RESPONSE OF LINEAR MECHANICAL SYSTEMS UNDER NONSTATIONARY RANDOM EXCITATIONS

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ABSTRACT

This thesis presents an analytical investigation of linear mechanical systems subjected to nonstationary type of random excitations. Expressions for the mean square values of the responses are derived using impulse response and, in some cases, frequency response characteristics of the system.

The input excitation is considered as a product of a modulating component and a stationary white noise stochastic component of zero mean. Under such representation, the autocorrelation of the input excitation is a delta function with a specified strength function. Purely harmonic, harmonic with exponential decay, and simple linear variations are considered for the strength function to simulate a wide variety of nonstationary forces.

Both single-degree and two-degree-of-freedom systems are investigated. The variation of the maximum mean square amplitudes and their phase angles against frequency ratios are presented in the form of plots for different values of the system parameters. From these results, the resonance regions are identified and conclusions are drawn on the behaviour of mechanical systems under nonstationary random excitations.

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A	Maximum Amplitude of Input Strength Function
a	Arbitrary Value
b	Arbitrary Value
$C_{xx}(t_1,t_2)$	Autocovariance of Process X(t)
c	Arbitrary Value; Constant of Viscous Damper
d	Arbitrary Value
E{ }	Expected Value of { }
e	Exponential
F(t)	Excitation Force
$H(j\omega)$	Frequency Response Function
$H^*(j\omega)$	Complex Conjugate of $H(j\omega)$
h(t)	Impulse Response Function
I	Arbitrary Function
I*	Complex Conjugate of I
$I(t_1)$	Strength Function of Input Autocorrelation
Im[ ]	Imaginary Part of [ ]
j	Indicated Imaginary Value ($j^2 = -1$)
k	Spring Stiffness
m	Mass of System
P	Amplitude Function
p[ ]	Probability Density of [ ]
p	Frequency
Q	Amplitude Function
R	Amplitude Function
Re[ ]	Real Part of [ ]

$R_{xx}(t_1,t_2)$	Autocorrelation of Process X(t)
$S_{xx}(\omega_1,\omega_2)$	Power Spectral Density (or Spectral Density) of Process X(t)
T	Constant of Time
t	Time Variable
$X(t)$	Response of the System
$\dot{X}(t)$	Time Derivate of X(t)
$\ddot{X}(t)$	Second Derivative of X(t) with respect to time
$Z(t)$	F(t)/m
$\|(\ )\|$	Absolute Value of ( )
α	Exponential Decay Parameter
$\delta(.)$	Dirac Delta Function
Δ	Determinant
ζ	Damping Ratio
μ	Mass Ratio
τ	Natural Period of System; Time Variable
$\sigma_x^2(t)$	Variance of Process X(t)
Φ	Phase Angle
ω	Frequency
ω_n	Natural Frequency of System
ω_d	Damped Frequency of System

CHAPTER 1

INTRODUCTION

Mechanical systems are generally classified as linear or nonlinear depending on the nature of the mathematical model used to describe them. A large number of mechanical systems can be represented by linear models over their normal operating ranges.

The external excitation on mechanical systems may be deterministic or random depending on the source of the input and the operational environment of the system. When the input forces are random, the response of the system can only be described in terms of probabilistic quantities. A random excitation can further be classified into stationary or nonstationary process depending on its probabilistic characteristics. For a stationary process, the statistical properties do not change with time, but depend only on the time difference, whereas a nonstationary process has all its statistical properties, such as mean value, autocorrelation, etc... described as function of time. The input excitations considered in this thesis are of the latter nature.

Most of the random processes occuring in reality are essentially nonstationary in character. Examples of such physical random processes are forces due to explosion, shock, and earthquake, gust response, vibration environment of vehicles, forces arising from rapid acceleration or deceleration, and similar transient phenomena. It is then important to have a knowledge of the response of mechanical systems under such random excitation environments.

For a linear system subjected to random excitations, the response is also a random process. If the excitation is nonstationary, the response may also be expected to be nonstationary. The behaviour of a linear mechanical system under stationary random excitations has been extensively investigated by many [2,5,7,10]. Of these, the contribution of Crandall and Mark [2] is extremely useful because they present a systematic approach to the solution of one and two-degree-of-freedom linear mechanical system under stationary random forces. Similar investigations, when the excitation is nonstationary, are very few. The concept of representing nonstationary forces through a modulating part and a stationary white noise process was first introduced by Roberts [3]. Using a corresponding autocorrelation described by a strength function, he presented analytical techniques giving the necessary response statistics of a simple linear system. Further investigations by Caughey [1], Lin [6], and Roberts [4] show mathematical difficulties in modelling nonstationary random processes for application to mechanical problems. All the above mentioned researchers considered only single-degree-of-freedom systems which are subjected to forces having a simple harmonic strength function for the autocorrelation of the input process.

The present investigation considers both one-degree and two-degree-of-freedom mechanical linear systems under a variety of strength functions describing different types of nonstationary excitations. The basic model for the non-stationary force is taken as a product of a modulating component representing the nonstationarity and a Gaussian delta-correlated stationary component as suggested by Roberts [3]. This model yields a delta correlation for the excitation process with a specified strength function. Different strength functions, namely linear, harmonic, and harmonic with exponential decay are considered in this thesis.

General expressions relating the input force and the probabilistic descriptions of the response for a single-degree-of-freedom linear system are derived in chapter 2 using the concept of impulse response. The maximum mean square amplitudes of the responses of the system and their corresponding phase angles are obtained in chapter 3 when external excitation has autocorrelations described by different strength functions as mentioned previously. The results are presented in terms of non-dimensional plots. The method is extended in chapter 4 to two-degree-of-freedom linear mechanical systems under similar nonstationary forces. Here, unlike the previous case, the expressions for the response probabilities are derived using the frequency response function of the system. Special cases arising from the complex receptance functions and their influence on the mean square response of the system are also discussed in detail. In all the cases considered in this investigation, the results are checked and compared with those of the previous investigations dealing with the stationary type of input force such as the one given by Crandall and Mark [2].

The symbols used in this thesis are defined in the nomenclature and are also described in the text when they appear for the first time. Figures and Tables mentioned in the text of the thesis are presented at the end of each chapter.

CHAPTER 2

ONE-DEGREE-OF-FREEDOM LINEAR MECHANICAL SYSTEMS UNDER RANDOM EXCITATIONS

2.1 Transient and Steady State Responses

Consider a damped mass spring system, as shown in Fig. 2.1. The equation of motion for the system is

$$m\ddot{X}(t) + c\dot{X}(t) + kX(t) = F(t) , \qquad t \geq t_o \tag{2.1}$$

with initial conditions

$$\left.\begin{aligned} X(t_o) &= a \\ \dot{X}(t_o) &= b \end{aligned}\right\} \tag{2.2}$$

Here, m : mass of oscillator

c : constant of viscous damper

k : linear spring constant

$F(t)$: random excitation

$X(t)$: displacement of mass m.

Let,

$$\left.\begin{aligned} c/m &= 2\zeta\omega_n \\ k/m &= \omega_n^2 \\ F(t)/m &= Z(t) \end{aligned}\right\} \tag{2.3}$$

where ω_n : the natural frequency of the system,

ζ : damping ratio.

Eq.(2.1) now takes the form

$$\ddot{X}(t) + 2\zeta\omega_n\dot{X}(t) + \omega_n^2X(t) = Z(t) \tag{2.4}$$

If $Z(t)$ is stochastic in nature, then $X(t)$ is also a stochastic process. Further, if $Z(t)$ is Gaussian, then $X(t)$

is also Gaussian distributed because the system is linear [8].

The solution for the response X(t) is made up of a transient part governed by the initial conditions and a steady state part governed by the excitation Z(t). The standard form of the solution for Eq.(2.4) is [9]

$$X(t) = ae^{-\zeta\omega_n(t-t_o)}\left[\cos\omega_d(t-t_o) + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d(t-t_o)\right] + \frac{b}{\omega_d}e^{-\zeta\omega_n(t-t_o)}\left[\sin\omega_d(t-t_o)\right] + \int_{t_o}^{t} h(t-\tau)Z(\tau)d\tau \tag{2.5}$$

where

ω_d : damped frequency of the system

$$= \omega_n(1-\zeta^2)^{\frac{1}{2}} \tag{2.6}$$

h(t) : impulse response of the system

$$= \frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{2.7}$$

Setting

$$\left.\begin{aligned} X_1(t-t_o) &= e^{-\zeta\omega_n(t-t_o)}\left[\cos\omega_d(t-t_o) + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d(t-t_o)\right] \\ X_2(t-t_o) &= \frac{1}{\omega_d}e^{-\zeta\omega_n(t-t_o)}\left[\sin\omega_d(t-t_o)\right] \end{aligned}\right\} \tag{2.8}$$

which depend only on the deterministic properties of the system and not on the input excitation, the input-output relation (2.5) then takes the form

$$X(t) = aX_1(t-t_o) + bX_2(t-t_o) + \int_{t_o}^{t} h(t-\tau)Z(\tau)d\tau \qquad (2.9)$$

where $X(t)$ and $Z(\tau)$ are stochastic processes, $X_1(t-t_o)$ and $X_2(t-t_o)$ are deterministic functions.

2.2 Mean Value of the Response X(t)

Taking expected value of both sides of Eq.(2.9)

$$E\{X(t)\} = aX_1(t-t_o) + bX_2(t-t_o) + \int_{t_o}^{t} h(t-\tau)E\{Z(\tau)\}d\tau \qquad (2.10)$$

where $E\{X(t)\}$ and $E\{Z(t)\}$ are the mean values of $X(t)$ and $Z(t)$ respectively.

Using a new variable $\xi = t-\tau$, Eq.(2.10) becomes

$$E\{X(t)\} = aX_1(t-t_o) + bX_2(t-t_o) + \int_{0}^{t-t_o} h(\xi)E\{Z(t-\xi)\}d\xi \qquad (2.11)$$

For infinite operating time systems, for which $t_o = -\infty$, the transient response dies out and the mean value of the output $X(t)$ takes the form

$$E\{X(t)\} = \int_{0}^{\infty} h(\xi)E\{Z(t-\xi)\}\, d\xi \qquad (2.12)$$

Case of Stationary Z(t)

If the excitation $Z(t)$ is stationary in character then its mean value is independent of time. That is,

$E\{Z(t)\} = E\{Z(t-\xi)\} = E\{Z\} = \text{constant}.$

Then,

$$E\{X(t)\} = aX_1(t-t_o) + bX_2(t-t_o) + E\{Z\}\int_0^{t-t_o} h(\xi)d\xi \qquad (2.13)$$

And for infinite operating time systems, where $t_o = -\infty$, the mean value of the output $E\{X(t)\}$ is also found to be independent of time

$$E\{X\} = E\{Z\}\int_0^{\infty} h(\xi)d\xi \qquad (2.14)$$

2.3 Autocovariance of the Response X(t)

By definition, the autocovariance of a process X(t) valid for time range t_1, t_2 is given by

$$C_{xx}(t_1,t_2) = E\left\{\left[X(t_1)-E\{X(t_1)\}\right]\left[X(t_2)-E\{X(t_2)\}\right]\right\} \qquad (2.15)$$

Substituting for X (t) and E{X(t)} from Eqs.(2.9) and (2.10)

$$C_{xx}(t_1,t_2) = E\left\{\left(\int_{t_o}^{t_1} h(t_1-\tau_1)Z(\tau_1)d\tau_1 - \int_{t_o}^{t_1} h(t_1-\tau_1)E\{Z(\tau_1)\}d\tau_1\right)\right.$$

$$\left.\left(\int_{t_o}^{t_2} h(t_2-\tau_2)Z(\tau_2)d\tau_2 - \int_{t_o}^{t_2} h(t_2-\tau_2)E\{Z(\tau_2)\}d\tau_2\right)\right\}$$

$$= E\left\{\int_{t_o}^{t_1}\int_{t_o}^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)\left[Z(\tau_1)-E\{Z(\tau_1)\}\right]\right.$$

$$\left.\left[Z(\tau_2)-E\{Z(\tau_2)\}\right]d\tau_1 d\tau_2\right\}$$

$$= \int_{t_o}^{t_1}\int_{t_o}^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)C_{zz}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{2.16}$$

where $C_{zz}(\tau_1,\tau_2)$ is the autocovariance of the input excitation $Z(t)$ given by

$$C_{zz}(\tau_1,\tau_2) = E\Big\{\Big[Z_1(\tau_1)-E\{Z(\tau_1)\}\Big]\Big[Z(\tau_2)-E\{Z(\tau_2)\}\Big]\Big\}.$$

If $\xi_1 = t_1-\tau_1$ and $\xi_2 = t_2-\tau_2$,

$$C_{xx}(t_1,t_2) = \int_0^{t_1-t_o}\int_0^{t_2-t_o} h(\xi_1)h(\xi_2)C_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2 \tag{2.17}$$

For infinite operating time systems,the autocovariance of the output in Eq.(2.17) reduces to

$$C_{xx}(t_1,t_2) = \int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)C_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2 \tag{2.18}$$

Case of Stationary Z(t)

If the input excitation $Z(t)$ is stationary, its autocovariance $C_{zz}(t_1-\xi_1,t_2-\xi_2)$ depends only on the time difference $\tau' = (t_2-\xi_2)-(t_1-\xi_1)$. Then Eq.(2.17) becomes

$$C_{xx}(t_1,t_2) = \int_0^{t_1-t_o}\int_0^{t_2-t_o} h(\xi_1)h(\xi_2)C_{zz}(\tau')d\xi_1 d\xi_2 \tag{2.19}$$

And for infinite operating time systems,

$$C_{xx}(\tau) = \int_0^\infty \int_0^\infty h(\xi_1)h(\xi_2)C_{zz}(\tau')d\xi_1 d\xi_2 \tag{2.20}$$

where $\tau = t_2 - t_1$.

