Overall Network Reliability and Chromatic Polynomials of Special Structures

Sophocles Lee Katsademas

A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements
for the Degree Master of Science at
Concordia University
Montréal, Québec, Canada

December 1986

ISBN 0-315-35525-5

ABSTRACT

Overall Network Reliability
and Chromatic Polynomials
of Special Structures

Sophocles Lee Katsademas

Let G=(V,E) be a graph whose edges may fail with known probabilities and let K a subset of V be specified. The overall reliability of G, denoted by R(G), is the probability that all vertices in K=V communicate with each other. We have two types of graphs, s-p reducible and s-p complex, depending on whether after series-parallel reductions we end up with a single edge or not. A number of s-p reducible graphs are presented and expressions that evaluate their overall reliability are introduced. Chromatic polynomials of a number of s-p reducible graphs are evaluated and with their help the domination and parity, two graph invariants, of these are calculated. A number of results on another graqph invariant, extended domination are proved.

ACKNOWLEDGEMENTS

I am deeply grateful to Professor Zohel Khalil for his constant support and infinite patience. Without his encouragement and help it would have been harder to complete this task.

I would also like to thank my family for the moral support it has given me throughout the last few years.

CONTENTS

INTRODUCTION

Our daily environment is the union of a number of systems. Each of these systems can be viewed as a network. Measures can be evaluated on these networks with reliability being the one most often referred to in todays "aim for perfection" highly computerized society. Reliability of a network given an event of interest is the probability that the network functions, given this event. So if X is our system, 1 or 0 are two states that denote functioning and not functioning respectively and U is an event of interest, then

$$\text{Reliability} = \Pr(X=1 \mid U) \quad .$$

The general network we are going to consider is an undirected graph $G=(V,E)$ with a vertex set $V=\{v_1,\ldots,v_n\}$ and an edge set $E=\{e_1,\ldots,e_m\}$. The existence of an edge e between v_i and v_j implies that communication exists between these two vertices. All vertices in the network are assumed to be functioning at all times. Each edge though may either function or it may not function, with probability being p_e for the former and $q_e=1-p_e$ for the latter. The event of interest mentioned above may be specified by a set K of distinguished vertices such that $|K| \quad |V|$ and $K \quad V$. The network reliability problem is then to compute the probability that all vertices in K communicate with each other. This is referred to as the K-terminal reliability problem, denoted by $R_K(G)$. Special cases are:

i. the source-to-terminal reliability problem, where $K=\{v_i,v_j\}$ denoted by $R_{st}(G)$. Here the problem is to calculate the probability that, say, v_i designated as the source communicates with v_j which is designated as the terminal,

ii. the all-terminal or overall reliability problem, where K=V denoted by R(G). Here the problem is to calculate the probability that all vertices in the network communicate with each other.

This thesis is stuctured as follows:

In Chapter 1 methods to evaluate network reliability ae presented along with domination, extended domination, parity of a graph and classical and extended chromatic polynomials.

In Chapter 2 recursive formulas are established for the overall reliability of a number of s-p reducible graphs. These are incorporated in an algorithm that computes the overall reliability of an s-p complex graph.

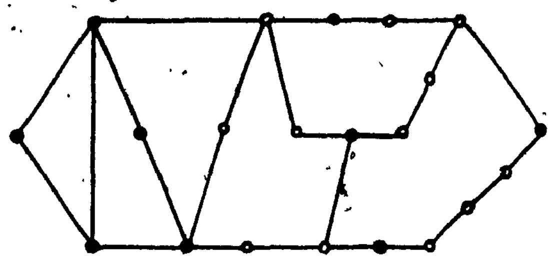
In Chapter 3 chromatic polynomials for some special structures are evaluated. From these we evaluate signed dominations and parities of the special structures.

In Chapter 4 some results are proved about extended domination.

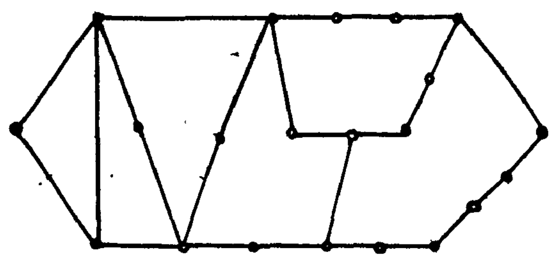
In Appendix I we calculate the number of edges of a ladder and a wheel.

In Appendix II a recursive formula for the number of spanning trees of a ladder is proved and from this a direct formula is derived.

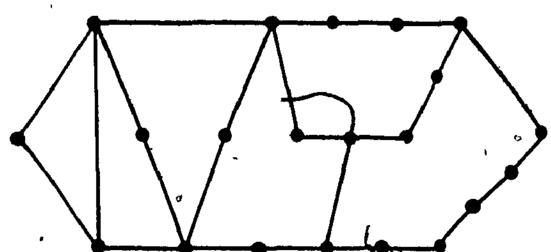
The Advanced Research Projects Network as a



K-terminal reliability problem



source-to-terminal reliability problem



overall reliability problem

where • are vertices belonging to K

and • are vertices not belonging to K .

CHAPTER 1

PRELIMINARIES

A cutvertex of a graph is a vertex of the graph that will cause the graph to be disconnected if it were to be removed. A nonseperable graph is a connected graph that has no cutvertices.

A success set is a minimal set of edges of G such that the vertices in K communicate with each other and is minimal in the sense that the removal of any one single edge will result in breakdown of communication between vertices in K. In terms of the graph of G this translates to a tree of G that contains all points of K. A tree of this kind will be called a K-tree. Edges of G that are in no K-tree of G do not contribute to the reliability expression $R_K(G)$ and are therefore called irrelevant. We will concentrate our attention to relevant edges of G. If every edge of G is in some K-tree of G with respect always to some K, then we shall call this graph a K-graph. A formation of a K-graph is a set of K-trees of G, the union of which results in the graph G itself. A formation is classified as odd or even if the number of K-trees in the formation is odd or even respectively. The signed domination, denoted by $d_K(G)$, of a K-graph G is the number of odd minus the number of even formations of G. The domination, denoted by $D_K(G)$, of a K-graph G is the absolute value of the signed domination, i.e. $D_K(G) = |d_K(G)|$.

1.1. Inclusion-Exclusion method

Many different algorithms have been suggested to calculate network reliability and many of these are based on success sets and/or failure sets. In general it is neither necessary nor desirable to find the family of success or failure sets.

Now the reliability of a graph G with a set K of distinguished vertices is the probability $R_K(G)$ that all edges of at least one success set are functional. Let A_i denote the event that all edges in the i-th success set are functional and let p be the total number of success sets.

Based on the inclusion-exclusion principle the reliability will be given by

$$R_K(G) = \Pr\left(\bigcup_{i=1}^{p} A_i\right) = \sum_{i=1}^{p} \Pr\left(A_i\right) - \\ - \sum_{i=1}^{p}\sum_{j<i}^{p} \Pr\left(A_i A_j\right) + \ldots + (-1)^{p-1} \Pr\left(A_1 A_2 \ldots A_p\right) \qquad (1)$$

This expression has 2^{p-1} terms. It may happen though that the intersections of different A_i's will topologically, with respect to the graph, be the same event. Now if these intersections have different signs (or one has an even number of success sets A_i while the other has an odd number of success sets A_j) they will cancel each other.

When all the cancelations have been performed the coefficient to event $G_p = A_1 A_2 \ldots A_p$, which may correspond to a number of intersections, is going to be the signed domination.

A forest H of G is called a K-forest if H contains all points of K and every point in H that has degree zero or one is in K. Let $F(G,K,j)$ be the collection of all K-forests of G that have exactly j components. A subset F of $F(G,K,j)$ is called a j-formation of G if every edge of G is in some K-forest from F. A j-formation is classified as odd or even if the number of forests is odd or even respectively.

The extended signed domination, denoted by $d(G,K,j)$, of G with respect to some K and a number j, is the number of odd minus the number of even j-formations of G. The extended domination, denoted by $D(G,K,j)$, is the absolute value of the extended signed domination, i.e. $D(G,K,j) = |d(G,K,j)|$.

For j=1 we have the signed domination and domination that we defined earlier. Therefore $F(G,K,1)$ is the collection of all K-trees of G. Now let formations F_i be a subset of $F(G,K,1)$ such that the union of all K-trees in F_i result in a K-subgraph of G that has exactly i edges. There exists a one-to-one correspondance between the terms in (1) and all possible formations F_i of K-subgraphs of G. Since expression (1) has 2^{P-1} terms, there are 2^{P-1} possible F_i formations. Let us call the set of all possible formations H. From any F_i we get one K-subgraph of G, so it may happen that the K-subgraphs that result from two different formations F_i^1 and F_i^2 are identical or they are different but nevertheless have i edges. Let us take partition $\theta = \{\theta_1, \ldots, \theta_h\}$ of H in such a way that $F_i \in \theta_j$, $j=1,\ldots,h$ results in the K-subgraph G_j of G. Therefore $G_1, \ldots, G_h$ is the set of all possible

K-subgraphs of G and the reliability expression can now be written as

$$R_K(G) = \sum_{j=1}^{h} d(G_j, K, 1) Pr(G_j) \qquad (2)$$

Let $S_i = \{ G_1 , \ldots , G_n \}$ be the set of all subgraphs of G such that each subgraph $G_j \in S_i$ has exactly i edges.

Satyanarayana and Khalil introduced the following graph invariant. The i-parity, denoted by $P_i(G)$, is the sum of the signed dominations of all subgraphs S_i , i.e.

$$P_i(G) = \sum_{G_j \in S_i} d_K(G) = \sum_{G_j \in S_i} d(G, K, 1)$$

1.2. Pivotal decomposition or Factoring method

If $R_K(G|e)$ is the reliability of G provided edge e is functional and $R_K(G|e')$ is the reliability of G provided edge e is not functional, then applying the pivotal decomposition introduced by Barlow and Proschan we have

$$R_K(G) = p_e R_K(G|e) + (1-p_e) R_K(G|e') \qquad (3)$$

To compute now the reliability $R_K(G)$ of any graph G we can apply (3) repeatedly. Topologically, if all vertices are assumed to be functional, graph $G|e$ is the same as G_e , where G_e is the graph obtained from G by omitting edge e and coalescing its two end vertices. Similarly, $G|e'$ is the same as $G-e$, where $G-e$ is the graph obtained from G by omitting edge e. Therefore (3) can be rewritten as

$$R_K(G) = p_e R_K(G_e) + (1-p_e) R_K(G-e) \qquad (4)$$

In this form the decomposition is referred to as the

Factoring Theorem.

1.2.1. Factoring for undirected graphs

Topologically after omitting edge e to create G-e and after coalescing the end vertices of e to create G_e, it may happen that these graphs have parallel or series edges. Two edges are parallel edges if they have the same end vertices. Two edges that are not parallel are adjacent if they are incident on the same vertex. If the common vertex has degree-2 the edges are series edges. Replacing a pair of parallel (series) edges by one single edge is called a parallel (series) replacement. Now we may apply some simple reductions to graph G.

Parallel reduction: Let e_1 and e_2 be parallel edges. A parallel reduction replaces edges e_1 and e_2 with one single edge e such that $p_e = 1 - q_{e_1}q_{e_2}$ and $R_K(G) = R_K(G')$, where $G'=(V',E')$ is the new graph that results after the parallel replacement and $V' = V$, $E' = E-e_1-e_2+e$.

Series reduction: Let e_1 and e_2 be series edges with w as their common vertex, where deg(w)=2 and $w \notin K$. A series reduction replaces edges e_1 and e_2 with one single edge e which connects the noncommon vertices of e_1 and e_2 such that $p_e = p_{e_1}p_{e_2}$ and $R_K(G) = R_K(G')$, where $G'=(V',E')$ is the new graph that results after the series replacement and $V' = V-w$, $E' = E-e_1-e_2+e$.

Degree-2 reduction: Let e_1 and e_2 be series edges with w as their common vertex, where deg(w)=2 and all the vertices they are incident to, are K-vertices. A **degree-2 reduction** replaces edges e_1 and e_2 with one single edge e which connects the noncommon vertices of e_1 and e_2 such that $p_e = p_{e_1} p_{e_2} / (1 - q_{e_1} q_{e_2})$ and

$$R_K(G) = (1 - q_{e_1} q_{e_2}) R_{K'}(G'), \text{ where}$$

$G'=(V',E')$ is the new graph that results after the degree-2 replacement and $V' = V-w$, $E' = E-e_1-e_2+e$ and $K' = K-w$.

Now if a graph G can be reduced to a tree, without taking into consideration the set K, by successive series and parallel replacements then it is called a **series-parallel graph**. But now taking into consideration the set K series-parallel graph may or may not take a simple reduction If by successive simple reductions we end up with a sinlge edge then the graph is called an **s-p reducible graph**. If it is not possible to end up with a single edge the graph is called an **s-p complex graph**.

1.2.2. Polygons to chains

Satyanarayana and Wood introduced the concept of chains and polygons in the context of network reliability.[14]

A **chain of length** n is a sequence of n edges and n+1 distinct vertices that alternate in such a way that the internal vertices have degree 2 and the end vertices have degree greater than or equal to 2.

