REDUCED STIFFNESS MATRIX FOR PANELIZED STRUCTURES

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ABSTRACT

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The static properties of panelized structures have already been successfully investigated using the finite element method. Here a preliminary attempt is made to analyze a panelized structure for dynamic response using well-known structural theory and programming procedures. Since earthquake ground motions produce mainly lateral forces, it is desirable to develop a procedure which eliminates all degrees of freedom but the lateral ones. Therefore, a reduced stiffness matrix which relates only lateral forces and displacements is developed.

A computer program is described which first assembles the complete structural stiffness matrix of a panelized structure and then derives the reduced stiffness matrix applicable for lateral degrees of freedom.

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CHAPTER 1

INTRODUCTION

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INTRODUCTION

Panelized structures are a new type of building now in the process of being experimentally and theoretically investigated. Figure 1.1 shows a typical example.

To achieve results similar to those obtained in investigations into framed structures, the finite element theory has to be applied. The large variety of different element types offers analytically nearly an unlimited number of possibilities. Considerable work has already been done for the determination of stresses due to static loading [1]; whereas, the application of dynamic loading is only now beginning to be investigated.

This report is directed toward the analysis of dynamic behaviour of panelized structures under earthquake type lateral loading.

The fundamental equation in structural dynamics is the equation of motion:

$$M\ddot{X} + C\dot{X} + KX = F$$

where M, C and K represent the mass matrix, the damping matrix and the stiffness matrix. F represents the applied forces.

The mass matrix M is easily found for any type of structure, by assuming lumped masses or rotational mass moments at storey levels or at joints. The damping matrix incorporates structural characteristics which are very difficult to predict such as joint slippage for panelized structures. It is therefore based on experimental values.

The stiffness matrix for panelized structures was already derived for the computation of static loading. For an earthquake analysis where only lateral inertia forces are created, it is desirable to establish a reduced stiffness matrix expressing only the lateral degrees of freedom. The main advantages of using a reduced stiffness matrix is the decreased size of that matrix. A lateral stiffness matrix for a 6 degrees of freedom system, is 1/36 of the size of the original stiffness matrix.

In this report, a first approach is described for the development of a computer program to derive the lateral stiffness matrix. Since a 6 degree of freedom system requires a large storage space and since no auxiliary storage was used, the application of the program is rather limited to a small size of panelized structures.

The sequence of the chapters, in this report, corresponds to the set-up of the computer program. (See the flow chart in the Appendix). After the information input a 24 by 24 element stiffness matrix is established. This matrix is renumbered and then assembled into the overall structure stiffness matrix. The structure stiffness matrix has to be arranged in a partitioned form to make the necessary reduction procedure possible, in order to derive the lateral stiffness matrix.

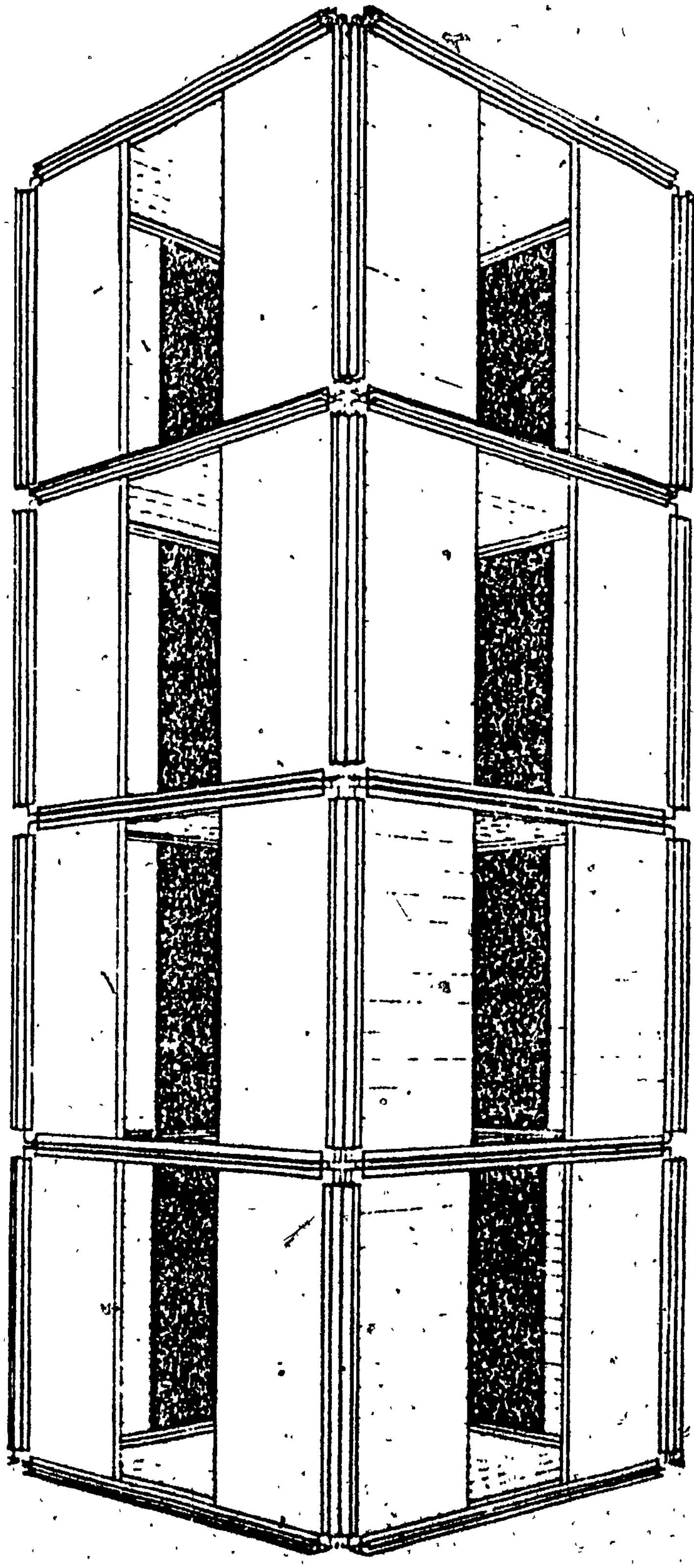


FIG. 1.1 A TYPICAL PANELIZED STRUCTURE

CHAPTER 2

INPUT ROUTINES, NUMBERING ARRAYS AND RESTRAINT LIST

CHAPTER 2

INPUT ROUTINES, NUMBERING ARRAYS AND RESTRAINT LIST

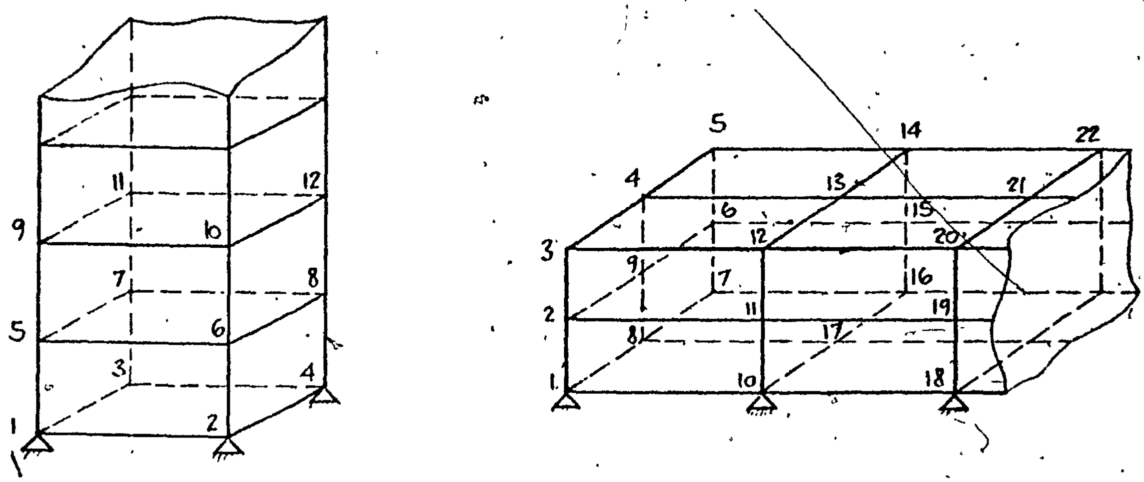
2.1 STRUCTURE INFORMATION

2.1.1 Partitioning of Structure

In general, panelized structures are already divided into rectangular elements. The natural joints are at the same time the structural joints or the mesh joints, thus building the basis for the finite element analysis. Openings can be easily accommodated in the program as long as they fit the mesh. It is then only a question of defining the properties of the element which represents the opening.

