

$\Longrightarrow \mathscr{C}(Xc, c') \cong \mathscr{C}(c, X^\bullet c')$ $\because$ TH a 5.1

$(f: Xc \longrightarrow c') \longleftrightarrow (f^\bullet: c \longrightarrow X^\bullet c')$

Then we have:

Transposition principle $f: a' \longrightarrow a$, $g: Xa \longrightarrow b$,

$h: b \longrightarrow b'$, $k = h.g.Xf : Xa' \longrightarrow b'$

$\Longrightarrow k^\bullet = X^\bullet h.g^\bullet.f : a' \longrightarrow X^\bullet b'$

Proof: See [9] page 320.

By DN a 4, $\forall$ category $\mathcal{C}$ $\exists$ the dual (opposite) category $\mathcal{C}^{op} \ni /\mathcal{C}/ = /\mathcal{C}^{op}/$ and $f^{op}: a \rightarrowtail b$ corresponds to $f: b \longrightarrow a$. $\forall$ functor $F: \mathcal{C} \longrightarrow \mathcal{C}$, F defines a functor $F^{op}: \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}$.

$$\begin{array}{lcl} \text{object} & c \longmapsto F^{op}c = Fc & \\ \text{morphism} & f^{op}: a \rightarrowtail b \longmapsto F^{op}f^{op}: Fa \rightarrowtail Fb & \text{corresponding to } Ff: Fb \longrightarrow Fa \end{array}$$

i.e. $F^{op}f^{op} = (Ff)^{op}$.

<u>TH 8.</u> $X \dashv X^{\bullet}: \mathcal{C} \longrightarrow \mathcal{C}$,
$U: \text{DYN}\,X \longrightarrow \mathcal{C}$ and $U': \text{Dyn}\,(X^{\bullet op}) \longrightarrow C^{op}$
are forgetful functors
$\Longrightarrow$ the correspondance $\delta: Xc \longrightarrow c$, $\delta^{\bullet op}: X^{\bullet}c \rightarrowtail c$
is an isomorphism of categories i.e.
$\text{DYN}^{\circ}\,X \cong \text{DYN}\,(X^{\bullet op})$ rendering commutative the diagram

$$\begin{tikzcd} (\text{DYN}\,X)^{op} \arrow[rr, phantom, "\cong"] \arrow[dr, "U^{op}"'] & & \text{DYN}\,(X^{\bullet op}) \arrow[dl, "U'"] \\ & C^{op} & \end{tikzcd}$$

<u>Proof:</u> $X \dashv X^{\bullet}: \mathcal{C} \longrightarrow \mathcal{C}$ $\quad\therefore \forall c \in \mathcal{C}$

$$\mathcal{C}^{op}(c, Xc) \cong \mathcal{C}(Xc, c) \cong \mathcal{C}(c, X^{\bullet}c) \cong \mathcal{C}^{op}(X^{\bullet}c, c)$$

$$(\delta^{op}: c \rightarrowtail Xc) \longleftrightarrow (\delta: Xc \longrightarrow c) \longleftrightarrow (\delta^{\bullet}: c \longrightarrow X^{\bullet}c) \longleftrightarrow (\delta^{\bullet op}: X^{\bullet}c \rightarrowtail c)$$

i.e. $\forall \delta^{op} \in /(\text{DYN}\,X)^{op}/ \;\exists$ unique $\delta^{\bullet op} \in /\text{DYN}(X^{\bullet op})$
and vice-versa, (1)

Consider $f : c \longrightarrow c'$, $\delta : Xc \longrightarrow c$, $\delta' : Xc' \longrightarrow c'$

By lemma 12 $\exists$ a 1 - 1 correspondence between $Id_{c'} . \delta' . Xf$ and $X' Id_{c'} . \delta' . f = Id_{Xc'} . \delta' . f$

and between $f . \delta . X\, Id_c = f . \delta . Id_{Xc}$ and $X' f . \delta' . Id_c$.

$$\begin{aligned}
\therefore\ f \in \text{DYN } X &\iff \delta' . Xf = f . \delta \qquad \because \text{DN 3 and DN 4} \\
&\iff Id_{c'} . \delta' . Xf = f . \delta . Id_{Xc} \\
&\iff Id_{Xc'} . \delta' . f = X' f . \delta' . Id_c \\
&\iff \delta' . f = X' f . \delta' \\
&\iff f^{op} . \delta'^{.op} = \delta^{.op} . (X^{.op}) f^{op} \\
&\iff f^{op} \in \text{DYN } (X')^{op} \qquad (2)
\end{aligned}$$

$\therefore$ (1) and (2) $\Rightarrow$ $(\text{DYN } X)^{op} \cong \text{DYN } (X^{.op})$

Besides $U^{op} \delta^{op} = c = U' \delta^{.op} \forall \delta \in |\text{DYN } X|$

and $U^{op} f^{op} = f^{op} = U' f^{op} \forall f \in \text{DYN } X$,

$\therefore$ the given diagram commutes as required.

<u>Corollary.</u> U has a left (right) adjoint $\iff$
U' has a right (left) adjoint,
i.e. X is an input (output) process $\iff$
X' is an output (input) process;
$\therefore$ X is a state-behavior process $\iff$ X' is .

TH 9. $\mathcal{C}$ has countable products and coproducts,
$X : \mathcal{C} \longrightarrow \mathcal{C}$ is an adjoint process
$\Longrightarrow$ X is a state-behavior process.

