ON THE DERIVATIVES OF POLYNOMIALS

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ABSTRACT

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Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n in complex domain and $P'(z)$ denote its derivative. The problem is to estimate $|P'(z)|$ on the unit disk $|z| \le 1$ under different conditions on $P(z)$. This study originated with the work of S. Bernstein from where it is established that $\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|$. Later P. Erdös suggested to investigate the influence of zeros on the estimate of $|P'(z)|$. Several interesting results are known in this direction, in particular when $P(z)$ has no zeros in $|z| < 1$ or $|z| < K$, $K \geq 1$. These results have wide applications and have been used to obtain a variety of results concerning polynomials. There are several cases which are still unsolved, in particular when $P(z)$ has no zeros in $|z| < k < 1$. Though lots of results concerning the influence of zeros of $P(z)$ on the behaviour of $P'(z)$ are known, a definite relation that would settle open problems is yet to be discovered.

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INTRODUCTION

In 1912, S. Bernstein studied the estimate concerning the derivative of polynomials which led to the following celebrated result known as:

Bernstein's Theorem: Let $P(z)$ be a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$. Then

$$\max_{|z|=1} |P'(z)| \le n .$$

In Chapter I, we present the original work of Bernstein along with the works of other mathematicians related to this result.

Later as a new direction, P. Erdös suggested to investigate the influence of zeros of a polynomial on the estimate of its derivative and proposed a conjecture that was proved by P.D. Lax. This result is known as:

Erdös-Lax Theorem: If $P(z)$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$ and $P(z)$ has no zeros on $|z| < 1$. Then

$$|P'(z)| \le \frac{n}{2}$$

for $|z| \le 1$.

Chapter II is mainly concerned with various proofs of Erdös-Lax Theorem and with its extensions such as in L_p-norm. This theorem has wide applications. These are discussed in Chapter III where we also give a new result (Theorem 3.1).

In Chapter IV, we present a generalization of Erdös-Lax Theorem given by M.A. Malik when the zeros of $P(z)$ do not lie in $|z| < K$, $K \geq 1$. He proved:

If $P(z)$ is a polynomial of degree n with $|P(z)| \leq 1$ on $|z| \leq 1$ and $P(z)$ has no zeros on $|z| < K$, $K \geq 1$. Then

$$|P'(z)| \leq \frac{n}{1+K}$$

for $|z| \leq 1$.

Several results in this direction are also presented in Chapter IV; in particular see Theorem 4.5 where some of the coefficients vanish in addition to the restriction on the zeros of a polynomial.

The case corresponding to $k < 1$ is not yet settled. To this problem, we present some observations involving computer calculation and also prove a result (Theorem 4.12) for polynomials of degree 3.

An isolated and partial result related to a conjecture due to E.B. Saff is duscussed in Chapter V when $P(z)$ has no zeros in $\operatorname{Re} z < 1$.

The thesis is mostly an historical survey of the above-mentioned results and we have attempted even to include proofs of partial results which led to the final form of the theorem; in particular see details of Chapter I. Whenever a result is our own; it is marked by an asterisk *, e.g. Theorem 3.1*, Theorem 4.5* , Theorem 4.12*, Figure 4.2*, etc.

CHAPTER I

BERNSTEIN'S THEOREM

In 1912, S. Bernstein studied the estimate concerning the derivatives of polynomials and established the following celebrated result now referred to as the Bernstein's Theorem.

Theorem 1.1: Let $P(z)$ be a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$. Then

$$\max_{|z|=1} |P'(z)| \le n. \tag{1.1}$$

At first, Bernstein's Theorem did not appear in the form as stated in Theorem 1.1. In fact Bernstein established a less precise estimate; see Theorem 1.5, concerning the derivative of trigonometric polynomials that follows from Theorem 1.2, 1.3 and 1.4. Later mathematicians refined this result and established the best possible result on the estimate of the derivative of trigonometric polynomials; see Theorem 1.6. There are several proofs of Theorem 1.6. In our discussion, we present the proof due to F. Riesz given in 1914 and another proof due to R.P. Boas given in 1969. Theorem 1.1 follows immediately from Theorem 1.6. However, a direct proof of Theorem 1.1 given by N.G. DeBruijn is interesting and we include it in the following discussion.

§1.1. THE WORK OF S. BERNSTEIN

In this section we present the original work of Bernstein [3] to see the actual process of development in this direction. Since Bernstein's work is not readily available in standard books, this section might stand for special interest to those who are interested in the historical background of the subject.

In the work of Bernstein, the polynomials under consideration are of real coefficients. We now present the work of Bernstein:

Theorem 1.2: In the interval $[-1,1]$, let $P_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n such that $\max_{-1\le x\le 1} |P_n'(x)\sqrt{1-x^2}| = M$. Then $|P_n(x)| < M/n$ for all $-1 \le x \le 1$ is not always true, i.e., there exists some x_0 with $-1 \le x_0 \le 1$ such that

$$|P_n(x_0)| \ge M/n . \tag{1.2}$$

Proof of Theorem 1.2: Let $P(x)$ be a polynomial among all polynomials $P_n(x)$ satisfying $\max_{-1\le x\le 1} |P_n'(x)\sqrt{1-x^2}| = M$ and has least deviation from zero[1]. Suppose $\max_{-1\le x\le 1} |P(x)| = L$ and let $x_1, x_2, \ldots, x_k$ be points in $[-1,1]$ such that

$$|P(x_i)| = L \qquad i = 1,2,\ldots,k .$$

Also let ξ be $-1 < \xi < 1$ such that

$$|P'(\xi)\sqrt{1-\xi^2}| = M .$$

---

[1] The existence of such a polynomial follows from well known results in Approximation Theory [16].

We claim that there exists no polynomial $F_n(x)$ of degree n which satisfies

$$P(x_1) = F_n(x_1); P(x_2) = F_n(x_2); \ldots; P(x_k) = F_n(x_k) \text{ and } F'_n(\xi) = 0. \quad (1.3)$$

Let us suppose contrary that such an $F_n(x)$ exists. For each x_i, let A_i be an interval so that $P(x)$ and $F_n(x)$ do not change sign on each of A_i. If we delete A_i from $[-1,1]$, the maximum of $|P(x)|$ on the remaining set is smaller than L, so we have

$$|P(x)| \leq L' < L \qquad \text{for } x \in [-1,1] \setminus \bigcup_{i=1}^{k} A_i .$$

Let $\delta = L-L'$ and choose λ small enough to have $|\lambda F_n(x)| < \delta$. Consider the polynomial $P - \lambda F_n$. Since on $\bigcup_{i=1}^{k} A_i$ the sign of P and F_n are the same, this gives $|P - \lambda F_n| < |P| \leq L$; on $[-1,1] \setminus \bigcup_{i=1}^{k} A_i$ then $|P - \lambda F_n| < L' + \delta = L$. Thus we always have

$$|P(x) - \lambda F_n(x)| < L \qquad \text{for } x \in [-1,1] .$$

On the other hand, $[P'(\xi) - \lambda F'_n(\xi)]\sqrt{1-\xi^2} = M$. If $\max_{-1\leq x\leq 1} |[P'(x) - \lambda F'_n(x)]\sqrt{1-x^2}| = M_1$ then $M_1 \geq M$. Thus the polynomial

$\mathcal{P}(x) = \frac{M}{M_1}(P - \lambda F_n)$ belongs to the class of polynomial satisfying

$\max_{-1\leq x\leq 1} |\mathcal{P}'(x)\sqrt{1-x^2}| = M$ but with its modules strictly less than L, which means that the deviation of $\mathcal{P}(x)$ from zero is less than that of $P(x)$, this contradicts the choice of $P(x)$.

Next we want to show that $k > n-1$. Again, suppose on the contrary that $k \leq n-1$. Using the Lagrange interpolation formula we construct a polynomial

$$Q(x) = \frac{(x-x_2)(x-x_3)\dots(x-x_k)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_k)} P(x_1) + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{k-1})}{(x_k-x_1)(x_k-x_2)\dots(x_k-x_{k-1})} P(x_k)$$

which satisfies the first k equations in (1.3). Putting

$$R(x) = (x-x_1)(x-x_2)\dots(x-x_k)$$

we see that the polynomial

$$F_{k+1}(x) = Q(x) + (Ax+B)R(x)$$

of degree $k+1$ also satisfies the first k equations in (1.3), now we choose A and B in order to have

$$F'_{k+1}(\xi) = Q'(\xi) + AR(\xi) + (A\xi+B)R'(\xi) = 0.$$

This is possible because $R(x)$ has no double zeros and so one cannot have $R'(\xi) = R(\xi) = 0$ at the same time. In consequence, if one has $k \le n-1$, the polynomial $F_{k+1}(x)$ of degree not exceeding n satisfies all the equations in (1.3); whose incompatibility is already established. Thus $k > n-1$.

Let us see the case when $k = n$. If it is so, we can construct a polynomial

$$F_n(x) = Q(x) + BR(x)$$

of degree $n = k$, satisfying the first k equations of (1.3) and the last equation as well if B satisfies

$$Q'(\xi) + BR'(\xi) = 0.$$

For the last equation of (1.3) not to be satisfied, it is therefore necessary to have

$$R'(\xi) = 0 \tag{1.4}$$

At this stage, recall that $k = n$ and note that all of x_j, $j = 1,2,\ldots,n$ cannot be in $(-1,1)$, the interior of $[-1,1]$ because if it is so, $P(x)$ would attain its maximum at n points resulting in $P'(x) = 0$ at n points, but $P'(x)$ being a polynomial of degree $n-1$ cannot have more than $n-1$ zeros. This further implies that $\max_{-1\le x\le 1} |P(x)| = L$ is also attained at either or both end points where $P'(x) \neq 0$. In the case when the maximum is attained at both the end points $+1$ and -1, the $n-2$ remaining x_j are in fact zeros of $P'(x)$. Thus we have

$$R(x) = c.\frac{(x^2-1)P'(x)}{x-\beta} \tag{1.5}$$

where c and β are constants. Note that β is a zero of $P'(x)$ where $|P(x)| \neq L$. In the other case, when $x = +1$ (or -1) is a point when $\max_{-1\le x\le 1} |P(x)| = L$ is attained we may take $\beta = -1$ (or $+1$). Consequently, (1.5) remains valid for all cases.

From (1.4) we get,

$$R'(\xi) = \frac{d}{dx}\left(\frac{x^2-1}{x-\beta}\,P'(x)\right)\bigg|_{x=\xi} = 0 . \tag{1.6}$$

Set $P^*(x) = \sqrt{1-x^2}\,P'(x)$ in (1.6) to get

$$\frac{d}{dx}\left(\frac{\sqrt{1-x^2}}{x-\beta}\,P^*(x)\right)\bigg|_{x=\xi} = P^*(x)\,\frac{d}{dx}\left(\frac{\sqrt{1-x^2}}{x-\beta}\right)\bigg|_{x=\xi} + \frac{\sqrt{1-x^2}}{x-\beta}\,P^{*\prime}(x)\bigg|_{x=\xi} = 0 \tag{1.7}$$

Since $P^*(x)$ attains its local maximum at $x = \xi$, $P^{*\prime}(\xi) = 0$, and so from (1.7) we have

$$\frac{d}{dx}\frac{\sqrt{1-x^2}}{x-\beta}\bigg|_{x=\xi} = \frac{-\xi(+\xi-\beta)-(1-\xi^2)}{(\xi-\beta)^2\sqrt{1-\xi^2}}$$

$$= \frac{\xi\beta-1}{(\xi-\beta)^2\sqrt{1-\xi^2}}$$

$$= 0$$

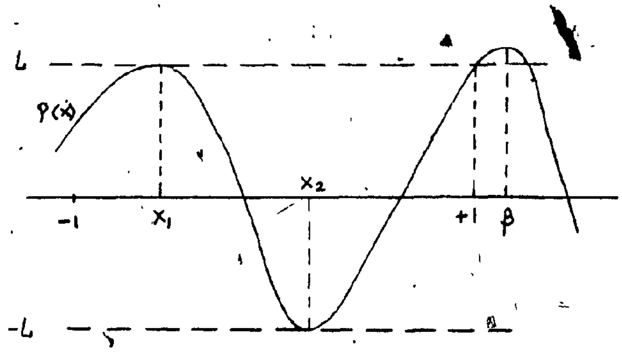
from where we have $\xi = 1/\beta$ and $|\beta|>1$.

From the preceding discussion, we claim that

$$L^2 - P^2(x) = \frac{P'^2(x)(1-x^2)(ax^2+bx+c)}{(x-\beta)^2} \tag{1.8}$$

where a,b,c being constants. To justify (1.8), note that $L^2 - P^2$ is a polynomial of degree 2n which admits double zeros at $x_1,x_2,\ldots,x_n$ which are different from ±1 (because they are where P' vanishes) and simple zeros at ±1. Therefore the polynomial $L^2 - P^2(x)$ of degree 2n is divisible by the polynomial $\frac{P'^2(x)(1-x^2)}{(x-\beta)^2}$ of degree 2n-2. The quotient results in ax^2+bx+c as in (1.8).

We shall now show that the equation $ax^2+bx+c=0$ has two real zeros, both of the same sign as of β and in absolute value greater than β. Suppose β positive, therefore $\beta>1$. When x passes 1 and approaches β (where P' vanishes), P^2 will increase to a number L_1^2 which is greater than L^2. Since $P'(x)$ has no more zeros in $x>\beta$, $P'(x)$ does not change its sign. This together with the fact that β is a simple zero of $P'(x)$ further implies that $P(x)$ decreases on $x>\beta$ if $P(\beta)>0$ or $P(x)$ increases on $x>\beta$ if $P(\beta)<0$. As a result, on $[\beta,\infty)$, $P^2(x)$ first decreases and after vanishing, it increases up to infinity; see Fig. 1.1.



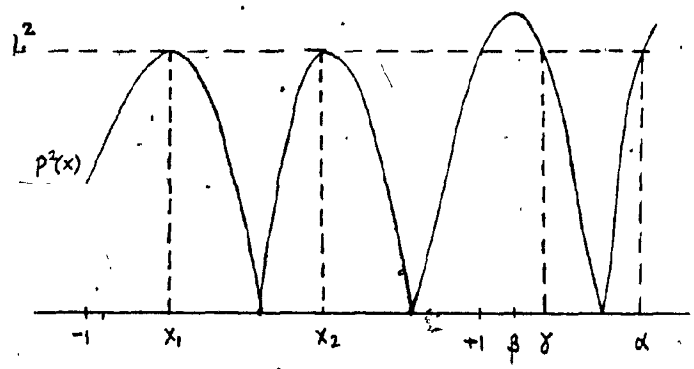


Fig. 1.1

In consequence, P^2 attains the value L^2 twice at $x = \gamma$ and $x = \alpha$, and $\gamma > \beta$, $\alpha > \beta$. γ, α must necessarily be the zeros of the equation $ax^2 + bx + c = 0$. Note that the coefficient of the highest degree term of P' is n time the coefficient of the highest degree term of P. (1.8) can be written as

$$L^2 - P^2 = \frac{(1-x^2)(x-\gamma)(x-\alpha)}{n^2(x-\beta)^2} P'^2 .$$

Since $\gamma > \beta > 1$, $\alpha > \beta > 1$, for every x within $-1 \le x \le 1$ we have

$$L^2 > \left|\frac{(1-x^2)P'^2}{n^2}\right|$$

or

$$L > M/n .$$

This proves the first part of the theorem.

Now consider the case when $k = n+1$ (we know that k cannot be greater than $n+1$ because if $k = n+2$, then in the interval $[-1,1]$, $|P(x)|$ must attain L as a local maxima or minima at least n times implying $P'(x)$ of degree $n-1$ vanishes n times). In this case, $n-1$ of x_i are the zeros of $P'(x)$ and the other two x_i's must be $+1$ and -1. Therefore, instead of (1.8), P satisfies the differential equation

$$L^2 - P^2 = \frac{(1-x^2)P'^2}{n^2}$$

which implies

$$\frac{n}{\sqrt{1-x^2}} = \frac{P'}{\sqrt{L^2-P^2}} ,$$

so

$$n \text{ arc cos } x = \text{arc cos } P(x)/L .$$

Consequently

$$P(x) = L \cos n \text{ arc cos } x$$

is the Tchebischeff polynomial. Since

$$P'(x) = n L \sin n \text{ arc cos } x \cdot \frac{1}{\sqrt{1-x^2}}$$

we conclude

$$L = M/n .$$

It is therefore the Tchebischeff polynomial which among all polynomials considered deviates least from zero. □

Theorem 1.3: If a polynomial $P_n(x)$ of degree n is such that the function $\left|[P_n(x)\sqrt{1-x^2}]'\sqrt{1-x^2}\right|$ attains the value M on the segment $[-1,1]$, then $\left|P_n(x)\sqrt{1-x^2}\right| < M/(n+1)$ for $-1 \le x \le 1$ is not always true, i.e., there exists some x_0 with $-1 \le x_0 \le 1$ such that

$$\left|P_n(x_0)\sqrt{1-x_0^2}\right| \ge M/(n+1) .$$

Proof of Theorem 1.3: The proof follows the same direction as of Theorem 1.2, therefore we only give an outline:

We see as before that the polynomial $P(x)$ giving the minimum deviation from zero and satisfying $\left|[P_n(x)\sqrt{1-x^2}]'\sqrt{1-x^2}\right| = M$ should exist and that the number k of points $x_1, x_2, \ldots, x_k$ where the maximum L of $\left|P(x)\sqrt{1-x^2}\right|$ is attained is at least equal to n. Because using a similar argument as in the proof of Theorem 1.2 on the function $P(x)\sqrt{1-x^2}$ instead of $P(x)$, we know that it is impossible to find a polynomial F_n of degree n satisfying the equations

$$P(x_1) = F_n(x_1);\ P(x_2) = F_n(x_2);\ldots;\ P(x_k) = F_n(x_k) \text{ and } [F_n(x)\sqrt{1-x^2}]'\Big|_{x=\xi} = 0 \tag{1.9}$$

where ξ is a point where $\left|[P(x)\sqrt{1-x^2}]'\sqrt{1-x^2}\right|$ attains its maximum. The case $k \le n-1$ yields a contradiction as in the preceding theorem. Let $k = n$, equations in (1.9) should be inconsistent and we can construct the polynomial

$$F_n(x) = Q(x) + BR(x)$$

of degree $n = k$ where

$$Q(x) = \frac{(x-x_2)(x-x_3)\dots(x-x_k)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_k)}P(x_1)+\dots+\frac{(x-x_1)(x-x_2)\dots(x-x_{k-1})}{(x_k-x_1)(x_k-x_2)\dots(x_k-x_{k-1})}P(x_k)$$

and

$$R(x) = (x-x_1)(x-x_2)\dots(x-x_k) .$$

$F_n(x)$ satisfies the k first equation of (1.9) and the last equation as well if

$$\left[(Q(x) + BR(x))\sqrt{1-x^2}\right]'\Big|_{x=\xi} = 0$$

which implies

$$\left[(Q(x)\sqrt{1-x^2})' + B(R(x)\sqrt{1-x^2})'\right]_{x=\xi} = 0 .$$

But this should be impossible whenever

$$[R(x)\sqrt{1-x^2}]'\Big|_{x=\xi} = 0 . \tag{1.10}$$

The zeros of $R(x)$ are the points where the derivative of $P(x)\sqrt{1-x^2}$ vanishes because at that point $\left|P(x)\sqrt{1-x^2}\right|$ attains its maximum. Since

$$(P(x)\sqrt{1-x^2})'\ \sqrt{1-x^2} = (1-x^2)P'(x)-xP(x) = T(x)$$

is again a polynomial of degree $n+1$, we can write

$$R(x) = c\,\frac{(P(x)\sqrt{1-x^2})'\ \sqrt{1-x^2}}{x-\beta}$$

$$= c\,\frac{T(x)}{x-\beta}$$

where β is some real number. On the other hand, ξ satisfies $T'(x) = 0$. (1.10) is therefore reduced to

$$\left(R(x)\sqrt{1-x^2}\right)'\Big|_{x=\xi} = \left[\frac{T(x)\sqrt{1-x^2}}{x-\beta}\right]'_{x=\xi} = 0$$

or

$$1 - \beta\xi = 0,$$

implying

$$|\beta| > 1 .$$

Now, $S(x) = P(x)\sqrt{1-x^2}$, attains its maximum modulus L, n times. The Polynomial $S^2 - L^2$ of degree $2n+2$ is divisible by

$$\left(\frac{S'(x)\sqrt{1-x^2}}{x-\beta}\right)^2 = \frac{S'^2(x)(1-x^2)}{(x-\beta)^2}$$

and consequently S satisfies the differential equation

$$S^2 - L^2 = \frac{S'^2(x^2-1)(x-\gamma)(x-\alpha)}{(n+1)^2\ (x-\beta)^2} ,$$

because the leading coefficient of $\left(S'\sqrt{1-x^2}\right)^2$ is $(n+1)^2$ times the leading coefficient of S.

Now, we note that γ and α are a pair of complex numbers and conjugate to each other, also $|\gamma| > |\beta|$ and $|\alpha| > |\beta|$. To verify this we recall that $S^2 - L^2$ is a polynomial of degree $2n+2$. Then

$$(S^2 - L^2)' = 2SS' = 2P(x)[(1-x^2)P'(x) - xP(x)]$$

is a polynomial of degree $2n+1$. Suppose $\beta > 1$, the first $2n$ zeros of $(S^2 - L^2)'$ appears as in Fig. 1.2.

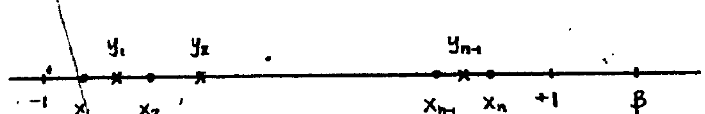


Fig. 1.2

where y_i, $i=1,\ldots,n-1$, are points that S^2-L^2 attains its local maxima or minima, and x_i being the double zeros of S^2-L^2 is again (single) zeros of S'. Now let us look at the curve of S^2-L^2. At $x=+1$, $S^2-L^2=-L^2$ which is a negative quantity, from $x=+1$ to $x=\beta$, S^2-L^2 keeps on decreasing up to a certain quantity $-L_1^2$. Then it increases again because β is a zero of $(S^2-L^2)'$. But as $S^2-L^2=P^2(x)(1-x^2)-L^2$ is always negative, the curve of S^2-L^2 will not cut the x-axis. Since when $x\to\infty$, $S^2-L^2\to-\infty$, there must be one turning point ω of S^2-L^2, which make up the $2n+1$ zeros of $(S^2-L^2)'$.

As mentioned above, the curve of S^2-L^2 does not cut the x-axis anymore at $x>\beta$. The zeros γ,α of S^2-L^2 must be a pair of conjugate complex numbers. The fact that $|\gamma|=|\alpha|>|\beta|$ follows from the Guass-Lucas Theorem; see Theorem 1.7.

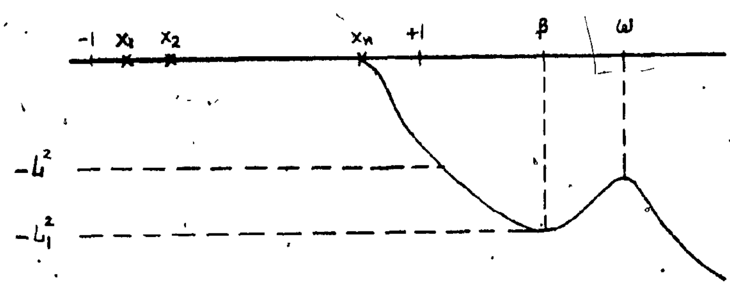


Fig. 1.3

Now, we can conclude that for $-1\le x\le 1$

$$L>\frac{|S'\sqrt{1-x^2}|}{n+1}\ge\frac{M}{n+1}.$$

In the case $k = n+1$, we obtain

$$S^2 - L^2 = \frac{S'^2(x^2-1)}{(n+1)^2} ,$$

and consequently

$$S = L \sin(n+1) \text{ arc cos } x$$

and

$$L = \frac{M}{n+1} . \quad \square$$

With a simple logic negation argument, we give a different version of Theorem 1.2 and 1.3 which we shall use later.

<u>Theorem 1.2'</u>: <u>In the interval</u> $[-1,1]$, <u>let</u> $P_n(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ <u>be a polynomial of degree</u> n <u>such that</u> $|P_n(x)| \le L$ <u>for</u> $-1 \le x \le 1$. <u>Then</u>

$$\max_{-1\le x\le 1} \left|P_n'(x)\sqrt{1-x^2}\right| \le nL .$$

<u>Theorem 1.3'</u>: <u>In the interval</u> $[-1,1]$, <u>if a polynomial</u> $P_n(x)$ <u>of degree</u> n <u>is such that</u> $\left|P_n(x)\sqrt{1-x^2}\right| \le L$ <u>for</u> $-1 \le x \le 1$. <u>Then</u>

$$\max_{-1\le x\le 1} \left|\left[P_n(x)\sqrt{1-x^2}\right]'\sqrt{1-x^2}\right| \le (n+1)L .$$

Now we shall apply Theorem 1.2' and Theorem 1.3' to trigonometric polynomials. First note that any polynomial $p_o \cos^n\theta + \ldots + p_n$ of degree n is identical to a cosine 'series' $a_o + a_1 \cos\theta + \ldots + a_n \cos n\theta$ of $n+1$ terms. Also the product $\sin\theta[p_o \cos^n\theta + \ldots + p_n]$ is identical to a sine 'series' $b_1 \sin\theta + \ldots + b_{n+1} \sin(n+1)\theta$ of $n+1$ terms. In Theorem 1.2' and 1.3', if we let $x = \cos\theta$, we can easily deduce the following:

Theorem 1.4: *If the trigonometric series*

$$a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta \qquad (1.11)$$

or

$$b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_n \sin n\theta \qquad (1.12)$$

is in absolute value less than L, *then its derivative is in absolute value less than* nL.

Proof of Theorem 1.4 : First we write the trigonometric polynomial (1.11) in an algebraic polynomial $P(x)$ of degree n where $x = \cos\theta$. As $|P(\cos\theta)| \le L$, from Theorem 1.2',

$\left|P'(\cos\theta)\sqrt{1-\cos^2\theta}\right| = \left|P'(\cos\theta)\sin\theta\right| \le nL$. But $P'(\cos\theta)\sin\theta$ can be written as a trigonometric polynomial which is the derivative of (1.11), the assertion for (1.11) is verified. Now we write (1.12) in the form $P(\cos\theta)\sin\theta$, i.e. $P(x)\sqrt{1-x^2}$ for $x = \cos\theta$. Apply Theorem 1.3', $\left|\left[P(x)\sqrt{1-x^2}\right]'\sqrt{1-x^2}\right| = \left|\left[P(\cos\theta)\sin\theta\right]'\right| \le nL$. This verified the assertion for (1.12). □

Bernstein then considered arbitrary trigonometric polynomials

$$T(\theta) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos\nu\theta + b_\nu \sin\nu\theta)$$

of order n and established:

Theorem 1.5: *If* $T(\theta)$ *is a trigonometric polynomial of order* n *with* $\max_{-\pi\le\theta\le\pi} |T(\theta)| \le L$. *Then*.

$$\max_{-\pi\le\theta\le\pi} |T'(\theta)| \le 2nL . \qquad (1.13)$$

Proof of Theorem 1.5: Let $T_1(\theta) = a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta$, $T_2(\theta) = b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_n \sin n\theta$, and $\max_{-\pi\le\theta\le\pi} |T_1(\theta)| = L_1$,

$\max_{-\pi\le\theta\le\pi} |T_2(\theta)| = L_2$. It is clear that the maximum modulus of $T(\theta)$ is at least equal to the larger of the two numbers L_1 and L_2. Because for example if we take L_1 and suppose L_1 is attained by $|T_1(\theta)|$ at $\theta = \pm\theta_1$, $T_2(\theta)$ being then equal to $+k$ or $-k$ at $\theta = \pm\theta_1$, the corresponding values of $T(\theta)$, $L_1 + k$ and $L_1 - k$ so is no less than L_1. But from Theorem 1.4 we have

$$|T_1'(\theta)| \le nL_1 ,$$

$$|T_2'(\theta)| \le nL_2$$

so

$$|T'(\theta)| = |T_1'(\theta) - T_2'(\theta)| \le n(L_1 + L_2)$$

$$< 2nL. \quad \square$$

§1.2. THE WORK OF F. RIESZ

Inequality (1.13) in Theorem 1.5 is not best possible. But since it was enough for the goal which Bernstein was going to pursue later, he did not pursue any further to refine the result. Realising this fact, Riesz [20] gave a refinement of Theorem 1.5 in 1914 and replaced the constant $2n$ by n; moreover he allowed the coefficients to be complex. He proved:

__Theorem 1.6__: __If__ $T(\varphi) = a_0 + \sum_{k=1}^{n} (a_k \cos k\varphi + b_k \sin k\varphi)$ __is a trigonometric polynomial of order__ n __with__ $\max_{-\pi\le\varphi\le\pi} |T(\varphi)| \le L$. __Then__

$$\max_{-\pi\le\varphi\le\pi} |T'(\varphi)| \le nL . \qquad (1.14)$$

Proof of Theorem 1.6: Using the orthogonality of trigonometric functions $\sin k\theta$, and $\cos k\theta$, it is easily seen that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} T(\theta)\cos k\theta d\theta ,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} T(\theta)\sin k\theta d\theta$$

and therefore

$$T'(\varphi) = \sum_{k=1}^{n} (-a_k k \sin k\varphi + b_k k \cos k\varphi)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\theta) \sum_{k=1}^{n} k\left\{-\cos k\theta \sin k\varphi + \sin k\theta \cos k\varphi\right\} d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\theta) \sum_{k=1}^{n} k \sin k(\theta-\varphi) d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\varphi+\theta)\left\{\sum_{k=1}^{n} k \sin k\theta\right\} d\theta .$$

Since T is a trigonometric polynomial of degree n, by the orthogonality of trigonometric functions we may add to the kernel terms $\cos k\theta$ and $\sin k\theta$ where $k > n$, without changing the value of the integral. Thus

$$T'(\varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} T(\varphi+\theta)\left\{n \sin n\theta + \sum_{k=1}^{n-1} k[\sin k\theta + \sin(2n-k)\theta]\right\} d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\varphi+\theta)\left\{n \sin n\theta + \sum_{k=1}^{n-1} k2[\sin n\theta \cos(n-k)\theta]\right\} d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\varphi+\theta) \sin n\theta[n+2 \sum_{k=1}^{n-1} (n-k) \cos k\theta] d\theta . \quad (1.15)$$

Note that $n + 2 \sum_{k=1}^{n-1} (n-k)\cos k\theta = \frac{1-\cos n\theta}{1-\cos\theta}$[1] is always positive.