2.4 Variance of the Response X(t)

The variance $\sigma_x^2(t)$ of the output X(t) is obtained by setting $t_1 = t_2 = t$ in the expression for the autocovariance $C_{xx}(t_1, t_2)$ of X(t) given in Eq.(2.16)

$$\sigma_x^2(t) = C_{xx}(t,t) = \int_{t_o}^{t} \int_{t_o}^{t} h(t-\tau_1)h(t-\tau_2)C_{zz}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{2.21}$$

or from Eq.(2.17),

$$\sigma_x^2(t) = C_{xx}(t,t) = \int_0^{t-t_o} \int_0^{t-t_o} h(\xi_1)h(\xi_2)C_{zz}(t-\xi_1, t-\xi_2)d\xi_1 d\xi_2 \tag{2.22}$$

when $t_o = -\infty$,

$$\sigma_x^2(t) = \int_0^\infty \int_0^\infty h(\xi_1)h(\xi_2)C_{zz}(t-\xi_1, t-\xi_2)d\xi_1 d\xi_2 \tag{2.23}$$

Case of Stationary Z(t)

In this case, the variance of the output X(t) is

$$\sigma_x^2(t) = \int_0^{t-t_o} \int_0^{t-t_o} h(\xi_1)h(\xi_2)C_{zz}(\xi_2-\xi_1)d\xi_1 d\xi_2 \tag{2.24}$$

And for infinite operating time systems, this becomes

$$\sigma_x^2 = \int_0^\infty \int_0^\infty h(\xi_1)h(\xi_2)C_{zz}(\xi_2-\xi_1)d\xi_1 d\xi_2 \tag{2.25}$$

2.5 Autocorrelation of the Response X(t)

By definition, the autocorrelation of the process X(t) is

$$R_{xx}(t_1,t_2) = E\{X(t_1)X(t_2)\} \tag{2.26}$$

Substituting for X(t) from Eq.(2.9),

$$R_{xx}(t_1,t_2) = E\Big\{\Big[aX_1(t_1-t_o) + bX_2(t_1-t_o) + \int_{t_o}^{t_1} h(t_1-\tau_1)Z(\tau_1)d\tau_1\Big] \Big[aX_1(t_2-t_o) + bX_2(t_2-t_o) + \int_{t_o}^{t_2} h(t_2-\tau_2)Z(\tau_2)d\tau_2\Big]\Big\} \tag{2.27}$$

Setting $\xi_1 = t_1-\tau_1$, $\xi_2 = t_2-\tau_2$, and carrying out the multiplication,

$$\begin{aligned} R_{xx}(t_1,t_2) = {} & \big[aX_1(t_1-t_o)+bX_2(t_1-t_o)\big]\big[aX_1(t_2-t_o)+bX_2(t_2-t_o)\big] \\ & + \big[aX_1(t_1-t_o)+bX_2(t_1-t_o)\big]\int_0^{t_2-t_o} h(\xi_2)E\{Z(t_2-\xi_2)\}d\xi_2 \\ & + \big[aX_1(t_2-t_o)+bX_2(t_2-t_o)\big]\int_0^{t_1-t_o} h(\xi_1)E\{Z(t_1-\xi_1)\}d\xi_1 \\ & + \int_0^{t_1-t_o}\int_0^{t_2-t_o} h(\xi_1)h(\xi_2)R_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2 \end{aligned} \tag{2.28}$$

where $R_{zz}(t_1-\xi_1,t_2-\xi_2)$ is the autocorrelation of Z(t).

For infinite operating time systems with $t_o = -\infty$, the transient response represented by the first three terms on the left hand side of Eq.(2.28) vanish . The expression for the autocorrelation of the output X(t) will then be

$$R_{xx}(t_1,t_2) = \int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2 \tag{2.29}$$

Case of Stationary Z(t)

If the input excitation Z(t) is stationary, its mean value is constant and its autocorrelation is a function of the time difference only. That is,

$E\{Z(t_2-\xi_2)\} = E\{Z(t_1-\xi_1)\} = E\{Z\}$ = constant,

and

$R_{zz}(t_1-\xi_1,t_2-\xi_2) = R_{zz}(\tau')$ where $\tau' = (t_2-\xi_2)-(t_1-\xi_1)$.

The autocorrelation of the output X(t) given in Eq.(2.28) takes the form

$$\begin{aligned} R_{xx}(t_1,t_2) = &\left[aX_1(t_1-t_o)+bX_2(t_1-t_o)\right]\left[aX_1(t_2-t_o)+bX_2(t_2-t_o)\right] \\ &+\left[aX_1(t_1-t_o)+bX_2(t_1-t_o)\right] E\{Z\}\int_0^{t_2-t_o} h(\xi_2)d\xi_2 \\ &+\left[aX_1(t_2-t_o)+bX_2(t_2-t_o)\right] E\{Z\}\int_0^{t_1-t_o} h(\xi_1)d\xi_1 \\ &+\int_0^{t_1-t_o}\int_0^{t_2-t_o} h(\xi_1)h(\xi_2)R_{zz}(\tau')d\xi_1 d\xi_2 \end{aligned} \tag{2.30}$$

Further, when $t_o = -\infty$,

$$R_{xx}(\tau) = \int_0^\infty \int_0^\infty h(\xi_1)h(\xi_2)R_{zz}(\tau')d\xi_1 d\xi_2 \tag{2.31}$$

where $\tau = t_2 - t_1$.

2.6 Mean Square Value of the Response X(t)

The mean square value of the output X(t) is obtained by setting $t_1 = t_2 = t$ in the expression for the autocorrelation $R_{xx}(t_1, t_2)$ in Eq.(2.28). Therefore,

$$E\{X^2(t)\} = R_{xx}(t,t) \tag{2.32}$$

$$= \left[aX_1(t-t_o) + bX_2(t-t_o)\right]^2$$

$$+ \left[aX_1(t-t_o) + bX_2(t-t_o)\right]\left[\int_{t_o}^{t} h(t-\tau_2)E\{Z(\tau_2)\}d\tau_2 + \int_{t_o}^{t} h(t-\tau_1)E\{Z(\tau_1)\}d\tau_1\right]$$

$$+ \left[\int_{t_o}^{t} \int_{t_o}^{t} h(t-\tau_1)h(t-\tau_2)R_{zz}(\tau_1,\tau_2)d\tau_1 d\tau_2\right]$$

(2.33)

$$= \left[aX_1(t-t_o) + bX_2(t-t_o)\right]^2$$

$$+ \left[aX_1(t-t_o) + bX_2(t-t_o)\right]\left[\int_0^{t-t_o} h(\xi_2)E\{Z(t-\xi_2)\}d\xi_2 + \int_0^{t-t_o} h(\xi_1)E\{Z(t-\xi_1)\}d\xi_1\right] + \ldots$$

$$\ldots + \int_0^{t-t_o}\int_0^{t-t_o} h(\xi_1)h(\xi_2)R_{zz}(t-\xi_1,t-\xi_2)d\xi_1 d\xi_2 \tag{2.34}$$

For infinite operating time systems, this expression reduces to

$$E\{X^2(t)\} = \int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(t-\xi_1,t-\xi_2)d\xi_1 d\xi_2 \tag{2.35}$$

Case of Stationary Z(t)

For stationary input Z(t), the mean square value of the output X(t) is obtained by setting $t_1 = t_2 = t$ in Eq.(2.30)

$$\begin{aligned} E\{X^2(t)\} = & \left[aX_1(t-t_o) + bX_2(t-t_o)\right]^2 \\ & + 2\left[aX_1(t-t_o) + bX_2(t-t_o)\right] E\{Z\}\int_0^{t-t_o} h(\xi)d\xi \\ & + \int_0^{t-t_o}\int_0^{t-t_o} h(\xi_1)h(\xi_2)R_{zz}(\xi_1-\xi_2)d\xi_1 d\xi_2 \end{aligned} \tag{2.36}$$

For $t_o = -\infty$,

$$E\{X^2(t)\} = \int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(\xi_1-\xi_2)d\xi_1 d\xi_2 \tag{2.37}$$

2.7 Power Spectral Density of the Response X(t)

The power spectral density of a random process X(t) is defined by the double Fourier transform of the autocorrelation of X(t) [8],

$$S_{xx}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)e^{-j(\omega_1 t_1-\omega_2 t_2)}dt_1dt_2 \qquad (2.38)$$

The inverse of Eq.(2.38) is given by

$$R_{xx}(t_1,t_2) = \frac{1}{4\pi^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(\omega_1,\omega_2)e^{j(\omega_1 t_1-\omega_2 t_2)}d\omega_1 d\omega_2\right] \qquad (2.39)$$

Substituting the autocorrelation $R_{xx}(t_1,t_2)$ from Eq.(2.29) of an infinite operating time system, into Eq.(2.38)

$$S_{xx}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2\right] e^{-j(\omega_1 t_1-\omega_2 t_2)}dt_1dt_2 \qquad (2.40)$$

Further, setting $\tau_1 = t_1-\xi_1$, $\tau_2 = t_2-\xi_2$, Eq.(2.40) becomes

$$S_{xx}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(\tau_1,\tau_2)d\xi_1 d\xi_2\right] e^{-j\left[\omega_1(\tau_1+\xi_1)-\omega_2(\tau_2+\xi_2)\right]}d\tau_1 d\tau_2 \qquad (2.41)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_0^{\infty} h(\xi_1)e^{-j\omega_1\xi_1}d\xi_1\right]\left[\int_0^{\infty} h(\xi_2)e^{j\omega_2\xi_2}d\xi_2\right] R_{zz}(\tau_1,\tau_2)e^{-j(\omega_1\tau_1-\omega_2\tau_2)}d\tau_1 d\tau_2 \qquad (2.42)$$

Defining the receptance of the system [7],

$$H(j\omega) = \int_0^{\infty} h(\xi) e^{-j\omega\xi} d\xi \tag{2.43}$$

Eq.(2.42) takes the form

$$S_{xx}(\omega_1,\omega_2) = H(j\omega_1)H^*(j\omega_2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{zz}(\tau_1,\tau_2)e^{-j(\omega_1\tau_1-\omega_2\tau_2)} d\tau_1 d\tau_2$$

$$= H(j\omega_1)H^*(j\omega_2)S_{zz}(\omega_1,\omega_2) \tag{2.44}$$

where

$$S_{zz}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{zz}(\tau_1,\tau_2)e^{-j(\omega_1\tau_1-\omega_2\tau_2)} d\tau_1 d\tau_2 \tag{2.45}$$

is the power spectral density of the input excitation Z(t).

<u>Case of Stationary Z(t)</u>

If the input excitation Z(t) is stationary [8],

$$\left.\begin{aligned} R_{zz}(\tau_1,\tau_2) &= R_{zz}(\tau_2-\tau_1) = R_{zz}(\tau) \\ S_{zz}(\omega_1,\omega_2) &= S_{zz}(\omega_1)\delta(\omega_1-\omega_2) \end{aligned}\right\} \tag{2.46}$$

where $\delta(.)$ is the Dirac delta function.

Thus, Eq.(2.44) becomes

$$S_{xx}(\omega_1,\omega_2) = H(j\omega_1)H^*(j\omega_2)S_{zz}(\omega_1)\delta(\omega_1-\omega_2) \tag{2.47}$$

or,

$$S_{xx}(\omega) = S_{zz}(\omega)\left|H(j\omega)\right|^2 \tag{2.48}$$

2.8 Probability Density of the Response X(t)

For linear systems with Gaussian input, the probability density of the output is also Gaussian. In such cases, the probability density of the output X(t) is given by [8]

$$p[X(t)] = \frac{1}{\sigma_x(t)(2\pi)^{\frac{1}{2}}} \exp\left[\frac{-\left[X(t)-E\{X(t)\}\right]^2}{2\sigma_x^2(t)}\right] \tag{2.49}$$

where $\sigma_x^2(t)$ and $E\{X(t)\}$ are the variance and the mean value of the output X(t) respectively.