2.1.2 Numbering and Location of Nodes

The numbering of the joints is basically arbitrary and the program accepts any kind of numbering. But it is a well known fact that the way of numbering the joints of a structure influences the bandwidth of the stiffness matrix directly and consequently, the required space to store this matrix. This can be achieved by numbering the joints in sequence along the narrow dimension of the structure, to minimize coupling [2]. In Figure 2.1, numbering systems are shown which create the smallest possible bandwidth in the stiffness matrix.



Vertical Numbering

Horizontal Numbering

FIG. 2.1 JOINT NUMBERING FOR NARROW BANDWIDTH

The location of each node is fixed by a 3-dimensional coordinate system, x-y-z.

2.1.3 Degrees of Freedom

For a 3-dimensional structure, where rigid moment connections are desired, 6 degrees of freedom are necessary.

3 Displacements or forces, u ; F_x

v ; F_y

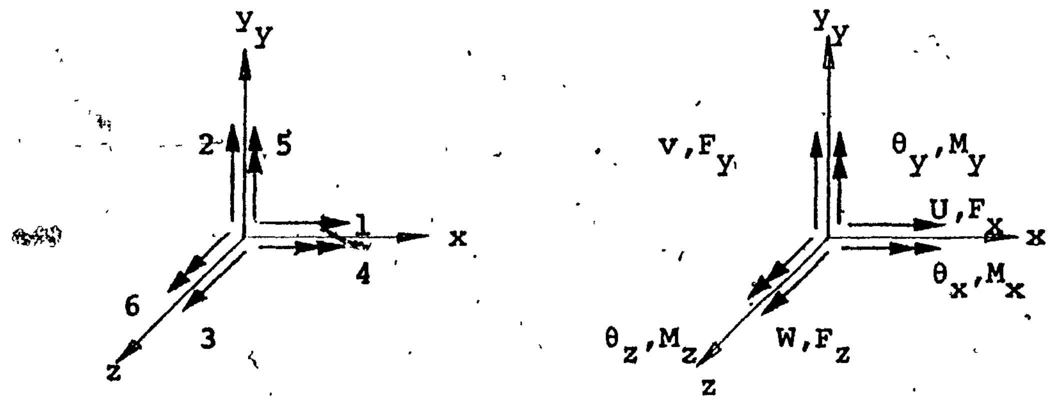
w ; F_z

3 Rotations or moments, θ_x ; M_x

θ_y ; M_y

θ_z ; M_z

(See also Figure 2.2). A rectangular element with 4-corner nodes has therefore a total of 24 degrees of freedom. The basic element numbering is shown in Figure (2.3).



(a) Numbering (b) Displacement and Forces

FIG. 2.2 BASIC NUMBERING OF DEGREES OF FREEDOM ACTING ON A NODE

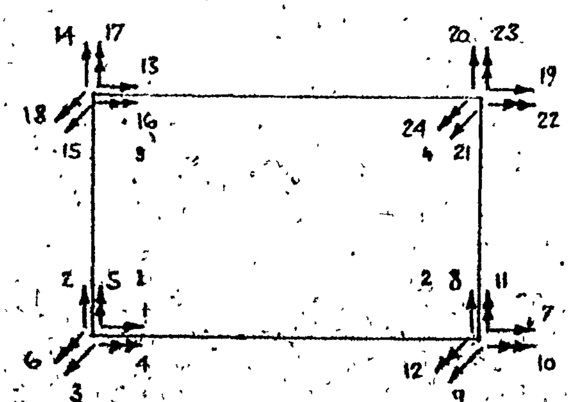


FIG. 2.3 BASIC ELEMENT NUMBERING

2.1.4 Support Conditions

Supports can only occur at nodes, and according to the 6 degrees of freedom, up to 6 restraints can be assumed per node. The support restraints of a structure have to be chosen in such a way to resist all applied external loads and the resulting overturning moments.

The input which is necessary for support restraints is demonstrated in Section 2.2.

2.1.5 Degrees of Freedom to Which the Structure Will be Reduced

Since this paper is dealing with the lateral stiffness matrix, we are mainly concerned with one degree of freedom - displacements in x or y direction - whereas, the other 5 degrees of freedom are to be eliminated. However, the program is written in such a way that any combination of the 6 degrees of freedom can be chosen. In general, all those degrees of freedom are eliminated where no corresponding external loads are expected. Examples for different combinations of chosen degrees of freedom are shown in Table 1, where the basic joint numbering is the same as in Figure 2.2.a. NE and NEA(I) are input symbols as explained in Section 2.4. N represents, in this example, the number of free joints.

2.1.6 Renumbering Array

Inside the input routine, it is a necessary step to create a new element numbering which numbers the reduced degrees of freedom first, then the others. This new numbering array is an extension of the input array NEA(I). In Table 2, NEA(I) is developed for examples 1 to 4 from Table 1. Figure 2.4 shows the typical renumbered element for lateral displacements in the z-direction (example 2).

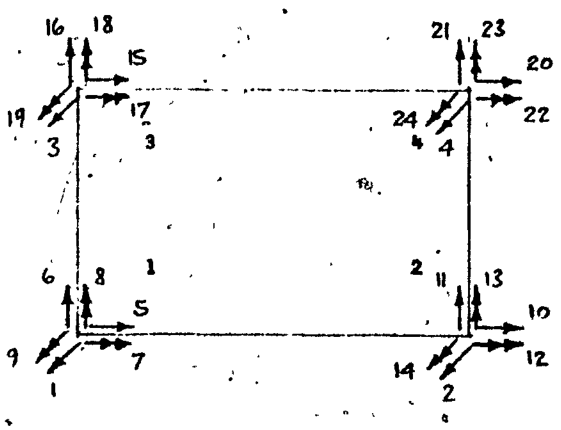


FIG. 2.4 RENUMBERED ELEMENT FOR LATERAL DEGREES OF FREEDOM (See example 2 in Table 1)

<u>TABLE 1</u>

EXAMPLES FOR REDUCED DEGREES OF FREEDOM

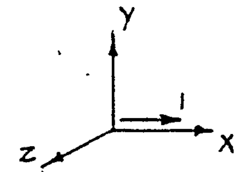
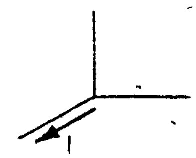
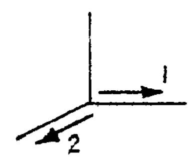
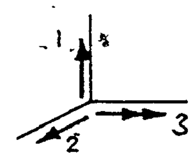
Example	Number of Reduced Degrees of Freedom	Array of Reduced Degrees of Freedom	Size of Reduced Stiffness Matrix	Explanatory Sketch	Remarks
1	NE = 1	NEA(1)=1	N x N		reduced to lateral displacement in x-direction
2	NE = 1	NEA(1)=3	N x N		reduced to lateral displacement in z-direction
3	NE = 2	NEA(1)=1 NEA(2)=3	(2N)x(2N)		reduced to lateral displacement in x and y direction
4	NE = 3	NEA(1)=2 NEA(2)=3 NEA(3)=4	(3N)x(3N)		fully restrained in x-y plane

TABLE 2

REARRANGED NUMBERING FOR EXAMPLES SHOWN IN TABLE 1

Basic Numbering See Fig.2.2	Example 1	Example 2	Example 3	Example 4
I	NEA(I)	NEA(I)	NEA(I)	NEA(I)
1	1	3	1	2
2	2	1	3	3
3	3	2	2	4
4	4	4	4	1
5	5	5	5	5
6	6	6	6	6

2.2 RESTRAINT LIST AND CUMULATIVE RESTRAINT LIST

This routine follows the same principle as it is derived by Gere and Weaver [3] for framed structures.

