Relative Injectivity for Groups and Modules

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ABSTRACT

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This thesis is a study of article [1] concerning absolute direct summand (a.d.s.) modules and article [7] concerning quasi-projective abelian groups. The a.d.s. modules are characterized by the property that for every decomposition $M = U \oplus V$, V is U-injective. These modules and quasi-continuous modules over a right noetherian ring of dimension one are characterized.

Quasi-projective modules are defined dually to quasi-injective modules and these are characterized in the case of abelian groups.

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INTRODUCTION

The object of this thesis is the study of absolute direct summand R-modules ([1]), R a ring, and quasi-projective abelian groups ([7]). The former were first defined by L. Fuchs ([5, p.73]) as those modules M with the property that every complement of every direct summand of M is itself a direct summand of M. In [5, p.93, Exercise 11(b)] he has given a complete description of the non-mixed absolute direct summand abelian groups. The absolute direct summand property is more general than that of quasi-continuity ([10]) and the implications injectivity → quasi-injectivity → continuity → quasi-continuity → absolute direct summand, hold. An important property of quasi-continuous modules, that which states that any two complementary direct summands are relatively injective is in fact a characterizing property of the absolute direct summand modules.

It is shown in [1] that when R is right noetherian, an absolute direct summand module has the form $V \oplus U$ where V is injective, U reduced and there are no non-zero partial monomorphisms $V \to U$. If U is also absolute direct summand and the word "monomorphism" is replaced by "homomorphism" these conditions become sufficient. Further, it is shown that a homogeneous, decomposable, absolute direct summand module is quasi-injective. Using the above, a complete description of these modules over a commutative noetherian domain of dimension one is given. The case of a right artinian ring also admits a description of its absolute direct summand modules. The techniques developed apply also to quasi-continuous modules over a commutative noetherian domain of dimension one and a characterization of such modules is given as well.

In [7], a complete description of the quasi-projective abelian groups is derived after a few lemmas which are stated and proved for modules over an arbitrary ring. By dualizing some of these lemmas, the torsion quasi-injective abelian groups are easily obtained. Finally, using the above, the absolute direct summand groups can be described. It is seen that the existence of absolute direct summand abelian groups that are not quasi-continuous is due to the existence of indecomposable abelian groups that are not uniform.

In what follows, all rings are rings with unit and all modules are right modules. For an R-module M, E(M), End M and Aut M denote the injective envelope, R-endomorphism and R-automorphism rings of M respectively. ann(X) denotes the right annihilator of a subset X of a module.

CHAPTER 1

General Results on Absolute Direct Summand Modules

Definition 1.1 Let M be an R-module.

i) M is called an absolute direct summand module (a.d.s.) if for every module decomposition $M = U \oplus V$ and for every complement submodule W of U in M, $M = U \oplus W$ holds.

ii) M is called quasi-continuous if M is invariant under all projectors of $E(M)$.

Proposition 1.2 Let R be a ring and M an R-module. Then

i) If M is indecomposable it is a.d.s.

ii) If M is quasi-continuous it is a.d.s.

iii) If $M = \oplus_{\Lambda} M_{\alpha}$ is a.d.s. then for each $\alpha \in \Lambda$ M_{α} is also a.d.s.

Proof: i) is obvious. ii) Let $M = U \oplus V$ and let W be a complement of U in M; then $E(M) = E(U + W) \oplus E' = E(U) \oplus E(W) \oplus E'$ for some submodule E' of $E(M)$. If π_1, π_2 and π_3 are the canonical projections of $E(M)$ onto $E(U)$, $E(W)$ and E' respectively, then $M \subseteq \pi_1(M) + \pi_2(M) + \pi_3(M) \subseteq (M \cap E(U)) + (M \cap E(W)) + M \cap E' = U + W + M \cap E'$ since U is a direct summand and W is closed in M, hence $M = U \oplus W \oplus (M \cap E')$. Now $U \cap (W \oplus M \cap E') = 0$ implies $M \cap E' = 0$ so $M = U \oplus W$.
iii) Fix $\alpha \in \Lambda$, let $M_{\alpha} = U \oplus V$ and let C be a complement of

U in M_α. Let $D = C \oplus (\bigoplus_{\alpha \neq \beta \in \Lambda} M_\beta)$ so that $U \cap D = 0$ and let E be a complement of U in M containing D. Then $M = U \oplus E$ so $M_\alpha = (U + E) \cap M_\alpha = U + (E \cap M_\alpha)$ by the modular law since $U \subseteq M_\alpha$. $U \cap E = 0$ implies $U \cap (E \cap M_\alpha) = 0$ and $E \supseteq D$ implies $E \cap M_\alpha \supseteq D \cap M_\alpha \supseteq C \cap M_\alpha = C$ and so by the maximality of C, $E \cap M_\alpha = C$. Thus $M_\alpha = U \oplus C$. □

<u>Definition 1.3</u> Let M and N be R-modules. N is called M-injective if every partial homomorphism $M \to N$ (that is, a homomorphism of a submodule of M into N) extends to homomorphism $M \to N$.

<u>Proposition 1.4</u> Let R be a ring and M an R-module. M is a.d.s. if and only if for every decomposition $M = U \oplus V$, V is U-injective.

<u>Proof:</u> Suppose M is a.d.s., let $U' \subseteq U$ be a submodule and let $\phi: U' \to V$ be a homomorphism. Put $X = \{u - \phi(u) \mid u \in U'\}$; then X is a submodule of M and if $x \in X \cap V$, $x = u - \phi(u)$ for some $u \in U'$, $u = x + \phi(u) \in V \cap U' = 0$ since $\phi(u) \in V$ and so $u = 0$ and $x = 0$. So $X \cap V = 0$, there is a complement C of V in M containing X and then $M = C \oplus V$. If $u \in U$, $u = c + v$, $c \in C$, $v \in V$ (uniquely) and the map $\bar{\phi}: U \to V$ defined by $\bar{\phi}(u) = v$ if a homomorphism. If $u \in U'$, $u = u - \phi(u) + \phi(u)$, $u - \phi(u) \in C$, so $\bar{\phi}(u) = \phi(u)$ and so $\bar{\phi}$ is an extension of ϕ.

