Mathematical, Philosophical, Religious and Spontaneous Students' Explanations of the Paradox of Achilles and the Tortoise

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ABSTRACT

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Certain areas in mathematics seem to possess deep secrets. Such are the areas of mathematics that deal with the concepts of infinity. The concepts of infinity have always stirred great emotions and produced seemingly unsovable paradoxes. One such paradox is Zeno's paradox of Achilles.

We begin our research by examining the times in which Zeno lived, the intellectual arguments of that time, and the reasons why Zeno formulated his paradoxes. We will also examine what effects this paradox had on the development of mathematics. This analysis will include Aristotle's formulation of the paradox and the general problems of actual infinity. The two views on the structure of matter and how the paradox is dealt with according to each view will also be covered. Next, we will examine various ways the paradox could be explained: From a mathematical point of view we will examine the paradox in terms of limits, transfinite numbers and through geometric proofs. Then we will examine some of the philosophical explanations, and how the two views on the structure of matter explain the paradox. Finally we will examine how the concept of infinity is dealt with in Jewish philosophy and what bearing this may have on the explanation of the paradox. We will conclude by listening to two pairs of students' spontaneous explanation of the paradox and examining if students background may have any affect on the way they explain and understand the paradox.

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The Basic Principle of all basic principles and the pillar of all sciences is to realize that there is a Fitst being who brought every existing thing into being.
Moses Maimonides (Misneh Torah, Book I Chapter I)

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INTRODUCTION

Most people view mathematics as a science with no ambiguities. It is believed that in mathematics everything can be proved and solved in a clear cut way. True, most areas of mathematics could be viewed in this light, but there are many paradoxes that have crept their way into the development of mathematical theories. Some have been solved while other linger on still sticking around like an annoying virus that won't go away. Many mathematicians believe that there is no real point wasting time dealing with these paradoxes if they have no practical effect on mathematical theory and application.

One such paradox is the paradox of Achilles and the Tortoise. This paradox was posed in the fifth century BC by Zeno of Elea. The present thesis examines the paradox in its historical context and attempts to understand what was its objective. It will look at the influence that this paradox has had on mathematics and how it is resolved from both a mathematical and philosophical point of view.

To this analysis, based mainly on the study of the existing literature, I have added two original elements: a religious explanation of the paradox and accounts of two pairs' of students spontaneous explanation of it: one pair with a strong Jewish religious background and another with a calculus background.

In conclusion, I will examine if there is anything to be gained by introducing the paradox of Achilles and the Tortoise into the teaching of mathematics. At what age does it become appropriate, if at all, to present this and other paradoxes of infinity to the students? How would introduction of these paradoxes affect students' intuitive understanding of infinity?

Many students say that they love mathematics. (If this is really loving mathematics, is a separate question). They argue "You always know if you got the correct answer. In mathematics you always know exactly what is being asked of you, it's all so cut and dry". For the most part, this is true but fortunately there are some areas in mathematics that are not at all so clear and not always is there unanimity amongst leading mathematicians as to what the final answer or best approach is.

With the development of Calculus many such problems have occurred. Most Calculus teachers, when introducing the notion of limit, division by zero or infinity, choose to handle these topics with a very quick and general overview of the philosophical angle and proceed to instruct their students to simply accept the concept of limit as being that number that a function approaches and "for all intents and purposes" is that number.

Is this really the best approach to introduce students to the exciting and wonderful world of mathematics. After all, as math teachers, we are given the opportunity to excite our students and give them something to think about. Should we allow the opportunity to arouse the passions of our students to pass us by? By introducing students to paradoxes are we giving the student a better understanding of the subject? Are we forcing them to think, really deep and hard about what is being done, or, are we just confusing them, causing students to throw up their arms in frustration, driving them to give up and declaring the problem ludicrous and not worthy of any contemplation?

The bottom line is, is it beneficial to introduce paradoxes of infinity to mathematics students? When should we begin? If we as teachers currently choose not to, why are we doing so?

As mentioned above, the paradox that I will focus on in this study will be the paradox of Achilles and the Tortoise. The paradox simply stated is the following:

> " Suppose that Achilles who runs twice as fast as his friend the Tortoise, lets him start a certain distance ahead of him in a race. Then before he can overtake her, he must reach the point at which she starts, by which time she will have advanced half the distance initially separating them. Achilles must now make up this distance, but by the time he does so the Tortoise will have advanced again. And so on *ad infinitum*. It seems that Achilles can never overtake the Tortoise. On the other hand, given the speeds and distances involved, we can calculate precisely how long it will take him to do so from the start of the race (Moore, 1990, p.3)."

The first reference we have to this paradox is from Aristotle's Physics. Aristotle formulates the paradox in the following way:

" The so called 'Achilles', viz. that the slower thing will never be caught up by the quickest; for the pursuer has to first arrive at the point from which the pursued had started, so that the slower is always ahead (Ross, 1936, p. 416)."

Essentially Moore is saying the same thing as Aristotle but in a more concise form.

CHAPTER I

THE HISTORICAL AND PHILOSOPHICAL CONTEXTS OF ZENO'S PARADOXES

In this chapter we give a short biography of Zeno, and we discuss the intellectual problems of his time. We reflect on the possible reasons for this formulation of the paradoxes and the effect they had on the development of mathematics.

Life of Zeno of Elea

Zeno of Elea was born in the fifth century BC. He was a student of the philosopher Parmenides whose central thesis was the denial of plurality.

> "Parmenides arrived at the conclusion that being is one, and that it is incapable of accommodating 'many-ness' even as parts of itself (Shamshi, 1973, p.19)"

Parmenides was the major influence on Zeno's thinking and work. Zeno was a philosopher and a logician but not a mathematician.

In the year 449 BC he accompanied his teacher on a trip to Athens where he met Socrates and made quite an impression on the young philosopher. Unfortunately there are no complete manuscripts of any of the books written by Zeno. Our knowledge of his work and life is limited to a very small subset of the wide array of scientific and philosophical fields that he mastered.

Later on in his life he became active in politics which at that time proved to be a more hazardous profession than bull fighting. Zeno was accused of conspiring against the city's tyrant Nearchus, and was tortured to death.

Zeno's claim to fame is two-fold. First of all he is credited with the creation of a form of argument called the dialectic. This form of argument which relies on the procedure of *reductio ad absurdum*, involves one arguer supporting a given idea while the other arguer tries to prove that the given view is absurd. Zeno's second great accomplishment are his four marvelous paradoxes which have stumped mathematicians for centuries: The Dichotomy, the Achilles and the Tortoise, the Arrow, and the Stadium.

It is important to remember that none of Zeno's actual writings have survived. We know of his work through their presentation by his proponents and critics. The three main sources that we draw upon are Plato, Aristotle and Simplicius all of which were born after Zeno.

The Intellectual Climate of the Epoch

Zeno's paradoxes were born in an environment of polemics around various fundamental philosophical issues. The Pythagoreans and the Eleatics argued about the nature of being: is being constituted by a dynamic plurality of primary objets, or is being a one, a logically consistent unity?

Regarding the question of the structure of matter, atomists competed with those who believed in the infinite divisibility of matter. Related to this were the questions about the existence and definition of infinity. The discussion of infinity led to the necessity of distinction between sensible things and "forms", "intelligibles", "ideas", as well as mathematical objects. There can be infinite divisibility in a geometric line, but physical matter may have an atomic structure.

The time of Zeno's paradoxes was also the time of the birth of scientific thinking, and of the first divisions between empiricist approaches, such as that of Democritus, and rationalist approaches, of which Parmenides was quite an orthodox partisan.

What Were Zeno's Paradoxes aiming at?

It is supposed that the reason Zeno composed the paradoxes was to answer to the criticism of his teacher's philosophy, the denial of plurality. A major rival of the Parmenidean philosophy was a group called the Pythagoreans. The Pythagoreans were a strange religious cult type of "school" that were led by their master Pythagoras. It is important to try to understand the period and how speculations about the world and philosophies of existence affected these two schools of thought.

The whole philosophy and way of thinking of the Pythagoreans was based on the notion of rational number. The Pythagorean lifestyle was organized so as to follow God. They claimed that by studying philosophy one invokes God. They believed that the Universe is organized according to various properties of number, and numbers and being were one. They held the view that " Number in all its plurality was the basic stuff behind phenomena (Boyer, 1968, p. 82)."

They rejected the monolithic idea of a single and primordial being that created and gave existence to the world but believed that existence involved opposing principles and they used the numbers as the tool to explain all being.

Each number was discussed both in relation to a mathematical and non mathematical property. The Pythagoreans were only really concerned with the first 10 numbers since they could construct all remaining numbers from the first 10. Within the first 10 they distinguished between the odd numbers - Monad and the even numbers - Dyod.

Their theory of number started with the one which was not considered a number itself. From this one come numbers which, of course, could be related to physical things, so one generated all things and numbers. Each of the successive numbers leading to number 10 had some relationship to the world. They were what actually made up the fabric of the world. These numbers were considered to be discreet units.

The Pythagoreans had a rosy outlook on the world. They were passionate in their belief that the world was essentially good and that it was a representation of harmony and order. To them the world was an organization built out of a void. This organization was composed out of a system of numerical terms all coming together in perfect harmony.

This perfect system or organization was to be represented using the set of natural numbers. With these numbers, more specifically using their finite ratios the Pythagoreans were able to represent the harmony in the world. They believed that everything could be represented in terms of natural number.

This optimistic view of the world, however, was soon to come to a sudden halt when the Pythagoreans were to discover, using the famous Pythagorean Theorem, that it was impossible to express the ratio for the length of the diagonal of a square to the length of its side. Simply stated there are no natural numbers p and q, such that $p/q=\sqrt{2}$, which means that $\sqrt{2}$ is not a rational number.

To us this may seem insignificant. However, to the Pythagoreans this was a tragedy, for they could no longer state that everything could be expressed in terms of natural numbers. Whether or not they liked it, these new numbers had arrived and they were here to stay. There was no way of denying that in addition to the ratios of natural numbers that they had acknowledged there were other numbers that could not be so expressed.

It is very clear how important the concept of plurality was to the Pythagoreans and it is precisely this strong belief in plurality that made their theory upsetting to the Eleatics.

The Eleatics` view was very different from that of the Pythagoreans. They held that all being is one and there is only one existence, the one. Through the works of Zeno using his new and very effective form of arguementation - the dialectic - the Eleatics were very successful in arguing against the theories of the Pythagoreans.

The Eleatic school of thought was based on the theories of Parmenides of Elea, a disgruntled ex-Pythagorean who revolted against the Pythagorean principles because he was unable to accept the notion that the world was a system of structures within a void.

"He believed that reality - The One - must be autonomous and explicable in its own terms, a perfect unified self-subsistent whole. In other words... though not, in Parmenides' own language, he believed that reality must be metaphysically infinite (Moore, 1990, p. 23)"

If you choose to view all the actual things in the world which exist at this moment you may conclude that they are finite. However if you think of the world in non-temporal terms but as a whole, in all time, and not in actual fact, then you can conclude that there are infinitely many things. If you think of the world as one, you cannot say it is in respect to time for it must but for all time. If you take the world and view it for all time then obviously there will be infinitely many things.

Zeno, a disciple of Parmenides, argued his position by taking the point of view of his opponent and accepting their theory and proceeded to prove that if he stuck to those theories he would inevitably draw illogical and false conclusions.

As mentioned earlier, Zeno's master, Parmenides, was attacked by the Pythagoreans. Zeno defended his master's view by proving that motion does not exist. But according to Tannery (Cajori,1915, p.3) in his paradoxes Zeno was attacking the Pythagoreans' claim of the existence of a mathematical point as a *unity having position*. According to Tannery, Zeno wasn't attempting to deny motion but was simply trying to show its inconsistency if we were to view space as a sum of points.

Based on an analysis of Simplicius' writings, Tannery develops the following interpretation of the purpose of Zeno's paradox. The paradox was meant as a response to the Pythagoreans' claim that a finite quantity can be regarded as the sum of indivisible parts. Zeno presents the impasse that is to arise, that, if we are to take quantity as being infinitely divisible, then the successive terms become smaller and smaller till their last term is zero. However the sum of these zeros is still zero. To which the Pythagoreans would ask, "why may the indivisible parts not be different from 0 and have magnitude?" To which Zeno would reply that if there are an infinite number of parts having a magnitude not equal to zero then the sum of these parts must be infinite which contradicts our original assumption that the quantity is finite. This analysis of Zeno's paradoxes raises their importance; however, this view of the paradoxes is not universally accepted.

The two views of what the paradoxes were aiming at have a strong association. The view that the aim of the paradoxes was the refutation of the Pythagorean pluralistic view of being, as well as the the view that the aim was to show the inconsistencies in the Pythagorean notion of point, could be viewed as one in the same. For, to have points, you must have many points. Having points and saying that the world is built up of points is part of the pluralistic world view.

In general what opposed the Eleatics and the Pythagoreans was that to the Pythagoreans everything is number and number is not one hence everything can be counted. In particular, time and space can be counted. To the Eleactics everything is one and space and time cannot be counted because they are continuous (indivisible into discrete parts). They felt that the assumption that space and time could be counted leads to paradoxes.

The Philosophical Questions underlying Zeno's Paradoxes

Tension Between Two Views on the Structure of Matter

I will now view the notion of plurality and later, in the section dealing with the philosophical explanations of the paradox, I will discuss the implication Zeno's paradoxes are meant to have upon the different ways of viewing magnitudes in a pluralistic system, for, as mathematicians, we view magnitudes in this light. We must examine if the paradoxes really cause any dilemma in our reality.

If we accept the Parmenidean principle, rejecting plurality, then we reject the possibility of motion, for motion involves the movement from one point to another. The rejection of plurality lies at the heart of Zeno's paradoxes.

By accepting plurality, we accept that there are many beings and points in our universe. There are two ways of viewing magnitudes in the pluralistic system.

- The Atomistic System: This is the view that any divisible unit is composed of a finite number of indivisible units.

- The Infinitely Divisible System. This is the view that any unit is composed of divisible units that are divisible into further units without end.

The Parmenidean doctrine was set to prove that pluralism is impossible or does not exist because it is full of inconsistencies. In order to appreciate the kind of arguments that the Eleatics would use, let us take, as an example, the problem of the structure of a line segment. What would this structure be from (a) the Atomistic point of view, (b) the infinite divisibility point of view? How would Parmenides refute each of these views?

According to the Atomistic point of view the line can be divided into a finite number of units (otherwise we would be dealing with the Infinite Divisibility point of view). Now, does not the division of the line into a finite number of units take away the unity of the line segment? What we appear to have is a collection of finite segments but do we actually have a complete unit or just a collection of many individual parts? For the collection of units to form a unit, something must intervene between them (Shamshi, 1973, p.21); otherwise one is led to a paradox like the paradox of the arrow (not be discussed in this thesis). But if something intervenes between units, then the structure of the line is no longer Atomistic.

The Atomists did admit that the atoms are physically indivisible, but they allowed for them to be geometrically divisible (Anglin, 1977, p.9).

The best way to deal with this is to understand that although these arguments may hold ground when dealing with physical objects, when dealing with magnitudes of space and time there is no problem with saying that a line is composed of several smaller lines or even atomic units. There is no need for space and time, as mental constructs, to be constrained by the same restrictions that limit matter.

According to the Infinitely Divisible point of view, there is continuity between the parts making up the line segment. This resolves our prior dilemma of how could the individual parts compose a complete unit, i.e. the line segment. We are, however, faced with a new problem. Are the parts that compose our line segment real units and do they have any real magnitude? If we say they have any magnitude then we would be able to further divide these parts. Therefore, we must assume that the parts have no magnitude. If the parts have no magnitude then how could parts with no magnitude really compose a line and where then does this line begin and end?

These problems could be resolved as follows. Once again, as in the Atomistic view, we could say that the argument that the units are not units because they lack any magnitude is an argument that applies only to matter and not when dealing with magnitudes of space and time as mental constructs. As to the question of where does the line begin and end, this is no real question at all, for the dilemma arises only when we view the line as part of a bigger line but not when the line is viewed as a unit in itself.

The Problem of The Existence of Infinity

It is of interest to compare the difference of opinion between Aristotle and Democritus on the question,"does Infinity exist?" Democritus represents the Atomistic point of view, while Aristotle represents the Infinite Divisibility point of view. I will examine how each of these philosophers would treat the concept infinity in sensible and in intelligible things.