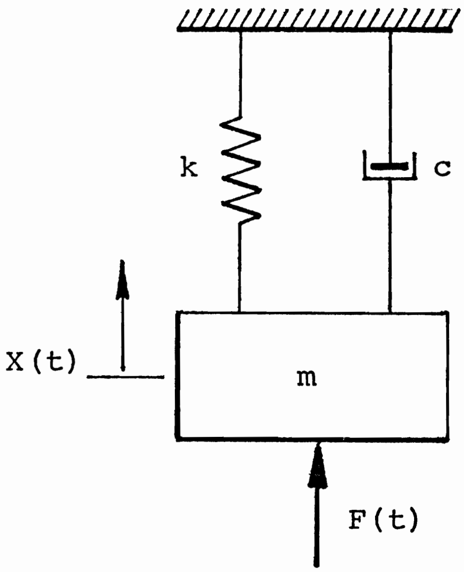

Fig. 2.1. One-Degree-Of-Freedom System

CHAPTER 3

RESPONSE STATISTICS OF ONE-DEGREE-OF-FREEDOM MECHANICAL SYSTEMS TO NONSTATIONARY FORCES WITH DIFFERENT STRENGTH FUNCTIONS

3.1 Preliminaries

The theoretical derivations of chapter 2 for generalized random inputs are used here to obtain response of one-degree-of-freedom linear system under different types of nonstationary forces. The type of excitation on the system is specified in terms of the correlation function. Certain typical correlation functions are considered and the responses of the system are evaluated using the input-output relation of chapter 2.

Generally, a nonstationary excitation $Z(t)$ is expressed in terms of a modulating function $Z_1(t)$ and a stationary excitation $F(t)$ of zero mean in the form

$$Z(t) = Z_1(t)F(t) \tag{3.1}$$

Such a definition will yield an autocorrelation function of the form

$$\begin{aligned} R_{zz}(t_1,t_2) &= E\{Z_1(t_1)Z_1(t_2)F(t_1)F(t_2)\} \\ &= I(t_1)\delta(t_1-t_2) \end{aligned} \tag{3.2}$$

where the excitation component $F(t)$ is assumed to be a stationary white noise process, $\delta(.)$is a Dirac delta function. $I(t_1)$ is then known as the strength function [3] of the autocorrelation of the excitation force. When $I(t_1)$ is a constant, $Z(t)$ is the well-known stationary white noise excitation; on the other hand, if $I(t_1)$ is a function of time, $Z(t)$ yields a nonstationary white noise process [3]. In this chapter, different strength functions of the autocorrelation of the input excitation are considered and the corresponding response statistics for the system are worked out.

Three important types of strength function are considered. They are : (i) harmonic function with exponential decay, (ii) pure exponential decay, and (iii) simple linear decay with a constant value. In types (i) and (ii), the decay rate is specified by a parameter α. Mathematically the three types of strength function may be expressed in the following form

$$1.\quad I(t_1) = Ae^{-\alpha p|t_1|}\cos pt_1 \tag{3.3a}$$

$$2.\quad I(t_1) = Ae^{-\alpha|t_1|} \tag{3.3b}$$

$$3.\quad I(t_1) = A\left(1-\frac{|t_1|}{T}\right) \tag{3.3c}$$

These strength functions are illustrated in Figs. 3.1, 3.2 and 3.3 respectively.

3.2 Physical Significance of the Strength Functions

Since the autocorrelation of the input process is defined by a delta function with a given strength $I(t_1)$, it is difficult to directly understand the physical nature of the excitation. It is then desirable to present the characteristics of the input in terms of the spectral density. The power spectral densities of the nonstationary excitations represented by strength functions in Eqs. (3.3a), (3.3b) and (3.3c) are calculated using Fourier transform relation (2.38)

<u>For $I(t_1) = Ae^{-\alpha p|t_1|}\cos pt_1$</u>

$$S_{zz}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ae^{-\alpha p|t_1|}\cos pt_1\,\delta(t_1-t_2)\,e^{-j(\omega_1 t_1-\omega_2 t_2)}dt_1 dt_2$$

$$= A\int_{-\infty}^{\infty} e^{-\alpha p|t_2|}\cos pt_2\,e^{-j(\omega_1-\omega_2)t_2}dt_2 \qquad (3.4)$$

Considering only the real part of Eq.(3.4) for physical interpretation,

$$S_{zz}(\omega_1,\omega_2) = A\int_{-\infty}^{\infty} e^{-\alpha p|t_2|}\cos pt_2\cdot\cos(\omega_1-\omega_2)t_2 dt_2$$

$$= \lim_{t\to\infty} A\int_{-t}^{t} e^{-\alpha p|t_2|}\cos pt_2\cdot\cos(\omega_1-\omega_2)t_2 dt_2$$

(3.5)

$$= \lim_{t\to\infty} A \left[\frac{\alpha p}{(\alpha p)^2+[p-(\omega_1-\omega_2)]^2} + \frac{\alpha p}{(\alpha p)^2+[p+(\omega_1-\omega_2)]^2} \right.$$

$$+ \frac{e^{-\alpha pt}\ [p-(\omega_1-\omega_2)]\sin[p-(\omega_1-\omega_2)]t - \alpha p\cos[p-(\omega_1-\omega_2)]t}{(\alpha p)^2+[p-(\omega_1-\omega_2)]^2}$$

$$\left. + \frac{e^{-\alpha pt}\ [p+(\omega_1-\omega_2)]\sin[p+(\omega_1-\omega_2)]t - \alpha p\cos[p+(\omega_1-\omega_2)]t}{(\alpha p)^2+[p+(\omega_1-\omega_2)]^2} \right] \tag{3.6}$$

The expression within brackets is plotted as a function of $(\omega_1-\omega_2)t$ in Figs. 3.4 and 3.5 for different values of the parameters α and pt. Data for these curves are taken from computer Program 0.A in the Appendix. The curves presented in these figures show that the generalized power spectral density of the input process has a maximum value at t=0 and decays to small values in an oscillatory fashion as t increases.

For $I(t_1)= Ae^{-\alpha|t_1|}$

Employing the same procedure for this case, the real part of power spectral density of the input process Z(t) is

$$S_{zz}(\omega_1,\omega_2) = \int_{-\infty}^{\infty} Ae^{-\alpha|t_2|}\cos(\omega_1-\omega_2)t_2dt_2$$

or,

$$S_{zz}(\omega_1,\omega_2) = \lim_{t\to\infty} A \int_{-t}^{t} e^{-\alpha|t_2|} \cos(\omega_1-\omega_2)t_2 dt_2 \tag{3.7}$$

$$= \lim_{t\to\infty} 2A \left[\frac{\alpha - e^{-\alpha t}[\alpha\cos(\omega_1-\omega_2)t - (\omega_1-\omega_2)\sin(\omega_1-\omega_2)t]}{\alpha^2+(\omega_1-\omega_2)^2} \right] \tag{3.8}$$

and this is illustrated in Fig. 3.6 using Program 0.B in Appendix. Here, the value of the generalized power spectral density $S_{zz}(\omega_1,\omega_2)$ is plotted against $(\omega_1-\omega_2)t$ for values of α varying from 0 to 3.

When $\alpha=0$, Eq.(3.8) takes the form

$$S_{zz}(\omega_1,\omega_2) = \lim_{t\to\infty} 2A \left[\frac{\sin(\omega_1-\omega_2)t}{\omega_1-\omega_2} \right]$$

$$= 2A.\delta(\omega_1-\omega_2)\pi \tag{3.9}$$

where $\delta(.)$ is Dirac delta function. This result is same as the one obtained by Roberts [3].

<u>For $I(t_1) = A(1 - \frac{|t_1|}{T})$</u>

In the same maner, the real part of the power spectral density of the input process for this case is

$$S_{zz}(\omega_1,\omega_2) = \int_{-\infty}^{\infty} A(1- \frac{|t_2|}{T}) \cos(\omega_1-\omega_2)t_2 dt_2$$

or,

$$S_{zz}(\omega_1,\omega_2) = \lim_{t\to\infty} A \int_{-t}^{t} (1- \frac{|t_2|}{T}) \cos(\omega_1-\omega_2)t_2 dt_2 \tag{3.10}$$

$$= \lim_{t\to\infty} 2A \left[\frac{\sin(\omega_1-\omega_2)t}{\omega_1-\omega_2} + \frac{1-[\cos(\omega_1-\omega_2)t + (\omega_1-\omega_2)\sin(\omega_1-\omega_2)t]}{(\omega_1-\omega_2)^2 T} \right] \tag{3.11}$$

Taking $\frac{T}{t} = a$, the generalized power spectral density in Eq.(3.11) is plotted against $(\omega_1-\omega_2)t$ for different values of the parameter a in Fig. 3.7. Data for these curves is obtained from Program 0.C in Appendix. As $T\to\infty$, the second term on right hand side of Eq.(3.11) vanishes and the input power spectral density takes the form of Eq.(3.9).

3.3 System Response Under Harmonically Varying Strength Function With an Exponential Decay

In this case, the autocorrelation of the input Z(t) has the form

$$R_{zz}(t_1,t_2) = Ae^{-\alpha p|t_1|} \cos pt_1 \delta(t_1-t_2) \tag{3.12}$$

where A, α, p are positive constants. For infinite operating time systems, the autocorrelation of the response X(t) is given by Eq.(2.29) as

$$R_{xx}(t_1,t_2) = \int_0^{\infty}\int_0^{\infty} h(\xi_1)h(\xi_2)R_{zz}(t_1-\xi_1,t_2-\xi_2)d\xi_1 d\xi_2 \qquad (3.13)$$

Substituting in Eq.(3.13) for $h(\xi)$ from Eq.(2.7), $R_{zz}(t_1,t_2)$ from Eq.(3.12), and integrating, the autocorrelation of the output $X(t)$ of the system may be obtained as

$$R_{xx}(t_1,t_2) = \frac{Ae^{-\zeta\omega_n|t_2-t_1|-\alpha p|t_1|}}{2\omega_d^2}\Bigg[\frac{2a.(2\zeta\omega_n-\alpha p)(2\omega_d)p + 2b.(2\omega_d)[(2\zeta\omega_n-\alpha p)^2+(2\omega_d)^2+p^2]}{[(2\zeta\omega_n-\alpha p)^2+(2\omega_d-p)^2][(2\zeta\omega_n-\alpha p)^2+(2\omega_d+p)^2]}$$

$$-\frac{c.p[(2\zeta\omega_n-\alpha p)^2+p^2-(2\omega_d)^2]}{[(2\zeta\omega_n-\alpha p)^2+(2\omega_d-p)^2][(2\zeta\omega_n-\alpha p)^2+(2\omega_d+p)^2]}$$

$$-\frac{d.(2\zeta\omega_n-\alpha p)[(2\zeta\omega_n-\alpha p)^2+p^2+(2\omega_d)^2]}{[(2\zeta\omega_n-\alpha p)^2+(2\omega_d-p)^2][(2\zeta\omega_n-\alpha p)^2+(2\omega_d+p)^2]}$$

$$+\frac{c.p + d.(2\zeta\omega_n-\alpha p)}{[(2\zeta\omega_n-\alpha p)^2+p^2]}\Bigg] \qquad (3.14)$$

where,

$$\left.\begin{aligned} a &= \sin\omega_d(t_2-t_1)\sin pt_1 \\ b &= \sin\omega_d(t_2-t_1)\cos pt_1 \\ c &= \cos\omega_d(t_2-t_1)\sin pt_1 \\ d &= \cos\omega_d(t_2-t_1)\cos pt_1 \end{aligned}\right\} \qquad (3.15)$$

The Mean Square Value of the Response X(t)

The mean square value of X(t) may be obtained by setting $t_1 = t_2 = t$ in Eq.(3.14),

$$E\{X^2(t)\} = R_{xx}(t,t)$$

$$= \frac{Ae^{-\alpha p|t|}}{2\omega_n^3\sqrt{1-\zeta^2}}\left\{\left[\frac{p'}{(2\zeta-\alpha p')^2+p'^2}\right.\right.$$

$$\left.-\frac{p'[(2\zeta-\alpha p')^2+p'^2-4(1-\zeta^2)]}{[(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}-p')^2][(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}+p')^2]}\right]\text{sinpt}$$

$$+\left[\frac{(2\zeta-\alpha p')}{(2\zeta-\alpha p')^2+p'^2}\right.$$

$$\left.\left.-\frac{(2\zeta-\alpha p')[(2\zeta-\alpha p')^2+p'^2+4(1-\zeta^2)]}{[(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}-p')^2][(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}+p')^2]}\right]\text{cospt}\right\}$$

(3.16)

where $p' = \dfrac{p}{\omega_n}$.

