On the Naimark Extension for a Commutative Positive Operator Valued Measure on a Differential Manifold

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ABSTRACT

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We consider a commutative positive-operator-valued (POV)-measure a which assigns to the Borel-subsets $\mathcal{B}(M)$ of a differential manifold M positive bounded linear operators on $L^2(M, \nu)$, where ν denotes a smooth Borel-measure on M. Assuming that the POV-measure a is informationally equivalent to the canonical projection-valued (PV)-measure for M on $L^2(M, \nu)$ we deduce that the von-Neumann algebra generated by the family $\{a(E) \mid E \in \mathcal{B}(M)\}$ is maximal abelian. The operators forming the POV-measure are therefore found to be point-dependent multiplication-operators. Having thus established the necessary form of a, we perform the Naimark extension of a to a projection-valued measure on an extended Hilbert-space $\mathcal{H}$ of which $L^2(M, \nu)$ is a proper subspace, and discuss the structure of $\mathcal{H}$. Conversely, we recover a POV-measure for M on $L^2(M, \nu)$ in case that we are given a PV-measure on $L^2(T^*M, \Lambda)$.

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CHAPTER 1 INTRODUCTION

The thesis presented here deals with the Naimark extension of a certain family of operators, a so-called positive operator valued (POV)-measure based on a locally compact space, more specifically a differential n-manifold M. (In brief, a POV-measure is an operator-valued map

$$a : \mathfrak{B}(M) \to L(\mathfrak{H})^{+}$$

where $\mathfrak{B}(M)$ denotes the Borel-subsets of M, and $L(\mathfrak{H})^{+}$ denotes the positive linear bounded operators on some separable complex Hilbert space $\mathfrak{H}$). The extension carried out here establishes a means for recovering a projection-valued (PV) measure P (i.e. a mapping $P : \mathfrak{B}(M) \to$ {projection operators on $\mathfrak{H}$}) however living on a larger Hilbert-space H such that $\mathfrak{H}$ is a proper subspace of H, from the POV measure a. The question dealt with is one of determining what the enlarged Hilbert-space H looks like in a special situation, namely when all the operators which establish the POV-measure a commute. An additional restriction of informational equivalence of the POV-measure a with the so-called canonical PV-measure (which acts as multiplication by the characteristic function of the set) is imposed.

The origin of this question lies in standard quantum mechanics. There, one encounters the following situation: Let S be a physical system, say a particle. Then its configuration space is normally given as a differential manifold M which locally looks like R^n for appropriately chosen n. Its states are considered as elements of a Hilbert-space, in our case given as $L^2(M, \nu)$ where ν is a measure on M. One focuses initially on giving a meaning to the statement, "The particle is localised in a region of M given that it is in a particular quantum state". Mathematically one introduces at this point a PV-measure $P : \mathcal{B}(M) \to \mathcal{L}(L^2(M, \nu))^+$. The square of the absolute value of an element $\psi \in L^2(M, \nu)$ at a given point m is normally considered as a probability density and the quantity

$$(\psi, P(E)\psi) = \int_M \psi(m)\overline{(P(E)\psi)(m)}\, d\nu(m)$$

is interpreted as the probability of finding the particle localized in $E \subset M$ in the state ν. From this interpretation one can easily deduce that the family P be positive: the requirement of the P(E)'s to be projections only comes in when imposing von Neumann's repeatability postulate (saying that the outcome of the position measurement should be the same if one performs the same experiment a second time). If one abandons this postulate, one will only arrive at a POV-measure a. This seems to be in accord with physical reality, if one considers eg. a photon where for determining its position the photon has to be destroyed, so that clearly this kind of measurement is not repeatable. Mathematically, life becomes a little bit harder as well, since by replacing the requirement of the localization operators to be projections by the requirement of them to be positive one loses some convenient properties, eg. the spectral theorem cannot be exploited any longer, so that eventually one is forced to answer questions of the type: " Will the operator $\int \lambda\, d\, a(\lambda)$ be self adjoint?", which formerly did not arise. If one still wishes to deal with POV-measures rather than PV-measures, one way out of answering questions of this type is given by extending the underlying Hilbert-space. This is the point where the Naimark-extension comes in.

The material in this thesis is organized as follows: Chapter 2 introduces some generalities on von Neumann - and C* - algebras, which are needed in later chapters (a general reference is [2]) as well as some results on differential manifolds together with a decomposition of $L^2(M, \nu)$ into an orthogonal direct sum (general reference is [4]).

In chapter 3 we define the notions of a POV - and a PV-measure and discuss the consequences of the requirement of informational equivalence of a commutative POV-measure with the canonical PV-measure. As it turns out this requirement leads to maximal abelianness of the von Neumann-algebra generated by the POV-measure a and to the deduction that each one of the operators a(E) acts as a point - dependent multiplication operator. We then perform the Naimark extension of such a POV-measure for an arbitrary locally compact space X taking values in $L(L^2(R^n, \kappa))^+$. The extended Hilbert-space turns out to be isomorphic to a direct integral over certain Hilbert-spaces.

In chapter 4 we then apply the result obtained in chapter 3 to find the Naimark extension of a commutative POV-measure a for a manifold M based on $L^2(M, \nu)$, using the specific decomposition of $L^2(M, \nu)$ obtained in chapter 2. In the most general case, the extended space will be a two-fold direct sum of Hilbert-spaces; in the most simple case, it turns out to be the space of square-integrable functions on the cotangent manifold T*M with respect to a product-measure. For the latter case we as well go the reverse way, namely starting with a PV-measure on $L^2(T^*M, \Lambda)$, we can recover a POV-measure on $L^2(M, \nu)$ under specific requirements on Λ .

CHAPTER 2 MATHEMATICAL PRELIMINARIES

2.1 Hilbert-space, C*-Algebras and von Neumann-Algebras

In what follows $\mathfrak{H}$ will always denote a separable complex Hilbert-space and $L(\mathfrak{H})$ the set of all linear bounded operators on $\mathfrak{H}$ which in a natural manner is an involutive Banach-algebra with the usual operator norm.

Definition 2.1.1 An operator $A \in L(\mathfrak{H})$ is said to be of trace-class iff

$$\|A\|_{tr} = \sum_{n=1}^{\infty} \lambda_n < \infty \tag{1}$$

where $\{\lambda_n\}_{n=1}^{\infty} \subset R_0^+$ is the set of eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ arranged in decreasing order and repeated according to their multiplicity.

The set $T(\mathfrak{H})$ of all trace-class operators on $\mathfrak{H}$ is a linear space which is a Banach-space when endowed with $\|\cdot\|_{tr}$. Moreover, if $\{x_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{H}$, then the sum

$$\operatorname{tr} A = \sum_{n=1}^{\infty} (Ax_n, x_n) \qquad \forall A \in T(\mathfrak{H}) \tag{2}$$

defines a linear functional $\operatorname{tr} : T(\mathfrak{H}) \to C$ on $T(\mathfrak{H})$. Furthermore, $T(\mathfrak{H})$ is a closed ideal of $L(\mathfrak{H})$ and under the pairing

$$\langle A, X \rangle = \operatorname{tr}(A \cdot X) \qquad \forall A \in T(\mathfrak{H}),\ X \in L(\mathfrak{H}) \tag{3}$$

one can identify $L(\mathfrak{H})$ with the dual $T^*(\mathfrak{H})$ of $T(\mathfrak{H})$.

Definition 2.1.2 Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $L(\mathfrak{H})$. Then $\{A_n\}_{n=1}^{\infty}$ is said to converge to $A \in L(\mathfrak{H})$

- in the uniform topology, iff

$$\lim_{n\to\infty} \| A_n - A \| = 0 \tag{4}$$

- in the strong topology, iff

$$\lim_{n\to\infty} \| (A_n - A)x \| = 0 \quad \forall\, x \in \mathfrak{H} \tag{5}$$

- in the weak topology, iff

$$\lim_{n\to\infty} (A_n x, y) = (Ax, y) \quad \forall\, x, y \in \mathfrak{H} \tag{6}$$

For these three topologies, one has the following relation: The uniform topology is finer than the strong topology which is finer than the weak topology.

For a subset $\mathfrak{M}$ of $L(\mathfrak{H})$ the collection of all operators in $L(\mathfrak{H})$ commuting with each element in $\mathfrak{M}$ is called the commutant of $\mathfrak{M}$, denoted by $\mathfrak{M}'$.

Definition 2.1.3 A von Neumann-algebra on $\mathfrak{H}$ is a subalgebra $\mathfrak{M}$ of $L(\mathfrak{H})$ which is invariant under the involution (i.e. a *-subalgebra) and such that

$$\mathfrak{M} = (\mathfrak{M}')' = \mathfrak{M}''$$

The algebra $\mathfrak{M}$ is called abelian, if $\mathfrak{M} \subset \mathfrak{M}'$, it is called maximal abelian, if $\mathfrak{M} = \mathfrak{M}'$. A C*-algebra of operators on $\mathfrak{H}$ is a non-degenerate *-subalgebra of $L(\mathfrak{H})$, which is closed under the uniform topology.

More generally, a C*-algebra $\mathfrak{A}$ is defined to be an involutive Banach-algebra over C for which involution satisfies

$$\|X^*X\| = \|X^*\| \cdot \|X\| \qquad \forall\, X \in \mathfrak{A} .$$

For an arbitrary locally compact space X, the set $C_\infty(X)$ of all continuous functions on X vanishing at ∞ is a commutative C*-algebra under the sup-norm with involution given by complex conjugation.

Trivially, $L(\mathfrak{H})$ is both a C*-algebra, as well as a von Neumann-algebra. Let S denote the closed unit-ball of $\mathfrak{M}$. Then, by the von Neumann density theorem, S is both weakly and strongly closed. Furthermore, the weak closure of any *-algebra $\mathfrak{N}$ with identity coincides with the von Neumann algebra generated by $\mathfrak{N}$.