Proof: C has countable coproduct and X adjoint
$\Longrightarrow X \dashv X^{\bullet} : \mathcal{C} \rightleftarrows \mathcal{C}$ $\because$ DN 38,
X is an input process and $X^{*}c = \coprod_{n \in \mathbb{N}} X^n c$

$\because$ Corollary of proposition 9
$\Longleftrightarrow$ $X^{\bullet}$ is an output process with
$X^{\bullet}_{*} c = \prod_{n \in \mathbb{N}} (X^{\bullet})^n c$ $\because$ Corollary of TH 8

and duality.

Besides $\mathcal{C}$ has product $\Longrightarrow$ C^{op} has coproduct which are preserved by $X^{\bullet op}$ as $X^{\bullet op} \dashv X^{op}$ (as $X \dashv X^{\bullet}$),
$\Longrightarrow$ $X^{\bullet op}$ is an input-process
$\Longrightarrow$ $X^{\bullet op}$ is state behavior
$\Longleftrightarrow$ $X^{\bullet}$ is state behavior
$\Longleftrightarrow$ X is state behavior $\because$ corollary of TH 8

TH 10. $X : \mathcal{C} \longrightarrow \mathcal{C}$ a state-behavior process
$\Longrightarrow X^{*} \dashv X_{*} : \mathcal{C} \rightleftarrows \mathcal{C}$

Proof: X a state-behavior process
$\Longrightarrow$ $V \dashv U : \mathcal{C} \rightleftarrows \text{Dyn } X$
and $U \dashv V' : \text{Dyn } X \rightleftarrows C$, $\therefore$, by TH a 10,

We have $U.V = X^{*} \dashv X_{*} = U.V'$.

$\therefore\ \forall\ c, c' \in \mathfrak{C}$ we have

$\mathfrak{C}(X^{*}c \longrightarrow c') \cong \mathfrak{C}(c \longrightarrow X_{*}c')$ i.e. a 1 - 1

correspondence $f \longleftrightarrow f'$

or $g, \longleftrightarrow g$

Hence we have the following <u>adjointness table for state behavior machines</u>.

Concept for M	Adjoint concept for M
Run morphism	State-behavior morphism
$\delta^{*} : X^{*}s \longrightarrow s$	$\varphi_{s} : s \longrightarrow X_{*}s$ (see note 15)
Full response morphism	Observability morphism
$\lambda . \delta^{*} = \ : X^{*}s \longrightarrow y$	$\omega : s \longrightarrow X_{*}y$
Reachability morphism	Adjoint reachability morphism
$\delta^{+} : x^{*} = X^{*}i \longrightarrow s$	$\delta^{+'} : i \longrightarrow X_{*}s$
External behavior	Adjoint external behavior
$\lambda . \delta^{+} = EM : x^{*} \longrightarrow y$	$EM' : i \longrightarrow X_{*}y$

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2. <u>Dual of a machine</u>.

Recall that, by TH 8, if $X \dashv X' : \mathfrak{C} \rightleftarrows \mathfrak{C}$

$(\mathrm{Dyn}\ X)^{op} \cong \mathrm{Dyn}\ (X^{\cdot op})$

and that $X^{\cdot op} \dashv X^{op} : \mathfrak{C}^{op} \rightleftarrows \mathfrak{C}^{op}$

$\therefore$ We can define the dual of M, M^{op} by the following:

( )

DN 39. $\mathscr{C}$ is a category with countable products and co-products;
$M = (X, s, i, y, \delta, \sigma, \lambda)$ is an adjoint machine
(i.e. $X \dashv X^{\cdot}$)
Then $M^{op} = (X^{\cdot op}, s, y, i, \delta^{\cdot op}, \lambda^{op}, \sigma^{op})$

M^{op} is an adjoint machine in C^{op} with initial object y and output object i. Besides $(M^{op})^{op} = M$.

As X is an adjoint process then $X^{*}s = \coprod_{n \in \mathbb{N}} X^n s$

while $X_{*}s = \prod_{n \in \mathbb{N}} (X^{\cdot})^n s$. Thus as we pass from $\mathscr{C}$ to $\mathscr{C}^{op}$
$\coprod X^n s$ becomes $\prod X^n s = (X^{\cdot op})_{*}s$; while $\prod (X^{\cdot})^n s$
becomes $\coprod (X^{\cdot})^n s = (X^{\cdot op})^{*} s$ since we interchange products and coproducts in opposite categories.

We have the following table.

M concept in $\mathscr{C}$	M^{op} concept in $\mathscr{C}^{op}$
initial state morph.: $\sigma: i \longrightarrow s$	Output morph. $\sigma^{op}: s \rightarrowtail i$
output morph. $\lambda : s \longrightarrow y$	Initial state morph. $\lambda^{op}: y \rightarrowtail s$
Adjoint process $X : \mathscr{C} \longrightarrow \mathscr{C}$	Adjoint process $X^{\cdot op}: \mathscr{C}^{op} \rightarrowtail \mathscr{C}^{op}$
Dynamics $\delta : Xs \longrightarrow s$	Dynamics $\delta^{\cdot op} : X^{\cdot op} s \rightarrowtail s$
$X^{*}s = \coprod X^n s$	$(X^{\cdot op})_{*} s = \prod X^n s$
$\delta_s : X X^{*}s \longrightarrow X^{*}s$	$\delta_s^{\cdot op} : X^{\cdot op} (X^{\cdot op})_{*}s \rightarrowtail (X^{\cdot op})_{*} s$
$\eta_s : s \longrightarrow X^{*}s$	$\eta_s^{\cdot op}: (X^{\cdot op})_{*}s \rightarrowtail s$
$X_{*}s = \prod (X^{\cdot})^n s$	$(X^{\cdot op})^{*} s = \coprod (X^{\cdot})^n s$
$\Delta_s : X X_{*}s \longrightarrow X_{*}s$	$\Delta_s^{\cdot op} : X^{\cdot op}(X^{\cdot op})^{*}s \rightarrowtail (X^{\cdot op})^{*}$
$\Lambda_s : X_{*} s \longrightarrow s$	$\Lambda_s^{\cdot op} : s \rightarrowtail (X^{\cdot op})^{*}s$