Aristotle rejected the Atomist view. He admitted the infinite divisibility and made a distinction between two kinds of infinity, the actual and the potential. Aristotle held that the actual infinite does not exist among sensible things, neither in the large, nor in the small. As for actual infinity in intelligible things, Aristotle stated that:

> "The infinity of numbering may be understood in terms of the continual bisection of a line, [which guarantees the endlessness of the process]. Consequently, the infinity of numbering is potential; it is never actual, but consists in the fact that a number can always be found which is greater than any number suggested (Aristotle, Book III, Chapter 7, 204b)"

So, according to Aristotle, actual infinity doesn't exist in intelligible things either.

Aristotle held that Infinity, large and small, existed potentially both in sensible and in intelligible things. For sensible things he claimed that "spatial magnitude is not actually infinite but is infinitely divisible (Chapter 6, 206a). For intelligible things he stated that:

"in counting we may proceed from a minimum to an ever greater number; but that, contrariwise, a magnitude is infinitely divisible but not infinite in prolonged dimensionality. (Physics, Book III Chapter 7, 207b)

Democritus claimed that there is infinity among sensible things, both in the large and in the small. In intelligible things he "claimed that everything is made of tiny, physical atoms. These Atoms are physically (but not geometrically) indivisible. The number of atoms is infinite, and the empty space containing them is also infinite.(Anglin, 1997, p.9)." So, to Democritus Infinity also existed in intelligible things. Democritus never made the distinction between the actual and potential infinite the way that Aristotle does.

Aristotle, like Zeno, rejected the Atomistic view since he rejected the notion of an infinite number. His definition of number was that which you could arrive at by counting. For an infinite number to exist, one would have to count an infinite series of numbers. Unlike the atomists, Aristotle claimed that there is no infinity among sensible things (Physics, Book III), but he accepted the potential infinity of numbers (which is obtained through "addition") and the infinite divisibility of a line segment.

The Problems of The definition and Acceptance of Infinity.

To define the infinite is a paradoxical task. By defining something you necessarily limit it, to your definition and nothing else. Once something is limited in any way, it cannot truly be unlimited or "infinite". This is the natural dilemma that we are faced with, when trying to define the infinite in any way, shape or form. If we are unable to even define the word "infinite" it could become "infinitely" difficult to address the issue and examine this concept. In any exercise dealing with the infinite, we will be faced with the various paradoxes and antinomies that have traditionally haunted the subject.

This being said, we could inquire. Is it at all possible to define the infinite? Throughout history, this subject has been a hot topic among mathematicians. Generally, however, we could say that there have been two opinions on how to approach the concept wich do not seem to reflect the modern mathematical way of looking at infinity. The first school of thought treats the infinite as that which is boundless, illimited or immeasurable. The second school of thought views the infinite as a state or amount of completeness. It is the all encompassing, that which comprises all.

According to the first, the infinite is viewed as something that only exists potentially or dynamically and could never exist in an actual form. According to the second, the infinite could actually be defined and does exist in actual form. It was Aristotle who first introduced the notion of potential infinite.

It is very important to understand Aristotle's view on the infinite since it is mainly through his writing that we learn about the ideas and works of that time.

Aristotle distinguishes those philosophers who, like the Pythagoreans and Plato, claimed that infinity is something independent, a "primary being", from those who, like Anaxagoras and Democritus treated infinity as an attribute or "a character of something different from infinity" (Aristotle, Physics, Chapter 4, 203a).

Those who believed that infinity is something independent, were led to this view through one or more of the following "five lines of investigation"; in Aristotle's Words:

> "(1) time is infinite; (2) magnitudes are divisible in such a way that mathematicians, too, work with the infinite; (3) things come to be and pass away ceaselessly only because the source of their generation [and destruction] is infinite; (4) the finite is always limited by something else so that it must be without any limit that something must always be limited by something else; and (5) there is, above all, the problem which everybody raises that thought has no stopping-place in dealing with numbers, with mathematical magnitudes, or with what is beyond the boundary of the universe. In particular, the infinite beyond the boundary of the universe suggests an infinite body and infinite worlds. Why in the void, would body be here rather than there? If there is mass anywhere it must be everywhere. Even [aside from this argument if there is void or infinite place, still, there must be infinite body also, for there is no difference in the eternal between what may be and what is (Aristotle, Physics, Book 3, Chapter 4, 203b)"

Aristotle strongly criticized the view that the infinite could exist as an independent primary being. If the infinite was a primary being, then, Aristotle argued, it would be indivisible. His argument went as follows: He admitted that if A is divisible then A must be a magnitude or a plurality. Now if infinity is a primary being, then it is neither a magnitude nor a plurality. Hence by the above, infinity is indivisible. But indivisible things are not infinite. Hence infinity is not infinite, a contradiction.

I believe, what is meant by this is that if something is indivisible it cannot be infinite because the infinite must have in it an infinity of something. Therefore the infinite could only be viewed as an attribute of primary objects. (Aristotle, Physics, Chapter 5, 204a).

But this view is no more tenable than the previous one, says Aristotle. For infinity would have to be a characteristic of an (actually) infinite entity, and such he argued, does not exist (Ibid.). Since we are never able to deal with something as an infinity, we will only be able to deal with parts of it so it is not an infinite element of reality. If we take time, for example, we will never be able to deal with an infinite amount of time or an element called time which is infinite. We will only be able to deal with a fraction of time. So , we will always be dealing with something that is limited.

Although it was Aristotle's view that there could be no actual infinite existing substance he did not consider it sensible to completely reject the infinite. He said, looking for a compromise:

> " It is also clear that, if we deny the infinite altogether, many impossible consequences would follow. Thus, time would have a starting-point and a stopping-point; there would be magnitudes not divisible into magnitudes; and numbering would not be unlimited. Since, then, neither of the alternatives appears possible [that there should be or that there should not be such a thing as the infinite], we must mediate between these two views and distinguish how the infinite is and how the infinite is not. Now, "to be" may mean to be potentially or to be actually, and there 'is' an infinite by addition and one by subtraction... (Aristotle, Physics, Book 3, Chapter 6, 206a)"

The compromise took the form, as mentioned in the previous section, of the distinction between the actual and the potential infinity. This is how Aristotle explained his use of the word "potentiality":

> " However we must not take the infinite as being potential in the ordinary meaning of the word potentiality which may be completely actualized, as the bronze which is potentially a statue may become an actual statue. But 'to be' has many meanings, and the infinite accordingly has the kind of being which the day has or the games have, namely, inasmuch as one after another continually comes into being: for these, too, 'are' potentially or actually; thus, there 'are' Olympic games both in asmuch as they may be held and in asmuch as they are being held. Then, too, the infinite in time and in the

generations of man clearly differs from the infinite in division of magnitudes; although the infinite has in general the kind of being which a continually repeated process has, finite on each occasion [65], but always different. (Aristotle, Physics, Book 3, Chapter 6, 206a)"

Finally Aristotle defined his own concept of infinity as:

"the infinite is contrary to what is usually described as such: there is an infinite, not when there is nothing left over and beyond, but when there is always something over and beyond! (Aristotle, Physics, Book 3, Chapter 6, 206b)"

Thus what is infinite must necessarily be unlimited, boundless without end. This is why Aristotle rejected the infinity in the large: something that would be even potentially large would go beyond the universe, but universe for Aristotle, was the bound of everything. The conception of infinity as something unlimited is also a source of the difficulty in accepting the conventional mathematical explanations of the paradox of Achilles and the Tortoise: they are based on the notion that an infinite sequence can have a limit. The explanation itself is thus paradoxical or contradictory, so how can it resolve the paradox?

What Belongs to Mathematics and what Belongs to Physics.

It is sometimes very easy to confuse the study of mathematics and physics, since there is much common ground covered in these two areas of study. According to Aristotle, the mathematician differs from the natural philosopher in that he studies objects in abstraction from change (Ross, 1936, p.350). The mathematician has the luxury of viewing things theoretically and not having to worry about the real life applications of his theories. On the other hand, the physicist is more concerned with the study of matter which yields many more practical problems.

According to Aristotle, infinity is the object of study for the physicist rather than for the mathematician.(Ross, 1936, p. 359)

"The physicist studies changes and therefore he must study the infinite because change is continuous and it is in the continuous that infinity is most evident. Probably because continuity means infinite divisibility (Sierpinska, unpublished, 1997)."

Considering this view, it is pertinent to examine the concepts of time and space. According to Aristotle, time and space would be considered as part of reality, since he held the empiricist view that mathematics describes the real world (Sierpiska, unpublished, 1997). Since in mathematics we could divide a line segment into infinitely many parts, so too can you divide space into infinitely many parts. But are time and space part of nature or are they theoretical notions that we have adopted as practical tools to describe our reality?

It becomes pertinent with respect to our paradox to try to understand what exactly does time and space mean to a mathematician. Is time and space something real or, are only the real constituents of time and space real? Bernard Bolzano examines this issue in his work "Theory of Science" (Bolzano, 1972).

He states that if time and space were real things then they would necessarily need to have certain effects and no two moments or points could ever be exactly equal, because no two existing things are exactly equal. On the other hand, if we are to take the view of the mathematician that space and time are real and that there do exist two points that are exactly equal we are faced with a minor dilemma; that since all time and space have the same inner attributes there could be no reason for why any object should be at any given point at any given time. Therefore it becomes difficult to assert that time and space are real.

According to Bolzano, this leads to the conclusion that the class of all infinite time or moments, and the class of all infinite space or points are not real either. Since intuitions must have an existing object, we cannot say that time or space are intuitions but pure concepts. Bolzano states that:

> "Locations of (real) things are those determinations of these things that we must think in addition to their forces in order to comprehend the changes that they cause in one another (Bolzano, 1972, p.111)".

To Bolzano, time is a necessary characteristic for the existence of any real thing and space is defined by the class of all possible locations.

Other philosophers such as Kant and Schultz treated space and time not as concepts but as intuitions as they pertained to the ideas of total infinite space and time (Bolzano, 1972). The arguments made by these two schools of thought are very difficult to make out. It seems that they are looking at the concept with completely different perspectives and use the same proof to prove opposite views. For instance, the fact that a single point in space cannot be determined solely by concept is used by Bolzano to prove that space is not real and not an intuition and by Schultz to prove the opposite.

As it pertains to the infinite divisibility of line, these two views again yield very differing opinions. If we are to take the view that the ideas of time and space are intuitions then every finite line, being infinitely divisible, would have to consist of infinitely many parts. According to Schultz this is a contradiction for it is impossible for something that contains an infinite number of parts to be considered finite. Incidentally, it seems that this problem still exists if we take time and space to be intuitions.

Of course, the above stated contradiction is not at all obvious and certainly not subscribed to by all mathematicians. That there exist totalities that could contain an infinite number of elements is taken for granted by mathematicians, whether or not they could prove the accuracy of this assumption.

Discussion about What Constitutes Knowledge.

What constitutes knowledge is the interest of the area of philosophy called epistemology. Epistemology asks questions such as: " what are the sources and aims of scientific knowledge? " These questions can be asked in a general way or can be examined for some specific domain of scientific study.

There are generally two categories for the origins of knowledge, rationalism and empiricism.

> "The central point of controversy between rationalism and empiricism was the extent to which understanding of the world could be arrived at by a *priori* means - by the exercise of pure reason which is characteristic of mathematics (Moore, 1990, p. 75)."

The main concern of Rationalism was to find an initial concept from which the rest of mathematics could be built and all mathematical truth could be derived.

As it pertains to the study of mathematical infinity, the rationalists were more friendly to the notion of actual infinity. Because they were not confined by experience, they were able to deal with its existence. Despite the fact that we are unable to experience or encounter actual infinity , rationalists view that we have a clear idea of what it is within us and this idea creates a fundamental insight into reality.

Rationalists stated that the idea of the existence of the infinite made much more logical sense than its denial. Of course, this argument had much to do with the belief in God, who is the only true infinite. Descartes argued that it is God that has ingrained in our mind the idea of infinite. (As the many religions believe that man is a part of God, it makes sense to assume that it is because we are parts of the infinite, we strive to understand it.) Although we are finite, and thus unable to grasp the notion of infinitude this should not preclude its existence. Other rationalists, such as Pierre Gassendi, argued that our idea of the infinite was really only based on our rejection of the finitist theory. Spinoza rejected this view and held a view that was more in line with Descartes and corresponded to the Parmenidean principle of the denial of plurality. Spinoza agreed with Descartes that God was absolutely infinite. This was something that, although we could grasp with our understanding, we could never grasp with our imagination, due to the fact that we are essentially limited beings. Spinoza stated that God was all that existed and all that existed was God. God existed in a unified indivisible way and he is the only true endless entity.

The empiricist's view differed from that of the rationalists in that they stated that it is not possible to have any knowledge that did not come to pass in our thinking through experience. This type of reasoning is very difficult to handle for a mathematician. If the only way that we can admit the truth of an axiom is through experience we will never be able to confirm the accuracy of any axiom, since we will never be able to test the axiom for every possible value of its variables.

If experience is to be a pre-requisite to our ability to understand something then the infinite could not exist. According to this view of knowledge we could never really experience the infinite. Empiricists, such as Hobbes, stated that we could not conceive the infinite and whenever we referred to the infinite it was nothing more than a statement that we were unable to conceive due to our own inability.

More moderate empiricists, such as John Locke, held views that were more in line with the theories of Zeno and Aristotle. Locke's view was that we could not have a real idea of what the infinite was, however we were able to recognize the possibility of increasing perpetually and dividing space and time without limit. Essentially what he meant was that we could not really think of infinity as a consummated thing in our minds. We could only think about the process that leads to the perpetual generation or infinite divisibility.

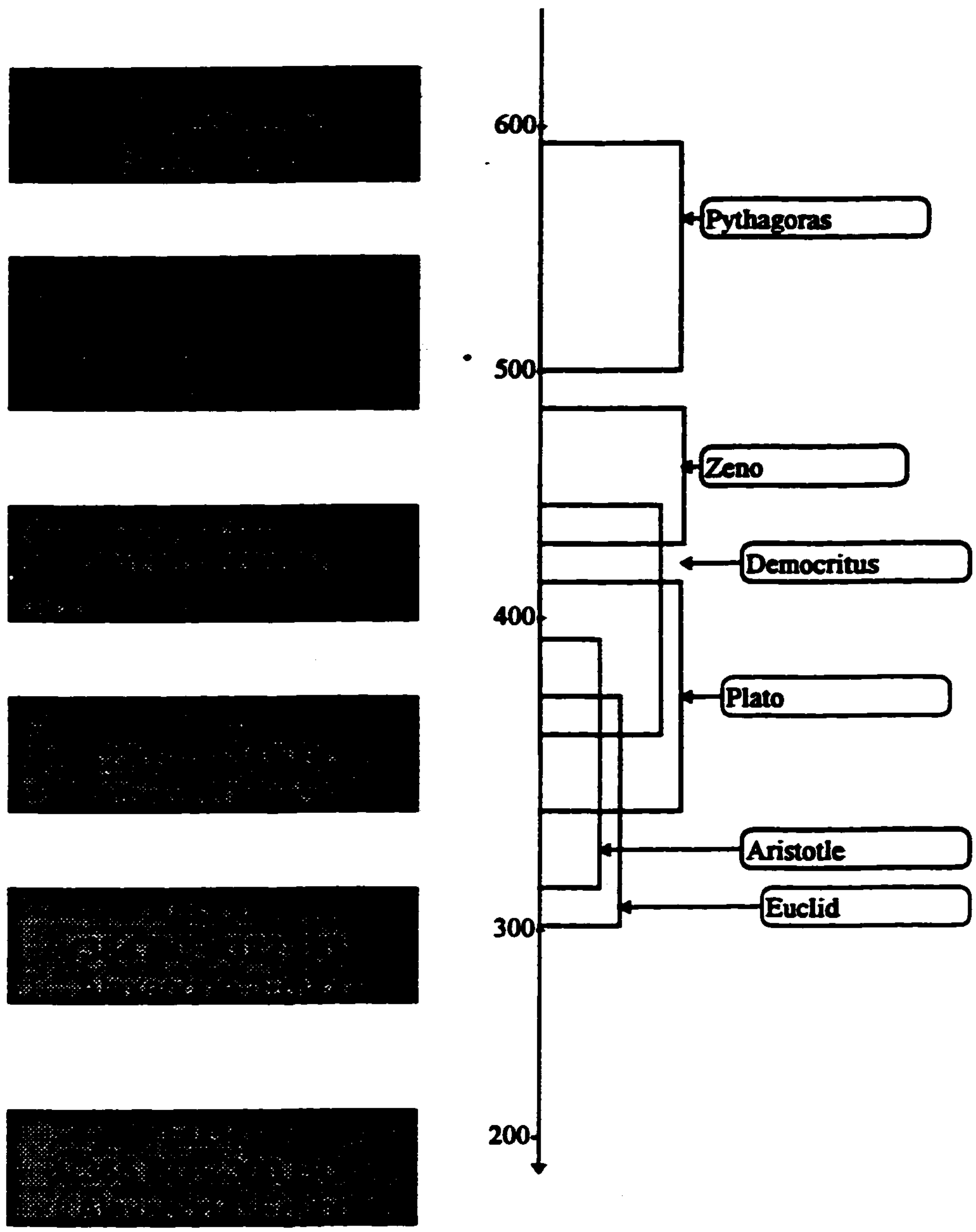
According to Parmenides, knowledge should be concerned only with invariable things or thoughts, or even logical relations between statements. Aristotle's view was much in line with what is considered the empiricist view.

