ON THE NUMERICAL INTEGRATION OF NEUTRAL-TYPE DIFFERENCE - DIFFERENTIAL EQUATIONS

by

SWAPAN MUKHERJEE

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ABSTRACT

In this study Picard's method of successive approximation is applied to functional differential equations of neutral type. The results are compared with those obtained by (i) using a fourth order Runge-Kutta method and (ii) solving the equation successively over intervals of fixed length. These comparisons indicate that (i) the accuracy of solutions obtained by Picard's method is comparable to that obtained by the other two methods and (ii) Picard's method in general requires more computational time.

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CHAPTER 1

INTRODUCTION

The objective of this study is to determine the applicability of Picard's successive approximation technique to the numerical solution of functional differential equations of neutral type, and to compare the results with those obtained by the fourth order Runge-Kutta method and a difference approximation method.

Chapter 2 provides the reader with a brief introduction to functional differential equations and their applications, as well as to existing numerical methods of solution.

Chapter 3 provides the details of Picard's method for ordinary and functional differential equations as well as some convergence theorems.

Chapter 4 provides the numerical results obtained by applying the various methods to a first order functional differential equation. Several different sets of initial conditions and forcing functions are tried. The results are plotted to facilitate comparison of the behaviour of the methods.

Chapter 5 concludes this study by presenting an evaluation of Picard's method as compared to the Runge-Kutta and difference approximation methods, as well as suggestions for future investigations.

CHAPTER 2

EXISTING NUMERICAL METHODS

2.1. Introduction

The objective of the present study is to compare Picard's method of successive approximation with the fourth order Runge-Kutta method and a difference approximation method. It is therefore considered worthwhile to present an introduction to functional differential equations and their applications and subsequently a brief survey of the numerical methods that are available for such equations. In Section 2.2 various types of functional differential equations are discussed. In Section 2.3 methods for Initial Value Problems are discussed. Finally in Section 2.4 methods for Boundary Value Problems are discussed.

2.2. Functional Differential Equations

It has been known for a number of years that functional differential equations play an important role in the modelling of mechanical and electrical systems [1] [2]. Interest in these equations has continued to grow in recent years, as it has become apparent that they are also of importance in the areas of modelling physiological and hormonal control systems [3], theory of epidemics [4] and population growth [5] [6]. A commonly encountered type of functional differential equation is of the form

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t))) \qquad (2.1)$$

In the above equation if $\tau(t) > 0, \forall t$ the functional differential equation is known as retarded type. If f depends explicity on $\dot{x}(t - \tau(t))$ then the delay-differential equation is known as neutral type. The latter is the type of equation under investigation in this study.

2.3. Methods for Initial Value Problems

The initial value problem for neutral differential equations can be stated as

$$\dot{x}(t) = f\,\{t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t))\}\ ;\ 0 \le t \le T \qquad (2.2a)$$

$$x(t) = r(t), \quad t \in [-b, 0] \qquad (2.2b)$$

where

$$b = \inf_{t \in [0,\, T]} \{\tau(t) - t\}$$

An important special case of (2.2) is the initial value probelm for delay-differential equations, namely

$$\dot{x}(t) = f\,\{t, x(t), x(t - \tau(t))\}\ ;\ 0 \le t \le T \qquad (2.3)$$

Another special case of (2.2) is the initial value problem for delay-differential equations with constant delay, namely

$$\dot{x}(t) = f\,\{t, x(t), x(t - \bar{\tau}), \dot{x}(t - \bar{\tau})\}\ ;\ 0 < t < T$$

$$x_0 = \phi\ ;\quad t \in [-\bar{\tau}, 0]\ ;\ \tau > 0 \qquad (2.4)$$

The problem

$$\dot{x}(t) = \dot{x}(t/2) + 1\ ;\quad 0 < t < 1$$

$$x(0) = 0$$

does not have a solution, while the problem

$$\dot{x}(t) = \dot{x}(t/2) \quad , \quad 0 < t < 1$$

$$x(0) = 0$$

has the one-parameter family of solutions $x(t) = ct$. To avoid these difficulties it is hereafter assumed that

$$\bar{\tau} \le \tau(t) \le \tau_m$$

which ensures that (2.2) has a unique solution which continuously depends on ϕ [17].

2.3.1 Euler's Method

In the case of problem (2.4) let h be the step size given by

$$h = \bar{\tau}/m \qquad (2.5)$$

for some integer m.

Define

$$t_k = k \cdot h$$

Then Euler's method can be stated as follows [7]:

$$x^h(t_{k+1}) - x^h(t_k) = h\, f(t_k, x^h(t_k), x^h(t_{k-m})), \quad 0 \le k \le N$$

$$x^h(t_k) = \phi(t_k), \quad -m \le k \le 0 \qquad (2.6)$$

where we introduce the superscript h to denote the dependence of the approximate solution on h. Note that when using (2.6) to compute x^h, it is necessary to store the last m computed values. Let

$$e^h = x - x^h$$

Then it can be shown $e^h = O(h)$ provided $x \in C^2[0, T]$

In the case of problem (2.3) a grid could be constructed such that for each k, $1 \leq k \leq N + 1$ either

$$t_k - \tau(t_k) \leq 0$$

or

$$t_k - \tau(t_k) = t_{q(k)}, \quad q(k) < k.$$

Euler's method then takes the form

$$x^h(t_{k+1}) - x^h(t_k) = \begin{cases} h_k f(t_k, x^h(t_k), \phi(t_k - \tau(t_k))) & \text{if } t_k - \tau(t_k) \leq 0 \\ h_k f(t_k, x^h(t_k), x^h(t_{q(k)})) & \text{otherwise} \end{cases} \quad 0 \leq k \leq N$$

$$x^h(t_0) = \phi \tag{2.7}$$

The convergence properties of this method were studied by Feldstein [9] who also introduced three other versions of the above method.

2.3.2 One Step Method

Tavernini [10] developed a general one-step method for functional differential equations of retarded type which is applicable to retarded differential equations. He considers equations of the form

$$\dot{x}(t) = f(t, x_t), \quad 0 < t < T$$
$$x_0 = \phi,$$

where the notation of Hale [11] is used. A one-step method for this was of the form

$$x^h(t_k + rh_k) - x^h(t_k) = r_k^h \beta(r, h_k, t_k, x^h_{t_k}),$$

$$0 \le k \le N,$$

$$0 \le r \le 1$$

$$x_0^h = \phi^h \tag{2.8}$$

Hutchison [12] developed a general theory of multistep methods for neutral differential equations which includes one-step method as a special case.

2.3.3 Multi-Step Methods

Consider the problem

$$\dot{x}(t) = x(t-1) \quad , \quad t > 0$$

$$x(t) = 1 \quad , \quad -1 \le t \le 0 \tag{2.9}$$

It can be easily seen that $\dot{x}(t)$ is discontinuous at $t = 0$, that $\ddot{x}(t)$ is discontinuous at $t = 1$ and that $x^{(p)}(t)$ is discontinuous at $t = p - 1$. Thus $x(t)$ is initially non-smooth but becomes smoother with increasing t.

Consider now the problem

$$\dot{x}(t) = \dot{x}(t-1) \quad , \quad t \ge 0$$

$$x(t) = 1 \quad , \quad -1 \le t \le 0 \tag{2.10}$$

Since $x(t)$ is discontinuous at $t = 0$ it follows that $\dot{x}(t)$ is discontinuous whenever t is an integer.