Run morph. $\delta^*: X^*s \to s$	State-behavior morph. $\delta^{*op}: s \rightarrowtail (X^{\cdot op})_* s$
State behavior $\varphi_s: s \to X_* s$	Run map $\varphi_s^{op}: (X^{\cdot op})^* s \rightarrowtail s$
reachability m. $\delta^+: x^* \to s$	Observability m. $\delta^{+op}: s \rightarrowtail (X^{\cdot op})_* i$
Observability m. $\omega: s \to X_* y$	Reachability m. $\omega^{op}: (X^{\cdot op})^* y \rightarrowtail s$
External behavior EM : $EM: x^* \to y$	$(EM)^{op}: y \rightarrowtail (X^{\cdot op})_* i$
Adjoint ext. beh. $EM': i \to X_* y$	External behavior $E(M^{op}): (X^{\cdot op})^* y \rightarrowtail i$ $= \sigma^{op}. \omega^{op}$
Full response $\omega_0 = \lambda.\delta^*: X^*s \to y$	Adjoint reachability m. $\omega_0^{op}: y \rightarrowtail (X^{\cdot op})_* s$
Adj. reachability m. $\delta^{+\cdot}: i \to X_* s$	Full response $\delta^{+\cdot op} = \sigma^{op}.\varphi_s^{op}: (X^{\cdot op})^* s \rightarrowtail i$

Therefore we have the following principle for adjoint machines:

<u>reachability and observability are dual;</u>

<u>run and state-behavior are dual.</u>

<u>DN 40.1</u> A category $\mathfrak{C}$ is <u>self-adjoint for the process X</u> if $\exists$ an isomorphism $\psi: \mathfrak{C} \to \mathfrak{C}^{op}$ which is the identity on objects and is reflexive; i.e. $\forall f: a \to b$ it provides a bijection $f \begin{array}{c} a \\ \downarrow \\ b \end{array} \longleftrightarrow \begin{array}{c} b \\ \downarrow \\ a \end{array} f^\dagger$ $\ni$

$(Id_a)^\dagger = Id_a,\ (f.g)^\dagger = g^\dagger.f^\dagger$ and $f^{\dagger\dagger} = f$

2. In such a situation X is <u>respectful</u> if $X \dashv \Psi^{-1} \cdot X\Psi$.
We write $\Psi^{-1} \cdot X\Psi = {}'X$

3. X a respectful functor we consider as the dual of a machine M, the machine $M^{\dagger} = \Psi^{-1}(M^{op})$, back in C.

We have then the usual result:

<u>TH 11.</u> $\mathfrak{C}$ is self-adjoint for a respectful process X, $M = (X, i, s, y, \delta, \sigma, \lambda)$ a machine in $\mathfrak{C}$ $\Longrightarrow$:
<u>M reachable $\Longleftrightarrow$ δ^{+} epi $\Longleftrightarrow$ $(\delta^{+})^{\dagger}$ monic $\Longleftrightarrow$ $M^{\dagger}$ observable</u>
<u>M observable $\Longleftrightarrow$ ω monic $\Longleftrightarrow$ $\omega^{\dagger}$ epi $\Longleftrightarrow$ $M^{\dagger}$ reachable.</u>

Three of the main concepts of system theory are: reachability, observability and controllability. As we have seen, Arbib and Manes have defined the first two in categorical terms. An interesting topic of further research would be to find a categorical definition for controllability which, in set theoritical terms, consists in:

Do there exist controls steering a given system from any of its states to one particular state (usually the origin)?

This completes the systematic exposé of the Arbib and Manes theory of machines in a category, as we know it up to this day. To stress the very wide range of this theory, let us mention that besides the four examples discussed in this work, other systems such as stochastic, topological and metric automata can be put in this context (see [7] and [9]).

Therefore we are strongly inclined to make ours the conclusion of these two authors in [7] :

"Given this vigorous growth [of category theory applied to machines] , we may expect the study of machines in a category both to feed back into the study of algebraic structure per se and also to have repercussions in many phases of the computer, information and system sciences from programming language studies to control theory."

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ANNEX

DN 1. A *category* $\mathcal{C}$ is a class $/\mathcal{C}/$ of objects together with $\forall\, a, b \in /C/$, a class $\mathcal{C}(a, b)$ of morphisms $a \longrightarrow b$, and $\forall\, a, b, c \in /C/$, an operation, called composition of morphisms, $\cdot : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \longrightarrow \mathcal{C}(a, c)$

$$(f, g) \longmapsto g.f$$

such that:

1. $\forall\, a \in /\mathcal{C}/ \ \exists$ identity $Id_a \in \mathcal{C}(a, a) \ni \forall\, g : b \longrightarrow a$
$Id_a.g = g$ and $\forall\, f : a \longrightarrow c$, $f.Id_a = f$;

2. $\forall\, a, b, c, d \in /C/$, $f: a \longrightarrow b$, $g: b \longrightarrow c$,
$h : c \longrightarrow d$, $f.(g.h) = (f.g).h$

Notation: $f: a \longrightarrow b$, $a = \mathrm{dom}\, f$, $b = \mathrm{codom}\, f$
We can identify a and $Id_a : a \longrightarrow a$;
then $/\mathcal{C}/ \subseteq U_{a,b}\ \mathcal{C}(a, b) = \mathcal{C}$.