2.2.1 Restraint Degrees of Freedom of a Node

The restraints of each restrained node are read in the same sequence as the basic numbering (Figure 2,2).

Examples:

node i :	ND =	1	2	3	4	5	6	}	restraint against lateral displacements and uplift
	R(ND) =	1	1	1	0	0	0		
node j :	ND =	1	2	3	4	5	6	}	fully restrained node
	R(ND) =	1	1	1	1	1	1		

2.2.2 Restraint List for the Entire Structure

Using the information for restraints (Section 2.2.1) and applying a numbering sequence as explained before (Section 2.1.6), the restraint list RL(I) for the whole structure is constructed, as demonstrated in an example in Figure 2.5. The same array RL(I) is later used in the assembly routine since restrained degrees of freedom have to be separated from the free movable ones.

2.2.3 Cumulative Restraint List

The cumulative restraint list CRL(I) is directly derived from the restraint list RL(I) as shown in Figure 2.5. The following formulae are applied [3] :

```
CRL(1) = RL(1)
CRL(I) = CRL(I-1) + RL(I)   for  I = 2 to ND*NJ
```

The array CRL(I) is later used to determine the location of each element when assembling the structural stiffness matrix.

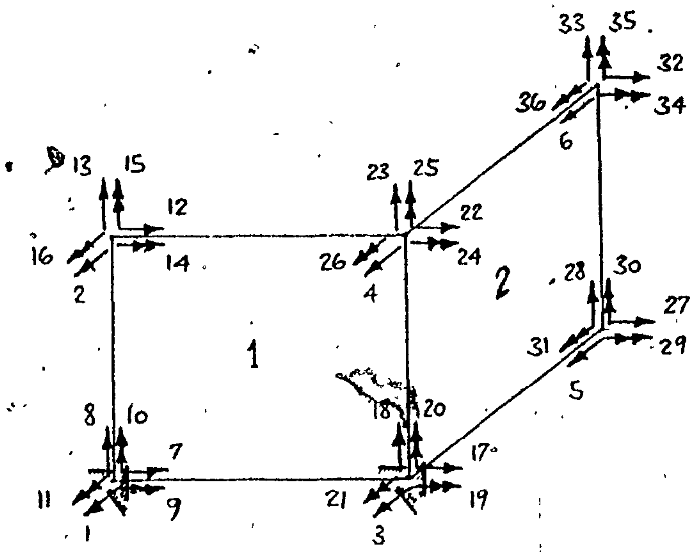
2.3 ELEMENT INFORMATION

2.3.1 Properties

Properties of an element are modulus of elasticity, Poisson's ratio and the thickness. Three arrays store this information, so that each element can have different properties. High rigidity of an element can be simulated by increasing the thickness or the modulus of elasticity. Openings are simulated by extremely flexible elements, i.e. the modulus of elasticity or the thickness is nearly zero.

2.3.2 Numbering of Element Nodes

The numbering of the element nodes corresponds to the structural numbering as derived in Section 2.1.2.



FREE MOVABLE DEGREES OF FREEDOM:

N = NJ*ND - NR
= 6 x 6 - 8 = 28

I	RL(I)	CRL(I)
1	1	1
2	0	1
3	1	2
4	0	2
5	0	2
6	0	2
7	1	3
8	1	4
9	1	5
10	0	5
11	0	5
12	0	5
13	0	5
14	0	5
15	0	5
16	0	5
17	1	6
18	1	7
19	1	8
20	0	8
21	0	8
22	0	8
23	0	8
24	0	8
25	0	8
26	0	8
27	0	8
28	0	8
29	0	8
30	0	8
31	0	8
32	0	8
33	0	8
34	0	8
35	0	8
36	0	8

FIG. 2.5 DERIVATION OF RESTRAINT LIST AND CUMULATIVE RESTRAINT LIST FOR SIMPLIFIED PANELIZED STRUCTURE

The four nodes are labelled with the symbols

JI(I), JJ(I), JK(I), JL(I)

and the numbering is read in the same sequence. See Figure 2.6.

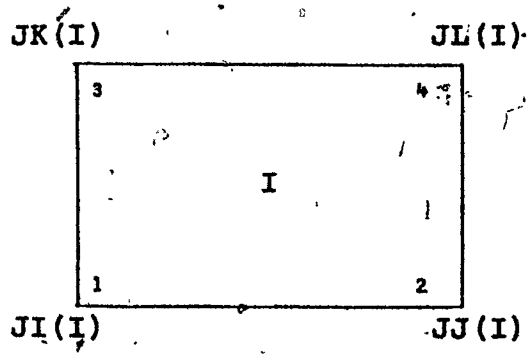


FIG. 2.6 NUMBERING SEQUENCE FOR ELEMENT NODES

2.4 INPUT SYMBOLS

The following symbols represent the input data in the form of single numbers or arrays. It could be a real number RE or an integer IN.

M	Number of elements	IN
NJ	Number of joints	IN
X(I) Y(I) Z(I)	Coordinates of joint I	RE
NRJ	Number of restraint joints	IN
ND	Number of degrees of freedom per joint. For a 3-dimensional structure ND = 6	IN
NR	Number of restraints. When only fully restrained joints are occurring, NR = NRJ*ND	IN
NE	Number of degrees of freedom to which the structure will be reduced. For lateral stiffness matrix NE = 1	IN
NEA(I)	Indicates which degrees of freedom are the reduced ones. For I = 1 to NE	IN
R(I)	Support restraints, for I = 1 to ND	IN
E(I)	Modulus of elasticity for element I	RE
P(I)	Poisson's ratio for element I	RE
T(I)	Thickness for element I	RE
JI(I) JJ(I) JK(I) JL(I)	Numbering for element I	IN

CHAPTER 3

ELEMENT STIFFNESS MATRIX

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ELEMENT STIFFNESS MATRIX

With the available element information, it is now possible to construct for each element the element stiffness matrix. In a loop which includes all the steps of Chapters 3 and 4, the element stiffness matrix of each element is established, rearranged and assembled into the structure stiffness matrix. The result of this is that only one storage space is continuously used for all element stiffness matrices. (See also the flow chart in the Appendix).

3.1 DETERMINATION OF PLANE AND DIMENSIONS

In a panelized structure, there are basically three element locations, the x-y, x-z and y-z plane. Using the coordinates for each element node, length a and width b for the element are calculated. At the same time, a code IP is determined to identify the plane of the element.

IP = 1	x-y plane
IP = 2	x-z plane
IP = 3	y-z plane

(See also Figure 3.1).