Conversely, suppose $M = U \oplus V$, V U-injective and let C be a complement submodule of V in M. Let $U' = U \cap (V \oplus C)$; if $u \in U'$, $u = c + v$, $c \in C$, $v \in V$ so we can define a homomorphism $\phi : U' \to V$ by $\phi(u) = v$ which extends to $\psi : U \to V$ by assumption. Let $u \in U$ and put $D = (u - \psi(u))R + C$; if $v \in D \cap V$ then $v = ur - \psi(u)r + c$ for some $r \in R$, $c \in C$ hence $ur = v + \psi(u)r - c$ with $v + \psi(u)r \in V$, $ur \in U'$ and hence $\psi(ur) = v + \psi(u)r$ showing that $v = 0$. Thus $D \cap V = 0$ and since $D \supseteq C$, the maximality of C shows that $D = C$ so that for each $u \in U$, $u - \psi(u) \in C$. If $x \in M$ we have $x = u + v = u - \psi(u) + \psi(u) + v$ with $u \in U$, $v \in V$, $u - \psi(u) \in C$, $\psi(u) + v \in V$ thus $M = C \oplus V$. □

Remarks i) Imitating the proof of [3, p.23] we will show that V is U-injective if and only if $HU \subseteq V$ where $H = \mathrm{Hom}_R(E(U), E(V))$: Suppose first that $HU \subseteq V$ and let $\phi : U' \to V$ be a partial homomorphism $U \to V$; considering ϕ as a map into $E(V)$ it extends to $\bar{\phi} : E(U) \to E(V)$ and then $\bar{\phi}|_U$ is the required extension of ϕ. Conversely, assume V in U-injective and let $h \in H$. Let $Q = \{x \in U \mid h(x) \in V\}$ a submodule of U. The homomorphism $\bar{h} = h|_Q : Q \to V$ extends to $\bar{h}_1 : U \to V$ and $\bar{h}_1$ gives rise in turn to $\bar{h}_2 \in H$ such that $\bar{h}_2|_U = \bar{h}_1$. If $Q \neq U$ then $(\bar{h}_2 - h)U \neq 0$ since $\bar{h}_2(U) = \bar{h}_1(U) \subseteq V$. In this case since V is essential in $E(V)$, $V \cap (\bar{h}_2 - h)U \neq 0$; then there exists a non-zero $y \in V$ and an $x \in U$ such that $y = (\bar{h}_2 - h)(x)$ and so since $\bar{h}_2(x) = \bar{h}_1(x) \in V$ we have $h(x) = \bar{h}_1(x) - y \in V$ $\Rightarrow x \in Q \Rightarrow y = 0$ a contradiction. Hence $Q = U$.

ii) An a.d.s. module M satisfies a weak form of the exchange property: If $K = M \oplus N = \oplus_\Lambda M_\alpha$ where each M_α is quasi-injective and isomorphic to a direct summand of M then there exist submodules $M'_\alpha \subseteq M_\alpha$ such that $K = M \oplus (\oplus_\Lambda M'_\alpha)$. This can be shown easily by replacing quasi-injectivity with relative injectivity in the proof of [6, Theorem 3]: Assume $N \neq 0$. There is an $\alpha \in \Lambda$ and a submodule $M'_\alpha \neq 0$ of M_α such that $M \cap M'_\alpha = 0$; otherwise, $M \cap M_\alpha$ would be essential in M_α for all $\alpha \in \Lambda$, then $\oplus_\Lambda (M \cap M_\alpha)$ would be essential in K and since $\oplus_\Lambda (M \cap M_\alpha) \subseteq M$, M would be essential in K contrary to assumption. Let M' be a submodule of K maximal with respect to the properties a) $M' = \oplus_\Lambda M'_\alpha$, M'_α submodule of M_α and b) $M \cap M' = 0$. The claim is that $K = M \oplus M'$. Let $\phi: K \to K/M'$ be the natural epimorphism. Since $M \cap M' = 0$, $\phi|_M$ is a monomorphism so that $\phi(M)$ is an a.d.s. submodule of K/M'. In fact, $\phi(M)$ is essential in K/M'; for, if there is a $k \in K$ such that $0 \neq k + M' \in K/M'$ and $(kR + M') \cap \phi(M) = 0$ in K/M' then $(kR + M') \cap M = 0$ in K and if $k = k_{\alpha_1} + \ldots + k_{\alpha_n}$, $\alpha_i \in \Lambda$, $k_{\alpha_i} \in M_{\alpha_i}$, $1 \leq i \leq n$, and $\Lambda' = \Lambda - \{\alpha_1, \ldots, \alpha_n\}$, then $kR + M' = \oplus_{\alpha \in \Lambda'} M'_\alpha \; \oplus_1^n (M'_{\alpha_i} + k_{\alpha_i} R)$ contrary to maximality of M'.. Hence $K/M' \subset E(\phi(M))$ and for each $\alpha \in \Lambda$, $\phi(M_\alpha) \subseteq K/M' \subseteq E(\phi(M))$. For each $\alpha \in \Lambda$ let ψ_α be an isomorphism from M_α onto a direct summand of $\phi(M)$ and put $\phi(M) = \psi_\alpha(M_\alpha) \oplus \bar{M}_\alpha$ so that $E(\phi(M)) = E(\psi_\alpha(M_\alpha)) \oplus E(\bar{M}_\alpha)$. For each $\alpha \in \Lambda$, $\phi\psi_\alpha^{-1} : \psi_\alpha(M_\alpha) \to E(\phi(M))$ and if π_1^α, π_2^α denote the canonical projections of $E(\phi(M))$ onto $E(\psi_\alpha(M_\alpha))$, $E(\bar{M}_\alpha)$ respectively, we have $\pi_1^\alpha \phi \psi_\alpha^{-1} : \psi_\alpha(M_\alpha) \to E(\psi_\alpha(M_\alpha))$ and $\pi_2^\alpha \phi \psi_\alpha^{-1} : \psi_\alpha(M_\alpha) \to E(\bar{M}_\alpha)$. Since $\psi_\alpha(M_\alpha)$ is quasi-injective and M a.d.s., in fact, $\pi_1^\alpha \phi(M_\alpha) \subseteq \psi_\alpha(M_\alpha)$ and $\pi_2^\alpha \phi(M_\alpha) \subseteq \bar{M}_\alpha$ and so $\phi(M_\alpha) \subseteq \phi(M)$ for each $\alpha \in \Lambda$. Hence $\phi(M) = K/M'$ and it follows that M and M' generate K .

In general, a.d.s. modules do not satisfy the exchange property. For example, Z as a module over itself is a.d.s. but does not satisfy the exchange property in view of the fact that $\mathrm{End}_Z Z \simeq Z$ and [15, Proposition 1].

Corollary 1.5 If $M = U \oplus V$ is an a.d.s. R-module where U has a torsion-free element then V is injective.

Proof: Baer's criterion for injectivity states that an R-module N is injective if and only if N is R_R-injective. If $u \in U$ is torsion free then $R \simeq uR \subseteq U$ and by U-injectivity of V and Baer's criterion it follows that V is injective. □

Remarks i) This corollary shows that a ring R is semisimple artinian if and only if every R-module is a.d.s. For, if R is semisimple artinian, every R-module is injective hence a.d.s. and if every R-module is a.d.s. and M is an R-module then $R_R \oplus M$ is a.d.s. hence M is injective showing that R is semisimple artinian.

ii) If every idempotent of R is central then R_R is a.d.s.