DN 2. $i \in /\mathcal{C}/$, i is *initial* if $\#\,\mathcal{C}(i, c) = 1 \quad \forall\, c\ /\mathcal{C}/$
$t \in /\mathcal{C}/$, t is *terminal* if $\#\,\mathcal{C}(c, t) = 1 \quad \forall\, c\ /\mathcal{C}/$.

DN 3.1 $m : a \longrightarrow b$ is *monic* in $\mathcal{C}$ if $\forall\, f, g : d \longrightarrow a$
$m.f = m.g \Longrightarrow f = g$ (i.e. m is left-cancelable).

2 $e : a \longrightarrow b$ is *epi* in $\mathcal{C}$ if $\forall\, f, g : b \longrightarrow c$
$f.e = g.e \Longrightarrow f = g$ (i.e. e is right-cancelable).

3 If $f.g = Id$ then f is <u>split-epi</u> (a retraction of g)
g is <u>split-monic</u> (a section of f)
$h = g.f$ is defined and indempotent.

4 $i : a \longrightarrow b$ is an <u>isomorphism</u> if $\exists\, i^{-1} : b \longrightarrow a \ni$ $i^{-1}.i = Id_a$, $i.i^{-1} = Id_b$; we say a is isomorphic to b and write $a \cong b$; i^{-1} is called the inverse of i.

<u>TH 1.1.</u> f and g are epi (split-epi, monic, split-monic, iso) $\Longrightarrow$ $f.g$ is epi (split-epi, monic, split-monic, iso respectively).

2. Split-epi $\Longrightarrow$ epi; split-monic $\Longrightarrow$ monic
f iso $\Longleftrightarrow$ f monic and split-epi.

3. $g.f$ monic $\Longrightarrow$ f monic,
$g.f$ epi $\Longrightarrow$ g epi.

<u>TH 2.1.</u> In $\mathcal{S}$, the category of sets with maps as morphisms,
f onto $\Longleftrightarrow$ split-epi $\Longleftrightarrow$ epi
f 1 - 1 $\Longleftrightarrow$ split-monic $\Longleftrightarrow$ monic
f 1 - 1 andonto $\Longleftrightarrow$ f iso.

2. In Gr (category of groups with group homomorphisms), R-Mod (category of R-Modules with linear transformation)
f onto $\Longleftrightarrow$ epi
f 1 - 1 $\Longleftrightarrow$ monic.

DN 4. The opposite (dual) category $\mathcal{C}^{op}$ of $\mathcal{C}$ is defined by:

$/\mathcal{C}^{op}/ = /\mathcal{C}/$, $\mathcal{C}^{op}(a, b) \cong \mathcal{C}(b, a)$, $f^{op}.g^{op} = (g.f)^{op}$
where $f^{op} \in \mathcal{C}^{op}(a, b)$ corresponds to $f \in \mathcal{C}(b, a)$.

We write $f^{op} : a \rightarrowtail b$ corresponding to $f : b \longrightarrow a$

DUALITY PRINCIPLE.

Let S be a statement valid in a category $\mathcal{C}$ then the corresponding statement S^{op} in $\mathcal{C}^{op}$ is called its dual statement S^{op}.

$\forall \mathcal{C}$, $\mathcal{C}$ a category $\Longrightarrow \mathcal{C}^{op}$ is a category, and
$(\mathcal{C}^{op})^{op} = \mathcal{C}$

$\therefore$ $\forall$ statement S true $\forall$ categories $\Longrightarrow S^{op}$ is also true $\forall$ categories.

Examples:

S	S^{op}
$f: a \longrightarrow b$	$f^{op}: b \rightarrowtail a$
$a = \text{dom } f$	$a = \text{cod } f^{op}$
$i = Id_a$	$i^{op} = Id_a$
$h = g.f$	$h^{op} = f^{op}.g^{op}$
f is monic	f^{op} is epi
f is split-monic	f^{op} is split-epi
f is iso	f^{op} is iso
t is a terminal object	t is an initial object

DN 5. $b \underset{g}{\overset{f}{\rightrightarrows}} c$ are a pair of $\mathcal{C}$ morphisms.

1. $e : a \longrightarrow b$ is an <u>equalizer</u> of f,g,
   $e = \text{equ}\,(f,g)$, if :

   a) $f.e = g.e$
   b) $\forall\ e' : a' \longrightarrow b \ni f.e' = g.e'$ $\exists$ a unique morphism $\varphi : a' \longrightarrow a \ni e' = e.\varphi$.

2. $k : c \longrightarrow d$ is a coequalizer of f,g,
   $k = \text{coeq}\,(f,g)$, if:
   a) $k.f = k.g$
   b) $\forall\ k' : c \longrightarrow d' \ni k'.f = k'.g$ $\exists$ a unique morphism $\psi : d' \longrightarrow d \ni k' = \psi.k$.

   i.e.
$$\begin{tikzcd}
a \arrow[r, "e"] & b \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c \arrow[r, "k"] \arrow[dr, "\forall k'"] & d \\
a' \arrow[u, "\exists\varphi"] \arrow[ur, "\forall e'"'] & & & d' \arrow[u, "\exists\psi"']
\end{tikzcd}$$
   commutes .

   Equ Coeq

TH 3.
1. e an equalizer $\Longrightarrow$ e monic.
2. k a coequalizer $\Longrightarrow$ k epi.
3. e and e' are equ. of $f,g \Longrightarrow a \cong a'$,
   k and k' are coeq. of $f,g \Longrightarrow d \cong d'$,
   i.e. an equ (a coeq) is unique up to isomorphism .
4. $\mathcal{S}$, Gr, R-Mod have equalizers (coequalizers) i.e.
   $\forall$ pair of morphisms with common dom and codom has an equ. (a coeq.).
5. In $\mathcal{S}$, Gr, R-Mod, f onto $\Longleftrightarrow$ f a coequalizer.