Let $\mathfrak{A}$ be an abelian C*-algebra of operators on $\mathfrak{H}$ and denote by $\Omega(\mathfrak{A})$ the set of all non-zero homomorphisms of $\mathfrak{A}$ onto C. Then $\Omega(\mathfrak{A})$ is contained in the unit ball S* of the conjugate space $\mathfrak{A}^*$ of $\mathfrak{A}$ and is locally compact with respect to the $\sigma(\mathfrak{A}^*, \mathfrak{A})$ topology.

If one defines for each $X \in \mathfrak{A}$ a function X on $\Omega(\mathfrak{A})$ by setting

$$\hat{X}(\omega) = \omega(X) \qquad \forall\, \omega \in \Omega(\mathfrak{A}) \tag{7}$$

then $\hat{X} : \Omega(\mathfrak{A}) \to C$ is continuous. Furthermore, it can be shown that $\hat{X}$ vanishes at ∞, hence $\hat{X} \in C_\infty(\Omega(\mathfrak{A}))$. Let $F : \mathfrak{A} \to C_\infty(\Omega(\mathfrak{A}))$ denote the mapping $X \to \hat{X}$. Then F is an isometric isomorphism of $\mathfrak{A}$ onto $C_\infty(\Omega(\mathfrak{A}))$ which preserves the *-operation.

Definition 2.1.4 The space $\Omega(\mathfrak{A})$ is called the spectrum of $\mathfrak{A}$ and the mapping $F : \mathfrak{A} \longrightarrow C_\infty(\Omega(\mathfrak{A}))$ is named the Gelfand isomorphism.

Via the Gelfand isomorphism it is thus possible to identify any abelian C^*-algebra of operators on $\mathfrak{H}$ with the C^*-algebra $C_\infty(\Omega)$ for some appropriately chosen locally compact space Ω.

<u>Definition 2.1.5</u> Let $\mathfrak{A}$ be a C^*-algebra. A <u>representation</u> of $\mathfrak{A}$ is a $*$-homomorphism π of $\mathfrak{A}$ into the C^*-algebra $L(\mathfrak{H})$. $\mathfrak{H}$ is called the representation space.

For an arbitrary C^*-algebra $\mathfrak{A}$ one has

<u>Lemma 2.1.1</u> Let $\mathfrak{A}$ be a C^*-algebra and π a representation of $\mathfrak{A}$ on $\mathfrak{H}$. Let $\mathfrak{M}(\pi)$ denote the von Neumann-algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$, then there is a unique linear map $\tilde{\pi}$ of the second conjugate space $\mathfrak{A}^{**}$ of $\mathfrak{A}$ onto $\mathfrak{M}(\pi)$ with the following properties:

(i) The diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\pi"] \arrow[d, "i"] & \mathfrak{M}(\pi) \\ \mathfrak{A}^{**} \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd} \qquad (8)$$

commutes, where i denotes the canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^{**}$.

(ii) $\tilde{\pi}$ is continuous with respect to the $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^{*})$-topology and the σ-weak topology of $\mathfrak{M}(\pi)$ (hence the weak topology of $\mathfrak{M}(\pi)$).

(iii) $\tilde{\pi}$ maps the unit ball S^{**} of $\mathfrak{A}^{**}$ onto the unit ball S of $\mathfrak{M}(\pi)$.

<u>Proof</u> See [2], pp. 121-122 ∞

In case one considers a commutative C*-algebra, which can be identified with $C_\infty(X)$ for some locally compact space X, the diagram (8) becomes:

$$\begin{tikzcd} C_\infty(X) \arrow[r, "\pi"] \arrow[d, "i"] & \mathfrak{M}(\pi) \\ L^\infty(X, \hat{\lambda}) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

where $\hat{\lambda}$ is a basic measure on X. The map $\tilde{\pi}$ then is a homomorphism (see [12] , pp. 127); so that the Gelfand-isomorphism F extends to a map

$$F : \mathfrak{M}(\pi) \to L^\infty(X, \lambda).$$

For completing the list of notions and results needed from the theory of operator algebras, we as well need to know what it means for a linear map $\phi : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras, to be completely positive. For this purpose we define the set $M_n(\mathfrak{A})$ to be the set of all n × n-matrices $X = [X_{ij}]_{i,j=1,..,n}$ with entries in $\mathfrak{A}$. $M_n(\mathfrak{A})$ is once again a C*-algebra and as usual an element $X \in M_n(\mathfrak{A})$ will be called positive, if there is an element $B \in M_n(\mathfrak{A})$ such that

$$X = B^*B \tag{9}$$

Recall as well that a linear map ϕ of a C*-algebra $\mathfrak{A}$ into a C*-algebra $\mathfrak{B}$ is called positive, if ϕ maps $\mathfrak{A}^+$ into $\mathfrak{B}^+$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and ϕ a linear map of $\mathfrak{A}$ into $\mathfrak{B}$. For each $n \in N$ we define $\phi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ by

$$\phi_n(A) = [\phi(A_{ij})]_{i,j=1,..,n} \tag{10}$$

<u>Definition 2.1.6</u> $\phi : \mathfrak{A} \to \mathfrak{B}$ is said to be n-positive, if ϕ_n is positive. If ϕ is n-positive for each $n \in N$, then ϕ is called <u>completely positive</u>.

If $\mathfrak{A}$ is an abelian C*-algebra, then one has

<u>Proposition 2.1.2</u> Any positive linear map ϕ of $\mathfrak{A}$ into another C*-algebra $\mathfrak{B}$ is completely positive.

<u>Proof</u> See [2], pp. 199 - 200 ∞

We will use proposition 2.1.2 in the following form:

<u>Proposition 2.1.3</u> Any positive linear map ϕ of $C_\infty(X)$ into $L(\mathfrak{H})$ is completely positive.

<u>2.2 Some Generalities on Differential Manifolds</u>

Let M denote a topological space, which is Hausdorff and satisfies the second axiom of countability.

<u>Definition 2.2.1</u> A triple (U, γ, V) where $U \subset M$ is an open subset of M, V is an open subset of R^n for some n, and $\gamma : U \to V$ is a homomorphism, is called a <u>(local) chart</u> of M.

An <u>atlas</u> A of M is a family of charts $\{(U_i, \gamma_i, V_i) \mid i \in N\}$ such that

$$\bigcup_{i \in N} U_i = M$$

The pair (M, A) is called a <u>topological manifold</u>. It is called an n-manifold, iff $\forall i \in N \quad V_i \subset R^n$.

Definition 2.2.2 Let (M, A) be an n-manifold. The pair (M, A) is called a differential n-manifold, iff the mappings

$$\gamma_i \circ \gamma_j^{-1} : \gamma_j(U_i \cap U_j) \to \gamma_i(U_i \cap U_j)$$

are C^∞-mappings whenever $U_i \cap U_j \neq \emptyset$.

Without loss of generality one can assume the atlas A to be maximal, that is, A is not contained in any other atlas as a proper subset. Thus, in what follows we shall use the symbol M to specify a differential n-manifold without regard to the atlas. Taking into consideration that M is a topological manifold, one has

Lemma 2.2.1 M is paracompact.

Proof See [3] ∞

Consequently, as a topological space, M is locally compact. Let $\mathfrak{B}(M)$ denote the σ-algebra of all Borel-subsets of M and let ν be a Borel-measure on M.

Definition 2.2.3 A Borel-measure ν on M is called smooth, if for each local chart (U, γ, V) of M there is a strictly positive, infinitely often differentiable function $k : V \to R^+$ such that

$$\nu \circ \gamma^{-1} = k \cdot \lambda^n \quad \text{on } V$$

where λ^n is the Lebesgue-measure on R^n and $k \cdot \lambda^n(G) = \int_G k(x)\, d\lambda^n(x)$ for $G \subset V$.

For a differential n-manifold one has

<u>Theorem 2.2.2</u> a) On any differential n-manifold there exist smooth Borel-measures.

b) All smooth Borel-measures are σ-finite.

c) Any two smooth Borel-measures ν and $\tilde{\nu}$ on M are equivalent.

d) If ν is a smooth Borel-measure on M and ϕ is a diffeomorphism on M, then $\nu \circ \phi^{-1}$ is a smooth Borel-measure on M.

e) If ν is a smooth Borel-measure on M, then any two continuous functions, which are equal almost everywhere, are equal everywhere.

<u>Proof</u> See [4] ∞

So that the definition 2.2.3 is non-void.

For a fixed Borel-measure ν on M we set

$$L^2(M, \nu) = \{\psi : M \to C \mid \int_M \psi(m)\, \overline{\psi(m)}\, d\nu(m) < \infty \}$$

$L^2(M, \nu)$ is a Hilbert-space in a natural way. Considering a local chart (U, γ, V) of M and restricting ν to $\mathcal{B}(M) \cap U$ one has

$$L^2(U, \nu|_U) = \{\psi : U \to C \mid \int_{\tilde{U}} \psi(m)\, \overline{\psi(m)}\, d\nu|_U(m) < \infty \}$$

$$\underline{\sim} \{\psi \circ \gamma^{-1} : V \to C \mid \int_V \psi(\gamma^{-1}(v))\, \overline{\psi(\gamma^{-1}(v))}\, d\nu|_U (\gamma^{-1} v) < \infty \}$$

$$= \{\hat{\psi} : V \to C \mid \int_V \hat{\psi}(v)\, \overline{\hat{\psi}(v)}\, k(v)\, d\lambda^n(v) < \infty \}$$

$$= L^2(V, k\lambda^n)$$

If one looks upon k as a function on R^n which has support in V, the isometric isomorphism between $L^2(U, \nu|_U)$ and $L^2(V, k\lambda^n)$ yields

$$L^2(U, \nu|_U) \simeq L^2(R^n, k\lambda^n). \tag{11}$$

By means of the atlas for M, one can construct a cover of M by disjoint Borel-sets in the following way

$$\begin{aligned} \hat{U}_1 &= U_1 \\ \hat{U}_k &= U_k \setminus \bigcup_{i=1}^{k-1} \hat{U}_i \quad \text{for} \quad k \geq 2 . \end{aligned} \tag{12}$$

Exploiting the smoothness of ν, once can then represent $L^2(M, \nu)$ as follows:

$$L^2(M, \nu) = \bigoplus_{k=1}^{\infty} L^2(\hat{U}_k, \nu_k) \tag{13}$$

where $\nu_k = \nu|\mathcal{B}(M) \cap \hat{U}_k$.