A **polygon** is two chains that have the same end vertices

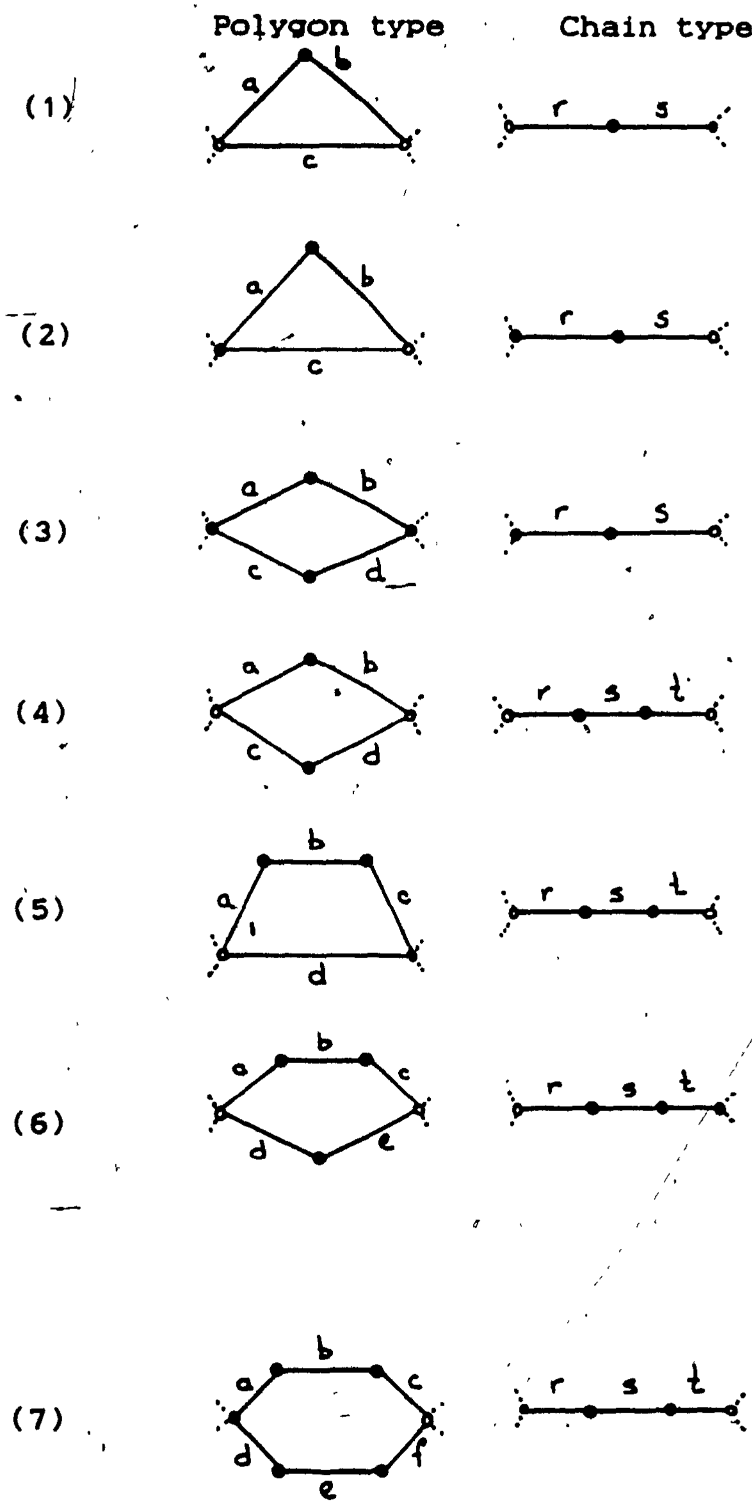


TABLE 1

Polygon-to-chain reductions

TABLE 1 (Continued)

<table>
<tr><th></th><th>Reduction formulas</th><th>New edge reliabilities</th></tr>
<tr><td>(1)</td><td>$\alpha = q_a p_b q_c$
$\beta = p_a q_b q_c$
$\gamma = p_a p_b p_c \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b}\right)$</td><td rowspan="3">$p_r = \frac{\gamma}{\alpha + \gamma}$

$p_s = \frac{\gamma}{\beta + \gamma}$

$\Omega = \frac{(\alpha+\gamma)\cdot(\beta+\gamma)}{\gamma}$</td></tr>
<tr><td>(2)</td><td>$\alpha = q_a p_b q_c$
$\beta = p_a q_b q_c$
$\gamma = p_a p_b p_c \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b}\right)$</td></tr>
<tr><td>(3)</td><td>$\alpha = p_a q_b q_c p_d + q_a p_b p_c p_d$
$\beta = p_a q_b p_c q_d$
$\gamma = p_a p_b p_c p_d \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d}\right)$</td></tr>
<tr><td>(4)</td><td>$\alpha = q_a q_b p_c p_d$
$\beta = p_a q_b q_c p_d + q_a p_b p_c q_d$
$\delta = p_a p_b q_c q_d$
$\gamma = p_a p_b p_c \left(1 + p_d\left(\frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c}\right)\right)$</td><td rowspan="2">$p_r = \frac{\gamma}{\alpha + \gamma}$

$p_s = \frac{\gamma}{\beta + \gamma}$

$p_t = \frac{\gamma}{\beta + \gamma}$</td></tr>
<tr><td>(5)</td><td>$\alpha = q_a p_b p_c q_d$
$\beta = p_a q_b p_c q_d$
$\delta = p_a p_b q_c q_d$
$\gamma = p_a p_b p_c p_d \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d}\right)$</td></tr>
<tr><td>(6)</td><td>$\alpha = q_a p_b p_c q_d$
$\beta = p_a q_b p_c (p_d q_e + q_d p_e) +$
$+ p_b (q_a p_c p_d q_e + p_a q_c q_d p_e)$
$\delta = p_a p_b q_c p_d q_e$</td><td>$\Omega = \frac{(\alpha+\gamma)\cdot(\beta+\gamma)\cdot(\delta+\gamma)}{\gamma^2}$</td></tr>
</table>

TABLE 1 (Continued)

(6)	$\gamma = p_a p_b p_c p_d p_e \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d} + \frac{q_e}{p_e}\right)$	
(7)	$\alpha = q_a p_b p_c q_d p_e p_f$ $\beta = p_a q_b p_c (q_d p_e p_f + p_d q_e p_f + p_d p_e q_f)$ $+ p_a p_b q_c p_f (p_d q_e + q_d p_e)$ $+ q_a p_b p_c q_d (q_e p_f + p_e q_f)$ $\delta = p_a p_b q_c p_d p_e q_f$ $\gamma = p_a p_b p_c p_d p_e p_f \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d} + \frac{q_e}{p_e} + \frac{q_f}{p_f}\right)$	$p_r = \frac{\gamma}{\alpha + \gamma}$ $p_s = \frac{\gamma}{\beta + \gamma}$ $p_t = \frac{\gamma}{\beta + \gamma}$ $\Omega = \frac{(\alpha + \gamma) \cdot (\beta + \gamma) \cdot (\delta + \gamma)}{\gamma^2}$

They prove the following property:

Property 1 : Let G with respect to some K be a graph on which no simple reductions can be performed. If G contains a polygon then it is one of the seven types in Table 1.

So the replacement of a polygon by a chain via the polygon-to-chain reduction has as follows:

Theorem 1.2.2.1. [14]

Suppose G with respect to some K contains a type j polygon. Let G' with respect to some K' be the graph that results after replacing the polygon Δ_j by the chain X_j with new edge probabilities and let Ω_j be the multiplication factor, from Table 1. Then $R_K(G) = \Omega_j R_{K'}(G')$.●

Now for an s-p complex graph they show that:

Property 2 : Let G with respect to some K be an s-p complex graph. Then on G we must be able to perform either a simple reduction or one of the seven polygon-to-chain reductions of Table 1.

This property shows that $R_K(G)$ can be computed in polynomial time for s-p complex graphs. In Chapter 2 an algorithm is presented that combines these results with some new ones to evaluate the reliability.

1.3. Domination theory

As mentioned earlier the domination of a graph G with respect to some K, is the number of odd minus the number of even formations of G.

A theorem proved by Satyanarayana and Chang showed the analogy between domination and the Factoring Theorem.

Theorem 1.3.1. [11]

Let G be a K-graph with respect to some K. If G-e is the graph obtained from G by omitting edge e and G_e is the graph obtained from G by omitting edge e and coalescing its two end vertices, then

$$D_K(G) = D_K(G-e) + D_K(G_e)$$

where $D_K(G)$ is the domination of G with respect to some K.●

Recursive application of the Factoring Theorem leads to the generation of 2^m leaves in a binary structure which corresponds to the enumeration of the states of the graph. But by the appropriate choice of the edge used in the Factoring Theorem, graphs G-e and G_e may happen to contain parallel and/or series edges, which can be reduced using the appropriate reduction. This results in making graphs G-e and G_e smaller and at the same time reducing the number of leaves in the binary structure. Now because the computations required to generate the binary structure are proportional to the number of leaves it has, the optimal binary structure would be the one with the least number of leaves.

Proposition 1.3.1. [11]

Suppose G is a K-graph with respect to some K. Then for any edge in G, at least one of G-e and G_e is a K-graph.●

Proposition 1.3.2.[11]

$D_K(G) \neq 0$ iff G is a K-graph with respect to some K.

Proposition 1.3.3.[11]

$D_K(G)$ remains invariant under series and parallel reductions.

Proposition 1.3.4.[11]

Let G be a K-graph with respect to some K. $D_K(G) = 1$ iff G is either a tree or an s-p reducible graph with respect to some K.

Theorem 1.3.2.[11]

Let G be a complex K-graph with $D_K(G) > 1$. Then G has an edge e such that $D_K(G-e) \neq 0$ and $D_K(G) \neq 0$.

Now according to the above when we pick an edge e which satisfies Theorem 1.3.2. we are going to generate the binary structure with the least number of leaves. This number is $D_K(G)$, the domination of the graph.

1.4. Chromatic Polynomials

A k-vertex coloring of a graph, referred to from now on as k-coloring, is the selection of k distinct colors and their application to the vertices of a graph. A coloring is proper if, when the assignment of colors to the vertices is done, two vertices that are adjacent are not assigned the same color. A graph is k-vertex colorable or k-colorable from now on, if it has a proper k-coloring.

The **chromatic number** of a graph is the minimum number k for which the graph is k-colorable.

If a graph is k-colorable, this means that we can possibly color it in a number of ways, always using these k colors, simply because two colorings are taken to be different if at least one vertex in the graph has two different colors in two colorings.

Birkhoff and Lewis [5] introduced the **chromatic polynomial**, denoted by $P(G;x)$. It is defined as the number of distinct x-colorings that we can possibly assign to graph G, given integer x, the number of colors available.

Here we give two elementary but important results about chromatic polynomials. They concern K_n, the complete graph on n vertices and I_n, the empty graph (with no edges) on n vertices:

Proposition 1.4.1.

(a) $P(K_n;x) = (x-1)(x-2)\cdots(x-n+1)$.

(b) $P(I_n;x) = x^n$. ●

We will now present two ways of calculating the chromatic polynomial of a graph G, by calculating the chromatic polynomial of graphs with fewer number of edges. Continuing in this fashion we will sooner or later come to a complete graph, an empty graph or a graph whose chromatic polynomial has been earlier calculated.

Theorem 1.4.1. (Two-Pieces Theorem)

Let G be a graph that is composed of graphs G_1 and G_2

such that G_1 and G_2 have no vertex in common. Then

$$P(G;x) = P(G_1;x) \cdot P(G_2;x) \quad \bullet$$

Theorem 1.4.2. (Fundamental Reduction)

$$P(G;x) = P(G-e;x) - P(G_e;x) \quad \bullet$$

Here are some of the properties of the chromatic polynomial.

Theorem 1.4.3.

Suppose G is a graph on n vertices and m edges and

$$P(G;x) = a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0 \quad .$$

(a) The degree of $P(G;x)$ is n, i.e. $p = n$.

(b) The coefficient of x^n is 1, i.e. $a_n = 1$.

(c) The absolute value of the coefficient of x^{n-1} is the number of edges of G, i.e. $a_{p-1} = a_{n-1} = m$.

(d) The constant term is 0, i.e. $a_0 = 0$.

(e) Either $P(G;x) = x^n$ or the sum of the coefficients in $P(G;x)$ is 0. $\bullet$

Theorem 1.4.4.

$P(G;x)$ is the sum of consecutive powers of x the coefficients of which alternate in sign, i.e.

$$P(G;x) = x^n - a_{n-1} x^{n-1} + \ldots + (-1)^{n+1} x \quad . \quad \bullet$$

Satyanarayana and Tindell showed that the following relationship exists between the chromatic polynomial and domination.

Proposition 1.4.2. [13]

$$\left|\frac{P(G;x)}{x}\Big|_{x=0}\right| = D_V(G) \quad \bullet$$

They also introduced the concept of the extended chromatic polynomial.

Definition 1.4.1. Given a graph G and some K a K-acyclic orientation of G is an acyclic orientation with all sources and sinks in K.

Definition 1.4.2. Given a graph G and some K a proper x-coloring of K within G is a pair (D,a) where a:K [x] is a coloring and D is a K-acyclic orientation of G such that if u and v are in K and there is a directed path in D from u to v, then $a(u) > a(v)$.

Definition 1.4.3. Given a graph G and some K, the **extended chromatic polynomial**, denoted by $P(G,K;x)$, is defined as the product of $(-1)^{|V|-|K|+i}$ and the number of distinct proper x-colorings of K within G, where i is the number of isolated points of G which are not in K.

The extended chromatic polynomial is related to the classical chromatic polynomial in the following way:

Proposition 1.4.3. [13]

For any graph G :

$$P(G,V;x) = P(G;x) \quad \bullet$$

They also showed that the extended chromatic polynomial is related to the signed domination.

Theorem 1.4.5. [13]

If $G=(V,E)$ is a graph and K is a subset of V, then

$$\left.\frac{P(G,k;x)}{x}\right|_{x=0} = (-1)^{|E|}\, d_K(G)$$

or equivalently

$$d_K(G) = (-1)^{|E|} \left.\frac{P(G,K;x)}{x}\right|_{x=0}$$

CHAPTER 2

OVERALL RELIABILITY OF SPECIAL STRUCTURES

2.1. Definitions

A ladder of order n , denoted by L_n , is a chain of length n-1 such that each of the vertices communicate with a common vertex. (Figure 2.1)

Let v_1 and v_2 be two vertices such that there are k edges connecting them. The structure is called a fan of order k , denoted by F_k . (Figure 2.2)

Let v_1 , v_2 , v_3 be three vertices such that :

i. v_1 is of degree 3 with one edge communicating with v_2 and two edges communicating with v_3 .

ii. v_2 is of degree k+1 with one edge communicating with v_1 and k edges communicating with v_3 .

iii. v_3 is of degree k+2 with two edges communicating with v_1 and k edges communicating with v_2 .

This structure is called an extended fan of order k , denoted by EF_k , where k is the number of edges that connect vertices v_2 and v_3 . (Figure 2.3)

A ladder-fan of order n,k , denoted by L_n^k , is a ladder of order n such that between vertices v_1 and v_2 instead of a single edge there exists a fan of order k . (Figure 2.4)

A wheel of order n , denoted by W_n , is a ladder of order n such that the end vertices in the chain are connected with each other. (Figure 2.5)

A wheel-fan of order n,k , denoted by W^k , is a wheel of order n such that between vertices v_1 and v_2 instead of a single edge there exists a fan of order k. (Figure 2.6)