LEMMA 1. ([4] p. 723) Coequalizer-monofactorization (i.e. $f = m.k$, k a coeq., m monic) are unique to isomorphism.

DN 6. 1. A functor $F : \mathcal{B} \longrightarrow \mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are categories, is a function:

on objects $b \longmapsto Fb \quad \forall b \in |\mathcal{B}|$

on morphisms $\begin{matrix} b \\ f\downarrow \\ b' \end{matrix} \longmapsto \begin{matrix} Fb \\ Ff\downarrow \\ Fb' \end{matrix} \quad \forall f \in \mathcal{B}$.

$\ni F(f.g) = Ff.\,Fg$ and $FId_b = Id_{Fb}$,

2. A functor $F : \mathcal{B} \longrightarrow \mathcal{C}$ is faithful if F restricted to $\mathcal{B}(b, b')$ is 1 - 1 $\forall b, b' \in |\mathcal{B}|$.
A functor $F : \mathcal{B} \longrightarrow \mathcal{C}$ is full if $\forall b, b' \in |\mathcal{B}|$ $F(\mathcal{B}(b, b')) = \mathcal{C}(Fb, Fb')$ i.e. $F/_{\mathcal{B}(b,b')}$ is onto.

3. Composition of functors:
$F : \mathcal{B} \longrightarrow \mathcal{C}$, $G : \mathcal{A} \longrightarrow \mathcal{B}$ are functors
$\Longrightarrow F.G : \mathcal{A} \longrightarrow \mathcal{C}$ is a functor $\begin{cases} F.G(a) = F(G(a)) \\ F.G(f) = F(G(f)). \end{cases}$

4. The identity functor $Id_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ is defined by $Id_{\mathcal{C}}\ c = c, \forall c \in |\mathcal{C}|$
$Id_{\mathcal{C}}\ f = f, \forall f \in \mathcal{C}$.

5. The metacategory CAT is the category whose objects are categories and morphisms are functors.

6. A functor $F : \mathcal{B} \longrightarrow \mathcal{C}$ is an isomorphism if either of the following conditions holds:

(1) $F : b \longrightarrow Fb$ is a bijection on objects

$F : f \longrightarrow Ff$ is a bijection on morphism

(2) $\exists$ a functor $F^{-1} : \mathcal{C} \longrightarrow \mathcal{B} \ni$ the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "Id_{\mathcal{B}}"] \arrow[dr, "F"'] & & \mathcal{B} \\ & \mathcal{C} \arrow[ur, "F^{-1}"'] & \end{tikzcd}$$

commutes

We write $\mathcal{B} \cong \mathcal{C}$.

DN 7. 1. A natural transformation $\eta : F \Longrightarrow G : \mathcal{B} \longrightarrow \mathcal{C}$ where $F, G : \mathcal{B} \longrightarrow \mathcal{C}$ are functors, is a collection $\eta = \{\eta_b \in \mathcal{C} / b \in /\mathcal{B}/, \eta_b : Fb \longrightarrow Gb\} \ni$ $\forall f : b \longrightarrow b'$ in $\mathcal{B}$, $Gf.\eta_b = \eta_{b'}.Ff$.

2. A natural equivalence (or natural isomorphism) is a natural transformation $\eta \ni \forall \eta_b, \eta_b$ is an isomorphism.

3. The composite transformation $\eta.\varepsilon : F \Longrightarrow H : \mathcal{B} \longrightarrow \mathcal{C}$, where $\eta : G \Longrightarrow H : \mathcal{B} \longrightarrow \mathcal{C}$ and $\varepsilon : F \Longrightarrow G : \mathcal{B} \longrightarrow \mathcal{C}$ are natural transformations, is defined by:

$$\eta.\varepsilon = \{\eta_b.\varepsilon_b / b \in /\mathcal{B}/, \eta_b.\varepsilon_b : Fb \longrightarrow Hb\}.$$

4. Functors: $\mathcal{B} \longrightarrow \mathcal{C}$ and corresponding natural transformations from a category called the <u>functor category $\mathcal{C}^{\mathcal{B}}$</u> with identity

$$Id_F : F \Longrightarrow F : \mathcal{B} \longrightarrow \mathcal{C} \ni (Id_F)_b = Id_{Fb} : Fb \longrightarrow Fb .$$

5. $\eta : G \Longrightarrow H : \mathcal{B} \longrightarrow \mathcal{C}$ a natural transformation

a) $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor

$F\eta : F.G \Longrightarrow F.H : \mathcal{B} \longrightarrow \mathcal{D}$ is defined by

$$(F\eta)_b = F\eta_b : F.Gb \longrightarrow F.Hb$$

b) $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor

$\eta F : G.F \Longrightarrow H.F : \mathcal{A} \longrightarrow \mathcal{C}$ is defined by

$$(\eta F)_a = \eta_{Fa} : G.Fa \longrightarrow G.Ha .$$

DN 8. $F : \mathcal{B} \longrightarrow \mathcal{C}$ is <u>LEFT-ADJOINT of</u> $G : \mathcal{C} \longrightarrow \mathcal{B}$ (F and G two functors), or G is <u>right-adjoint</u> of F if $\exists$ a natural isomorphism

$$\varphi : \mathcal{C}(F_*, .) \Longleftrightarrow \mathcal{B}(_*, G.) : \mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{S}$$

<u>Notation</u>: $F \dashv G : \mathcal{B} \rightleftarrows \mathcal{C}$

TH 4. Any two left-adjoints (or right-adjoints) of a functor F are naturally isomorphic.