By the same argument as used for deriving (11) we thus find

$$L^2(M, \nu) \simeq \bigoplus_{k=1}^{\infty} L^2(R^n, \kappa_k \lambda^n) \tag{14}$$

where

$$\kappa_k : R^N \to R_0^+, \ \text{supp}\ \kappa_k \subset \gamma_k(U_k) \subset R^n . \tag{15}$$

(Note that $\kappa_k \in C^\infty(\gamma_k(U_k))$.) and

$$\nu_k \circ \gamma_k^{-1} = \kappa_k \cdot \lambda^n \qquad \text{on} \quad \gamma_k(U_k) \tag{16}$$

for all $k \in N$.

Note however that the supports of the functions κ_k in general are not disjoint, so that the spaces $L^2(R^n, \kappa_k \lambda^n)$ in the decomposition (14) are not necessarily orthogonal as subspaces of $L^2(R^n, \lambda^n)$, whereas this is the case in the decomposition (13). In case the supports are disjoint, (14) can be simplified to

$$L^2(M, \nu) \simeq L^2(R^n, \kappa \lambda^n) \tag{17}$$

where

$$\kappa = \sum_{k=1}^{\infty} \chi_{\gamma_k(\hat{U}_k)} \kappa_k \tag{18}$$

In the sequel we will use the representation (14) rather than $L^2(M, \nu)$ itself.

CHAPTER 3 PV - AND POV-MEASURES

3.1 Localization in Quantum - Mechanics

Consider a physical system S whose configuration space is given by some locally compact topological space X and suppose that the system is a non-relativistic quantum system.

One assumes then, that the set of all propositions of the type

"S has the property A"

generates an orthocomplemented σ-lattice $\mathcal{L}(S)$ and that every state of S, leads to a unique function

$$p : \mathcal{L}(S) \to [0,1]$$

such that p(A) can be interpreted as the probability for obtaining the outcome "S has the property A".

One then requires p to have the following properties:

a) $p(\phi) = 0$, $p(X) = 1$

b) $p(\bigcup_{i\in I} A_i) = \sum_{i\in I} p(A_i)$

whenever $A_i \leq A_k^{\perp}$ $i \neq k$, where $\leq$ denotes the partial ordering in $\mathcal{L}(S)$ and $\perp$ stands for the operation of taking the ortho-complement.

The function p is called a <u>state of $\mathcal{L}(S)$</u>.

An observable consequently is defined to be a σ-homomorphism from $\mathcal{B}(R)$ into $\mathcal{L}(S)$, (see [5]) which in quantum mechanics, where $\mathcal{L}(S)$ is realized as the orthocomplemented σ-lattice $\mathcal{L}(\mathfrak{H})$ of projection operators on a Hilbert space $\mathfrak{H}$, can be represented as a self-adjoint linear operator on $\mathfrak{H}$. The states further on are identified with positive trace-class-

-operators of unit trace, so that the probability that a measurement of the observable A in the state T yields a result in $\Delta \in \mathcal{B}(R)$ is given by

$$\mathrm{tr}(T \circ E^A(\Delta))$$

where $E^A(\Delta)$ is the element corresponding to Δ of the spectral-family of A.

If $x \in \mathfrak{H}$, $x \neq 0$, then the operator $T_x : \mathfrak{H} \to \mathfrak{H}$

$$T_x y = \frac{1}{\|x\|^2} (y, x)\, x$$

is a positive trace-class-operator of unit trace, and one has

$$\mathrm{tr}(T_x \circ E^A(\Delta)) = \frac{1}{\|x\|} \cdot (E^A(\Delta)x, x).$$

By the spectral-theorem for bounded self-adjoint operators one can therefore conclude that the quantity

$$\frac{1}{\|x\|^2} (Ax, x)$$

yields the expectation value of the distribution corresponding to A in T_x.

The operator T_x is called a <u>pure state</u> of S.

Note that the set of all positive trace-class-operators of unit trace forms a convex set, with the pure states being its extreme points. We can thus identify it with the closed unit ball of $\mathfrak{H}$, if necessary.

Our concern for the rest of this exposition will be the observation of the position of S on X. For doing so, we assume a sufficiently large number of regions of X to be associated with physical observables, "position observables". $\mathcal{B}(X)$, the σ-algebra of Borel-subsets of X, will give us this sufficient amount.

From what has been said above, each Borel subset B of X will then give rise to a self-adjoint operator E(B) on $\mathfrak{H}$. For a normalized vector $x \in \mathfrak{H}$ the quantity

$$\langle E(B)x, x\rangle = \mu_{T_x}(B)$$

will then consequently be interpreted as the probability of finding S in the state T_x localized in B.

This forces E(B) to be a bounded positive operator on $\mathfrak{H}$ for any $B \in \mathfrak{B}(X)$. For fixed T_x one assumes the map

$$\mu_{T_x} : \mathfrak{B}(X) \to R^+$$

to be a probability measure on $\mathfrak{B}(X)$.

Therefore one is led to impose the following properties on E(B):

a) $\|E(B)\| < \infty$, $E(B) \geq 0 \quad \forall B \in \mathfrak{B}(X)$

b) $E(\phi) = 0$, $E(X) = 1$

c) $E(\bigcup_{i=1}^{\infty} B_i) = \lim_{N\to\infty} \sum_{i=1}^{N} E(B_i)$ strongly for mutually disjoint sets.

From what has been said so far, on sees that $\{E(B) \mid B \in \mathfrak{B}(X)\}$ forms a positive-operator valued (POV) measure (the justification of this term will be given in the next chapter).

From repeatability arguments for measurements, one normally imposes the further requirement that the possible outcome of an experiment related to position measurement be 0 or 1, i.e. each of the operators E(B) be a projection operator.

This leads to adding as a fourth property of E(B)

d) $E(B) \cdot E(B') = E(B \cap B')$.

A family $\{E(B) \mid B \in \mathcal{B}(X)\}$ satisfying (a) - (d) is called a projection-valued (PV) measure.

In the following part of this chapter, we are going to derive the justification of the term "measure" for these families of operators and study their interrelation.

3.2 PV-Measures and POV-Measures and their Relation

Let once again X be some locally compact space, $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, and $\mathfrak{H}$ an arbitrary separable Hilbert-space. Consider a family of operators $\{E(B) \mid B \in \mathcal{B}(X)\} \subset L(\mathfrak{H})$ which has the following properties

$$0 \leq E(B) \leq id_{\mathfrak{H}}$$

$$E(\phi) = 0, \; E(X) = id_{\mathfrak{H}} \tag{1}$$

$$E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i) \quad \text{for } B_i \cap B_k = \phi, \; i \neq k$$

with the summation on the right hand side being understood as a weak limit.

For a fixed normalized vector $x \in \mathfrak{H}$ we will determine the properties of the mapping

$$\mu_x : \mathcal{B}(X) \to R_0^+$$

defined by

$$\mu_x(B) := (E(B)x, x) \tag{2}$$

First of all, for any $B \in \mathcal{B}(X)$ one has that

$$0 \leq \mu_x(B) \leq 1$$

and secondly,

$$\mu_x(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu_x(B_i) \qquad \text{for} \quad B_i \cap B_k = \phi \,, \quad i \neq k$$

from the definition of the family $\{E(B) \mid B \in \mathfrak{B}(X)\}$.
Furthermore, for $B = X$ one has

$$\mu_x(X) = 1$$

Hence the mapping $\mu_x : \mathfrak{B}(X) \to [0\,,1]$ is a probability measure on $\mathfrak{B}(X)$.
For a non-normalized vector y one will by the same reasoning always find a finite measure.
Making use of the polarization identity in a complex Hilbert-space, one will thus find a complex "finite" measure $\mu_{x,y}$ for any two non-zero vectors by setting

$$\mu_{x,y}(B) = \frac{1}{4}\{\mu_{x+y}(B) - \mu_{x-y}(B) + i\,\mu_{x+iy}(B) - i\,\mu_{x-iy}(B)\} \tag{3}$$

Because of this relation the following definition makes sense.

Definition 3.2.1. Let X be a locally compact space, $\mathfrak{H}$ a complex separable Hilbert-space, and $\{E(B) \mid B \in \mathfrak{B}(X)\}$ a family of positive operators on $\mathfrak{H}$ satifying (1).
Then $\{E(B) \mid B \in \mathfrak{B}(X)\}$ is called a normalized positive-operator valued (POV)-measure for $\mathfrak{B}(X)$ on $\mathfrak{H}$.
If furthermore, for all B and B' in $\mathfrak{B}(X)$ the relation

$$E(B)\,E(B') = E(B')\,E(B) \tag{4}$$

is valid, $\{E(B) \mid B \in \mathfrak{B}(X)\}$ is said to be a commutative, normalized POV-measure.
A commutative normalized POV-measure is a projection-valued (PV)-measure,

if,

$$E(B)\,E(B') = E(B \cap B') \tag{5}$$

holds for all B, $B' \in \mathfrak{B}(X)$.

By the von Neumann density theorem the weak limit occuring in (1) can be replaced by the strong limit of the sum. Hence we are back to the situation described in chapter 3.1.