These two problems are typical of retarded and neutral differential equations respectively, and when considering such equations it is therefore of importance to take account of the possibility of non-smooth solutions. Zverkina [13] developed multi-step methods which can handle non-smooth solutions. The location of the points of discontinuity of the solution x and the magnitude of the jumps δ in the derivatives of x can be computed. Using this information the linear multi-step method is written in the form

$$\sum_{j=0}^{\ell} \alpha_j \, x^h(t_{k+j}) = h \sum_{j=0}^{n} \beta_j \, \dot{x}^h(t_{k+j}) + T_k(\delta)$$

$$\dot{x}^h(t_{k+j}) = f(t_{k+j}, x^h(t_{k+j}), x^h(t_{k+j} - \tau(t_{k+j})), \dot{x}^h(t_{k+j} - \tau(t_{k+j}))) \quad (2.11)$$

where $T_k(\delta)$ depends on the jump δ and is such that the order of the method is preserved.

The work of Zverkina has been extended in several ways by Hutchison [12]. Instead of (2.11) she considers modified methods of the form

$$\sum_{j=0}^{\ell} \alpha_j \, x^h(t_{k+j}) = h \, f\{h, t_k, x^h(t_k), \ldots, x^h(t_{k+j}), f\} + T_k(\delta) \quad (2.12)$$

Other forms of extension can be found in Hutchison's [12] work.

Zverkina in her papers has asserted that multistep methods are more efficient than Runge-Kutta methods. The assertion is on the basis of storage needed, number of computations involved and accuracy. The conclusions are based upon theoretical considerations and no computational evidence is presented.

2.3.4 Runge-Kutta Methods

Runge-Kutta methods for ordinary differential equations are easily extended to functional differential equations. Consider an ordinary differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{2.13}$$

Let h be the step size and t_0 be the initial time. Then a fourth order Runge-Kutta method is given by

$$x_0(t_0 + h) = x(t_0) + 1/6\,[\Delta_1 + 2(\Delta_2 + \Delta_3) + \Delta_4] \tag{2.14a}$$

where

$$\Delta_1 = h\, f(t_0, x(t_0)) \tag{2.14b}$$

$$\Delta_2 = h\, f(t_0 + h/2,\ x(t_0) + \Delta_1/2) \tag{2.14c}$$

$$\Delta_3 = h\, f(t_0 + h/2,\ x(t_0) + \Delta_2/2) \tag{2.14d}$$

$$\Delta_4 = h\, f(t_0 + h,\ x(t_0) + \Delta_3) \tag{2.14e}$$

Let $t_1 = t_0 + h$. Then

$$x(t_1 + h) = x(t_1) + 1/6\,[\Delta_1 + 2(\Delta_2 + \Delta_3) + \Delta_4] \tag{2.15}$$

and continuing in this manner

$$x(t_k + h) = x(t_k) + 1/6\,[\Delta_1 + 2(\Delta_2 + \Delta_3) + \Delta_4] \tag{2.16}$$

where Δ_1, Δ_2, Δ_3, Δ_4 in (2.15) and (2.16) are defined in a manner similar (2.14b), (2.14d) (2.14d), (2.14e).

It is clear that a Runge-Kutta method applied to neutral type equations would generate solutions in time intervals of length τ. Thus the solution for the entire interval is obtained in steps of size τ i.e. first the solution is obtained in the interval $[0, \tau]$, from which the solution in the interval $[\tau, 2\tau]$ is generated, and subsequently for the entire interval by continuing the process.

2.3.5 Difference Approximation Methods

The first work on the stability of finite difference approximations to retarded and neutral differential equations is that of Brayton and Willoughby [14]. They consider the equation

$$\dot{x}(t) + A_1 x(t) + A_2 x(t - \tau) + A_3 \dot{x}(t - \tau) = 0 \qquad (2.17)$$

where A_1, A_2 and A_3 are symmetric nxn matrices such that $I \pm A_3$ and $A_1 \pm A_2$ are positive definite. Divide the interval τ so that the step size h equals τ/m for some integer m, and let $\mu \in [0, 1]$ be constant. Then the difference approximation for (2.17) takes the form

$$\begin{aligned} x(t_{k+1} - x(t_k) = & - A_3 [x(t_{k-m+1}) - x(t_{k-m})] \\ & - A_1 h [\mu x(t_{k+1}) + (1-\mu) x(t_k)] - \\ & A_2 h [\mu x(t_{k-m+1}) + \\ & (1 - \mu) x(t_{k-m})] \end{aligned} \qquad (2.18)$$

Note that if $m = 0$ and $\mu = 0$ then (2.18) reduces to Euler's method. It can be shown that if, in addition to the conditions on A_1, A_2, A_3 stated earlier, $(I \pm A_3) - (1/2 - \mu).\ h(A_1 \pm A_2)$ are positive definite, then $x(t_k) \to 0$ as $t_k \to \infty$.

2.4 Methods for Boundary Value Probelms

2.4.1 Shooting Methods

Consider the scalar ordinary two point boundary value problem

$$\ddot{x}(t) = f(t,\ x(t)),\ 0 < t < T$$

$$x(0) = x(T) = 0 \tag{2.19}$$

One approach to solving (2.19) is to consider the family of initial value probelms

$$\ddot{x}(t,\ s) = f(t\ ,x(t)),\ 0 < t < T$$

$$x(0,\ s) = 0$$

$$\dot{x}(0,\ s) = 0 \tag{2.20}$$

Solving equation (2.19) is equivalent to finding a value of s, say $\hat{s}$ such that

$$\phi(\hat{s}) = 0$$

where $\phi(s) = x(T,\ s)$

In the shooting method, a numerical method is chosen to compute approximate solutions $x^h(t,\ s)$ to the initial value problem (2.20). Then $\hat{s}^h$ is computed such that

$$\phi^h(s^h) = 0$$

where $\phi^h(s) = x^h(T, s)$. The approximate solution to (2.19) is taken to be $x^h(t, \hat{s})$. de Nevers and Schmitt [15] have studied problems of the form

$$\begin{aligned} \ddot{x}(t) &= f\{t, x(t), x(t - \tau(t))\}, \quad 0 < t < T \\ x_0 &= \phi \in C\,([-\bar{\tau}, 0], R^1) \\ x(T) &= \psi \in R^1 \end{aligned} \tag{2.21}$$

using the shooting method

2.4.2 Finite Difference Method

Cryer [16] considers the finite difference method for problems of the form

$$\begin{aligned} \ddot{x}(t) &= f\{t, x(t)\} + (Fx)(t), \quad 0 < t < T \\ x(0) &= 0 \\ x(T) &= 0 \end{aligned} \tag{2.22}$$

where $F : C[0, T] \to C\,[0, T]$

It should be mentioned that in the case of ordinary differential equations the shooting method and the method of finite differences complement one another in the sense that the shooting method gives good results when the finite difference method give poor results and vice versa. The same appears to be true in the case of functional differential equations also.

Methods for initial value probelms such as Splines and Hermite, Chebyshev series, Extrapolation and Deferred Correction as well as methods for boundary value problems which are not detailed here can be found in [17].