TH 5. $F \dashv G : \mathcal{B} \rightleftarrows \mathcal{C}$ is equivalent to each of the following:

1. $\forall\, b \in /\mathcal{B}/,\ c \in /C/,\ \mathcal{B}(b, Gc) \cong \mathcal{C}(Fb, c)$.

2. $\forall\, b \in /\mathcal{B}/,\ \exists\, (c, \eta_b : b \longrightarrow Gc)$ <u>universal from b to G</u> (or <u>free over b w. r. to G</u>)

   i.e. such that $\forall\, (c' \in /C/,\ f : b \longrightarrow Gc')$

   $\exists$ unique $\varphi : c \longrightarrow c'$ $\ni$

$$\begin{tikzcd} b \arrow[r, "\eta_b"] \arrow[dr, "f"'] & Gc \arrow[d, "G\varphi"] \\ & Gc' \end{tikzcd} \quad \text{commutes}$$

In that case, <u>given G</u>, its left-adjoint F is defined by the following:

Let $(c, \eta_b : b \longrightarrow Gc)$ be universal from b to G,

$(c', \eta_{b'} : b' \longrightarrow Gc')$ be universal from b' to G

then F: <u>objects</u> $b \longmapsto Fb = c$

$\forall$ <u>morphisms</u>

$$\begin{tikzcd} b \arrow[d, "f"'] & c = Fb \arrow[d, "\text{unique } \varphi = Ff"] \\ b' \arrow[r, phantom, "\longmapsto"] & c' = Fb' \end{tikzcd} \ \ni$$

$$\begin{tikzcd} b \arrow[r, "\eta_b"] \arrow[d, "f"'] \arrow[dr, "\eta_{b'} \cdot f"] & G.Fb = Gc \arrow[d, "G\varphi"] \\ b' \arrow[r, "\eta_{b'}"'] & G.Fb' = Gc' \end{tikzcd} \quad \text{commutes}$$

i.e. $G\varphi \cdot \eta_b = \eta_{b'} \cdot f$

or written in terminology of (i);

$$\mathcal{B}(b,\ G(Fb' = c')) \cong \mathcal{C}(Fb, Fb')$$

$$\eta_{b'} \cdot f \longleftrightarrow Ff = \varphi$$

3. Dually $\forall\, c \in /\mathcal{C}/$
$\exists\ (b \in /\mathcal{B}/,\ \varepsilon_c : Fb \longrightarrow c)$ universal from F to b
ie. such that $\forall\ (b' \in /\mathcal{B}/,\ f' : Fb' \longrightarrow c)\ \exists$ unique
$\varphi' : b' \longrightarrow b \ni$

$$\begin{tikzcd} Fb \arrow[r] & c \\ Fb' \arrow[u, "F\varphi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.

In this case, given F, its right-adjoint G is defined by the following:

Let $(b, \varepsilon_c : Fb \longrightarrow c)$ be universal from F to c
$(b', \varepsilon_{c'} : Fb' \longrightarrow c')$ be universal from F to c'

Then G : objects $c \longmapsto Gc = b$

morphisms $\begin{array}{c} c' \\ f' \downarrow \\ c \end{array} \longmapsto \begin{array}{c} b' = Gc' \\ \downarrow \\ b = Gc \end{array}$ unique $\varphi' = Gf' \ni$

$$\begin{tikzcd} Fb = F.Gc \arrow[r, "\varepsilon_c"] & c \\ Fb' = F.Gc' \arrow[u, "F\varphi'"] \arrow[r, "\varepsilon_{c'}"'] \arrow[ur, dashed, "f'.\varepsilon_{c'}"] & c' \arrow[u, "f'"'] \end{tikzcd}$$

commutes, i.e. $f'.\varepsilon_{c'} = \varepsilon_c.F\varphi'$

or, in the terminology of (1)
$\mathcal{C}(F(Gc' = b'), c) \cong \mathcal{B}(Gc', Gc)$
$f'.\varepsilon_{c'} \longleftrightarrow \varphi' = Gf'$

4. $\exists$ natural transformations $\eta : Id_{\mathcal{B}} \Longrightarrow G.F : \mathcal{B} \longrightarrow \mathcal{B}$ and $\varepsilon : F.G \Longrightarrow Id_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$, called the unit and co-unit of the adjunction, such that:

$$\begin{tikzcd} & F.G.F \arrow[dr, Rightarrow, "\varepsilon F"] & \\ F = F.Id_{\mathcal{B}} \arrow[ur, Rightarrow, "F\eta"] \arrow[rr, Rightarrow, "Id_F"'] & & Id_{\mathcal{C}}.F = F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G.F.G. \arrow[dr, Rightarrow, "G\varepsilon"] & \\ Id_{\mathcal{B}}.G = G \arrow[ur, Rightarrow, "\eta G"] \arrow[rr, Rightarrow, "Id_G"'] & & G = G.Id_{\mathcal{C}} \end{tikzcd} \quad \text{commute .}$$

In that case, $\forall b \in /\mathcal{B}/$, $(Fb, \eta_b : b \longrightarrow G.Fb)$ is universal from b to G ,

$\forall c \in /C/$, $(Gc, \mathcal{E}_c : F.Gc \longrightarrow c)$ is universal from F to c.

<u>DN 9.</u>

1. $\mathcal{B}$ is a <u>subcategory of $\mathcal{C}$</u> if $/\mathcal{B}/ \subseteq /C/$ and they have same composition of morphisms.

2. $\mathcal{B} \subseteq \mathcal{C}$ is <u>full</u> if $\forall$ b, b' $\in /\mathcal{B}/$, $\mathcal{B}(b, b') \cong \mathcal{C}(b, b')$, i.e. $\mathcal{B}$ is completly described by its objects.

