REGULAR SELF-INJECTIVE RINGS

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ABSTRACT

REGULAR SELF-INJECTIVE RINGS

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This thesis studies regular right self-injective rings with identity. First, the basic results of the theory are proved. Second, structure theorems are obtained. In particular, we show that any regular, right self-injective ring has a unique decomposition $R = R_1 \times R_2 \times R_3$ such that R_1 has a lot of abelian idempotents, R_2 has a lot of quasi-abelian idempotents but no nonzero abelian idempotents, and R_3 has no quasi-abelian idempotents. Finally, prime ideals are studied, and characterizations are obtained for biregular, quasi-abelian regular and quasi-biregular right self-injective rings.

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Chapter I

Regular self-injective rings

This chapter introduces the basic results on regular right self-injective rings. In particular, proposition 1.2 shows that they have a large supply of central idempotents.

Lemma 1.1 If e is an idempotent in a semiprime ring R, then the following conditions are equivalent:

(a) e is central.

(b) e commutes with every idempotent in R.

(c) $(1 - e)Re = 0$.

(d) eR is a two-sided ideal of R.

(e) $eR(1 - e) = 0$.

(f) Re is a two-sided ideal of R.

Proof (a)$\Rightarrow$(b) a priori.

(b)$\Rightarrow$(c): Let $x \in (1 - e)Re$. Then $e + x$ is an idempotent, and so $e(e + x) = (e + x)e$, whence $e = e + x$. Therefore $x = 0$.

(c)$\Rightarrow$(d): Since $(1 - e)Re = 0$, we have that $Re \subseteq r(R(1 - e)) = eR$, whence eR is a two-sided ideal of R.

(d)$\Rightarrow$(a): Assume eR is a left ideal. Then, $Re \subseteq eR$ and $eR(1 - e)$ is a nilpotent left ideal of R. It follows that $(1 - e)Re = 0$ and $eR(1 - e) = 0$, whence $er = ere = re$ for any $r \in R$. Therefore e is central.

(a)$\Leftrightarrow$(e)$\Leftrightarrow$(f) by symmetry.

<u>Proposition 1.2</u> Let R be a regular, right self-injective ring, and let J be a two-sided ideal of R. Then the following hold:

(a) There is a unique $u \in B(R)$ for which $J_R \leq_e (uR)_R$.

(b) $(R/J)_R$ is nonsingular if and only if J is generated by a central idempotent.

Proof (a) Since R is right self-injective, there exists an idempotent $u \in R$ such that $J_R \leq_e uR$. We will show that $u \in B(R)$. Let x be any element in R. Then, $xJ \leq J \leq uR$, and so $(1 - u)xJ = 0$. This implies that the map $g: R/J \to (1 - u)xR$ given by $g(r + J) = (1 - u)xr$ is well-defined. Hence, $(1 - u)xuR$ is a homomorphic image of the singular module uR/J (6, proposition 1.21), and so is singular by (6, proposition 1.22), whence $(1 - u)xuR = 0$. Thus, $(1 - u)Ru = 0$; hence lemma 1.1 shows that $u \in B(R)$. It remains to show uniqueness.

If $J_R \leq_e (fR)_R$ for some $f \in B(R)$, then $J_R \leq_e fR \cap uR = ufR$. Since ufR is a submodule of uR and fR, we see that $fR = ufR = uR$. Therefore $f = u$.

(b) First assume that $(R/J)_R$ is a nonsingular R-module. According to (a), there exists $u \in B(R)$ such that $J_R \leq_e uR$. This implies that uR/J is singular by (6, proposition 1.21). Now, since uR/J is a submodule of the nonsingular module R/J, we also have that uR/J is nonsingular. Therefore, $uR/J = 0$, i.e., $uR = J$.

Conversely, assume $J = uR$ for some $u \in B(R)$. Then $R/J \simeq (1 - u)R$. Thus, $(R/J)_R$ is nonsingular.

<u>Definition 1.1</u> Let R be a regular, right self-injective ring, and let $x \in R$. The central cover of x, written $c(x)$, is the unique central

idempotent $u \in R$ satisfying $(RxR)_R \leq_e (uR)_R$ (this exists by proposition 1.2).

<u>Proposition 1.3</u> Let R be a nonzero, regular, right self-injective ring. Then R is indecomposable (as a ring) if and only if R is a prime ring.

Proof Assume R is a prime and $u \in B(R)$. Then $(uR)(1 - u)R = 0$, and so $uR = 0$ or $(1 - u)R = 0$, whence $u = 0$ or $u = 1$. Therefore, R is indecomposable.

Conversely, if R is not prime, then R has nonzero two-sided ideals J and K for which $JK = 0$. Set $L = \{r \in R \mid Jr = 0\}$, and note that L is a nontrivial two-sided ideal. We will show that $(R/L)_R$ is nonsingular. Assume there is a large right ideal I such that $(a + L)I = 0$. Then $aI \leq L$, and so $(Ja)I = 0$, whence $Ja = 0$, i.e., $a \in L$. Thus, $(R/L)_R$ is a nonsingular R-module. According to proposition 1.2, $L = uR$ for some $u \in B(R)$, and since L is nontrivial we have that $u \notin \{0, 1\}$. Thus, R is decomposable.

Recall that for any ring R, B(R) is a Boolean algebra in which $e \wedge f = ef$, $e \vee f = e + f - ef$ and $e' = 1 - e$.

<u>Proposition 1.4</u> If R is a regular, right self-injective ring, then B(R) is a complete Boolean algebra. For any $\{u_i\} \subseteq B(R)$, we have $(\bigwedge u_i)R = \cap u_iR$ and $(\bigvee u_i)R = E((\sum u_iR)_R)$.

Proof Let $\{u_i\}$ be a family of central idempotents. The obvious map $(R/\cap u_iR)_R \longrightarrow \Pi(1 - u_i)R$ is one-one, and so $(R/\cap u_iR)_R$ is nonsingular.

According to proposition 1.2, $\cap u_i R = uR$ for some $u \in B(R)$, and it is clear that $u = \bigwedge u_i$.

Again by lemma 1.2, there is a $f \in B(R)$ such that $(\sum u_i R)_R \leq_e (fR)_R$, whence $(fR)_R = E((\sum u_i R)_R)$. Obviously $u_i \leq f$ for all i. If $g \in B(R)$ such that $u_i \leq g$ for all i, then $(1 - g)(\sum u_i R) = 0$. Since $\sum u_i R \oplus (1 - f)R$ is a large right ideal of the ring R, we see that $((1 - g)fR)_R$ is a singular right R-module, and so $(1 - g)f = 0$, i.e., $f \leq g$. Therefore $f = \bigvee u_i$.

The following proposition is often used to decompose regular self-injective rings into a direct product of rings.

<u>Proposition 1.5</u> Let R be a regular, right self-injective ring. Let X be a nonempty subset of B(R), and let $h: R \longrightarrow \prod_{u \in X} uR$ be the natural ring map. Then the following hold:

(a) $\operatorname{Ker} h = (1 - \bigvee X)R$.

(b) h is surjective if and only if the elements of X are pairwise orthogonal.

Proof (a) According to proposition 1.4, $\operatorname{Ker} h = \bigcap_{u \in X} (1 - u)R = (\bigwedge_{u \in X} (1 - u))R = (1 - (\bigvee X))R$.

(b) First assume that h is surjective, and let u_1, u_2 be distinct elements of X. Define $x \in \prod_{u \in X} uR$ so that $x(u_1) = u_1$ and $x(u) = 0$ for all $u \neq u_1$. Since h is surjective, there exists $y \in R$ such that $hy = x$, whence $u_1 y = u_1$ and $u_2 y = 0$. Thus, $u_1 u_2 = (u_1 y)u_2 = u_1(u_2 y) = 0$.

Conversely, assume X is an orthogonal family. According to (a), $\operatorname{Ker} h = uR$ for some $u \in B(R)$. This implies that hR is isomorphic to the ring $(1 - u)R$, and so hR is a right self-injective ring. Thus, by

(6, proposition 1.8), it is enough to show that $(hR)_{hR} \leq_e (\prod_{u \in X} uR)_{hR}$ to get that $hR = \prod_{u \in X} uR$.

Let x be a nonzero element of $\prod_{u \in X} uR$. Then there exists a $v \in X$ such that $x(v) \neq 0$. Since $x(hv)(v) = x(v) = h(x(v))(v)$ and since $x(hv)(u) = 0 = h(x(v))(u)$ for all $u \in x$ such that $u \neq v$, we see that $x(hv) = h(x(v))$, whence $x(hv)$ is a nonzero element of hR. Hence $x(hR) \cap hR \neq 0$. Therefore $(hR)_{hR} \leq_e (\prod_{u \in X} uR)_{hR}$.

<u>Definition 1.2</u> Let R be a ring. A minimal nonzero element of B(R) is called an atom. The ring B(R) is called atomic if for every nonzero $u \in B(R)$ there exists an atom a such that $a \leq u$.

<u>Corollary 1.6</u> Let R be a regular, right self-injective ring. Then R is isomorphic to a direct product of prime rings if and only if B(R) is atomic.

Proof Necessity is obvious. Conversely, assume that B(R) is atomic, and let X be the set of atoms in B(R). Since $\bigvee X = 1$ and since atoms are pairwise orthogonal, proposition 1.5 shows that the natural ring map $R \rightarrow \prod_{a \in X} aR$ is an isomorphism. According to proposition 1.3, aR is a prime ring for each $a \in X$. Thus, R is isomorphic to a direct product of prime rings.

<u>Proposition 1.7</u> Let R be a regular, right self-injective ring. Then the following hold:

(a) If $x \in R$, then $r(xR) = (1 - c(x))R$.

(b) Let J be a right ideal, and let $S = \{c(y) \mid y \in J\}$.

Then $r(J) = (1 - \bigvee S)R$.

(c) For any $x, y \in R$, $xRy = 0$ is equivalent to $c(x)c(y) = 0$.

(d) If e is an idempotent in R and $f \in B(eRe)$, then $f = c(f)e$.

Proof (a) By definition 1.1, $RxR \leq_e c(x)R$, whence $r(xR) = r(RxR) = r(c(x)R) = (1 - c(x))R$.

(b) Since $r(J) = r(RJ) = r(\sum_{y \in J} RyR) = \bigcap_{y \in J} r(RyR)$, (a) shows that $r(J) = \bigcap_{y \in J}(1 - c(y))R$. Then, $r(J) = (\bigwedge_{y \in J}(1 - c(y))R = (1 - \bigvee S)R$ by proposition 1.4.