Recognizing Eq.(3.16) in the form

$$E\{X^2(t)\} = \frac{Ae^{-\alpha p|t|}}{\omega_n^3}(P\ \text{sinpt} + Q\ \text{cospt}) \tag{3.17}$$

where

$$P = \frac{1}{2\sqrt{1-\zeta^2}}\left[\frac{p'}{(2\zeta-\alpha p')^2+p'^2}\right.$$

$$\left. - \frac{p'[(2\zeta-\alpha p')^2+p'^2-4(1-\zeta^2)]}{[(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}-p')^2][(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}+p')^2]}\right] \tag{3.18a}$$

$$Q = \frac{1}{2\sqrt{1-\zeta^2}}\left[\frac{(2\zeta-\alpha p')}{(2\zeta-\alpha p')^2+p'^2}\right.$$

$$\left. - \frac{(2\zeta-\alpha p')[(2\zeta-\alpha p')^2+p'^2+4(1-\zeta^2)]}{[(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}-p')^2][(2\zeta-\alpha p')^2+(2\sqrt{1-\zeta^2}+p')^2]}\right] \tag{3.18b}$$

Alternately,

$$E\{X^2(t)\} = \frac{Ae^{-\alpha p|\dot{t}|}}{\omega_n^3} R\cos(p't+\Phi) \tag{3.19}$$

where

$$R = [\,P^2+Q^2]^{\frac{1}{2}} \tag{3.20a}$$

$$\Phi = \tan^{-1}(\frac{P}{Q}) \tag{3.20b}$$

R and Φ may be considered as the maximum amplitude and phase angle components of the mean square response of the system.

For a given value of α, the quantities R and Φ of the mean square value of the response X(t) are computed as function of frequency ratio p/ω_n and damping ratio ζ using Program 1 in Appendix. Computed results for R and Φ are plotted against p/ω_n for different values of ζ . Figs. 3.8 and 3.9 give the response for α=0.2, and Figs. 3.10 and 3.11 show the response when α=1.0.

From Fig. 3.8, it may be noted that large values for the maximum mean square amplitude of the system occur in the regions $\frac{p}{\omega_n} \simeq 0$ and $\frac{p}{\omega_n} \simeq 2.0$. The peaks are more pronounced for small damping ratios, especially when ζ is less than 0.4. For certain damping ratios, large peak amplitudes are noted for very low values of p/ω_n with comparatively smaller peak amplitudes in the region of $p \simeq 2\omega_n$. The opposite occurs for certain other damping ratios as can be seen from the figure. From this, it may be concluded that there are two distinct "resonance regions" of which one is more critical. For high damping ratios ($\zeta \geq 0.4$), the maximum amplitudes of the mean square response are subdued for all values of p/ω_n. It is also important to note that for $p/\omega_n > 3.0$, all the amplitudes approach to zero value for all ζ.

Fig. 3.9 shows the variation of phase angle Φ in function of frequency ratio p/ω_n for different values of ζ. It may be noted from this figure that the mean square amplitude is in phase with the strength function for p = 0, and all the curves pass through $\Phi = \frac{\pi}{2}$ in the region $p \simeq \omega_n$.

For low damping ratios, the phase angles asymptotically approach $-\pi/2$, whereas for slighly higher ζ, they reach asymptotically $3\pi/2$ when $p \gg \omega_n$.

Similar plots for $\alpha=1.0$ are given in Figs. 3.10 and 3.11. The behaviour of the system is essentially the same as for $\alpha=0.2$ except that the peak amplitudes are relatively closely spaced. Also, unlike the plots for $\alpha=0.2$, there is only one dominant peak or "resonance region" in this case for every ζ value.

The "resonance frequencies" corresponding to the peak amplitudes of $E\{X^2(t)\}$ shown in Figs. 3.8 and 3.10 may be exactly determined by differentiating Eq. (3.20a) with respect to p' and setting the resulting expression to zero. The values of these "resonance frequencies" are evaluated using Newton-Raphson numerical method (Program 1.A in Appendix) for the value of the decay parameter α equal to 0.2 and 1.0. The results are presented in Tables 3.1 and 3.2. Using these results, the locus of the "resonance amplitudes" of $E\{X^2(t)\}$ are indicated in Figs. 3.8 and 3.10 by dotted lines.

It may also be interesting to know how the maximum amplitude of the mean square response of X(t) varies with the decay parameter α of the strength function for a given system with a fixed damping ζ. Figs. 3.12, 3.13 and 3.14 show this variation with respect to the frequency ratio p/ω_n for a range of values of α when $\zeta=.05$, $\zeta=.25$ and $\zeta=.45$ respectively. Data for these curves are taken from Program 1 in Appendix.

From these figures, it may be seen that all the curves start at the same point when $p/\omega_n = 0$. This means that the maximum mean square response of the system is a constant regardless of the values of the decay rate α when the correlation frequency of the excitation is zero. The "resonance regions" corresponding to peak values of $E\{X^2(t)\}$ are influenced by both values of the decay rate α and the damping ratio ζ. The peaks are also shifted to the each other as the value of each parameter increases.

3.3.1 Special Case 1 : $\alpha = 0$

When $\alpha = 0$, the input autocorrelation $R_{zz}(t_1,t_2)$ has harmonically varying strength function $I(t_1)$,

$$R_{zz}(t_1,t_2) = A \cos pt_1 \delta(t_1-t_2) \tag{3.21}$$

as shown in Fig. 3.15.

Setting $\alpha = 0$ in Eq.(3.14), the autocorrelation of the response $X(t)$ becomes

$$R_{xx}(t_1,t_2) = \frac{Ae^{-\zeta\omega_n\left|t_2-t_1\right|}}{2\omega_d^2}\Bigg[$$

$$\frac{2a(2\zeta\omega_n)(2\omega_d)p + 2b(2\omega_d)[(2\zeta\omega_n)^2+(2\omega_d)^2+p^2]}{[(2\zeta\omega_n)^2+(2\omega_d-p)^2][(2\zeta\omega_n)^2+(2\omega_d+p)^2]} - \cdots$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots \quad - \frac{cp[(2\zeta\omega_n)^2+p^2-(2\omega_d)^2]}{[(2\zeta\omega_n)^2+(2\omega_d-p)^2][(2\zeta\omega_n)^2+(2\omega_d+p)^2]}$$

$$- \frac{d(2\zeta\omega_n)[(2\zeta\omega_n)^2+p^2+(2\omega_d)^2]}{[(2\zeta\omega_n)^2+(2\omega_d-p)^2][(2\zeta\omega_n)^2+(2\omega_d+p)^2]}$$

$$+ \left. \frac{cp + d(2\zeta\omega_n)}{[(2\zeta\omega_n)^2+p^2]} \right] \tag{3.22}$$

where the expressions for a, b, c, and d are given in Eq.(3.15).

The mean square value of the response X(t) is now obtained either by setting $\alpha = 0$ in Eq.(3.16) or by setting $t_1 = t_2 = t$ in Eq.(3.22),

$$E\{X^2(t)\} = \frac{A}{2\omega_n^3\sqrt{1-\zeta^2}} \Bigg\{$$

$$\left[\frac{p'}{(2\zeta)^2+p'^2} - \frac{p'[(2\zeta)^2+p'^2-4(1-\zeta^2)]}{(4+p'^2)^2-(4p'\sqrt{1-\zeta^2})^2}\right] \sin pt$$

$$\left[\frac{2\zeta}{(2\zeta)^2+p'^2} - \frac{2\zeta(4+p'^2)}{(4+p'^2)^2-(4p'\sqrt{1-\zeta^2})^2}\right] \cos pt \Bigg\}$$

where $p' = p/\omega_n$.

(3.23)

This can also be written in the form

$$E\{X^2(t)\} = \frac{A}{\omega_n^3}\,[\,P_o \sin pt + Q_o \cos pt\,] \tag{3.34}$$

where

$$P_o = P\Big|_{\alpha=0} = \frac{1}{2\sqrt{1-\zeta^2}}\left[\frac{p'}{(2\zeta)^2+p'^2} - \frac{p'[(2\zeta)^2+p'^2-4(1-\zeta^2)]}{(4+p'^2)^2-(4p'\sqrt{1-\zeta^2})^2}\right] \tag{3.35a}$$

$$Q_o = Q\Big|_{\alpha=0} = \frac{1}{2\sqrt{1-\zeta^2}}\left[\frac{2\zeta}{(2\zeta)^2+p'^2} - \frac{2\zeta(4+p'^2)}{(4+p'^2)^2-(4p'\sqrt{1-\zeta^2})^2}\right] \tag{3.35b}$$

And alternately,

$$E\{X^2(t)\} = \frac{A}{\omega_n^3} R_o \cos(p't + \Phi_o) \tag{3.36}$$

where

$$R_o = (\,P_o^2 + Q_o^2\,)^{\frac{1}{2}} \tag{3.37a}$$

$$\Phi_o = \tan^{-1}\left(\frac{P_o}{Q_o}\right) \tag{3.37b}$$

The quantities R_o and Φ_o are computed using Program 1 in Appendix and plotted in Figs. 3.17 and 3.18 against p/ω_n for a range of values of damping ratio ζ. The results for this particular case are identical to to those obtained by Roberts [3,4].

The basic characteristics of the plots in Figs. 3.17 and 3.18 are quite similar to those in Figs. 3.8 to 3.11. Since $\alpha = 0$ in this case, the peak amplitudes are more pronounced and reach almost infinite values at $\frac{p}{\omega_n} \simeq 0$ and $\frac{p}{\omega_n} \simeq 2$ for small values of ζ. The interval between two peaks has maximum value when ζ is small. For $p/\omega_n > 3.0$ the amplitudes asymptotically approach zero. The phase angles start at zero value for all damping ratio ζ, pass through $\pi/2$ when $p = \omega_n$, and asymptotically reach $3\pi/2$ for large values of p/ω_n.

The values of the frequency ratio p/ω_n for which $E\{X^2(t)\}$ is a maximum are determined by differentiating Eq.(3.37a) with respect to p/ω_n and setting the resulting expression to zero. The results are presented in Table 3.3 using Program 1.A in Appendix. From this data, the locus of the maximum mean square response of the system may be indicated as shown by dotted lines in Fig. 3.17.

3.3.2 Special Case 2 : p = 0

When $p = 0$, the autocorrelation of the input $Z(t)$ in Eq.(3.12) becomes

$$R_{zz}(t_1,t_2) = A\,\delta(t_1-t_2) \tag{3.38}$$

or,

$$R_{zz}(\tau) = A\ \delta(\tau) \tag{3.39}$$

where $\tau = \left|t_1 - t_2\right|$, and the input process $Z(t)$ in this case becomes a stationary white noise process. The strength function $I(t_1)$ is a constant as shown in Fig. 3.16.

From Eq.(3.14), when $p = 0$, the autocorrelation of the response $X(t)$ takes the form

$$R_{xx}(t_1,t_2) = \frac{Ae^{-\zeta\omega_n\left|t_2-t_1\right|}}{4\zeta\omega_n^3}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d(t_2-t_1) + \cos\omega_d(t_2-t_1)\right] \tag{3.40}$$

Letting $\tau = \left|t_1-t_2\right|$, Eq.(3.40) becomes

$$R_{xx}(\tau) = \frac{Ae^{-\zeta\omega_n\tau}}{4\zeta\omega_n^3}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\tau + \cos\omega_d\tau\right] \tag{3.41}$$

Usually for white noise excitation, the amplitude of strength function A is written in the form $A = 2\pi S_o$, where S_o is the constant value of the spectral density of the excitation. Therefore Eq.(3.41) may be written as

$$R_{xx}(\tau) = \frac{\pi S_o e^{-\zeta\omega_n\tau}}{2\zeta\omega_n^3}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\tau + \cos\omega_d\tau\right] \tag{3.42}$$

which is the same expression as that obtained by Crandall a and Mark [2].

The mean square response of the system may be obtained by setting p = 0 in Eq.(3.16) or by setting $\tau = 0$ in the expression for the autocorrelation $R_{xx}(\tau)$ in Eq.(3.42) and is

$$E\{X^2\} = \frac{\pi}{2}\frac{S_o}{\zeta\omega_n^3} \tag{3.43}$$

This value is independant of time and is the mean square response of one-degree-of-freedom linear system subjected to a stationary white noise excitation [2].

3.4 Response Under an Exponentially Decaying Strength Function

In this case, the autocorrelation of the excitation is of exponentially decaying form,

$$R_{zz}(t_1,t_2) = A\ e^{-\alpha|t_1|}\,\delta(t_1-t_2) \tag{3.44}$$

and shown in Fig. 3.2 . The spectral characteristics of this type of excitation is given in Fig. 3.6. For infinite operating time systems, the autocorrelation of the output X(t) is given in Eq.(3.13). Substituting for $h(\xi)$ and $R_{zz}(t_1,t_2)$ from Eq.(2.7) and Eq.(3.44) into Eq.(3.13), and integrating, the autocorrelation of the response X(t) is obtained as

$$R_{xx}(t_1,t_2) = \frac{Ae^{-\zeta\omega_n|t_2-t_1|-\alpha|t_1|}}{(2\zeta\omega_n-\alpha)^2+(2\omega_d)^2}\left[\frac{1}{\omega_d}\sin\omega_d(t_2-t_1)+\frac{2}{2\zeta\omega_n-\alpha}\cos\omega_d(t_2-t_1)\right] \tag{3.45}$$

<u>The Mean Square Value of the Response X(t)</u>

The mean square value of the output $X(t)$ is obtained by setting $t_1 = t_2 = t$ in the expression for the autocorrelation $R_{xx}(t_1,t_2)$ in Eq.(3.45)

$$E\{X^2(t)\} = \frac{2Ae^{-\alpha|t|}}{\omega_n^3(2\zeta-\alpha')(\alpha'^2-4\zeta\alpha'+4)} \tag{3.46}$$

where $\alpha' = \alpha/\omega_n$.