<u>DN 10.</u>

1. Categories $\mathcal{B}$ and $\mathcal{C}$ are <u>equivalent</u> if $\exists$ a functor $F: \mathcal{B} \longrightarrow \mathcal{C}$ which is full and faithfull (i.e. $\mathcal{B}(b, b') = \mathcal{C}(Fb, Fb')$ $\forall$ $b, b' \in \mathcal{B}$) and $\forall$ $c \in /C/$, $c \cong Fb$ for some $b \in /\mathcal{B}/$).

2. $\mathcal{B}$ is an equivalent subcategory of $\mathcal{C}$ if the inclusion functor satisfies this condition (i.e. $\forall c \in /\mathcal{B}/$, $\exists$ $\mathcal{B}(c, d') \cong \mathcal{C}(c, c')$)

TH 6. $\mathcal{B}$ and $\mathcal{C}$ are equivalent
$\Longleftrightarrow$ $\exists$ functors $\mathcal{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$ $\ni$ F.G and G.F are both naturally isomorphic to $Id_{\mathcal{C}}$ and $Id_{\mathcal{B}}$ respectively or $\Longleftrightarrow$ $\exists$ adjunction $F \dashv G : \mathcal{B} \longrightarrow \mathcal{C}$ whose unit η and co-unit ε are both natural isomorphisms.

DN 11. $\mathcal{B} \subseteq \mathcal{C}$ is <u>reflective</u> in $\mathcal{C}$ if the inclusion functor has a left-adjoint called the reflector R.

TH 7. $\mathcal{B} \subseteq \mathcal{C}$ is reflective $\Longleftrightarrow$ $\forall$ $c \in /\mathcal{C}/$
$\exists\, Rc \in /\mathcal{B}/$ and $\eta_c : c \longrightarrow Rc$ in $\mathcal{C}$ $\ni$ $\forall$ $g : c \longrightarrow b$
$\exists$ a unique $f : Rc \longrightarrow b$ $\ni$ $g = f.\eta_c$

LEMMA 2. ([1] , page 373)

$$\left.\begin{array}{l} \mathcal{A} \text{ an equivalent subcategory of } \mathcal{B} \\ \mathcal{B} \subseteq \mathcal{C} \\ \mathcal{A} \text{ a reflective subcategory of } \mathcal{C} \end{array}\right\} \Rightarrow \mathcal{B} \text{ is a reflective subcategory of } \mathcal{C}.$$

LEMMA 3. $F \dashv G : \mathcal{B} \rightleftarrows \mathcal{C}$
$\mathcal{B}_0$ and $\mathcal{C}_0$ are full subcategories of $\mathcal{B}$ and $\mathcal{C}$ respectively
$\ni$: $F/_{\mathcal{B}_0}$ factors through $\mathcal{C}_0$
$G/_{\mathcal{C}_0}$ factors through $\mathcal{B}_0$
yielding $F_0 : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$ and $G_0 : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$
respectively $\Longrightarrow$ $F_0 \dashv G_0 : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$.

<u>DN 12.</u> $\{c_\alpha \mid \alpha \in I\}$. is a family of objects in a category $\mathcal{C}$.

1. A <u>product</u> of $\{c_\alpha\}$ is an object $c \in /\mathcal{C}/$ together with a family $\{p_\alpha : c \longrightarrow c_\alpha\}$ of $\mathcal{C}$ morphisms $\ni \forall$ families $\{f_\alpha : x \longrightarrow c_\alpha\}$ of $\mathcal{C}$-morphisms, $x \in /C/$, $\exists$ unique $\mathcal{C}$-morphism $f : x \longrightarrow c \ni$

$$\begin{tikzcd} c_\alpha & c \arrow[l, "p_\alpha"'] \\ & x \arrow[u, "f"'] \arrow[ul, "f_\alpha"] \end{tikzcd}$$

commutes ,

we write $c = \prod_{\alpha \in I} c_\alpha$

2. A <u>co-product</u> of $\{c_\alpha\}$ is $c' \in /C/$ with a family $\{in_\alpha : c_\alpha \longrightarrow c'\} \ni \forall$ families $\{g_\alpha : c_\alpha \longrightarrow x\}$ $\exists$ a unique $\mathcal{C}$-morphism $g : c' \longrightarrow x \ni$

$$\begin{tikzcd} c_\alpha \arrow[r, "in_\alpha"] \arrow[dr, "g_\alpha"'] & c' \arrow[d, "g"] \\ & x \end{tikzcd}$$

commutes

we write $c' = \coprod_{\alpha \in I} c_\alpha$

<u>NOTE:</u> $\prod c_\alpha$ and $\coprod c_\alpha$ are unique up to isomorphism.

DN 13. 1. $\forall n \in \mathbb{N}$, $\exists$ $u_n = u \in /\mathcal{C}/$

the countable copower of u is $u^+ = \coprod_{n \in \mathbb{N}} u_n$

$\Longrightarrow \exists$ a unique $\mathcal{C}$-morphism

$z : u^+ \longrightarrow u^+ \ni z.\,in_n = in_{n+1}$

i.e. $\ni$

$$\begin{tikzcd} u \arrow[r, "in_n"] \arrow[dr, "in_{n+1}"'] & u^+ \arrow[d, "z"] \\ & u^+ \end{tikzcd}$$

commutes $\forall n \in \mathbb{N}$.

