Mathematical Induction
An Epistemological Study with Consequences for Teaching

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ABSTRACT

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Mathematical induction is considered to be a difficult topic for students. The ability to reason using recursion informally, which according to Poincaré is the basis for mathematical induction, is the subject of this thesis. The main method of investigation was a clinical study designed to examine students' thinking as they reason using recursion, both formally and informally.

The subjects of the study, six university students, were engaged in tasks requiring understanding and use of recursion, both informally, and formalized as mathematical induction. It was found that recursive reasoning was used informally by all subjects, but mathematical induction on a formal level was used only by three. A lack of connection was observed between informal recursive reasoning and mathematical induction as a method of formal proof. It was concluded that teaching which takes advantage of the abilities of students to reason informally, and makes a connection between these abilities and mathematical induction, would be more successful.

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INTRODUCTION

Mathematical Induction (MI) is considered to be a difficult topic for students (Sfard 1988; Ernest 1984; Dubinsky 1986). While this difficulty is due in part to the difficulties associated with proof and algebra, an inability on the part of students to connect a formal presentation of MI with already existing understandings might contribute as well. Students who have an ability to reason using recursive argument on an informal level might not associate MI with such arguments. There is little point in introducing MI in a way which does not develop a useful understanding of MI in the learner, and such a presentation is all too easy for topics related to proof (Schoenfeld 1985). If we wish for students to use MI to validate conjectures for themselves rather than as a procedure used when required on examinations, then care must be taken to present MI in a way which allows the learner to integrate MI with his/her ability to reason recursively, producing a way of understanding which is as close to the mathematical meaning of MI as possible.

The aim of this thesis is to explore the different ways of understanding MI which students develop, in order to clarify the nature of the difficulties associated with them. The primary research question is: What different understandings of MI might students develop? In addition several more specific questions will be investigated: How does the understanding of MI in one way affect the development of other understandings? How are different understandings of MI related to recursive reasoning in other contexts? Do different students use recursive reasoning in the same way? What implications for the teaching of MI do these possible understandings present?

In an effort to answer these questions two methods of inquiry were used: a historical survey of the development and significance of MI in the mathematical community, and a clinical study of the understandings demonstrated by individual students.

The historical survey provided indications of different possible ways of understanding MI, by producing examples of mathematicians who have understood MI in different ways. The ways in which recursive reasoning was used to validate statements, and the way in which its use was justified, offered possible answers to some of the research questions.

Interviews were used in the clinical study in order to allow investigation of aspects of the students' ways of understanding which are not visible from written work. While a student may be able to identify, follow, and compose proofs using MI, his/her way of understanding may be limited to producing the required behaviors. In Vygotsky's (1962) terminology the student may possess a **complex** related to MI rather than a **concept** of MI.

Two related investigations were made, each composed of two stages. In the first investigation four students enrolled in university mathematics courses were involved. They were grouped in pairs. The intention was to explore the ways of understanding MI possessed by these students, as they tried to use MI, and tried to communicate what they were doing to their peers. In the first stage one group learned MI in a fairly straight forward manner. They practiced using MI, attempted proofs, and were involved in a discussion comparing MI with empirical induction. As the statements proven were generally accepted by the students, MI was used as a method of explanation rather than proof. The other group was exposed to statements which on the basis of empirical examination appear true but which turn out to be false. This was intended to foster skepticism with inductive reasoning.

In the second stage a pair of students, one from each group, worked together to determine the validity of a set of statements. As the students in the second group were expected to be skeptical of the truth of the statements, they were expected to see MI as a method of proof. Techniques used by the students were observed, and further discussion of the justifications of MI occurred.

The second investigation involved two humanities students with mathematical backgrounds including algebra, but without recent experiences with academic mathematics. In this case the objective was to study the extent of students ability to construct an informal, or semi-formal, proof by MI, both without assistance, to determine their functional ability, and with assistance, to determine their zone of proximal development (Vygotsky 1962).

In the first stage one student was given several statements and asked to determine their validity. Several statements which can be easily proven using MI were included. The student was encouraged to develop a proof by MI with hints being given if needed. The second student was presented with material similar to that seen by the second group in the first investigation. The remainder of the investigation followed the procedures described above for the first investigation.

This thesis consists of four chapters. Chapter I explores the role of MI in teaching and in mathematics. Its role in teaching is explored through an examination of textbook and classroom presentations. Its role in mathematics is explored through a survey of the development and significance of MI in the history of mathematics. Chapter II describes the organization of the clinical study including a description of the students involved, the reasoning behind the questions presented to them, and the actual unfolding ("déroulement") of the study. Chapter III consists of the analysis of the information derived from the clinical study. Each student is discussed separately, and then a more global analysis is made. Chapter IV presents conclusions based on the historical survey and the clinical study, and makes some suggestions for further research.

CHAPTER I

MATHEMATICAL INDUCTION IN MATHEMATICS AND TEACHING

Most educators writing on the subject define MI as "a method of proof" and add that it involves a recursive generalization of instances from previous instances (Ernest 1982; Woodall 1975). Woodall describes four different forms that MI can take: **simple induction**, **strong induction**, **method of descent**, and **practical induction**.

Simple induction is the form in which MI is usually presented in text books. It is also the form that most closely resembles Peano's axiom. An example of simple induction is the treatment in Durbin (1988):

> **Principle of Mathematical Induction**
> For each positive integer $\mathbf{n}$ let $\mathbf{P}_n$ represent a statement depending on $\mathbf{n}$. If
> (a) $\mathbf{P}_1$ is true, and
> (b) the truth of $\mathbf{P}_k$ implies the truth of $\mathbf{P}_{k+1}$ for each positive integer $\mathbf{k}$,
> Then $\mathbf{P}_n$ is true for every positive integer $\mathbf{n}$.

This is followed with a model based on an infinite sequence of dominoes, and this instruction:

> To apply the Principle of Mathematical Induction we must verify both parts, (a) and (b). Notice that to verify (b) we must prove that **if** $\mathbf{P}_k$ is true, **then** $\mathbf{P}_{k+1}$ is true; that is, we must establish $\mathbf{P}_{k+1}$ based on the assumption of $\mathbf{P}_k$.

Strong induction is similar to simple induction but allows the assumption of $P_1, \ldots, P_n$. That is, it states: **P_1 & $[P_1, \ldots, P_n] \Rightarrow P_{n+1}$ implies P_n for all n.**

The method of descent begins with the assumption that P_n is not true, and that **n** is the smallest **n** for which P_n is not true. The proof is then by contradiction, often by finding some other $m < n$ for which P_m is not true.

Practical induction, Woodall claims, is the form most often used in practice. As he states it:

> If we can prove the truth of P_n (assuming, if necessary, the truth of $P_1, P_2, \ldots, P_{n-1}$) for each $n \geq 1$, then it will follow that P_n is true for all $n \geq 1$. (Woodall 1975, 65)

There is in Woodall's wording a possibility which could render such a technique not inductive at all. This possibility is that it will never be necessary to assume the truth of $P_1, P_2, \ldots, P_{n-1}$. In this case what we have is no longer a proof by induction. It is this necessity of assuming the truth of preceding cases which partially defines MI. The second definitive element of MI is often forgotten by students (Young 1908, 148; Ernest 1984, 182). It is that, at some point, it is necessary to demonstrate the truth the **basis:** a specific instance, or several instances, preceding all other instances. Taken together these two elements are sufficient to define MI as requiring:

> A) The recursive proof of a general instance, on the basis of preceding instances.
> B) The proof of a specific instance, or specific

instances, preceding all other instances.

The History of MI

In the history o⊥ mathematics MI has taken several different forms. The habit of the mathematical community to rework its communications into as concise a package as possible makes it impossible to know, except in a few cases, the thoughts that went into the production of a piece of mathematics; however, there are a few clues.

Early in the history of mathematics occur the paradoxes of Zeno concerning motion. Two of these, the "Achilles", and the "Dichotomy" or "Race Course", depend on a recursive sequence of events to produce an infinity. In the "Achilles", Achilles and a tortoise have a race. Because the tortoise is slower it gets a head start. Achilles can never catch the tortoise because in the time it takes Achilles to reach the point were the tortoise started, the tortoise has moved to another point. In the time it takes Achilles to reach this second point the tortoise has moved again. As the tortoise will always be a step ahead Achilles can never catch the tortoise. The argument is implicitly recursive. While this is not MI in any formal way, it is recursive argument and so could be considered informal MI.

In Euclid [Proposition 20, Book IX] there appears a proof of the infinitude of primes, or so Euclid claims. An examination of Euclid's proof reveals that it is in fact a proof that there exist four prime numbers if there exist three, followed by an assertion that this proves that there exists an infinitude of primes. How then is this a proof? Euclid was working without any notation for a general number. He possessed the idea but no way of writing it. To overcome this he used the case of three primes to stand for a general number of primes. Nothing in what he does in his proof makes use of the "threeness" of 3. It uses only the fact that 3 is a number and can be manipulated like one. Euclid's proof requires the reader to understand the recursive nature of the argument based on the prompt of the assertion which follows. This is, implicitly, a proof by MI (Ernest 1982).

The explicit development of MI had to wait for Francesco Maurolico (1494-1575). In his Opuscula Mathematica published in 1575 he presented what is often considered the first proof by MI (Gussett 1986, 145; Bussey, 1917). He writes:

> By a previous proposition [XIII] the first square number added to the following odd number makes the following square number; and this second square number added to the third odd number makes the third square number; and likewise the third square number added to the fourth odd number makes the forth square number; and so successively to infinity the proposition is demonstrated by repeated application of proposition XIII. (Bussey 1917, 203)

Note that for Maurolico a proof by MI occurs, rather than is. That is, his proofs require time. Maurolico's proof demonstrates the general truth of his assertion only by showing the equivalent of what would now be called the induction step for several specific cases. He demonstrates only what Sfard (1987) would call an operational understanding. Once more recursive reasoning is being used but the notation for a general number is lacking.

Pascal (1623-1662) had read Maurolico and, in describing the proof of a theorem involving triangular numbers he says "Cela est aisé par Maurolic" (Bussey 1917, 203). He uses MI in his proof of Corollary 12 in his book on what is now known as Pascal's Triangle. He is much more explicit in his use of MI than Maurolico was. He begins by stating two Lemmas. The first establishes the corollary in the second row of the triangle. The second Lemma establishes that if the corollary is true in any row then it will hold in the next row. He states that from these two lemmas "it will be seen that this proportion is necessary in all the rows: for it is so in the second row by the first lemma: hence by the second, it is in the third row, hence in the fourth, and so on to infinity" (Smith 1959, 72). His initial presentation is structural; the two lemmas alone establish the corollary. However, his justification demonstrates an operational understanding. It seems that the bridge from an argument "for any" and an argument "for all" is intuitive for Pascal.

Throughout the history of mathematics informal MI appears to have been used. It was the introduction of symbolic notation in the nineteenth century that allowed its formalization. With the formalist movements of the late nineteenth century MI became an axiom of arithmetic. Dedekind included it in his set of axioms published in 1888 and in 1889 Peano published his axioms of arithmetic including MI. This axiom is now seen as the justification for the use of MI in mathematical papers. No longer is it necessary to appeal to the intuitions of the readers, or to provide operational justifications in the style of Pascal. Of course, in less formal contexts intuitive and operational understandings continue to be used.

The history of mathematics provides several examples of recursive reasoning processes which could be called MI (Ernest 1982); however, as these processes differ in important respects, MI must be taken to mean different things at different times. In addition to these differing meanings, MI has also differed in its importance and role, its significance, in mathematics.

The current significance of MI in formal mathematics is as an axiom. In the development of an axiom system for arithmetic Peano included MI as one of his basic axioms of formal number theory. As such MI plays a part in the proofs of such important principles as the associativity and commutativity of addition. As an axiom MI is not a part of the reasoning used, but just another property of the natural numbers.

Historically MI has played a different role: one of the reasoning processes used in proofs. In actual mathematical practice MI continues to be seen this way. Larson writes: "Mathematical induction is the most important proof technique" in areas such as discrete mathematics (1985, 373). Clearly, he is thinking of an inferential process, not an axiom.

When Peano first proposed his axioms, his treatment of MI was opposed by Henri Poincaré (Styazhkin 1969, 280) In Science et Méthode, La Valeur de la science, and La Science et l'Hypothèse Poincaré discusses the role MI plays in mathematical reasoning, and in mathematical creativity. In so doing he gives MI a special place in mathematics. His assertion "Nous ne pouvons nous élever que par l'induction mathématique, qui seule peut nous apprendre quelque chose de nouveau." (Poincaré 1943, 28) provoked much discussion at the time (eg Young 1908, 146) and an examination of his writings on MI suggests a broader meaning for it than is usually given.

For Poincaré the main components of mathematical thinking are **intuition** and **logic**. They are differentiated in that logic provides validations for mathematical statements. From intuition comes the creative force which produces new mathematics. Both of these components are needed in the reasoning of a mathematician, but Poincaré provides examples of mathematicians who have possessed one or the other of these in a larger proportion (Poincaré 1900, chapter 1).

Intuition provides the creativity needed to advance mathematics beyond its current bounds in unobvious and original ways. But intuition alone cannot provide a basis for mathematics for "L'intuition ne peut nous donner la rigueur, ni même la certitude" (Poincaré 1900, 17). The history of mathematics shows the gradual increase in the role of logic in mathematical thinking, to the extent that in La Valeur de la science Poincaré writes "Nous croyons dans nos raisonnements ne plus faire appel á l'intuition" (1900, 20) and "On peut dire qu'aujourd'hui la rigueur absolue est atteinte." (1900, 23) In Poincaré's time mathematicians such as Russell, Frege, Hilbert, and Peano were attempting to demonstrate that this was in fact the case. They endeavored to show that mathematics is a field in which every statement can be verified by logic.

If mathematics can be based solely on a logical structure, and creation is ascribed solely to the unrigorous intuition, how then can creativity be a part of mathematics? Poincaré's answer to this lies in examining more closely the nature of intuition. He writes: "Nous avons ... plusieurs sortes d'intuition: d'abord l'appel aux sens et à l'imagination; ensuite la généralisation par induction, calquée pour ainsi dire sur les procédés des science expérimentales; nous avons enfin l'intuition du nombre pur [whence derives MI] qui peut engendere le véritable raisonnement mathématique" (Poincaré 1900, 22).

What then does Poincaré mean by "l'intuition du nombre pur"? He does not say. Daval & Guilbaud (1945) discuss this point in Le Raisonnement Mathématique. They maintain that Poincaré refers to the idea of a general number; "l'intuition du nombre **n**, c'est-à-dire du nombre arbitraire, du nombre quelconque, par opposition aux intuitions de chaque nombre en particulier" (Daval and Guilbaud 1945, 31).

The idea of a general number is closely related to what Poincaré calls "raisonnement par récurrence". "Le caractère du raisonnement par récurrence c'est qu'il contient, condensés ... en une formule unique, une infinité de syllogismes" (Poincaré 1943, 20). Poincaré illustrates this with a "cascade" of syllogisms, each one's conclusion providing the minor premise of the next, and each one possessing a major premise of the same form: If it is true in this case, then it is true in the next. The idea of a general number is based on recursive reasoning applied to the counting numbers 1, 2, 3, ... Poincaré goes on to show that neither logic nor the intuitions of sense and experience can provide a basis for recursive reasoning (Poincaré 1943, 21-22). That basis he finds in "la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible. L'esprit a de cette puissance une intuition directe et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience." (Poincaré 1943, 23-24) Recursive reasoning is a power of mind, an intuition based on reasoning about reasoning and provides, for Poincaré, the basis for MI.

The questioning of the foundations of mathematics, which gave rise to both formalist approaches and to Poincaré's investigations, also inspired the "Intuitionist" school. The role of MI in intuitionist writing differs from that assigned to it in formalist approaches in two important ways.

First, the proofs of the intuitionists are constructive in nature, that is to say no entities are proposed, or conclusions drawn which cannot be produced in a finite number of determined steps. In a traditional proof by MI the conclusion is that a statement is proven for **all** natural numbers, based on the existence of a method of proving the statement for **any** natural number. Constructivist techniques do not permit this final conclusion. Instead, for the intuitionists, MI is simply a method of proving a statement for **any** number, but not for **all** (Heyting 1966). This is equivalent to denying what is known as the ω-rule. The ω-rule is a rule of inference which states that from the infinite number of premises P(1), P(2), P(3), ... one can conclude $\forall x\ P(x)$ (Vinner 1976). In the formalist approach this rule is subsumed within the principle of mathematical induction.

Second, MI is derived as a theorem by intuitionists. This is permitted by the structure of the natural numbers, and by the weaker form of MI used by the intuitionists.

The Significance of Mathematical Induction in Teaching

The significance of MI in education has varied through recent history. Through the first seven decades of the twentieth century MI gradually received more attention from text book writers and school curricula (Ernest 1984). The "New Math" movement advocated the early introduction of proof, and the use of proof as a teaching tool (Hanna 1983). By the late 70s high school textbooks used in Canada and Britain were devoting considerable space to MI. A comparison of textbooks used in British high schools reveals that prior to 1960 the texts examined devoted two or less pages to MI. In contrast, texts printed after 1965 devoted between 4 and 10 pages to MI (Ernest 1984, 187).

The 80s saw a decrease in the significance of methods of proof in high school education. There is now no requirement that MI be studied at the high school level in Québec. Meanwhile, university mathematics courses assume

MI as a known technique. One possible reason for this curricular change is the criticisms of the New Math, which decreased the study of abstractions, in favor of a study of more "basic" topics. Some topics seen as uniquely mathematical have perhaps been removed to make room for applications intended to increase the utility and relevance of mathematics to students.

Even when MI is taught the presentation of it may not facilitate the formation of a mathematical understanding. As an example, in a college level course given at Concordia University, MI received a total of one hour of lecture time. In the description of this lecture numbers in parentheses "(1)" serve to indicate lines referred to in the discussion which follows. Ellipses "..." indicate phrases which were omitted. These include statements which were not completed, or which were only related to the material in an indirect way, for example reminders concerning the correct way to substitute a number in an expression.

The presentation began with a problem being written on the board:

> Prove that
> $$1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$$
> for every $n > 0$ n an integer

Meanwhile the lecturer said:

> The way I'm going to do it, I'll just give an example and you'll see what the problem is. [wrote example on board] ... Before I would give you this formula, take it, and your instinct, most of you

would take it, well now, only because we need to learn mathematical induction do we learn how to prove this. ... This is a statement linked to an integer n, we'll call it P_n [wrote this on board] ... P_5 is this statement: $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 2^6 - 1$ [wrote this on board] ... I'll give you another example [wrote out P_4] ... the whole idea is to understand what this is for specific n ... This is something you can verify on a calculator, but we are not now trying to prove it, in a second we will prove it. ... What's P_7? [wrote it on the board as the class dictated] ... So this statement in here [P_n] when we say it's valid for each n, an integer, we are saying that all those statements are correct. This is correct, this is correct, this is correct, P_{100} is correct, P_{1000} is correct, and so on. ... proving this statement is in fact proving all of these other statements. This is what we mean by each integer. Now the question is to (1) convince you that this is correct. I'll say how we do it and then elaborate more. First what is P to the k? [a student answered] So P to the k is $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$ [wrote this on the board] What's P to the k+1? [a student answered and P_{k+1} was written on the board] ... What's P to the 1, or P_1, the statement P_1? [students answered] 1 + 2 equals what? $2^{1+1} - 1$ [wrote P_1: $1 + 2 = 2^{1+1} - 1$ on the board] Now, is this correct? This is 3 and 4 - 1, which is 3. Now we know that P_1 is true, we know that the first statement, we know that this is true for n equals 1. Now the problem is to prove that it's true for any one. ... To prove it rigidly you should prove it for every single n, but there's no way to stop. You can't do it. (2) But mathematical induction is this, to prove it for each one, by a trick, and that trick is the following. We proved that P_1 is true, so this is what we call "a" [wrote "a)" in front of P_1] and then the second thing, we prove the following. We prove that P_k implies P_{k+1}. ... P_k is true implies P_{k+1} is true [wrote "P_k is true => P_{k+1} is true" on the board] ... The way to prove this statement was, to prove ... (3) P_1 is true, P_2 is true, P_3 is true, P_4 is true, and so on, and so on. And we said there are millions of them. So if we prove that ... P_1 is true, and we are going to prove that whenever you have a statement which is true, then the statement which is next to it ... (4) Whenever P_k is true then the one next to it, after it, is true. ... once we establish this link, between one statement and the one after it then we have the proof. [a break was called] ... [wrote out $P_k \Rightarrow P_{k+1}$ in full] ... now the proof in here is not induction, it's just computations, it's not the principle itself. ... Now we'll prove this and then later I'll explain why this proves that the whole statement is correct. ... This argument does not depend on k. ... Now I will tell you why this proves the statement. ... We were trying to prove this equality for each n. ... What we've done was the following: we proved that, (5) first we said that saying that this statement is true amounts to saying that millions of such statements are true. Whenever you replace n by any value you get a true statement and all those statements are true. ... (6) We said that P_1 was true and we said, by this, that if something is true then the next one is true, well since 1 is true the next one is true which is 2, and the next one is true which is 3, and so on. [described the domino model as given in the textbook (Durbin, 1988, 435)] ...If you have a family, and you take one and you say this is a stupid one. And you number them, you have a a large number, (7) you have ten people, and you say this is first, second third, and so on. (8) You say the first is stupid, and then you prove that, you say that if given that someone is stupid then his brother is stupid. Now what does this mean? They're all stupid. ... You always use the same thing: you suppose P_1 is true, P_k true implies P_{k+1}, you're finished. Now the second thing about mathematical induction is you can't use it everywhere. ... [explained that MI can be used to prove that the number of roots of a polynomial is equal to the degree of the polynomial] ... [gave proving the area formula for a triangle as an example of a case where MI cannot be used] ... the polynomial statements, they are related to a certain integer, and we prove it using this integer. ... there is always this n ... [Did another example :

Prove that: $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

after which he gave another model:] ... (9) Now if you want a process which is infinite, we talked about brothers, now think of the son. Well, we will do it the other way. Think of the father as smart, and think that if you are smart, your kid is smart. Suppose this is true, and then what is this saying? Well this is saying that from here to eternity you are getting smart generations. ... [Worked another example: $2^n > n^2$ for $n \geq 5$, pointing out that the initial case need not be $n = 1$] ... Now since you are a little bit familiar with induction, ... the reason we call it mathematical induction because there is in fact a different kind of induction, in science. Usually in science, like biology, chemistry, something like that, they don't prove things as we do. We do not prove anything in biology. The way we do it is by induction. So this is completely different form of induction than this. Induction there is you assume that, for example, you say "the sun has risen, for eternity, and this implies that it rises tomorrow." The implication is not proved. The implication is based on our experience. We induce that this would happen. For example, another example, how do we prove for example that is you transplant a kidney that it works? You have no way of proving it. The way you've done it, you prove that because you've done it so much and it works, so you generalize it. You say that if you've done an experiment on a bacteria, if this experiment works then it's going to work for every bacteria under the same circumstances. And then it doesn't and so you're proven wrong. Everything in physics, science and in social science is not based on mathematical deduction, on mathematical proofs. It's based on experience. ... This is not math. In math we would prove everything. The induction here is very rigid on the other side, the mathematical induction. ... The special cases are treated in us by this P_1. P_1 is our special case. We look at it and then we prove the connection. (10) We prove that if you see n, 5 bacteria that replicate then the sixth that you see will replicate. We prove that. We prove this connection and then you say any bacteria that you see will reproduce by replication or every cell. [Spent the rest of the lecture on Arithmetic Progressions, the next topic]

While this is not a typical example of a lecture on MI (I hope) it does present examples of several concepts related to MI which are difficult to teach well, and of which students develop various understandings.

The lecturer did not justify MI by referring to an axiom, nor was he very clear in presenting intuitive justifications in the style of Poincaré. Instead he described MI as relying on "a trick" (2). Later he simply asserted: (4) "Whenever P_k is true then the one next to it, after it, is true. ... once we establish this link, between one statement and the one after it then we have the proof." His most complete statement of the underlying principle was: (6) "We said that P_1 was true and we said, ... that if something is true then the next one is true, well since 1 is true the next one is true which is 2, and the next one is true which is 3, and so on." Finally, in the context of his discussion of empirical induction, he used a single case to establish the "link": (10) "We prove that if you see n, 5 bacteria that replicate then the sixth that you see will replicate. ... We prove this connection and then you say any bacteria that you see will reproduce by replication ..." While some students might have connected these justifications with their own intuitions concerning recursion, it is not certain that all of them did, and these others might have come away believing that MI relies on trickery, or the examination of a few cases. Both of these bases are sufficient if the only purpose of a proof by MI is to : (1) "convince you that this is correct."

The lecturer also offered some models of MI. The first of these is the domino model given in the text and also discussed by Ernest (1984, 184) The only danger with this model is that students might not extend the finite situation of a line of dominos to infinite situations such as the natural numbers. The second model offered by the lecturer is explicitly finite: (7) "you have ten people" and also deals with a set which is not ordered. The lecturer attempted to deal with this second problem by numbering each member of the family, but then ignores the numbering: (8) "You say the first is stupid, and then ... you say that if given that someone is stupid then his brother is stupid. Now what does this mean? They're all stupid." He seems to have been aware of these problems, and corrected them in his third model: (9) "Now if you want a process which is infinite, ... Think of the father as smart, and think that if you are smart, your kid is smart. Suppose this is true, and then what is this saying? Well this is saying that from here to eternity you are getting smart generations." There were other times in the lecture at which the applicability of MI to infinite sets was not made clear: (3) "P_1 is true, P_2 is true, P_3 is true, P_4 is true, and so on, and so on. And we said there are millions of them." (10) "first we said that saying that this statement is true amounts to saying that millions of such statements are true." This colloquial use of "million", combined with the finite models given, might have caused some students to understand that MI is limited to proving results on finite sets.

An example of the difficulties students might have in following a lecture on MI at this level is given in Shaw (1978). In that case the students were school mathematics teachers doing further training in a university course. Given the difficulties encountered by them, it seems likely that students at the high school and college level face a considerable challenge in developing understandings of MI which are compatible with those of their teachers.

Research related to Students' Understandings of Mathematical Induction

There have been several theoretical discussions of the difficulties involved in the learning of MI, and several research projects. Avital & Hansen (1976), Ernest (1984), Woodall (1975), and Young (1908) present theoretical discussions. Of these Ernest is the most complete. The research projects are reported in Dubinsky (1986, 1989), Sfard (1988), and Whitton (1978).

Ernest discusses the significance of MI, the structure of a correct proof by MI, some prerequisites to learning MI, and some problems students encounter. His discussion of the understanding of MI is divided into two parts. The first deals with those outward behaviors which a student must be able to exhibit in order to produce a proof by MI. The second deals with the mathematical and logical concepts which must be understood.

Of the concepts which Ernest feels are required for an understanding of MI the first, and most important, is that of logical implication. Students must understand what constitutes a valid implication and how to prove an implication. In addition to being able to prove, students must also understand what constitutes a proof in order to understand that a proof by MI actually is one.

As the proof claims to demonstrate that the natural numbers possess a particular property the idea of a property of natural numbers is also essential to the concept of MI.

Ernest also mentions recursion and the ordering of the natural numbers as essential to MI and likewise essential to the understanding of it. These concepts are seen by Ernest as basic prerequisites. "The justification for the method of MI, and the mechanism by which it works, are to be found in the well ordering of the natural numbers and their construction by recurring succession." (Ernest 1984, 179)

Ernest also describes some problems which students often encounter when learning to prove using MI. These provide some insight into the different ways of understanding MI students develop.

One of the problems he mentions is the confusion of MI with induction, in the sense of generalization from particulars to the general.

Students have difficulty with the induction step as it requires the assumption of **P(k)**, which looks suspiciously like what they are trying to prove. The reasoning can appear to be circular due to this misapprehension of the nature of **P(k)**.

Ernest also mentions a problem which Young and Whitton also comment on, the omission of one of the steps of the argument.

A non-understanding or void understanding of MI arises if a student does not see the justification for the principle. If the development of MI is based on taking the principle of mathematical induction as an axiom there is no deductive proof of it which can be offered. The best which can be hoped for is to derive it from a related, but more acceptable, axiom. Ernest suggests using the ordering of natural numbers to justify the method of descent, which can then be used to prove the principle of MI by contradiction. On the other hand, if MI is being treated as an intuitive deductive process, as suggested by Poincaré, its rejection cannot be contradicted in any way. A third possibility is that the student accepts recursive reasoning as valid, but does not connect it with MI, leading to a rejection of MI as unjustified. In this case the making of this connection is sufficient to produce some understanding.

The variable used in the induction step of a formal proof by MI has a complicated status. Ernest describes the logical structure of the induction step as follows:

> The proof of the induction step is usually carried out in the simplest possible way, first by adopting the assumption P(**n**) known as the <u>inductive hypothesis</u>. This is followed by the derivation of P(**n** + 1). This permits the assertion of P(**n**) -> P(**n** + 1) and, finally, of ∀**n**[P(**n**) -> P(**n** + 1)], provided that the variable **n** first occurs freely in the inductive hypothesis. This last step, rather a logical nicety, is almost always taken for granted and is theoretically dispensable in a free variable form of PMI [the principle of mathematical induction]. (Ernest 1984, 175)

Ernest also notes that some problems students encounter as they develop their understandings of MI are related to its logical structure. First, students confuse the assumption of P(**n**) in the induction step with an assumption of ∀**n**[P(**n**)], which is what they are trying to prove. Second the sheer complexity of this logical structure, and the use of quantifiers within the proof can be a source of difficulty (Ernest 1984, 181-182).

The possible difficulties associated with the induction step can be thought of as related to different roles assigned to the variable **k**. In the following the form of the induction step is reinterpreted in terms of its possible structures in the minds of students, rather than its logical form. In the terms of logic, what is described here as a general number would be called a universally quantified variable. What is described here as an arbitrary number would be described as a free variable. In most proofs the variables used play the role of general numbers. In the induction hypothesis, however, **k** cannot be taken to be a general number, as that would be equivalent to assuming the truth of the statement being proven, which would immediately imply the truth of P(**k**+1). Instead **k** is an arbitrary number, that is, **k** is some individual unspecified number. Throughout the proof of the induction step **k** plays this role. However, once the induction step is proven, the implication P(**n**) -> P(**n** + 1) is generalized (in logical terms, universally quantified) to prove the original statement. It is this switch of roles, which corresponds to the "logical nicety" mentioned by Ernest, which seems to be at the root of some students rejection of MI.

MI has also been the object of some experimental research. This research has been concentrated on testing different methods of teaching MI. Such studies have been done by Dubinsky (1986, 1989), Sfard (1988), and Whitton (1978). Dubinsky and Sfard both used computer programming to develop the skills necessary to produce proofs employing MI. Whitton developed these skills through the use of worksheets providing skeletons of proofs by MI.

Dubinsky's teaching model is based on an interpretation of Piaget's epistemology. Dubinsky develops a "genetic decomposition" of MI into the subconcepts he believes are essential to understanding MI. He defines MI as "a method of proof" (Dubinsky and Lewen 1986, 65) and he means this in a formalist way. He describes proving by MI in terms of the proof of the basis followed by the proof of the induction step, with the conclusion justified by the principle of mathematical induction. The proof depends on the principle, but in Dubinsky's genetic decomposition, based on the procedures used by students, this principle is not developed. Instead MI is developed from the "coordination" of two concepts: modus ponens and implication-valued function. The specific implication-valued function referred to is n -> (P(n) => P(n+1)). The result of this coordination is "the detailed statement of how one uses the idea of induction to know that the statement is true for n equal to 10 or for n equal to a million". That is, a procedure for proving a statement for any specific number. Dubinsky presents a decomposition of a way of understanding MI which is not the same as the one he defines as depending on the principle of mathematical induction. He describes a procedural, constructive proof while the principle of mathematical induction defines a static, general proof. While the formalist definition of MI Dubinsky gives would suggest otherwise, the decomposition never develops a structural view of MI as a part of an axiomatic system.

Sfard believes that "abstract notions [...] can be conceived either structurally (as static constructs) or operationally (as processes rather than objects)"(Sfard 1987, 162) and "that formation of an operational conception is, in many cases, the first step in the acquisition of a new concept"(Sfard 1988, 560). With this in mind she describes a computer based method of teaching MI intended to develop both an operational and a structural understanding. Her experimental results indicate that students taught in this way did do better on written test items than a control group taught in the standard, structural, manner.

A reservation which applies to both Dubinsky and Sfard's work is in their use of computer programming experiences to teach an abstract concept. It is quite possible that the understanding gained is procedural only, of the kind described by Ernest as "behavioral" (Ernest 1984, 176). It is difficult to tell whether an improvement on written tests indicates more than a procedural understanding. The students' understanding may be nothing more than a set of behaviors. In Vygotsky's terminology the students may have a complex of MI but not the concept (Vygotsky 1962). While Sfard is aware of the importance of both an operational and a structural understanding, Dubinsky seems not to be, and both fail to show that students possess any meaning for MI beyond that of a procedure.

Whitton's work seems to be entirely oriented towards the development of a set of behaviors which she equates with understanding of MI. That some students did develop these behaviors can be taken as a argument that her teaching did develop at least a procedural understanding. What other ways of understanding might also have been present is not determined.

The studies which have been discussed can be divided into theoretical studies of students' learning of MI, and experimental studies of teaching methods. No studies have been done which attempted to determine students' understandings and learning based on clinical studies outside of the context of evaluating a particular teaching method. The closest thing to such research so far attempted is Dubinsky and Lewen's (1986) study. Some features of interest may have been obscured, however, by their concentration on observing levels of development in a Piagetian structure. It was the intention of this study to try to fill in a part of this gap by studying a few students as they went through the process of learning MI, and as they justified their understandings to peers.

CHAPTER II

DESIGN AND "DEROULEMENT" OF THE CLINICAL STUDY

Design of the study

The clinical study was used to investigate in detail the ways different students understood MI. This form of investigation was selected because of its suitability for the investigation of the students' thinking while they were using MI. It was intended to allow investigation of aspects of the students' ways of understanding which are not visible from written work. The insights which can be gained in this way, into the processes by which MI is applied to a given situation, cannot readily be gained using other investigative techniques. While a student may be able to identify, follow, and compose proofs using MI on paper, his/her way of understanding may be limited to producing the required behaviors. It should be noted that the use of a clinical study limited the number of students who could be considered, and the intention of this research was not to provide a general picture of how thinking about MI occurs in the whole population of students. Instead the intention was to provide a picture of the thinking of some individuals, at a sufficient level of detail to be useful in considering future research.

Subjects

The clinical study was designed around the idea of using two groups of students, one mathematically more experienced than the other. The more experienced group was to consist of four students selected from a first year university linear algebra course. The students were to be selected on the basis of their strong performance in this course. No attempt was to be made to determine whether or not these students had previously encountered MI, in order to avoid indicating to them the nature of the study. They were to be informed only that it related to mathematical reasoning. As it is unusual for students at this level in Quebec to have taken courses in which MI is taught it was expected that most would not have prior experience with it. The mathematically less experienced group of students was to consist of two upper level undergraduate students taking courses in the humanities. It was expected that these students would have taken no mathematics at the university level, and would never have been taught MI. It was expected that their mathematical skills would extend to high school algebra, at least.

The use of these two groups was intended to serve several purposes: (1) revealing the understanding of concepts related to MI possessed by the students prior to encounteı g MI, and drawing contrasts between students based on the level of their mathematical achievement. (2) indicating the ways in which students in each of the two groups adapted their cognitive structures in light of experiences involving MI, specifically in terms of developing and using methods of proof analogous to MI. (3) exploring the ways in which the students differed in using MI and concepts related to MI, when communicating with their peers. (4) exploring the different zones of proximal development of the students in terms of the possibility of learning to use formal MI with assistance.

Organization of the Study

The organization of the study differed slightly for the two groups. The general organization was as follows. For the first stage of the study each group was divided in two. Half of the group was exposed to situations in which MI could be used, and was shown how MI could be used in these situations, if they were not able to do so independently. The second half of each group was exposed to situations in which empirical inductions led to incorrect conclusions, in order to develop skepticism on the part of the students for arguments which relied on specific cases, and to provoke a desire for proof. The first stage was intended to serve as preparation for the second. In the second stage of the study students who had been shown MI were paired with students who had been encouraged to be skeptical. These pairs worked together on proving several problems using MI. The interaction of the students who had seen MI used to prove statements, and those who had been taught to be skeptical of inductive proofs, was expected to reveal the nature of the students' understandings of MI and of concepts related to MI.

Two investigators participated in the study, the author and his thesis advisor. This permitted the two parts of stage one the be conducted simultaneously, and provided additional insight during stage two.

Some modifications to this general plan were made for each of the two groups. A general outline of these differences is included here. The scripts used for each session of the study are included in Appendix A, and a detailed discussion of the rational behind each item in the scripts appears below. It should be noted that these scripts represent the intentions of the investigators, and that it was necessary to deviate from the scripts to adapt to discoveries made during the sessions.

Outline of Study as It Was Designed

The script designed for the students from the mathematically more experienced group who were to be exposed to MI in stage one can be divided into several steps. As it is designed the students are: (1) asked to prove two statements. It was expected that their arguments would approximate MI. (2) asked to describe the method they had used and to compare it with the principle of mathematical induction, which was shown to them as an axiom of the natural numbers. (3) asked how the principle related to the methods they had used. (4) asked to prove three other statements, one which can be proven by MI, one which is false, and one for which no proof is known, but which appears to be true. (5) involved in a discussion of their attitudes towards MI, and how MI relates to empirical induction.

One of the students from the group of mathematically less experienced students was also exposed to MI in stage one. In the case of this student the script designed was quite different from that designed for the students from the mathematically more experienced group. The stage was broken into three sessions, both to permit more time for the student to develop ideas, and to allow time for modification of the scripts based on progress made in previous sessions. This time for modification was considered necessary because it was difficult to anticipate the exact difficulties a mathematically less experienced student might have in the context of proving statements employing recursion.

The first session was intended to develop a feeling in the student for the need to prove in mathematics. The script calls for the student to be: (1) introduced to the Fibonacci sequence, and asked to find the pattern in it. (2) asked to formulate the pattern algebraically. (3) asked to generalize three properties of the Fibonacci sequence based on single examples. In one case the property is quite general, in another the property is general but slightly different from the property which would be generalized from a single example, and the last is not general but appears true based on examination of a few examples. (4) asked to look at further examples in order to become more confident of the truth of the statements. (5) asked to check the false generalization for an example which disproves it. (6) asked about methods for becoming sure of the truth of a statement. (7) asked to prove a fourth generalization about the Fibonacci numbers.

The second session was intended to investigate the students ability to construct a proof without assistance. The script calls for the student to be: (1) introduced to the notation F_n for the **n**th Fibonacci number and to the recursive definition of the Fibonacci numbers, and the generalizations made in session one in this notation. (2) asked to prove two generalizations about Fibonacci numbers. Both generalizations in this case are true.

The third session was intended to introduce the student to MI. The script calls for the student to be: (1) asked to consider one statement from session two again and to check some aspects of it which appeared interesting based on the attempted proof in session two. (2) prove a new statement, geometric in nature. If the student had difficulty the investigator was to assist. (3) consider the statement from session two again, and to prove it with assistance from the investigator. (4) asked to define a phrase used in session two.

The scripts for mathematically more experienced and less experienced groups are similar in the case of the students who were to be developing a skeptical attitude towards induction. The script calls for them to be: (1) introduced to the Fibonacci numbers. (2) asked if they think four statements are true, of which three are false but appear true on the basis of some initial cases. (3) shown counter examples to the three false statements. (4) asked if they still believe the single true statement is true and asked what criteria they would use to determine truth.

The scripts designed for stage two differed slightly between the two groups. In the case of the mathematically more experienced students they are asked to prove three statements and to comment on the proof given of a fourth. The first statement requires modification as it is false in the form given but leads easily to a true generalization. In the case of the mathematically less experienced students they are first asked to judge the validity of a proof, and then asked to prove a statement, given some empirical evidence for it. They are then asked to evaluate another proof and to prove two more statements. Again they are given some empirical evidence for the statements they are to prove.

Rationales for the Scripts

Scripts Used with the Mathematically Experienced Students

Stage One

Group A: Students Who Were to be Introduced to MI

Item 1:

1. Find the relation between the number n of sides of a convex polygon and the sum A_n of its internal angles.

Item 1 was designed to encourage inductive problem solving, involving consideration of the angle sums of polygons with varying number of sides. This would then serve as a starting point for the proof required by item 2. Use of non-inductive techniques to answer item 1 would indicate a preference for such techniques which would have to be overcome in order to present MI as a method of proof in item 2.

Item 2

2. You have probably found that:

$$A_n = (n-2)\pi \text{ for } n \geq 3$$

Give a mathematical proof of this relation (assume as an axiom that the sum of angles of a triangle equals π).

Item 2 clarifies the nature of the relation in item 1, in case the students work is unclear. In the event that the students do not attempt an inductive proof, the investigator can provide hints encouraging such a proof.

Item 3:

3. Prove that n straight lines on the plane divide the plane into no more than 2^n parts.

Item 3 provides another opportunity for the students to work through an inductive proof, with assistance if needed.

Item 4:

4. Could you describe your method of proving the above two statements?
Could you identify the steps of your reasoning?
Were you, in a further step of your reasoning, referring to some previous step of reasoning?

Item 4 serves to draw the students' attention to important aspects of their proofs. First, to the similar method used in each case. Second, to the steps employed in arriving at the conclusion. And finally, to the recursive nature of the argument. If these aspects of the proofs are not understood the investigator continues the questioning to clarify the students thoughts at this point. This reflection on their own reasoning is an activity which is linked to Poincaré's basis for MI (see page 15).

Items 5, 6 and 7:

5. In the so-called Arithmetic of natural numbers we admit, among others, the following axioms:
I. 1 is a natural number
II. If n is a natural number then $n + 1$ is a natural number.
III. Let $S(1), S(2), S(3), \ldots, S(n), \ldots$ be statements about natural numbers. If the statement $S(n_0)$ holds true for some n_0, and, for any $n \geq n_0$, the truth of $S(n)$ implies the truth of $S(n+1)$, then all the infinite number of statements $S(n_0)$, $S(n_0+1)$, $S(n_0+2)$, ... hold true.

6. This last axiom is called "the principle of mathematical induction" and provides us with a method of proving theorems about natural numbers that start with "for all n ...", or with "for any n ...".

7. Is there any link between this principle and your reasoning in problems 2 and 3?

Items 5, 6 and 7 provide a mathematical context for the reasoning the students have been employing, pointing out the axiomatic nature of the principle of mathematical induction, and the sort of theorems which are proved by it. The students' own proofs serve as examples.

Item 8:

8 Let the sequence $A = (a_n, b_n)$ of natural numbers be defined as follows:
$a_1 = 1 \quad b_1 = 1$
$a_{n+1} = b_n$
$b_{n+1} = a_n + b_n$
Let the sequences $C = (c_n, d_n)$ of natural numbers be defined as follows:
$c_1 = c_2 = 1$
$d_1 = 1 \quad d_2 = 2$
$c_{n+2} = c_n + c_{n+1}$
$d_{n+2} = d_n + d_{n+1}$
Show that the sequences A and C are identical.

Item 8 provides another opportunity to observe the students use of MI. It was intended that the students should need no assistance, and that this proof would serve to evaluate their ability to construct such proofs independently.

Item 9:

9. Is the following statement true:
For all natural numbers n:
$$p(n) = n(n+1) + 11$$

is a prime number.

Item 9 exposes the students to one feature of MI. Attempting a proof by MI does not necessarily indicate that a statement is false. While one might come to such a conclusion after continuing failures to prove the induction step, MI itself does not lead to disproofs. In this case, if the students become embroiled in the proof and fail to see that the statement is not true, the investigator can assist them.

Item 10

10. Is the following statement true:
    Every even number is the sum of two primes.
[may be left out]

Item 10 is of the same nature in that it exposes a limitation of MI, one that is shared by all forms of proof. The statement is of the Goldbach conjecture, which has not been proven, but appears to be true on the basis of considerable empirical evidence. The investigator may wish to leave this question out, depending on the comprehension of proof attained by the students at this point.

Items 11, 12 and 13:

11. Do you accept mathematical induction as obvious or reasonable?
Do proofs by mathematical induction convince you?
Does a proof by mathematical induction raise the degree of certainty of a conjecture?

12. In empirical sciences, we also speak of induction; we speak about inductive inference. For example:

Experiment has shown that:
sugar s_1 dissolves in water,
sugar s_2 dissolves in water,
...
...
sugar s_{20} dissolves in water,
Conclusion: all kinds of sugar dissolve in water.
Or:
The sun has always risen in the east, as far as I and other people can remember. Conclusion: the sun will always rise in the east.

13. What is the difference or differences between this kind of induction and mathematical induction?

Item 11 directly asks the students' opinions on the nature of MI and their trust in it. This is contrasted in items 12 and 13 with reasoning by empirical induction, in order to clarify the difference between the two for the students.

Group B - Students Who Were Encouraged to be Skeptical.

This session was divided into two parts, each with a different intention. The first part develops several propositions. The second refutes most of these propositions and introduces a discussion of the need for skepticism.

Q1 and Q2:

INTRODUCTION to Fibonacci numbers
I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers. [If "yes" I: what do you know about it/them?] This is the beginning of the sequence
[present Table 1:
1 1 2 3 5 8 13 21 34 ...]
[Talk about Fibonacci]
[Q1] Can you find the pattern in this sequence? [If "no" I:

try adding up pairs of consecutive numbers [more prompts if needed]]

I: [Q2] Can you formulate a rule for producing the Fibonacci sequence? [prompt if needed] How would you write your rule algebraically? [give if needed, test understanding]

Q1 was designed to indicate the students' abilities to use recursion to define a sequence. Q2 was designed to indicate the students' ability to represent a recursive relationship algebraically. Both of these skills are needed in the construction of a proof by MI, and also to investigate the questions which follow.

Q3, Q4 and Q5:

DEVELOPMENT OF PROPOSITIONS 1
I: [Q3] Consider this question:
[S1a: "Is there anything special about the sum of the first *n* Fibonacci numbers? Is it related to the sequence in any way?"]

I: [Q4] Is this statement true?
[S1b: For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$]

I: Here is another statement about the Fibonacci numbers:
[S2: The Fibonacci sequence is given by:

$$F_n = \left[\left[\left[\left[\frac{1}{60}n - \frac{7}{24}\right]n + 2\right]n - \frac{149}{24}\right]n + \frac{569}{60}\right]n - 4$$

[Q5] Is this statement true?

Q3 was designed to test the students' ability to discover recursive relationships of more complexity. Q4 and Q5 then indicated the criteria used by the students' to determine the truth of a statement.

Q6, Q7 and Q8:

INTRODUCTION to prime numbers
I: [Q6] Do you know what primes numbers are? [if "no" explain].
Here is a list of the prime numbers less than 10000. [Table 2]
DEVELOPMENT OF PROPOSITIONS 2
I: Here is another statement:
[S3: For all n: $P = n^2 + n + 41$ is prime]
[Q7] Is this statement true?

I: Here is a statement about prime numbers and Fibonacci numbers:
[S4: "For all n: If n is prime then F_n is prime."]
[Q8] Is this statement true?

Q6 determined only whether the students' had the background knowledge necessary for the next questions. Q7 and Q8 then further indicated the criteria used by the students' to determine the truth of a statement.

Q9, Q10 and Q11

DEVELOPMENT OF SKEPTICISM: Proofs and refutations
I: Have a look at the formula for producing Fibonacci numbers. [S2] Try n=7 [if not already done] [Q9] Would you now say that this property only produces Fibonacci numbers? Why not?

I: Have a look at the formula for producing prime numbers. [S3] Try n=40 [if not already done] [Q10] Would you now say that this property only produces prime numbers? Why not?

I: Look again at the fourth statement. [S4] 19 is prime isn't it? [Q11] Is the 19th Fibonacci number prime?

Q9, Q10, and Q11 were intended to indicate the students' reactions to counter examples. It was anticipated that the students would reject the last three statements refuted by the counter-examples, and become skeptical of the truth of the first statement.

Q12:

I: [Q12] Would you now say that the first statement is true for all Fibonacci numbers? If I assure you that it is how would you go about showing that it is true?

Q12 then indicates whether the students have become skeptical of the truth of the first statement, and how the students' would ensure that such a statement is true. Without using MI the best argument they could produce would be a large number of examples. It was expected that this would not satisfy them.

Q13:

I: What would you say about trying to find out if statements are true?
[In the context of this discussion:]
We have seen that in mathematics it is extremely unwise to assume that something is true for all n just because it is true for some n.
You've just seen several cases where even though there are lots of examples, the statement turns out not to be true.
In the next interview you will be looking at some other statements, some, perhaps all, of which are not true.
Remember to be careful about accepting a statement unless you have a valid explanation.
[Q13] What would you say would be a valid explanation?

Q13 was intended to test whether the students' had developed a rigorous standard for judging statements, and a skepticism of statements which had not been proven in some rigorous way.

Stage Two

The script of stage two takes the form of four statements. The fourth statement is offered with a proof.

For the other statements the students are asked to try to provide a proof.

First Statement:

I: Consider this statement. Try to determine whether it is correct or not and explain why it is correct or why it is not.
[Statement A:
For all $n \geq 3$,

$$(F_n)^2 = (F_{n-1})(F_{n+1}) + 1$$

(F_n is the n^{th} Fibonacci number)]

The first statement is not true as written. If the students discover that it can be modified to make it true they would then have been instructed to try to prove the modified statement. If they did not discover the possible modification of the statement then it is left as an early example to encourage a skeptical approach to the other statements.

Second and Third Statements

I: Consider this statement. Try to decide whether it is correct or not and explain why or why not.
[Present statement B:
B: The number of diagonals in a convex polygon of n sides is

$$\frac{n(n-3)}{2}$$]

I: Here is a sequence of numbers:
[Table 3: the **B** sequence:
1 1 3 5 11 21 43 85 ....]
Can you find the pattern in this sequence?
[If they answer $B_n + B_{n+1} = 2^k$ then skip to α below]
I: Look this statement [statement C1 :$B_n + B_{n-1} = 2^{n-1}$]
Do you think it is correct? Explain why you think so, or why you don't.
[skip to β below]
[α]
I: Look at this statement [statement C2 :$2B_n + B_{n+1} = B_{n+2}$]

Do you think it is correct? Explain why you think so, or why you don't.

[β]

The second statement is true and is easily proven by MI. The third statement can take two forms. The students are expected to propose one of the two forms, which is then assumed as a definition of the **B** sequence. The task is then to prove the equivalence of the second form.

Fourth Statement:

I: Consider this statement and its proof. Try to determine whether it is correct or not and explain why it is correct or why it is not.

Statement D:
For all $n \geq 0$:

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

where F_n is the n^{th} Fibonacci number

The final statement is the Binet formula for the Fibonacci numbers. It is offered with a proof by MI (see Appendix A for the full proof). This was introduced to indicate the students' ability to recognize it as similar to their arguments and to indicate what aspects of the proof each of them felt required attention.

Scripts Used With the Mathematically Less Experienced Students

Stage One

A: The Student Who Was to be Introduced to MI

Session One:

Q1:

I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers.
If "yes"
I: what do you know about it/them?
I: A merchant named Leonardo Fibonacci of Pisa studied this sequence in connection with a problem he was trying to solve. Since then many mathematicians and scientists have found applications of the sequence in a variety of contexts. This is the beginning of the sequence
[present figure 1:
1 1 2 3 5 8 13 21 34 ...]
Q1: Can you find the pattern in this sequence?
If "no"
I: try adding up pairs of consecutive numbers [more prompts if needed]
I: Can you formulate a rule for producing the Fibonacci sequence?
[prompt if needed]

I: Q2: how would you write your rule algebraically?

Q1 and Q2 are identical to the first questions presented to the "skeptical" students in the mathematically more experienced group. The intention of the questions was the same in this context. As noted above Q1 was designed to indicate the students' abilities to use recursion to define a sequence. Q2 was designed to indicate the students' ability to represent a recursive relationship algebraically. Both of these skills are needed in the construction of a proof by MI, and also to investigate the questions which follow.

Q3, Q4 and Q5:

I: let's play around a bit and see if we can find out anything about the Fibonacci sequence. Pick one of the numbers in the sequence. [response] Now square it [response]
Now take the two numbers before and after the number you picked and multiply them together. What do you get? [response]
How is that related to the square of the number you picked?
[The square is one more/less than the product]
Q3: If that relationship were true for all Fibonacci numbers how would you write a general rule describing this property?

I: Let's try something else. Do you know what prime numbers are?
If "no" explain
I: pick a prime number. Now count along the Fibonacci sequence that many numbers. What number do you land on? Is that prime?
Q4: If that relationship were true for all Fibonacci numbers how would you write a general rule describing this property?

I: let's try one more: Add up the first ten Fibonacci numbers. What do you get? Now multiply the seventh Fibonacci number by 11. What do you get?
Q5: If that worked for any set of 10 Fibonacci numbers how would you write a general rule describing this property?

Q3, Q4, and Q5 all require the student to make a generalization from a single case. The student's response to such questions might indicate skill in generalizing, willingness to do so from small samples, and ease of acceptance of such generalizations.

Q6:

I: Q6: Do you think any of these properties hold for all Fibonacci numbers?
I: Why?
I: How could you become more confident?
[expected response: more examples]

Q6 specifically inquires after the student's willingness to accept generalizations, and the reasons the student might use to justify such generalizations.

Q7:

I: Q7: try looking at a few more examples
[work through examples. The first proposition is likely to provoke a reaction when it fails but the subject will probably suggest the (correct) general property soon after. By now confidence should be high]

Q7 was intended to result in the student rejecting the first statement (from Q3), as it does not hold half the time. It was expected that the student would modify the generalization, and accept the modified statement on the basis of several examples. Confidence in all three statements (from Q3, Q4, and Q5) was expected to be high at this point.

Q8 and Q9:

I: Q8: 19 is prime isn't it? Is the 19th Fibonacci number prime?

I: Q9: Would you now say that this property is true for all Fibonacci numbers? What about the other properties?

Q8 was intended to undermine the student's confidence by presenting her with a counter-example to the second statement. Q9 then explicitly asks the student to appraise her confidence in the other statements. One of the intended effects of the line of questioning followed in this session was to undermine the student's faith in empirical proof, while maintaining a desire to accept the statement. This question was included to provide an indication of this effect.

Q10:

I: Q10: Can you think of some way to definitely establish the truth of these properties, if they are true?

Q10 was intended to indicate what sorts of arguments the student felt provided certainty. This was asked both to further expand on the information gleaned from the answer to Q9, and also to lead into a possible discussion of the features of a proof, for use in designing the later sessions.

Q11:

I: Q11: How would you establish the truth of this property: "The sum of any set of consecutive Fibonacci numbers is a Fibonacci number"?

Q11 was intended to remind the student of the possibility that even likely looking statements might be false, and so to encourage the checking of some examples even if the possibility of a counter example seems slight.

Session Two: The second session was intended to provide the student with an opportunity to prove statements. The general intent was to see if the student could construct a proof of an inductive nature, under these circumstances.

First sections:

I: Remember the Fibonacci numbers? Here's a list of the first 35. Remember how we wrote the first Fibonacci number as F_1 and the second as F_2 and so on? If n is some number then F_n would be the Fibonacci number in position n in the list. How would you write the Fibonacci number which came right after F_n?
[help if needed]
I: Do you remember how you said to use two Fibonacci numbers to get the next one? [yes...] How could you write that using F_n and F_{n+1} ?
[help if needed]
I: Here are the statements we looked at last time written with F_n's. Do they make some kind of sense? can you see where the symbols come from?

The first sections concern the development of a notation for the Fibonacci numbers and the establishment of a symbolic form of the recursive rule defining them. This was intended to provide a basis for the student's attempts to prove, as such a basis had been lacking in the first session.

The two statements:

I: How would you write this statement in symbols?
[show figure S1a:
The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

The response should look like :
$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$]
I: Do you think this statement is true? How would you convince me that it is? [prompt for a proof]

I: How about this statement?
[Show figure S1b: The sum of the first **n** even indexed Fibonacci numbers is one less than the odd indexed Fibonacci number after the ones added up.
$F_2 + F_4 + \ldots + F_{2n} = F_{2n+1} - 1$]
I: Is this one true?... How would you convince me?

The two statements are very similar in form and in proof. It was intended that if the student did succeed in proving the first statement then the second would provide reinforcement of the student's method, in preparation for a generalization of it later.