Eq.(3.46) may also be written in the form

$$E\{X^2(t)\} = E\{X^2(0)\}e^{-\alpha|t|}. \tag{3.47}$$

where

$$E\{X^2(0)\} = \frac{2A}{\omega_n^3(2\zeta-\alpha')(\alpha'-4\zeta\alpha'+4)} \tag{3.48}$$

which is the mean square response of the system at $t = 0$.

The quantity $E\{X^2(0)\}$ in Eq.(3.48) is computed in terms of ζ and α/ω_n using Program 2 in Appendix, and plotted in Fig. 3.19 against α/ω_n for different values of $\zeta \leq 1.0$

The quantity $E\{X^2(0)\}$ has unbounded values when the denominator in Eq.(3.48) has zero value. This happens when $\alpha_1' = 2\zeta$, $\alpha_2' = 2\zeta+\sqrt{\zeta^2-1}$, and $\alpha_3' = 2\zeta-\sqrt{\zeta^2-1}$. The values of α_2' and α_3' are complex if the damping ratio is less than 1.0. Therefore, for $\zeta \leq 1.0$ i.e. for real values of α', the quantity $E\{X^2(0)\}$ will have infinite value when $\alpha' = 2\zeta$ as may be seen from Fig. 3.19. From the expression (3.48), it may also be seen that $E\{X^2(0)\}$ is positive only when $\alpha' \leq 2\zeta$. Since the mean square response of the system cannot be negative, the values of the expression (3.48) for $\alpha' > 2\zeta$ are discarded. Unlike the previous cases where for finite values of ζ, $E\{X^2(t)\}$ is always bounded and has finite magnitude, the peak values of the mean square response in the present case can be infinity, hence unbounded, even for finite values of damping ratio ζ if $\alpha' = 2\zeta$ i.e. if $\alpha = 2\zeta\omega_n = c/m$, where c is actual vicous damping of the system and m is the mass.

As a particular case, suppose $\alpha = 0$. The autocorrelation of the input process in Eq.(3.44) is exactly the same as in Eq.(3.38) giving a stationary white noise process. The mean square value of the response X(t) of the system in this case will be the same as in Eq.(3.43), namely

$$E\{X^2\} = \frac{\pi}{2} \frac{S_o}{\zeta\omega_n^3} \tag{3.49}$$

This expression can be directly obtained by setting $\alpha = 0$

and $A = 2\pi S_o$ in Eq.(3.46).

3.5 Response Under Linearly Decaying Strength Function

In this case, the autocorrelation of the input process $Z(t)$ is taken in the form

$$R_{zz}(t_1,t_2) = A\left(1 - \frac{|t_1|}{T}\right)\delta(t_1-t_2) \tag{3.50}$$

where T is a constant value of time at which $I(t_1) = 0$. This is illustrated in Fig. 3.3 and the power spectral density of the excitation is shown in Fig. 3.7.

Substituting $R_{zz}(t_1,t_2)$ from Eq.(3.50) and $h(\xi)$ from Eq.(2.7) into Eq.(3.13) and integrating, the autocorrelation of the output X(t) for an one-degree-of-freedom system with infinite operating time is given by the following expression

$$R_{xx}(t_1,t_2) = \frac{Ae^{-\zeta\omega_n|t_2-t_1|}}{4\zeta\omega_n^3}\Bigg[\frac{\zeta}{\sqrt{1-\zeta^2}}\left[1-\frac{|t_1|}{T}+\frac{\zeta}{\omega_n T}\right]\sin\omega_d(t_2-t_1)$$
$$+\left[1-\frac{|t_1|}{T}+\frac{1+2\zeta^2}{2\zeta\omega_n T}\right]\cos\omega_d(t_2-t_1)\Bigg] \tag{3.51}$$

Mean Square Value of the Response X(t)

The mean square value of the output X(t) is obtained by setting $t_1 = t_2 = t$ in the expression for the autocorrelation in Eq.(3.51)

$$E\{X^2(t)\} = \frac{A}{4\zeta\omega_n^3}\left[1 + \frac{1+2\zeta^2}{2\zeta\omega_n T} - \frac{|t|}{T}\right] \tag{3.52}$$

which is essentially a linear function of time.

By assuming $T = \frac{\tau}{n}$, where τ is the natural period of the system and n is a proportional constant, Eq.(3.52) may be written as

$$E\{X^2(t)\} = \frac{A}{4\zeta\omega_n^3}\left[1 + \frac{n(1+2\zeta^2)}{4\pi\zeta} - n(\frac{|t|}{\tau})\right] \tag{3.53}$$

For any given value of n, the mean square response of X(t) may be evaluated in terms of t/τ for different values of damping ratio ζ using Program 2.A in Appendix. Some of these results for $E\{X^2(t)\}$ are plotted against t/τ for different values of ζ. Figs. 3.20 and 3.21 give the responses when n = 0.5 and n = 1.0 respectively.

From these figures, it is seen that the mean square amplitudes of the response of the system are linearly decreased from a maximum value to zero for all values of ζ. Here also, only positive values of $E\{X^2(t)\}$ are considered. As expected, these amplitudes are restricted by an amount of the damping coefficient presented in the system with maximum values occuring at the origin.

It is noted that when $T \to \infty$, $n \to 0$, the strength function of the input autocorrelation in Eq. (3.50) becomes a constant, and the mean square value of the response X(t) in Eq. (3.53) takes the form

$$E\{X^2\} = \frac{A}{4\zeta\omega_n^3} = \frac{\pi}{2}\frac{S_o}{\zeta\omega_n^3} \tag{3.54}$$

This is the standard mean square response of the system under a stationary white noise excitation.

Table 3.1. Values of p/ω_n where $E\{X^2(t)\}$ has Peak Amplitudes

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-0.2p|t_1|}\cos pt_1$
Governing Equation : (3.20a)

Damping Ratio ζ	Values of p/ω_n	
	Region 1	Region 2
.05	.019409	1.869610
.15	.062600	1.952478
.25	.119752	1.949162
.35	.202839	1.856331
.45	.326057	1.618507
.55	.484605	-
.65	.572547	-
.75	.551039	-
.85	.488217	-
.95	.417425	-

Table 3.2. Values of p/ω_n where $E\{X^2(t)\}$ has Peak Amplitudes

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-p|t_1|}\cos pt_1$
Governing Equation : (3.20a)

Damping Ratio ζ	Values of p/ω_n	
	Region 1	Region 2
.05	.050125	-
.15	.153456	-
.25	.266836	-
.35	.401218	-
.45	.588842	-
.55	-	1.267490
.65	-	1.396239
.75	-	1.403421
.85	-	1.284383
.95	-	-

Table 3.3. Values of p/ω_n where $E\{X^2(t)\}$ has Peak Amplitudes

One-Degree-Of-Freedom System
Strength Function of Excitation = A $\cos pt_1$
Governing Equation : (3.37a)

Damping Ratio ζ	Values of p/ω_n	
	Region 1	Region 2
.05	0	1.989936
.15	0	1.904705
.25	0	1.676012
.35	0	-
.45	0	-
.55	0	-
.65	0	-
.75	0	-
.85	0	-
.95	0	-