(c) Obviously, $c(x)c(y) = 0$ always implies $xRy = 0$. Conversely, assume $xRy = 0$. According to (a), $r(xR) = uR$ for some $u \in B(R)$. Since $y \in r(xR)$, we see that $yu = y$, whence $c(y) \leq u$ (by definition 1.1). Also $xu = 0$, and so $x(1 - u) = x$, whence $c(x) \leq 1 - u$. Therefore $c(x)c(y) = 0$.

(d) Let $x \in R$. Since $f \in eRe$, we have that $fxf = (fe)x(ef) = f(exe)f = ff(exe) = fxe$, whence $fR(e - f) = 0$. By part (a), $c(f)c(e - f) = 0$. Hence, $c(f)(e - f) = c(f)c(e - f)(e - f) = 0$. Therefore, $c(f)e = c(f)f = f$.

<u>Definition 1.3</u> Idempotents e, f in a ring R are equivalent, written $e \sim f$, if $eR \simeq fR$. Note that $e \sim f$ if and only if there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$ (10, p. 21, theorem 14).

<u>Lemma 1.8</u> Let R be a regular ring, and let e, f be two idempotents in R. If $eRf \neq 0$, then there exist nonzero idempotents $e_1 \in eRe$ and $f_1 \in fRf$ such that $e_1 \sim f_1$.

Proof Consider any nonzero $x \in eRf$. Since R is regular, there exists $y \in R$ such that $x = xyx$. Put $e_1 = x(fye)$ and $f_1 = (fye)x$, and note that e_1 and f_1 are nonzero. Then $e_1 \sim f_1$.

<u>Notation</u>: If A and B are modules, then the notation $A \lesssim B$ means that A is isomorphic to a submodule of B.

We are now in a position to prove the most useful property of regular self-injective rings, which will be used extensively from this point on, namely general comparability.

<u>Theorem 1.9</u> Let R be a regular, right self-injective ring, and let e, f be two idempotents in R. Then there exists a $u \in B(R)$ such that:

(a) $euR \lesssim fuR$.

(b) $f(1 - u)R \lesssim e(1 - u)R$.

Proof Let S denote the collection of all triples (X, Y, h) such that $X \leq eR$, $Y \leq fR$, and $h: X \longrightarrow Y$ an isomorphism. Define a partial order on S by setting $(X, Y, h) \leq (X', Y', h')$ whenever $X \leq X'$, $Y \leq Y'$, and h' is an extension of h. By Zorn's lemma, there is a maximal element (X_0, Y_0, h_0) in S.

Inasmuch as eR and fR are injective, eR contains an injective hull X_1 for X_0 and fR contains an injective hull Y_1 for Y_0, and h_0 extends to an isomorphism $h_1: X_1 \longrightarrow Y_1$. By maximality, $(X_0, Y_0, h_0) = (X_1, Y_1, h_1)$, whence X_0 and Y_0 are injective. Thus, X_0 is a direct summand of eR and Y_0 is a direct summand of fR. This implies that there exist idempotents g, q, k, $m \in R$ such that

$$X_0 = gR,\ e = g + k,\ gk = kg = 0,$$

$$Y_0 = qR,\ f = q + m,\ qm = mq = 0.$$

Using lemma 1.8 and the maximality of the triple (X_0, Y_0, h_0), it is not hard to show that $mRk = 0$. According to part (c) of proposition 1.7, we then have that $c(m)c(k) = 0$. Put $u = c(m)$. Then $ku = kc(k)c(m) = 0$, and so

$$eu = gu \text{ and } fu = qu + m.$$

Since $gR \simeq qR$, we have that $guR \simeq quR$, hence $euR \simeq quR$, whence $euR \lesssim fuR$. It remains to show that (b) also holds.

Using the relations $e = g + k$ and $f = q + m$, we obtain $e(1 - u) = g(1 - u) + k(1 - u)$ and $f(1 - u) = q(1 - u)$. Hence, since $gR \simeq qR$, $f(1 - u)R = q(1 - u)R \simeq g(1 - u)R \lesssim e(1 - u)R$. Therefore $f(1 - u)R \lesssim e(1 - u)R$.

<u>Lemma 1.10</u> Let e, f be equivalent idempotents in a ring R. If I is a two-sided ideal containing e, then $f \in I$.

Proof Since $e \sim f$, there exist $x, y \in R$ such that $xy = e$ and $yx = f$. Then $f = yx = yxyx = yex \in I$.

<u>Proposition 1.11</u> Let R be a regular, right self-injective ring, and let M be a maximal two-sided ideal of R. If I is a two-sided ideal of R not contained in M, then $I - M$ contains a central idempotent.

Proof Since $M + I = R$, $1 = x + i$ for some $x \in M$ and some $i \in I$, and so there exists $x \in M$ such that $1 - x \in I$. Let $h \in R$ be an idempotent such that $hR = xR$. Then $(1 - x) - (h(1 - x)) \in I$, whence $1 - h \in I$.

According to general comparability, there exists a central idempotent u such that:

(a) $huR \lesssim (1 - h)uR$.

(b) $(1 - h)(1 - u)R \lesssim h(1 - u)R$.

Since $(1 - h)u \in I$, it follows by lemma 1.10 and the relation (a) that $hu \in I$. Hence $u = (1 - h)u + hu \in I$. Similarly, $1 - u \in M$. Therefore $u \in I - M$.

Definition 1.4 A ring R is said to be directly finite if $xy = 1$ implies $yx = 1$ for any $x, y \in R$. An idempotent e in R is called directly finite if eRe is directly finite.

Theorem 1.12 Let R be a directly finite, regular, right self-injective ring. If I is a nonzero two-sided ideal of R, then I contains a nonzero central idempotent.

Proof Let $e_1 \neq 0$ be an idempotent in I, and let $fA = \bigoplus_{i=1}^{n} e_iR$ be a maximal direct sum of right ideals isomorphic to e_1R, where f, e_2, e_3,..., e_n are idempotents (7, proposition 5.7). Note that lemma 1.10 implies $f \in I$. According to theorem 1.9, there exists a $u \in B(R)$ such that:

(a) $(1 - f)uR \lesssim e_1uR$,

(b) $e_1(1 - u)R \lesssim (1 - f)(1 - u)R$.

Because of the relation (b) and the maximality of the e_iR's, we see that $u \neq 0$. Also, lemma 1.10 shows that $(1 - f)u \in Re_1uR \leq I$. Therefore $u = fu + (1 - f)u$ is a nonzero element of I.

The converse of theorem 1.12 is false. A regular, right self-injective ring R has the property $I \cap B(R) = 0$ for every two-sided ideal if and only if it contains a $u \in B(R)$ such that uR is finite and $(1 - u)R$ is quasi-abelian, as we show in chapter VI.

Chapter II

Quasi-abelian idempotents in regular self-injective rings

In this chapter, we show that any regular, right self-injective ring has a decomposition $R = R_1 \times R_2 \times R_3$ (Types I, $\overline{\text{II}}$, $\overline{\text{III}}$) such that R_1 has a lot of abelian idempotents, R_2 has a lot of quasi-abelian idempotents but no nonzero abelian idempotents, and R_3 has no quasi-abelian idempotents. At the end of the chapter, we give characterizations for Type $\overline{\text{II}}$ and Type $\overline{\text{III}}$ rings.

Definition 2.1 A ring is said to be abelian if all its idempotents are central. An idempotent e is called abelian if eRe is abelian.

Definition 2.2 A ring R is said to be quasi-abelian if for any idempotent e in R there exists $u \in B(R)$ such that $e \sim u$. An idempotent e is called quasi-abelian if eRe is quasi-abelian.

For example, let V be an infinite-dimensional vector space over a field F, and set $Q = \text{End}_F(V)$ and $M = \{x \in Q \mid \dim_F(xV) < \dim_F(V)\}$. Then Q/M is a simple quasi-abelian regular ring, which is not abelian. Moreover, Q/M is not right self-injective by (12, section 3).

Proposition 2.1 Let R be a regular, right self-injective ring, and let $e^2 = e \in R$. Then the following conditions are equivalent:

(a) e is quasi-abelian.

(b) $f^2 = f \in eR$ implies $fe \sim ec(f)$.

(c) $f^2 = f \in eRe$ implies $f \sim ec(f)$.

Proof (a)$\Rightarrow$(b): Let $f^2 = f \in eR$. Then $fe \sim u$ for some $u \in B(eRe)$. By proposition 1.7, $u = c(u)e$. Also, since $fR = feR \simeq uR$, we have that $RfR = RuR$ by lemma 1.10, whence $c(f) = c(u)$. Therefore, $fe \sim u = c(u)e = c(f)e$.

(b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a): Suppose $f^2 = f \in eRe$. Then $f \sim ec(f)$ relative to R, and so there exist $x \in fRec(f)$ and $y \in ec(f)Rf$ such that $xy = f$ and $yx = ec(f)$. Since $x, y \in eRe$, $f \sim ec(f)$ relative to eRe. Thus e is quasi-abelian.

<u>Corollary 2.2</u> Let R be a quasi-abelian, regular, right self-injective ring. If $e^2 = e \in R$, then eRe is quasi-abelian.

Proof Let $f^2 = f \in eRe$. By proposition 2.1, we see that $e \sim c(e)$ and $f \sim c(f)$. This implies $ec(f) \sim c(e)c(f)$ and $fc(e) \sim c(e)c(f)$, whence $f = fc(e) \sim ec(f)$. Thus e is quasi-abelian by proposition 2.1.

<u>Definition 2.3</u> A Baer ring is a ring in which every right (and left) annihilator ideal is generated by an idempotent.

Note that proposition 2.1 and corollary 2.2 also hold for the more general class of semiprime Baer rings.

<u>Definition 2.4</u> Let R be a regular, right self-injective ring and let $e^2 = e \in R$. Then e is faithful if 0 is the only central idempotent of

R which is orthogonal to e.

Lemma 2.3 Let R be a regular, right self-injective ring, and let e be an idempotent in R. Then the following conditions are equivalent:

(a) e is faithful.

(b) $c(e) = 1$.

(c) Re is a faithful left R-module.

(d) eR is a faithful right R-module.

(e) $\mathrm{Hom}_R(eR, J) \neq 0$ for all nonzero right ideals J of R.

Proof (a)$\Rightarrow$(b): According to proposition 1.2, there exists $u \in B(R)$ such that $(ReR)_R \leq_e (uR)_R$. Then $(1 - u)e = 0$, whence $1 - u = 0$. Thus, $(ReR)_R \leq_e R_R$.