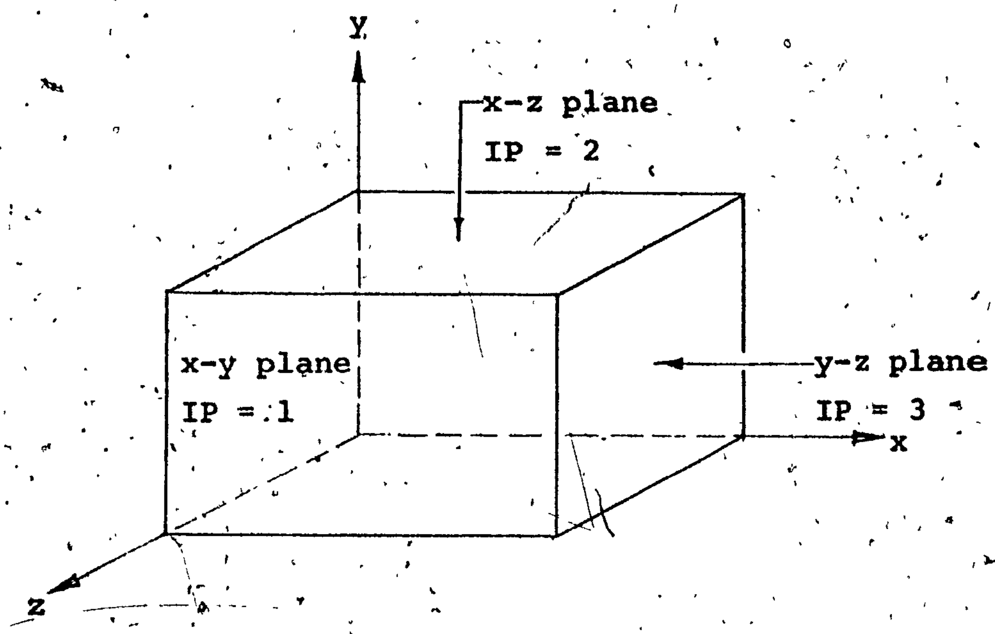


FIG. 3.1 IDENTIFICATION OF PLANES

3.2 ASSEMBLY OF ELEMENT STIFFNESS MATRIX

Due to the law of superposition in the theory of elasticity, it is possible to build up a rectangular element having 24 degrees of freedom which describes bending, as well as membrane action. This is true only for small deformations, which means that both actions are uncoupled. Stiffness matrices for practically any kind of elements are described in several finite element publications [4],[5].

3.2.1 Bending Stiffness Matrix $[K_b]$

The element bending stiffness matrix can be derived as a 12 x 12 matrix for compatible and for non-compatible slopes along the edges. In this program, a matrix was used based on non-compatible slopes published by Przemieniecki [5]. This matrix is shown in Figure 3.2. The different numbering system is adapted by rearrangement.

3.2.2 Membrane Stiffness Matrix $[K_m]$

An 8 x 8 matrix published by the same author [5] was used as the membrane stiffness matrix, see Figure 3.3. The numbering sequence shown is the same as in Figure 3.2.

3.2.3 Superposition of Bending and Membrane Action in One Matrix

Bending and membrane actions so far represent 5 degrees of freedom and not 6, as is required for a space structure. The missing in-plane rotations θ_z have to be introduced by a 4 x 4 matrix $[K_z]$, see Figure 3.4. The values of this matrix can be arbitrary but have to be very small. The additional requirements for every stiffness matrix - summation over moments and forces in each column has to be zero - can be checked easily. Values for γ are suggested in publications [4]. According to Ha [1] γ was chosen to be $3*10^{-6}$.

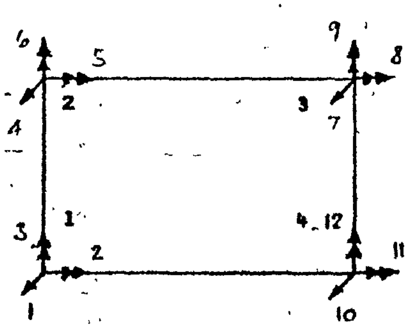
The superposition of the three uncoupled actions can be represented by the following matrix equation:

$$[K] = \begin{bmatrix} K_b & 0 & 0 \\ 0 & K_m & 0 \\ 0 & 0 & K_z \end{bmatrix}$$

At the end of this assembly subroutine, a rearrangement is performed which leads to the basic element numbering used in the program (Figure 2.3).

	1	2	3	4	5	6
1	$4(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$			Symmetric		
2	$[2\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[\frac{4}{3}\beta^{-2}+\frac{4}{15}(1-\nu)]b^2$				
3	$-[2\beta^2+\frac{1}{5}(1+4\nu)]a$	$-\nu ab$	$[\frac{4}{3}\beta^2+\frac{4}{15}(1-\nu)]a^2$			
4	$2(\beta^2-2\beta^{-2}) - \frac{1}{5}(14-4\nu)$	$-[2\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[-\beta^2+\frac{1}{5}(1+4\nu)]a$	$4(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$		
5	$[2\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[\frac{2}{3}\beta^{-2}-\frac{1}{15}(1-\nu)]b^2$	0	$-[2\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[\frac{4}{3}\beta^{-2}+\frac{4}{15}(1-\nu)]b^2$	
6	$[-\beta^2+\frac{1}{5}(1+4\nu)]a$	0	$[\frac{2}{3}\beta^2-\frac{4}{15}(1-\nu)]a^2$	$-[2\beta^2+\frac{1}{5}(1+4\nu)]a$	νab	$[\frac{4}{3}\beta^2+\frac{4}{15}(1-\nu)]a^2$
7	$-2(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$	$[-\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[\beta^2-\frac{1}{5}(1-\nu)]a$	$-2(2\beta^2-\beta^{-2}) - \frac{1}{5}(14-4\nu)$	$[-\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[2\beta^2+\frac{1}{5}(1-\nu)]a$
8	$[\beta^{-2}-\frac{1}{5}(1-\nu)]b$	$[\frac{1}{3}\beta^{-2}+\frac{1}{15}(1-\nu)]b^2$	0	$[-\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[\frac{2}{3}\beta^{-2}-\frac{4}{15}(1-\nu)]b^2$	0
9	$[-\beta^2+\frac{1}{5}(1-\nu)]a$	0	$[\frac{1}{3}\beta^2+\frac{1}{15}(1-\nu)]a^2$	$-[2\beta^2+\frac{1}{5}(1-\nu)]a$	0	$[\frac{2}{3}\beta^2-\frac{1}{15}(1-\nu)]a^2$
10	$-2(2\beta^2-\beta^{-2}) - \frac{1}{5}(14-4\nu)$	$[-\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[2\beta^2+\frac{1}{5}(1-\nu)]a$	$-2(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$	$[\beta^{-2}-\frac{1}{5}(1-\nu)]b$	$[\beta^2-\frac{1}{5}(1-\nu)]a$
11	$[\beta^{-2}-\frac{1}{5}(1+4\nu)]b$	$[\frac{2}{3}\beta^{-2}-\frac{4}{15}(1-\nu)]b^2$	0	$[-\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[\frac{1}{3}\beta^{-2}+\frac{1}{15}(1-\nu)]b^2$	0
12	$-[2\beta^2+\frac{1}{5}(1-\nu)]a$	0	$[\frac{2}{3}\beta^2-\frac{1}{15}(1-\nu)]a^2$	$[-\beta^2+\frac{1}{5}(1-\nu)]a$	0	$[\frac{1}{3}\beta^2+\frac{1}{15}(1-\nu)]a^2$

	7	8	9	10	11	12
7	$4(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$					
8	$-[2\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[\frac{4}{3}\beta^{-2}+\frac{4}{15}(1-\nu)]b^2$		Symmetric		
9	$[2\beta^2+\frac{1}{5}(1+4\nu)]a$	$-\nu ab$	$[\frac{4}{3}\beta^2+\frac{4}{15}(1-\nu)]a^2$			
10	$2(\beta^2-2\beta^{-2}) - \frac{1}{5}(14-4\nu)$	$[2\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[\beta^2-\frac{1}{5}(1+4\nu)]a$	$4(\beta^2+\beta^{-2}) + \frac{1}{5}(14-4\nu)$		
11	$-[2\beta^{-2}+\frac{1}{5}(1-\nu)]b$	$[\frac{2}{3}\beta^{-2}-\frac{4}{15}(1-\nu)]b^2$	0	$[2\beta^{-2}+\frac{1}{5}(1+4\nu)]b$	$[\frac{4}{3}\beta^{-2}+\frac{4}{15}(1-\nu)]b^2$	
12	$[\beta^2-\frac{1}{5}(1+4\nu)]a$	0	$[\frac{2}{3}\beta^2-\frac{4}{15}(1-\nu)]a^2$	$[2\beta^2+\frac{1}{5}(1+4\nu)]a$	νab	$[\frac{4}{3}\beta^2+\frac{4}{15}(1-\nu)]a^2$