The influence of Zeno's Paradoxes on the Development of Mathematics.

Although never really fully resolved, many believe that Zeno's paradoxes have had a tremendous influence on the development of mathematics. His paradoxes would create grave dilemmas for mathematicians that would linger on till this very day. As Carl Boyer states, it is probably due to Zeno's paradoxes that our method of treating magnitude has changed. Initially, magnitudes were symbolized by a process of using pebbles. However, Euclid treated magnitudes as inseparable wholes having different aspects like shape or size, which associated magnitudes with line segments. This shift of treating magnitudes through the geometric method was probably due to the influence of Zeno.

Tannery's impression of Zeno's paradox is that they were serious efforts conducted with logical rigor (Cajori, 1915). However, not all mathematicians agree about the importance of his role. Aristotle, for instance, didn't agree that Zeno's paradox was linked with the discovery of incommensurability. Many mathematicians question whether Zeno had any effect on the development of mathematics.

Chronological Axis



CHAPTER 2

THE MATHEMATICAL EXPLANATIONS OF THE PARADOX OF ACHILLES.

In this chapter we will look at some of the mathematical explanations of the Paradox of Achilles. We will briefly take a look at Cantorial set theory and examine how it is used in the explanation of the paradox. We will also look at geometric explanation of the paradox.

The Notion of Limit

The mathematical explanations of Zeno's paradoxes are normally based on the notion of limit. Let us look at this notion first.

Before the nineteenth century, mathematicians did not have a clear concept of functions and limits. There was much debate about whether a function could ever reach it's limit. Calculus was independently developed by two men, Sir Isaac Newton and G.W. Leibniz. Although their ideas gave concrete results that yielded definite and precise solutions, their explanations for why their proofs worked were full of unanswered questions. Their methods involved accepting that a quantity in a series that got closer and closer to zero at each successive iteration would become equal to zero in the end. Essentially what was to be known as taking the limit, would be what would cause the quantity to "equal" zero. In essence by making use of the infinitesimally small they were forced to deal with questions of the existence of these quantities.

Leibniz never really gave these "quantities" much thought and simply decided to make practical use of these quantities which yielded precise solutions.

The logical foundation for the works of Newton and Leibniz was provided by Cauchy who tried to clarify the concepts of function, limit and continuity but did not establish the real number system thus leaving many unanswered questions which would not be "fully" resolved till the work of George Cantor was presented.

> "The completed work found Calculus arithmetized and freed from a foundation in intuitions of space and motion. It involved a construction of a linear continuum of real numbers identical in structure to the geometric straight line (Salmon, 1970, p. 21)."

Understanding that the process of calculus works in practice is not enough for a mathematician, although many are content simply accepting the fact that the concept of limit "works" without dwelling into the particulars of whether it is the actual or potential infinite that is at stake.

But Cantor was interested in these questions and his set theory aimed at giving the notion of actual infinity a solid logical and mathematical foundation. As this notion plays an important role in mathematical explanations of the paradox of Achilles, let us now look at how Cantor's theory dealt with it.

Actual Infinity in Cantor's Set Theory.

Set theory is a relatively new area in mathematics that developed rather quickly and is almost exclusively attributed to mathematician, Georg Cantor who lived in the 19th century. He revolutionized the way we think about infinity. His main interest was in the area of infinite sets. His theory allows to say that the set of natural numbers has the same number of elements as the set of even numbers, which has the same number of elements as the set of squares of natural numbers and rational numbers. His theory allows to prove that the set of points in a line segment is equipotent to the set of points in an infinite line.

Cantor proved that not only did infinite exist as a consummated or completed thing, but there are many levels of infinity with each level greater in size than the one preceding it. His goal was to create a hierarchy for the various infinities which resulted in his transfinite arithmetic. The rules for this mathematics were generally simple. Two sets, finite or infinite, are said be equal in size if they could be placed into a 1-1 correspondence with each other. If they can be matched in this way, then they are said to have the same cardinality.

The first level of infinity, the infinity of natural numbers, was given the symbol $\aleph_0$ which is the first transfinite cardinal number. What was to follow was a set of rules to manipulate these transfinite numbers which differed from the arithmetic of real numbers. For Example $\aleph_0+1=\aleph_0$; or, even more fascinating, $\aleph_0+\aleph_0=\aleph_0$.

Cantor proved that the set of rational numbers, integers, and natural numbers all had the same cardinality. These sets were called denumerable. However, he was also able to prove that the set of real numbers contained more elements than the above mentioned sets and thus had a greater cardinality and was non denumerable. The cardinality of the set of real numbers was called the continuum denoted by C. Cantor didn't stop there, he went on to find additional infinities with larger cardinality than the continuum. For example the set of all subsets of real numbers has cardinality 2^C and $2^C > C$.

The continuum is the cardinality of the number line, but it is also the cardinality of the interval (0,1). Following this, it is possible to provide geometric proof that the number of points in any given finite interval could be placed a 1-1 relationship with the number of points in another finite interval. This goes to the heart of Zeno's Paradox. In spite of the set of natural numbers and rational numbers being both denumerable, they are subject to different orderings. While the ordering in natural numbers is discrete, rational numbers form a dense set, which means that between every two elements there is another element.

Explanation in Terms of Types of Infinity

Zeno's arguments for the impossibility of motion was based on the assumption of the fact that for distance to be traversed in an infinite series the elements can be counted one after the other in order of position. According to Zeno, motion required continuous space and time. According to Harold N. Lee,

> "Zeno's analysis of the conditions is inadequate because the infinity involved in an adequate analysis of continuity is of a different type of power than the type he assumes and which was made definitive in Aristotle's statements of the paradoxes (Lee, 1965, p.564)."

Essentially what this means is, that without Cantor's model of linear continuum Zeno was stuck analyzing a continuous line segment as a sum of discrete and denumerable positions which clearly makes motion impossible.

Explanation in Terms of The Sum of Geometric Series.

Zeno's paradox of Achilles can be seen as questioning the possibility of a sequence to ever reach it's limit. We are able to interpret the paradox in terms of infinite series. The definition for the sum of an infinite series as given by Spivak (1980) is:

> "The sequence $\{a_n\}$ is summable if the sequence $\{s_n\}$ converges, where
>
> $s_n = a_1 + \ldots + a_n$
>
> In this case Lim s_n (as $n \rightarrow \infty$) is denoted by
>
> $\sum a_n$ (or, less formally, $a_1+a_2+a_3+\ldots.$)
>
> and is called the sum of the sequence $\{a_n\}$(Spivak, 1980, p. 439)"

Using this definition, it makes sense to say that infinitely many terms of a convergent series do sum up to a finite number.

We could now view the paradox of Achilles and the Tortoise in this light. In order for Achilles to catch up to the Tortoise he must cover a series of distances, each of which is positive but smaller than the distance preceding it. For if Achilles first need to cover 100m then 10m then 1m then 1/10m etc., the total distance he must cover is the sum of the series 100 + 10 + 1 + 1/10 +.... This is a convergent geometric series whose sum is easy to calculate. It would then appear that

Zeno's dilemma was that he thought it impossible for the sum of an infinite series to converge to some number.

To better illustrate the problem let's break it down as follows. If Achilles runs at a rate of 10m/mn and the Tortoise runs at a rate of 5m/mn and Achilles gives the Tortoise a 2 minute head start, the following table illustrates the series of half steps that take place in the race.

Time	Tortoise Position	Achilles Position	Difference
0	10	0	10
1	15	10	5
1+1/2	15+2.5=17.5	15	2.5
1+1/2+1/4	17.5+1.25=18.75	15+2.5=17.5	1.25
1+1/2+1/4+1/8	18.75+.625= 19.375	17.5+1.25=18.75	.625
1+1/2+1/4+1/8+ 1/16	19.375+.3125= 19.6875	19.375	.3125
1+1/2+1/4+1/8+ 1/16+ 1/32	19.84375	19.6875	.15625

What we now want to do is to take the limit as the number of terms approach infinity so that we can see when the distance between Achilles and the Tortoise is equal to 0.

To do this we begin by defining our series as follows

$S_n = 1 + 1/2 + 1/2^2 + 1/2^3 + ... + 1/2^n$

In order to calculate the sum we first divide both sides of the equation by 2 and subtract the two expressions. Since $S_n/2 = 1/2 + 1/2^2 + ... 1/2^n + 1/2^{n+1}$

$S_n - 1/2S_n = 1 - 1/2^{n+1}$

$1/2S_n = 1 - 1/2^{n+1}$

$S_n = 2(1 - 1/2^{n+1})$

Now we have the ability to take the limit of the series S_n as n approaches infinity and thus compute that in exactly 2 minutes, Achilles will reach the Tortoise and then pass him.

The reason that we have no real difficulty handling this dilemma is that we see fit to manipulate infinite sums . But at Zeno's time the theory of infinite series was not available. Zeno played on the common belief that whenever adding infinitely many terms, no matter how small these terms may be, the sum of the series would always be an infinite number.

Explanation Based on a Geometric Representation of a Geometric Series

There is a very clear geometric proof by mathematician Gregorius a San Vincentio. I will now explain his proof as reconstructed by Dr. A. Sierpinska (unpublished manuscript) from a presentation of his work by Dhombres (1994).

The first step in this proof is to construct a geometric progression that will be used to represent the steps taken by Achilles and the Tortoise.

We know that the terms of a geometric progression will always be a function of a constant ratio and that all the terms of the progression can be determined by the first two terms in the series.

Let us now take a ray R with starting point A (figure , Appendix A). Now put two points D_1 and D_2 where the segments AD_1 and $D_1 D_2$ will represent the first two terms of a geometric progression (figure 2).
Let us set the constant ratio of progression to be q, where $q = D_1 D_2 / AD_1$.

Our next step is to find the position of the third point which will correspond to the geometric progression. The way we will do this is as follows:

- Draw a ray, R' from A (figure 3)

- Draw a line through D_1 that will intersect R' at some point A_1 : $A_1 <> A$ (figure 4)

- Draw a line parallel to A_1D_1 going from D_2 to R' intersecting it at A_2 (figure 5).

- Draw a line parallel to R from A_1 intersecting A_2D_2at C_2 (figure 6)

- Draw a line through D_1 and C_2 which will intersect R' at 0 (figure 7)

- Draw a line parallel to R from A_2 , intersecting D_10 at C_3 (figure 8)

- Draw a line parallel to A_1D_1 through C_3 intersecting AD_1 in D_3 (figure 9)

We must now prove that, using this method of construction, the point D_3 will be the correct third term in this geometric progression.

Using Thales theorem we can see that for the angle OD_1D_2 whose sides are cut by the parallel lines C_2D_2 and C_3D_3

$D_2D_3/D_1D_2 = C_2C_3/D_1C_2$ (1)

Do the same for angles AOD_1 whose sides are cut by parallel lines A_2C_3 and A_1C_2 we see that

$A_1A_2/AA_1 = C_2C_3/AD_1$ (2)

By the properties of proportion we are able to obtain the following result

$D_2D_3 / D_1D_2 = A_1A_2 / AA_1$ (3)

Now examine the angle A_2D_2 whose sides are cut by the parallel lines A_1D_1, A_2D_2 and conclude that:

$A_1A_2/AA_1 = D_1D_2/AD_1$ (4)

Using the above results we can conclude that

$D_2D_3/D_1D_2 = D_1D_2/AD_1$

Now that we have proven the point D_3 satisfies the ratio of our series we are ready to continue the construction of the series.

The sum of this geometric series is simply the total of all its parts ie: sum $= AD_1 + D_1D_2 + D_2D_3 + \ldots$

We can prove that if we are to draw a line parallel to A_1D_1 from O intersecting R at B then AB = S .

Proof: By construction $AA_1 + A_1A_2 + A_2A_3 + \ldots$ is a geometric series with the same ratio as $AD_1 + D_1D_2 + D_2D_3 + \ldots$

If we take $AD_1/AA_1 = k$ then we have:

$D_1D_2/A_1A_2 = k$, $D_2D_3/A_2A_3 = k$ etc.,

$AB / AO = k$

Hence, $AD_1 + D_1D_2 + D_2D_3 + ... = k\,(AA_1 + A_1A_2 + A_2A_3 + ...)$

To show that AB=S it is enough for us to prove that $AA_1 + A_1A_2 + A_2A_3 + ... = AO$.

This could be done as follows: since $A_1 <> A$ then angle $AD_1A_1 <> O$. In general, since $A_nC_{n+1} \| A_1D_1$ then the angle $A_nC_{n+1}A_{n+1} = AD_1A_1$ which is non -zero therefore $A_nA_{n+1} <> O$ for all n.

Therefore the sequence $S^a_n = AA_1 + A_1A_2 + A_2A_3 + ... A_nA_{n+1}$ is increasing yet < AO for all n .

We see that S^a_n is a convergent series whose limit is AO. The sum of the series cannot be larger or smaller than AO.

From this construction we can derive the generally known formula for the sums of an infinite series $S = a/(1-q)$.

We can use this formulation to explain the paradox of Achilles and the Tortoise. The way we do this is by putting Achilles and the Tortoise on our Ray R. Achilles will start at point A while the Tortoise will start at point D_1. Since we assume that each of the runners are maintaining a constant speed, the distance they will traverse will maintain a constant ratio for given intervals of time. By the time Achilles is at D_1 the Tortoise will be at D_2 .

If the distance separating Achilles and the Tortoise becomes zero then they will be at the same position at the same time and Achilles will have caught up to the Tortoise. The distances separating them are represented by AD_1, D_1D_2, D_2D_3, etc. which are equal to A_1C_2, A_2C_3, A_3C_4, etc. respectively. From the diagram it is clear that after the infinite number of steps have been traversed, the distance is equal to zero. Since the path is finite and their speed is positive, Achilles does not need an infinite amount of time to overtake the Tortoise.

This explanation is a reconstruction of Anna Sierpinska's reconstruction of Gregorius a san Vincentio's paper " Opus geometricum quadraturae circuli et sectionum coni, decem libris comprehensum " (1647, Leiden).

Explanation in Terms of Transfinite Numbers

Yet another interesting analysis of this paradox is given by Bashmakova, I. G., (1975). This explanation of the paradox goes as follows: Assume that the distance separating Achilles and the Tortoise is equal to *a* and that Achilles runs k times faster than the Tortoise. This would mean that for every distance *a* covered by Achilles, the Tortoise will only cover a distance of a/k and whenever Achilles covers a distance of a/k, the Tortoise will only cover a distance of a/k^2. We could easily generalize that whenever Achilles covers a distance of a/k^n the Tortoise will cover a distance of a/k^{n+1}. This means that the distance covered by :

Achilles is : $S_a = a + a/k + a/k^2 + \ldots$

Tortoise is : $S_t = a/k + a/k^2 + \ldots$

According to Zeno, since he had no access to theories of infinite number, the distance separating Achilles and the Tortoise would always be greater than 0 and thus Achilles would never catch up to the Tortoise.

Today since we have access to Cantor's transfinite number, we could say that Achilles will reach Tortoise at finite time = t_ω , where ω is the first transfinite cardinal number.

According to Bashmakova , the paradox is solved by examining the number of path segments that are needed to be covered. Achilles must traverse the same segments as the Tortoise if he is to catch up. However, the Tortoise will always be one segment ahead of Achilles. If the Tortoise has covered m segments, Achilles would have covered m+1 segments. The only way we could understand how Achilles catches up with the Tortoise is by having m = m + 1, meaning that a part is equal to a whole which is only understood set theory.

The paradox is also explained in "the Mathematical Experience" by Davis, Hersh, and Marchisotto. They contend the paradox is simply due to "irrelevant parameterization". It is unimportant to examine the fact that Achilles is always behind the Tortoise at the convergent infinite sequence of times. Why should we limit our discussion to the sequence of times that Achilles is behind the Tortoise and not to the sequence of time that includes that time in which Achilles catches up to the Tortoise?

CHAPTER 3

PHILOSOPHICAL EXPLANATIONS OF THE PARADOX OF ACHILLES

In this chapter we examine the various philosophical explanations of the paradox with respect to the various views of infinity and the structure of matter. We will examine Aristotle's explanation and how the paradox is explained through the two way that magnitudes could be treated in a pluralistic system. We will also mention some of the unresolved philosophical problems that remain.