CHAPTER 3

PICARD'S METHOD

3.1 Introduction

To motivate Picard's method, an ordinary differential equation (Section 3.2) is used as an example to give the basic idea. It is subsequently (Section 3.3) extended to a delay differential equation and finally to a neutral differential equation (Section 3.4).

3.2 Ordinary Differential Equation

Consider an ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x_0 = x(0) \tag{3.1}$$

The initial value problem can be equivalently formulated in terms of the following integral equation:

$$x(t) = x_0 + \int_0^t f\{s, x(s)\}\, ds \tag{3.2}$$

Then Picard's iteration formula for arbitrary $x_0(t)$ is

$$x_{k+1}(t) = x_0(t) + \int_0^t f(s, x_k(s))\, ds \tag{3.3}$$

$k = 0, 1, \ldots$

which as $k \to \infty$ converges to the solution of the differential equation (3.1) [18].

3.3 Delay Differential Equation

Picard's method can be extended to delay-differential equations, as follows:

Theorem 3.1 [19] Consider a delay differential equation

$$\dot{x}(t) = A(t)\, x(t) + f\,\{x(t-\tau)\} \tag{3.4a}$$

$$x(t) = c(t),\ t \in [-\tau,\ 0] \tag{3.4b}$$

where A(t) is piecewise continuous and f satisfies a global Lipschitz condition. Then for any finite T and arbitrary continuous function r(t), $t \in [0, T]$, the sequence of functions $\{x_k(.)\}$ defined by

$$x_0(t) = \begin{cases} c(t), & -\tau \le t \le 0 \\ r(t), & 0 \le t \end{cases} \tag{3.5a}$$

$$x_k(t) = \begin{cases} c(t), & \tau \le t \le 0 \\ \phi(t,0)\,c(0) + \int_0^t \phi(t,s)\, f\,\{x_{k-1}(s-\tau)\}\, ds \end{cases} \tag{3.5b}$$

where ϕ is the transition matrix corresponding to A, converges uniformly to the unique solution of the differential equation (3.4) satisfying the given intial condition.

Proof

Consider $\{x_k(.)\}$ as a sequence in $C^n[-\tau, T]$, the space of continuous n-vector valued functions, with the usual supremum norm. Suppose the Lipschitz norm of f is F, so that for all v_1 and v_2 in R^n, we have

$$||f(v_1) - f(v_2)|| \le F\,||v_1 - v_2|| \tag{3.6}$$

Let $\psi(t, \xi)$ denote the induced norm of the matrix $\phi(t, \xi)$, and define

$$M = \sup_{t,\ \xi\ \in [0,\ T]} \psi(t, \xi) \tag{3.7}$$

From 3.5b we can write

$$x_1(t) = \phi(t, 0)\, c(0) + \int_0^t \phi(t, s) f\{x_0(s-\tau)\}\, ds$$

$$x_2(t) = \phi(t, 0)\, c(0) + \int_0^t \phi(t, s) f\{x_1(s-\tau)\}\, ds$$

$$\vdots$$

$$x_{k+1}(t) = \phi(t, 0)\, c(0) + \int_0^t \phi(t, s) f\{x_k(s-\tau)\}\, ds$$

Take the difference between (k+1)st and k th equation and then the norm to obtain

$$||x_{k+1}(t) - x_k(t)|| = ||\int_0^t \phi(t, s)[\, f\{x_k(s-\tau)\}$$

$$- \; f\{x_{k-1}(s-\tau)\}]ds||$$

$$\leq \int_0^t ||\phi(t, s)[\, f\{x_k(s-\tau)\} - f\{x_{k-1}(s-\tau)\}]|| \, ds$$

$$\leq \int_0^t ||\phi(t, s)||.||f\{x_k(s-\tau)\}-f\{x_{k-1}(s-\tau)\}|| \, ds$$

$$\leq \int_0^t M \, ||f\{x_k(s-\tau)\}-f\{x_{k-1}(s-\tau)\}|| \, ds$$

$$\leq \int_0^t M \, F \, ||x_k(s-\tau) - x_{k-1}(s-\tau)|| \, ds \qquad (3.8)$$

From the first step in 3.5b

$$||x_1(t) - x_0(t)|| \leq \int_0^t ||\phi(t, s)||.||f\{x_0(s-\tau)\}|| \, ds$$

Also

$$||x_2(t) - x_1(t)|| \le \int_0^t ||\phi(t, s)||.||f(x_1(s-\tau))-f(x_0(s-\tau))|| \, ds$$

$$\le \int_0^t M\, F\, ||x_1(s-\tau) - x_0(s-\tau)|| \, ds$$

$$\le M\, F\, t\, ||x_1(.) - x_0(.)||_c \tag{3.9}$$

where $||.||_c$ denotes the norm in the space $C^n[-\tau, T]$. Going another step we obtain

$$||x_3(t) - x_2(t)|| \le \int_0^t M\, F\, ||x_2(s-\tau) - x_1(s-\tau)|| \, ds$$

$$\le \int_0^t M\, F\, .\, MF(s-\tau)||x_1(.) - x_0(.)||_c \, ds$$

$$\le \frac{M^2F^2t^2}{2} ||x_1(.) - x_0(.)||_c \tag{3.10}$$

Then by induction, it can be easily shown that

$$||x_{k+1}(t) - x_k(t)|| \le \frac{(M\, F\, t)^k}{k!} ||x_1(.) - x_0(.)||_c \tag{3.11}$$

We now claim that the sequence $\{x_0(.), x_1(.) \ldots.\}$ is a Cauchy sequence. Indeed let ℓ be any positive integer, then

$$||x_{k+\ell}(.) - x_k(.)|| = ||\{x_{k+\ell}(.) - x_{k+\ell-1}(.)\} + \{x_{k+\ell-1}(.)$$

$$- x_{k+\ell-2}(.)\} + \ldots.\, \{x_{k+1}(.) - x_k(.)\}||$$

Applying the triange inequality, we get

$$||x_{k+\ell}(.) - x_k(.)|| \le \sum_{j=1}^{\ell} ||x_{k+j}(.) - x_{k+j-1}(.)||_c$$

$$\leq \sum_{j=1}^{\ell} \frac{(M\,F\,t)^{k+j-1}}{(k+j-1)!} \cdot ||x_1(.) - x_0(.)||_c$$

(3.1?)

$$\leq ||x_1(.) - x_0(.)||_c \frac{(M\,F\,t)^k}{k!} \sum_{j=1}^{\ell} \frac{(M\,F\,t)^{j-1}}{(j-1)!}$$

because

$$\frac{1}{(k+j-1)!} \leq \frac{1}{k!} \cdot \frac{1}{(j-1)!}$$

Keeping in view that

$$e^{M\,F\,t} = \sum_{j=0}^{\infty} \frac{(M\,F\,t)^j}{j!}$$

we have

$$||x_{k+\ell}(.) - x_k(.)|| \leq ||x_1(.) - x_0(.)||_c \frac{(M\,F\,t)^k}{k!} \exp[MFt]$$

(3.13)

This shows however large ℓ is, as $k \to \infty$ $||x_{k+\ell}(.) - x_k(.)||_c \to 0$. Thus the sequence $\{x_0(.), x_1(.) \ldots.\}$ is a Cauchy sequence. Since the space $C^n[-\tau, T]$ with the defined norm is complete, the sequence $\{x_k(.)\}$ converges. It can be easily verified that this limit is indeed the unique solution of (3.4).