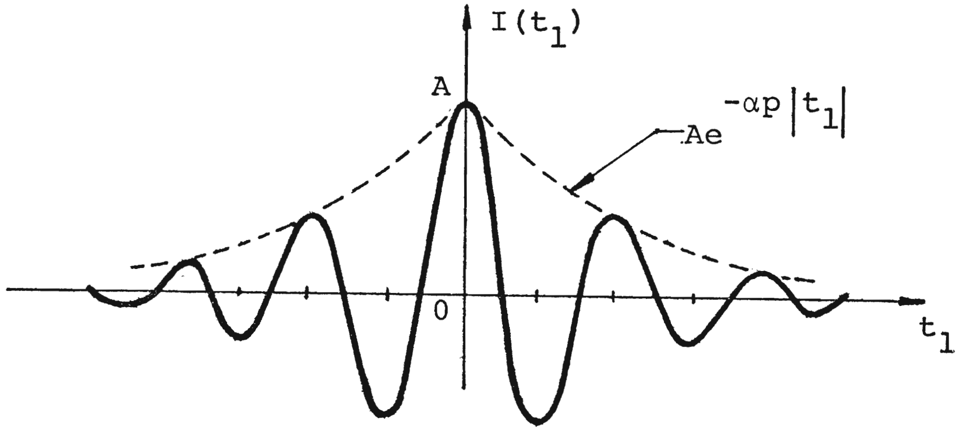

Fig. 3.1. Strength Function $I(t_1) = Ae^{-\alpha p|t_1|}\cos pt_1$

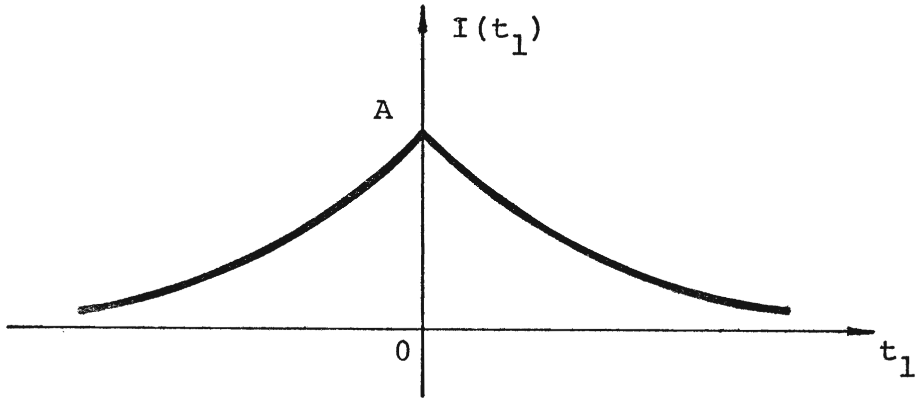

Fig. 3.2. Strength Function $I(t_1) = Ae^{-|t_1|}$

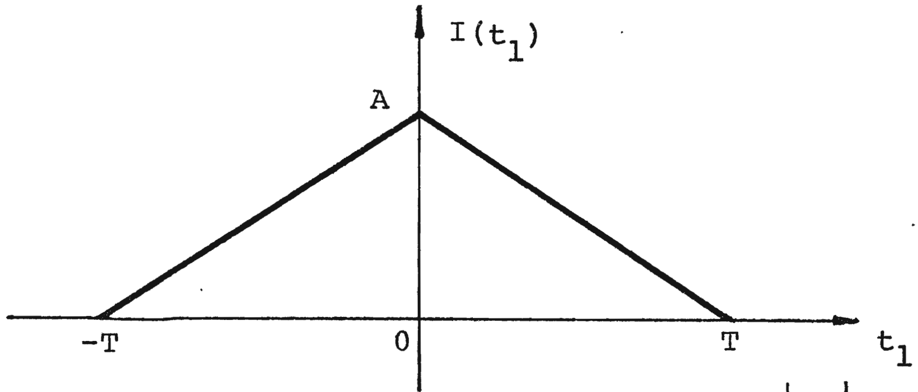

Fig. 3.3. Strength Function $I(t_1) = A(1 - \frac{|t_1|}{T})$

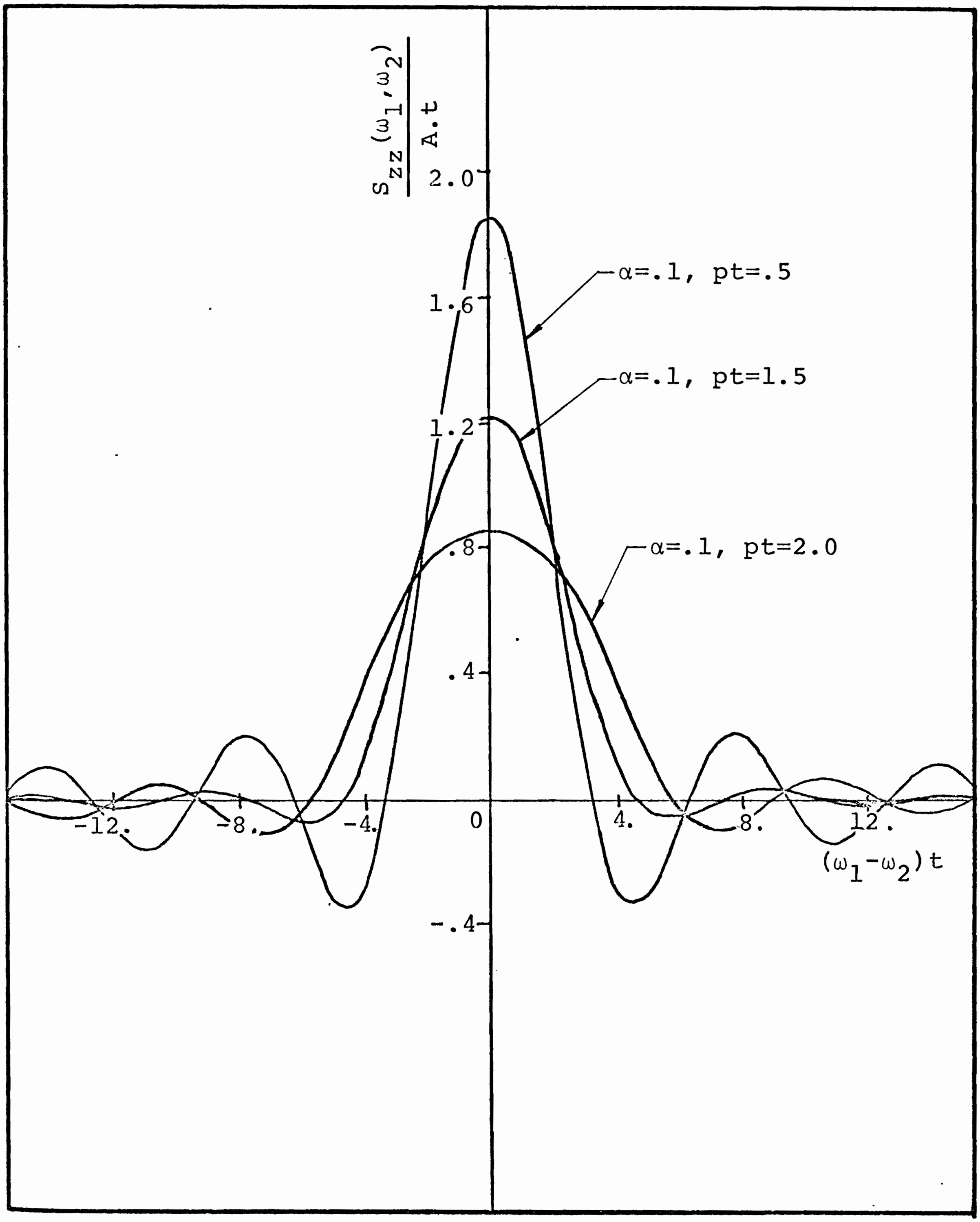

Fig. 3.4. Generalized Power Spectral Density of Excitation Z(t)

Strength Function of Excitation = $Ae^{-\alpha|t_1|}\cos pt_1$
Governing Equation : (3.6)

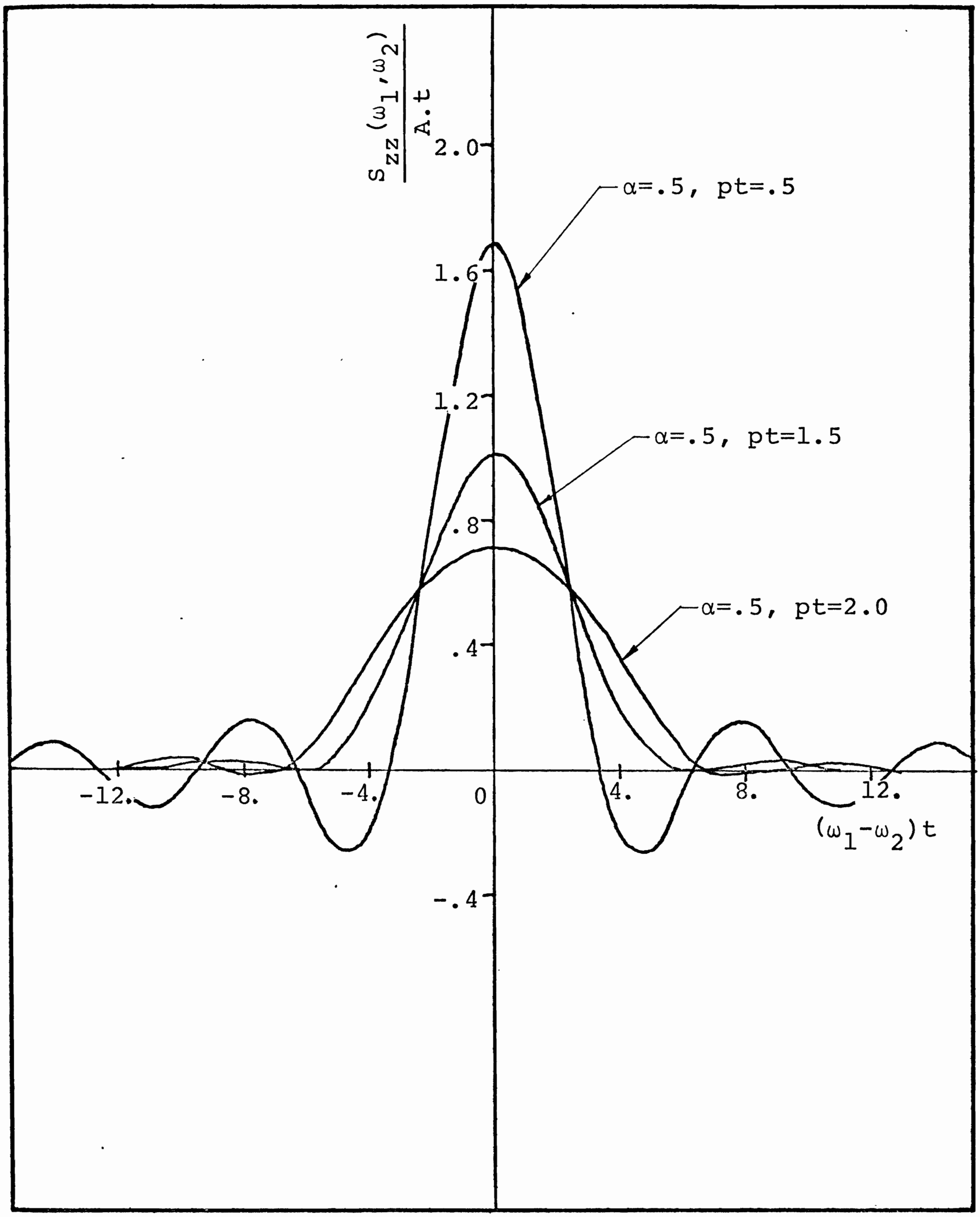

Fig. 3.5. Generalized Power Spectral Density of Excitation Z(t)

Strength Function of Excitation = $Ae^{-\alpha|t_1|}\cos pt_1$
Governing Equation : (3.6)

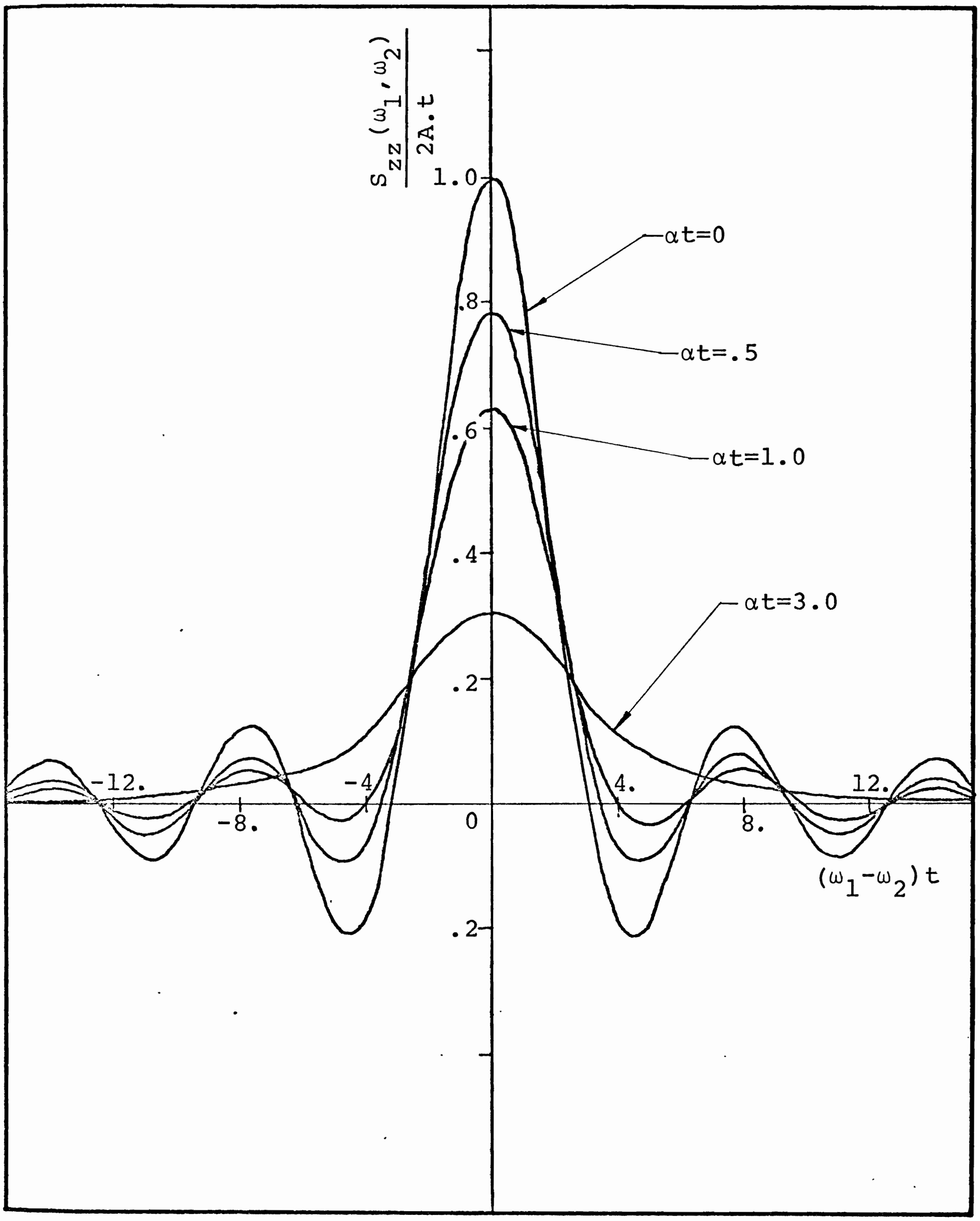

Fig. 3.6. Generalized Power Spectral Density of Excitation Z(t)

Strength Function of Excitation = $Ae^{-\alpha|t_1|}$
Governing Equation : (3.8)

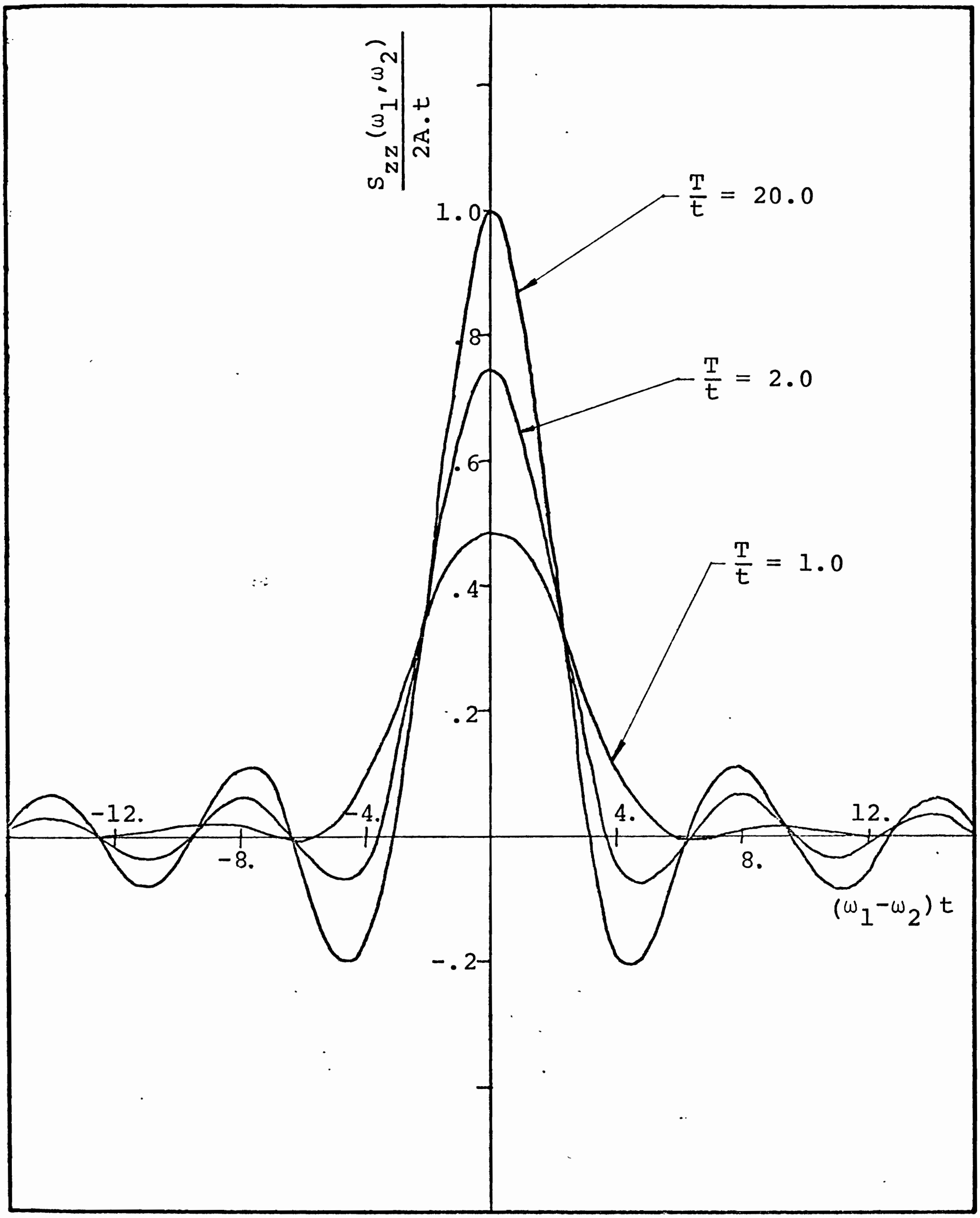

Fig. 3.7. Generalized Power Spectral Density of Excitation Z(t)