The explanation of Aristotle

In his explanations, Aristotle assumes that the race takes place on a finite path (Ross,1936, p.416). Aristotle states that Zeno's claim that since the Tortoise is ahead, Achilles will never catch up, is erroneous. True, Achilles does not catch up while the Tortoise is ahead, but if we accept that a finite line could be traversed to its end then Achilles will catch up. In Aristotle's words:

> "The claim that that which is ahead is not caught up is false; it is not caught up when it is ahead, but it *is* caught up, if one allows that a finite line can be traversed to the end (Aristotle, Physics, Book 6, Chapter 9, 239b)."

Aristotle also analyzes the problem from the perspective of proportionality in uniform movement, between space and time. According to Aristotle, since magnitudes are infinitely divisible, faster moving objects will traverse a greater distance than slower moving objets. His analysis is based on the fact that in the time the faster object moved a certain distance the slower object could only move a certain fraction of the distance covered by the faster object because the faster object will undergo a change before the slower object does (Aristotle, Physics, Book 6, Chapter 2, 232a-233a).

He also concludes that since all motion occurs at some given time any object may move faster or slower at any given time, it follows that time as well as space is infinite. Consequently Zeno's paradox is inaccurate because it presumes that it is impossible to traverse an infinite number of points in a finite amount of time. The inaccuracy lies in the fact that Zeno doesn't take into account the fact that, both, time and magnitude are infinitely divisible. Granted, that it is not possible for any object to pass over an infinite amount of points in a finite amount of time, if we only view them as quantities, however, it can if we are dealing with infinite divisibility, because time is also infinitely divisible. Since time is infinitely divisible, the object is really traversing an infinite amount of points in a finite amount of time, so to speak.

Essentially, in order to dispute Zeno's paradox we must refute Zeno's idea of motion and magnitudes. The paradoxes could only be resolved if we are to accept motion as a continuous magnitude, that the finite distance could be traversed as a whole.

According to Aristotle, the only way to gain a satisfactory understanding of the refutation of Zeno's paradox is through understanding the concept of continuity. Aristotle's view was that we cannot generate an infinite number of points or indivisible by dividing a continuum. In a continuum, points have no independent existence apart from their connection to the points at their extremities. The paradox is resolved if we view the process as Achilles traversing a certain prescribed distance.

Atomistic Explanations.

Another philosophical explanation for the paradox is to view it through the two ways that spatio-temporal magnitudes could be treated in a pluralistic system. As mentioned earlier the two ways that magnitudes could be viewed in this system are the Atomistic and the Infinitely Divisible System.

If we are to examine the paradox of Achilles in the context of the Atomistic theory we have to conclude that there is no real contradiction because according to the Atomistic view we would say that time and space are divisible into a finite number of indivisible units. The problem occurs because of an error in our definition of the concept of motion. If we view motion as movement of some thing from one point to the other, as it should be in the Atomistic system, we will end up with the question that if at every indivisible unit of time it is at another indivisible unit of space, when does it make its movement? And can it be at rest and in motion at the same time? All this could be avoided if only we correct our definition of motion to reflect the more accurate reality of the Atomistic system.

Motion must de defined as being at one indivisible unit of space at one indivisible unit of time and being at another unit of space at another unit of time.

If we are to take this view of motion, the paradox could be resolved by simply accepting the notion, that a state of motion does not involve being at a different position at every different unit of time. With this definition of motion, the paradox could be resolved by simply accepting that since Achilles is moving at, say, twice the speed of the Tortoise, then he would be at each of the indivisible units of space for half as long as the Tortoise and would thus catch up and overtake the Tortoise.

Explanations in the Infinitely Divisible System

Now let's examine the paradoxes of Achilles with reference to the infinitely divisible view. This view states that every magnitude could be divided into an infinite number of points. Take any given finite magnitude. This finite magnitude could be divided into a infinite number of parts. Now there are two possibilities of how we could view the individual parts. Either the parts are finite in which case Zeno would hold that given magnitude is infinite, or we could view the individual parts as having no magnitude in which case Zeno would say that the magnitude is equal to zero. By this view Zeno would be of the opinion that everything is either infinite or zero in magnitude.

According to Shamshi (1973 p.23), the problem lies in the assumption that an infinitely divisible magnitude necessarily could be divided into an infinite number of parts. Infinite divisibility does not imply division into an infinite number of parts. This means that the distance that Achilles must traverse could only be divided into some finite number of parts, but what this finite number is, is not determined, it could be any finite number.

If we are to assume that it is possible for an infinitely divisible quantity to not necessarily have an infinite number of parts, the paradox of Achilles fails to create as much of a dilemma. The reason we think that Achilles will never catch up to the Tortoise is because we think that if we divide a given magnitude of time and space infinitely we would be able to correlate these two for all their infinite elements.

Philosophical Problems Left Unresolved by Mathematical Explanations

From another perspective we could also choose to accept our definition for the sum of an infinite series as the resolution to the paradox and say that Zeno's inability to accept this definition left him with the paradox.

On the other hand we could argue that this does not completely resolve our problem. The entire problem appears to me as a problem of perspective. If we view the problem simply as dealing with the sum of a convergent series, the paradox poses no major difficulty. If you examine the process from the mathematical physics point of view you may run into some problems. Max Black , a philosopher who wrote a detailed paper on the subject (Salmon, 1970), argues that the problem is not solved by our definition of the sum of an infinite series.

"The crucial problem, according to Black and Thomson, is whether it makes sense to suppose that anyone has completed an infinite sequence of runs..... The mathematical operation of summing the infinite series will tell us where and when Achilles will catch the Tortoise *if* he can catch the Tortoise at all, that is a big "if" (Salmon, 1970, p 27-28)

True, the sum of the distances covered by Achilles are convergent but the paradox is trying to question whether it is at all possible for Achilles to ever catch the Tortoise, so assuming that he can catch up and then figuring out when this will occur will not bring us any closer to resolving our dilemma.

Black argues that it is impossible to say that Achilles has **completed** an infinite number of steps. He uses the argument of an infinity machine to make his point. Assume that we have a desk lamp with a push button switch. Each push of the switch causes it to go on or off depending on its prior state. Now, if someone were to push the switch an infinite amount of times by completing the first thrust in one-half minute the second in one quarter minute and so on, what will be the final state of the lamp? After an infinite amount of depressions the lamp can be neither on nor off because for each on push there was an off push and vice-versa.

The real issue here obviously is not whether Achilles could or will catch up to the Tortoise because we know he will. What is being argued here is whether or not it makes any sense to describe the process of how Achilles catches up to the Tortoise as an infinite sequence of tasks. Can we realistically make the connection between mathematical theory, as it applies to infinite series and real life physical situations?

Black's conclusion is that our fallacy is in believing that what Achilles is doing really involves an infinite number of steps. It is his contention that the notion of an infinite series of acts is untenable since Achilles will catch up to the Tortoise, therefore the number of steps that he takes is in reality finite. Although we could divide the distance into an infinite number of intervals and make a 1-1 correspondence between each of our intervals and the steps Achilles must cover, this is not a realistic description of physical distances that Achilles must traverse.

> " The class of what will then.be called 'distances' will be series of pairs of numbers, not an infinite series of spatio-temporal things (Salmon, 1970, Pg. 80)

This argument seems to be grounded in the fact that it is not always possible to make a rational connection between theoretical mathematics and practical application.

This particular angle of dealing with the problem may not satisfy everyone. Another approach to the problem is by viewing it using the tools of Cantor's set theory. The real nature of the paradox lies in the fact that it appears as though it is not possible for Achilles to overtake the Tortoise, for if he does , all the steps traversed by the Tortoise will be part of the steps that are covered by Achilles. Now since they are both running the course we must be able to establish a one to one relationship between the steps of Achilles and those of the Tortoise. Thus there should be less steps covered by the Tortoise, so how is it possible to establish a 1-1 relationship between the Tortoise steps and those of Achilles?

Of course this dilemma has been resolved above, in my examination of Cantor's work. It has been proven that the set of even numbers is equipotent with the set of naturals even though the latter set may appear to be much larger. Because Zeno falsely assumes that the whole and the part cannot be placed into a 1-1 relationship, he ends up with this paradox. This is an error of assuming that the rules that govern finite mathematics apply to infinite sets.

The reason for this confusion lay not in the fact that Zeno was unprepared to accept that when dealing with infinite sets there was a whole new set of rules but rather because Zeno was unprepared to even accept the possibility of the existence of infinite sets.

CHAPTER 4

RELIGIOUS EXPLANATIONS OF ZENO'S PARADOXES

In this chapter we will briefly examine how philosophers have handled the notion of infinity from a religious point of view. We will also look at how the notion of infinity is dealt with by Jewish philosophy, in particular, Chabad philosophy.

Early Religious Philosophers

In any discussion of the infinite, it is logical to expect a religious angle. Infinity and divinity have always been seen to go hand in hand. Aristotle correlated infinity with divinity. The next great thinker to shape the way we think about the subject was Thomas Aquinas. Aquinas believed in the Metaphysical infinitude of God whom he held to be perfect. He believed that nothing in creation was actually infinite, yet he accepted the potential infinite in Aristotelian terms.

The philosopher Spinoza believed that God was infinite in a way that we could understand but not imagine. He held that God was the unified whole and nothing else could exist apart from him, views reminiscent of Parmenides.

Reconciling God with the Universe (Transition From Infinite to Finite)

I would now like to examine the way the subject of infinity and infinite divisibility is treated by Jewish Mysticism; specifically in the Chabad Lubavitch tradition, as presented by Rabbi Schneur Zalman of Liadi in his book of Tanya.

In the chapter called "Unity and Faith "he deals with the Unity of God and the world and how we can reconcile the fact that an infinite God should be able to create a finite world. Tanya explains creation in terms of a "Theory of Emanationisms."

> "by means of a chain of successive emanations from "higher" to "lower" the finite evolved from the infinite, and matter evolved from spirit.(Schochet, 1988, p.47)"

Before the creation of the world, the only thing that existed was God. Nothing could be postulated about God except that he is without limit. When God decided to create the Universe He did so by withdrawing or contracting some of His infinitude into Himself. In the space that was brought about by this contraction the Universe was created. The important thing to note is that this contraction caused no change in God. He is still complete and the existence of the world has no effect on him.

The Universe still does receive some of His infinitude in order to survive. However, this divine power must be in a very concealed and contracted form otherwise we as finite beings would be consumed by His infinitude.

The main idea of the Tanya is that the way the world is sustained is through God's continued recreation of the world. The analogy is made between a silversmith that makes a cup out of silver and an individual that throws a stone in the air. Once the silversmith has completed his task, the cup has an independent existence and no longer needs the silversmith. The stone thrown in the air, however, will only remain in flight so long as the power of the thrower is in it.

The world is like the flight of the stone. It always needs God in order to exist and without his continuous creation and recreation of the world there would be no world, no remnant or memory that any world has ever existed.

To the question : "Can God create an object which He cannot move?" Rabbi Schneurson answers : "Yes, for to say He can't, is to limit Him." The book of Kabbalah states that to say that God is infinite limits Him for you are saying that He is not able to be finite. The definition of God must therefore be that which is at the same time finite and infinite.

Infinity in Chabad Philosophy and the Parmenidean Principle

In another work of Chabad in " Sefer Halikutim ", Rabbi Menachem Mendel Schneurson says that from a collection of finite units, you can never obtain an infinite number.

He uses the analogy of human generations, that even if the world would persist without end and there would be generation after generation reproducing, there would still not be an infinite number of generations only, that the number of generations simply would not have an end.

The way I view this is by saying that not having an end and infinite are not the same thing which is very close to Aristotle's view of the potential infinite.

Rabbi Schneurson's view is whatever has either a beginning or an end cannot be infinite. Although you may be able to continue the process of division infinitely, space is nonetheless finite and viewed as the sum of finite things.

Essentially, if we are to compare this to the Parmenidean principle of denial of plurality, we see that according to the Tanya there is no existence apart from God, for we do not have any independent existence from God. The Universe must exist in a perfect unity with God. Although from our perspective we appear to have an independence, this is only due to the fact that there has been a contraction and concealment of the Godliness and we are unable to see the Godliness that sustains us.

Relating Chabad Philosophy to The Achilles Paradox.

In the context of the Achilles paradox, I believe that the paradox could be interpreted from the point of view that we are trying to place the concept of infinity in the form of infinite divisibility into a finite position. This continuous division which is not really an infinity is being assessed as if it actually exists.

I am not sure how the Chabad philosophy would deal with the dilemma of Achilles and the Tortoise. It could be explained by the fact that since the space is limited it is erroneous to make the analysis of infinite divisibility or simply by stating that since everything is one with God there is no paradox.

As in Jewish philosophy it is accepted that the only true infinite is God, by trying to deal with the subject of infinite divisibility, irrespective of whether we are able to perform such an operation or not, we are dealing with the concept of

God. God as the infinite doesn't need to follow the rule of nature. The book of Tanya states that nature is merely a tool that is used to conceal Godliness. Rules that we accept as pure truths like 2+2=4, are not necessarily rules to God. To God 2+2 need not equal 4. This kind of rule is but a restriction we, as finite beings, have.

It is said in Talmud (Tractate Megillah) that in the first temple there was an area called the holy of holies. In this area stood the ark of the covenant which was constructed by Moses. The Ark was the manifestation of Godliness in this world. It was God's dwelling place in this world, so to speak. The tractate states that the dimensions of the Holy of Holies was 20 amah by 20 amah and the dimensions of the Ark were 10 by 10 amah. The ark stood in the middle of the Holy of Holies. The Talmud states that there were 10 amah between the walls of the Ark and the walls of the Holy of Holies which means that the ark did not take up any room.

It seems that the reason that the ark did not take up any room was that it was a manifestation of Godliness. Godliness has no limitation and need not be limited by space. Whenever dealing with the notion of infinity we are in a sense dealing with the notion of God and when dealing with God, the rules of space and time need not be followed. When Zeno is talking about infinitely dividing space, he is talking about a divine characteristic, and a divine characteristic need not take any place, as the Ark of the Covenant didn't take any place in the Holy of Holies.

This is my interpretation of Zeno's paradox based upon the concepts of Jewish philosophy as I understand them.

CHAPTER 5

SPONTANEOUS STUDENTS' EXPLANATIONS OF ZENO'S PARADOX OF ACHILLES

In this section we will examine two pairs of student's spontaneous explanations of the paradox. The first pair of students had a strong background in Jewish studies, having spent 6 years of post-secondary level education at a rabbinical college. The second pair were students with a calculus background. These two classmates had taken Calculus I and II together. Each pair was presented the paradox and allowed to work it out amongst themselves for about a half hour. Occasionally I interjected in order to push forward the conversation or to gain further insight into the students' ideas by suggesting alternate explanations.

Explanation By Students with a Background in Jewish Studies.

I first presented the paradox to the pair of rabbinical students U and W. This Group had no real formal secular education past grade 8 or 9. They had a basic knowledge of arithmetic but had not studied anything but Jewish studies in at least the past 10 years.

It was quickly evident to me that they were treating the implications of the paradox and the notion of infinity in a rather different way than I would have expected from university students.

The first thing that caught my attention was how quickly the grasped what the paradox was saying and how promptly they engaged in discussion about the paradox and its relation to the notion infinite divisibility. Another point that was quite particular to this group that they related the paradox to the theories that they were familiar with in Jewish philosophy. The notion of infinity didn't seem foreign to them and U had a very clear opinions on the subject. W was less philosophical and was initially making the statement that "practically the theory won't work" I think this was his attempt at explaining the paradox. However, the two quickly realized that this sort of logic would not resolve the dilemma.

The pair came up with some very original and imaginative ways to explain the paradox. W explained the paradox in terms of an analogy with age. If A is half the age of B, although the gap in percentage will always decrease the actual age of B will always be greater than that of A. U dismissed this argument and countered it by presenting the case of two men, one smarter and the other less intelligent. If the smarter man were to ease off on his learning, and the less intelligent man were to advance by working harder and longer at his studies, he would eventually overtake the "smarter" man.

I found this "explanation" very interesting and imaginative even if it could be subject to much interpretation. U did however end this example with the statement " The question is how is he going to pass him?" I assume he was referring to Achilles though he may have been talking about the "less intelligent" man.

Another argument that was suggested as a possible explanation, was the length of the head start but was quickly dismissed. U stated that "eventually he's going to catch up to him and pass him, that is the fact."