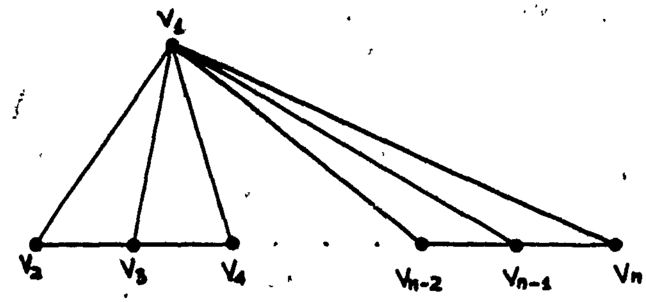


FIGURE 2.1

FIGURE 2.2

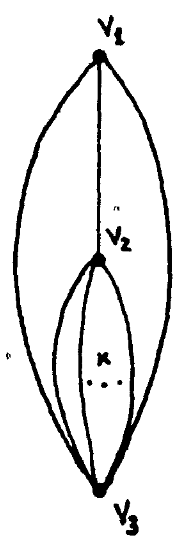


FIGURE 2.3

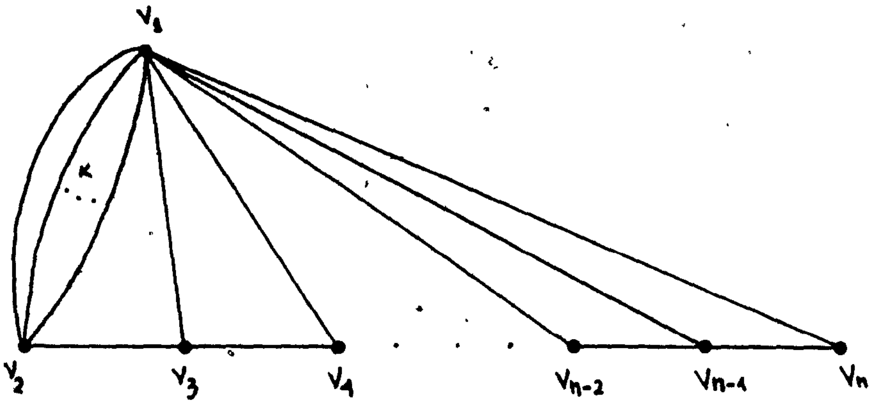


FIGURE 2.4

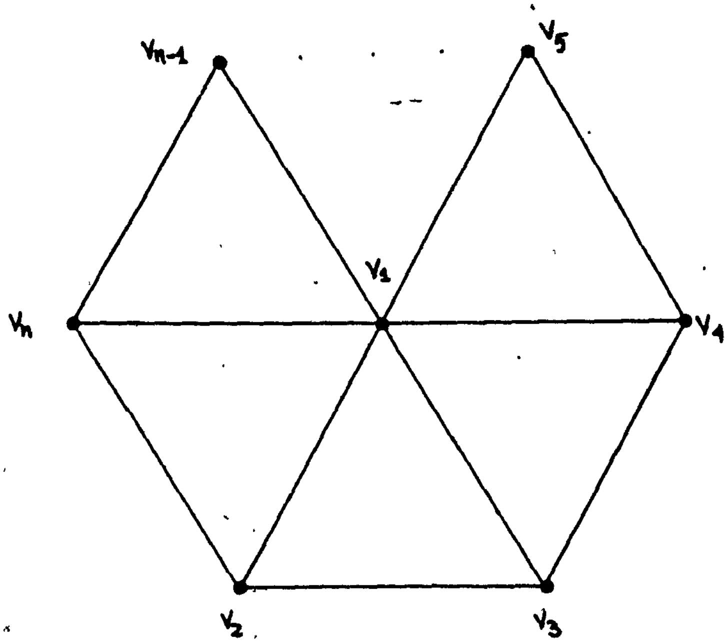


FIGURE 2.5

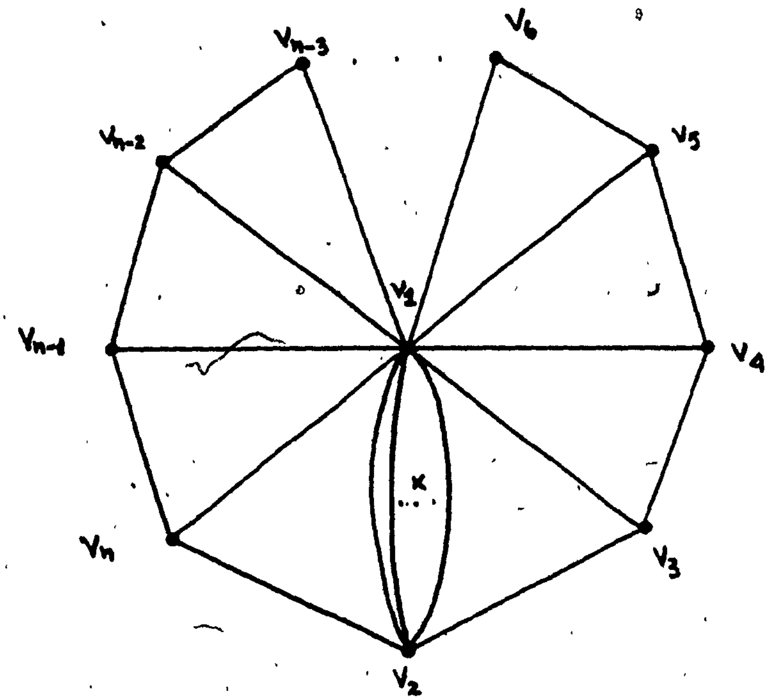


FIGURE 2.6

2.2. Examples

Example 2.2.1. Let us consider a ladder of order 3 (Figure 2.7). Using the Factoring theorem, degree-2 and parallel reductions we have

$$R(L_3) = pR(G_e) + qR(G-e)$$

$$= p(1-q^2) + q(1-q^2)\frac{p^2}{1-q^2}$$

$$= p(1-q^2) + p^2q$$

Example 2.2.2. Let us consider a ladder of order 4 (Figure 2.8). Using the Factoring theorem, degree-2 and parallel reductions we have

$$R(L_4) = pR(G_{e_1}) + qR(G-e_1)$$

$$= p \quad pR(G_{e_1e_2}) + qR(G_{e_1}-e_2) \quad + qR(G-e_1)$$

$$= p^2R(G_{e_1e_2}) + pqR(G_{e_1}-e_2) + qR(G-e_1)$$

$$= p^2(1-q^3) + pq\ p(1-q^2) + q\ pR(L_3)$$

$$= p^2(1-q^3) + p^2q(1-q^2) + pqR(L_3)$$

Example 2.2.3. Let us consider a ladder of order 5 (Figure 2.9). In the same way as above we have

$$R(L_5) = pR(G_{e_1}) + qR(G-e_1)$$

$$= p \quad pR(G_{e_1e_2}) + qR(G_{e_1}-e_2) \quad + qR(G-e_1)$$

$$= p^2R(G_{e_1e_2}) + pqR(G_{e_1}-e_2) + qR(G-e_1)$$

$$= p^2 \quad pR(G_{e_1e_2e_3}) + qR(G_{e_1e_2}-e_3) + pqR(G_{e_1}-e_2) + qR(G-e_1)$$

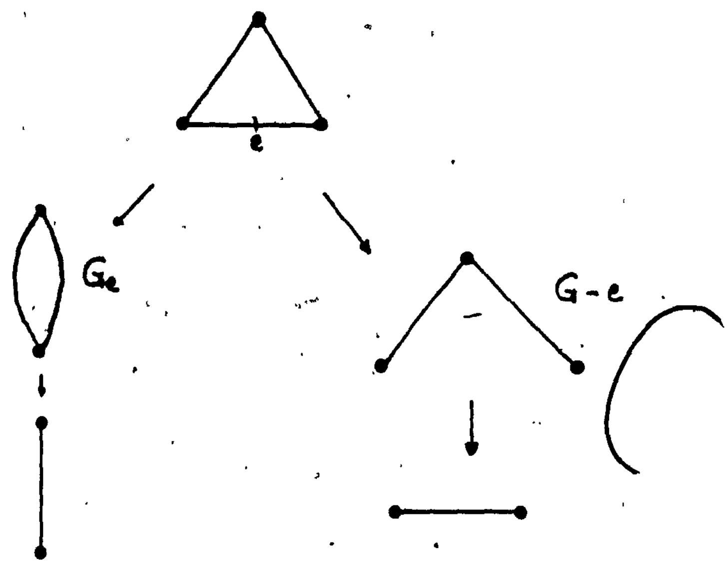


FIGURE 2.7

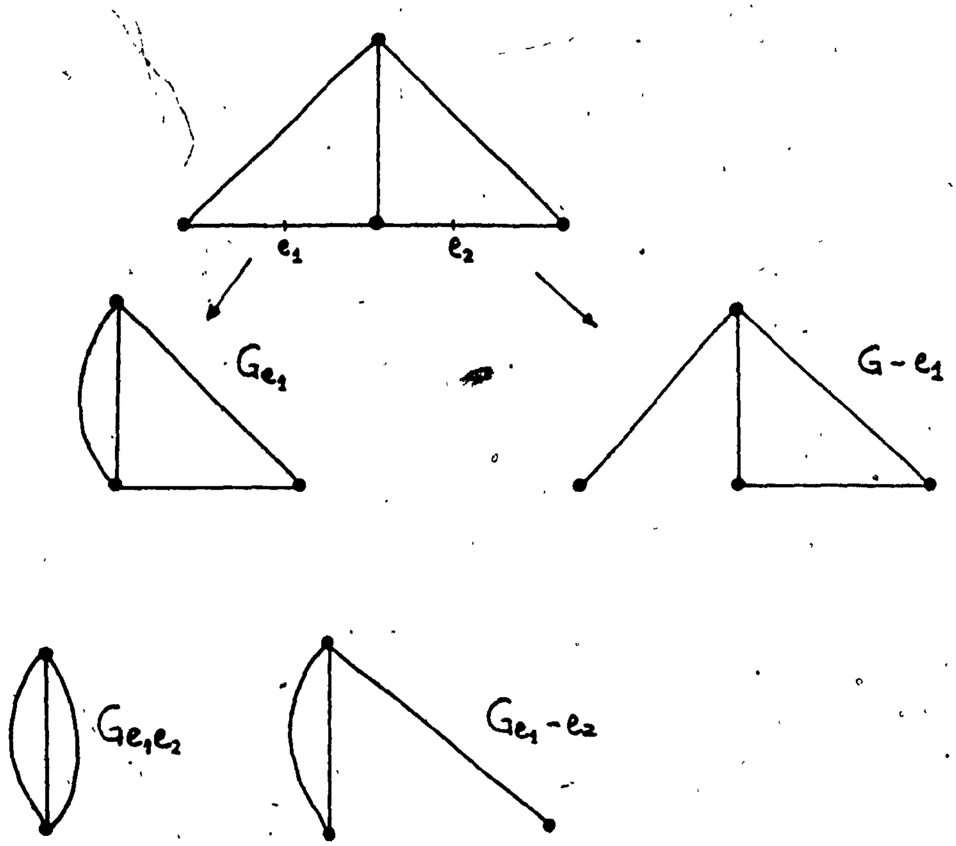


FIGURE 2.8

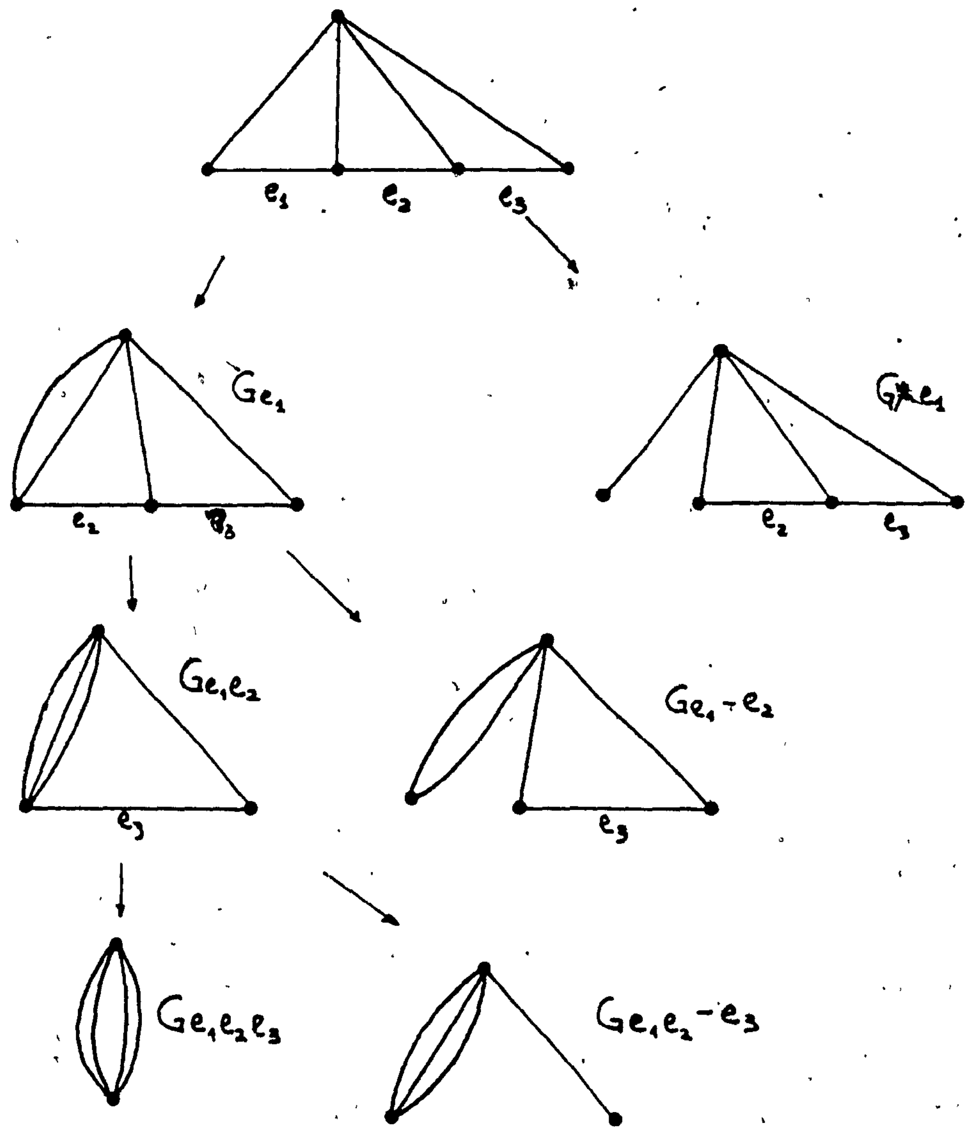


FIGURE 2.9

$$= p^3R(G_{e_1e_2e_3}) + p^2qR(G_{e_1e_2}-e_3) + pqR(G_{e_1}-e_2) + qR(G-e_1)$$

$$= p^3(1-q^4) + p^2q\ p(1-q^3) + pq\ (1-q^2)R(L_3) + q\ pR(L_4)$$

$$= p^3(1-q^4) + p^3q(1-q^3) + pq(1-q^2)R(L_3) + pqR(L_4)$$

2.3. Overall Reliability of ladders, fans and extended fans

Proposition 2.3.1.