A PV-measure for $\mathfrak{B}(X)$ on $\mathfrak{H}$ gives rise to a POV-measure for $\mathfrak{B}(X)$ in a natural way:

Let $H \subset \mathfrak{H}$ be a proper subspace of $\mathfrak{H}$ and let $\mathbb{P}$ denote the orthogonal projection of $\mathfrak{H}$ onto H. For each $B \in \mathfrak{B}(X)$ set

$$a(B) := \mathbb{P}\,E(B)\,\mathbb{P} \tag{6}$$

Then one has

$$a(X) = \mathbb{P}\,E(X)\,\mathbb{P} = \mathbb{P} = \mathrm{id}_H$$
$$0 \le a(B) \le \mathbb{P} = \mathrm{id}_H$$

and

$$a\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \mathbb{P}\,E\Big(\bigcup_{i=1}^{\infty} B_i\Big)\mathbb{P} = \mathbb{P}\Big(\sum_{i=1}^{\infty} E(B_i)\Big)\mathbb{P}$$
$$= \sum_{i=1}^{\infty} \mathbb{P}\,E(B_i)\,\mathbb{P}$$
$$= \sum_{i=1}^{\infty} a(B_i)$$

whenever $B_i \cap B_k = \phi$. Once again the sum is to be understood as the weak limit of $\sum_{i=1}^{n} a(B_i)$.

As, in general, the projection $\mathbb{P}$ fails to commute with all of the $E(B)$'s, the family $\{a(B) \mid B \in \mathcal{B}(X)\}$ turns out to be a normalized POV-measure for $\mathcal{B}(X)$ on H.

On the other hand, starting out with a normalized POV-measure a for $\mathcal{B}(X)$ on some Hilbert space $\mathfrak{H}$, the Naimark extension theorem [6] ensures the existence of an enlarged Hilbert space $\hat{H}$ into which $\mathfrak{H}$ can be isometrically embedded, and the existence of a PV-measure P on $\hat{H}$ for $\mathcal{B}(X)$, such that the composition of the projection $P : \hat{H} \to \mathfrak{H}$ with the PV-measure on $\hat{H}$ gives back the original POV-measure on $\mathfrak{H}$.

Furthermore, $\hat{H}$ can be chosen to be minimal in the sense that $\hat{H}$ is spanned by the set $\{P(B)x \mid B \in \mathcal{B}(X),\ x \in \mathfrak{H}\}$.

The main objective of this treatise will be to exhibit what the Naimark extension of a commutative normalized POV-measure for $\mathcal{B}(M)$ on $L^2(M, \nu)$ looks like (for the notation see chapter 2). For the time being however the additional structure of a manifold can be discarded. Only the Borel structure is needed.

Consequently, we are going to restrict our investigation to normalized, commutative POV-measures, which will be denoted by $\{a(E) \mid E \in \mathcal{B}(X)\}$ for the rest of this chapter.

For auxiliary purposes we shall as well need a PV-measure on $\mathfrak{H}$, denoted by $\{P(E) \mid E \in \mathcal{B}(X)\}$.

Let A(P) denote the von-Neumann-algebra generated by this PV-measure and assume that A(P) is maximal abelian. By maximal abelianness of A(P) it follows that there is a unitary map

$$U : \mathfrak{H} \to L^2(X, \lambda) \tag{7}$$

where λ is a σ-finite Borel-measure on X such that for all $B \in \mathcal{B}(X)$

$$(U\, P(B)\, U^{-1}\, \psi)(x) = \chi_B(x) \cdot \psi(x) \tag{8}$$

for all $\psi \in L^2(X, \lambda)$, where χ_B denotes the charateristic function of the set B. (see e.g. [7]). Therefore, we will from now on assume that $\mathfrak{H}$ is realized as $L^2(X, \lambda)$ and that P(B) acts as multiplication by χ_B. The above defined PV-measure P is called the <u>canonical PV-measure</u> for $\mathfrak{B}(X)$ on $L^2(X, \lambda)$.

We wish furthermore to consider only POV-measures which carry the same "amount" of information as the canonical PV-measure. This additional property is mathematically incorporated in the following definition (stated for arbitrary POV-measures):

<u>Definition 3.2.2.</u> Let $\{a(B) \mid B \in \mathfrak{B}(X)\}$ and $\{\hat{a}(B) \mid B \in \mathfrak{B}(X)\}$ be POV-measures for $\mathfrak{B}(X)$ on $\mathfrak{H}$. Then a and $\hat{a}$ are said to be <u>informationally equivalent</u> iff for $\rho \in T(\mathfrak{H})$

$$\begin{array}{c} \mathrm{tr}(a(B)\rho) = 0 \qquad \forall\, B \in \mathfrak{B}(X) \\ \Longleftrightarrow \\ \mathrm{tr}(\hat{a}(B)\rho) = 0 \qquad \forall\, B \in \mathfrak{B}(X) \end{array} \tag{10}$$

Following definition 3.2.2. we require the POV-measure $\{a(B) \mid B \in \mathfrak{B}(X)\}$ for $\mathfrak{B}(X)$ on $L^2(X, \lambda)$ to be informationally equivalent to the canonical PV-measure for $\mathfrak{B}(X)$. An immediate consequence of this requirement is

<u>Lemma 3.2.1.</u> The von Neumann-algebra $A(a)$ generated by the family $\{a(B) \mid B \in \mathfrak{B}(X)\}$ is maximal abelian.

<u>Proof</u> . Let T*(a) denote the norm-closed linear span of the set $\{a(B) \mid B \in \mathfrak{B}(X)\}$ and T*(P) the norm-closed linear span of $\{P(B) \mid B \in \mathfrak{B}(X)\}$, their elements being considered as linear functionals on $T(L^2(X, \lambda))$. Then T*(a) regarded as a sub-set of $L(L^2(X, \lambda))$ is a

subset of $A(a)$, and $T^*(P) = A(P) = A(P)' = A(P)''$. Let $\rho \in T(L^2(X, \lambda))$ be such that for all $f \in T^*(a)$

$$\langle \rho , f \rangle = 0$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between $T(L^2(X, \lambda))$ and its dual. But then one has

$$\langle \rho, a(B) \rangle = 0$$

for all $B \in \mathcal{B}(X)$.

The latter however by informational equivalence (10) of a and P is equivalent to

$$\langle \rho , P(B) \rangle = 0$$

for all $B \in \mathcal{B}(X)$.

Hence one finds that for all $g \in T^*(P) = A(P)$,

$$\langle \rho , g \rangle = 0$$

For a closed sub-space $B \subset L(L^2(X, \lambda))$ define $B^\perp$ to be the annihilator of B in $L(L^2(X, \lambda))^*$ and ${}^\perp B = B^\perp \cap T(L^2(X, \lambda))$. Thus the equations above lead to

$${}^\perp T^*(a) \subset {}^\perp T^*(P) = {}^\perp A(P)$$

Exploiting the fact that $({}^\perp T^*(a))^\perp = T^*(a)$ and $({}^\perp A(P))^\perp = A(P)$ (see e.g. [14]) we arrive at

$$T^*(a) \supset A(P)$$

so that

$$A(a) \supset A(P) \ .$$

$A(a)$ is a commutative von Neumann algebra which is not equal to $L(L^2(X, \lambda))$. As $A(P)$ is maximal abelian, one thus concludes that $A(P) = A(a)$ which establishes the result. ∎

Let $C_\infty(X)$ denote the C*-algebra of all continuous functions on X vanishing at infinity and let $A_c(a)$ denote the C*-algebra generated by $\{\hat{a}(f) = \int_X f(x)\, da(x) \mid f \in C_\infty(X)\}$. The algebra $A_c(a)$ is clearly commutative. One then has

Lemma 3.2.2. The von Neumann-algebra $A(a)$ coincides with the weak closure of $A_c(a)$, i.e. $A(a) = A_c(a)''$, and $A_c(a)$ is contained densely in $A(a)$ in the weak operator topology.

Proof see [8] lemma 6. ∎

The most important result of this chapter is contained in the following proposition:

Proposition 3.2.3. Let a be informationally equivalent to the canonical PV measure for $\mathcal{B}(X)$. Then for each $B \in \mathcal{B}(X)$ the operator $a(B)$ is given as a multiplication operator on $L^2(X, \lambda)$

$$(a(B)\psi)(x) = \mu_x(B)\psi(x) \qquad \forall\, x \in X \tag{11}$$

where for fixed $x \in X$, $\mu_x : \mathcal{B}(X) \to [0, 1]$ is a probability measure.

Proof Consider the C*-algebra $A_c(a)$ introduced above. Then by the Gelfand isomorphism we have a mapping

$$F : A_c(a) \to C_\infty(X)$$

which can be extended to a bijective isometry

$$F : A(a) \to L^{\infty}(X, \hat{\lambda})$$

by the weak deneness of $A_c(a)$ in $A(a)$ (Lemma 2.1.1). ($\hat{\lambda}$ is a basic measure on X.)

For $y \in X$ define a mapping μ_y of the abelian C*-algebra $C_\infty(X)$ into $\mathbb{C}$ by

$$\mu_y(f) = [F(\hat{a}(f))](y)$$

By linearity of $\hat{a} : C_\infty(X) \to A_c(a)$, μ_y is linear and for $f \geq 0$ one observes that by the definition of $\hat{a}(f), \hat{a}(f) \geq 0$, so that μ_y is positive.

Let $\{f_n\} \subset C_\infty(X)$ be an increasing sequence converging pointwise to $1 : f_n \nearrow 1$, then

$$\mu_y(f_n) \nearrow 1 .$$

Therefore μ_y is a finite Radon-measure and is furthermore normalized.

Let $B \in \mathcal{B}(X)$ be a fixed Borel set, then

$$a(B) = \int_X \chi_B(x)\, da(x)$$

and by lemma 3.2.2 there is a sequence $\{f_n\} \subset C_\infty(X)$ such that $f_n \nearrow \chi_B$ and consequently

$$\hat{a}(f_n) \nearrow a(B) \qquad \text{weakly .}$$

By the dominated convergence theorem (see e.g. [9]) it follows that

$$\mu_y(f_n) \nearrow \mu_y(\chi_B) =: \mu_y(B)$$

for each $y \in X$.