3.4 Neutral Differential Equations

Lemma 3.1 The solution of

$$y(t) = F(t) + E(t)\, y(t-\tau),\ t \geq 0$$
$$y(t) = c(t) \text{ for } t \in [-\tau, 0] \tag{3.14}$$

is given by

$$\begin{aligned} y(t) = {} & F(t)+E(t) \,.\, F(t-\tau)+E(t) \,.\, E(t-\tau) \,.\, F(t-2\tau) \\ & + E(t) \,.\, E(t-\tau) \,.\, E(t-2\tau) \,.\, F(t-3\tau) \\ & + \ldots\ldots\ldots\ldots \\ & + E(t) \,.\, E(t-\tau) \ldots\ldots\ldots E(t-(p-1)\tau) \,.\, F(t-p\tau) \\ & + E(t) \,.\, E(t-\tau) \ldots\ldots\ldots E(t-p\tau)\, F\, (t-(p+1)\tau) \\ & + E(t) \,.\, E(t-\tau) \,.\, E(t-2\tau) \ldots\ldots E(t-(p+1)\tau) \,.\, c(t-(p+2)\tau) \end{aligned} \tag{3.15}$$

where (p+1) is the integer part of the quantity t/τ.

Proof

The proof of the lemma proceeds by substituting for the delay term $y(t-\tau)$ in (3.14) in terms of $y(t-2\tau)$ and repeating the process (p+1) times. For example if p=2 we have

$$\begin{aligned} y(t) &= F(t) + E(t)\, y(t-\tau) \\ &= F(t) + E(t)\, F(t-\tau) + E(t)\, E(t-\tau)\, [\\ & \quad F(t-2\tau) + E(t-2\tau)\, c(t)] \end{aligned} \tag{3.16}$$

<u>Theorem 3.2</u> Let $x(t)$ be the solution of

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,U(t) + C(t)\,x(t-\tau)$$

$$+ E(t)\,\dot{x}(t-\tau)\ ,\ t \in [0, T] \qquad (3.17)$$

satisfying the initial conditions

$$x(t) = r(t)\ ,\ \dot{x}(t) = \dot{r}(t)\ ;\ t \in [-\tau, 0]$$

where $r(t)$ and $\dot{r}(t)$ are continuous over the time interval $[-\tau, 0]$. Then $x(t)$ is also the solution of

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,U(t) + C(t)\,x(t-\tau)$$

$$+ G(t, \tau) + v(t) \qquad (3.18)$$

where

$$v(t) = \prod_{i=0}^{p-1} E\{t-(i-p-1)\tau\}\dot{x}\{t-(p+2)\tau\}$$

$$+ \sum_{j=0}^{p} \prod_{i=0}^{p} E\{t+(i-p)\tau\}B\{t+(j-p-1)\tau\}\ \cdot$$

$$U\{t+(j-p-1)\tau\} \qquad (3.18a)$$

$$G(t,\tau) = \prod_{i=0}^{p} E\{t+(i-p)\tau\}[A\{t-(i-p-1)\tau\}\ \cdot$$

$$x\{t-(i-p-1)\tau\} + C\{t-(i-p-1\tau\} \cdot$$

$$x[t - (i-p)\tau]] \tag{3.18b}$$

and $(p+1)$ is the integer part of t/τ.

Proof

Define

$$F(t, \tau) = A(t)\, x(t) + B(t)\, U(t) + C(t)\, x(t-\tau) \tag{3.19}$$

Hence, (3.17) can be put in the form

$$\dot{x}(t) = F(t, \tau) + E(t)\, \dot{x}(t-\tau) \tag{3.20}$$

which is of the form (3.14).

Hence direct application of Lemma 3.1 yields (3.18), (3.18a), (3.18b).

Note the important fact that $v(t)$ is known beforehand since it depends only on the initial conditions $\dot{x}(t)$ and is bounded. Hence as time increases, $v(t)$ remains bounded if $||E(t)|| \leq \rho < 1$. Note also that it is not necessary to use the description of the neutral system as defined by (3.18), (3.18a), (3.18b). Theorem 3.2 states that the neutral system (3.17) can be put in the form (3.4). Since the convergence of Picard's method for (3.4) is already established by Theorem 3.1, it follows that Picard's method can be applied to neutral differential equations.

CHAPTER 4

NUMERICAL RESULTS

4.1 Introduction

To demonstrate the applicability of the method described in chapter 3, a first order neutral type difference differential equation is chosen as an example. Necessary equations are developed for Picard's, Runge-Kutta and difference approximation methods. Numerical solutions, are obtained for different initial conditions and forcing functions, and the results are displayed in graphical form. The numerical solutions were obtained on CDC 6400/Cyber System.

4.2 Runge-Kutta Method

Consider a first order neutral type difference differential equation

$$\dot{x}(t) = -3x(t) - x(t-\tau) - 0.5\dot{x}(t-\tau) + u(t) \quad (4.1)$$

Equation (4.1) was solved by using fourth order Runge-Kutta method, with the following choices

$$\tau = 1.0\ ; \quad h = 0.1 \quad (4.2)$$

Initial function

$$c(t) = 1.0\ ; \ -1 \leq t \leq 0$$

The result for the interval $0 \leq t \leq 1$ is shown in Table 4.1, and the result for $0 \leq t \leq 5$ is shown in Figure 4.1.

Time	$x(t)$	$\dot{x}(t)$
0.1	0.654	-2.963
0.2	0.398	-2.195
0.3	0.209	-1.626
0.4	0.068	-1.204
0.5	-0.036	-0.893
0.6	-0.113	-0.661
0.7	-0.170	-0.490
0.8	-0.212	-0.362
0.9	-0.243	-0.269
1.0	-0.267	-0.199

Table 4.1

4.3 Difference Approximation Method

Following Section 2.3.5 and choosing $m = 10$, $u = 0$, $\tau = 1$, we obtain

$$h = 1/10$$

Equation (4.1) then reduces to

$$x(t_{k+1}) - x(t_k) = -0.5\,[x(t_{k-10+1}) - x(t_{k-10})] - 3\times 0.1\,x(t_k) - 0.1\,x(t_{k-10}) + 0.1\,u(t_k)$$

which yields

$$x(t_{k+1}) = 0.7\,x(t_k) - 0.5\,x(t_{k-9}) + 0.4\,x(t_{k-10}) + 0.1\,u(t_k) \tag{4.9}$$

Let

$$t_{k+1} \triangleq k+1$$

to obtain

$$x(k+1) = 0.7\,x(k) - 0.5\,x(k-9) + 0.4\,x(k-10) + 0.1\,u(k) \tag{4.10}$$

Choose

$$u(t) \equiv 0$$

The result for the interval $0 \le t \le 1$ is shown in Table 4.2, and the result for $0 \le t \le 5$ is shown in Figure 4.1.