The overall reliability of a ladder of order n , n>2 is given by

$$R(L_n)= p^{n-2}(1-q^{n-1}) +p^{n-2}q(1-q^{n-2})+ \sum_{i=1}^{n-3} p^{i-1}q(1-q^i)R(L_{n-i})$$

Proof: For n=3 , n=4 , n=5 from examples 2.2.1. 2.2.2. and 2.2.3. we have

$$R(L_3)=p(1-q^2) + p^2q$$

$$=p^{3-2}(1-q^{3-1}) + pqp$$

$$=p^{3-2}(1-q^{3-1}) + p^{3-2}q(1-q^{3-2})$$

$$R(L_4)=p^2(1-q^3) + p^2q(1-q^2) + pqR(L_3)$$

$$=p^{4-2}(1-q^{4-1}) + p^{4-2}q(1-q^{4-2}) + p^0q(1-q)R(L_3)$$

$$=p^{4-2}(1-q^{4-1})+p^{4-2}q(1-q^{4-2}) + \sum_{i=1}^{4-3} p^{i-1}q(1-q^i)R(L_{4-i})$$

$$R(L_5)=p^3(1-q^4) + p^3q(1-q^3) + pq(1-q^2)R(L_3) + pqR(L_4)$$

$$=p^{5-2}(1-q^{5-1})+p^{5-2}q(1-q^{5-2})+p^0q(1-q)R(L_4)+pq(1-q^2)R(L_3)$$

$$p^{5-2}(1-q^{5-1}) +p^{5-2}q(1-q^{5-2})+ \sum_{i=1}^{5-3} p^{i-1}q(1-q^i)R(L_{5-i})$$

Therefore it is true for these values of n. Let it be true for n=k and all n<k, i.e.

$$R(L_k)= p^{k-2}(1-q^{k-1}) +p^{k-2}q(1-q^{k-2})+ \sum_{i=1}^{k-3} p^{i-1}q(1-q^i)R(L_{n-i})$$

Let us consider L_{k+1} (Figure 2.10). By successively applying the Factoring Theorem, degree-2 and parallel reductions we have

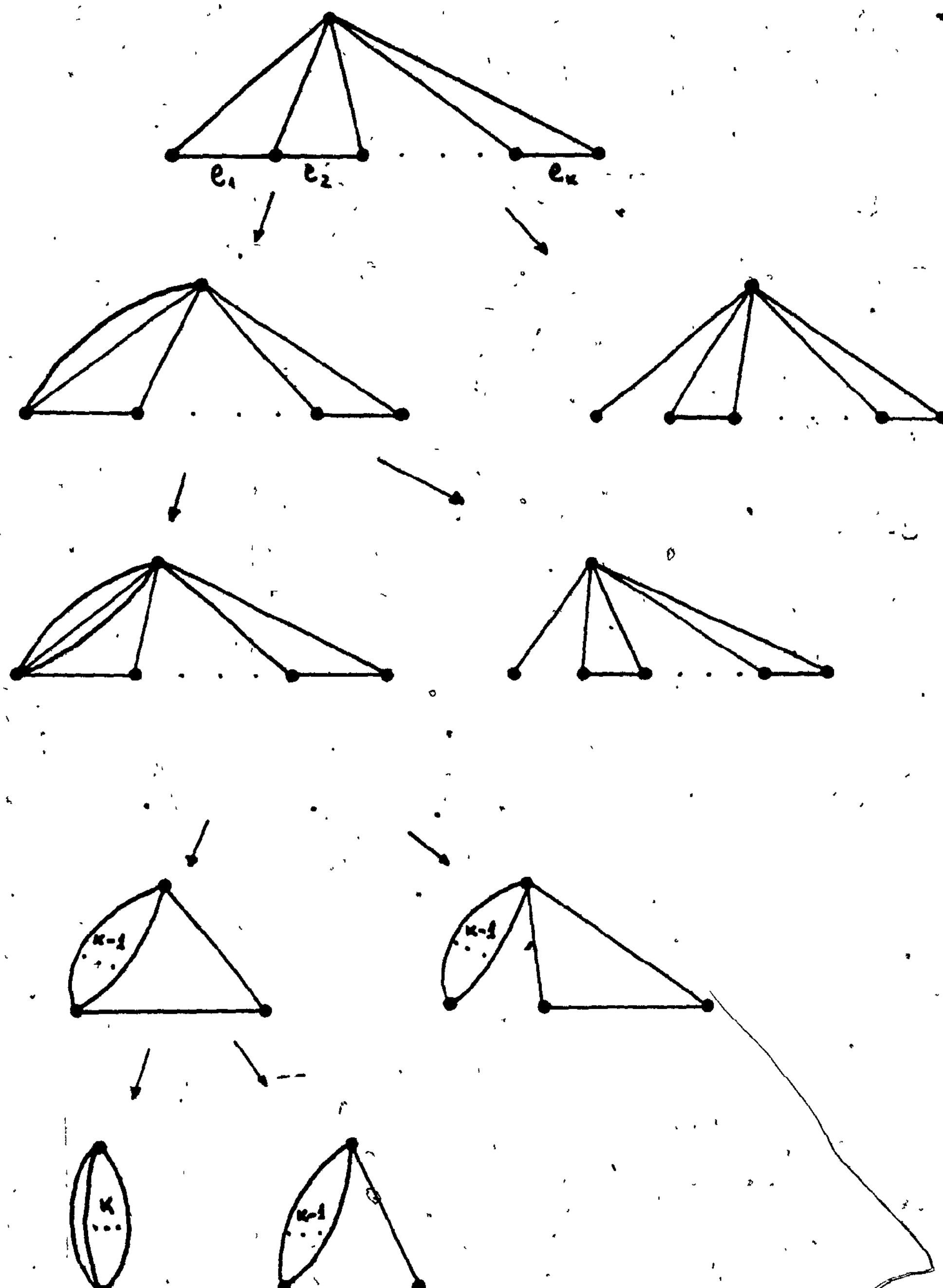


FIGURE 2.10

$$
\begin{aligned}
R(L_{k+1}) &= pR(L_{k+1}\cdot e_1) + qR(L_{k+1}-e_1) \\
&= p\Big(pR(L_{k+1}\cdot e_1e_2) + qR(L_{k+1}\cdot e_1-e_2)\Big) + qR(L_{k+1}-e_1) \\
&= p^2R(L_{k+1}\cdot e_1e_2) + pqR(L_{k+1}\cdot e_1-e_2) + qR(L_{k+1}-e_1) \\
&= p^2R(L_{k+1}\cdot e_1e_2) + pq(1-q^2)R(L_{k-1}) + qpR(L_k) \\
&= \ \cdot \quad \cdot \quad \cdot \\
&= p^{k-2}\Big(p(1-q^k)+qp(1-q^{k-1})\Big) + \sum_{i=1}^{k-2} p^{i-1}q(1-q^i)R(L_{k+1-i}) \\
&= p^{(k+1)-2}(1-q^{(k+1)-1}) + p^{(k+1)-2}q(1-q^{(k+1)-2}) + \\
&\qquad + \sum_{i=1}^{(k+1)-3} p^{i-1}q(1-q^i)R(L_{(k+1)-i})
\end{aligned}
$$

Since it is true for $n=k+1$, by the induction hypothesis, it is true for all n, $n>2$.

The overall reliability of a fan of order k, as a parallel structure is

$$\overline{R}(F_k) = 1 - q^k \; .$$

Proposition 2.3.2.

— The overall reliability of an extended fan of order k, $k>2$ is given by

$$
\begin{aligned}
R(EF_k) &= (1 - q^{k+1})\left(1 - q^2\left(1 - \frac{p(1 - q^k)}{1 - q^{k+1}}\right)\right) \\
&= 1 - q^3 - q^{k+1}(1 + q - 2q^2)
\end{aligned}
$$

Proof: Let us consider EF_k (Figure 2.11). By the previous observation $p_a = 1 - q^k$ and by a parallel reduction $p_b = 1 - q^2$.

By a degree-2 reduction $p_c = \dfrac{p(1 - q^k)}{1 - q^{k+1}}$

By a parallel reduction $p_d = 1 - q^2 \; 1 - \dfrac{p(1 - q^k)}{1 - q^{k+1}}$

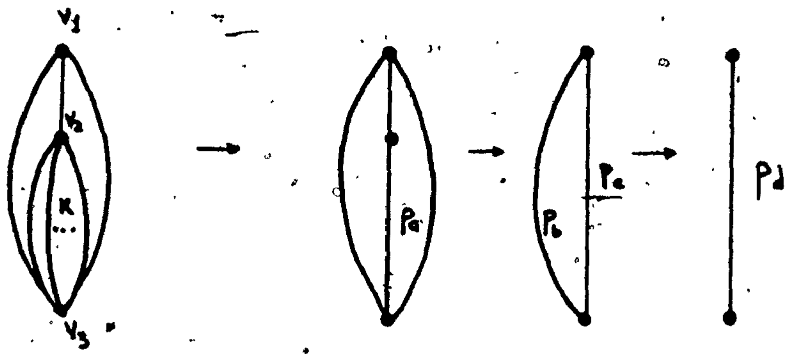


FIGURE 2.11

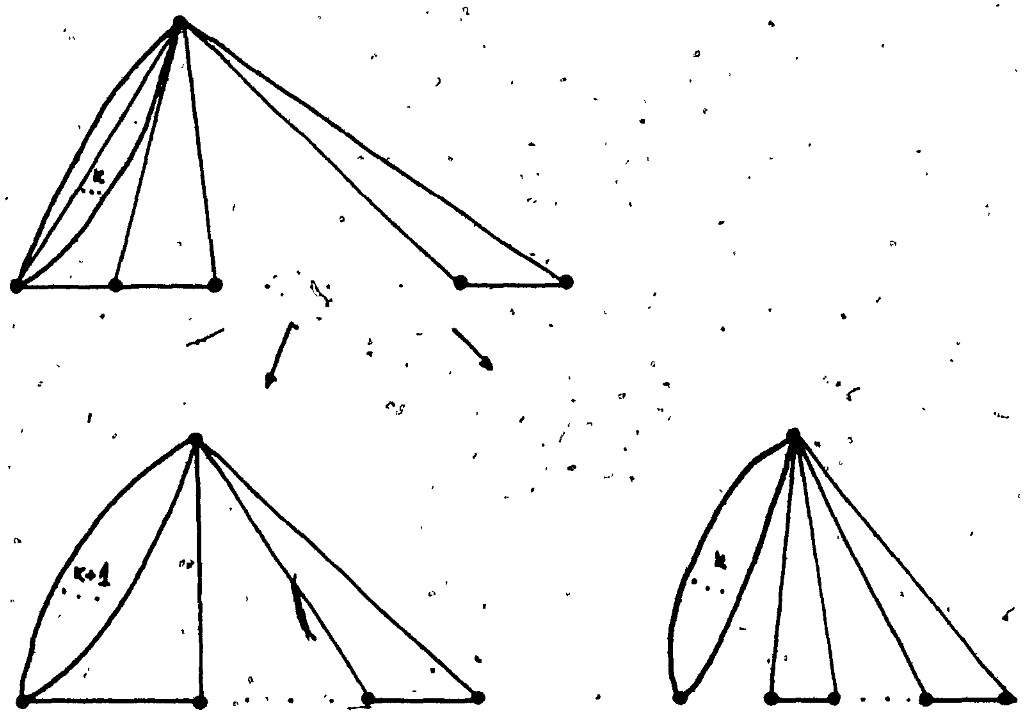


FIGURE 2.12

Therefore

$$R(EF_k) = (1 - q^{k+1})\left(1 - q^2\left(1 - \frac{p(1 - q^k)}{1 - q^{k+1}}\right)\right)$$

$$= (1 - q^{k+1})\left(\frac{1 - q^{k+1} - q^2\left(1 - q^{k+1} - p(1 - q^k)\right)}{1 - q^{k+1}}\right)$$

$$= 1 - q^{k+1} - q^2 + q^{k+3} + pq^2 - pq^{k+2}$$

$$= 1 - q^{k+1} - q^2 + q^{k+3} + q^2 - q^3 - q^{k+2} + q^{k+3}$$

$$= 1 - q^3 - q^{k+1}(1 + q - 2q^2) \quad . \bullet$$

2.4. Overall Reliability of ladder-fans, wheel-fans, wheels

Proposition 2.4.1.

The overall reliability of a ladder-fan of order n,k, $n>3$ is given by

$$R(L_n^k) = pR(L_{n-1}^{k+1}) + q(1 - q^k)R(L_{n-1})$$

$$= p^{n-3}R(L_3^{k+n-3}) + \sum_{i=0}^{n-4} p^i(1 - q^{k+i})R(L_{n-i-1})$$

Proof: Let us consider L_n^k (Figure 2.12). Using the Factoring theorem we have two new graphs, one being a ladder-fan of order $n-1,k+1$ and the other being a ladder of order $n-1$ with a fan of order k attatched to it.

Therefore

$$R(L_n^k) = pR(L_{n-1}^{k+1}) + qR(F_k)R(L_{n-1})$$

$$= pR(L_{n-1}^{k+1}) + p(1 - q^k)R(L_{n-1}) \qquad (5)$$

The second part is proved by induction.

For $n=4$ using (5) we have

$$R(L_4^k) = pR(L_{4-1}^{k+1}) + q(1 - q^k)R(L_{4-1})$$

$$= p^{4-3}R(L_3^{k+4-3}) + p^0q(1 - q^{k+0})R(L_{4-0-1})$$

$$= p^{4-3}R(L_3^{k+4-3}) + \sum_{i=0}^{4-4} p^iq(1 - q^{k+i})R(L_{4-i-1})$$

Therefore it is true for $n=4$. Let it be true for $n=m$ and all $n<m$, i.e.

$$R(L_m^k) = p^{m-3}R(L_3^{k+m-3}) + \sum_{i=0}^{m-4} p^i q(1 - q^{k+i})R(L_{m-i-1})$$

Let us consider L_{m+1}^k. Then by (5)

$$\begin{aligned}
R(L_{m+1}^k) &= pR(L_m^{k+1}) + q(1 - q^k)R(L_{(m+1)-1}) \\
&= p\Big(p^{m-3}R(L_3^{k+1+m-3}) + \sum_{i=0}^{m-4} p^i q(1 - q^{k+i+1})R(L_{m-i-1})\Big) + \\
&\qquad + q(1 - q^k)R(L_{(m+1)-1}) \\
&= p^{m+1-3}R(L_3^{k+m+1-3}) + \sum_{i=0}^{m-4} p^{i+1} q(1 - q^{k+i+1})R(L_{m-i-1}) + \\
&\qquad + p^{-1+1}q(1 - q^{k-1+1})R(L_{m-(-1)-1}) \\
&= p^{m+1-3}R(L_3^{k+m+1-3}) + \sum_{i=-1}^{m-4} p^{i+1} q(1 - q^{k+i+1})R(L_{m-i-1}) \\
&= p^{m+1-3}R(L_3^{k+m+1-3}) + \sum_{i=0}^{(m+1)-4} p^i q(1 - q^{k+i})R(L_{m+1-i-1})
\end{aligned}$$

Since it is true for n=m+1 , by the induction hypothesis, it is true for all n, n>3.

Proposition 2.4.2.

The overall reliability of a wheel-fan of order n,k , n>3 is given by

$$\begin{aligned}
R(W_n^k) &= pR(W_{n-1}^{k+1}) + qR(L_n^k) \\
&= p^{n-3}R(EF_{k+n-3}) + \sum_{i=0}^{n-4} p^i qR(L_{n-i}^{k+i})
\end{aligned}$$