The reason for this is that the assumption implies in essence that if $e \in R$ is an idempotent, $\mathrm{Hom}_R(E(eR), E((1-e)R)) = 0 = \mathrm{Hom}_R(E((1-e)R), E(eR))$ so that there are no non-zero partial homomorphisms between any two complementary direct summands of R_R.

Corollary 1.6 If $M = U \oplus V$ is a.d.s. and there is a non zero monomorphism $\phi: U \to V$ then U is quasi-injective.

Proof: Let U' be a submodule of U and $\psi: U' \to U$ a homomorphism. The map $\psi\phi^{-1}: \phi(U') \to U$ extends to $\alpha: \phi(U) \to U$ since $\phi(U') \subseteq \phi(U) \subseteq V$ and U is V-injective. Then if $u \in U'$, $\alpha\phi(u) = \psi\phi^{-1}\phi(u) = \psi(u)$ hence $\alpha\phi$ is the required extension of ψ. □

Remarks This corollary permits us to generalize corollary 4.4 of [10]. If M is an R-module then M is $\sum$-quasi-injective if and only if $M^{(N)}$ is a.d.s. if and only if $M^{(I)}$ is a.d.s. for every index set I: If M is $\sum$-quasi-injective the $M^{(N)}$ is quasi-injective by [2, Corollary 2] hence a.d.s. Now suppose that $M^{(N)}$ is a.d.s. but that there is an infinite index set I such that $M^{(I)}$ is not a.d.s. Then by [2, proposition 1], $M^{(N)}$ is not quasi-injective a contradiction in view of the fact that $M^{(N)} \simeq M^{(N)} \oplus M^{(N)}$ and corollary 1.6. If $M^{(I)}$ is a.d.s. for every I, then $M^{(I)} \oplus M^{(I)}$ is a.d.s. hence $M^{(I)}$ is quasi-injective showing that M is Σ-quasi-injective. The same for corollary 4.5 of [10] : A ring R is quasi-Frobenius if and only if $R_R^{(N)}$ is a.d.s. If R_R is quasi-Frobenius then $R_R^{(N)}$ is in fact injective hence a.d.s. Conversely, if $R_R^{(N)}$ is a.d.s. then $R_R^{(I)}$ is a.d.s. for every I so that by corollary 1.5 $R_R^{(I)}$ is injective for each I . If P is a projective R-module there is an index set I and an exact sequence $R_R^{(I)} \to P \to 0$ which splits showing that P is injective. Then by [4, Theorem A] R is quasi-Frobenius.

If R is a right noetherian ring then every R-module has a maximal injective submodule [14, Exercise 4.1] which is a direct summand and so its complementary direct summand is reduced, that is, it contains no injective submodules except 0 . In this case the following result holds.

Corollary 1.7 Let R be a right Noetherian ring and M an a.d.s. R-module. Then $M = U \oplus V$ where V is injective, U reduced and there is no non-zero partial monomorphism $V \to U$.

Proof: The first part was stated above. If $U \neq 0$ let $f' : V' \to U$ be a partial monomorphism $V \to U$ and assume $f' \neq 0$; then f' extends to $f : V \to U$ and $f|_{E(V')}$ is a monomorphism since V' is essential in $E(V')$. Then $f(E(V'))$ is a non zero submodule of U a contradiction. □

Corollary 1.8 Let R be a right noetherian ring such that R_R is uniform and let M be an a.d.s. R-module. If M is decomposable, not injective and has a torsion-free element then $M = U \oplus V$ where U is reduced indecomposable, V injective torsion and there is no non-zero partial monomorphism $V \to U$.

Proof: $U \neq 0$ and it only needs to be shown that it is indecomposable and V torsion. If V has a torsion-free element then U would be injective by corollary 1.5 a contradiction since U is reduced. If $x = u + v \in M$, $u \in U$, $v \in V$ is a torsion-free element then $\text{ann}(u) \cap \text{ann}(v) = 0$ and since R_R is uniform either $\text{ann}(u) = 0$ or $\text{ann}(v) = 0$. Since V is torsion, $\text{ann}(v) \neq 0$, so U has a torsion-free element. If $U = U_1 \oplus U_2$, $U_i \neq 0$, $i = 1, 2$ again if $u = u_1 + u_2 \in U$, $u_i \in U_i$ is a torsion-free element either $\text{ann}(u_1) = 0$ or $\text{ann}(u_2) = 0$ so that either U_2 is injective or U_1 is injective since U is a.d.s. . In both cases we have a contradiction thus U is indecomposable. □

CHAPTER 2

The Case of a Right Noetherian Ring

We will now assume that R is a right Noetherian ring. We know from corollary 1.7 that an R-module M which is a.d.s. has the form $M = U \oplus V$, U reduced, V injective and there is no non zero partial monomorphism $V \to U$. If $U \neq 0$ then V is necessarily torsion. By replacing the word "monomorphism" by "homomorphism" below, we obtain conditions that are sufficient for a module to be a.d.s.

<u>Proposition 2.1</u> Let R be a right noetherian ring and M an R-module, where $M = U \oplus V$, U a.d.s. and reduced and V injective such that there is no non-zero partial homomorphism $V \to U$. Then M is a.d.s.

<u>Proof</u>: Let $M = C \oplus D$; we can write $C = C_1 \oplus C_2$, $D = D_1 \oplus D_2$ where C_1, D_1 are injective and C_2, D_2 reduced. By [12, Lemma 1.1] $C_2 \oplus D_2$ is reduced and the projection π sending V into $C_1 \oplus D_1$ is a monomorphism. Then $C_1 \oplus D_1 = \pi(V) \oplus V'$ for some injective submodule V' and since $V \cap (C_1 \oplus D_1) \subseteq \pi(V)$, $V' \cap V = 0$ hence maximality of V shows $V' = 0$. Then $V \cong C_1 \oplus D_1$ and consequently $U \cong C_2 \oplus D_2$. We now show that D is C-injective by using remark (i) following proposition 1.4. Let $\phi: C_1 \oplus E(C_2) \to D_1 \oplus E(D_2)$ be a homomorphism by the matrix

of homomorphisms $\phi_{ij} = \pi_j \phi i_i$ $i = 1, 2$, where π_j, i_i are the relevant canonical projections and injections respectively. Then $\phi_{11}(C_1) \subseteq D_1$, $\phi_{12} = 0$ since ϕ_{12} corresponds to a partial homomorphism $V \to U$: if $\phi_{12} \neq 0$, since D_2 is essential in $E(D_2)$, $D_2 \cap \phi(C_1) \neq 0$ and if $X = \{c \in C_1 \mid \phi_{12}(c) \in D_2\}$ then X is a non-zero submodule of C_1 and $\phi_{12} \mid X : X \to D_2$ is a non-zero partial homomorphism $V \to D_2$. Also $\phi_{21}(C_2) \subseteq D_1$ and finally $\phi_{22}(C_2) \subseteq D_2$ since U is a.d.s. Hence $\phi: C_1 \oplus C_2 \to D_1 \oplus D_2$. □

Example If R is a commutative noetherian domain and $P \neq 0$ is a prime ideal of R then $R \oplus E(R/P)$ is a.d.s. . R_R is reduced since $R_R \neq E(R_R)$ = field of fractions of R and R_R is uniform, $E(R/P)$ is injective and torsion hence there is no non-zero partial homomorphism $E(R/P) \to R$. For example the Z-module $Z \oplus Z(p^\infty)$ is a.d.s., $p \in Z$ a prime.