Thus for each $\psi \in L^2(X, \hat{\lambda})$ one finds that

$$(a(B)\psi)(x) = \mu_x(B) \cdot \psi(x) \qquad \forall x \in X$$

As furthermore the Borel-measure λ lies in the same measure class as $\hat{\lambda}$ we can conclude that

$$(a(B)\psi)(x) = \mu_x(B)\, \psi(x) \qquad \forall x \in X \ , \quad \psi \in L^2(X, \lambda)$$

The measure μ_x now is a probability measure obtained from the Radon measure μ_x .

Altogether we have shown that by assuming the commutative normalized POV-measure a to be informationally equivalent to the canonical PV-measure P for $\mathcal{B}(X)$, it is given as multiplication operators on $L^2(X, \lambda)$. For a POV-measure of this type it is quite easy to deduce, what its Naimark extension looks like.

As our main concern for the next chapter will be to find the Naimark extension of a commutative, normalized POV-measure on a manifold we shall assume that we are given an isometric isomorphism Γ between $L^2(X, \lambda)$ and $L^2(R^n, \kappa)$ where κ is a Borel measure on R^n which is induced by a homeomorphism $\gamma : X \to R^n$.

That is, we have a one-to-one correspondence

$$\hat{\psi} \in L^2(X, \lambda) \to \psi \in L^2(R^n, \kappa)$$

given by (12)

$$\psi(x) = (\Gamma \hat{\psi})(x) = \hat{\psi}(\gamma^{-1} x) \qquad \forall x \in R^n .$$

The commutative normalized POV-measure $\{a(B) \mid B \in \mathcal{B}(X)\}$ then gives rise to a commutative normalized POV-measure $\{\tilde{a}(B) \mid B \in \mathcal{B}(X)\}$ on

$L^2(R^n, \kappa)$ by

$$
\begin{aligned}
(\tilde{a}(B)\psi)(x) &:= (\Gamma a(B) \Gamma^{-1} \psi)(x) \\
&= \mu_{\gamma^{-1}(x)}(B)\, \psi(x) \qquad (13) \\
&= \rho_x(B)\, \psi(x)
\end{aligned}
$$

for $x \in R^n$, where for any $x \in R^n$, ρ_x is a probability measure on $\mathcal{B}(X)$ and one has

<u>Proposition 3.2.4.</u> Let X be a locally compact topological space and let $a : \mathcal{B}(X) \to L(L^2(R^n, \kappa))^+$ be a commutative normalized POV-measure given as

$$(a(B)\psi)(x) = \rho_x(B)\, \psi(x)$$

for $\psi \in L^2(R^n, \kappa)$ and all $x \in R^n$, where $\rho_x : \mathcal{B}(X) \to [0, 1]$ is a probability measure for each $x \in R^n$. Then there is a PV-measure $P : \mathcal{B}(X) \to L(H)^+$ where

$$H = \int_{R^n} L^2(X, \rho_y) \otimes C \; d\kappa(y) \qquad (14)$$

and a projection $\mathbb{P} : H \to L^2(R^n, \kappa)$ such that for all $B \in \mathcal{B}(X)$

$$a(B) = \mathbb{P}\, P(B)\, \mathbb{P} \; .$$

<u>Proof</u> Let $\mathfrak{A}$ denote the C*-algebra of all bounded λ-measurable functions from X into C.

Define a mapping $\mu : \mathfrak{A} \to L(L^2(R^n, \kappa))$ by

$$(\mu(f)\psi, \psi) := \int_{R^n} \int_X f(x)\, d\rho_y(x)\; |\psi(y)|^2 \; d\kappa(y)$$

for $f \in \mathfrak{A}$ and $\psi \in L^2(R^n, \kappa)$.

If $f \in \mathfrak{A}^+$, $(\mu(f)\psi, \psi) \geq 0$ for any $\psi \in L^2(R^n, \kappa)$, so that $\mu : \mathfrak{A}^+ \to L(L^2(R^n, \kappa))^+$, and μ is clearly linear, hence μ is completely positive by proposition 2.1.3'. Let $\mathfrak{A} \odot L^2(R^n, \kappa)$ denote the algebraic tensor-product of $\mathfrak{A}$ and $L^2(R^n, \kappa)$ and define a mapping

$$\langle\ ,\ \rangle : (\mathfrak{A} \odot L^2(R^n, \kappa)) \times (\mathfrak{A} \odot L^2(R^n, \kappa)) \to C$$

by

$$\langle f \odot \psi,\ g \odot \phi \rangle := (\mu(g^*f)\psi, \phi)$$

for $f \odot \psi, g \odot \phi \in \mathfrak{A} \odot L^2(R^n, \kappa)$. (Observe that the right hand side is well-defined by applying the polarization identity (see (3) of this chapter) in $L^2(R^n, \kappa)$ to the defining relation of μ above and $g^*(x) = \overline{g(x)}$) As μ is completely positive, $\langle\ ,\ \rangle$ extends to a positive sesquilinear form on $\mathfrak{A} \odot L^2(R^n, \kappa)$

Let

$$N := \{\xi \in \mathfrak{A} \odot L^2(R^n, \kappa) \mid \langle \xi, \xi \rangle = 0\}$$

and set

$$K := \overline{\mathfrak{A} \odot L^2(R^n, \kappa)/N}^{\langle\ ,\ \rangle}$$

K is a Hilbert-space in a natural way (see proof of theorem 3.6. chapter IV in [2])

Define a *-representation β of $\mathfrak{A}$ in $L(K)$ via

$$\beta(f)(g \odot \psi) := f \cdot g \odot \psi \qquad \forall\, f \in \mathfrak{A},\ g \odot \psi \in K.$$

In fact, for $f_1, f_2 \in \mathfrak{A}$ one has

$$\begin{aligned}\beta(f_1 f_2)(g \odot \psi) &= f_1 f_2\, g \odot \psi \\ &= \beta(f_1)(f_2\, g \odot \psi)\end{aligned}$$

$$= \beta(f_1)(\beta(f_2)(g \otimes \psi))$$

$$= (\beta(f_1)\, \beta(f_2))(g \otimes \psi)$$

$$\begin{aligned}
\langle \beta(f)^*(g \otimes \psi), h \otimes \phi \rangle &= \langle g \otimes \psi, \beta(f)(h \otimes \phi) \rangle \\
&= \langle g \otimes \psi, fh \otimes \phi \rangle \\
&= (\mu((fh)^* g)\psi , \phi) \\
&= (\mu(h^* f^* g)\psi , \phi) \\
&= \langle f^* g \otimes \psi, h \otimes \phi \rangle \\
&= \langle \beta(f^*)(g \otimes \psi), h \otimes \phi \rangle
\end{aligned}$$

Furthermore, let $\{f_n\} \subset \mathfrak{A}$ be a sequence converging to $f \in \mathfrak{A}$, then

$$\langle \beta(f_n)(g \otimes \phi), h \otimes \psi \rangle = \int_{R^n} \int_X h(x)\, f_n(x)\, g(x)\, d\rho_y(x)\ \phi(y)\, \overline{\psi(y)}\, d\kappa(y)$$

$$\xrightarrow[n\to\infty]{} \int_{R^n} \int_X h(x)\, f(x)\, g(x)\ d\rho_y(x)\, \phi(y)\, \overline{\psi(y)}\, d\kappa(y)$$

$$= \langle \beta(f)(g \otimes \phi), h \otimes \psi \rangle$$

by Lebesgue's dominated convergence theorem.

We now embed $L^2(R^n, \kappa)$ into K by setting

$$\gamma(\psi) := 1 \otimes \psi \qquad \forall\, \psi \in L^2(R^n, \kappa)$$

where $1(x) = 1$ for all $x \in X$.

$\{1 \otimes \psi \mid \psi \in L^2(R^n, \kappa)\}$ is a proper subspace of K and one has

$$\| 1 \otimes \psi \|_K^2 = \int_{R^n} \int_X d\rho_y(x) \quad |\psi(y)|^2\, d\kappa(y)$$

$$= \int_{R^n} |\psi(y)|^2\, d\kappa(y)$$

$$= \|\psi\|^2_{L^2(R^n,\kappa)}$$

as ρ_y is a probability measure for each $y \in R^n$. Hence γ is an isometric isomorphism between $L^2(R^n,\kappa)$ and the subspace $1 \otimes L^2(R^n,\kappa)$ of K. Moreover, for each $f \in \mathfrak{A}$, and $\phi, \psi \in L^2(R^n,\kappa)$ one has

$$\begin{aligned}(\gamma^* \beta(f)\gamma\,\psi,\phi) &= \langle \beta(f)\gamma\,\psi, \gamma\,\phi\rangle \\ &= \langle \beta(f)(1 \otimes \psi), 1 \otimes \phi\rangle \\ &= \langle f \otimes \psi, 1 \otimes \phi\rangle \\ &= (\mu(f)\psi, \phi)\end{aligned}$$

so that

$$\gamma^*\beta(f)\gamma = \mu(f) \qquad \forall f \in \mathfrak{A}\ .$$

Define a mapping $P : \mathcal{B}(X) \to L(K)$ by

$$P(B) = \beta(\chi_B) \qquad \forall B \in \mathcal{B}(X)$$

where χ_B denotes the characteristic function of the set B.

Then

$$\begin{aligned}&P(X) = 1\ ,\ P(\emptyset) = 0 \\ &P(B \cap B') = \beta(\chi_{B \cap B'}) = \beta(\chi_B)\ \beta(\chi_{B'}) = P(B)\,P(B') \\ &P(B)^* = \beta(\chi_B)^* = \beta(\chi_B) = P(B)\end{aligned}$$

and

$$\begin{aligned}P\left(\bigcup_{i=1}^{\infty} B_i\right) = \beta\left(\chi_{\bigcup_{i=1}^{\infty} B_i}\right) &= \lim_{n\to\infty} \beta\left(\chi_{\bigcup_{i=1}^{n} B_i}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)\end{aligned}$$

$$= \sum_{i=1}^{\infty} P(B_i) \quad \text{weakly}$$

for $B_i \cap B_k = \emptyset$, $i \neq k$.