(b)$\Rightarrow$(c): Since R_R is nonsingular and ReR is a large right ideal of R, we have that $xReR \neq 0$ for all nonzero $x \in R$. Thus, $xRe \neq 0$ for all nonzero $x \in R$.

(c)$\Rightarrow$(d): Let $K = \{x \in R \mid eRx = 0\}$. Then $(KRe)^2 = 0$, and so $KRe = 0$. Therefore, $K = 0$.

(d)$\Rightarrow$(e): Let $x \neq 0 \in J$. By assumption, there exists $y \in eR$ such that $yx \neq 0$. Define the epimorphism $f: xR \longrightarrow yxR$ by $f(xr) = yxr$. Since yxR is projective there is a nonzero map $h: yxR \longrightarrow xR$ such that $fh = 1$. Since yxR is a direct summand of eR, h extends to a nonzero map $eR \longrightarrow xR \leq J$.

(e)$\Rightarrow$(a): If $u \in B(R)$ and $eu = 0$, then $\mathrm{Hom}_R(eR, uR) = 0$. Thus $u = 0$.

Definition 2.5 A regular, right self-injective ring R is said to be of Type $\overline{\mathrm{II}}$ if R contains a faithful quasi-abelian idempotent but R contains no nonzero abelian idempotents.

Theorem 2.4 Let R be a regular, right self-injective ring, and assume that R contains no nonzero abelian idempotents. Then the following conditions are equivalent:

(a) R is Type $\overline{\text{II}}$.

(b) Every nonzero right ideal of R contains a nonzero quasi-abelian idempotent.

(c) The two-sided ideal generated by the quasi-abelian idempotents of R is essential as a right ideal.

Proof (a)$\Rightarrow$(b): Let e be a faithful quasi-abelian idempotent in R, let J be a nonzero right ideal, and let f be a nonzero idempotent in J. Then, $c(e)c(f) = c(f) \neq 0$ by lemma 2.3. According to proposition 1.7, it follows that $eRf \neq 0$. Consequently, lemma 1.8 implies that there exist nonzero idempotents $e_1 \in eRe$ and $f_1 \in fRf$ such that $e_1 \sim f_1$. Thus, $f_1Rf_1 \simeq \text{End}(f_1R) \simeq \text{End}(e_1R) \simeq e_1Re_1 \subseteq eRe$, and so f_1Rf_1 is a quasi-abelian ring by corollary 2.2. Therefore f_1 is a nonzero quasi-abelian idempotent contained in J.

(b)$\Rightarrow$(c) is clear.

(c)$\Rightarrow$(b): Let $x \neq 0 \in R$, and let $y \in R$ such that $x = xyx$. By assumption, there exist $s, s_1, s_2, \ldots, s_n, r_1, r_2, \ldots, r_n \in R$ and quasi-abelian idempotents $e_1, e_2, \ldots, e_n$ such that $xys = r_1e_1s_1 + r_2e_2s_2 + \ldots + r_ne_ns_n \neq 0$. This implies that $xre \neq 0$ for some $r \in R$ and some quasi-abelian idempotent e in R. Now, let $z \in R$ such that $(xre)z(xre) = xre$, and consider the idempotents $f = xrez$ and $g = ezxre$. Since $(xre)(ezxrez) = f$ and $(ezxrez)(xre) = g$, we see that $f \sim g$. Thus, $fRf \simeq \text{End}(fR) \simeq \text{End}(gR) \simeq gRg \subseteq eRe$, and so fRf is a quasi-abelian ring by corollary 2.2. Therefore f is a nonzero quasi-abelian idempotent contained in xR.

(b)$\Rightarrow$(a): Set $X = \{u \in B(R) \mid uR \text{ is Type } \overline{II}\}$ and consider a maximal orthogonal family $\{u_i\} \subseteq X$. We will show $\bigvee u_i = 1$. Assume $\bigvee u_i = f \neq 1$. Then, there exists a nonzero quasi-abelian idempotent g in the ring $(1 - f)R$. Since g is faithful in the ring $c(g)R \subseteq (1 - f)R$ and since $c(g)R$ contains no nonzero abelian idempotents, we have $c(g)R$ is Type $\overline{II}$, i.e., $c(g) \in X$. Then $c(g) \in fR \cap (1 - f)R = 0$, which is a contradiction. Therefore $f = 1$.

Now, for each i there is a faithful quasi-abelian idempotent h_i in the ring u_iR. Since $(h_i)\,\Pi\, u_iR(h_i) \simeq \Pi\, h_i(u_iR)h_i$, (h_i) is a faithful quasi-abelian idempotent in the ring $\Pi\, u_iR$. Moreover, $\Pi\, u_iR$ has no nonzero abelian idempotents, and so $\Pi\, u_iR$ is Type $\overline{II}$. According to proposition 1.5, $R \simeq \Pi\, u_iR$. Therefore R is Type $\overline{II}$.

<u>Definition 2.6</u> A regular, right self-injective ring is said to be Type I if R contains a faithful abelian idempotent.

<u>Theorem 2.5</u> Let R be a regular, right self-injective ring. Then the following conditions are equivalent:

(a) R is Type I.

(b) Every nonzero right ideal of R contains a nonzero abelian idempotent.

(c) The two-sided ideal generated by the abelian idempotents of R is essential as a right ideal.

Proof Analogous to theorem 2.4.

<u>Lemma 2.6</u> Let R be a ring, let $u \in B(R)$, and let $X \subseteq uR$. If

$(RXR)_R \leq_e (uR)_R$, then $((uR)X(uR))_{uR} \leq_e (uR)_{uR}$.

Proof Assume $0 \neq J \leq (uR)_{uR}$. Then $0 \neq J \leq (uR)_R$, and so $RXR \cap J \neq 0$, whence $(uR)X(uR) \cap J \neq 0$. Therefore $((uR)X(uR))_{uR} \leq_e (uR)_{uR}$.

Definition 2.7 A regular, right self-injective ring is Type $\overline{III}$ if it contains no nonzero quasi-abelian idempotents.

Theorem 2.7 Any regular, right self-injective ring R is uniquely a direct product of rings of Types I, $\overline{II}$, $\overline{III}$.

Proof Let X denote the collection of all abelian idempotents of R. By proposition 1.2, there exists a $u \in B(R)$ such that $(RXR)_R \leq_e (u_1R)_R$. Consequently, u_1R is Type I by theorem 2.5 and lemma 2.6. Note that since $(1 - u_1)R \cap X = 0$, the ring $(1 - u_1)R$ has no nonzero abelian idempotents. Now, let Y denote the collection of all quasi-abelian idempotents in $(1 - u_1)R$. By proposition 1.2, there exists a unique $u_2 \in B(R)$ such that $(RYR)_R \leq_e (u_2R)R$. Since $Y \subseteq (1 - u_1)R$, it follows from the uniqueness of u_2 that $u_2(1 - u_1) = u_2$, whence u_2R contains no nonzero abelian idempotents. Thus, by theorem 2.4 and lemma 2.6, u_2R is Type $\overline{II}$. Set $u_3 = 1 - u_1 - u_2$. Then since $u_3R \cap Y = 0$, the ring u_3R is Type $\overline{III}$. Therefore, we have orthogonal central idempotents $u_1, u_2, u_3 \in R$ such that $u_1 + u_2 + u_3 = 1$ and the rings u_1R, u_2R, u_3R are Types I, $\overline{II}$, $\overline{III}$. It remains to show uniqueness.

Suppose $f_1, f_2, f_3 \in B(R)$ such that $f_1 + f_2 + f_3 = 1$ and f_1R, f_2R, f_3R are Types I, $\overline{II}$, $\overline{III}$ respectfully. Then, u_1f_2R is both Type I and Type $\overline{II}$ by theorem 2.4 and theorem 2.5, whence $u_1f_2 = 0$. Likewise,

$u_1f_3 = f_1u_2 = f_1u_3 = 0$. Hence, $u_1 = u_1(f_1 + f_2 + f_3) = u_1f_1 = f_1(u_1 + u_2 + u_3) = f_1$. Similarly, $f_2 = u_2$ and $f_3 = u_3$. Therefore the ring decomposition is unique.

Proposition 2.8 Let R be a regular ring. Then R is abelian if and only if R is quasi-abelian and R is directly finite.

Necessity is obvious. Conversely, let e be an idempotent in R. Since R is quasi-abelian, there exist elements $x, y \in R$ and $u \in B(R)$ such that $xy = e$ and $yx = u$. Clearly u is directly finite, whence $u = yx = uyx = (uyu)(uxu) = (uxu)(uyu) = xuy = x(yx)y = xy = e$. Therefore R is abelian.

Definition 2.8 A regular, right self-injective ring R is said to be Type II if R contains a faithful directly finite idempotent but R contains no nonzero abelian idempotents.

Lemma 2.9 Let R be a regular, right self-injective ring. If R is Type II and Type $\overline{\text{II}}$, then R = 0.

Proof Let e be a nonzero quasi-abelian idempotent in R. According to (7, corollary 10.9), eRe is Type II, and so there exists a nonzero directly finite idempotent f in eRe. Since e is quasi-abelian, we also have that f is quasi-abelian (by corollary 2.2). Then, proposition 2.8 implies that f is a nonzero abelian idempotent in R, which is a contradiction. Thus, R = 0.

Definition 2.9 A regular, right self-injective ring is Type III if it contains no nonzero directly finite idempotents.

<u>Lemma 2.10</u> Let R be a regular, right self-injective ring. If R is Type III and Type $\overline{\text{III}}$, then R = 0.

Proof This follows from (14, theorem 3.2 and lemme 3.3).

<u>Theorem 2.11</u> Let R be a regular, right self-injective ring. Then the following conditions are equivalent:

(a) R is Type $\overline{\text{III}}$.

(b) R is Type II.

(c) R has a faithful directly finite idempotent but no nonzero quasi-abelian idempotents.

Proof (a)$\Rightarrow$(b): According to (7, theorem 10.13), R has a unique decomposition into rings of Types I, II, III. Then R is Type II by lemma 2.10.

(b)$\Rightarrow$(a): According to theorem 2.7, R has a unique decomposition into rings of Types I, $\overline{\text{II}}$, $\overline{\text{III}}$. Then R is Type $\overline{\text{III}}$ by lemma 2.9.

(b)$\Leftrightarrow$(c): This follows from the equivalence of (a) and (b).

<u>Theorem 2.12</u> Let R be a regular, right self-injective ring. Then the following conditions are equivalent:

(a) R is Type $\overline{\text{II}}$.

(b) R is Type III.

(c) R has a faithful quasi-abelian idempotent but no nonzero finite idempotents.