Strength Function of Excitation = $A\ (1-\frac{|t_1|}{T})$
Governing Equation : (3.11)

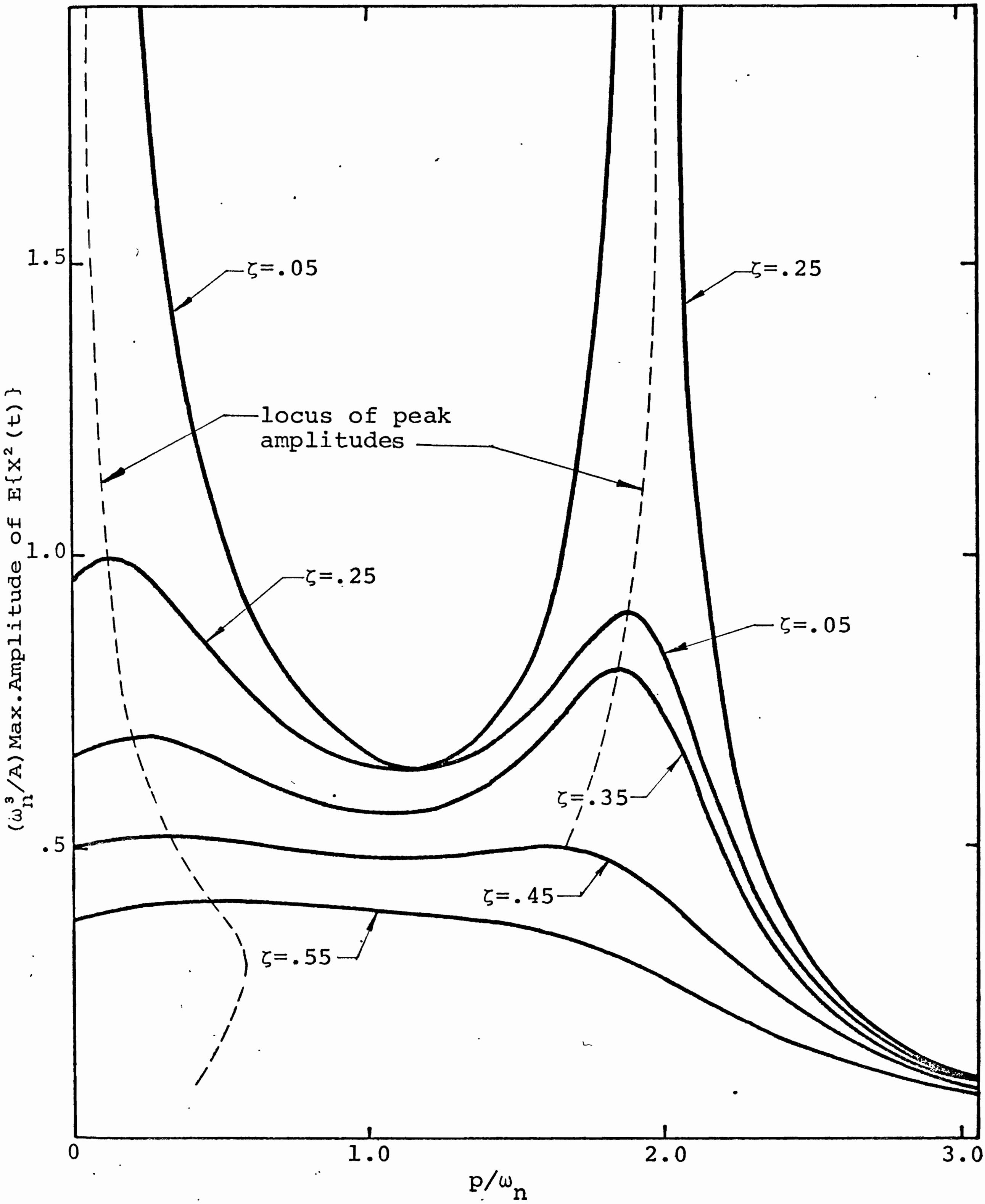

Fig. 3.8. Maximum Amplitude of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-0.2p|t_1|}\cos pt_1$
Governing Equation : (3.19)

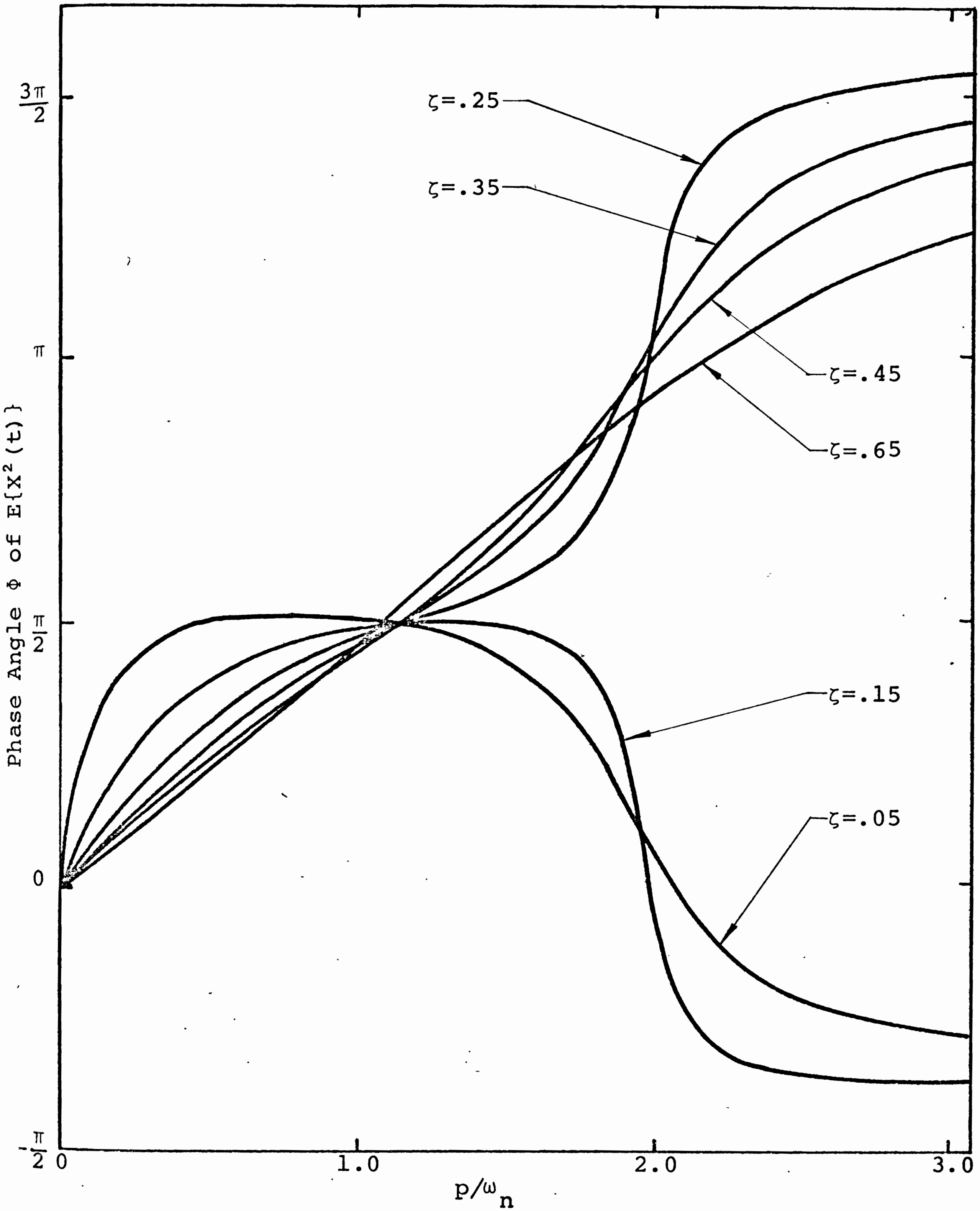

Fig. 3.9. Phase Angle of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-0.2p|t_1|}\cos pt_1$
Governing Equation : (3.19)

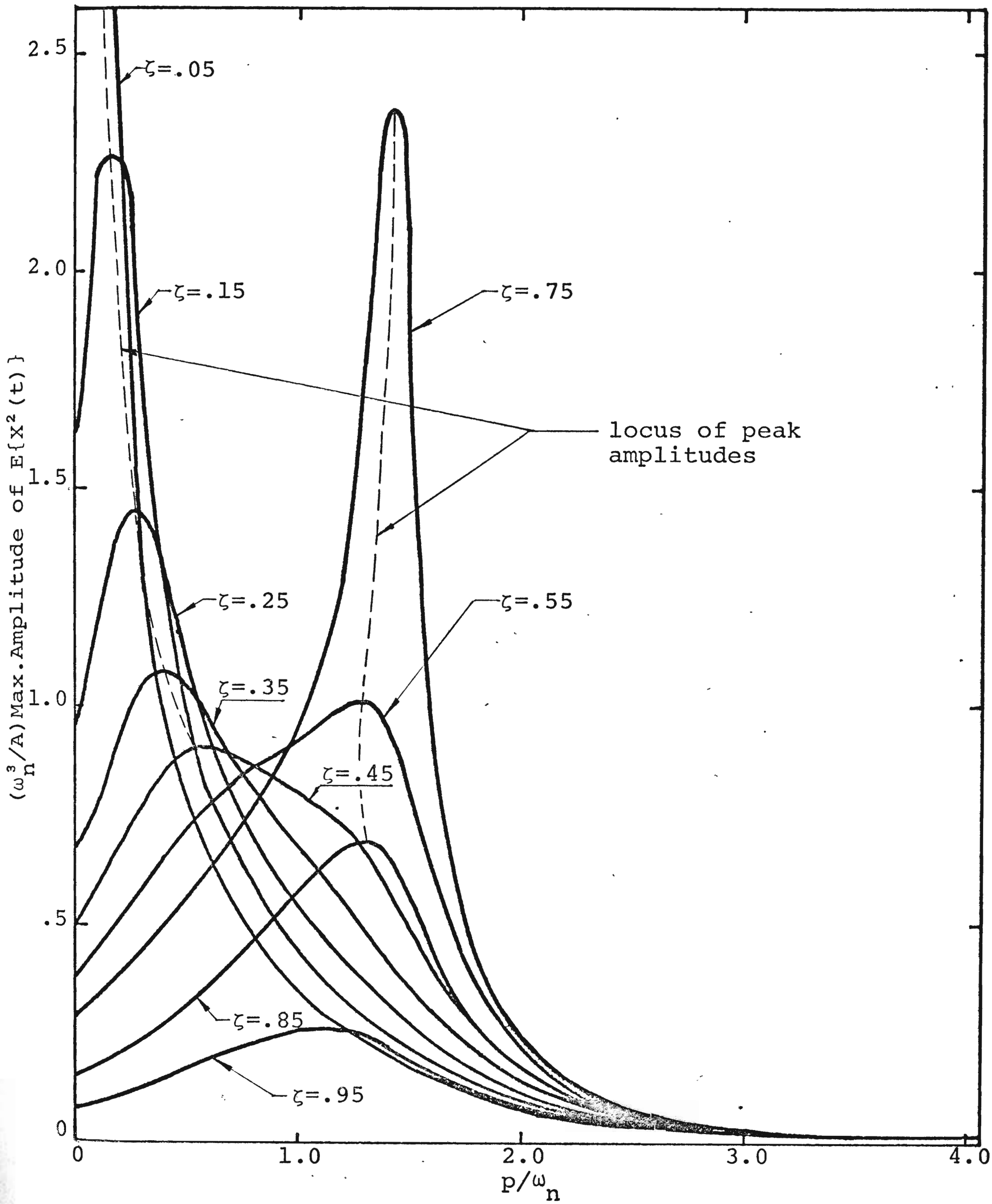

Fig. 3.10. Maximum Amplitude of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-p|t_1|}\cos pt_1$
Governing Equation : (3.19)