Proof: Let us consider W_n^k (Figure 2.13). Using the Factoring theorem we have two new graphs, one being a whell-fan of order n-1,k+1 and the other being a ladder-fan of order n,k . Therefore

$$R(W_n^k) = pR(W_{n-1}^{k+1}) + qR(L_n^k)$$

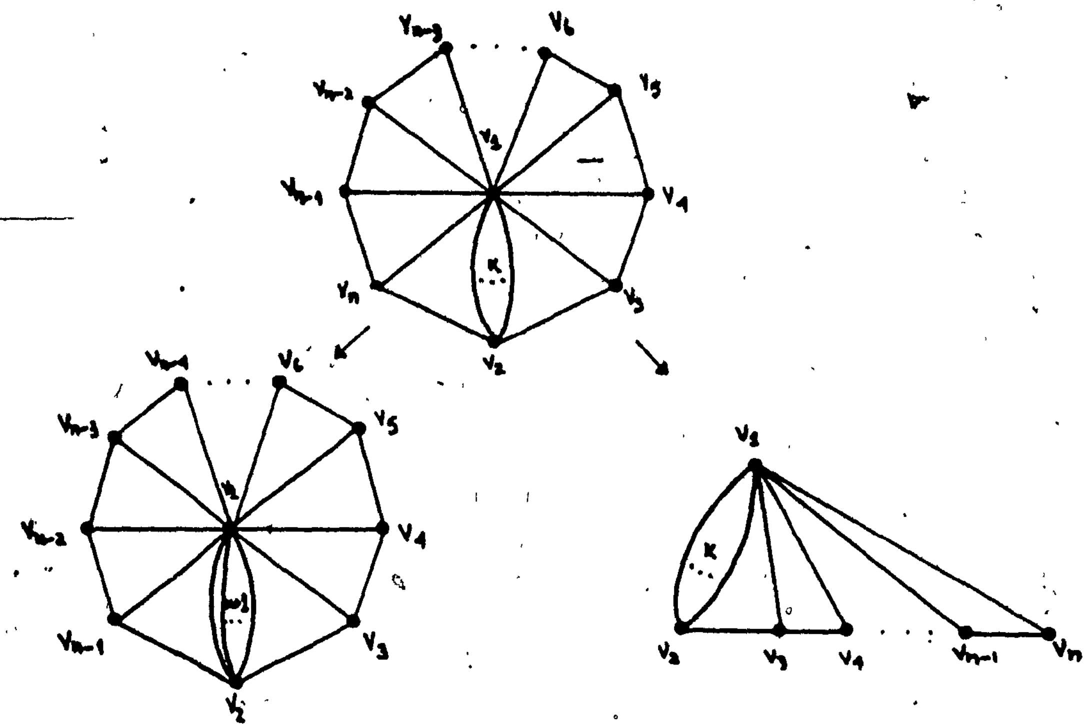


FIGURE 2.13

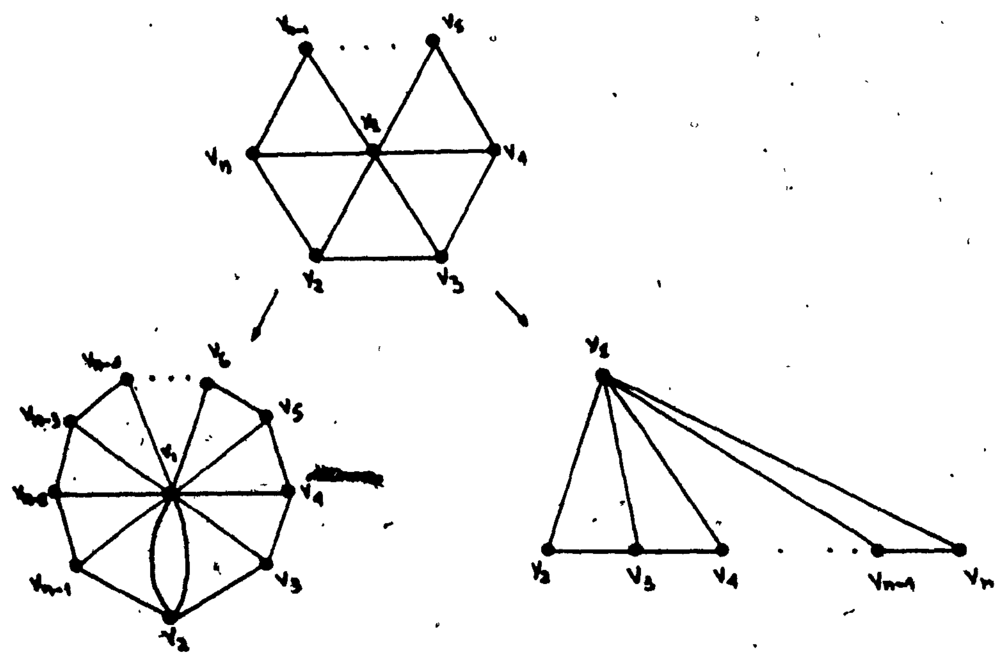


FIGURE 2.14

The second part is proved by induction.

For n=4 using (6) and the fact that $W_3^k = EF_k$ we have

$$R(W_4^k) = pR(W_3^{k+1}) + qR(L_4^k)$$
$$= p^{4-3}R(EF_{k+1}) + p^0 qR(L_4^k)$$
$$= p^{4-3}R(EF_{k+1}) + \sum_{i=0}^{4-4} p^i qR(L_{4-i}^{k+i}) \quad .$$

Therefore it is true for n=4. Let it be true for n=m and all n<m, i.e.

$$R(W_m^k) = p^{m-3}R(EF_{k+m-3}) + \sum_{i=0}^{m-4} p^i qR(L_{m-i}^{k+i})$$

Let us consider W_{m+1}^k . Then by (6)

$$R(W_{m+1}^k) = pR(W_m^{k+1}) + qR(L_{m+1}^k)$$
$$= p\left(p^{m-3}R(EF_{k+1+m-3}) + \sum_{i=0}^{m-4} p^i qR(L_{m-i}^{k+1+i})\right) + qR(L_{m+1}^k)$$
$$= p^{(m+1)-3}R(EF_{k+(m+1)-3}) + \sum_{i=0}^{m-4} p^{i+1} qR(L_{m-i}^{k+i+1}) + p^0 qR(L_{m+1}^k)$$
$$= p^{(m+1)-3}R(EF_{k+(m+1)-3}) + \sum_{i=-1}^{m-4} p^{i+1} qR(L_{(m+1)-(i+1)}^{k+(i+1)})$$
$$= p^{(m+1)-3}R(EF_{k+(m+1)-3}) + \sum_{i=0}^{(m+1)-3} p^i qR(L_{(m+1)-i}^{k+i})$$

Since it is true for n=m+1 , by the induction hypothesis, it is true for all n, n>3. •

Proposition 2.4.3.

The overall reliability of a wheel of order n , n>3 is given by

$$R(W_n) = pR(W_{n-1}^2) + qR(L_n)$$
$$= p^{n-3}R(EF_{n-2}) + \sum_{i=0}^{n-4} p^i qR(L_{n-i}^{i+1})$$

Proof: Let us consider W_n (Figure 2.14). Using the Factoring theorem we have two new graphs, one being a wheel-fan of order $n-1,2$ and the other being a ladder of order n. Therefore

$$R(W_n) = pR(W_{n-1}^2) + qR(L_n) \qquad (7)$$

From proposition 2.4.2. we have that

$$R(W_{n-1}^2) = p^{(n-1)-3}R(EF_{2+(n-1)-3}) + \sum_{i=0}^{(n-1)-4} p^i qR(L_{(n-1)-i}^{2+i})$$

Replacing in (7) we get

$$\begin{aligned} R(W_n) &= p\left(p^{n-4}R(EF_{n-2}) + \sum_{i=0}^{n-5} p^i qR(L_{n-(i+1)}^{(i+1)+1})\right) + qR(L_n^1) \\ &= p^{n-3}R(EF_{n-2}) + \sum_{i=0}^{n-5} p^{i+1} qR(L_{n-(i+1)}^{(i+1)+1}) + p^0 qR(L_{n-(-1+1)}^{(-1+1)+1}) \\ &= p^{n-3}R(EF_{n-2}) + \sum_{i=-1}^{n-5} p^{i+1} qR(L_{n-(i+1)}^{(i+1)+1}) \\ &= p^{n-3}R(EF_{n-2}) + \sum_{i=0}^{n-4} p^i qR(L_{n-i}^{i+1}) \quad . \end{aligned}$$

2.5. Algorithm for computing overall reliability

From what has been presented in Chapter 1 and what was established here we can construct the following algorithm for computing the overall reliability of an s-p complex graph :

```
OVERALL_RELIABILITY (G) ;
Begin
  REL := 1 ;
  While  G contains a special structure  do
  begin
    case  special structure  of
          ladder          :  REL := REL R(L_n)   ;
          fan             :  REL := REL R(F_k)   ;
          extended fan    :  REL := REL R(EF_k)  ;
          ladder-fan      :  REL := REL R(L_n^k) ;
          wheel-fan       :  REL := REL R(W_n^k) ;
          wheel           :  REL := REL R(W_n)   ;
    end;
  end;
  While  G is not a single edge  do
  begin
    case  reduction  of
          parallel           :  REL := REL ;
          series             :  REL := REL ;
          degree-2           :  REL := REL (p_a + p_b - p_a p_b) ;
          polygon-to-chain   :  REL := REL   _j ;
    end;
  end;
  OVERALL_RELIABILITY := REL ;
End.
```

CHAPTER 3

CHROMATIC POLYNOMIALS AND NETWORK RELIABILITY

3.1. Chromatic polynomials of ladders and wheels

Example 3.1.1. Let us consider a ladder of order 3 (Figure 3.1). Using the Two-Pieces theorem and the Fundamental Reduction theorem we have

$$\begin{aligned} P(L_3;x) &= P(G-e_1;x) - P(G_{e_1};x) \\ &= P(G-e_1-e_2;x) - P(G-e_1e_2;x) - P(G_{e_1};x) \\ &= x\ x(x-1) - x(x-1) - x(x-1) \\ &= x(x-1)(x-1-1) \\ &= x(x-1)(x-2) \end{aligned}$$

Proposition 3.1.1.

The chromatic polynomial $P(L_n;x)$ of a ladder of order n, $n>2$ is given by

$$P(L_n;x) = x(x-1)(x-2)^{n-2}$$

Proof: For $n=3$ from example 3.1.1. we have

$$P(L_3;x) = x(x-1)(x-2)$$

Therefore it is true for $n=3$. Let it be true for $n=k$, i.e.

$$P(L_k;x) = x(x-1)(x-2)^{k-2}$$

Let us consider L_{k+1} (Figure 3.2). By the Two-Pieces theorem and the Fundamental Reduction theorem we have

$$\begin{aligned} P(L_{k+1};x) &= P(G-e;x) - P(L_k;x) \\ &= x\ P(L_k;x) - P(L_k;x) - P(L_k;x) \\ &= (x-2)\ P(L_k;x) \\ &= (x-2)\ x(x-1)(x-2)^{k-2} \\ &= x(x-1)(x-2)^{(k+1)-2} \end{aligned}$$

Since it is true for $n=k+1$, by the induction hypothesis, it is true for all n, $\overline{n>2}$. ●

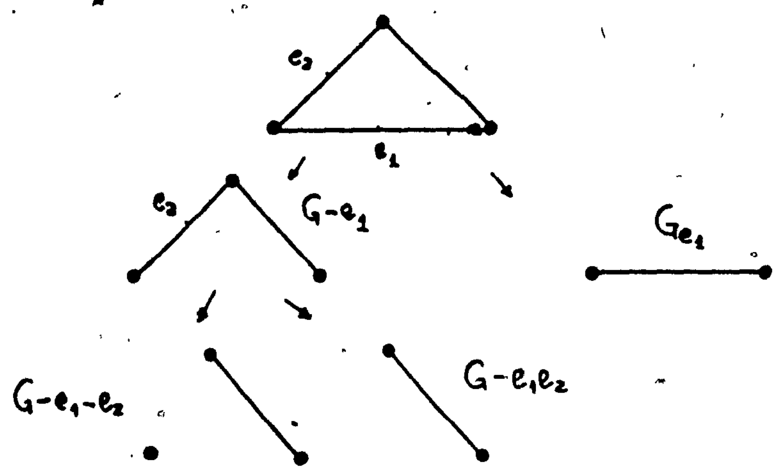


FIGURE 3.1

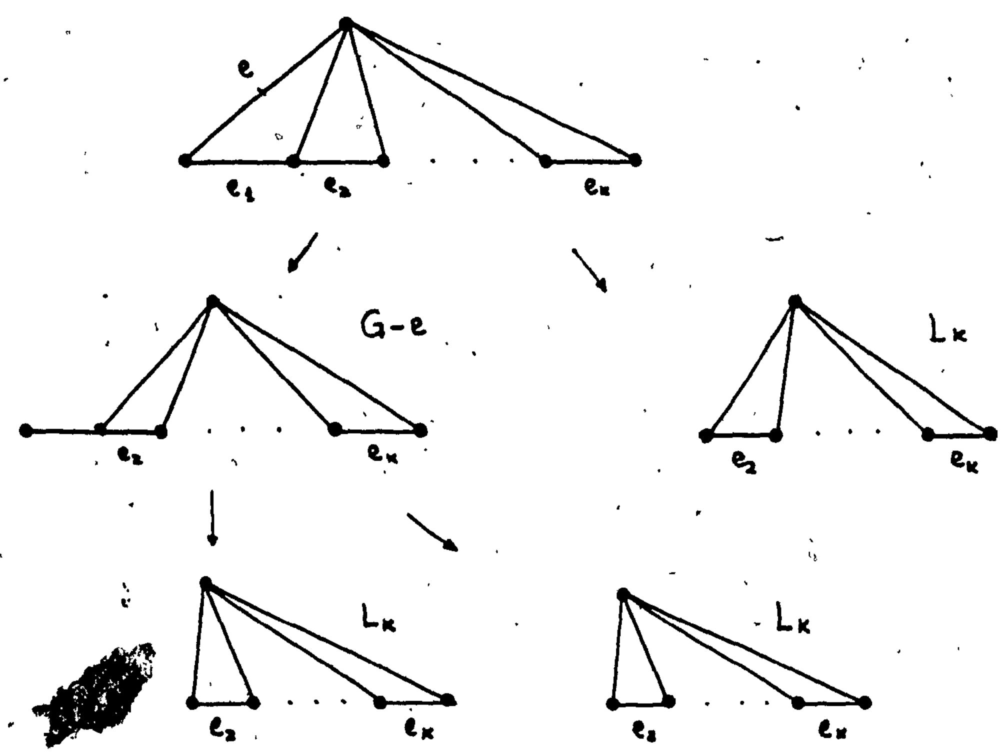


FIGURE 3.2

Example 3.1.2. Let us consider a wheel of order 4 (Figure 3.3). Using the Two-Pieces theorem, the Fundamental Reduction theorem and Proposition 3.1.1. we have

$$\begin{aligned} P(W_4;x) &= P(L_4;x) - P(L_3;x) \\ &= x(x-1)(x-2)^2 - x(x-1)(x-2) \\ &= x(x-1)(x-2)(x-3) \end{aligned}$$

Proposition 3.1.2.