Proof Analogous to 2.11.

Chapter III

Purely infinite rings

This chapter is concerned with regular, right self-injective rings, which do not have nonzero directly finite ring direct summands.

Definition 3.1 A regular, right self-injective ring is purely infinite if it has no nonzero central directly finite idempotents.

For example, every regular, right self-injective ring of Type III is purely infinite. For another example, let R be the endomorphism ring of any infinite-dimensional vector space. Since $B(R) = \{0, 1\}$ and R is directly infinite, we see that R is purely infinite.

For characterizations of purely infinite and quasi-abelian rings (chapter V and VI), we need the following results.

Proposition 3.1 If A and B are injective modules such that $A \lesssim B$ and $B \lesssim A$, then $A \simeq B$.

Proof By assumption, $A \lesssim B$ and B is isomorphic to a submodule C of A, and so there exists a monomorphism $f: A \to C$. Since C is injective, we have $A = H \oplus C$ for some H. We claim that $\{H, fH, f^2H, f^3H, \ldots\}$ is an independent family of submodule of A.

Obviously, $\{H\}$ is independent. Now, suppose that

$\{H, fH, f^2H, \ldots, f^nH\}$ is independent, for some nonnegative integer n. Since f is a monomorphism, it follows that $\{fH, f^2H, \ldots, f^{n+1}H\}$ is independent. Moreover, $H \cap (fH \oplus f^2H \oplus \ldots \oplus f^{n+1}H) \leq H \cap fA \leq H \cap C = 0$, whence $\{H, fH, f^2H, \ldots, f^{n+1}H\}$ is independent. Thus, the induction works, and so $\{H, fH, f^2H, \ldots\}$ is independent.

Now A has a submodule $P = H \oplus fH \oplus f^2H \oplus \ldots$, and we note that $P = H \oplus fP$. Since $fP \leq C$, we see that C contains an injective hull Q for fP. It follows that, $E(P) = E(H \oplus fP) = E(H) \oplus E(fP) = H \oplus Q$, and so $H \oplus Q$ is an injective hull for P. Inasmuch as $fP \simeq P$, we get that $Q = E(fP) \simeq E(P) = H \oplus Q$. Finally, $C = Q \oplus K$ for some K, whence $A = H \oplus C = H \oplus Q \oplus K \simeq Q \oplus K = C \simeq B$. Therefore $A \simeq B$.

<u>Lemma 3.2</u> Let R be a regular, right self-injective ring, and let A and B be nonsingular injective right R-modules. Let $X \subseteq B(R)$, and set $g = \bigvee X$.

(a) If $Au \lesssim B$ for all $u \in X$, then $Ag \lesssim Bg$.

(b) If $Au \simeq B$ for all $u \in X$, then $Ag \simeq Bg$.

Proof If X is empty, then $g = 0$, and so the results hold trivially. Thus, we can assume that X is nonempty.

(a) Set $Y = \{f \in B(R) \mid f \leq u \text{ for some } u \in x\}$, and note that $Af \lesssim Bf$ for all $f \in Y$. Now, consider a maximal orthogonal family $\{u_i\} \subseteq Y$, and let $\bigvee u_i = w \leq g$. Assume $w \neq g$. Then there exists $s \in Y$ such that $s(g - w) \neq 0$, and so $s(g - w)$ is a nonzero element of Y. This contradicts the maximality of the family $\{u_i\}$. Hence $\bigvee u_i = w = g$. We claim that $Ag = E(\oplus Au_i)$ and $Bg = E(\oplus Bu_i)$.

According to proposition 1.4, $(gR)_R = E((\oplus u_iR)_R)$, whence $gR/(\oplus u_iR)$ is a singular right R-module (6, proposition 1.21). Now, since the mapping $f_a : gR/(\oplus u_i R) \longrightarrow agR + (\oplus Au_i)/(\oplus Au_i)$ given by $f_a(\overline{gr}) = \overline{agr}$ is an epimorphism for any $a \in A$, we see that $\sum_{a \in A} agR + (\oplus Au_i)/(\oplus Au_i) = Ag/(\oplus Au_i)$ is an epimorphic image of a direct sum of copies of $gR/(\oplus u_i R)$. Hence $Ag/(\oplus Au_i)$ is singular as well. Consequently, $\oplus Au_i \leq_e Ag$ by (6, proposition 1.21). Thus $Ag = E(\oplus Au_i)$. Similarly $Bg = E(\oplus Bu_i)$. Finally, since $Au_i \lesssim Bu_i$ for all i, we have $\oplus Au_i \lesssim \oplus Bu_i$, whence $E(\oplus Au_i) \lesssim E(\oplus Bu_i)$. Thus $Ag \lesssim Bg$.

(b) This follows automatically from (a), in view of 3.1.

<u>Theorem 3.3</u> Let R be a regular, right self-injective ring R. Then the following conditions are equivalent:

(a) R is purely infinite.

(b) $nR_R \lesssim R_R$ for some $n \geq 2$.

(c) $nR_R \simeq R_R$ for all positive integers n.

(d) $E(\aleph_0 R_R) \simeq R_R$.

Proof (d)$\Rightarrow$(c): Let n be a positive integer. Then $R_R \simeq E(\aleph_0 R_R) \simeq E((n-1)R_R \oplus \aleph_0 R_R) = (n-1)R_R \oplus E(\aleph_0 R_R) \simeq nR_R$. Thus $R \simeq nR_R$.

(c)$\Rightarrow$(b) is clear.

(b)$\Rightarrow$(d): Since $2R_R \lesssim nR_R$, we see that $2R_R \lesssim R_R$. As a result, we can construct right ideals $A_1, A_2, \ldots$ and $B_0 = R, B_1, B_2, \ldots$ of R such that for all k, $A_k \oplus B_k \leq B_{k-1}$ and $A_k \simeq B_k \simeq R_R$. Since the A_k are independent, we have that $\aleph_0 R_R \simeq \oplus A_k \leq R_R$, whence $E(\aleph_0 R_R) \lesssim R$. According to proposition 3.1, $E(\aleph_0 R_R) \simeq R_R$.

(b)$\Rightarrow$(a): Let $u \in B(R)$ be directly finite. By assumption, $2R_R \lesssim R_R$, and so $2(uR)_R \lesssim (uR)_R$. According to proposition 3.1, $2(uR)_R \simeq (uR)_R$. Since u is directly finite, we must have $u = 0$. Thus, R is purely infinite.

(a) $\Rightarrow$ (b): By proposition 1.4, there exists $g \in B(R)$ such that $1 - g$ is the supremum of all $u \in B(R)$ for which $2(uR) \lesssim uR$. Then, lemma 3.2 implies $2(1-g)R \lesssim (1-g)R$. We claim that $g = 0$.

Assume $g \neq 0$. By assumption, g is directly infinite, and so gR contains a nonzero idempotent f such that $2(fR) \simeq fR$ (7, proposition 5.7). Now, choose a maximal independent family $\{J_i\}$ among those right ideals of gR which are isomorphic to fR. Then $gR = E(\oplus J_i) \oplus kR$ for some $k^2 = k$ such that $fR \not\lesssim kR$, and note that $\oplus J_i \neq 0$. According to theorem 1.9, there exists $h \in B(R)$ such that $hkR \lesssim hfR$ and $(1-h)fR \lesssim (1-h)kR$. Then, since $hgR = E(\oplus hJ_i) \oplus hkR$, we have $hgR \lesssim E(\oplus hJ_i) \oplus hfR \simeq E(\bigoplus_{i \neq s} hJ_i) \oplus hJ_s \oplus hfR \simeq E(\bigoplus_{i\neq s} hJ_i) \oplus 2(hfR) \simeq E(\bigoplus_{i\neq s} hJ_i) \oplus hfR \simeq E(\bigoplus_{i\neq s} hJ_i) \oplus hJ_s \simeq E(\oplus hJ_i)$, that is, $hgR \lesssim E(\oplus hJ_i)$. This implies that $2(hgR) \lesssim 2E(\oplus hJ_i) \simeq E(\oplus 2(hJ_i)) \simeq E(\oplus hJ_i) \leq hgR$. Hence $(1-g)hg = hg$, whence $hg = 0$. Hence $fR = gfR = (1-h)gfR = (1-h)fR \lesssim (1-h)kR \leq kR$, which is false. Therefore $g = 0$. Thus $2R_R \lesssim R_R$.

<u>Theorem 3.4</u> Any regular, right self-injective ring R is uniquely a direct product of a directly finite ring and a purely infinite ring.

Proof Let $X = \{u \in B(R) \mid u \text{ is directly finite}\}$, and set $f = \bigvee X$. According to proposition 1.5, the kernel of the natural ring map $h: R \to \prod_{u \in X} uR$ is $(1-f)R$. Since $fR \simeq R/(1-f)R = R/\ker h \simeq hR$, we see that fR is isomorphic to a subring of the directly finite ring $\prod_{u\in X} uR$, and so is directly finite as well. Also, $(1-f)R \cap X = 0$, whence $(1-f)R$ contains no nonzero directly finite central idempotents. Therefore, $(1-f)R$ is purely infinite and fR is directly finite. It remains to show uniqueness.

Assume, $g \in B(R)$ such that gR is directly finite and $(1 - g)R$ is purely infinite. Then $f(1 - g)$ is a directly finite central idempotent in the purely infinite ring $(1 - g)R$, whence $f(1 - g) = 0$. Also, $(1 - f)g$ is a directly finite central idempotent in the purely infinite ring $(1 - f)R$, whence $(1 - f)g = 0$. Hence $f = fg = g$. Therefore the decomposition is unique.

In a similar manner one can show that any regular, right self-injective ring R is uniquely a direct product of an abelian ring and a ring without nonzero central abelian idempotents (7, proposition 13.14). It is interesting that the corresponding result also holds for central quasi-abelian idempotents, as we show in chapter VI.

Definition 3.2 A regular, right self-injective ring R is Type I_F if R is Type I and directly finite; Type I_∞ if R is Type I and purely infinite; Type II_F if R is Type II and directly finite; Type II_∞ if R is Type II and purely infinite.

Theorem 3.5 Any regular, right self-injective ring is uniquely a direct product of rings of Types I_F, I_∞, II_F, II_∞, III.

Proof Theorems 2.7, 2.11, 2.12 and 3.4.

Chapter IV

Prime ideals in regular self-injective rings

In this chapter we study prime ideals in a regular ring with comparability. In particular, we show that a two-sided ideal P is prime if and only if $P \cap B(R)$ is a maximal ideal in $B(R)$.