Session Three: As the student did not produce a proof by MI in session two, but did show some indications that instruction might serve to lead her to one, the third session was intended to introduce MI to her.

Q1:

Q1 Look at this statement again:
Figure 3:
The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.
$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$
Check it for n=6
Check it for n=7
Check it for n=8

Q1 was intended to remind the student of the previous session and the statement. In session two the student used a recursive technique to calculate each new sum, to avoid having to repeat calculations. By asking the student to calculate some specific sums it was hoped that this method would be remembered so it could be used in the proof later.

Q2:

Q2 Consider a pancake. What is the maximum number of pieces you can cut with 1 cut? With 2 cuts? With 3 cuts? Would you agree with the statement:
" **k** cuts will never produce more than 2^k pieces"
Could you show that this is true? [guide if necessary]

Q2 asks the student to prove a geometric statement. The intention was to see if a non-algebraic context would be more suited to the student's attempts to find a proof. If the student was unable to find a proof, then this problem would be used to demonstrate proof by MI for the first time. MI was to be introduced through a chain of implications, as the historical analysis had indicated that this was likely to be an accessible form.

Q3:

Q3 Let us return to this statement. Can an argument similar to the one we just used be applied in this case? [guide if necessary]

Q3 was intended provide a context for a further example of proof by MI, in the familiar context of the statement the student had been trying to prove in session two.

Q4:

Q4 What did you mean in the last session by the phrase "Prove itself"?

Q4 was intended to shed light on a phrase used by the student in session two, which seemed likely to indicate something of the student's attitude towards proof.

B: the student who was to be made skeptical

This session was divided into two parts, each with a different intention. The first part develops several propositions. The second refutes most of these propositions and introduces a discussion of the need for skepticism.

Q1:

INTRODUCTION to prime numbers
I: [Q1] Do you know what primes numbers are? [if "no" explain].
Here is a list of the prime numbers less than 10000. [Table 2]

Q1 was intended to ensure that the student was acquainted with prime numbers in preparation for Q2 and Q5.

Q2:

DEVELOPMENT OF PROPOSITION 1
I: Here is a statement:
[Figure 8: $P = n^2 + n + 41$]
[Q2] If n can be any whole number what can you say about P?

Q2 is the first of the questions which were designed with the intention of causing the student to make untrue generalizations, in this case that **P** is always prime.

Q3:

Introduction to Fibonacci numbers
I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers.
If "yes"
I: what do you know about it/them?
I: A merchant named Leonardo Fibonacci of Pisa studied this sequence in connection with a problem he was trying to solve. Since then many mathematicians and scientists have found applications of the sequence in a variety of contexts. This is the beginning of the sequence
[present figure 1:
1 1 2 3 5 8 13 21 34 ...]
[Q3] Can you find the pattern in this sequence?
If "no"
I: try adding up pairs of consecutive numbers [more prompts if needed]
I: Can you formulate a rule for producing the Fibonacci sequence?
[prompt if needed]
I: We can write the first Fibonacci number as F_1 and the second as F_2 and so on. If n is some number then F_n would be the Fibonacci number in position n in the list. How would you write the Fibonacci number which came right after F_n?
[help if needed]
I: Can you write the rule for producing Fibonacci numbers using F_n and F_{n+1} ? [help if needed]

Q3 is in two parts, which are similar to the first questions presented to the "skeptical" students in the mathematically more experienced group and to the other student in the mathematically less experienced group. The intention of the questions was the same in this context. As noted above one was designed to indicate the students' abilities to use recursion to define a sequence. The second was designed to indicate the students' ability to represent a recursive relationship algebraically. Both of these skills are needed to investigate the questions which follow.

Q4:

I: [Q4] How would you answer this question? [Figure 3a: "Is there anything special about F_n when **n** is a multiple of 3?"] [Guide to discovery]
I: Do you think this property holds for all Fibonacci numbers?
I: why?
I: How could you become more confident?
[expected response: more examples]
I: try looking at a few more examples
[work through examples.]

Q4 was designed with the intention of allowing the student make a true generalization. The students confidence in the truth of this generalization is later (Q8) used to indicate how skeptical the student has become.

Q5:

I: [Q5] Let's try something else. How would you answer this question? [Figure 3: "Is there anything special about F_n when **n** is prime"] [Guide to discovery]
I: Do you think this property holds for all Fibonacci numbers?
I: why?
I: How could you become more confident?
[expected response: more examples]
I: try looking at a few more examples
[work through examples.]

Q5 is similar to Q3, in that it was designed with the intention of causing the student to make an untrue generalization, in this case that $\mathbf{F}_p$ is prime if **p** is prime.

Q6:

DEVELOPMENT OF SKEPTICISM: Proofs and refutations
I: [Q6] Have a look at the formula for producing prime numbers. [Figure 8] Try n=40 [if not already done] Would you now say that this property only produces prime numbers? Why not?

The second part of the session refutes two of the three generalizations just made, and introduces a discussion of the need for skepticism. Q6 is the first question in the second part and refutes the generalization made in Q2.

Q7:

I: Look again at the third statement. [Figure 3] 19 is prime isn't it? [Q7] Is the 19th Fibonacci number prime?
I: Would you now say that this property is true for all Fibonacci numbers?

Q7 refutes the generalization made in Q5.

Q8:

I: [Q8] Would you now say that the second statement [Figure 3a: "Is there anything special about F_n when n is a multiple of 3?"] is true for all Fibonacci numbers?

Q8 was intended to serve as an indication of how skeptical the student had become. It was expected that at this point the student would no longer feel sure of the statement made in Q4.

Q9 and Q10:

I: [Q9] What would you say about trying to find out if statements are true?
[In the context of this discussion:]
We have seen that in mathematics it is extremely unwise to assume that something is true for all n just because it is true for some n.
You've just seen several cases where even though there are lots of examples, the statement turns out not to be true.
In the next interview you will be looking at some other statements, some, perhaps all, of which are not true.
Remember to be careful about accepting a statement unless you have a valid explanation.
[Q10] What would you say would be a valid explanation?

Q9 and Q10 frame the discussion of the need for skepticism. The discussion was intended to remind the student of the possibility that even likely looking statements might be false, and so to encourage the checking of some examples even if the possibility of a counter example seems slight.

Stage Two

The script for stage two was designed with the students' reactions in stage one in mind. As a result several of the activities relate to activities in the previous stage. Investigator interventions were to be limited in stage two. The script consists of five activities which the students were expected to work on independently. The possibility that the students might not finish all of them was considered, and they were to be told to take as much time as they felt was needed.

Introduction and Activity SA1:

I'm going to give you several different activities. For each one please discuss what you are thinking and doing thoroughly with each other to make sure that you are in agreement. In addition to any written work you might do in exploring these activities, please write down the conclusions which you agree on in the end.

Activity SA1: Here is a statement and a proof:
[Statement SA1:
For any number **n**, if a circular region (like a crêpe) is cut into pieces by **n** straight lines which cut all the way across, then the number of pieces produced (P_n) is at most 2^n.

$$P_n \leq 2^n$$

Proof:
One line produces exactly 2 pieces, so the statement is correct for n = 1 as $P_1 = 2 \leq 2^1$

Each time a new cut is made by a new line the new line, at most, cuts every piece into two pieces, doubling the number of pieces. This means $P_{n+1} \leq 2P_n$. If $P_n \leq 2^n$ then $P_{n+1} \leq 2P_n \leq 2(2^n) = 2^{n+1}$

This proves the statement $P_n \leq 2^n$ for all numbers **n**]
and some questions:
[Questions SA1:
1. Do you agree with the statement? Why or why not?
2. Can you give an example where the statement is correct, or an example where it is incorrect?
3. Do you agree with the proof? Why or why not?
4. What would you say is the most problematic thing about this proof?
5. Could you make the proof better somehow?]

Activity SA1 asks the students to evaluate a proof. The proof is a formalization of the argument that *A* was shown in the third session of stage one. It was expected that she would remember this experience and use it to explain the operation of the proof to *B*. *B*'s presumably skeptical reactions would then encourage further justification on the part of *A*. This particular proof was chosen as the first one because of its geometric nature, which was thought to be more accessible to the students.

The induction hypothesis $P_n \le 2^n$ was deliberately cast in the same form as the statement to be proven, to determine if this would create a difficulty.

Activity SA2:

[Activity SA2
Diagonals of a polygon are lines joining the vertices (corners) other than the sides. For example here are the diagonals of a pentagon:
[a figure was included]
The number of diagonals D_n depends on the number of sides **n** of the polygon:

Polygon	Sides **n**	Diagonals D_n	Example
Triangle	3	0	
Quadrilateral	4	2	
Pentagon	5	5	
Hexagon	6	9	

[figures were included under "Example"]
Show that the number of diagonals of a polygon is always:

$$D_n = \frac{n(n-3)}{2}$$
]

Activity SA2 asks the students to construct a proof of their own. It was intended that the students would use the previous proof as a model, and the inclusion of several examples was intended to facilitate this. It was also expected that the student's conversation while creating the proof would provide further insight into their understandings of MI.

Activity SA3:

Here is a table of the Fibonacci numbers
[a table of Fibonacci numbers was included]
and a statement with a proof :
[Statement SA3:
For all **n**, if **n** is a multiple of three, then F_n is even.

Proof:
F_3 is 2, so the statement is true for **n** = 3
We will need to refer to the following:

A. Even numbers are all multiples of 2.
B. Odd numbers are all 1 more than a multiple of 2
C. If you add an even number and an odd number ie 2n + (2m+1) you get 2n + 2m +1 which is odd.
D. If you add two odd numbers ie (2n+1) + (2m+1) you get 2n + 2m + 2 which is even.

If you have two consecutive Fibonacci numbers F_{k-1}, and F_k; and F_{k-1} is odd; and F_k is even; and **k** is a multiple of 3:
Then $F_{k-1} + F_k$ is odd (by C, above);
So F_{k+1} is odd because $F_{k-1} + F_k = F_{k+1}$;
And $F_k + F_{k+1}$ is odd (by C above);
So F_{k+2} is odd because $F_k + F_{k+1} = F_{k+2}$;
And $F_{k+1} + F_{k+2}$ is even (by B above);
So F_{k+3} is even because $F_{k+1} + F_{k+2} = F_{k+3}$;
Note that if **k** is a multiple of 3 then **k+3** is the next multiple of 3.
We can conclude that: IF it happens that F_{k-1} is odd, and F_k is even, and **k** is a multiple of 3 THEN **k+3** will be the next multiple of 3 and F_{k+3} will be even.
This proves the statement:
For all **n**, if **n** is a multiple of three, then F_n is even.]
and some questions:
[Questions SA3:
1. Do you agree with the statement? Why or why not?
2. Can you give an example where the statement is correct, or an example where it is incorrect?
3. Do you agree with the proof? Why or why not?
4. What would you say is the most problematic thing about this proof?
5. Could you make the proof better somehow?
]

Activity SA3 asks the students to evaluate another proof, this one a formalization of the reasoning used by *B* in preferring the statement "F_{3n} is even" to the others she encountered in stage one. It was expected that this activity would allow *B* to associate the methods used by A in the first two activities with her own experience. This would permit her to question aspects of the proof, which was intended to indicate what elements of MI she felt required more clarification.

Activity SA4 and Activity SA5:

[Activity SA4:
Consider the following:

1 + 1	= 2	= 3 - 1	= $F_4 - 1$
1 + 1 + 2	= 4	= 5 - 1	= $F_5 - 1$
1 + 1 + 2 + 3	= 7	= 8 - 1	= $F_6 - 1$
1 + 1 + 2 + 3 + 5	= 12	= 13 - 1	= $F_7 - 1$
1 + 1 + 2 + 3 + 5 + 8	= 20	= 21 - 1	= $F_8 - 1$

Show that in general:
The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

$$F_1 + F_2 + \dots + F_n = F_{n+2} - 1$$]

[Activity SA5:
The rule: $B_1 = 1$
$B_2 = 1$
$B_n = 2B_{n-2} + B_{n-1}$
defines the **B** sequence. These are the first terms of the sequence:
[a table was included]
Consider the following:

1 + 1	= 2	= 2^1
1 + 3	= 4	= 2^2
3 + 5	= 8	= 2^3
5 + 11	= 16	= 2^4
11 + 21	= 32	= 2^5

Show that in general:

$$B_n + B_{n+1} = 2^n$$]

Activities SA4 and SA5 ask the students to create proofs. It was expected that the presence of examples, and their previous experiences with the proofs in the first three activities, would allow them to construct proofs by MI for the two statements presented in these activities.

"Déroulement"

The Unfolding of the Study

There would be no point in conducting clinical studies if the responses to the questions asked could be anticipated fully. In the course of the study several unanticipated responses resulted in changes being made to the course of questioning. The way in which the study was actually carried out is described in this section, with explanation as to the reasons for the changes.

Subjects

Three of the four students in the mathematically more experienced group were, as intended, drawn from a first year linear algebra course. They are indicated by the initials *E*, *G* and *H* in the discussions which follow. The fourth, who will be referred to as *J*, was a friend of *E*. *E* and *J* are female. *G* and *H* are male. *J* turned out to be mathematically the most experienced of the four. The expectation that the students would not have seen MI in the past turned out to be erroneous. *E*, *G* and *J* all had experience with MI as a method of proof prior to the study. The effects this had on the organization of the study are discussed below.

The two students in the mathematically less experienced group, *A* and *B*, are both female. *A*'s highest level mathematics course was a college calculus course, taken seven years prior to the study. *B*'s highest level mathematics course was a high school pre-calculus course, take five years prior to the study. Both of these students had limited algebraic skills and concepts, which interfered with some of their attempts at proof. No major modifications to the design of the study were necessitated by this, although more active intervention on the part of the investigator was required than might otherwise have been the case.

Modifications to the Organization of the Study

The largest change made to the organization of the study came about because of the prior experiences the three students in the mathematically more experienced group had had with MI. For the first stage *E* and *G* formed the pair who were to be shown MI, and *J* and *H* formed the pair who were to be made skeptical. The original plan was to have *E* and *H* form a pair for stage two, and *G* and *J* form the second pair. *E* and *J* displayed a thorough understanding of MI as it is usually taught, and were dropped from the study after stage one. Much of the interaction which had been planned for stage two occurred in stage one as *E* and *J* attempted to explain MI to *G* and *H*. After the first stage *G*, who was to have been the proponent of MI in stage two, was clearly skeptical, and *H*, who had been cast in the role of the skeptic, was enthusiastic about MI. As a result stage two proceeded using *G* and *H* as the only pair, and in the opposite of their anticipated roles.

Changes were also made to order of the scripts in stage one, especially in group A, who were meant to be learning MI. The students made slow progress in finding the angle sum formula in item one, and the investigator intervened, guiding them to an informal proof using MI. The students also had to be guided to a proof for the second statement. *E* identified both of these proofs as MI, and *G* also noted that he had experience with MI, hence it did not make much sense to go through the sequence causing them to compare their proofs with the axiom. Items 4 through 7 (which dealt with the axioms of the natural numbers and the place of MI in them) were skipped and the students were asked complete the proof in item 8 independently. Again the investigator had to intervene. The investigator skipped to item 11 in order to explore *G*'s difficulties with MI, with item 9 (the statement "n(n+1) + 11 is always prime") introduced as an example at one point. Item 13, discussing the differences between MI and empirical induction, was also used, although with a different example than that included in item 12.

In group B the script was followed, for the most part. The students never found the relation they were expected to in Q3 and so it was given to them. Q9 through Q11 were rendered redundant, as the students found counter examples to all three false statements by themselves. It seems that their skepticism was already fairly well developed, especially in the case of *J*. Q12 was unexpectedly answered by *J*'s proof using MI, and *H*'s wholehearted acceptance of it. A false counter example was manufactured by the investigator to test the students' faith in *J*'s proof and the counter-example was rejected.

As noted above *G* and *H* participated in stage two, with *H* expected to accept the use of MI and *G* expected to be skeptical. They did not make the expected modification of the first "almost true" statement, and constructed a proof of the second without any use of MI. *H* and *G* did not discover either of the recursive definitions of the B sequence which were expected. Instead they discovered two more. The question was modified to proving the equivalence of the two statements they had found, using MI. It turned out that the proof is quite difficult, and this problem was abandoned. The students were then given the problem of showing that one of the anticipated definitions was equivalent to the definition *H* had found. This was expected to be much easier to show. *H* did show it easily, but without using MI. The discussion of the proof of the Binet formula was expanded to include a variety of issues which had been raised during the session.

It should not be assumed that the modifications indicated above completely undermined the original intent of the study. In fact, the students' previous experiences with MI allowed insights into the actual effects of the teaching of MI on the minds of these students. In effect much of what was to be accomplished in stage two was accomplished in stage one, and stage two became an opportunity to closely study the opinions of *H* and *G* regarding MI.

As a certain amount of modification was anticipated in the design of the scripts for the mathematically less experienced group, less actually occurred during the sessions. *A* was to be taught about MI, and *B* was to be made skeptical. The first two sessions involving A were preliminary in nature. In the first session the extent of A's difficulty with algebraic notation was revealed and the design of session two reflects this. Session two revealed that algebraic manipulations in general were difficult for A and the inclusion of geometric activities in session three and stage two was a result of this. Session three proceeded according to the script with the exception that A asked about a listing of the **B** sequence which was among the investigator's papers and this lead to her being asked to find a pattern in it. She eventually found three, arriving at one of the expected ones only after considerable guidance from the investigator.

The only surprise in the stage one session with *B* was the extent to which she was already skeptical. This was primarily a measure of her confidence in mathematics, but it was clear that she could be relied upon not to accept as true almost any statement. A short digression from the main script occurred as the investigator attempted to lead *B* to a proof that the sum of two odd numbers is even.

In stage two the amount of investigator intervention required was greater than had been anticipated. This was a result of the students' limited memories of their experiences in stage one. The students did not develop a sufficiently clear understanding of the proof in activity SA1 to use it as a model in SA2. The investigator had to guide their steps on several occasions. These two activities took most of the time allotted to the session, and the student's opportunity to discuss SA3 was limited as a result. SA4 and SA5 were not done.

While the sessions with the mathematically less experienced students did not result in much formal MI, there was a surprising amount of informal activity relating to MI in their work, even before they had seen MI. The material which was derived from the mathematically less experienced group has a very different character than that derived from the more experienced group, but it is no less interesting.

CHAPTER III

RESULTS

This chapter relates the results of the clinical study. It is divided into two sections. The first section outlines each students' behavior during the study, with analysis of the understandings revealed or indicated by this behavior. The second section compares the students' understandings of MI and of several related concepts .

Throughout the following, references to the transcripts of the sessions are made by enclosing line numbers in parentheses, eg (A123). In the quotations ellipses (...) indicate that the quotation has been edited to improve readability. Generally, repeated words, unfinished phrases and null words such as "yeah" and "um" have been omitted. The full line quoted appears in the transcripts in Appendix B. Three dashes (---) indicate a short pause in the speech. Capital letters are used to indicate the names of variables, even when they would be written in lower case. This is done both to make them more visible, and to evade the problem of determining which of several possible expressions might be intended.

Section A

Analysis of Each Student's Behavior and Understandings

This section describes for each student the behaviors related to MI they exhibited during the study. In each case this behavior is analyzed first in terms of what ways of understanding MI were indicated and then in terms of the ways of understanding various related concepts which were also suggested by the student's behavior. The related concepts discussed are not the same for every student, as none of the students had exactly the same experiences throughout the study. Specifically, there is less which can be said about *J* and *E* with regard to their understandings of related concepts, as they spent less time involved in the study than the other students.

Analysis of *J*'s Behavior and Understandings

Summary of *J*'s Concept of MI

J was acquainted with MI before the study. Unfortunately there was not much opportunity to discover how she would characterize MI. Her one comment on that subject was:

> J189: You use induction when you prove the truth of some statement for N using the fact that the previous statement was true. I am trying to prove that K plus 1 is true using the fact that K is true

This description seems tied to the procedure of using MI as a proof. *J* mentioned explicitly the usual subscripts K and K+1. This may be because she was trying to describe the procedure rather than the principle behind it, or she may have understood MI to consist of the procedure only. Her use of the word "fact" may indicate some awareness of the underlying principle, as in the procedure there is only an assumption that the previous statement is true, and a verification of a specific case. An understanding of the underlying principle, especially if based of the idea of "chaining" would justify referring to the previous statement as a fact. Her initial explanation of why her proof of "For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$" actually proves the statement was limited:

> J153: Because it's proven.
> J154: Well, that's the formal proof.

Her explanation may reflect a belief that it is the "formal proof" status of the procedure of proving by MI which give it legitimacy. At least it indicates that the underlying principle of MI was not called to mind to explain the functioning of the proof.

J certainly accepted MI as a method of proof, given that she used it as such to prove "For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$" (J139-152), and also given her faith in the correctness of her proof:

> J182: No, there cannot be anything, anything wrong with this proof.

J also recognized the problem of proving that $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ as suitable for proof by MI. This was the only context in which she tried to use MI and as it came late in the session it is hard to say how accurate she is in recognizing appropriate contexts for MI. That she used MI without any prompting certainly indicates some ability to do so.

J's Understanding of Concepts Related to MI

Infinity

J realized that proving that $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ by checking examples would be an infinite process:

> J137: It can go forever --- so there must be a way to prove it

and her use of MI as a method of proof of this statement indicates an awareness that MI does indeed prove an infinite number of cases. In general, J was willing to make conjectures on the basis of examples, but did not ascribe such conjectures the same validity that she did a statement proven by MI.

Use of specific versus general techniques

There were many opportunities to observe J's preference between consideration of specific cases and searching for general rules. In general her behavior seemed to be based on her description of how one determines the truth of a statement:

> J159: First you have to check ... if it is true, like, if you cannot find any counter-example ...
> R122: How far do you go searching for counter-examples?
> J160: Through every natural number
> J162: and then you must think ... why it works, ... how it works and ... how you can prove it

She considered specific cases when investigating problems on several occasions. She first did so when she accepted $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ based on checking a few cases (J38-41). In this case she appears to have been inclined to believe the statement to begin with, and a few substantiating examples were enough to convince her. It was only after she had considered some statements which held true for several cases but were not generally true that she tried to prove this statement.

When she encountered Statement 2, a polynomial formula which was falsely claimed to give the Fibonacci sequence, she immediately began checking specific cases. In this case it seems she did not think that the statement is true. In fact she later commented:

> J174: ... I was just wondering
> J175: ... if there was any formula for this, [the Fibonacci sequence] ...
> J177: ... You have to calculate everything from the beginning, ... and so to get from this point to 87 would take you some long time

indicating that she felt it unlikely that there exists a general formula for the Fibonacci numbers.

When trying to determine the truth of Statement 3: "For all n: P = n(n + 1) + 41 is prime" she commented:

> J66: ... if there was a way to get a multiple of 41 in here then ... then it wouldn't be a prime number anymore ...

She then began checking values:

> J73: 41, 6, 7 times 8, 6 [writing "2, 6, 12, 20, 30, 42, 56, 72"]
> R58: What are you doing?
> J74: No I'm just trying to know if this expression, can ever give me like a multiple of 41

In this case she was systematic in searching for a counter-example, first reducing the statement to "$x^2 + x$ is not a multiple of 41" and then checking values.

Her immediate reaction to the statement: "For all n: if n is prime then F_n is prime" was "No" (J105). She then began looking for a counter-example:

> J107: 1 2 11
> H122: What are you doing? ...
> J109: I'm trying to check them

On only one occasion did *J* deal with a problem entirely in a general way. This was in response to the question:

> Is there anything special about the sum of the first **n** Fibonacci numbers? Is it related to the sequence in any way?

In response she began by working with a general sum: "F1 + F2 + F3 + F4 + F_n" which she reduced to: "F1 + F2 + {F1 + F2} + {F2 + F3} + {F3 + F4} + {F_{n-2} + F_{n-1}}" (*J* did not include the usual "..." to indicate the uncertain length of the series). She continued working in this way until it was suggested that she look at specific sums. Even then she saw no pattern in the sums (J29, J37). As noted above, when shown the statement $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ she accepted it based on checking a few cases (J38-41). Her preference for searching for a general rule in this case, without checking specific cases, may be due to the different nature of this problem in comparison to the others she encountered. This problem is the only one in which a rule or formula must be derived, rather than merely verified. The only other occasion when a derivation might have been required was in determining the defining rule for the Fibonacci sequence, but in that case *J* already knew the rule. This is also the only occasion in which a judgement can be made concerning *J*'s preference for reduction of a problem to a simpler case, versus induction of a solution from simple cases. Her first preference was to reduce a general case to a simpler form (J11-14), but she later recognized that MI could be used (J139). This may indicate that she saw that an inductive approach might be fruitful, but it could be that she simply recognized the situation as suitable for MI without thinking of MI as inductive in nature.

J's initial preference for searching for a general rule may also be related to an expectation that there was one to be found. She indicated that she believed that there must be some relation when she expressed surprise at the possibility that there might not be:

> J25: ... you should be able to find the relation.
> ...
> R22: If there is one.
> J26: If there is one ... mmm

Treatment of counter-examples

J's attitude towards counter-examples seemed to depend on the context in which they are found. In general she rejected any statement when a counter-example was found. In the case where a proof also existed, however, she recognized that either the proof or the counter-example must be rejected.

She recognized that a statement cannot be true if a counter-example exists (J159-160) and rejected the statement "For all n: if n is prime then F_n is prime" when she found a counter-example:

> J115: It's not true
> H133: What? Is it true?
> J116: No it's not, because I found a counter-example

This overcomes the opinion she had developed previously:

> J112: Yeah, by observation, this seems to be right

However, she almost ignored the counter-example found by *H* to the statement: "For all n: $P = n(n + 1) + 41$ is prime". This may have been due to *H*'s presentation. He began by explaining the process by which he found the counter-example, concluding:

H86: It could be any K here, actually, there's still K here because I'm just taking an arbitrary N right, could be any N, show I'm just working so it came out to 1, so try, plug in 40, gives me, gives me, 40 squared plus 40 plus 41 gives that [wrote "1681"] and dividing by 41 gives 41, as it should.

J80: I still don't understand, what you are saying is N squared,

It seems that the discovery of a counter-example (n = 40) was not as interesting to *J* as the process by which it was discovered. *J*'s confusion led to another explanation of the derivation of the counter-example. At the end of this explanation *J* first commented on the counter-example itself:

J85: So you want to find N for which this could happen, Ya? --- and your N is 40?

At this point she seems to have realized that the derivation was not so important as the existence of this counter-example. While continuing to listen politely to *H*'s explanations, she turned her attention to confirming for herself that 40 was in fact a counter example (J92).

The last counter-example she encountered was in the claim that the statement: " $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$" is not true for n = 87. This claim was made to her after she had proven the statement by MI, and she rejected the counter-example, noting that she would "ask for an explanation, a detailed explanation, (J169) Why?! Why does it work? (J170)". She went on to suggest that the discovery of such a counter-example was probably a result of a computer miscalculation (J172).

Analysis of *H*'s Behavior and Understandings

Description of *H*'s Understanding of MI As it Developed in the Course of the Study

H had not seen MI prior to the study (H221). He had no problem with the recursive definition of the Fibonacci sequence, and pointed out the need for initial values (H3-4), indicating a certain level of comfort with recursion and awareness of the way recursion works.

He first encountered MI when *J* proved the statement: "For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$" using MI. He was not aware of the usual forms of the proof as he had not seen it before, and questioned the assumption of the induction hypothesis (H178). Once the proof was complete he accepted it without reservation (H180). He could not explain why the proof worked (H181-182) and may have thought that the induction hypothesis had been proven within the induction step (H183). He recognized that all that was needed for the proof was the proof of the basis and the proof of the induction step (H185) and gave a chain of implications as a justification for accepting the proof:

> H186: ... If it works for F of 8, ok, then it works for F of 9, and also works for F of 10, it works for F 11, and so on and so forth. It works for everything, right.

Considering this was his first exposure to MI, *H* understood it both as a formal proof and in an informal way quite quickly.

He was very confident of the proof, and when it was suggested that there might be a counter-example to the statement he rejected the idea (H203) and stated that if the proof was incorrect "then there's something extremely wrong with our minds" (H204) He seemed to understand the nature of the variable **k** in the induction step as well (H222).

To justify MI *H* referred to the axiom that every number has a successor:

> H229: If it works for 1, and it works for K plus 1; since we know all numbers exist, that every number goes 1 plus 1 and you get all numbers; if you add 1, you know it ...

and that a chain of implications could be generated:

> H230: works for some K, ... and it works for some K plus 1. If it works for a value then it also works for the value K plus 1, and it works for all values K plus 1. You want some value N and it works for the one plus that, so then that one has to work, and the one plus that has to work too, and so on, and so on, right?

The reference to the successor of a number may have been related to the need for **k**+1 to have meaning in the induction step, or it may have been referring to the well-ordering of the natural numbers. He repeated this reference to the successor of a number in stage two (H696-698).

Stage two began with the problem of proving the diagonal formula for polygons. *H* and *G* proved the formula deductively, without considering using MI (H434-468). When presented with the **B** sequence *H* had no trouble developing a recursive definition for it (H476). His definition was based on doubling the previous term and adding or subtracting 1 depending on the parity of **n**.

He mentioned again that he had not used MI before (H501) but claimed that he could recognize it (H503). He was not totally comfortable with MI:

> H504: ... I'm just not really at ease with it so I try and do something else unless I ---

When trying to use MI to prove the equivalence of the two definitions of the **B** sequence he and *G* had produced, *H* began by starting with a "low one" (H507), his basis. He then tried to set up the induction step, making use of the notation "F_n" which had been used it the proof he saw in stage one (H525). In describing MI he described the procedure followed when constructing a proof by MI (H539, H542):

> H542: ... for each you find one that works, and then you say if B K works then, you should prove that B K plus 1 works then it works for all B.

His plan for the proof correctly made use of MI (H545) taking into account the need to work with **k**+2 because of the existence of an odd and an even case (H567). At one point (H587) he tried to substitute **u** for both **k**-1 and **k**+1 in order to establish the induction step:

H587: and K minus 1 here, is it the same as saying, it's U here, and U here? [writing **u** over **k-1** in the upper expression in figure 1] because it's the same, because these are the same ones right?
H588: these are these are the same expressions
H589: Now here it's again the same expression [indicating the lower expression in figure 1]

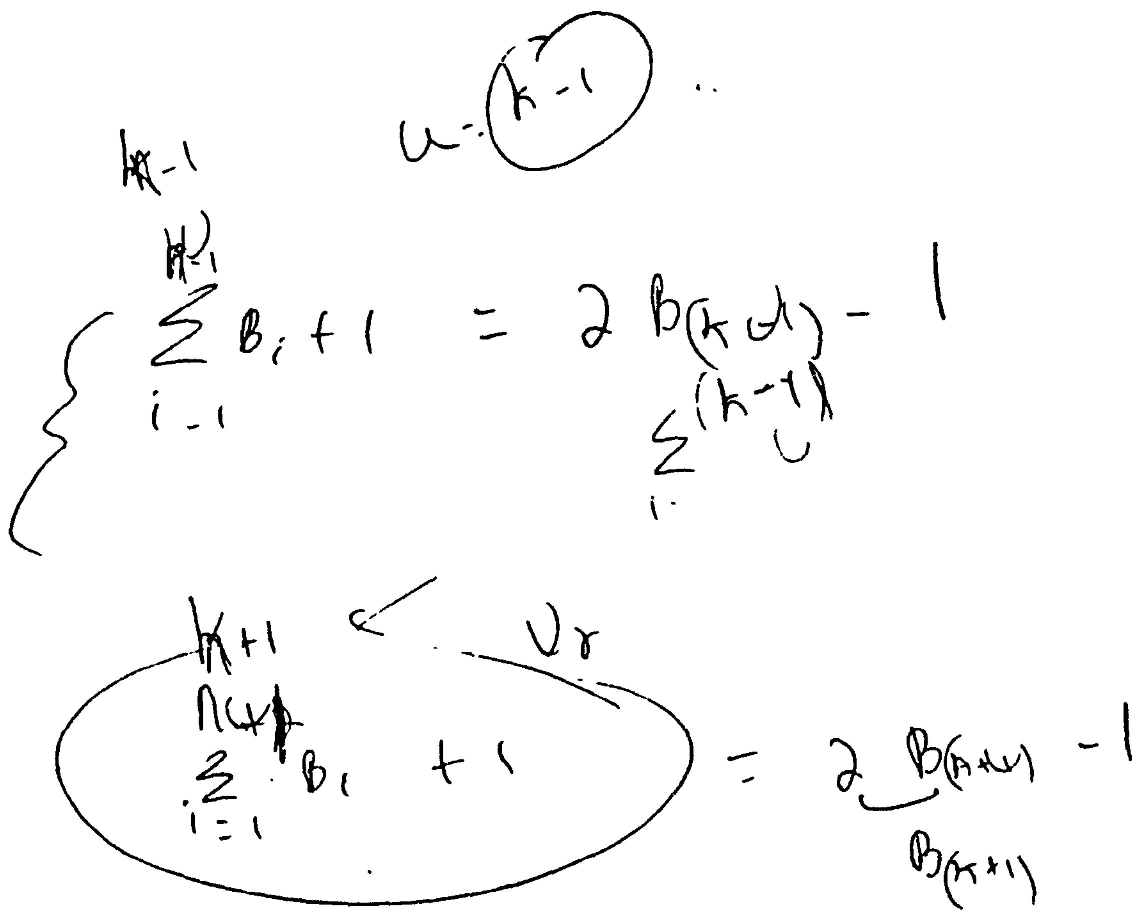

Figure 1: *H*'s writings referred to in line H589, involving the substitution of **U** for **k-1** and **k+1**

H has substituted **u** for **k-1** and then substituted **k+1** for **u**. As such a substitution would be acceptable in a straight forward proof without the complexity of a hypothesis, he may simply have forgotten. On the other hand a comment he made later (H727) indicated that he was not sure whether **k** could be treated as a general number. This confusion, if it existed, does not seem to have affected his use of MI.

When *G* described his idea of MI, based on backtracking, and outlined a proof of the equivalence of the two definitions, *H* may have been aware that G's proof was not MI. His comment is suggestive but not clear:

H612: It's not like a really, but I mean the idea.

H passed up another opportunity to use induction when asked to prove $2B_{n-1} + B_n = B_{n+1}$. In spite of the fact that he had been trying to use MI to prove a very similar statement his proof was straightforward and deductive and he correctly stated that his proof did not employ MI:

H648: This isn't really induction
S508: Well, are you relying on some assumption that it works for some previous Ns, do you do that?
S509: ... or is it a straightforward ---
H651: well no, it's very straight forward. ...

It should be noted that the investigators had not closely followed his proof, and had assumed that it employed MI. *H*, however, was sure that it did not.

H and *G* were shown a proof of the Binet formula for the Fibonacci numbers, along with a proof by MI. *H* followed this proof without objection (H660-690).

In the closing discussion *H* justified MI by noting that the natural numbers don't "skip" :

> H696: Well because you don't skip any numbers. You can't skip numbers. They're all there. Every number exists.

as well as again making reference to the axiom that every number has a successor:

> H697: ... If it works for any number, and it works for the number plus that, then it works for all numbers, because all numbers has a plus, K has one more,

He was then told that MI is based on an axiom of the natural numbers. He wondered of this axiom "how does it come to that conclusion?" (H704), which may indicate that he was unsure of the status of axioms in the logical structure of mathematics. When the axiom was stated he responded: "that's what I said" (H707) and in fact many of his descriptions of MI had been very close to the form of the axiom. They lacked the explicit reference to it being an axiom, however, which denied them the freedom from the need for proof enjoyed by axioms.

In response to *G*'s conviction that exceptions could still occur if a statement had been proven by MI, *H* asserted that no exceptions could occur (H708, H710-711) He also stated that the implication proven in the induction step was general (H712) and that if a statement is not true then a proof by MI simply will not work (H714). He later combined these two ideas in pointing out that if exceptions existed then it would be impossible to prove the induction step:

> H722: yeah it can't ... or else you can't prove that it works for all ... K plus 1 ...

In general *H* seems to have understood MI in a manner which was clear and self-consistent. The only indications that he might have been uncertain about MI were his unwillingness to use MI and some indications that he was uncertain of the generality of the variable in the induction step. This has been mentioned above (H587) and is also indicated by these quotations:

> H727: ... let's say if you find, ... say 10 can you backtrack, can you assume that all the ones lower than that too are true?
> H728: ... you've only proved one, ... Is it automatically true that the ones behind it are true? That's what I mean.

At this point *H* must be wondering if **k** can be treated as a general variable, or confused about the converse of the induction step.

<u>Summary of *H*'s understanding of MI</u>

In spite of having no memory of seeing MI in the past *H* accepted it immediately when he saw it and exhibited considerable confidence in MI as a method of proof. He asserted that *J*'s proof, relating to the sum of the first **n** Fibonacci numbers, was correct on several occasions (H216, H222) and stated that if a counter-example were produced then: "The person who's telling me that calculated incorrectly" (H206).

In the course of the study *H* seems to have developed a good grasp of the nature of MI. He was aware that the existence of a single counter-example to a general rule would make proof by MI impossible. He was also aware of the need for the basis. *H* characterized MI in several ways at different times; as a chain of implications, as a procedure, and as a formal method of proof. All of these characterizations are consistent with what is traditionally considered to be a good understanding of MI.

<u>*H*'s Understandings of Other Concepts Related to MI</u>

Use of specific versus general methods

H worked with specific examples, or with general forms, depending on the nature of the problem he was considering. He used specific examples in three ways: as examples when explaining a general principle, when searching for a pattern, and when testing a statement which he doubted. He worked with general forms when constructing proofs, and to search for patterns when specific examples failed to reveal any.

When he was explaining to *J* that the final term of $F_1 + F_2 + \dots +F_n$, when decomposed into the Fibonacci pairs which make up each term, would not be a duplicate of another term, he worked through a specific example:

J19: All of them will be counted twice
H17: Well not all of them, the last one you won't

count. Let's say if you did it for a certain set here ---

H18: This one would not be counted in, at all. This one would be counted once, would be counted in twice. See it would be like this, from here to here would be 12. This which is equal to 1 plus 1 and you add this one again plus 1 both twice, because they'd both be counted twice, plus 2, hmm, 2 plus 1 for this one because you're counting this one, like this [Pointing to "1 + 1 + 1 + 1 + 2 + 1 + 2 + 3", his decomposition of 1 + 1 + 2 + 3 + 5]

He had been working with these series while searching for a pattern in the way the sum was determined. *H* later (H20) switched to a general notation to continue his search.

H's main use of specific examples was in testing conjectures he doubted. He did this in stage one to test Statement 2 (the false polynomial formula for F_n) (H52-66) and agreed with *J* that testing Statement 4 (For all n: if n is prime then F_n is prime) was a good idea (H123). In stage two he tested Statement A ($(F_n)^2 = (F_{n-1})(F_{n+1}) + 1$) and Statement B (The number of diagonals in a convex polygon of n sides is $(n)(n-3)/2$) (H409 and figure 34).

H's preferred method of searching for patterns was through general forms. As mentioned above he switched to general forms in searching for a pattern in the way the sum was determined (H20, H21). He continued in this way on returning to the problem later (H165) in spite of seeing it as complicated: "It'd be too long to do all that. I don't want to do it that way" (H163). He also used general forms in trying to determine the truth of Statement 3 (For all n: $P = n(n + 1) + 41$ is prime) (H71) and, in stage two, Statement A ($(F_n)^2 = (F_{n-1})(F_{n+1}) + 1$) (H402, H419).

When attempting to prove a statement *H* used general forms exclusively. His proof of Statement C2 (For all $n > 2 : 2B_n + B_{n+1} = B_{n+2}$) (H642-646) is entirely general:

S504: Did you check for some ... small Ns or not?
H645: No, I didn't
S505: No you didn't
H646: I think that in general I don't ever bother, I just ---

In fact *H* seems to dislike looking at specific cases in some contexts:

R23: Have either of you calculated any of the sums ... to just know what they are?
H23: Ah no I didn't want to actually, ...

R72: If you only have 40 cases to prove, you may as well prove every case.
H112: No that's too long, I wouldn't.

This disinclination might have affected his willingness to use MI, as Mi involves the use of at least one specific case.

Expectation of order in mathematics

H was inclined to assume that there would be a pattern to be found while working on a problem:

H33: No I think there probably is a pattern.

H172: You see I'm taking this, see I'm trying to find the pattern. ...

This seems to reflect a basic belief concerning determining the truth of a statement:

> H193: ... Mostly what you have to do is try to find some pattern, for all the examples.

Once he saw a pattern, or some order in the situation he was investigating, *H* expected it to continue. For example, having found that 40 is a counter-example to Statement 3: (For all n: $P = n(n + 1) + 41$ is prime) he conjectured that there were no counter-examples less than 40. He had not, in fact, determined that the statement was true for any specific values, but he expected that the counter-example which he had discovered was the least (H105). He also expected there to be a connection between Statement 4 (For all n: if n is prime then Fn is prime) and Statement 1b (For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$):

> H139: F of N minus 1 ... What was the, that thing, that you showed us before there, the, let's say that the summation of each one is equal to F of N plus 2 minus 1.

His reaction to *G*'s suggestion that they look for a counter-example to Statement A ($(F_n)^2 = (F_{n-1})(F_{n+1}) + 1$) also indicates an expectation that an orderly arrangement such as that suggested by the statement was likely:

> H421: Well I don't know. See the problem is ... if you do that, if it's not true, fine. If it's true ---

When *H* and *G* arrived at different rules to generate the **B** sequence *H* commented:

> H482: yeah but, I'm pretty sure there's a relationship between them, I'm sure you can figure out why.

and

> H488: I mean it comes down to the same thing. ... It's just another way of doing it. ... It has to be.

A final indication of *H*'s expectation of order came in the development of the proof of the Binet formula. *H*'s reaction to Lemma 1, before seeing the proof, was:

> H674: Which is, you know, very true, I mean, obviously, I can see that.

Treatment of counter-examples

H displayed a good understanding of counter-examples in general. He rejected four statements on the basis of counter-examples (H66, H77, H144, H431) and he stated quite explicitly:

> H192: You got to prove it for all cases that it's true, actually every case. Prove that there is no single case that i. not true.

When faced with a hypothetical situation in which a counter-example was found to a statement which had been proven by MI he commented:

> H204: If I knew, and if I found out that it was true, then I'd go "Well, shit", what can I say? I mean, obviously from that [the proof] it's true, end of story, it's true, It should work for everything. If it doesn't then there's something extremely wrong with our minds, because we don't understand what we're

doing.

H215: ... if that [the counter-example] was true then there is something wrong with this [the proof] for sure, there'd be no question about that. If you prove something that ... is not true then there's something wrong with it. ...

Use and attitude towards empirical induction

H made use of empirical induction in investigating mathematical statements. For example he seems to have accepted Statement S1b (For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$) on the basis of a few examples:

H45: What is, what did you say it is 12? yeah it is, let me see it, yeah it is, yeah it's true

When investigating the incorrect polynomial formula for the Fibonacci numbers he empirically tested the accuracy of his calculator keypresses:

H66: ... I'm getting 50 for a while, ... I'll do it one last time and then I'm saying no. But I'll make sure I'm doing it right ... [calculates] ... no, it [the formula] doesn't work.

He also accepted the validity of the two rules *H* and *G* invented for the **B** sequence on the basis of empirical induction:

H505: well anyway, these two, this is right, this is right, now what you want us to do is make sure these two things are the same?

H repeated that the two statements are correct on other occasions (H522, H563, H565, H632). He was aware, however, that empirical induction is not proof in the mathematical sense. Of the rules they invented he said:

R548: Did you ever prove either of those?
H732: Not like, no

Analysis of *G*'s Behavior and Understandings

Description of *G*'s Understanding of MI As It Developed in the Course of the Study

G had seen MI in the past, in a course on discrete mathematics. In the first problem he faced, determining the angle sum of a polygon with n sides, he followed the inductive investigation indicated by the investigator, and provided an informal induction step (G102), on the basis of which the formula $\pi(n-2)$ was eventually accepted. *E* noted that the proof could be done by induction, at which point *G* was asked if he had studied it. He replied that he had "touched up on it" (G109) and roughly outlined the procedure (G108-109). He used a triangle as the basis for the induction (G115), and then stated the induction hypothesis very formally:

$$\exists k \ni k > 3 \rightarrow (k-2)\pi = A_k$$

Figure 2: *G*'s induction hypothesis for the angle sum formula, written about line E135.

As a result of suggestions from *E* he switched from this formal pattern to a more informal approach describing the induction step in terms of going back to the one before (G133). *G* accepted the final informal proof by MI as correct (G152).

The second problem was that of showing that 2^n is an upper bound to the number of regions produced by **n** lines in the plane. He observed the doubling which is at the root of the induction step (G199, G202) and accepted it as an informal proof (G206). He stated that if the induction step is accepted then the statement is "obvious" (G207-208).

G did express some misgivings about MI, in spite of his acceptance of the informal proofs mentioned above. His first comment:

> G211: you use the hypothesis, I can't see why they use that hypothesis within their a, their assumption, ok? they assume something and then they use it within their proof

seems to indicate that his difficulty lay in the use of the inductive hypothesis. He may have seen the variable used as being general in nature, rather than being an arbitrary value. On the other hand the following quotation suggests that he saw the variable as being specific, and wondered how the statement can be assumed true for some value, without proof.

> G213: Ok, you don't, --- I have trouble seeing it. Why should that assumption be true, when applied to your ---

Yet another possibility is suggested by this comment:

> G215: ... You can go to as many steps as you want and your hypothesis is, I don't ---

The investigator had raised the issue of empirical induction, and *G* seems to have added that to the problems he experiences with MI. He may have been rejecting the idea that empirical induction provides the justification for the assumption of the truth of the statement, which is then used in the proof by MI of the statement. In fact, this is very close to what actually happens in MI. Empirical testing provides one case in which the statement is true, which is then used, by way of the reasoning outlined in the inductive step, to pass to a second case. The process then repeats. This is, of course, a dynamic model of MI; as a chain of implications. In the static form *G* has encountered the process is collapsed, and the justification for the assumption of the inductive hypothesis becomes more obscure.

G made a distinction between the proofs he had accepted, which had been proven by an informal use of MI, and formal proofs by MI. In fact he didn't consider the proofs he had just seen to be proofs by MI:

> G222: I'm convinced through word not through induction, and if this is indeed the case when you showed it here, in this particular case, I can see it.

If the proofs they had done had indeed been by MI then in this case *G* could see MI working, but he had not thought of them as proofs by MI. He gave an example of the sort of thing he would expect to prove by MI: the formula for the sum of squares (G251). This, and the formal way in which he began to prove the angle sum formula, indicate that *G* had thought of MI as being a very formal process, and this may have contributed to his uncertainty as to the role played by the variable in the induction step.

When *G* discussed proving the equivalence of the two recursively defined sequences, **A** and **C** (see page 41), he again indicated an understanding of MI closely tied to proving formulas for sequences and series. He suggested proving the equivalence as follows:

> G251: No, it's prove by induction that your formula does indeed give you this recursive sequence, and then do like wise with this one, give a general formula, ...

He did not see the application of MI to proving the equivalence directly. Instead he suggested finding a formula for each sequence and using MI to prove these formulae. Such a task is possible in this case, as recursive definitions had been given for the sequences, but the direct proof is much simpler to find. Later (H563) when working with *H* on a similar problem, he suggested the same method, in spite of the fact that in that case the general formulae could not be proven, except with reference to each other.

In considering the equivalence of the two sequences *G* began by noting the importance of the equivalence of the first terms, which was given as part of the definitions (G263). He also began his attempt to construct an inductive proof by establishing the basis (G273-274). Throughout the remainder of the construction of the proof *G* referred to the inductive hypothesis as an "assumption" (G275, G294). This may merely reflect the choice of word of whoever first taught him MI, or it may be an indication that *G* felt that the hypothesis is an assumption, in the sense of an unjustified conclusion.

When the proof was completed *G* professed to being convinced (G305) but his next comment:

> G306: I'm convinced. Well for the next step I can see it as well.

raises another possible role *G* might have seen the variable in the induction step playing. If the variable is taken to be general, then there is no point in considering whether the induction argument works for the next case as well. In MI the variable is arbitrary, but only within the induction step. Once the induction step is proven the variable is treated as general variable, and so the induction step applies equally for every number. If *G* saw a need to check the induction step for **k** implies **k**+1 and for **k**+1 implies **k**+2, and so forth, then he was not changing the role of **k** from arbitrary to general. If this was the case then for a proof by MI to be seen as correct by *G* the induction step would have to have been proven for every **k**. Further indications that this might have been the case (G578, G588, G591-592) are discussed below.

In a general discussion of MI which followed the completion of the proof of the equality of the sequences *G* again described the difference he felt existed between two different kinds of proof by MI. Some proofs, such as the informal ones which began the session, he felt were obvious, while others he could not accept:

> G314: ... some I find trouble seeing it, with others it's evident, it's obvious. But some of them I ask why do you prove? It's obvious.

He again treated MI as if it were empirical induction, when he wondered how long a proof by MI would have to go on to show that π is irrational (G325). This supports the suggestion made above that *G* felt that the induction step must be proven for each **k**, and also indicates that *G* understood that MI is best suited to proofs involving infinity.

In lines G337-338 *G* first mentioned the idea of going back to the previous case to establish the induction hypothesis. This idea was developed more fully in stage two, and is discussed below. This comment occurred in the context of a discussion of the expression $n^2 + n + 11$ which produces prime numbers for $n<10$. *G* wondered how MI could be used to prove such a statement (G338) and then how to disprove it using MI (G340). He seems to expect that the process of proving by MI would reveal counter-examples, in the same way as a proof by empirical induction would (G343, G345). If, as was suggested above, *G* believes that MI involves the proof of the induction step for every **k** then this would be that case.

G may have a problem with MI in that MI does not take the form of most of the proofs seen in mathematics classes; that is, the reduction of an equation to a blatant identity. The following quotation indicates that *G* is uncomfortable with proofs which do not "come to a final agreement":

> G351: I can move forwards, I can move backwards, but I have to move somewhere to show my conclusion, now it's either I can prove it wrong or I can['t]. Like I said we can imply things from right to left or from left to right but we're going to come to, we're going to come to a final agreement, but this doesn't, you have no, no concrete way of knowing that this is indeed proof enough. That's what I don't understand.

Stage two began with the problem of proving the diagonal formula for polygons. *G* and *H* proved the formula deductively, without considering using MI (G430-484). When presented with the **B** sequence *G* had no trouble developing a recursive definition for it (G492-493). His definition required the summing of all previous terms in the sequence to determine the next term.

G suggested using MI to prove the equivalence of the definitions he and *H* had found for the **B** sequence (G510).

The way in which both definitions differed for even and odd cases seems to have cause him some trouble. At one point (G536) he tried to go from **n** to **n**+1 in spite of the fact that he was concentrating on odd cases. This may be a result of attempting to preserve the form in which he had seen MI in the past. He later agreed with *H*'s procedural description of MI (G543). Both of these acts indicate a concentration on the form of MI which is to be expected if the underlying structure is not very clear to him.

Proving the equivalence of two definitions of the **B** sequence is a similar problem to that of proving the equivalence of the two definitions *G* had seen in stage one. *G* attempted to solve the problem in the same way, by proving a general formula for each sequence and then showing the formulae were equivalent (G563). In stage one this strategy was inefficient. In the case of the **B** sequence it was impossible, as no other definitions had been provided. The combination of sequences with MI in a situation seems to have strongly indicated to *G* that he should use MI to prove a formula for the general term of the sequence. Presumably, this is a product of his training in the past.

In the course of trying to prove the equivalence of the two definitions *G*, expanded on his idea of MI as backtracking to the basis:

> G578: ... and then it'll take you to K minus 1,
> and that one will take you to back, back, back

> all the way to the first step, this is the way I proved mine, ...

He repeated the same sort of description on several other occasions (G588, G591-592). This process is reductive rather than inductive, and may be related to recursive computer programming techniques, in which the basis, instead of being the starting point, is the "stop rule". It is not clear that *G* meant this backtracking to be general.

Shortly after these comments about backtracking he switched to building up from the basis, but he was only considering specific cases (G597, G610). Notice in the following that each induction step must be done separately:

> G609: It's an imply-ance, it's I think, it's you imply the first, knowing that the first one is true, you can imply that the second one is true, knowing that the second one is true, you can imply that the third one is true, and so forth, but you can't, I couldn't go from 3 to 10 for instance, without having to go through, 4 to 9, ...

This building up seems to be the process of proving the induction step for each **k**. *G* later returned to describing what he was doing as backtracking (G663, G665, G669, G671-672). He may have been combining backtracking with the idea of proving each step, and have meant that backtracking is simply a standard way of testing a specific case.

From the above discussions it seems clear that *G* was not comfortable with the idea of using MI to prove general statements. He indicated that he did feel this way (G673-

674, G679), and also mentioned that it was the possibility of exceptions which disturbed him (G680, G682) The possibility of exceptions was a result of: "not checking every one" (G684).

The sort of understanding *G* had of MI at this point can be seen by working through the examples he suggested:

> G685: ... You gave a sequence, but assume that you gave us a formula that works for everything, but it doesn't work this one, ...
> H713: You can't come to the conclusion that it doesn't work for all N plus 1
> G686: Is that true, I don't think that's true, I think you can assume it, that it is true, and just by backtracking, check your initial hypothesis because your hypothesis was based on the first few numbers, and it won't be correct for the entire sequence,

G was asking *H* to consider a sequence such as 2, 4. 6, 7, 10, 12 and a formula such as $S_n = S_{n-1} + 2$. In this case *G*'s backtracking will work to prove that the formula defines the sequence for 2 4 6 10 and 12, but not 7 as *G* did not see backtracking as providing information about any element of the sequence other than the one being tested (G687). The failure of 7 to conform to the formula will not be discovered unless it is checked specifically (G688). When he combined these problems with the infinite sets covered by general rules he arrived at the impossibility of testing every case, and so the impossibility of using MI to prove general statements (G689-690).

If the assumption made in the induction step could be proven in some way then *G* would have no problem with using

MI as a general proof (G702-3). Otherwise, *G* was not willing to accept MI as any more reliable than empirical induction (G706).

Summary of *G*'s Understanding of MI

G seems to have treated informal and formal arguments by MI differently. He was willing to accept informal arguments, but didn't see them as MI. He rejected formal proofs, on the grounds that they made an unjustified assumption of the inductive hypothesis. This reduced these proofs to being empirical in nature, consisting of an assumption that the statement is true for a particular **k** and then a, possibly recursive, verification of this. As *G* rejected empirical proofs, he also rejected formal proofs by MI.

MI as empirical proof was one of the characterizations *G* employed during the study; the other was MI as a procedure. *G* also displayed a preference for seeing MI as beginning at a case and proceeding backwards to the basis.

G's Understanding of Concepts Related to MI

Infinity

G was aware that the sequence of numbers involved in a proof by MI is infinite. That is part of his problem with the method of proof:

G689: Fine, but you're not going to go---. If there's an infinite sequence of numbers you won't check every single one.

Use of specific versus general methods

G seemed to prefer investigating specific cases when problem solving. He did so when investigating angle sums (G10), regions produced by lines (G160, G204), sequences defined by recursive rules (G233), relations between Fibonacci numbers (G407, G409, G419), diagonals of a polygon (G437, G450), and the relationship between two recursive rules (G506, G595). He used general procedures only when investigating relations between Fibonacci numbers (G403) and the relationship between two recursive rules (G618-620). These two situations are exclusively algebraic, where many of the others are geometric in nature. Also in these cases *G* was working with *H*, who showed a preference for general formulae.

Treatment of counter-examples

G accepted counter-examples as disproofs of general statements:

S255: This, but this counter-example is proof enough?
G352: For me? that's fine

He rejected statements due to counter-examples at lines G425, G429, G488 and G623 as well. He was aware of the difficulties associated with proving by searching for counter-examples:

> G417: Or would it be wiser to look for a sequence, up until we can find something that contradicts it, but well, that might take forever

Use of and attitude towards empirical induction

G generalized based on a few examples:

> G23: the sum of each ---
> G24: is 180 degrees, so for every 2 sides we have a triangle [wrote "8 sides; 4 triangles"]

> G35: n divided by 2 times 180

He generalized in similar ways in lines G75-76, G419-420, G486, G491-494, and G627. *G* was aware that these generalizations were not proven:

> G706: ... for instance, ... you give us these numbers. ... Now ... assume our formula works for this particular sequence, but from 24 on there's a continuing sequence ... but they completely ... [differ] from our formula. We checked for this case here, then we've already set up our [formulae]. We've assumed that what we've written down is proof enough, but that's not proof.