Time	x(t)	ẋ(t)
0.1	0.6000	-2.8000
0.2	0.3200	-1.9600
0.3	0.1240	-1.7320
0.4	0.0132	-0.9604
0.5	-0.1092	-0.6722
0.6	-0.1765	-0.4706
0.7	-0.2235	-0.3294
0.8	-0.2564	-0.2305
0.9	-0.2795	-0.1614
1.0	-0.2956	-0.1129

Table 4.2

4.4 Picard's Method

Following Section 3.4 and chosing $\tau = 1$, (4.1) is solved as follows:

Let

$$t \in [0, 1]$$

Then the integer part of t/τ is 0 and

$$v(t) = -0.5\, \dot{x}(t-1)$$

$$G(t, \tau) = 0$$

Now let

$$t \in [1, 2]$$

The integer part of t/τ now is 1 and

$$v(t) = -0.5\, \dot{x}(t-2) - 0.5\, u(t-1)$$

$$G(t, \tau) = 0.5\, \{3\, x(t-1) + x(t)\}$$

The process is continued in the same manner for the entire interval.

Trapezoidal rule of integration was used for the solution of (4.1). Choosing the initial approximating function

$$x_0(t) = 1 \quad ; \quad -1 \le t < \infty$$

and

$$u(t) \equiv 0$$

a total of 20 iterations were performed. The results for the interval $0 \le t \le 1$ are shown in Table 4.3, and the result for $0 \le t \le 5$ are shown in Figure 4.1.

Time	$x(t)$	$\dot{x}(t)$
0.1	0.652	-2.960
0.2	0.395	-2.190
0.3	0.205	-1.620
0.4	-0.065	-1.190
0.5	-0.039	-0.882
0.6	-0.116	-0.652
0.7	-0.173	-0.482
0.8	-0.215	-0.356
0.9	-0.246	-0.263
1.0	-0.268	-0.195

Table 4.3

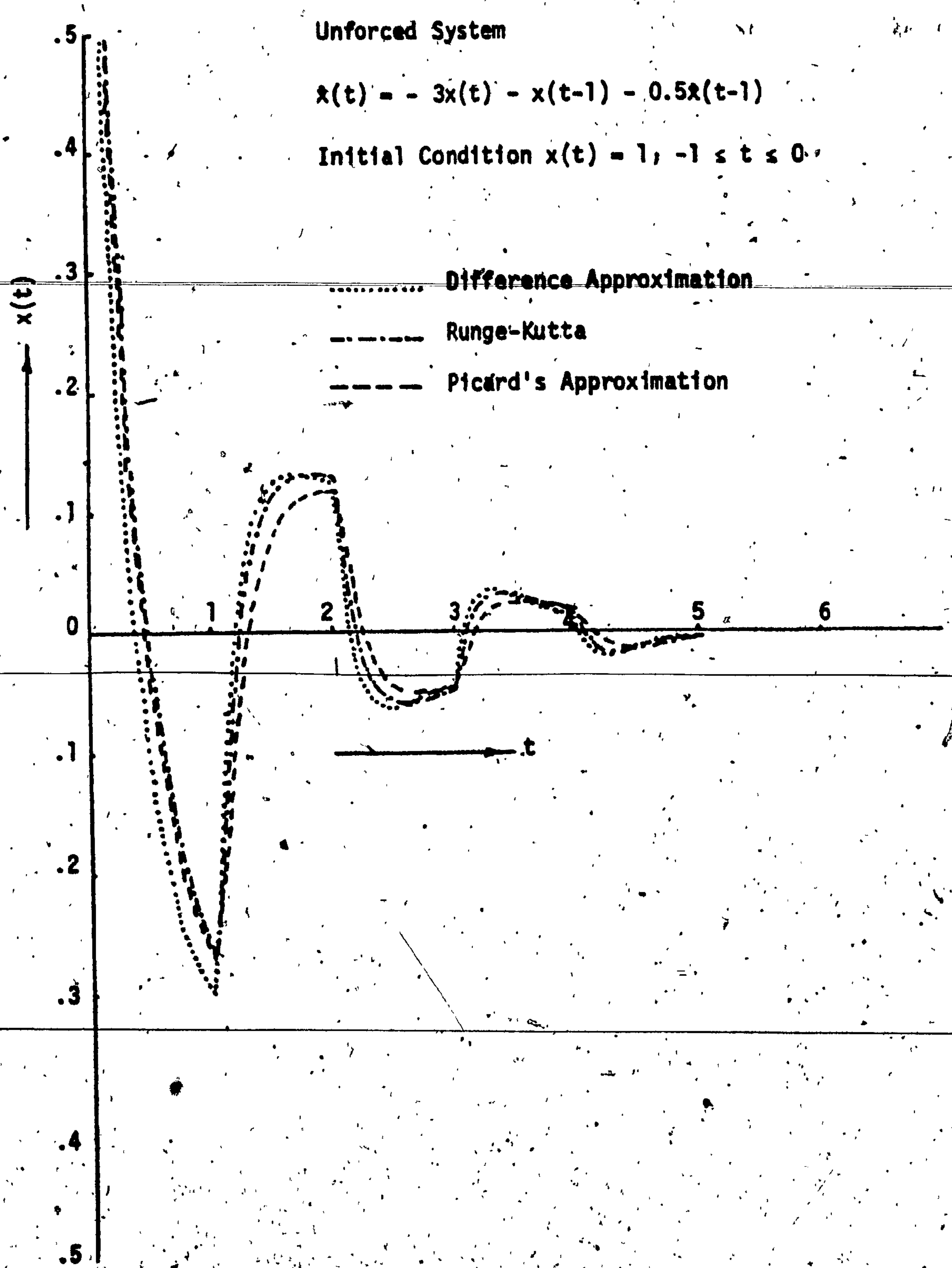

Unforced System
ẋ(t) = - 3x(t) - x(t-1) - 0.5ẋ(t-1)
Initial Condition x(t) = 1, -1 ≤ t ≤ 0
Difference Approximation
Runge-Kutta
Picard's Approximation
x(t)
t
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
1
2
3
5
6