Thus the family $\{P(B) \mid B \in \mathcal{B}(X)\}$ is a PV-measure for $\mathcal{B}(X)$ on K.

Furthermore, $\gamma^* P(B)\gamma = \mu(\chi_B)$, but

$$(\mu(\chi_B)\psi, \phi) = \int_{R^n} \int_X \chi_B(x)\, d\rho_y(x)\, \psi(y)\, \phi(y)\, d\kappa(y)$$

$$= \int_{R^n} \rho_y(B)\psi(y)\phi(y)d\kappa(y)$$

$$= (a(B)\psi, \phi),$$

therefore $\gamma^* P(B)\gamma = a(B)$.

Hence by identifying $L^2(R^n, \kappa)$ with its image under γ and defining $\mathbb{P} = \gamma^*$, one has that

$$\mathbb{P}\, P(B)\, \mathbb{P} = a(B)$$

and $\mathbb{P} : K \to L^2(R^n, \kappa)$.

Furthermore, since the characteristic functions are dense in $\mathfrak{A}$, one sees that K is generated by $\{\chi_B \otimes \psi \mid \psi \in L^2(R^n, \kappa)\ B \in \mathcal{B}(X)\}$, which means that the extension is minimal. The functions $f \in \mathfrak{A}$ are square-integrable with respect to ρ_y for any $y \in R^n$ by construction, so that for any fixed y, the family $\{f \otimes \psi \mid f \in \mathfrak{A},\ \psi \in L^2(R^n, \kappa)\}$ generates a Hilbert-space

$$L^2(X, \rho_y) \otimes C = H_y .$$

The family $\{H_y \mid y \in R^n\}$ of Hilbert-spaces is κ-measurable in the sense of Definition 8.9 in [2], chapter IV:

A countable basis for K is given by

$\{\xi_{ij} = f_i \otimes \phi_j \,|\, \{f_i\}, \{\phi_j\}$ orthonormal bases in $\mathfrak{A}$ and $L^2(R^n, \kappa)$ respectively $\}$

and a basis for H_y is given by $\{f_i(\cdot) \cdot \phi_j(y)\}$.

The mapping

$$y \to \int f_i(x)\, \overline{f_j(x)}\, d\rho_y(x)\, \phi_k(y)\, \overline{\phi_{k'}(y)}$$

is clearly κ-measurable for any i, j, k, k', as the functions ϕ_k, $\phi_{k'}$ are measurable.

Thus the set

$$\mathcal{M} = \{\xi \in \prod_{y \in R^n} H_y \,\Big|\, \int_X \xi(x,y)\, f_i(x)\, \phi_j(y)\, d\rho_y(x) < \infty \quad \forall i, j, \quad \forall y \in R^n\}$$

together with the field $\{H_y \mid y \in R^n\}$ constitutes a κ-measurable field of Hilbert spaces.

For the norm in K one has

$$\|\xi\|^2 = \int_{R^n} \|\xi(y)\|^2_{H_y}\, d\kappa(y)$$

so that letting y run through all of R^n, we hence find that

$$K \simeq \int_{R^n}^{\oplus} L^2(X, \rho_y) \otimes C \quad d\kappa(y) =: H$$

which establishes the result. ∎

<u>Corollary 3.2.5</u>

$$H \simeq L^2(X \times R^n, \rho \otimes \kappa) \quad \text{iff} \tag{15}$$

there is a probability measure ρ on X such that for all $y \in X$

$L^2(X, \rho_y) \simeq L^2(X, \rho)$

Proof In this case one has

$$\begin{aligned} H &= \int_{R^n}^{\oplus} L^2(X, \rho_y) \otimes C \quad d\kappa(y) \\ &\simeq \int_{R^n}^{\oplus} L^2(X, \rho) \otimes C \quad d\kappa(y) \\ &= L^2(X, \rho) \otimes L^2(R^n, \kappa) \simeq L^2(X \times R^n, \rho \otimes \kappa) \end{aligned}$$

The situation of corollary 3.2.5 occurs e.g. if the measures ρ_y have support on all of X and lie within the same measure class. (see e.g.[10], chapter 9).

One might note, that for the construction carried out in proposition 3.2.4 the fact that the POV-measure a is normalized plays quite an important role for ensuring isometry. If one drops this restriction, one can't find an isometric embedding of the Hilbert-space $L^2(R^n, \kappa)$ any longer, unless a(X) is some multiple of the identity operator, this case, however, restricts to the case of a normalized POV-measure and is therefore of no interest. If a does not act as an operator of multiplication, the construction can still be carried out by defining the mapping

$$\mu : \mathfrak{A} \to L(L^2(R^n, \kappa))$$

by

$$(\mu(f)\psi, \psi) = \int_X f(x)(a(dx)\psi, \psi) .$$

The mapping of course still gives rise to a Hilbert-space K which however does not allow any further insight into its structure, except that once again the functions $f \in \mathfrak{A}$ are square-integrable with respect to $(a(\cdot)\psi, \psi)$ for any ψ . (For this more general situation see [11], where the Naimark extension is carried out for a system of covariance. The POV-measure is not required to be commutative any longer, however the existence of a group action on the locally compact space is imposed).

In the given framework the "shape" of the POV-measure is determined by physical reasons (see definition 3.2.2 and chapter 3.1). In the next chapter we will specify the locally compact space somewhat more, by considering a differential manifold M and derive the Naimark extension in this particular case.

CHAPTER 4 THE NAIMARK EXTENSION FOR A COMMUTATIVE, NORMALIZED POV-MEASURE BASED ON A DIFFERENTIAL MANIFOLD

We will now assume that the physical system under consideration has a differential-manifold M as its configuration space. We will adopt the notation introduced in chapter 2.2, and assume that the underlying Hilbert-space of the system is given as $L^2(M, \nu)$ where ν is a smooth Borel-measure on $\mathfrak{B}(M)$.

The commutative, normalized POV-measure for M is hence a mapping

$$a : \mathfrak{B}(M) \to \mathcal{L}(L^2(M, \nu))^+ \tag{1}$$

with the properties defined in definition 3.2.1.
Furthermore, we suppose that a is informationally equivalent to the canonical PV-measure $P : \mathfrak{B}(M) \to \mathcal{L}(L^2(M, \nu))^+$ given as

$$(P(E)\psi)(m) = \chi_E(m)\ \psi(m) \tag{2}$$

for all $E \in \mathfrak{B}(M)$, $m \in M$, $\psi \in L^2(M, \nu)$.
By lemma 3.2.1 it follows that the von Neumann-algebra $A(a)$ generated by $\{a(E) \mid E \in \mathfrak{B}(M)\}$ is maximal abelian and therefore we conclude with proposition 3.2.3 that for all $m \in M$, $E \in \mathfrak{B}(M)$ and $\psi \in L^2(M, \nu)$

$$(a(E)\psi)(m) = \mu_m(E)\ \psi(m) \tag{3}$$

where $\mu_m : \mathfrak{B}(M) \to [0, 1]$ is a probability measure for each $m \in M$.
As $A(a)$ is maximal abelian, we in particular know that for all Borel-sets $E, F \in \mathfrak{B}(M)$

$$[a(E), P(F)] = 0 . \tag{4}$$

For a fixed Borel-set $F \in \mathfrak{B}(M)$ one sees that

$$P(F)[L^2(M,\nu)] = \{\chi_F\psi \mid \psi \in L^2(M,\nu)\}$$
$$= \{\tilde{\psi} \mid \int_F |\tilde{\psi}(m)|^2 \, d\nu(m) < \infty\}$$
$$= L^2(F, \nu|_F)$$

where $\nu|_F = \nu|\mathfrak{B}(M) \cap F = \nu|\mathfrak{B}(F)$.

Because of (4) this means that $\{a(E) \mid E \in \mathfrak{B}(M)\}$ gives rise to a commutative normalized POV-measure on each subspace of $L^2(M,\nu)$.

In chapter 2.2 we have constructed a disjoint covering of M by Borel-sets $\hat{U}_k$ such that

$$L^2(M,\nu) = \bigoplus_{k=1}^{\infty} L^2(\hat{U}_k, \nu_k) \tag{5}$$

and

$$L^2(\hat{U}_k, \nu_k) \simeq L^2(R^n, \kappa_k \lambda^n) \qquad \forall k \in N. \tag{6}$$

where $\kappa_k : R^n \to R_0^+$, supp $\kappa_k \in \gamma_k(\hat{U}_k)$, $\kappa_k \in C^\infty(\gamma_k(\hat{U}_k))$ and $\gamma_k : U_k \to R^n$, so that $\nu_k \circ (\gamma_k \mid \hat{U}_k)^{-1} = \kappa_k \lambda^n$ on $\gamma_k(\hat{U}_k)$.

As the direct sum in (5) is an orthogonal direct sum, we furthermore have

$$\sum_{k=1}^{\infty} P(\hat{U}_k) = id_{L^2(M,\nu)} \tag{7}$$

(which, by the way, of course follows as well from the definition of a PV-measure)

and for all $E \in \mathfrak{B}(M)$

$$a(E) = \sum_{k=1}^{\infty}{}^{\oplus} P(\hat{U}_k)\, a(E)\, P(\hat{U}_k)\,. \tag{8}$$

Let

$$a_k(E) := P(\hat{U}_k)\, a(E)\, P(\hat{U}_k)$$

for all $k \in N$.