Notation: Let R be a ring. We write $L_2(R)$ to denote the lattice of two-sided ideals of R (partially ordered by inclusion).

Definition 4.1 A regular ring R is said to satisfy the comparability axiom provided that, for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$.

Note that a regular ring satisfies the comparability axiom if and only if, for any idempotents $e, f \in R$, there exist elements $s \in eRf$ and $t \in fRe$ such that either $st = e$ or $ts = f$.

Proposition 4.1 Let R be a regular ring which satisfies the comparability axiom then the following hold:

(a) $L_2(R)$ is linearly ordered.

(b) All proper two-sided ideals of R are prime.

Proof (a) Let I, J be two-sided ideals of R and let $j \in J$. Assume there exists an element x in I - J. Since R is regular, there exist idempotents e and f in R such that $xR = eR$ and $jR = fR$. It follows from comparability that $eR \lesssim fR$ or $fR \lesssim eR$. By lemma 1.10, $eR \not\lesssim fR$, hence $fR \lesssim eR$. Another application of lemma 1.10 shows that $f \in I$, and so $j \in I$. Thus $J \subseteq I$. Therefore $L_2(R)$ is linearly ordered.

(b) Let P be a proper two-sided ideal of R, and consider any two-sided ideals I and K of R which properly contain P. According to (a), either $J \leq K$ or $K \leq J$, say $J \leq K$. It follows that $P < J = J^2 \leq JK$. Thus P is prime.

<u>Theorem 4.2</u> Let R be a regular ring satisfying general comparability, and let P be a proper two-sided ideal of R. Then the following conditions are equivalent:

(a) P is a prime ideal of R.

(b) $P \cap B(R)$ is a maximal ideal of $B(R)$.

(c) For all $u \in B(R)$, either $u \in P$ or $1 - u \in P$.

(d) $L_2(R/P)$ is linearly ordered.

Proof (a)$\Rightarrow$(b): Let $u \in B(R)$. Then $uR(1 - u)R = 0 \subseteq P$. Since P is prime, either $uR \subseteq P$ or $(1 - u)R \subseteq P$, i.e., $u \in P$ or $1 - u \in P$. By (11, p. 32, proposition 2), $P \cap B(R)$ is a maximal ideal of $B(R)$.

(b)$\Rightarrow$(c) by (11, p. 32, proposition 2).

(c)$\Rightarrow$(d): First we claim that R/P satisfies comparability. Let $\bar{e}$ and $\bar{f} \in R/P$ with e and f two idempotents. Since R satisfies general comparability, there exists a $u \in B(R)$ such that $ueR \lesssim ufR$ and $(1 - u)fR \lesssim (1 - u)eR$. It follows that $\overline{ueR}/P \lesssim \overline{ufR}/P$ and $(\overline{1 - u})\bar{f}R/P \lesssim (\overline{1 - u})\bar{e}R/P$. By assumption, either $\bar{u} = 1$ or $\overline{1 - u} = 1$, whence $\bar{e}R/P \lesssim \bar{f}R/P$ or $\bar{f}R/P \lesssim \bar{e}R/P$. Thus, R/P satisfies the comparability axiom. Then, proposition 4.1 implies that R/P is linearly ordered.

(d)$\Rightarrow$(a) as in 4.1 (b).

Corollary 4.3 Let R be a regular ring satisfying general comparability, and let $P \leq Q$ be proper two-sided ideals of R. If P is prime, then Q is prime.

Proof This follows from the equivalence of (a) and (b) in theorem 4.2.

Corollary 4.4 Let R be a regular ring satisfying general comparability. If M is a maximal ideal of B(R), then MR is a minimal prime ideal of R.

Proof First, assume MR is not a proper ideal. Then $e_1x_1 + e_2x_2 + \ldots + e_nx_n = 1$ for some $e_i \in M$ and some $x_i \in R$. Set $e = e_1 \vee e_2 \vee \ldots \vee e_n$, and note that $e \in M$. Since $ee_i = e_i$ for all i, $e = e1 = e(e_1x_1 + e_2x_2 + \ldots + e_nx_n) = e_1x_1 + e_2x_2 + \ldots + e_nx_n = 1$, whence $e = 1 \in M$, which is impossible. Thus MR is proper. It remains to show that MR is a minimal prime ideal.

Given any $f \in B(R)$, either $f \in M$ or $1 - f \in M$, and so either $f \in MR$ or $1 - f \in MR$. According to theorem 4.2, MR is a prime ideal of R. Consider any prime ideal $P \leq MR$. If $e \in M$, then $1 - e \notin MR$, whence $1 - e \notin P$, whence $e \in P$. Then $M \subseteq P$, and so $P = MR$. Therefore MR is a minimal prime ideal.

Corollary 4.5 Let R be a regular ring satisfying general comparability, and let P be a prime ideal of R. Then P is contained in a unique maximal two-sided ideal of R, and P contains a unique minimal prime ideal of R.

Proof By theorem 4.2, $L_2(R/P)$ is linearly ordered, whence R/P has a unique maximal two-sided ideal M/P. This implies that M is the unique maximal two-sided ideal of R which contains P. It remains to show that P contains a unique minimal prime ideal.

Inasmuch as $P \cap B(R)$ is a maximal ideal of B(R), corollary 4.4 implies that $Q = (P \cap B(R))R$ is a minimal prime ideal of R contained in P. If K is another minimal prime ideal contained in P, then $K \cap B(R) \subseteq P \cap B(R)$. Then, since $K \cap B(R)$ and $P \cap B(R)$ are maximal in B(R), it follows that $K \cap B(R) = P \cap B(R)$. As a result, $Q = (P \cap B(R))R = (K \cap B(R))R \leq K$, whence K = Q by minimality of K. Therefore Q is unique.

<u>Definition 4.2</u> A ring is said to be biregular if for each $x \in R$, there exists a $u \in B(R)$ such that the two-sided ideal $RxR = uR$.

<u>Corollary 4.6</u> Let R be a regular ring satisfying general comparability. Then R is biregular if and only if all prime ideals of R are maximal.

Proof Assume R is biregular, and let P be a prime ideal of R. Clearly R/P is biregular, and so every two-sided ideal of R/P contains a central idempotent. Now, since R/P is also prime, 1 is the only nonzero central idempotent of R/P. Hence every nonzero two-sided ideal contains 1, whence R/P is simple. Thus P is maximal.

Conversely, if R is not biregular, then there is some $x \in R$ such that RxR is not generated by a central idempotent. Set $J = \{y \in R \mid yRx = 0\}$, and note that J is a two-sided ideal. Since $J(RxR) = 0$, we have that $J \cap RxR = 0$ (by the semiprimeness of R). As a result, since RxR is not generated by a central idempotent, we must have $RxR \oplus J \neq R$. Hence,

$R x R \oplus J$ is contained in some maximal two-sided ideal M of R. According to theorem 4.2 and corollary 4.4, $P = (M \cap B(R))R$ is a prime ideal of R. We claim that $P \neq M$.

If $P = M$, then $x \in RxR \subseteq M = P$, and so $x = e_1x_1 + e_2x_2 + \ldots + e_nx_n$ for some $e_i \in M \cap B(R)$ and $x_i \in R$. Setting $e = e_1 \vee e_2 \vee \ldots \vee e_n$, we see that $e \in M \cap B(R)$ and $(1 - e)Rx = 0$. But then $1 - e \in J \leq M$, which is impossible. Therefore $P \neq M$. Thus, P is a nonmaximal prime ideal of R.

Proposition 4.7 Let R be a regular, right self-injective ring. Then, any prime ideal of R is either essential or closed.

Proof Assume P is a prime ideal of R. According to proposition 1.2, there exists a $u \in B(R)$ such that $P \leq_e uR$. Since P is a prime ideal, we have that $u \in P$ or $1 - u \in P$, whence $P = uR$ or $P \leq_e R$. Therefore, P is either closed or essential.

Notation: Let R be a ring. We write Spec(R) for the set of all prime ideals of R.

Recall that Spec(R) becomes a topological space (Zarisky topology), if as open sets we take all sets of the form $\{P \in \text{Spec}(R) \mid X \nsubseteq P\}$ for any $X \subseteq R$.

Definition 4.3 The Boolean spectrum of a ring R, denoted BS(R), is Spec(B(R)).

In chapter V (proposition 5.6), we will show that BS(R) is a compact, Hausdorff, totally disconnected space.

Definition 4.4 The maximal spectrum of a ring R, denoted MaxSpec(R), is the subspace of Spec(R) consisting of all maximal two-sided ideals of R.

Theorem 4.8 If R is a regular ring satisfying general comparability, then the rule $M \to M \cap B(R)$ defines a homeomorphism of MaxSpec(R) onto BS(R).

Proof For all $M \in$ MaxSpec(R), set $f(M) = M \cap B(R) \in$ BS(R). If M, $N \in$ MaxSpec(R) such that $f(M) = f(N)$, then M and N both contain $(f(M))R$. According to corollary 4.4, $(f(M))R$ is a prime ideal of R, whence $N = M$ by corollary 4.5. Thus f is injective.

Let $M \in$ BS(R). By corollary 4.4, MR is a proper ideal of R, whence MR is contained in a maximal two-sided ideal N. Then $M \subseteq N \cap B(R) = f(N)$ and so $M = f(N)$, by maximality of M in B(R). Therefore f is a bijection.

If X is any closed subset of BS(R), then $X = \{M \in B(R) \mid Y \subseteq M\}$ for some $Y \subseteq B(R)$. It is not hard to see that $f^{-1}(X) = \{N \in \text{MaxSpec}(R) \mid Y \subseteq N\}$, which is closed in MaxSpec(R). Thus, f is continuous. Since f is a continuous bijection of a compact space onto a Hausdorff space, we have f is a homeomorphism.

Chapter V

Quasi-abelian rings

In this chapter and the next, we study two classes of rings which have a lot of importance in the structure theory of regular self-injective rings, namely quasi-abelian regular rings, and rings in which every two-sided ideal contains a nonzero central idempotent.

Proposition 5.1 For a regular ring R, the following conditions are equivalent:

(a) R is quasi-abelian.

(b) Every principal right (left) ideal is isomorphic to a two-sided ideal (as R-modules).

(c) $\{ xy^2x \mid xy \in B(R)\}$ is the set of all idempotents in R.

Proof (a)$\Rightarrow$(b): Let I be a principal right ideal. Since R is regular, we have $I = eR$ for some idempotent e. By assumption, there exists a $u \in B(R)$ such that $I = eR \simeq uR$. Therefore I is isomorphic to a two-sided ideal.