Fig. 4.1

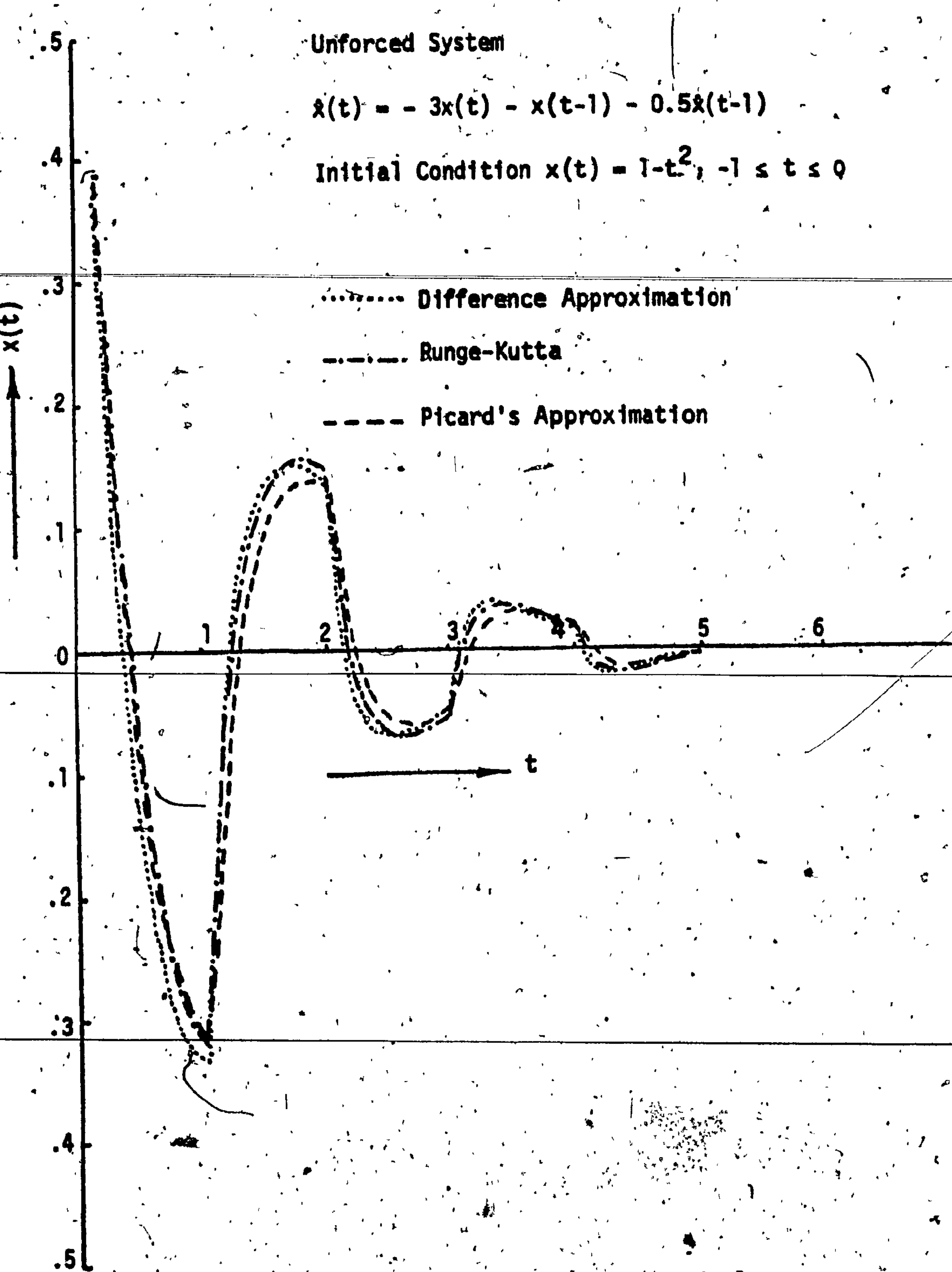
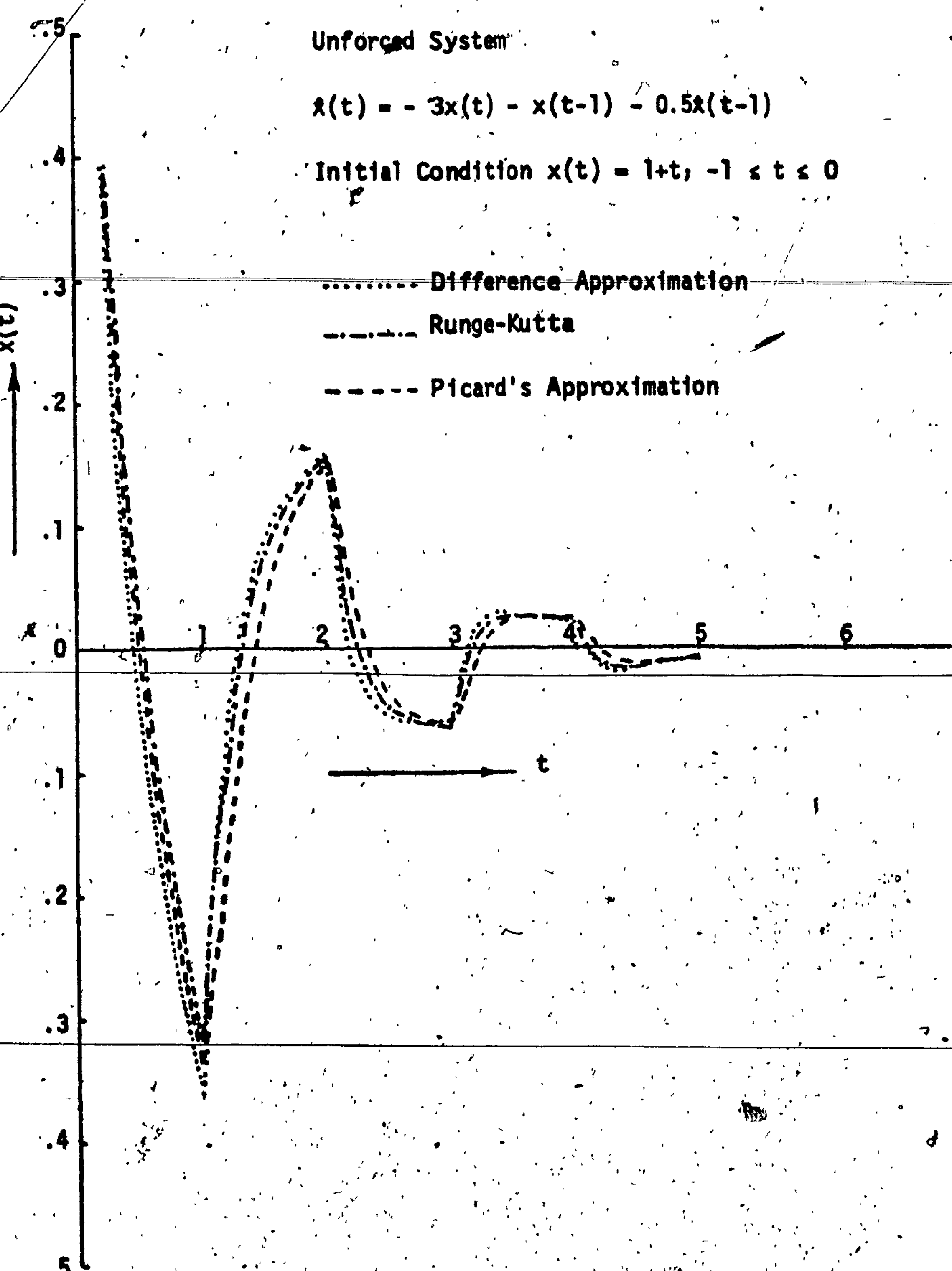
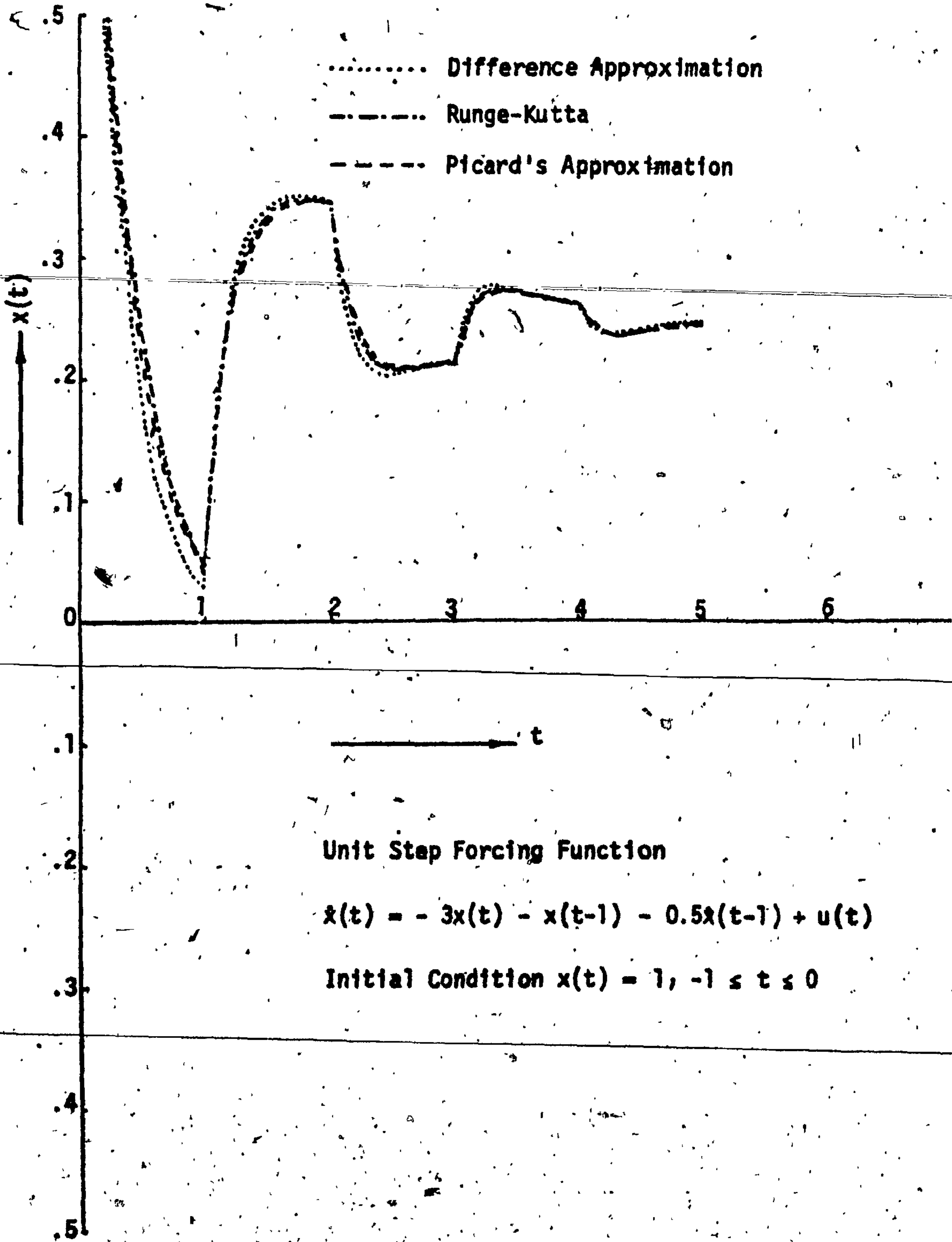