When discussing the possibility of proving that his formula for the **B** sequence produced the same sequence as *H*'s formula, *G* was determined to prove that the formulae themselves were correct, rather than just proving them equivalent. He stated that even if the two were proven equal, "that wasn't proof enough" (G716). There are two possible explanations for this assertion. *G* might not have distinguished the proof of an implication from the proof of a conjunction or *G* might have thought that the terms he had seen of the **B** sequence were sufficient to define it, hence his expectation that he could prove his formula:

> R550: Could you prove this? [that *G* and *H*'s formulae define the same sequence]
> G711: Not for an infinite amount of numbers. No, unless you can tie it to the fact that it is a continuing sequence such as the natural numbers,
> H735: What are you saying? that after this it is not part of the sequence anymore?, say you put any number here?
> H736: Say I put 27 after this, say B 25 is 27
> G713: yeah something that,
> H737: Yeah, ok, I get what you mean, ok, now I understand what your problem, yeah
> R553: well, could you prove that the sequence determined by your formula is the same as the sequence determined by his formula
> G714: Yeah, but you have to prove one in order to show that the other is also true.
> G715: Just because we tie ours together we might not come up with the same conclusion that's not necessarily the right conclusion.
> G716: So even though we tried to tie it together, that wasn't proof enough. I don't think that was proof. Proving that mine was indeed true for this sequence, which if we check each and every single one then I could say it's proof enough.

Analysis of *E*'s Behavior and Understandings

<u>Summary of *E*'s Concept of MI</u>

E had studied MI previously and she characterized MI in three ways. Her first characterization was procedural, and came as a continuation of *G*'s description of the procedure:

> G109: well I've touched up on it, Prove your basic stuff, prove it for
> E112: For the first one
> G110: yeah for the first one
> E113: For the n equal three
> E115: And the second one you assume its true for probably K and use this assumption to prove it's true for n equal [K + 1]

Her second characterization involved a model:

> E225: It's very simply, if you have a very very long line of people and the last one kicks anybody then the next one wants to kick one before and again and again, you know

This model is reminiscent of the usual domino model, and may be something she remembered from explanations offered her when she first learned about MI. That she remembered it argues for it having some significance in her understanding of MI, although this is not apparent in her other comments.

Soon after using her model to describe MI, *E* gave a different explanation:

> E226: Because probably induction, we have to start from the middle. Every time we start with one line, with the first case and after that we start from the middle.

While this is not perfectly clear, one aspect is: *E* was describing MI as having two stages, one at the start and one in the middle. In this respect this explanation differs from her model. The model describes one continuous, cascading, process beginning with the basis and proceeding from there through all the elements of the set. This is quite different from a process in which "we start ... with the first case and after that we start from the middle". In this case the connection between the basis and the rest of the set is not so clear. *E* may have been basing her characterization not on the underlying process of MI but rather on the procedure of constructing a proof by MI, as described in her exchange with *G* (G109-E117 above). These two ideas may have been two unconnected parts of her understanding of MI. On the other hand she may not have distinguished this procedure from the recursive process at all.

E had no difficulty accepting MI as a legitimate method of proof. After having produced a proof of the angle sum formula she was asked what she had shown and replied: "We proved that our rule, that our formula, is good for all n greater or equal to 3" (E159). She may only have been offering the truism "When you have made a proof you have proved" but her disagreement with *G* when he refused to accept MI as a method of proof (G210-E230) argues otherwise.

While *E* accepted MI as a method of proof, she was not entirely happy with it. She seems to have felt that it is complicated:

E110: ... it's too long, it's terrible

E304: But it is, it's too long! ... I started to do it. You have so many indexes here.

E had no trouble recognizing appropriate contexts in which to use MI. She identified MI as appropriate on several occasions. First when asked how to formulate an argument for the sum of the angles of a polygon being $(n-2)\pi$ (E110). Second, when she identified the argument setting an upper bound on the number of regions created by n lines (E219) as being inductive in nature. Third, in discussing how to show that two sequences are identical (E273). In each of these three cases she recognized that MI is a suitable method of proof for the situation. In a fourth situation she recognized a context in which MI is inappropriate. The statement under discussion was: "n(n+1)+11 is always prime" which had been shown false by counter-example.

G340: Ok I can see the contradiction but how would you prove it false through induction?
E333: you can never prove it by induction.

She recognized that induction cannot be used to show that a statement is false, only that one is true.

E's Understandings of Concepts Related to MI

Infinity

While it is hardly conclusive, *E*'s model of MI as: "a very very long line of people" (E225) may indicate some confusion on her part. She might believe that a proof by MI demonstrates the truth of a statement only over a large set, rather than an infinite set.

Use of specific versus general methods

In working on problems *E* considered both specific cases and general rules applying to the problem. In working on the problem of determining a formula for the angle sum of a polygon she considered first the case of a triangle (E14). Later, when working of the problem of proving that 2^n is the upper bound of the number of regions produced by n lines, she began:

> E166: If you have one line you have two parts, if you have 2 lines you have 4 parts, ya?

In both cases she was considering specific cases for which the statement in question is easily tested. When she first examined the two recursively defined sequences, **A** and **C** in item 8, however, she began by trying to understand how the rules defining the sequences work in general. She did this before determining any specific values in the sequence, even though she was asked:

> S145: Have you any idea what the two sequences are?
> E235: From the previous element of this sequence we take the second coordinate and put it in the

first place, and the second place is obtained
by summing both of the coordinates

Here, rather that determining what the sequence is, she tried to see how it is generated.

Treatment of counter-examples

E was certain that a counter-example was sufficient to disprove a statement:

S20: You have not n minus 3 you have n minus ...
G100: 2
E99: OK, something, something is wrong

In this case she had deduced n-3 and while she acknowledged that her deduction was incorrect she continued to wonder why until it was explained:

E105: Why, I took this 3? I don't know, I have to check 1 2 3
S29: You just subtracted, but you have to add a Pi
E106: oh ok

Use of and attitude towards empirical induction

Although she didn't accept small samplings as sufficient to prove the truth of a statement *E* did use them to generate statements which she then tried to prove. This indicates both some trust in small samplings reflecting general laws, and a belief that general rules predominate in mathematics. She used small samplings in this way twice. First she used a table of values as evidence of a constant increase in the angle sum of a polygon as the number of sides increases:

E135: If we make the figure bigger then we, look at this, we have one Pi more
S52: you are showing the table
E137: Ya

Later she drew a conclusion from the first five terms of a sequence: "we get the same sequence" (E251)

Analysis of *B*'s Behavior and Understandings

Description of *B*'s Understanding of MI As It Developed in the Course of the Study

B began the study with no previous knowledge of MI, and no knowledge of methods of verification other than empirical tests (B16), which she recognized as limited (B9-14). She had no problem discovering the recursive rule for the Fibonacci sequence (B18, B27).

When she was asked to justify her assertion that F_{3n} is always even she gave an argument making reference to previous cases:

> B30: --- because ... , when you add two odd numbers together you make an even number ... the first two numbers are odd and so when they add up to the one that's a multiple of three then they become even

She was not very sure of this argument (B31). She did prefer it to completely empirical arguments (B42), possibly because it was related to other knowledge in which she had faith (B43). Later she said of it: "I'll say that until I'm proved wrong" (B57) but she thought it was likely she would be proved wrong (B58).

Her insecurity about the truth of this statement might have been increased by the empirical nature of the underlying assumption, that the sum of two odd numbers is even (B59). She later constructed a more complete induction step, accounting for the odd parity of the two intervening Fibonacci numbers:

B65: Because, ... if each Fibonacci number is the first one plus the second one equals the third one, ---... it starts out, ... you would be adding two odd numbers together and get an even number, ... then the next one then is odd, so you'd add that to the even and then you'd come out to another odd, but then I don't necessarily know that the, that the next number after an even number would be odd so
R66b: can you think of any reason why the next one after an even number should be odd?
B66: --- because the one before the even number was odd.
R67b: How does that make the one after the even number odd?
B67: Because if you add an even number to an odd number then it comes out as odd.

The only problem *B* had with this proof is that she could not justify her statement in B67. It was just a verbal understanding, something she had been taught (B68). This undermined her confidence in the proof as a whole.

In stage two she was shown a formal proof by MI that the number of regions produced by **n** lines in the plane is less than or equal to 2^n. She accepted this proof, but her acceptance of the formal aspect of it may have been due to a lack of understanding of its logical structure (see the discussion of *B*'s behavior regarding proof, below). She does seem to have understood the situation though, and to have accepted the argument on an informal level.

B204: ... but it works with what it said up here.
B205: Or did you say it? how it doubles each piece?
B206: ... yeah because it's the number of pieces produced it at most 2 times ...
B207: the cuts

This is an informal description of the induction step of the argument, which seems to have established the truth of the statement for *B*.

B accepted A's proof that the diagonals formula worked for D_5 as a general proof (B336), although she noted that it depended on the assumption that the relation $D_n = D_{n-1} + n-2$ worked in general (B341) and pointed out that it required proof (B344-345). This serves as an example of her somewhat uneven knowledge of the requirements of a proof. She expressed a desire to generalize A's proof (B347) and when this was done she was concerned about its premises (B390). She didn't seem to have any problem with assuming the consequent in a proof. She was asked:

> R620: During the course of doing this you used this formula that you're trying to prove to figure out what D N minus 1 is.
> R622: Is this a problem?
> B397: Is it? Because if it weren't true then would it have worked out?

She made a similar comment later (B424). She also seems to have seen the variable used in the induction step as being general in nature (B430, B434). She did make the interesting comment:

> B437: ... But then that's the same thing. If we ... worked it out for a triangle --- the formula works for a triangle it works for a hexagon and it works for something that has 3 more sides ...

She was extending the idea of the induction step to one where the proof is of $P_k \Rightarrow P_{K+3}$. This would suggest that the idea of an induction step had been generalized to some degree.

Summary of *B*'s understanding of MI

B had never seen MI before the study, and does not seem to have abstracted much from the proofs she saw. She did make use of informal proofs by MI on several occasions. Although she was not very confident of these proofs, she preferred them to empirical arguments.

B's Understanding of Concepts Related to MI

Use of specific versus general methods

B used specific cases and tried to formulate general rules in appropriate contexts. When faced with the task of proving the formula for the number of diagonals of a polygon she began by checking the examples she had been given to make sure they worked (B256). When A noticed the pattern $D_n = D_{n-1} + n-2$ for small n, *B* began attempting to produce a general statement of this relation:

> B270: wait, the diagonals are equal to the sides minus 1
> B284: ... the diagonal equals the side minus 2 plus --- the diagonal

Later the investigator provided them with two statements in symbolic form: $D_5 = D_4 + 3$ and $D_6 = D_5 + 4$. Again *B* saw the necessity of producing a general form:

> B347: OK, so how can we reduce, ... this to ... substitute Ns and little letters for everything in this? ... how can we reduce this to a formula instead of something with numbers? If we can reduce this to a formula and then still prove it with that then,

B's continuing attempts to generalize the relation indicate that she was aware of the need to do so. In her last comment in B347, unfortunately cut off in conversation, she began to describe the use she saw for the general form: as a part of a proof.

Expectation of order in mathematics

B expected to find order and patterns in mathematics:

B45: --- I have a really difficult time reasoning
out mathematics. --- but they do see~ to
develop patterns

She saw patterns in many of the situations she was exposed to. In most cases the patterns found were put there as part of the study. For example, she found patterns in the Fibonacci numbers (B18, B21), in the values of $n^2 + n + 41$ (B12), and in the values of F_p for **p** prime (B35, B38, B40). She also found patterns which were not intended, and which were not there. When she checked F_{19} her first reaction was to claim that it produced the same thing as $n^2 + n + 41$ had for $n = 40$:

R49b: Ok, have a look at this one again. What
happens if you check the 19th prime number?
B49: There we go.
R50b: There we go what?
B50: It's the same as this formula.
R51b: Is it?
B51: It's the same number.
R52b: I don't think so.
B52: --- No it's not.
R53b: You're determined to find some patterns here
somewhere.
B53: I am yeah.

This easy acceptance of patterns might indicate that no great need of proof was felt by *B*. Her attitude towards counter-examples and empirical induction do indicate that she considered a pattern insufficient to guarantee a statement, but that she would accept a statement until it was proved wrong.

Treatment of counter-examples

B rejected the false statements in stage one on the basis of counter-examples. Concerning the statement that F_p is prime when **p** is prime she commented:

> B55: --- well, I'd love to find another pattern here. [laughter] I can't ... say anything about it I suppose. Or is this an exception, or presumably if there is one exception there will be others.

She made a distinction between a single exception and a number of exceptions, perhaps meaning that a single exception could simply be excluded from a general statement while a number of exceptions would require its rejection.

Use of and attitude towards empirical induction

B's expectation of patterns may have been related to her use of empirical induction to produce generalizations. She used empirical induction to conclude that $n^2 + n + 41$ always produces prime numbers (B9, B13-16), that F_{3n} is always even (B28), that F_p is prime if **p** is prime (B40), and that the number of diagonals D_n of a polygon is D_{n-1} +

n-2 (B271). She was aware that empirical induction does not produce certainty. She stated that a demonstration that a statement is true for n=1,...,100 would not convince her:

> B80: No, because again that's just plugging in numbers. I'd have to come to some sort of understanding. I have no idea how, why when you plug the numbers in it, it works out to be whatever it works out to be. ...

She also commented (B344, B345) that the relation $D_n = D_{n-1} + n-2$ had not been proven, although she had accepted it earlier (B271).

Her experiences in stage one seem to have been partially responsible for her distrust of empirically arrived at conclusions in stage two, but this carried over to statements which had been proven deductively. When, in stage two, she was shown a formal version of her proof that F_{3n} is even she stated that there might still be a counter-example (B457, B458), because she remembered the statement as one of those disproven in stage one (B460).

B did distinguish between the level of confidence she had in an empirical demonstration versus one with an underlying reason. As mentioned above she felt more confident of the statement concerning F_{3n} than of the others she saw in stage one (B42). She stated her reasons for this preference as follows:

> B60: It is because ... I could see the pattern outside of just the denotation whereas with this one I was looking, here I couldn't figure out why, ... when N is a multiple of 3 I

could see the pattern ... in the actual numbers instead of just looking at the charts and, do you understand?

B63: I would say you ... have to actually ... be able to ... see. With these ones it's basically a matter of plugging in numbers ... and seeing what you come out with, and looking at those ... numbers but, ... I never actually figured out why it was doing that. ... That's the way it worked, whereas with this one it has another, ... [is] backed up by another rule? ... you actually have to understand why, ... it is doing that, --- ... for example the N 2 plus N plus 41, ... I didn't know why it was doing that. ... the numbers seemed to be working out to prime numbers. They just were doing that. I didn't know why. Whereas this one, at least I think I know why, the multiples of three work out to be even because ..., when you add the Fibonacci numbers the other two are odd and then so it would come out to be even.

B also recognized in one case that her empirical evidence could be turned into a proof by cases:

B68: ... I haven't tested out every single odd number adding it to every other odd number, ... --- But then I guess you'd only ... have to do the first 1 through 9, because that's what they're all going to end up with anyway.

R69b: Why do you just have to do the first 1 through 9?

B69: Because anything after that will also end with 1 through 9

Use of variables

B at one point indicated that she felt that a variable **n** could be considered a general number:

B434: ... but if we've taken it into a general statement using N instead of numbers, and it works, then isn't that a general statement about all polygons?

Understanding of proof

B's understanding of proof in general was unstable. While she understood certain aspects of the requirements of a proof, she was unaware of others. Most importantly in the context of MI, she seemed unaware of the fallacy of assuming the consequent. She did not have the problem some students encounter with MI, that of seeing the assumption of P_k in the induction step as an assumption of the statement to be proven, because she did not see any problem with assuming the statement to begin with, at least in formal contexts where algebraic complexity hid the structure of the proof.

Most of the examples below occurred in the context of *A* and *B*'s attempt to understand the proof given that the number of regions P_n produced by **n** lines in the plane is less than or equal to 2^n. The proof included this paragraph:

> Each time a new cut is made by a new line the new line, at most, cuts every piece into two pieces, doubling the number of pieces. This means $P_{n+1} \le 2P_n$. If $P_n \le 2^n$ then $P_{n+1} \le 2P_n \le 2(2^n) = 2^{n+1}$

It was anticipated that the assumption "If $P_n \le 2^n$" would give rise to an objection on the grounds that this was exactly the statement to be proven. However, the major difficulty for *B* lay in the inequality $P_{n+1} \le 2P_n \le 2(2^n) = 2^{n+1}$. After careful consideration she came to the conclusion that the deduction of $P_{n+1} \le 2^{n+1}$ from the inequality was justified, at least if the intermediate inequalities were correct:

> B153: Yeah, well that makes sense
> B154: if all of these things are true.

> B185: ... It works logically, ... if you agree with all of these statements then it makes sense.

> B190: This makes sense, but it's just that you have to agree with every step along the way.

She examined each step (B159, B194-201) and so concluded that the statement was correct. She had no problem using $P_n \leq 2^n$ within the inequality. This would indicate either that she understood that P_n in this case was not expressing a general case but rather an arbitrary one, or that she did not see that what she was assuming was in fact the statement she was trying to prove. The generally literal approach *B* took to symbolism makes it unlikely that she took P_n to have two different meanings within the same proof, leaving the most likely conclusion that she did not realize that she had assumed her conclusion, or saw no problem in this.

She later gave two possible reasons for accepting the statement: that it had not be disproven, and that it made sense at each step:

> B236: So then is it true because we can't disprove it? --- ---
> B237: Well, ... It makes sense to me, like this is the proof is estab[lished]
> B238: It makes sense to me in each step

Her comment in B236 seems to be related to her criteria for judging an empirical proof. As has been noted above she ascribed tentative validity to any statement which displayed a pattern and which had not been contradicted. Here she used a similar criterion for a deductive proof. Later *A* produced a demonstration that $2(2^{n-1}) = 2^n$ which ended with the statement $2^n = 2^n$. *B* commented:

B245: ... I like that ... sort of proof.

This is perhaps the form of deductive proof with which *B* was most familiar, and so she might treat it with more confidence than an inductive proof, which in turn she might treat with more confidence than an empirical proof. When A produced the equality 10 = 10 in the course of proving the diagonals formula for D_5 based on an assumption that it works for D_4, *B* was quick to accept it in spite of A's uncertainty:

A797: ... So 2 times 2 plus 3 is equal to 10 so 2 times 5 is equal to 10. So I don't know what we've proved, but we have something that seems. This works. Ok. So I don't know.
B336: So, well we've shown it.

In fact *B* seems to have meant that A's proof proved the formula for any **n** not just 5. The convincing form of the proof may have led her to assume its generality without any further indications.

B did see one problem in A's proof. She was aware that the proof relied on an assumption that $D_5 = D_4 + 3$, which they knew for that specific case, but not in general.

She commented:

> B341: So we've proved that ... in relation to one another they're true, but in relation to the truth
>
> B344: Because the only the way we proved this [$D_n = D_{n-1} + n-2$]
> B345: Was to look at the patterns in this. [The examples they had]

She later insisted on seeing a proof of $D_n = D_{n-1} + n-2$ (B408). After they had been guided through the general formulation of the proof she made a comment which provides more evidence that she was not aware of the fallacy of assuming the consequent, at least in this case:

> B424: ... assuming this was right [$D_n = D_{n-1} + n-2$], and assuming this was right [the formula], because if it hadn't been right then it wouldn't have worked out or something like that, we substitute this for the D N then worked it all out.

Any proof which assumes its conclusion can be made to "work out". That this one did so in a way which is logically correct is something with which more mathematically sophisticated students encounter difficulties, but *B* avoided the problem by being unaware of it.

Analysis of A's Behavior and Understandings

Description of A's Understanding of MI As it Developed in the Course of the Study

A began the study with no prior experience with MI, and limited algebra skills which hampered her ability to formalize generalizations. She could use the concept of recursion, however, as evidenced by her creation of a recursive definition of the Fibonacci numbers. She could also use recursion to justify statements, such as her assertion that there was no upper bound to the number of Fibonacci numbers:

> R45: Are Fibonacci numbers unlimited?
> A46: Yes, because then you keep multiplying [she means adding] to the right. You can always have numbers to multiply and you get it ...

Here the recursive definition of the Fibonacci numbers provides the equivalent of induction step in a proof by MI.

In session two of stage one it seems that her idea of proof as an algebraic manipulation culminating in an identity prevented her from looking for recursive justifications when explicitly asked for a proof of $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ (A69-92). She did observe that she could recursively apply previous steps to test the statement for specific values (A102). At this point she was guided through the process of proving the statement for the case of n=36 from an assumption of it for the case of n=35.

In session three she saw the same statement and again used previous results recursively to derive new ones (A126-133). In her attempt to prove that 2^n is an upper bound on the number of regions produced by **n** lines in the plane she derived an incorrect induction step, involving adding 2 regions with each line (A168), which she recognized could be repeated recursively forever (A169).

> A168: each time that you add another line you are bringing it up sort of x number of cuts
> A169: but then it it doesn't change its nothing weird happens each time you add a line there's only so many more pieces you can add its it doesn't seem that anything unexpected is going to happen when you get to 50 slices

When it was determined that a new line can at most double the number of pieces A asserted that the number of pieces will always be less than 2^n (A178). In this way the statement was proven for her, indicating an informal acceptance of MI as a method of proof:

> A184: its the same sort of it's the same relationship over and over. It doesn't matter if it's two slices or fifteen slices when you look at it. When you look at sort of the pattern of what goes on when you cut ...

After she was shown that her induction step was in error she then doubted the truth of the statement (A194).

She was then reminded that the number of regions at most doubles, and was shown the first five of a chain of implications. She recognized this as an unending process (A221). She was shown a similar chain of implications for the statement $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$, and followed its development (A249-250). When she was shown the induction step in general form she commented:

> A286: That the sum of the Fibonacci numbers to F K plus 1, we could just as easily call it F K

which seems to indicate some confusion as to the arbitrary rather than general nature of **k** in the induction step. She also associated the way the proof had worked with the Fibonacci numbers rather than seeing it as a general principle which could be applied in various circumstances (A304).

When prompted she described a chain of implications beginning at F_{34} and continuing indefinitely:

> A309: Yeah, and then we could say that we know it for the 34th, because we know it for the 33rd, we know it for the 34th, and if we know it for the 34th, then we know it for the 35th, and if we know it for the 35th, then we know it for the 36th, and so on, and so on, and so on, and so on, and so on.

This idea of MI seems to have been clear to her. She also asserted that such a chain proved the statement for all Fibonacci numbers (A310) indicating a belief in something equivalent to the ω-rule (see page 16). Immediately afterward she expressed doubt in her assertion that the chain would continue, possibly due to uncertainty about the generality of the induction step (A311). She did feel that: "It doesn't seem logically that it should [fail] though" (A314) and after a reconsideration of the induction step (A316-318) she became more confident.

Of the chain argument A said:" to me ... at a certain point it just comes down to trust. " (A327). The basis for her belief that the induction step is general was this trust:

> A329: we've gone through them all, they follow this pattern, OK but you trust that they will just continue to follow this pattern, that I mean, that somehow randomness or some crazy fluke of nature won't happen so that, for some reason, like Fibonacci number 17 million decides that it's going to do something completely wonky because it just stops working for whatever reason.

She explicitly stated that if the induction step worked for all Fibonacci numbers: "then this is true" (A333). It would seem that the induction step itself, rather that the recursive application of it, was A's main problem. She explored for herself the circumstances under which the statement could fail to be true:

> A337: but then if it does stop working then it's not a Fibonacci number, and if it's not a Fibonacci number then it doesn't have to work along according to that pattern.

A recognized that the proof of $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ is similar to the proof shown her that 2^n is an upper bound to the number of regions in the plane produced by **n** lines (A341-345). She described both in terms of a chain of implications:

> R359: Then do you know it for the seventh Fibonacci number?
> A359: Yes and the eighth, yes, yes, and that's how we come to this belief in it going on forever.
> A360: And the same thing with the slices of pie. You know it for the first, so then you know it for the second, and you know it, therefore you

> know it for the third, therefore you know it for the fourth.

She continued to have doubts however, perhaps because of knowledge that empirical induction is not certain. Her trust in the induction step may have been undermined by this knowledge:

> A366: ... I find it incredible to say that it would work, for all of them because you go from one to the other and to the other. Somehow that doesn't prove it to me.

In stage two A recognized the statement that 2^n is an upper bound to the number of regions in the plane produced by **n** lines but made no connection between the formal proof of it which was given and the chain of implications used to show it in stage one (A518). She again suffered the same confusion between 2^n and 2n (A525) and may have been partially remembering what she had done before. In following the proof given A used the inductive hypothesis without comment (A549-554) and may have assumed that it was a premise of the proof:

> A549: So then if your number of pieces ... then is maximumly smaller than or equal to this 2 to the N we said up here.

The induction hypothesis was presented in exactly the same form as the statement to be proven and it was intended that A and *B* would object to its use in the proof on the grounds that it was exactly what they were trying to prove.

A was uncertain about how to go about verifying the proof but eventually concluded that it was correct (A624, A638) after contempla ɔn of some algebraic manipulations (A619). She did not make the connection with recursion or a chain of implications. She commented only that the statement was proven:

> A653: Because it seems to make, to be a logical proof.
> A654: It seems to describe the situation. When we play around with it things seem to fit into each other. When we tried it for a few sample numbers it worked. --- I don't know why, why.

It may be that it was easier for A to use reasoning related to MI that it was for her to identify it when shown formally.

When considering the problem of proving the formula for the number of diagonals of a polygon she exhibited considerably more use of recursion than in her consideration of the proof just discussed. A noted that a simple starting point would be helpful (A707) and on considering the examples they had been provided with discovered a recursive relationship between them (A711) This recursive relationship later served as the induction step for the proof. She checked the relation empirically, and accepted it based on her empirical test (A715-716) Once this was done she felt the need to formalize it (A725), but was unable to do so without help. Her comment:

> A745: Like we can predict right now, how many side we can, without using this formula you and I can say, how many sides the next one is going to have, right?

indicates that she recognized that the relation she had found could be applied as an induction step.

Once she was given a formal version of her relation for the case of n=5 she was able to prove the diagonal formula for D_5 based on its assumption for D_4 (A766-796). She was uncertain as to whether this was sufficient to prove the statement in general (A800). She did state that the truth of the induction step would imply the truth of the diagonals formula:

> A803: ...I think that if we decide that this [the recursive relation] is true as a statement about polygons ...
> A804: then we've proved with this, that this [the formula] is true.

When she was given a general form of her relation she was able to use it in a way analogous to her use of a specific form. She had no objection to the assumption of the truth of the diagonals formula for a previous case (A837) but was surprised when her work culminated in the proof of the induction step (A864). She was, however, very confident that her proof proved the statement (A867):

> A867: And we're back to what we, so we got, that's a proof. This constitutes a proof. This is a proof.

She did comment that the assumption of the induction hypothesis bothered her, both in her proof and in the previous proof:

> A881: ... This comes down to like ... what I think is my bone about the other one we did. I think ultima[tely] some way you're assuming that something is true aren't you? To do this?

She did seem to be comfortable with the chain idea, however, commenting:

> A890: Yeah if it works for quadrilaterals we can work from there using this like, using our statements up, can we not?

She recognized the infinite nature of this chain:

> R639: OK --- could you keep doing that?
> B407: Forever and ever? ---
> A896: I think so, yeah

She repeated the chain idea again later (A928-929) but she was still uncertain that the chain would continue on forever (A934). This may have been related, as noted above, to an association to empirical induction (A935), or uncertainty about infinite processes in general (A975).

She recognized the proof that F_{3n} is even as being the same as the sort of proof they had done for the diagonals formula (A969), but expressed the same uncertainty about the infinite continuation of the chain of implications:

> A975: NO! but it could go wonky. Who knows what's going to happen in the infinite universe?

<u>Summary of A's Understanding of MI</u>

A had not encountered MI before the study. She accepted informal proofs using MI, and the justification of these proofs by a chain of implications, in both stages of the study. She did have difficulty accepting that such a chain would continue indefinitely. This may have been related to an association of such chains with empirical induction, which she knew to be uncertain (see below). She did feel, however, that "logically" (A314) it should continue. She associated acceptance of such informal arguments with "trust" in the generality of the induction step (A329).

A recognized the similar nature of different arguments based on MI on several occasions. In stage one she saw that the two proofs, of the Fibonacci relation and of the upper bound in partitioning the plane, were of the same type (A360). In stage two she recognized that the technique used to prove that F_{3n} is even was the same as the technique used in proving the diagonals formula (A969). This seems to indicate that she had abstracted some elements of MI from the different situations.

<u>A's Understandings of Other Concepts Related to MI</u>

Infinity

A's ideas concerning infinity are not easily categorized. On one hand she was aware that the Fibonacci numbers have no upper bound (A44-46) and that chains of implications can also go on forever (A896). On the other hand infinity seems to be a place where rules break down. For example an infinite number of lines cutting the plane:

> A164: It'd be a guess. Because it works for the first four and there seem to be a pattern involved I guess that I expect that the pattern will continue, but then like when I think about like when you do something like infinite cuts.

A seemed to use physical analogues for infinity, which may have affected the way she thought about infinite processes:

> A935: ... If the highway goes straight for as far as the eye can see does that mean that [it's] forever straight? I don't know. ---

On the whole nothing she said clearly indicated a classification for her idea of infinity. It was perhaps very dependent of the context in which she is working.

Use of specific versus general methods

A was more likely to use specific examples that general forms in solving problems. She used general forms only when she first tried to state a recursive rule for the B sequence and when she first began trying to prove the diagonals formula:

> A408: ... the sum of the B sequence, up to a even number, ... gives you a, the following number. If you work up to a, the sum of up to an odd number then it gives you the following number, I think plus 1

> A704: ... 2 D N is equal to N, N minus 3. dces that work? if you're trying to multiply up can you do that? OK. but --- ... --- --- --- what happens if you divide D N by N?

The difficulties she encountered in A704 might indicate the reason for her disinclination to reason using general forms. The notation usually used for such reasoning is algebraic, and manipulation of algebraic expressions was not something A felt very confident about.

She used specific examples on many occasions: to finish her search for a B sequence formula (A411-412); when trying to see why $2(2^n) = 2^{n+1}$ was true (A601-612); when she first found the pattern $D_n = D_{n-1} + n-2$ in the diagonals problem (A710); and when she verified the diagonals formula for D_5 (A766-776). Her extensive use of specific examples was probably due to her limited ability in algebra, which hampered her attempts to work generally.

Expectation of order in mathematics

For A mathematics is orderly and patterned, and so she is inclined to accept statements based simply on their expression of some pattern. She accepted the statements offered in stage one, session one because:

> A41: The whole time, because its math I'm going to say yes, all the time.

but she wasn't absolutely sure:

> R41: Ok, so you're sure that I could never come up with an example were it wouldn't.
> A42: No, I'm not.

In stage one, session two A was tempted to relate the properties of the Fibonacci numbers she had seen before (those shown her in session one) to the statement $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$:

> A69: ... I was thinking I could try and use one of these formulas --- ???, to see if I could, prove [do] something with them, ...

She seems to have had a preference for believing that the statement is true, and was surprised at the investigator's suggestion that it might not be:

> A71: or a non proof? ok --- aaagh! Ok, just tell me something, what I'm asking for in fact is a clue, can I have a clue, should I be using these, problems [from session one] ?

The final statement offered in session one: "The sum of any set of consecutive Fibonacci numbers is a Fibonacci number" she accepted immediately, only to quickly find a counter-example. Her acceptance was based on the way the statement fit with the way Fibonacci numbers work, the way the patterns seemed similar, rather that any empirical or logical evidence.

In addition to expecting order A is also fairly good at recognizing patterns. She quickly saw the recursive rule for the Fibonacci numbers (A1), a recursive rule for the **B** sequence (A413), and found the relation $D_n = D_{n-1} + n-2$ when working on a proof for the diagonals formula in stage two.

Problems with algebra

A's algebra skills were not very good. She was not comfortable with equations (A75, A76, A83, A683) and exponents (A596). Her understanding of variables seems to have been largely based on the appearance of expressions. For example she was not comfortable using the same **n** that had been used in previous formulae in her attempt to prove the statement $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$:

> A53: ... I don't like using the same n, did you choose n in particular?

One of the techniques she used in her attempted proof was the manipulation of an algebraic formula, without any guiding principle:

> A97: ... I think that my problem is that I'm trying to, think about --- ways of manipulating symbols, so that they mean something, that that they show, that this side of the equation is equal to this side of the equation, ...

> A69: ... So, is there some way you can do this?, so I look back to my ???, and I, what is there to manipulate in this formula?, that would make it prove itself? ...

She repeated this phrase "Prove itself" on two other occasions (A85, A655) and when she was asked what she meant by it. She referred to proofs she had seen done in the past:

> A384: I was desperately trying to think back to my calculus classes, and the proof she was writing on the board, in which she would take this and then build down from it, or build up from it, and it would prove itself. ...
> A385: ... in the course of this calculation, from this flipping things around in the formula, putting something here, or adding something, ... substituting another formula into it, which is what we eventually did, [we used another one?], that the formula would prove itself, but in fact, it's not proving itself. It, You're proving it using other, we, working it with something else, or with other elements of it, ...

This does not seem to be the description of an algebraic process that would be given by someone who felt in control of the process. A's manipulations were an attempt to mimic the proofs she had seen, which consisted of a series of equations culminating in an identity.

Treatment of counter-examples and
Use of and attitude towards empirical induction

A only referred to counter-examples on a few occasions (A69, A619, A715) but on these occasions she seemed aware that a single counter-example would serve to disprove a general statement. Her expectation of order inclined her to not expect to find counter-examples, however, and she never actually searched for one:

> A69: ... there's no way I'm going to sit here all afternoon, and crunch numbers, just so I can find out whether its wrong --- especially as I suspect that I'm not going to find one

She was also inclined to accept statements based on empirical evidence. She did so in verifying that F_p is prime if p is prime (A24); in concluding that $F_1 + F_2 + \ldots + F_n + 1 = F_{n+2}$ (A90); in concluding that $2(2^n) = 2^{n+1}$ (A612, A614); in verifying the proof of that the number of regions produced by **n** line is $\leq 2^n$ (A654); and in verifying the relation $D_n = D_{n-1} + n-2$ (A715).

In the case of the sum of $F_1 + F_2 + \dots + F_9$ she was uncertain of a result she had derived algebraically because she had not verified it empirically:

A66: I don't know, I didn't test it

She eventually concluded that $F_1 + F_2 + \dots + F_n + 1 = F_{n+2}$ was true based on the examination of a few cases (A90).

She did not consider empirical methods to be proofs however:

A68: well so far, so good, but I don't know, I don't think that that's the most successful way to figure this out. If I was to just sit here going, like I said before, it's not as if I can, try every single Fibonacci number I can possibly think of, ...

Use of variables

A's treatment of variables was unstable. In most contexts she did not indicate that her understanding of the variables was different from that intended by the investigator, but on some occasions she may have been thinking of a different role for the variables involved that was expected. When she first saw the statement: "The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up." she said:

A51: I can't understand what you wrote --- the sum of the first N Fibonacci numbers, so N being we don't know how many numbers, we don't know how many Fibonacci numbers we're dealing with.

She may have been indicating that **n** was a general number, or she may have meant that **n** was an unknown, and assumed that the problem would be to find **n**. When she first saw a semi-formal proof of the above statement she seems to have taken the variable **k** to be totally general:

> A286: That the sum of the Fibonacci numbers to F K plus 1, we could just as easily call it F K

She is correct in saying this only if she believes **k** is intended to be general; for only then could any other symbol be substituted for it without affecting the argument. Later, however, she does not see the argument as general, as she suggests that there might exist numbers for which the argument would not hold:

> A311: ... I mean there obviously ... it could just change for some reason, something could just go wrong in the whole pattern in question and it could just not work.

The role of **k** in the induction step is an unusual one. While A seems not to have a standard understanding of the role of **k** in the two examples above she did at one point seem to grasp it. In the following she is discussing the relationship between P_n and P_{n+1} in the partitioning problem. Although her language is a bit uncertain she seems to be describing P_{n+1} as the number of pieces obtained from an additional cut made to P_n pieces:

> A540: ... P N is the total number of pieces. ... So the P N is the total cuts and you add another one, ...
> A543: one more cut, so that's plus 1. You're going to get double the number of pieces, ...
> A545: That's the number of pieces plus one more

cut.

Understanding of proof

A began the clinical study with not much faith in her ability to understand a proof:

> A88: I don't see how anything could prove it. I've never understood proofs in all my life. I shouldn't give up ---Is there a way? Can I write an actual formula that's going to prove it?

A seems to have a basis philosophical problem with mathematical proofs as well:

> A674: Yeah. --- Well, I'm suspicious of anything that reduces reality into mathematical formulas, and then asks me whether it's true or not. I just, I think that that's the problem, I'm just suspicious of this this this thing. --- That's not a problem with the proof that's a problem with me, and my suspicions.

Her idea of proof seems to be based on those she has seen in the past which primarily took the form of algebraic equations being manipulated in such a way that eventually a blatant identity was produced (A384). She attempted such a proof of: $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$

> A92: I'm thinking about how I can ... write this down so that you get, an equation that gives you Fq is equal to Fq, and I think that would prove it ------ that's all that I'm doing here, I'm saying 13 is equal to 13. That's this one right. I can break this down, and work it out so that, Fq equals Fq, that's really what I have to do to prove it ---

When such an identity was produced in the proving of the diagonals formula she was certain of the proof (A867)

A seems also to have been aware of the fallacy of assuming the consequent, and this made her wary of the formal proofs by MI which she saw. In stage two she made a comment concerning the proof of the diagonal formula which indicates such an awareness:

> A881: I don't know --- ... This comes down to like ..., what I think is my bone about the other one we did [the proof of $P_n \le 2^n$]. I think ultima[tely] some way you're assuming that something is true aren't you? To do this?

Shortly afterward she accepted the proof, based of the idea of a chain of implications. It seems that she had trouble with a more formal phrasing which hides the idea of a chain and explicitly makes an assumption of the induction hypothesis. At the same time she could accept the reasoning if she saw the chain in it:

> A890: Yeah if it works for quadrilaterals we can work from there using this like, using our statements up, can we not?

Section B

Comparisons of Students' Understandings, Attitudes and Abilities

This section makes a comparison of the students in terms of their abilities, attitudes and understandings of MI and concepts related to MI. The discussion of MI includes within it ideas such as recursion, acceptance of informal and formal MI, and ability to construct arguments using informal and formal MI, and the students' characterizations of MI. The discussion of related concepts include discussions of the equation of a proof for **any** with a proof for **all**, use of specific versus general problem solving methods, understandings of the way the variable is used in the induction step, and the students' understandings of and abilities to produce specific proofs.

Mathematical Induction

Three of the students (*E*, *G*, and *J*) had studied MI prior to their participation in the study. The other three (*A*, *B*, and *H*) had not.

Recursion

All of the students studied had no trouble with recursion, at least in the context of recursive definitions. *A*, *B*, *G*, *H*, and *J* all accepted and worked with the recursive definition of the Fibonacci numbers. *E* and *G* accepted and worked with the recursive definitions of the two sequences **A** and **C** shown them in stage one. Finally, *A*, *G*, and *H* created recursive definitions of the **B** sequence.

Every student except *E* was asked to define the Fibonacci numbers by their defining rule. In the cases of *G*, and *J* they did so on the basis of the name alone, as they had previous knowledge of them. *H* accepted the definition given by *J*, noting that F_1 and F_2 must be defined explicitly. *H* was the only one to make this observation. *A* and *B* had not encountered the Fibonacci numbers before, and derived the rule for themselves. The recursive nature of the definition did not seem to be a problem for any of the students.

The **B** sequence is interesting in that it admits a number of different defining rules. It was the expectation of the investigators that the students would arrive either at the rule $2B_n + B_{n+1} = B_{n+2}$ or at the rule $B_n + B_{n+1} = 2^n$. In fact the students discovered two other rules, both based on treating even and odd cases separately. One of these was the first rule discovered by all three students who encountered the sequence. The rule is:

If **n** is odd: $B_{n+1} = \Sigma B_i$ for **i** = 1 to **n**

If **n** is even: $B_{n+1} = \Sigma B_i + 1$ for **i** = 1 to **n**

H first conjectured that the rule for the odd case was true in general and then rejected it when he found it didn't work for even cases (H474). *G* and A, on the other hand, restricted it to the odd case and then found the other rule for the even case (G492, A413). *H* went on to find the rule $B_n = 2\ B_{n-1} - (-1)^n$ (H476). A almost found the same rule, noting that it was "doubling, sort of" (A414).

The four rules each have certain features of interest. As noted before, the rules found by the students must treat the even and odd cases differently, while those expected by the investigators apply to both cases. *H*'s rule ($B_n = 2B_{n-1} - (-1)^n$) and the rule $B_n + B_{n+1} = 2^n$ are similar in that they are based only on the term immediately preceding the term being defined, and in that they employ another sequence as part of the definition. The rule $2B_n + B_{n+1} = B_{n+2}$ is similar to the Fibonacci sequence in that it is based on the two terms preceding that being defined, and makes reference to no outside sequence. The rule most popular with the students is based on the sum of all the preceding terms, as well as on the parity of the term in question. The explanation for this variation in rules discovered is not immediately obvious. The choice of rules which are based on all previous terms is interesting, however, as it shows that students are willing to accept such a recursion, which is that employed in the strong induction described by Woodall (1975, 64).

Informal Mathematical Induction

Informal MI is distinguished here from formal MI by its lack of a precise structure. While formal MI requires nothing more than a single instance and a proof of the induction step, informal MI often includes several instances and working of the induction step for specific values. The induction step may never be proven generally; instead a general pattern is inferred from specific steps and extended with "and so on" or some similar phrase.

Most of the students (*A*, *B*, *E*, and *G*) were exposed to informal arguments based on MI. The two students in the mathematically more experienced group, *E* and *G*, accepted these arguments without difficulty.

The two students in the mathematically less experienced group expressed reservations. For *A* informal MI was associated with empirical induction, and her acceptance of MI was limited by the knowledge that in the case of empirical induction an infinite chain of occurrences can break down:

> A311: ... Something could just go wrong in the whole pattern in question and it could just not work.

A did feel that MI was more reliable:

> A314: It doesn't seem logically that it should [break down] though ...

For *B* MI was also more reliable, but not much more so than empirical induction. This might have been a result of the design of the study, which intended to make *B* skeptical.

The remaining two students, *J* and *H*, were not exposed to informal proofs by MI, although *H*'s description of MI as a chain of implications would seem to indicate that he would accept an informal argument based on such a chain.

Formal Mathematical Induction

All of the students were exposed to formal proofs by MI; that is, proofs which followed the pattern of MI as laid out in text books. Three of the four students in the mathematically more experienced group (*J*, *H*, and *E*) accepted formal proofs by MI. *J* actually produced such a proof herself, which *H* accepted immediately as correct. *E* attempted to construct a proof in the textbook form for the equivalence of the two sequences **A** and **C**, and agreed with the one finally arrived at.

G also agreed with the proof of the equivalence of the two sequences **A** and **C**, but in his other comments it is plain that he does not agree with such formal proofs in general. He said as much:

> G220: I have a lot of problems being convinced by induction.

As has been noted above these "problems" do not extend to informal MI.

The two students in the mathematically less experienced group accepted formal proofs by MI, but seem not to have understood them. In their discussions they justified the use of the induction hypothesis within the induction step by noting that they had been given it, without noting that it was the statement they were meant to be proving (A549). Their acceptance of this formal argument may have been based more on the construction of an informal argument justifying the statement than on the formal argument itself.

Ability to Construct Formal Proof Using MI.

Not every student was asked to produce a formal proof using MI, but their abilities to do so became apparent during the study. Of the four students in the mathematically more experienced group *E* and *J* could produce such proofs. *J* actually did so when proving the statement: "For all n: $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$". *E* made considerable progress towards such a proof of the equivalence of the two sequences **A** and **C** and her descriptions of MI indicated an awareness of the basic steps in constructing such a proof.

The other two students in the mathematically more experienced group, *G* and *H*, had difficulty in constructing such proofs, and were disinclined to do so. The reasons for this, however, were different for each of them. *H*'s only experience with MI before stage two had been looking at *J*'s proof in stage one. He lacked confidence, quite understandably, in his ability to construct such a proof independently. *G*, on the other hand, had constructed such proofs in the past, but did not trust them. As a result he too was disinclined to use formal proofs by MI. The one occasion when they did attempt such a proof was their attempt to prove the equivalence of the two definitions of the **B** sequence they had discovered. The proof required is quite complex, and the fact that they made some progress seems to indicate that in a simpler situation they would have been able to construct a correct formal proof using MI.

The two mathematically less experienced students, *A* and *B*, are probably not able to construct such a proof. They did not recognize the essential parts of such a proof when shown one, as *H* did. As well, they lack the algebra skills needed in most cases for the construction of the induction step.

Ability to Use MI Informally as a Method of Proof

The students' relative abilities to use MI informally differ considerably from their abilities to do so formally. The students in the mathematically less experienced group, *A* and *B*, used informal MI spontaneously in simple situations, and followed arguments based on a chain of implications.

Of the students in the mathematically more experienced group, *G* and *E* actually used informal MI to prove statements. *H* described MI as a chain of implications, which might indicate he could use MI in this way, although he seems to have a marked preference for general problem solving techniques, which would limit the likelihood of his realizing such a chain could be used. *J*'s ability in using formal MI is such that, at least in algebraic situations, she would be more likely to approach a proof formally than informally.

Characterizations of MI

The students in the study used a variety of ways to characterize MI. Among these were MI as a procedure, MI as formal proof, MI as a chain of implications, MI as analogous to a physical process (such as dominos falling), and MI as empirical induction.

All of the students in the mathematically more experienced group described MI in terms of the procedure one follows in order to produce a proof by MI. *J* produced a proof which followed the standard procedure exactly and all of them gave descriptions such as this:

> H542: ... for each you find one that works, and then you say if B K works then, you should prove that B K plus 1 works, then it works for all B.

For the three students who accepted MI as a formal proof, "formal proof" was one of their characterizations of MI. *J* justified her proof by saying: "Well, that's the formal proof." (J154).

H was the only student who spontaneously used a chain of implications to characterize MI, but A adopted this characterization when it was suggested to her.

E was the only student to suggest a physical model:

> E225: It's very simply, if you have a very very long line of people and the last one kicks anybody then the next one wants to kick one before and again and again, you know

That she characterized MI in this way does not necessarily mean that this is the way *E* understands MI. It is possible that she felt that this model would be useful for bringing *G* to a certain understanding of MI, without confusing the physical process with MI itself.

G and A seem to have had reservations about the use of MI related to a characterization as similar to empirical induction. For example *G*'s comment:

> G690: ... I think that, you haven't accounted for every single one, unless you've taken every one sequentially.

indicates this sort of characterization of MI.

<u>The Transition From Proving For Any to Proving For All</u>

The principle of mathematical induction provides justification of a transition from an infinite number of cases to an assertion that a statement holds for all such cases. That this transition is not justified by other logical properties of the natural number system is indicated by the inclusion of principles such as the principle of mathematical induction, or the so-called ω-rule (see page 16), among its axioms. Acceptance of such principles as logically warranted is not automatic. While some of the students studied displayed an acceptance of such a principle, others did not.

J, *E* and *H* seemed to be completely at ease with MI and also the transition from an infinite number of cases to all. Their acceptance of MI is the strongest evidence of this. *H*'s repeated statements that MI means that: "If it works for any number, and it works for the number plus that, then it works for all numbers" (H697) also indicates a belief in such a principle.

G seems not to have accepted such a principle. When he was asked about it, he expressed reservations:

> R537: ... you can do it for any number. If I give you a number you can do it, but does that show that you can do it for all numbers?
> G674: Not necessarily, I don't, see that's what [I] have a problem seeing.

A and *B* also had trouble accepting this principle, but it is difficult to say whether this relates to the principle itself, of to their lack of confidence about mathematics and infinite processes in general.

<u>Use of Specific Cases Versus Use of General Methods</u>

The situations in which the students used general or specific techniques can be described as testing, investigating, and proving. Testing situations are those in which a statement is checked in a specific case. As such, any testing which was done, was done specifically. Not all students were as inclined to test statements however. While *A*, *B*, *E*, *G* and *J* all tested many of the statements presented to them, *H* did so in fewer cases.

The contrast between *H* and the other students he worked with is interesting. In stage one *H* and *J* were presented with four statements and asked to determine if they were true. They reacted identically to the first two, accepting the first without testing and testing the second. *J* also tested the third and fourth statements, while *H* instead investigated them using general methods. In stage 2 *H* began investigating the truth of Statement A in a general way as did *G*. It was *G* who first tried testing a specific example (G407) and *H* followed his lead. Thereafter they continued to work in similar ways, although *G* worked more with specific cases in testing the rules they discovered for generating the **B** sequence.

Investigating a problem can be done using either specific cases or general forms. As noted above *H* tended to use general forms for investigating problems, as did *J*. The other students tended to work with specific cases. *G* did use general forms when investigating problems, but only those which are exclusively algebraic in nature, such as those involving Fibonacci numbers (G403) or the **B** sequence (G618). In these cases he was working with *H* which may also have affected his methods. While working with *E* he rarely used general forms in his investigations.

Proving must necessarily be general, unless the proof is of an empirical nature. All the students attempted to use general forms when trying to prove statements. *A* and *B* had difficulty in doing so due to their weak algebra skills; however, *B* on several occasions noted the need for a general form to be used in a proof (B270, B284, B347). On the other hand, *A* worked through a proof an induction step for a specific example (A766-776) and *B* accepted this as a proof (B336).

In general *A* and *B* could be described as being the most inclined to the use of specific cases, due to their lack of algebra skills, and *H* the most inclined to general forms, to the extent that he might attempt to prove statements without ever checking to see that they were true in a single case (H646).

Understandings of the Variable in the Induction Step

Several difficulties with the role of **k** in the induction step were observed in the students studied. Some students treated **k** as a general number, and rejected proofs by MI of the grounds that they assumed their consequent. On the other hand, **k** was sometimes treated generally, but no objection to the assumption of the consequent was raised. In some cases students considered **k** to be arbitrary, both in the induction step and in the proof as a whole, leading to the possibility that a statement proven by MI might fail to be true for some other arbitrary **k**. Finally some students accepted the change of role of **k**, and so accepted MI, possibly without being aware of the issue at all.

G, *A*, *B*, and *H* all made comments indicating a belief that **k** is used as a general number within the induction step. At several times *G* seems to have felt this way. This lead him to wonder how this could be justified:

> G211: ... I can't see why they use that hypothesis within ... They assume something and they they use it within their proof.
> G221: ... just because you assume something, why are you assuming it to be true in every case?

A noted that she had misgivings about the induction hypothesis used in proving the diagonals formula in stage two. Her comments are similar to those of *G*:

> A881: I don't know --- ... This comes down to like ..., what I think is my bone about the other one we did [the proof of $P_n \leq 2^n$]. I think ultima[tely] some way you're assuming that something is true aren't you? To do this?

A had not objected to the use of the induction hypothesis during her consideration of the proof of $P_n \leq 2^n$. In the proof the induction step used the same variable **n** as did the original statement to be proven. It was expected that an objection would be raised to this assumption of the statement to be proven, but in fact neither *A* nor *B* suggested it was a problem at the time. This may indicate that they were not thinking of the logical structure of the proof and so were unaware of the possible fallacy of assuming the consequent. If this was the case then it provides an example in which the variable in the induction step is taken as being general, without resulting in any difficulties at all. A's differing attitudes while considering the proof and when she referred to it later may be a result of further consideration as the study progressed, or she may have felt misgivings earlier and not expressed them.

In stage one there occurred a similar example in which A did not object to the induction hypothesis even though she seems to have taken the variable **k** to be totally general. When she was shown a semi-formal proof of the statement $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$, she indicated that **k** was general:

> A286: That the sum of the Fibonacci numbers to F K plus 1, we could just as easily call it F K

The substitution of **k** for **k** + 1 can only be justified if **k** is a general number. She did not object to the assumption of P(**k**) in this case, perhaps because it was in the context of the development of a chain of implications and so she had seen specific examples of the induction step before being introduced to it's arbitrary form.

In a similar way *B* indicated at one point that she felt that a variable **n**, used in the induction step, could be considered a general number:

> B430: That works in general. Then --- if this works for N minus 1 and it works for N --- then shouldn't it work for --- N minus 2 or N plus 2 or whatever? If this always works? ---

While most of his comments indicate that *H* understood MI in a way compatible to the logical structure described above, on two occasions he seemed to be ascribing a general role to the variable in the induction step. While attempting to prove that his formula for the **B** sequence was equivalent to *G*'s, *H* made the following argument:

> H586: Am I right in saying, if this is equal OK? for A K minus 1
> H587: and K minus 1 here, is it the same as saying, it's U here, and U here? [writing **u** above **k-1** in the upper expression on figure 3] because it's the same, because these are the same ones right?
> H588: these are these are the same expressions
> H589: Now here it's again the same expression [the lower expression in figure 3]

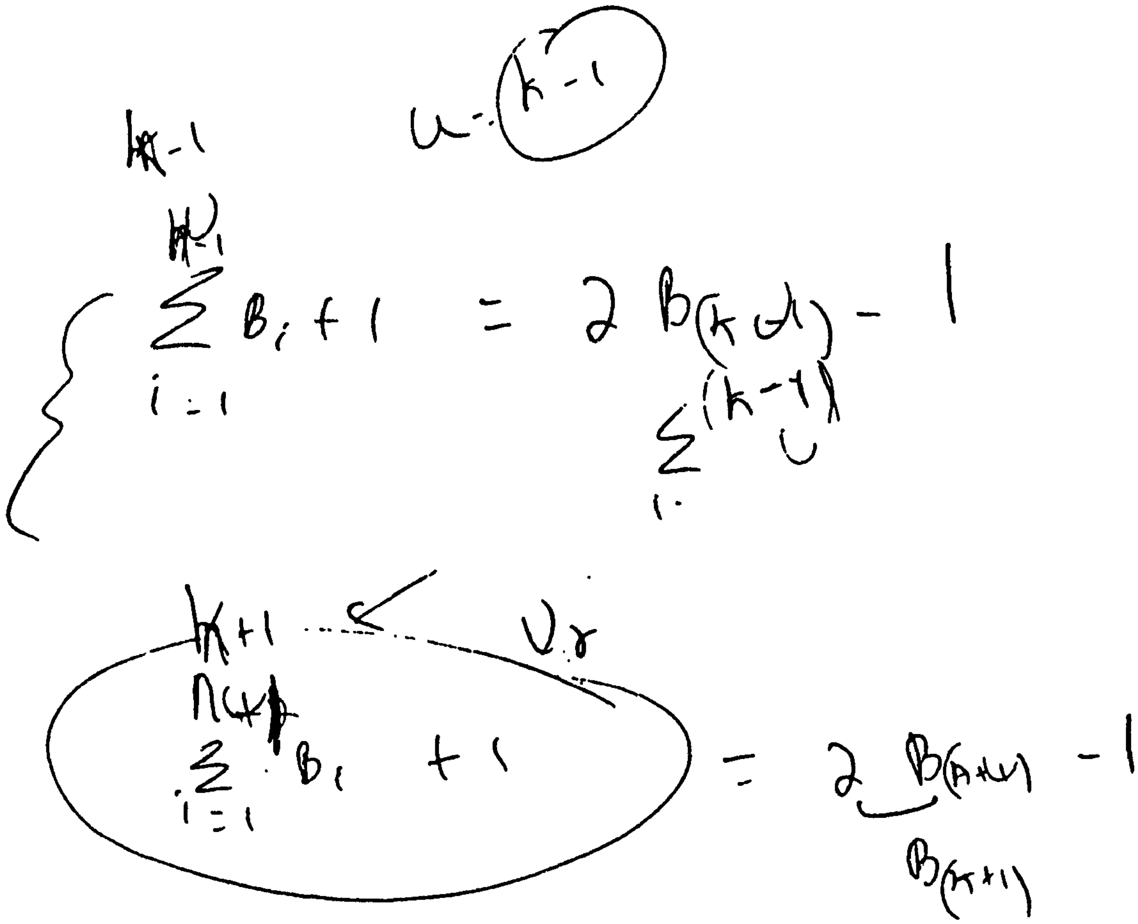

Figure 3: *H*'s writings referred to in line H589, involving the substitution of **U** for **k-1** and **k+1**

H is indicating that **k**-1 be replaced with a general variable **u** which, because it is general, can show the truth of an equation involving **k**+1. In essence he is treating **k** as a general number. He later asked a question indicating that he felt that it was possible for the proof of P(**k**) => P(**k**+1) to imply the truth of P(**m**) for **m** < **k**:

> H727: A question, let's say if you find, if you do that, and you find, say 10 can you backtrack, can you assume that all the ones lower than that too are true?