The chromatic polynomial $P(W_n;x)$ of a wheel of order n, $n>3$ is given by

$$\begin{aligned} P(W_n;x) &= (-1)^n x(x-1)(x-1)(x-3) + \sum_{i=3}^{n-2} (-1)^{n-i} x(x-1)(x-2)^i \\ &= (-1)^n x(x-1)(x-2)(x-3) + (-1)^n x(2-x)^3 \left(1-(2-x)^{n-4}\right) \end{aligned}$$

Proof: For $n=4$ from example 3.1.2. we have

$$P(W_4;x) = x(x-1)(x-2)(x-3)$$

Therefore it is true for $n=4$. Let it be true for $n=k$, i.e.

$$P(W_k;x) = (-1)^n x(x-1)(x-2)(x-3) + \sum_{i=3}^{k-2} (-1)^{k-i} x(x-1)(x-2)^i$$

Let us consider W_{k+1} (Figure 3.4). By the Fundamental Reduction theorem and Proposition 3.1.1. we have

$$\begin{aligned} P(W_{k+1};x) &= P(L_{k+1};x) - P(W_k;x) \\ &= x(x-1)(x-2)^{(k+1)-2} - (-1)^k x(x-1)(x-2)(x-3) - \\ &\qquad - \sum_{i=3}^{k-2} (-1)^{k-i} x(x-1)(x-2)^i \\ &= (-1)^{k+1} x(x-1)(x-2)(x-3) + \\ &\qquad + \sum_{i=3}^{(k+1)-2} (-1)^{(k+1)-i} x(x-1)(x-2)^i \end{aligned}$$

Since it is true for $n=k+1$, by the induction hypothesis,

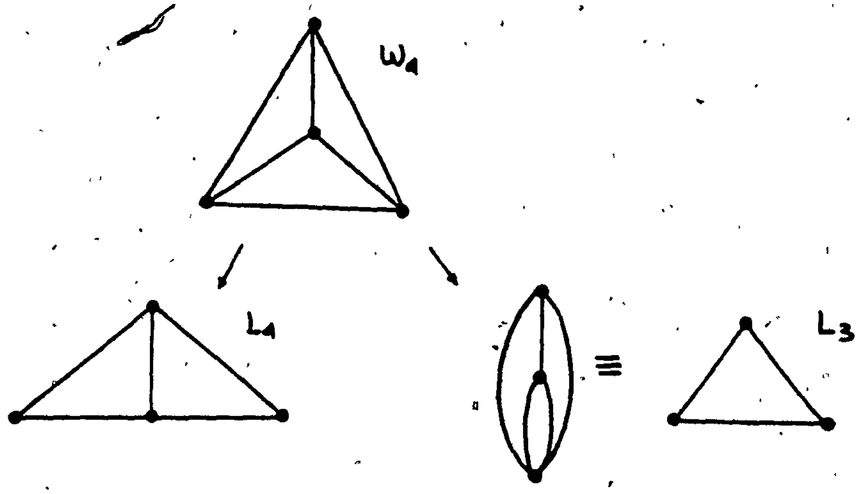


FIGURE 3.3

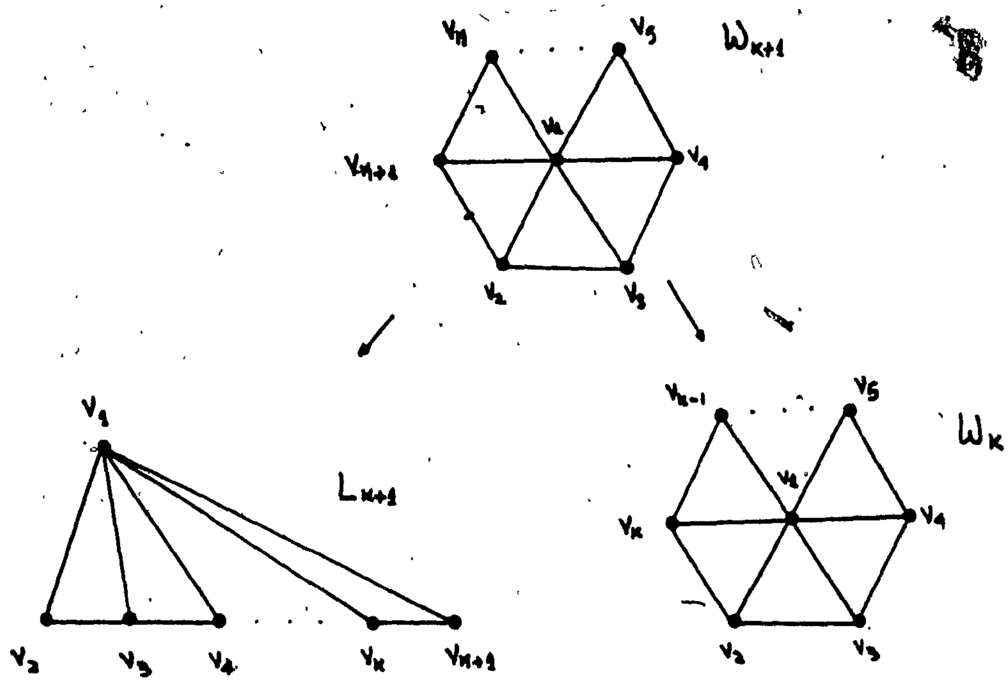


FIGURE 3.4

it is true for all n, n>3.

We now have to show that

$$\sum_{i=3}^{n-2} (-1)^{n-i}x(x-1)(x-2)^{i} = (-1)^{n}x(2-x)^{3}\left(1-(2-x)^{n-4}\right)$$

Therefore taking the left hand side we have

$$\begin{aligned}\sum_{i=3}^{n-2} (-1)^{n-i}x(x-1)(x-2)^{i} &= x(x-1) \sum_{i=3}^{n-2} (-1)^{n}(-1)^{-i}(x-2)^{i} \\ &= (-1)^{n}x(x-1) \sum_{i=3}^{n-2} (2-x)^{i} \\ &= (-1)^{n}x(x-1)(2-x)^{3} \sum_{i=0}^{n-5} (2-x)^{i} \\ &= (-1)^{n}x(x-1)(2-x)^{3}\cdot \frac{1-(2-x)^{(n-5)+1}}{1-(2-x)} \\ &= (-1)^{n}x(x-1)(2-x)^{3}\cdot \frac{1-(2-x)^{n-4}}{x-1} \\ &= (-1)^{n}x(2-x)^{3}\left(1-(2-x)^{n-4}\right) \bullet\end{aligned}$$

3.2. Signed domination, chromatic polynomials and network reliability

Based on the results of Satyanarayana and Tindell we have the following.

Corollary 3.2.1.

For any graph $G=(V,E)$.

$$d(G) = (-1)^{|E|} \left. \frac{P(G;x)}{x} \right|_{x=0}$$

Proof: Immediate from Proposition 1.4.3. and Theorem 1.4.5. . •

Corollary 3.2.2.

If $G=(V,E)$ is a graph and K is a subset of V , then

$$P_i(G) = \sum_{G_j \in S_i} (-1)^{|E|} \left. \frac{P(G_j,K;x)}{x} \right|_{x=0}$$

Proof: Immediate from the definition of $P_i(G)$ and theorem 1.4.5. . •

Corollary 3.2.3.

For any graph $G=(V,E)$

$$P_i(G) = \sum_{G_j \in S_i} (-1)^{|E|} \left. \frac{P(G_j;x)}{x} \right|_{x=0}$$

Proof: Immediate from Proposition 1.4.3. and Corollary 3.2.2. . •

Satyanarayana and Khalil showed the following.

Theorem 3.2.1.

The reliability of a graph $G=(V,E)$ is given by

$$R_K(G) = \sum_i P_i(G) \cdot p^i$$

Based on this we have

Corollary 3.2.4.

The reliability of any graph $G=(V,E)$ is given by

$$R_K(G) = \sum_i \sum_{G_j \in S_i} (-1)^{|E|} \left. \frac{P(G_j,K;x)}{x} \right|_{x=0} \cdot p^i$$

Proof: Immediate from Theorem 3.2.1. and Corollary 3.2.2. . •

Corollary 3.2.5.

For any graph $G=(V,E)$

$$R(G) = \sum_i \sum_{G_j \in S_i} (-1)^{|E|} \left. \frac{P(G_j;x)}{x} \right|_{x=0} \cdot p^i$$

Proof: Immediate from Proposition 1.4.3. and Corollary 3.2.4. ●

Now using these formulas we can come to expressions for the signed domination of ladders and wheels.

Proposition 3.2.1.

The signed domination of a ladder of order n , $n>2$ is given by

$$d(L_n) = (-1)^n 2^{n-2}$$

Proof: By Corollary 3.2.1. we have

$$d(L_n) = (-1)^{|E|} \left.\frac{P(L_n;x)}{x}\right|_{x=0}$$

By Proposition 3.1.1.

$$P(L_n;x) = x(x-1)(x-2)^{n-2}$$

By Proposition I.1. $|E|=2n-3$

Combining these we have

$$\begin{aligned} d(L_n;x) &= (-1)^{2n-3} \left.\frac{x(x-1)(x-2)^{n-2}}{x}\right|_{x=0} \\ &= (-1)^{2n-3}(-1)(-2)^{n-2} \\ &= (-1)^{2n}(-1)^3(-1)(-1)^n(-1)^{-2}2^{n-2} \\ &= (-1)^n 2^{n-2} \end{aligned}$$ ●

Proposition 3.2.2.

The signed domination of a wheel of order n, $n>3$ is given by

$$d(W_n) = (-1)^n(2-2^{n-1})$$

Proof: By Corollary 3.2.1. we have

$$d(W_n) = (-1)^{|E|} \left. \frac{P(W_n;x)}{x} \right|_{x=0}$$

By Proposition 3.1.2.

$$P(W_n;x) = (-1)^n x(x-1)(x-2)(x-3) + (-1)^n x(2-x)^3\left(1-(2-x)^{n-4}\right)$$

By Proposition I.2. $|E|=2n-2$

Combining these we have

$$d(W_n)=(-1)^{2n-2} \left. \frac{(-1)^n x(x-1)(x-2)(x-3) + (-1)^n x(2-x)^3\left(1-(2-x)^{n-4}\right)}{x} \right|_{x=0}$$

$$= (-1)^n(-1)(-2)(-3) + (-1)^n 2^3(1-2^{n-4})$$

$$= (-1)^n(-6+8-2^{n-1})$$

$$= (-1)^n(2-2^{n-1}) \quad \bullet$$

3.3. Parities of ladders and wheels

Let us partition the $2n-3$ edges of a ladder of order n, into three types. (Figure 3.5)

For $n>2$, we will call an edge of type a if it connects two vertices, one of degree two and the other of degree at least two. There are 4 such edges.

For $n>3$, an edge will be of type b if it connects two vertices, one of degree three and the other of degree at least three. There are $n-3$ such edges.

For $n>4$, we will call an edge of type c if it connects two vertices that are of degree three. There are $n-4$ such edges.

We will now evaluate the chromatic polynomials of ladders with one edge removed and combining them according to Corollary 3.2.3. we will estabish a formula for

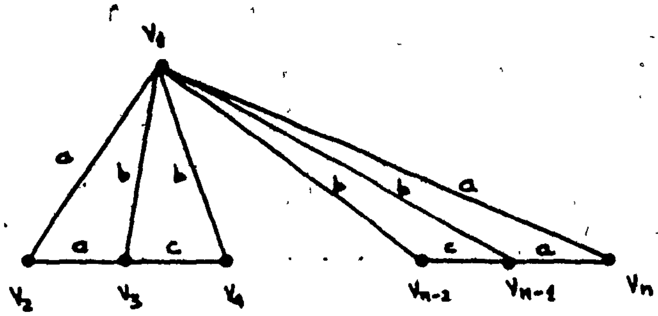


FIGURE 3.5

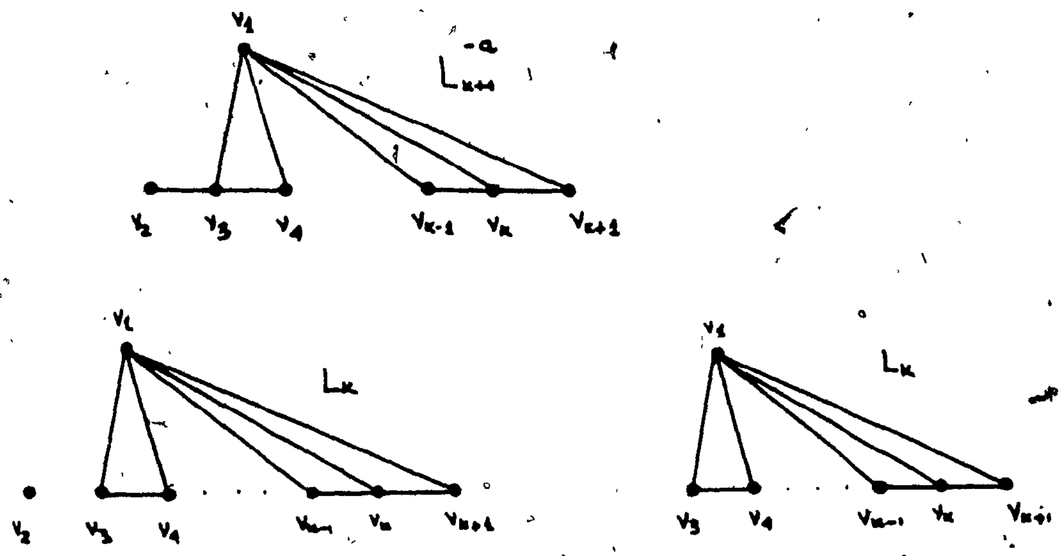


FIGURE 3.6

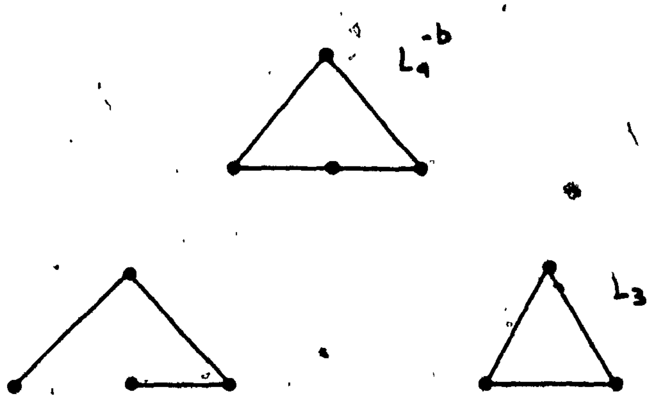


FIGURE 3.7

$P_{(2n-3)-1}(L_n)$.

Let us therefore denote by L_n^{-a} , L_n^{-b} , L_n^{-c} the ladders of order n with one edge of type a , b , c removed respectively.

Proposition 3.3.1.

The chromatic polynomial $P(L_n^{-a};x)$ of a ladder of order n , n>2 with one edge of type a removed is given by

$$P(L_n^{-a};x) = x(x-1)^2(x-2)^{n-3}$$

Proof: For n=3 it is obvious that

$$P(L_3^{-a};x) = x(x-1)^2$$

Therefore it is true for n=3. Let it be true for n=k , i.e.

$$P(L_k^{-a};x) = x(x-1)^2(x-2)^{k-3}$$

Let us consider L_{k+1}^{-a} (Figure 3.6) . By the Two-Pieces theorem, the Fundamental Reduction theorem and Proposition 3.1.1. we have

$$\begin{aligned} P(L_{k+1}^{-a};x) &= x\ P(L_k;x) - P(L_k;x) \\ &= (x-1)\ P(L_k;x) \\ &= (x-1)\ x(x-1)(x-2)^{k-2} \\ &= x(x-1)^2(x-2)^{(k+1)-3} \end{aligned}$$

Since it is true for n=k+1 , by the induction hypothesis, it is true for all n, n>2. •

Proposition 3.3.2.

The chromatic polynomial $P(L_n^{-b};x)$ of a ladder of order n , n>3 with one edge of type b removed is given by

$$P(L_n^{-b};x) = x(x-1)(x-2)^{n-4}\Big((x-1)^2-(x-2)\Big)$$

Proof: Let us take L_4^{-b} (Figure 3.7). By the Fundamental Reduction theorem and Proposition 3.1.1. we have

$$\begin{aligned}P(L_4^{-b};x) &= P(G-e;x) - P(G_e;x)\\ &= x(x-1)^3 - P(L_3;x)\\ &= x(x-1)^3 - x(x-1)(x-2)\\ &= x(x-1)\left((x-1)^2-(x-2)\right)\end{aligned}$$

Therefore it is true for n=4. Let it be true for n=k , i.e.

$$P(L_k^{-b};x) = x(x-1)(x-2)^{k-4}\left((x-1)^2-(x-2)\right)$$

Let us consider L_{k+1}^{-b} (Figure 3.8). By the Two-Pieces theorem and the Fundamental reduction theorem we have

$$\begin{aligned}P(L_{k+1}^{-b};x) &= x\ P(L_k^{-b};x) - P(L_k^{-b};x) - P(L_k^{-b};x)\\ &= (x-2)\ P(L_k^{-b};x)\\ &= (x-2)\ x(x-1)(x-2)^{k-4}\left((x-1)^2-(x-2)\right)\\ &= x(x-1)(x-2)^{(k+1)-4}\left((x-1)^2-(x-2)\right)\end{aligned}$$

Since it is true for n=k+1 , by the induction hypothesis, it is true for all n, n>3. •

In the theory of chromatic polynomials we have the following theorem.