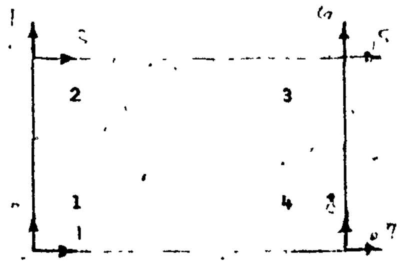


All coefficients to be multipled by

$$Et^3/12(1-\nu^2)ab$$

FIG. 3.2 STIFFNESS MATRIX FOR RECTANGULAR PLATE IN BENDING, BASED ON NONCOMPATIBLE DEFLECTIONS

	1	2	3	4	5	6	7	8
1	$4\beta + 2(1-\nu)\beta^{-1}$							Symmetric
2	$\frac{3}{2}(1+\nu)$	$4\beta^{-1} + 2(1-\nu)\beta$						
3	$2\beta - 2(1-\nu)\beta^{-1}$	$-\frac{3}{2}(1-3\nu)$	$4\beta + 2(1-\nu)\beta^{-1}$					
4	$\frac{3}{2}(1-3\nu)$	$-4\beta^{-1} + (1-\nu)\beta$	$-\frac{3}{2}(1+\nu)$	$4\beta^{-1} + 2(1-\nu)\beta$				
5	$2\beta - (1-\nu)\beta^{-1}$	$\frac{3}{2}(1+\nu)$	$4\beta - (1-\nu)\beta^{-1}$	$\frac{3}{2}(1-3\nu)$	$4\beta + 2(1-\nu)\beta^{-1}$			
6	$-\frac{3}{2}(1+\nu)$	$-2\beta^{-1} - (1-\nu)\beta$	$\frac{3}{2}(1-3\nu)$	$2\beta^{-1} - 2(1-\nu)\beta$	$\frac{3}{2}(1+\nu)$	$4\beta^{-1} + 2(1-\nu)\beta$		
7	$4\beta + (1-\nu)\beta^{-1}$	$\frac{3}{2}(1-3\nu)$	$2\beta - (1-\nu)\beta^{-1}$	$\frac{3}{2}(1+\nu)$	$2\beta - 2(1-\nu)\beta^{-1}$	$\frac{3}{2}(1-3\nu)$	$4\beta + 2(1-\nu)\beta^{-1}$	
8	$\frac{3}{2}(1-3\nu)$	$2\beta^{-1} - 2(1-\nu)\beta$	$\frac{3}{2}(1+\nu)$	$2\beta^{-1} - (1-\nu)\beta$	$\frac{3}{2}(1-3\nu)$	$-4\beta^{-1} + (1-\nu)\beta$	$-\frac{3}{2}(1+\nu)$	$4\beta^{-1} + 2(1-\nu)\beta$



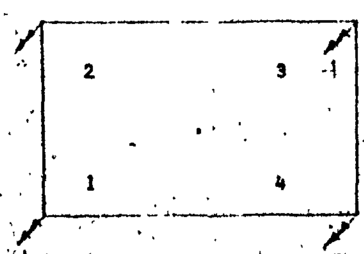
All coefficients to be multiplied by

$$Et(1-\nu^2)$$

Numbering sequence

FIG. 3.3 STIFFNESS MATRIX FOR RECTANGULAR PLATE - MEMBRANE ACTIONS

	1	2	3	4
1	1			Sym.
2	- 1/3	1		
3	- 1/3	- 1/3	1	
4	- 1/3	- 1/3	- 1/3	1



All coefficients to be multiplied by

$$\gamma Eab$$

Numbering sequence

FIG. 3.4 STIFFNESS MATRIX FOR RECTANGULAR PLATE - IN PLANE ROTATIONS

CHAPTER 4

ASSEMBLY OF STRUCTURE STIFFNESS MATRIX

CHAPTER 4

ASSEMBLY OF STRUCTURE STIFFNESS MATRIX

4.1 COORDINATE TRANSFORMATION

To make the forces at a node compatible, so that they can be added up, a transformation of the element stiffness matrix has to be performed. The code IP which specifies the location in the plane initiates this coordinate transformation. The general form of a transformation matrix for a 3-dimensional coordinate system at node i is

$$[LB]_i = \begin{bmatrix} \cos(x_i, x_i^*) & \cos(x_i, y_i^*) & \cos(x_i, z_i^*) \\ \cos(y_i, x_i^*) & \cos(y_i, y_i^*) & \cos(y_i, z_i^*) \\ \cos(t_i, x_i^*) & \cos(z_i, y_i^*) & \cos(z_i, z_i^*) \end{bmatrix}$$

where x_i^*, y_i^* and z_i^* are the structure or global axes. For a panelized structure, the following 3 matrices have to be created:

for x-y plane, IP = 1, $[LB]_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

for x-z plane, IP = 2, $[LB]_i = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

for y-z plane, IP = 3, $[LB]_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

If IP = 1, the identity matrix is created, which means there is no transformation for the x-y plane. To achieve a complete transformation of the 24 x 24 element stiffness matrix, the following operations are necessary:

for one node: $[L]_i = \begin{bmatrix} LB_i & 0 \\ 0 & LB_i \end{bmatrix}$

for all four nodes: $[L] = \begin{bmatrix} L_i & 0 & 0 & 0 \\ 0 & L_i & 0 & 0 \\ 0 & 0 & L_i & 0 \\ 0 & 0 & 0 & L_i \end{bmatrix}$

and

$$[K_*] = [L]^T[K][L]$$

where $[K_*]$ is the transformed element stiffness matrix. Figure 4.1 shows the transformation of a node in 3 different planes, with corresponding renumbering arrays.

4.2 REARRANGING OF ELEMENT NUMBERING

The renumbered array which was explained in Section 2.1.6 is now used to rearrange the element stiffness matrix. All the reduced degrees of freedom are assembled in the upper