U then stated that "there's nothing wrong with the story, the question is, if there's something wrong with the theory". He tried to make the allowance that the Tortoise wasn't really advancing but maybe he was losing. He said that since Achilles was moving faster than the Tortoise, the Tortoise was irrelevant, "in the world of Achilles, in his sense of speed, B does not exist." He related this to the speed of light, that it travels so fast that in the world of humans this speed doesn't really exist since our eyes cannot appreciate it.

W seemed to continue to look at it in terms of proportion of distance traversed by each runner. He went into an elaborate explanation about how Achilles will overtake the Tortoise but didn't explain why he would be able to so.

I then presented them with a chart representing the steps that each of the runner took in terms as a sum of infinite terms i.e. 1/2 + 1/4 + 1/8 + ... W immediately concluded that the final term would eventually be zero, just as in a calculator, and justified this by saying "if you take a calculator and keep pressing the square root, eventually the square root is going to get down to one and eventually it equals up." U rejected this analysis.

The discussion got into whether it made sense to say that numbers are infinite. Initially there was a difference of opinion, U thought that numbers are infinite while W said they were not. W's justification was that since you could subtract a number from infinity and this subtraction would have an effect on the rest of all the infinite numbers, therefore it was unreasonable to say that numbers were infinite. W wasn't totally convinced and the discussion ensued. U reiterated his point by stating that "the very fact that I could subtract a number from this infinite amount of numbers, I can't tell you what the final total is going to be, but I know for a fact that it's definitely going to change the total, although there is no total because the total goes on. W was satisfied with this argument and agreed that the numbers were not infinite.

The discussion then began to take a more philosophical twist. U talked about how in "chassidus" - Chabad Philosophy- there is a concept called "bli-gvul" -without boundry- . This concept cannot be characterized for by characterizing it we necessarily limit it. Since we, as humans, are limited beings with limited minds and we are able to count, the concept of number could not be infinite. The infinite is necessarily something that we cannot fathom. W got back to the point that since we were infinitely dividing something that is limited we would eventually end up with nothing, like with a piece of cake. I used this to bring up the question: If I add an infinite amount of very small things, could I have an infinite amount? The consensus was that it would not be possible to obtain a finite thing from an infinite collection. However, this again brought about another discussion about whether numbers were infinite or not and the distinction was made between being finite and having no end. W felt that having no end was the definition of infinite while U claimed this was not the case.

I then asked them if they felt that a horizontal line extending indefinitely to the right and left was infinite. U said no, W said yes. W then brought up a very interesting point. He said that "we need to first examine how did Achilles even take the first step." It seems to me as if U inadvertently stumbled across Zeno's paradox of The Runner but was having difficulty putting it into word. He realized the paradox involved in concept of movement when dealing with this kind of logic.

Both felt that space was infinitely divisible yet it was still finite. They reconciled this by resorting to scripture that there was the possibility for many finites coming together and forming something that has no end.

U felt that the problem with my story was that I was dealing with numbers as infinite. To him numbers and space we finite. He said "If you deal with the concept that numbers are infinite you're never going to be able to get anywhere". Since, to U, space was finite Achilles was tearing into the Tortoise lead as soon as he started running because there were only a finite amount of steps for Achilles to traverse in order to catch up. He summed it up as a problem of "trying to bring together the infinite and the finite."

W was more confused about the whole notion and never really came to a firm conclusion that he stated with any authority or certainty.

Since U had claimed that a horizontal line was not infinite and divisible into a finite number of units I asked him if this did not take away from the unity of the line. I asked if his thinking didn't make the line look like a collection of units and not a line. U had no problem with this. He said that "it just appears to be a line just as numbers appear to go on forever." It was only the similarity of the points that was tying them together. He extended this further by saying that though the world appears to be continuous it is not, for God is recreating it at every moment.

It was very clear that U took command of this discussion and was more confident about his Ideas than W. U was ready to use Chabad philosophy to explain the concept of infinity, while W was wavering between accepting the paradox in terms of what he had learnt and what his limited amount of mathematics training allowed him to.

Explanation by Students with background In Calculus.

For the second interview I used two students with no background in Jewish philosophy. These students had completed Calculus I and II at Cegep but were not intending to pursue any further studies in mathematics.

I had expected this interview to be a bit faster paced being that the students had a background in Calculus and were familiar with the concepts of limit and infinite series. The interview did not go at all as I had expected. There were many long periods of silence. It took these students much longer to understand the paradox and its implications. These students showed no real desire to draw upon the knowledge they acquired in their calculus courses to explain the paradox and when they did they seemed quite uncertain about what they were taught.

These students also used various ways to try to explain the paradox. B remembered what appeared to be a form of the Runner paradox, that was taught to them in school, "Its like that problem where you walk half the way home, you know? So you're only walking half...". She brought up this point at more than one occasion in the interview because it was familiar to her, but it was not successfully used to explain the problem at hand. B also suggested that it was like a race on a curved race track, where it appears as if one runner is getting a head start, but this explanation was rejected by both students.

For the next little while they tried to explain the paradox by stating that Achilles will catch up merely be saying Achilles will catch up because he has the momentum. Once making this statement the conversation died down and after a quite prolonged silence I again reiterated the point that Achilles was required to first reach the point the Tortoise was at when the race began and once again the conversation flared up. Again B tried to explain away the paradox in terms of Achilles' momentum. R rejected this.

Again the conversation centered around the fact that in real life people run races. B said "The people that get the better start don't always win, you know?" The conversation centered around this premise for a bit. I am wondering if at this point they really were appreciating the point of the Zeno's argument.

Another explanation that was brought up, though I am not sure if it was meant seriously, was that Achilles had more energy than the Tortoise because he started later. This obviously went nowhere.

At this point the conversation was slowing down. It was taking longer for them to make any comments. I think there was a bit of frustration due to the fact that they were having difficulty reconciling what they knew with what was being implied. B said ".. since he is running faster he has to pass him. I can't see why he wouldn't." The conversation now stopped so I asked how long it would take to traverse an infinite amount of steps to which I was given the response, an infinite amount of time. I related this to the paradox and asked them to discuss amongst themselves.

R felt that the problem was with my infinity while B continued trying to explain the problem in terms of what she knew would happen and she said "I don't see how it could be an infinite amount of steps". R disputed this point for the next little while maintaining that it was indeed an infinite amount of points or steps.

Somehow the conversation turned, and B was now defending the position that there were infinitely many points and thus it would take infinitely long for Achilles to catch up. R was arguing the converse. R stated that "there is no such thing as an infinite amount of space."

Momentum was again brought up as a possible solution that both agreed upon but for some reason they abandoned this route. Both students were now starting to get frustrated. B said "I have no idea, I have no Idea.".

Finally they started to discuss the fact that since the differences were getting smaller and smaller at one point the difference would be negligible. They started talking about the infinitely small. R started to explain it as "it's infinite but its converging into something that's becoming smaller so you're not like, extrapolating into infinity, you're converging." At this point B realized that it was a "limit problem."

Once they realized that it was a limit problem they started to discuss the fact that limits were only approximations. R wanted to resolve this by saying that "it approximates the truth." This discussion ensued but nothing was resolved.

I suggested that they try and explain the paradox in terms of infinite series, where the series 1 + 1/2 + 1/4 +.. represented the distance to be covered by Achilles. I could see they were not eager to use this dreaded tool when B described infinite series by saying "those horrible words". Again, this led them to argue that the series was just an approximation but they didn't remember much about series and what they did remember, I don't think they really understood. I asked them, if the series represented the steps that Achilles must traverse in order to catch up to the Tortoise, and if it was only an approximation, how was it that Achilles actually overtook the Tortoise.

I don't think they ever really grasped what I was trying to ask and I didn't want to push this point. R did answer me by saying that Achilles wasn't really running an infinite series because eventually the distance would become negligible. Since the distance was finite she could not see it as an infinity. She said that since the area was bounded it didn't really have an infinite amount of points. That was why Achilles was able to overtake the Tortoise.

To counter this claim, that there were not really an infinite amount of point in a finite segments, I presented them, a geometric proof of how a finite line segment could be placed into a one to one relation with the infinite line. Since they accepted that the infinite line had an infinite amount of point it should follow that there were also an infinite amount of points in the finite segment.

They tried to argue the proof by suggesting that maybe the points on the finite line segment were smaller than those on the infinite line. B simply didn't agree but couldn't justify it. R grudgingly accepted the proof but proceed once again to explain the paradox in terms of the finite line segment being bounded, so not really an infinity.

Difference Between the Two Groups.

I found that students with background in Jewish philosophy were less intimidated by the paradox and its implications. They were familiar with the notion of infinity, albeit in a non-mathematical sense. This familiarity with the notion greatly improved their understanding of the paradox.

I believe that they understood the paradox and its implications and were very ready to present possible solution to the impasse. The solutions they presented, however, were not very mathematical . They were mostly based on Aristotelian principles that we as mathematicians have rejected.

To these students the paradox didn't create as much of a dilemma as for the calculus students, because their notion of infinity differed from its commonly accepted mathematical definition. In their view the problem lay not with the story but with the mathematicians that have an inaccurate definition of the infinite.

The calculus students, on the other hand, have dealt with the notion of infinity in a formal mathematical sense but have only received a very cursory explanation of the concept. They have studied infinite series and limits. They have accepted the definitions that they were taught, but never really bothered to look deeper into the subject to see if they totally agreed with what they had accepted.

The major difference between the two groups of students was in their definition of infinity. For an individual that accepts the precept of Chabad philosophy the paradox poses no major difficulty. The paradox could be explained away as simply an error in the way we view the world, nothing more nothing less.

I must admit that I found the arguments rather convincing and would have had a difficult time arguing against these points. Although the theories of Chabad have a strong Aristotelian flavor to them, I still find that they had a rather convincing effect on me. To admit that the set of natural numbers is infinite just because we say so is rather hard to swallow. Why should anyone accept such a notion?

A whole area of mathematics has been built upon the notion of various levels of infinity, concepts that are in complete disagreement with Chabad philosophy, and most people's logic. I don't see why we have to accept its validity. We could collapse the whole theory simply by rejecting the notion that the set of natural numbers is an infinite set.

This is what students with a strong background in this philosophy would say with strong conviction. They would not accept any of the explanations based on infinite series, because their notion of infinity differs from its generally accepted definition.

The calculus students have the advantage of formal mathematics. Although these tools are at their disposal I didn't find that they were ready or willing to use them, probably because they never really understood why they work. They simply became accustomed to perform them mechanically.

CONCLUSION

Mathematics needs to stimulate student's minds. If students only use mathematics mechanically as an assembly line workers who repeat their assigned task without thinking about how and why they are doing what they are doing, they are missing the point. Students need to have the opportunity to examine the paradoxes of infinity in order to find out how they view the concept.

That being said, I must say that I don't see the paradox of Achilles as a relevant part of a calculus class. I don't think that students that are taking calculus need to be bogged down by more confusion when they are being taught a completely new form of mathematics that deals with the notion of infinity in a way that the students have not seen before.

I think that these students are too busy simply trying to understand how to do what needs to be done, and are unable to jump into thinking about the philosophical problems involved in doing so.

The paradox of Achilles, in my opinion is more relevant as a section in a class on mathematical thinking like the Math 216 class being offered at Concordia. When a student is learning how to <u>think</u> mathematically, this is the best time to introduce questions that require him to think about whether it makes any sense to think the way he does.

I do think that most students will not change their preconceived notions about infinity due to this paradox. Nonetheless, by forcing themselves to understand the paradox, the will have a better understanding of what it is that they are doing when taking a limit or finding the sum of an infinite series. This better understanding is essential for anyone wishing to pursue further studies in mathematics.

It is my opinion that for students who take calculus simply to satisfy the requirement of some program in which calculus will never be used as tool, the paradox has limited use. It may be usefull to motivate them to think about inifinite series however, it makes no sense to spend too much time on this paradox with these students. Advanced mathematics is not for everyone. There are many other talents and skills that students could excel in, without a deep understanding of mathematics. For students wishing to pursue these other areas of study, we should not drive them crazy with deep mathematical thought that they are neither interested in nor capable of understanding.

Finally, I also believe that students' backgrounds will have a great deal of influence on how they will treat the paradox. Students with a strong background in Jewish philosophy will most likely see infinity in Aristotelian terms. For someone with a strong Jewish belief, God is the only true infinite and this must have some effect on the person's definition of the notion, when presented in mathematical terms.

A question that is not handled in this thesis, and which may be of interest is to see how students with other religious background such as Christianity, Islam, Buddhism etc. would explain this paradox.

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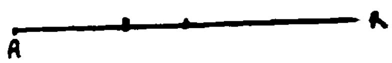
APPENDIX A

Figures



Fugure 1

Figure 2

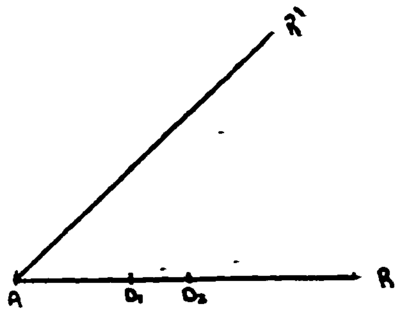
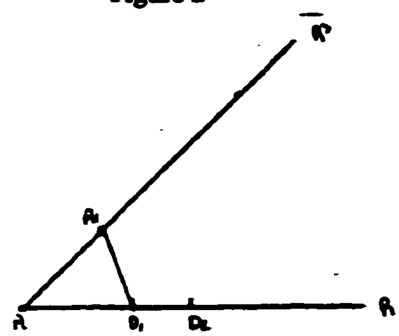


Figure 3



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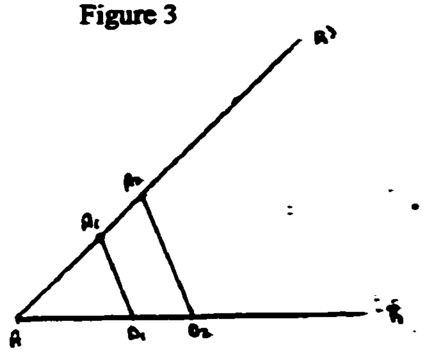