(b)$\Rightarrow$(c): Let e be an idempotent. Then eR is isomorphic to a two-sided ideal aR. Since R is regular, $aR = fR$ for some idempotent f. By lemma 1.1, we have $f \in B(R)$. Now since $e \sim f$, there exist $x, y \in R$ such that $e = yx$ and $xy = f$, whence $e = yx = y(xy)x = (xy)(yx) = xy^2x$. Thus $e \in \{ xy^2x \mid xy \in B(R)\}$. For the reverse inclusion, if $xy \in B(R)$, then $(xy^2x)(xy^2x) = (xy)(yx)(xy)(yx) = (xy)(yx)(yx) = (xy)y(xy)x =$

$xyyx = xy^2x$. Thus, $\{xy^2x \mid xy \in B(R)\}$ is the set of all idempotents in R.

(c)$\Rightarrow$(a): Suppose e is an idempotent. Then there exist $x, y \in R$ such that $e = xy^2x$ and $xy \in B(R)$. It follows that $e = (xy)(yx) = y(xy)x = y(xyx)$ and $xy = (xy)(xy) = (xyx)y$; whence $e \sim xy \in B(R)$. Thus, R is quasi-abelian.

<u>Proposition 5.2</u> Let R be a quasi-abelian regular ring. Then R satisfies general comparability.

Proof Let e and f be idempotents in R. Then $e \sim u$ and $f \sim v$ for some $u, v \in B(R)$. Since $ev \sim uv \sim fu = fuv$ and $f(1 - v) = 0$, it follows that

(a) $evR \lesssim fvR$,

(b) $f(1 - v)R \lesssim e(1 - v)R$.

Thus, R satisfies general comparability.

It is obvious that the class of all quasi-abelian regular rings is closed under direct products. In the next proposition, we show that it is also closed under factor rings.

<u>Proposition 5.3</u> Let R be a quasi-abelian regular ring and let I be a two-sided ideal of R. Then R/I is a quasi-abelian regular ring.

Proof Clearly R/I is a regular ring. Let $\bar{a}$ be an idempotent in R/I. Since every idempotent in R/I comes from an idempotent in R, there exists an idempotent $e \in R$ such that $\bar{e} = \bar{a}$. By assumption, $e \sim u$ for some $u \in B(R)$. This implies that there exist $x, y \in R$ such that $xy = e$

and $yx = u$. It follows that $\overline{xy} = \overline{e} = \overline{a}$ and $\overline{yx} = \overline{u} \in B(R/I)$, whence $\overline{a} \sim \overline{u}$. Therefore R/I is a quasi-abelian regular ring.

Proposition 5.4 If R is a quasi-abelian regular ring, then R is biregular.

Proof Let x be an element in R, and let $y \in R$ such that $x = xyx$. Since xy is an idempotent, there exists a $u \in B(R)$ such that $xy \sim u$. According to lemma 1.10, this implies that $RxyR = uR$. Since $RxyR \subseteq RxR = RxyxR \subseteq RxyR$, we have that $RxR = uR$. Therefore R is biregular.

Corollary 5.5 If R is a quasi-abelian regular ring, then every prime ideal of R is maximal.

Proof According to proposition 5.2 and proposition 5.4, R is a biregular ring which satisfies general comparability. Consequently, every prime ideal of R is maximal by corollary 4.6.

Proposition 5.6 If R is a quasi-abelian regular ring, then Spec(R) is a compact, Hausdorff, totally disconnected space.

Proof Since R is a ring with unit, Spec(R) is compact. Now, consider any distinct $P, Q \in \mathrm{Spec}(R)$. According to corollary 5.5, P and Q are maximal, and so there exists an element x such that $x \in P - Q$. This implies that $P - Q$ contains an idempotent (R is regular). Let $e = e^2 \in P - Q$. By assumption, $e \sim u$ for some $u \in B(R)$, whence $P - Q$ also contains the idempotent u (by lemma 1.10). Since u is central, it is not hard to see that $V = \{M \in Spec(R) \mid u \notin M\}$ and $W = \{M \in Spec(R) \mid 1 - u \notin M\}$ are disjoint open subsets of Spec(R) such that $Q \in V$ and $P \in W$. Thus, Spec(R) is Hausdorff. It remains to show that Spec(R) is totally disconnected.

We will show that Spec(R) has a basis of clopen sets. Let $V = \{M \in Spec(R) \mid X \not\subseteq M\}$ be an open set, and let $Q \in V$. Then there exists an $x \in X - Q$, whence $W = \{M \in Spec(R) \mid x \notin M\}$ is an open subset of Spec(R) such that $Q \in W \subseteq V$. Since R is regular there exists an idempotent e such that $eR = xR$, whence $W = \{M \in Spec(R) \mid e \notin M\}$. Moreover, $e \sim u$ for some $u \in B(R)$, and so lemma 1.10 implies that $W = \{M \in Spec(R) \mid u \notin M\}$. Since $u \in B(R)$, we also have that $W = \{M \in Spec(R) \mid 1 - u \in M\}$, which is closed. Thus, W is a clopen subset of Spec(R) such that $Q \in W \subseteq V$. Therefore Spec(R) has a basis of clopen sets and so is totally disconnected.

<u>Proposition 5.7</u> Let R be a regular right self-injective ring, and let $\{e_i\}$ be a maximal infinite set of orthogonal equivalent idempotents which contains e_1. Then there exists a nonzero $u \in B(R)$ such that $uR = E(\bigoplus h_i R)$, where $h_1 = e_1 u$ and $h_i \sim h_1$ for all $i \in I$.

Proof Let $eR = E(\bigoplus e_i R)$. By general comparability, there exists a $u \in B(R)$ such that

(a) $(1 - e)uR \lesssim e_1 uR$,

(b) $e_1(1 - u)R \lesssim (1 - e)(1 - u)R$.

Note that by the maximality of the family $\{e_i\}$ and the relation (b), we have that $u \neq 0$. Since $(1 - e)uR \lesssim e_1 uR$, there exist idempotents

f_1 and g_1 such that $e_1uR = f_1R \oplus g_1R$ and $f_1 \sim (1 - e)u$. Consequently, since $e_iu \sim e_1u$, there exist idempotents f_i and g_i such that $e_iuR = f_iR \oplus g_iR$, $f_i \sim (1 - e)u$ and $g_i \sim g_1$ for each $i \neq 1$. Now, consider the direct sum $S' = (1 - e)uR \oplus (\oplus f_iR) \oplus (\oplus g_iR)$. Since the cardinality of the set $I \cup \{1\}$ is equal to the cardinality of I, we see that $S' = \oplus h_iR$ with $h_1 = e_1u$ and $h_i \sim e_1u$ for all $i \in I$. It remains to show that $uR = E(S')$.

Since $euR = E(\oplus e_iuR)$, it follows that $uR = (1 - e)uR \oplus euR = E((1 - e)uR) \oplus E(\oplus e_iuR) = E((1 - e)uR \oplus (\oplus e_iuR)) = E(S')$. Therefore $uR = E(\oplus h_iR)$ with $h_1 = e_1u$ and $h_i \sim h_1$ for all $i \in I$.

Lemma 5.8 Let R be a regular ring, let e be an idempotent in R, and let $x, y \in R$. Then $xeyR \lesssim eR$.

Proof Since R is regular, there exists $s \in R$ such that $xey = xeysxey$. Then, $xeyR = xeysR = (xeysxey)sR = (xeysx)(eys)R \simeq (eys)(xeysx)R \leq eR$. Thus $xeyR \lesssim eR$.

Proposition 5.9 For a regular, right self-injective ring R, the following conditions are equivalent:

(a) R is quasi-abelian.

(b) For every noncentral idempotent e of R, there exists an infinite set $\{f_i\}$ of pairwise equivalent orthogonal idempotents such that $c(e) - e \in \{f_i\}$.

(c) For every noncentral idempotent e of R, there exists a nonzero $u \in B(R)$ such that $eu \sim u$.

(d) $\{e \in R \mid e^2 = e$ and $xey = c(e)$ for some $x, y \in R\}$ is the set of all idempotents of R.

Proof (a)$\Rightarrow$(b): Let e be a noncentral idempotent. By assumption, there exist $x, y \in R$ such that $yx = e$ and $xy = c(e)$. Set $e_{ij} = (y^{i-1}x^{j-1} - y^i x^j)xy$ for all $i, j = 1, 2, 3, \ldots$. Whenever $j \leq k$, we have that $e_{ij}y^k = (y^{i-1}x^{j-1}y^k - y^i x^j y^k)xy = (y^{i-1}y^{k-j+1} - y^i y^{k-j})xy = 0$. Hence, $e_{ij}e_{kn} = (e_{ij}y^{k-1}x^{n-1} - e_{ij}y^k x^n)xy = 0$ whenever $j \leq k - 1$, and $e_{ij}e_{jn} = (e_{ij}y^{j-1}x^{n-1} - e_{ij}y^j x^n)xy = ((y^{i-1}x^{j-1} - y^i x^j)y^{j-1}x^{n-1})xy = (y^{i-1}x^{n-1} - y^i x x^{n-1})xy = e_{in}$. Since $x^k e_{kn} = (x^k y^{k-1}x^{n-1} - x^k y^k x^n)xy = (xx^{n-1} - x^n)xy = 0$, we also have $x^j e_{kn} = 0$ for all $j \geq k$, whence $e_{ij}e_{kn} = y^{i-1}x^{j-1}e_{kn} - y^i x^j e_{kn} = 0$ whenever $j \geq k + 1$.

Now, since $e_{1j}e_{ii}e_{i1} = e_{11} = xy - yx \neq 0$, we see that all $e_{ii} \neq 0$. Also, from the above, the nonzero e_{ii} are pairwise orthogonal and are pairwise equivalent. Thus, $e_{11} = (1 - yx)xy = xy - yx = c(e) - e$ is part of an infinite set of pairwise equivalent orthogonal idempotents.

(b)$\Rightarrow$(c): Let e be a noncentral idempotent, and let $\{f_i\}$ be an infinite set of pairwise equivalent orthogonal idempotents such that $c(e) - e \in \{f_i\}$. According to proposition 5.7, there exists a $u \in B(R)$ such that $uR = E(\oplus h_i R)$ with $h_1 = u(c(e) - e)$ and $h_i \sim h_1$ for all i. Then, $uc(e)R = E(\oplus h_i c(e)R)$, and so $(1 - h_1)c(e)uR = E(\bigoplus_{i \neq 1} h_i c(e)R) \simeq uc(e)R$. Hence, $euc(e) = (1 - c(e)u + eu)c(e)u = (1 - h_1)c(e)u \sim uc(e)$. Thus, $euc(e) \sim uc(e)$ with $uc(e) \neq 0$.