In (1.15) $T(\varphi+\theta)\left[n + 2 \sum_{k=1}^{n-1} (n-k)\cos k\theta\right]$ is a trigonometric polynomial of order $2n-1$ and therefore again by the orthogonality of trigonometric functions

$$\int_{-\pi}^{\pi} T(\varphi+\theta)\left[n + 2 \sum_{k=1}^{n-1} (n-k)\cos k\theta\right]\sin mn\theta d\theta = 0 \quad \text{for } m = 2,3,\ldots$$

Thus $\sin n\theta$ can be replaced by $\sin n\theta - r^2 \sin 3n\theta d\theta + \ldots$ in the integrand of (1.15) without changing the value of $T'(\varphi)$, so we have

$$T'(\varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} T(\varphi+\theta)(\sin n\theta - r^2 \sin 3n\theta + r^4 \sin 5n\theta - \ldots)\left[n + 2 \sum_{k=1}^{n-1} (n-k)\cos k\theta\right]d\theta . \tag{1.16}$$

Now $\sin n\theta - r^2 \sin 3n\theta + r^4 \sin 5n\theta - \ldots = \frac{(1-r^2)\sin n\theta}{1+2r^2\cos 2n\theta + r^4}$. It is an

---

1

$$n + 2 \sum_{k=1}^{n-1} (n-k)\cos k\theta$$

$$= n + 2[(n-1)\cos\theta] + \ldots + 2[2\cos(n-2)\theta] + 2[\cos(n-1)\theta]$$

$$= \cos(n-1)\theta - i\sin(n-1)\theta + 2[\cos(n-2)\theta - i\sin(n-2)\theta] + \ldots + n$$
$$+ \ldots + 2[\cos(n-2)\theta + i\sin(n-2)\theta] + \cos(n-1)\theta + i\sin(n-1)\theta$$

$$= e^{-i(n-1)\theta} + 2e^{-i(n-2)\theta} + \ldots + n + \ldots + 2e^{i(n-2)\theta} + e^{i(n-1)\theta}$$

$$= \sum_{k=-(n-1)}^{n-1} e^{ik\theta} + \sum_{k=-(n-2)}^{n-2} e^{ik\theta} + \ldots + \sum_{k=-1}^{1} e^{ik} + 1$$

$$= \sum_{k=o}^{n-1} \sum_{\ell=-k}^{k} e^{i\ell\theta}$$

$$= \sum_{k=o}^{n-1} \frac{\sin(2\ell+1)\theta/2}{\sin\theta/2} = \left[\frac{\sin n\theta/2}{\sin\theta/2}\right]^2 = \frac{1-\cos n\theta}{1-\cos\theta} .$$

odd function of period 2π which is always positive for $0 < n\theta < \pi$ and negative for $\pi < n\theta < 2\pi$. After taking the absolute value we have

$$|\sin n\theta - r^2 \sin 3n\theta + r^4 \sin 5n\theta - \dots| = \frac{1-r^2}{1+2r^2 \cos 2n\theta + r^4} |\sin n\theta|$$

which becomes an even function with Fourier series representation

$$A_o + \sum_{k=1}^{\infty} A_k \cos k\theta \tag{1.17}$$

where

$$\begin{aligned} A_o &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin n\theta - r^2 \sin 3n\theta + r^4 \sin 5n\theta - \dots| d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} (\sin n\theta - r^2 \sin 3n\theta + r^4 \sin 5n\theta - \dots) d\theta \\ &= \frac{2}{\pi} \left(1 - \frac{r^2}{3} + \frac{r^4}{5} - \dots\right) \\ &= \frac{2}{r\pi} \text{arc tan } r \end{aligned}$$

and

$$A_k = \frac{1}{\pi} \int_0^{\pi} \frac{1-r^2}{1+2r^2 \cos 2n\theta + r^4} |\sin n\theta| \cos k\theta d\theta \tag{1.18}$$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi/n} D(\theta) \sin n\theta \cos k\theta d\theta - \int_{\pi/n}^{2\pi/n} D(\theta) \sin n\theta \cos k\theta d\theta \right.$$

$$\left. + \dots + (-1)^{n-1} \int_{(n-1)\pi/n}^{\pi} D(\theta) \sin n\theta \cos k\theta d\theta \right\}.$$

Here $D(\theta) = \frac{1-r^2}{1+2r^2 \cos 2n\theta + r^4}$. Since $D(\theta)$ is invariant under the transformation $\theta = \alpha + \frac{\nu\pi}{n}$ where $\nu = 1, \dots, n-1$ we have

$$A_k = \frac{1}{\pi} \left\{ \int_0^{\pi/n} D(\alpha) \sin n\alpha \left[\cos k\alpha + \cos k(\alpha + \tfrac{\pi}{n}) + \dots + \cos k(\alpha + \tfrac{n-1}{n}\pi)\right]\right\} d\alpha$$

and

$$\cos k\alpha + \cos k(\alpha+\pi/n) + \ldots + \cos k(\alpha + \frac{n-1}{n}\pi)$$

$$= \mathrm{Re}\left[e^{ik\alpha} + e^{ik(\alpha+\pi/n)} + \ldots + e^{ik(\alpha+\frac{n-1}{n}\pi)}\right]$$

$$= \mathrm{Re}\ e^{ik\alpha}\left[1 + e^{ik\ \pi/n} + \left(e^{ik\ \pi/n}\right)^2 + \ldots + \left(e^{ik\ \pi/n}\right)^{n-1}\right]$$

$$= \mathrm{Re}\ e^{ik\alpha}\left[\frac{1 - e^{ik\pi}}{1 - e^{ik\ \pi/n}}\right]$$

which is zero when k even except possibly for those k being multiples of n. For k odd, from (1.18) we have

$$A_k = \frac{1}{\pi}\left\{\int_0^{\pi/2} \frac{(1-r^2)|\sin n\theta|}{1+2r^2\cos 2n\theta+r^4}\cos k\theta d\theta + \int_{\pi/2}^{\pi} \frac{(1-r^2)|\sin n\theta|}{1+2r^2\cos 2n\theta+r^4}\cos k\theta d\theta\right\}.$$

But with the transformation $\theta = \pi - \alpha$ we have

$$\int_{\pi/2}^{\pi} \frac{(1-r^2)|\sin n\theta|}{1+2r^2\cos 2n\theta+r^4}\cos k\theta d\theta = -\int_0^{\pi/2} \frac{(1-r^2)|\sin n\alpha|}{1+2r^2\cos 2n\alpha+r^4}\cos k\alpha d\alpha$$

which implies $A_k = 0$ for every k odd. Therefore we know that expression (1.17) starts with the constant term A_0 and then continues with terms of order $>n-1$. Thus from (1.16), using the orthogonality of trigonometric functions and letting $r \to 1$ we have

$$|T'(\varphi)| \le \frac{L}{\pi}\int_{-\pi}^{\pi} \left(\frac{2}{r\pi}\text{arc tan } r + \ldots\right)(n+\ldots)d\theta$$

$$= \frac{4nL}{r\pi}\text{arc tan } r$$

$$\to \frac{4nL}{\pi}\text{arc tan } 1$$

$$= nL. \qquad \square$$

§1.3. A GEOMETRIC ARGUMENT BY R.P. BOAS

In 1969, Boas [4] gave a short and elegant proof of Theorem 1.6. The proof is interesting in itself because it only demands very little machinery and illustrates how results can sometimes be obtained from simple considerations. We need the following:

Lemma 1.1: Suppose ϕ and f are real-valued continuous functions on $0 \le \theta \le 2\pi$, that ϕ takes the value ± 1 at $k+1$ points θ_j with alternating signs and that $|f(\theta_j)| < 1$ for all θ_j. Then there are at least k points where $f(\theta) = \phi(\theta)$.

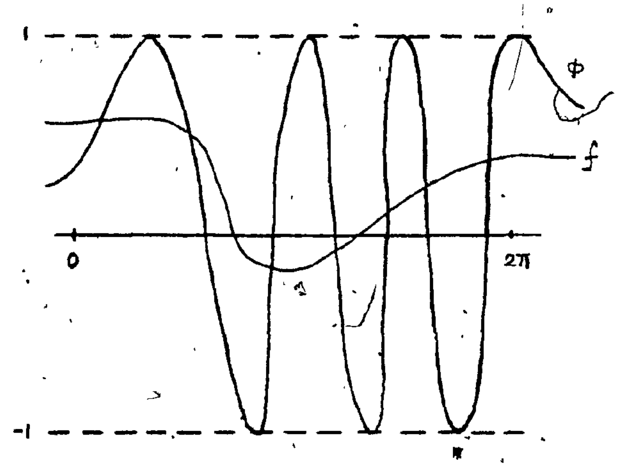


Fig. 1.4

Proof of Lemma 1.1: If we have, say, $\phi(\theta_j) = +1$ and $\phi(\theta_{j+1}) = -1$, then $\phi(\theta) - f(\theta)$ is positive at θ_j and negative at θ_{j+1} which implies that there is a zero of $\phi(\theta) - f(\theta)$ somewhere between θ_j and θ_{j+1}. Thus if the graph of ϕ has k arcs connecting the lines $y = 1$ and $y = -1$, the graph of f and ϕ

must have k intersections. In other words $\phi(\theta) - f(\theta)$ must have k zeros.

Lemma 1.2: Suppose that ϕ and f are real-valued continuous functions on $0 \le \theta \le 2\pi$, that ϕ takes the value ± 1 at $k+1$ points θ_j with alternating signs and that $|f(\theta_j)| < 1$ for all θ_j. If, further, the graph of f crosses the graph of ϕ from below to above on an arc that rises from -1 to +1, then the graphs of f and ϕ cross at at least $k+2$ points.

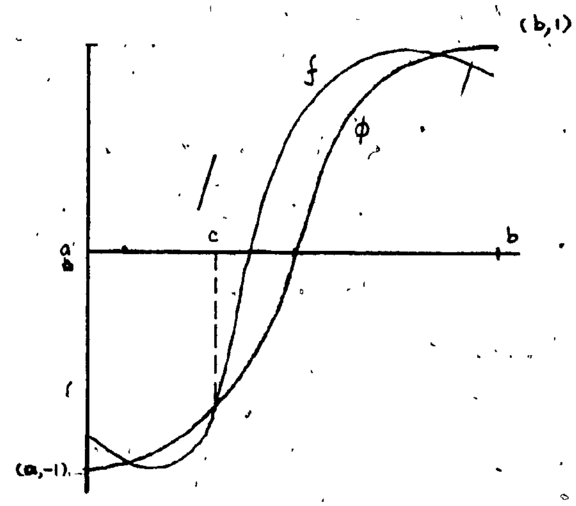


Fig. 1.5

Proof of Lemma 1.2: As in Fig. 1.5, let the arc connecting (a,-1) and (b,+1) be the arc specified in the lemma, and let the crossing of f and ϕ be at c. Since the graph of f is above that of ϕ at $x=a$ and below that of ϕ at $x=b$, so between $x=a$ and $x=b$ there are at least three crossings on this arc. We also know that there are at least $k-1$ remaining crossings of the two graphs, thus there must be at least $k+2$ intersections of the graphs of ϕ and f. In other words, $\phi(\theta) - f(\theta)$ has at least $k+2$ zeros.

Proof of Theorem 1.6: We begin with a real trigonometric polynomial. Given a real trogonometric polynomial $T(\theta)$ with $|T(\theta)| \le 1$, we shall suppose

$$T'(\theta_0) > n \tag{1.19}$$

for some θ_0 and obtain a contradiction. Since we can replace $T(\theta)$ by $\lambda T(\theta)$ with λ slightly less than 1 and still have $\lambda T'(\theta_0) > n$, we may assume $|T(\theta)| < 1$.

Take the graph of $\sin n\theta$ and shift it horizontally until one of its arcs with positive slope meet the graph of $T(\theta)$ at θ_0. Note that, the shifted graph of $\sin n\theta$ has the form $\sin n(\theta - \theta^*)$ and takes the value ± 1 at $2n$ points with alternating sign; similar to that of $\sin n\theta$. At θ_0, from (1.19) we know that $T(\theta)$ has a larger slope than $\sin n(\theta - \theta^*)$ (whose slope is at most n). Now we have a similar situation as shown in Fig. 1.5. By Lemma 1.2, $\sin n(\theta - \theta^*)$ and $T(\theta)$ meet at at least $2n+2$ points between 0 and 2π. This implies that the trigonometric polynomial $T(\theta) - \sin n(\theta - \theta^*)$ has $2n+2$ zeros which is identically zero because any trogonometric polynomial of order n

$$\begin{aligned} S(\theta) &= a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \\ &= \sum_{-n}^{n} C_k e^{ik\theta} \\ &= e^{-in\theta} \sum_{k=0}^{2n} C_{k-n} e^{ik\theta} \end{aligned}$$

can have at most 2n zeros as of $\left|\sum_{k=0}^{2n} c_{k-n} z^k\right|$. Consequently, $T(\theta)$ is itself a shifted sine curve. A contradiction since $|T(\theta)| < 1$.

Now we consider trigonometric polynomials with complex coefficients. Let $T(\theta) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$ be a trigonometric polynomial of degree n with $|T(\theta)| \le 1$. $T(\theta)$ has only $2n+1$ number of coefficients whose possible values are in a bounded set since $|T(\theta)| \le 1$. For each θ_0,

$$|T'(\theta_0)| = \left|\sum_{\nu=1}^{n} (-\nu a_\nu \sin \nu\theta + \nu b_\nu \cos \nu\theta)\right|$$

is a real-valued continuous function defined over a compact set (the $(2n+1)$-dimensional set of possible coefficients) and hence attains its maximum. For θ_0, let such a trigonometric polynomial be denoted by $T_0(\theta)$, i.e., $|T_0(\theta)| \le 1$ and $|T_0'(\theta_0)|$ has the largest possible value. Since $T(\theta - \theta_0)$ is a trigonometric polynomial if $T(\theta)$ is, there is no loss of generality in assuming that $\theta_0 = 0$. Then, take an $T_0(\theta)$ such that $|T_0'(0)|$ has the largest possible value; we have to show that $|T_0'(0)| \le n$.

Since we can choose a real λ such that $e^{i\lambda} T_0'(0) > 0$, and

$$|\mathrm{Re}(e^{i\lambda} T_0(\theta))| \le |e^{i\lambda} T_0(\theta)| \le 1 ;$$

the derivative of $\mathrm{Re}\, e^{i\lambda} T_0(\theta)$ also has absolute value not exceeding n. From what we already know; in particular, this is true at $\theta = 0$, so that $0 < e^{i\lambda} T_0'(0) \le n$. Hence

$$|T_0'(0)| \le n ,$$

as asserted. □

As can be easily seen with the trigonometric polynomial $f(\varphi) = \sin n\varphi$, Theorem 1.6 is best possible. Now we consider a complex algebraic polynomial $P(z)$. Unless otherwise specified, $P(z)$ is of degree n. Suppose we have

$$|P(z)| = |P(e^{i\theta})| = |\sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}| \le 1 \, .$$

Let $T(\theta) = \sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}$, $T(\theta)$ is a trigonometric polynomial of degree n and by Theorem 1.6 we have $|T'(\theta)| \le n$. Since

$$\begin{aligned} |T'(\theta)| &= \left|\sum_{\nu=0}^{n} i\nu\, a_\nu e^{i\nu\theta}\right| \\ &= \left|\sum_{\nu=0}^{n} \nu a_\nu\, e^{i(\nu-1)\theta}\right| \left|ie^{i\theta}\right| \\ &= \left|\sum_{\nu=0}^{n} \nu a_\nu\, z^{\nu-1}\right| \\ &= |P'(z)| \end{aligned}$$

on $|z| = 1$, we have $|P'(z)| \le n$.

Therefore we have a complete proof of Theorem 1.1.

Remark: The inequality (1.1) is best possible. The extremal polynomial for the Bernstein Theorem is $P(z) = \alpha z^n$, $|\alpha| = 1$ for which there is equality in (1.1).

§1.4. A DIRECT PROOF OF THE BERNSTEIN'S THEOREM

In 1947, using the well known Gauss-Lucas Theorem on the location of zeros of the derivative of a polynomial, N.G. DeBruijn [5] gave a direct, short and simple proof of Theorem 1.1. In fact DeBruijn proved a more general result from where Theorem 1.1 follows. First we present:

Theorem 1.7: (Gauss-Lucas) The closed convex hull that contains all zeros of a polynomial $P(z)$ also contains all zeros of its derivative $P'(z)$.

Proof of Theorem 1.7: Let the degree of $P(z)$ be n and $z_1, z_2, \ldots, z_n$ be the n zeros of $P(z)$. Suppose that the closed convex hull containing the zeros of $P(z)$ is a closed half-plane H. Since given any closed half-plane H, there exists $c \in \mathbb{C}$, $c \neq 0$ and $\alpha \in \mathbb{R}$ such that

$$z \in H \text{ if and only if } \operatorname{Re} cz \geq \alpha \, .$$

then we have

$$\operatorname{Re} cz_\nu \geq \alpha \qquad \nu = 1,2,\ldots,n \, .$$

Let $z \notin H$, we wish to show that $P'(z) \neq 0$. Because $\operatorname{Re} cz < \alpha$, we have

$$\operatorname{Re} c(z - z_\nu) < 0 \qquad \nu = 1,2,\ldots,n \, .$$

hence

$$\operatorname{Re} \frac{\bar{c}}{z - z_\nu} < 0 \qquad \nu = 1,2,\ldots,n \, .$$

and

$$\operatorname{Re} \bar{c} \sum_{\nu=1}^{n} \frac{1}{z - z_\nu} < 0 \, .$$

It follows that

$$\frac{P'(z)}{P(z)} = \sum_{\nu=1}^{n} \frac{1}{z-z_\nu} \neq 0$$

and therefore we have $P'(z) \neq 0$. From the preceding discussion, we conclude that the zeros of $P'(z)$ are contained in the intersection of all closed half-planes containing the zeros of $P(z)$ and the intersection of these closed half-planes is the closed convex hull of the zeros of $P(z)$. □

Theorem 1.8: Let C be a convex region in the z-plane and ∂C be its boundary. Let $P(z)$ and $Q(z)$ be polynomials. Suppose $Q(z) \neq 0$ on $(C \cup \partial C)^c$, and the degree of $P(z)$ does not exceed that of $Q(z)$. Now if, $|P(z)| \leq |Q(z)|$ on ∂C, then

$$|P'(z)| \leq |Q'(z)| \qquad \text{on } \partial C.$$

Proof of Theorem 1.8: Let $D = (C \cup \partial C)^c$ and $f(z) = \frac{P(z)}{Q(z)}$. Then f is analytic on D (including at infinity) because $Q(z) \neq 0$ on D and degree of $P(z)$ does not exceed that of $Q(z)$. Since $|P(z)| \leq |Q(z)|$ on ∂C one has $|f(z)| \leq 1$ on ∂C and so by maximum modulus Theorem we have $|f(z)| \leq 1$ on $\partial C \cup D$. This implies that the zeros of $P(z) - \lambda Q(z)$, for $|\lambda|>1$ belongs to C because if α is a zero of $P(z) - \lambda Q(z)$, then $\left|\frac{P(\alpha)}{Q(\alpha)}\right| = |\lambda|>1$ implying $\alpha \notin \partial C \cup D$. By Gauss-Lucas Theorem all the zeros of $P'(z) - \lambda Q'(z)$ also belong to C a convex hull which contains the zeros of $P(z) - \lambda Q(z)$, for any λ with $|\lambda|>1$.

Suppose contrary, if for any $z \notin C$, $|P'(z)|>|Q'(z)|$. We can choose a λ with $|\lambda|>1$ for which $P'(z) - \lambda Q'(z) = 0$. This contradicts the foregoing conclusion that all the zeros of

$P'(z) - \lambda Q'(z)$ for any $|\lambda|>1$ lie in C. Consequently $|P'(z)| \leq |Q'(z)|$ for $z \in D$ and by maximum modulus Theorem along with the fact that $Q'(z) \neq 0$ in D we have

$$|P'(z)| \leq |Q'(z)| \qquad \text{for } z \in \partial C. \quad \square$$

Theoren 1.1 can now be easily derived from Theorem 1.8. Let $Q(z) = z^n$ and $C = \{z : |z|<1\}$, the condition in Theorem 1.8 can be written as $|P(z)| \leq |z^n|$ on $|z| = 1$. Moreover the degree of $P(z)$ which is n does not exceed that of $Q(z) = z^n$ and all the zeros of $Q(z)$ are in $|z|<1$. Since

$$|Q'(z)| = n|z^{n-1}| \leq n \qquad \text{for } |z| = 1,$$

we get

$$|P'(z)| \leq n \qquad \text{for } |z| \leq 1.$$

§1.5. BERNSTEIN'S THEOREM IN L_p-NORM

In 1932, A. Zymund [30] proved a generalization of the Bernstein's Theorem in which he considered L_p-norm of $P(z)$ rather than sup-norm.

Theorem 1.9: Let $P(z)$ be a polynomial of degree n. Then for $p \geq 1$

$$\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta \leq n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \qquad (1.20)$$

The result is best possible and equality in (1.20) holds for $P(z) = \alpha z^n$, $|\alpha| = 1$.

In fact, Zymund proved a corresponding result for trigonometric polynomials in:

Theorem 1.10: Let $f(\theta)$ be an arbitrary trigonometric polynomial of order n and let $\phi(u)$ $(u \geq 0)$ be a convex function, increasing with u. If

$$\int_0^{2\pi} \phi(|f(\theta)|)d\theta = M$$

then

$$\int_0^{2\pi} \phi\left(\left|\frac{f'(\theta)}{n}\right|\right) d\theta \leq M . \qquad (1.21)$$

If ϕ is strictly increasing, the result is best possible and equality in (1.21) holds for $f(\theta) = A \sin n(\theta-\alpha)$ for any constant A and α.

Proof of Theorem 1.10: Let $f(\theta) = \sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}$, we note that

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta+u)[\sin u + 2 \sin 2u + \ldots + n \sin nu]du$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\left[\frac{e^{iu} - e^{-iu}}{2i} + 2\,\frac{e^{i2u} - e^{-i2u}}{2i} + \ldots + n\,\frac{e^{inu} - e^{-inu}}{2i}\right]du$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\,\frac{e^{iu} - e^{-iu}}{2i}\,du + \frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\,2\,\frac{e^{i2u} - e^{-i2u}}{2i}\,du$$

$$+ \ldots + \frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\,n\,\frac{e^{inu} - e^{-inu}}{2i}\,du$$

and

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\ \frac{e^{iu}-e^{-iu}}{2i}\ du$$

$$=\frac{1}{\pi}\int_0^{2\pi}\sum_{\nu=-n}^{n} a_\nu\ e^{i\nu(\theta+u)}\ \frac{e^{iu}-e^{-iu}}{2i}\ du$$

$$=\frac{1}{\pi}\int_0^{2\pi}\left(a_{-n}\ e^{-in(\theta+u)}+\ldots+a_0+a_1e^{i(\theta+u)}+\ldots+a_ne^{in(\theta+u)}\right)\cdot$$

$$\cdot\ \frac{e^{iu}-e^{-iu}}{2}\ du$$

$$=\frac{1}{\pi}\int_0^{2\pi} a_{-n}e^{-in(\theta+u)}\ \frac{e^{iu}-e^{-iu}}{2i}\ du+\ldots+\frac{1}{\pi}\int_0^{2\pi} a_0\ \frac{e^{iu}-e^{-iu}}{2i}\ du$$

$$+\frac{1}{\pi}\int_0^{2\pi} a_1e^{i(\theta+u)}\ \frac{e^{iu}-e^{-iu}}{2i}\ du+\ldots+\frac{1}{\pi}\int_0^{2\pi} a_ne^{in(\theta+u)}\ \frac{e^{iu}-e^{-iu}}{2}\ du$$

$$=\frac{a_1}{2\pi i}\int_0^{2\pi}\left(e^{i\theta+i2u}-e^{i\theta}\right)du^{1}$$

$$=ia_1e^{i\theta}\ .$$

Similarly, we have

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta+u)\ 2\ \frac{e^{i2u}-e^{-i2u}}{2i}\ du\ =\ 2ia_2e^{i2\theta}$$

$$\vdots$$

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta+u)n\ \frac{e^{inu}-e^{-inu}}{2i}\ du\ =\ nia_ne^{in\theta}\ .$$

1.
$$\frac{a_{-n}}{2\pi i}\int_0^{2\pi}\left(e^{-in\theta-i(n-1)u}-e^{-in\theta-i(n+1)u}\right)du=\frac{a_{-n}e^{-in\theta}}{2\pi i}$$

$$\left[\int_0^{2\pi} e^{-i(n-1)u}\ du-\int_0^{2\pi} e^{-i(n+1)u}du\right]=0.$$

Hence we have the following identity.

$$f'(\theta) = i\sum_{\nu=1}^{n} \nu\, a_\nu\, e^{i\nu\theta}$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\theta+u)[\sin u + 2\sin 2u + \ldots + n \sin nu]du \,. \tag{1.22}$$

Add the polynomial $(n-1)\sin(n+1)u + (n-2)\sin(n+2)u + \ldots + \sin(2n-1)u$, to the expression in the above square bracket and take together $k \sin ku$ and $k \sin(2n-k)u$; this gives

$$[\sin u + \sin(2n-1)u] + 2[\sin 2u + \sin 2(n-1)u]$$

$$+ \ldots + (n-1)[\sin(n+1)u + \sin(n-1)u] + n \sin nu$$

$$= 2\Big\{\sin nu \cos(n-1)u + 2[\sin nu \cos(n-2)u]$$

$$+ \ldots + (n-1)(\sin nu \cos u) + \frac{n}{2}\sin nu\Big\}$$

$$= 2 \sin nu\Big\{\cos(n-1)u + 2\cos(n-2)u + \ldots + (n-1)\cos u + \frac{n}{2}\Big\}$$

$$= 2 \sin nu\Big\{\cos u + [\cos u + \cos 2u] + \ldots + [\cos u + \cos 2u$$

$$+ \ldots + \cos(n-1)u] + \frac{n}{2}\Big\}$$

$$= 2 \sin nu\Big\{\frac{n}{2} + \sum_{k=1}^{n-1}(\cos u + \cos 2u + \ldots + \cos ku)\Big\}. \tag{1.23}$$

Note that

$$\cos u + \cos 2u + \ldots + \cos nu$$

$$= \frac{e^{iu}+e^{-iu}}{2} + \frac{e^{i2u}+e^{-i2u}}{2} + \ldots + \frac{e^{inu}+e^{-inu}}{2}$$

$$= \frac{1}{2}\left\{\frac{e^{iu}(1-e^{inu})}{1-e^{iu}} + \frac{e^{-iu}(1-e^{-inu})}{1-e^{-iu}}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{iu}(1-e^{inu})}{e^{iu/2}(e^{-iu/2}-e^{iu/2})} + \frac{e^{-iu}(1-e^{-inu})}{e^{-iu/2}(e^{iu/2}-e^{-iu/2})}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{iu/2}(1-e^{inu})}{e^{-iu/2}-e^{iu/2}} + \frac{e^{-iu/2}(1-e^{-inu})}{e^{iu/2}-e^{-iu/2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{-e^{iu/2}(1-e^{inu}) + e^{-iu/2}(1-e^{-inu})}{2i\sin\frac{u}{2}}\right\}$$

$$= \frac{1}{2}\,\frac{-\sin\frac{u}{2} + \sin(\frac{2n+1}{2})u}{\sin\frac{u}{2}}$$

$$= \frac{1}{2}\left(-1 + \frac{\sin(\frac{2n+1}{2})u}{\sin\frac{u}{2}}\right),$$

therefore (1.23) becomes

$$2\sin nu\left\{\frac{1}{2}\left(-n + \sum_{k=0}^{n-1}\frac{\sin(\frac{2k+1}{2})u}{\sin\frac{u}{2}}\right) + \frac{n}{2}\right\}$$

$$= 2\sin nu \sum_{k=0}^{n-1}\frac{\sin(\frac{2k+1}{2})u}{\sin\frac{u}{2}}$$

$$= 2n\sin nu\, K_{n-1}(u)$$

where $K_{n-1}(u)$ is the Fejer's Kernel. Hence from (1.22) we have

$$f'(\theta) = \frac{2n}{\pi}\int_0^{2\pi} f(\theta+u)\sin nu\, K_{n-1}(u)du.$$

Let $\lambda(u) = f(\theta+u)K_{n-1}(u)$, an integration by parts[1] gives

$$f'(\theta) = \frac{2}{\pi}\int_0^{2\pi} \lambda'(u)\cos nu\,du\ . \tag{1.24}$$

In this integral $\lambda'(u)$ is a trigonometric polynomial of degree $2n-1$ and so $\cos nu$ may be replaced by any function whose Fourier Series is of the form

$$\cos nu + a_{2n}\cos 2nu + a_{2n+1}\cos(2n+1)u + \ldots .$$

Let

$$\psi(u) = \begin{cases} 1 & 0 \le u < \frac{\pi}{2} \\ 0 & u = \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < u \le \pi \end{cases}$$

which is an even function with period 2π. The Fourier Series expansion[2] of ψ is given by

$$\psi(u) = \frac{2}{\pi}\left[\cos u - \frac{1}{3}\cos 3u + \ldots\right]$$

---

1. $\int_0^{2\pi}\lambda(u)\sin nu\,du = -\frac{1}{n}\int_0^{2\pi}\lambda(u)d\cos nu$

$$= -\frac{1}{n}\left[\lambda(u)\cos nu\Big|_0^{2\pi} - \int_0^{2\pi}\cos nu\,\lambda'(u)du\right]$$

$$= \frac{1}{n}\int_0^{2\pi}\lambda'(u)\cos nu\,du.$$

2. Consider

$$a_n = \int_{-\pi}^{\pi}\psi(x)\sin nx\,dx = \int_{-\pi}^{0}\psi(x)\sin nx\,dx + \int_0^{\pi}\psi(x)\sin nx\,dx = 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(x)\cos nx\,dx = \frac{1}{2\pi}\left[\int_0^{\pi/2}\cos nx\,dx - \int_{\pi/2}^{\pi}\cos nx\,dx - \int_{\pi}^{3\pi/2}\cos nx\,dx + \int_{3\pi/2}^{2\pi}\cos nx\,dx\right] = \frac{1}{2\pi n}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} - \sin n\frac{3\pi}{2} - \sin n\frac{3\pi}{2}\right],$$

therefore $n=1$, $b_1 = \frac{2}{\pi}$; $n=2$, $b_2 = 0$; $n=3$, $b_3 = \frac{-2}{3\pi}$; ....

and

$$\psi(nu) = \frac{2}{\pi}\left[\cos nu - \frac{1}{3}\cos 3nu + \frac{1}{5}\cos 5nu - \dots\right]$$

In (1.24), the factor $2\frac{\cos nu}{\pi}$ can be replaced by $\psi(nu)$, i.e.,

$$f'(\theta) = \int_0^{2\pi} \lambda'(u)\psi(nu)du \ .$$

An integration by parts[1] and putting back $\lambda(u) = f(\theta+u)K_{n-1}(u)$ gives

$$f'(\theta) = -\int_0^{2\pi} f(\theta+u)K_{n-1}(u)d\psi(nu) \ . \qquad (1.25)$$

Putting $\theta = 0$, $f(u) = \sin nu$ we have

$$1 = -\int_0^{2\pi} \frac{K_{n-1}(u)}{n} \sin nu \, d\psi(nu) \ .$$

On the interval $[0,2\pi]$, for any partition $\{u_i\}$ within the norm $\frac{\pi}{n}$, we note that $\psi(nu_i) - \psi(nu_{i-1}) < 0$ if $[u_{i-1},u_i]$ contains a point $\eta = \frac{2k+1}{2n}\pi$ for which $\sin n\frac{2k+1}{2n}\pi = 1$ and $\psi(nu_i) - \psi(nu_{i-1}) > 0$ if $[u_{i-1},u_i]$ contains a point $\eta = \frac{2k+1}{2n}\pi$ for which $\sin n\frac{2k+1}{2n}\pi = -1$. If no such point lies in the interval $[u_{i-1},u_i]$, then $\psi(nu_i) - \psi(nu_{i-1}) = 0$. Moreover, the Fejer's Kernel $K_{n-1}(u) > 0$. Thus from (1.25) we have

---

1. $\int_0^{2\pi} \lambda'(u)\psi(nu)du = \int_0^{2\pi} \psi(nu)d\lambda(u) = \psi(nu)\lambda(u)\Big|_0^{2\pi} - \int_0^{2\pi} \lambda(u)d\psi(nu)$.

$$1 = \int_0^{2\pi} -\frac{K_{n-1}(u)}{n} \sin nu \, d\,\psi(nu)$$

$$= \int_0^{2\pi} K_{n-1}(u) \left|\frac{d\psi(nu)}{n}\right|$$

$$= \sum_{i=0}^{2n} \alpha_i$$

where $\alpha_i = K_{n-1}\left(\frac{2k+1}{2n} \pi \right) \geq 0$.