Proposition 2.2 Let R be a right noetherian ring. Suppose $M = \oplus_\Lambda M_\alpha$ where for each $\alpha \in \Lambda$ M_α is a.d.s. and for $\alpha \neq \beta$ in Λ, $\mathrm{Hom}_R(E(M_\alpha), E(M_\beta)) = 0$. Then M is a.d.s.

Proof: Assume $M = U \oplus V$. The condition $\mathrm{Hom}_R(E(M_\alpha), E(M_\beta)) = 0$, $\alpha \neq \beta$, implies that there is no non-zero partial homomorphism $M_\alpha \to M_\beta$. For each $m \in M_\beta$, we have $m = u + v$ where $u = \sum m_\alpha \in U$, $v = \sum u_\alpha \in V$, m_α, $n_\alpha \in M_\alpha$. If $\alpha \neq \beta$ the correspondence $m \to m_\alpha$ is a homomorphism $M_\beta \to M_\alpha$ which is therefore zero so for each $\alpha \in \Lambda$, $\alpha \neq \beta$, we have $m_\alpha = 0 = n_\alpha$, and so $m_\beta \in U$, $n_\beta \in V$ which implies that $M_\beta = (M_\beta \cap U) \oplus (M_\beta \cap V)$. It follows that $V = \oplus_\Lambda (M_\alpha \cap V)$, $U = \oplus_\Lambda (M_\alpha \cap U)$. We must show that V is U-injective. Since R is right noetherian, $E(U) \oplus E(V) = [\oplus_\Lambda E(M_\alpha \cap U)] \oplus [\oplus_\Lambda E(M_\alpha \cap V)]$; let $\phi : E(U) \to E(V)$ be a homomorphism and put $\phi_\alpha = \phi i_\alpha$ where i_α are the relevant canonical injections so that $\phi_\alpha : E(M_\alpha \cap U) \to E(V)$. If $\beta \neq \alpha$ and $\pi_\beta : E(V) \to E(M_\beta \cap V)$ is the canonical projection, $\pi_\beta \phi_\alpha : E(M_\alpha \cap U) \to E(M_\beta \cap V)$ so that $\pi_\beta \phi_\alpha = 0$ and hence $\phi_\alpha(E(M_\alpha \cap U)) \subseteq E(M_\alpha \cap V)$. Since M_α is a.d.s. $\phi_\alpha(M_\alpha \cap U) \subseteq M_\alpha \cap V$ and hence $\phi(U) \subseteq V$ showing V U-injective. □

Note that the above proposition is a partial converse to proposition 1.2 (iii).

<u>Corollary 2.3</u> Let $R = \oplus_1^k Re_i$ be a right noetherian ring where e_i, $i = 1, \ldots, k$ are central orthogonal idempotents and $\sum_1^k e_i = 1$. Then an R-module M is a.d.s. if and only if Me_i is a.d.s. for each $i = 1, \ldots, k$.

<u>Proof</u>: If $m \in M$, $M = m \cdot 1 = \sum_1^k me_i$ so $M = \sum_1^k Me_i$. If $1 \leq j \leq k$ and $m \in Me_j \cap \sum_{j \neq i = 1}^{k} Me_i$, $m = \sum_{j \neq i=1}^{k} m_i e_i = m_j e_j$, $m_i, m_j \in M$ and $m = m_j e_j = m_j e_j^2 = \sum_{j \neq i=1}^{k} Me_i e_j = 0$ so that $M = \oplus_1^k Me_i$. $E(M) = \oplus_1^k E(M)e_i$ so that $E(M)e_i$ is injective for each $i = 1, \ldots, k$. Also $Me_i \subset E(M)e_i$ and in fact since M is essential in E(M) and for each $1 \leq j \leq k$,

$E(M)e_j \cap \sum_{j \neq i=1}^{k} Me_i = 0$, Me_j is essential in $E(M)e_j$ so $E(Me_j) = E(M)e_j$. Since clearly $\mathrm{Hom}_R(E(M)e_i, E(M)e_j) = 0$ if $i \neq j$, by propositions 1.2 and 2.2 the conclusion follows. □

<u>Definition 2.4</u> Let M be an R-module (R right noetherian). Then by [12, Theorem 2.5] $E(M) = \oplus_\Lambda E_\alpha$ where each E_α is injective indecomposable. We say that M is a homogeneous module if the E_α are pairwise isomorphic.

We will show that a homogeneous a.d.s. R-module is either indecomposable of quasi-injective. Let $M = U \oplus V$ be a homogeneous a.d.s. module. We can write $E(M) = E(U) \oplus E(V)$ where, say, $E(U) = \oplus_\Lambda E_\alpha$, $E(V) = \oplus_\Gamma E_\beta$ the E_α, E_β being indecomposable injective and $E_\alpha \cong E_\beta$ for all $\alpha, \beta \in \Lambda \cup \Gamma$ [12 Theorem 2.7]. Put $U_\alpha = E_\alpha \cap U$, $V_\beta = E_\beta \cap V$; then $E(U_\alpha) = E_\alpha$, $E(V_\beta) = E_\beta$ and since R is right noetherian $E(\oplus_\Lambda U_\alpha) = \oplus_\Lambda E_\alpha = E(U)$ so that $\oplus_\Lambda U_\alpha$ is essential in U. Similarly $\oplus_\Gamma V_\beta$ is essential in V. Now fix this notation.

<u>Lemma 2.5</u> $U_\alpha \cong V_\beta$ for all $\alpha \in \Lambda$ and for all $\beta \in \Gamma$.

<u>Proof:</u> Let $\alpha \in \Lambda$, $\beta \in \Gamma$ and $\psi_{\alpha\beta}: E_\alpha \cong E_\beta$. $\psi_{\alpha\beta}$ is the restriction of a homomorphism $E(U) \to E(V)$ hence $\bar{\psi}_{\alpha\beta} = \psi_{\alpha\beta}|U_\alpha : U_\alpha \to V_\beta$. Similarly $\psi_{\alpha\beta}^{-1}$ is the restriction of a homomorphism $E(V) \to E(U)$ so $\bar{\psi}_{\alpha\beta}^{-1} = \psi_{\alpha\beta}^{-1}|V_\beta : V_\beta \to U_\alpha$.