(c)$\Rightarrow$(a): Suppose e is a noncentral idempotent. Set $v = \bigvee\{u \in B(R) \mid eu \sim u\}$. According to lemma 3.2, $ev \sim v$. We will show that $ev = e$. Assume $ev \neq e$. Then $e(1 - v) \neq 0$, and so $e(1 - v)h \sim h$ for some nonzero $h \in B(R)$. This implies that $e(1 - v)h \sim (1 - v)h$ and $(1 - v)h = h \neq 0$. Since $v = \bigvee\{u \in B(R) \mid eu \sim u\}$, we see that

$(1 - v)h = (1 - v)hv = 0$, which is a contradiction. Therefore $e = ev \sim v$. Thus, R is a quasi-abelian ring.

(a)$\Rightarrow$(d): Let f be an idempotent. Since $f \sim c(f)$, there exist $x, y \in R$ such that $c(f) = xfy$ and $f = yc(f)x$. Thus, $f \in \{e \in R \mid e^2 = e$ and $xey = c(e)$ for some $x, y \in R\}$.

(d)$\Rightarrow$(a): Let e be an idempotent. Then $xey = c(e)$ for some $x, y \in R$. According to lemma 5.8, $c(e)R \lesssim eR$. Consequently, $e \sim c(e)$ by proposition 3.1.

In the following chapter, we will give other characterizations of quasi-abelian, regular right self-injective rings.

We end this chapter with a generalization of (13, corollaire 1.2).

<u>Proposition 5.10</u> Let R be a regular right self-injective ring, and let e and f be two faithful quasi-abelian idempotents in R. Then $e \sim f$.

Proof Set $u = \bigvee\{v \in B(R) \mid ev \sim fv\}$, and note that $eu \sim fu$ by lemma 3.2. We claim that $u = 1$. Assume $u \neq 1$. Since $Re(1 - u)R \leq_e c(e)(1 - u)R = 1(1 - u)R = (1 - u)R$, we see that $c(e(1 - u)) = 1 - u \neq 0$, and so $c(e(1 - u))c(f) \neq 0$. Hence, proposition 1.7 part (c) implies that $e(1 - u)Rf \neq 0$. It follows by lemma 1.8, that there exist a nonzero idempotent $g \in e(1 - u)Re(1 - u)$ and a nonzero idempotent $h \in fRf$ such that $g \sim h$. According to proposition 2.1, $g \sim e(1 - u)c(g)$ and $h \sim fc(h)$, whence $e(1 - u)c(g) \sim fc(h)$. Since $c(g) = c(h)$ and $c(g)(1 - u) = c(g)$, we get that $e(1 - u)c(g) \sim f(1 - u)c(g)$. This implies $e(u + (1 - u)c(g)) \sim f(u + (1 - u)c(g))$ with $u \lneqq u + (1 - u)c(g)$, which is a contradiction. Therefore $u = 1$. Thus $e \sim f$.

Chapter VI

Quasi-biregular rings

The purpose of this chapter is to study regular right self-injective rings in which every two-sided ideal contains a nonzero central idempotent. By way of example, theorem 6.9 shows if R is a regular right self-injective ring, then every two-sided ideal of R contains a nonzero central idempotent if and only if there is a ring decomposition $R = R_1 \times R_2$ such that R_1 is directly finite and R_2 is quasi-abelian.

Definition 6.1 A ring is quasi-biregular if every nonzero two-sided ideal of R contains a nonzero central idempotent. An idempotent e is quasi-biregular in R if eRe is quasi-biregular.

Obviously, any biregular ring is quasi-biregular. For another example, let R be a regular ring whose primitive factor rings are artinian. According to (7, theorem 6.6), R is quasi-biregular. ∎

Proposition 6.1 Let R be a quasi-biregular ring. Then R is right and left nonsingular.

Proof Suppose that $xJ = 0$ for some nonzero $x \in R$ and some large right ideal J. Since $RxR \cap B(R) \neq 0$, there exists a nonzero central idempotent u such that $u = r_1xs_1 + r_2xs_2 + \ldots + r_nxs_n$ for some $r_i, s_i \in R$. Set $I = s_1^{-1}J \cap s_2^{-1}J \cap \ldots \cap s_n^{-1}J$. Then $uI = 0$, whence $I \cap uR = 0$.

Since I is large in R, we get $uR = 0$, which is a contradiction. Thus, R_R is nonsingular.

It follows from this proposition that a quasi-biregular right self-injective ring is regular.

<u>Proposition 6.2</u> If R is a regular, right self-injective ring, then the following are equivalent:

(a) R is quasi-biregular.

(b) For every nonzero nilpotent element x of R, $RxR \cap B(R) \neq 0$.

(c) For every noncentral idempotent e of R, $ReR \cap B(R) \neq 0$.

(d) For every noncentral idempotent e of R, $\bigvee(ReR \cap B(R)) = c(e)$.

(e) The intersection of the maximal two-sided ideals of R is zero.

Proof (a)$\Rightarrow$(b) a priori.

(b)$\Rightarrow$(c): Let e be a noncentral idempotent. According to lemma 1.1, there exists an element b in R such that $eb(1 - e) \neq 0$. Since $eb(1 - e)$ is nilpotent, $Reb(1 - e)R \cap B(R) \neq 0$. Thus $ReR \cap B(R) \neq 0$.

(c)$\Rightarrow$(d): Let e be a noncentral idempotent and let $\{u_i\}$ be a maximal orthogonal family of central idempotents in the two-sided ideal ReR. Set $v = \bigvee u_i$. Since $ReR \leq_e c(e)R$, we see that $v \leq c(e)$. Assume $v \neq c(e)$. Then $(c(e) - v)R$ is a nonzero submodule of $c(e)R$, whence $ReR \cap (c(e) - v)R \neq 0$. Let f be a nonzero idempotent in $ReR \cap (c(e) - v)R$. By assumption, $RfR \cap B(R) \neq 0$, whence there exists a nonzero central idempotent u in $ReR \cap (c(e) - v)R$. This is a contradiction on the maximality of the set $\{u_i\}$. Hence $v = c(e)$. Therefore $\bigvee(ReR \cap B(R)) = c(e)$.

(d)$\Rightarrow$(e): Suppose x is a nonzero element contained in the intersection of the maximal two-sided ideals of R. Because R is regular, there exists an idempotent e in R such that ReR = RxR. By assumption, $\bigvee(ReR \cap B(R)) = c(e) \neq 0$, whence there exists a nonzero central idempotent u in ReR = RxR. It follows that 1 - u is not contained in any maximal two-sided ideal, which is a contradiction. Thus the intersection of the maximal two-sided ideals is zero.

(e)$\Rightarrow$(a): Let I be a nonzero two-sided ideal of R. Since the intersection of the maximal two-sided ideals is zero, there exists a maximal two-sided ideal M which does not contain I. According to proposition 1.11, there exists a central idempotent u in I - M. Thus R is quasi-biregular.

<u>Definition 6.2</u> Let I be a two-sided ideal in a ring R. We call I quasi-biregular if $RxR \cap B(R) \neq 0$ for all $x \in I$.

<u>Proposition 6.3</u> Let R be a regular, right self-injective ring. Then there is a unique ring decomposition $R = S \times T$ such that S is quasi-biregular and T has no nonzero quasi-biregular central idempotents.

Proof Let J be the sum of all quasi-biregular ideals of R. We show that r(l(J)) is quasi-biregular, from which it will follow that J = r(l(J)). Let x be a nonzero element in r(l(J)). Then xR = eR for some nonzero idempotent e in R. Assume $eR \cap I = 0$ for all quasi-biregular ideals I, then $eRI \leq eR \cap I = 0$ implies that $e \in l(J)$, and hence $e = ee \in l(J)r(l(J)) = 0$, which is a contradiction. Therefore $eR \cap I \neq 0$ for some quasi-biregular ideal I. Since $eR \subseteq ReR$, we have that $ReR \cap I \neq 0$, whence ReR = RxR contains a nonzero central idempotent. Hence, r(l(J)) is quasi-biregular, and so $r(l(J)) \leq J$, whence

$r(l(J)) = J$. According to proposition 1.7, $J = uR$ for some $u \in B(R)$. Therefore $R = uR \oplus (1 - u)R$, with uR a quasi-biregular ring and $(1 - u)R$ a ring without nonzero quasi-biregular central idempotents.

Now, suppose that we also have $v \in B(R)$ such that vR is a quasi-biregular ring and $(1 - v)R$ a ring without nonzero quasi-biregular central idempotents. Then $u(1 - v)R$ is a quasi-biregular ring with no nonzero central quasi-biregular idempotents, whence $u(1 - v) = 0$. Similarly, $v(1 - u) = 0$; hence $u = v$. Therefore the decomposition is unique.

<u>Proposition 6.4</u> Let R be a regular, right self-injective ring which contains a faithful finite idempotent. Then the following conditions are equivalent:

(a) R is quasi-biregular.

(b) R is finite.

Proof (a)$\Rightarrow$(b): Let e be a faithful directly finite idempotent and let u be a nonzero central idempotent in ReR. Note that by proposition 6.2, $\bigvee(ReR \cap B(R)) = c(e) = 1$. We will show that u is directly finite. Let $u = x_1ey_1 + x_2ey_2 + \ldots + x_ney_n$ for some $x_i, y_i \in R$. By lemma 5.8, $x_iey_iR \lesssim eR$ for all i, whence $x_iey_i \in \{a \in R \mid aR \text{ is a directly finite right R-module}\}$ for all i. According to (7, corollary 9.21), u is directly finite. Now, since $\bigvee(ReR \cap B(R)) = 1$, ReR contains a family of pairwise orthogonal central idempotents $\{u_i\}$ such that $\bigvee u_i = 1$. According to proposition 1.5, $R \simeq \Pi u_iR$. By the above, R is isomorphic to a product of directly finite rings. Therefore R is directly finite.

(b)$\Rightarrow$(a) by theorem 1.12.

Corollary 6.5 If R is a quasi-abelian, regular, right self-injective ring of Type I, then R is abelian.

Proof According to proposition 6.4, R is also finite. Hence, proposition 2.8 shows that R is abelian.

Lemma 6.6 Let R be a regular, right self-injective ring of Type III, and let e, g_1, g_2 be idempotents in R. If $g_1R \lesssim eR$ and $g_2R \lesssim eR$, then $g_1R + g_2R \lesssim eR$.