Theorem 3.3.1.

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs, such that $V_1\ V_2 = \{v\}$ one single edge. Then

$$P(G_1\ G_2;x) = \frac{P(G_1;x)\ P(G_2;x)}{x} \quad \bullet$$

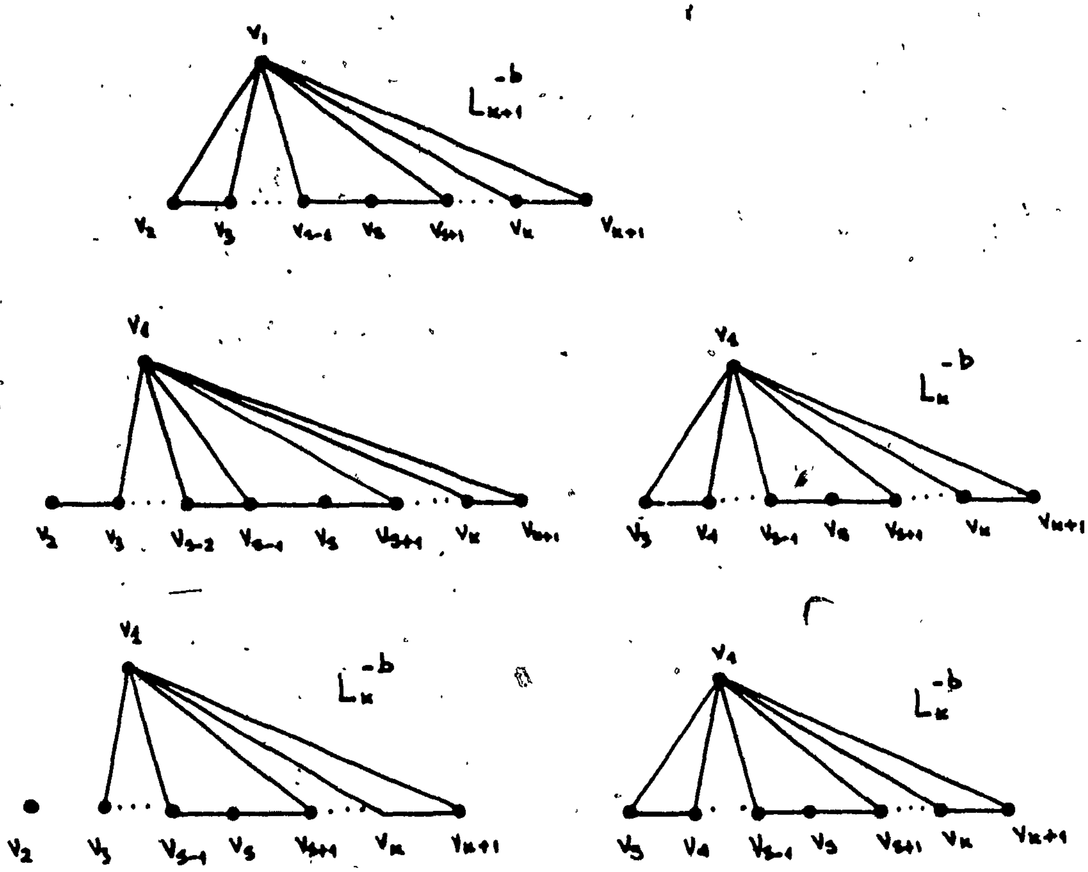


FIGURE 3.8

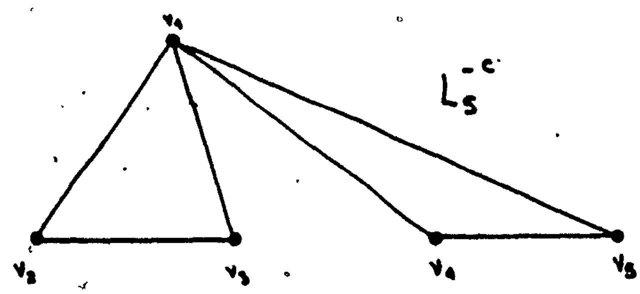


FIGURE 3.9

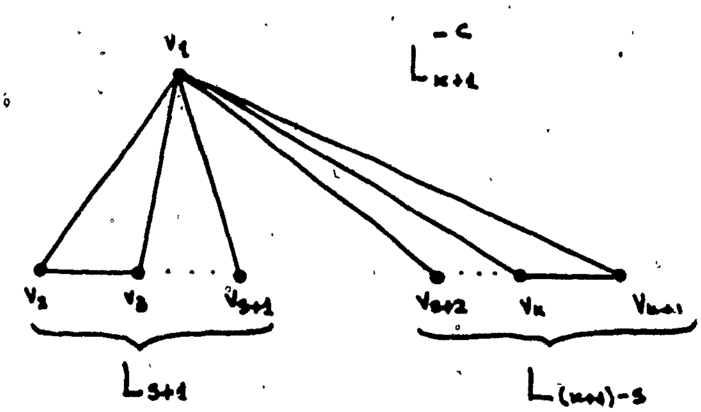


FIGURE 3.10

Proposition 3.3.3.

The chromatic polynomial $P(L_n^{-c};x)$ of a ladder of order n , n>4 with one edge of type c removed is given by

$$P(L_n^{-c};x) = x(x-1)^2(x-2)^{n-3}$$

Proof: Let us take L_5^{-c} (Figure 3.9). But L_5^{-c} is the union of two ladders of order 3 whose intersection is vertex v_1 . Therefore by Theorem 3.3.1. and Proposition 3.1.1. we have

$$\begin{aligned} P(L_5^{-c};x) &= \frac{P(L_3;x)\cdot P(L_3;x)}{x} \\ &= \frac{x(x-1)(x-2)\ x(x-1)(x-2)}{x} \\ &= x(x-1)^2(x-2)^2 \end{aligned}$$

Therefore it is true for n=5. Let it be true for n=k , i.e.

$$P(L_k^{-c};x) = x(x-1)^2(x-2)^{k-3}$$

Let us consider L_{k+1}^{-c} (Figure 3.10). But L_{k+1}^{-c} is the union of two ladders of order (s+1) and (k+1)-s , whose intersection is vertex v_1 . Therefore by Theorem 3.3.1. and Proposition 3.1.1. we have

$$\begin{aligned} P(L_{k+1}^{-c};x) &= \frac{P(L_{s+1};x)\ P(L_{(k+1)-s};x)}{x} \\ &= \frac{x(x-1)(x-2)^{(s+1)-2}\cdot x(x-1)(x-2)^{(k+1)-s-2}}{x} \\ &= x(x-1)^2(x-2)^{k-2} \\ &= x(x-1)^2(x-2)^{(k+1)-3} \end{aligned}$$

Since it is true for n=k+1 , by the induction hypothesis, it is true for all n, n>4. ●

Proposition 3.3.4.

For a ladder of order n , n>2

$$P_{2n-4}(L_n) = (9-5n)(-2)^{n-4}$$

Proof: By Corollary 3.2.3

$$P_{2n-4}(L_n) = \sum_{G_j \in S_{2n-4}}' (-1)^{|E|} \left.\frac{P(G_j;x)}{x}\right|_{x=0}$$

We are now taking one edge off a ladder of order n, so we end up with L_n^{-a} , L_n^{-b} , L_n^{-c} of which there are 4 , n-3 and n-4 respectively. Therefore with Propositions 3.3.1. , 3.3.2. and 3.3.3. we have

$$P_{2n-4}(L_n) = 4\ (-1)^{2n-4} \left.\frac{x(x-1)^2(x-2)^{n-3}}{x}\right|_{x=0} +$$

$$+ (n-3)\ (-1)^{2n-4} \left.\frac{x(x-1)(x-2)^{n-4}\big((x-1)^2-(x-2)\big)}{x}\right|_{x=0}$$

$$+ (n-4)\ (-1)^{2n-4} \left.\frac{x(x-1)^2(x-2)^{n-3}}{x}\right|_{x=0}$$

$$= 4(-1)^2(-2)^{n-3} + (n-3)(-1)(-2)^{n-4}\big((-1)^2-(-2)\big) +$$

$$+ (n-4)\ (-1)^2(-2)^{n-3}$$

$$= 4(-2)^{n-3} + (3n-9)(-1)(-2)^{n-4} + (n-4)(-2)^{n-3}$$

$$= (-8)(-2)^{n-4} + (9-3n)(-2)^{n-4} + (8-2n)(-2)^{n-4}$$

$$= (9-5n)(-2)^{n-4} \quad \bullet$$

Let us now partition the 2n-2 edges of a wheel of order n , into two types (Figure 3.11).

We will call an edge of type r if it connects two vertices that are of degree three. There are n-1 such edges.

We will call an edge of type s if it connects two

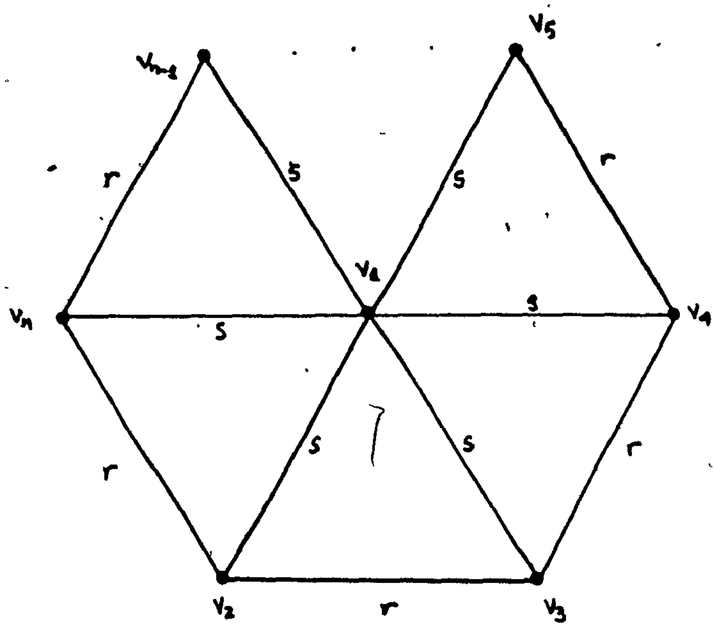


FIGURE 3.11

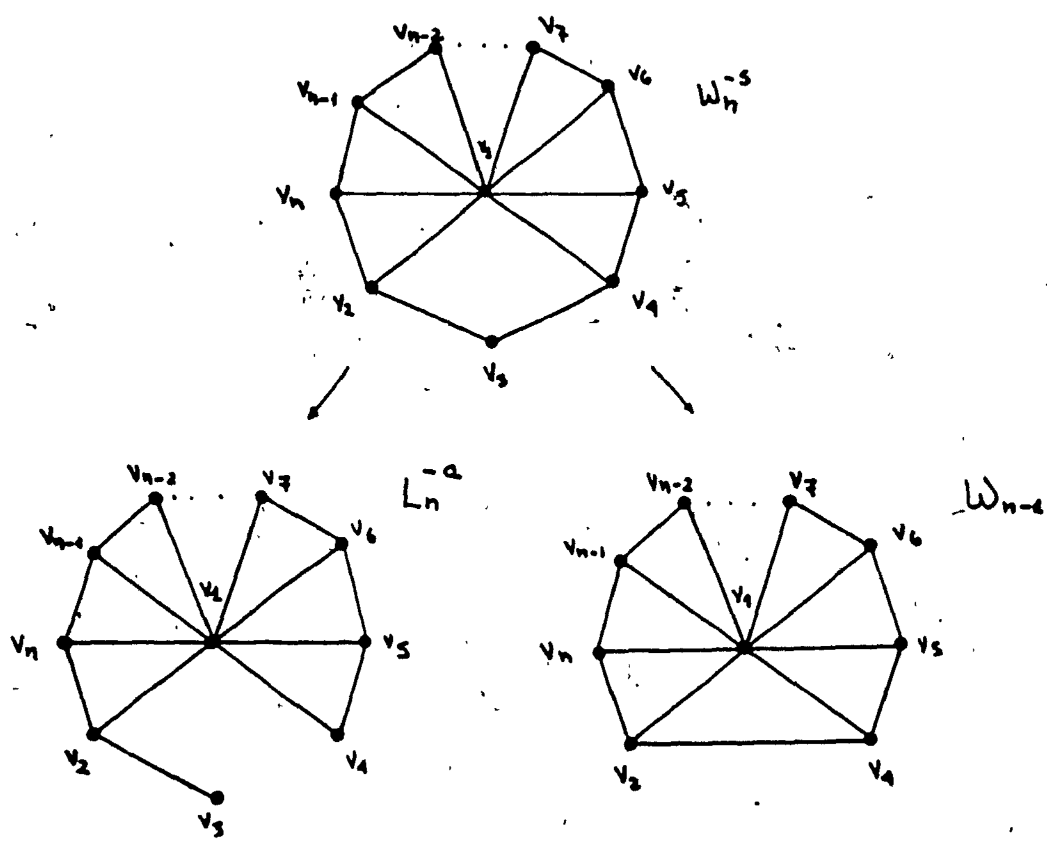


FIGURE 3.12

vertices, only one of which is of degree three while the other is of degree $n-1$. There are $n-1$ such edges.

We will now evaluate the chromatic polynomials of wheels with one edge removed and combining them according to Corollary 3.2.3. we will establish a formula for $P_{(2n-2)-1}(W_n)$.

Let us therefore denote by W_n^{-r}, W_n^{-s} the wheels of order n with one edge of type r, s removed respectively.

Proposition 3.3.5.

The chromatic polynomial $P(W_n^{-r};x)$ of a wheel of order n, $n>3$ with one edge of type r removed is given by

$$P(W_n^{-r};x) = x(x-1)(x-2)^{n-2}$$

Proof: Let us consider W_n^{-r}. It is obvious that $W_n^{-r} = L_n$. But by Proposition 3.1.1.

$$P(L_n;x) = x(x-1)(x-2)^{n-2}$$

and therefore

$$P(W_n^{-r};x) = x(x-1)(x-2)^{n-2} \bullet$$

Proposition 3.3.6.

The chromatic polynomial $P(W_n^{-s};x)$ of a wheel of order n with one edge of type s removed is given by

$$P(W_n^{-s};x) = \begin{cases} P(L_4;x) & , n=4 \\ P(L_n^{-a};x) - P(W_{n-1};x) & , n>4 \end{cases}$$

Proof: For n=4 it is obvious that $W_n^{-s} = L_4$. Let us consider W_n^{-s}, $n>4$ (Figure 3.12). By the Fundamental Reduction theorem we have

$$P(W_n^{-s};x) = P(L_n^{-a};x) - P(W_{n-1};x)$$

Corollary 3.3.1.