If

$$\Gamma_k : L^2(\hat{U}_k, \nu_k) \to L^2(R^n, \kappa_k \lambda^n)$$
$$(\Gamma_k \psi)(x) := \psi(g_k^{-1}(x)) \tag{9}$$

with

$$g_k := \gamma_k \mid \hat{U}_k$$

denotes the isometric isomorphism between $L^2(\hat{U}_k, \nu_k)$ and $L^2(R^n, \kappa_k \lambda^n)$, then the following diagram is commutative:

$$\begin{array}{ccc} L^2(\hat{U}_k, \nu_k) & \xrightarrow{a_k(E)} & L^2(\hat{U}_k, \nu_k) \\ \downarrow \Gamma_k & & \downarrow \Gamma_k \\ L^2(R^n, \kappa_k \lambda^n) & \xrightarrow{\hat{a}_k(E)} & L^2(R^n, \kappa_k \lambda^n) \end{array} \tag{10}$$

Hence for each $k \in N$, the commutative, normalized POV-measure a_k induces a commutative, normalized POV-measure $\hat{a}_k$ on $L^2(R^n, \kappa_k \lambda^n)$ for $\mathcal{B}(M)$ defined as

$$\hat{a}_k(E) := \Gamma_k \cdot a_k(E) \cdot \Gamma_k^{-1} \tag{11}$$

which satisfies

$$(\hat{a}_k(E)\, \hat{\psi})(x) = \mu_{g_k^{-1}(x)}(E)\, \hat{\psi}(x) \tag{12}$$

for all $\hat{\psi} \in L^2(R^n, \kappa_k \lambda^n)$, all $x \in g_k(\hat{U}_k)$ and $E \in \mathcal{B}(M)$, and for any x,

$\mu_{g_k^{-1}(x)} : \mathcal{B}(M) \to R$ is a probability measure.

Herewith, we are in the position to apply proposition 3.2.4: There is a PV-measure $\tilde{P} : \mathcal{B}(M) \to L(H_k)^+$ where

$$H_k = \int_{R^n}^{\oplus} L^2(M, \mu_{g_k^{-1}(x)}) \otimes C\, \kappa_k(x)\, d\lambda^n(x) \tag{13}$$

and a projection

$$P_k : H_k' \to L^2(R^n, \kappa_k \lambda^n)$$

such that H_k is the minimal extension space in the sense of Naimark and

$$\hat{a}_k(E) = P_k \tilde{P}(E) P_k$$

for all $E \in \mathcal{B}(M)$.

For $a_k(E)$ one finds by using (11)

$$a_k(E) := \Gamma_k^{-1} P_k \tilde{P}(E) P_k \Gamma_k \tag{14}$$

Note that, by construction, $\tilde{P}(E)$ acts on H_k as multiplication by χ_E.

Having carried out this construction for any k, we arrive at the following conclusion:

<u>Theorem 4.1</u> Let $\{a(E) \mid E \in \mathcal{B}(M)\}$ be a commutative, normalized POV-measure for $\mathcal{B}(M)$ which is given as

$$(a(E)\psi)(m) = \mu_m(E)\psi(m)$$

where μ_m is a probability measure for each $m \in M$.

Then there is a PV-measure $\tilde{P} : \mathcal{B}(M) \to L(H)$ where

$$H = \bigoplus_{k=1}^{\infty} H_k \,, \tag{15}$$

with H_k as in (13), and a family of projections $P_k : H_k \to L^2(R^n, \kappa_k \lambda^n)$ such that

$$a(E) = \sum_{k=1}^{\infty} \Gamma_k^{-1} \; P_k \tilde{P}(E) P_k \Gamma_k \;, \tag{16}$$

the extension being minimal in the sense of Naimark in each constituent of the direct sum (15).

Written out in full, this direct sum reads:

$$H = \bigoplus_{k=1}^{\infty} \int_{R^n}^{\oplus} L^2(M, \mu_{g_k^{-1}(x)}) \otimes C\, \kappa_k(x)\, d\lambda^n(x) \;. \tag{15'}$$

As in Corollary 3.2.5, this representation can be simplified further to

$$H = \bigoplus_{k=1}^{\infty} L^2(M \times R^n, \mu_k \otimes \kappa_k \lambda^n) \tag{15''}$$

provided that for each fixed $k \in N$ the condition

$$L^2(M, \mu_{g_k^{-1}(x)}) \simeq L^2(M, \mu_k)$$

is satisfied for each $x \in g_k(\hat{U}_k)$.

Under certain circumstances, the spaces $L^2(M \times R^n, \mu_k \otimes \kappa_k \lambda^n)$ can be given an interpretation which is interesting from the physical point of view:

Suppose that the cotangent-bundle T^*M is globally trivialisable, (for the def. see [3]) i.e., $T^*M \simeq M \times R^n$ and one has a measure Λ_k on

$\mathcal{B}(T^*M)$ which is such that it can be expressed as a product measure:

$$\Lambda_k(E \times G) = \rho_k(E) \cdot \lambda_k(G)$$

where ρ_k is a probability measure on $\mathcal{B}(M)$ and λ_k a σ-finite measure on $\mathcal{B}(R^n)$, in case, ρ_k is equivalent to μ_k and λ_k is equivalent to $\kappa_k\lambda^n$, we find

$$L^2(M \times R^n, \mu_k \otimes \kappa_k\lambda^n) \simeq L^2(T^*M, \Lambda_k) \tag{17}$$

which means, in physical terms, that we consider functions depending on position and momentum.

Supposing that (17) holds for all k one finds

$$H \simeq \bigoplus_{k=1}^{\infty} L^2(T^*M, \Lambda_k) \tag{18}$$

Analysing (15") a little bit more, one has

$$H \simeq \bigoplus_{k=1}^{\infty} L^2(M, \mu_k) \otimes L^2(R^n, \kappa_k\lambda^n)$$

If the μ_k's have support on all of M, and if they all lie in the same measure class, we are led to

$$H \simeq \bigoplus_{k=1}^{\infty} L^2(\tilde{M}, \mu) \otimes L^2(R^n, \kappa_k\lambda^n)$$

$$\simeq \bigoplus_{k=1}^{\infty} L^2(M \times R^n, \mu \otimes \kappa_k\lambda^n). \tag{15'''}$$

If furthermore, the κ_k's have disjoint supports V_k in R^n, this expression can be simplified to

$$H \simeq L^2(M \times R^n, \mu \otimes \kappa\lambda^n) \tag{19}$$

where κ is an abbreviation for $\sum_{k=1}^{\infty} \kappa_k$.

With the same assumptions as used for deriving (18) we can then interpret H as a Hilbert-space of square integrable functions on the total space T^*M of the cotangent-bundle for M:

$$H \simeq L^2(T^*M, \Lambda) \tag{20}$$

with

$$\Lambda(E \times G) = \hat{\mu}(E) \cdot \hat{\kappa}(G)$$

and

$$\hat{\mu} \sim \mu , \hat{\kappa} \sim \kappa\lambda^n .$$

It is well possible, that one asks for too much by assuming that T^*M be homeomorphic to $M \times R^n$, this requirement has only been adopted for the sake of clarity.

Apparently, as T^*M in any case is locally trivial, one can arrive at the same result by passing from T^*M to a disjoint union of Borel-sets U_i of T^*M which can be identified with some $V_i \times R^n$ for each i, and then impose requirements on the measure while passing from $L^2(U_i, \Lambda|U_i)$ to $L^2(V_i \times R^n, \hat{\Lambda}_i)$ where $\hat{\Lambda}_i$ is the image of the measure $\Lambda|U_i$.

As a matter of fact, for making an identification as done in (20) it suffices that the spaces $L^2(M \times R^n, \mu \otimes \kappa\lambda^n)$ and $L^2(T^*M, \Lambda)$ be isometrically isomorphic.

As we have already seen in chapter 3.2, there is always the possibility of constructing a POV-measure from a PV-measure simply by projecting onto a proper subspace of the Hilbert-space on which the given PV-measure lives.

We wish of course to recover a POV-measure on $L^2(M, \nu) \simeq \overset{\infty}{\underset{k=1}{\oplus}} L^2(R^n, \kappa_k \lambda^n)$.
Given H as in (15') and keeping k fixed for the time being let P_k denote the projection of

$$\int_{R^n}^{\oplus} L^2(M, \mu_{g_k^{-1}(x)}) \otimes C\, \kappa_k(x)\, d\lambda^n(x)$$

onto $L^2(R^n, \kappa_k\, d\lambda^n)$ where κ_k once again is supported in $g_k(\hat{U}_k)$ given as

$$P_k(f \otimes \psi_k)(x) = \int_M d\mu_{g_k^{-1}(x)}(m)\, \psi_k(x)$$
$$= \psi_k(x)$$

as $\mu_{g_k^{-1}(x)}(M) = 1$.

Consequently

$$(P_k P(E) P_k)(f \otimes \psi_k)(x) = (\hat{a}_k(E)\psi_k)(x) = \mu_{g_k^{-1}(x)}(E)\, \psi_k(x)$$

so that, locally, we get back to $L^2(\hat{U}_k, \nu_k)$ by using the isometric isomorphism Γ_k as defined in (9) and setting

$$a_k(E) = \Gamma_k^{-1}\, \hat{a}_k(E)\, \Gamma_k$$

for arbitrary $E \in \mathcal{B}(M)$.

As the sets $\hat{U}_k$ by construction are disjoint we can then define

$$a(E) := \sum_{k=1}^{\infty} \Gamma_k^{-1}\, \hat{a}_k(E)\, \Gamma_k$$

which gives us a commutative normalized POV-measure on $L^2(M, \nu)$.

This construction will always work if $L^2(M, \nu)$ or equivalently $\bigoplus_{k=1}^{\infty} L^2(R^n, \kappa_k\lambda^n)$ can be isometrically embedded into the given Hilbert-space H. However, in general, the resulting POV-measure might fail to be commutative.