Figure 5

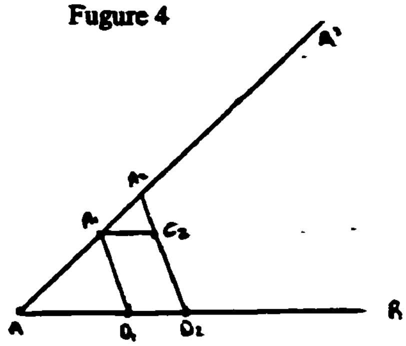

Figure 6

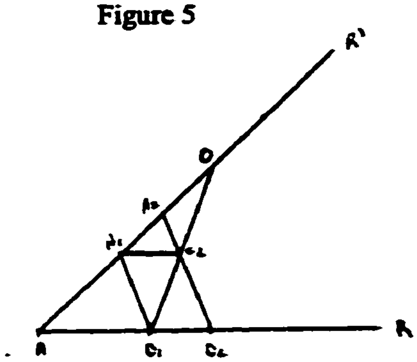

Figure 7

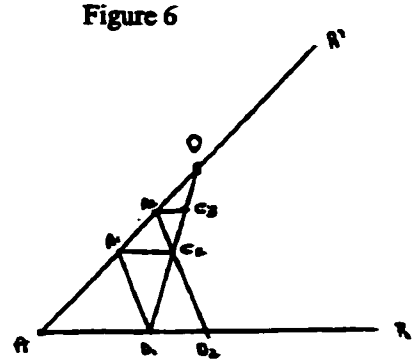

Figure 8

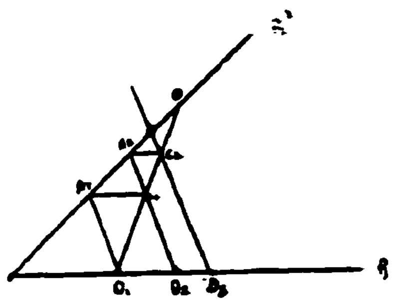

Figure 9

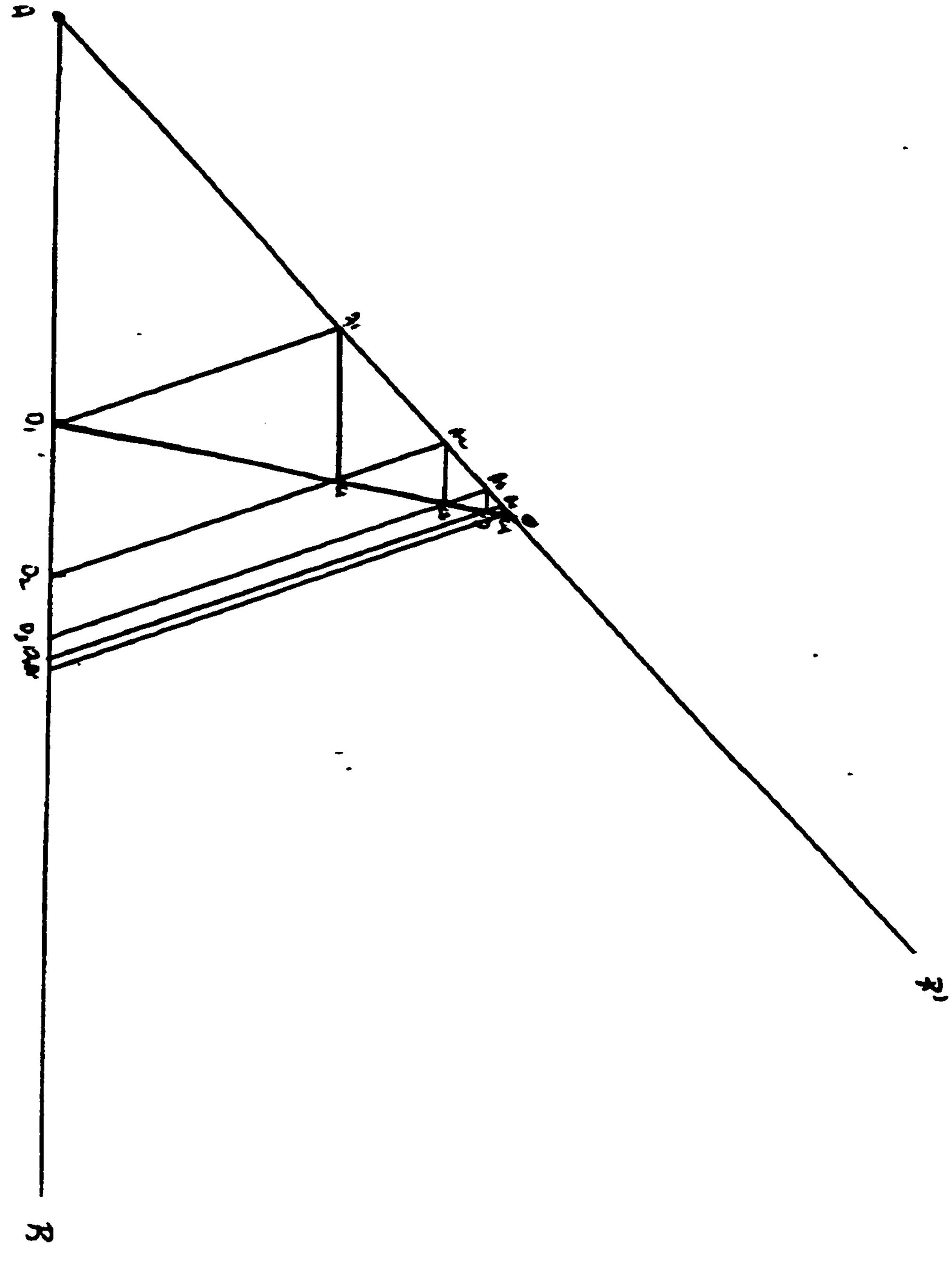

Figure 10

APPENDIX B

Protocols

The following is a transcript of students' discussions after I presented them the paradox and asked them if there was anything wrong with my story.

Interview of Students with Rabbinical Background.

U: The fact is that he would pass him.

W: But through that logic, it is impossible because however minute the time is, the seconds, the half seconds, the quarter seconds, there is always that millimeter where he is always getting behind.

W: Isn't there something about this idea in Chassidus?

U: In Chassidus?

W: Yeah, something with that same idea.

U: You see what I don't understand is the Achilles, in fact, everyone agrees will pass the Tortoise. In fact, in reality, he's going to pass him because at one point the Tortoise.

W: In practical, he's going to pass him according to the logic that he said, and if you're using that logic all the way you're always saying 2, 2, then half is one. So now as long as it takes to get to the two you can't go 2 because he's slower so he'll get half so eventually it will get down to the time that a person can't understand but it's still physically a certain amount of time.

U: Repeat that point again?

W: That if you're saying that here's two feet, let's say now as long as they stay within two feet he's slower so he advances only one foot. When now he goes one foot, he's slower so he only advances 1/2 foot. So he's always going to get less and less so eventually, according to practicality he'll catch up to him because he's always going less and less. But according to that theory, he'll never catch up to him because eventhough it's less and less, he's always advancing a little bit further than what he is. So if he's going a half an inch he's going a quarter of an inch, if he's going a quarter he's going an eighth but there's always going to be that practical distance.

U: So how does he pass him? According to what you're saying, he could never catch him!

W: According to what I'm saying, he doesn't catch him. That the point he's trying to bring in practicality we see it won't work because...

U: Practically it does work, practically he passes him.

W: Yah right, I'm saying, practically the theory doesn't work. In theory it makes sense that he won't pass him.

U: Why doesn't theory make sense that he'll pass him?

W: Because you're saying that he's always going to. To make it equal let's say that he's always going half as fast as he is. So Achilles goes a mile while Tortoise goes half a mile. When Achilles goes half a mile, Tortoise goes a quarter of a mile. So that distance will always show. It's the same thing, I'll give you a perfect example. In age, right? you're 10 someone is 5. You're double his age. Now 10 years later you're 20, he's only 15. You're not double his age anymore. When he's 100 he will be 105, so percentage wise, there is always an age difference of 5 years but the percentage wise will always go less and less. So theoretically you say, he's double, now he's a quarter so why can't he ever pass him in age? But you can't because there is always that gap.

U: The difference in age is that when I'm going to be 20, you're still going to be 5 years younger than me. You're never going to advance on me in time. If I'm one year old and you're 10 years old, when you're going to be 20 years old, I'm going to be 10 years old. I'm never going to advance on you in time.

W: No, this is just the percentage.

U: There's no advancement.

W: Yah, yah.

U: There is a percentage. Over here Achilles is advancing on the Tortoise. If there's a smart man and not such a smart man and the smart man decides to stop learning, but the not so smart man decides to really indulge in knowledge and he works on himself and he sharpens his brain and slowly he's reaching himself to the level, to the IQ level of that very smart man that we started off the story with. Eventually if the smart man whom we first started talking about is still at his pace and he doesn't work on himself and he doesn't try to achieve more knowledge, the not so smart man will catch up to him and will advance. Because when he's advancing, he's taking off, he's advancing on that man's knowledge even if the smart man is learning at a slower pace. The same idea with the running. He's learning at a slower pace but the other one is putting in 20 hours a day and he is really advancing and developing his mind. Eventually, his IQ, or his sharpness in a certain area let's say like in learning mathematics, one will know more than the other. The question is how he is going to pass him?

W: Yah, but because you see the question is because you were looking at a small distance. We're looking at one mile let's say. But let's say he gives him an advancement of 5 million miles?

U: I don't think that's relevant.

W: It is because the distance, because you're looking at the distance that...

U: So 5 million miles is just a question of time?

W: Time but it.

U: It will just take a couple of years more for the whatever to catch up to the Tortoise.

W: Fine, but he'll never because he's even at the time when there's a bit of a difference as soon as he goes a little he's still going a little.

U: When we're dealing with reality, as much as a head start he'll give to the Tortoise, I don't care if it's 5 million miles, in reality, eventually, he's going to catch up to him and pass him, that is the fact. I don't think it's relevant how much of a head start he gives, whether it's a mile or a thousand miles, he's going to pass him. The question is according to this theory, if this theory is correct, that he will always advance one step ahead, um, how is he going to pass him?

W: So what is the problem with this story? How do we put together what we know and this story? You see, according to this story, according to that theory, he'll never pass him, but in practice he will pass him. So how do we make the theory and what we know work together? Is there something wrong with the story?

U: There's nothing wrong with the story. The question is if there's something wrong with the theory. Is the Tortoise constantly advancing or is he losing? Maybe he's constantly. He never really is advancing although in reality it seems like when Achilles gets to this point Tortoise will go a bit further.

W: It's actually less and less.

A: It could be that from the very start, even when he gets the head start, he's in the level of number A, he's declining. To his level he is really declining, because the proof of that is that when Achilles starts running, he will eventually pass him.

W: The distance will get less and less.

U: That's how I feel.

W: Yah, it makes sense.

U: You see in his world. Let's say there is A and there is B. In the world of A and B is never advancing.

W: It's just how long he takes to get to that level.

U: That's why he could give him a head start.

W: What do you mean by that?

U: You have A that can go very quick and you have B that it takes him a very long time. In the world of A, in his sense of speed, B does not exist.

W: One second, I have another theory.

U: One second, one second. In the world of A the speed. Ok, let's take another example, a more simple example. The speed of light. I don't think we could see the speed of light. When you turn on the light in your bedroom from the time that the light goes through the bulb till it reflects in the wall, that speed is so quick that to the human eye, in the world of the human, that speed almost does not exist. He can't see it, he can't feel it. It's not tangible. And I think that the reason why, um, the reason why the speed of light is so much quicker than the speed of our eyesight is because in the world of the speed of light, our speed is so slow that we're almost meaningless. You can't even compare it to the speed of our eyes. It's not a question of one or two it's a question of can you compare or can you not compare. That's why the speed of light could give the speed of our eyesight as much as of a head start as we want because these are two elements that cannot be compared. Now the same idea with A and B. In the world of A the speed of B doesn't really matter. It doesn't really concern A. So therefore as much of a head start B will get, I don't care if it's 100 million miles, A will eventually pass him because they're not compatable.

W: The same thing you could say mathematically, the theory is wrong. If you have fractions and you want to get the common denominator of them, so if you have a half and you have 20/24 or whatever; so eventually there has to be something in common on the bottom that they'll both be divisible of. So even though for every 10 miles he goes 2 miles it doesn't matter because eventually for every three he's going to eventually catch up to him and go ahead of him.

Because in the theory of how we are trying to understand this story or the way I understood it before was that for every distance he goes half of that so he will always be a half, a half, a half. The other way to look at it is that for every certain x amount of miles he's going x amount of miles so that some x amount of miles is always eventually going to catch up with him. If he's going 10 per 3 it doesn't matter what the percentage is but he is going a certain amount for every certain amount that he is going; if it's 10 to 3 so by the time he has got the head start of 10 and he goes 20, eventually he will go into his thing and go ahead of him. It will take some time but eventually he will get there.

U: So what's your point?

W: That's why the theory is no good because he was saying why it is infinite.

U: That's why who's theory is no good?

W: The theory of the story is no good, of the race because he was saying that infinitely there will always be less and less of a gap but there will always be a gap. But there really won't.

U: Why?

W: Because eventually if you look at it that you had the head start, let's say he starts 20 miles ahead. Now you're saying that his speed, that for every 10 miles he goes three miles, so if he starts at 20 and he's starting at zero: now he goes and he's at 23, he goes 10 so he's 10. Now he goes another 3 so he's 26, he's 10, and he's 20. He goes another 3 so he's 29 and he is 30, he beat him. Because for every certain percentage he is going every certain amount of miles, he's going a greater distance. Because the way the theory made me believe was why it made sense.

U: But why is it that when he got to this point B, let's say B got to point 10 miles and. (silence)...

At this point I showed them the chart of how the distance separating the two runners will get smaller and smaller (figure A) yet there still is a difference.

W: No, there won't be a difference eventually. If you take a calculator and you keep passing the square root eventually the square root is going to get down to one and eventually it will equal up. It may take a long amount of time but eventually it equals up.

The way I see it is that at time zero, the faster one, Achilles, is at zero and the Tortoise is at 10. Now if he is advancing double then at point number 1 he advanced 5 units, he advanced 10 units, at point number 2 again he advanced 5 and he advanced 10 so he caught up.

U: You're jumping ahead, you're saying what will happen once they catch up.

W: No, I'm taking one step at a time. One time he went 5 and he went 10 and he went double him.

U: But you're just saying what happened after they caught up. The question is, one second. In numbers, if we deal with the simple numbers would the numbers catch up?

W: No! it depends on how you look at it. If you look at it like the chart no, because numbers are infinity because it's always going to be .0000007. So you could always add zeros till a million till trillions, numbers are infinity. There is no limit for number; that's a fact. Numbers are infinity so no matter how much you divide it in half there will always be a half of that half. There could be millions and millions of halves but numbers are infinity. There is no end to numbers. So if you take a difference and divide it in half, you will always be able to divide.

U: I don't think numbers are infinity

W: Of course they are.

U: No.

W: That I mean.

U: Why do you think numbers are infinity?

W: Because for humans that which time and space is.

U: Now, one second. If I minus number one from this infinite amount of numbers would there be a minus one?

W: But there is still an infinite amount.

U: But would I be able to minus one number?

W: Yes. But you don't know what that infinite number is to minus.

U: But whatever it is, I have to minus one from that infinity. So this one number is having an effect on the rest and all the infinite numbers. It will have an effect on all the infinite numbers.

W: Yah, but you still don't know.

U: The very fact that I could minus one number.

W: What do you mean it has an effect on the infinite number?

U: Let me explain. If I take 100 million and I minus one, what would the number be?

W: 999 000 000

U: So that minus one had a tremendous effect on that large sum.

W: So no matter what the sum is, it is always going to have an effect. Yes.

U: So do you agree that no matter what the sum is, I'm going to have to take minus one number?

W: But you said if I take one away from infinite.

U: I didn't say one out of infinite, I said one out of the infinite amount of numbers. Do you agree?

W: That whatever the infinite number is, if you take one it will have an effect on it. Yes.

U: Because if you deal with infinity according to "Chassidus" there is no such thing as minusing one or two or minusing anything.

I then asked them to clarify what they meant by the statement that by taking away one from the infinite amount of numbers there would be an effect on those numbers.

U: How do we get to the infinite amount of numbers? You keep on counting.

W: Yah, there is no end, you just keep on counting.

U: Let's deal with the issue of someone who has no end. The world of numbers has no end. But the world of numbers has a beginning. We have to start from number one.

W: Well, not really; we have 0, and -1...

U: Ok, let's take both sides. In the world of numbers,there is plus and minus that go infinitly. Can I deduct one number? Theoretically, I deduct one number of the world of numbers that go plus or minus.

W: Theoretically yes. But you don't know, but yah, it's possible.

U: I don't know what the outcome is going to be. I can't say it's going to be 9.999. I don't know what the outcome will be.

W: What you're saying is that you can take any number and subtract something from it. Yes I agree.

U: Ok, so I could subtract. The very fact that I could subtract a number from this infinite amount of numbers, I can't tell you what the final total is going to be, but I know for a fact that it's definitely going to have to change the total, although there is no total because it goes on.

W: Yah, makes sense. But it's irrelevant to this theory because here you're taking half of what the previous step was. It's always half of half of half. In other words here you're always going at the same rate 5, 5, 5, eventually he'll catch up. Because he is going half that so eventually he will catch up. Here you're saying half, now half of that so of course he will never catch up, because it will always be half of what that half is. That's where the theory comes in with infinity. Because it's always a less number no matter what, eventhough we don't understand what that number is, it's always going to be less.

U: You see, according to Chassidus, there's a concept called "bli-gvul" - infinity - Now, infinity cannot be characterized because once you characterize it.

W: Then it's not infinity anymore.

U: Then it's not infinity. The very fact that a human being in his limited mind could characterize something makes it finite, because we're limited people. The very fact that we could count, I don't care if we could count forever; but if we could conceive in our limited minds numbers, then the idea of infinity of numbers can't really exist. Because if it really does exist then we are dealing with the infinite and finite coming together.

W: No, they're not coming together, they're just...

U: According to "Chassidus" infinite and finite do not come together unless there is a very strong powerful force that unites them.

W: I'll tell you what's wrong with this theory because you're taking half of half of half. So of course it's going to be infinite because you're always taking something and dividing it in half and then taking half of that, so it's always going to be like anything. If you take a piece of cake and divide it in half and then take half of that, according to a normal person you'll eventually say it's nothing. But if you think of something that is infinite like number, you see in a cake you're right, the theory won't work because a cake is something which is limited.

Eli: How about space?

W: Space also, anything which is infinite won't work. Something that is finite is half, half, half. Eventually you'll get to nothing. Something which is infinite, there is never any half because the numbers are forever so there is always something going less.

U: So you're saying infinity with numbers is not the same as infinity in space.

W: No, it is, space and number it is.

U: A minute ago, you said space.

W: Wait one second; infinity and space, it's very hard.

U: You see, when you apply this theory in numbers it makes alot of sense. The numbers will never catch up to each other.

W: But, they will, it depends how you look at it.

U: The numbers will never, you know why the numbers will never catch up? because.

W: Why not?

U: Because numbers go on forever.

W: So what?! It caught up!

U: No, but numbers go on forever.

W: But it will pass it.

U: It won't pass it.