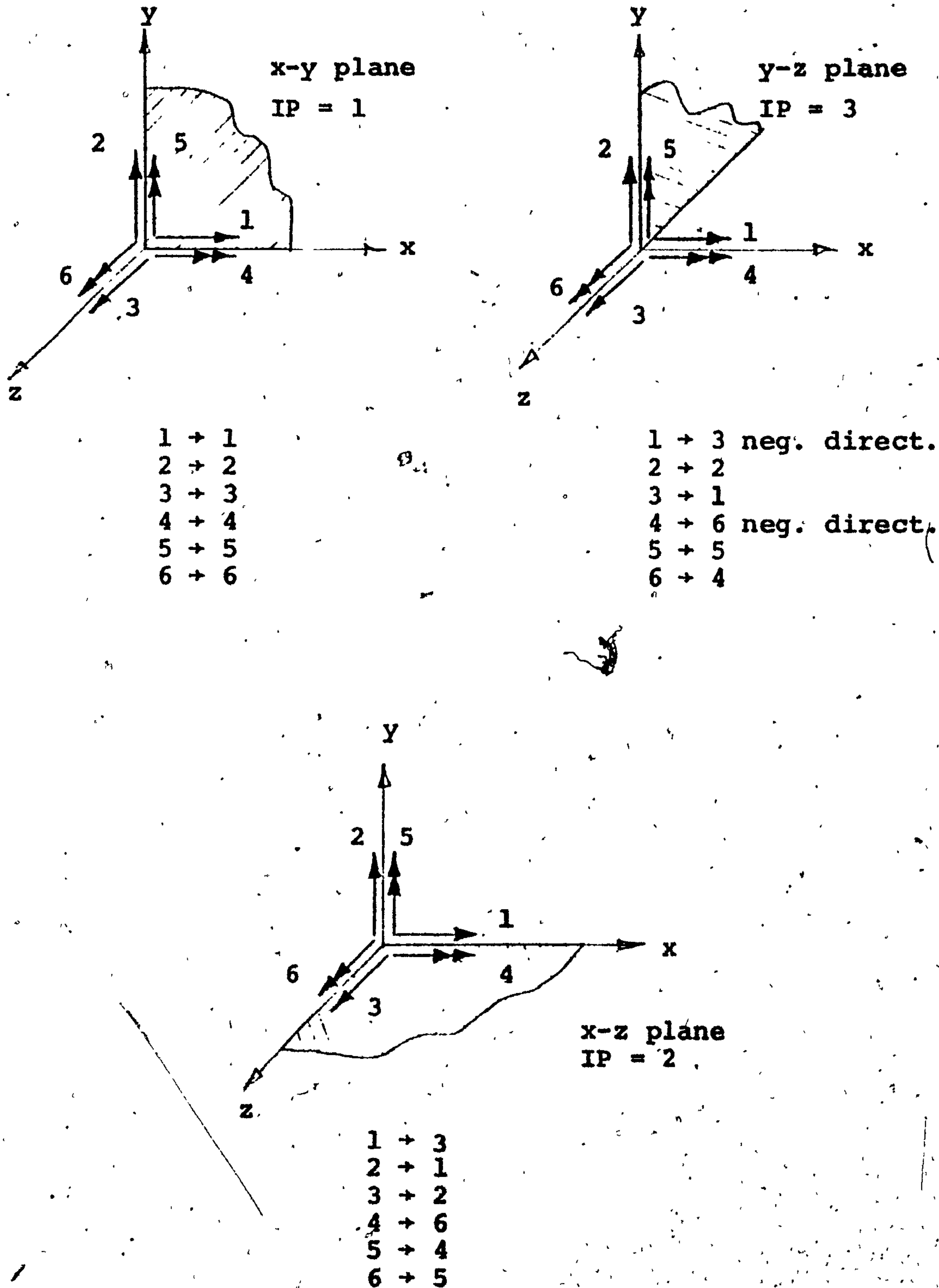


FIG. 4.1 RENUMBERING OF A NODE BY COORDINATE TRANSFORMATIONS

left quadrant of the element stiffness matrix. The eliminated ones are assembled in the lower right quadrant. The interrelated degrees of freedom occupy both the lower left and the upper right rectangle.

4.3 GLOBAL NUMBERING

Up to now the numbering was done locally on the element, i.e. numbering indices were used only from 1 to 24. A new numbering array is now created, which uses the input information from Section 2.3.2. All degrees of freedom are renumbered and prepared for assembly into the overall stiffness matrix. The new numbering is temporarily stored in an array called NDF (I), for I = 1 to 24.(See column 4, in Table 3).

4.4 ASSEMBLY ROUTINE

This routine creates the overall joint stiffness matrix or structural stiffness matrix for all free movable joints or non-restrained degrees of freedom. All restrained ones are eliminated. In a further developed program, the matrix elements representing the restrained degrees of freedom could be easily taken into account if support displacements have to be considered.

The assembly routine takes each element of the element stiffness matrix SM(I,J) and adds its value to a corresponding element in the structural stiffness matrix

S(ROW,COL), of which the location has to be determined. Three arrays are necessary to determine that new location: RL(I), CRL(I) from Section 2.2, and NDF(I) from Section 4.3. The assembly procedure is based on two formulas, originally used for framed structures [3]:

```
        ROW  or  COL = NDF(I) - CRL(NDF(I))   for free movable
                                              degrees of free-
                                              dom, when RL(I)=0

and     ROW  or  COL = N + CRL(NDF(I))        for restrained
           for     N = NJ*ND - NR             degrees of free-
                                              dom, when RL(I)=1
```

Table 3 demonstrates the determination of the new location (ROW,COL), using the 2 elements of the example shown in Figure 2.5.

An example for an assembly of a 2-dimensional plate bending problem of which the lateral stiffness matrix shall be developed is shown in Figure 4.2. Since only the lateral displacements are of interest, and since bending actions and membrane actions are uncoupled, only 3 degrees of freedom have to be considered. To demonstrate a 3-dimensional structure having 6 degrees of freedom would be too complex to use as an example.

The described renumbering of the element indices leads to an assembly of the upper triangle of the structure stiffness matrix, as shown in Figure 4.2(a). The lower

TABLE 3

PROCEDURE OF STRUCTURE NUMBERING (FOR ELEMENTS SHOWN IN FIGURE 2.5)

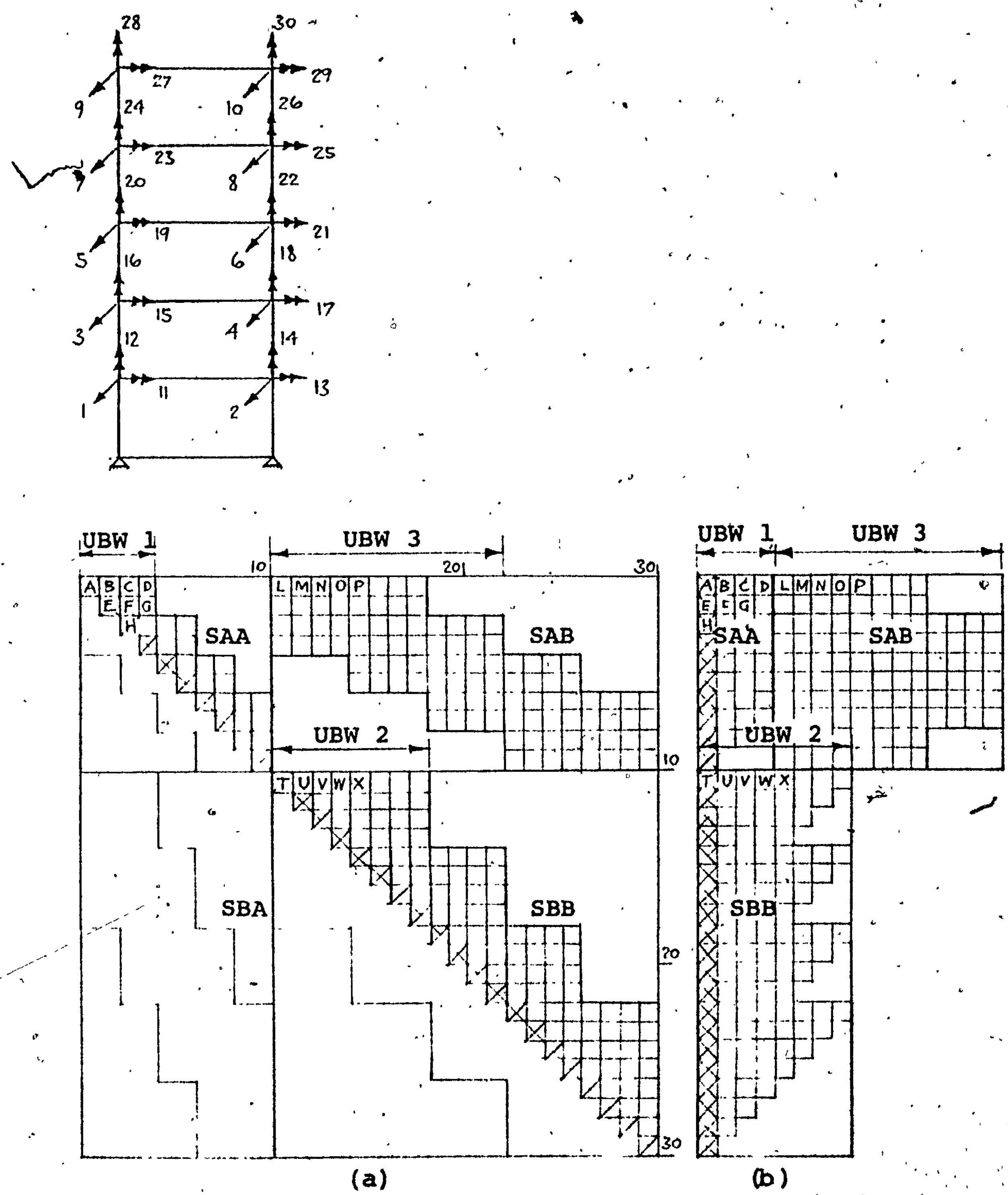
Element No.	Joint Number	I	NDF(1)	RL(NDF(I))	CRL(NDF(I))	Index for Renumber-ing
1	JI(1) = 1	1	7	1	3	[31]*
	"	2	8	1	4	[32]
	"	3	1	1	1	[29]
	"	4	9	1	5	[33]
	"	5	10	0	5	5
	"	6	11	0	5	6
	JJ(1) = 3	7	17	1	6	[34]
	"	8	18	1	7	[35]
	"	9	3	1	2	[30]
	"	10	19	1	8	[36]
	"	11	20	0	8	12
	"	12	21	0	8	13
	JK(1) = 2	13	12	0	5	7
	"	14	13	0	5	8
	"	15	2	0	1	1
	"	16	14	0	5	9
	"	17	15	0	5	10
	"	18	16	0	5	11
	JL(1) = 4	19	22	0	8	14
	"	20	23	0	8	15
	"	21	4	0	2	2
	"	22	24	0	8	16
	"	23	25	0	8	17
	"	24	26	0	8	18

* Elements in brackets are not assembled.