From (1.25) it follows that

$$\left|\frac{f'(\theta)}{n}\right| \leq \int_0^{2\pi} |f(\theta+u)| \; k_{n-1}(u) \left|\frac{d\psi(nu)}{n}\right|$$

$$= \sum_{i=1}^{2n} \alpha_i \; |f(\theta+u_i)| \; , \tag{1.26}$$

here $u_1, \ldots, u_{2n}$ denotes the consecutive points of discontinuity of $\psi(nu)$ in $(0,2\pi)$. Since $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n} = 1$ and ϕ is convex, by applying Jensen's inequality to (1.26), we have

$$\phi\left(\left|\frac{f'(\theta)}{n}\right|\right) \leq \phi\left(\sum_{i=1}^{2n} \alpha_i \left|f(\theta+u_i)\right|\right)$$

$$\leq \sum_{i=1}^{2n} \alpha_i \; \phi(|f(\theta+u_i)|) \; . \tag{1.27}$$

Integrating (1.27), we get

$$\int_0^{2\pi} \phi\left(\left|\frac{f'(\theta)}{n}\right|\right) d\theta \leq \sum_{i=1}^{n} \alpha_i \int_0^{2\pi} \phi(|f(\theta+u_i)|) d\theta$$

$$= \int_0^{2\pi} \phi(|f(\theta)|) \; d\theta \; . \qquad \square$$

Now we return to complete the

Proof of Theorem 1.9: Knowing that $|u|^p$; $p \geq 1$ is a convex function increasing with u and $P(e^{i\theta}) = \sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}$ can be considered as a trigonometric polynomial of order n, Theorem 1.9 follows immediately from Theorem 1.10. □

§1.6. AN ESTIMATE OF $|P'(z)|$ ON AN ELLIPSE.

In [5], De Bruijn had pointed out that analogous results may be obtained for general convex domains using Theorem 1.8. But so far, for domains with non-empty interior other than $|z| \leq 1$, we do not have precise estimates of $|P'(z)|$. For the sake of interest, in this section, we present an isolated result in this direction which is due to W.E. Sewell [26].

Theorem 1.11: Let E be an ellipse in the z-plane with semi-axes a and b, $a>b$, centered at the origin. Let $P(z)$ be a polynomial of degree n with $|P(z)| \leq M$ for $z \in E$. Then for $z \in E$

$$|P'(z)| \leq Mn/b \ .$$

Proof of Theorem 1.11: If $z \in E$ then

$$z = x + iy = a \cos\theta + ib \sin\theta \ , \qquad (1.28)$$

so $P(z)$ for $z \in E$ can be written as a trigonometric polynomial of order n as follows

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

$$= \sum_{\nu=0}^{n} (A_\nu \cos \nu\theta + B_\nu \sin \nu\theta)$$

$$\doteq Q(\theta).$$

Thus

$$P'(z) = \frac{d}{dz} P(z) = \frac{d}{d\theta} Q(\theta) \frac{d\theta}{dz} = Q'(\theta) \frac{d\theta}{dz} \tag{1.29}$$

If $|P(z)| \le M$ on E then $|Q(\theta)| \le M$. By Theorem 1.6 we know that

$$|Q'(\theta)| \le Mn . \tag{1.30}$$

Also from (1.28) we have

$$\frac{dz}{d\theta} = -a \sin \theta + ib \cos \theta$$

whence

$$\left|\frac{dz}{d\theta}\right|^2 = a^2 \sin^2\theta + b^2\cos^2\theta$$

$$= a^2 - (a^2 - b^2) \cos^2\theta$$

$$\ge b^2 ,$$

and implies

$$\left|\frac{dz}{d\theta}\right| \ge b$$

or

$$\left|\frac{d\theta}{dz}\right| \le \frac{1}{b} . \tag{1.31}$$

Take the absolute value of (1.29), together with (1.30) and (1.31) we get

$$|P'(z)| \le \frac{Mn}{b} \qquad \text{for } z \text{ on } E. \qquad \square$$

CHAPTER II

ERDÖS-LAX THEOREM

The extremal polynomial for Bernstein's Theorem is $P(z) = Az^n$ which has all its zeros at the origin. So if we restrict ourselves to the class of polynomials having no zeros at the origin and excluding even the cases where the limiting polynomial has the form $P(z) = Az^n$, the constant 'n' in the inequality (1.1) is expected to be replaced by a constant smaller than n. With this observation P. Erdös suggested to study the influence of zeros on the estimates of $|P'(z)|$ and proposed a conjecture which was later proved by P.D. Lax [13] in the following:

Theorem 2.1: (Erdös-Lax Theorem) If $P(z)$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$ and $P(z)$ has no zeros on $|z| < 1$. Then

$$|P'(z)| \le n/2 \tag{2.1}$$

for $|z| \le 1$. The result is best possible and equality in (2.1) holds for $P(z) = \frac{z^n+1}{2}$.

§2.1. THE WORKS OF POLYÁ AND SZEGÖ

Erdös' conjecture was first studies by G. Polyá and G. Szegö. They both assumed the zeros of $P(z)$ are on $|z| = 1$ rather than in $|z| \ge 1$. In fact they proved:

<u>Theorem 2.2: If $P(z)$ is a polynomial of degree n with</u> $\max_{|z|=1} |P(z)| = 1$ <u>and all its zeros on $|z| = 1$, then</u>

$$|P'(z)| \le n/2 \tag{2.2}$$

<u>for</u> $|z| \le 1$.

<u>Proof of Theorem 2.2</u>: (due to G. Szegö) Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$ and $|z_\nu| = 1$ for every $\nu = 1,2,\ldots,n$. Suppose $\max_{|z|=1} |P'(z)| = |P'(z_0)|$. By maximum modulus Theorem, $|z_0| = 1$.

Case I: Suppose $P(z_0) \neq 0$. Since $P'(z_0 e^{i\theta})\, \overline{P}'(\overline{z}_0 e^{-i\theta})$ is maximum for $\theta = 0$ and because

$$\frac{d}{d\theta}[P'(z_0 e^{i\theta})\, \overline{P}'(\overline{z}_0 e^{-i\theta})]$$

$$= \overline{P}'(\overline{z}_0 e^{-i\theta})\, P''(z_0 e^{i\theta})\, z_0 e^{i\theta} i - P'(z_0 e^{i\theta}) \overline{P}''(\overline{z}_0 e^{-i\theta})\, \overline{z}_0 e^{-i\theta} i$$

$$= 0$$

for $\theta = 0$, one has $\overline{P}'(\overline{z}_0) P''(z_0) z_0 = P'(z_0) \overline{P}''(\overline{z}_0) \overline{z}_0$ and therefore we know that $z_0 \dfrac{P''(z_0)}{P'(z_0)}$ is real. Now

$$z \frac{P'(z)}{P(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu}$$

$$z^2 \frac{P''(z)}{P(z)} = \left(\sum_{\nu=1}^{n} \frac{z}{z - z_\nu}\right)^2 - \sum_{\nu=1}^{n} \left(\frac{z}{z - z_\nu}\right)^2 \tag{2.3}$$

thus

$$z \frac{P''(z)}{P'(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu} - \frac{\sum_{\nu=1}^{n} \left(\frac{z}{z - z_\nu}\right)^2}{\sum_{\nu=1}^{n} \frac{z}{z - z_\nu}}$$

Let $\frac{z_0}{z_0 - z_\nu} = \frac{1}{2} + it_\nu$, then

$$z_0 \frac{P''(z_0)}{P'(z_0)} = \frac{n}{2} + i \sum_{\nu=1}^{n} t_\nu - \frac{\sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right) + i \sum_{\nu=1}^{n} t_\nu}{\frac{n}{2} + i \sum_{\nu=1}^{n} t_\nu}$$

$$= \frac{n}{2} + i \sum_{\nu=1}^{n} t_\nu - \left\{\frac{n}{2} \sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right) + \left(\sum_{\nu=1}^{n} t_\nu\right)^2 + i\left[\frac{n}{2} \sum_{\nu=1}^{n} t_\nu - \left(\sum_{\nu=1}^{n} t_\nu\right)\left(\sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right)\right)\right]\right\} \Big/ \left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2 ,$$

because $z_0 \frac{P''(z_0)}{P'(z_0)}$ is real, this gives

$$\sum_{\nu=1}^{n} t_\nu = \frac{\frac{n}{2} \sum_{\nu=1}^{n} t_\nu - \left(\sum_{\nu=1}^{n} t_\nu\right)\left(\sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right)\right)}{\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2} \tag{2.4}$$

and

$$z_0 \frac{P''(z_0)}{P(z_0)} = \frac{n}{2} - \frac{\frac{n}{2} \sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right) + \left(\sum_{\nu=1}^{n} t_\nu\right)^2}{\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2} \tag{2.5}$$

Case Ia. Suppose $\sum_{\nu=1}^{n} t_\nu = 0$, from (2.3)

$$z_0 \frac{P'(z_0)}{P(z_0)} = \frac{n}{2} + i \sum_{\nu=1}^{n} t_\nu = \frac{n}{2} ,$$

and

$$|P'(z_0)| = \frac{n}{2} \, |P(z_0)| \le \frac{n}{2} .$$

Case Ib: Suppose $\sum_{\nu=0}^{n} t_\nu \neq 0$, from (2.4)

$$\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2 = \frac{n}{2} - \sum_{\nu=1}^{n}\left(\frac{1}{4} - t_\nu^2\right) .$$

Hence from (2.5) we have

$$z_0 \frac{P''(z_0)}{P(z_0)} = \frac{n}{2} - \frac{\left(\frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^3 - \left(\frac{n}{2}\right)\left(\sum_{\nu=1}^{n} t_\nu\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2}{\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2}$$

$$= \frac{n}{2} - \frac{\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2 - \left(\frac{n}{2}\right)\left[\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2\right]}{\left(\frac{n}{2}\right)^2 + \left(\sum_{\nu=1}^{n} t_\nu\right)^2}$$

$$= \frac{n}{2} + \frac{n}{2} - 1$$

$$= n - 1$$

which implies

$$|P''(z_0)| = (n-1) \max_{|z|=1} |P'(z)| .$$

According to Bernstein's Theorem; see Remark 4.3, we know that (2.5) is possible only when

$$P'(z) = \text{constant } z^{n-1}$$

or

$$P(z) = \text{constant } (z^n + c) ,$$

$|c| = 1$ because all the zeros of $P(z)$ are assumed to be on $|z| = 1$. Since $\max_{|z|=1} |P(z)| = 1$, constant $= \frac{1}{2}$, this gives (2.2).

Case II. Suppose $P(z_o) = 0$. Since $P'(z_o) \neq 0$, z_o is a simple zero of $P(z)$, let $z_o = z_\mu$. Now

$$z_o \frac{P''(z_o)}{P'(z_o)} = \lim_{z \to z_\mu} z \frac{P''(z)}{P'(z)}$$

$$= \lim_{z \to z_\mu} \frac{\left(\sum_{\nu=1}^{n} \frac{z}{z - z_\nu}\right)^2 - \sum_{\nu=1}^{n} \left(\frac{z}{z - z_\nu}\right)^2}{\sum_{\nu=1}^{n} \frac{z}{z - z_\nu}}$$

$$= 2 \sum_{\nu \neq \mu} \frac{z_\mu}{z_\mu - z_\nu}$$

$$= (n-1) + 2i \sum_{\nu \neq \mu} t_\nu$$

which is real, hence equal to $(n-1)$; see Case Ib. □

<u>Proof of Theorem 2.2</u>: (due to G. Polyá) Let $c = |c|e^{i\gamma}$, $|c| \leq 1$, $z_\nu = e^{i\theta_\nu}$, then

$$h(\theta) = i^n e^{-i\gamma} \prod_{\nu=1}^{n} \left(e^{-i\frac{\theta_\nu}{2}}\right) P(e^{i2\theta})\, e^{-in\theta}$$

$$= i^n e^{-i\gamma}\, c \prod_{\nu=1}^{n} e^{-i\frac{\theta_\nu}{2}} \left(e^{i2\theta} - e^{i\theta_\nu}\right) e^{-in\theta}$$

$$= i^n |c| \prod_{\nu=1}^{n} \left(e^{i2\theta - i\frac{\theta_\nu}{2}} - e^{i\frac{\theta_\nu}{2}}\right) e^{-in\theta}$$

$$= |c| \prod_{\nu=1}^{n} \left(ie^{i(\theta - \frac{\theta_\nu}{2})} - ie^{-i(\theta - \frac{\theta_\nu}{2})}\right)$$

is a trigonometric polynomial of degree n with real coefficients. From a result of Van der Corput and Schaake [25], which is quite involving and uses the theory of homogeneous polynomials of two variables; Polyá established[1]:

$$h'^2(\theta) + n^2 h^2(\theta) \le n^2 , \tag{2.6}$$

and so

$$|2P'(e^{i2\theta})| = \left|\frac{d}{d\theta} P(e^{i2\theta})\right| = \left|\left[i^{-n} e^{+i\gamma} \Pi(e^{+i\theta_\nu/2})\right] \frac{d}{d\theta} e^{in\theta} h(\theta)\right|$$

$$= |h'(\theta) + inh(\theta)| \le n$$

Consequently,

$$|P'(z)| \le \frac{n}{2}$$

for $|z| \le 1$. □

§2.2. THE ERDÖS-LAX THEOREM

Finally in 1944, P.D. Lax [13] proved the Erdös' conjecture in the full form and the conjecture is now referred to as the Erdös-Lax Theorem. In the proof, Lax first introduced the polynomial $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, called the conjugate polynomial of $P(z)$. It is easy to see that $Q(z)$ is also a polynomial of degree n and if $re^{i\alpha}$ is a zero of $P(z)$, then $\frac{1}{r} e^{i\alpha}$ is a zero of $Q(z)$. Further, we note that if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, then

[1] Q.I. Rahman [19] gave an easier proof of (2.6) using a result like (1.2) for real trigonometric polynomials.

$$
\begin{aligned}
Q(z) &= z^n \overline{P\left(\frac{1}{\bar{z}}\right)} \\
&= z^n \overline{\sum_{\nu=0}^{n} a_\nu \left(\frac{1}{\bar{z}}\right)^\nu} \\
&= z^n \sum_{\nu=1}^{n} \bar{a}_\nu \left(\frac{1}{z^\nu}\right) \\
&= \sum_{\nu=1}^{n} \bar{a}_\nu \, z^{n-\nu} \\
&= \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \ldots + \bar{a}_n .
\end{aligned}
$$

In the following, we state two interesting relations between $P(z)$ and $Q(z)$ as two lemmas. By the use of these two lemmas and Theorem 2.2, Lax proved the Erdös' conjecture.

Lemma 2.1: If $P(z)$ is a polynomial of degree n having no zeros inside $|z|<1$, the polynomial $P(z) + \lambda Q(z)$, $|\lambda| = 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, will have all its zeros on $|z| = 1$.

Proof of Lemma 2.1: Since $P(z)$ has no zero inside $|z|<1$, for $|z|<1$, $\frac{Q(z)}{P(z)}$ is analytic and will attain its maximum on $|z| = 1$. Moreover we have

$$\left| \frac{Q(z)}{P(z)} \right| = 1 \qquad \text{on } |z| = 1$$

because $|Q(e^{i\theta})| = \left| e^{in\theta} \overline{P\left(\frac{1}{e^{-i\theta}}\right)} \right| = \left| \overline{P(e^{i\theta})} \right| = \left| P(e^{i\theta}) \right|$. Therefore by maximum modulus Theorem

$$\left| \frac{Q(z)}{P(z)} \right| < 1 \qquad \text{on } |z|<1 . \tag{2.7}$$

Hence for any λ, with $|\lambda| = 1$, we have $P(z) + \lambda Q(z) \neq 0$ on $|z|<1$. Otherwize, if there exists z_0, $|z_0|<1$ such that $P(z_0) + \lambda Q(z_0) = 0$, then $|P(z_0)| = |\lambda||Q(z_0)|$ which implies $\left|\frac{Q(z_0)}{P(z_0)}\right| = |\lambda| = 1$. This contradicts with (2.7). On the other hand, as $Q(z)$ has no zeros in $|z|>1$, $\frac{P(z)}{Q(z)}$ is analytic for $|z|>1$ and will also attain its maximum on $|z| = 1$. Again

$$\left|\frac{P(z)}{Q(z)}\right| = 1 \qquad \text{on } |z| = 1$$

and

$$\left|\frac{P(z)}{Q(z)}\right| < 1 \qquad \text{on } |z|>1 .$$

So, using the same argument as above, for any $|\lambda| = 1$, $P(z) + \lambda Q(z) \neq 0$ on $|z|>1$. Hence $P(z) + \lambda Q(z)$ must have all its zeros on $|z| = 1$.

<u>Lemma 2.2</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>having no zeros inside</u> $|z|<1$. <u>We have</u> $|P'(z)| \leq |Q'(z)|$ <u>on</u> $|z| = 1$.

<u>Proof of Lemma 2.2</u>: Let z_ν, $\nu = 1,2,\ldots,n$ be the zeros of $P(z)$. Let $z \neq z_\nu$ $A_\nu = z^{-1} z_\nu$ $\forall_\nu$. We find for $|z| = 1$

$$\left|\frac{P'(z)}{P(z)}\right| = \left|\sum_{\nu=1}^{n} \frac{1}{z - z_\nu}\right| = \left|\sum_{\nu=1}^{n} \frac{1}{1 - A_\nu}\right|$$

$$\left|\frac{Q'(z)}{Q(z)}\right| = \left|\sum_{\nu=1}^{n} \frac{1}{z - \bar{z}_\nu^{-1}}\right| = \left|\sum_{\nu=1}^{n} \frac{A_\nu}{1 - A_\nu}\right| .$$

Since on $|z| = 1$, $|A_\nu| \geq 1$, $A_\nu \neq 1$, we can suppose

$$A_\nu = r \cos \theta_\nu + ir \sin \theta_\nu$$

then

$$\frac{1}{1-A_\nu} = \frac{1}{1-r\cos\theta_\nu - ir\sin\theta_\nu}$$

$$= \frac{(1-r\cos\theta_\nu) + ir\sin\theta_\nu}{(1-r\cos\theta_\nu)^2 + (r\sin\theta_\nu)^2}$$

$$\operatorname{Re}\left(\frac{1}{1-A_\nu}\right) = \frac{1-r\cos\theta_\nu}{1-2r\cos\theta_\nu + r^2} \le \frac{1-r\cos\theta_\nu}{2(1-r\cos\theta_\nu)} = \frac{1}{2} .$$

Hence $\operatorname{Re} \sum_{\nu=1}^{n} \frac{1}{1-A_\nu} \le \frac{n}{2}$, also

$$\left| \operatorname{Re} \sum_{\nu=1}^{n} \frac{1}{1-A_\nu} \right| \le \left| \operatorname{Re} \sum_{\nu=1}^{n} \frac{1}{1-A_\nu} - n \right|$$

$$= \left| \operatorname{Re} \sum_{\nu=1}^{n} \left(\frac{1}{1-A_\nu} - 1\right) \right| \tag{2.8}$$

$$= \left| \operatorname{Re} \sum_{\nu=1}^{n} \frac{A_\nu}{1-A_\nu} \right| .$$

On the other hand

$$\left| \operatorname{Im} \sum_{\nu=1}^{n} \frac{1}{1-A_\nu} \right| = \left| \operatorname{Im} \sum_{\nu=1}^{n} \frac{1}{1-A_\nu} - n \right|$$

$$= \left| \operatorname{Im} \sum_{\nu=1}^{n} \left(\frac{1}{1-A_\nu} - 1\right) \right| \tag{2.9}$$

$$= \left| \operatorname{Im} \sum_{\nu=1}^{n} \frac{A_\nu}{1-A_\nu} \right| .$$

From (2.8) and (2.9) we have $\left|\operatorname{Re} \frac{P'(z)}{P(z)}\right| \le \left|\operatorname{Re} \frac{Q'(z)}{Q(z)}\right|$ and $\left|\operatorname{Im} \frac{P'(z)}{P(z)}\right| = \left|\operatorname{Im} \frac{Q'(z)}{Q(z)}\right|$, therefore we can conclude the lemma. □

Proof of Theorem 2.1: Let $P(z)$ be a polynomial as stated in the Theorem and $Q(z) = z^n \overline{P(\frac{1}{\bar z})}$, we define

$$q(z) = \frac{P(z) + \lambda Q(z)}{2} \qquad |\lambda| = 1.$$

Then

$$|q(z)| = \frac{|P(z) + \lambda Q(z)|}{2}$$

$$\le \frac{|P(z)| + |\lambda||Q(z)|}{2}$$

$$\le 1 \qquad |z| \le 1 .$$

By Lemma 2.1, $q(z)$ must have all its zeros on $|z| = 1$. By Theorem 2.2 we have

$$\max_{|z|=1} |q'(z)| \le \frac{n}{2} . \tag{2.10}$$

Now suppose $\max_{|z|=1} |P'(z)| = |P'(z_0)|$, $|z_0| = 1$. We can choose a particular λ with $|\lambda| = 1$ such that $\arg \lambda = -\arg \frac{Q'(z_0)}{P'(z_0)}$.

Hence by Lemma 2.2,

$$\max_{|z|=1} |q'(z)| \geq |q'(z_0)|$$

$$= \frac{|P'(z_0) + \lambda Q'(z_0)|}{2}$$

$$= \left|\frac{|P'(z_0)|+|Q'(z_0)|\lambda e^{i\arg Q'(z_0)/P'(z_0)}}{2}\right|$$

$$= \frac{|P'(z_0)|}{2} + \frac{|Q'(z_0)|}{2}$$

$$\geq |P'(z_0)|$$

$$= \max_{|z|=1} |P'(z)| .$$

Together with (2.10), the proof of Theorem 2.1 is established.

To show that $P(z) = \frac{1+z^n}{2}$ is extremal, we note that it has no zeros in $|z|<1$ and $\max_{|z|=1} |P(z)| = |P(1)| = 1$ and also $|P'(z)| = \left|n \frac{z^{n-1}}{2}\right| = \frac{n}{2}$. □

Concerning the estimate of $|P'(z)|$, P. Turán [27] established a result in the opposite direction as the following:

<u>Theorem 2.3</u>: (Turán) <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>with</u> $\max_{|z|=1} |P(z)| = 1$ <u>and</u> $P(z)$ <u>has no zeros in</u> $|z|>1$, <u>then</u>

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} . \tag{2.11}$$

The results is best possible and equality in (2.11) holds for $P(z) = (\alpha+\beta z^n)/2$ where $|\alpha| = |\beta| = 1$.

Proof of Theorem 2.3: Let $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$ where $|z_\nu| \le 1$, $\nu = 1,2,\ldots,n$ and suppose $|P(z)|$ attains its maximum at 1, i.e.,

$$|P(1)| = \max_{|z|=1} |P(z)| = 1.$$

Then, we have

$$\begin{aligned}
\max_{|z|=1} |P'(z)| &\ge |P'(1)| \\
&= \left|\frac{P'(1)}{P(1)}\right| \\
&= \left|\sum_{\nu=1}^{n} \frac{1}{1-z_\nu}\right| \\
&\ge \left|\mathrm{Re} \sum_{\nu=1}^{n} \frac{1}{1-z_\nu}\right| \\
&= \left|\mathrm{Re} \sum_{\nu=1}^{n} \frac{1-\bar{z}_\nu}{|1-z_\nu|^2}\right| \\
&= \left|\sum_{\nu=1}^{n} \frac{1-\mathrm{Re}z_\nu}{|1-z_\nu|^2}\right| . \qquad (2.12)
\end{aligned}$$

Since $1 - \mathrm{Re}z_\nu \ge 0$, the absolute value sign in (2.12) can be taken off. Further for each ν

$$\begin{aligned}
|1 - z_\nu|^2 &= |1 - \mathrm{Re}z_\nu - i\,\mathrm{Im}z_\nu|^2 \\
&= (1 - \mathrm{Re}z_\nu)^2 + (\mathrm{Im}z_\nu)^2 \\
&= 1 - 2\mathrm{Re}z_\nu + (\mathrm{Re}z_\nu)^2 + (\mathrm{Im}z_\nu)^2 \\
&\le 2 - 2\mathrm{Re}z_\nu \\
&= 2(1 - \mathrm{Re}z_\nu) .
\end{aligned}$$

Therefore, for each ν

$$\frac{1 - \mathrm{Re} z_\nu}{|1-z_\nu|^2} \geq \frac{1}{2} ,$$

and the proof follows from (2.12). □

Returning to the Erdös-Lax Theorem, we note that the extremal polynomial is not only $P(z) = (z^n + 1)/2$ but any polynomial having all its zeros on $|z|=1$ and of maximum modulus one on $|z|\leq 1$ is an extremal polynomial for Erdös-Lax Theorem. This is viewed by combining Erdös-Lax Theorem with Turán's Theorem. In fact if $P(z)$ has all its zeros on $|z| = 1$, we have

$$\frac{n}{2} \leq \max_{|z|=1} |P'(z)| \leq \frac{n}{2}$$

where the left inequality follows from Theorem 2.3 and the right inequality follows from Theorem 2.1.

§2.3. SELF-INVERSE POLYNOMIAL AND SELF-RECIPROCAL POLYNOMIAL

Equality in (2.1) may also be attained for polynomials not necessarily having all its zeros on $|z| = 1$. In fact, if $P(z)$ is a self-inverse polynomial, i.e., $P(z) = \mu Q(z)$; $|\mu| = 1$,

$Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then we always have equality in (2.1). This observation is reported by Saff and Sheil-Small in [23]. They proved:

<u>Theorem 2.4</u>: <u>If</u> $P(z)$ <u>is a self-inverse polynomial of degree</u> n <u>and</u> $|P(z)| \leq 1$ <u>on</u> $|z| \leq 1$, <u>then</u>

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} .$$

<u>Proof of Theorem 2.4</u>: Let $R(z) = P(z) - e^{i\alpha}$, $0 \leq \alpha < 2\pi$ and $T(z) = z^n \overline{R(\frac{1}{\bar z})} = z^n \overline{P(\frac{1}{\bar z})} - z^n e^{-i\alpha} = P(z) - z^n e^{-i\alpha}$. Since the maximum of $|P(z)|$ is one, $R(z)$ has no zeros lie in $|z| < 1$, we can apply Lemma 2.2 to get $|R'(z)| \leq |T'(z)|$ for $|z| = 1$. This means

$$|P'(z)| \leq |P'(z) - nz^{n-1}e^{i\alpha}| \qquad (2.13)$$

for $|z| = 1$. For a suitable choice of α, we get

$$|P'(z) - nz^{n-1}e^{i\alpha}| = n - |P'(z)| \qquad (2.14)$$

(2.13), (2.14) together yield

$$2|P'(z)| \leq n .$$

Now we need only to show that $\max_{|z|=1} |P'(z)| \geq \frac{n}{2}$. Let z_0, $|z_0| = 1$, be such that $|P(z_0)| = 1$, and let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$. Since the zeros of $P(z)$ are symmetric with respect to $|z| = 1$, we have for $|z_\nu| \neq 1$

$$\mathrm{Re}\left(\frac{z}{z-z_\nu} + \frac{z}{z-1/\bar z_\nu}\right) = \mathrm{Re}\left(\frac{1}{1-\bar z\, z_\nu} - \frac{\overline{\bar z\, z_\nu}}{1-\bar z\, z_\nu}\right) = 1 .$$

and for $|z_\nu| = 1$, $\mathrm{Re}\,\frac{z}{z-z_\nu} = \frac{1}{2}$. Hence

$$|P'(z_0)| \geq \mathrm{Re}\left[z_0 \frac{P'(z_0)}{P(z_0)}\right] = \sum_{\nu=1}^{n} \mathrm{Re}\,\frac{z_0}{z_0-z_\nu} \geq \frac{n}{2}$$

so that $|P'(z_0)| \geq \frac{n}{2}$. This completes the proof. □

In this direction, Q.I, Rahman defined self-reciprocal polynomial $P(z) = z^n P(\frac{1}{z})$ and proposed to study the estimate of $|P'(z)|$ for this class of polynomials. He proposed the following conjecture:

If $P(z)$ is a self-reciprocal polynomial of degree n such that $|P(z)| \leq 1$ for $|z| \leq 1$, then

$$|P'(z)| \leq \frac{n}{\sqrt{2}}$$

for $|z| \leq 1$.

If this conjecture were true then the polynomial $P(z) = z^n + 2iz^{\frac{n}{2}} + 1$, n even, would be an extremal polynomial. This can be seen in the following:

Since $P(z) = z^n + 2iz^{\frac{n}{2}} + 1$, $P'(z) = nz^{\frac{n}{2}-1}\left[z^{\frac{n}{2}} + i\right]$, then

$$\max_{|z|=1} |P(z)| = \max_{0\leq\theta\leq 2\pi} \left|\cos n\theta + i \sin n\theta + 2i(\cos \tfrac{n}{2}\theta + i \sin \tfrac{n}{2}\theta) + 1\right|$$

$$= \max_{0\leq\theta\leq 2\pi} \left|\left(\cos n\theta - 2 \sin \tfrac{n}{2}\theta + 1\right) + i\left(\sin n\theta + 2 \cos \tfrac{n}{2}\theta\right)\right|$$

$$= \max_{0\leq\theta\leq 2\pi} \left[\left(\cos n\theta - 2 \sin \tfrac{n}{2}\theta + 1\right)^2 + \left(\sin n\theta + 2 \cos \tfrac{n}{2}\theta\right)^2\right]^{1/2}$$

$$= \max_{0\leq\theta\leq 2\pi} \left[6+4\left(\sin n\theta \cos \tfrac{n}{2}\theta - \cos n\theta \sin \tfrac{n}{2}\theta\right) + 2 \cos n\theta - 4 \sin \tfrac{n}{2}\theta\right]^{1/2}$$

$$= \max_{0\leq\theta\leq 2\pi} \left[6 + 4 \sin \tfrac{n}{2}\theta + 2 \cos n\theta - 4 \sin \tfrac{n}{2}\theta\right]^{1/2}$$

$$= \sqrt{8}$$

and

$$\max_{|z|=1} |P'(z)| = n \max_{0\le\theta\le 2\pi} \left|\cos\frac{n}{2}\theta + i \sin\frac{n}{2}\theta + i\right|$$

$$= n \max_{0\le\theta\le 2\pi} \left[\cos^2\frac{n}{2}\theta + \left(\sin\frac{n}{2}\theta + 1\right)^2\right]^{1/2}$$

$$= n \max_{0\le\theta\le 2\pi} \left|\sin\frac{n}{2}\theta + 1\right|^{1/2} \sqrt{2}$$

$$= 2n . \quad \square$$

Rahman's conjecture is still an open problem in the study of polynomials. Some partial results are known in this direction [8].

§2.4. THE ERDÖS-LAX THEOREM IN L_2-NORM

In [13], Lax also proved the following analogy of Theorem 2.1, using L_2-norm rather than sup-norm.