Clearly $\bar{\psi}^{-1}_{\alpha\beta} = \bar{\psi}^{-1}_{\alpha\beta}$ so $\psi_{\alpha\beta} : U_\alpha \cong V_\beta$. □

We will need the following result. Let R be a right noetherian ring and A_α, $\alpha \in \Lambda$, B_β, $\beta \in \Gamma$ R-modules such that A_α is B_β-injective for all $\alpha \in \Lambda$ and for all $\beta \in \Gamma$. Then $\oplus_\Lambda A_\alpha$ is $\oplus_\Gamma B_\beta$-injective: Let $h \in \mathrm{Hom}_R(E(\oplus_\Gamma B_\beta), E(\oplus_\Lambda A_\alpha))$; since R is right noetherian $E(\oplus_\Gamma B_\beta) = \oplus_\Gamma E(B_\beta)$, $E(\oplus_\Lambda A_\alpha) = \oplus_\Lambda E(A_\alpha)$ and if $h_{\alpha\beta} = \pi_\alpha h i_\beta : E(B_\beta) \to E(A_\alpha)$ where π_α, i_β are the relevant canonical projection and injections, $h_{\alpha\beta}(B_\beta) \subseteq A_\alpha$ since A_α is B_β-injective hence $h(\oplus_\Gamma B_\beta) \subseteq \oplus_\Lambda A_\alpha$.

<u>Proposition 2.6</u> Let R be a right noetherian ring and M a homogeneous a.d.s. R-module. If M is decomposable then it is quasi-injective.

<u>Proof:</u> We use again the terminology established above. We can assume without loss of generality that $\mathrm{Card}\,\Lambda \le \mathrm{Card}\,\Gamma$; then there is a subset $\Gamma' \subseteq \Gamma$ with $\mathrm{Card}\,\Gamma' = \mathrm{Card}\,\Lambda$ and an isomorphism $\oplus_\Lambda U_\alpha \to \oplus_{\Gamma'} V_\beta$ which extends to a monomorphism $U \to V$ since M in a.d.s. and $\oplus_\Lambda U_\alpha$ essential in U. Then by corollary 1.6 U is quasi-injective hence $U = \oplus_\Lambda (U \cap E_\alpha) = \oplus_\Lambda U_\alpha$ and each U_α is quasi-injective. Now let $v \in V$, $v = e_1 + \ldots + e_n$, $e_i \in E_{\beta_i}$, $\beta_i \in \Gamma$. Since $V \oplus V_{\beta_i}$ is a.d.s. for each i, the identity $V_{\beta_i} \to V_{\beta_i}$ extends to $\sigma_i : V \to V_{\beta_i}$ and then the homomorphism $\sigma : V \to \overset{n}{\underset{1}{\oplus}} V_{\beta_i}$ defined by $\sigma(v) = \sum \sigma_i(v)$ is an extensioan of the identity $\overset{n}{\underset{1}{\oplus}} V_{\beta_i} \to \overset{n}{\underset{1}{\oplus}} V_{\beta_i}$; it follows that $V = \overset{n}{\underset{1}{\oplus}} V_{\beta_i} \oplus \mathrm{Ker}\,\sigma$

and we have $v = \sigma(v) + (v - \sigma(v))$. Since $\oplus_\Gamma V_\beta$ is essential in V, $vR \cap (\overset{n}{\underset{1}{\oplus}} V_{\beta_i}) \neq 0$ so $\sigma(v) \neq 0$; then $v - \sigma(v) \in \mathrm{Ker}\sigma \cap (\overset{n}{\underset{1}{\oplus}} E_{\beta_i})$ hence $v - \sigma(v) = 0$ since $\overset{n}{\underset{1}{\oplus}} V_{\beta_i}$ is essential in $\overset{n}{\underset{1}{\oplus}} E_{\beta_i}$. Thus $v = \sigma(v)$, $e_i \in V_{\beta_i}$, $1 \leq i \leq n$ and $V = \oplus_\Gamma V_\beta$. This shows that M is a direct sum of isomorphic quasi-injective submodules and by the result above M is quasi-injective. □

We can now give a complete description of a.d.s. modules over two kinds of rings. First we need the following lemmas.

<u>Lemma 2.7</u> Let R be a commutative noetherian domain of dimension one. If M is a torsion R-module then $M = \oplus_{\mathcal{M}} M_P$ where $\mathcal{M}$ is the family of maximal ideals of R and $M_P = \{x \in M \mid \mathrm{ann}(x) \text{ is a } P\text{-primary ideal}\}$.

<u>Proof</u>: That M_P is a submodule for each $P \in \mathcal{M}$ follows from [13, p.153 corollary 1]. We now show that $M = \sum_{\mathcal{M}} M_P$. Let $0 \neq x \in M$ then $0 \neq \mathrm{ann}(x) \subsetneq R$ and $\mathrm{ann}(x) = Q_1 \cap Q_2 \ldots \cap Q_n$ where the Q_i primary with distinct radicals, say $\sqrt{Q_i} = P_i$, $1 \leq i \leq n$. Any two distinct P_i are comaximal so $P_i + P_j = R$ and $Q_i + Q_j = R$ if $i \neq j$. Then it is easy to see that for each i, $2 \leq i \leq n$, we have $Q_1 Q_2 \ldots Q_{i-1} + Q_i = R$ and then $R = (Q_1 + Q_2)(Q_1 Q_2 + Q_3)\ldots(Q_1 Q_2 \ldots Q_{n-1} + Q_n) = Q_2 Q_3 \ldots Q_n + Q_1 Q_3 \ldots Q_n + \ldots Q_1 Q_2 \ldots Q_{n-1}$ as can easily be shown by induction. Hence $1 = \sum_1^n q_i$ where $q_i \in \prod_{i \neq j = 1}^{n} Q_j$ and $x = \sum_1^n x q_i$. For each i,

$1 \leq i \leq n$, $\mathrm{ann}(xq_i) \supseteq Q_i$ so that again by [13, p.153 corollary 1] $\mathrm{ann}(xq_i)$ is P_i-primary and hence $xq_i \in M_{P_i}$. Thus $M = \sum_{\mathcal{M}} M_P$. That this sum is direct follows from maximality of $P \in \mathcal{M}$. □

We remark here that $E(M)_P = E(M_P)$ is equal to the P-component of the decomposition of $E(M)$ so that M_P is homogeneous for each $P \in \mathcal{M}$ [12, Theorem 3.3]. If $P, P' \in \mathcal{M}$ and $P \neq P'$ then it is easy to see that $\mathrm{Hom}_R(E(M)_P, E(M)_{P'}) = 0$ hence by propositions 1.2 and 2.2 M is a.d.s. if and only if M_P is a.d.s. for each $P \in \mathcal{M}$.