In stage two *G* seems to have concluded that **k** is not meant to be general within the inductive step, but he then gives it an arbitrary role in the final statement. In other words he accepts that MI proves a statement for a particular, unspecified, case, but does consider the statement proven for any other case. This permits the possibility that there might be a **k** for which the statement does not hold, which was missed in the working of the proof:

G687: Yeah, but let's assume that your K was chosen outside of that.

A also saw **k** as arbitrary at times. For example in the following, where she is discussing the relationship between P_n and P_{n+1} in the partitioning problem. Although her language is a bit uncertain she seems to be describing P_{n+1} as the number of pieces obtained from an additional cut made to P_n pieces suggesting that **n** is arbitrary, not general:

A540: ... P N is the total number of pieces. ... So the P N is the total cuts and you add another one, ...
A543: one more cut, so that's plus 1. You're going to get double the number of pieces, ...
A545: That's the number of pieces plus one more cut.

Earlier she rejected an argument based on MI, and suggests that there might exist numbers for which the argument would not hold:

A311: ... I mean there obviously ... it could just change for some reason, something could just go wrong in the whole pattern in question and it could just not work.

The arbitrary role of **k** here results in a rejection of the argument because of the possibility of **k** behaving differently at some other point. As with *G*, the arbitrary role for **k** causes a problem in seeing the proof as general.

Of the students studied *G* and A are the most interesting with regard to their understandings of the variable in the induction step. The changing role of **k** was not a part of their understandings of MI. They both saw the variable as being always general on some occasions, while on others they saw it as being always arbitrary. A also differed in her reactions to the problems introduced by assigning a fixed role to **k**. At times she rejected MI, while at other times she ignored the problems entirely. One difficulty, the treatment of **k** as a general variable within the induction step, was shared by *B* and *H* as well as *G* and A. This difficulty was also noted by Ernest (1984, 181). *E* and *J* displayed no difficulties with the role of the variable in the induction step, although this may have been due to the limited opportunities they had to discuss the structure of proofs by MI.

<u>Understandings of, and Abilities to Produce, Proofs</u>

While proof by MI is the specific focus of this thesis the different methods of proof used by the students provide interesting insights into their attitudes towards proofs in general.

Proof of the diagonals formula

Four students were faced with the task of proving the formula $D_n = n(n-3)/2$ for the number of diagonals of a n sided polygon. *H* and *G* worked together, as did A and *B*.

The proof of *H* and *G* was entirely deductive. Both examined an octagon. Working separately they each verified the formula except for a particular feature. *H* could not see why (n-3) appeared in the numerator; he felt it should have been (n-2) (H442). *G* was unable to arrive at a justification for the 2 in the denominator (G438, G454). After *G* had explained the (n-3) to *H*, *H* accepted the formula, but did not describe his reasoning (H445). *G* realized the reasoning behind the 2 and accepted the formula somewhat later (G483). Throughout their reasoning was based on simple deduction, without any attempt to build the formula recursively from specific cases. *G* does not seem to have checked the formula at all. It is not clear from *H*'s diagrams whether or not he did so.

A and *B* were given examples for n = 3, 4, 5, and 6 to begin with, but it seems likely that they would have generated examples of their own. They did extend those given to them to n = 7 and 8. The proof they developed was hampered by difficulties in expressing general relations. *A* discovered the relation $D_n = D_{n-1} + n-2$ but was unable to make use of it until the investigator provided formalizations for the specific cases of $n = 5$ and $n = 6$. They were attempting to produce such formalizations (B270), but were unable to do so due to their weak algebra skills. Equipped with these relations *A* succeeded in proving the formula for D_5 based on an assumption of the formula for D_4. *B* suggested generalizing A's proof (B347) but was unable to do so until the investigator provided a general version of the relation $D_n = D_{n-1} + n-2$. They worked through the proof and arrived at the desired conclusion. For A this proof was conclusive (A867). *B* had noted that the relation used in the proof also needed proof, and the investigator provided one.

While there was considerable interference from the investigator in the case of *A* and *B*, the differences between the two approaches to the problem seem to be more closely related to the backgrounds and beliefs of *G* and *H* versus those of A and *B*. *G* and *H* were quite skilled at deductive proof, but inexperienced with MI, and *G* had certain fundamental problems with such proofs. A and *B*, on the other hand, lacked the algebraic skill needed for most deductive proofs, and approached MI from an informal perspective based, at least in the case of *A*, on the idea of a chain of deductions rather that of a proof procedure. This explains the methods employed by each pair. The superior deductive skills of *G* and *H* discouraged them from the kind of empirical investigations which led A and *B* to the relation which is at the heart of proof by MI.

The upper bound in the partitioning of the plane by n lines

Four of the students encountered the problem of proving that 2^n is an upper bound to the number of partitions of the plane produced by **n** lines. *A* saw this problem twice, once in stage one, session three, and once in stage two.

G and *E* worked on the problem together. They both considered n = 1, 2 and 3, at *E*'s suggestion (E166, G160-164). *G* noted the possibility that two lines could be equal, which temporarily confused *E* (E174). *G* also spent considerable time considering every possibility, not just the maximal arrangements (G169-179), although *E* noted that only the maximal arrangements were relevant (E171, E187). Both recognized the importance of doubling (G199, E215-218). They arrived at an inductive proof, which *E* saw as being based on MI (E219).

The first time *A* saw the problem she was guided through it. She had some difficulty with it as she often interpreted 2^n as 2n. She saw an additive, rather than a multiplicative pattern in the transition from n=4 to n=5 (A168), but was led to the idea of doubling (A174). The use of informal MI in the proof was not a problem for her (A183).

The second time *A* saw the problem it was presented along with a formal proof by MI. She worked on it with *B*. *A* never explicitly related the proof to her previous experience, aside from noting that she had worked on it before (A518). The importance of doubling was noted by both *A* and *B* (A541-543, B198). The formal proof was never really understood by them, as indicated by their concentration on details, and implicit assumption of the consequent. The proof was accepted by them primarily on the basis of the idea of doubling.

In all the above cases an informal proof by MI was accepted by the students. In the case of *A* and *B* this occurred in spite of the informal proof being hidden in a more formal proof which they did not understand. As is noted elsewhere *G* did not accept formal proofs by MI, but here accepted an informal proof. One could conclude that the elements of proof by MI which he rejected are not present in an informal version.

Notions and Use of Empirical Induction

A, *B*, *G*, *H*, and *J* all encountered statements which they accepted on the basis of empirical evidence alone. In some cases this evidence seems to have been quite scanty. For example *H* and *J* seem to have accepted the truth of the equation $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$ on the basis of a single example (H45, J38-41). *E* was never in a position to judge a statement based on empirical evidence, but she did empirically arrive at the conclusion that the two sequences **A** and **C** are equal based on five cases (E251).

All of the students did not consider empirical evidence to guarantee the truth of a statement. *A* was unusual in that she used empirical techniques to confirm deduced results (A66). *A*'s understanding of deductive proof did not ascribe to it much more reliability than she ascribed to empirical proof. This may have been a result of the algebraic nature of many of the deductive proofs, as she was somewhat more confident of informal deductive proofs (A881 versus A890).

A and *G* were alike in perceiving a strong similarity between MI and empirical induction, which undermined their faith in MI (A329, A935, G609, G682, G684). This perception is discussed in more detail in the comparison of understandings of MI (above). *B* differed from *A* and *G* in this respect as she actively differentiated MI from empirical induction (B59-60) and expressed more trust in statements which were supported by a MI type argument (B42, B60).

Treatment of Counter-examples

Every one of the students in the study accepted counter-examples as disproofs of general statements. The only one whose doubts were indicated at any time was *B*, who commented: "Or is this an exception, or presumably if there is one exception there will be others" (B55). *H* and *J* were specifically asked whether a counter-example to an accepted proof could be admitted. Both indicated that the proof or the counter-example must be in error.

Not every student was equally likely to look for counter-examples, however. All students checked statements empirically at some time, but only *J* engaged in a systematic search for a counter-example (J109). *G* considered doing so at one point (G417) but rejected the idea, and *H* discovered a counter-example through a generalized method of searching, after *J* suggested the conditions a counter-example would have to meet.

CHAPTER IV

CONCLUSIONS

Several interesting ideas are raised by the above analyses. Some of these have application to the teaching of MI, while others suggest interesting possibilities for further research. The major findings relate to the use of MI on an informal level, the accessibility of a chain of implications as a characterization of MI, the importance of making the role of the variable in the induction step clear, the influence that the form of a proof can have on it's acceptance, and connections between MI and other concepts.

Use of MI on an Informal Level

The history of mathematics provides evidence that MI was accepted as a method of proof long before it was named and formalized in the nineteenth century. Informal proofs by MI, such as Euclid's proof of the infinitude of primes, might be understood by students at the high school level. In fact, Freudenthal (1973, 172) gives an example of a girl in a kindergarten who became aware of the recursive nature of the positional notation for the natural numbers, and so concluded "it goes on".

The results of the clinical study also indicated that MI is accessible even to students without much mathematical training, at least on an informal level. Both of the students in the mathematically less experienced group, *A* and *B*, used MI spontaneously to prove statements like "The number of Fibonacci numbers is infinite". Their lack of confidence in their proofs seems to have been more a product of their lack of confidence in mathematics in general than specifically associated with MI. Drawing attention to and validating the informal proofs they created might contribute to increasing their confidence in them, and in contexts for which they are mathematically prepared they might be able to apply MI on a more formal level.

Accessibility of a Chain of Implications

The first proof which is generally accepted to involve MI, Maurolico's proof of $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, takes the form of a chain of implications. Pascal, although his use of MI is closer in form to modern usage, also uses a chain of implications to justify his conclusion. The same sort of justification is used by Poincaré. This seems to indicate that such a chain serves as a more accessible version of MI, which is more easily accepted. Sfard's (1987) contention that such operational conceptions are more easily accepted by students would also support this idea.

In the clinical study such a characterization was spontaneously adopted by *H*, a mathematically more experienced student who had not seen MI before. In his case exposure to a formal proof employing MI resulted in the reformulation of it into a chain of implications. This characterization also seems to have been accessible to A.

The teaching of a purely structural form of MI seems less likely to succeed than a characterization as a chain of implications, and perhaps many of *G*'s difficulties with formal MI are related to such teaching. It may be that a presentation of MI which makes the recurrence of the induction step more apparent should be adopted and used by students until they are ready to collapse the implications into a single induction step.

This is not to say that such a chain of implications is without potential problems. Ernest (1984) suggests that implications themselves are a cause of difficulties. There is some support for this suggestion in the behavior of *G* and *H*. *G* felt that proving the equivalence of two recursive rules was insufficient. This may have been due to confusion between implications and conjunctions. *H*'s query as to whether the induction step proved statements for numbers preceding the basis may indicate that he was assuming the converse of the implication.

Importance of the Variable in the Induction Step

One difficulty students had with MI during the clinical study was the different roles that the variable plays in the induction step. The different ways in which variables are employed in mathematics generally is difficult for students to comprehend, and the lack of explicit instruction concerning variables cannot help. In the context of MI, because of its logical structure, this problem of determining the role of the variables used is quite complex. Most proofs seen in high schools and colleges involve a sequence of algebraic manipulations. Throughout these manipulations the variables used act as general numbers, and in the final conclusion they are again general. In a proof by MI the variable employed in the induction step plays a complicated role. In the induction step the variable is not general, but arbitrary; it stands for some unspecified number. It is treated as a particular value, not as a place holder for a whole set of numbers. The proof in the induction step of P(**k**) => P(**k**+1) is then generalized to state that this implication holds for <u>every</u> **k**. In this final conclusion of the statement the variable is treated as a general number. This situation occurs in any context in which an implication is proven as a part of a larger proof, but such proofs are rarely seen. MI is likely to be the first time students experience such a change in role, and it may be too much to expect them to spontaneously understand what is going on. The problems experienced by students with implications reported by Ernest (1984) may be also be related to this difficulty with the variable.

Influence of the Form of the Proof

Another difficulty students have with MI which was revealed during the study is the form of the proof. Most proofs students see involve the manipulation of algebraic expressions to produce a blatant identity. MI does not produce such proofs, although they are sometimes seen in the proof of the induction step. *G* specifically indicated that this failure to "come to a final agreement" (G351) was one of his objections to MI. *A* and *B* were also much more satisfied with proofs which came to such an agreement, and had more confidence in them.

This difficulty with the form of the proof may be partly responsible for a difficulty suggested by Ernest (1984, 182), the tendency to leave out the basis. The induction step of a proof by MI is often a proof of the usual sort, arriving at an identity. If the proof is reduced to the induction step only, then the difficulty with the form of the proof is avoided. If the students prior experiences with proof have taught them that nothing is important except coming to that final agreement then they might be expected to discard the basis as irrelevant.

Relation of MI to Problem Solving

Polya (1954, 111) suggests that problems which are solved inductively lead naturally to the use of MI to prove conjectures. In the study, *H* used completely general techniques in his attempt solve a problem which could easily have been approached inductively. He was also the only student who never spontaneously used MI, either formally or informally, to prove statements. The two students who made the least use of general methods, *A* and *B*, used MI informally on several occasions. While they are hardly conclusive, these results suggest that there may be some relation between a students' use of induction in problem solving and use of informal MI.

Recursion

All of the students in the study were able to work with recursive definitions without difficulty. This, and the use of informal MI by most of them, seems to indicate that recursion is not at the root of the difficulties these students had with MI. This suggests that experiences with recursion and recursive computer programming would not have a significant effect on students' understanding of MI. In fact, Word (1988) found this to be the case. *G*'s references to "backtracking" might have been related to prior experiences with computer programming, which seem to have actively interfered with his understanding of MI, perhaps due to the "downward" orientation of such recursions, as opposed to the "upward" direction of the recursion on MI (Leron and Zazkis 1986).

Implications for Teaching

The above conclusions indicate several ways in which teaching MI might be improved. If students have some ability to use recursion in proofs already then a connection should be made between these informal proofs and formal proofs by MI. This would allow the students to connect formal MI to their existing cognitive structures more easily, and lessen the likelihood of forming incomplete associations with incompatible forms of proof. Exploring these informal uses of MI, and characterizing them as chains of implications, would provide a foundation for the introduction of a formal use of MI.

Students' difficulties with the role of the variable in the induction step is but one of the many difficulties students have with variables. In general these difficulties might be addressed by making explicit the different roles variables can play in mathematics. Specifically in the context of MI, connection can be made between the induction step and the implications in a chain of implications, with the induction step seen as a generalization of the implications. This would be a part of the process mentioned above, of making a connection between the students' informal understanding of MI, and the formal use of MI.

Students' difficulties with the form of a proof by MI seem to be rooted in the experiences with proof the students have had in the past. A formal understanding of MI requires both a knowledge of proof, to permit the proof of the induction step, and a knowledge of the variety of different forms proofs can take. In teaching many forms of proof are avoided, perhaps with an intention of preventing possible confusion on the part of students. This results in a characterization of proof by students which is limited, resulting in problems in the introduction of different forms of proof later in the students' education.

The difficulties associated with MI might be responsible for the neglect of it in the curricula current in Québec, and elsewhere in North America. There are, however, sound reasons for teaching MI at an informal way, perhaps in a geometric context, at the high school level. One of these reasons has already been mentioned above. The limiting of proofs to only the simplest forms leaves students with a characterization of proof which interferes with their learning at later stages. Other reasons include the relation of proof by MI to inductive problem solving, students' use of informal MI, and the philosophical importance of MI to mathematics.

Polya (1954) discusses inductive problem solving at length, and notes that while empirical investigations can discover relations, they cannot prove them. If such inductive investigations are to be a part of schooling, and current opinion is strongly in favor of problem solving as a part of mathematics, then there are three choices for verifying relations once found. The first is accepting the empirical evidence as sufficient. This seems unsatisfactory if problem solving in mathematics is to retain the deductive nature of mathematics as a whole. The second is using proofs which do not employ MI. Such proofs require that the proving process be considered separately from the discovery process, as the specific examples found in the discovery of a relation play no role in the construction of a proof. The third is proof by MI, which makes use of the discovery process, and which can often be seen within the process itself. As noted above, several of the students who made use of specific examples in problem solving, also made use of informal arguments based on MI to prove the relations they had discovered. *A*, *B*, *E* and *G* all exhibited this behavior.

That students use MI informally argues that they feel it has some value as a method of validation of their conjectures. Ignoring MI solely on the grounds that students find the current presentation of it difficult to follow seems unreasonable in light of this informal use of MI. If a modified presentation can permit students to associate part of the reasoning the use informally with the reasoning used in mathematics, this seems a valuable opportunity.

The deductive structure of mathematics, and the processes of discovery and proof of statements by mathematicians is sometimes considered relevant only for those students who will go on to become mathematicians themselves. Mathematics, however, is a part of popular culture, in the same way as literature and history. It has long been accepted that the study of these topics is more than learning the skills of writing and reading, and similarly mathematics is more than the learning of the skills of arithmetic and algebra. While much of what is taught will never be needed by the students, unless they specialize in these fields in the future, by learning it they learn about the popular culture and about their place in it. Mathematics, as an aspect of popular culture, has far reaching effects in a technological age. The technological and information oriented character of contemporary living increasingly exposes people to situations which are, at least partially, mathematical in nature. As important as understanding how mathematics is used in such situations is understand **why** mathematics is being used and what conclusions can be drawn from the use of mathematics in a particular situation. The setting of insurance rates, for example, is often justified on the basis of statistical inferences. The use of mathematics to determine these rates makes them seem to be more objective than if rates were determined on the basis of a personal interview, or by some other method. It takes an understanding that mathematics is a human creation, and of the standards used within mathematics to judge new creations, to understand the subjective nature of mathematics in application.

Suggestions for further research

This study attempted to fill a part of the gap in research on MI between studies of a theoretical nature and empirical studies of teaching methods. In the course of it a reminder of the uncertainties of clinical studies was encountered, in the form of the unexpected reversal of roles in the case of *G* and *H* in stage two. In addition several possibilities for further research were raised. There remains considerable work to be done.

The students who participated in this study were all enrolled in university. Although two of them had never studied mathematics at the university level, their academic performance in other levels was above average. It remains to be seen whether the same use of informal MI would be found in students at the high school level. A similar study involving high school students would serve to indicate whether the conclusions reached here can only be applied to some students, or whether high school students in general can use informal MI.

It would be interesting to know at what age this informal ability begins to develop, if in fact it is generally present by high school age. A comparative study of students from different age groups would be worthwhile for this reason.

It was hypothesized above that making a connection between students' informal use of recursive reasoning and formal proof by MI would make acceptance of MI as a proof technique more likely. This too could be tested, through a teaching experiment. The relationship between students' understandings of variables, algebraic skills, and MI could also be explored in more detail, in such a context.

The students in this study showed an interesting mix of inductive and deductive approaches to problems and proofs. The interferences between these two modes, and the possible transfer of inductive or deductive techniques from problem solving to proof and back, would also be interesting as topics of study.

Finally, even the students in this study who had not studied mathematics at the university level had a strong notion of what characterizes a proof. The development of students' understandings of proof during their high school education might make a useful focus for a general study of proof, including proof by MI. In connection with this the possible interaction between the learning of different kinds of proofs could be examined, including consideration of the order of teaching different forms of proof.

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APPENDICES

APPENDIX A

SCRIPTS

The tables of the Fibonacci and **B** sequences referred to in the scripts are included after the scripts.

Scripts for Mathematically More Experienced Group

Scripts for Stage 1

Script for students being introduced to MI

1. Find the relation between the number n of sides of a convex polygon and the sum A_n of its internal angles.

2. You have probably found that:

$$A_n = (n-2)\pi \text{ for } n \geq 3$$

Give a mathematical proof of this relation (assume as an axiom that the sum of angles of a triangle equals π).

3. Prove that n straight lines on the plane divide the plane into no more than 2^n parts.

4. Could you describe your method of proving the above two statements?
Could you identify the steps of your reasoning?
Were you, in a further step of your reasoning, referring to some previous step of reasoning?

5. In the so-called Arithmetic of natural numbers we admit, among others, the following axioms:
I. 1 is a natural number
II. If n is a natural number then $n + 1$ is a natural number.
III. Let S(1), S(2), S(3), ..., S(n), ... be statements about natural numbers. If the statement $S(n_0)$ holds true for some n_0 , and, for any $n \geq n_0$, then truth of S(n) implies the truth of S(n+1), then all the infinite number of statements $S(n_0)$, $S(n_0+1)$, $S(n_0+2)$, ... hold true.

6. This last axiom is called "the principle of mathematical induction" and provides us with a method of proving theorems about natural numbers that start with "for all n ...", or with "for any n ...".

7. Is there any link between this principle and your reasoning in problems 2 and 3?

8 Let the sequence $A = (a_n, b_n)$ of natural numbers be defined as follows:

$a_1 = 1 \quad b_1 = 1$

$a_{n+1} = b_n$

$b_{n+1} = a_n + b_n$

Let the sequence $C = (c_n, d_n)$ of natural numbers be defined as follows:

$c_1 = c_2 = 1$

$d_1 = 1 \quad d_2 = 2$

$c_{n+2} = c_n + c_{n+1}$

$d_{n+2} = d_n + d_{n+1}$

Show that the sequences A and C are identical.

9. Is the following statement true:

For all natural numbers n:

$$p(n) = n(n+1) + 11$$

is a prime number.

10. Is the following statement true:

Every even number is the sum of two primes.

[may be left out]

11. Do you accept mathematical induction as obvious or reasonable?

Do proofs by mathematical induction convince you?

Does a proof by mathematical induction raise the degree of certainty of a conjecture?

12. In empirical sciences, we also speak of induction; we speak about inductive inference. For example:

Experiment has shown that:

sugar s_1 dissolves in water,

sugar s_2 dissolves in water,

...

...

sugar s_{20} dissolves in water,

Conclusion: all kinds of sugar dissolve in water.

Or:

The sun has always risen in the east, as for as I and other people can remember. Conclusion: the sun will always rise in the east.

13. What is the difference or differences between this kind of induction and mathematical induction?

Script for Students Being Made Skeptical

SCRIPT for Stage 1 Group B

INTRODUCTION to Fibonacci numbers

I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers. [If "yes" I: what do you know about it/them?] This is the beginning of the sequence

[present Table 1:

1 1 2 3 5 8 13 21 34 ...]

[Talk about Fibonacci]

[Q1] Can you find the pattern in this sequence? [If "no" I: try adding up pairs of consecutive numbers [more prompts if needed]]

I: [Q2] Can you formulate a rule for producing the Fibonacci sequence? [prompt if needed] How would you write your rule algebraically? [give if needed, test understanding]

DEVELOPMENT OF PROPOSITIONS 1

I: [Q3] Consider this question:

[S1a: "Is there anything special about the sum of the first *n* Fibonacci numbers? Is it related to the sequence in any way?"]

I: [Q4] Is this statement true?

[S1b: For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$]

I: Here is another statement about the Fibonacci numbers:

[S2: The Fibonacci sequence is given by:

$$F_n = \left[\left[\left[\left[\frac{1}{60} n - \frac{7}{24}\right] n + 2\right] n - \frac{149}{24}\right] n + \frac{569}{60}\right] n - 4$$

[Q5] Is this statement true?

<u>INTRODUCTION to prime numbers</u>

I: [Q6] Do you know what primes numbers are? [if "no" explain]. Here is a list of the prime numbers less than 10000. [Table 2]

<u>DEVELOPMENT OF PROPOSITIONS 2</u>

I: Here is another statement:

[S3: For all n: $P = n^2 + n + 41$ is prime]

[Q7] Is this statement true?

I: Here is a statement about prime numbers and Fibonacci numbers:

[S4: "For all n: If n is prime then F_n is prime."]

[Q8] Is this statement true?

DEVELOPMENT OF SKEPTICISM: Proofs and refutations

I: Have a look at the formula for producing Fibonacci numbers. [S2] Try n=7 [if not already done] [Q9] Would you now say that this property only produces Fibonacci numbers? Why not?

I: Have a look at the formula for producing prime numbers. [S3] Try n=40 [if not already done] [Q10] Would you now say that this property only produces prime numbers? Why not?

I: Look again at the fourth statement. [S4] 19 is prime isn't it? [Q11] Is the 19th Fibonacci number prime?

I: [Q12] Would you now say that the first statement is true for all Fibonacci numbers? If I assure you that it is how would you go about showing that it is true?

I: What would you say about trying to find out if statements are true?

[In the context of this discussion:]

We have seen that it mathematics it is extremely unwise to assume that something is true for all n just because it is true for some n.

You've just seen several cases where even though there are lots of examples, the statement turns out not to be true.

In the next interview you will be looking at some other statements, some, perhaps all, of which are not true.

Remember to be careful about accepting a statement unless you have a valid explanation.

[Q13] What would you say would be a valid explanation?

Script for Stage 2

I: Consider this statement. Try to determine whether it is correct or not and explain why it is correct or why it is not.

[Statement A:

For all $n \geq 3$,

$$F_n^2 = (F_{n-1})(F_{n+1}) + 1$$

(F_n is the n^{th} Fibonacci number)

]

I: Consider this statement. Try to decide whether it is correct or not and explain why or why not.

[Present statement B:

B: The number of diagonals in a convex polygon of n sides is

$$\frac{n(n - 3)}{2}$$

]

I: Here is a sequence of numbers:

[Table 3: the B sequence:

1 1 3 5 11 21 43 85 ....]

Can you find the pattern in this sequence?

[If they answer $B_n + B_{n+1} = 2^k$ then skip to α]

I: Look this statement [statement C1 : $B_n + B_{n-1} = 2^{n-2}$]

Do you think it is correct? Explain why you think so, or why you don't.

[skip to ß]

[α]

I: Look at this statement [statement C2 : $2B_n + B_{n+1} = B_{n+2}$]

Do you think it is correct? Explain why you think so, or why you don't.

[ß]

I: Consider this statement and its proof. Try to determine whether it is correct or not and explain why it is correct or why it is not.

Statement D:

For all $n \geq 0$:

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

where F_n is the n^{th} Fibonacci number

Lemma 1: For all $n \geq 0$ there exist c and d such that:

$$\left[\frac{1+\sqrt{5}}{2}\right]^n = c + \frac{1+\sqrt{5}}{2} d$$

Proof of Lemma 1 by induction:

The lemma is true for the case $n = 0$ as:

$$\left[\frac{1+\sqrt{5}}{2}\right]^0 = 1 + \frac{1+\sqrt{5}}{2} (0)$$

Induction Hypothesis:

$$\left[\frac{1+\sqrt{5}}{2}\right]^k = c + \frac{1+\sqrt{5}}{2} d$$

Induction step:

$$\left[\frac{1+\sqrt{5}}{2}\right]^{k+1} = \left[c + \frac{1+\sqrt{5}}{2} d\right]\left[\frac{1+\sqrt{5}}{2}\right]$$

by I.H.

$$= c\left[\frac{1+\sqrt{5}}{2}\right] + \left[\frac{1 + 2(\sqrt{5}) + 5}{4}\right] d$$

$$= c\left[\frac{1+\sqrt{5}}{2}\right] + \left[\frac{3+\sqrt{5}}{2}\right] d$$

$$= c\left[\frac{1+\sqrt{5}}{2}\right] + \left[\frac{2}{2}+\frac{1+\sqrt{5}}{2}\right]d$$

$$= c\left[\frac{1+\sqrt{5}}{2}\right] + d + \left[\frac{1+\sqrt{5}}{2}\right]d$$

$$= d + \left[\frac{1+\sqrt{5}}{2}\right](c+d)$$

which is of the required form.

Lemma 2:

Define the sequences c_n and d_n by:

$$\left[\frac{1+\sqrt{5}}{2}\right]^n = c_n + \frac{1+\sqrt{5}}{2}d_n$$

then $d_n = F_n$ for all $n \geq 0$.

Proof of Lemma 2:

It suffices to show: (i) $d_0 = F_0 = 0$,
(ii) $d_1 = F_1 = 1$,
and (iii) $d_n = d_{n-1} + d_{n-2}$ for $n > 1$.

(i)

$$\left[\frac{1+\sqrt{5}}{2}\right]^0 = 1 + \frac{1+\sqrt{5}}{2}(0) \qquad \text{so } d_0 = 0$$

(ii)

$$\left[\frac{1+\sqrt{5}}{2}\right]^1 = 0 + \frac{1+\sqrt{5}}{2}(1) \qquad \text{so } d_1 = 1$$

(iii) From the proof of Lemma 1 we have: $c_{n+1} = d_n$
and $d_{n+1} = d_n + c_n$

Therefore: $d_{n+2} = d_{n+1} + c_{n+1} = d_{n+1} + d_n$

Lemma 3:

Define the sequences c_n and d_n by:

$$\left[\frac{1-\sqrt{5}}{2}\right]^n = c_n + \frac{1-\sqrt{5}}{2} d_n$$

then $d_n = F_n$ for all $n \geq 0$.

The proof of Lemma 3 is similar to that of Lemma 2.

Proof of Theorem:

$$\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n =$$

$$\frac{1}{\sqrt{5}}\left[c_n + \frac{1+\sqrt{5}}{2} d_n\right] - \frac{1}{\sqrt{5}}\left[c_n + \frac{1-\sqrt{5}}{2} d_n\right]$$

$$\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}d_n\right] - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}d_n\right] =$$

$$\left[\frac{1+\sqrt{5}}{2(\sqrt{5})}d_n\right] - \left[\frac{1-\sqrt{5}}{2(\sqrt{5})}d_n\right] =$$

$$\frac{d_n}{2(\sqrt{5})} + \frac{\sqrt{5}}{2(\sqrt{5})}d_n - \frac{d_n}{2(\sqrt{5})} + \frac{\sqrt{5}}{2(\sqrt{5})}d_n =$$

$$\frac{1}{2}d_n + \frac{1}{2}d_n = d_n = F_n$$

Scripts for Mathematically Less Experienced Group

Scripts for Stage 1

Scripts for A

Script for Session 1

I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers.

If "yes"

I: what do you know about it/them?

I: A merchant named Leonardo Fibonacci of Pisa studied this sequence in connection with a problem he was trying to solve. Since then many mathematicians and scientists have found applications of the sequence in a variety of contexts. This is the beginning of the sequence

[present figure 1:

1 1 2 3 5 8 13 21 34 ...]

Q1: Can you find the pattern in this sequence?

If "no"

I: try adding up pairs of consecutive numbers [more prompts if needed]

I: Can you formulate a rule for producing the Fibonacci sequence?

[prompt if needed]

I: Q2: how would you write your rule algebraically?

I: let's play around a bit and see if we can find out anything about the Fibonacci sequence. Pick one of the numbers in the sequence. [response] Now square it

[response]

now take the two numbers before and after the number you picked and multiply them together. what do you get?

[response]

how is that related to the square of the number you picked?

[the square is one more/less than the product]

Q3: If that relationship were true for all Fibonacci numbers how would you write a general rule describing this property?

I: Let's try something else. Do you know what prime numbers are?

i' "no" explain

I: pick a prime number. Now count along the Fibonacci sequence that many numbers. What number do you land on? Is that prime?

Q4: If that relationship were true for all Fibonacci numbers how would you write a general rule describing this property?

I: let's try one more: Add up the first ten Fibonacci numbers. What do you get? Now multiply the seventh Fibonacci number by 11. What do you get?

Q5: If that worked for any set of 10 Fibonacci numbers how would you write a general rule describing this property?

I: Q6: Do you think any of these properties hold for all Fibonacci numbers?

I: why?

I: How could you become more confident?

[expected response: more examples]

I: Q7: try looking at a few more examples

[work through examples. The first proposition is likely to provoke a reaction when it fails but the subject will probably suggest the (correct) general property soon after. by now confidence should be high]

I: Q8: 19 is prime isn't it? Is the 19th Fibonacci number prime?

I: Q9: Would you now say that this property is true for all Fibonacci numbers? What about the other properties?

I: Q10: can you think of some way to definitely establish the truth of these properties, if they are true?

I: Q11: How would you establish the truth of this property: "The sum of any set of consecutive Fibonacci numbers is a Fibonacci number"?

Script for Session 2

I: Remember the Fibonacci numbers? Here's a list of the first 35. Remember how we wrote the first Fibonacci number as F_1 and the second as F_2 and so on? If n is some number then F_n would be the Fibonacci number in position n in the list. How would you write the Fibonacci number which came right after F_n?

[help if needed]

I: Do you remember how you said to use two Fibonacci numbers to get the next one? [yes...] How could you write that using F_n and F_{n+1} ?

[help if needed]

I: Here are the statements we looked at last time written with F_ns. Do they make some kind of sense? can you see where the symbols come from?

I: How would you write this statement in symbols?

[show figure S1a:

The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

The response should look like :

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$$]

I: Do you think this statement is true? How would you convince me that it is? [prompt for a proof]

I: How about this statement?

[Show figure S1b: The sum of the first **n** even indexed Fibonacci numbers is one less than the odd indexed Fibonacci number after the ones added up.

$F_2 + F_4 + \ldots + F_{2n} = F_{2n+1} - 1$]

I: Is this one true?... How would you convince me?

Script for Session 3

Q1 Look at this statement again:

Figure 3:

D: The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$$

Check it for n=6

Check it for n=7

Check it for n=8

Q2 Consider a pancake. What is the maximum number of pieces you can cut with 1 cut? With 2 cuts? With 3 cuts? Would you agree with the statement:

" **k** cuts will never produce more than 2^k pieces"

Could you show that this is true? [guide if necessary]

Q3 Let us return to this statement. Can an argument similar to the one we just used be applied in this case? [guide if necessary]

Q4 What did you mean in the last session by the phrase "Prove itself"?

Script for B

INTRODUCTION to prime numbers

I: [Q1] Do you know what primes numbers are? [if "no" explain]. Here is a list of the prime numbers less than 10000. [Table 2]

DEVELOPMENT OF PROPOSITION 1

I: Here is a statement:

[Figure 8: $P = n^2 + n + 41$]

[Q2] If n can be any whole number what can you say about P?

Introduction to Fibonacci numbers

I: We are going to be looking at a sequence of numbers called the Fibonacci sequence. Have you heard of the Fibonacci sequence or Fibonacci numbers? The numbers in the sequence are sometimes called Fibonacci numbers.

If "yes"

I: what do you know about it/them?

I: A merchant named Leonardo Fibonacci of Pisa studied this sequence in connection with a problem he was trying to solve. Since then many mathematicians and scientists have found applications of the sequence in a variety of contexts. This is the beginning of the sequence

[present figure 1:

1 1 2 3 5 8 13 21 34 ...]

[Q3] Can you find the pattern in this sequence?

If "no"

I: try adding up pairs of consecutive numbers [more prompts if needed]

I: Can you formulate a rule for producing the Fibonacci sequence?

[prompt if needed]

I: We can write the first Fibonacci number as F_1 and the second as F_2 and so on. If n is some number then F_n would be the Fibonacci number in position n in the list. How would you write the Fibonacci number which came right after F_n?

[help if needed]

I: Can you write the rule for producing Fibonacci numbers using F_n and F_{n+1} ? [help if needed]

I: [Q4] How would you answer this question? [Figure 3a: "Is there anything special about F_n when n is a multiple of 3?"] [Guide to discovery]

I: Do you think this property holds for all Fibonacci numbers?

I: why?

I: How could you become more confident?

[expected response: more examples]

I: try looking at a few more examples

[work through examples.]

I: [Q5] Let's try something else. How would you answer this question? [Figure 3: "Is there anything special about F_n when n is prime"] [Guide to discovery]

I: Do you think this property holds for all Fibonacci numbers?

I: why?

I: How could you become more confident?

[expected response: more examples]

I: try looking at a few more examples

[work through examples.]

<u>DEVELOPMENT OF SKEPTICISM: Proofs and refutations</u>

I: [Q6] Have a look at the formula for producing prime numbers. [Figure 8] Try n=40 [if not already done] [Q10] Would you now say that this property only produces prime numbers? Why not?

I: Look again at the third statement. [Figure 3] 19 is prime isn't it? [Q7] Is the 19th Fibonacci number prime?

I: Would you now say that this property is true for all Fibonacci numbers?

I: [Q8] Would you now say that the second statement [Figure 3a: "Is there anything special about Fn when n is a multiple of 3?"] is true for all Fibonacci numbers?

I: [Q9] What would you say about trying to find out if statements are true?

[In the context of this discussion:]

We have seen that it mathematics it is extremely unwise to assume that something is true for all n just because it is true for some n.

You've just seen several cases where even though there are lots of examples, the statement turns out not to be true.

In the next interview you will be looking at some other statements, some, perhaps all, of which are not true.

Remember to be careful about accepting a statement unless you have a valid explanation.

[Q10] What would you say would be a valid explanation?

Script for Stage 2

I'm going to give you several different activities. For each one please discuss what you are thinking and doing thoroughly with each other to make sure that you are in agreement. In addition to any written work you might do in exploring these activities, please write down the conclusions which you agree on in the end.

Activity SA1: Here is a statement and a proof:

[Statement SA1:

For any number **n**, if a circular region (like a crêpe) is cut into pieces by **n** straight lines which cut all the way across, then the number of pieces produced (P_n) is at most 2^n.

$$P_n \leq 2^n$$

Proof:

One line produces exactly 2 pieces, so the statement is correct for n = 1 as $P_1 = 2 \leq 2^1$

Each time a new cut is made by a new line the new line, at most, cuts every piece into two pieces, doubling the number of pieces. This means $P_{n+1} \leq 2P_n$. If $P_n \leq 2^n$ then $P_{n+1} \leq 2P_n \leq 2(2^n) = 2^{n+1}$

This proves the statement $P_n \leq 2^n$ for all numbers **n**

]

and some questions:

[Questions SA1:

1. Do you agree with the statement? Why or why not?
2. Can you give an example where the statement is correct, or an example where it is incorrect?
3. Do you agree with the proof? Why or why not?
4. What would you say is the most problematic thing about this proof?
5. Could you make the proof better somehow?

]

[Activity SA2

Diagonals of a polygon are lines joining the vertices (corners) other than the sides. For example here are the diagonals of a pentagon:

The number of diagonals D_n depends on the number of sides n of the polygon:

Polygon	Sides n	Diagonals D_n	Example
Triangle	3	0	
Quadrilateral	4	2	
Pentagon	5	5	
Hexagon	6	9	

Show that the number of diagonals of a polygon is always:

$$D_n = \frac{n(n-3)}{2}$$

Here is a table of the Fibonacci numbers
[Table is included after the scripts]
and a statement with a proof :
[Statement SA3:
For all **n**, if **n** is a multiple of three, then F_n is even.

<u>Proof:</u>
F_3 is 2, so the statement is true for **n** = 3

We will need to refer to the following:

A. Even numbers are all multiples of 2.
B. Odd numbers are all 1 more than a multiple of 2
C. If you add an even number and an odd number ie 2n + (2m+1) you get 2n + 2m +1 which is odd.
D. If you add two odd numbers ie (2n+1) + (2m+1) you get 2n + 2m + 2 which is even.

If you have two consecutive Fibonacci numbers F_{k-1}, and F_k; and F_{k-1} is odd; and F_k is even; and **k** is a multiple of 3:
Then $F_{k-1} + F_k$ is odd (by C, above);
So F_{k+1} is odd because $F_{k-1} + F_k = F_{k+1}$;
And $F_k + F_{k+1}$ is odd (by C above);
So F_{k+2} is odd because $F_k + F_{k+1} = F_{k+2}$;
And $F_{k+1} + F_{k+2}$ is even (by B above);
So F_{k+3} is even because $F_{k+1} + F_{k+2} = F_{k+3}$;
Note that if **k** is a multiple of 3 then **k+3** is the next multiple of 3.
We can conclude that: IF it happens that F_{k-1} is odd, and F_k is even, and **k** is a multiple of 3 THEN **k+3** will be the next multiple of 3 and F_{k+3} will be even.

This proves the statement:
For all **n**, if **n** is a multiple of three, then F_n is even.
]

and some questions:
[Questions SA3:
1. Do you agree with the statement? Why or why not?
2. Can you give an example where the statement is correct, or an example where it is incorrect?
3. Do you agree with the proof? Why or why not?
4. What would you say is the most problematic thing about this proof?
5. Could you make the proof better somehow?
]

[Activity SA4:

Consider the following:

1 + 1	= 2	= 3 - 1	= $F_4 - 1$
1 + 1 + 2	= 4	= 5 - 1	= $F_5 - 1$
1 + 1 + 2 + 3	= 7	= 8 - 1	= $F_6 - 1$
1 + 1 + 2 + 3 + 5	= 12	= 13 - 1	= $F_7 - 1$
1 + 1 + 2 + 3 + 5 + 8	= 20	= 21 - 1	= $F_8 - 1$

Show that in general:

> The sum of the first **n** Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$$

]

[Activity SA5:

The rule: $B_1 = 1$

$B_2 = 1$

$B_n = 2B_{n-2} + B_{n-1}$

defines the B sequence. These are the first 24 terms of the sequence:

[see table below]

Consider the following:

1 + 1	= 2	= 2^1
1 + 3	= 4	= 2^2
3 + 5	= 8	= 2^3
5 + 11	= 16	= 2^4
11 + 21	= 32	= 2^5

Show that in general:

$$B_n + B_{n+1} = 2^n$$

]

Table of Fibonacci Numbers

	Fibonacci numbers
F_1	1
F_2	1
F_3	2
F_4	3
F_5	5
F_6	8
F_7	13
F_8	21
F_9	34
F_{10}	55
F_{11}	89
F_{12}	144
F_{13}	233
F_{14}	377
F_{15}	610
F_{16}	987
F_{17}	1597
F_{18}	2584
F_{19}	4181
F_{20}	6765
F_{21}	10946
F_{22}	17711
F_{23}	28657
F_{24}	46368
F_{25}	75025
F_{26}	121393
F_{27}	196418
F_{28}	317811
F_{29}	514229
F_{30}	832040
F_{31}	1346269
F_{32}	2178309
F_{33}	3524578
F_{34}	5702887
F_{35}	9227465

Table of the B Sequence

The B sequence:

B_1	1
B_2	1
B_3	3
B_4	5
B_5	11
B_6	21
B_7	43
B_8	85
B_9	171
B_{10}	341
B_{11}	683
B_{12}	1365
B_{13}	2731
B_{14}	5461
B_{15}	10923
B_{16}	21845
B_{17}	43691
B_{18}	87381
B_{19}	174763
B_{20}	349525
B_{21}	699051
B_{22}	1398101
B_{23}	2796203
B_{24}	5592405

APPENDIX B

TRANSCRIPTS

Each session of the clinical study was recorded on audio tape, and transcribed for analysis. This appendix includes abridged transcripts of all sessions. Parts of sessions which did not provide information relevant to the study have been summarized. In some cases short utterances such as "yeah", or "uhhuh" have been omitted where they do not have importance. In the transcripts three dashes "---" indicate a pause of about 1 second. Question marks in groups represent words which could not be understood on the tape. In some cases a possible interpretation of partially audible words is offered, enclosed in brackets, and ending with a question mark. Line numbers beginning with "R" indicate utterances of the author. Line numbers beginning with "S" indicate utterances of Professor Sierpinska, the author's thesis supervisor.

<u>Stage One, Group Learning MI</u>

The students were first shown item 1:

1. Find the relation between the number n of sides of a convex polygon and the sum A_n of its internal angles.

E was uncertain of the meaning of convex polygon. S explained. G was also uncertain about convex. S and E explained.

E6: [reading] find the relation between the number n of sides
E7: and the sum
G10: of a convex polygon and the sum A N of its internal angles. OK assuming we have so many sides

Figure 4: G's sketch at line G10.

E and G debated whether the number of angles equalled the number of sides. E gave triangle and rectangle as examples.

E14: we have to think about this angle. We know 1 very interesting thing that for triangle we have the sum of angles is 180 degrees

E and G suggested to each other breaking the octagon up into triangles.

G23: the sum of each ---
E21: is 180
G24: is 180 degrees, so for every <u>2</u> sides we have a triangle [see figure 4]
G25: So assuming we have 8 sides
E23: yeah
G26: ok we know for an even number of sides, we have 4 triangles within it.
E24: yes and we have to multiply it by 180, ya?
G27: correct

They then wrote down their conclusion for n even.

G35: n divided by 2 times 180
E33: wait a moment --- triangles, ok?

G36: sub-triangles
E34: you are right
G37: triangles
E35: yes, and the sum, ya?
G38: the sum, that's right
E36: An   is equal 180 multiplied by n by 2. ok?

G then saw that their rule did not work because the triangles did not cover all of four angles. E suggested placing all the triangles with a vertex on one vertex of the polygon. They made a drawing.

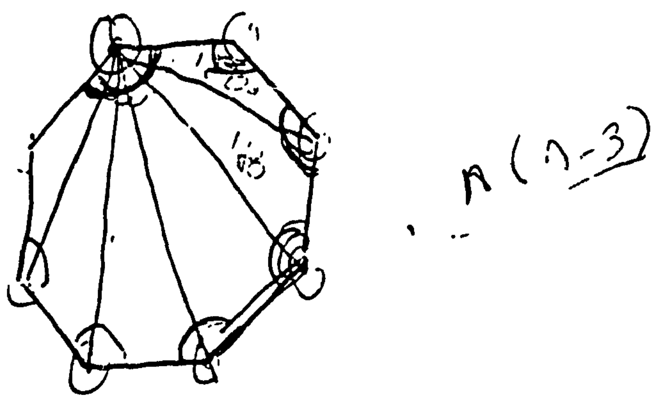

Figure 5: E and G's sketching of sub-triangles of an octagon.

G agreed to E's plan but felt the process would have to be repeated with every vertex to determine the angle sum of the polygon. They counted the number of triangles and the number of sides. The number of sides was counted incorrectly. G generalized n-3 from the miscounted sides. E attempted to explain that the process did not have to be redone for each vertex of the polygon.

At this point S intervened and suggested starting with a triangle.

G93: a triangle, that's what I was thinking
S12: How many, how, what's the sum?
E91: 180 that's all
S13: Ok and the
E92: OK It was one
S14: No
E93: Sorry three and
G94: 180
E94: and one hundred---
S15: Pi, or Pi
G95: or Pi [see figure 7]
S16: Now how do you make, how do you make .. a quadrangle
G96: Ok that's right

E96: we have two pieces more
G97: Ok then I see what you're building
S17: So for?
G98: 2 Pi
E97: 2 Pi
S20: You don't have Pi but you have not n minus 3 you have n minus ---
G100: 2
E99: OK, something, something is wrong
S21: Something is lost here
E100: Ya ya umhmm
S22: so how do you build a, a pentagon
E101: OK OK OK
G101: well..
S23: how do you build a pentagon from that?

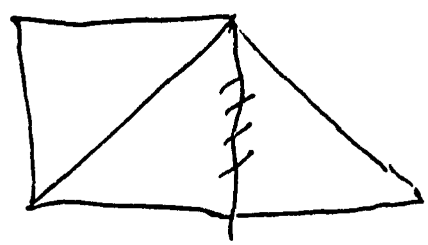

Figure 6: G's sketching at line G102

G102: Well you, all you have to do is add an extra two
E102: No if you mmm--- ok It works, It works only for even number uh uh, odd number ---. [nacht???]
G103: Can't you build the edges like this?
S24: yeah but you don't mean that this and this lies on the same line
G104: No they don't
S25: so you have 1 2 3 4 5, so for 5
G105: umhmm
E103: 3
G106: Yeah, interesting --- [?] finish off the relationship then
S26: What is it?
G107: For
S27: n
E104: n minus 2 uh n minus 2 multiplied by Pi. umhmm ya

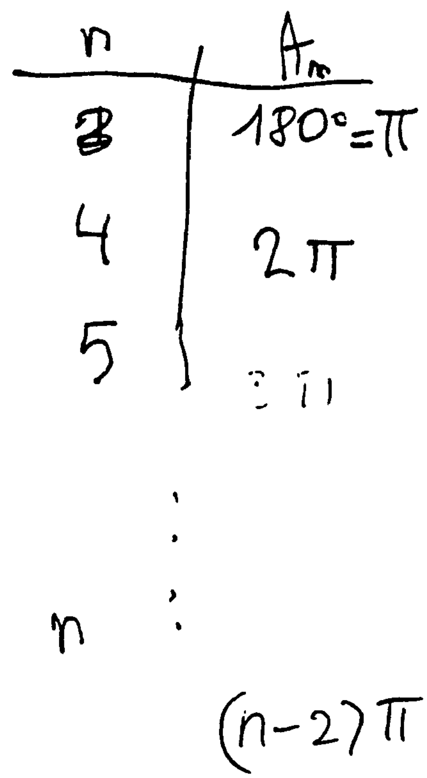

Figure 7: Table of angle sums.

S28: OK
E105: Why, why, why I took this 3? I don't know, I have to check 1 2 3
S29: You just subtracted, but you have to add a Pi
E106: oh ok
S30: because you're subtracting the three but this has to be added
E107: umhmm ya
S31: Ok so now the second problem --- So --- you can still look at that
E108: [reading] give a mathematical proof of this relation assume as an axiom
S32: Because what you've one just now is discovered it
E109: mmm?
S33: How would you formulate the argument?
E110: By, by induction. it's too long, it's terrible
S34: By induction?
E111: Ya
S35: Do you know mathematical induction?
G108: yeah
S36: you do?
G109: well I've touched up on it, Prove your basic stuff, prove it for

E112: For the first one
G110: yeah for the first one
E113: For the n equal three
G111: That's your hypothesis [?? hard to hear] prove for uh prove for n greater than three
E114: No the first step is to prove for n equal 3
G112: For, yeah it's you basic step
E115: And the second one you assume its uh true for probably K and to
G113: where K is, yeah
E116: and use this assumption to prove
S37: OK
E117: it's true for n equal

S asked E to let G describe the proof.

G114: well I'd go for the basic step
E122: N umhmm ok
G115: Ok n is equal to 3. Then we show that n-2 equals Pi So, we, that's evident as such or do you want a graphical, can we give a geometrical ---

S instructed them to take the angle sum of a triangle as an axiom. She mentioned that this is not always so. E and G wrote out the beginnings of their proof.

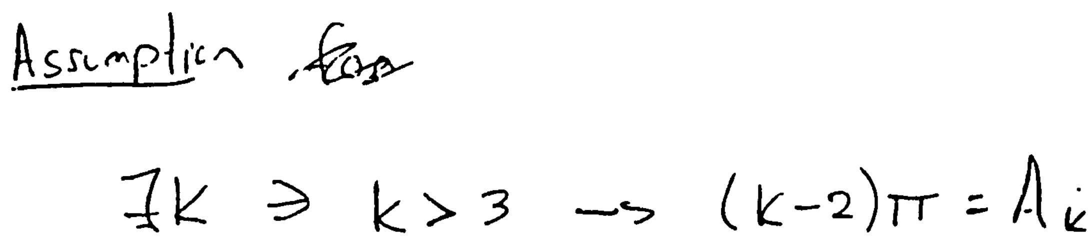

Figure 8: G's writing at line E135

E135: Ok, then, it's really simply because every time if we uh greater, No, if we make the figure bigger
G131: that's right
E136: you know, then we, look at this we have one Pi more
S51: you show that table [see figure 7]
G132: all right
S52: you are showing the table
E137: Ya
S53: You are not showing this --- the diagram
E138: yes, yes
G133: for every increase in K the one before we're always getting a Pi less that's. ---

S then prompted them for a proof of the induction step. G suggested proving it by summation. E rejected the idea. G expressed confusion about the status of "The sum of the angles of a triangle is π". He was not sure if it was an assumption, an axiom, or a theorem. S assured him it was an axiom. E stated that a formal approach was needed, but G and S rejected the idea. G claimed that the axiom proved the angle sum formula, but was not clear how. E stated that a polygon could be divided into triangles, or built up from triangles. S expressed the inductive step of building up an (n+1)-gon from an n-gon. G agreed with this idea and felt it proved the formula. E also agreed, describing the triangle as the basis, and referring to an assumption.

G152: it's proved for that
E156: If we add
S73: that's what you have to show
E157: Is this good? A K plus 1 is equal ---. k minus 2 by our assumption ok? plus Pi so it's equal how much? K minus 1 Pi

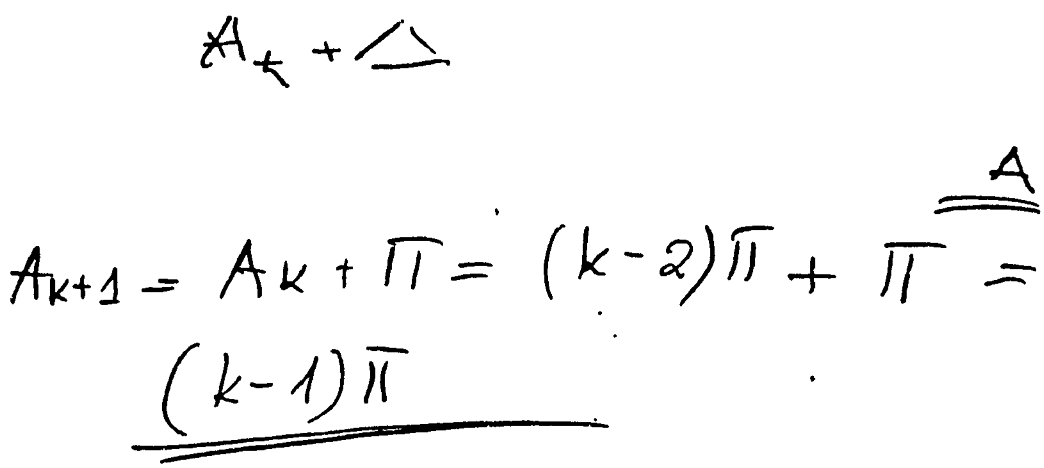

Figure 9: E's writing at line E157

G153: yeah
S74: So what have you shown?
E158: What we have, We proved.
S75: Proved what?
E159: we proved that our rule, that our mmm
S76: formula
E160: formula, is good
G154: holds for K greater than
E161: Is good for all n greater or equal to 3
G155: 3

S then showed E and G item 3, involving determining that 2^n is an upper bound to the number of partitions of the plane produced by n lines. G commented that he had never done geometry. They read over the item, and E commented that the previous problem had shown them that they should begin by making a table. G agreed. S told them to look at the problem geometrically instead.