Theorem 2.5: If $P(z)$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$ and $P(z) \ne 0$ on $|z| < 1$, then

$$\int_{-\pi}^{\pi} |P'(e^{i\theta})|^2 \, d\theta \le \frac{n^2}{2} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \, d\theta .$$

Proof of Theorem 2.5: By Lemma 2.2 the following inequality holds

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P'(e^{i\theta})|^2 \, d\theta \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q'(e^{i\theta})|^2 \, d\theta$$

where $Q(z) = \sum_{\nu=0}^{n} \bar{\alpha}_{n-\nu} z^{\nu}$. Therefore

$$\left(\frac{1}{\pi} - \frac{1}{2\pi}\right) \int_{-\pi}^{\pi} |P'(e^{i\theta})|^2 \, d\theta \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q'(e^{i\theta})|^2 d\theta$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|P'(e^{i\theta})|^2 d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|Q'(e^{i\theta})|^2 d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}|P'(e^{i\theta})|^2\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\nu=1}^{n}\nu\, \bar{\alpha}_{n-\nu}e^{i\nu\theta}\right)\overline{\left(\sum_{\nu=1}^{n}\nu\, \bar{\alpha}_{n-\nu}e^{i\nu\theta}\right)}\, d\theta$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\nu=0}^{n-1}(n-\nu)\alpha_{n-\nu}e^{i(n-\nu)\theta}\right)\cdot \overline{\left(\sum_{\nu=0}^{n-1}(n-\nu)\alpha_{n-\nu}e^{i(n-\nu)\theta}\right)}\, d\theta$$

$$= \sum_{\nu=0}^{n}(\nu^2+(n-\nu)^2)|\alpha_\nu|^2 .^1$$

Since the greatest of the numbers $\nu^2 + (n-\nu)^2$, $\nu = 1,2,\ldots,n$, is n^2, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|P'(e^{i\theta})|^2\, d\theta \le n^2 \sum_{\nu=0}^{n}|\alpha_\nu|^2$$

which implies

$$\int_{-\pi}^{\pi}|P'(e^{i\theta})|^2\, d\theta \le \frac{n^2}{2}\int_{-\pi}^{\pi}|P(e^{i\theta})|^2\, d\theta .^2 \qquad \square$$

---

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\nu=1}^{n}\nu\bar{\alpha}_{n-\nu}e^{i\nu\theta}\right)\left(\sum_{\nu=1}^{n}\nu\alpha_{n-\nu}e^{-i\nu\theta}\right)d\theta = \sum_{\nu=1}^{n}\nu^2|\alpha_\nu|^2\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta = \sum_{\nu=1}^{n}\nu^2|\alpha_\nu|^2$

   because all other terms involving $e^{i\nu\theta}$ or $e^{-i\nu\theta}$ will vanish after integrating.

2. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|P(e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\nu=0}^{n}\alpha_\nu e^{i\nu\theta}\right)\left(\sum_{\nu=0}^{n}\bar{\alpha}_\nu\, e^{-i\nu\theta}\right)d\theta$

   $= \sum_{\nu=0}^{n}|\alpha_\nu|^2\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta = \sum_{\nu=1}^{n}|\alpha_\nu|^2 .$

Note that equality in Theorem 2.5 holds for $P(z) = 1+z^n$, since

$$\int_{-\pi}^{\pi} |P'(e^{i\theta})|^2 d\theta = \int_{-\pi}^{\pi} n^2 \, d\theta = n^2 \, 2\pi$$

and

$$\frac{n^2}{2} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta = \frac{n^2}{2} \int_{-\pi}^{\pi} (1+e^{in\theta})(1+e^{-in\theta}) d\theta = n^2 \, 2\pi .$$

§2.5. THE ERDÖS-LAX THEOREM IN L_p-NORM

In 1947, instead of the sup-norm used in Theorem 2.1 and the L_2-norm used in Theorem 2.5, DeBruijn [5] made a further generalization by considering the L_p-norm. The result is also a refinement of Zymund's result in Theorem 1.9, by considering polynomials having no zeros in $|z| < 1$.

Theorem 2.6: *If the polynomial* $P(z)$ *of degree* n *has no zeros in* $|z|<1$, *then we have, for* $p \geq 1$,

$$\int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta \leq n^p \, C_p \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \qquad (2.15)$$

where $C_p = 2\pi / \int_0^{2\pi} |1+e^{i\eta}|^p \, d\eta = \dfrac{\sqrt{\pi}\, \Gamma(\frac{1}{2} p+1)}{2^p \Gamma(\frac{1}{2} p + \frac{1}{2})}$. *The result is best possible and equality in* (2.15) *holds for* $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Remark 2.1: Erdös-Lax Theorem is a special case of Theorem 2.6, namely when $p \to \infty$. It is also interesting to note that the extremal polynomials of the Erdös-Lax Theorem can have their zeros arbitrarily on $|z| = 1$. But for this case, the zeros of the extremal polynomials are further restricted on $|z| = 1$.

In order to prove Theorem 2.6, we need the following two theorems.

Theorem 2.7: (Schaake-Van der Corput) *If*

$$f(z_1,z_2,\ldots,z_n) = a_0+a_1\Sigma z_1+a_2\Sigma z_1z_2+\ldots+a_\mu\Sigma z_1z_2\ldots z_\mu+\ldots+a_nz_1\ldots z_n$$

where $\Sigma z_1z_2\ldots z_\mu$ *is the elementary symmetric function of* $z_1,z_2,\ldots,z_\mu$, *and if we put* $\lambda_n(z_1,z_2,\ldots,z_n) = \frac{1}{n}\sum_{\mu=0}^{n-1}\binom{n}{\mu}^{-1}\Sigma z_1z_2\ldots z_\mu$, *then we have the identity*

$$f(z_1,z_2,\ldots,z_n) = \sum_p \lambda_n\left(\frac{z_1}{p},\frac{z_2}{p},\ldots,\frac{z_n}{p}\right) f(p,p,\ldots,p)$$

where p *runs through the n-th roots of* $z_1z_2\ldots z_n$. *Further*

$$\sum_p \lambda_n\left(\frac{z_1}{p},\frac{z_2}{p},\ldots,\frac{z_n}{p}\right) = 1 \qquad (2.16)$$

and if $|z_1| = |z_2| = \ldots = |z_n| = 1$, *we have*

$$\lambda_n\left(\frac{z_1}{p},\frac{z_2}{p},\ldots,\frac{z_n}{p}\right) \geq 0 \ . \qquad (2.17)$$

Proof of Theorem 2.7: Let $0 \leq \rho \leq n$, $0 \leq \mu \leq n-1$, then

$$\sum_p p^{\rho-\mu} = \begin{cases} n & \text{if } \rho = \mu \\ nz_1\ldots z_n & \text{if } \rho = n,\ \mu = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_p \frac{f(p,p,\ldots,p)}{p^\mu} = \sum_p \frac{1}{p^\mu} \sum_{\rho=0}^{n} a_\rho \binom{n}{\rho} p^\rho$$

$$= \sum_{\rho=0}^{n} a_\rho \binom{n}{\rho} \sum_p p^{\rho-\mu}$$

$$= \begin{cases} a_\mu \binom{n}{\mu} n & \text{if } \mu \neq 0 \\ a_0 n + a_n n z_1 \ldots z_n & \text{if } \mu = 0 . \end{cases}$$

Hence

$$\sum_p \lambda_n \left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) f(p,\ldots,p)$$

$$= \frac{1}{n} \sum_p \sum_{\mu=0}^{n-1} \frac{1}{\binom{n}{\mu}} \left(\sum \frac{z_1 \ldots z_\mu}{p^\mu}\right) f(p,\ldots,p)$$

$$= \frac{1}{n} \sum_{\mu=0}^{n-1} \frac{1}{\binom{n}{\mu}} \left(\sum z_1 \ldots z_\mu\right) \sum_p \frac{f(p,\ldots,p)}{p^\mu}$$

$$= \frac{1}{n} \sum_{\mu=0}^{n-1} \frac{1}{\binom{n}{\mu}} \left(\sum z_1 \ldots z_\mu\right) a_\mu \binom{n}{\mu} n$$

$$+ \frac{1}{n} \cdot \frac{1}{\binom{n}{0}} .1. a_n n z_1 \ldots z_n$$

$$= f(z_1,z_2,\ldots,z_n)$$

and identity follows since

$$\sum_p \lambda_n\left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) = \sum_p \frac{1}{n}\sum_{\mu=0}^{n-1} \frac{1}{\binom{n}{\mu}} \sum \frac{z_1}{p}\cdots\frac{z_\mu}{p}$$

$$= \frac{1}{n}\sum_{\mu=0}^{n-1} \frac{1}{\binom{n}{\mu}} \sum \left(z_1\ldots z_\mu \sum \frac{1}{p^\mu}\right)$$

$$= 1 .$$

Now, the difficulty lies in proving that $\lambda_n \geq 0$ if all z_i have modulus 1. Putting $z_i = p\xi_i$ we have to establish that $\lambda_n(\xi_1,\ldots\xi_n) \geq 0$ if $|\xi_1| = \ldots = |\xi_n| = 1$, $\xi_1\ldots\xi_n = 1$. Let $b_\mu = \Sigma\, \xi_1\ldots\xi_\mu$ then

$$\bar{b}_\mu = \Sigma\, \overline{\xi_1\ldots\xi_\mu}$$

$$= \Sigma\overline{(\xi_{\mu+1}\ldots\xi_n)^{-1}}$$

$$= \Sigma\, \xi_{\mu+1}\ldots\xi_n$$

$$= b_{n-\mu} ,$$

also $\binom{n}{\mu} = \binom{n}{n-\mu}$, consequently λ_n is real. To show that λ_n cannot be negative we take

$$P(z) = z + z^2 + \ldots + z^{n-1} + \delta z^n$$

and

$$Q(z) = z^n - b_1 z^{n-1} + \ldots = (z-\xi_1)\ldots(z-\xi_n) .$$

Then define

$$\{P,Q\} = \frac{1}{\binom{n}{1}} b_1 + \frac{1}{\binom{n}{2}} b_2 + \ldots + \frac{1}{\binom{n}{n-1}} b_{n-1} + \delta b_n$$

$$= 1 + \frac{1}{\binom{n}{1}} \Sigma\xi_1 + \frac{1}{\binom{n}{2}} \Sigma\xi_1\xi_2 + \ldots + \delta\text{-}1$$

$$= n\lambda_n(\xi_1,\ldots,\xi_n) + \delta - 1 .$$

According to Kakeya's Theorem[1], if $\delta > 1$, $P(z)$ has no zeros for $|z| \geq 1$. Now the zeros of $P(z)$ are in $|z| < 1$ whereas the zeros of $Q(z)$ are in $|z| \geq 1$, so by Grace Apolarity Theorem[2], $\{P,Q\} \neq 0$. Thus if λ_n is negative we can suitably choose $\delta > 1$ such that $n\lambda_n + \delta - 1 = 0$ which is a contradiction. □

Theorem 2.8: If $f(z_1,\ldots,z_n) = a_0 + a_1\Sigma z_1 + \ldots + a_\mu \Sigma z_1 \ldots z_\mu + \ldots + a_n z_1 \ldots z_n$ and if $\phi(\omega)$ is a real and convex function of the complex variable ω, i.e.,

$$\phi(\alpha\omega_1 + \beta\omega_2) \leq \alpha\phi(\omega_1) + \beta\phi(\omega_2)$$

for $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$, then we have, for $|z_1| \leq 1, \ldots, |z_n| \leq 1$

---

[1] Kakeya's Theorem: If $P(z)$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$$

then $P(z)$ does not vanish in $|z| > 1$.

[2] Grace Apolarity Theorem: If $n \geq 1$, and

$$P(z) = a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \ldots + \binom{n}{n} a_n z^n$$

$$Q(z) = b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \ldots + \binom{n}{n} b_n z^n$$

and if $P(z)$ has no zeros in a circular domain C which contains all the zeros of $Q(z)$, then we have

$$\{P,Q\} = a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \ldots + (-1)^n \binom{n}{n} a_n b_0 \neq 0.$$

$$\int_0^{2\pi} \phi\left\{f(z_1e^{i\theta},\ldots,z_ne^{i\theta})\right\}d\theta \le \int_0^{2\pi} \phi\left\{f(e^{i\theta},\ldots,e^{i\theta})\right\}d\theta . \qquad (2.18)$$

<u>Proof of Theorem 2.8</u>: Since $f(z_1e^{i\theta},\ldots,z_ne^{i\theta})$ is a linear function of z_1, the left hand side of (2.18) is a convex function of z_1. Consequently its maximum for $|z_1| \le 1$ is attained at the boundary $|z_1| = 1$. The same applies to $z_2,\ldots,z_n$, and hence it is sufficient to prove (2.18) for the case

$$|z_1| = \ldots = |z_n| = 1.$$

By Theorem 2.7 we have

$$f(z_1e^{i\theta},\ldots,z_ne^{i\theta}) = \sum_p \lambda_n\left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) f(pe^{i\theta},\ldots,pe^{i\theta})$$

where p runs through the n-th roots of $z_1\ldots z_n$. λ_n satisfies (2.16) and (2.17). Since ϕ convex, we have

$$\phi\left\{f(z_1e^{i\theta},\ldots,z_ne^{i\theta})\right\} = \phi\left\{\sum_p \lambda_n\left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) f(pe^{i\theta},\ldots,pe^{i\theta})\right\}$$

$$\le \sum_p \lambda_n\left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) \phi\left\{f(e^{i\theta},\ldots,e^{i\theta})\right\}$$

$$\le \sum_p \lambda_n\left(\frac{z_1}{p},\ldots,\frac{z_n}{p}\right) \phi\left\{f(e^{i\theta},\ldots,e^{i\theta})\right\} .$$

On integrating with respect to θ from 0 to 2π, using (2.16) we obtain (2.18). □

<u>Proof of Theorem 2.6</u>: Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < 1$, $f(z_1,\ldots,z_n)$ be such that $f(z,\ldots,z) = P(z)$. We know that

$$f(z,\ldots,z,\xi) = P(z) + (\xi - z)\frac{P'(z)}{n} .$$

With Theorem 2.8, take $z_1 = z_2 = \ldots = z_{n-1} = 1$, $z_n = e^{i\eta}$ where η is real, and $\phi(\omega) = |\omega|^p$ $(p\geq 1)$ we have

$$\int_0^{2\pi} \phi\left\{f(z_1e^{i\theta},\ldots,z_{n-1}e^{i\theta},z_ne^{i\theta})\right\} d\theta$$

$$= \int_0^{2\pi} \phi\left\{f(e^{i\theta},\ldots,e^{i\theta}, e^{i(\eta+\theta)})\right\} d\theta$$

$$= \int_0^{2\pi} \left|P(e^{i\theta}) + e^{i(\eta+\theta)} \frac{P'(e^{i\theta})}{n} - e^{i\theta} \frac{P'(e^{i\theta})}{n}\right|^p d\theta$$

$$\leq \int_0^{2\pi} \phi\left\{f(e^{i\theta},\ldots,e^{i\theta})\right\} d\theta$$

$$= \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta .$$

Putting $A(\theta) = P(e^{i\theta}) - e^{i\theta} \frac{P'(e^{i\theta})}{n}$, $B(\theta) = e^{i\theta} \frac{P'(e^{i\theta})}{n}$ and integrating with respect to η we have

$$\int_0^{2\pi}\int_0^{2\pi} \left|A(\theta) + e^{i\eta} B(\theta)\right|^p d\eta d\theta \leq 2\pi \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \quad . . \quad (2.19)$$

Since by Lemma 4.7, $A(\theta) + \xi B(\theta) \neq 0$ for $|\xi|<1$, so

$$|B(\theta)| \leq |A(\theta)| .^1$$

---

1. $A(\theta) + \xi B(\theta) \neq 0$ implies $A(\theta) \neq -\xi B(\theta)$ which means with a suitable choice of the argument of ξ we have either

$$|A(\theta)| < |\xi||B(\theta)|$$

or $\qquad\qquad (2.20)$

$$|A(\theta)| > |\xi||B(\theta)| .$$

But a sufficiently small value of $|\xi|$ contradicts (2.20). Take $|\xi|\to 1$.

Also for $|a| \geq |b|$, we have

$$\int_0^{2\pi} \left|a + be^{i\eta}\right|^p d\eta \geq |b|^p \int_0^{2\pi} \left|1 + e^{i\eta}\right|^p d\eta .$$

Therefore, from (2.19),

$$\int_0^{2\pi} |B(\theta)|^p d\theta \cdot \int_0^{2\pi} \left|1 + e^{i\eta}\right|^p d\eta$$

$$\leq \int_0^{2\pi} \int_0^{2\pi} \left|A(\theta) + e^{i\eta} B(\theta)\right|^p d\eta \, d\theta$$

$$\leq 2\pi \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta ,$$

which implies

$$\int_0^{2\pi} \left|\frac{P'(e^{i\theta})}{n}\right|^p d\theta \leq \left(2\pi \Big/ \int_0^{2\pi} \left|1 + e^{i\eta}\right|^p d\eta\right) . \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta .$$

To show the equality holds for $P(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$ we note

$$\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta = \int_0^{2\pi} \left|\alpha + \beta e^{in\theta}\right|^p d\theta$$

$$= \int_0^{2\pi} \left||\beta| + |\beta| e^{i(n\theta+\beta)}\right|^p d\theta$$

$$= |\beta|^p \int_0^{2\pi} \left|1 + e^{i(n\theta+\beta-\alpha)}\right|^p d\theta$$

$$= |\beta|^p \int_0^{2\pi} \left|1 + e^{i\eta}\right|^p d\eta$$

and

$$\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta = 2\pi n^p |\beta|^p . \qquad \square$$

CHAPTER III

APPLICATIONS OF THE ERDÖS-LAX THEOREM

Erdös-Lax Theorem has been used to obtain results in different disciplines of the theory of polynomials. We shall present some of these results in this chapter. Using the Erdös-Lax Theorem we give some results of our own in section 3.1. In section 3.2, we present a result on integral mean estimates for algebraic and trigonometric polynomials with restricted zeros due to Saff and Sheil-Small. In section 3.3 we present the work of Lachance, Saff and Varga, concerning the relationship of two problems of polynomials having a prescribed zero at $z=1$. In section 3.4, we present two results due to Ankeny and Rivlin concering the estimates of $|P(z)|$ on $|z|=R$ with different restriction on the zeros of $P(z)$. Also we give some new observations in this direction using geometric arguments.

§3.1 CONCERNING THE MAXIMUM MODULUS OF A POLYNOMIAL

First we give an observation of our own:

<u>Theorem 3.1*</u>: <u>If</u> $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ <u>with</u> $c_n = 1$ <u>has no zeros in</u> $|z| < 1$ <u>and</u> $\max_{|z|=1} |P(z)| = 2$, <u>then</u> $P(z) = z^n + \alpha$ <u>where</u> $|\alpha| = 1$.

Before giving the proof, we need a known result due to C. Visser [28].

<u>Lemma 3.1</u>. <u>If</u> $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$, <u>then</u> $|c_0| + |c_n| \leq \max_{|z|=1} |P(z)|$.

<u>Proof of Lemma 3.1</u>: Let $c_o = |c_o|e^{i\alpha}$, $c_n = |c_n|e^{i\beta}$, and $\omega_1, \omega_2, \ldots, \omega_n$ be the n-th root of $e^{-i(\beta-\alpha)}$. Substituting each ω_i into $P(z)$ we have

$$P(\omega_1) = c_o + c_1\omega_1 + c_2\omega_1^2 + \ldots + c_n\omega_1^n$$
$$\vdots$$
$$P(\omega_n) = c_o + c_1\omega_n + c_2\omega_n^2 + \ldots + c_n\omega_n^n$$

respectively, and therefore

$$\begin{aligned} &|n\, c_o + c_1(\omega_1 + \ldots + \omega_n) + c_2(\omega_1^2 + \ldots + \omega_n^2) + \ldots + c_n(\omega_1^n + \ldots + \omega_n^n)| \\ &= |P(\omega_1) + \ldots + P(\omega_n)| \\ &\le n \max_{|z|=1} |P(z)| . \end{aligned} \tag{3.1}$$

From a property of the n-th root of unity we know that $\omega_1^i + \omega_2^i + \ldots + \omega_n^i = 0$ for each i, $1 \le i \le n-1$ and (3.1) can be reduced to

$$\left| c_o + c_n e^{-i(\beta-\alpha)} \right| \le \max_{|z|=1} |P(z)| ,$$

which implies

$$\left| |c_o|e^{i\alpha} + |c_n|e^{i\beta - i(\beta-\alpha)} \right| \le \max_{|z|=1} |P(z)|$$

and

$$|c_o| + |c_n| \le \max_{|z|=1} |P(z)| . \quad \square$$

Proof of Theorem 3.1: Since $\max_{|z|=1} |P(z)| = 2$, from Erdös-Lax Theorem $\max_{|z|=1} |P'(z)| = n$. Using Lemma 3.1 for the polynomial $P'(z) = \sum_{\nu=1}^{n} \nu c_\nu z^{\nu-1}$ one gets $n + |c_1| \le n$, but this implies $c_1 = 0$. Apply Lemma 3.1 again to the polynomial $\frac{P'(z)}{z} = \sum_{\nu=2}^{n} \nu c_\nu z^{\nu-2}$ one concludes that $c_2 = 0$. Consequently, all the coefficients $c_\nu = 0$ except c_n and c_0. Since $c_n = 1$ and $\max_{|z|=1} |P(z)| = 2$, one must have $c_0 = \alpha$ where $|\alpha| = 1$. □

For the next result we mainly consider polynomial $P(z)$ which has all its zeros on $|z| = 1$. We find that $P(z)$ and $P'(z)$ attain their maximum at the same point on $|z| = 1$, and if $P(z)$ attains its maximum n times on $|z| = 1$, then $P(z)$ has the form $\alpha z^n + \beta$, $|\alpha| = |\beta|$. Some related observations are also discussed. Now we present:

Theorem 3.2: Let $P(z)$ be a polynomial of degree n. If $P(z)$ has all its zeros on $|z| = 1$ and $|P(z)|$ attains its maximum on $|z| = 1$ at n points on $|z| = 1$, then

$$P(z) = \alpha z^n + \beta$$

where $|\alpha| = |\beta|$.

After we established Theorem 3.2, it was brought to our attention by Professor Q.I. Rahman that Z. Rubinstein has also conceived this result recently though yet unpublished; this is why we do not put * with this theorem. However, we present our own proof here and further observe that Theorem 3.2 is best possible in the sense that the condition that all the zeros are on $|z| = 1$ cannot be relaxed.

For proof we need the following lemmas, the first one is due to our own observation and the second one is well known.

<u>Lemma 3.2*</u> : <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>having all its zeros on</u> $|z| = 1$ <u>and</u> $\max_{|z|=1} |P(z)| = |P(z_0)|$, <u>then</u>

$$|P'(z_0)| = \frac{n}{2} |P(z_0)| .$$

<u>Proof of Lemma 3.2</u>: Since $P(z)$ has all its zeros on $|z| = 1$, by Erdös-Lax Theorem and the pointwise inequality (4.30) taking $k = 1$ we have

$$\max_{|z|=1} |P(z)| \frac{n}{2} \geq |P'(z)| \geq \frac{n}{2} |P(z)| \qquad (3.2)$$

on $|z| \leq 1$. If we put $z = z_0$ in (3.2) immediately we have

$$|P(z_0)| \frac{n}{2} = |P'(z_0)| = \frac{n}{2} |P(z_0)| . \qquad \square$$

<u>Lemma 3.3</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>and</u> $|P(z)|$ <u>attains its maximum at</u> $n+1$ <u>points on</u> $|z| = 1$, <u>then</u> $P(z) = \alpha z^n$.

<u>Proof of Lemma 3.3</u>: Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$. Suppose $|P(z)|$ attains its maximum M on $|z| = 1$ at $(n+1)$ points. Then

$$\left.\begin{aligned} & M^2 - |P(z)|^2 \\ &= M^2 - P(z)\overline{P(z)} \\ &= M^2 - \sum_{\nu=0}^{n} a_\nu z^\nu \cdot \sum_{\nu=0}^{n} \overline{a}_\nu \overline{z}^\nu \end{aligned}\right\} \qquad (3.3)$$

is a trigonometric polynomial of order n with the form

$$\sum_{\nu=-n}^{n} \alpha_\nu e^{i\nu\theta} = e^{-in\theta} \sum_{\nu=0}^{2n} \alpha_\nu^* e^{i\nu\theta} \qquad (3.4)$$

which is real and positive for every θ. We also note that (3.3) vanishes at $n+1$ points on $|z|=1$, hence every zero of (3.4) is a zero of even order and that (3.4) has at least $2n+2$ zeros. For a trigonometric polynomial of order n but with $2n+2$ zeros, by the Fundamental Theorem of algebra, the polynomial is identically equal to zero. Thus

$$M^2 - |P(z)|^2 \equiv 0$$

on $|z|=1$. Again we consider

$$P(z)\,\overline{P(z)} \equiv M^2$$

on $|z|=1$, using the same technique as above we have

$$e^{-in\theta} \sum_{\nu=0}^{2n} \beta_\nu e^{i\nu\theta} \equiv M^2$$

for every θ. This implies

$$\sum_{\nu=0}^{2n} \beta_\nu e^{i\nu\theta} \equiv M^2 e^{in\theta}$$

on $|z|=1$. As we compare the coefficient, we have

$$\beta_{2n} = \bar{a}_0\, a_n = 0 ,$$

since $a_n \neq 0$ we know that $a_0 = 0$ and

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z$$

$$= z(a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_2 z + a_1)$$

$$= zQ(z)$$

where $Q(z)$ is a polynomial of degree $n-1$. Repeat the whole process using $Q(z)$, we get $a_1 = 0$. Thus $a_{n-1} = a_{n-2} = \ldots = a_1 = 0$ and

$$P(z) = a_n z^n . \quad \square$$

Proof of Theorem 3.2*: From Lemma 3.2 we know that if $P(z)$ has all its zeros on $|z| = 1$ and z_0 is a point where $|P(z_0)| = 1$ then

$$|P'(z_0)| = \frac{n}{2} |P(z_0)| ,$$

i.e., $|P'(z)|$ attains its maximum at the same point as of $|P(z)|$. This implies that $P'(z)$, a polynomial of degree $n-1$, attains its maximum on $|z| = 1$ at n points on $|z| = 1$. Then by Lemma 3.3 we have

$$P'(z) = Az^{n-1}$$

which implies

$$P(z) = \alpha z^n + \beta .$$

Since $P(z)$ has all its zeros on $|z| = 1$, we must have $|\alpha| = |\beta|$. $\square$

Now we return to show that the result is best possible:

Example 3.1*: Let $P(z) = (z^2+1)(z^2-\rho^2)$ where ρ is suitably chosen being greater than or equal to one. If $\rho = 1$, it is obvious that $|P(z)|$ attains its maximum 2 at four points, namely $e^{ik\pi/4}$, $k = 1,3,5,7$. For $\rho > 1$, let the maximum be attained in the first quadrant at $e^{i\lambda}$, where $0 < \lambda < \frac{\pi}{2}$. By symmetry as shown in Fig. 3.1, the maximum is also attained in other three quadrants. In fact

$$\left|P(e^{i\lambda})\right| = \left|(e^{i\lambda}+i)(e^{i\lambda}-i)(e^{i\lambda}+\rho)(e^{i\lambda}-\rho)\right|$$

$$= \left|(e^{i(\pi-\lambda)}-i)(e^{i(\pi-\lambda)}+i)(e^{i(\pi-\lambda)}-\rho)(e^{i(\pi-\lambda)}+\rho)\right|$$

$$= \left|P(e^{i(\pi-\lambda)})\right|.$$

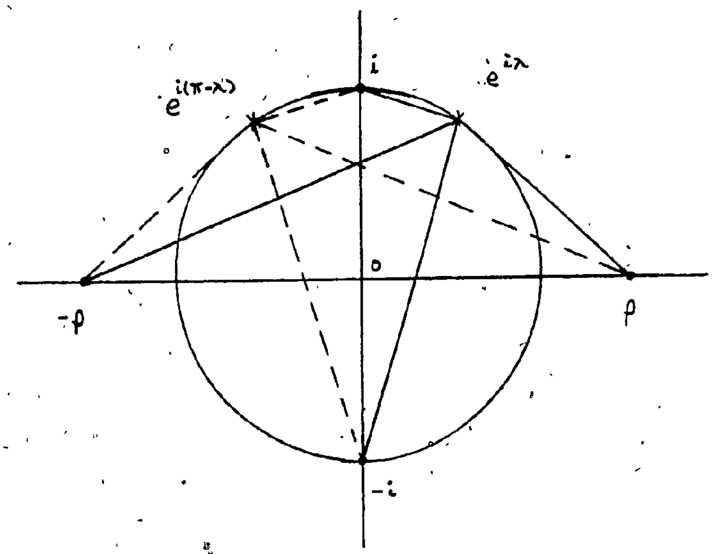


Fig. 3.1

We also note that for polynomial $P(z)$ of degree 2^m with zeros symmetrically distributed on $|z| = 1$, we can always choose n suitable zeros be moved out from $|z| = 1$ and $P(z)$ still attains its maximum at 2^m points on $|z| = 1$.

However in view of Theorem 3.1 and 3.3 the following conjecture seems plausible:

Let $P(z)$ be a polynomial of degree n with $\max_{|z|=1} |P(z)| = 2$ and $|P(z)|$ attains its maximum on $|z| = 1$ at n points. If one zero of $P(z)$ lies on $|z| = 1$, then $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta| = 1$.

§3.2. A RESULT ON INTEGRAL MEAN ESTIMATES FOR ALGEBRAIC AND TRIGONOMETRIC POLYNOMIALS WITH RESTRICTED ZEROS.

In 1940, P. Erdös [6] proposed the following conjecture:

Let $T(\theta)$ be a trigonometric polynomial of degree n having all zeros being real, i.e., $T(\theta)$ has $2n$ zeros in $[0,2\pi)$ and let

$$M = \max_{0\le\theta<2\pi} |T(\theta)| .$$

Then

$$\int_0^{2\pi} |T(\theta)|\, d\theta \le 4M .$$

This conjecture remained open for over 30 years. In 1973, using the Erdös-Lax Theorem and a result on principle of subordination due to Rogosinski, E.B. Saff and T. Sheil-Small [23] gave a complete proof of the above conjecture. Before presenting their works, for the sake of completeness, we first recall some concept about the principle of subordination.

Definition 3.1: Let D be a domain. A function f is called univalent in D if it is analytic and one-one in D.

Definition 3.2: Let f and F be analytic on $D = \{z: |z|<1\}$ with range $f(D)$ and $F(D)$ respectively. f is said to be subordinate to F in D if f and F satisfy the following conditions:

1. $f(D) \subset F(D)$
2. F is univalent in D
3. $f(0) = F(0)$.

f is called the subordinate function of F.

The following Theorem 3.3 is due to W.W. Rogosinski. For a proof see [11].

Theorem 3.3: If f is a subordinate function of F, $p > 0$, then

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \le \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta , \qquad r \le 1 .$$

In solving the conjecture, Saff and Sheil-Small established the following two theorems.

Theorem 3.4: If $P(z)$ is a polynomial of degree n having all its zeros on $|z| = 1$ and $\max_{|z|=1} |P(z)| = M$, then for each $p > 0$ we have

$$\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \le A_p \left(\frac{M}{2}\right)^p \qquad (3.5)$$

where

$$A_p = \int_0^{2\pi} |1+e^{i\theta}|^p \, d\theta = 2^{p+1} \sqrt{\pi} \, \frac{\Gamma\left(\frac{1}{2}p + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}p + 1\right)} .$$

Furthermore, the result is best possible and equality in (3.5) holds if and only if $P(z) = M(\lambda z^n + \mu)/2$, $|\lambda| = |\mu| = 1$.