Proof According to (7, theorem 2.3), $g_1R \cap g_2R = hR$ for some idempotent h. Since hR is an injective R-submodule of g_1R and g_2R, we get that $g_1R = hR \oplus h_1R$ and $g_2R = hR \oplus h_2R$ for some idempotents h_1 and h_2. It follows that $g_1R + g_2R = hR \oplus h_1R \oplus h_2R \lesssim 3eR$. According to (7, corollary 10.17), $g_1R + g_2R \lesssim eR$.

Proposition 6.7 Let R be a regular, right self-injective ring of Type III. Then the following conditions are equivalent:

(a) R is quasi-biregular.

(b) R is quasi-abelian.

(c) R is biregular.

Proof (a)$\Rightarrow$(b): Let e be a nonzero idempotent and let u be a nonzero central idempotent in ReR. Then $u = r_1eus_1 + r_2eus_2 + \ldots + r_neus_n$ for some $r_i, s_i \in R$. According to lemma 5.8 and lemma 6.6, $uR \lesssim euR$. Consequently, proposition 3.1 shows that $uR \simeq euR$. By proposition 5.9, R is quasi-abelian.

(b) $\Rightarrow$ (c) by proposition 5.4.

(c) $\Rightarrow$ (a) is obvious.

<u>Theorem 6.8</u> Let R be a regular, right self-injective ring. Then there is a unique ring decomposition $R = S \times T$ such that S is a quasi-abelian ring and T has no nonzero quasi-abelian central idempotents.

Proof Propositions 2.8, 6.3, 6.4, 6.7 and (7, proposition 13.14).

In the following proposition we obtain another characterization of quasi-biregular, right self-injective rings.

<u>Theorem 6.9</u> Let R be a regular, right self-injective ring. Then R is quasi-biregular if and only if there is a ring decomposition $R = S \times T$ such that S is directly finite and T is quasi-abelian.

Proof First assume that R is quasi-biregular. Since R is a regular right self-injective ring, there exists a $u \in B(R)$ such that uR has a faithful finite idempotent and $(1 - u)R$ is Type III. According to proposition 6.4 and proposition 6.7, uR is finite and $(1 - u)R$ is quasi-abelian.

Conversely, assume there exists a directly finite $u \in B(R)$ such that $1 - u$ is quasi-abelian. Then $R = uR \oplus (1 - u)R$ is quasi-biregular by theorem 1.12 and proposition 5.4.

We are now in a position to extend the results of (7, theorem 3.2) for quasi-abelian regular, right self-injective rings.

<u>Theorem 6.10</u> Let R be a regular, right self-injective ring. Then the following conditions are equivalent:

(a) R is quasi-abelian.

(b) For every nilpotent element x, there exists a $u \in B(R)$ such that $xR \simeq uR$.

(c) Every nonzero right ideal of R contains a nonzero idempotent e such that $e \sim u$ for some $u \in B(R)$.

(d) R/P is a quasi-abelian ring for all prime ideals P of R.

Proof (a)$\Rightarrow$(b) a priori.

(b)$\Rightarrow$(c): Let I be a right ideal of R which does not contain a central idempotent, and let e be an idempotent in I. According to lemma 1.1, we have that $eR(1 - e) \neq 0$, whence I contains a nonzero nilpotent element x. By assumption, $xR \simeq uR$ for some $u \in B(R)$. Since R is regular, we also have $fR = xR$ for some idempotent f. Therefore, $f \in I$ and $f \sim u$.

(c)$\Rightarrow$(a): Let x be a nonzero element of R. Then xR contains a nonzero idempotent e such that $e \sim v$ for some $v \in B(R)$. According to lemma 1.10, we see that $v \in RxR$. Thus R is a quasi-biregular ring. According to proposition 6.9, there exists a $u \in B(R)$ such that uR is a directly finite ring and $(1 - u)R$ is a quasi-abelian ring. It remains to show that uR is an abelian ring. Let I be a nonzero right ideal in the ring uR. Then there exists an idempotent $f \in I$ such that $f \sim u_1$ for some $u_1 \in B(R)$. Since uR is finite, we have that $f = u_1$, whence I contains a central idempotent. According to (7, theorem 3.2), uR is abelian. Therefore $R = uR \oplus (1 - u)R$ is a quasi-abelian ring.

(a)$\Rightarrow$(d) by proposition 5.3.

(d)$\Rightarrow$(a): Let P be a prime ideal in R. Then R/P is a prime quasi-abelian regular ring. This implies that R/P is simple by proposition 5.4, and so P is a maximal ideal. According to corollary 4.6, we get that R is a biregular ring. Now, let $u \in R$ such that uR has a faithful directly finite idempotent and $(1 - u)R$ is a Type III ring. By proposition 6.7, $(1 - u)R$ is a quasi-abelian regular ring. We claim that uR is also a quasi-abelian ring. Note that $K = \{uM \mid M$ is a maximal ideal in R and $u \notin M\}$ is the set of maximal ideals of uR.

According to proposition 6.4, uR is finite. Let $uM \in K$. Then uR/uM is finite by (7, proposition 10.26). Moreover, since uR/uM is an homomorphic image of the quasi-abelian ring R/M, we also have that uR/uM is a quasi-abelian ring. Hence, uR/uM is a division ring for all $uM \in K$. By (7, theorem 3.2), we have that uR is abelian. Therefore $R = uR \oplus (1 - u)R$ is a quasi-abelian ring.

<u>Proposition 6.11</u> The maximal right (left) ring of quotients of a quasi-abelian regular ring is a quasi-abelian ring.

Proof Let R be a quasi-abelian ring, and let Q be its maximal right quotient ring. First, we claim that $B(R) \subseteq B(Q)$. Let $u \in B(R)$, let $b \in Q$, and let $f: R_R \longrightarrow Q_R$ be the R-module homomorphism defined by $f(r) = br$. Since $R_R \leq_e Q_R$, we have $f^{-1}R \leq_e R_R$ by (6, proposition 1.1). It follows that for any $s \in f^{-1}R$, u commutes with bs as well as s, whence $ubs = bsu = bus$. Hence $(ub - bu)f^{-1}R = 0$, and so $ub = bu$ (because $f^{-1}R \leq_e R$). Thus $u \in B(Q)$.

Now, let K be any nonzero right ideal of Q_Q. Since $R_R \leq_e Q_R$, we have $K \cap R \neq 0$, and so $K \cap R$ contains a nonzero idempotent e. Because R is quasi-abelian, there exist $x, y \in R$ and $v \in B(R)$ such that $xy = v$ and

$yx = e$. By the claim above, $v \in B(Q)$, whence $e \sim v$ in Q for some $v \in B(Q)$. Thus K contains a nonzero idempotent e such that $e \sim v$ for some $v \in B(Q)$. According to theorem 6.10, Q is quasi-abelian.

Proposition 6.12 The maximal right (left) ring of quotients of a quasi-biregular ring is a quasi-biregular ring.

Proof Let Q be the maximal right ring of quotients of R. Since R_R is right nonsingular (proposition 6.1), we have that Q is regular, right self-injective ring. Also, note that as in proposition 6.11, $B(R) \subseteq B(Q)$. Now, let I be a two-sided ideal in Q. Since $R_R \leq_e Q_R$, we see that $I \cap R$ is a nonzero two-sided ideal in R. By assumption, $I \cap R$ contains a nonzero $u \in B(R) \subseteq B(Q)$. Hence I contains a nonzero central idempotent in Q. Thus Q is quasi-biregular.

Proposition 6.13 Let R be a regular, right self-injective ring and let e be a quasi-abelian idempotent in R. Then there exists a nonzero $u \in B(R)$ such that $uR = E(\oplus h_j R)$ with $h_j \sim eu$ for all j.

Proof Let $S = \oplus e_i R$ be a maximal direct sum of right ideals isomorphic to eR, with $\{e_i\}$ a set of idempotents which contains e, and let $fR = E(S)$. If $fR = R$, put $u = 1$. Hence, we can assume $fR \neq R$. By general comparability, there exists a $v \in B(R)$ such that

$$(1 - f)vR \lesssim evR,$$

$$e(1 - v)R \lesssim (1 - f)(1 - v)R.$$

In view of the maximality of the direct sum S, we see that $v \neq 0$. Now, if $(1 - f)v = 0$, then $vR = fvR = E(\oplus ve_i R)$, and so $u = v$ does the trick.

It remains to show that the result is true if $(1 - f)v \neq 0$.

Since $(1 - f)vR \lesssim evR$, there exists an idempotent $k \in eR$ such that $k \sim (1 - f)v$. Set $u = c(k) = c((1 - f)v)$ and $T = (1 - f)uR \oplus (\oplus e_i uR)$. Note that $u \neq 0$ and $uv = u$. According to proposition 2.1, $ke \sim eu$, whence $(1 - f)uR = (1 - f)uvR \simeq kuR = kR = keR \simeq euR$. Moreover, since $(1 - f)R \oplus (\oplus e_i R) \leq_e R$, we have that $T \leq_e uR$, whence $uR = E(T)$. Therefore, u is a nonzero central idempotent such that $uR = E(\oplus h_j R)$ with $h_j \sim eu$ for all j.

We conclude this chapter by using proposition 1.5 and proposition 6.13 to decompose rings of Type I and Type III.

<u>Theorem 6.14</u> Let R be a regular, right self-injective ring with a faithful quasi-abelian idempotent. Then R is isomorphic to a direct product of rings R_i, where R_i is the injective hull of a direct sum of isomorphic right ideals generated by quasi-abelian idempotents.

Proof Let $\{u_i\}$ be a maximal set of orthogonal central idempotents which satisfy proposition 6.13, and let $f: R \longrightarrow \Pi u_i R$ be the natural ring map. According to proposition 1.5, f is onto and Ker $f = (1 - \bigvee u_i)R$. We will show that f is injective. Suppose Ker $f \neq 0$. As in theorem 2.4, Ker f contains a nonzero quasi-abelian idempotent e. Hence, proposition 6.13 implies that there exists a nonzero central idempotent u such that $uR = E(\oplus h_j R)$ with $h_j \sim eu$ for all j. According to lemma 1.10, $\oplus h_j R \leq$ Ker f, whence $u \in$ Ker f (again by lemma 1.10). This contradicts the maximality of the set $\{u_i\}$. Therefore f is injective. Therefore f is an isomorphism.

Corollary 6.15 Let R be a regular, right self-injective ring of Type I (III). Then R is isomorphic to a direct product of rings R_i, where R_i is the injective hull of a direct sum of isomorphic right ideals generated by abelian idempotents (quasi-abelian idempotents of Type III).

Proof Corollary 6.5 and theorem 6.14.

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