E166: If you have one line you have two parts, if you have 2 lines you have 4 parts, ya? and if you
S81: Oh? Have you? Two lines?
E167: Two lines
G160: two lines you have 3 parts
S82: you may have 3 parts, or 4. When do you have 3 parts?
G161: or 4
E168: for 2 lines?
S83: yes
G162: well the lines they may be parallel and onto each other
E169: no no
S84: yes
E170: ok
G163: You may have
S85: If they are parallel, just count the numbers
G164: so can we make an assumption that not, no two lines are, are the same?
S86: Why should you? Those, you just have to show that the plane makes,
E171: not more than
S87: into not more than
E172: 2 power n
S88: you don't have to calculate the exact number
G165: n is the number of [parts? points? planes?]
E173: yes
G166: unless you did this, we have two lines they're onto so for
E174: I don't understand your sketching

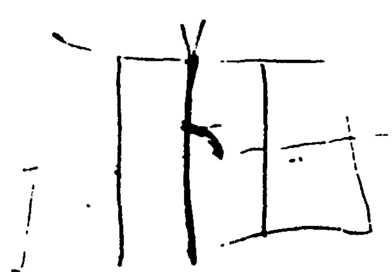

Figure 10: G's sketching at line G167

G167: well this is the plane we have, two lines one over the other
E175: Ok

G168: assuming we have two lines here, ok?
E176: Umhmm
G169: then for, fcr this case in which the two lines are parallel , are parallel and onto each other, the same line, we only have two parts
E177: We ---
S89: These are two lines?
E178: No
G170: Yeah it's one on top of the other so we'll make this one a different color [see figure 10]
S90: Why?
G171: we're dividing a plane with 2 lines, those 2 lines can be the same line can't they?
S91: yes
G172: A line on the same
S92: you have the base of one
G173: The two lines are equal
E179: Ok but we have, we have to
S93: ??? Because he is finding the extreme cases so here's two lines
E180: Yes but
S94: suppose they're equal, then you have two parts, suppose they're not equal, parallel,
E181: OK
S95: three parts, if they intersect
G174: Now this line, in order to have 4 parts this one has to be
E182: ya
G175: Not necessarily perpendicular, [see figure 10]
E183: No yes I understand
G176: but at an angle to this
S96: You are mixing your pens
G177: line here, are we assuming that the plane
E184: Ya
G178: Sorry ah there must be some angle I shouldn't put, so given that there's an angle between the two, and they are not parallel, then it splits it into
E185: Ok
G179: n uh, n plus, n plus 1, n plus 1 sides, correct? Or I'm assuming that, that if, lines are parallel then we split this, assuming we have this and this and what do we have, three cases

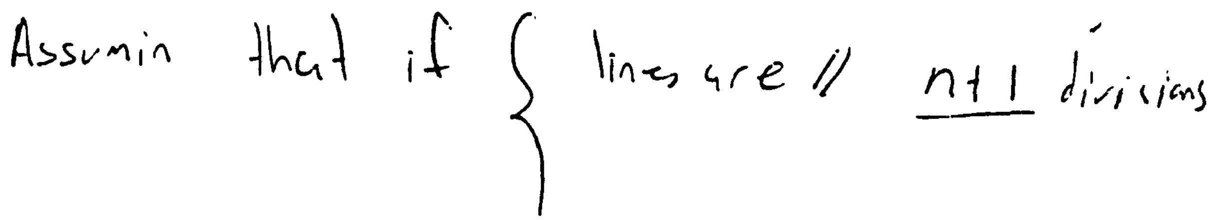

Figure 11: G's sketching at line G179

E186: Ok but but
G180: If we put another line 3 we have four cases, so we're going to has n plus 1
E187: OK look, but we have to consider only one extremum situation, when you have the biggest number of parts, never mind what happens for parallel or now not parallel I think the most interesting thing is when the line, line,
S97: Intersects
E188: Intersects in one, in one point

E claimed that the maximum number of partitions is produced if all the lines share a common point. G only claimed that no two should be parallel. He did however insist on looking at the minimum case as well as the maximum. He claimed that if all the lines intersect the number of partitions would be 2^n. E presented the possibility of three lines forming 7 partitions.

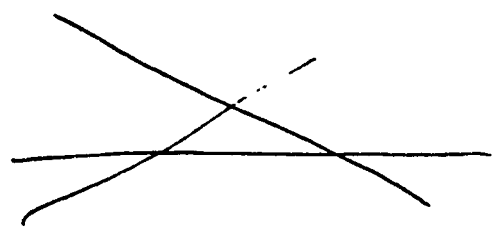

Figure 12: E's sketch of 7 partitions formed by 3 lines

S asked if this situation produced more or less than the case of all lines intersecting in a point. E counted, found 7, and G stated that this was "one less". They then considered that case of four lines. Returning to the case of three lines E commented that it should be 8 not 7. S asked why and pointed out that 2^n is an upper bound, not the actual value. They agreed that all lines intersecting in a point did not produce the maximum possible. S brought them back to the problem of showing that 2^n is the upper bound.

G199: It's kind of evident but it's kind of hard to show, it's hard to show that either that line will intersect all those other lines and intersect what those other lines aren't partitions of 2 which will double it and yet it still won't exceed that
E212: ya
G200: that value but it will be equal to it.
S110: How many more, parts more will you get when you add another line?

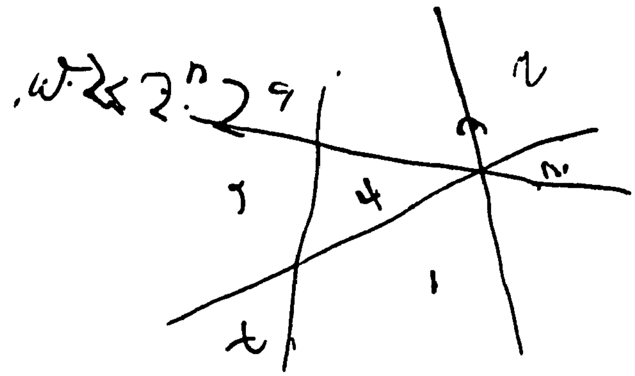

Figure 13: S's sketching at line S110

G201: when you add another line?
S111: yes
G202: maximum, multiply the number you had by 2, maximum
S112: yes that's it, because it can intersect in at most two parts
E213: Umhmm
G203: two parts
S113: each of those can intersect ---
G204: For instance you have let's say 6 lines
E214: if we, probably, if we have n ---
S114: lines
E215: No no no, n lines it gaves us, gaves us, the double you parts
S115: yes
E216: then if we add one line it can not divide
S116: more
E217: more than had by half
S117: so at the most you double
G205: double
E218: times 2 [S says at same time]
S118: so it gives you the proof
G206: I think that's proof enough. I can't, a could see ---
E219: by induction, you know
G207: assuming that your maximum is n and adding an extra line that extra line you add cut through and intersects each and every one of the others, at a particular point then the maximum you can have is 2 times that value, yeah
S119: yes
G208: that's obvious to me
S120: What is it? This is the induction step
E220: Umhmm
S121: That's what you, that's what you [need?] because assume that this is less than or equal to 2 power n for

n lines
E221: umhmm
S122: then you add another line and then it just doubles
G209a: doubles
E222: by two this side and this side
S123: this would be 2 power n plus 1
E223: plus one
G209b: that's right
S124: This is the induction step and that gives you
E224: the [conclusion?]
G210: --- which --- I have, I have trouble with induction, I can't ---
S125: that's a ---
G211: you use the hypothesis, I can't see why they use that hypothesis within their a, their assumption, ok? they assume something and they they use it within their proof
S126: and you don't believe it.
G212: and you don't believe it, why
E225: It's very simply, if you have a very very long line of people, and the last one kicks [everybody? anybody?] then the next one wants to kick one before and again and again, you know
G213: Ok, you don't, --- I have trouble seeing it. Why should that assumption be true, when applied to your ---
E226: Because probably induction, we have to start from the middle, you, you, we start --- Every time we start with one line you know, with the first, the first case and after that we start from the middle, of, of this ---
S127: you would propose what? to check for some 5 or 6, that would give you more confidence?
E227: No no no no no then, this, in this case ---
G214: How many do you need to build up that confidence? That gives you, well here's your proof now?
E228: No
G215: Yeah, that's the way I see it. How else would you see it? you can go to as many steps as you want and your hypothesis is, I don't ---
S128: Suppose we have proved that, then we can prove it for, what you are doing in the induction step I think is you're showing how to prove each step.
G216: Each individual but you have---
S129: Yeah
G217: to show for that many steps on, it does hold
S130: It's done
E229: But you make, [on this crazy?]
G218: Proof by induction, induction proof by, all the others I understand, induction I have problem understand, I'm sorry I can't
S131: But that's what the whole experiment is all about.
G219: Is it?

S132: because people have trouble with being convinced there is
G220: I have a lot of problems being convinced by induction.
S133: Yeah
G221: you can show me a lot of points, but what, what just because you assume something, why are you assuming it to be true in every case?
E230: But you made it without any problem here
S134: were you convinced by those, those ---
G222: I'm convinced through word not through induction, and if this is indeed the case when you showed it here, in this particular case, I can see it.
S135: Umhmm
G223: But in several other proofs that I've done, I can't see it. I have problems --- [working?] ---
S136: Let's well, where were you convinced in this case?
G224: Because for each multiple on this side you multiply the other [see figure 13]
S137: Because I think the only one the inductive argument has convinced you have seen what you have seen is the pattern of proof for each individual case
G225: [???]
S138: Because you can prove it for n equal, I don't know, 1 and 2 and 3
G226: Ok
S139: You see how, you might use the --- concrete Ns, you could show it at one, you know, one shot, to prove an infinite number of a, of assertions --- I think that because you know so much about induction
G227: there's one in particular is proving that K squared, is equal to K, what is it 2 K plus 1 K,

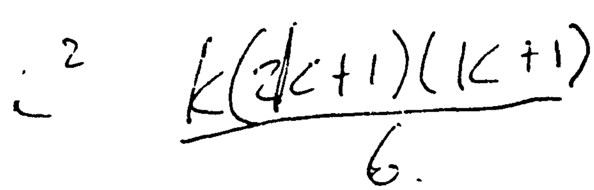

Figure 14: G's writing at line G227

S140: What is it?
G228: K squared will be 2 K plus 1 K plus 1 no? is that right?
E231: Are you thinking about the sum?
G229: over 6
E232: about the sum?
G230: showing that this, yeah the summation, that this is ---
E233: squared

S141: And you don't, you're not ---
G231: No I can't this I had problems with, I remember doing this
S142: So lets' try this problem now, ok? ---. we have this [Item 8, the two sequences A and C]---
[long silence]
S143: Here's a definition by recursion
E234: Umhmm
S144: You know what the first are, and then you know what the next is when you know what, do you understand
G232: Yeah ---
S145: Have you any idea what the two sequences are?
G233: 1 2 just give me a minute
S146: calculate
E235: if you, if we have a sequence from the previous ah element of this sequence we take the second coordinate and put, it in the first place, and the second place is obtained by
S147: summing
E236: summing both of the coordinates
S148: Ok what are, could you write some of the first, terms
E237: It will be the first one, ya? ok the first one is even by this, the second element of this sequence is given by, I take this one and put it here and the second one is given by adding, ya? Ok?
S149: this goes
E238: This goes here and, this one is obtained from this and this by
S150: summing
E239: summing

E then began to determine the actual terms of the sequences. She began with the A sequence. After finding several terms she switched to the C sequence:

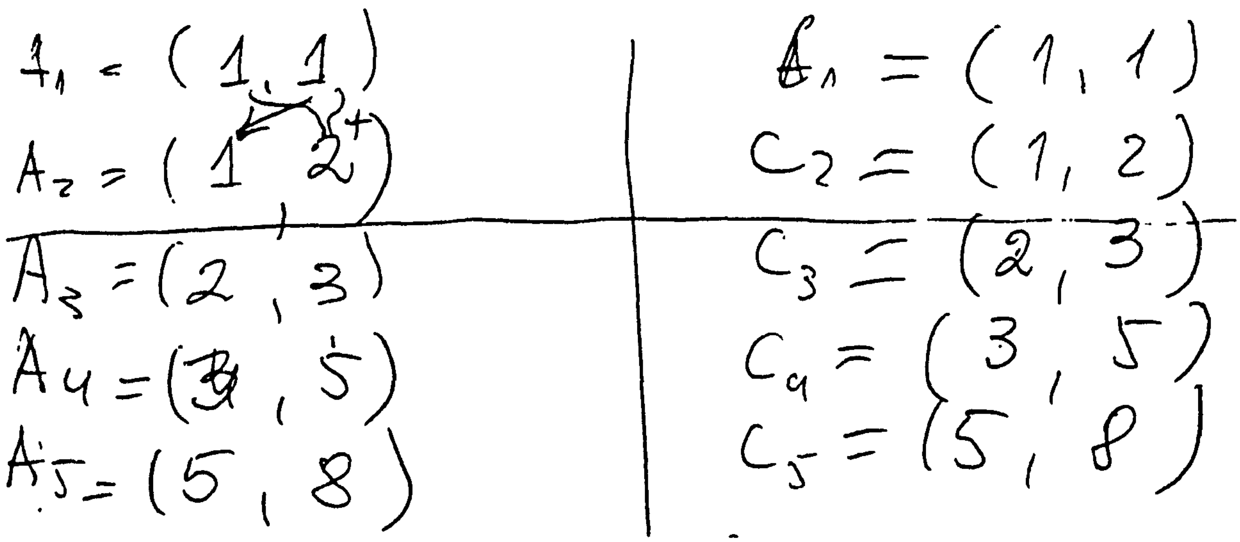

Figure 15: E's calculation of the first terms of the A and C sequences.

E248: Ok we add in this way, this one and this one
S156: the previous two
E249: ya 2 and here 3 and again this one and this one gives us 3 and 5, this one and this one give us 5 and 8
G240: now its 5
S157: Ok so what did you get?
E250: we, we get the same thing, the same
G241: well write it down in general
E251: we get the same sequence
S158: Then if you can define it a different way and still get the same thing how can you?
G242: recursively, isn't it,
E252: No I think we can find, we have to find a formula to to come from one sequence to, to the second --- isn't it?

E was uncertain how to begin. G suggested that the sequence was related to 2n-2. He then found he was wrong.

S162: What are you trying to do? Have a general formula?
G247: Yeah, I'm trying to find the general formula for this one here
S163: And ---
G248: And generalize this one
S164: And show that
G249: And show that they're equal
E258: umhmm
G250: so if we prove a general formula for this, prove by induction through here
E259: But you are ---

G251: No it's prove by induction that your formula does indeed give you this recursive sequence, and then do like wise with this one give a general formula, prove that it does indeed show through induction for n greater than
S165: Two inductions
G252: two inductions, yeah and if the formulas hold then if they're equal well you can get that they're equal
S166: You can use one single induction
G253: You can use one induction?
S167: Ya
G254: On both, because your assumption is that they're both equal --- yeah that they're equal
S168: No, Yeah but you can show this by showing that A 1 equals C 1 and then assuming that A N is equal to C N show that A N plus 1 is equal to C N plus 1, and then you move on. Maybe you're not going to [be?] convince --- hm?

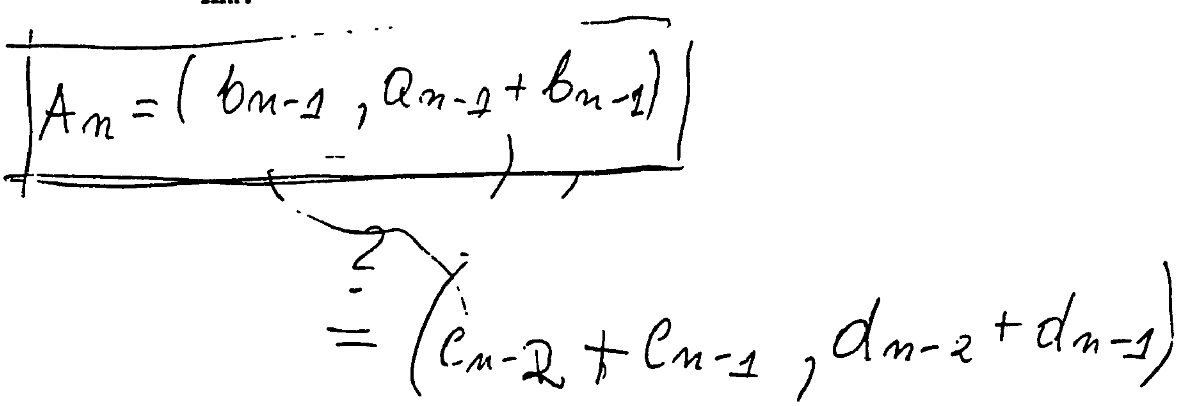

Figure 16: E's writing at line S168

E now suggested finding a general form for each sequence. S said this was not a good idea. G wrote:

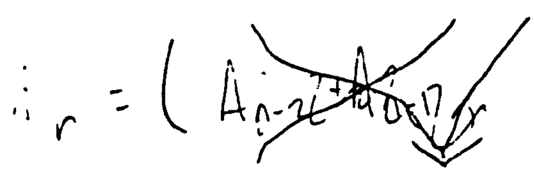

Figure 17: G's writing at line G259

This was considered useful.

G259: Case that they both start off at the same place so their sequence will indeed be the same, no? --- Given that C 1 C 2 starts with 1, A 1 B 1 starts with 1
E266: Ok but
G260: No?
E267: no, I don't think so, ok,

G261: I I
E268: You know that they are the same and you are trying to, hm?
G262: I think that if you can show this case and you can show this case the only thing that differs in this and this are their---
E269: But you write the same ---
G263: Yeah but you might assume that you start at different, you might start differently here, as opposed to here, where here you can start with 2 2, but since they both start with 1 1 then you can show that these two are the same, can't you?

E considered to herself. G continued to try to see a way to build up from the beginning of each sequence.

G268: I think you should show this for this and you should show this one here for this one
S176: Oh
G269: that's what I was saying
S177: so umhmm
G270: 'cause if you can show this to be true for this case and this to be true for this case then you've already implied that this one you've taken both sides, and I think it's proof enough instead of trying to go directly from here to here with some, by formulating it ---
E273: But how are you going to do it? You can see, you can make it only by induction I think, There's no, there's no other
G271: You have to show that, you have to show that both sequences are
S178: Yeah, but even in that, in that solution you have to use induction
E274: Ya, there's no
G272: You have to use induction to show that the sequence is the same, or you can prove inductively that this, these two sequences will yield the same
E275: umhmm, then there's formula for A N and there's formula for C N OK? and we have to show that the first
G273: Start with the basic,
E276: Ya
G274: the basic case
E277: the basic case
S179: the basic case is, you have it
E278: Ya, the basic case we have
S180: You don't worry about it
E279: We don't have
S181: You have everything there, from 1 to 5 ---
G275: OK, our assumption is ---
E280: Yes, and we have to extend about 1, about 1, each of these elements

G276: yes

They conferred.

$$t_{n+1} = (b_n, a_n + b_n)$$

$$= c_{n+1} = (c_{n-1} + c_n, d_{n-1} + d_n)$$

$$b_n = a_{n-1} + b_{n-1})$$

$$a_n + b_n = b_{n-1} + (a_{n-1} + b_{n-1})$$

Figure 18: E's writing after line G276

E did most of the writing while G agreed. G objected to the use of an assumption at one point. E explained that the important thing was the method of going from one term to the next which was always the same in each sequence.

E303: It is possible probably to make it formal [activity?]
S192: yes
E304: But it is, it's too long! --- [Oh god?] because you know, I started to do it you have so many indexes here ---
G294: Let's assume this is true

They continued to work on the problem. S then led them through the argument, summarizing the relevant points in what they had done.

E316: that's the whole thing
S201: the so that you can use this induction step and show this formula because the previous are OK hmm?
G301: yeah I can see that
S202: But, you can't---
G302: I can't, I don't know how to put it down in,
S203: Ok you can see it
G303: yeah I can see it
S204: well that's just a matter of some technicalities
G304: That I ---

S205: But are you convinced by this?
G305: I am convinced
S206: You will be if you proven that
G306: I'm convinced well for the next step I can see it as well
S207: Umhmm you see the method
G307: I see the method
S208: You see the method
G308: yeah but again, proving by, induction ---

S then switched to items 11 and 13:

S210: doing mathematical induction well, we have already discussed this. I didn't expect you to talk so much, before I asked this question. so let us come to this, you know, in empirical sciences we are using something that's called induction, which I suppose someone is doing some ethnographic research enters a tribe and sees three people, they all have blue eyes, so he jumps to the conclusion, Oh all the people in this tribe have blue eyes. How does this kind of reasoning differ from the mathematical induction?
G310: well we have to ---
S211: We start from some previous, some [yes this extension???] we start from some, small numbers, a few cases, yeah
G311: I'm not, personally I'm not convinced with it so
S212: You're not, not as much as with, the empirical as well as mathematical induction doesn't convince you
G312: [not much at all???]
S213: but you were convinced in some cases
G313: In some cases yes, I was
S214: well I don't think you have to make it because just as with this assumption that the sum of angles in a triangle is Pi --- This is an assumption that we do because it works in certain domains of experience. It can be applied in, you know, local places of the earth where you have, on the flat, on the flat you have, you know, the sum of angles is Pi As you, suppose you take very big triangles on the earth as, as a globe, as a sphere then you get those triangles the sum of angles of which is bigger. So this geometry it is a theory of local parts of the earth, which look flat, and I think that with Mathematical induction is, has the same status, a status of an axiom, this is an axiom of our, and it have the same status as our assumptions about logic, as about geometry. You have Euclidean, you have non-Euclidean geometries so you don't have to be convinced, about it so much, because there may be theories in which this doesn't work ---
G314: I'm curious, when you mentioned about the tribe, I'm, I don't know, I find it ambiguous sometimes, obvious I

can't see why you're doing, I can't see it for all cases for instance, some I find trouble seeing it, with others it's evident, it's obvious, but some of them I ask why do you prove, it's obvious
S215: And sometimes not obvious, even after proving
G315: that's right, yeah.

S then asked E her opinion of MI. E felt that one problem was the lack of experience many students have with MI. G asked how much experience E had had, and whether she had studied here (in Montréal). E replied she had not. S then showed them item 9: "Show n(n+1) + 11 is a prime number". They plugged in numbers. S pointed out all they got were prime numbers.

G325: but how far do you go, in the case of proving that, Pi is, or any [ir?]rational number for instance, doesn't, doesn't have, for instance the, how do you know it doesn't end somewhere along the line, and how do you prove that by induction, for instance, In that case
S225: to prove
G326: prove that, prove that Pi
S226: Pi is, is not rational
G327: yes
S227: do you prove that by induction? No no I think that to prove that it is not rational is somewhat difficult, it is one of the biggest theorems of algebra, you have to prove that

S commented on the difficulty of such a proof, and E tried to remember if it had been among the proofs she had seen done by induction. S said that it couldn't have been. She then returned to n(n+1)+11, and asked them to try n=10. She pointed out that MI avoided such situations, of being unsure of the truth of statements, by providing a way of proof. G asked why they didn't see more MI in schools. S said that the difficulty of the method, and the lack of need in early grades, kept MI from being used extensively. G commented that he felt most of the students in the class he saw MI in hadn't understood it. E mentioned that she felt there was a general lack of study of logic in her classmates. G felt that he generally understood proofs, it was just lack of practice with MI which caused him problems when working with it. S said she though that the assumption in the induction step often caused students difficulty, and G agreed that this was a problem.

G337: Well exactly when, if you're not sure your assumption right, but if it's based on a previous assumption
S240: Yeah
G338: and that assumption yeah, it's recursive but it's based on a previous assumption, for instance how would

you prove this? would you

S241: Well you can't prove it because it's false, it's false

G339: How would you prove it false?

S242: by a counter example

G340: Ok I can see the contradiction but how would you prove it false through induction?

S243: No no

E333: you can never prove it by induction you have to prove

S244: I'm not showing that all of them, none of them is a prime number, I am showing that one of them is not a prime number

G341: well, that's easy, that's a straight forward

S245: That's because this is the, the contradiction, the negative of this theorem is there exists

G342: Oh I understand, I completely understand but proof by contradiction, that's proof by contradiction

S246: No this is not. proof by counter example

G343: How about if someone just started proving this through induction, where would he fall upon that false case, for instance 10? What method?

S247: well, No, No methcd. You know, mathematical induction doesn't give you even the idea, the idea has to be discovered, like that

G344: No it doesn't, it doesn't. then

S248: you have to disprove it

G345: How do you know, you know you've gone wrong somewhere along that way

S249: well, you know, that's kind of chance through ingenuity you have to have insight, you know, to prove that, you know you can have 41 here and it's more difficult so to discover the, Great mathematicians have come up with this formula saying this will give us all the prime numbers or an infinite number, this proves that we have an infinite number of prime numbers, and one century later someone discovered that for 40 it doesn't work

S discussed the precess of discovery in mathematics, and the place of proofs in providing certainty for empirically derived conjectures. She mentioned that they had seen how MI could be used in this way.

G348: It doesn't make much sense

S252: Mathematics doesn't make sense?

G349: No, mathematic makes sense. mathematics is very logical,

S253: Except this, except this

G350: Exactly, I don't like this at all

S254: Maybe we should stop the experiment

G351: I can move forwards, I can move backwards, but I have to move somewhere to show my conclusion, now it's either I can prove it wrong or I can Like I said we can imply things from right to left or from left to right but we're going to come to, we're going to come to a final agreement, but this doesn't, you have no, no concrete way of knowing that this is indeed proof enough that's what I don't understand

S255: This, but this counter example is proof enough?

G352: For me? that's fine

The session ended at this point.

Stage One, Group Being Made Skeptical

The session began with some questions about the Fibonacci sequence. Both J and H said they had heard of the Fibonacci sequence. J also knew the rule for generating the sequence and was asked to explain it to H.

J3: OK so, the next one is the sum of the two previous ones
H3: Oh, I see it, I see it yeah
R5: OK, how would you write that as a, an expression or a formula, algebraically
J4: should I write it here?
R6: yeah
J5: OK F n plus 1 F n plus F n minus 1 --- so that's for
R7: Does that make sense?
J6: That's for the term, like each term
R8: Umhmm
H4: That's right. You should put A N is equal to 1, but I mean it's right
J7: Ya Ya and the condition that F F 1 and F 2 is equal to 1, because otherwise it wouldn't work, how to start

H and J were then given the question: "Is there anything special about the sum of the first *n* Fibonacci numbers? Is it related to the sequence in any way?" They read over the question and H asked for a clarification of it. R restated the question for the specific cases n=3 and n=4.

J11: we must [?] first terms [?] second term in terms of all these terms then you have F 1 plus F 2
J12: then you have [?] F 2 then you have --- then you have F 2 plus F 3, plus F 4 and then you have F 3 plus F 4
H9: Funny how the first, if you add like this, It's like you're adding
J13: and so you get
H10: 2 times each one
J14: 2 ---

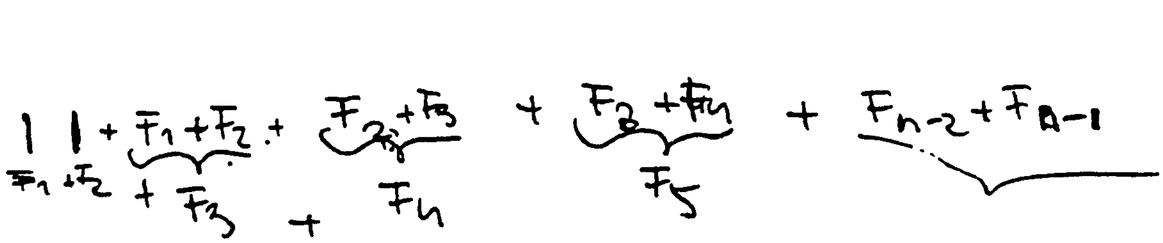

Figure 19: J's writing at line J14

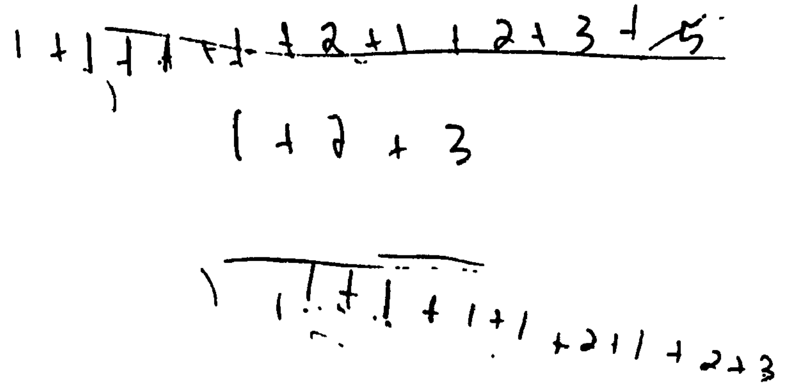

Figure 20: H's writing at line J14

H and J had been working separately, and were asked to compare their work at this point. They felt that they were doing the same thing.

H13: I'm not writing anything down, just a bit but that's what I was thinking .. like adding these things to some sequence --- Why?
R14: Just wondering ---
H14: So F,
R15: Is this getting you anywhere?
J17: Not, not really because, mm I cannot write formula for this, for the sum, because there's like nothing , we know that each of them repeats two times, you're adding every number, every previous number in the sequence two times to get the, the sum of the N Fibonacci number, numbers but I don't know --- hmm let's see --- they're, you just take mm minus --- N minus 1 --- --- well I don't know how, to formally, describe my, my idea but, ---
R16: Perhaps if the two of you compare your ideas you'd be able to put something together here?
H15: It's the same, it's the same thing see

J18: like all
H16: adding these two terms
J19: Ya all of them will, will be counted like twice
H17: well not all of them, the end, the end points, the last one, see you won't count, like let's say if you did it say for a certain set here
J20: Ok
H18: This one would not be counted in, at all, this one would be counted once, would be counted in twice, see it would be like this, from here to here would be 12, this which is equal to 1 plus 1 and you add this one again plus 1 both twice, because they'd both be counted twice, plus 2, hmm, 2 plus 1 for this one because you're counting this one, like this [see figure 20]
J21: Umhmm equals 5
H19: and then 3 plus 2 would equal 5 and and this would be right here, I screwed up this 5 12 7, so, [see figure 20]
R17: Is that the sum of the first 5?
H20: This one right here yes, first, first 5 let's see not this, I don't think so yeah 7, 1 2 3 4 5, but this, here it's counted 1 2, this is like, as well, and we've proven, this series, this series of numbers when you do this do you, is it A 1 and A 2 are equal to 1 or is it that, they assume that A, A 1 is equal to 1, this is equal to like 1 plus 0, I'm just curious, ah it's irrelevant actually

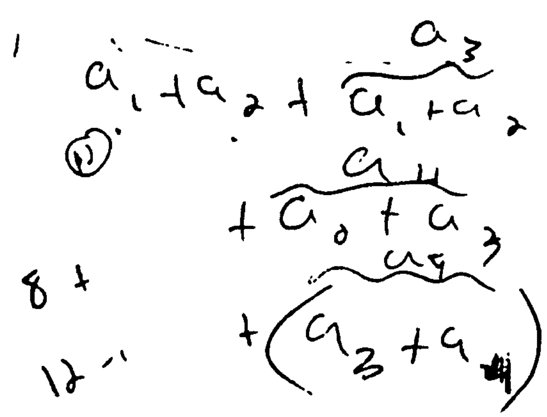

Figure 21: H's writing at line H20

R18: Yeah, Just define A 1 and A 2 as being equal to 1 is the easiest way to do it
H21: A 1 equals --- so A 1 plus A 2 plus --- plus A 1 plus A 2 --- plus A 2 plus A 3
J22: [Reading] is it related to the sequence in any way. by sequence we mean like geometric sequence or ..
R19: No No to the sequence of Fibonacci numbers itself

J23: To the sequence ah ha
R20: Yeah is the sum related to the sequence, in any way other that being the sum of certain terms of the sequence
J24: The sum
R21: The obvious way
J25: so this right here should, you should be able to find the relation. found
R22: If there is one
J26: If there is one --- mmm
H22: A 4
R23: Have either of you calculated any of the sums to, to just know what they are?
H23: Ah no I didn't want to actually, I did this right here, I did the first 5 the sum of 5 to see what it is in some expression, to see how it comes out and, --- it does this, so

At this point J checked the first five sums, while H continued to work on the problem generally.

R27: the sum of the first 5 is 12
H28: Right
R28: Is there anything special about 12?
J28: mmm
H29: No, why, Want do you mean, I don't think so, well, it's just a number
J29: No
H30: Is that what you want? Yeah, it's just a number. Is there anything special about 12? It's just a number.
J30: No
H31: Now if you want a pattern to get any sum of any N, that's different
J31: No I don't think so, I don't think there's
H32: There's a pattern?
J32: There's a pattern
H33: No I think there probably is a pattern

They were then asked to consider particular sums, and whether there was anything special about the sums.

H37: No ------ 2 --- I mean, I guess you can make a pattern for this, but no there's nothing special about it
R33: What would be the next one?
H38: uh
J36: 20 --- you are adding 1 ---
H39: 20
J37: OK the difference between this and this is, no, that's nothing
H40: 2 3 5 --- 2 2 [something too quiet to hear] --- No I don't think there's a pattern, I mean, let me see this, 2 4 7 12 8 12 is 2 12 4 ---

R34: Is this a true statement?

[For all n: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$]

J38: uhhuh --- F plus 1 minus 1 --- Yeah it is --- umhmm ---

R35: Do you think that that's a true statement?

H41: Uh,

J39: No, it is

H42: 2 4 7 12 8 20 --- is

J40: mmm 1

H43: N Is that the summation of the series or is that like each term

R36: Well F 1 is the first one and F 2 is the second one, and F 3 is the third one

H44: Oh OK

R37: and F 1 plus F 2 plus F 3 is the summation

J41: yeah that's true --- that's it

H45: What is, what did you say it is 12? yeah it is, let me see it, yeah it is, yeah it's true

R38: Ok you'll buy it?

H46: yeah

They were then shown the statement:

The Fibonacci sequence is given by:

$$F_n = \left[\left[\left[\left[\frac{1}{60}n - \frac{7}{24}\right]n + 2\right]n - \frac{149}{24}\right]n + \frac{569}{60}\right]n - 4$$

H brought out a calculator. J began checking for n=1. H began with n=8. After considerable calculating:

H52: I might have made a mistake. But, my calculator says no, but did I make a mistake, that's the question ---

R44: Which N did you try it for, on the calculator?

H53: 8, I got 23, but I may, I may easily have plugged this wrong ---[tried again] --- No, I must have done something wrong, I get something completely different this time, again --- [tried again] ---

J50: --- [still working on n=1] --- 6

H54: I keep on getting different answers, anyway

R45: Maybe 8 is cursed or something

H55: I'll try something else, it might be --- 12, 12 divided by 60 ---

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J51: It doesn't seem to be working for ---

H56: No, I don't think it's working, unless I'm doing something wrong in particular

J52: It does work for, I was trying to check it for 1

H57: No, Ah, is it?

J53: It doesn't say that it doesn't work for 1, so it should

be working for 1 also if it's valid
R46: Umhmm, Yeah it should work for any N
J54: For any N, there's no restriction, so when you try to add , 1 60 pluc 1, minus 24, see that's 2, ---
R47: So you tried it for 12 and 8
H58: Yeah it didn't work, but perhaps I might have made a mistake, I guess not, Is it, Is it true?
J55: 569 60
R48: well you're supposed to figure that out
J56: minus 4 OK
H59: Yeah, but I could easily be making a mistake, Oh, you put it in for 1, did it work
J57: No it didn't, but maybe, maybe I made a mistake
H60: 60 minus --- plus 2 --- ah shit, for 1 it does [?] work
J58: It does work?
H61: well I mean, we know, It worked for this one but maybe not fr this one [pointing at a later number on list of Fibonacci numbers] we have to take one, arbitrary one
J59: Maybe you made a mistake somewhere? So it does work for 1
H62: Ok let me try this for 8, try it for 8 again, 8 --- [calculating] --- time 8 plus 2 ---
R49: You can just hit equals to close each set of brackets if you wanted to
H63: What?
R50: do it that way. The reason it's set up this way, actually, is so if you're doing it on a calculator you can do this calculation, hit equals to close this bracket, and then
H64: Yeah, I, that's what I did the first time, I thought it'd be easier
R51: Just in case, the brackets are messing you up somehow
H65: --- plus 2 ---
J60: equal, ya
H66: why do I keep on getting, I'm getting 50 for a while, that's what puzzles me, obviously it's not 50 --- I'll do it one last time and then I'm saying no. But I'll make sure I'm doing it right ---[calculating] --- no, it doesn't work, unless I'm pressing it wrong it doesn't work, no. Ok I get F 8 I get 50, that's the answer F 8 is 21

They were at this point given a table of prime numbers less than 10000, and the statement: " For all n: P = n(n + 1) + 41 is prime ". They were asked if they thought the statement was true.

They copied it down, H accidentally writing 2n for n^2. They were asked what they were thinking of.

H71: Ah I, I'm thinking of, all pair numbers, if you square them with give you a pair, something that's pair, is dividable, and adding up to a pair is going to give you impair, it's going to be odd, Now I'm looking, I'm thinking, I'm just thinking in terms of

J65: No

H72: when you, I'm determining when you, odds number what happens when you add them up, and so

J66: must, if there was a way to get a multiple of 41 in here then you'd, then it wouldn't be a prime number anymore then sum of these two

H73: What? What are you saying?

J67: If you were to get, a multiple of 41 here, in the start,

They then began to work separately on J's idea of checking n(n + 1) to see if it could be a multiple of 41. After some time H announced:

H77: No it's, not true

R57: Why not

J71: why don't you think it's true?

H78: What I did is, but you're right, well, I mean, at first what I did, but then I used your idea, see I did N squared plus N plus 41, ok you're saying that this is a multiple of 41, you're right, so I did N squared plus N is equal to N 41 N

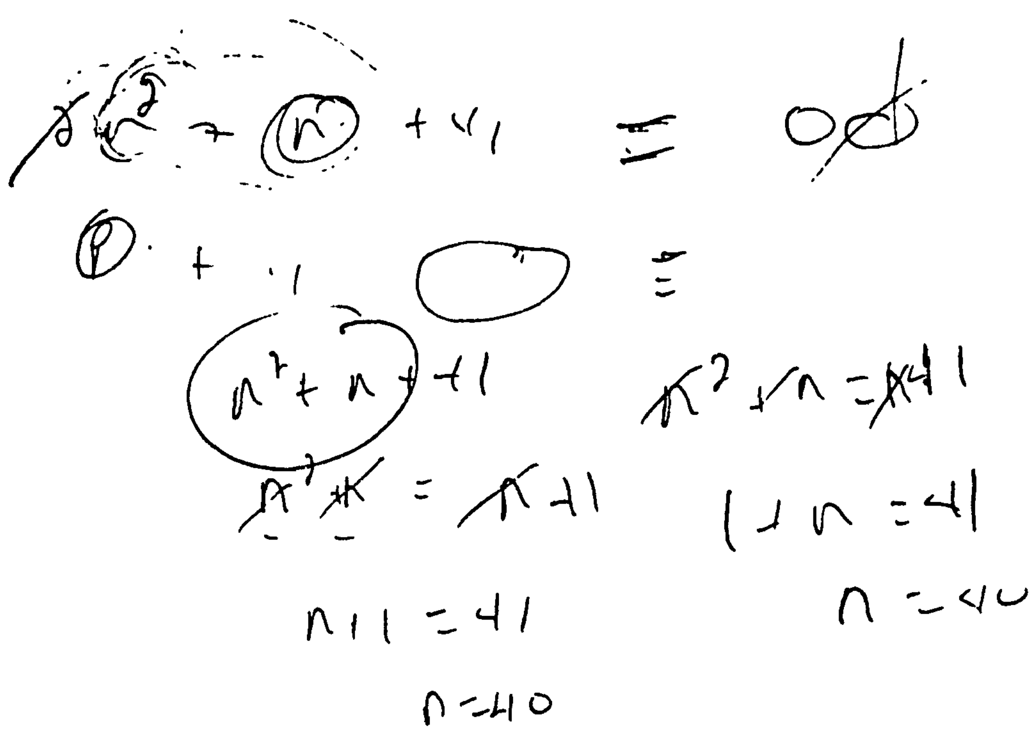

Figure 22: H's writing at line H78

J72: No

H79: dividing by N is, 1 plus n equal to 41
J73: 41 6 7 times 8, 6

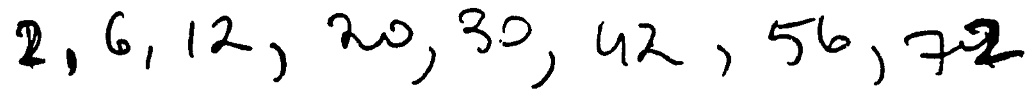

Figure 23: E's checking at line J73

H80: let's try N is equal to 40, plug it in and it's equal to 41
R58: What are you doing?
J74: No I'm just trying to know if this expression, can ever give me like a multiple of 41

H resumed his explanation of what he had been doing. His work began with the equation: $n^2 + n = 41n$

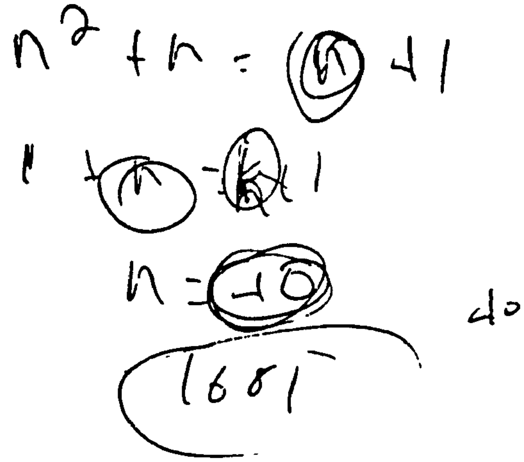

Figure 24: H's writing at line H85

H85: Ok so divided by N I get 1 plus N is equal to 41, so I tried the number N is equal to 40, just to see it, yeah? Or I mean, it could be any N for that matter,
J79: Why?
H86: It could be any K here, actually, there's still K here because I'm just taking an arbitrary N right, could be any N, show I'm just working so it came out to 1, so try, plug in 40, gives me, gives me, 40 squared plus 40 plus 41 gives that ["1681"] and dividing by 41 gives 41, as it should
J80: I still don't understand, what you are saying is N squared

H repeated his explanation. This time he represented a multiple of 41 as "41k" rather than "41n", but then decided "k" could be "n" after all.

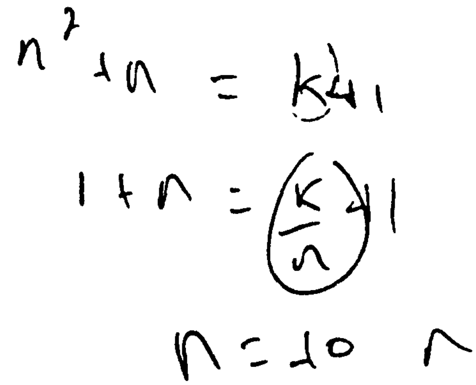

Figure 25: H's writing at line J84

J84: Ok so you want to find
H91: In fact it's the square, this squared
J85: So you want to find N for which this could happen, Ya?
--- and your N is 40?

H agreed and then repeated his explanation again.

J91: umhmm, so your N is 40?
H98: see if I took
J92: So your N is 40 and you're getting, ok when you
multiply 40 by 41

H then showed that the same procedure would give 81 as a counter example. He also mentioned that there would be others, and J agreed.

R61: Are there any values less than 40 for which,
H103: It's going to be
J97: No No
H104: Of N less than 40 you mean
R62: Umhmm
J98: of N less than 40 Ya, I don't think so
R63: Why not? you don't think so. You don't know
J99: Because here you, the number up here that you get is,
[converginous???] 41 times 41
R64: What are you
J100: No No no
R65: when N is equal to 40 Yeah,
J101: When N is equal to 40 yeah
H105: My guess, Id' say, Would be No either, but that's just

a total guess, less than 40
J102: No there could be, there could be other numbers less than 40

They were then asked how they would find out if there were counter examples less than 40. H tried to determine something based on the same procedure which gave him 41. After some time R told them that they are all prime. H stated that he was trying to modify his method for multiples of numbers other than 41.

R72: OK, general math tip, if you only have 40 cases to prove you may as well prove every case
H112: No that's too long, I wouldn't
R73: I mean, that would be
H113: Isn't there a method of doing it?
R74: It's the only method I know
H114: Oh, OK. I'm sure there's another method, I knew it wasn't a multiple of 41, any of these things, unless, but I did not know if it was a multiple of anything else

R than gave a bit of history, of the empirical discovery of the statement, its tentative acceptance as a formula for primes, and its eventual disproof. He then showed them the statement: " For all n: if n is prime then F_n is prime " H began with the rule for generating Fibonacci numbers and asked for confirmation that he had understood the problem.

R80: yeah, Is that true, Is that a true statement?
J105: No
H120: Yes it is, I think so
J106: [OK?]
H121: yes it is, Am, Am I right? I'm almost sure this is right, it should, let me think about this, if, say this is not a [multiple?] number --- 2 ---
J107: 1 2 11
H122: 19 --- what are you doing? are you doing
J108: I'm just trying
H123: Actually it's a good idea to check them
J109: I'm trying to check them ---

R reminded them that they had a list of all the prime numbers less than 10000. H asserted that it was true, but was unsure. He gave an argument based on the pattern of even and odd numbers in the Fibonacci sequence. R asked if he had confused prime with "pair". He said he had not, but that he knew that prime numbers would be odd. He than saw an unspecified flaw in his argument. Meanwhile J had been checking using the table of primes.

J112: Yeah, by observation, this seems to be right
H130: Uh
J113: Correct, it seems to be true --- um but ---
H131: --- 2 2 2
J114: oh no --- so
H132: No No it's not true
J115: It's not true
H133: What? Is it true?
J116: No it's not, because I found a counter-example
H134: Ah, you saw an example that it's not true?
J117: Ya
H135: Now wait
J118: Because 17 is prime isn't it?

J had not seen 1597 in the table of primes. R pointed it out. She then checked F_{19}, 4181. H continued looking for a general proof or disproof.

H139: F of N minus 1 --- What was the, that thing, that you showed us before there, the, let's say that the summation of each one is equal to F of N plus 2 minus 1
R88: Umhmm
H140: and that's summation of F of N?
R89: Yeah ---
J122: I found another counter example, 4181
H141: Is it, is it prime?
J123: No it's not and 19 is prime
H142: What's the number? F 19
J124: 4181
H143: 4181
J125: 81 yeah
H144: Ok well it isn't then, it's not true, she found one, I guess, wait, it does seem, it does seem to skip it --- Ok it's not true because we found a case which is not true
R90: Ok, so we've looked at
H145: Is it true though? or is it not true? Ah I, no
R91: What do you think?
H146: It's not true, but maybe. It's not true. but, did we make a mistake?
J126: then you shouldn't ask, you shouldn't ask
H147: But did we make a mistake looking at it, Is it true of is it not true?

H's confusion was based on uncertainty about whether J had missed 4181 in the table. On checking for himself he became sure it is not true. R then reviewed the four statements they had looked at, noting that three had been found to be wrong, and the first one had been accepted without proof. They were asked if observation had been their only evidence, and said that it had been. R pointed out that observation had misled them in the case of the prime-Fibonacci statement they had just seen. They returned to the first statement to try to prove it.

H163: Oh it's the summation of the thing, oh, it'd be too long to do all that I don't want to do it that way
R107: Ok, well how many do you want to do before you're absolutely sure?
J137: It can go forever --- so there must be a way to prove it
H164: Oh yeah, let's just think about this logically, summation F of N is you're adding, you're adding A 1
J138: N minus, N plus 2 minus 1
H165: plus A 1 plus A 2 plus --- A 2 plus A 3 plus --- plus A 4 is equal to A 5, A 3, A 2, A 3, that's A 3, A 4, no this, shit, A 3
J139: We can try to prove it by induction

J then wrote the proof:

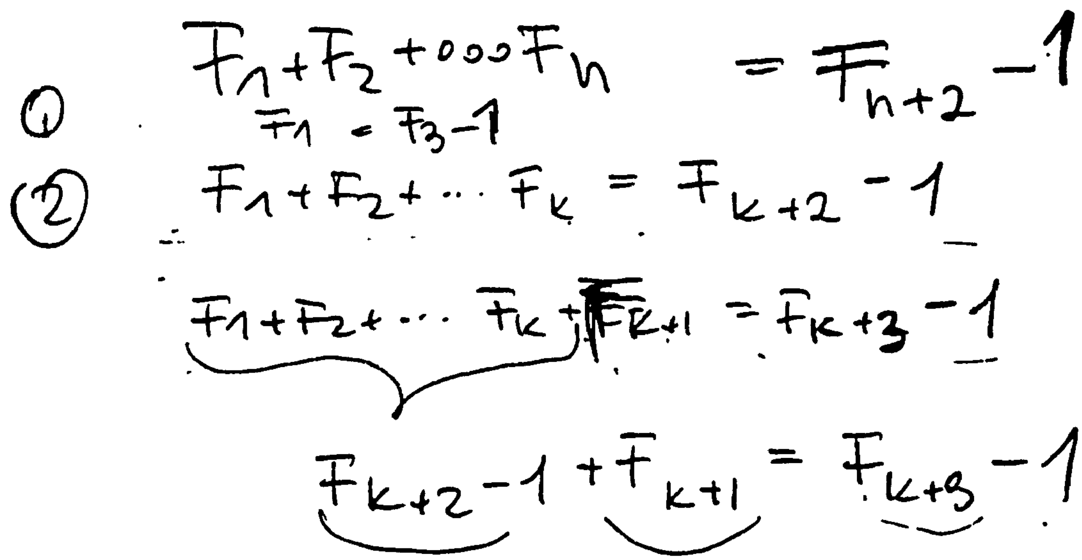

Figure 26: J's proof by MI

Meanwhile H continued in the way he had begun, trying to break down the sum into previous terms of the sequence.

R108: What are you trying to prove here?

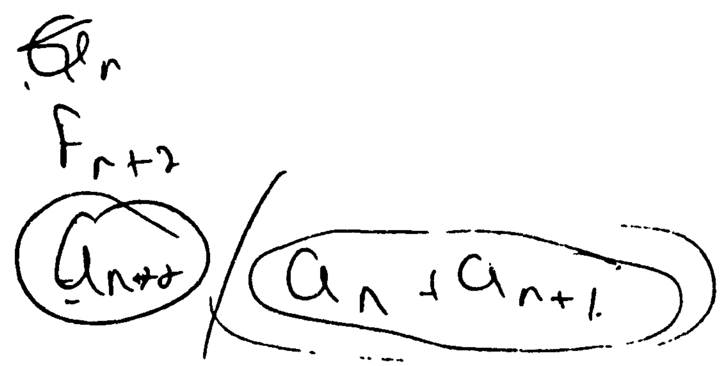

Figure 27: H's writing at line H172

H172: You see I'm taking this, see I'm trying to find the pattern, you see, I know this is always true, and this is like the summation till, till F of N, till the value of N right, Now I know what F of N plus 2 is equal to this ok and I know that it is also possible to break this down to small expressions, same with this one, and them up and see if this is equal to this minus, is this equal to this plus 1, so like, basically we just
J146: Ok this is
H173: It might be true, but it's long
J147: I think I can prove it by induction
R109: OK, she thinks she's got a proof
H174: Oh she's got---
R110: Are you going to look at it?
H175: Fine with me, I know I could work it out here, I just don't feel like doing it, well, I mean, I could, but, see if this works it would be very nice
J148: So the, the first, the base step would be proving this for 1
H176: yeah
J149: it obviously works because when F is, when N is 1
H177: It's good
J150: it's only 1, 2 minus 1 is 1, then you, you write the inductive, you have the induction step, write this in this form and assume that this is right, this one is right for K it must be right for K plus 1, so [see figure 26]
H178: OK, let's see is it right for K?
J151: Ya, if it's, if it's right, right for K, it must be right for K plus 1, so you just write K plus 1, so here you will have K plus 3 --- minus 1, then you replace this, by the expression direct inside, and what you have left here is F K plus 1, it is obvious that when you take this one and this one [see figure 26]

H179: it's F of K 3
J152: K 3 and you have this minus 1 here what is exactly
H180: yeah it works, it's good
R111: Why does it work?
H181: well, I mean, because it works
[laughs]
J153: Because it's proven
H182: Because that's the proof, because it makes sense, it's obvious it's true
R112: well
J154: well that's the formal proof
R113: Ok, isn't this proof based on an assumption that it works for K?
H183: Well didn't she prove that it works for K? I
R114: yeah but she assumed that it worked for K
H184: No No , shouldn't you assume for like F of
J155: mm well
H185: I don't know, to me, I, I mean if it works for one F of K then it works for all F of Ks period, and we know it works for some so if it works for one, I mean, it works for all, that's right
J156: yeah we know that it works for
R115: well the prime one worked for a bunch of them and then it stopped
H186: no, yeah but we didn't prove it, this is true, this is obviously true, I mean, we did, like, 8 and if it works for 8 it works for all, I mean, this is true, the proof is fine, I mean, if it works for F of 8, ok, then it works for F of 9, and also works for F of 10, it works for F 11, and so on and so forth, it works for everything, right

Each was then asked whether they believed the other person understood the proof. They felt they both did.

R120: OK, what do you have to do, in math, to figure out whether or not a statement is true?
H191: [once you get er five???] well, if it's true?
R121: Yeah, I mean I've been giving you statements and saying is this true and,
H192: You got to prove it for all cases that its true, actually every case, prove that there is no single case that is not true
J159: first you have to check if it's all, if it is true, like, if you cannot find any counter example that's the way I think --- It's very
R122: How far do you go searching fo counter examples?
J160: Through every natural number
H193: eventually, you can make an , you can take a pretty good guess but, mostly what you have to do is try to find some pattern to find, for all the examples
J161: and the you can

H194: [??]
J162: and then you must think why, why it works, why, how it works and why, how you can prove it

The distinction between proving "why" something is true and "how" something is true, was then discussed. H said that it depended on the way of proving. J stated that an existence proof which involves the actually construction of the thing which is proven to exist, showed both how and why. H said that if you understand a proof then you understand both why and how, but if you don't understand then you don't know why.

R127: If I tell you that the sum of the first 87 Fibonacci numbers is not equal to one less than the 89th Fibonacci number
H202: yeah, what would I say?
R128: What would you say?
H203: You're wrong
[laughs]
H204: and if, and if I knew, and if I found out that it was true then I'd go "Well, shit", what can I say? I mean, obviously from that it's true, end of story, I mean, it's true, It should work for everything, I mean, if it doesn't then there's something extremely wrong with our minds, because we don't understand what we're doing, but no, it should work, right?
R129: Would that be your reaction? If you found that for the 87th
H205: If somebody told you
R130: then with 87 this doesn't work anymore?
J169: ya, I would, I would ask for an explanation, a detailed explanation
R131: An explanation of what?
J170: Why?! Why does it work?
R132: Where would you think the problem was? Where would you guess the problem would be?
H206: The person who's telling me that calculated incorrectly
J171: There was something wrong in the computer
H207: what?
J172: There was something wrong in the computer when you were calculating this, this 87th Fibonacci number Because there isn't any formula for formula for like Nth Fibonacci number, you would have to calculate all of them
H208: What?
J173: When you want to get a certain number F 87 you have to
H209: You have to work them through of course, it'll take a long time
J174: Ya so the, I was just wondering
H210: you can't get just one, just

J175: If there was, if there was any formula for this, like
H211: Ah, just to get the one out? You mean, just take
J176: Just to, ya
H212: Just to get that without having to calculate the one before
J177: You have, you have to calculate everything from the beginning, then adding them extending from here, to get the next one you have to add this and so to get from this point to 87 would take you some long time
R133: Well it only took me about two minutes to do this, but I just wrote a computer program and it did it.
J178: Ya
R134: If it did, If you were absolutely sure that yeah, it didn't work for the 87th Fibonacci number, would you then conclude that there had to be something wrong in the way you had done your proof?
H213: Of course
J179: mmm
H214: But I don't see anything wrong with it. Do you see anything wrong with it?
J180: No, there cannot be anything wrong with this proof
H215: yeah but, yeah, but if that was true then there is something wrong with this for sure, there'd be no question about that, If you prove something that, you know, is not true then there's something wrong with it. I mean, if it's not,
R135: is there something wrong with that proof?
J181: No, no
H216: No there's nothing wrong with that proof, it's fine. I think so, let me see. No, that's fine
J182: No, there cannot be anything, anything wrong with this proof, like,

They were then asked for their criteria in determining whether a proof is correct. J mentioned a lack of weak points, a step by step process, and a correct application of certain rules. H only said that a proof is correct if its validity cannot be questioned. J was asked if her proof went from step to step. She mentioned the basis, and the "rules of induction".

R138: Ok, you're using the rules of induction, what are these rules of induction?
H221: I [wouldn't do?] that anyway, I never saw that before, It's a pretty good trick. Induction, what's that?
R139: You've never seen a proof like this before but you believe it's true
H222: No no no no but I mean, it's true, I mean, looking at it you can tell it's true. i mean you can see, yeah, it's true, It works for K plus 1, I mean, obviously it works for all of them then. I mean, you try any value

of K then it works. I mean the, the K, I mean, I mean, it is true

J187: No

H223: But, I mean I've never seen this before

R140: Ok, so you're saying

J188: [That it's actually?]

R141: that there's a rule, and you're saying, well it's obvious

H224: Well, I mean, this is true

R142: so you're both looking a different proof here

H225: Well, I mean, this is true, there's no question that it's true for K plus 1, now induction now, I get

J189: You use induction when you are trying to find, when you prove the truth of some statement for N using the fact that, that the statement, that the previous statement was true, the statement for, for, here you, I am trying to prove that K plus 1 is true using the fact that K is true

R143: OK, how does that help you to prove something?

H226: What, sorry

R144: How does what she just said help to prove, to prove something?

H227: well, she knows [laughs]

R145: Well you might be able to help

H228: what was the thing?

J190: How this fact, helps me to prove.