Proof of Theorem 3.4: Suppose $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, since all the zeros of $P(z)$ are on $|z|=1$, there exists a constant u[1], $|u|=1$ such that

$$a_k = u\,\overline{a}_{n-k} \qquad \text{for } k=0,1,2,\ldots,n .$$

From the above relation we know that

$$P(z) = \frac{zP'(z) + uQ(z)}{n} \; {}^{2}$$

where $Q(z) = z^{n-1}\,\overline{P'\left(\frac{1}{\overline{z}}\right)}$. Let

$$\omega(z) = \frac{zP'(z)}{uQ(z)}$$

then

$$P(z) = \frac{uQ(z)}{n}(1+\omega(z)),$$

and since $|Q(z)| = |P'(z)|$ for $|z|=1$, it follows from Theorem 2.1 that

---

1. If the zeros of $P(z)$ are $e^{i\alpha_\nu}$, $\nu=1,2,\ldots,n$, then $u = e^{i(\alpha_1+\ldots+\alpha_n)}$

2. $P'(z) = a_1+2a_2z+\ldots+(n-1)a_{n-1}z^{n-2}+na_nz^{n-1}$

$Q(z) = \overline{a}_1z^{n-1}+2\overline{a}_2z^{n-2}+\ldots+(n-1)\overline{a}_{n-1}z+n\overline{a}_n$

then $zP'(z)+Q(z) = (a_1z+2a_2z^2+\ldots\ldots+(n-1)a_{n-1}z^{n-1}+na_nz^n)$

$+ (u\overline{a}_1z^{n-1}+2u\overline{a}_2z^{n-2}+\ldots+(n-1)u\overline{a}_{n-1}z+nu\overline{a}_n)$

$= n(a_0+a_1z+a_2z^2+\ldots+a_{n-1}z^{n-1}+a_nz^n)$.

$$|P(e^{i\theta})| = \frac{|u\, Q(e^{i\theta})|}{n} \left|1+\omega(e^{i\theta})\right|$$

$$= \frac{|P'(e^{i\theta})|}{n} \left|1+\omega(e^{i\theta})\right| \tag{3.6}$$

$$\leq \frac{M}{2} \left|1+\omega(e^{i\theta})\right|$$

By Gauss-Lucas Theorem, all the zeros of $P'(z)$ lie in $|z| \leq 1$, which implies all zeros of $Q(z)$ lie in $|z| \geq 1$. Hence $\omega(z)$ is analytic on $|z| \leq 1$. Furthermore, $\omega(0) = 0$, $|\omega(z)| = 1$ for $|z| = 1$. Thus $1+\omega(z)$ is subordinate to $1+z$ in $|z| < 1$ which is univalent. By (3.6) and Theorem 3.3 we have

$$\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \leq \left(\frac{M}{2}\right)^p \int_0^{2\pi} |1+\omega(e^{i\theta})|^p \, d\theta$$

$$\leq \left(\frac{M}{2}\right)^p \int_0^{2\pi} |1+e^{i\theta}|^p \, d\theta$$

$$= \left(\frac{M}{2}\right)^p A_p$$

where $A_p = \int_0^{2\pi} |1+e^{i\theta}|^p \, d\theta = 2^{p+1} \sqrt{\pi} \, \frac{\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}$.

Now suppose equality holds in (3.5). Then from (3.6) we must have $|P'(e^{i\theta})| = \frac{Mn}{2}$ for all θ. Since $P'(z)$ is a polynomial of degree $n-1$, by Lemma 3.3 we know that

$$P'(z) = \lambda \frac{Mn\, z^{n-1}}{2}$$

where $|\lambda| = 1$. But $P(z)$ must have all its zeros on $|z| = 1$, so

$$P(z) = M \frac{\lambda z^n + \mu}{2}$$

where $|\lambda| = |\mu| = 1$. Finally, for every polynomial of the form, we have

$$\int_0^{2\pi} \left| M \frac{\lambda e^{i\theta} + \mu}{2} \right|^p d\theta$$

$$= \left(\frac{M}{2}\right)^p \int_0^{2\pi} |\lambda e^{in\theta} + \mu|^p \, d\theta$$

$$= \left(\frac{M}{2}\right)^p A_p \; . \qquad \square$$

As an immediate consequence of Theorem 3.4, we have

Theorem 3.5: Let $T(\theta)$ be a trigonometric polynomial of degree n, all of whose zeros are real. Let $M = \max_{0 \le \theta \le 2\pi} |T(\theta)|$. Then for each $p > 0$ we have

$$\int_0^{2\pi} |T(\theta)|^p \, d\theta \le A_p \, (M/2)^p \tag{3.7}$$

The result is best possible and equality in (3.7) holds iff $T(\theta) = Me^{i\phi} \cos(n\theta + \tau)$ where ϕ and τ are real constants.

Proof of Theorem 3.5: Suppose $T(\theta) = \sum_{\nu=-n}^{n} \alpha_\nu e^{i\nu\theta}$, then

$$T(\theta) = e^{-in\theta} \sum_{\nu=0}^{2n} \alpha_{-n+\nu} \, e^{i\nu\theta}$$

$$= e^{-in\theta} \, P(e^{i\theta}) \; ,$$

where $P(z)$ is an algebraic polynomial of degree $2n$ having all its zeros on $|z| = 1$. Thus (3.7) follows from (3.5) and equality holds if and only if $P(z) = M \dfrac{(\lambda z^{2n} + \mu)}{2}$. But

$$e^{-in\theta}P(e^{i\theta}) = e^{-in\theta} M \cdot \frac{\lambda e^{i2n\theta} + \mu}{2}$$

$$= M \cdot \frac{\lambda e^{in\theta} + \mu e^{-in\theta}}{2}$$

$$= M \cdot \frac{e^{i(n\theta + \arg\lambda)} + e^{-i(n\theta - \arg\mu)}}{2}$$

$$= e^{i\phi} M \frac{e^{i(n\theta + \arg\lambda - \phi)} + e^{-i(n\theta - \arg\mu + \phi)}}{2}$$

$$= e^{i\phi} M \frac{e^{i(n\theta+\tau)} + e^{-i(n\theta+\tau)}}{2}$$

$$= e^{i\phi} M \cos(n\theta+\tau)$$

where ϕ, τ are real constants such that $\arg\lambda - \phi = \tau$ and $-\arg\mu + \phi = \tau$. □

The conjecture is a special case of Theorem 3.5. If we take $p = 1$ then

$$A_1 = 2^2\sqrt{\pi}\,\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}$$

$$= 2^3\sqrt{\pi}\,\frac{1}{\Gamma(\frac{1}{2})}$$

$$= 2^3$$

$$= 8 ,$$

and the conjecture follows.

§3.3. POLYNOMIALS WITH PRESCRIBED ZEROS

In 1978, in solving an estimation problem of finding $\min_{p\in C} \max_{|z|=1} |p(z)|$, where C is the class of polynomials of degree n having a prescribed zero at 1, M. Lachance, E.B. Saff and R.S. Varga [12] established an interesting relation between the following two problems. The relation is stated in Theorem 3.6.

Problem I: For any non-negative integer s,m, find

$$e_{s,m} = \min\left\{ \max_{|z|=1} |p(z)| : p(z) = (z-1)^s q(z),\ q(z) = z^m + \ldots \text{ is a polynomial of degree } m \right\}.$$

Problem II: For any non-negative integer s,m, find

$$E_{s,m} = \min\left\{ \max_{|z|=1} |P(z)| : P(z) = (z-1)^s Q(z),\ Q(z) = z^m + \ldots \text{ is a polynomial of degree } m \text{ having all its zeros on } |z|=1 \right\}.$$

In fact, the proof of the result depends heavily on the Erdös-Lax Theorem. We present their work in the following. We also need a lemma which is a slight generalization of Lemma 2.1.

Lemma 3.4: Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le 1$, $Q(z) = z^n P\left(\frac{1}{z}\right)$. Then the polynomial

$$z^k P(z) + e^{i\theta} Q(z),$$

for any non-negative integer k, and θ real, has all its zeros on $|z|=1$.

Theorem 3.6: For each pair s, m of non-negative integers, let $p_{s,m}(z) = (z-1)^s(z^m+\ldots)$ be the unique solution to Problem I, and let $P_{s+1,m}(z) = (z-1)^{s+1}(z^m+\ldots)$ be any polynomial of degree $s+m+1$ having all its zeros on $|z|=1$ for which $\max_{|z|=1}\left|P_{s+1,m}(z)\right| = E_{s+1,m}$. Then

$$p_{s,m}(z) \equiv P'_{s+1,m}(z)/(s+m+1) , \tag{3.8}$$

$$e_{s,m} = E_{s+1,m}/2 . \tag{3.9}$$

Consequently, $P_{s+1,m}(z)$ is unique.

Proof of Theorem 3.6: The uniqueness of the solution of Problem I is established by Walsh [29]. Since $p_{s,m}(z)$ is unique, its coefficients must be real, and so its nonreal zeros occur in conjugate pairs. $p_{s,m}(z)$ has a zero of precise multiplicity s at $z=1$, we now want to show that its remaining m zeros lie in $|z|<1$.

For $m \geq 1$, write $p_{s,m}(z) = (z-1)^s \prod_{\nu=1}^{m} (z-\alpha_\nu)$, and assume for some ν we have $|\alpha_\nu| \geq 1$. If, for $\delta > 0$, we set

$$r(z;\delta) = p_{s,m}(z)\left(\frac{z-(1-\delta)\alpha_\nu}{z-\alpha_\nu}\right),$$

then for δ sufficiently small

$$\max_{|z|=}|r(z;\delta)| < \max_{|z|=1}|p_{s,m}(z)|$$

$$= e_{s,m}$$

which is a contradiction to the definition of $e_{s,m}$. Hence $|\alpha_\nu| < 1$ for $1 \le \nu \le m$.

Next we define the polynomial $Q(z)$ by

$$Q(z) = zp_{s,m}(z) + (-1)^{s+1}q_{s,m}(z) \tag{3.10}$$

where $q_{s,m}(z) = z^{s+m}\,\overline{p_{s,m}\left(\frac{1}{\bar{z}}\right)}$. Actually

$$Q(z) = z(z-1)^s \prod_{\nu=1}^{m}(z-\alpha_\nu) + (-1)^{s+1}(1-z)^s \prod_{\nu=1}^{m}(1-\bar{\alpha}_\nu z)$$

$$= z(z-1)^s \prod_{\nu=1}^{m}(z-\alpha_\nu) - (z-1)^s \prod_{\nu=1}^{m}(1-\bar{\alpha}_\nu z)\,.$$

$$= (z-1)^s\left[z\prod_{\nu=1}^{m}(z-\alpha_\nu) - \prod_{\nu=1}^{m}(1-\bar{\alpha}_\nu z)\right].$$

Since the nonreal zeros of $p_{s,m}(z)$ occur in conjugate pairs we know that $Q(z)$ has a zero of multiplicity $s+1$ at $z=1$. Also $Q(z)$ is a monic polynomial of degree exactly $s+m+1$. Because all zeros of $p_{s,m}(z)$ lie in $|z| \le 1$, it follows from Lemma 3.4 that all zeros of $Q(z)$ lie on $|z|=1$.

Now, as $Q'(z)/(s+m+1) = (z-1)^s(z^m+\ldots)$ is monic, it is a competitor of $p_{s,m}(z)$. By Erdös-Lax Theorem, we obtain

$$e_{s,m} = \max_{|z|=1} |p_{s,m}(z)| \le \max_{|z|=1} \left| \frac{Q'(z)}{s+m+1} \right|$$

$$= \frac{1}{2} \max_{|z|=1} |Q(z)| \qquad (3.11)$$

$$\le \max_{|z|=1} |p_{s,m}(z)|$$

where the last inequality follows by applying triangular inequality to (3.10). Since $p_{s,m}(z)$ is unique, we have

$$p_{s,m}(z) \equiv \frac{Q'(z)}{s+m+1} \qquad (3.12)$$

Next, we want to show $Q(z) \equiv P_{s+1,m}(z)$. Since $P_{s+1,m}(z)$ is extremal for Problem II, we have

$$E_{s+1,m} = \max_{|z|=1} |P_{s+1,m}(z)| \le \max_{|z|=1} |Q(z)| \, . \qquad (3.13)$$

On the other hand, $P'_{s+1,m}(z)/(s+m+1)$ is an admissible polynomial for Problem I, we have from (3.12) that

$$e_{s,m} = \max_{|z|=1} |p_{s,m}(z)| = \max_{|z|=1} \left| \frac{Q'(z)}{s+m+1} \right| \le \max_{|z|=1} \left| \frac{P'_{s+1,m}(z)}{s+m+1} \right| . \qquad (3.14)$$

But all the zeros of $Q(z)$ and $P_{s+1,m}(z)$ lie on $|z|=1$, therefore, it follows from (3.14) and Erdös-Lax Theorem that

$$\max_{|z|=1} |Q(z)| \le \max_{|z|=1} |P_{s+1,m}(z)| \, . \qquad (3.15)$$

Consequently, from (3.13), (3.14), (3.15) we have

$$\max_{|z|=1} |Q(z)| = \max_{|z|=1} |P_{s+1,m}(z)|$$

and

$$\max_{|z|=1} |p_{s,m}(z)| = \max_{|z|=1} \left| \frac{Q'(z)}{s+m+1} \right| = \max_{|z|=1} \left| \frac{P'_{s+1,m}(z)}{s+m+1} \right| .$$

Thus from the uniqueness of solutions to Problem I we have

$$Q'(z) \equiv P'_{s+1,m}(z)$$

and

$$Q(z) = (s+m+1)\int_1^z p_{s,m}(t)dt = P_{s+1,m}(z) \qquad (3.16)$$

which proves that $P_{s+1,m}(z)$ is unique and (3.8). Finally (3.9) follows from (3.16) and (3.11). □

§3.4. ESTIMATES OF $\max_{|z|=R} |P(z)|$

So far, we have been discussing estimates of $|P'(z)|$ for $|z| \leq 1$. But estimates of $|P(z)|$ for $|z|=r$, $r \in \mathbb{R}$, with different restrictions on the zeros of $P(z)$, are also of great interest to mathematicians. Results were obtained in this direction. In this section, we denote $\max_{|z|=R} |P(z)|$ by $M(R)$. First we present a result due to Bernstein:

<u>Theorem 3.7</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>with</u> $|P(z)| \leq 1$ <u>for</u> $|z| \leq 1$, <u>then for</u> $R > 1$

$$M(R) \leq R^n .$$

<u>The result is best possible and equality holds for</u> $P(z) = \lambda z^n$, $|\lambda| = 1$.

<u>Proof of Theorem 3.7</u>: Let $f(z) = P(z)/z^n$. $f(z)$ is an analytic function on any region including infinity but not containing the origin. By maximum modulus principle we have

$$\max_{|z|=R} |f(z)| \le \max_{|z|=1} |f(z)| . \tag{3.14}$$

Since $\max_{|z|=1} |P(z)| = 1$, and so $\max_{|z|=1} |f(z)| = 1$, from (3.14) we have

$$\max_{|z|=R} |P(z)| \le R^n . \quad \square$$

In the following, we present a result which is due to N.C. Ankeny and T.J. Rivlin [1]. We also present some of our own observations.

Theorem 3.8: *If* $P(z)$ *is a polynomial of degree* n *with* $|P(z)| \le 1$ *for* $|z| \le 1$, *having no zeros in* $|z| \le 1$, *then for* $R > 1$

$$M(R) \le \frac{1+R^n}{2} . \tag{3.15}$$

This result is best possible and equality in (3.15) *holds for* $P(z) = (\lambda + \mu z^n)/2$, *where* $|\lambda| = |\mu| = 1$.

Proof of Theorem 3.8: Since $P(z)$ has no zero in $|z| \le 1$, by Erdös-Lax Theorem we have

$$|P'(e^{i\theta})| \le n/2 \qquad 0 \le \theta < 2\pi$$

which implies

$$|P'(re^{i\theta})| \le n/2\, r^{n-1} \qquad 0 \le \theta < 2\pi , \ r > 1.$$

But for each $0 \le \theta < 2\pi$, we have

$$P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(re^{i\theta})dr .$$

Hence

$$|P(Re^{i\theta})-P(e^{i\theta})| \le \int_1^R |P'(re^{i\theta})|\,dr$$

$$\le \int_1^R r^{n-1}dr \,.\, \max_{0\le\theta\le2\pi} |P'(e^{i\theta})|$$

$$\le \frac{n}{2}\int_1^R r^{n-1}dr$$

$$= \frac{R^n-1}{2},$$

and

$$|P(Re^{i\theta})| \le \frac{R^n-1}{2} + |P(e^{i\theta})|$$

$$\le \frac{R^n-1}{2} + 1$$

$$= \frac{1+R^n}{2} .$$

If $P(z) = (\lambda+\mu z^n)/2$, $|\lambda| = |\mu| = 1$, then $P(z) = \frac{1+R^n}{2}$ for $|z|\le R$, $R>1$.

□

The above result was obtained in 1954 by Ankeny and Rivlin [1], see also [18]. In 1960 Rivlin [21] also proved the following inequality which is in the opposite direction.

<u>Theorem 3.9</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>with</u> $|P(z)|\le 1$ <u>for</u> $|z|\le 1$, <u>having no zeros in</u> $|z|\le 1$, <u>then for</u> $r\le 1$

$$M(r) \ge \left(\frac{1+r}{2}\right)^n . \tag{3.16}$$

The result is best possible and equality in (3.16) holds for

$P(z) = \left(\frac{1+z}{2}\right)^n$.

Proof of Theorem 3.9: Let $P(e^{i\theta}) = \prod_{\nu=1}^{n}\left(e^{i\theta}-k_\nu e^{i\alpha_\nu}\right)$, $k_\nu \geq 1$, $\nu = 1,2,\ldots,n$, then

$$\left|\frac{P(re^{i\theta})}{P(e^{i\theta})}\right| = \prod_{\nu=1}^{n}\left|\frac{re^{i\theta}-k_\nu e^{i\alpha_\nu}}{e^{i\theta}-k_\nu e^{i\alpha_\nu}}\right| .$$

For each ν and $r \leq 1$, we consider

$$\left|\frac{re^{i\theta}-k_\nu e^{i\alpha_\nu}}{e^{i\theta}-k_\nu e^{i\alpha_\nu}}\right| = \left|\frac{r-k_\nu e^{i(\alpha_\nu-\theta)}}{1-k_\nu e^{i(\alpha_\nu-\theta)}}\right|$$

$$= \left(\frac{r^2+k_\nu^2-2rk_\nu\cos(\theta-\alpha_\nu)}{1+k_\nu^2-2k_\nu\cos(\theta-\alpha_\nu)}\right)^{\frac{1}{2}} . \qquad (3.17)$$

Suppose for each ν, $\left|\frac{re^{i\theta}-k_\nu e^{i\alpha_\nu}}{e^{i\theta}-k_\nu e^{i\alpha_\nu}}\right| \geq \frac{1+r}{2}$, we shall see whether this inequality is valid in the following. From (3.17) we have

$$\frac{r^2+k_\nu^2-2rk_\nu\cos(\theta-\alpha_\nu)}{1+k_\nu^2-2k_\nu\cos(\theta-\alpha_\nu)} \geq \frac{1+2r+r^2}{4}$$

$$\begin{aligned}4r^2+4k_\nu^2-8rk_\nu\cos(\theta-\alpha_\nu) \geq\ & 1+k_\nu^2-2k_\nu\cos(\theta-\alpha_\nu)+2r\\ &+ 2rk_\nu^2-4rk_\nu\cos(\theta-\theta_\nu)\\ &+ r^2+r^2k_\nu^2-2r^2k_\nu\cos(\theta-\theta_\nu)\end{aligned}$$

$$3r^2 + 3k_\nu^2 - 1 - 2r - 2rk_\nu^2 - r^2k_\nu^2 \geq (4rk_\nu - 2k_\nu - 2r^2k_\nu)\cos(\theta-\alpha_\nu)$$

$$(3r^2 - 2r - 1) + k_\nu^2(3 - 2r - r^2) \geq (2r - 1 - r^2)2k_\nu \cos(\theta-\alpha_\nu)$$

$$(3r^2 - 2r - 1) + k_\nu^2(3 - 2r - r^2) \geq 2k_\nu(2r - 1 - r^2) \qquad (3.18)$$

$$k_\nu^2(3 - 2r - r^2) - 2k_\nu(1-2r + r^2) + (3r^2 - 2r - 1) \geq 0 .$$

If we take the equality sign in (3.18), we obtain a quadratic equation in k_ν. Therefore

$$k_\nu = \frac{(1-2r+r^2) \pm \sqrt{(1-2r+r^2)^2 - (3r^2-2r-1)(3-2r-r^2)}}{(3 - 2r - r^2)} .$$

After some technical simplicification of the expression under the squareroot sign in (3.19) we have

$$k_\nu = \frac{(1-2r+r^2) \pm 2(1-r^2)}{3 - 2r - r^2}$$

$$= 1 \quad \text{or} \quad \frac{-1 - 2r + 3r^2}{3 - 2r - r^2} \leq 1 .$$

Thus we know that (3.18) is true which also implies (3.17) is true. Since (3.17) is true for every ν, taking the product over $\nu=1,2,\ldots,n$, we have

$$\frac{|P(re^{i\theta})|}{|P(e^{i\theta})|} \geq \prod_{\nu=1}^{n} \left(\frac{1+r}{2}\right)$$

$$= \left(\frac{1+r}{2}\right)^n .$$

Hence $M(r) \geq \left(\frac{1+r}{2}\right)^n$.

It is easily seen that equality is attained for $P(z) = \left(\frac{1+z}{2}\right)^n$. □

It is interesting to note that Theorem 3.9 can be easily established using geometric argument.

<u>Alternate Proof of Theorem 3.9*:</u> Consider

$$\left| \frac{re^{i\theta} - k_\nu e^{i\alpha\nu}}{e^{i\theta} - k_\nu e^{i\alpha\nu}} \right| \geq \frac{1+r}{2}$$

which is equivalent to

$$\frac{|re^{i\theta} - k_\nu e^{i\alpha\nu}|}{1+r} \geq \frac{|e^{i\theta} - k_\nu e^{i\alpha\nu}|}{2} ,$$

and this inequality can be expressed in the following geometric diagram as

$$\frac{CD}{AC} \geq \frac{BD}{AB} .$$

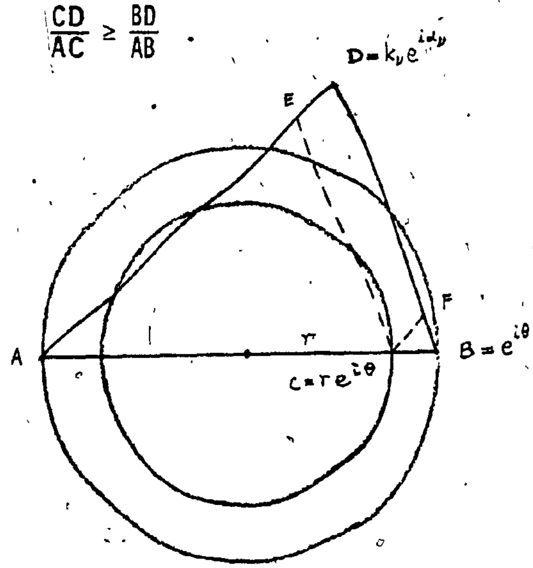


Figure 3.2

Since <ADB is an acute angle[1], in the parallelogram CFDE, the diagonal CD is greater than any of its sides CE or CF. Comparing triangles ΔACE and ΔABD we get

[1] Whenever D lies outside the circle, <ADB is acute.

$$\frac{EC}{AC} = \frac{BD}{AB}$$

But $CD > EC$, therefore we have

$$\frac{CD}{AC} > \frac{BD}{AB}$$

which implies

$$\frac{CD}{BD} > \frac{AC}{AB} .$$

Consequently

$$\left| \frac{P(re^{i\theta})}{P(e^{i\theta})} \right| \geq \left(\frac{1+r}{2}\right)^n$$

and

$$M(r) \geq \left(\frac{1+r}{2}\right)^n . \qquad \square$$

Using the same underlying method in the alternate proof of Theorem 3.9, we can obtain an analogous result of Theorem 3.9, which is a further restriction to the zeros being in $|z| \leq K$, $K>1$.

Theorem 3.10*: If $P(z)$ is a polynomial of degree n with $|P(z)| \leq 1$ for $|z| \leq 1$ having no zeros in $|z| \leq K$, $K>1$ then for $r \leq 1$

$$M(r) \geq \left(\frac{r+K}{1+K}\right)^n . \tag{3.19}$$

The result is best possible and equality in (3.19) holds for $P(z) = \left(\frac{z+K}{1+K}\right)^n$.

Proof of Theorem 3.10: Since

$$\left| \frac{P(re^{i\theta})}{P(e^{i\theta})} \right| = \prod_{\nu=1}^{n} \left| \frac{re^{i\theta} - k_\nu e^{i\alpha\nu}}{e^{i\theta} - k_\nu e^{i\alpha\nu}} \right| ,$$

it is sufficient to prove the following inequality

$$\left| \frac{re^{i\theta} - k_\nu e^{i\alpha\nu}}{e^{i\theta} - k_\nu e^{i\alpha\nu}} \right| \geq \frac{r+K}{1+K} . \qquad (3.20)$$

But (3.20) can be expressed in the following geometric diagram as

$$\frac{CD}{BD} > \frac{AC}{AB}$$

which is equivalent to

$$\frac{CD}{AC} > \frac{BD}{AB} .$$

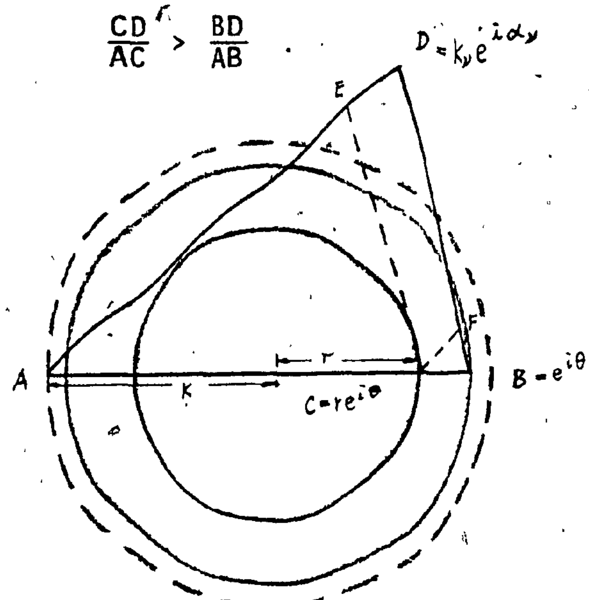


Fig. 3.3

We know that <ADB is an acute angle, then in the parallelogram CFDE, the diagonal CD is greater than any of its sides CE or CF.

Comparing triangles ΔACE and ΔABD we

$$\frac{EC}{AC} = \frac{BD}{AB} .$$

But $CD > EC$ hence we have

$$\frac{CD}{AC} > \frac{BD}{AB}$$

which implies

$$\frac{CD}{BD} > \frac{AC}{AB} .$$

Consequently we have (3.20) and taking the product over ν we get

$$M(r) \geq \left(\frac{r+K}{1+K}\right)^n . \qquad \square$$

The result corresponding to Theorem 3.9 when $P(z)$ has no zeros in $|z| < k$, $k \geq 1$ is not yet known. Recently, Aziz and Mohammed [2] announced that:

If $P(z)$ is a polynomial of degree n with $|P(z)| \leq 1$ on $|z| \leq 1$ and has no zeros in $|z| \leq K$, $K \geq 1$ then

$$|P(Re^{i\theta})| \leq \frac{R^n+K^n}{1+K^n} \qquad \text{for} \quad R > k^2$$

and

$$|P(Re^{i\theta})| \leq \left(\frac{R+K}{1+K}\right)^n \qquad \text{for} \quad 1 \leq R \leq k^2 .$$

The proof of this assertion they gave has an error.

CHAPTER IV

GENERALIZATION OF THE ERDÖS-LAX THEOREM

In Chapter II, the restriction on the location of zeros is that the polynomials have no zeros inside the unit disk. So if we put a further restriction on the zeros of $P(z)$ in considering the class of polynomials having no zeros in $|z| < K$, where $K \neq 1$, the constant $n/2$ in (2.1) should be replaced by another constant. For the case $K > 1$, several results on the derivative of polynomials are known and we present them in section 4.1 along with some new observations in section 4.2, 4.3, 4.4, 4.5 and, in particular see Theorem 4.5. When $K < 1$, the corresponding result to Erdös-Lax Theorem is not yet known. For this case we present a discussion involving computer calculation and a result for polynomials of degree 3 in Theorem 4.12.

§4.1 POLYNOMIALS HAVING NO ZEROS IN $|z| < K$, $K \geq 1$

In generalizing the Erdös-Lax Theorem, for $K \geq 1$, M.A. Malik [14] established the following Theorem in 1969.

Theorem 4.1: If $P(z)$ is a polynomial of degree n with $|P(z)| \leq 1$ on $|z| \leq 1$ and has no zeros in $|z| < K$, $K \geq 1$, then

$$|p'(z)| \leq \frac{n}{1+K} \tag{4.1}$$

for $|z| \leq 1$. The result is best possible and equality in (4.1) holds for $P(z) = \left(\frac{z+K}{1+K}\right)^n$.

The proof of Theorem 4.1 is based on a result due to Laguerre [5], which is also used to prove other theorems in this Chapter. For the sake of immediate reference we include Languerre's Theorem with its proof in the following:

Lemma 4.1: If $P(z)$ is a polynomial of degree n having no zeros in a circular domain C, then for any $\xi \in C$ and $z \in C$

$$(\xi - z)\, P'(z) + n\, P(z) \neq 0 \tag{4.2}$$

Proof of Lemma 4.1: Let z_ν, $\nu = 1,2,\ldots,n$ be the zeros of $P(z)$. The left hand side of (4.2) is

$$(\xi - z)\, P'(z) + n\, P(z)$$

$$= P(z)\left[(\xi - z)\frac{P'(z)}{P(z)} + n\right]$$

$$= P(z)\left[(\xi - z)\sum_{\nu=1}^{n}\frac{1}{z - z_\nu} + n\right]$$

$$= P(z)\left[\sum_{\nu=1}^{n}\left(\frac{\xi - z}{z - z_\nu} + 1\right)\right]$$

$$= P(z)\left[\sum_{\nu=1}^{n}\frac{\xi - z_\nu}{z - z_\nu}\right]$$

Since for every $\nu = 1,2,\ldots,n$, z_ν is not contained in C which contains ξ, z, the linear transformation $\frac{\xi-\lambda}{z-\lambda}$ maps the outside of C into a circular domain C' which contains neither zero nor infinity, so maps z_ν into a convex domain. Hence the centre of gravity $\frac{1}{n}\sum_{\nu=1}^{n}\frac{\xi - z_\nu}{z - z_\nu}$ cannot be at zero for fixed $\xi, z \in C$, which implies

$$(\xi - z)\, P'(z) + n\, P(z) \neq 0$$

for $\xi, z \in C$ .. □

Remark 4.1: A circular domain is the image of the unit disk (open or closed) under a linear transformation.

Now we return to the proof of Theorem 4.1 as given by Malik.