TABLE 3 (continued)

PROCEDURE OF STRUCTURE NUMBERING (FOR ELEMENTS SHOWN IN FIGURE 2.5)

Element No.	Joint Number	I	NDF(I)	RL(NDF(I))	CRL(NDF(I))	Index for Renumbering
2	JI(2) = 3	1	3	1	2	[30]
	"	2	18	1	7	[35]
	"	3	17	1	6	[34]
	"	4	21	0	8	13
	"	5	20	0	8	12
	"	6	19	1	8	[36]
	JJ(2) = 5	7	5	0	2	3
	"	8	28	0	8	20
	"	9	27	0	8	19
	"	10	31	0	8	23
	"	11	30	0	8	22
	"	12	29	0	8	21
	JK(2) = 4	13	4	0	2	2
	"	14	23	0	8	15
	"	15	22	0	8	14
	"	16	24	0	8	16
	"	17	25	0	8	17
	"	18	26	0	8	18
	JL(2) = 6	19	6	0	2	4
	"	20	33	0	8	25
	"	21	32	0	8	24
	"	22	34	0	8	26
	"	23	35	0	8	27
	"	24	36	0	8	28



(a) Complete Structure Stiffness Matrix
UBW = Upper Band Width
(b) Condensed Storage
(c) Storage Arrangement in Core

(c)

FIG. 4.2 ASSEMBLY OF STIFFNESS MATRIX FOR 3-DEGREES OF FREEDOM SYSTEM triangle is created by symmetry. The upper left quadrant SAA represents lateral degrees of freedom; the lower right quadrant SBB represents the eliminated degrees of freedom and the rest of the matrix SAB, SBA shows a combination of the two. This partitioned arrangement is necessary for the reduction procedure which is explained in Chapter 5.

Symmetry and the banded form of this matrix can be used to decrease the core storage (see Figure 4.2(b)). Each of the three blocks is then stored as a one-dimensional array applying a certain mapping pattern, Figure 4.2(c). Matrix SBA is derived by transposing SAB = SBA = SAB^T.

CHAPTER 5

REDUCTION ROUTINE

CHAPTER 5

REDUCTION ROUTINE

5.1 STATIC CONDENSATION

The transformation from the overall joint stiffness matrix to the lateral stiffness matrix is done by reduction or static condensation [4],[6]. This transformation is usually applied, when the stiffness matrix includes more degrees of freedom than are required for the solution of the problem. To achieve a condensation, a partitioned set of a stiffness matrix K, displacements r and forces R are necessary.

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Symbols with index 1 represent a set in which one wants to establish a condensed stiffness matrix.

Multiplication leads to

$$K_{11}r_1 + K_{12}r_2 = R_1$$

$$K_{21}r_1 + K_{22}r_2 = R_2$$

Eliminating r_2

$$(K_1 - K_{21} K_2^{-1} K_{12}) r_1 = R_1 - K_{12} K_{22}^{-1} R_2$$

$$K^* r_1 = R^*$$

$$K^* = K_1 - K_{21} K_{22}^{-1} K_{12}$$ reduced stiffness matrix

$$R^* = R_1 - K_{12} K_{22}^{-1} R_2$$

Normally $R_2 = 0$

$$R^* = R_1$$

5.2 MATRIX OPERATIONS

The partitioned stiffness matrix shown in Figure 4.2(a) suits the requirements for a condensation. Applying the formula derived in Section 5.1 leads to the lateral stiffness matrix.

$$K^* = SAA - SAB\ SBB^{-1}\ SBA$$

To evaluate this formula requires matrix inversion, matrix multiplication and matrix subtraction. The efficiency of the program depends mainly on the method used to invert SBB. A procedure commonly used is inversion by Gauss-Seidel method or by Gauss elimination. The first is used in this program. Since a stiffness matrix is symmetric and "positive definite", usually a more efficient procedure is applied, known as the "Choleski decomposition" [7],[8]. Positive definite means, that a decomposition of a matrix A into two triangular matrices is possible : $A = U^T U$.

If the condensed form of Figure 4.2(b) is used, the same inversion procedures can be applied [8]. Multiplication and subtraction for the condensed matrices can be performed, using the corresponding mappings.

The described methods can be used so far only for a rather limited size of structure. The inversion makes it necessary to keep the complete matrix SBB in core storage; that means the size of the available storage in the computer limits the size of the structure. In a further development of this program, the main emphasis should be put into finding a way to invert SBB in a partitioned form, where different parts of SBB are stored in auxiliary storage and only one part occupies the core. A second possibility to be investigated is the development of an elimination procedure by forward and backward substitution, similar to that which is usually known as "solution by substructures" [7],[9].

5.3 LATERAL STIFFNESS MATRIX

The described reduction process leads in this case, to the lateral stiffness matrix. All properties of that matrix correspond to those of a structural stiffness matrix:

1) Symmetry
2) Positive definite

3) Sum of forces and moments for each column has to be zero - except for element nodes where the element is connected to a support restraint.

This matrix furnishes correct solutions, when only lateral forces are involved, or if R* (see Section 5.1) is calculated. In earthquake problems, lateral inertia forces arise due to horizontal accelerations of points where the lumped masses are assumed to be concentrated. These lateral forces correspond to the stiffness factors represented by the lateral stiffness matrix. For earthquake problems, the lateral stiffness matrix is therefore sufficient to study the dynamic behaviour of a structure.

The real advantage of the lateral stiffness matrix is the reduced storage which is required. The storage space of a 6 degree of freedom structure, where only one degree of freedom is selected is 1/36 of the original size. The size of a stiffness matrix for a panelized building with N free movable joints is N*N.

CHAPTER 6

SUMMARY AND CONCLUSIONS

CHAPTER 6

SUMMARY AND CONCLUSIONS

A computer program to derive a reduced stiffness matrix for a panelized structure is described. The main emphasis is put on the derivation of the lateral stiffness matrix. Rectangular elements are used combining bending and membrane action.