W: Sure it will.

U: No, it will go point, point, point,

W: No, that's if you do that theory, but by my theory is the same half distance.

U: So, if you're dealing with numbers, it is very possible to split number into an infinite amount.

W: That's what I just said, exactly. So you're dealing with numbers is infinite so it's always going to be something multiple of that. It's always a multiple of that; it's always a multiple of something.

Eli: If I add an infinite amount of small finite things, could I have a finite amount?

W: No!

U: I don't believe that infinite could form finite.

W: It's impossible. It's infinite: how could you create something finite?

U: It's hard to conceive that something that is infinite could create finite. One second. Although we are dealing with numbers that have no end, I don't think that that gains the characteristic of infinity.

W: Why not? Infinity is something to which there is no end.

U: Because we both just agreed that you can't take a bunch of finites and an infinite thing.

W: Right!

U: And numbers are a bunch of finites?!

W: No they're not.

U: Sure they are. Each number is a finite, take each number for itself. 1 is not 2, 2 is not 3... But I don't believe that numbers gain the characteristic of infinity. I think there is an infinite amount of numbers but I don't think that makes them infinity.

W: Why not, if there's an infinite amount they are infinity.

U: I think that numbers have no end.

W: But that's what infinity means.

U: No, the concept of having no end, I don't believe makes something infinity.

W: So what makes something infinity?

U: I think what makes something infinity is that it has no character of finite, there is no character of finite at all. Let's take for example, numbers are only numbers. A number is not orange juice, a number is not a table so is there a character of finite in numbers?

W: According to that yah.

U: Numbers have the boundary of the concept that they are only numbers; it's not this and it's not that, infinity is not bounded. When you talk about infinity, there isn't a character that you could say this is it! Infinity is something. That us, as limited creatures, it's very hard to understand what infinity is because everything we know and see is finite.

W: I always understood infinity to mean no end and numbers have no end, each individual number has an end it itself; 5 is 5.

U: No. It has no end but why call infinity?

W: Well that's what infinity is.

U: I don't think so.

" To try and dispel this impass I decided to ask if they felt that a line, extending indefinitely to the right and to the left, was infinite".

Eli: Is this line infinite?

W: Yah! Two parrallel lines could never meet.

U: No!

W: Why not? Of course it is!

U: The line has no beginning and no end but I don't think it's infinite.

W: That's what infinite means.

U: I don't think that's what infinite means. I think that our definition of what infinity is has to be explained more.

W: As far as I'm concerned that line is infinite.

U: I think we need to first examine how Achilles even took the first step. Forget about the point of where he passes him, let's not deal with that , let's deal with when the Tortoise is at step 10 Achilles takes his first step. Ok, now the Achilles just tore into his lead by one ste,; so instead of him being 10 steps ahead he is now 9 steps ahead. How did he do that?

W: Why is he 9 steps ahead? He was 10 steps ahead when he went to the first one?!

U: How did he do that?

W: He went ahead so he is not 9 steps ahead. Say that again!

U: How did he take the first step if the Tortoise is constantly moving ahead? How did he tear into his lead?

W: Because he is going faster. I don't understand your question.

U: What do you mean he is going faster?

W: If he is going at a faster speed than him.

U: No, no. How did he first? I'm having a hard time getting this down in words. Ok, what are we talking about? What is the problem? How is he passing him?, how could he constantly, if the turtle is always going ahead? How is he passing him? I think that the question starts from the very beginning. Really, it's not how he passes him that bothers me. According to this whole theory, how is it that Achilles takes that first step and catches up that instead of being 10 steps ahead he is now only 9 steps ahead? How did he reach into the Tortoise's lead? The turtle is always going?!

W: Yah, but at a lesser speed.

U: Again we are dealing with a theory over here of numbers. That at it's point, point, he is always going to get ahead.

W: Yeah, but by less and less and less.

U: But how, how did he take that first step? How is it possible for Achilles to deduct what the Tortoise is gaining?

W: Ok, I think I understand what you're saying but, anyways, how does this help you explain the problem?

W: I think that if you deal with that theory, that it's infinity, if you deal with the concept that numbers are infinite you're never going be able to get anywhere, that's how I feel.

Let's take a piece of paper. How many times can you cut this piece of paper? But the fact is that when I cut a piece of paper in half, I am deducting half of this paper. I'm taking it out of the game. This was once a whole piece of paper and I had the power to deduct from it half. I took away a very large sum of this paper. The fact that I could take away one piece tells me that it can't be infinite; and I think the same thing with numbers. The very fact that I could take away one niumber even if there is no end to numbers, the very fact that I can theoretically minus a number and take it away and put it in a whole different world tells me that it is not really infinite.

There is a famous saying in Chassidus that there can't be "gvul - unlimited" - on one corner and "bli - gvul " - unlimited on the other corner.

W: Yeah, I remember that .

U: And " Rambam " - Maimonides deals with this in " Moreh - Nevuchim" - " Guide for the Perplexed. I actually saw this lately that's it's almost impossible to say that in one corner of the world there is finity living and in the other corner there is infinity, because if there is one corner of something that is finite then the other corner is going to suffer because of that. Because if I deduct one corner of a page, this page is no more a full page, it's a page minus a corner. I feel that if I were going to go with the theory that the numbers be infinities, space is infinity, paper is infinite. Yah, you can't take that first step, how could you take that first step? Because how could you take that first step into infinity when you're dealing with a finite? This is what I feel is the problem with the story.

W: So that's your resolution.

U: Yes.

U: How about you (Wenger)?

W: Well I can't exactly grasp it but there is some kind of difference how you see the theory with anything. If we just forget about the halves of the halves; just take 2 numbers and double that number. I can't explain it exactly; there's just two ways to look at it.

W: But then how do you resolve the theory? Do you agree that he will eventually catch up to him?

U: Yah, I agree that he will eventually catch up.

W: Do you agree that the points are infinite?, a half of a half of a half, that's infinite and he will never catch up.

U: No.

W: Why not?

U: No, because when you're dealing with numbers there is no end. Correct? But, when you're dealing with space, in fact, he will catch up because when he is taking that first step he really is tearing into his lead. When Achilles takes his first step he tears into the Tortoise's lead because he never really took an infinite amount of steps; and he is not taking an infinite amount of steps. He is taking a very finite amount of steps so really what we are dealing with are two finites. One happens to be faster so one will pass. So the question is: " Which finite is faster? "

W: So another way to say is that we are dealing with numbers and space; number is infinite and space is not.

U: No, no. I'm saying numbers don't have an end, you could count forever so you could always say point, point, point and you won't catch up.

W: But how about space? Is space infinite? In other words, if you take a certain amount of space, can you divide forever or will you eventually have to stop?

U: It has to come to an end!

W: So that's exactly the answer.

Eli: What happens at that end?

U: I'm sorry. Let me rephrase. It doesn't have to come to an end but it can be deducted. It can be deducted meaning, that in a way when I take a part or a piece off of this space I'm deducting from the space that's there.

W: So can you keep on deducting forever or eventually you'll have to stop deducting?

U: The very fact that I could deduct one part of this space tells me that...

W: That it's finite.

U: That it's finite. I might be able to cut it up into an infinite amount of pieces but it's finite.

W: Yah, I understand! There may be no end to how many pieces but the thing itself is finite.

Eli: Do you see a problem with the fact that you could infinitely divide it, yet it is still finite?

U: No, I don't see a problem with that. The Torah writes (Genesis 49:19) " Gad gedoud yegoudenu " - Gad, a troop shall troop upon him -. The commentary says it is a troop without number, no end. There is number that one could say there is a thousand, a million, a billion... there is no final number that one could say that there is one amount of angels in the sky. But there are groups of angels; each group is finite with it's specific amount but there is no number to these amounts of groups. We find this in the Torah, that the concept of finite subjects coming together and creating a, not a, I wouldn't say an infinite, the passage doesn't call it infinite, the passage calls it " It has no end ". There is no end to it's amount. Yeah, so I agree, you see that in the Torah an accumulation of finites can create the concept of no end. But I don't believe it gains the characteristic of infinity.

Eli: So in a few short sentences how do you treat this story?

W: The whole problem is that you're trying to bring together infinite and finite.

U: As I said before, you're not really dealing with an infinite amount of steps.

Eli: When according to you is there the jump from right before infinite to right into infinite.

U: According to Chassidus, there is no jump. There is, a, between the world of infinite and the world of finite there is no attachment therefore there cannot be any jump. They can't get close.

Eli: Didn't God create the world? Isn't that the connection?

U: God is God. God could be anything.

Eli: Isn't God the connection? He is infinite and the world is finite, created by an infinite God.

U: but in the world of God, infinite and finite are the same.

Eli: But would there not need to be the same sort of leap between the infinite and finite otherwise we would be infinite.

U: God is not infinite and finite therefore God doesn't have to take that leap. God is God, infinite and finite are the same to Him; so creating the world and not creating the world are the same. He didn't have to take a leap because He is both infinite and finite; He is beyond that. you can't compare Him to either infinite or finite. God is beyond our understanding.

Eli: Let us again take this line that extends indefinitely into the left and to the right. How is it that you mantain that it is finite?

U: By the very fact that I could cut it, I could cut a piece out of it.

W: No, then each part is not infinite but together as a group they are infinite.

U: No, together they create a line.that goes on forever.

W: Which is an infinite line?

U: Not an infinite line; I'll tell you why it's not infinite. It's not infinite because the line could only this way but it can't go this way and not that way and it's only a line. Yah, the line goes on forever but is not infinite.

Eli: So what then is infinite?

U: You can't describe it. If I could describe it then it's finite.

W: I think it is something that has no end.

U: I don't think that something that is infinite could be described by a finite.

Eli: If the line is not infinitely divisible and it is not made up of a finite number of units, does not the division of the line into finite units take away from the unity of the line? We appear to have not a line, but a collection of finite segments.

U: Yes, it does. Your line is made up of finite units. It just appears to be a line, just as numbera appear to go on forever but every singlenumber has it's own individuality and characteristic. The very fact that I could minus one number tells me that the so called infinity of numbers is really just an accumulation of many numbers.

W: So what is tying the points together?

U: There is a pattern and all these finites are forming a certain pattern which is giving us the impression that there is a line here and what ties them together is the fact that they are very similar. They may appear together but it is only their similarity that holds them together.

Eli: But how about when I move my pen across a piece of paper to form a line. Is this line composed of many points or is it one single line?

W: But it appears to be a continuous action.

U: It does appear just like this world appears to be continuous but it's not really. God is recreating every instant from new.

W: Yah, I see. It makes perfect sense. According to Chassidus this is all explained.

Interview of Students with Calculus Background.

R: The distance is going to get shorter each time.

B: But can't he overtake the Tortoise?

R: But you can't overtake an infinite amount of steps. It can't be an infinite amount of steps if he is running at a constant pace and Achilles is running faster than him. There has to be a set difference. you know it can't be infinite.

B: It's a set point between where the Tortoise is. Like let's say Achilles starts up here and the Tortoise is down here; so then whatever he's going to move up, but he's running faster so the distance is getting smaller. It's definitely a defined distance.

R: What if they are running in a circle? (both laugh)

B: It's like that problem where you walk half the way home, you know?. so you're only walking half so even if you're standing at your front door you still have half of the space to go when in fact you're inside; like that infinite amount of space; like you know how we did that one like even if there is that space you haven't covered. It doesn't matter because it is so small like negligible. He has to catch up if he is running at a faster pace. Doesn't he Rebecca? (both laugh)

R: Oh, um.

B: He will catch up, he should catch up; I can catch up to my kids when they start up before me.

R: It's just that, makes sense that if he is going faster he will catch up.

B: Depends on how much faster.

R: No.

B: Like if the race is from x to z and he starts off from here.

R: But he should catch him eventually.

B: It's like those races when they are running the 200 they have, that big curve, and then one guy starts off in front but they are both actually running the same distance, so they will eventually be at the same point.

R: no, this isn't anything like that.

B: Why. you have them starting off at 2 different points; do you know what I mean?, like instead of the head start?

R: I think you're supposed to assume this is the same course.

R: But, at one point he is going to run and be at the spot before the Tortoise gets there. It only makes sense and he will have already passed him.

B: so then the distance between them, like,

R: So he may never catch him at the same point at the same time but he will go passed him.

B: Yah, the first time will be like a positive distance between them and then.

R: It's like Victors.

B: That's a horrible class. (both laugh)

R: But at every point he reaches the Tortoise, the Tortoise will advance but at one point when he catches up to the Tortoise he will be going faster then the Tortoise so the Tortoise will be advancing as much-No?

R: Is the speed of the Tortoise a factor?

B: But at one point the distance between Achilles and the Tortoise will be so small that by the time Achilles reaches the Tortoise, the Tortoise won't have advanced that much.

R: Yah, it's going to become like a spec.

B: He is advancing at a slower pace and once he advances, then Achilles is advancing at the same time the Tortoise is.

R: And Achilles is not stopping where the Tortoise started and he has the momentum to keep going.

B: And he is running faster.

R: It's not like he comes to a dead stop waiting for the Tortoise to move up and then starting again.

(long wait)

Eli: But does he not have to first reach the point where the Tortoise was?

B: He is going to reach the point where the Tortoise is but at the point where he reaches the Tortoise, his momentum is going to carry him faster to the finish then the Tortoise momentum. If we say that they will reach the same point then the Tortoise will advance but the Tortoise is advancing at such a pace that he will reach him again and he'll pass him before the Tortoise has a chance to advance-No?, because he is slower.

R: Well, basically what he is saying is that the Tortoise is always going to be a bit ahead, some specific distance ahead. Achilles is going to catch up and in that time the Tortoise will have moved up a bit more and so Achilles will have to catch up to him and the Tortoise will be ahead again.

B: But realistically, you see people running races. The people that get the better starts don't always win, you know? And even if someone has a bad start, they can even win.

R: But in the end, the person who has got the advanced start will have gone further at that moment like, they will cross the finishing line.

B: They won't cross the finish line first; that's the whole deal. Like if you have a bad start in the 100 meter race you won't necessarily win. Like they're going to cross the 100 meters in less time, right?

R: OK, but over the distance, if you average the time, like;

B: Like where he is at 3 seconds.

R: No, let's say he has only 10 meters to go and the other guy got 100 and even if he gets there. no, that makes no sense.

B: I understand this point about catching up, and then he's going to go and he's going to catch up and he's going to go but the deal is: let's say there is 50 meters in between them, separating them and so Achilles is going to start running he running, he's running, he's running and the Tortoise is going at whatever his pace is and then they only have 5 meters separating them and then whatever Achilles is going to run, run, run and then there is 10 meters separating them and then he is going to run, run, run and there is like no meters separating them because Achilles will be ahead. Isn't that the point because, like you're saying, there is always a distance in between the two, like there is always that infinite space in between Achilles and the Tortoise.

R: Yah, the Tortoise has more energy; he hasn't worked so much. (both laugh)

R: No, but in terms of, I don't know, in. Well it depends what kind of race. We could say philosophically that he has actually won because he hasn't put as much effort.

B: No! Like a regular race like a race where you're running the 100 meters; it's a straight line which is what we want, right, we don't want any curves. Achilles goes to the Tortoise: " You can have a better start than me so you're going to run 10 meters in the time. I'm going to run 10 right?, so this guy is going like his 20 meters per second and Achilles starts out running at 30 meters per second in the small space of time that it takes. You know what I'm talking about?

R: Yeah.

B: Like for a small time Achilles will be behind but since he is running faster he has to pass him, he just has to. I can't understand why he wouldn't.

R: Because Achilles is always trying to get to the distance that the Tortoise has already been to.

B: But at one point the Tortoise is going to have to get to the distance Achilles is going to because he is going faster. Like the whole point is that if they are running at a steady pace obviously the Tortoise is going to be first but Achilles is never going to reach the point where the Tortoise is but the whole fact.

R: No, but it also depends on where you put the finish line.

B: Yah, like if you put the Tortoise right in front of the finish line

(long silence)

Eli: How long does it take to traverse an infinite amount of steps?

R & B: An infinite amount of time.

E: If it does take an infinite amount of time to traverse an infinite amount of steps, how many steps must Achilles traverse in order to catch up to the Tortoise?

B: But it is not an infinite space, that's my whole problem with it.

Eli: According to my story, how many steps will Achilles need to traverse in order to catch up to the Tortoise?

R: Infinite plus one.

Eli: Do you now see a problem with the story?

R: Because it seems like Achilles must pass an infinite amount of steps.

B: But it's not an infinite space, though, it's a concrete space between Achilles and the Tortoise. No?

R: It's not like that argument 2 parallel lines won't pass.

B: No, it's not because they are not crossing. I just am not sure how it could be an infinite amount of space if there are two people and they are running.