Lemma 4.2: If $P(z)$ is a polynomial of degree n having no zeros in $|z| \le K$, $K \ge 1$ then for $|z| = 1$

$$K|P'(z)| < |Q'(z)|$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof of Lemma 4.2: Let z_ν, $\nu = 1,2,\ldots,n$ be the zeros of $P(z)$. Then

$$z \frac{P'(z)}{P(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu}$$

and

$$\bar{z}\, \overline{Q'(z)} / \overline{Q(z)} = \sum_{\nu=1}^{n} \frac{-\bar{z}\, z_\nu}{1 - \bar{z}\, z_\nu} .$$

Since $P(z) \neq 0$ in $|z| \le K$, it follows from Lemma 4.1 that $(\xi - z)\, P'(z) + nP(z) \neq 0$ for all $|z| \le K$, all $|\xi| \le K$. Equivalently we have

$$\xi P'(z) \neq zP'(z) - nP(z)$$

or

$$\xi \frac{P'(z)}{P(z)} \neq z \frac{P'(z)}{P(z)} - n \qquad (4.3)$$

if $|z| \le K$, $|\xi| \le K$. Since for $|z| = 1$,

$$z\,\frac{P'(z)}{P(z)} - n = \sum_{\nu=1}^{n} \left(\frac{z}{z-z_\nu} - 1\right)$$

$$= \sum_{\nu=1}^{n} \frac{z_\nu}{z-z_\nu}$$

$$= \sum_{\nu=1}^{n} \frac{z_\nu/z}{1 - z_\nu/z}$$

$$= \sum_{\nu=1}^{n} \frac{\bar{z}\,z_\nu}{1 - \bar{z}\,z_\nu} = -\bar{z}\,\overline{\frac{Q'(z)}{Q(z)}} \quad [1] \qquad (4.4)$$

it follows from (4.3), (4.4) that

$$\xi\,\frac{P'(z)}{P(z)} \ne -\bar{z}\,\overline{Q'(z)}\Big/\overline{Q(z)}\ .$$

Hence for an appropriate choice of arg ξ we get

$$\left|\xi\,\frac{P'(z)}{P(z)}\right| \ne \left|\frac{Q'(z)}{Q(z)}\right|$$

for $|z| = 1$ and $|\xi| \le K$. Thus either

$$\left|\xi\,\frac{P'(z)}{P(z)}\right| < \left|\frac{Q'(z)}{Q(z)}\right| \qquad (4.5)$$

or

$$\left|\xi\,\frac{P'(z)}{P(z)}\right| > \left|\frac{Q'(z)}{Q(z)}\right| \qquad (4.6)$$

for $|z| = 1$ and $|\xi| \le K$. But a sufficiently small value of $|\xi|$ contradicts (4.6). Taking $|\xi| \to K$, the proof of the lemma follows from (4.5) as $|P(z)| = |Q(z)|$ on $|z| = 1$. □

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[1] This give $|Q'(z)| = |zP'(z)-nP(z)|$ on $|z|=1$, which will be used frequently and referred to being (4.4).

As an immediate consequence of Lemma 4.2 (by continuity) we have

Lemma 4.3: If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$ then for $|z| = 1$

$$K|P'(z)| \leq |Q'(z)|$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof of Theorem 4.1: Let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, $R(z) = P(z) - e^{i\alpha}$, $0 < \alpha \leq 2\pi$ and $T(z) = z^n \overline{R\left(\frac{1}{\bar{z}}\right)} = Q(z) - z^n e^{-i\alpha}$. Since $R(z)$ has no zeros in $|z| < 1$, from Lemma 4.3 with $K = 1$ we have $|R'(z)| \leq |T'(z)|$ for $|z| = 1$. This implies

$$|P'(z)| \leq |Q'(z) - nz^{n-1}e^{-i\alpha}| \tag{4.7}$$

for $|z| = 1$; for a suitable choice of α we get

$$|Q'(z) - nz^{n-1}e^{-i\alpha}| = n - |Q'(z)| .^{1} \tag{4.8}$$

Inequality (4.7) and (4.8) yields

$$|P'(z)| + |Q'(z)| \leq n . \tag{4.9}$$

---

1. $|Q'(z) - nz^{n-1}e^{-i\alpha}|$

$= \left||Q'(z)|e^{i\varphi_1} - ne^{i\varphi_2}e^{i\alpha}\right|$ where $\varphi_1 = \arg Q'(z)$; $\varphi_2 = (n-1)\arg z$

$= \left|e^{i\varphi_1}\right|\left||Q'(z)| - ne^{i(\varphi_2 - \varphi_1 - \alpha)}\right|$

$= ||Q'(z)| - n|$ if we choose $\alpha = \varphi_2 - \varphi_1$

$= n - |Q'(z)|$.

Hence by Lemma 4.3, we obtain

$$(1+K)|P'(z)| \le |P'(z)| + |Q'(z)| ;$$

consequently

$$|P'(z)| \le \frac{n}{1+K}$$

for $|z| \le 1$.

Now we consider the polynomial $P(z) = \left(\frac{z+K}{1+K}\right)^n$, it satisfies the hypothesis of the theorem and

$$P'(z) = \frac{n}{1+K}\left(\frac{z+K}{1+K}\right)^{n-1} .$$

Hence, it is easily seen that

$$\max_{|z|=1} |P'(z)| = \frac{n}{1+K} . \quad \square$$

Using Theorem 4.1 Malik also proved the following result which is a generalization of Turán's result in Theorem 2.3.

Theorem 4.2: If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ on $|z| \le 1$ and $P(z)$ has all its zeros in $|z| \le k$, $k \le 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k} . \tag{4.10}$$

The result is best possible and equality in (4.10) holds for $P(z) = \left(\frac{z+k}{1+k}\right)^n$.

Proof of Theorem 4.2: Let $Q_1(z) = z^n P(\frac{1}{z})$, it satisfies Theorem 4.1 with $K = \frac{1}{k}$. Since

$$Q_1'(z) = nz^{n-1}P(\frac{1}{z}) - z^{n-2}P'(\frac{1}{z})$$

we have

$$z^{n-2}P'(\frac{1}{z}) = nz^{n-1}P(\frac{1}{z}) - Q_1'(z) ,$$

and so

$$\begin{aligned} |z^{n-2}P'(\frac{1}{z})| &= |nz^{n-1}P(\frac{1}{z}) - Q_1'(z)| \\ &\geq |nz^{n-1}P(\frac{1}{z})| - |Q_1'(z)| \end{aligned} \tag{4.11}$$

From (4.11) we can conclude that

$$\begin{aligned} \max_{|z|=1} |P'(z)| &\geq n - \max_{|z|=1} |Q_1'(z)| \\ &\geq n - \frac{n}{1+\frac{1}{k}} \\ &= \frac{n}{1+k} . \quad \square \end{aligned}$$

§4.2. DISCUSSIONS ON THE EXTERNAL POLYNOMIALS

In the case of Erdös-Lax Theorem (i.e., $K = 1$ in Theorem 4.1), we know that all the polynomials having their zeros on $|z| = 1$ are extremals. But such is not the case when $K > 1$: N.K. Govil, Q.I. Rahman and G. Schmeisser [9] observed that in the case when $K > 1$, the only polynomial for which equality in (4.1) is attained is

$$P(z) = \left(\frac{z+K}{1+K}\right)^n . \tag{4.12}$$

If we let $c_0 = \left(\frac{K}{1+K}\right)^n$, (4.12) can be written as

$$\begin{aligned} P(z) &= c_0 \left(1 + \frac{z}{K}\right)^n \\ &= c_0 \left\{1 + \binom{n}{1} \frac{z}{K} + \ldots + \binom{n}{\nu}\left(\frac{z}{K}\right)^\nu + \ldots + \left(\frac{z}{K}\right)^n\right\} \end{aligned} \tag{4.13}$$

In fact, they proved:

Theorem 4.3: If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$ and having all its zeros in $|z| \ge K$, $K \ge 1$, then

$$|P'(z)| \le n \cdot \frac{n|c_0| + K^2|c_1|}{(1+K^2)n|c_0| + 2K^2|c_1|} . \tag{4.14}$$

Proof of Theorem 4.3: Since $P(z) \neq 0$ in $|z| < K$, by Lemma (4.1) we have

$$nP(z) + (\xi - z)P'(z) \neq 0$$

for $|\xi| < K$, $|z| < K$, i.e.

$$nP(z) - zP'(z) \neq -\xi P'(z) \tag{4.15}$$

for $|\xi| < K$, $|z| < K$. Consequently with a suitable choice of the argument of ξ, we have either

$$|nP(z) - zP'(z)| < |\xi P'(z)| \tag{4.16}$$

or

$$|nP(z) - zP'(z)| > |\xi P'(z)| . \tag{4.17}$$

But sufficiently small value of $|\xi|$ contradicts (4.16). Hence (4.17) is true and taking $|\xi| \to K$ in (4.17) we have

$$K|P'(z)| \le |nP(z) - zP'(z)|$$

and

$$\left| \frac{P'(z)}{nP(z) - zP'(z)} \right| \le \frac{1}{K} \tag{4.18}$$

for $|z| \le K$. Note that $nP(z) - zP'(z) \neq 0$ in $|z| < K$. If we define

$$f(z) = \frac{KP'(Kz)}{nP(Kz) - KzP'(Kz)} ,$$

then from (4.18) we know that $|f(z)| \le 1$ for $|z| \le 1$. Also we have $f(0) = \frac{K}{n} \frac{c_1}{c_0}$. From (4.15), for $\xi = 0$ we have $nP(z) - zP'(z) \neq 0$ in $|z| < K$, thus $f(z)$ is analytic in $|z| \le 1$. So by the generalized Schwarz Lemma [17 , p. 167] we have

$$|f(z)| \le \frac{|z| + |f(0)|}{|f(0)||z| + 1}$$

$$= \frac{|z| + \frac{K}{n} \left| \frac{c_1}{c_0} \right|}{\frac{K}{n}\left| \frac{c_1}{c_0} \right| |z| + 1}$$

for $|z| < 1$. Thus, in particular for $|z| = 1$

$$|P'(z)| \le \frac{1}{K^2} \frac{1 + \frac{K^2}{n} \left| \frac{c_1}{c_0} \right|}{\frac{1}{n}\left| \frac{c_1}{c_0} \right| + 1} |nP(z) - zP'(z)| .$$

Let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z| = 1$, $|nP(z) - zP'(z)| \equiv |Q'(z)|$ and therefore for $|z| = 1$

$$|P'(z)| \le \frac{1}{K^2} \frac{1 + \frac{K^2}{n}\left|\frac{c_1}{c_0}\right|}{\frac{1}{n}\left|\frac{c_1}{c_0}\right| + 1} \left|Q'(z)\right| .$$

Combining this with the inequality (4.9)

$$|P'(z)| + |Q'(z)| \le n$$

which is valid for all polynomial $P(z)$ of degree at most n and $|P(z)| \le 1$ on $|z| \le 1$, we get

$$|P'(z)| . \left[1 + \frac{K^2\left(\frac{1}{n}\left|\frac{c_1}{c_0}\right| + 1\right)}{1 + \frac{K^2}{n}\left|\frac{c_1}{c_0}\right|}\right] \le n .$$

Hence after simplifying we have

$$|P'(z)| \le n . \frac{n|c_0| + K^2|c_1|}{n|c_0|(1+K^2) + 2K^2|c_1|}$$

for $|z| \le 1$. □

Inequality (4.14) is best possible for even n, and equality holds for

$$P(z) = c_0 \frac{1}{K^n}\left(ze^{i\gamma} + Ke^{i\alpha}\right)^{n/2}\left(ze^{i\gamma} + Ke^{-i\alpha}\right)^{n/2} \qquad (4.18)$$

where γ and α are arbitrary real numbers. This can be seen as follows: Since

$$P(z) = c_0 \frac{1}{K^n}\left[z^2 e^{i2\gamma} + ze^{i\gamma} 2K \cos \alpha + K^2\right]^{n/2}$$

$$= c_0\left\{1 + \frac{n}{K} (\cos \alpha)\, ze^{i\gamma} + \dots \right\},$$

therefore, $c_1 = c_0 \frac{n}{K} (\cos \alpha) e^{i\gamma}$ and also,

$$P'(z) = c_0 \frac{1}{K^n}\left[\frac{n}{2} (ze^{i\gamma} + Ke^{i\alpha})^{n/2-1}(ze^{i\gamma} + Ke^{-i\alpha})^{n/2}\right.$$

$$\left. + \frac{n}{2} (ze^{i\gamma} + Ke^{i\alpha})^{n/2}(ze^{i\gamma} + Ke^{-i\alpha})^{n/2-1}\right] e^{i\gamma}$$

$$= \frac{n}{2} c_0 \frac{1}{K^n} (ze^{i\gamma} + Ke^{i\alpha})^{n/2-1}(ze^{i\gamma} + Ke^{-i\alpha})^{n/2-1} e^{i\gamma}$$

$$(ze^{i\gamma} + Ke^{i\alpha} + ze^{i\gamma} + Ke^{-i\alpha})$$

$$= \frac{n}{2} c_0 \frac{1}{K^n}(ze^{i\gamma} + Ke^{i\alpha})^{n/2-1}(ze^{i\gamma} + Ke^{-i\alpha})^{n/2-1} e^{i\gamma}$$

$$(2ze^{i\gamma} + 2K \cos \alpha). \tag{4.19}$$

From (4.18) and (4.19) we know that $\max_{|z|=1} |P(z)|$ and $\max_{|z|=1} |P'(z)|$ are both attained at the point $z = e^{-i\gamma}$ or $-e^{-i\gamma}$. Thus

$$\frac{\max_{|z|=1} |P'(z)|}{\max_{|z|=1} |P(z)|} = \frac{\left|\frac{n}{2} c_0 \frac{1}{K^n} (1+Ke^{i\alpha})^{n/2-1}(1+Ke^{-i\alpha})^{n/2-1} (2+2K \cos \alpha)\right|}{\left|c_0 \frac{1}{K^n} (1+Ke^{i\alpha})^{n/2} (1+Ke^{-i\alpha})^{n/2}\right|}$$

$$= \frac{n|1+K \cos \alpha|}{(1+Ke^{i\alpha})(1+Ke^{-i\alpha})}$$

$$= n \frac{|1+K \cos \alpha|}{1+K^2+2K \cos \alpha}$$

which is equal to the left hand side of (4.14) when $c_1 = c_0 \frac{n}{K} (\cos\alpha) e^{i\gamma}$.

Since we have

$$\frac{n|c_0| + K^2|c_1|}{(1+K^2)n|c_0| + 2K^2|c_1|} \le \frac{1}{1+K},$$

and equality holds when

$$\frac{|c_1|}{|c_0|} = \frac{n}{K}, \tag{4.20}$$

we know that equality holds in (4.1) if the coefficients c_0 and c_1 of $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$, which has all its zeros lie in $|z| \ge K$, $K \ge 1$, satisfy (4.20). On the other hand, as we shall show in the following that if the coefficients c_0, c_1 of $P(z)$ satisfies (4.20), $P(z)$ must be of the form (4.13).

Suppose $P(z)$ is not of the form (4.13), we let

$$P(z) = \prod_{\nu=1}^{n} \left(z - K_\nu e^{i\alpha_\nu}\right)$$

where K_ν, α_ν is any real number and $K_\nu > K$. Thus we have

$$(-1)^n c_0 = e^{i(\alpha_1 + \ldots + \alpha_n)} \prod_{\nu=1}^{n} K_\nu,$$

$$(-1)^{n-1} c_1 = \sum_{\binom{n}{n-1}} e^{i(\alpha_1 + \ldots + \alpha_{n-1})} \prod_{\nu=1}^{n-1} K_\nu.$$

and

$$\left|\frac{c_1}{c_0}\right| = \left|\sum_{\nu=1}^{n} \frac{1}{K_\nu e^{i\alpha_\nu}}\right|$$

$$< \sum_{\nu=1}^{n} \frac{1}{K_\nu}$$

$$< \sum_{\nu=1}^{n} \frac{1}{K}$$

$$= \frac{n}{K} .$$

This contradiction shows that $P(z)$ must be of the form (4.13). From the above discussion, we can formulate the following remark.

Remark 4.2: If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$ having all its zeros on $|z| \ge K$, $K > 1$, then there is equality in (4.1) if and only if $\frac{|c_1|}{|c_0|} = \frac{n}{K}$.

§4.3. POLYNOMIALS HAVING ALL ITS ZEROS IN $|z| \le K$, $K \ge 1$

In 1973, N.K. Govil [7] considered the problem of finding the estimate of $|P'(z)|$ on $|z| = 1$ when all the zeros are in $|z| \le K$, $K \ge 1$. He proved:

Theorem 4.4: *If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ and having all its zeros in $|z| \le K$, $K \ge 1$, then*

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+K^n} \qquad (4.21)$$

The result is best possible and equality in (4.21) holds for

$$P(z) = \frac{z^n + K^n}{1 + K^n} .$$

For the proof of the theorem, we need the following lemmas.

Lemma 4.4: If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le K$, $K \ge 1$, then for $0 \le \theta \le 2\pi$,

$$|P'(K^2 e^{i\theta})| \ge K^{n-2} |Q'(e^{i\theta})|$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof of Lemma 4.4: Let $P(z) = c \prod_{\nu=1}^{n} (z - z_\nu)$. Then

$$P_1(z) = P(Kz) = c \prod_{\nu=1}^{n} (Kz - z_\nu)$$

has all its zeros lie in $|z| \le 1$, and the polynomial

$$Q_1(z) = z^n \overline{P_1\left(\frac{1}{\bar{z}}\right)}$$

$$= z^n \overline{P\left(\frac{K}{\bar{z}}\right)}$$

$$= K^n Q\left(\frac{z}{K}\right)$$

has all its zeros lie in $|z| \ge 1$. Since $|P_1(z)| = |Q_1(z)|$ on $|z| = 1$, it follows that $|P_1(z)| \ge |Q_1(z)|$, for $|z| \ge 1$. Hence, if $|\lambda| > 1$, we have $Q_1(z) - \lambda P_1(z)$ has all its zeros in $|z| < 1$. By Gauss-Lucas Theorem we know that the zeros of $Q_1'(z) - \lambda P_1'(z)$ also lie in $|z| < 1$, which implies

$$|Q_1'(z)| \le |P_1'(z)|$$

for $|z| \geq 1$[1]. In particular

$$K^{n-1}\ |Q_1'(e^{i\theta})| \leq K|P'(K^2 e^{i\theta})| . \qquad \square$$

Lemma 4.5: If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then

$$\max_{|z|=1} |Q'(z)| \leq K^n \max_{|z|=1} |P'(z)| .$$

where $Q(z) = z^n\ \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof of Lemma 4.5: By Lemma 4.4, we have

$$\max_{|z|=1} |Q'(z)| \leq \frac{1}{K^{n-2}} \max_{|z|=K^2} |P'(z)| .$$

Let $g(z) = \frac{P(z)}{z^n}$, and $M = \max_{|z|=1} |g(z)| = \max_{|z|=1} |P(z)|$, then $g(z)$ is analytic on $|z| > 0$. Since $K > 1$, let $M(K) = \max_{|z|=K} |g(z)| = \frac{1}{K^n} \cdot \max_{|z|=K} |P(z)|$, then we have

$$M(K) \leq M ,$$

which implies

$$\max_{|z|=K} |P(z)| \leq K^n \max_{|z|=1} |P(z)| .$$

The above inequality is valid for any polynomial of degree n without

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1. Since $Q_1'(z) - \lambda P_1'(z) \neq 0$ for $|z| \geq 1$, with a suitable choice of the argument of λ, we have $|Q_1'(z)| \neq |\lambda||P_1'(z)|$. If $|Q_1'(z)| > |P_1'(z)|$, we always choose $|\lambda|$ so that $|Q_1'(z)| = |\lambda||P_1'(z)|$ which is a contradiction.

any restriction on its zeros. Now for $P'(z)$ is a polynomial of degree $n-1$, we have

$$\max_{|z|=K^2} |P'(z)| \le K^{2n-2} \max_{|z|=1} |P'(z)| .$$

Thus

$$\max_{|z|=1} |Q'(z)| \le \frac{1}{K^{n-2}} \cdot K^{2n-2} \max_{|z|=1} |P'(z)|$$

$$= K^n \max_{|z|=1} |P'(z)| . \qquad \square$$

Lemma 4.6: If $P(z)$ is a polynomial of degree n with $|P(z)| \le 1$ on $|z| \le 1$, $P(z) \equiv Q(z)$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} .$$

Proof of Lemma 4.6: Since $P(z) \equiv Q(z)$, and $|P'(z)| + |Q'(z)| \le n$, we have

$$\max_{|z|=1} |P'(z)| = \max_{|z|=1} |Q'(z)|$$

$$\le \frac{n}{2} .$$

On the other hand, on $|z|=1$, from (4.4) we have

$$|Q'(z)| = |nP(z) - zP'(z)| \tag{4.22}$$

which implies

$$\frac{n}{2} \geq |Q'(z)| = |nP(z) - zP'(z)|$$

$$\geq n|P(z)| - |P'(z)| .$$

Choosing z on $|z| = 1$ for which $|P(z)|$ becomes maximum, we get

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2}$$

and the lemma follows. See also Theorem 2.4 □

<u>Proof of Theorem 4.4</u>: Let $P^*(z) = \frac{1}{2}\left\{P(z) + Q(z)\right\}$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. Then $P^*(z)$ satisfies

$$P^*(z) \equiv z^n \overline{P^*\left(\frac{1}{\bar{z}}\right)}$$

and

$$|P^*(z)| = 1$$

for $|z| \leq 1$. Hence by Lemma 4.6 we have

$$\max_{|z|=1} |P'(z) + Q'(z)| = n$$

which implies

$$\max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)| \geq n .$$

Apply Lemma 4.5 we get,

$$\max_{|z|=1} |P'(z)| + K^n \max_{|z|=1} |P'(z)| \geq n$$

or

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1 + K^n}$$

To see that equality holds for $P(z) = \dfrac{z^n + K^n}{1 + K^n}$, we know that first $\max_{|z|=1} |P(z)| = 1$ and $P'(z) = \dfrac{nz^{n-1}}{1+K^n}$. Therefore we have

$$\max_{|z|=1} |P'(z)| = \frac{n}{1 + K^n} . \qquad \square$$

A combination of Theorem 4.1 and Theorem 4.4 gives:

If $P(z)$ is a polynomial of degree n with $\max_{|z|=1} |P(z)| = 1$ having all its zeros on $|z| = K$, $K \ge 1$, then

$$\frac{n}{1+K^n} \le \max_{|z|=1} |P'(z)| \le \frac{n}{1+K} . \tag{4.23}$$

This observation led us to ask the question: how does the distribution of the zeros of $P(z)$ on $|z| = K$ influence $\max_{|z|=1} |P'(z)|$ to vary from $\frac{n}{1+K}$ to $\frac{n}{1+K^n}$ in (4.23)? In fact, the zeros of $P(z)$ should be distributed so that some of the coefficients of $P(z)$ vanish. We prove:

<u>Theorem 4.5*</u>: <u>If</u> $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ <u>is a polynomial of degree</u> n <u>with</u> $|P(z)| \le 1$ on $|z| \le 1$ <u>having no zeros in</u> $|z| < K$, $K \ge 1$, <u>and</u> $a_\nu = 0$ <u>for</u> $1 \le \nu \le \mu-1 < n$, <u>then</u>

$$|P'(z)| \le \frac{n}{1+K^\mu} \tag{4.24}$$

<u>for</u> $|z| \le 1$. <u>The result is best possible for each</u> μ <u>and equality holds for</u> $P(z) = \left(\dfrac{z^\mu + K^\mu}{1 + K^\mu}\right)^{n/\mu}$ <u>where</u> n <u>is a multiple of</u> μ.

<u>Proof of Theorem 4.5</u>: Since $P(z) \neq 0$ in $|z| < K$, by Lemma 4.1 we have

$$(\xi - z)\, P'(z) + nP(z) \neq 0$$

or

$$nP(z) - zP'(z) \neq -\xi\, P'(z)$$

for $|z| < K$, $|\xi| < K$. Choosing a suitable argument of ξ and with a similar argument as used in the proof of Lemma 4.2 we have

$$|nP(z) - zP'(z)| > |\xi P'(z)| . \tag{4.25}$$

Taking $|\xi| \to K$ in (4.25) we have

$$\left| \frac{KP'(z)}{nP(z) - zP'(z)} \right| \le 1$$

for $|z| < K$. Note that $nP(z) - zP'(z) \neq 0$ in $|z| < K$. Writing out $P'(z)$ explicitly we get

$$\left| \frac{K \sum_{\nu=\mu}^{n} \nu\, a_\nu z^{\nu-1}}{nP(z) - zP'(z)} \right| \le 1$$

for $|z| < K$ or

$$\left| \frac{\rho \sum_{\nu=\mu}^{n} \nu\, a_\nu (\rho z)^{\nu-1}}{nP(\rho z) - \rho z\, P'(\rho z)} \right| \le 1$$

for $|z| \le 1$ and $\rho < K$, which implies

$$\left| \frac{\rho\rho^{\mu-1} \sum_{\nu=\mu}^{n} \nu a_\nu (\rho z)^{\nu-\mu}}{nP(\rho z) - \rho z\, P'(\rho z)} \right| \le 1 \tag{4.26}$$

for $|z| = 1$ and applying maximum modulus theorem we know that (4.26) also holds for $|z| < 1$. Letting $\rho \to K$, we can replace z by $\frac{\lambda}{K}$ in (4.26) where $|\lambda| = 1$ because $|z| = |\frac{\lambda}{K}| < 1$. This gives

$$\left| \frac{K^{\mu} \sum_{\nu=\mu}^{n} \nu a_\nu \lambda^{\nu-\mu}}{nP(\lambda) - \lambda P'(\lambda)} \right| \le 1 . \tag{4.27}$$

Now multiply the left hand side of (4.27) by $\lambda^{\mu-1}$ of absolute value 1 to get

$$\left| \frac{K^{\mu} \sum_{\nu=\mu}^{n} \nu a_\nu \lambda^{\nu-1}}{nP(\lambda) - \lambda P'(\lambda)} \right| \le 1 ,$$

$|\lambda| = 1$ or

$$\left| \frac{K^{\mu} P'(z)}{nP(z) - zP'(z)} \right| \le 1$$

for $|z| = 1$. Consequently we have

$$K^{\mu} |P'(z)| \le |nP(z) - zP'(z)|$$

for $|z| = 1$. As (4.4) we know that $|Q'(z)| = |nP(z) - zP'(z)|$ where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$ and since $|P'(z)| + |Q'(z)| \le n$ is always true, we have

$$|P'(z)| + |nP(z) - zP'(z)| \le n ,$$

from where we conclude that

$$|P'(z)| \le \frac{n}{1+K^{\mu}}$$

for $|z| \le 1$.

To show that there is equality in (4.24) for $P(z) = \left(\frac{z^{\mu}+K^{\mu}}{1+K^{\mu}}\right)^{n/\mu}$ we note

$$\max_{|z|=1} |P(z)| = |P(1)| = 1$$

and

$$\max_{|z|=1} |P'(z)|$$

$$= \max_{|z|=1} \left| \frac{n}{\mu} \mu \frac{z^{\mu-1}}{1+K^{\mu}} \left(\frac{z^{\mu}+K^{\mu}}{1+K^{\mu}}\right)^{\frac{n}{\mu}-1} \right| = |P'(1)|$$

$$= \frac{n}{1+K^{\mu}} . \qquad \square$$

§4.4. APPLICATIONS OF LAGUERRES THEOREM

In the study of the influence of zeros on the estimates of the derivative polynomial, the application of Laguerre Theorem (Lemma 4.1) was first given by N.G. De Bruijn [5]. Let $P(z)$ be a polynomial of degree n, the image of the unit disk under $P(z)$ must be contained in a certain point set S. Choosing $\lambda \notin S$ and applying Laguerre Theorem to the polynomial $P(z) - \lambda$, he established:

Lemma 4.7: Let C be a circular domain in the z-plane and S an arbitrary point set in ω-plane. If the polynomial $P(z)$ of degree n satisfies $P(z) = \omega \in S$ for any $z \in C$, then we have, for any $z \in C$ and $\xi \in C$

$$\frac{\xi}{n} P'(z) + P(z) - \frac{zP'(z)}{n} \in S \tag{4.28}$$

Proof of Lemma 4.7: Choose any $\lambda \notin S$, we have $P(z) \neq \lambda$ for $z \in C$, which implies the polynomial $P(z) - \lambda$ has no zeros in C. Applying Lemma 4.1 to $P(z) - \lambda$ we have

$$(\xi - z)\ P'(z) + n[P(z) - \lambda] \neq 0$$

for ξ, $z \in C$. Thus

$$(\xi - z)\ P'(z) + nP(z) \neq n\lambda$$

or

$$\frac{\xi}{n} P'(z) + P(z) - \frac{zP'(z)}{n} \neq \lambda$$

for ξ, $z \in C$. This proves the lemma. □

Lemma 4.7 is very useful. In fact, short and simple proof of most of the results in our previous discussion can be obtained by applying Laguerre's Theorem and Lemma 4.7. In the following we give:

Alternate Proof of Bernstein's Theorem. In Lemma 4.7, we let C be the unit disk $|z| \le 1$ and S be the disk $|\omega| \le 1$. Since $|P(z)| \le 1$ for $|z| \le 1$, as ξ varies in the unit disk (4.28) implies a disk of radius $\left|\frac{P'(z)}{n}\right|$ and centre $z\,\frac{P'(z)}{n} - P(z)$ is entirely contained in S. Obviously the largest disk that can be contained in S is S itself which is of radius one.

Thus

$$\left|\frac{P'(z)}{n}\right| \le 1$$

for $|z| \le 1$. □

<u>Remark 4.3*</u>: Incidentally, we observe that Lemma 4.7 can be used to establish the fact that equality in (1.1) holds only for $P(z) = \alpha z^n$, $|\alpha| = 1$; an already known result. To show this let $P'(z_0) = ne^{i\alpha}$ for some $|z_0| = 1$. From (4.28), for any ξ in the unit disk,

$$\xi e^{i\alpha} + P(z_0) - z_0 \frac{P'(z_0)}{n} \in S$$

where $S = \{P(z) \mid |z| \le 1\}$. But this implies that a disk with centre $P(z_0) - z_0 \dfrac{P'(z_0)}{n}$ and radius one (i.e. $|e^{i\alpha}| = 1$) is entirely contained in S. This is possible only when S is the unit disk itself which further implies that $|P(z)|$ attains its maximum one infinitely many times on $|z| = 1$. Hence $P(z) = \alpha z^n$, $|\alpha| = 1$.

De Bruijn also gave a short and easy proof of the Erdös-Lax Theorem.