<u>Theorem 2.8</u> [1]. Let R be a commutative noetherian domain of dimension one. Then an R-module M is a.d.s. if and only if it satisfies one of the following conditions:

i) M is indecomposable or injective.

ii) M is torsion and for every maximal ideal P of R, M_P is indecomposable or quasi-injective.

iii) M has a torsion-free element and $M = U \oplus V$ where U is reduced indecomposable, V injective torsion and there is no non-zero partial homomorphism $V \to U$.

<u>Proof:</u> The modules in (i), (ii) and (iii) are a.d.s. by propositions 1.2, 2.1 and 2.2. Conversely, if M is not injective, decomposable and a.d.s. we have two cases. If

M is torsion we get (ii) from proposition 2.6 and Lemma 2.7. If M has a torsion free element then corollary 1.8 applies, so $M = U \oplus V$ where $U \neq 0$ is reduced indecomosable, V injective torsion and there are no non-zero partial monomorphisms $V \to U$. Assume there is a non-zero partial homomorphism $V \to U$ then it extends to a non-zero homomorphism $V \to U$ whose restriction ϕ on some indecomposable injective summand E of V is non-zero. Also there is an indecomposable injective summand E' of E(U) such that if $\pi' : E(U) \to E'$ is the canonical projection, $\pi'\phi(E) \cap E' \neq 0$. $E' \cong E(R/P)$ for a maximal ideal P of R [12, Proposition 3.1] and if we consider R/P as a submodule of E', $\pi'\phi(E) \supseteq R/P$ and $U \supseteq R/P$. $W = \{x \in E \mid \pi'\phi(x) \in R/P\}$ is a submodule of E and if $\bar{\phi} = \pi'\phi|_W$, $\bar{\phi}(W) = R/P$, $W/\mathrm{Ker}\bar{\phi} \cong R/P$ and hence $W/\mathrm{Ker}\bar{\phi}$ is irreducible. Then by [8, Lemma 2], $\mathrm{Hom}_R(W/\mathrm{Ker}\bar{\phi}, W) \neq 0$ thus $W/\mathrm{Ker}\bar{\phi}$ is isomorphic to a submodule of W and we have a non-zero partial monomorphism $V \to U$ a contradiction. □

Proposition 2.9 Let R be a commutative artinian ring, $R = \bigoplus_1^k Re_i$, $e_i = e_i^2$, where each Re_i is a local ring with maximal ideal P_i. Then an R-module M is a.d.s. if and only if Me_i is indecomposable or quasi-injective for each i.

Proof: The existence of such a decomposition for R is given for example by [9, Theorem 7.13]. If $i \neq j$ then

$Hom_R(E(Me_i), E(Me_j)) = 0$ and for each i, every indecomposable injective direct summand of $E(Me_i)$ is R-isomorphic to $E(Re_i/P_i)$ so that Me_i is homogeneous. The conclusion then follows from propositions 1.2, 2.2 and 2.6 . □

<u>Theorem 2.10</u> Let R be a commutative noetherian domain of dimension one. Then an R-module M is quasi-continuous if and only if

i) M is quasi-injective, or

ii) $M = U \oplus V$, V injective torsion, U torsion-free uniform.

<u>Proof</u>: First it is clear that if M is indecomposable then it is quasi-continuous if and only if it is uniform. Assume that M is indecomposable and quasi-continuous. If M has a torsion-free element then it is torsion-free hence we have (ii). If M is torsion, since E(M) is indecomposable, $E(M) \cong E(R/P)$, P a maximal ideal of R hence by [8, Lemma 2] we have (i). Now assume M is decomposable and quasi-continuous. If it has a torsion-free element and is not quasi-injective we have (ii) by Theorem 2.8. If it is torsion, we have $M = \oplus_{\mathcal{M}} M_p$ where $\mathcal{M}$ = set of maximal ideals of R. Since each M_p is also quasi-continuous, by Theorem 2.8 each M_p is quasi-injective. Then M is quasi-injective because if $P, P' \in \mathcal{M}$ and $P \neq P'$ then

$Hom_R(E(M_p), E(M_{p'})) = 0$. Conversely, if M is as in (i) then it is quasi-continuous [10] . Assume M as in (ii). Let $E(M) = E_1 \oplus E_2$ and π_1, π_2 the corresponding canonical projections; also $E(M) = V \oplus E(U)$ and since E(U) is indecomposable we can assume without loss of generality that E(U) is a direct summand of E_1 . Then

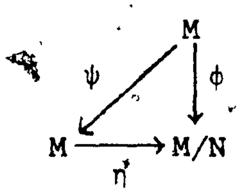
$\pi_1(U + V) = \pi_1 U + \pi_1 V = U + \pi_1 V \subseteq U + V \cap E_1 \subseteq M$ and

$\pi_2(U + V) = \pi_2 V \subseteq V \cap E_2 \subseteq M$ hence M is quasi-continuous. □

CHAPTER 3

Quasi projective abelian groups

<u>Definition 3.1</u> An R-module M is called quasi-projective if for every submodule N of M and for every R-homomorphism $\phi: M \to M/N$ there is an R-endomorphism ψ of M such that the diagram



is commutative where η is the natural epimorphism.

<u>Example</u>: Projective and completely reducible modules are quasi-projective.

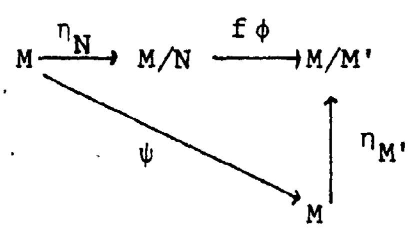
<u>Lemma 3.2</u> Every direct summand of a quasi-projective module is quasi-projective.

<u>Proof</u>: Let $M = M_1 \oplus M_2$ be a quasi-projective R-module, let $N \subseteq M_1$ be a submodule and $\phi: M_1 \to M_1/N$ a homomorphism. If $\pi: M \to M_1$ is the canonical projection then $\phi\pi: M \to M_1/N \subseteq M/N$ and there is a $\psi \in End_R\, M$ such that $\eta\psi = \phi\pi$ where $\eta: M \to M/N$ is the natural epimorphism. If $i: M_1 \to M$ is the canonical injection then $\eta\psi i = \phi\pi i = \phi$, $\psi i: M_1 \to M_1$ and is the required homomorphism. □

<u>Lemma 3.3</u> If M is quasi-projective and $N \subseteq M$ is a fully

invariant submodule then M/N is quasi-projective.