Unforced System
ẋ(t) = - 3x(t) - x(t-1) - 0.5ẋ(t-1)
Initial Condition x(t) = 1-t², -1 ≤ t ≤ 0
Difference Approximation
Runge-Kutta
Picard's Approximation
x(t)
t
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
1
2
3
4
5
6

Fig. 4.2



Unforced System
ẋ(t) = - 3x(t) - x(t-1) - 0.5ẋ(t-1)
Initial Condition x(t) = 1+t, -1 ≤ t ≤ 0
Difference Approximation
Runge-Kutta
Picard's Approximation
x(t)
t
.5
.4
.3
.2
.1
0
1
2
3
4
5
6

Fig. 4.3

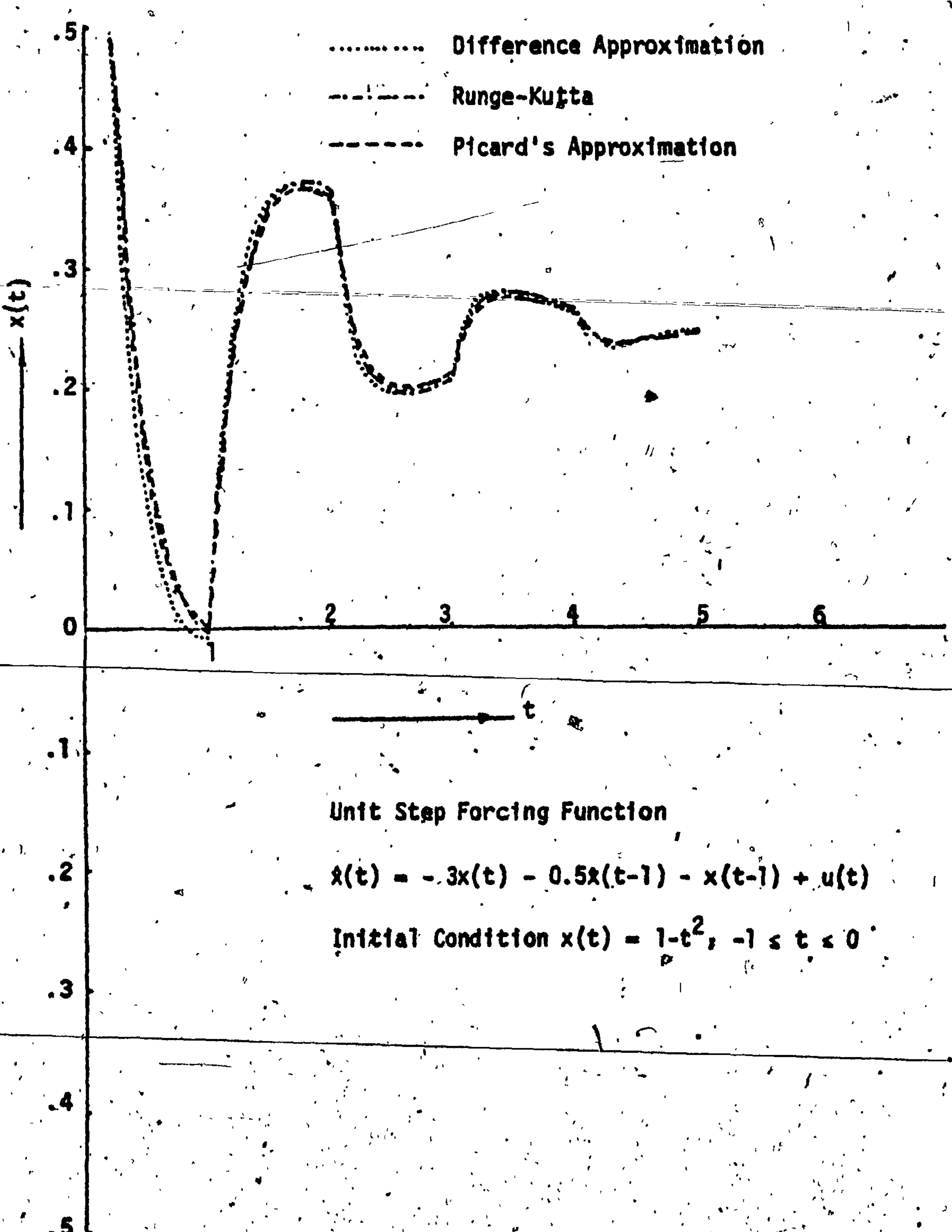


Unit Step Forcing Function

$\dot{x}(t) = -3x(t) - x(t-1) - 0.5\dot{x}(t-1) + u(t)$

Initial Condition $x(t) = 1, \ -1 \le t \le 0$

Fig. 4.4

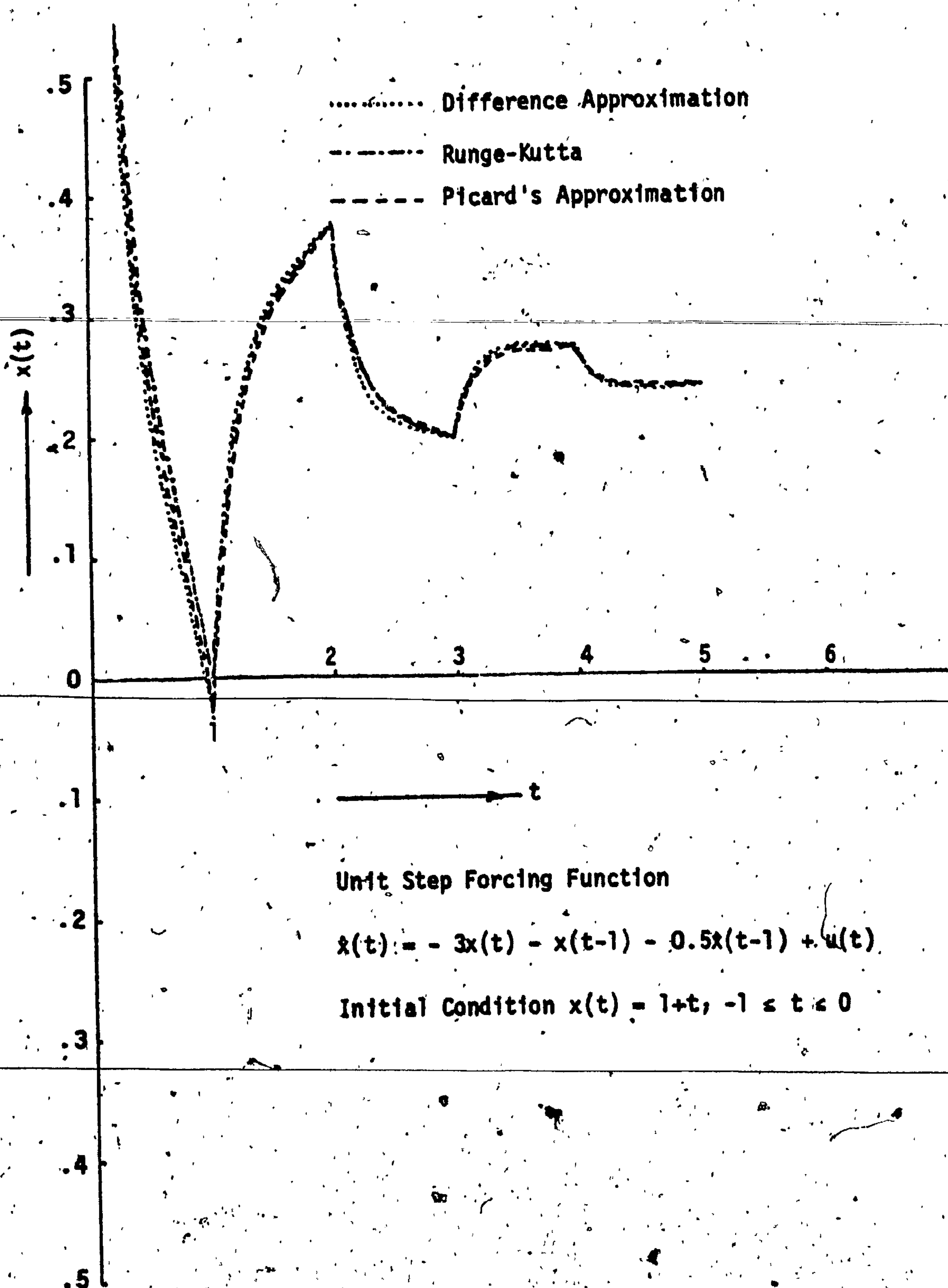


Unit Step Forcing Function

$\dot{x}(t) = -.3x(t) - 0.5\dot{x}(t-1) - x(t-1) + u(t)$

Initial Condition $x(t) = 1-t^2, \; -1 \leq t \leq 0$

Fig. 4.5



Unit Step Forcing Function

$\dot{x}(t) = -3x(t) - x(t-1) - 0.5\dot{x}(t-1) + u(t)$

Initial Condition $x(t) = 1+t, \; -1 \le t \le 0$

Fig. 4.6

4.5 Discussion of Results

Solutions of (4.1) were obtained by Picard's method, Runge-Kutta method and difference approximation method for the following cases:

1. Unforced System with the initial conditions

   (i) $\phi = 1.0 \; ; \quad -1 \leq t \leq 0$

   (ii) $\phi = 1-t^2 \; ; \quad -1 \leq t \leq 0$

   (iii) $\phi = 1+t \; ; \quad -1 < t < 0$

2. Step forcing function with the initial conditions

   (i) $\phi = 1.0 \; ; \quad -1 \leq t \leq 0$

   (ii) $\phi = 1-t^2 \; ; \quad -1 \leq t \leq 0$

   (iii) $\phi = 1+t \; ; \quad -1 \leq t \leq 0$

For an unforced system and initial conditions (i) 1.0, (ii) $1-t^2$, (iii) 1+t the results are shown in Figs. 4.1, 4.2, and 4.3, respectively. The figures show that Picard's iteration gives solutions very close to those obtained by the other two methods. Also, the solutions show that the system reaches steady state when t equals 5, although the solutions were obtained in the interval [0, 10].