<u>Alternate Proof of Erdös-Lax Theorem</u>. Since $P(z)$ has no zeros in $|z| < 1$ and $|P(z)| < 1$ for $|z| < 1$, we note that the set $S = \{P(z) \mid |z| < 1\}$ is entirely contained in the unit disk but does not contain the origin. As for any $|\xi| < 1$ and $|z| < 1$

$$\xi \frac{P'(z)}{n} - z \frac{P'(z)}{n} + P(z) \in S \quad ,$$

we observe that for any z the disk of radius $\left|\frac{P'(z)}{n}\right|$ and centre $z\frac{P'(z)}{n} - P(z)$ is entirely contained in S. But the radius of the largest disk that can be contained in S but does not have the origin cannot exceed $\frac{1}{2}$ we have $|P'(z)| < \frac{n}{2}$. Making $|z| \to 1$ we get $|P'(z)| \le \frac{n}{2}$ for $|z| \le 1$. □

<u>Alternate proof of Theorem 4.2*</u>: Since $P(z)$ has all its zeros in the disk $|z| \le k < 1$ it follows from Lemma 4.1 that for any $|\xi| > k$ and $|z| = 1$ we have

$$(\xi - z)\, P'(z) \neq -nP(z)\,. \tag{4.29}$$

We note that for any fixed z, $|z| = 1$, a number ξ lying on the circle $c_\rho : |\xi - z| = 1 + \rho > 1 + k$ does not belong to the disk $|\xi| \le k$. Moreover we can find ξ on c_ρ such that $\arg(\xi - z)\, P'(z) = \arg(-nP(z))$, so from (4.29) we have

$$(1+\rho)|P'(z)| = |\xi - z|\,|P'(z)| > n|P(z)|\,, \qquad \text{for } \rho > k\,.$$

Taking $\rho \to k$, consequently

$$|P'(z)| \ge \frac{n}{1+k}\,|P(z)|\,. \tag{4.30}$$

This implies (4.10). □

If $P(z)$ has all its zeros in $|z| \le k \le 1$, following Gauss-Lucas Theorem, the zeros of the successive derivatives $P^{(\nu)}(z)$ also lie in $|z| \le k$. With this observation the following result is immediate:

<u>Theorem 4.6</u>: <u>If</u> $P(z)$ <u>has all its zeros in</u> $|z| \le k \le 1$ and $\max_{|z|=1} |P(z)| = 1$, <u>then</u>

$$\max_{|z|=1} |P^{(\nu)}(z)| \geq \frac{n(n-1)\ldots(n-\nu+1)}{(1+k)^{\nu}} \qquad (4.31)$$

<u>The result is best possible and equality in</u> (4.31) <u>holds for</u>

$P(z) = \left(\frac{z+k}{1+k}\right)^n$.

We have already use (4.30) for $k=1$ in establishing Lemma 3.2. We further note that when $P(z)$ has all its zeros on $|z|=1$, there is an interesting relation between $\arg P'(z_0)$ and $\arg P(z_0)$ where $|P(z_0)| = \max_{|z|=1} |P(z)|$. We present:

<u>Theorem 4.7</u>*: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>having all its zeros on</u> $|z|=1$ <u>and</u> $|P(z_0)| = \max_{|z|=1} |P(z)| = 1$, <u>then</u>

$$\arg z_0 P'(z_0) = \arg P(z_0) .$$

<u>Proof of Theorem 4.7</u>: By Laguerre Theorem (Lemma 4.1), for all z and ξ with $|z|<1$ and $|\xi|<1$ (or $|z|>1$, $|\xi|>1$) we have

$$\xi \frac{P'(z)}{n} \neq z\frac{P'(z)}{n} - P(z) .$$

With a suitable choice of argument of ξ, we get

$$|\xi| \left|\frac{P'(z)}{n}\right| \neq \left|z \frac{P'(z)}{n} - P(z)\right| .$$

This implies either

$$|\xi| \left|\frac{P'(z)}{n}\right| > \left|z \frac{P'(z)}{n} - P(z)\right| \qquad (4.32)$$

$$|\xi| \left|\frac{P'(z)}{n}\right| < \left|z \frac{P'(z)}{n} - P(z)\right| . \qquad (4.33)$$

But a sufficiently small value of $|\xi|$ contradicts (4.32), so (4.33) is true and letting $|\xi| \to 1$ and $|z| \uparrow 1$ we have

$$\left|\frac{P'(z)}{n}\right| \le \left|\frac{zP'(z)}{n} - P(z)\right| . \tag{4.34}$$

On the other hand, for sufficiently large value of $|\xi|$ and $|z| \downarrow 1$ we have (4.32) is true and

$$\left|\frac{P'(z)}{n}\right| \ge \left|z\,\frac{P'(z)}{n} - P(z)\right| . \tag{4.35}$$

Thus from (4.34), (4.35) on $|z| = 1$ we have

$$\left|\frac{P'(z)}{n}\right| = \left|z\,\frac{P'(z)}{n} - P(z)\right|$$

and so

$$\left|z\,\frac{P'(z)}{n}\right| = \left|z\,\frac{P'(z)}{n} - P(z)\right| . \tag{4.36}$$

Putting $z = z_0$ in (4.36) we have

$$\left|z_0\,\frac{P'(z_0)}{n}\right| = \left|z_0\,\frac{P'(z_0)}{n} - P(z_0)\right| . \tag{4.37}$$

From Lemma 3.2 with $|P(z_0)| = 1$ we know that (4.37) can be true only when

$$\arg z_0\,\frac{P'(z_0)}{n} = \arg P(z_0)$$

which implies

$$\arg z_0\, P'(z_0) = \arg P(z_0) \quad . \qquad \square$$

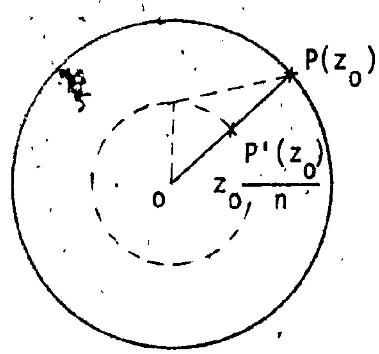

Fig. 4.1

§4.5. GENERALIZATION OF LAGUARRE'S THEOREM AND ITS APPLICATIONS

Let $P_\xi(z) = (\xi - z)\,P'(z) + nP(z)$ where ξ is any complex number. $P_\xi(z)$ is called the polar derivative of $P(z)$ with respect to ξ. We note that by taking the successive polar derivative $P_{\xi_1,\ldots,\xi_m}(z)$ of $P(z)$ and apply Lemma 4.1 and Lemma 4.7 to it we can easily obtain the following two lemmas.

Lemma 4.8: *If $P(z)$ is a polynomial of degree n having no zeros in a circular domain C, then for any $\xi_1,\ldots,\xi_m \in C$, $m \le n$ and $z \in C$, we have,*

$$P_{\xi_1,\ldots,\xi_m}(z) = \frac{(\xi_1-z)\ldots(\xi_m-z)P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} + \ldots + \sum_{\binom{m}{\nu}} \frac{(\xi_1-z)\ldots(\xi_\nu-z)P^{(\nu)}(z)}{n(n-1)\ldots(n-\nu+1)}$$

$$+ \ldots + \sum_{\binom{m}{1}} \frac{(\xi_1-z)P'(z)}{n} + P(z) \neq 0 \qquad (4.38)$$

where $\sum_{\binom{m}{\nu}} (\xi_1 - z)\ldots(\xi_\nu - z)$ represents the sum of the products of $(\xi_j - z)$, $j = 1,2,\ldots,m$ taken ν at a time.

Lemma 4.9: Let C be a circular domain in the z-plane and S be an arbitrary point set in the ω-plane. If the polynomial $P(z)$ of degree n satisfies $P(z) = \omega \in S$ for any $z \in C$, then for any $\xi_1,\ldots,\xi_m \in C$, $m \le n$ and $z \in C$

$$\frac{(\xi_1-z)\ldots(\xi_m-z)P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} + \sum_{\binom{m}{m-1}} \frac{(\xi_1-z)\ldots(\xi_{m-1}-z)P^{(m-1)}(z)}{n(n-1)\ldots(n-m+2)}$$

$$+\ldots+ \sum_{\binom{m}{\nu}} \frac{(\xi_1-z)\ldots(\xi_\nu-z)P^{(\nu)}(z)}{n(n-1)\ldots(n-\nu+1)} +\ldots+ \sum_{\binom{m}{1}} \frac{(\xi_1-z)P'(z)}{n}$$

$$+ P(z) \in S .$$

We now give a theorem involving the second derivative and with the bound $\frac{n}{1+K}$ in the following:

Theorem 4.8* : If $P(z)$ has no zeros in $|z| < K$, $K \ge 1$, then for $|z| \le 1$

$$\left|\frac{P^{(2)}(z)}{n-1}\right| + \left|\frac{zP^{(2)}(z)}{n-1} - P'(z)\right| \le \frac{n}{1+K} \max_{|z|=1} |P(z)| . \qquad (4.39)$$

The result is best possible and equality in (4.39) holds for $P(z) = (z+K)^n$.

Proof of Theorem 4.8: Since $P(z) \neq 0$ in $|z| < K$, from Lemma 4.8 we have

$$\frac{(\xi-z)(\mu-z)P^{(2)}(z)}{n(n-1)} + \frac{(\xi-z)P'(z)}{n} + \frac{(\mu-z)P'(z)}{n} + P(z) \neq 0$$

for $|\xi|$, $|\mu|$, $|z| < K$. From where

$$\xi\left(\mu \frac{P^{(2)}(z)}{n(n-1)} - z\frac{P^{(2)}(z)}{n(n-1)} + \frac{P'(z)}{n}\right) + \lambda(\mu,z) \neq 0 \qquad (4.40)$$

where $\lambda(\mu,z) = -\frac{z(\mu-z)P^{(2)}(z)}{n(n-1)} + \frac{(\mu-z)P'(z)}{n} - z\frac{P'(z)}{n} + P(z)$.

Consequently for fixed $|\mu| = 1$, $|z| = 1$ and $|\xi| \to K$ from (4.40) we have

$$K\left|\mu \frac{P^{(2)}(z)}{n(n-1)} - \frac{zP^{(2)}(z)}{n(n-1)} + \frac{P'(z)}{n}\right| \le |\lambda(\mu,z)| . \qquad (4.41)$$

On the other hand from Lemma 4.9,

$$\left|\mu \frac{P^{(2)}(z)}{n(n-1)} - z\frac{P^{(2)}(z)}{n(n-1)} + \frac{P'(z)}{n}\right| + |\lambda(\mu,z)| \le \max_{|z|=1} |P(z)| \qquad (4.42)$$

for $|\mu| = 1$, $|z| = 1$. (4.41) and (4.42) together imply that

$$\left|\mu \frac{P^{(2)}(z)}{n(n-1)} - \frac{zP^{(2)}(z)}{n(n-1)} + \frac{P'(z)}{n}\right| \le \frac{1}{1+K} \max_{|z|=1} |P(z)| .$$

Now choosing the $\arg\mu$ suitably, one get (4.39). □

From Lemma 4.8, we can also deduce the following:

Theorem 4.9*: Let $P(z)$ be a polynomial of degree n having no zeros in a circular domain C which contains all the zeros of the polynomial $Q(z)$ of degree m, then

$$\sum_{\nu=0}^{m-1} \frac{(-1)^{m-\nu} Q^{(\nu)}(z) P^{(m-\nu)}(z)}{\nu! n(n-1)\ldots(n-(m-\nu-1))} + P(z) \neq 0 \qquad (4.43)$$

<u>for</u> $z \in C$ <u>if</u> $m < n$ <u>and for all</u> z <u>if</u> $m = n$.

<u>Proof of Theorem 4.9</u>: Let $\xi_1, \ldots, \xi_m \in C$ be the zeros of $Q(z) = \sum_{\nu=0}^{m} b_\nu z^\nu$, i.e., $Q(z) = (-1)^m b_m(\xi_1 - z)\ldots(\xi_m - z)$. Obviously we see that

$$(m-\nu)!\, b_m \sum_{\binom{m}{\nu}} (\xi_1 - z)\ldots(\xi_\nu - z) = (-1)^{m-\nu} Q^{(m-\nu)}(z) \ . \qquad (4.44)$$

Substitute (4.44) into (4.38), we get (4.43) for $z \in C$, $0 \le m < n$. □

<u>Remark 4.4</u>: Now if $P(z)$ and $Q(z)$ are polynomials of degree n satisfying the hypothesis of Theorem 4.9, then (4.44) becomes

$$\sum_{\nu=0}^{n} (-1)^{n-\nu} Q^{(\nu)}(z) P^{(n-\nu)}(z) \neq 0 \qquad (4.45)$$

Setting $z = 0$ in (4.45), we get the Grace Apolarity Theorem [5] .

In 1969, N.K. Govil and Q.I. Rahman could prove a result concerning the estimate of $|P^{(\nu)}(z)|$ when the zeros of $P(z)$ are prescribed. Using Lemma 4.8 and Lemma 4.9 we present an alternate proof of their result stated in the following:

<u>Theorem 4.10</u>: <u>If</u> $P(z)$ <u>is a polynomial of degree</u> n <u>having no zeros in</u> $|z| < K$, $K \ge 1$ <u>and</u> $|P(z)| \le 1$ <u>on</u> $|z| \le 1$, <u>then</u>

$$|P^{(m)}(z)| \le \frac{n(n-1)\ldots(n-m+1)}{1+K^m} \qquad (4.46)$$

<u>for</u> $|z| \le 1$;

Proof of Theorem 4.10*: Let the circular domain C in Lemma 4.8 be $|z| < K$, $K \geq 1$ and $\xi_1, \ldots, \xi_m$ be the m-th root of λ^m where $|\lambda| < K$ and $|z| = 1$. Thus from (4.38) in Lemma 4.8, we have

$$\frac{\lambda^m \, P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} + Q(z) \neq 0$$

where $Q(z)$ is independent of λ. For a suitable choice of argument of λ we get

$$\frac{|\lambda|^m \, |P^{(m)}(z)|}{n(n-1)\ldots(n-m+1)} \neq |Q(z)|$$

from where follows

$$\frac{|\lambda|^m \, |P^{(m)}(z)|}{n(n-1)\ldots(n-m+1)} < |Q(z)|$$

because the other possibility is violated for sufficiently small value of $|\lambda|$. Taking $|\lambda| \to K$ we have

$$\frac{K^m |P^{(m)}(z)|}{n(n-1)\ldots(n-m+1)} \leq |Q(z)| .$$

On the other hand, from Lemma 4.9 for $|\lambda| \leq 1$ and $|z| = 1$, we have

$$\left| \frac{\lambda \quad P^{(m)}(z)}{n(n-1)\ldots(n-m+1)} + Q(z) \right| \leq 1 .$$

This means that a disk with centre $Q(z)$ (which also belongs to the unit disk $|\omega| \leq 1$, $\lambda = 0$) and radius $K^m|P^{(m)}(z)| / n(n-1)\ldots(n-m+1)$ is entirely contained in $|\omega| \leq 1$. But the centre $Q(z)$ itself is at a distance greater than $K^m|P^{(m)}(z)|/n(n-1)\ldots(n-m+1)$ from the origin. Consequently

$$\frac{(1+K^m)|P^{(m)}(z)|}{n(n-1)\dots(n-m+1)} \le 1.$$

This completes the proof. □

Inequality (4.46) is best possible when $m = 1$. When $m = 1$, (4.46) becomes exactly the same as (4.1). For other values of m, we do not expect (4.46) to be also best possible because the location of the zeros of the first derivative $P'(z)$ heavily influences the inequality. In fact, for any $K \ge 1$, although $P(z)$ has all its zeros lie in $|z| \geqslant K$, $P'(z)$ might still has zeros lying very close to the origin. This involves the investigation of the problem considering polynomials of degree n having no zeros in $|z| < k$, $k < 1$.

§4.6. THE CASE WHEN $k < 1$

For the problem of estimating $|P'(z)|$ for a polynomial $P(z)$ of degree n with $|P(z)| \le 1$ on $|z| \le 1$ and having all its zeros lie in $|z| > k$, $k < 1$, it has been quite a while that the inequality $|P'(z)| \le \frac{n}{1+k^n}$ for $|z| \le 1$ was expected to be true. But to everyone's surprise, recently E.B. Saff [22] constructed a polynomial of degree 2 for which

$$\max_{|z|=1} |P'(z)| > \frac{n}{1+k^n} \max_{|z|=1} |P(z)| .$$

<u>Saff's example</u>: $P(z) = (z-\frac{1}{2})(z+\frac{1}{3})$, for this particular polynomial, it has no zeros in $|z| < \frac{1}{3}$, and we have

$$|P(e^{i\theta})|^2 = \left|(\cos\theta + i\sin\theta)^2 - \frac{1}{6}(\cos\theta + i\sin\theta) - \frac{1}{6}\right|^2$$

$$= \left|\cos 2\theta + i\sin 2\theta - \frac{1}{6}\cos\theta - i\,\frac{1}{6}\sin\theta - \frac{1}{6}\right|^2$$

$$= \left|(\cos 2\theta - \frac{1}{6}\cos\theta - \frac{1}{6}) + i(\sin 2\theta - \frac{1}{6}\sin\theta)\right|^2$$

$$= (\cos 2\theta - \frac{1}{6}\cos\theta - \frac{1}{6})^2 + (\sin 2\theta - \frac{1}{6}\sin\theta)^2$$

$$= \cos^2 2\theta + \frac{1}{36}\cos^2\theta + \frac{1}{36} - \frac{1}{3}\cos 2\theta\cos\theta$$

$$- \frac{1}{3}\cos 2\theta + \frac{1}{18}\cos\theta + \sin^2 2\theta - \frac{1}{3}\sin\theta\sin 2\theta$$

$$+ \frac{1}{36}\sin^2\theta$$

$$= 1 + \frac{2}{36} - \frac{1}{3}(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$$

$$+ \frac{1}{18}\cos\theta - \frac{1}{3}\cos 2\theta$$

$$= \frac{19}{18} - \frac{1}{3}\cos\theta + \frac{1}{18}\cos\theta - \frac{1}{3}\cos 2\theta$$

$$= \frac{19}{18} - \frac{5}{18}\cos\theta - \frac{1}{3}(2\cos^2\theta - 1) \qquad (4.47)$$

and hence

$$\frac{d}{d\theta}\left|P(e^{i\theta})\right|^2 = \frac{5}{18}\sin\theta + \frac{4}{3}\sin\theta\cos\theta . \qquad (4.48)$$

We set (4.48) equals to zero to obtain

$$\frac{5}{6}\sin\theta + 4\sin\theta\cos\theta = 0$$

which gives

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = -\frac{5}{24} .$$

This implies that $\theta = 0$ or $\theta = \cos^{-1}(-\frac{5}{24})$ is the point where $|P(e^{i\theta})|^2$ attains its maximum. Since $|P(e^{i\theta})|^2$ and $|P(e^{i\theta})|$ attain their maximum at the same point, and $\theta = 0$ gives a smaller quantity in (4.47), we have

$$\max_{0\le\theta<2\pi} |P(e^{i\theta})| = \sqrt{\frac{19}{18} - \frac{5}{18}\left(-\frac{5}{24}\right) - \frac{1}{3}\left[2\left(-\frac{5}{24}\right)^2 - 1\right]}$$

$$= 1.91 .$$

On the other hand, $P'(z) = 2z - \frac{1}{6}$ and hence

$$\max_{0\le\theta<2\pi} |P'(e^{i\theta})| = \left|-2 - \frac{1}{6}\right| = 2.167 .$$

Therefore

$$\frac{1}{2}\left(\max_{0\le\theta<2\pi} |P'(e^{i\theta})| \Big/ \max_{0\le\theta<2\pi} |P(e^{i\theta})|\right) = 0.910$$

$$> 0.9$$

$$= \frac{1}{1 + (\frac{1}{3})^2} .$$

So the polynomial $P(z) = z^n + k^n$ is not an extremal polynomial for the Erdös-Lax Theorem when $k < 1$. Inspite of the effort of those working in this area, there is not yet any suggestion what would be the form of the extremal polynomial in this case. Several mathematicians including R.P. Boas and A. Zymund admit the difficulty of this problem. To add to the difficulty, we note that even for a polynomial of degree two with real zeros the behaviour of $\frac{1}{2}\left(\max_{|z|=1} P'(z) \Big/ \max_{|z|=1} |P(z)|\right)$ is "strange". We present this observation in the following:

Fpr $P(z) = (z-\alpha)(z-\beta)$ where α,β are real, we let

$$f_\alpha(\beta)=\frac{1}{2}\left(\max_{|z|=1}|P'(z)|/\max_{|z|=1}|P(z)|\right)$$

for some fixed α. With the help of a digital computer (CDC Cyber 174). We obtain the graph of $f_\alpha(\beta)$ on $-1\le\beta\le 1$ as in Fig. 4.2*

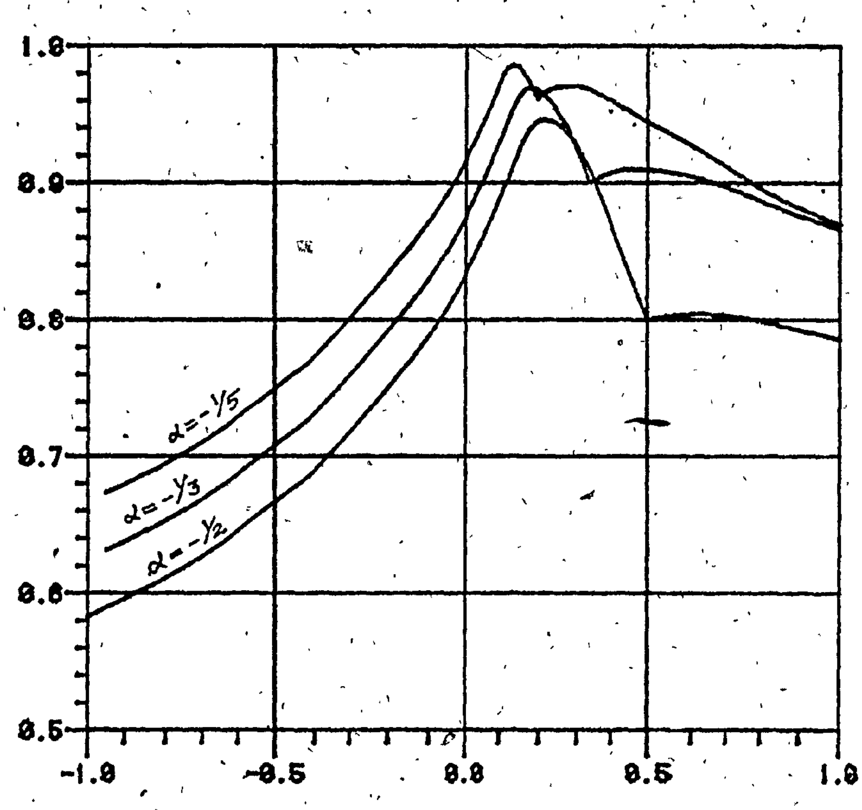

Fig. 4.2*

Theorem 4.11* : Let $P(z) = (z-\alpha)(z-\beta)$ where $\alpha \ge 0$, $\beta \le 0$ $|(\alpha+\beta)(1+\alpha\beta)| \le 4|\alpha\beta|$, then

$$\frac{1}{2}\left(\max_{|z|=1}|P'(z)| / \max_{|z|=1}|P(z)|\right) = \frac{(2+|\alpha+\beta|)\sqrt{|\alpha\beta|}}{|(\alpha-\beta)(1-\alpha\beta)|} .$$

Proof of Theorem 4.11: Since

$$\begin{aligned}
|P(e^{i\theta})|^2 &= \left|(\cos\theta + i\sin\theta)^2 - (\alpha+\beta)(\cos\theta + i\sin\theta) + \alpha\beta\right|^2 \\
&= |\cos 2\theta + i\sin 2\theta - (\alpha+\beta)\cos\theta - i(\alpha+\beta)\sin\theta + \alpha\beta|^2 \\
&= \left[\cos 2\theta - (\alpha+\beta)\cos\theta + \alpha\beta\right]^2 + \left[\sin 2\theta - (\alpha+\beta)\sin\theta\right]^2 \\
&= \cos^2 2\theta + (\alpha+\beta)^2\cos^2\theta + (\alpha\beta)^2 - 2(\alpha+\beta)\cos 2\theta\cos\theta \\
&\quad - 2(\alpha+\beta)\alpha\beta\cos\theta + 2\alpha\beta\cos 2\theta + \sin^2 2\theta \\
&\quad + (\alpha+\beta)^2\sin^2\theta - 2(\alpha+\beta)\sin 2\theta\sin\theta \\
&= 1 + (\alpha+\beta)^2 - 2(\alpha+\beta)\cos\theta - 2(\alpha+\beta)\alpha\beta\cos\theta \\
&\quad + 2\alpha\beta\cos 2\theta + (\alpha\beta)^2 \\
&= 1 + (\alpha+\beta)^2 + (\alpha\beta)^2 - 2(\alpha+\beta)(1+\alpha\beta)\cos\theta \\
&\quad + 2\alpha\beta(2\cos^2\theta - 1) .
\end{aligned}$$

If

$$\frac{d}{d\theta}|P(e^{i\theta})|^2 = 2(\alpha+\beta)(1+\alpha\beta)\sin\theta - 4\alpha\beta\sin 2\theta = 0 ,$$

then we have

$$(\alpha+\beta)(1+\alpha\beta)\sin\theta = 2\alpha\beta\sin 2\theta ,$$

which implies either $\sin\theta = 0$ or $\cos\theta = \frac{(\alpha+\beta)(1+\alpha\beta)}{4\alpha\beta}$. We know that $\theta = \cos^{-1}\left[\frac{(\alpha+\beta)(1+\alpha\beta)}{4\alpha\beta}\right]$ is the point where $|P(e^{i\theta})|^2$ attains its maximum. Thus

$$\max_{0\le\theta<2\pi} |P(e^{i\theta})|^2 = 1+(\alpha+\beta)^2+(\alpha\beta)^2 - \frac{2(\alpha+\beta)^2(1+\alpha\beta)^2}{4\alpha\beta} + 2\alpha\beta\left[2\,\frac{(\alpha+\beta)^2(1+\alpha\beta)^2}{16(\alpha\beta)^2} - 1\right]$$

$$= 1+(\alpha+\beta)^2+(\alpha\beta)^2 - \frac{2(\alpha+\beta)^2(1+\alpha\beta)^2}{4\alpha\beta} + \frac{(\alpha+\beta)^2(1+\alpha\beta)^2}{4\alpha\beta} - 2\alpha\beta$$

$$= 1+\alpha^2+2\alpha\beta+\beta^2+(\alpha\beta)^2 - 2\alpha\beta - \frac{(\alpha+\beta)^2(1+\alpha\beta)^2}{4\alpha\beta}$$

$$= \Big[4\alpha\beta + 4\alpha^3\beta + 4\alpha\beta^3 + 4(\alpha\beta)^3 - \alpha^2 - 2\alpha\beta - \beta^2 - 2\alpha^3\beta - 4(\alpha\beta)^2 - 2\alpha\beta^3 - \alpha^4\beta^2 - 2(\alpha\beta)^3 - \alpha^2\beta^4\Big] / 4\alpha\beta$$

$$= -\frac{(\alpha-\beta)^2(1-\alpha\beta)^2}{4\alpha\beta}.$$

Since $|P(e^{i\theta})|^2$ and $|P(e^{i\theta})|$ attain their maximum at the same point, therefore we have

$$\max_{|z|=1} |P(z)| = \sqrt{\frac{(\alpha-\beta)^2(1-\alpha\beta)^2}{4|\alpha\beta|}} = \frac{|(\alpha-\beta)(1-\alpha\beta)|}{2\sqrt{|\alpha\beta|}}.$$

Now $P'(z) = 2z - (\alpha+\beta)$, $\max_{|z|=1} |P'(z)| = 2+|\alpha+\beta|$. Thus

$$\max_{|z|=1} |P'(z)| \Big/ 2\max_{|z|=1} |P(z)| = \frac{(2+|\alpha+\beta|)\sqrt{|\alpha\beta|}}{|(\alpha-\beta)(1-\alpha\beta)|}. \qquad \square$$

Remark 4.4: If $\alpha,\beta \geq 0$, then $|P(-1)| = \max_{|z|=1} |P(z)|$ and if $\alpha,\beta \leq 0$ then $|P(1)| = \max_{|z|=1} |P(z)|$; in both the cases $\left|\frac{(\alpha+\beta)(1+\alpha\beta)}{4\alpha\beta}\right| \geqslant 1$.

Moreover, if $\alpha > 0$ and $\beta < 0$, then for $\frac{(\alpha+\beta)(1+\alpha\beta)}{4\alpha\beta} > 1$.

$|P(1)| = \max_{|z|=1} |P(z)|$ and for $\frac{(\alpha+\beta)(1+\alpha\beta)}{4\alpha\beta} < -1$, $|P(-1)| = \max_{|z|=1} |P(z)|$.

However, we do expect that the polynomial $P(z) = z^n + k^n$, $k \leq 1$ would be an extremal for the case when $P(z)$ has all its zeros on $|z| = k$. When $n = 2$ we establish the following:

Theorem 4.12* : If $P(z)$ is a polynomial of degree 2 having both of its zeros on $|z| = k \leq 1$, then for $|z| \leq 1$

$$|P'(z)| \leq \frac{2}{1+k^2} \max_{|z|=1} |P(z)| \tag{4.49}$$

The result is best possible and equality in (4.49) holds for $P(z) = z^2 + k^2$.

Proof of Theorem 4.12: Let $ke^{i\alpha}$ and $ke^{i\beta}$ be the zeros of $P(z)$. Consider the polynomial $R(z) = P(e^{i(\alpha+\beta)/2} z)$. It is obvious that $\max_{|z|=1} |R(z)| = \max_{|z|=1} |P(z)|$. Since

$$|R(z)| = \left|\left(e^{i(\alpha+\beta)/2} z - ke^{i\alpha}\right)\left(e^{i(\alpha+\beta)/2} z - ke^{i\beta}\right)\right|$$

$$= \left|\left(z - ke^{i(\alpha-\beta)/2}\right)\left(z - ke^{-i(\alpha-\beta)/2}\right)\right|$$

$$= \left|z^2 - 2kz \cos\left(\frac{\alpha-\beta}{2}\right) + k^2\right|$$

one gets

$$\max_{|z|=1} |P(z)| = 1 + 2k\left|\cos\left(\frac{\alpha-\beta}{2}\right)\right| + k^2 \qquad (4.50)$$

We also have

$$e^{-i\left(\frac{\alpha+\beta}{2}\right)} P'(z) = e^{-i\left(\frac{\alpha+\beta}{2}\right)}\left(2z - k(e^{i\alpha} + e^{i\beta})\right)$$

$$= 2e^{-i\left(\frac{\alpha+\beta}{2}\right)} z - k\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} + e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right)$$

$$= 2\left(e^{-i\left(\frac{\alpha+\beta}{2}\right)} z - k \cos\left(\frac{\alpha-\beta}{2}\right)\right)$$

which gives

$$\max_{|z|=1} |P'(z)| = 2\left(1 + k\left|\cos\left(\frac{\alpha-\beta}{2}\right)\right|\right). \qquad (4.51)$$

Consequently from (4.50) and (4.51) one gets

$$\left(\max_{|z|=1} |P'(z)| \Big/ \max_{|z|=1} |P(z)| \right) = \frac{2(1+k|\cos(\frac{\alpha-\beta}{2})|)}{1+2k|\cos(\frac{\alpha-\beta}{2})|+k^2}$$

$$\leq \frac{2}{1+k^2}$$

as $k \leq 1$. This proves (4.49).

CHAPTER V

A CONJECTURE OF SAFF

§5.1. THE SAFF CONJECTURE

In the previous chapters we considered the problems concerning the estimate of the derivatives of polynomials having no zeros inside a circular disk. It is natural to study also the problems when the polynomials have no zeros in a half-plane. In this direction, E.B. Saff proposed the following conjecture:

Let $P(z) = \prod_{\nu=1}^{n} (z-z_\nu)$ be a polynomial of degree n having all its zeros in $\operatorname{Re} z \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \sum_{\nu=1}^{n} \frac{1}{1+\operatorname{Re} z_\nu} \max_{|z|=1} |P(z)| . \tag{5.1}$$

Equality in (5.1) holds when all z_ν's are real.