Proof: Recall that N is a fully invariant submodule of M if for all $f \in End_R M$ we have $f(N) \subseteq N$. Let $M'/N \subseteq M/N$ be submodule where $N \subseteq M' \subseteq M$ and let $\phi : M/N \to (M/N)/(M'/N)$ be a homomorphism. Let $f: (M/N)/(M'/N) \cong M/M'$, $f((m + N) + M'/N) = m + M'$ and $\eta_N : M \to M/N$, $\eta_{M'} : M \to M/M'$, $\eta : M/N \to (M/N)/(M'/N)$ the natural epimorphisms. There is a $\psi \in End_R M$ making the diagram



commutative, that is, $\eta_{M'}\psi = f\phi\eta_N$. Since $\psi(N) \subseteq N$ we can define a homomorphism $\bar{\psi} : M/N \to M/N$ by $\bar{\psi}(m + N) = \psi(m) + N$, $m \in M$. We then have $f\eta\bar{\psi}\eta_N = \eta_{M'}\psi = f\phi\eta_N$ hence $\eta\bar{\psi} = \phi$ since f is an isomorphism and η_N an epimorphism and $\bar{\psi}$ is the required map. □

Example If $n \in Z$, $Z(n)$ is quasi-projective.

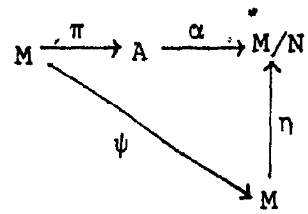
Lemma 3.4 If $M_i (i \in I)$ are quasi-projective R-modules such that for every submodule N of $M = \oplus_I M_i$ we have $N = \oplus_I (N \cap M_i)$, then M is quasi-projective.

Proof: If $N \subseteq M$ is a submodule then $M/N = \oplus_I (M_i/N_i)$ where $\forall i \in I$, $N_i = N \cap M_i$. If $i \neq j$ then every homomorphism $M_i \to M_j/N_j$ is zero; for, otherwise there exist submodules $N_i' \subseteq M_i$, $N_j' \subseteq M_j$ such that $f: M_i/N_i' \cong N_j'/N_j \neq 0$. Then the submodule $S = \{m + n \in M_i \oplus N_j' \mid f(m + N_i') = n + N_j\}$ is a subdirect sum of M_i and N_j'. On the other hand we have by assumption that $S = (M_i \cap S) \oplus (M_j \cap S) = N_i' \oplus N_j$ a contradiction since the projection of S into M_i and N_j' are onto.

Thus every homomorphism $\phi: \oplus_I M_i \to \oplus_I (M_i/N_i)$ acts componentwise and it is now easy to see that M is quasi-projective. □

Lemma 3.5 If N is a submodule of a quasi-projective module M such that M/N is isomorphic to a direct summand of M, then N is also a direct summand of M.

Proof: Let A be a direct summand of M with $\pi: M \to A$, $i: A \to M$ the projection and injection, and let $\alpha: A \cong M/N$. If $\eta: M \to M/N$ is the natural epimorphism, there is a $\psi \in \text{End}_R M$ making the diagram



commutative, that is, $\alpha\pi = \eta\psi$. The sequence $0 \to N \to M \overset{\eta}{\to} M/N \to 0$ is exact, the homomorphism $\psi i \alpha^{-1}$ sends M/N into M and $\eta \psi i \alpha^{-1} = \alpha \pi i \alpha^{-1}$ in the identity map of M/N hence the sequence splits and $M \cong N \oplus (M/N)$. □

<u>Lemma 3.6</u> Let N be a submodule of the quasi-projective module M such that there exists an epimorphism $\varepsilon : N \to M$. Then M is isomorphic to a direct summand of N.

<u>Proof</u>: Put $K = \operatorname{Ker} \varepsilon$ and let $\bar{\varepsilon} : N/K \cong M$ be the isomorphism induced by ε. Let $\alpha : M \to M/K$ be the injection defined by $M \overset{\bar{\varepsilon}^{-1}}{\to} N/K \subseteq M/K$ so that $\alpha \bar{\varepsilon}$ is the identity on N/K and $\eta : M \to M/K$ the natural epimorphism. Since M is quasi-projective there is a $\psi \in \operatorname{End}_R M$ such that $\eta\psi = \alpha$; since $\alpha(M) \subseteq N/K$, $\psi(M) \subseteq N$ hence $\psi\bar{\varepsilon}$ is a homomorphism $N/K \to N$ and $\eta \psi \bar{\varepsilon} = \alpha \bar{\varepsilon}$ is the identity on N/K. Thus the exact sequence $0 \to K \to N \overset{\eta}{\to} N/K \to 0$ splits and $N \cong K \oplus N/K \cong K \oplus M$. □

<u>Lemma 3.7</u> If N is a submodule of a quasi-projective module M then $\operatorname{Card}(\operatorname{End}_R(M/N)) \leq \operatorname{Card}(\operatorname{End}_R M)$.

<u>Proof</u>: Define $\psi : \operatorname{End}_R(M/N) \to \operatorname{End}_R M$ by letting $\psi(\alpha)$ be the R-endomorphism of M such that $\eta \psi(\alpha) = \alpha\eta$ which exists by quasi-projectivity of M, where $\eta : M \to M/N$ is the natural epimorphism. For $\alpha, \beta \in \operatorname{End}_R(M/N)$, $\psi(\alpha) = \psi(\beta)$ implies

$\alpha\eta = \eta\psi(\alpha) = \eta\psi(\beta) = \beta\eta$ implies $\alpha = \beta$ since η is epimorphic, hence ψ is an injection. □

Before giving a complete description of the quasi-projective abelian groups we state the following results all of which can be found in [5]

1) If F is a free abelian group and p^n a prime power then F/p^nF is a direct sum of cyclic groups each of order p^n.
2) A bounded pure subgroup of an abelian group G is a direct summand of G. [5, Theorem 24.5]
3) Every p-group contains a basic subgroup [5, Theorem 29.2].
4) If G is an arbitrary countable torsion abelian group then $\mathrm{Card}(\mathrm{Aut}\ G) = 2^{\aleph_0}$ [5, p.229, Exercise 21].
5) Every torsion abelian group is a direct sum of p-groups (Lemma 2.7).
6) For any torsion-free abelian group G, $\mathrm{rank}(G) \leq \mathrm{Card}(G) \leq \mathrm{rank}(G) \cdot \aleph_0$ [5, p.32].

Theorem 3.8 [7]. An abelian group A is quasi-projective if and only if it is:

i) free, or
ii) a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n.

Proof: Free groups are clearly quasi-projective. Now let A be as in (ii). For each prime p there is a free group F_p such that $A_p \cong F_p/p^{n_p} F_p$ for some $n_p \in N$. $p^{n_p} F_p$ is a fully invariant subgroup of F so by Lemma 3.3 $F_p/p^{n_p} F_p$ is quasi-projective. Then by Lemma 3.4, $A = \oplus_p A_p \cong \oplus_p (F_p/p^{n_p} F_p)$ is quasi-projective.