(long silence)

Eli: Do you agree that there are an infinite amount of steps to take?

B: If you're saying it's an infinite space then you have to say that it will take an infinite amount of time and it's going to take him an infinite amount of steps. But our problem is that we don't think it's an infinite space separating the two, that's why I think he is going to pass him. But it bothers me that if we look at the story the way he said, you know, Achilles catches up and in that time the Tortoise advances each time.

R: It just means that your infinity is not right.

B: Well, I have this picture in my mind of this person starting here and if this person is running faster, fast enough to make up the distance, he's going to pass him. I don't see how it could be an infinite amount of steps.

R: Well, if he's not going to catch up to him it's going to be an infinite amount of steps. You see, is there an infinite amount of points?

B: No, because there's the front of them, there's a whole bunch of points in between them but can't be an infinite amount of points in between them; it will round up to infinity. You see, if this is the distance separating the two of them...

R: Yah, but you could always split that distance, even if you say there is like a thousand points.

B: Well, if you say this is infinity well then what happens here if this is an infinite amount of points?

R: But that's also an infinite amount of points because if you're going to start chopping it up, you could chop it up an infinite amount of times.

B: There is an infinite amount of points, but each point isn't a step. You can get smaller and smaller and smaller, right? You can go like, what is it?, zero over a number is infinity. Right?

R: No, a number over zero. Where were you going with that?

(both giggle)

B: Trying to be sure. Well you see, there is going to be a whole bunch of points. You could divide it into two and divide it into two and so forth; there will be a whole bunch of points but each point doesn't correlate into a step. Right? How small are his feet?

(both laugh)

R: Yah, but it just means that there is an infinite amount of instances. Even if you have a certain specific bounded distance like 5 meters, you could still always chop up the distance.

B: I understand.

R: I realize you're saying that his feet are bigger.

(both laugh)

B: That's my whole problem, that he is a person, he is a man running. That was the basis of the problem: he is a man running and he is going to catch up to the Tortoise.

R: But the Tortoise has smaller feet so it's infinity.

(both laugh)

B: I understand the whole point you're saying; I do?

(long silence)

Eli: if you could infinitely divide, how many points are there? - (discuss that)

R & B: Infinite.

R: But within that infinite amount of points there could be a concrete point where Achilles could step above that distance.

B: But how, if he has to take an infinite amount of steps?

R: It's like an asymptote, the curve is going to converge.

B: It's like when you put your two fingers together and you could feel they one together, you still know there is space in between them. You know, like I'm sure you've done this with someone. It looks like there is space in between them, right?, but there is like you could know that because, like space, like everything exists with space and time, blah, blah, blah... There is going to be like at the point where they look like where Achilles looks like he is at the same point as the Tortoise. The Tortoise will be that little bit ahead of him, you know?

R: Yah!

B: You see, there is that space. Whatever that space is, there are all those points in this space. This space is going to get smaller and smaller and smaller but there is still going to be an infinite amount of points in this space. At some point they are going to look like they are like parallel but they won't be because there is still all those points that heck didn't pass. You know what I'm saying? You see, I can't do it with those things, it doesn't, I can't. You know, there's going to be whatever, that little point.

R: Even if there are infinite divisions, he is still going to overtake him at one point because they're just even.

B: But if we're saying there are infinite divisions, that means that it will take an infinite amount of time for him to catch up to him; so he can't catch up because there is no such thing as an infinite amount of time.

R: Yah, but there's no such thing as an infinite amount of space.

B: But if there is infinite division, there is going to be an infinite amount of points and an infinite amount of steps.

R: And not an infinite amount of time; and you're saying that more of it will even intersect because you can't get to infinity.

B: But that's odd because we know that he is going to pass him, like that's the whole problem. You know what I mean, we know he is going to pass him.

R: Maybe he is just going to pass him because of the momentum; it's a physics thing. Maybe it doesn't have anything to do with cutting up space, maybe he's just got the wind or something.

B: Maybe it'll just look like he's passing him.

(both laugh)

B: Because, well, this is saying that he can't pass him, but he is going to.

R: I say it's the momentum.

B: How does that help? I don't get it.

R: Because he is not stopping, there's no time to cut it up because he's like.

B: The momentum is what? Hiss mass times his velocity square?

R: Not squared.

B: So his momentum is going to carry him to the finish line.

R: Yah, that's why he's going to pass him.

B: At the finish line.

R: Well, you could make that the finish line, he may pass him well before.

B: But it's an infinite amount of steps in a finite amount of time so he can't; there will always be that one little step that he hasn't passed.

R: But he does pass him, so how does he pass him? If there is an infinite amount of steps and it happens, you have to explain it.

B: I have no idea, I have no idea.

R: Maybe, well, mass times velocity he is going at a speed and it's a force and that force is pushing him forward.

B: It still doesn't explain it how he is going to pass an infinite number of steps, unless he is not actually passing him; unless it just looks like he's at the end. You know what I mean like it's a whole mirage, like you know.

R: Yah, but we know he is going to pass him and it doesn't just look like he is going to pass him. My question is: "How is he going to pass him if he hasn't overcome that infinite point?

B: I don't know.

R: Maybe he takes the infinite number of steps.

B: He can't take the infinite number of steps in a finite amount of time unless you say it's an infinite amount of time.

R: Even if there are an infinite amount of points, they are getting smaller and eventually his body is going to be bigger than one point and he's going to pass him. You know that little speck and his foot is going to cover that little speck and go beyond it.

B: but then you're saying, OK, his foot is going to cover it, but now you're saying that he's done the impossible; he's covered an infinite amount of steps in a finite amount of time and that just doesn't make sense. You know?, like I agree with you, I know he is going to pass him but I don't know how, I don't know why.

R: But it's not an infinite in that it's large, it's an infinite in that it's getting smaller. It's like it's infinitely small. If you keep chopping up into smaller and smaller pieces so you can't say he is running across an infinite amount of space. It's perception because you're saying it's infinitely large but it's not infinitely small; it's getting smaller and smaller.

B: But no matter what kind of infinity. whether large or small, it's still going to take an infinite amount of time because it's infinity.

R: But not if it's a small infinity.

B: If you're saying it's an infinite amount of steps, or infinite amount of points it has to be an infinite amount of time, you know, but it's not that the whole problem.

R: But the thing is; it's infinite but it's converging into something that's becoming smaller so you're not like extrapolating into infinity, you're converging. This is horrible calculus.

B: You can't converge to infinity.

R: Well, you're converging to zero, you're taking infinity and you're converging to zero.

B: It's like a limit problem.

R: Yah, whatever, yah. And you can converge it to zero.

B: But that's an approximation when you use differentials but that's like the dx or whatever but it's not really an approximation - it's correct; it's only when you make dx approximately equal to delta x.

B: But this is a delta x.

R: Maybe it's that dx thing, I don't know. But it is a finite distance.

B: But that's the whole problem, it's a finite distance with a finite amount of points in that finite distance, you know what I mean.

R: When you look at it right away, there are an infinite amount of points but as he gets closer it's converging to a finite number of points which is zero and when it becomes finite, he can pass him in an infinite amount of time.

B: That's a limit though.

R: So?

B: But a limit is always an approximation.

R: But it approximates the truth. I mean you can't get by with that; it's just what happens. The limit may be an approximation but it's going to happen. I mean, he's running.

B: We know it's going to happen. The point is how is he going to do this?

R: The question isn't: " Is it going to take him 5 seconds or approximately 4.53? "

B: Then, I'm not getting it across right, I'm saying. We now know you can divide one amount of space into an infinite amount of points, right? We also know to traverse an infinite amount of points is going to take an infinite amount of time.

R: But you're perceiving infinity as something that keeps going. It's infinite before you start. But once you start going, you could only chop up this space into so many spaces.

B: You can always chop it.

R: OK, but at this point your limit is going to zero, it's not going to infinity. Because we know he is going to pass him, so we can't expect the turtle to be going to infinity. He's going to have to at some point pass him so if we think of that as the limit at the end, then it's just getting closer. So I guess that's why he can pass him because it becomes it becomes finite.

B: I understand what you're saying but see, you have this man standing here, right? and you're this turtle and you're saying this is a finite distance into an infinite amount of points but we know it's a finite distance so we know it's going to take a real number of seconds, that the problem, you could divide it into an infinite amount of points but it's still four meters or whatever it is.

R: Yah, except that the distance is growing because the turtle is moving up.

B: And the guy is moving up also.

R: The actual distance between them is

B: getting smaller.

R: Yah.

B: Infinitely smaller.

(both laugh)

B: I know he's going to pass him and I know you could divide any distance into an infinite amount of parts, but I also know that it's an infinite distance between them and that's why he could pass him. That's what makes sense, right?

R: Yah, because it's a set limit on how big as where the start that's the most that could be between them.

B: And it's only going to get smaller.

R: And it's within a smaller boundary, it's not going to get any bigger.

B: It can't get bigger because he's running faster. But you could divide an infinite distance into an infinite amount of points.

R: Well, you could infinitely divide it. Yah, I guess you could divide it into an infinite amount of points. (long silence)

Eli: Now do you think you could explain this as the sum of an infinite series? If Achilles is going twice the speed of the Tortoise, initially one unit will separate them, then half a until then a quarter and so on. So, for Achilles to catch up, he will have to traverse all these progressively smaller units. So think of it as the sum of this infinite series.

B: Those are horrible words.

R: That was the worst part of my calculus course.

B: I don't remember much about series, we had a really bad teacher.

R: Remember the P series, one over p to the r or something. A harmonic series or something.

B: When we replace the n with a.

R: Well, yah, when n is one it's a harmonic series. When n is two, whatever, sorry, wrong one.

B: Well, if we look at the one half, one quarter, one eighths... it's just like the problem when you walk home.

R: Doesn't it converge into 2 or something? (both laugh)

(long silence)

B: But the end of the series is going to become zero because it's one over a very large number.

R: Yah, one over infinity is a very large number.

B: Yah right? So you approximate it to zero. I think the series converge but I don't remember.

(long silence)

Eli: If the distance Achilles must traverse to catch up to the Tortoise could be represented as the sum of series with an infinite amount of terms , what does this tell you about this series?

R: I will be 1 plus many smaller terms which will eventually go to zero.

(long silence)

R: It's going to be a number less than 2.

B: So the sum of this infinite series is a finite number.

Eli: Do you see this infinite series as taking the limit of the series one over 2 to the n?

B & R: Yes.

Eli: Does the function ever reach it's limit?

R: Well, series never reach their limit, you could only just look at the limit as it heads towards something but it never actually gets to it so you can't get to infinity.

Eli: So then how does Achilles surpass the Tortoise if he never reaches the limit?

(long silence)

R: The deal is he's not running an infinite series.

Eli: But didn't you say that this series (1+ 1/2 + 1/4)only approaches its limit, so how is it that Achilles doesn't just approach the Tortoise but actually catches up to him.

R: But he is going in a finite place. It becomes so negligible.

B: Like I understand this and I know that Achilles is going to catch up to the Tortoise.

Eli: how do you reconcile the fact that a series will only approach its' limit yet Achilles will actually reach the Tortoise?

B: I can't.

Eli: Take your time and discuss amongst yourselves.

(long wait)

B: I can't.

B: Well, isn't this space so small that it will look like he's at the same space as him?

R: I think the point is that this is negligible and because it is negligible.

B: It becomes finite.

R: It becomes finite.

B: And then he could run it. Didn't we just used to cross it out and make zero?

R: Well, it's a function but not a series; a series just goes on.

R: Well, that's the only way I could rationalize; it is that the distance becomes negligible and in that way it converges.

Eli: But if this series only approaches two and it represents the steps Achilles must take, how does Achilles ever catch up?

R: Maybe relative to his foot size, I don't know.

(long silence)

Eli: Are you still convinced that space is infinitely divisible?

B: So maybe it's not. (laugh)

R: This whole series is assuming that you could go to infinity but again he is running within a finite distance. This is not going to infinity. Even though you could cut it up into infinity it's getting shorter; it's a finite distance.

B: Two is a finite distance is what we are saying. The sum of series is 1.999 with the little line on top.

Eli: Is 1.999 with a line on top equal to 2 or not?

R: It's like one of those questions we did in grade eleven.

B: It has to equal two for our purposes. You can never measure the .0001 that it doesn't equal to, so for us it equals two.

R: The way I learnt in school is that with series you just accept that it can converge. (both laugh)

B: You just accept it.

R: Well, it goes. It converges to two, therefore it's convergent, it's not the point.

B: It's not equal to two; it can't be.

R: The whole idea of adding an infinite amounts of terms is so abstract that I don't think you could say that it converges to a specific number. You can't grasp the whole concept.

B: I have problems grasping most things.

Eli: But if the series doesn't reach 2, how does Achilles ever reach the Tortoise?

B: But for us it does reach two. We pretend it reaches two, it will approximately reach two.

V: I think what he is saying is if there is an infinite amount of points, it doesn't ever actually even converge; it could only tend towards a certain valve. Just like this will never actually converge to 2, it will tend to two; but it never actually gets to two. And if the distance never actually converges then he never actually catches up to him.

B: But we know he does.

R: You see, since it's a one over function, even though it's increasing it's decreasing which is the same thing as the distance between them is decreasing.

B: it's that house problem all over again, it really is. Because you're in your house but you still have that half a thing to go. Like you're there but there is still that half thing you didn't pass. Even if you pass that half a distance, there is still that other little half. We did that in grade 11 in physics with Bovin. I wonder how she explained that though.

R: Probably not well. (both laugh)

So anyway, I think it is possible to rationalize it in that even though you keep adding something you're actually adding a smaller and smaller number and eventually you're going to get to zero.

B: Like a number so small you can't even think about.

Eli: Do you still believe that a limit is an approximation?

B & R: Yes.

R: As it's defined as an approximation. But for the series, the terms eventually become negligible and I think that's OK to do.

B: I think that's OK too.

V: But the problem with your story is that you're not considering the fact that these infinite divisions at some point will become negligible because they are so small that it actually converges it at that point where he is going to be at the same place as him.

Eli: How do you view infinity?

R: Like this black space.

B: Space?

R: That just keeps going.

B: The Universe.

B: Unbounded.

V: Yah, I guess as long as it just continues.

Eli: By converges, do you mean equal;

R: Yes.

Eli: So why does this series never reach 2?

B: It's approximately equal.

Eli: So should you just approximately reach the Tortoise?

R: Well, I think if it were actually possible to reach infinity, he would actually pass.

Eli: Is it possible to reach actual infinity?

R: No.

Eli: So how does he pass him?

R: Well, I guess if you chopped up this area into an infinite amount of space, because it's confined he could actually pass him because even if it's an infinite amount of spaces, you put boundaries.

B: I think the problem is you know he is going to pass him.

R: You see, even though you chop it into an infinite amount of spaces it doesn't really have an infinite amount of points because it is bounded; and since it is not really an infinite amount of points he could pass.

Eli: Is this line that extends indefinitely to the right and to the left infinite?

R & B: Yes.

(I showed her how I could make a 1-1 correspondence between the points in the infinite line and the points in the finite line by:

B: But it doesn't.

R: You see there is less points on this line (shorter one).This is a specific distance, it could only have a certain amount of points.

Eli: (I explain the proof again).

R: If you associate all the points on this line with this line one, you're really using up all the points on this line (longer one). I could see how you're using all the points on the shorter line.

B: Maybe it has to do with the angle?

R: No, if you sweep across it will probably even out.

B: I still don't think there are as many points on this line (shorter line).

R: Well, if we say that we could cut up a line into as many points as we want, then it is true that we could associate every point on this curve to a point on the line. But I don't think that that means that there are exactly as many points.

B: But that doesn't make sense either.

R: Maybe we could say that there are as many points on the curve as on the line but I think that

B: Maybe the points on the points on the line are bigger than the points on the curve.

R: No, but if we just sweep, I don't think that's it.

B: I don't agree but I can't justify it.

(very long silence)

R: You see, this line is only infinity because we could infinitely divide it.

B: But isn't that still infinity?

R: I guess then. If you look at this picture, I guess, there are as many points, I guess, but it just doesn't seem right.

(silence)

Eli: So then how could Achilles overtake the Tortoise if there are an infinite amount of points?

R: Because it is bounded.

B: You see then you could associate any point in space with this one line (infinite) but it doesn't make every point infinite.

R: I'm confused. You see the distance only gets shorter; I still agree with what I said before because I don't think that this drawing proves that this line is infinite because the further you get with the hypotenuse the less significant it becomes. So he could pass him because even though you could divide the distance into an infinite amount of points but that infinity is decreasing as Achilles gets closer and it is bounded by that distance; it could only get smaller and it's not really an infinity so he can get passed him.

B: Yeah!