The interesting fact about Saff conjecture is that in (5.1) each zero is supposed to make a contribution which is independent of other zeros as well as of the degree of the polynomial. In most of the problems we previously discussed, the estimate of $|P'(z)|$ is always given by the degree of the polynomial and the constant concerning the restriction on zeros, but is free from the exact position of zeros. The Saff conjecture is still unresolved. In 1979, A. Giroux, Q.I. Rahman and G. Schmeisser [10] proved the Saff conjecture for polynomials of degree 2 in the following:

<u>Theorem 5.1</u>: <u>Let</u> $P(z) = (z-z_1)(z-z_2)$ <u>be a polynomial with</u> $\mathrm{Re}\,z_1 \ge 1$ <u>and</u> $\mathrm{Re}\,z_2 \ge 1$, <u>then</u>

$$\max_{|z|=1} |P'(z)| \le \left(\frac{1}{1+\mathrm{Re}z_1} + \frac{1}{1+\mathrm{Re}z_2}\right) \max_{|z|=1} |P(z)|. \qquad (5.2)$$

<u>Proof of Theorem 5.1</u>: Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, $x_1 \ge 1$, $x_2 \ge 1$, $y_1, y_2 \in \mathbb{R}$. Then $P'(z) = 2z-(z_1+z_2)$ vanishes at $\zeta+i\eta$, where

$$\zeta = \frac{x_1+x_2}{2}, \quad \eta = \frac{y_1+y_2}{2} .$$

We may assume that $\eta \ge 0$ and $x_1 \le x_2$. Since

$$\max_{|z|=1} |P'(z)| \le |P(-1)| \left(\frac{1}{1+x_1} + \frac{1}{1+x_2}\right) \qquad (5.3)$$

implies

$$\max_{|z|=1} |P'(z)| \le \max_{|z|=1} |P(z)| \left(\frac{1}{1+x_1} + \frac{1}{1+x_2}\right),$$

in order to prove (5.2), it is enough to show (5.3). For fixed x_1, x_2 and η consider the family

$$P_{x_1,x_2,\eta} = \left\{f_\lambda(z) = (z-x_1-i(\eta-\lambda))(z-x_2-i(\eta+\lambda)) : \lambda \in \mathbb{R}\right\} .$$

With $\lambda = (y_2-y_1)/2$, we know that $P(z) \in P_{x_1,x_2,\eta}$. Also, since

$$\begin{aligned} f'_\lambda(z) &= 2z - \left[(x_1+i(\eta-\lambda)) + (x_2+i(\eta+\lambda))\right] \\ &= 2z - x_1 - x_2 - 2i\eta \end{aligned}$$

is independent of λ, we know that $\max_{|z|=1} |f'_\lambda(z)|$ is the same for each member $f_\lambda(z)$ in $P_{x_1,x_2,\eta}$. Therefore, it is sufficient to prove (5.3) for the polynomial $f_\lambda(z)$ in $P_{x_1,x_2,\eta}$ for which $|f_\lambda(-1)|$ is smallest. Setting

$$A(\lambda) = \left|\sqrt{(1+x_1)^2 + (\eta-\lambda)^2}\right| ,$$

$$B(\lambda) = \left|\sqrt{(1+x_2)^2 + (\eta+\lambda)^2}\right|$$

then $\left|f_\lambda(-1)\right| = A(\lambda)\ B(\lambda)$. Taking the derivative of $|f_\lambda(-1)|$ with respect to λ we have

$$\frac{d}{d\lambda}|f_\lambda(-1)| = A(\lambda)\ B'(\lambda) + B(\lambda)\ A'(\lambda)$$

$$= \frac{A(\lambda)(\eta+\lambda)}{B(\lambda)} - \frac{B(\lambda)(\eta-\lambda)}{A(\lambda)} .$$

Thus we see that $|f_\lambda(-1)|$ is smallest when

$$\frac{B(\lambda)(\eta-\lambda)}{A(\lambda)} = \frac{A(\lambda)(\eta+\lambda)}{B(\lambda)} ,$$

i.e. (see Fig. 5.1).

$$B(\lambda)\sin\varphi_1 = A(\lambda)\sin\varphi_2 = u \quad \text{(say)}. \tag{5.4}$$

If we denote by $|C,D|$ the distance between two points C,D, then in fact $A(\lambda) = |A,z_1|$ and $B(\lambda) = |A,z_2|$.

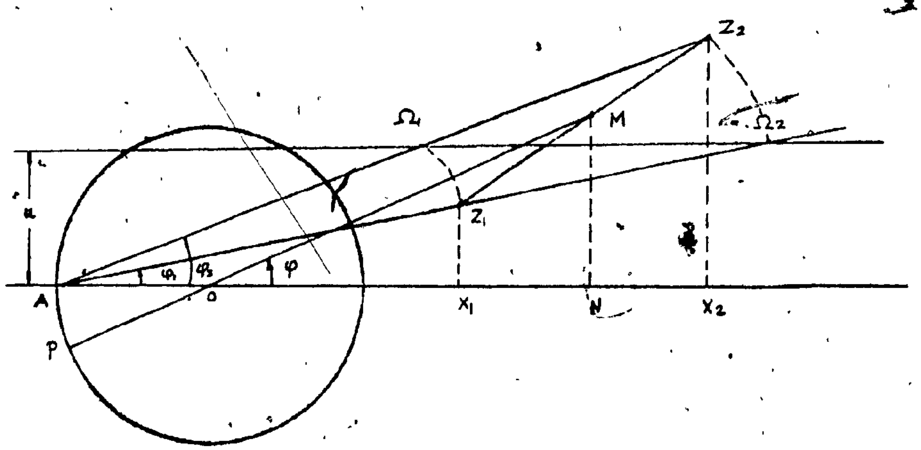

Fig. 5.1

In other words, the line passing through the points

$$\Omega_1 = (A(\lambda)\ \cos\varphi_2 - 1) + iA(\lambda)\sin\varphi_2 = A(\lambda)e^{i\varphi_2} - 1,$$

and

$$\Omega_2 = (B(\lambda)\cos\varphi_1 - 1) + iB(\lambda)\sin\varphi_1 = B(\lambda)e^{i\varphi_1} - 1$$

should be parallel to the real axis. Let $Z_1, Z_2, M, A, 0, X_1, N$ and X_2 of the complex plane correspond to $x_1 + i(\eta-\lambda)$, $x_2 + i(\eta+\lambda)$, $(x_1+x_2)/2 + i\eta$, -1, 0, x_1, $(x_1+x_2)/2$ and x_2 respectively. Then we have to prove that

$$2\,\frac{|A,0| + |0,M|}{|A,Z_1|\,.\,|A,Z_2|} \le \frac{2(|A,0| + |0,X_1|/2 + |0,X_2|/2)}{|A,X_1|\,.\,|A,X_2|}\ ^{1} \qquad (5.5)$$

or equivalently

$$\cos\varphi_1\cos\varphi_2\,(|A,0| + |0,M|)/(|A,0| + |0,N|) \le 1 \qquad (5.6)$$

where $\cos\varphi_1 = |A,X_1|/|A,Z_1|$ and $\cos\varphi_2 = |A,X_2|/|A,Z_2|$. Since $|A,X_1| \ge 2$, we may write $|A,0| = \sigma|A,X_1|$ for some $\sigma \le \frac{1}{2}$ and (5.6) becomes

$$\cos\varphi_1\cos\varphi_2\left(\sigma|A,X_1| + \sqrt{|M,N|^2 + (|A,N| - \sigma|A,X_1|)^2}\right)\Big/|A,N| \le 1. \qquad (5.7)$$

---

1. Since $f_\lambda'(z) = 2(z - ((x_1+x_2)/2 + i(y_1+y_2)/2)$, thus

$$\max_{|z|=1} f_\lambda'(z)/|f_\lambda(-1)| \le 1/(1+x_1) + 1/(1+x_2)$$
$$= 2\left(1 + \frac{x_1}{2} + \frac{x_2}{2}\right)\Big/(1+x_1)(1+x_2)$$

is equivalent to (5.5). Also note $|AO|=|PO|$.

But clearly

$$|A,\Omega_2| = u/\sin\varphi_1 , \qquad |Z_2,X_2| = u\sin\varphi_2/\sin\varphi_1 ,$$

$$|A,\Omega_1| = u/\sin\varphi_2 , \qquad |Z_1,X_1| = u\sin\varphi_1/\sin\varphi_2 ,$$

$$|A,X_2| = u\cos\varphi_2/\sin\varphi_1 , \quad |A,X_1| = u\cos\varphi_1/\sin\varphi_2 ,$$

$$|M,N| = (u/2)(\sin\varphi_2/\sin\varphi_1 + \sin\varphi_1/\sin\varphi_2) ,$$

$$|A,N| = (u/2)(\cos\varphi_2/\sin\varphi_1 + \cos\varphi_1/\sin\varphi_2) .$$

Hence (5.7) is equivalent to

$$\cos\varphi_1 \cos\varphi_2\left(2\sigma\sin 2\varphi_1 + \sqrt{4(\sin^2\varphi_1+\sin^2\varphi_2)^2+(\sin2\varphi_1+\sin2\varphi_2-2\sigma\sin2\varphi_1)^2}\right)\Big/ (\sin 2\varphi_1+\sin 2\varphi_2) \le 1. \qquad (5.8)$$

Let $F(\varphi_1, \varphi_2, \sigma)$ denotes the left hand side of (5.8), then

$$\frac{\partial F}{\partial\sigma} = \frac{2\cos\varphi_1\cos\varphi_2\sin2\varphi_1}{\sin2\varphi_1+\sin2\varphi_2}\left[1-\frac{2(\sin2\varphi_1+\sin2\varphi_2-2\sigma\sin2\varphi_1)}{\sqrt{4(\sin^2\varphi_1+\sin^2\varphi_2)^2+(\sin2\varphi_1+\sin2\varphi_2-2\sigma\sin2\varphi_1)^2}}\right]$$

We see that $\frac{\partial F}{\partial\sigma}$ is positive if we replace σ by $\frac{1}{2}$, which implies the left hand side of (5.8) increases if we replace σ by $\frac{1}{2}$. Hence it will be enough to prove the inequality

$$\cos\varphi_1\cos\varphi_2\left(\sin2\varphi_1+2\sqrt{\sin^2\varphi_2+2\sin^2\varphi_1\sin^2\varphi_2+\sin^4\varphi_1}\right)\Big/ (\sin2\varphi_1+\sin2\varphi_2) \le 1$$

which is equivalent to

$$\frac{1}{2}(\sin 2\varphi_1 \sin 2\varphi_2)^2 + (\sin 2\varphi_1 \sin\varphi_1 \cos\varphi_2)^2 \le \sin^2 2\varphi_1 .$$

$$(1-\cos\varphi_1 \cos\varphi_2)^2 + 2\sin 2\varphi_1 \sin 2\varphi_2(1-\cos\varphi_1 \cos\varphi_2) + (\sin 2\varphi_2 \sin\varphi_1)^2 . \tag{5.9}$$

(5.9) will be proved if we show that

$$(\sin 2\varphi_1 \sin\varphi_1 \cos\varphi_2)^2 \le (\sin 2\varphi_2 \sin\varphi_1)^2 \tag{5.10}$$

and

$$\frac{1}{2}(\sin 2\varphi_1 \sin 2\varphi_2)^2 \le 2\sin 2\varphi_1 \sin 2\varphi_2(1-\cos\varphi_1 \cos\varphi_2). \tag{5.11}$$

In fact, the sum of the left-hand sides of (5.10), (5.11) is equal to the left-hand side of (5.9) whereas the sum of the right-hand sides of (5.10), (5.11) is smaller than the right-hand side of (5.9). Now, as far as inequality (5.10) is concerned it is obvious. As for inequality (5.11) we have

$$\frac{1}{4}\sin 2\varphi_1 \sin 2\varphi_2 \le 1-\cos\varphi_1 \cos\varphi_2$$

or $$\sin\varphi_1 \cos\varphi_1 \sin\varphi_2 \cos\varphi_2 \le 1-\cos\varphi_1 \cos\varphi_2$$

or $$\cos\varphi_1 \cos\varphi_2(1+\sin\varphi_1 \sin\varphi_2) \le 1.$$

It is equivalent to

$$\left\{\cos(\varphi_1-\varphi_2)+1\right\}^2 - \left\{\cos(\varphi_1+\varphi_2)-1\right\}^2 \le 4$$

which is certainly true. With this the proof of the theorem is completed. □

In [10], although they could not prove the conjecture in the full form, they did show considerably more by imposing further restrictions on the polynomial P(z). They proved:

Theorem 5.2: Let $P(z) = \prod_{\nu=1}^{n} (z - z_\nu)$ be a polynomial which is real for real z, then

$$\max_{|z|=1} |P'(z)| \le \sum_{\nu=1}^{n} \frac{1}{1+|z_\nu|} \max_{|z|=1} |P(z)| \tag{5.12}$$

provided all the zeros are in $D = \{z \in \mathbb{C} : \text{Re } z \ge 0,\ |z| \ge 1\}$.

Proof of Theorem 5.2: Since $P(z)$ is real for real z, its complex zeros occur in conjugate pairs. The polynomial $P'(z)$ is also real for real z, its complex zeros also occur in conjugate pairs. Besides, the zeros of $P'(z)$ all lie on the right half-plane. Hence

$$\max_{|z|=1} |P'(z)| = |P'(-1)|.$$

Also we can rearrange the terms of $P(z)$ and write

$$P(z) = \prod_{\nu=1}^{m} (z-z_\nu)(z-\bar{z}_\nu) \prod_{\nu=2m+1}^{n} (z-z_\nu)$$

where z_ν, $\nu = 2m+1, \ldots, n$ are real and ≥ 1. Thus

$$\max_{|z|=1} |P'(z)| = |P'(-1)|$$

$$= |P(-1)| \left| \sum_{\nu=1}^{m} \left\{ \frac{1}{-1-z_\nu} + \frac{1}{-1-\bar{z}_\nu} \right\} + \sum_{\nu=2m+1}^{n} \frac{1}{-1-z_\nu} \right|$$

$$\le \max_{|z|=1} |P(z)| \left(\left| \sum_{\nu=1}^{n} \frac{1}{1+z_\nu} + \frac{1}{1+\bar{z}_\nu} \right| + \sum_{\nu=2m+1}^{n} \frac{1}{1+z_\nu} \right).$$

Note that $\left| \frac{1}{1+z_\nu} + \frac{1}{1+\bar{z}_\nu} \right| \le \frac{2}{1+|z_\nu|}$[1] if $z_\nu \in D$ and hence the desired result follows. □

1. $\left| \frac{1}{1+z_\nu} + \frac{1}{1+\bar{z}_\nu} \right| = \frac{2(1+\text{Re} z_\nu)}{1+2\text{Re} z_\nu + |z_\nu|^2} \le \frac{2}{1+|z_\nu|}$ is equivalent to $\text{Re} z_\nu \le |z_\nu|$ which is obvious.

§5.2. DISCUSSION ON THE SAFF CONJECTURE

We note that inequality (5.12) in Theorem 5.2 is best possible and equality holds for polynomial $P(z)$ having all its zeros on the positive real axis. By rotation, it is easily seen that the inequality (5.12) still holds if all the zeros of $P(z)$ are on a ray $re^{i\alpha}$, for $r \geq 0$ and any fixed α, $-\pi/2 \leq \alpha \leq \pi/2$, or 'conjugate' with respect to $re^{i\alpha}$.

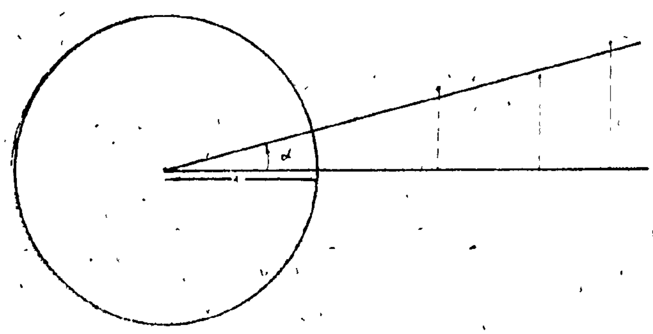

Fig. 5.2

However, if the zeros of $P(z)$ are not on $re^{i\alpha}$ but very near to the ray (as in Fig. 5.2), the right hand side of (5.12) should not vary too much. Now $P(z)$ does not belong to the class of polynomials considered in Theorem 5.2, and the bound suggested by Saff conjecture when compared to the right hand side of (5.12) appears to be too large.

Anyway, we note the following observations. Let $z_\nu = \rho_\nu e^{i\theta_\nu}$ be the zeros of $P(z)$. It is easily seen that if $|P(e^{i\theta})|$ attains its maximum at $\theta = \alpha$,

$$\frac{d}{d\theta}|P(e^{i\theta})|^2\Big|_{\theta=\alpha} = \frac{d}{d\theta} P(e^{i\theta})\overline{P(e^{i\theta})}\Big|_{\theta=\alpha}$$

$$= 2\ \mathrm{Im}\, e^{i\alpha} P'(e^{i\alpha})\overline{P(e^{i\alpha})}$$

$$= 0\ .$$

Therefore we have for $z = e^{i\alpha}$

$$\frac{1}{|P(e^{i\alpha})|^2}\ \mathrm{Im}\ e^{i\alpha} P'(e^{i\alpha})\overline{P(e^{i\alpha})} = \mathrm{Im}\ e^{i\alpha} P'(e^{i\alpha})/P(e^{i\alpha})$$

$$= \mathrm{Im}\left[e^{i\alpha}\frac{P'(e^{i\alpha})}{P(e^{i\alpha})} - n\right]$$

$$= \mathrm{Im}\left[\sum_{\nu=1}^{n}\frac{z_\nu}{z-z_\nu}\right]$$

$$= \mathrm{Im}\left[\sum_{\nu=1}^{n}\frac{\bar{z}\, z_\nu}{1-\bar{z}\, z_\nu}\right]$$

$$= \mathrm{Im}\left[\sum_{\nu=1}^{n}\frac{\rho_\nu e^{i(\theta_\nu-\alpha)}}{1-\rho_\nu e^{i(\theta_\nu-\alpha)}}\right] \qquad (5.13)$$

$$= \mathrm{Im}\left[\sum_{\nu=1}^{n}\frac{\rho_\nu e^{i(\theta_\nu-\alpha)} - \rho_\nu^2}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}\right]$$

$$= \sum_{\nu=1}^{n}\frac{\rho_\nu \sin(\theta_\nu-\alpha)}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}$$

$$= 0$$

Since $P'(z)/P(z) = \sum_{\nu=1}^{n} (\bar{z}-\bar{z}_\nu)/|z-z_\nu|^2$, we observed that if considered at the point $z = e^{i\alpha}$ where $|P(z)|$ attains it maximum we have

$$\left|\frac{P'(z)}{P(z)}\right| = \left|\sum_{\nu=1}^{n} \frac{\bar{z}-z_\nu}{|z-z_\nu|^2}\right|$$

$$= \left|\frac{\bar{z}}{|z|^2} \sum_{\nu=1}^{n} \frac{1-z\,\bar{z}_\nu}{|1-\bar{z}\,z_\nu|^2}\right|$$

$$= \left|\sum_{\nu=1}^{n} \frac{1-z\,\bar{z}_\nu}{|1-\bar{z}\,z_\nu|^2}\right|$$

$$= \left|\sum_{\nu=1}^{n} \frac{1-\rho_\nu \cos(\theta_\nu-\alpha) - i\rho_\nu \sin(\theta_\nu-\alpha)}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}\right|$$

$$= \left\{\left[\sum_{\nu=1}^{n} \frac{1-\rho_\nu \cos(\theta_\nu-\alpha)}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}\right]^2 + \left[\sum_{\nu=1}^{n} \frac{\rho_\nu \sin(\theta_\nu-\alpha)}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}\right]^2\right\}^{\frac{1}{2}},$$

applying (5.13) we obtain

$$\frac{|P'(e^{i\alpha})|}{|P(e^{i\alpha})|} = \sum_{\nu=1}^{n} \frac{|1-\rho_\nu \cos(\theta_\nu-\alpha)|}{\left|1-\rho_\nu e^{i(\theta_\nu-\alpha)}\right|^2}$$

$$\le \sum_{\nu=1}^{n} \frac{1-\rho_\nu \cos(\theta_\nu-\alpha)}{\left[1-\rho_\nu \cos(\theta_\nu-\alpha)\right]^2}$$

$$= \sum_{\nu=1}^{n} \frac{1}{1-\rho_\nu \cos(\theta_\nu-\alpha)} \qquad (5.14)$$

Therefore we have the following:

Theorem 5.3*: Let $P(z) = \prod_{\nu=1}^{n} (z-z_\nu)$ be a polynomial. If $|P(z)|$ and $|P'(z)|$ both attain their maximum at $e^{i\pi}$, then

$$\max_{|z|=1} |P'(z)| \le \sum_{\nu=1}^{n} \frac{1}{1+\text{Re}z_\nu} \max_{|z|=1} |P(z)| .$$

Proof of Theorem 5.3: In (5.14) if $\alpha = \pi$ then $\rho_\nu \cos(\theta_\nu - \alpha) = -\rho_\nu \cos\theta_\nu = -\text{Re}z_\nu$. □

The hypothesis in Theorem 5.3 is very strong, because (5.14) only holds at the point $\theta = \alpha$ where $|P(z)|$ attains its maximum. But we may propose the following conjecture.

Let $P(e^{i\theta}) = \prod_{\nu=1}^{n} (e^{i\theta} - \rho_\nu e^{i\theta_\nu})$ be a polynomial having all its zeros in $\text{Re}z \ge 1$. If $|P(e^{i\theta})|$ attains its maximum at $\theta = \alpha + \pi$, when α satisfies

$$\sum_{\nu=1}^{n} \frac{\rho_\nu \sin(\theta_\nu - \alpha)}{\left|1+\rho_\nu e^{i(\theta_\nu - \alpha)}\right|^2} = 0 , \tag{5.15}$$

then

$$\max_{|z|=1} |P'(z)| \le \sum_{\nu=1}^{n} \frac{1}{1+\rho_\nu \cos(\theta_\nu - \alpha)} \max_{|z|=1} |P(z)| . \tag{5.16}$$

When $n = 2$, we observe that the above conjecture can be verified exactly along the same direction as in Theorem 5.1; see Fig. 5.3.

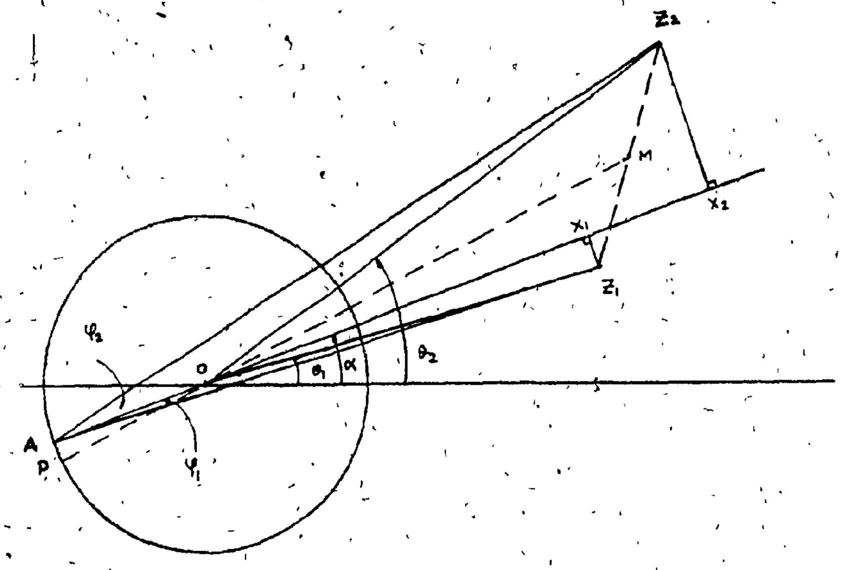

Fig. 5,3

Here z_1, z_2 are the two zeros of the polynomial $P(z)$; A, P are the points where $|P(z)|$ and $|P'(z)|$ attain their maximum respectively. In the following, we again use $|c, D|$ to denote the length of the line segment between two points c and D.

Now from (5.15) we have

$$\frac{\rho_1 \sin(\theta_1-\alpha)}{\left|1+\rho_1 e^{i(\theta_1-\alpha)}\right|^2} = \frac{\rho_2 \sin(\theta_2-\alpha)}{\left|1+\rho_2 e^{i(\theta_2-\alpha)}\right|^2}$$

which implies

$$\frac{|z_1, x_1|}{\left|1+\rho_1 e^{i(\theta_1-\alpha)}\right|^2} = \frac{|z_2, x_2|}{\left|1+\rho_2 e^{i(\theta_2-\alpha)}\right|^2}$$

or

$$\frac{\sin\varphi_1}{\left|1+\rho_1 e^{i(\theta_1-\alpha)}\right|} = \frac{\sin\varphi_2}{\left|1+\rho_2 e^{i(\theta_2-\alpha)}\right|} . \tag{5.17}$$

Note that (5.17) is equivalent to (5.4). On the other hand, the relation

$$\max_{|z|=1} |P'(z)| \Big/ \max_{|z|=1} |P(z)| \le \frac{1}{1+\rho_1 \cos(\theta_1-\alpha)} + \frac{1}{1+\rho_2 \cos(\theta_2-\alpha)} \tag{5.18}$$

can be interpreted as

$$\frac{2|P,M|}{|A,z_1||A,z_2|} \le \frac{1}{|A,0|+|0,x_1|} + \frac{1}{|A,0|+|0,x_2|}$$

$$= \frac{2\left[|A,0|+|0,x_1|/2 + |0,x_2|/2\right]}{|A,x_1| \; |A,x_2|}$$

or

$$\frac{|A,0|+|0,M|}{|A,z_1||A,z_2|} \le \frac{|A,0| + |0,x_1|/2 + |0,x_2|/2}{|A,x_1| \; |A,x_2|} . \tag{5.19}$$

Since (5.17) is equivalent to (5.4), and (5.18) can be interpreted as (5.19) which is exactly the same as (5.2) being interpreted as (5.5), using the technique in section 5.1, we can prove (5.18).

In fact, we do not know whether the proposed conjecture can be regarded as better than the Saff conjecture.

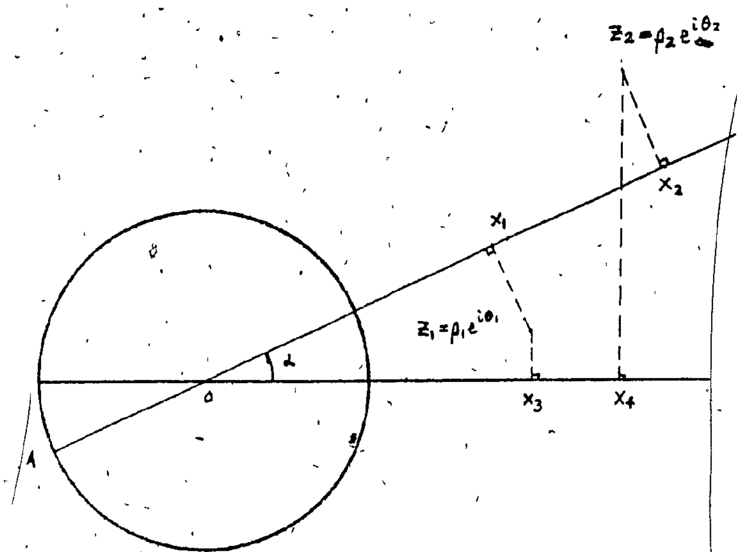

Fig. 5.4

For such an undecided situation see Fig. 5.4. Here we can easily construct a polynomial $P(z)$ of degree 2 with the position of zeros satisfying $|z_1,x_3| < |z_1,x_1|$ which implies $|0,x_3| > |0,x_1|$.

Suppose A is the point where $|P(z)|$ attains its maximum, because the ray joining A and the origin 0 must lie between z_1 and z_2, we always have $|0,x_4| < |0,x_2|$. We are unable to decide which of the following quantities

$$\frac{1}{1+|0,x_3|} + \frac{1}{1+|0,x_4|} = \frac{1}{1+\text{Re}z_1} + \frac{1}{1+\text{Re}z_2} ,$$

$$\frac{1}{1+|0,x_1|} + \frac{1}{1+|0,x_2|} = \frac{1}{1+\rho_1 \cos(\theta_2-\alpha)} + \frac{1}{1+\rho_2 \cos(\theta_2-\alpha)} \qquad (5.20)$$

is smaller.

However, we can give an example to show that the estimate suggested by Saff conjecture is too large.

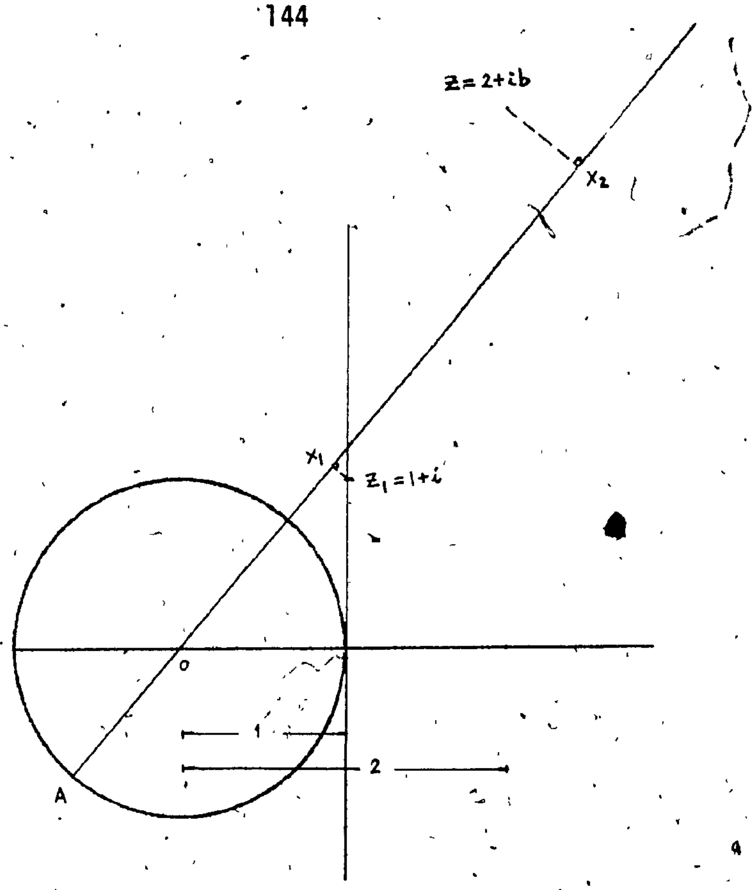

Fig. 5.5

Letting $z_1 = 1+i$, $z_2 = 2+ib$, we have a polynomial $P(z)$ of degree 2 with one fixed zero at $z_1 = 1+i$; see Fig. 5.5. The bound suggested by Saff conjecture is

$$\frac{1}{1+\mathrm{Re}z_1} + \frac{1}{1+\mathrm{Re}z_2} = \frac{1}{1+1} + \frac{1}{1+2} = \frac{5}{6} \tag{5.21}$$

and the bound for (5.20) is

$$\frac{1}{1+|0,x_1|} + \frac{1}{1+|0,x_2|} = \frac{1}{1+\sqrt{2}\cos(\pi/4-\alpha)} + \frac{1}{1+\sqrt{4+b^2}\cos(d-\alpha)} \tag{5.22}$$

where $d = \tan^{-1} b/2$. Obviously (5.21) is independent of b. Now if we increase b, $|0,x_1|$ will decrease and $|0,x_2|$ will increase in contrast. But $|0,x_1|$ is bounded below by 1 and $|0,x_2|$ is not bound above.

Thus if we increase b to a very large value and keeping $|0,x_1| = 1$, in (5.22) we get

$$\frac{1}{2} + \frac{1}{1+\sqrt{4+b^2}\cos(d-\alpha)} \doteq \frac{1}{2}$$

which is very much less than $\frac{5}{6}$.

In conclusion, the constant suggested by Saff conjecture seem to be too large, but unfortunately it does not make the problem simple.

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