H229: I mean, oh ok, I mean, well if it works for 1, Ok? and it works for K plus 1, since we know all numbers exist, you know, that every number goes 1 plus 1 and you get all numbers, I mean, it's all numbers, all values of, all the numbers you can get, you can get all numbers, right? I mean, if you add 1, you know it works for any K,

J191: umhmm

H230: works for some K, sorry, some K, not any K, some K, and it works for some K plus 1, and if it works for that, if it works for a value then it also works for the value K plus 1, and it works for all values K plus 1, K plus 1 you can consider this, you want some value N and it works for the one plus that, so then that one has to work, and the one plus that has to work too, and so on, and so on, right?

H231: And that is how she proved it. that is how she really proved it.

The session ended at this point. In some discussion afterward H seemed uncertain as to the truth of the statement they had proven, and wanted assurance from R. R refused to answer and asked what H thought. H then asserted that the statement is true.

Stage Two

The first statement was given to G and H:

For all $n \geq 3$,

$$(F_n)^2 = (F_{n-1})(F_{n+1}) + 1$$

(F_n is the n^{th} Fibonacci number)

H had a bit of trouble remembering the recursive rule for the Fibonacci sequence. G knew the first six terms.

H402: I'm just looking and saying, ok, this is saying F of N is, square root of that, and I'm just looking if you multiply 2. I don't know what to do, [with this?] sequence --- 3 --- Because if you put a 3 this would be 2 times 3, would you say?

H403: No that's ok

G403: well, we can work this one out and, maybe use, for any value N, we have a value N minus 1 times N plus 1, we can multiply those two values, and, kind of, find an equation out of that. you want me to start writing it now, N squared, assuming N is the value we're looking at, N minus 1, would be again N minus 1, N plus 1, multiply that through, you get N squared minus 1 plus 1 yields N squared which is plain and simple, so this is a true statement

$$(n-1)$$
$$(n+1)$$
$$n^2 - 1 + 1$$
$$= n^2 = n^2$$

True

Figure 28: G's writing at line G403

H was asked if G's argument made sense to him and G repeated it for him. H pointed out G's confusion of the indexes with the actual numbers.

G407: Yeah this will be true for the first 3 but it won't be true for these ones here for these numbers here

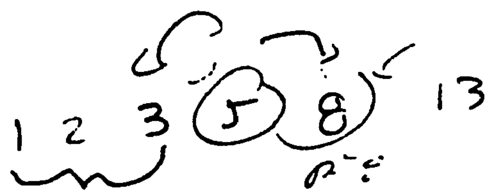

Figure 29: G's writing at line G407

H407: well why won't it it only starts at
G408: Because, yeah
H408: 3 anyway, but why won't it be true for these ones?
G409: Because it's not exactly N minus, this value here is not exactly, assuming we're at 8, N is equal to 8
H409: No No No, put in 3, N is equal to 3, I mean, it starts with N is equal to 3, put 3, is 2 times 4 plus 1, 9, square root of 9 is 3, so it's right
G410: yeah
H410: Put in, put in 4, you
G411: yeah but
H411: No No No put in 3 here --- this means like N is equal to 3, 4, 5 whatever

R reviewed the meaning of the indexes giving a specific example of F_3 H then tried to work through the statement for n=4. He asked what F_{n+1} indicated and G told him it was "the following number".

H418: F of N minus 1, is equal to F of N minus 2 plus F of N minus 3, right?

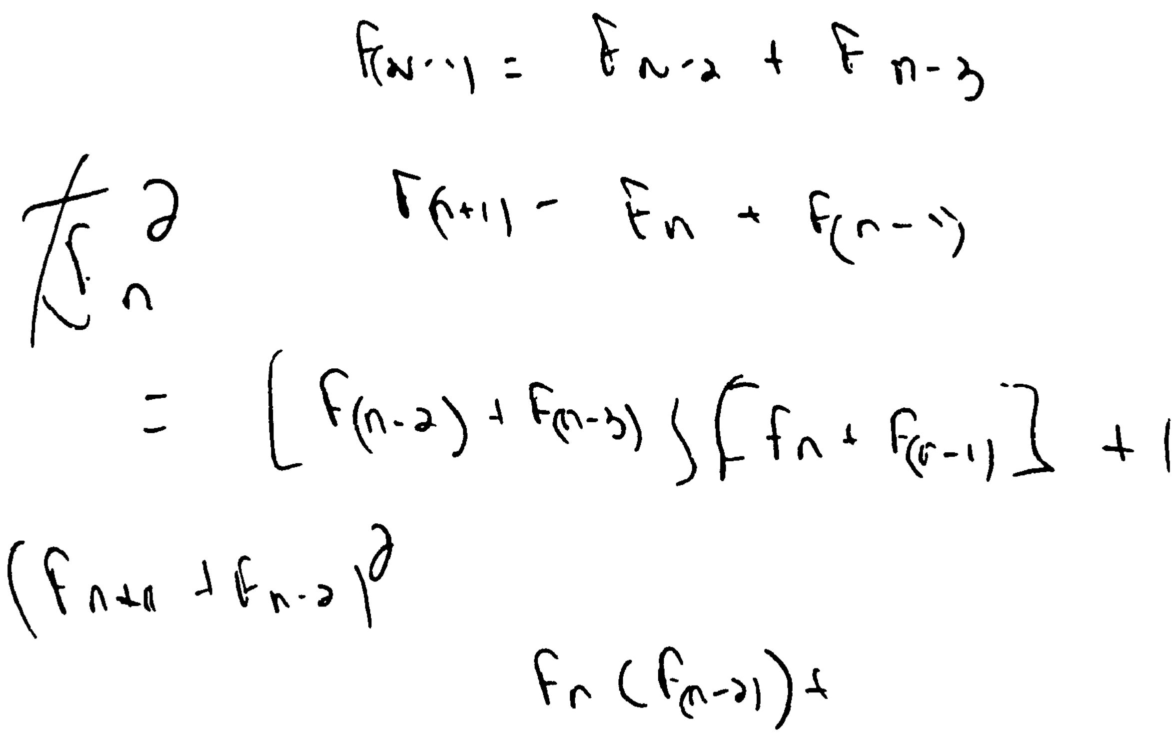

Figure 30: H's writing at line H418

G416: N plus 1
H419: F of N plus 1 is equal to F of N plus F of N minus 1, is that true? yeah, now multiplying these two out together, N minus 2, --- N minus 1, N plus 1 is equal to F of N squared that's what they're saying, [it's squared in here?] F of N squared is equal to F of N plus 1, no minus 1, plus F of N minus 2 squared. Which is, Now, F of N times F of N --- this is getting a bit boring, minus 2, and then plus, do you want to continue working this? I don't know if it's going to get anywhere, I think it should be able to ---
S403: What's your idea?
H420: I just want to see if I can get some expression here, to equate this thing and I'll, I'll take some arbitrary ones and see if it comes out something here, --- [generally?], and if it works I can equate, generalize it.
G417: Or would it be wiser to look for a sequence, up until we can find something that contradicts it, but well, that might take forever
H421: well I don't know, see the problem is you can, if you do that if it's not true fine, if it's true
G418: yeah
H422: there's a problem. You're, you're through, it'll take a little while , I I mean

S404: I think you should follow your idea, and you should follow yours for a while
H423: Let me, let me just, you know, waste time
S405: Umhmm, ok
H424: Probably,
S406: Your idea was to try the first few
G419: yeah the first 3 here, yeah, assuming N is equal to 2
S407: so that's true?
G420: yeah it does work for this

S asked what G was referring to, and G answered n=2. H pointed out that the statement was only meant to work for n>2. They then tried it for n=3 and found that it didn't work.

$$3^2 = 2(5) + 1.$$

$$9 \neq 11$$

Figure 31: G's writing at line H430

H430: 11 square root 11
G427: yeah,
H431: is not equal to 3, that's if you read it like that, I read it like, I don't know, so it's Ok, it's not true, you guys,

G commented that it shouldn't have taken them so long to find it didn't work. They were then shown the second statement:

The number of diagonals in a convex polygon of n sides is

$$\frac{n(n - 3)}{2}$$

G430: the number of diagonals, [reading] of a convex polygon of n sides --- ---
R412: Do you have any questions about the statement
G431: Is diagonals referring to slant,
H434: Going from one point to another, diagonal
G432: yeah
R413: Within a figure
H435: Yeah, within the figure --- ---
G433: [?]
S411: a hexagon isn't it?

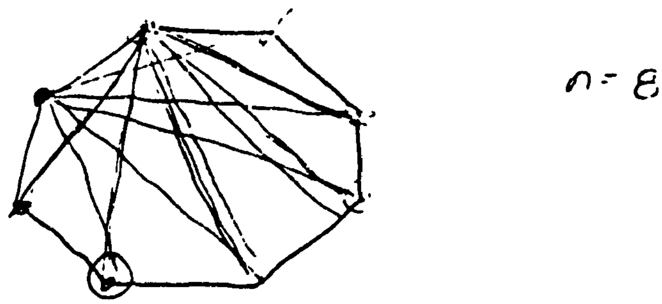

Figure 32: G's sketching at line G433

R414: No an octagon
S412: Octagon, the greek names, they're so confusing --- how many do you have?
G434: But is it to each individual, vertex?
S413: Yeah, from each to each
G435: From each to each
S414: Umhmm
G436: So, not including this and this
S415: No because it's a side
G437: So we have N times N, we have, assume 1 2 3 4 5 out of the 8, so it's N minus 3, and then we have N vertices --- but that's, how many diagonals --- a vertex, vertex, right [wondering about pronunciation]
R415: Umhmm
G438: yeah, the number of diagonals in a convex polygon --- so why is it divided by 2?
S416: If you start from this --- is it the same a starting form there?
G439: Yeah, we're back to the same diagonal
S417: yes
G440: OK Minus --- minus, we come here, but then we've already got one, so this goes down to, no that's not necessarily true, the first one, it'll be N, times N minus 3 plus N minus 1, N minus 3

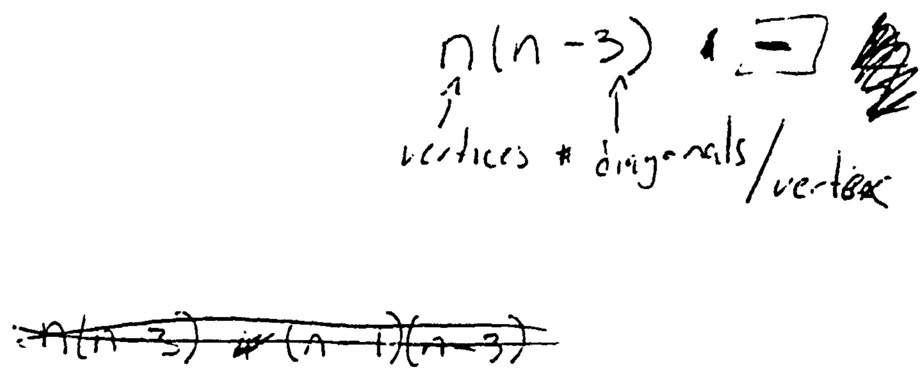

Figure 33: G's sketching at line G440

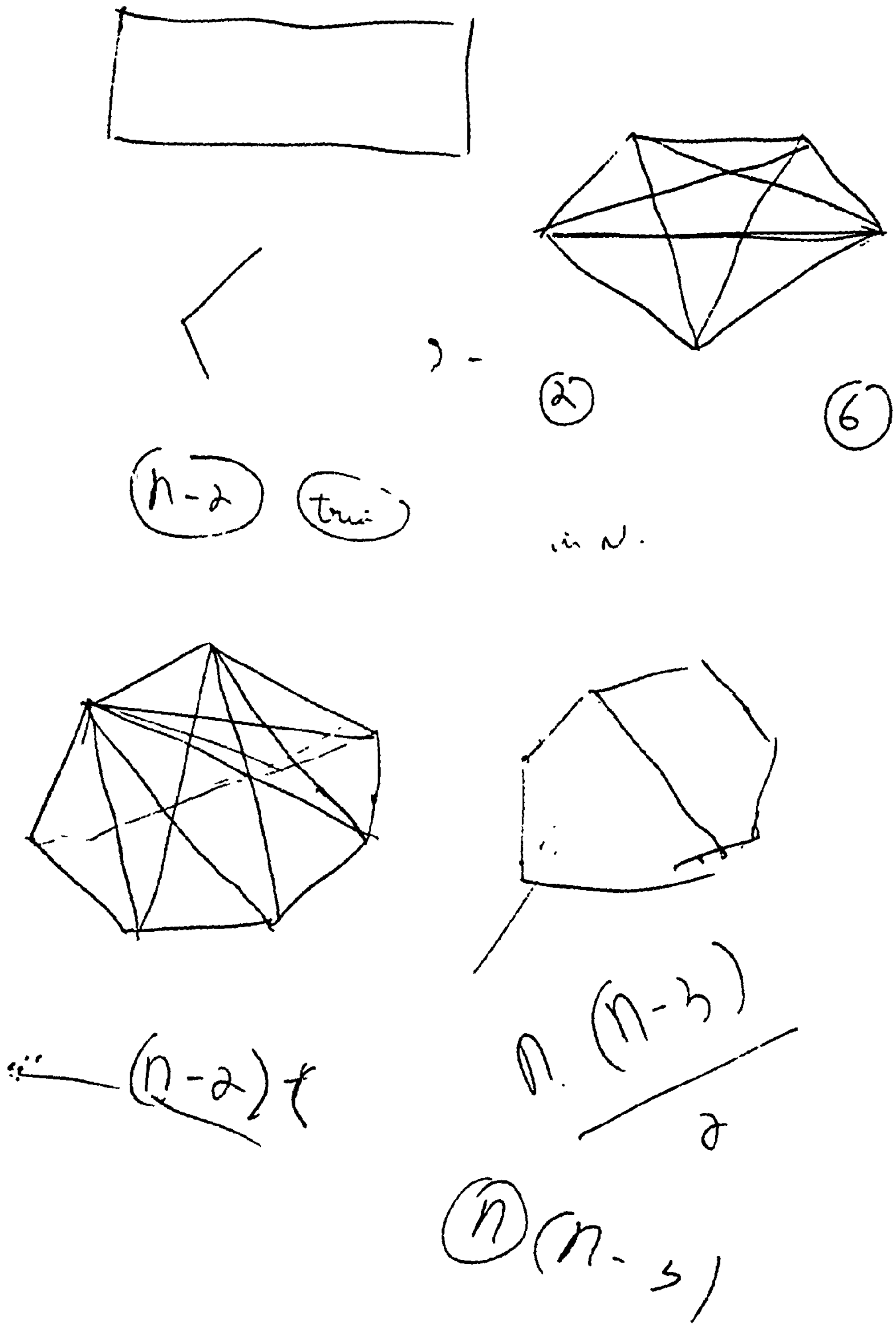

Figure 34: H's sketching prior to line R416

S418: Why are you adding?
G441: Hold on, for the second one, --- equals 3, then assuming we've taken this one here, we've got 1 2 3 4 5 again, ---
R416: Maybe the two of you should compare notes to see if,
G442: yeah this is, it's a true statement but, it's just proving that once we get to half of them then
R417: Why do you think it's true?
G443: all of then they're going to start repeating, at that point
H436: yeah, well that's the problem here,
G444: yeah
H437: but why, why do you say it's a true statement? --- ---

G445: OK, what I have here is what I, what I assume to be the total number of vertices not including, this includes, having the same vertex crossing over twice,
H438: But you have
G446: but this is not necessarily
H439: For each N
G447: so for every N
H440: Ok it's true
G448: For every diagonal
H441: No No, let me see it, wait a minute. N minus 3, It's the minus 3 that's bothering me
G449: No the 3 is fine
H442: I would get this to be N times N minus 2 divided by 2
G450: No because let's say you're taking this one here, you're taking the first one here, 2 3 4 5 Ok you're not including these 3 vertices here, it's N minus 3
H443: you're not including which ones?
G451: This one this one and this one, these three vertices
H444: well you aren't doing this one, you're doing this one
G452: Yeah, that's 1 this is 2 this is 3, this is 4, and this is 5
H445: Oh Ok I mean, I get it, It's true then, yeah
G453: yeah fine, well, then, it doesn't explain the
H446: wait wait wait
G454: dividing that by 2
H447: N minus 3, N, each N, times N minus 3
G455: Oh Yeah, this point will intersect with the first divided by 2, this one 1 2 minus this,
H448: This is just half of one
G453: yeah fine, well, then, it doesn't explain the
H446: wait wait wait
G454: dividing that by 2
H447: N minus 3, N, each N, times N minus 3
G455: Oh Yeah, this point will intersect with the first divided by 2, this one 1 2 minus this,
H448: This is just half of one
G456: maybe I can, --- can I use different colours here? do we have a time limit on each one?

R418: Yes
S419: No
G457: We do, we don't, or
R419: no

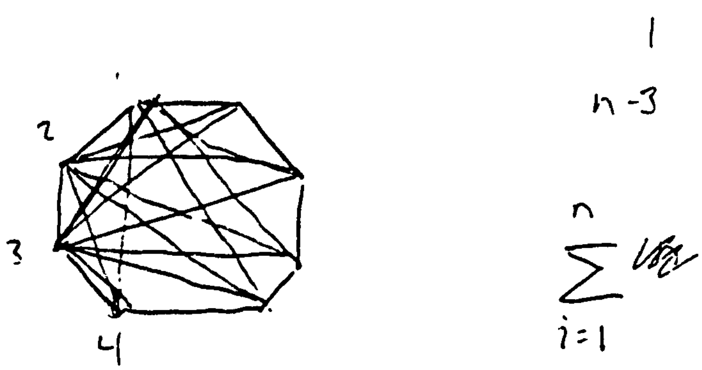

Figure 35: G's sketching at line G458

G458: --- assume this is true for the first, how am I going to visualize this, I 'll just go this way --- stupid diagram, assume for this has it's --- oh god, --- N minus 5 as well --- let me just use this --- 8, divided by 2 you said
H449: It's right anyway
G459: It's right
[laughs]
G460: It's true, let's just take it. Yeah, but why?
H450: why?
G461: If you know for every one you hit over here
H451: see
S420: Why did you divide by 2?
H452: Why? because, well, I mean, Because, you get, because there's a repetition 2 times , repetition of each one, you'll repeat each one, or, say you have a polygon, you start at any single, well, I mean, at any convex thing. I thought it was minus 2, but I realized it was minus 3 here say, let's say you have here, you take it here, for each one you'll have N minus 3 like
S421: are you sure ---
H453: diagonals, for each one, now you can do this, this is true,
G462: N times
H454: wait let's say N times for each one, but only half of them because you'll repeat one, two times
G463: you're also repeating at the point
H455: and you're
G464: But how do you show that, do you want that shown? or
H456: But, you will, you will repeat, that's obvious, I mean

S422: Ya
H457: I think
G465: It is ob, well yeah you can see
R419a: Descartes liked to say "it's obvious" too, but nobody believed him.
S423: well I think it's by construction, that's how you construct the
H458: wait wait but you will repeat half exactly I mean, It's
R420: Why exactly half?
H459: well because you'll take each, you'll take one point, see you take one point here, you go here, then eventually you're going to have to come back to this point, but this point you've, by the time you've taken every other point you've taken this point, everything here, you'll do that for half of the points. wait, I don't know
R421: Do you buy that G?
G466: I'm still working on mine here
H460: It's true
G467: yeah I can see it's true but,
H461: I mean, --- ok construction what, do you want us to make this elaborate thing? and start with like a three sided one and go upward?
S424: umhmm
R422: yeah?
G468: For 1 and 2, we have n minus 3, n minus 3, for, for 3 we have this one which is the same so its, these are all original, 1 is the same, 2, 3
S425: This is an octagon again [see figure 35]
G469: 5
S426: Is this
R423: yes
S427: Suppose when you start at the fourth vertex
G470: the what?
R424: if you try with just a, oh ok
G471: If we start with a fourth
S428: ya
G472: well we get repetitiveness on
S429: this one
G473: On one and here you'll get repetitiveness, on one and here you'll get repetitiveness on 1 all the way until that, so in essence you're you're you can subtract
S430: why subtract? --- ---
G474: the way I see it is you can write this formula another way, --- we have N --- if you have N , can you do this? Minus N divided by 2, for each individual point, and this one would be to that one, [too low to hear, working with figure 35] Isn't this sigma, for, in this case, for I is equal to, to 1 to N
H462: Anyway, it is true, there's no doubt it's true, why are we still on it?

G475: Because for each one you go through one has already been cover for that particular one, after the fourth
H463: Yeah what I was looking at was for impair, impair numbers, how do you say impair? odd, odd number of things but it still should work for that too, I mean, It works for everything
G476: yeah but how can you show that? do you want us to show it, or is it enough to
H464: No, we can explain it, it's easily explained, you take N, you have N sides, N dots, ok, like N things, ok
G477: That's right
H465: and N N N what are these points called again?
R425: Vertices
H466: Vertices, that's it, N vertices ok, well there will be, starting from each vertices there will be N minus 3 diagonals, right? now, so, N minus 3, now you have N of them, so times N, But, if you look at it an N sided, polygon, convex polygon, what you will, you will see recurring is that, you would have, by, I mean, It's, I mean, half of them will be repeated. I mean it's true
G478: yeah
H467: What do you want me to say?
G479: I can see it, yeah, I
R426: Does that explain it G, are you satisfied?
G480: yeah I'm satis, yeah the logic is the same, but I'm trying to, to show it
S431: You wanted to subtract here
G481: No this is wrong, this is not true, but what I
S432: He was trying to subtract some those that repeat themselves
G482: Yeah yes, It started, here 1 repeats then 2 start repeating, then 3 start repeating and, yeah, until half of them start repeating
S433: OK, then why didn't you just divide by 2
G483: So you could just divide by 2
R427: Ok
G484: Is that good enough?
R428: good enough
H468: you'll buy it now ---oh, no not these things again [on seeing a table of the beginning of the B sequence]

H and G were instructed to attempt to find the pattern in the B sequence. H noted that it is similar to the Fibonacci sequence. After some time, G stated that each term was the sum of all the previous terms. G noticed that this pattern does not apply to the third term and H rejected his suggestion.

H474: no I say that at first, I was wondering, I thought so but no ------ I'm seeing something very interesting that does recur, is you notice that, see here it's 10 for this one it's times 2 minus 1 but not for this one,

for this one here though it's times 2 minus 1, see this one here, times 2 minus 1 this one here, it's not, it happens for every other, it skips one every time it does that, so there's obviously something between there that says something, it's pretty [unreal?] 171 times 2 is 342 minus 1 is 341, let's try some numbers [further on?] from here, --- [?] times 2 minus 1 , yeah it works for that one too but it doesn't work for all of them, OK well, I mean, I see it, the other one is plus 2,
S435: Maybe it's minus or something
H475: What?
S436: Plus or minus
H476: yeah, plus or minus, yeah, actually, that's it plus or minus, put, to the end, what we've got here, between, [?] this is times 2 minus 1 times 2 plus 1
G489: 21 43, can I,
H477: times 2 plus 1, times 2 minus 1 times 2 plus 1, times 2 minus 1 times 2 plus 1, yeah that's it

H asked if this is the pattern, and S agreed that it is one of them.

G491: If it's odd, if it's odd, if it's odd, you add 1, you take all the previous ones except the
H479: You want to write it down?
S438: he's got another one, ok
G492: No for, for the odd what you're saying is that assuming 3 is odd, you take the previous 2 and add 1, so it would be,
H480: yeah, well, yeah, It's another way of doing it
G493: you add 1, and for B even you take all the subsequent ones, ah, not the, the previous ones, down to the first one
H481: Yeah, well in any case it's the
G494: the additive of all the previous ones down to the first one
R431: Now I don't understand, those sounded very different to me
H482: yeah but, I'm pretty sure there's a relationship between them, I'm sure you can figure out why

R asked G and H to write down their rules, in preparation to discussing the relationship between them. H checked some values to see if he had written down his rule correctly.

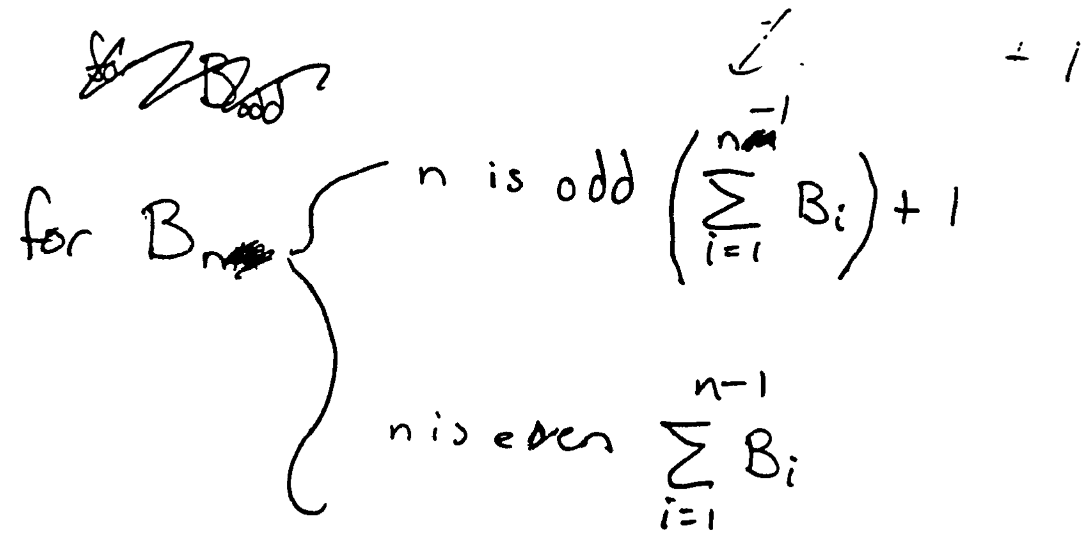

Figure 36: G's writing at line G500

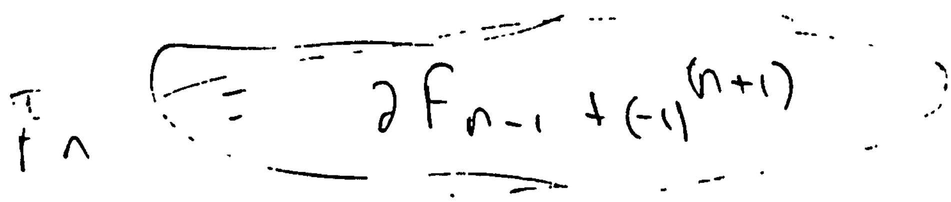

Figure 37: H's sketching at line G500

G500: I came up with this here, what did you come up with?
H487: But it's the, I'm, it's the same thing because the idea of it is behind this thing here is, it would
G501: yeah
H488: I mean it comes down to the same thing, I mean, it's just another way of doing it well, I mean, obviously, it has to be --- If it's, I'm just going to make sure if this comes out perfect, if so I'll just make sure, I might have you know, put this wrong, 3 --- for 3 it's going to be 3 times

S summarized their rules as "doubling and adding or subtracting 1" in the case of H, and "adding the previous" in the case of G. G noted that 1 needed to be added for the odd case. R then asked if the two rules meant the same thing. H immediately said they were, because they produce the same sequence. S then gave H and G the problem of showing that the rules did in fact produce the same sequence. They both began by rechecking their written rules for correctness.

G506: I just want to make sure of a few things First of all I made a mistake here, for N is odd, plus 1, assuming 5 we have 11, takes you to 8 9 10 plus 1 is 11, assuming for this sequence 43
H495: well anyway, this is right,
G507: 32 37
H496: now prove
G508: 40 41 42 plus 1 is 43, all right it works for that, I can take it for
H497: Ok that's good enough, did you want to work it through, did you check?
G509: yeah, let me just check maybe one more
G510: should we be working these out inductively or?
S446: oh yes please
R438: We thought that you would. but I guess we were wrong.
H500: To do these things, or do you mean
R439: Well all of these there are simple inductive things that,
H501: well to be honest I don't think I've ever really used induction, so I never used it, I've never, I I
S447: You never used induction
H502: I've never seen it, I've never, I mean, I've seen it I guess but I didn't really
G511: First year students haven't been introduced to induction, not in Vanier, not in Dawson College, they may have had some, you know, maybe some older
H503: Right, I've never seen it, so I know, Now when I look at it I know what it is
S448: but you had it at the
G512: yeah I had it in discrete mathematics
H504: I'm not, I'm not very, I'm just not really at ease with it so I try and do something else unless I
R440: well as a hint all of these things were chosen because they could be done fairly easily with mathematical induction, so if you're having trouble,
H505: well anyway, these two, this is right, this is right, now what you want us to do is make sure these two things are the same?
R441: Yeah
H506: why did you have to come up with this?
G513: let's see what's going on over here, 2
H507: N is odd, ok let's start with it, ok inductive, start with like a low one then you say n plus 1 is equal to that same thing, ok we'll do that now, say B equal, whoops, no we're not supposed to look at this right?

H was instructed that he could look at the listing of the first terms, but not to use it as part of his proof, as the table only gave a sample of the infinite terms of the B sequence. G then summarized his rule for H, and began to figure out how H's rule worked. G was uncertain how to begin, but noted that the same odd/even cases applied to both rules. H, meanwhile, was examining G's rule. He noted that the first term needed to be given explicitly to begin the sequence. There was some confusion as to the indexes in G's rule.

H522: what are you saying? It's odd so you sum 2 2 5. Now B 4 Ok Now, now let's work with induction, or whatever it's called, now, are you familiar with it?
G528: What's that?
H523: do you know what is?
G529: let's see, I am familiar with it, you could say,
H524: Ok well
G530: I never liked it, I never paid much attention to it
H525: Ok let's work it through, right, let's take a shot at this, --- F of N plus 1, I'll do here, F of N plus 1 is equal to, that's not going to get me anywhere, 2 F of

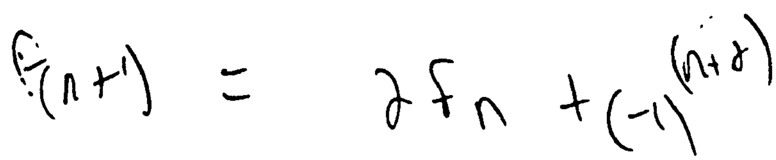

Figure 38: H's writing at line H525

G531: N
H526: N
G532: Plus
H527: plus minus 1, N plus 2, that didn't give me anything

S suggested splitting the odd and even cases and proving each separately. H suggested that n was always odd, but this was apparently a slip of the tongue as he corrected himself when S questioned his use of "always".

G535: We can take sigma here to N 5 plus 1 but, assume this is B N this has to be B N minus 1 minus 1 to simplify matters,
S462: Does it simplify matters?
H537: No not really, maybe I, see the idea of it being, I know what that is but I
G536: Adding 1 to this will take us to B N plus 1
S463: Are you am concerned that B, This being odd? maybe you're concerned with this being,
H538: No
G537: Well, this can be anything, this'll take us to all previous ones, adding, this is fine

H539: the way induction works you take 1 and then you choose that it works for any N plus 1
G538: yeah and
H540: any K plus 1, that's right
G539: yeah for any N plus 1 N is odd, this'll hold true

R suggested that G explain MI to H. Instead H explained it:

H542: I think I, is this you, you start, you say, if each for each you find one that works, and then you say if B K works then, you should prove that B K plus 1 works then it works for all B
G542: yeah
H543: all, right, is that it?
G543: That's the way I see it too
G544: You've assumed, you've assumed a hypothesis that if it works for a particular then it will work for that
H545: Now what we're going to do is take this thing, and I'll show, ok I know it works for at least 1, now I'll show if it works B K an odd one, I'll work with the odd one then I'll show that it works for the B K plus 2 which is still odd, which will come to this, [in? and?] both of the previous two are equal then it works for all odd numbers, then we'll try for the even, and see what it does

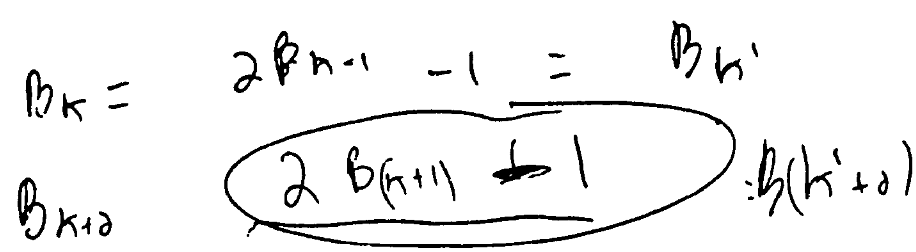

Figure 39: H's writing at line H545

G545: OK, let's go
[laughter]

H pointed out that they already had the basis of their proof as they had derived their rules based on the first cases. He also expressed his intention in the induction step of proving that if "it works" for B_k then it works for B_{k+2} K, in this case, he said was odd. He then wrote his rule for k+2. G watched and made suggestions. G noted that they could use k+1 instead of k+2 and show the even case based on the odd case. H said they would work out the even case later. R pointed out that they were only working with H's rule.

R455: you have to show that there's some kind of equivalency between them
H563: I want to see, do something with this
G561: You want us to find the equivalence between this and,
R456: That's all you're trying to prove is that these two things are equivalent
H564: I'm trying, see
R457: Now whether any of them relate to these numbers
G562: Yeah well I, that still won't prove, won't prove,
H565: No, if I can prove for all numbers
G563: that it works, I have to prove that mine works, he has to prove that his works,
R458: NoNoNoNoNoNoNoNoNoNoNoNoNoNoNoNo, you can't do that. you've only got 30, 24 of them there and there's an infinite
G564: Ok
R459: number, so there's no way you can be sure
H566: All I have to do now is prove that N, I don't, Ok this B K here is equal to your B K
S464: Yeah your B K just mark it by a prime or something
H567: Ok B K your B K prime here ok I know it's equal to your B K prime, now I want to prove that my B K 2 is equal to your B K prime plus 2, see that's what I got to prove, now ok, I assume, now your B K prime is equal to what? the summation, I is equal to 1, to N or N minus 1? [see figure 39]
G565: N minus 1, for B N
H568: Ok well, but B K is odd so plus 1, B odd, ok?

H produced G's formula for k odd, and G agreed that it was correct. He also produced G's formula for k+2. He stated that the objective was now to prove one from the other, and that he saw how it could be done. G commented that going from multiplying by 2 to a summation would be a difficulty in their proof.

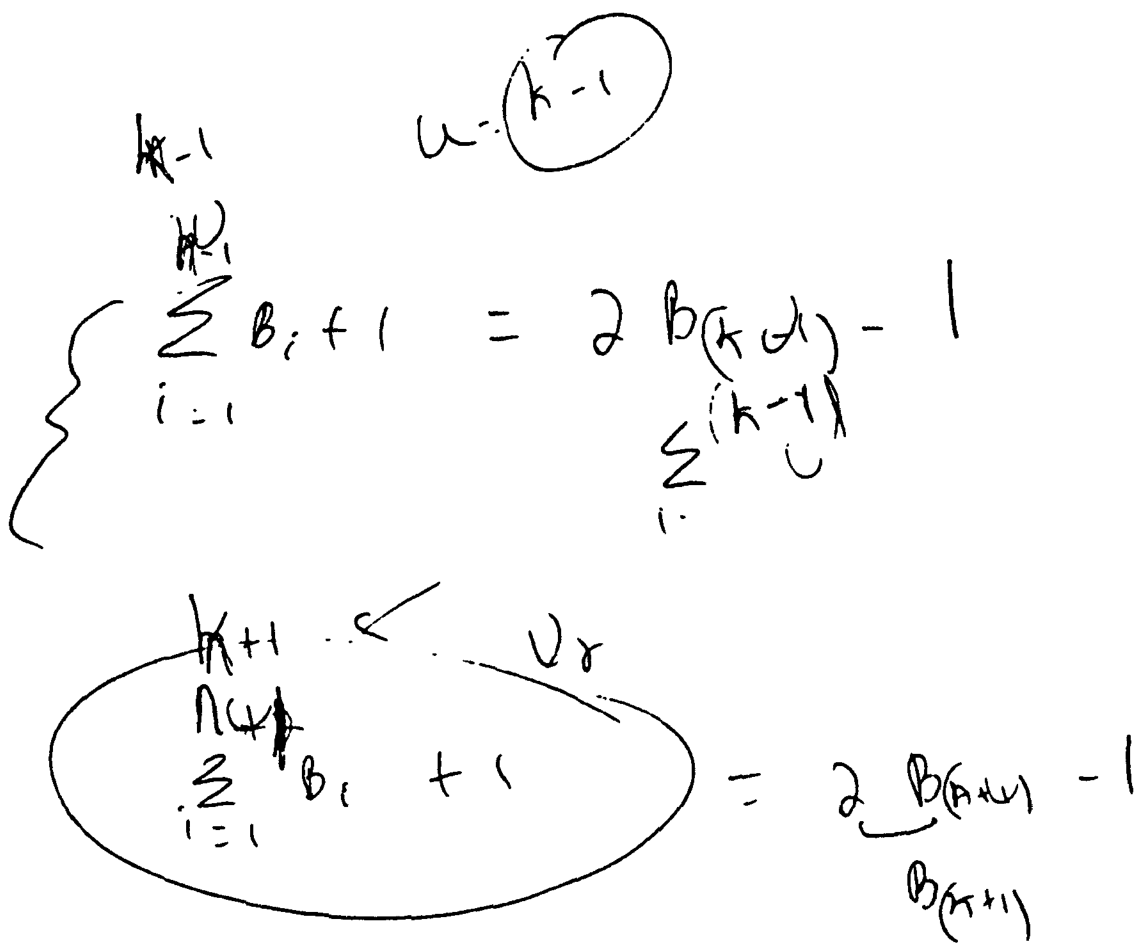

Figure 40: H's writing at line H573

H573: No, it's proven you see because here
S465: You can use the previous
H574: Ok here U, take U is equal to this is like this is the same thing here, U is equal to K minus 1, saying that U here or U here, now if you put here U again, like I'm saying, U, U 2 here is this is this the same thing if this is equal this has to be equal, you see what I mean? [see figure 40]
G570: yeah, that's for, for, a value
H575: Maybe, oh maybe not, hold on a second, --- ok maybe,
G571: would you care to explain the transition between here, K minus 1, you got this
H576: maybe I,
G572: what's happening here?
H577: well, I mean, It seems to be equal to me
G573: For, for a given value K plus 2, where K is odd, why did you take K plus 2?
H578: Why? because I wanted it to be odd again, the next odd one, ok, this is K plus 2

R461: The problem I see with this
G574: OK, but that's not the problem
R462: here, this is
H579: My assumption that,
R463: this is indicating the number of terms that you're adding up, this is indicating which term in the sequence you are, I don't see how the two of them , you can just say, well their the same thing, so
H580: Ok hold hold on, I do know this
R464: [we can just change it?]
H581: ok here look look, I do know this is N minus 1, right?, Ok, N minus 1 this is what, ok so now I'm getting all confused
G575: yeah, but it's a recursive sequence,
H582: This is N minus 1
G576: you can, can't you,
H583: K minus 1
G577: can't you take your basic step, can't you take,
H584: K minus 1, look, what I 'm saying is,
G578: a given value K, and through that you'd have to show that K is, satisfies, is satiefied through this, but this will take you, hold on it's , and then it'll take you to K minus 1, and that one will take you to back, back, back all the way to the first step, this is the way I proved mine, but his
G579: Hold on, I think I got something going here, it's kind of hard to explain,
H586: Am I right in saying, if this is equal OK? for A K minus 1
G580: Can I see your
H587: and K minus 1 here, is it the same as saying, it's U here, and U here? because it's the same, because these are the same ones right? [see figure 40]
H588: these are these are the same expressions
S467: Ya, sure
H589: Now here it's again the same expression

R pointed out that it is not always possible to substitute into variables. H saw the problem. H asked G if he had made any progress. G was uncertain about how H's rule worked. H explained that his rule was the same for both even and odd cases, and proposed working on the even case.

G586: My idea is that it, you you, for each one if, if it'll build up now, if you take something for his here, given, if you find a certain value ok, for through his formula then you can find this value, I, it's a recursive, hold on, let me try to, write this down concretely, ---

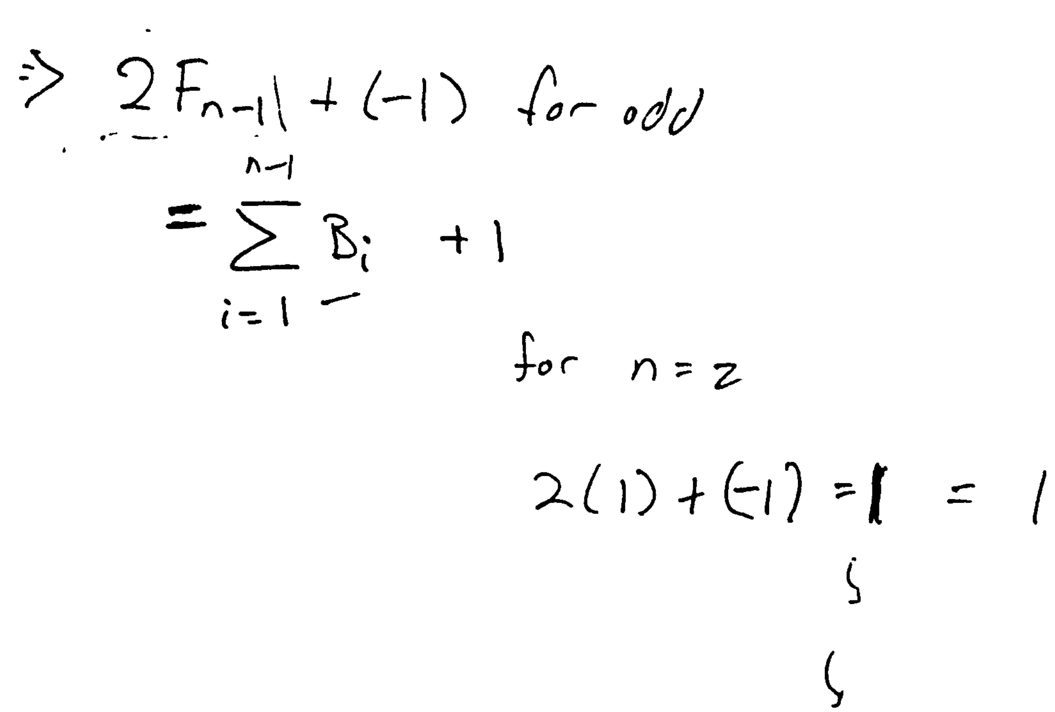

Figure 41: G's writing at line G586

H595: was yours here,
G587: summation I is equal to 1
S469: Is it because you can always rely on the previous?
G588: yeah you can always rely on the previous one, that's all it is, and, up until it takes you back to the basic, the basic one, your first one.
S470: You have it
G589: That's good enough? That's what I've been trying to do,
H596: So what are we trying to do?
G590: you're taking a particular value for, for yours OK, let's assume that we found it to be, F of 15 which is a thousand, 10923 ok
H597: Umhmm
G591: We found it through yours, now we can take, that and plug it into mine and find a value, before that one, which would be this one here, but then you can apply the same rule over and over, recursively, up until you come to your basic step which you've proved to be true, that's what I've been trying to say.

R and S discussed whether or not MI was being described.

S471: I think you've grasped the idea
G592: It's kind of taking, for mine it's kind of taking it back to the basic step
S472: the idea of mathematical induction, isn't that so?

R467: Does what he is saying connect with what you think
H598: Yeah, well, repeat it, what, I really didn't
G593: Ok assuming for the first one, yours applies ok, for N is equal to 2, [see figure 41]
H599: right
G594: right, we found it to be 1,
H600: ok, it works, ok
G595: Ok, for mine, again we found it to be 1, now for odd, assuming its N is equal to, for, what did we do here, even? Ok assuming for even numbers, N is equal to 4, ok, now for N is equal to 4 we assume that, N is equal to, for my sigma, what's N equal to 4 through your formula? after you're plugging it in? For N is equal to 4? 3 here, what would you come up with?
H601: Oh Oh ok
G596: You should come up with 5 right?
H602: Right, yeah, I guess
G597: Is equal to 5, well assuming that my basic step is true, and assuming that, we have to prove both of them, though, we have to prove the odd and the even at the same time so we can do mine recursively finding it true for my basic step finding it true for 2 finding it true for 3 and then I can solve for 4 because I'm using my sigma, I'm using sigma notation, so we'd have to, know indeed, that the previous ones are true, do you see? do you follow?

H said he did not follow. G described his process for the case of n=4. He built upward from the basis of n=1. He described using each rule to go from case to case, proving each as he went along.

G606: Is that proof enough?
S473: yeah but
H612: It's not like a really, but I mean the idea
S474: the idea is there, you prove that you can rely on the previous
G607: On the previous
S475: by using the general, Ns or Ks or whatever
H613: yeah
G608: yeah
S476: But the idea is, it's you can rely on this kind of reasoning, because it will come back, you know, because of this recursion
G609: it's an imply-ance, it's I think, it's you imply the first, knowing that the first one is true, you can imply that the second one is true, knowing that the second one is true, you can imply that the third one is true, and so forth, but you can't, I couldn't go from 3 to 10 for instance, without having to go through, 4 to 9, so I'd, I mean it's more of a implication, through my numbers

H614: yeah, it'll get all the numbers
G610: yeah. and we'd have to prove that, for each one of mine yours is true

R then asked if the following statement is true: "$2B_n + B_{n+1} = B_{n+2}$". H compared it to his rule, and noticed the similarity, that something doubles. G also worked from H's rule. G checked some examples to see if the rule worked.

S483: What is it that you are trying to prove? ---
G617: Actually I should
S484: I mean, the equivalence of H's formula and this?
G618: yeah --- what am I doing here, minus, --- B N is equal to B N plus 2 minus B N plus 1 over 2 --- He says that B N equal to ------ this couldn't possibly be true --- dividing by 2, this must always give you a sequence of even numbers --- these here, yields integers, in order for this to yield integers this top portion here itself, will have to be an integer
H625: No, it isn't true
G619: So
H626: what?
S485: even
G620: it's not true
S486: even
G621: ah, yeah an even number, an even integer

H then announced that it was not true, though he felt he might have made an error. G also expressed a belief that it might be false based on having shown the difference of two terms to be even.

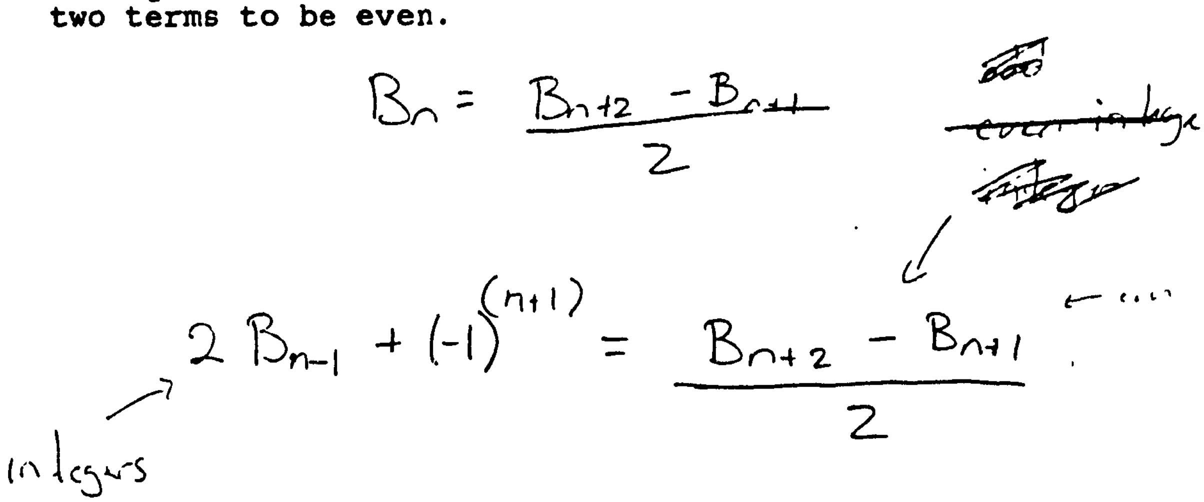

Figure 42: G's writing after line G621

S then asked R if the statement was false, and R replied that it is true. H and G then reconsidered their arguments.

R474: assume that H's is true
H632: well it is true

Based on the assumption that H's rule was correct they continued trying to prove the statement. S asked G to differentiate between Bs based on H's rule and Bs based on R's rule. G described his argument. He had arrived at an expression in which the difference of two terms was divided by 2. This, he asserted meant that all the terms had to be even. As not all terms are even (in fact none are) he felt the statement was false. R pointed out that all the terms are odd, so all the differences are even, which is what G's expression required. H then announced that the statement is correct. He described his proof which involved rewriting each term of R's rule in terms of H's rule.

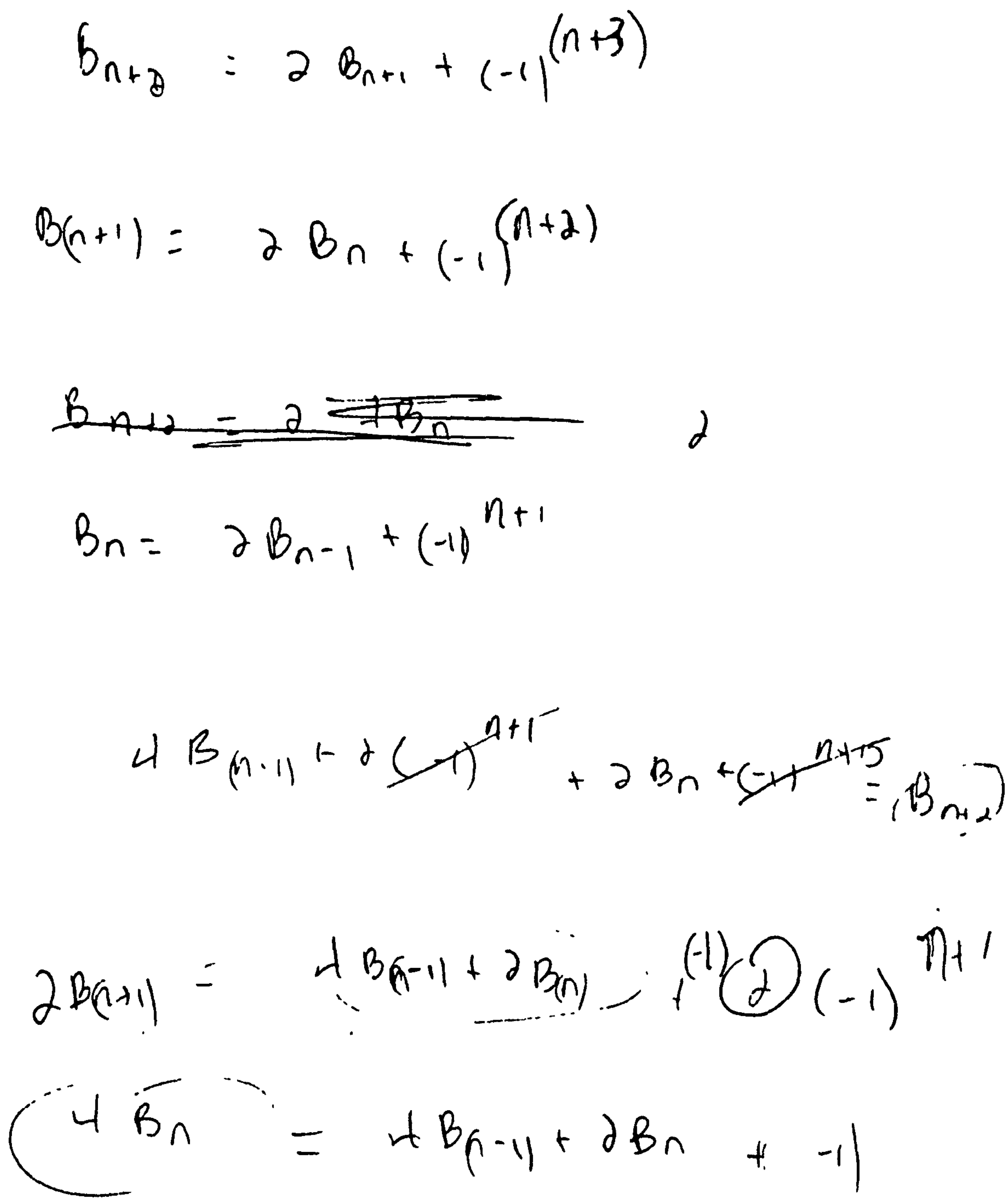

Figure 43: H's proof of R's rule for the B sequence.

H642: So I multiply by 2 I come out to this thing, came up here, I reduced this and I came into the, I came , to, let's see, then I had another pattern that B, this 2 B N plus 1, is that what I'm saying? no I lost myself

G638: No I
H643: It's right though, I mean, it made sense I just lost where I wrote this, because I, I do, wait here it is, 2 N B, here it is 2 B N plus 1 is equal to this, expression here, well actually, I'm missing one of the things, plus, there's plus a constant 2 here times 1 minus 1 to the power whatever, I mean, It was, N plus 1 I think now, shoot this in the other side, now which one did I have to shoot on the other side? --- oh no, here it is, this is it, I have it, I shot the plus, here, how did I get that again? [laughter] It worked because I came to the conclusion, I had B N is equal to 2 B N plus minus 1 which proved this.
S503: Ok the idea is there, I don't think we have to
G639: you had to multiply by
H644: It's just a matter of working it out again, But I mean, that's what I was planning to do, It's just long, and I have this problem that I'm not very neat
S504: So what is the idea of the proof, once more so, did you check for some N, small Ns or not?
H645: No, I didn't
S505: No you didn't
R481: so you don't know if this works at all
H646: I think that in general I don't ever bother, I just
S506: Just work the
H647: general
S507: Induction step and you don't bother about the first, whether the first
H648: This isn't really induction
R482: you've proved it in general but you don't know if it's true for anything
H649: It's not true for any specific one, ok, it's not true you know, but if it is, I assume it is because you wouldn't have put it, Ok, I assume that you guys are not just going to let us put 1 or 2 and then it's not going to give it, right? Anyway, so I work it out and I get it so I just find and expression for this, find an expression for this, and prove, since I know that this, I can prove that 2 times this plus this will equal to this, just by working it out, it's not very complicated to do it, you have this
S508: Well, are you relying on some assumption that it works for some previous Ns, do you do that?
H650: um
S509: did you rely on some, or is it a straight forward
H651: well no, it's very straight forward, because if this, if this true, ok, 2 times this plus this is equal to this, ok, and what you're, the statement is, what, what's the statement again? --- oh here it is, if the statement is this and it, you, you're saying that this statement, will produce this, well if this is equivalent, the N plus 2 is equal to this, well, I

mean, obviously, this reduces to yours. so then it is, it's not relying on any

R then summarized H's argument and commented on its lack of use of MI. This led to H mentioning his lack of experience with MI his desire to work with it more. R then began to introduce the Binet formula by recalling H's claim in the first session that there was no formula for the Fibonacci numbers. H did not remember having made such a statement.

R489: Ok this is it:

Statement D:

For all $n \geq 0$:

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

where F_n is the n^{th} Fibonacci number

G642: Can we do this for homework?
H660: Whoa
S517: the formula of Binet
H661: Of what? what's it called?
S518: Binet
H662: Binet? French guy? --- so what's the question?
R490: do you buy that, do you think it's the formula for the Fibonacci sequence?
H663: You just told me it was, I mean, I trust you
R491: Proof by reference to authority, ok that's, that works a lot of the time
S519: it doesn't look very friendly does it?
G643: No
H664: I doesn't really
G644: those square roots there
S520: It doesn't, the Fibonacci numbers are all integers
H665: yeah that's why
S521: and you have those square roots of five, you know
H666: your telling me it's true
R492: But on the first glance, you say "Oh yeah sure, that looks like it" ?
H667: well I don't, no, at first glance it doesn't look like

it at all, but, it could be true because, obviously you think

S522: why don't you just plug in 1?

R493: just to be sure

G645: well zero's , equal, would be, but that's evident

H668: 1 it's going to be, so 1 this is the same thing, conjugate, take that, 1 over, square root of 5, 1 plus square root of 5, 2, 2, minus one half plus square root of 5, 2 --- minus one half,

S523: You get rid of the square roots

H669: times square root of 5, oh one half plus, 1 yeah ok fine 1

S524: you get rid of the 5 the square root of 5

H670: yeah

S525: Show the proof

R494: I won't keep you, I have a very simple proof here, here's, the proof is in three lemmas followed by actual proof, this is

H671: What does that mean, a lemma?

S526: A lemma, an auxiliary theorem, that helps you to prove the main one

R495: Yes

G646: Because it's been proven

H672: why are you showing it to us anyway?

S527: just a special case, you know

R496: So you believe me

H673: yeah but how are we supposed to figure this out ourselves?