Conversely, assume A is quasi-projective and torsion so that $A = \oplus_p A_p$. By Lemma 3.2, every A_p is quasi-projective. If A_p is not reduced then it contains a subgroup isomorphic to $Z(p^\infty)$ which is a direct summand of A_p hence $Z(p^\infty)$ is quasi-projective; but then if $X \subsetneq Z(p^\infty)$ is a non-zero subgroup, $Z(p^\infty)/X \cong Z(p^\infty)$ and by Lemma 3.3 X is a direct summand of $Z(p^\infty)$ a contradiction since $Z(p^\infty)$ is indecomposable. Thus A_p is reduced. Also A_p cannot have direct summands of the form $Z(p^m) \oplus Z(p^n)$ for $n < m$: if it did, $Z(p^m) \oplus Z(p^n)$ would be quasi-projective and if $f: Z(p^m) \to Z(p^n)$ is the epimorphism $x \mapsto x(\text{mod } p^n)$ then $Z(p^n) \cong Z(p^m)/\text{Ker } f \cong (Z(p^m) \oplus Z(p^n)) / (\text{Ker } f \oplus Z(p^n))$ so by lemma 3.3 $\text{Ker } f \oplus Z(p^n)$ would be a direct summand of $Z(p^m) \oplus Z(p^n)$ and hence Ker f would be a direct summand of $Z(p^m)$ a contradiction since $Z(p^m)$ is indecomposable. Let B be a basic subgroup of A_p and assume it has direct summands of the form $Z(p^m)$ and $Z(p^n)$ with $n \neq m$. Let B_m be the direct sum of all the direct summands of B that are of the form $Z(p^m)$ and similarly for B_n. Then $B_m \oplus B_n$

is pure and bounded hence a direct summand of A_p a contradiction since then A_p has a direct summand of the form $Z(p^n) \oplus Z(p^m)$. Thus B is a direct sum of cyclic groups each of order p^n, $n \in N$, hence pure and bounded and hence a direct summand of A_p. If $A_p = B \oplus X$ then $X \cong A_p/B$ is injective so $X = 0$ since A_p is reduced and so $A_p = B$.

Now assume A is quasi-projective and torsion-free. If rank(A) = r let F be a free subgroup of A of rank r. $End_Z A$ is countable because A is countable and every endomorphism of A is determined by its restriction on F, hence by Lemma 3.7 $End_Z(A/F)$ is at most countable. Then by (4) above A/F is finite, so A is finitely generated; by the fundamental theorem for finitely generated abelian groups A is free. If rank(A) is infinite, by (6) above rank(A) = Card A, there is a free group F with rank(F) = Card A and an epimorphism $F \to A$. F is also isomorphic to a free subgroup of A of the same rank so by Lemma 3.6 A is isomorphic to a direct summand of F and hence is free.

Finally A cannot be mixed; otherwise, the torsion part T of A is non-zero and a fully invariant submodule so that A/T is quasi-projective; since A/T is also torsion-free it is free by above, so $A = T \oplus F$, $F \cong A/T$ (the exact sequence $0 \to T \to A \to A/T$ splits). Since $T \neq 0 \neq F$ by assumption, there is a cyclic direct summand $Z(p^n)$ of T and on epimorphism $\varepsilon: F \to Z(p^n)$ so that $Z(p^n) \simeq F/\mathrm{Ker}\,\varepsilon$. By Lemma 3.2 $F \oplus Z(p^n)$ is quasi-projective, $(F \oplus Z(p^n))/(\mathrm{Ker}\,\varepsilon \oplus Z(p^n)) \simeq Z(p^n)$, by Lemma 3.5 $\mathrm{Ker}\,\varepsilon \oplus Z(p^n)$ is a direct summand of $F \oplus Z(p^n)$ thus $\mathrm{Ker}\,\varepsilon$ is a direct summand of F; this is a contradiction because then, the exact sequence $0 \to \mathrm{Ker}\,\varepsilon \to F \to Z(p^n) \to 0$ splits.

□

<u>Remarks</u> We can use the dual of Lemma 3.5 to show that a torsion abelian group A is quasi-injective if and only if every p-component A_p is a direct sum of isomorphic cyclic or quasi-cyclic groups $Z(p^n)$, $(n \le \infty)$. The dual states: If M is a quasi-injective module and $N \subseteq M$ is a submodule isomorphic to a direct summand of M then N is itself a direct summand of M. The proof follows easily from that of Lemma 3.5 by reversing the arrows. Now if A is a quasi-injective, torsion abelian group so is each A_p (dual of Lemma 3.2). A_p cannot have direct summands of the form $Z(p^m) \oplus Z(p^n)$, $n < m$; otherwise, $Z(p^m) \oplus Z(p^n)$ would be quasi-injective. Then, since the subgroup $\{0, p^{m-n}, 2p^{m-n}, \ldots, (p^n - 1)p^{m-n}\}$ of $Z(p^m)$ is isomorphic to $Z(p^n)$, it is a direct summand of $Z(p^m) \oplus Z(p^n)$ and hence of $Z(p^m)$ a contradiction. The rest of the argument is as in the proof of Theorem 3.8. Conversely, if A is an abelian group such that each A_p

is a direct sum of isomorphic cyclic groups, it follows directly from the definition of quasi-injectivity that each A_p is quasi-injective; then A is quasi-injective because $Hom_Z(E(A_p), E(A_{p'})) = 0$ whenever $p \neq p'$.

It is easy to see that there is no quasi-injective submodule of Q containing Z except Q itself. It follows from this that a torsion-free abelian group A is quasi-injective if and only if it is isomorphic to $\oplus_I Q$ where Card I = rank A, that is, if and only if it is divisible.

Now, if A is a mixed quasi-injective abelian group, and T its torsion submodule, then $A = T \oplus F$ since T is closed in A, where F is torsion-free. Since F is also quasi-injective, it is divisible; also by Corollary 1.5, T is divisible so A is divisible. Thus we have that a non-torsion abelian group is quasi-injective if and only if it is divisible. This result is also an easy consequence of Bauer's criterion for injectivity and of [6, Lemma 2].

Now we can determine the a.d.s. abelian groups using the above, Theorem 2.8 and [5, Corollary 24.4]. They are the indecomposable groups, the quasi-injective groups and groups of the form $V \oplus U$ where V is divisible torsion (a direct sum of quasi-cyclic groups $Z(p^\infty)$), U reduced indecomposable with no non-zero partial homomorphisms (or monomorphisms) $V \to U$. (Equivalently we can say U indecomposable and not divisible.)

<u>Examples</u> (i) The Z-groups Z, $Z \oplus Z(p^\infty)$ are quasi-continuous but not quasi-injective, $p \in Z$ a prime. (ii) The additive group of p-adic integers is a.d.s. but not quasi-continuous: By [11, Theorem 18] it is indecomposable, torsion-free of infinite rank. It is also not divisible because it is not isomorphic to a direct sum of copies of Q. Hence by Theorem 2.10 it is not quasi-continuous, but it is a.d.s. since it is indecomposable. In fact the existence of a.d.s. abelian groups that are not quasi-continuous is due to the existence of indecomposable abelian groups that are not uniform.

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