For the step forcing function and initial conditions (i) 1.0, (ii) $1-t^2$, (iii) 1+t the results are shown in Figs. 4.4, 4.5, and 4.6, respectively. In this case also Picard's iteration yields solutions very close to the other methods. The numerical solutions indicate that, as far as accuracy of solution is concerned, Picard's method is as good as the Runge-Kutta and difference approximation methods.

As indicated earlier, a total of 20 iterations were required to obtain the solution by Picard's method. The total computer time for 20 iterations was on the average 50% more than that required for the solutions by either of the other two methods. It is therefore felt there exists ample scope for improvement as far as the efficiency of the method is concerned. For instance, reducing the number of iterations would lower time consumption, but at the cost of accuracy. Therefore there is apparently a trade-off between time consumed and accuracy of solution. However, it is felt that improvements can be made by proper choice of initial approximating function, for which further investigations are necessary. It is hoped that further investigation may put Picard's method on more favourable footing against the other two.

CHAPTER 5

CONCLUSIONS

The objective of the study was to apply Picard's method of successive approximation described in Chapter 3 to neutral type functional differential equations. The performance of the method as indicated by the solutions obtained for varying conditions on the differential equation, is very encouraging in terms of computation time and accuracy of solution in comparison to those obtained by the Runge-Kutta and difference approximation methods. Therefore, as far as the application of the method is concerned, it can be stated that Picard's method certainly provides an alternative to the other methods.

However, to determine its advantages over the methods already existing, it is considered worthwhile to continue this work in the areas of computer storage requirements, improvements in the choice of initial approximating function and guidelines in chosing this function.

Results of further investigations in these areas may provide definite answers regarding the efficiency of the method.

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