R497: No no no no it took me hours

S528: especially to type it into the computer ---

R498: to begin with we need a lemma to show that for all N greater than 0, this expression here, can be expressed in terms of an integer, C and D are both integers here, plus this horrible thing times another horrible thing

H674: Which is, you know, very true, I mean, obviously, I can see that

S529: very true

R499: It's very true.

H675: I can see it

R500: you can see it, ok

H676: this is seeable

R501: So we don't need to look at the proof any more

H677: No NO this is seeable, let's see what the rest of the proof is

R502: When you say it's obvious do you mean you understand what this means

H678: Yeah, I understand what it means

R503: Or it's true

H679: and it seems true to me, I mean, maybe I'd be surprised, well, the lemma is true for the case N is equal to 0

R504: Ok, we're going to do this by induction first we show

it's true for the case of N equal 0
H680: induction hypothesis
R505: trivial 1 equals 1
H681: umhmm yeah
R506: Ok then we're going to start off with this hypothesis: assume that this is true, for K, and then on to the rest of it,
G647: K plus 1
R507: then we show that K plus 1, is going to be of this form too, now, this, right
G648: C D yeah right
S531: --- and this is substituted ---
H682: times C times that
R508: and then it's algebra
G649: you worked it out, you got D plus, let's see, this thing, which is just another integer
R509: Ok, Lemma 1, that was easy
H683: oh there's three, there's three or four lemmas?
R510: there's three
H684: I thought that was the whole proof
R511: No no, it goes on for pages, here's lemma 2, define two new sequences, C and D, by,
H685: aha
R512: as C N being those C things that you get
G650: Ok
R513: and D N being those D things that you get, D N is always a Fibonacci number, in fact it's always Fibonacci number N, and this is shown by showing that it's true for D 0, for D 1, and then showing that the same recursive, the same
H686: induction yeah
R514: recursive rule that, applies for the Fibonacci numbers, applies [they studied the proof] Do you buy it so far? [they continued]
H687: here where C N, C N plus 1 is equal to D N
R515: Umhmm, if you notice when we did this, going from
S532: from here
R516: from here to here, C and in the next case became D, so this is the, the C for N, for K plus 1 is equal to the D for K
H688: Ok C N plus 1 is equal to D --- ---. D N plus 1 is equal to D N plus C N
R517: Umhmm
H689: OK yea', then D N plus 2 is equal to D N plus --- C N plus 1 is equal to D N
G651: What's that you have here
R518: Which is?
H690: the Fibonacci

R then continued with the proof (the full text of which appears in Appendix A). The proof of Lemma 3 was described as similar to the proofs of Lemmas 1 and 2, and was omitted.

The proof of the theorem itself was then introduced, and described as algebraic. S commented that the formula is not at all obvious, and everyone agreed. She felt this was a good example of an occasion in which MI is useful. G was asked if he believed the formula is true, based on the proof. He said yes, but that at first glance it didn't seem very likely.

R527: How does the inductive argument indicate that it's actually true?
G662: for this particular lemma here?, no, in the second case, where is it
R528: Where it's the proof of the lemma, yeah, here's the entire proof of lemma 1, the horrible inductive one,
H695: That means induction, I H means,
R529: yeah that's where it referring to this induction hypothesis here. ---
G663: well you prove that it is indeed a Fibonacci number, I think that it kind of backtracks
R530: umhmm
G664: to the fact that this is true, but it's not evident, it's just not evident
R531: Ok
S536: So mathematical induction's backtracking for you, isn't it?
G665: I, you could say that
S537: are you feeling better with mathematical induction now
G666: Yeah but, I think I'm going to start spending some time with it, I think I should start doing some exercises with it,
S538: using it
G667: Practicing it, using it yeah, just practical experience,
R532: One last question, why does mathematical induction work?
H696: Well because you don't skip any numbers, you can't skip, numbers, they're all there, every number exists,
G668: Why, did it work?
S539: yeah ---
H697: I mean if it works for K plus, if it works for any number, and it works for the number plus that, then it works for all numbers, because all numbers has a plus, K has one more,
G669: well, It just kind of backtracks, like if, you mentioned something about backtracking it's taking you back to your hypothesis, it's tracking back to your hypothesis
R533: well where are you starting? when you back track
G670: when you back track?
R534: yeah
G671: you're starting with your, K plus first step
R535: step

G672: for instance, and backtrack that down to you initial hypothesis, and you prove that to be true if you can show them to be the same,
R536: How do you do the jump from showing that you can do it for any number, to showing that you can do it for all numbers?
G673: that's the difficult part, [laughs] that's the part I hate
H698: If you show that you can do it for one number, and if you show that you can do it for the number plus, well then you can do it for all numbers
R537: Well then you can do it for any number, if I give you a number you can do it, but does that show that you can do it for all numbers?
G674: Not necessarily, I don't, see that what have a problem seeing

H said that the infinite number of numbers was related to the way MI works. S observed that this is not sufficient, and mentioned MI's status as an axiom of number theory. H asked what the axiom actually said.

R538: It basically says if you can show it is true for any natural number then, then you can show, then you can say it is true for all, and that's the axiom
H704: yeah but what, how does it say, how does it come to that, how does it come to that conclusion?
R539: it says, if you can prove something for 1,
H705: yeah
R540: and you can prove that if it is true for N then it is true for N plus 1,
H706: Yeah
R541: then you are justified in leaping to: It is true for all natural numbers
H707: yeah, that's what I'm saying, that's what I said, isn't that what I said?
G679: See that's what, the jump, this leap, that's what I, what I find, that's what I
S543: You can't believe it
G680: yeah, well I could, but there might be a number, somewhere within that sequence,
H708: No it can't be, it won't
G681: Why?
H709: It's saying every N plus 1
G682: No you have to take it step by step but I mean one implies the other, I don't see how you can make that, that jump for, you can prove something for N, ok let's say you could prove something for N plus 1 but there might be a number within that sequence, that does not work
H710: No but then you are skipping a number, then you're not saying it worked for all N plus 1

G683: Yeah but what
H711: If it does work for all N plus 1, then you're not, then you can't skip a number, you get all the natural numbers
G684: that's right, but you're not checking every one
H712: Yeah, but you don't have to, you just check it generally speaking, if it works N and it works for N plus 1, and it works for 1, then it works for all
G685: Not necess, yeah, see, you gave a sequence, but assume that you gave us a formula that works for everything, but it doesn't work this one, and we've applied, our
H713: you can't come to the conclusion that it doesn't work for all N plus 1
G686: is that true, I don't think that's true, I think you can assume it, that it is true, and just by backtracking, check your initial hypothesis because your hypothesis was based on the first few numbers, and it won't be correct for the entire sequence,
H714: No but then it won't be true for all N plus 1 either
S544: you will not be able to prove that it follows from the assumption that it's true for K because your K is anything
G687: Yeah, but let's assume that your K was chosen outside of that
S545: you are not choosing a K, a particular K
G688: But you're not checking every case either
H715: if you say it works for one 1 and it works for K plus 1, and it worked for 1 too
G689: Fine, but you're not going to go, if there's an infinite sequence of numbers you won't check every single one
H716: yeah, but you don't have to,
G690: you don't, see that's [finalized? I'm lost?] see I understand, I understand how it works, I'm not completely, I haven't used it and pract, you know, I think I need more practical experience with it, but I think that, you haven't accounted for every single one, unless you've taken every one sequentially
H717: You mean, one by one?
G691: yeah,
S546: for an infinite number
G692: yeah for an infinite number
H718: Forever
G693: of times, yeah,
H719: there's no resting, like, eternity, not for his life, for eternity

G commented that no one agreed with him. S said she didn't. H attempted to explain MI to G by restating the axiom as it had been stated to him.

S550: I have a proof, I have a, tool for proving, when I'm proving the induction step, you know, from K to K plus 1, I have sort of, in one, in one shot, I have proved, all those steps, you know, I've proved for all
G698: ok
S551: All those steps, each one from there to there, I can go, I know I can go, it's proven
G699: Ok yeah, ah, no
S552: Ok so I haven't, It can't happen
G700: it's a hypothe
S553: look, I've proved it
G701: Initially it's a hypothesis
S554: No, it's a proof
G702: Ok if it's a proof initially then that's fine
S555: I've proved that if I have, if this is, it holds true then I have, you know
G703: ok, ok then I, I will believe it
S556: it can't happen that something doesn't work for some K, particular K it can't happen
H722: yeah it can't , I mean, or else you can't prove that it works for all K plus, K plus 1 from, for
S557: Unless you can't prove that it works
H723: yeah

R asked if G had meant the induction hypothesis when he mentioned a hypothesis. G said that he had. R said that that was not proven in the beginning, it was assumed. H stated that the purpose of the proof is to prove the assumption. R then said that all that was actually proven is that if one case is proven then so is the next. S and H then added that it is necessary to start with something as well.

H727: A question, let's say if you find, if you do that, and you find, say 10 can you backtrack, can you assume that all the ones lower than that too are true?
R545: What do you think?
H728: What do I think? But you've only proved one, and let's say you have no means of, like, lets say you can't just theoretically if, is it automatically true that the ones behind it are true? That's what I mean.

S reminded H and G of a proof they had seen in class, which had been by induction. H didn't remember it. S then described the proof, which involved proving the cases n=0, n=1, and n=2, at which point it became clear how to prove the general induction step.

G706: yeah, but for instance, ok, you give us a particular sequence of numbers ok, this is what we start with, you give us these numbers, here we have a sequence, now our, assume our formula works for this particular sequence, but from 24 on there's a continuing sequence

of right, but they completely deter [differ] from our formula, we checked for this case here, then we've already set up our, we've assumed that what we've written down is proof enough, but that's not proof
S564: Ok, that's true,
G707: that's not proof, I'm not willing to say this is proof
H731: ok but that's not part. what are you saying? that this is not part of this sequence?

R asked if the two rules for the B sequence that H and G discovered had ever been proven.

R548: Did you ever prove either of those?
G710: No completely
H732: Not like, no
R549: Ok
H733: Who would, I think everything is a hypothesis
R550: Could you prove this?
H734: No, I mean, if it's, yeah,
G711: not for an infinite amount of numbers, No, unless you can tie it to the fact that it is a continuing sequence such as the natural numbers,
H735: What are you saying? that after this it is not part of the sequence anymore?, say you put any number here?
G712: assume
H736: Say I put 27 after this, say B 25 is 27
G713: yeah something that,
H737: Yeah, ok, I get what you mean, ok, now I understand what your problem, yeah
R551: OK
H738: it's not
R552: second question: could you prove that if your formula,
H739: is true
R553: well, could you prove that the sequence determined by your formula is the same a the sequence determined by his formula
H740: yeah, well that's pretty much what we were trying to do before
R554: Yes, but do you think it's possible, were we giving you a possible task, or an impossible task?
H741: I think it's possible
G714: yeah but you have to prove one in order to show that the other is also true
H742: no no
G715: Just because we tie ours together we might not come up with the same conclusion that's not necessarily the right conclusion
H743: No no
G716: So even though we tried to tie it together, that wasn't proof enough, I don't think that was proof, proving that mine was indeed true for this sequence, which if we check each and every single one then I

could say it's proof enough

S explained that her motivation in asked them not to look at the first terms of the sequence was to force them to concentrate on the equivalence of the sequences actually produced by their rules. H said that if they had not made a mistake in discovering their rules then they must produce the same sequence. He restated this, saying that it meant there must be a way of proving, but just finding two rules from the same finite sequence does not constitute a proof. The session ended at this point

First Session with A

R1: These are the first few Fibonacci numbers. Can you figure out what the pattern is to this sequence?

A1: hmm. Ok what I'm doing is right now I'm looking at, how it gets smaller, how they combine, how it's progressing, --- so it's gone up by 13. the next number down is 13. it gone up by --- 8! --- ok I see. What it is is that you, add the first number to the second to get the third number. You add the third number to the next number to get --- so it progresses.

R2: ok, um How would you formulate a rule for producing the Fibonacci sequence if you wanted to give specific instructions?

A2: why do I think you want me to do algebra? --- actually I couldn't write it in algebra. I could if I sat here for fifteen minutes and --- something like that.

R3: ok could you express it verbally?

A3: yeah I could --- you just take the first --- you start with the first number --- um, you double the first number --- you start with 1 --- you double it --- so you have two ones, --- you add 1 and 1 --- that starts you off in the sequence, you get the 2 --- then you add --- --- so you could say each numbers --- let's say its --- A B C --- ok what you're doing is --- A plus B --- gives C --- and then C plus B --- gives A --- in another sequence --- oh, no --- 'cause --- the number D --- so how would you explain that? --- start with two ones --- and then you --- add them together and that gives you the third one and then you add the third number --- your result the, the the, --- second number --- ??? up two steps and back one ---

R4: what do you mean "up two steps and back one?"

A4: if you look at this thing as a ladder progression --- you start by going up to here --- you add these two together you get this result --- and this result, go down one step and this is your addition --- and that gives you ??? --- goes up three --- and you go back down to your second number --- ??

A was then asked to square a Fibonacci number and compare the result to the product of its predecessor and successor in the sequence. She noted that the difference was 1 and expressed this as a general rule. She was then asked to pick a prime number. She picked 3. She was asked if the third Fibonacci number is prime. She responded "No" due to confusion about the definition of prime number. She was corrected.

R24: so you picked a prime number and you counted along the Fibonacci numbers, the Fibonacci sequence that many numbers and you came to another prime number. if that were true in general how would you state it as a general rule? that that works.

A24: and moved three along and its a prime number too --- so if I move 5 along --- 1 2 3 4 5 ?? --- 13 which ?? --- if I move 13 --- 1 2 3 4 5 6 7 8 9 10 11 12 13 ?? ---

R27: How would you make a general rule to say that that works? ---

A25: um I'd say that --- starting from the base point of 1 --- if --- I'm trying to get for myself what it is ?? --- each prime number has got a --- corresponding --- number, Fibonacci number --- which if you count along --- that chosen prime number starting from the base point 1 you reach --- another prime number ---

A was then asked to add up the first 10 Fibonacci numbers. She did so. She was asked to compare her result with the product of the seventh Fibonacci number (which is 13) and 11. She observed that the two were the same and described this as a general property. She was unsure about the generality of all three statements, especially the second, relating to prime numbers. Hr uncertainty turned out to relate to composite numbers. She was sure it worked for prime numbers. A also accepted the first statement, but not the third. The first statement seemed to her to fit with the way the Fibonacci numbers work. After trying another example she found that it did not work generally, but soon discovered the alternation of adding and subtracting 1 which makes it a general statement. At this point notation was discussed, using F_n to stand for the n^{th} Fibonacci number. A continued to express belief in the first statement in its modified form. After trying one other case for the third statement A was convinced that all were generally true.

R39: How confident do you feel?

A40: Very

R40: Very confident? Do you think it's true all the time? most of the time?

A41: The whole time, because its math I'm going to say yes, all the time

R41: ok, so you're sure that I could never come up with an example were it wouldn't.

A42: no I'm not

R42: so your not <u>that</u> sure

A43: well no. Actually, if you're just dealing with straight Fibonacci numbers but I don't know what happens, do you get into ??? Fibonacci numbers?

A discussion of the nature of truth in mathematics versus the nature of true in other fields ensued. Mathematics was described as determining truth based on the ability to derive statements from other statements based on an agreed upon set of rules. The nature of prime numbers was again discussed and the first six prime indexed Fibonacci numbers were tested for factors. Confidence in all statements was established, but the context of a study made A suspicious. When asked if the 19th Fibonacci number is prime A began by multiplying 19 by 19. She was confused at this point between the Fibonacci numbers themselves and their indexes. A then produced a list of the first 19 Fibonacci numbers. She stated that she was unsure of F_{19} being prime as she thought only Fibonacci numbers whose index is a prime Fibonacci number would be prime. The investigator indicated that all primes numbers were intended, not just prime Fibonacci numbers. The counter-example to this general rule did not effect A's confidence that it would work if the index is a prime Fibonacci number. Methods of becoming more confident of statements in mathematics were discussed. A suggested that a number of examples would increase her confidence. When pressed for a more sure method she said: "what you have to do is do a proof there". She remembered having done proofs in the past. The F_n notation was discussed again.

R43: is there a largest number that you could stick in it? [n in F_n]
A44: no
R44: why not?
A45: because there must be a larger number there's no limit ???
R45: Are Fibonacci numbers unlimited?
A46: Yes, because then you keep multiplying [she meant adding] to the right. You can always have numbers to multiply and you get it ???

The nature of truth in mathematics was again discussed, but in a general way. The statement: "the sum of any consecutive Fibonacci numbers is a Fibonacci number" was shown to A and she was asked if she accepted it. She believed it to be true because of its relation to the way Fibonacci numbers are generated. She then tried an example and disproved the statement.

Transcript of Second Session with A

The interview began with a discussion of the notation F_n for the n^{th} Fibonacci number. The recursive rule for generating the Fibonacci sequence was expressed in this notation. A manipulated the rule algebraically, to create a subtractive form. The three statements she had worked with in the first session were shown to her in symbolic form. A checked the statement "$(Fn)^2 = (Fn+1)(Fn-1) + 1$" again, and noted that the alternation of adding and subtracting 1 was missing from the statement. She was then asked about the statement: "The sum of the first *n* Fibonacci numbers is one less than the second Fibonacci number after the ones added up."

R52: How would you write this: with symbols
[laughter]
R53: maybe you should try a few actual examples before you --- try and write it
A51: I can't understand what you wrote --- the sum of the first N Fibonacci numbers, so N being we don't know how many numbers, we don't know how many Fibonacci numbers we're dealing with --- the sum of the first n Fibonacci numbers is, 1 less than the second Fibonacci number, after the ones added up [reading] ---
R54: Maybe you should try and example by picking a number --- for n to be
A53: should I take?, OK, n, say n, say of, says its 4, the first 4 Fibonacci numbers starting from, the beginning, so the sum of them is, 1, 2, 4 5 6 7, so we have, 1 plus, 1, what does n equal 4, 1 plus 4, 2 plus 3, is equal to ---, I
don't like using the same n, did you choose n in particular? ---
R56: no you could use any letter you want
A54: Q --- is equal to 1 less than the second Fibonacci number, after last one --- so that'd be 5 6 right --- --- so its 4, 1 2 3 4 5 6, 6, makes 7, and the sixth is 8, minus 1, is 7 so it works --- and how would I write this?, um, what we're doing, the number of Fibonacci numbers is Q so ---

A had some difficulties with notation when trying to write her statement which were cleared up at this point.

A63: the sum of the first, the first, so lets say you want 9, Fibonacci numbers, then Q is equal to 9, right --- Q is equal to 9 so then you'd have, so then --- so then you could just put it in automatically that its equal to 9, then you're looking at, F1, F2, F3 --- F4 F5 F6 to F Q, is equal to F Q plus 2, minus 1

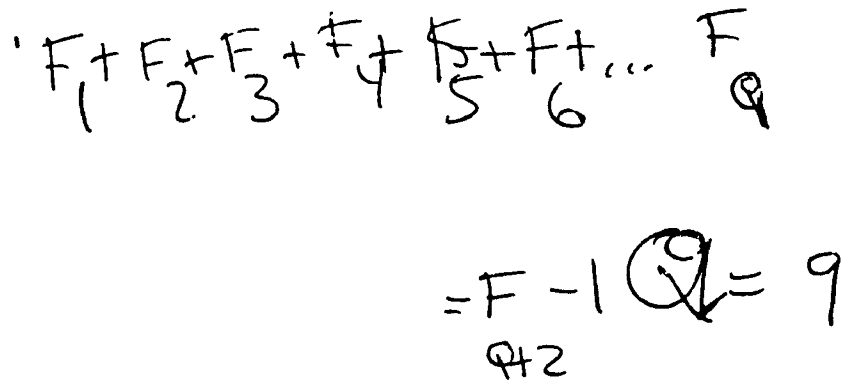

Figure 44: A's equation at line A63

R67: does that say it then?
A65: yeah ---
R68: do you think that this statement here is actually true?
A66: I don't know, I didn't test it
R69: try and figure out if its true or not
A67: , ???, so, 1 plus --- 1 plus 2 plus 3 plus --- ???, 89, that's --- Am I surprised? [laughs]
R70: what happened?
A68: well so far, so good, but I don't know, I don't think that that's the most successful way to figure this out, If I was to just sit here going, like I said before its not as if I can, try every single Fibonacci number I can possibly think of, so let's look at this first, --- ---
R71: It would help if you'd talk to me about what your doing --- so I'll have some idea of what you're thinking about ---
[laughs]
A69: ok I'm thinking to my self, ok here you have this equation right?, now you want me to tell you whether its true or not, ok so I'm thinking to myself, there's no way I'm going to sit here all afternoon, and crurch numbers, just so I can find out whether its wrong --- especially as I suspect that I'm not going to find one ---???, so, is there some way you can do this?, so I look back to my ??, and I, what is there to manipulate in this formula?, that would make it prove itself?, ?? --- I was thinking I could try and use one of these

formulas --- ???, to see if I could, prove [do] something with them, but I don't know, hoe long does this have to go one before I get to give up? ---
[laughs]
R72: well I'm hoping that you wont give up, that you'll get it
A70: that I'll get it? that I'll figure out a proof?
R73: or a non proof
A71: or a non proof? ok --- aaagh! ok, just tell me something, what I'm asking for in fact is a clue, can I have a clue, should I be using these, problems?
R74: it doesn't really relate to what's going on at all
A72: ok --- this is the base point ??? ---
R75: it looks like exactly what I would have written it --- this is my version, F1 plus F2 plus dot dot dot plus F N equals F N plus 2 minus 1, so your formula should look like [mine] and it does, I think you got that ok
A73: ok so now should, but you don't give a proof [looking at R's sheet]
R76: no
[laughs]
A74: let me just think ---

A took a short break to collect her thoughts at this point.

A75: F Q F 2 minus 1 --- if I want to get rid of the um, this --- minus 1 ok, and just reduce it to F Q plus 2, then the question is, I forget this, do I have to do F Q plus 2 minus 1 --- times F Q plus 2, or would I just go plus 1, plus one over here and, minus one from this side
R77: if you add one to the right hand side you'll also have to, add one to the left hand side, to make it balance out
A76: I can't remember does it change signs or it doesn't change signs???, so if I go minus 1, so I guess minus 1 and then, its like that plus 1?
R78: umhmm
A77: and that reduces this to F Q plus 2?
R79: yeah
A78: so then F Q minus F1, plus F2 plus F3 dot dot dot ??? --- F Q, plus 1, F1 plus F2 plus F3 ????, F3 plus that is equal to F Q, plus 2
R80: ok
A79: so now what do you get?, now if we do F Q plus 2 --- and we do, how would we get rid of this 2?, is there a ??? for that, can I do ???, can I ask you or do I have to do it myself?, ???, so I've got F Q plus 2, that f9 plus 2 lets say, we want to get rid of plus 2, so we want to --- subtract it, to get it to Q right
R81: well the Q there isn't, [interrupted by A] ---

A80: so let's get rid of this, get rid of this, so what we've got basically is, 1 plus 2 plus we don't know
R82: you can't really just subtract something, from an index ---
A81: ok, right, yeah, um ---, is there a way of simplifying this side of the equation so I don't have --- f1 plus f2 plus ---
R83: not really, there are simpler notations but
A82: no ---
R84: you don't want to fool with them
A83: ok um, let's see if this is true [the modified $1+F_1+F_2+...+F_q=F_{q+2}$], um, 1 plus 1 plus 2 plus 3 ---, plus 5 plus 8 --- plus 13 plus 21, plus 1 is ???, ???, 2, 2 4, 12, 32 --- 54, 54, 108 ---, 108 so that works too --- but that doesn't prove it, just for that number???, ok --- this is ridiculous, [laughs]

A second break occurred at this point

A84: F Q is an index, F Q index plus 2, is?? plus 1 --- right --- how to prove this, it's weird working with an index
R85: well would you be more comfortable looking at a few more examples, to get an idea about how ??, or looking back to where you first came up with, how the, when you first were using the indexes???
A85: um --- ???, the first plus the second equals the third is how it, is how it works, so the first plus the second would give you the third, the indexes ????, do that with that ??, ??, ???, the first and the second, this is the third one, ?, ?, it would prove itself if this F Q, were ?? this F Q, which it should
R86: you mean if the last Fibonacci number you added here was --- two before this Fibonacci there on this side
A86: yeah
R87: ok ???, say it could be, how does that prove it?
A87: um well to me that doesn't prove it
R88: ok
A88: I don't see how anything could prove it, I've never understood proofs in all my life, I shouldn't give up --- Is there a way?, can I write an actual formula that's going to prove it?
R89: maybe
A89: ohhhh! [frustrated]
R90: why don't you work with a few more examples?, pick Q to be a small number, and try a few
A90: Q is equal to 3 ok?, so 1 plus 1 plus 2, plus one is equal to ---equal to ---5, and, or on the other hand, or if you do it this way, 1 plus 1 plus 2 --- is equal to 5, and this is equal to 5 minus 1, so, I prefer this [pointing], cause this equals this, but --- so lets see, does that work?, 2 3 4 5 yeah, did this work?, 2 3

???, 2 3 4 is equal to 4, that works, yes, so let's say Q is equal to 5, in that case ??, 1 plus 1 --- 1 plus 1 plus 2 plus 3, plus 5 ---, 15 5 10, I've made a new rule, all the Fibonacci numbers added up, the sum of the, of the --- the sum of the first few Fibonacci numbers, plus 1 is equal to, the second Fibonacci number, after the ones added up., that we know too, what we don't know is how to prove this ---, that's , yeah ---, this reminds me of the puzzles with wire we used to play with

R91: yeah

A91: for some odd reason --- 3

R92: what are you thinking about now?

A92: I'm thinking about how I can figure how, write this down so that you get, an equation that gives you F Q is equal to F2 [F Q?], and I think that would prove it --- --- that's all that I'm doing here, I'm saying 13 is equal to 13, that's this one right, I can break this down, and work it out so that, F Q equals F Q, that's really what I have to do to prove it ---

R93: Now all you have to do is find the proof [???] [garbled]

A93: yeah, because it's, I don't know what to do with these numbers, and um, and how to deal with this sum factor, all these numbers, that's a question too --- --- I wonder what happens if I do this ---

R94: what are you doing? [???]

A94: ok from the beginning, what would happen if I do like this, right, like this ---, F1 plus F2 plus F3 plus f4, that left over, plus --- --- but we don't know if that's F Q, because its ---, I don't know david

R95: How do you mean separate?

A95: I don't know, get it on its own

R96: Each of these is just being added, you could add them in any order, if you wanted to add it last, you could do that ---

A96: but isn't it F1 and then all the way to, F Q, but then you've got this tree, and you don't know where it goes --- and you say yes, it goes off --- and it would get more weird if you went like this, Q ---I don't think I adding another variable to this would help any ---

R97: probably not

A97: ???, I don't know, I think that my problem is that I'm trying to, think about --- ways of manipulating symbols, so that they mean something, that that they show, that this side of the equation is equal to this side of the equation, I'm trying to do this as if its numbers, and not, but it becomes kind of dubious [???] --- lets go backwards some., the sum of the first n --- Fibonacci number is one less than the second --- Fibonacci number after the ones added up ---I don't really think that its right, I just want to check if its wrong

R98: well to do that you'll need to find a counter example --- maybe you should start looking

A98: for a counter example --- a counter example, I'm trying to figure out what, this would be like this --- --- does not equal

R99: yeah so you'd have to find a case, when its not equal ---there's lots of paper here

A99: yes [laughs]

R100: so far you've checked for which Fibonacci numbers? --- first you ---

A100: 2, didn't I do 2 3 4 and 5?

R101: yeah, maybe you should check it for 6 and 7

A101: ok --- I'll check it for 6 --- 1 plus 1 plus 2 plus 3 plus 5 plus 8, this is, that's 20, did we say its plus 1 or minus 1?, plus so it works, 1 plus 1 plus 2 plus 3 plus 5 plus 8 gives me 20 plus 13, would give me 33, and that's, that's right, oh wait a second [!!!], ha ha, ok, wait I almost got a flash of something, let's look at that again, 33, that means the sum of all these, is going to be, 10 ---immediately the sum of those, so 33, plus 34, is going to be, 3, 67 ---

R102: where did this 33?

A102: oh sorry 13, 33, plus 21, is going to be 53 --- uh oh, it didn't do it, let me try again, 1 plus 1 plus 2 plus 3 is, 1 plus 1 plus, 2 4 5 6 7 7 and 12 12 3 3 33, that's the 34 here because, two more is 34 and now it's true, so we go back to 33, 33 and 21 is --- 50 50 54 so it's right for this, well it mean actually something different, it means that ---, well I'm just looking at all of the figuring this out, now is I just, I really don't have to add all these numbers up --- because everything, I proved up to here, is just that, the fact that, this Fibonacci number minus 1 is going to be the sum of all those, before here, which is what I'm trying to prove, that, that ---

The remainder of the session was not recorded due to a technical error. The following summary is based on the notes taken during the session.

A realized that she didn't need to add up all the numbers because she knew the sum is one less than F_{q+2}. She adopted a procedure for determining if $\Sigma F_q = F_{q+2}-1$:
(1) Determine ΣF_{c-1} from $F_{q+1}-1$ (2) Add F_q to get ΣF_q
(3) Check F_{q+2} on list and subtract 1 (4) compare.
She used this to check $\Sigma F_q = F_{q+2}-1$ for q=33
She was then encouraged to try the next case.
She applied her procedure.
She eventually arrived at $F_{35}-1 + F_{34} = F_{36} - 1$ and recognized that this was true based on the formation rule, but she could not put together the steps to form an argument.

The session ended at this point.

Third Session With A

R110: remember that? [showing A statement]
The sum of the first n Fibonacci numbers is one less than the second Fibonacci number after the ones added up.

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$$

A was asked to check the statement for n = 6, 7, and 8. After taking a moment to recall what the statement meant she did so.

A121: oh right is one less than 8 so the second Fibonacci number after the ones added up and you did it for 6 and that's 8 so yeah this is this is the second one after all the additions to 6 so it's number 8 is the one we're talking about the one after the ones added up is is 8 and the Fibonacci number is 21 and the ones added up is 20 plus 1 is 21 so yes it works
R122: ok could you try it for 7?
A122: ok so, it should be 34? so it should equal minus 1 so it should be 33 the sum of, of the first seven Fibonacci numbers
R123: ok
A123: so 1 and 1 is 2 4 is 7 is 12 is 20 is 20 is 33 33 plus 1 is 34 so it works
R124: ok could you actually write out the sum so we can see it?
A124: ok whoops 2 plus 3 plus 5 plus 8 plus 13 is equal to F9 minus 1 which means wait let me figure out 34 minus 1
R125: ok how about if you wanted to do it for wanted to do it for 8?
A125: 8? 55 so that should equal that now we just want to check it
R126: umhmm
A126: so just do the sum of the first 8 so you do the last one was 33 so 33 which is the sum of every thing up to sss wait yeah, up to seven so 33 plus 21 which is 50, uh oh is that right?
R127: is something wrong?
A127: yeah
R128: what's wrong
A128: it's not working Is it because of the way that I did it?
R129: what is your difficulty?
A129: well instead of, oh no instead of adding them all up again I just decided that the sum of the first oh that's what's wrong
R130: what's wrong?
A130: no the first, the sum of the first 7 is 33

R131: umhmm
A131: right, I believe so yes the sum of the first 7 is 33 and now we're doing it for 8 so 33 plus 21 should equal the F10 minus 1
R132: ok
A132: and it doesn't
R133: ok what is 33 plus 21?
A133: it's fifty --- four oh wait it does work [laughs] 54 ok so now

A was then introduced to the problem of determining the number of pieces produced when the plane (represented by a pancake in this case) is cut by n lines. She initially confused the problem with a problem she remembered of determining the maximum number of pieces a pancake can be cut into with any number of lines. This was clarified. A was asked how many pieces could be produced by one line.

A147: one half, two pieces
R148: I was a little worried when you got infinity I thought the answer was two too [laughs] ok, how about if you do two slices two perfectly straight cuts
A148: then you can have let's see you could have three
R149: ok
A149: or you could have four
R150: ok
A150: if you put one on top of the other you could have two because you've cut in exactly the same place as you did before
R151: how would you have two oh ok
A151: if you're being sneaky
R152: ok but what's the most you could have?
A152: 4
R153: 4, ok how about if you made 3 cuts? What do you think would be the most that you could have?
A153: may I draw?
R154: sure that's why you've got a piece of paper and pen
A154: 2 3 1 2 3 1 2 3 the most you could have?
R155: umhmm
A155: 1 2 3 4 5 6

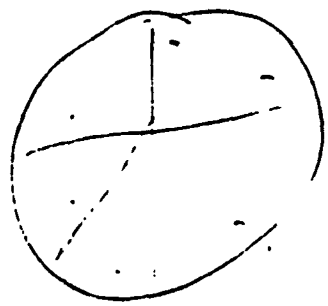

Figure 45: A's sketching at line A155

A was asked about exponents and seemed to understand them.

R159: what would you say to the statement that if you make say k cuts of your pancake you're never going to get more than 2 to the power k pieces --- [A writes] never gets you more than that
A159: well it works with 1
R160: ok
A160: no no wait it doesn't work for 1 isn't 1 to the power of oh right 1 to the power of 1 is 1 but then you get 2 cuts of the pancake from 1 to the power this isn't 1 to the power 1, 1?
R161: yeah but you should be doing 2 to the power of 1
A161: oh sorry right 2 to the power of 1 2 hmm well it works for 2 it works for 1 3 6 works for 3 4 8 4 works for 4 works for the first 4
R162: ok, can you see any reason why if it works for 4 it should work for 5?
A162: can I see any reason?
R163: umhmm
A163: yes
R164: what kind of reason?
A164: because it seem to be describing the way to figure out the maximum number of slices that you'd get like 8 is the maximum like it gives you it actually gives you that why? I don't know it's just a sense I don't under- -- It's hard to answer why It'd be a guess because it works for the first four and there seem to be a pattern involved I guess that I expect that the pattern will continue but then like when I think about like when you do something like infinite cuts
R165: hmm we'll deal with finite numbers of cuts
A165: ok well yeah yes I think so yes
R166: can you explain why you think so?
A166: um it's hard to explain --- because of the nature of pancakes and cutting? [laughs]

R167: ok that's an interesting idea
A167: because because because there's a pattern you can see it when you look at the pictures
R168: well how does the pattern work?
A168: each time that you add another line you are bringing it up sort of x number of cuts
R169: ok
A169: but then it it doesn't change its nothing weird happens each time you add a line there's only so many more pieces you can add its it doesn't seen that anything unexpected is going to happen when you get to 50 slices
R170: ok lets look at this pancake that you sliced up in nice parallel lines here
A170: ok
R171: if you added another cut
A171: uhhuh
R172: how would you put a cut so that you would get the most number of new pieces
A172: oh what an interesting thing so you get the most number of new pieces
R173: umhmm
A173: put one right in the middle
R174: ok what happened to the number of pieces?
A174: it's just doubled
R175: ok why would it double?
A175: because I just split the number of pieces in half making 2 a whole new set
R176: ok
A176: so you'd be looking at 2 sets I took this one set of of 4 and I made it two sets of 4
R177: if you had a pancake cut up any way at all and you cut it again could you ever more than double the number of pieces?
A177: no
R178: ok
A178: no you couldn't which is why this 2 K pieces works
R179: ok could you explain in a bit more detail why it works rather than just saying that whatever I just said is why it works
[laughs]
A179: slave driver ok because ok because 2 to the k is in fact doubling your number right?
R180: yeah if you're going from 2 to the k and then you're increasing k by 1 then you're doubling
A180: and so what you're saying in this for k cuts of pancake never gives you 2k pieces you're just saying what you just said to me and what we said which is that you can never more than double it
R181: ok the statement that you can never more than double it applies specifically to if you've got a pancake and you just put one cut

A181: yeah
R182: how can we generalize that to this statement which says that whatever k is
A182: ok
R183: does it follow?
A183: yeah because you're adding them 1 cut at a time um it does follow I don't know how to explain it um when no matter how many times you slice it you can't do more than double what you already have like if you even if you if you slice it 8 times you can't get more than that 16 and you can look at that by cutting if you're trying to get maximum slices the cake The number of cuts of pancake doesn't really matter
R184: why not?
A184: its the same sort of it's the same relationship over and over it doesn't matter if it's two slices or fifteen slices when you look at it when you look at sort of the pattern of what goes on when you cut if you look at maybe I don't know look at it as pancake, or cake and slicing it doesn't there's a sort of a maximum number that you can of slices that you can cut actually like if you I mean I don't know I'm having a hard time describing it what is it that you want me to try and say?
R185: that would be telling
[laughs]
A185: oh no Am I supposed to be describing

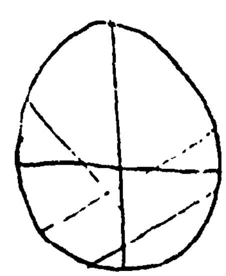

Figure 46: R's sketching at line R189. The two parallel lines were added later by A.

R186: maybe I'll confuse you a little bit instead um you said that for three cuts the maximum number of pieces was 6
A186: yeah, uh, no, did I? no I didn't
R187: you did. did you mean that?
A187: No I didn't I meant 8
R188: you meant 8
A188: or no 9 do I mean 9 or 8? hold on, let me see if you slice it 3 times there we go 1 2 3 4 no its not 1 2 3

yeah I guess it is 6 which makes sense what am I saying? Of course it makes sense because doubled 3 doubled is 6
R189: How many pieces?
A189: [laughs] Those aren't terribly straight lines [laughs]
R190: you can redraw it
A190: 7 it's 7 7 pieces
R191: ok
A191: hmm
R192: now, does that cause a problem with this statement?
A192: yes
R193: how? what is 2 to the third power?
A193: 2 to the third power is 6 2 times 2 is 4 no it's 8
R194: ok so it still says this is still within this range
A194: yeah but let's see now if we did something wonko with this kind of like um we did another line and we did like --- this right
R195: ok
A195: so that's 4 k equals 4 and that's 1 2 3 4 5 6 7 8 9 10 and 2 to the 4 is 2 times 2 is 4 times 2 is 8 times 2 is 16 hmm
R196: so we're still ok
A196: still ok
R197: but that doesn't necessarily prove anything
A197: no It might be that you'd have to get pretty damn creative about the way you cut
R198: ok if you got a creative as you could possibly get what would be the maximum increase in the number of pieces?
A199: the maximum number you could
R200: if you add another cut how much could you increase the number of pieces by?
A200: maximumly?
R201: yeah
A201: it would be if you could do an incredibly crazy cut and divide, bisect everything once and thus double the number of slices of pie
R202: ok
A202: would be the maximum I think
R203: ok the maximum would be double
A203: yeah
R204: ok if you start of with a pancake
A204: but you couldn't do that with a straight line in this case
R205: um not in this case no but perhaps you could if we had arranged the lines differently to begin with
A205: yeah
R206: but could you ever more than double it?
A206: then you'd have to go through some sections more than twice
R207: can you do that with a straight line?

A207: Here I don't know if it's a very strangely sort of shaped section I suppose you could right? no you couldn't no
R208: ok so on the first cut what would be the maximum number of pieces you could make?
A208: 2
R209: ok because you've got 1 piece to begin with
A209: right
R210: when you haven't made any slices you've got one piece so we do a cut I don't really care where it is
A210: yeah
R211: so after one cut we've got at most 2 after 2 cut what would be the maximum?
A211: 4
R212: after three what would be the maximum?
A212: um --- 5 no [laughs] not Fibonacci numbers ok
R213: we had four pieces
A213: 6 no 7 we've got 7 we've even got seven with that but maybe even more
R214: theoretically what would be the theoretical maximum
A214: theoretical maximum?
R215: I mean we start we've got 4 pieces from 2 cuts
A215: 6 but we know we can do more than that we can double 3 9
R216: what are we doubling when we do a new cut?
A216: the number of slices 8
R217: 8 ok so right this is cuts and this is slices

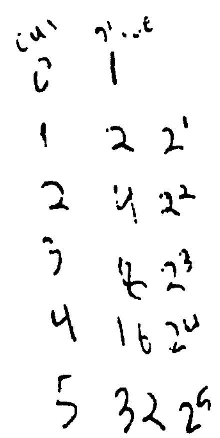

Figure 47: Chart made by R at lines R208-R220

A217: ok how about for 4? 12
R218: how do you get 12?
A218: oh sorry it looked like a 6 from here [the 8 from 3 cuts] 16
R219: ok and for 5?
A219: 32
R220: could I keep doing this?

A220: yes?
R221: how long?
A221: until we ended up not having any possibility of making any more slices because we'd run flat out of pancake
R222: ok, fine so could you use this to show that you could never have more than 2 to the power k pieces?
A222: I imagine you could, yes
R223: because 2 is 2 to the first power and 4 is two times 2 is two to the second power
A223: 2 times 2 times 2 is 2 to the third
R224: because each time we just multiply by 2
A224: yeah 2 to the fourth 2 to the fifth yeah
R230: If we do something analogous in this case to adding another slice and seeing what happens,
A230: umhmm
R231: In this case it would be adding the next Fibonacci number, and seeing what would happen. Here we're going from, say when we went from the 4 case to the 5 case
A231: right
R232: we were adding another slice
A232: right
R233: and it doubled
A233: right
R234: here when we're going from say 4 to 5, we're going from adding up the first four to adding up the first five.
A234: right
R235: and we're looking at how the sum changes, when we add up the first four we get 7
A235: right
R236: which happens to be 8 minus 1, we notice, which is neat because that means that this works out for 4, great.
A236: right
R237: I should put the plus signs in just so we can tell what we're doing
A237: and we add those up

$$1+1+2+3 = 7 = 8-1$$

$$\begin{aligned} 1+1+2+3+5 &= 7+5 \\ &= 8-1+5 \\ &= 5+8-1 \\ &= 13-1 \end{aligned}$$

$$\begin{aligned} 1+1+2+3+5+8 &= 8+13-1 \\ & \quad 21-1 \end{aligned}$$

$$\begin{aligned} \ldots +13 &= 21+13-1 \\ & \quad 34-1 \end{aligned}$$

Figure 48: R's chain of implications written about line R238

R238: Now we can add these up a couple of ways. we can add them up by going 1
A238: you get 12
R239: plus 1 is 2 and da da da da da
A239: or we can do 7 plus 5 is 12
R240: or we can do 7 plus 5 is 12. ok. Now where did that 7 come from? that 7 was
A240: the sum of the first four
R241: uhhuh and it was also 8 minus 1
A241: yeah
R242: ok. Let me rearrange this a little bit. [writing] like that.
A242: ok
R243: Now, what's interesting about 5 and 8? anything?
A243: Added up together they equal 13
R244: Ok what's interesting about 5 and 8 and 13?
A244: 13 is, 13 is the seventh Fibonacci number!
R245: Does it have anything at all to do with 5 and 8?

A245: yes, because, yes, this is the rule we did way back then. yes, now I remember. yes because 5 plus 8 gives the third Fibonacci number.
R247: ok, let's see what happens, now that we know that this works, if we add the next one on. --- we add it on there and we'll add it on here. ---
A247: you want to see what happens. like you add it all up and you see what happens?
R248: No no no no. what happens specifically when we add those two?
A248: the 13?
R249: umhmm
A249: we get 21, and 21 is, lo and behold, 8 plus 13, which means the seventh Fibonacci number plus the eighth Fibonacci number gives us the ninth Fibonacci number.
R250: ok, so what would
A250: which minus itself gives us, minus 1 gives us the sum of
R251: umhmm
A251: the first eight.
R252: right
A252: so the first, ok,
R253: if we look at
A253: and that's the eighth, is that the seventh? so the sum of the the seventh, did we say that 13 plus 8 was 13, so the sixth and the seventh the sum of then, gives us the, added up minus 1 gives you the sum of the first six
R255: if we add on the next one here, whatever the next Fibonacci number is, it's 13, so we have this whole mess, da da da, plus 13 should give us,
A255: let me see let me see, let me tell you should give us, what was this 21? all right, 21, plus 34. Oh sorry 21 plus 13
R256: ok because we added 13 to
R257: this junk over here so we should just add 13 to this junk over here and it'll still be equal
A257: that's right, and minus 1
R258: but we happen to know these two because they're consecutive Fibonacci numbers are going to give us the next Fibonacci number
A258: right which means you have to add 34 on to the other side
R259: no, the 34 was
A259: oh right
R260: 21 plus 13, it was already there
A260: but if you go up another one you have to
R261: yeah, when we go up to the next one we'll add 21 to there and we'll add 21 here and 21 plus 34 will give us the next Fibonacci number
A261: yeah
R262: and then the next one we'll add a Fibonacci number

here, we add a Fibonacci number there and we get the next one

A became distracted for a moment at this point and missed R's description of the general case.

If $1+1+2+\cdots F_K = F_{(K+2)}-1$

$$1+1+2+\cdots F_K + F_{K+1} = \underbrace{F_{K+1}+F_{K+2}}-1$$

then $1+1+2+\cdots F_K + F_{K+1} = F_{K+3}-1$

$$F_{K+1+2}-1$$

$$F_K + F_{K+1} = F_{K+2}$$

Figure 49: R's induction step at line R267

R267: OK, what I'm doing here is I'm following this pattern that we were developing here. We had some Fibonacci numbers, and we figured that they were equal to some number
A268: right
R268: and then we found that when we added on the next Fibonacci number, and the some number we had over here tot eh same Fibonacci number to make it all nice and equal
A269: we got that, and it worked
R269: we get the next Fibonacci number
A270: right
R270: because we happen to have two consecutive Fibonacci numbers over here
A271: yeah
R271: Now what this says is, say we've got a bunch of Fibonacci numbers
A272: to K yeah
R272: and they happen to add up to the next Fibonacci number 2 down

A273: Oh what, that 2 that 2
R273: that 2, this is just saying OK we go up 2
A274: OK Ok , see I was thinking
R274: from the Kth Fibonacci number
A275: I was thinking it was F K plus 2
R275: Ah no, this is all down here
A276: ok great
R276: this is just like 2 beyond that one
A277: right
R277: ok
A278: what's, minus 1, right.
R278: minus 1, just like it always has been. now we add the next one on, so that's K plus
A279: K plus 1
R279: 1, and we add it on to the other side as well
A280: right
R280: to make them, because if this is equal to this and we add the same thing to it then it will still be equal to that
A281: right
R281: if we add different things, we'll be in trouble so we'd better add the same thing. So we've added the same thing here
A282: right
R282: and then what do these two add up to?
A283: F K plus 3
R283: Ah Ha! ---
A284: minus 1
R284: right, so what we've shown is that if
A285: the sum of
R285: Uhhuh, you were saying?
A286: That the sum of the Fibonacci numbers to F K plus 1, we could just as easily call it F K
R286: ok, but I want to keep F K being this particular
A287: ok,
R287: F K whatever it is
A288: added up is F K plus 3
R288: ok, now
A289: minus 1
R289: yeah, that works, if this works. Do you see that kind of relationship?
A290: Yeah because you've just made, this was completely balanced so this works and therefore this works, you just done, you haven't changed it you just added that to both sides, it's equal, but then that actually equals this and if that equals this then that works because this works, yes
R290: ok, so this says that if we know, if we figure out somehow, that this silly statement I wrote is true for a given Fibonacci number K, then it will also work for the next one. Is that OK? Because this is the Fibonacci 2 beyond the last one here.

A291: right, Yeah
R291: ok, so we've shown that if it works for a Fibonacci number than it works for the next Fibonacci number
A292: That shows that it works for, that the 2, that the second one after though, not the next one but the one following that.
R292: How do you mean?
A293: well, it's because, we're dealing with like, the Fibonacci number, but the third one like
R293: ok
A294: not the next one
R294: But we want the second, we want
A295: but that's because we want the second one over here
R295: we want the second one after this one, now
A296: oh, right
R296: see we're looking at the next Fibonacci number, we've added on another Fibonacci number
A297: oh right
R297: to this side, so now we want to move from
A298: but where is, wait, we,
R298: the second after
A299: But we don't have the F K plus 2
R299: Ok, well we, what we have here is actually F K whoops, that's an F, K plus 1, because that's the last one in the list
A300: right, oh yeah
R300: right, plus 2
A301: Right. Plus F K plus 2, plus F, yeah, right, Got it
R301: ok
A302: ok yeah
R302: Now this algebraically shows that if we know it for one Fibonacci number then we know it for the next Fibonacci number, and we know that we know it for all these little ones that we've been playing with, can we then say that we know it for all Fibonacci numbers?
A303: --- --- My heart tells me to say yes.
R303: OK, your heart tells you to say yes. Well why is your heart telling you to say yes? is the next question
A304: Because what you've shown here, and what I've seen, and what we've been discussing, is a series of relationships, and this is a series of relationships, that makes up Fibonacci numbers, it makes them what they are
R304: OK
A305: So if you're doing, this is, yeah, this is the way they work, this is the way they develop, so I would say, that's why I would say, for any Fibonacci number it's nature, it's being a Fibonacci number, means that it will work along these these things that are in its nature as a Fibonacci number
R305: That sounds very mystical

[laughter]
A306: I know
R306: If I know that this statement works, well we worked it out for the sixth
A307: Fibonacci number and we have worked it out, we've worked these out all the way to the 32nd Fibonacci number, before
R307: OK, we worked it out to the 32nd Fibonacci number, does this kind of argument say that it should be true for the 33rd Fibonacci number?
A308: Yes
R308: So we could say that we know it for the 33rd Fibonacci number
A309: Yeah, and then we could say that we know it for the 34th, because we know it for the 33rd, we know it for the 34th, and if we know it for the 34th, then we know it for the 35th, and if we know it for the 35th, then we know it for the 36th, and so on, and so on, and so on, and so on, and so on
R309: Can we say that it works for every Fibonacci number?
A310: Yes
R310: Why?
A311: How do I explain that? because once, we've, we've actually worked it out, ---, well, I mean, it couldn't have just, I mean there obviously a thing that it could just change for some reason, something could just go wrong in the whole pattern in question and it could just not work
R311: ok and then we wouldn't know that
A312: no
R312: Do you think
A313: But, No, It doesn't seem logically
R313: that this ever goes wrong?
A314: It doesn't seem logically that it should though, because ---
R314: Well where would it go wrong?
A315: Ok, if, ok at the beginning here
R315: ok
A316: this, this makes absolute sense because it has,it sort of the basic way that Fibonacci numbers work, Now here, this F K, adding it to both sides. It wouldn't go wrong there because you're just adding a number to both sides, it's not, you're not changing the first one,
R316: ok
A317: ok, and this third one is just, is just a growth from the first, the second one. So if it, it would have to go wrong anywhere, it's going to go wrong, it would have to go wrong right at the root of the problem up up here.
R317: ok with the 'if we knew it for the first Fibonacci number, then we know it for the second'. The thing that could be wrong is that we didn't know it for the first

Fibonacci number
A318: Yeah
R318: ok we kind of have. We've got
A319: OK so if it's going to go wrong , it'll go wrong at the beginning
R319: this says, if this then
A320: But it won't, it won't go wrong at the beginning, because the beginning just describes what Fibonacci numbers are. ok So then that, The first one is the description of what Fibonacci numbers are, how they, how the pattern evolves the Fibonacci numbers and how they are going to develop
A322: so if it's going to go wrong because, the Fibonacci numbers don't work the way we think they do
A323: at all
R323: ok so if this was going to go wrong then this whole thing would have to not be true at some point
A324: yeah
R324: we know that it's true for
A325: a lot of Fibonacci numbers
R325: a lot of Fibonacci numbers and from that
A326: and we think it's true for all Fibonacci numbers
R326: Ok why do we think it's true for all Fibonacci numbers?
A327: Because, Because --- I don't know, to me just for, at a certain point it just comes down to trust.
[laughter]
R327: what are you trusting in?
A328: I'm trusting in that this development. Like, I'm trusting in the fact that it always grows in this particular manner charted
R328: ok
A329: we've gone through them all, they follow this pattern, OK but you trust that they will just continue to follow this pattern, that I mean, that somehow randomness or some crazy fluke of nature won't happen so that, for some reason, like Fibonacci number 17 million decides that it's going to do something completely wonky because it just stops working for whatever reason.
R330: The pattern that we're looking at, that we're talking about with the Fibonacci numbers is that of you add up a Fibonacci number and the next one
A331: it gives you
R331: the one after that
A332: the one after that yeah
R333: so as long as that works for all Fibonacci numbers
A333: then this is true
R334: then this is true
A334: yeah
R335: ok. That's the definition of the Fibonacci numbers so
A335: Yes it is
R336: it better bloody work for all of them

A336: exactly exactly, that's what I mean, but there is a bit of trust in there, don't you think?
R337: well, yeah I suppose so. --- ok
A337: but then if it does stop working then it's not a Fibonacci number, and if it's not a Fibonacci number then it doesn't have to work along according to that pattern.
R338: That sounds good to me
A338: It's a tautology isn't it?
R339: Is there anything similar, or anything, well there's lot's of things different, in this adding another slice to the pancake, and, or at least, is there anything similar, here we have an argument that said 'e never get more than 2 to the power K pieces
A339: umhmm
R340: and here we have an argument which says that this weird thing about Fibonacci numbers works.
A340: umhmm
R341: Is there anything related about the way our arguments went? ---Do you even remember how the argument went?
A341: yeah--- because, let's see, --- because, in both cases we started with the very simplest examples
R342: ok
A342: and we, we made some statements that were true about those very simple examples
R343: ok
A343: and they seemed to be true as they developed. Starting with our, one slice of our pancake and moving up to 2, and coming to the point that we noticed that 2 cuts of a pancake never gives you more that 4 slices, and how do you write that? and what does that mean. And in a similar way we started with the Fibonacci numbers because we started with our first Fibonacci number, and I way way back, and we started very simple and started building up
A344: until we started trying to, starting from the simple, building up to much more complicated, or not more complicated but more bigger ideas
R345: How did we go from the very simple cases to <u>everything</u> which is, a bit of a jump.
A345: Trust, No. Well basically what we did was we proved, well in the case of the Fibonacci numbers, we, we showed that the things that we, everything that was bigger that we did was just simply changing the way we made our first statement --- like these things are all developments of the first statement
R346: ok
A346: taking this, and reworking it and twisting it around into other things --- but it's all based on that
R347: ok, this transition from if we know it for K then we know it for K plus 1 is based on that
A347: yeah

R348: and here we had an argument that was based on cutting it, that if
A348: double it
R349: knew it for one, we could never more than double it so
A349: right
R350: so we would know it for the next one
A350: yeah
R351: ok, how did we go from those specific thing, if we knew it for one we knew it for the next one to the everything statement?
A351: ------ because, I don't know how to answer that.
R352: ok
A352: I mean I probably, if I really though about it for a while, that to me is the biggest thing, it's the hardest thing to describe in everything, it always comes down to you asking me is this for it all and I always have to give you this this vague answer because it's, I have a hard, I think it's very hard to explain that leap. In some ways it seem very simple but in some ways I find it very tricky
R353: If I said show that this is true for the 328th Fibonacci number
A353: I could
R354: Could you do it?
A354: I could show it, yeah
R355: How would you do it?
A355: I would just, If I wanted to do it the painstaking way with a calculator work it all out.
R356: OK, do you have to?
A356: NO, because I could Just fool around with the numbers and and actually knowing a few, a couple of numbers I could know the sum of all the numbers and figure it out
R357: ok
A357: prove it
R358: Do you know this for the sixth Fibonacci number?
A358: yes
R359: then do you know it for the seventh Fibonacci number?
A359: Yes and the eighth, yes yes and that's how we come to this belief in it going on forever
R360: ok
A360: and the same thing with the slices of pie. You know it for the first, so then you know it for the second, and you know it, therefore you know it for the third, therefore you know it for the fourth
R361: Therefore you know it for all of them
A361: Therefore you know it for all of them
R362: ok, so you can always make that, you do it one step at a time?
A362: yeah
R363: and you can work one step at a time from the little tiny cases
A363: all the way up to the higher ones, yeah

R364: ok, That's called the principle of mathematical induction
A364: Is it?
R365: It's very big in math
A365: yeah?
R366: yeah
A366: hmm, But you see to me there's a certain, a certain, I mean, maybe it's wrong, but I always, maybe it because of my, not being a mathematical person but I always think that some things just, I, I don't look for that, I don't look for that moment of irrationality where it just doesn't work, that inconsistancy, I find it incredibly to say that it would work, for all of them because you go from one to the other and to the other, somehow that doesn't prove it to me.
A367: Because practically, here on paper, in the short term, but
R368: Do you think it doesn't prove it to you because it, the same kind of argument wouldn't work in
A368: yeah
R369: real life for things like the sun.