2. $\forall n \in \mathbb{N}$, $\exists$ $y_n = y \in /\mathcal{C}/$

the countable power of y is $y^x = \prod_{n \in \mathbb{N}} y_n$

$\Longrightarrow \exists$ a unique morphism $z' : y^x \longrightarrow y^x \ni p_n.z' = p_{n+1}$

i.e. $\ni$

$$\begin{tikzcd} y & y^x \arrow[l, "p_n"'] \\ & y^x \arrow[u, "z'"'] \arrow[ul, "p_{n+1}"] \end{tikzcd}$$

commutes $\forall n \in \mathbb{N}$.

LEMMA 4. 1. $\exists$ u^+ the countable copower of $u \in /\mathcal{C}/$
([4] p. 708, 720)

$\Longrightarrow \forall$ pairs $(g : u \longrightarrow x,\ f : x \longrightarrow x)$ of $\mathcal{C}$-morphisms

$\exists$ a unique $\mathcal{C}$-morphism $\delta^+ : u^+ \longrightarrow x$ $\ni$

$$\begin{tikzcd} u \arrow[r, "in_o"] \arrow[dr, "g"'] & u^+ \arrow[d, "\delta^+"] \\ & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} u^+ \arrow[r, "z"] \arrow[d, "\delta^+"'] & u^+ \arrow[d, "\delta^+"] \\ x \arrow[r, "f"'] & x \end{tikzcd}$$

commute.

δ^+ is uniquely defined by $\delta^+.in_n = f^n.g$ $\forall n \in \mathbb{N}$,

i.e. $\ni \forall n \in \mathbb{N}$

$$\begin{tikzcd} u \arrow[r, "in_n"] \arrow[dr, "f^n.g"'] & u^+ \arrow[d, "\delta^+"] \\ & x \end{tikzcd}$$

commutes.

2. $\exists\, y^x$ the countable power of $y \in /C/$
$\Longrightarrow \forall$ pairs $(h : x \longrightarrow y,\ f : x \longrightarrow x)$ of $\mathcal{C}$-morphisms
$\exists$ a unique $\mathcal{C}$-morphism $\omega : x \longrightarrow y^x$ $\ni$

$$\begin{tikzcd} y & y^x \arrow[l, "p_0"'] \\ & x \arrow[u, "\omega"'] \arrow[ul, "h"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y^x & y^x \arrow[l, "z'"'] \\ x' \arrow[u, "\omega"] & x \arrow[l, "f"] \arrow[u, "\omega"'] \end{tikzcd} \quad \text{commute.}$$

ω is uniquely defined by
$p_n \cdot \omega = h.f^n$ i.e. $\ni$

$$\begin{tikzcd} y & y^+ \arrow[l, "p_n"'] \\ & x \arrow[u, "\omega"'] \arrow[ul, "h.f^n"] \end{tikzcd} \quad \text{commutes } \forall n \in \mathbb{N}.$$

<u>TH 8.</u> A functor with a right-adjoint preserves all co-products.

<u>DN 13.</u> 1. $\mathcal{E}$ is a class of epis in category $\mathcal{C}$, $\mathcal{M}$ a class of monics in $\mathcal{C}$, both closed under composition and $\forall\, i$ isomorphism in $\mathcal{C}$, $i \in \mathcal{E}$ and $i \in \mathcal{M}$.

2. $f : a \longrightarrow b$ has an $(\mathcal{E}, \mathcal{M})$ factorisation if $f = a \xrightarrow{e} \mathrm{Im}f \xrightarrow{m} b$, where $e \in \mathcal{E}$, $m \in \mathcal{M}$.
This factorization is said to be <u>unique</u> if we also have $f = a \xrightarrow{e'} (\mathrm{Im}f)' \xrightarrow{m'} b$, then $\exists$ an isomorphism $h : (\mathrm{Im}f)' \longrightarrow \mathrm{Im}f$ $\ni$
the diagram

$$\begin{tikzcd} & (\mathrm{Im}f)' \arrow[dr, "m'"] \arrow[dd, dashed, "h"] & \\ a \arrow[ur, "e'"] \arrow[dr, "e"'] & & b \\ & \mathrm{Im}f \arrow[ur, "m"'] & \end{tikzcd} \quad \text{commutes.}$$

3. $\mathcal{C}$ is called an <u>$(\mathcal{E}, \mathcal{M})$ category</u> if it is uniquely $(\mathcal{E}, \mathcal{M})$ factorizable, i.e. $\forall f \in \mathcal{C}$, $\exists$ a unique $(\mathcal{E}, \mathcal{M})$ factorization.

4. $\mathcal{C}$ is said to have the $(\mathcal{E}, \mathcal{M})$ diagonalization property provided that $\forall$ commutative square in $\mathcal{C}$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$,

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[d, "f"'] & b \arrow[d, "g"] \arrow[dl, dashed, "k"'] \\ a' \arrow[r, "m"'] & b' \end{tikzcd}$$

$\exists$ k $\ni$ the diagram commutes (k is unique as e is epi, or m is monic).

TH 9. $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category $\Longleftrightarrow$ $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$ factorizable and has the $(\mathcal{E}, \mathcal{M})$ diagonalization property.

NOTE: $\mathcal{S}$, R-Mod and Gr are $(\mathcal{E}, \mathcal{M})$ categories with $\mathcal{E} = \{ e \mid e \text{ is onto} \}$, $\mathcal{M} = \{ m \mid m \text{ is } 1-1 \}$, with e, m maps in $\mathcal{S}$, homo-morphisms in R-Mod or Gr.

TH 10. Adjunctions can be composed; specifically

$F \dashv G : \mathcal{A} \rightleftarrows \mathcal{B}$

$S \dashv T : \mathcal{B} \rightleftarrows \mathcal{C}$

$\Rightarrow$ $S.F \dashv G.T : \mathcal{A} \rightleftarrows \mathcal{C}$.

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