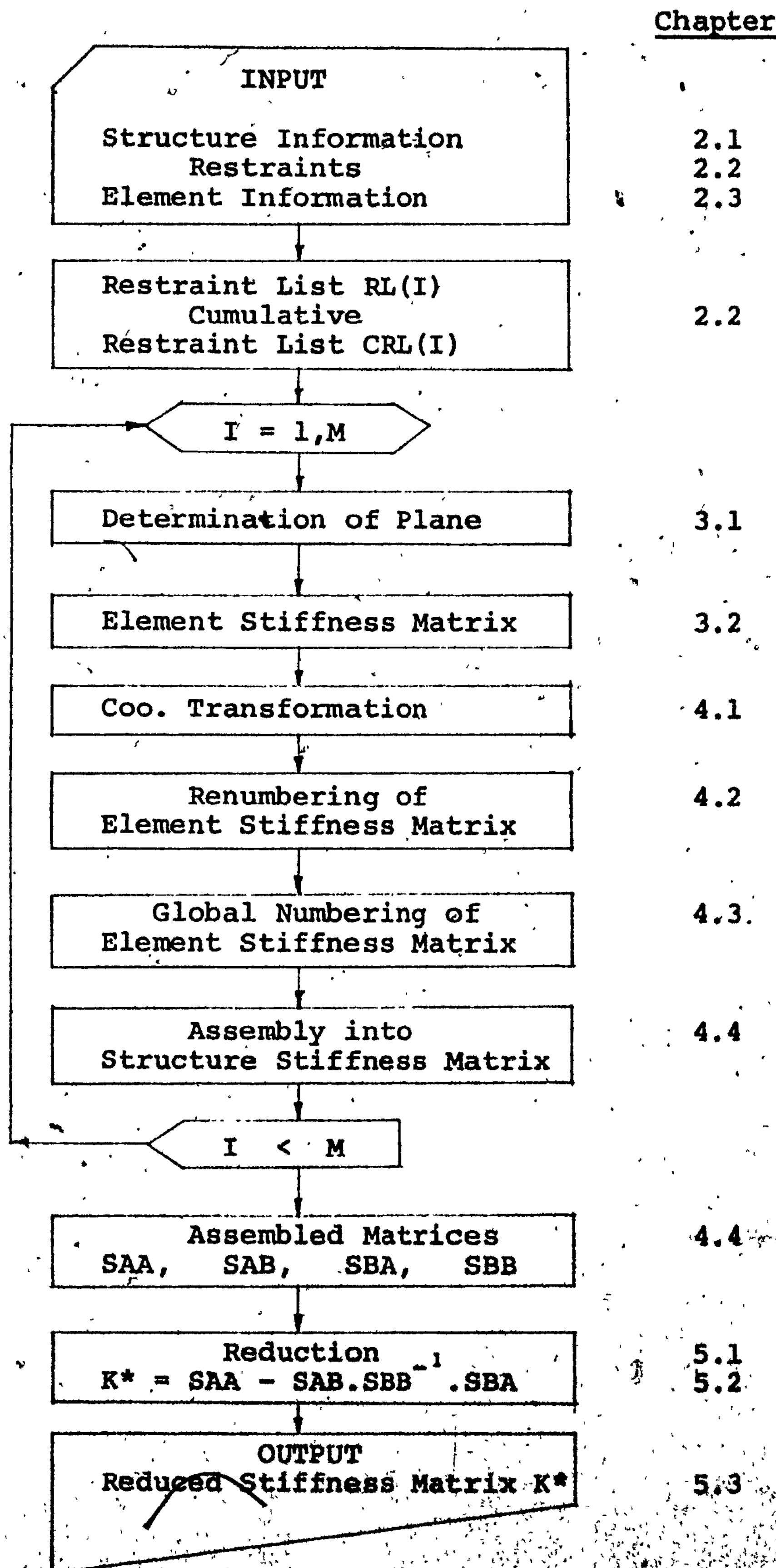
This report explains all fundamental steps - mainly numbering systems - which are necessary to write a program of that type. It builds, at the same time, the base for a computer program of larger capacity which is to be written for future work. This program is limited by the size of the matrix which has to be inverted. An elimination procedure where auxiliary storage is used needs further development.

The lateral stiffness matrix simplifies the equation of motion if it applies to the solution of earthquake problems where only lateral degrees of freedom are required.

Other important parts of the equation of motion are the mass and damping matrix, which are problems not considered here.

APPENDIX

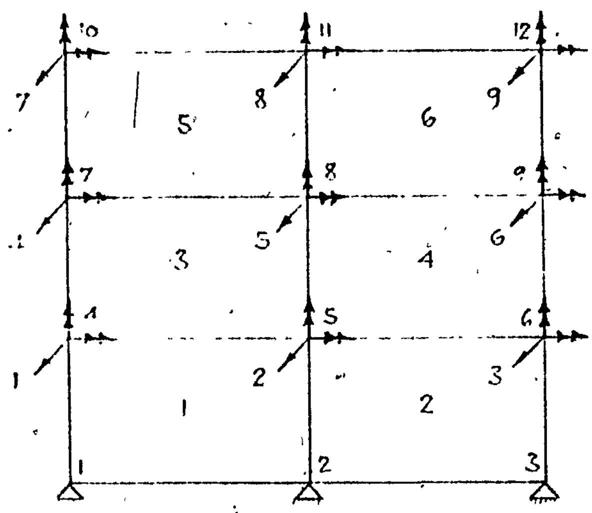
APPENDIX


Chapter
INPUT
Structure Information 2.1
Restraints 2.2
Element Information 2.3
Restraint List RL(I)
Cumulative
Restraint List CRL(I) 2.2
I = 1,M
Determination of Plane 3.1
Element Stiffness Matrix 3.2
Coo. Transformation 4.1
Renumbering of
Element Stiffness Matrix 4.2
Global Numbering of
Element Stiffness Matrix 4.3
Assembly into
Structure Stiffness Matrix 4.4
I < M
Assembled Matrices
SAA, SAB, SBA, SBB 4.4
Reduction 5.1
K* = SAA - SAB.SBB^-1.SBA 5.2
OUTPUT
Reduced Stiffness Matrix K* 5.3

APPENDIX

ILLUSTRATIVE EXAMPLE FOR PLATE BENDING

The 27-degree of freedom system will be reduced to a lateral 9-degrees of freedom system.



INPUT

M	NJ	NRJ	NR	ND	NE	NEA	E(ksi)	P	T(in)
:	:	:	:	:	:	:	:	:	:
6	12	3	9	3	1	1	$.3x10^5$	.3	.1

ELEMENT NO.	NODE JI	NODE JJ	NODE JK	NODE JL
1	1	2	4	5
2	2	3	5	6
3	4	5	7	8
4	5	6	8	9
5	7	8	10	11
6	8	9	11	12

APPENDIX

NODE NO.	X(in)	Y(in)
1	0.	0
2	3.	0
3	6.	0
4	0.	2.
5	3.	2.
6	6.	2.
7	0.	4.
8	3.	4.
9	6.	4.
10	0.	6.
11	3.	6.
12	6.	6.

NODE NO.	RESTRAINTS		
1	1	1	1
2	1	1	1
3	1	1	1

OUTPUT

REDUCED STIFFNESS MATRIX

	1	2	3	4	5
1	.8807E+01	-.5546E+00	.5853E-01	-.2876E+01	-.3526E+00
2	-.5546E+00	.1813E+02	-.5546E+00	-.2276E+00	-.6297E+01
3	.5853E-01	-.5546E+00	.8807E+01	.4043E-01	-.3526E+00
4	-.2876E+01	-.2276E+00	.4043E-01	.5103E+01	-.6115E+00
5	-.3526E+00	-.6297E+01	-.3526E+00	-.6115E+00	.1240E+02
6	.4043E-01	-.2276E+00	-.2876E+01	.2284E+00	-.6115E+00
7	.7497E+00	.3469E+00	.1972E-01	-.1966E+01	-.5525E-02
8	.2897E+00	.1404E+01	.2897E+00	.3205E+00	-.4163E+01
9	.1972E-01	.3469E+00	.7497E+00	-.1120E-01	-.5525E-02

	6	7	8	9
1	.4043E-01	.7497E+00	.2897E+00	.1972E-01
2	-.2276E+00	.3469E+00	.1404E+01	.3469E+00
3	-.2876E+01	.1972E-01	.2897E+00	.7497E+00
4	.2284E+00	-.1966E+01	.3205E+00	-.1120E-01
5	-.6115E+00	-.5525E-02	-.4163E+01	-.5525E-02
6	.5103E+01	-.1120E-01	.3205E+00	-.1966E+01
7	-.1120E-01	.1401E+01	-.7147E+00	.1804E+00
8	.3205E+00	-.7147E+00	.2968E+01	-.7147E+00
9	-.1966E+01	.1804E+00	-.7147E+00	.1401E+01

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REFERENCES

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