A discussion of empirical induction followed. A say that in the case of empirical induction there is no link between each consecutive event. She also saw that nothing unusual could happen in the case of MI, because of the link established between each step. It was also pointed out to her that the logic used in everyday life must account for errors in perception, while the logic of mathematics can be precise.

R383: that's that. I had one actual question. A couple of times when we were doing the last one, you were looking at, trying to get. Well, you had this written down basically, and you were fooling around with it, and you said that you were trying to get it to 'prove itself'
A383: umhmm
R384: What does it mean for a mathematical statement to 'prove itself' or do you know what you meant by that, or were you just saying it?
A384: I was desperately trying to think back to my calculus classes, and the proof she was writing on the board, in which she would take this and then build down from it, or build up from it, and it would prove itself. And that, starting from this she could, she would work it [in a search?] in such a way, what she did to this formula would, it would prove that this formula worked, that's what I meant.
R385: How would it prove itself?
A385: in this, in the course of this calculation, form this flipping things around in the formula, putting something here, or adding something, I don't know,

compare, connecting it, substituting another formula into it, which is what we eventually did, [we used another one?], that the formula would prove itself, but in fact, it's not proving itself. It, You're proving it using other, we, working it with something else, or with other elements of it, or, but then I guess, no

R386: What would be the final line in that kind of a proof? What would it look like?

A386: the final line?

R387: yeah

A387: like, the line you started with.

R388: The line that you ended up with that would, that, your teacher would stop at that point and say 'there'.

A388: I don't know, I don't remember, but I would imagine that it would be the line that she started with, at the very beginning, when she started the

R389: ok

A389: [working with it?] the base rc^t of it, which is what's been happening sort of to us here, with the Fibonacci numbers, like basically what I remember when we were working on, when I sort of had to prove it eventually what I came to was I, I think I managed to take it and by working with things that I had learned about Fibonacci numbers and putting them in with the formula that I had, that it actually came down to it could be simply, I think I simplified it down to, almost our first, this is what Fibonacci numbers are, equation.

R then pointed out the similarity between A's proof for the induction step in session 2, and the general proof of the induction step. The general plan of stage 2 was also described, and the session was declared finished. After some conversation, however, A caught sight of the Table of the B sequence and asked about it. The session resumed.

R406: can you find the pattern in the B sequence?

A406: it exactly like the Fibonacci numbers up to the 5

R407: ok

A407: and then something happens --- yeah, let's see--- yeah

R408: What's the pattern?

A408: --- whoops, wait, doesn't quite work ------- oh Ok, --- the sum of the B sequence, up to a even number, even numbered B sequence sequencer, gives you a, the following number. If you work up to a, the sum of up to an odd number then it gives you the following number, I think plus 1

R409: Can you give me an example?

A409: let's see like, for instance, B4 it'd be the first two, ok, their equals up to 2, so the, so it's 2 plus 1 is equal to 3

R410: ok

A410: and now the first three, plus 3 gives you 5, right?
R411: ok
A411: then the next, 4, 5 6 7 8 9 10, gives you 10 plus 1 is 11, and the next, 5, so you need to plus 1 for that,
R412: umhmm
A412: it's an odd, it was an even, and for the 5 you've got 11 plus 5 is 16 17 18 19 20 21, gives you it right on the nose so now we should expect that, 21 --- plus well, all these added together, 21 plus 21 is 43, 44, 42, 42 plus 1
R413: umhmm
A413: so 43 plus 43 is 86 so it stops working there. because it becomes again plus 1. 86 plus 86, they're they're doubling. sort of. sometimes doubling sometimes you have to add 1
R414: or subtract 1
A414: or subtract 1
R415: with 86
A415: yeah

R then attempted to lead A to the relation $B_n = B_{n-1} + 2B_{n-2}$. He eventually succeeded and the session ended.

First Session With B

The session began with a discussion of the definition of prime number. B was shown the statement: "$P = n^2 + n + 41$" and asked to try some values for n. She tried small numbers of which she knew the squares.

B9: --- so is this an exercise in this formula everything comes out as a prime number?
R11b: well, are all of those prime numbers?
B10: I don't know, I think those are --- so far --- try something big ---
R12b: You might want to do that one on the calculator because I don't know what 61 squared is
B11: How do you square numbers?
R13b: There's a button here somewhere that just does it, there
B12: --- a pattern is emerging.
R14b: Can you draw a conclusion?
B13: A number squared plus a number plus 41 all would have to be prime numbers. So far they have.
[laughter]
R15b: Do you think it works in general?
B14: um, I, probably, I don't know
R16b: what would you do if you had to be more sure?
B15: uh, myself, I have no idea. Presumably you could plug this into a computer and get it to do it for you.
[laughter]

B16: I could sit around for hours and plug in many numbers
R17b: ok That's about it for those, for the moment, and we'll shift to something else. Do you know what Fibonacci numbers are?
B17: No
R18b: A hasn't been talking about Fibonacci numbers? I've been hitting her over the head with Fibonacci numbers. let's see here, Fibonacci numbers, yeah I was going to give a history. A guy named Leonardo Fibonacci who lived in Pisa in the 13th century came up with these things, something to do with the breeding habits of rabbits. the exact details of which I forget right now, but he ended up when working with rabbits finding this sequence of numbers, here. These are the first 35, it does continue on after that. Can you find any kind of pattern in that sequence of numbers?
B18: --- The first numbers added equals the third, and those two equal the fourth number and so on and so forth
R19b: Ok, so if you wanted to give somebody a definite rule for producing the Fibonacci numbers, how would you phrase that?
B19: --- I don't know, in mathematical terms? like
R20b: well just
B20: just in general
R21b: any way if you were just explaining to somebody on the telephone how to write down the Fibonacci numbers and you didn't want to tell them what they were
B21: --- The first number plus the second number equals the third number; the second number plus the third number equals the fourth number --- And the third one plus the fourth equals the fifth and so on and so forth

The notation F_n for the n^{th} Fibonacci number was then introduced and discussed. After being shown F_{n+1} as the notation for the Fibonacci after the n^{th} B was asked how the next Fibonacci number would be represented, She answered correctly F_{n+2}.

R27b: ok, that sounds great. We don't know what it is, and it says it comes after F N plus 1 so that's all you really want. Can you think of a way to write the rule for producing the Fibonacci numbers using F N and F N plus 1?
B27: --- do something like F N plus F N plus 1 equals F N plus 2
R28b: Ok that sounds good. Do you want to write it down somewhere just in case we need to talk about it later? ---[garbled ???] ---
B28: I can't even see where the lines are going --- umhmm, they're all even numbers
R29b: do you think that's true for all the Fibonacci numbers when N's a multiple of 3?

B29: Up to 35 they are
R30b: ok, why do you think that's so?
B30: --- because the, when you add two odd numbers together you make an even number so the, if it were not a multiple the, the first two numbers are odd and so when they add up to the one that's a multiple of three then they become even
R31b: ok, that sounds great, Are you pretty sure that it's true for all the Fibonacci numbers
B31: No
R32b: What would make you more sure?
B32: Probably if I read it in a textbook somewhere and hey told me it was. [laughter]
R33b: ok, let's skip on to yet another question.
B33: --- [long pause] --- The Fibonacci, If N is prime then the Fibonacci number is also a prime number.
R34b: ok, Do you think that's true for every Fibonacci number when N is prime?
B34: I'll say yes
R35b: Ok, Why?
B35: Well we're working on a pattern here ---
R36b: ok, how could you be more confident that that was the case?
B36: I don't know --- ---
R37b: How did you become as confident as you are now?
B37: --- I don't know
[laughter]
R38b: Well reflect on the kind of thinking you were doing. From never having thought about it before now you you think it's true for every Fibonacci number. That's a change
B38: --- I guess something to do with numbers seem to go in patterns because you're always working with formulas to ---
R39b: ok, have you seen a pattern here?
B39: yes
R40b: What kind of a pattern?
B40: That they seem to be working in just the prime numbers seem to correspond
R41b: ok
B41: well, I mean, that's as far as I get anyway which is about here
R42b: ok, looking at the three things that we've looked at here. Are there any that you feel more or less confident about? compared to the other ones?
B42: --- Now I've just gone and forgotten what it meant. All right, ok. I think I feel more confident about the, the F N when N is a
multiple of 3.
R43b: why?
B43: because I think, a long long time ago I remember being told that if you add two odd numbers you get an even

number
R44b: can you think of any reason why that should be so?
B44: No
R45b: ok
B45: --- I have a really difficult time reasoning out mathematics. ---but they do seem to develop patterns
R46b: ok have a look at this one again. What do you get if you plug in 40 for that one?
B46: --- oh my, it doesn't work.
R47b: Would you now say that that formula only produces prime numbers?
B47: No
R48b: Why not?
B48: Because it didn't work for this one
R49b: ok, have a look at this one again. What happens if you check the 19th prime number?
B49: there we go
R50b: there we go what?
B50: it the same as this formula
R51b: is it?
B51: it's the same number
R52b: I don't think so
B52: --- No it's not
R53b: You're determined to find some patterns here somewhere
B53: I am yeah
R54b: Do you get a prime number?
B54: No
R55b: What would you say now about this statement? would you say that there's something special about F N when N is a prime number? or what would you say, if N is a prime number?
B55: --- well, I'd love to find another pattern here [laughter] I can't --- can't say anything about it I suppose. Or is this an exception, or presumably if there is one exception there will be others.
R56b: How about the second statement about when N is a multiple of 3?
B56: This is the one I liked most
R57b: yes, but would you still say that that's true for every Fibonacci number when N is a multiple of 3?
B57: I'll sav that until I'm proved wrong
R58b: ok do you think that, it's possible that you would be proved wrong, likely that you would be proved wrong?
B58: yes
R59b: Why would you be inclined to stick to it?
B59: --- Because of the, it seem to me that the, it still works out as a, with the two, two even, odd numbers adding up together to become an even number would still work in most cases, well yeah, would still work
R60b: ok is that kind of a pattern different from the pattern you found with the other two?
B60: --- It is because this is, because I could see the

pattern outside of just the denotation whereas with this one I was looking, here I couldn't figure out why, sorry this isn't making sense now, ---when N is a multiple of 3 I could see the pattern in the, in the actual numbers instead of just looking at the charts and, do you understand?

R61b: Yeah

B61: Yeah?

R62b: what would you say about trying to find out if a mathematical statement is true or not?

B62: what would I say about what?

R63b: about how you would go about trying to figure out whether something was true when you make a conjecture, like you made conjectures here about these three questions,

B63: --- I would say you would have, have to actually have a, --- be able to, to see, with these ones it's basically a matter of plugging in numbers and, and seeing what you come out with, and looking at those, at those numbers but, there's, I never actually figured out why it was doing that, but just that, that's the way it worked, whereas with this one it has another, --- I don't know, backed up by another rule? I, but it, --- you actually have to understand why, why it, it is doing that, --- for, like for example the N 2 plus N plus 41, I had no, I didn't know why it was doing that it just, why the, the numbers seemed to be working out to prime numbers they just were doing that. I didn't know why. Whereas this one, at least I think I know why, the multiples of three work out to be even because the the other two, when you add the Fibonacci numbers the other two are odd and then so it would come out to be even.

R64b: How do you know the other two are going to be odd?

B64: I don't --- that again is looking at the little charts and they seem to work out that way ---

R65b: So you've made . conjecture that, the two Fibonacci numbers before one that is a multiple of three will both be odd

B65: because, no, because you, if each Fibonacci number is the first one plus the second one equals the third one, --- the first, it starts out, well, then you would be adding two odd numbers together and get an even number, and then you add, oh, that's the same thing, I see, you'd say, then the next one then is odd, so you'd add that to the even and then you'd come out to another odd, but then I don't necessarily know that the, that the next number after an even number would be odd so ---

R66b: can you think of any reason why the next one after an even number should be odd?

B66: --- because the one before the even number was odd

R67b: How does that make the one after the even number odd?
B67: Because if you add an even number to an odd number then it comes out as odd
R68b: ok, how do you know that if you add two odd numbers together you get an even number?
B68: I don't know, I just, again something I was taught in grade 1, --- it seems to be the way it works, again I'm making, I haven't tested out every single odd number adding it to every other odd number, but, --- But then I guess you'd only have to, have to do the first 1 through 9, because that what there all going to end up with anyway.
R69b: Why do you just have to do the first 1 through 9?
B69: because anything after that will also end with 1 through 9
R70b: ok, do you have to do all of them from 1 through 9?
B70: --- I don't know. I'll say yes. It would make me feel better if I did it all for 1 through 9, I don't know if anyone else would have to.

The definition of "odd number" was discussed. B only knew which numbers were odd based on the final digit. She had learned which ones indicated even and which odd, but did not see divisibility by 2 as part of a definition.

R76b: this is kind of an interesting digression, but maybe I should go back to what I was originally going to talk about. What sort of justifications would you look for, What I want to do next is have you and A look at some statements together and try and figure out which statement are true and which ones are false. What kinds of things do you think you would try and do in that circumstance?
B76: --- I would try, to figure out why a statement would be true, for example these examples that you've given me, --- as I said before the ones that I was just plugging numbers into but I didn't know why that would work out so I'd have to try and figure out, why whatever statement I was making, or whatever pattern I was looking at would, would be the case.
R77b: What kind of an explanation would you need?
B77: --- I'm not sure ---
R78b: well would an explanation that something is true because it works for, all the numbers from 1 to 10 be sufficient?
B78: NO
R79b: why not
B79: because it didn't work for this one
R80b: ok, how about for 1 through 100?
B80: No, because again that's just plugging in numbers. I'd have to come to some sort of understanding, I have no idea how, why when you plug the numbers in it, it works

out to be whatever it works out to be. why that happens

R81b: ok. You say you have no idea how, How did you come to an explanation for this one?

B81: I don't know, I just made the, I just made the connection. It happened to work for that one. Whereas, for example, these ones I couldn't come up with a reason why that was working out

The session ended at this point.

Stage Two Session With A and B

A and B were instructed to work together, and to record whatever conclusions they reached on the paper provided. R was to remain quiet, serving only as a source of clarification of anything they had difficulty with. They were to take as long as they wished on each activity.

They began by reading the first proof. A asked about the meaning of 2 to the power n, and was answered by R.

A518: So P N is smaller than or equal to 2 N. We did this one before. Did you do this one before?

B102: No I don't think so.

A continued to read the proof. She had difficulty with the use of P_{n+1}. B also had a similar difficulty, wondering if P_{n+1} or P_n+1 was meant.

A524: So the pieces, and then every time you add a new one. That make sense.

R521: The N plus 1 is all one thing.

A525: is 2 time P N, it makes two more pieces every time? Is that what it means? ---

R522: It doubles the number of pieces each time.

A526: OK --- so we have this one little piece like this --- That makes it, now each time a new cut is made by a new line, the new line, cuts every piece into 2 pieces. That's right. Doubling, that's right. Cuts every piece into two pieces?

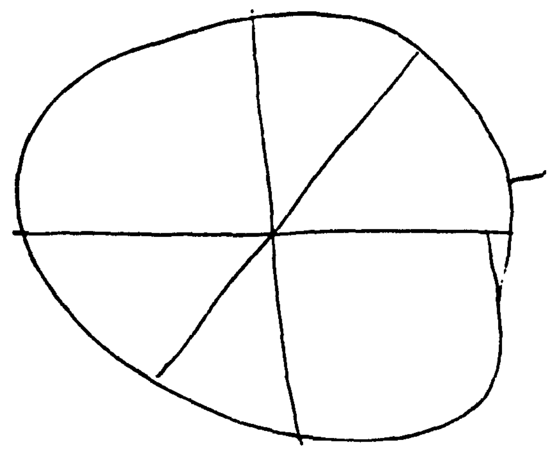

Figure 50: A's sketching at line A526

A was concerned that the line had not cut every piece into two pieces, but B pointed out that it didn't have to. A reread the proof and saw the relevance of the phrase "at most" They each rewrote the inequality, commenting that it was difficult to understand.

B115: And N is the total number of lines. Cut, right?
A540: Yeah --- and P N is the total number of pieces. --- Total pieces --- number of cuts --- [reading] Each of the new cuts made by a new line at most cuts each piece into two pieces. So the P N is the total cuts and you add another one, and it's going to be equal to 2 --- doubling, doubling the number of pieces. That makes sense right, you get this one?
B116: When, I still don't really understand the N plus 1
A541: Well look at this, basically you have, your number of pieces
A542: right, of your pie. Now every time you add one,
A543: one more cut, so that's plus 1. you're going to get double the number fo pieces, cause the most,
B119: Yeah
A544: So that's what the 2 P N
B120: So 2 P N plus 1 is that
A545: That's the number of pieces plus one more cut.
B121: So the number of pieces
B122: I think the plus, the number of pieces
A546: I think that's the one more cut
A547: I think that's P N
B124: OK but say
A548: instead of saying P N plus --- That's plus 1 more cut, I think . It gives you double the number of pieces.
A548a: It gives you as, it will give you as much or less than double the number fo pieces, at the maximum it will give you the double of the number of pieces.

B126: OK ---
A549: So then if your number of pieces --- is, back up, if your number of pieces then is maximumly smaller than or equal to this 2 to the N we said up here
A550: then this this P to the N
B128: plus 1
A551: Yeah this thing here that we decided that we'd agree on.
A552: right, is going to be smaller
B130: Than
A553: than double the ---
B131: than the pieces doubled
A554: which is, yeah, right
B132: which is right

At this point A asked about the inequality involving exponents. She saw no connection between it and the explanation which preceded it. R went through each step of the inequality explaining the meaning of each one in words. B wondered if the inequalities were transitive. R explained that they were. A and B then rewrote the combined inequality as four separate inequalities. B decided that the inequalities were transitive in their separated state. She felt, however, that the possibility of equality was lost if two inequalities were combined.

B152: Well no I suppose, yeah that's right it can be because if this is if this is equal to this and then this can be equal to this also then OK it can be.
A585: OK
B153: Yeah, well that makes sense.
A586: But I don't understand is how
B154: if all of these things are true
A587: I don't understand how we get to the 2 to the N plus 1. I understand how we get to this statement, I'm understand how we get to this statement, and I'm not sure how we get to this statement .
B155: This?
A588: Yeah
B156: From that?
A589: yeah
B157: Why the 2
A590: Yeah
B158: P N is greater than
A591: 2 N plus 1
B159: Because it's greater than or equal to 2 to the 2 N which is just 2 N plus 1
A592: But I don't see how this becomes equal
B160: Isn't it?
A593: At the end it's become equal, it's not greater than or equal to, is equal to---
B161: Well this, 2 to the 2 N is equal to

A594: Is equal to
B162: Is equal to 2 to the N plus 1
A595: I don't understand that
B163: Isn't that just timsing it out? timing it?
A596: Is it? --- I think I get a little freaked out when they start playing with things in the exponents. --- up here. ---
B164: So that's 3 becomes 6 12 --- ---
A597: N plus 1 is the number of times
B165: Yeah so that's
A598: plus 1 so 1 more cut
B166: 2 time 2 time 2 say times 2
A599: Plus 1, where's the 1 at the end? so what you're saying here is
B167: Wait wait wait how does 2
A600: I'm going to just do it in cuts, like I understand, if that's OK
B168: umhmm
A601: So, we're going to do 1 cut. OK? So it's 2 to the power of 1 plus 1 which is 2 to the power of 2, right? Which is 4. Right? And 2 --- I'm not getting 4 out of this. Like I'm trying, If we do this with cuts what happens here? let's say let's do 2 cuts, well let's just do 1 cut.
B169: Well no do 3 cuts
A602: Ok 1 2 3
B170: OK
A603: So we have 2 to the power of N, N is the number of cuts for any number N, is the certain number of cuts, all right
B171: Yeah
A604: 2 to the power of 3 plus 1 is equal to 2 times 2 to the power of 3
B172: No, 2 to the
A605: yeah 3 cuts
B173: Wait what's the difference between 2 N or 2 N
A606: well 2 N is 2 times 3, 2 N is 2 times 3, 2 N is 2 times 2 times 2
B174: OK Is it?
R538: Yeah
A607: OK, so what we end up with then is --- Now if I remember correctly, I'm going to ask you about algebra this means that it's 2 to the power of 4
R539: Yeah, 3 plus 1 is 4
A608: That's what I thought, OK so that's 2 to the power of 4 is what? 2 times 2 is 4, 4 times 2 is 8, 8 times 2 is sixteen, --- and then 2, ok that's 2 4 that's 6 2, 2 times 2 is 4, 4 times 2 is 8, that's 8, eight times 2 is sixteen. So it works for this one.
B175: Umhmm
A609: Let's try it for 1, because I don't think it worked for 1 --- 2 times 1 plus 1, 2 to 1

B176: Yeah
A610: So that's equal to
R540: 4
B177: yeah 4
A611: equal to 4, yeah
B178: But, OK
A612: So at least I understand how it works
B179: So if it works for the example
A613: Now I understand how it works, at least, but now what are we trying to do?
B180: Yeah
A614: but what happens if we use 352? Should we try it with 352?
B181: No
A615: Do you have an exponent? Do you have something like that on there?
R541: Umhmm
A616: Is it E to the X [on calculator]
R542: No it's Y to the X
B182: --- Yeah but I kept doing this last time and I tried about 10 things and then He'd say well try this one.
A617: Exactly
B183: and it didn't work, so.
A618: Unless we were incredibly lucky
B184: Yeah
A619: Like we happened to find the one case where it doesn't work. OK so. So now what? So now what maybe we should try and do is something like, let's, let's fool around with this then OK? Let's say that, can we substitute from here, and here into here? ---
B185: Well see, It works logically, if you just move down, if you agree with all of these statements then it makes sense
A620: Yeah
B186: that this is, that this is the case
A621: Umhmm
B187: like if, ok if this is if if ---
A622: Cause I mean if
B188: P N plus 1 is smaller than or equal to 2 P N and 2 P N is smaller than or equal to 2
A623: It works
B189: Yeah and and then that is that then yeah
A624: this makes sense
B190: this makes sense, but it's just that you have to agree with every step along the way.
A625: Exactly
B191: I'm not totally sure how
A626: Yeah
B192: How that works. so,
A627: So is the
B193: so of the proof is --- is right then it makes --- yeah like the proof is perfectly logical I don't have any

problem with that
A628: Yeah
B194: It's just that how each part of the proof --- works
A629: Moves on to each other, yeah --- let's see --- 2 to 2 N would mean, If you multiply this times 2, we've got 2, 2 N here right? So let's say we go P N I just want to see
B195: Well presumably wait --- let's look back at this
A630: you do that and I'm just going to do this. And we'll confer
B196: OK
A631: Cause I just want to try this while I have the chance. --- ok then 2 time 2 time 2 --- this is right, going backwards, OK forget it.
B197: Yeah so I'm just thinking of --- if you're always doubling
A632: Umhmm
B198: Pieces --- for, like if you were always doubling pieces it would be equal.
A633: Umhmm
B199: like everything equal, But you're not a always doubling pieces because you're sometime you're only cutting through 2 pieces.
A634: Yeah
B200: so then it's less than or equal to
A635: Yeah
B201: So that makes sense
A636: you can go like this --- --- Should we just say we agree with it?
B202: Yeah
A637: For the sake of whatever. What's the next question?
R543: Why?
A638: Can you give an example where the statement is correct or an example where it is incorrect? It is correct for 1 and 2, 3 slices OK Number 3 do you agree with the proof? Why or why not? I agree with the proof, because it's logical.
B203: [laughing] Oh god, all right. No --- well the proof makes sense --- because, yeah, because each step
A639: Yeah you can substitute each into each, you can fool around with it. and it still works.
B204: Very coherent statement --- and because if you draw little diagrams with crepes and make it smaller and smaller it still works. --- but it works with what it said up here.
A640: I can't
B205: Or did you say it? how it doubles each piece?
A641: it says it up there
B206: Or --- yeah because it's the number of pieces produced it at most 2 times the number the line
A642: The cuts
B207: the cuts

A and B then tried a couple of unusual examples, A trying 4 lines, B considering the importance of the line being straight. R presented them with three lines forming 7 regions. They counted, checked that 7 is less than 2^3, and inquired as to why this example had been shown them. R explained that they had always been drawing figures and talking in a way which suggested that the number of regions was only double the number of lines. The example was intended to remind them that 2^n was needed, not just 2n.

A650: OK --- --- what would you say was the , Ok do we agree with this proof? or no? do we agree with the statement?
B212: Well the statement
A651: With this statement? Yes
B213: Why?
A652: Why? why or why not?
B214: Well we agree with the statement because we agree with the proof. Why do we agree with the proof?
A653: Because it seems to make, to be a logical proof.
A654: It seems to describe the situation. When we play around with it things seem to fit into each other. When we tried it for a few sample numbers it worked. --- I don't know why, why.
B216: well --- straight lines or something --- --- ---
A655: Let's try something here. --- --- --- Can we go on to number 2? --- --- Can you tell us when you've learned enough that we could, we could just leave it at where we are? Can you give an example where, well we couldn't give an example where the statement's ccrrect. Do you agree with the proof? --- The proof proves itself, whether it proves anything besides itself I don't know ---
B217: Yeah
A656: And that would be what's most problematic about it [laughs]
R548: What do you mean by that?
A656a: I mean that, What we're proving here -- I mean yes, we have proved that P N to, That this , I don't know. I feel like that we have proved this little statement here can be taken in and this can be all made true of it, of this statement. But whether this statement correctly I mean, --- I don't know --- ---
R549: Do you have a problem, what you are trying to prove is that statement?
A657: Yeah I know
R550: You've just said the proof works just fine
A658: For that statement
R551: If you assume that statement's true
A659: Yeah
B218: --- --- I suppose if this proof is trying to prove the statement then

A660: then that's right, then I agree. I don't know
B219: Well I don't know ---

A and B returned to reading the proof. For a moment they thought that the example of 3 lines forming 7 regions was a counter-example, but then saw that it wasn't. They continued to read.

A667: That makes sense logically to me. It really does.
B225: Well it does, at most it cuts every
A668: every piece
B226: At least if it's a straight line, I mean unless you're drawing squigglies
B227: Because it can cut it into --- 2
A670: I mean, cause how could it. it's going to cut into 2 pieces. --- yeah, if it's a straight line. --- cut it into two more pieces ---
B228: Yeah, I mean it'll never cut it into more than
A671: Cuts every piece into 2 pieces. I, I agree with that. That seems to me to be completely reasonable. with a straight line. and doubling, and then doubling the number of pieces. yes. yes. ---

They concluded that they agreed with the statement, but without being able to say why.

A673: What would you say is the most problematic thing about this proof? --- --- It's the sneaking suspicion that we're being had. [laughs] ---
B231: That may just be R and may have nothing to do with the proof at all.
A674: Yeah. --- Well I'm suspicious of anything that reduces reality into mathematical formulas, and then asks me whether it's true or not. I just, I think that that's the problem, I'm just suspicious of this this this thing. --- That's not a problem with the proof that's a problem with me. and my suspicions.
A675: [reading] Could you make this proof better somehow? --- ---
A676: I don't know if I could make the proof better somehow.
B234: Well I probable could if I knew anything about this stuff, but since I don't then I can't. --- --- --- --- --- --- --- Yeah, all right what, do you want to go back? do you want, do we agree with this?
A677: I don't know, I think
B235: I mean, can you think of a way to disprove it?
A678: No, I cannot ---
B236: So then is it true because we can't disprove it? --- ---
A679: All I study in school is how there is no truth. ---
B237: Well, but, I mean, log, It makes sense to me, like this is the proof is estab estab

B238: It makes sense to me in each step
A680: and if you go back here and you say 2 times 2 to the N here then you go 2 times P N , yeah that's OK it's all there it's just taking this thing and moving it around and doing stuff with it.. It all. It's all the same as this. This is exactly. there's just no way of saying this. ---
B239: Can you start doing, if N
A681: If N equals
B240: But can N only equal an entire, a whole number?
A682: Let's see, what has to go with this, 2 over 2 N plus 1, 2 divided by 2,
B241: Presumably
A683: so you get rid of that, see is equal to, if I divide on this side do I have to divide on this side or multiply on this side?
B242: Divide
A684: Divide on this side? --- so that's 2 N is equal to --- 1 to the power of N plus 1? --- does that work? Is that right? Is that how you do this, to make sure that I'm not making an incorrect, a mathematically incorrect, but 2 to 2 N is equal to 2 N plus 1, right, if you go like that and like that, divide to get just 2 N right, would that, does that work?

A expressed confusion about the exponents and R explained. A then continued to work at showing $2(2^n)=2^{n+1}$.

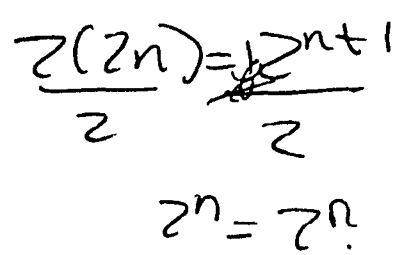

Figure 51: A's writing prior to line A691

B244: well see that works, when you have 2 N
A691: It equals 2 N
B245: 2 N I like that those sort of proof
A692: Apple equals an apple. No hesitation, no doubt.

A and B began activity 2. After clarifying the meanings of "diagonal" and "vertex" they read over the activity.

A703: The number of diagonals is equal to D N is what? The number of diagonals
B253: Yeah
A704: what's N? the number of sides. Is equal to the number

of sides times the number of sides minus 3 divided by 2. --- --- --- OK I want to see if I can figure out some kind of relationship between these. 5 6 9 hmm --- --- --- --- --- 2 D N is equal to N N minus 3. does that work? if you're trying to multiply up can you do that? OK. but --- um --- --- --- what happens if you divide D N by N?

R565: You get D N over N

A705: OK ---

R566: Because in one case the N is just a little subscript to tell you which D you are talking about

A706: So 2 times the number of lines is equal to the number of lines times, no, the number of lines minus 3 --- --- --- --- --- --- --- --- --- So what do we have to do with this? We have to show that the number of diagonal of a polygon is always

B254: Huh ---

A707: Is there something simple we can start with about polygons?

B255: Umhmm, all these

A708: I mean is there some kind of simple

B256: If you plug all of these in they work --- for every polygon

A709: But is there something simple we can start with? That we can say about the number of lines? besides this? like is there something simpler we can say about then besides something like this? like, what's simple, like we go from 3 sides which has none, 4 sides which has 2, 5 sides has 5, 6 sides has 9, how many does seven side have? according to this? 7 is equal to --- what's N again? the number of sides? D N is what?

R567: The number of diagonals.

A710: OK so D N is equal to 7 times 7 minus 3 so 4 divided by 2 so D N is equal to --- 28 divided by 2 so D N is equal to 14, right? so for something that's like whatever, something that's got 7 sides, we get 14. --- --- ---

B257: hmm --- --- --- --- ---

A711: Oh wait, hey, look at this.

B258: What?

A712: 2 plus 3 is 5, right?

B259: Umhmm

A713: 4 plus 5 is 9

B260: uhhuh

A714: 5 plus 9 is 14

B261: Oh, OH these are those numbers

A715: Right, OK let's see, let's see how many we get now. theoretically 5 we should get 6, 6 we should get 20 for 8 right? So let's see, if we do this D N thing OK so D N is equal to 8 times 3 minus 3 over 2, so D N is equal to 8 6 5 over 2, D N is equal to 30 over 2, D N is equal to 15? and it should be equal to I'm wrong, Did I

do this wrong? It was looking hopeful there for a second .
B262: Was 8 20?
A716: Yeah, I was hoping it would work out to 20 but I think, No, this 8 5 was equal to 40, sorry. Not 30, it was equal to 40. Yeah it's right, it works out. OK so 8 sides gives 20. So this means that, this is great, so that, we can write this a different way I think, than this to show this. So basically what we know then, we can make a guess and say that D N, so what we have here is, it starts by --- it goes from 0, to 2 --- to
B263: That doesn't work then
A717: Well I'll make the first one a bit, 2 times
B264: 2 plus 3
A718: 2 plus 3 is 5
B265: 4 plus 5 is
A719: 4 plus 5 is 9, 9 plus 6 is 14, um
B266: No 9 plus 5 is 14 and
A720: 9 plus 5
B267: 6 plus 14 is 20
A721: Yeah
B268: So it's, so diagonals are
A722: So it's 3 plus another one, ok so we've got 3 plus
B269: So it's the diagonals
A723: 2 plus 2 plus
B270: wait, the diagonals are equal to the sides minus 1
A724: Let's try this once more, 2 plus 3 is 5, 5 plus 4 is 9, 9 plus 5 is 14, 14 plus 6 is
B271: yeah, so that works
A725: But how can we write that in a mathematical formula? If we can write that in a mathematical formula
B272: well
A726: We can fit it, we can plug it into here and see if that still works. With that formula, and that will, that will be,

They continued to try to describe a general formula, without success. They also suggested that the relation they found might be related in some way to the formula they were supposed to prove.

B284: OK the diagonal equals --- --- --- --- the diagonal equals the side minus 2 plus --- the diagonal
A738: Oh wait! Wait wait wait, let me see this. what's happening here? --- --- --- --- --- --- No I don't see it. --- --- ---
B285: N minus 3 is 3 times N ---
A739: number of sides
B286: But work it this way, how does this work with your pattern, if you have N minus 3 is this times that, then divided by 2
A740: N minus 3 is --- N minus 3 times N --- 6 minus 3 is

3, which means you're back to there,
B287: yeah, no, I mean, it works
A741: Yeah right
B288: just by plugging numbers in
A742: Yeah it does
B289: But if you yeah, I mean plugging numbers in it works, but if you --- see it's 6 minus 3 or whatever 7 minus 3 but, see, I'm just saying, see, I'm trying to do visually, here
A743: OK
B290: So if it's there, and N times that again, divided by 2 it equals that. and that's plugging numbers in that makes sense, but with you're pattern of how
A744: These are developing
B291: Yeah
A745: Like we can predict right now, how many side we can, with out using this formula you and I can say, how many sides the next one is going to have, right?

A asked R how to write a the relation and R gave them the relation for D_5 and D_6.

A765: so then we can, if that's true, then let's see what happens if we take --- --- I like it better when it's 2 D N. What do you think? Yeah I like it better.
B307: Which way?
A766: I like it better like this. So if we've got D D 5 right 2 times D 5 the number of slices is 5 --- is equal to, 5 times, wait a second, what am I doing? --- Let's wait, wait a second, 2, How would you substitute this, OK then you would say, 2 times --- --- So you'd go for D 5 and then, then according to D, for D 5 then, then it should be 2 times D, 2 times --- D 4 plus 3 is equal to --- 5, is that right? 5 minus 3? Do you think that's true what I've written here? We're taking this as being D??? D 5 is equal so then we'll say for the number D 5 OK? well so proving this equation, the one he gave us, for D 5, for diagonals 5

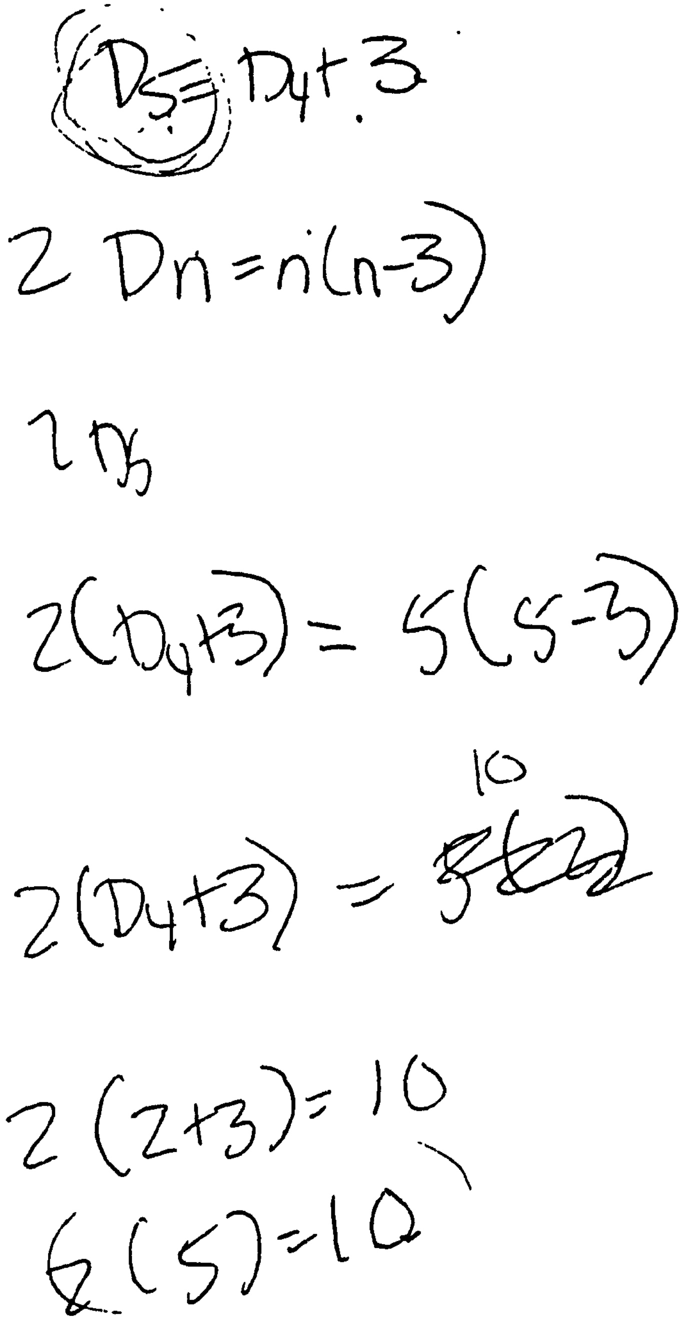

Figure 52: A's proof for n=5 about line A766

B308: Umhmm
A767: then 2 times D 5 right, is equal to, so that's equal to actually
B309: But that's not plugging in, No

A768: cause we're saying
B310: Cause you've got
A769: So then I plugged in this one right? So I've said 2 times D 4 Plus 3
B311: Uhhuh
A770: Right? and then equals, and then we're talking about the number of slices, 5, 5 times 5 minus 3 so that would work?
B312: No. Wait, no I don't understand what you've done
A771: Ok basically I've said
B313: because you've got to take this down to
A772: So what I've done here, we're saying 2, I think, I might be wrong about this. then we've got this one here D 5 is equal to D 4 plus 3 right? So if we've got Now let's solve this D N for D 5 so instead of D N we're going to solve it, like this D N is 5 OK? We're solving it for the diagonal with five sides, five sides OK? N here is, where are we? Um N is N is the number of sides.
B314: No No No wait because, yeah but
A773: So the D
B315: D 5 is the number of diagonals it's not, that's 5 diagonals it's not the number of sides.
A774: Right, right
R574: Well it's the number of diagonals you would have if you had 5 sides
A775: Yeah, I'm not saying, so that's
B316: Ok
A776: So what we're saying now is, so we can actually say so you can say D 5, so D 5 is equal, we're saying D 5, so instead of writing 2 times D 5 we're going to write 2 times D 4 plus 3, right? cause D 5 is the same thing as D 4 plus 3 That's what, OK That's what we've said
B317: OK
A777: So instead of saying 2 times D 5 we're going to say 2 times D 4 plus 3
B318: Umhmm
A778: Is equal to, Now what we're doing to do, is we're going to say Now, I, tell me just if I'm straying here but this should be the number of slices 5 right
R575: Umhmm
A779: cause we're talking D 5 the 5
B319: Umhmm
A780: So 5 and then we're still talking 5, 5 minus 3. So this should work --- OK?
B320: All right, I think I understand
A781: Hopefully, so let's see. --- So this side is easy to do
B321: So, D N you're saying D N is
A782: For this time I'm still calling it D 5, we're solving for D 5
B322: OK -- so D N is D 5

A783: And N is 5, yeah --- ---
B323: But then how do you get
A784: How do we find out
B324: 2 times 3 --- D 4 plus 3 is equal to this? Because haven't you just said that that's that?
A785: Before I try and explain this am I on the right track here? towards something helpful, or is this useless?
R576: Everything you are saying is correct.
A786: OK
B325: Ok
A787: So basically what I'm trying to do. I just don't want to bother trying to explain something that's totally a fallacy, it's really not worth it. So he's given us this equation, right
B326: Umhmm
A788: D N is equal to, well, 2 D N is equal to N over N minus 3, that's what he's given us, ok?
B327: Umhmm
A789: Now we, out of our own ingenuity by doing this, we have said this. Where is what you gave me here.
B328: D 5 is
A790: Yeah
B329: D 4
A791: D 5 is equal to D 4 plus 3, so --- so now, we know then that that's what, for D 5 we know that it's D 4 plus 3 OK so instead of calling it D 5 we're going to call it D 4 plus 3
B330: Uhhuh
A792: That's the same thing according to
B331: Ok ok ok ok I understand now ok
A793: And then we've just, what we've done is
B332: Ok got it
A794: We put it all in there. OK
B333: Ok
A795: So this is what we end up with --- So this is like 5 and 2 and this has got to be 2, we'll leave this complicated for the moment. So what is this, is going to equal 10. Should equal 10. Now we know that D 4 -- right
B334: Uhhuh
A796: --- D 4 is is 2 right?
B335: 2
A797: Right? So 2 times 2 plus 3 is equal to 10 so 2 times 5 is equal to 10 So I don't know what we've proved, but we have something that seems. This works. Ok. So I don't know.
B336: So, well we've shown it
A798: I don't know if that, I don't know if this
B337: Or you've shown it anyway
A799: But does this show that the number of diagonals. I don't know if this if this shows this though. This shows something. But does it show that the number of

diagonals of a polygon is always --- ---
R577: Do you know that this is always true?
A800: Well no, but it looks good to me. --- I mean. So we've got two things which work together, that seem to be true. --- It's a fine little game, but I don't think I proved anything. Oh well ---
B338: I think it's pretty impressive
A801: we played with some numbers a little bit --- Show that the number of diagonals of a polygon is always --- --- Well if you could say, if you say that this relationship describes the number, predicts the number of diagonal you're going to get
B339: Umhmm
A802: And then you can, What we've done is we've built a proof for this. and saying that this, Why did you give us this formula? ---
R578: To see if you could prove that that was the formula
A803: Ok, but we have if you, if we take this then I think, personally I think that if we decide that this is true as a statement about polygons then it
B340: this is also true
A804: then we've proved with this that this is true
B341: So we've proved that they yeah, in relation to one another they're true, but in relation to the truth
A805: Who knows [laughs] which seems to be the ultimate problem.

R then requested a recap of what A and B had accomplished. R also commented that they had made two assumptions. The assumption that the relation they had found was always true, and the assumption that the diagonal formula was correct for n=4. R commented that based on these two assumptions they had proven 10=10. A objected to this. B recognized that their relation was unproven:

B344: Because the only the way we proved this
B345: Was to look at the patterns in this OK
A813: This, but we came to this independently, right? And we said that independently we can know what the diagonal D 5 is by this, so then we substituted this into this equation and it still works out. As being true, wouldn't that doesn't that prove something besides 10 equals 10? ---
B346: Well 10 does equal 10 though
A814: But doesn't that prove, I mean
A815: I don't know
R585: You know that D 5 is in fact equal to D 4 plus 3 because you can look at this and say, oh look
A816: That's true
R586: D 5 is equal to D 4 plus 3
B347: OK, so how can we reduce, wait, what can we reduce this, to like substitute Ns and little letters for

everything in this? Letter and numbers, how can we reduce this to a formula instead of something with numbers? If we can reduce this to a formula and then still prove it with that then,

A and B then began trying to come up with a general form of their relation. R helped. A began to recreate her proof for n=5 in a general form. She had difficulty with the term D_{n-1}. B also began a general proof. They both had trouble with the role of the index of D_{n-1}.

R595: If you represented D N minus 1 using this formula here, like, currently you're ok, you've just got Ns, except you've got this D N minus 1
R596: Would it be OK to write D N minus 1 by using this formula to figure out what it is in terms of N? ---
A837: I'm going to work this out here
B362: Do you mean ---
A838: Where were we? D N is equal to, Yes we have this problem with this D N minus 1
B363: Can you in, if you put --- ---
A840: So this would be, if we wanted to get rid of, So D N minus 1 would be equal to this minus 1?

They both began to rewrite the diagonal formula in terms of n-1. Once this was done they proceeded with their proofs, working independently. They eventually compared their work and found that they had different expressions. This was partly due to A's changing the form of the formula, and partly due to algebra errors by both of them. R helped to correct these and B continued to work on A's proof.
B383: we go --- that's it. ?
A863: But now what do we do with that? ---
B384: Well that's D N is equal to that
A864: OH!
B385: That's D N
A865: That's the thing
B386: is N N minus 3
A866: So we got back to the, OK
B387: Over 2. So we did all this algebra and it was lots of fun
A867: And we're back to what we, so we got, that's a proof. This constitutes a proof. This is a proof.
B388: There.
A868: What did we start with? We started with our other equation?
B389: Wait
A869: Interesting
B390: Where did we start?
A870: But what did we start with? [laughs] Where did we start? We started like OK 2 right we, up here right

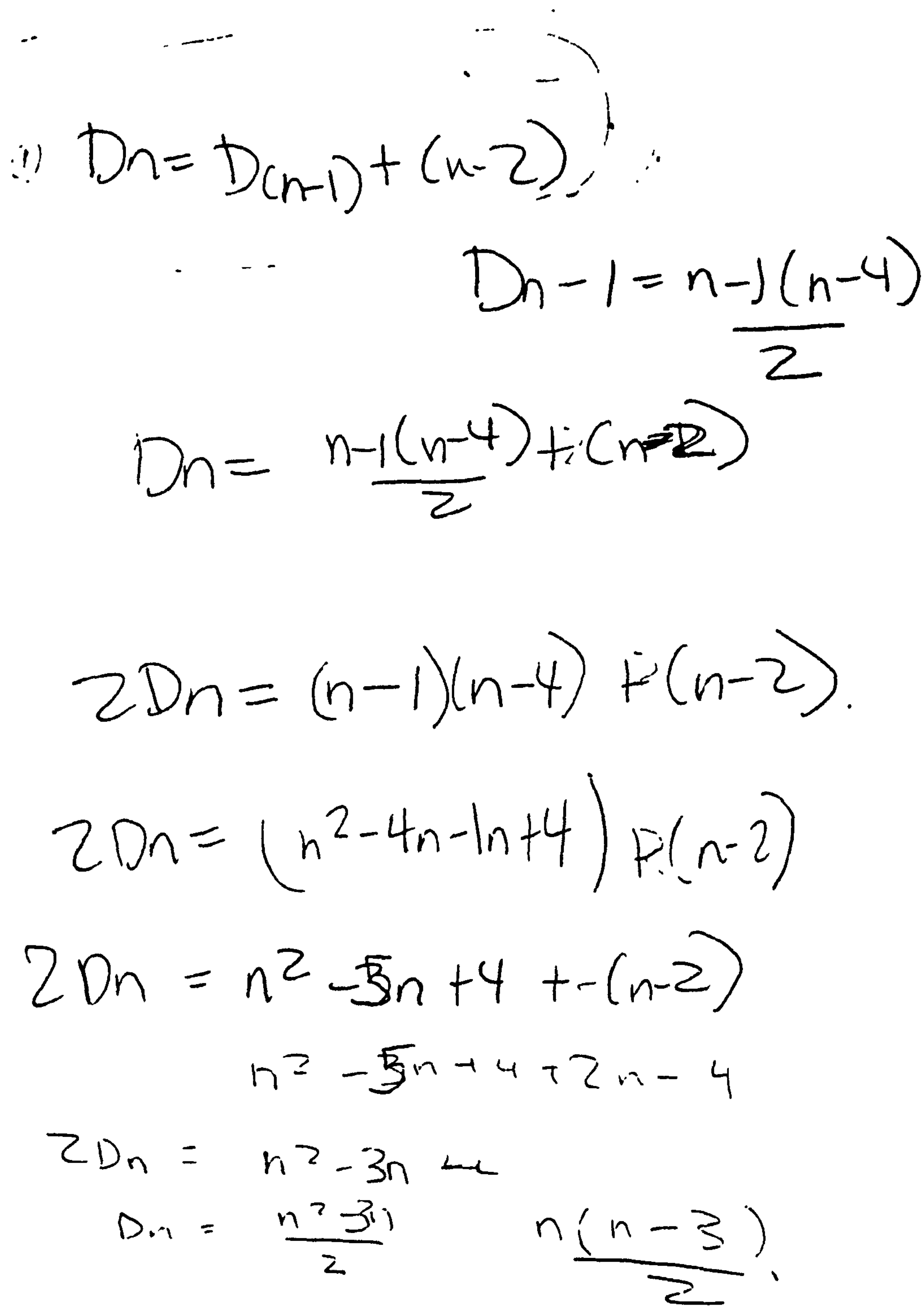

Figure 53: A and B's induction step for the diagonals formula, about line A863.

They reviewed their proof.
R620: During the course of doing this you used this formula that you're trying to prove to figure out what D N minus 1 is
A878: Umhmm
B395: Umhmm
R621: In other words to prove that for the case of N you assumed it was already true for the previous case. As you had proved that it worked for a hexagon by assuming already that it worked for a pentagon.
B396: Umhmm
R622: Is this a problem?
A879: Yes. I don't think so
B397: Is it, because if it weren't true then would it have worked out?
A880: No, I guess it wouldn't have worked out --- --- ---
R623: You don't think that it worked out because you assumed it was true in the middle of proving it? --- --- ---
A881: I don't know --- I mean basically I don't I mean, this is This comes down to like like, what I think is my bone about the other one we did. I think ultima some way you're assuming that something is true aren't you? To do this?

R asked if there was anything they could be sure of assuming. A noted they knew how many sides triangles, quadrangles, and pentagons have. R asked if they knew anything about the numbers of diagonals for those shapes. It was noted that they could count for a quadrilateral, and find it had 2 diagonals.

R632: That means that this formula works for quadrilaterals
A890: Yeah if it works for quadrilaterals we can work from there using this like, using our statements up, can we not?
R633: How would you do that?
A891: Didn't we just do that? ---
R634: Is that what you just did?
A892: I don't know --- sorry, he's asking you
B403: Did we just do that? What did we just do?
A893: We just did it for D 4 we did it for D 5, we did it for the pentagon
R635: ??? See if it works for a specific case. If N is 5 here
B405: Umhmm
R636: And you assume the formula works for N minus 1 which is 4, and you know that it works for N minus 1 which is 4 because you did an example
B406: Umhmm
R637: Then you could proceed through this just as you did in general and at the end you get that the formula works for 5

A894: Umhmm ---
R638: What would you do then if you wanted to prove it for 6?
A895: Go through the whole process again.
R639: OK --- could you keep doing that?
B407: Forever and ever? ---
A896: I think so, yeah

R then attempted to prove the relation they had found: $D_n = D_{n-1} + n-2$. The proof was confusing. R ended by assuring them that it is true:

R644: It is true that this
B408: But why is it true?
R645: But why is it true.
A900: Why is it true?

R attempted again to prove the relation, this time with more success. He then asked A and B to describe the reasoning behind their proof. They described the diagonal formula as just another way of stating the relation. They then described the substitution of the formula into the relation.

A913: But we know this is true Ok this is
B418: Yeah
A914: true, so when we --- put this OK, when we put this formula into here it was still true --- I mean, but then we ended up, back, I don't know, sorry, don't listen to me
B419: No because because, OK we took this, Ok from this we assumed this was true, we made it into this,
A915: yeah
B420: we put all the Ns in the right places, but,
A916: We had this problem with the N
B421: with the D N
A917: Minus 1
B422: But from this, it said that D N was this, was the,
A918: That D N minus 1 was
B423: times N N minus 3 whatever, and so,
A919: Yeah
B424: to see if this worked, to see if they yeah, assuming this was right, and assuming this was right, because if it hadn't been right then it wouldn't have worked out or something like that. We substitute this for the D N then worked it all out
A920: The D N minus 1 I think we did it. For the D N minus 1
B425: D N Minus 1
A921: yeah because we were having trouble with the D N minus 1
B426: Right
R661: So by substituting

A922: What we called
R662: This for D N minus 1
A923: Yeah, instead we called this , we called this thing here, we called this thing here, this. Which I believe we got from, where we got from that. and then we substituted in here, and we simplified it down, we were right back to that. Curiously enough.
R663: So by assuming that this worked for N minus 1
A924: Yeah
R664: And that this worked,
A925: yeah
R665: you showed that this works for N
A926: Yeah ---
R667: OK how does that show that it works in general? There were two assumptions, one was this, which fine I'll let you have. And the other one was that this works for N minus 1
B428: uhhuh
A928: But doesn't that have something to do with the way that we actually figured out that it does work for the quadrangle? because we know that's true.
R668: OK
A929: And we know this whole thing works on a works on a, like a working from what, from 1 equal, it builds off of one up to the other, so if you start off with a building block that you know is true you can keep , I don't know. I don't know
B429: But if that was true, --- and all this is D N is equal to D N D times N minus 1 plus N minus 2
R669: That's just saying that this works in general, yeah
B430: that works in general. Then --- if this works for N minus 1 and it works for N --- then shouldn't it work for --- N minus 2 or N plus 2 or whatever? if this always works? ---

R explained that all their proof showed was that the step can be made from one case to the next. He then described the way in which this allowed the creation of a chain of implications and asked if such a chain proved the statement for all numbers.

A931: If you fulfill that requirement?
R673: Umhmm --- ---
B432: so it always works if you've fulfilled the requirement
A932: yes yes
B433: For N minus 1
R674: Ok, then we know that it works for these four, is that enough to show that it works for everything?
A933: Well we've tried it for more than four, we've actually done it for six
R675: these six
A934: It works for these six, I mean, this is hard, I have

this problem
B434: But if it works, se if we've taken it down, not just plugging number in but if we've taken it into a general statement using N instead of numbers, and it works then isn't that a general statement about all polygons?
R676: well you've got a general statement here about the relationship between the number of diagonals for a polygon with N minus 1 sides versus a polygon with N sides, with 1 more side. ---
B435: OK so you're saying that if it were N minus 2 for example, it might not necessarily work
R677: All the, all that this does is say that if it works for a polygon then it will work for a polygon with one more side.
B436: Umhmm
R678: It doesn't say anything about a polygon with one less side or two more sides, --- --- but you have shown that if it works for a polygon then it will work for a polygon with one more side. --- --- --- --- If this says it works for a polygon, or you could calculate that it works for a polygon, with 8 sides, you could go through this argument with 8 and show that it works for a polygon with 9 sides. --- --- --- But is it true for every polygon?
A935: Well, ultimately, what eventually happens is either something bizarre happens and it's not true, like it's some bizarre number, I don't know, I --- If the highway goes straight for as far as the eye can see does that mean that for ever straight? I don't know. ---
B437: well could we work it out if --- --- --- But then that's the same thing, if we we worked it out for a triangle --- the formula works for a triangle it works for a hexagon and it works for something that has 3 more sides --- that ---

A said she was tired at this point, and R suggested that it might be a good time to stop. B expressed an interest in seeing the next activity and R showed it to them. B asked about the four statements concerning even and odd numbers, and R explained them. R then went through the proof using F_8 and F_9 as examples.

R717: Does that proof make sense? Is it OK to do that kind of [thing?]
A969: well that's what we just did with our thing, and I think it is.
R718: OK
B456: [whispering] but why?
A970: I'm not a philosophy student. --- I hate the question WHY? Oh it's such an awful question
A971: There's no answer to WHY? This is like, what?
R720: Do you think it works?

A972: Yes I do
B457: Except for 40
R721: Without caring about why or not
A973: Yeah I think it works
R722: OK
B458: Except for something like 42 or something, right?
R723: Do you think that it's going to freak out somewhere around 42 or do you think it's always going to just work?
A974: Well, you know, I --- I don't know. I don't know. I can't answer that. It's too difficult. It's very hard doing post-modernist thinking and all my television classes and coming down to math, because I'm asked to, it's a contradiction. --- Yes I think it will always work.
[laughs]
R724: What do you think?
A975: NO! but it could go wonky. Who knows what's going to happen in the infinite universe?
B459: Well I don't know. --- because I seem to recall last time, but I can't really remember, because it was such a horrible memory I've blocked it out. [laughs]
B460: That I did some little explanation about --- every third one is even or something. I can't remember. It was timsing or all the minusing or something. I can't remember what it was, and then you said try 40. and it didn't work. And so I'm suspicious that this may not work. Because Fibonacci numbers are weird. But since I can't really remember

R finished off by reminding B of the two statements she was confusing from stage one.