In the special case $H = L^2(T^*M, \Lambda) \simeq L^2(M \times R^n, \mu \otimes \alpha)$ one finds:

<u>Proposition 4.2</u> Let $P : \mathcal{B}(M) \to L(L^2(M \times R^n, \mu \otimes \alpha))^+$ be a PV-measure for M. Suppose that

$$\mu : \mathcal{B}(M) \to [0, 1]$$

is a probaility measure on $\mathcal{B}(M)$ and

$$\alpha : \mathcal{B}(R^n) \to R$$

is a σ-finite measure on R^n which is absolutely continuous with respect to Lebesgue-measure λ^n on $R^n : \alpha \to \kappa\lambda^n$, where α is a version of the Radon-Nikodym derivative of κ with respect to λ^n. Let $\{(V_k, g_k, U_k) | k \in N\}$ be a cover of M which is such that $V_k \cap V_i \neq 0$ and $U_k \cap U_i = 0$ for $i \neq k$. If κ admits a representation $\sum_{k=1}^{\infty} \kappa_k$ where supp $\kappa_k' \subset U_k$ and $\kappa_k \in C^\infty(U_k, R^+)$ for all k, then:
There is a projection $\mathbb{P} : L^2(M \times R^n, \mu \otimes \alpha) \to \bigoplus_{k=1}^{\infty} L^2(R^n, \kappa_k\lambda^n)$ such that $\{\mathbb{P} P(E) \mathbb{P} \mid E \in \mathcal{B}(M)\}$ is a commutative, normalized POV-measure for $\mathcal{B}(M)$ on $\bigoplus_{k=1}^{\infty} L^2(R^n, \kappa_k\lambda^n)$. Furthermore, if $\Gamma : \bigoplus_{k=1}^{\infty} L^2(R^n, \kappa_k\lambda^n) \to L^2(M, \nu)$ where $\nu = \sum_{k=1}^{\infty} (\kappa_k \cdot \lambda^n) \circ g_k$ is a smooth Borel-measure on M, is the isometric isomorphism defined by

$$\Gamma \mid L^2(R^n, \kappa_k\lambda^n) = \Gamma_k$$

then

$\{\Gamma\, \mathbb{P}\, P(E)\, \mathbb{P}\, \Gamma^{-1} \mid E \in \mathcal{B}(M)\}$ is a commutative, normalized POV-measure for $\mathcal{B}(M)$ on $L^2(M\,,\nu)$.

<u>Proof</u> By the assumptions imposed on the measure Λ one has that

$$L^2(T^*M\,,\Lambda) \simeq \bigoplus_{k=1}^{\infty} L^2(T^*M\,,\mu \otimes \kappa_k \lambda^n)$$

$$\simeq \bigoplus_{k=1}^{\infty} \int_{R^n}^{\oplus} L^2(M\,,\mu) \otimes C\, \kappa_k(x)\, d\lambda^{n(x)}$$

Let $P_k : L^2(T^*M\,,\mu \otimes \kappa_k \lambda^n) \to L^2(R^n\,,\kappa_k \lambda^n)$ be defined by

$$P_k(f \otimes \psi_k)(m\,,x) = 1_x(m) \otimes \psi_k(x).$$

As the sum above is an orthogonal direct sum we thus get the required result by setting

$$\mathbb{P} := \sum_{k=1}^{\infty}{}^{\oplus}\, P_k$$

Defining

$$\Gamma = \sum_{k=1}^{\infty}{}^{\oplus}\, \Gamma_k$$

where $\Gamma_k : L^2(R^n\,,\kappa_k \lambda^n) \to L^2(V_k\,,\nu_k)$, the composition $\Gamma \circ \mathbb{P} = \sum_{k=1}^{\infty} \Gamma_k P_k$ yields a map $\Gamma \circ \mathbb{P} : \bigoplus_{k=1}^{\infty} L^2(T^*M\,,\mu \otimes \kappa_k \lambda^n) \to \bigoplus_{k=1}^{\infty} L^2(V_k\,,\nu_k)$ and

$$a(E) = (\Gamma \circ \mathbb{P})\, P(E)\, \mathbb{P} \circ \Gamma^{-1}$$

for all $E \in \mathcal{B}(M)$.

Locally, a(E) thus defined acts as multiplication by $\mu_{g_k(m)}(E)$ at m. ∎

One might note right here, that this kind of construction depends highly on the character of the underlying manifold M. In the given proof, one actually has to assume that $L^2(M, \nu) \simeq L^2(R^n, \kappa\lambda^n)$ which means that what one does is considering R^n, that is M is an n-dimensional sub-manifold of some m-dimensional real vector-space. Hence proposition 4.2.1. is quite restrictive from a mathematical point of view.
In as far as the physical situation is concerned, this is rather common, as normally the systems under consideration (non-relativistic!) have some sub-manifold of an m-dimensional real vector-space as their configuration space.

CHAPTER 5 POSTFACE

As has been shown in the previous 2 chapters, the analysis of a commutative normalized POV-measure on a manifold M is quite satisfactory, if one pairs the mathematical structure given with the additional condition of informational equivalence with the canonical PV-measure on $\mathcal{B}(M)$. This is reasonable from the point of view of standard quantum mechanics, as one thus ensures that the position observables obtained from such a POV-measure are the same as the ones obtained in the usual approach. Then one can, at least for the simplest case, namely $H \simeq L^2(T^*M, \Lambda)$ summarize the result as follow. "Given a commutative normalized POV-measure a for $\mathcal{B}(M)$ on $L^2(M, \nu)$, the minimally extended Hilbert-space is $L^2(T^*M, \Lambda)$ and there is a PV-measure on this space such that when restricted to $L^2(M, \nu)$ which is isometrically embedded into $L^2(T^*M, \Lambda)$ it yields again the POV-measure a. Conversely, if $L^2(M, \nu)$ can be isometrically embedded into $L^2(T^*M, \Lambda)$ and one is given a PV-measure on $L^2(T^*M, \Lambda)$ the family $\{ \mathbb{P} P(E) \mathbb{P} \mid E \in \mathcal{B}(M) \}$ is a commutative normalized POV-measure for $\mathcal{B}(M)$ on $L^2(M, \nu)$, where

$$\mathbb{P} : L^2(T^*M, \Lambda) \rightarrow L^2(M, \nu)$$

denotes the projection onto $L^2(M, \nu)$."

The construction carried out depends however strongly on the maximal abelianness of the von Neumann-algebra $A(a)$. This has allowed us to show that each a(E) is given as a multiplication operator where one multiplies the function $\psi \in L^2(M, \nu)$ by a probability measure μ_m at each point of M. In case one has instead of only a commutative POV-measure a commutative system of covariance, which furthermore is transitive (i.e. $M = G/H$, where G is a topological group and H a closed subgroup of G.), then the requirement of $A(a)$ to be maximal abelian is superfluous. Then however one will have to consider induced systems of covariance on $L^2(R^n, \kappa_k\lambda^n)$ instead, which would not make life any simpler. For a treatment of this situation see [8] .

The fact that we were dealing with a differential manifold should not be disregarded either. Like this we were permitted to locally replace our Hilbert-space by an ordinary L^2-space over R^n with repect to some Borel-measure. For doing so the "differential" in front of the manifold was important, as it secured the existence of smooth Borel-measures on M.

As has already been pointed out, in the concluding remarks of chapter 3.2, neither the commutativity of the POV-measure nor the representation as multiplication operators is essential, in the sense of being able to carry out the extension. Unfortunately, in either of these cases one is stuck with the tensor-product shown in the proof of proposition 3.2.5, which leads to a loss of information. Similarly, if one can't identify the given Hilbert-space with some $L^2(R^n, \kappa)$, κ a Borel measure on R^n, or a direct sum of spaces of this type, we are led to

$$H \simeq \overline{\mathfrak{A}_\infty L^2(X, \rho)/N}^{\langle\,,\,\rangle}$$

which for the same reasons as above is unsatifactory. Dealing with a manifold rather than an arbitrary locally compact space, enables us to make this identificaiton without once again restricting our space.

The result in quotation marks mentioned above in any case will in general not occur. Furthermore, the occurrence of the total space of the cotangent bundle seems to fall upon us out of free space. It can only happen, as we are only concerned with the Borel-structure and do not care too much about the topological structure of T^*M. From this point of view, the quoted result is not too surprising.

REFERENCES

[1] Davies, E.B., Quantum Theory of Open Systems, Academic Press, London (1977)

[2] Takesaki, M., Theory of Operator Algebras I, Springer Verlag, New York (1979)

[3] Greub, W., Halperin, S. and Vanstone, R., Connections, Curvature and Cohomology I, Academic Press, New York (1972)

[4] Angermann, B., Über Quantisierungen lokalisierter Systeme-Physikalisch interpretierbare mathematische Modelle, Ph.D.-Thesis Clausthal (1983)

[5] Jauch, J.M., Foundations of Quantum Mechanis, Addison-Wesley Publishing Company, Reading (1972)

[6] Riesz, F., and Sz-Nagy, B., Functional Analysis, F. Ungar Publishing Company, New York (1960)

[7] Halmos, P.R., Introduction to Hilbert-Space and the Theory of Spectral Multiplicity, Chelsea Publishing Co., New York (1951)

[8] Ali, S.T., in Lecture Notes in Mathematics, No. 905, 207-228, ed. Doebner, H.D., Andersson, S.I. and Petry, H.R., Springer Verlag, Heidelberg (1980)

[9] Bauer, H., Probability Theory and Elements of Measure Theory Academic Press, London (1981)

[10] Varadarajan, V.S., Geometry of Quantum Theory II, van Nostrand Reinhold Company, New York (1970)

[11] Castrigiano, D.P.L. and Henrichs, R.W., Letters Math. Phys. 4, 169-175 (1980)

[12] Dixmier, J., von Neumann-Algebras, North Holland Publishing Company, Amsterdam (1981)

[13] Stinespring, W.F., Amer. Math. Soc. 6, 211-216 (1955)

[14] Luenberger, D.G., Optimization by Vector Space Methods, John Wiley & Sons, Inc., New York (1969)