The chromatic polynomial $P(W_n^{-s};x)$ os a wheel of order n with one edge of type s removed is given by

$$P(W_n^{-s};x) = \begin{cases} x(x-1)(x-2)^2 & , n=4 \\ x(x-1)^2(x-2)^{n-3} + (-1)^n x(x-2)\left((2-x)^{n-3}-1\right) & , n>4 \end{cases}$$

Proof: By Propositions 3.1.1. and 3.3.6. it is obvious that

$$P(W_n^{-s};x) = x(x-1)(x-2)^2$$

Now from Propositions 3.1.2. , 3.3.1. and 3.3.6. we have

$$\begin{aligned} P(W_n^{-s};x) &= P(L_n^{-a};x) - P(W_{n-1};x) \\ &= x(x-1)^2(x-2)^{n-3} - (-1)^{n-1}x(x-1)(x-2)(x-3) - \\ &\qquad - (-1)^{n-1}x(2-x)^3\left(1-(2-x)^{n-1-4}\right) \\ &= x(x-1)^2(x-2)^{n-3} + (-1)^n x(x-1)(x-2)(x-3) + \\ &\qquad + (-1)^n x(2-x)^3\left(1-(2-x)^{n-5}\right) \\ &= x(x-1)^2(x-2)^{n-3} + \\ &\qquad + (-1)^n x(x-2)\left((x-1)(x-3)+(2-x)^2\left((2-x)^{n-5}-1\right)\right) \\ &= x(x-1)^2(x-2)^{n-3} + \\ &\qquad + (-1)^n x(x-2)\left(x^2-4x+3+(2-x)^{n-3}-4+4x-x^2\right) \\ &= x(x-1)^2(x-2)^{n-3} + (-1)^n x(x-2)\left((2-x)^{n-3}-1\right) \quad \bullet \end{aligned}$$

Proposition 3.3.7.

For a wheel of order n

$$P_{2n-3}(W_n) = \begin{cases} 24 & , n=4 \\ (-1)^{3n-4}(n-1)(5\cdot 2^{n-3}-2) & , n>4 \end{cases}$$

Proof: By Corollary 3.2.3.

$$P_{2n-3}(W_n) = \sum_{G_j \in S_{2n-3}} (-1)^{|E|} \left.\frac{P(G_j;x)}{x}\right|_{x=0}$$

We are taking one edge off a wheel of order n, so we end up with a W^{-r}, W^{-s} of which there are $n-1$ and $n-1$ respectively.

Now for n=4 with Proposition 3.3.5. and Corollary 3.3.1. we have

$$P_{2\cdot 4-3}(W_4) = (4-1)(-1)^{2\cdot 4-3} \left.\frac{x(x-1)(x-2)^{4-2}}{x}\right|_{x=0} +$$

$$+ (4-1)(-1)^{2\cdot 4-3} \left.\frac{x(x-1)(x-2)^{2}}{x}\right|_{x=0}$$

$$= 3(-1)(-1)(-2)^2 + 3(-1)(-1)(-2)^2 = 24$$

Now for n>4 with Proposition 3.3.5. and Corollary 3.3.1. we have

$$P_{2n-3}(W_n) = (n-1)\ (-1)^{2n-3} \left.\frac{x(x-1)(x-2)^{n-2}}{x}\right|_{x=0} +$$

$$+ (n-1)(-1)^{2n-4} \left.\frac{x(x-1)^2(x-2)^{n-3} + (-1)^n x(x-2)\Big((2-x)^{n-3}-1\Big)}{x}\right|_{x=0}$$

$$= (n-1)(-1)^{2n-3}(-1)^{n-2} +$$

$$+ (n-1)(-1)^{2n-3}\ (-1)^2(-2)^{n-3} + (-1)^n(-2)(2^{n-3}-1)$$

$$= (-1)^{2n-3}(n-1)\Big((-1)(-1)^{n-2}2^{n-2} + (-1)^{n-3}2^{n-3} +$$

$$+ (-1)^{n+1}2^{n-2} + 2(-1)^n\Big)$$

$$= (-1)^{2n-3}(n-1)\Big((-1)^{n-1}2^{n-2} + (-1)^{n-1}2^{n-3} +$$

$$+ (-1)^{n-1}2^{n-2} + 2(-1)^n\Big)$$

$$= (-1)^{2n-3}(n-1)\Big(2^{n-3}\big(2(-1)^{n-1}+(-1)^{n-1}+2(-1)^{n-1}\big)+2(-1)^n\Big)$$

$$= (-1)^{2n-3}(n-1)\Big(2^{n-3}\cdot 5(-1)^{n-1} + 2(-1)^n\Big)$$

$$= (-1)^{2n-3}(n-1)(-1)^{n-1}(5\cdot 2^{n-3}-2)$$

$$= (-1)^{3n-4}(n-1)(5\cdot 2^{n-3}-2)\ .\ \bullet$$

CHAPTER 4

EXTENDED DOMINATION THEORY

Boesh, Satyanarayana and Suffel proved a pivot equation for extended domination similar to the one that holds for domination.

Theorem 4.1. [6]

$$D(G,K,j) = D(G_e,K,j) + D(G-e,K,j) \bullet$$

4.1. A characterization of K-graphs

Proposition 4.1.1.

$D(G,K,j) \neq 0$ iff G is a K-graph with respect to K.

Proof: $D(G,K,j) \neq 0$ implies that G has at least one j-formation. Hence every edge of G is in some K-forest and G is a K-graph.

Conversely, suppose G is a K-graph. By induction on the edges of G. If G consists of $(|V|-1)$ edges $D(G,K,j)=1$ and the result is true.

Assume it is true for all K-graphs with $b \geq |V|-1$ edges, and $j < b$. Consider a graph with $b+1$ edges. For some edge e in G, let $G-e$ be the graph obtained from G by omitting e and G_e be the graph obtained by coalescing the end vertices of e . Both $G-e$ and G_e contain b edges, and at least one of them is a K-graph. Hence by the induction hypothesis $D(G_e,K,j) + D(G-e,K,j) \neq 0$. Therefore $D(G,K,j) \neq 0$. $\bullet$

4.2. Series and Parallel reductions

Proposition 4.2.1.

Let G be a K-graph with respect to some K.

(a) Suppose e_1 and e_2 are parallel edges in G , then

$$D(G-e_1,K,j) = D(G-e_2,K,j) = D(G,K,j)$$

(b) Suppose e_1 and e_2 are series edges in G , such that their common vertex does not belong to K , then

$$D(G_{e_1},K,j) = D(G_{e_2},K,j) = D(G,K,j)$$

Proof: (a) G_{e_1} and G_{e_2} are not K-graphs because they contain self-loops e_2 and e_1 respectively. By Proposition 4.1.1. we have

$$D(G_{e_1},K,j) = D(G_{e_2},K,j) = 0$$

and therefore by Theorem 4.1. we have

$$D(G-e_1,K,j) = D(G-e_2,K,j) = D(G,K,j)$$

(b) $G-e_1$ and $G-e_2$ are not K-graphs since e_2 is in no K-tree of $G-e_1$ while e_1 is in no K-tree of $G-e_2$ respectively. By Proposition 4.1.1. we have

$$D(G-e_1,K,j) = D(G-e_2,K,j) = D(G,K,j)$$

and therefore by Theorem 4.1. we have

$$D(G_{e_1},K,j) = D(G_{e_2},K,j) = D(G,K,j) \quad \bullet$$

Corollary 4.2.1.

D(G,K,j) remains invariant under series and parallel reductions.

Proof: Immediate from Proposition 4.2.1. •

REFERENCES

(1) A.AGRAWAL and R.E.BARLOW, "A Survey of Network Reliability and Domination Theory", Oper. Res. 32 (1984) pp. 478-492.

(2) R.E.BARLOW and F.PROSCHAN, "Statistical Theory of Reliability and Life Testing", Halt, Rinehart and Winston (1975).

(3) S.D.BEDROSIAN, "Evaluation of Network Determinants via the Key Subgraph", Proc. First Allerton Conf. on Circuit and System Theory (1963) pp. 589-602.

(4) G.D.BIRKHOFF, "A Determinant Formula for the Number of Ways of Coloring a Map", Ann. Math. 14 (1912) pp. 42-46.

(5) G.D.BIRKHOFF and D.C.LEWIS, "Chromatic Polynomials", Trans. Amer. Math. Soc. 60 (1946) pp. 355-451.

(6) F.BOESCH, A.SATYANARAYANA and C.L.SUFFEL, "On some Alternate Characterizations of a Graph Invariant called Domination", submitted.

(7) F.HARARY, "Graph Theory", Addison-Wesley (1969).

(8) F.S.ROBERTS, "Applied Combinatorics", Prentice-Hall (1984).

(9) A.SATYANARAYANA, "Multi-terminal Network Reliability", Technical Report Number ORC 80-6, Operations Research Center, Univ. of California at Berkeley (1981).

(10) A.SATYANARAYANA, "A Unified Formula for Analysis of some Network Reliability Problems", IEEE Trans. Reliability R-31 (1982) pp. 23-32.

[11] A.SATYANARAYANA and M.K.CHANG, "Network Reliability and the Factoring Theorem", Technical Report Number ORC 81-12, Operations Research Center, Univ. of California at Berkeley (1981).

[12] A.SATYANARAYANA and Z.KHALIL, "On an Invariant of Graphs and the Reliability Polynomial", SIAM J.Alg. Disc.Meth. 7 (1986) pp. 399-403.

[13] A.SATYANARAYANA and R.TINDELL, "Chromatic Polynomials and Network Reliability", Technical Report Number 86-11, Department of Electical Engineering and Computer Science, Stevens Institute of Technology (1986).

[14] A.SATYANARAYANA and R.K.WOOD, "Polygon-to-chain Reductions and Network Reliability", Technical Report Number ORC 82-4, Operations Research Center, Univ. of California at Berkeley (1982).

[15] G.S.SUBRAMANIAN, "Bayesian Approaches to Evalute System Reliability", PhD Dissertation, Univ. of California at Berkeley (1984).

[16] M.N.S.SWAMY and K.THULASIRAMAN, "Graphs, Networks and Algorithms", Wiley (1981).

APPENDIX I

Proposition I.1.

Consider a ladder of order n. Then $|E|=2n-3$, $n>2$ where $|V|=n$.

Proof: For $n=3$, i.e. L_3 we have $|V|=3$ and $|E|=2*3-3=3$ which is true.

Let it be true for $n=k$, i.e. for L_k $|E|=2k-3$ where $|V|=k$.

Let us consider L_{k+1}. Now $|V|=k+1$ and we have added two edges to the number of edges of L_k.

Therefore $|E|=2k-3+2=2(k+1)-3$.

Since it is true for $n=k+1$, by the induction hypothesis, it is true for all n, $n>2$. ●

Proposition I.2.

Consider a wheel of order n. Then $|E|=2n-2$, $n>3$ where $|V|=n$.

Proof: For $n=4$, i.e. W_4 we have $|V|=4$ and $|E|=2*4-2=6$ which is true.

Let it be true for $n=k$, i.e. for W_k $|E|=2k-2$ where $|V|=k$.

Let us consider W_{k+1}. Now $|V|=k+1$ and we have added two edges to the number of edges of W_k.

Therefore $|E|=2k-2+2=2(k+1)-2$.

Since it is true for $n=k+1$, by the induction hypothesis, it is true for all n, $n>3$. ●

APPENDIX II

Proposition II.1.

The number s_n of spanning trees of a ladder of order n, is given by

$$s_n = 3s_{n-1} - s_{n-2} \quad n>4$$
$$s_3 = 3$$
$$s_4 = 8$$

Proof: For n=5 the reliability expression is

$$R(L_5) = 21p^4 - 44p^5 + 32p^6 - 8p^7$$

It is obvious that $s_5 = P_{5-1}(L_5) = 21$. But $21 = 3 \times 8 - 3$. Therefore it is true for n=5.

Let it be true for n=k and all n<k, i.e. a ladder of order k has $s_k = 3s_{k-1} - s_{k-2}$ spanning trees.

Let us consider a ladder of order k+1 (Figure II.1).

Let $ST(i)$ and $ST(i \cap j)$ denote the number of spanninng trees that include edge i and both i and j respectively.

Now the total number of spanning trees of L_{k+1} will come from adding to L_k:

(i) edge 1, with a contribution of s_k,

(ii) edge 2, with a contribution of s_k,

(iii) edges 1 and 2. By an inclusion-exclusion argument on edges 1, 2, 3, 4 (considering that 1 and 2 have to be included and that 1, 2, 3 cannot be included together because they form a cycle) we have that the contribution will be

$$ST(1) + ST(2) + ST(3) + ST(4) -$$

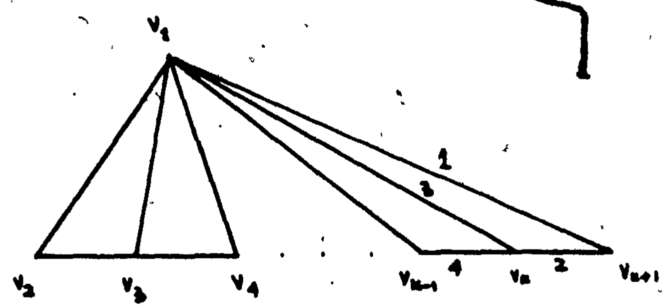


FIGURE II.1

$-ST(1\cap 2) - ST(1\cap 3) - ST(1\cap 4) - ST(2\cap 3) - ST(2\cap 4) - ST(3\cap 4)$

$+ ST(1\cap 2\cap 3) + ST(1\cap 2\cap 4) + ST(2\cap 3\cap 4) - ST(1\cap 2\cap 3\cap 4)$

$= ST(1\cap 2\cap 4) - ST(1\cap 2)$

$= s_k - s_{k-1}$

Therefore the total number of spanning trees for a ladder of order $k+1$ is given by

$$s_{k+1} = s_k + s_k + s_k - s_{k-1} = 3s_k - s_{k-1}$$

Since it is true for $n=k+1$, by the induction hypothesis, it is true for all n, $n>4$. •

Corollary II.1.

The number s_n of spanning trees of a ladder of order n, is given by

$$s_n = \left(\frac{15 + 7\sqrt{5}}{10}\right)\left(\frac{3 + \sqrt{5}}{2}\right)^{n-3} + \left(\frac{15 - 7\sqrt{5}}{10}\right)\left(\frac{3 - \sqrt{5}}{2}\right)^{n-3}, \quad n>2$$

Proof: With loss of correspondance let us find th solution of :

$$s_k = 3s_{k-1} - s_{k-2}, \quad k>1$$
$$s_0 = 3$$
$$s_1 = 8$$

We have the characteristic polynomial

$$a^2 - 3a + 1 = 0$$

with roots $a_1 = \frac{3 + \sqrt{5}}{2}$ and $a_2 = \frac{3 - \sqrt{5}}{2}$.

Therefore the general solution is

$$s_k = A_1 a_1{}^k + A_2 a_2{}^k$$

From the initial conditions we have

$$s_0 = 3 \qquad A_1 + A_2 = 3 \qquad A_1 = \frac{15 + 7\sqrt{5}}{10}$$

$$s_1 = 8 \qquad \frac{3 + \sqrt{5}}{2} A_1 + \frac{3 - \sqrt{5}}{2} A_2 = 8 \qquad A_2 = \frac{15 - 7\sqrt{5}}{10}$$

Therefore

$$s_k = \left(\frac{15 + 7\sqrt{5}}{10}\right)\left(\frac{3 + \sqrt{5}}{2}\right)^k + \left(\frac{15 - 7\sqrt{5}}{10}\right)\left(\frac{3 - \sqrt{5}}{2}\right)^k$$

which restoring the correspondance to the ladder is transformed to

$$s_n = \left(\frac{15 + 7\sqrt{5}}{10}\right)\left(\frac{3 + \sqrt{5}}{2}\right)^{n-3} + \left(\frac{15 - 7\sqrt{5}}{10}\right)\left(\frac{3 - \sqrt{5}}{2}\right)^{n-3}.$$