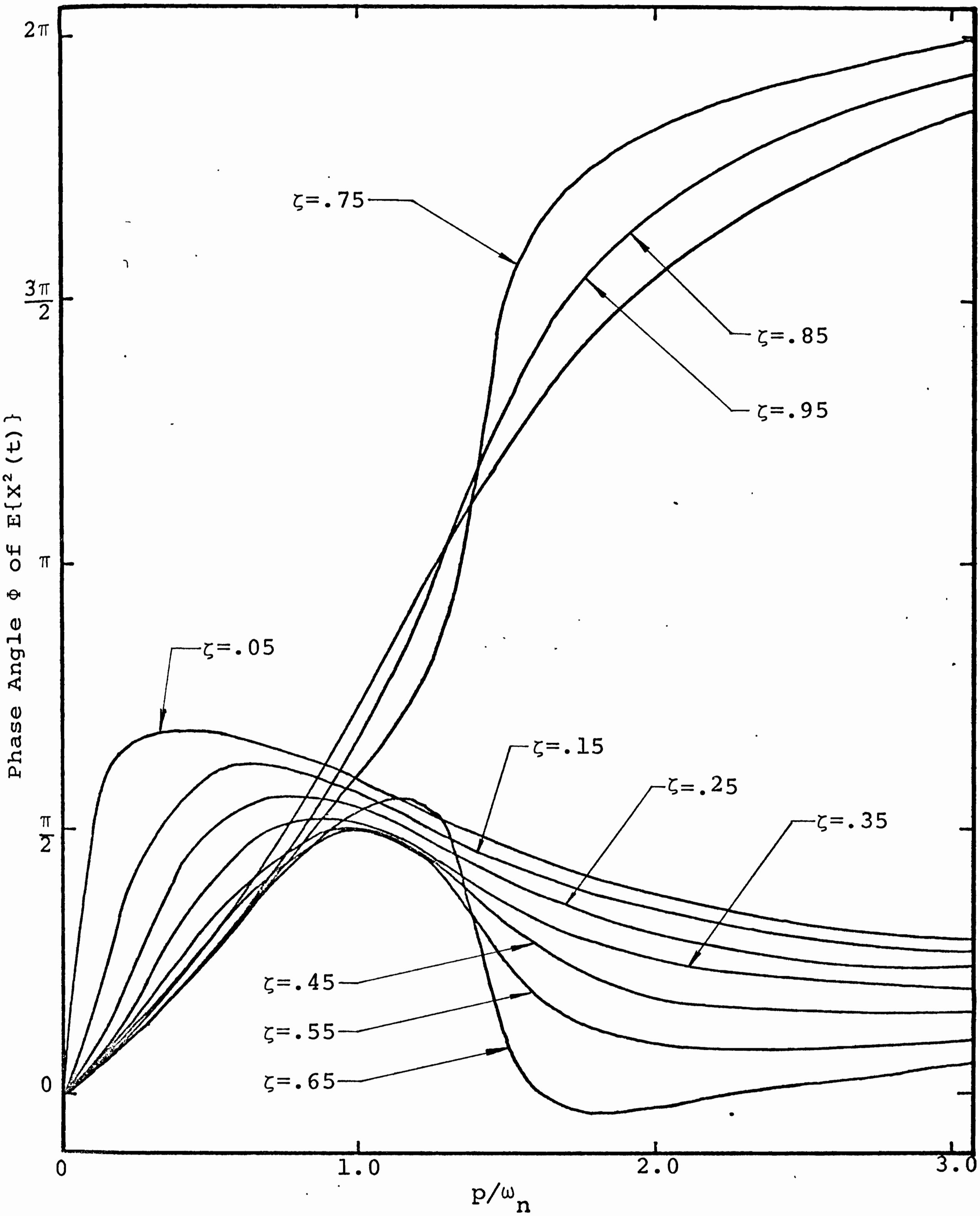

Fig. 3.11. Phase Angle of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation $= Ae^{-p|t_1|}\cos pt_1$
Governing Equation : (3.19)

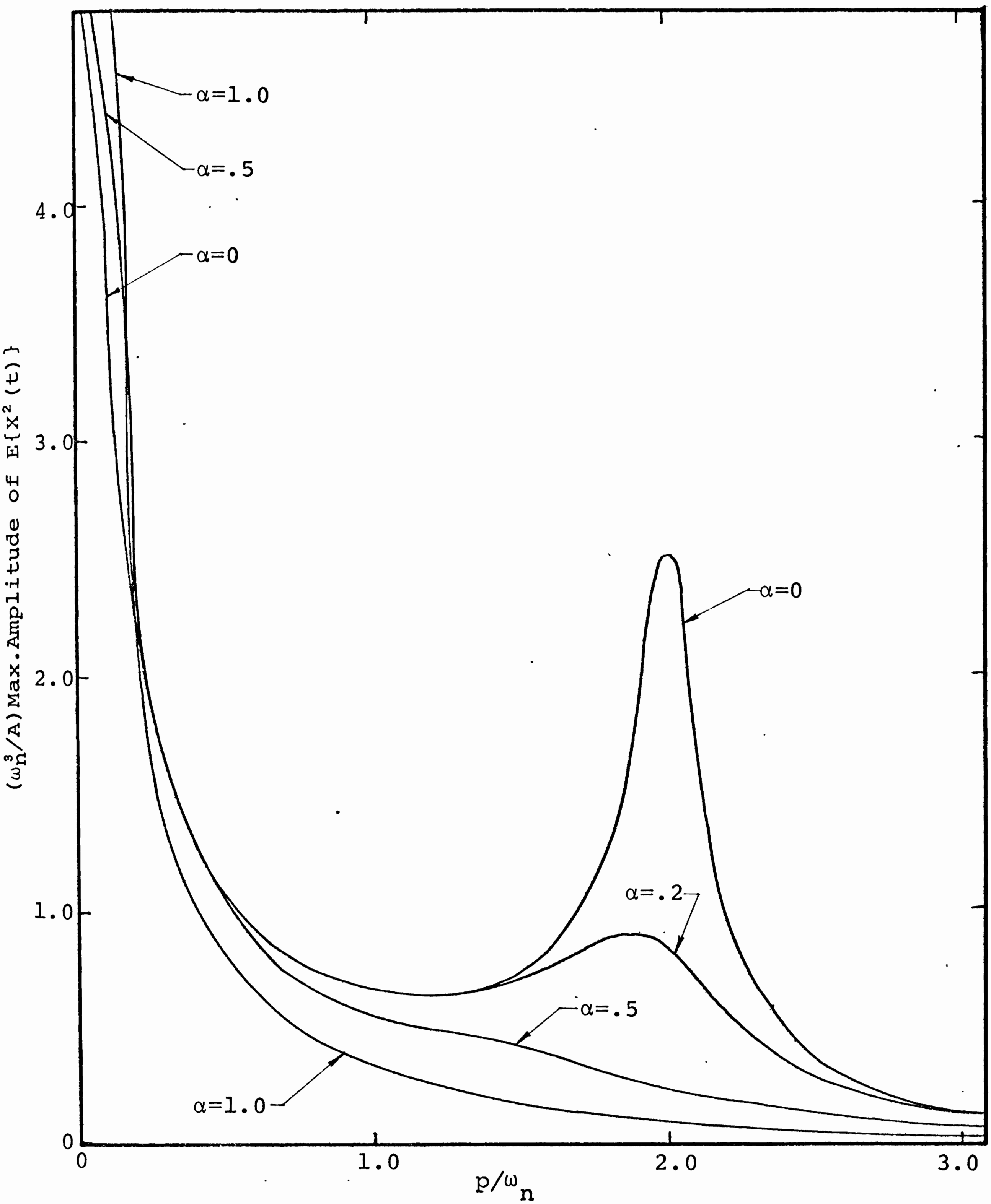

Fig. 3.12. Maximum Amplitude of $E\{X^2(t)\}$ against p/ω_n for $\zeta=.05$

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-\alpha p|t_1|}\cos pt_1$
Governing Equation : (3.19)

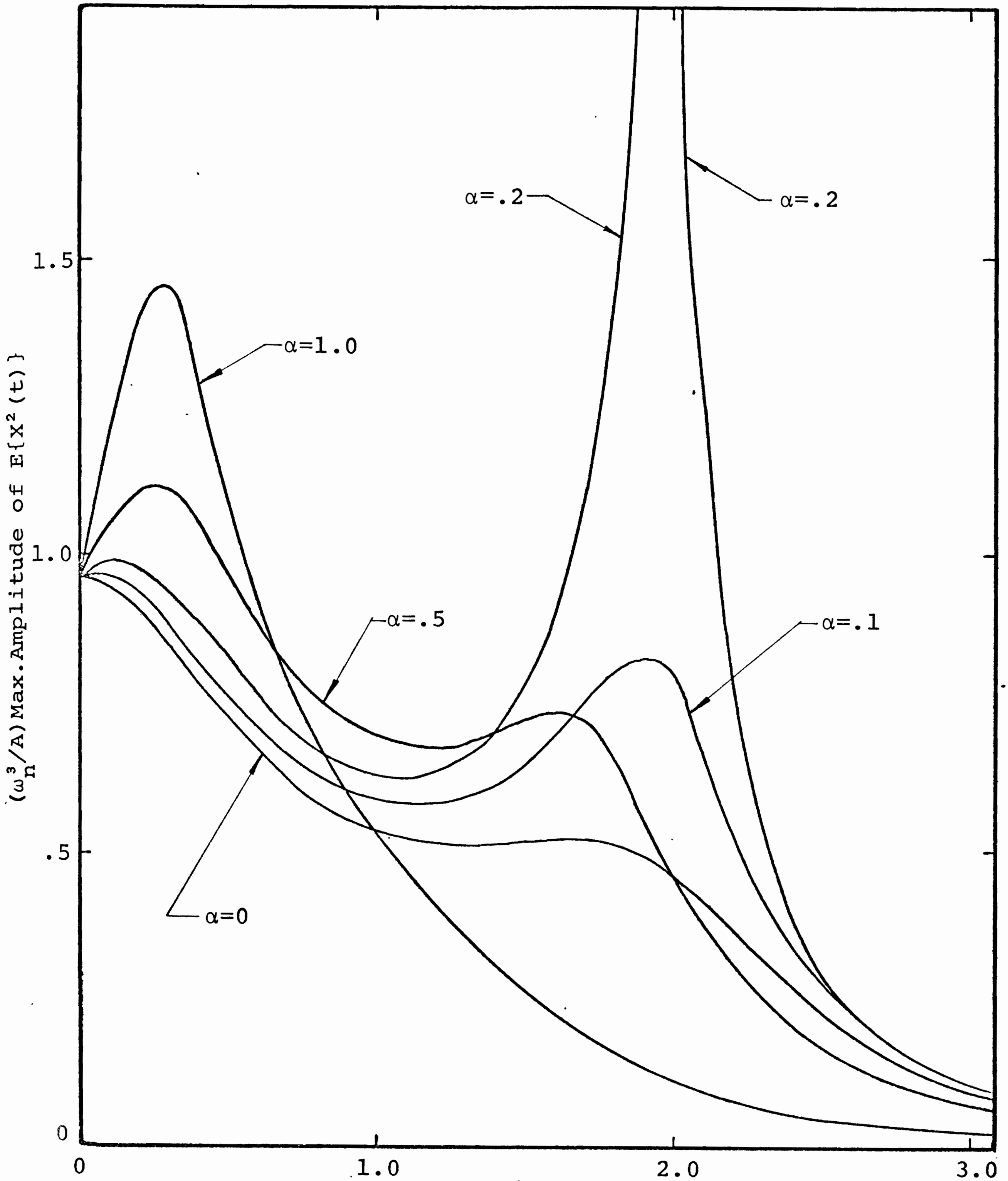

Fig. 3.13. Maximum Amplitude of $E\{X^2(t)\}$ against p/ω_n for $\zeta=.25$

One-Degree-Of-freedom System
Strength Function of Excitation = $Ae^{-\alpha p|t_1|}\cos pt_1$
Governing Equation : (3.19)

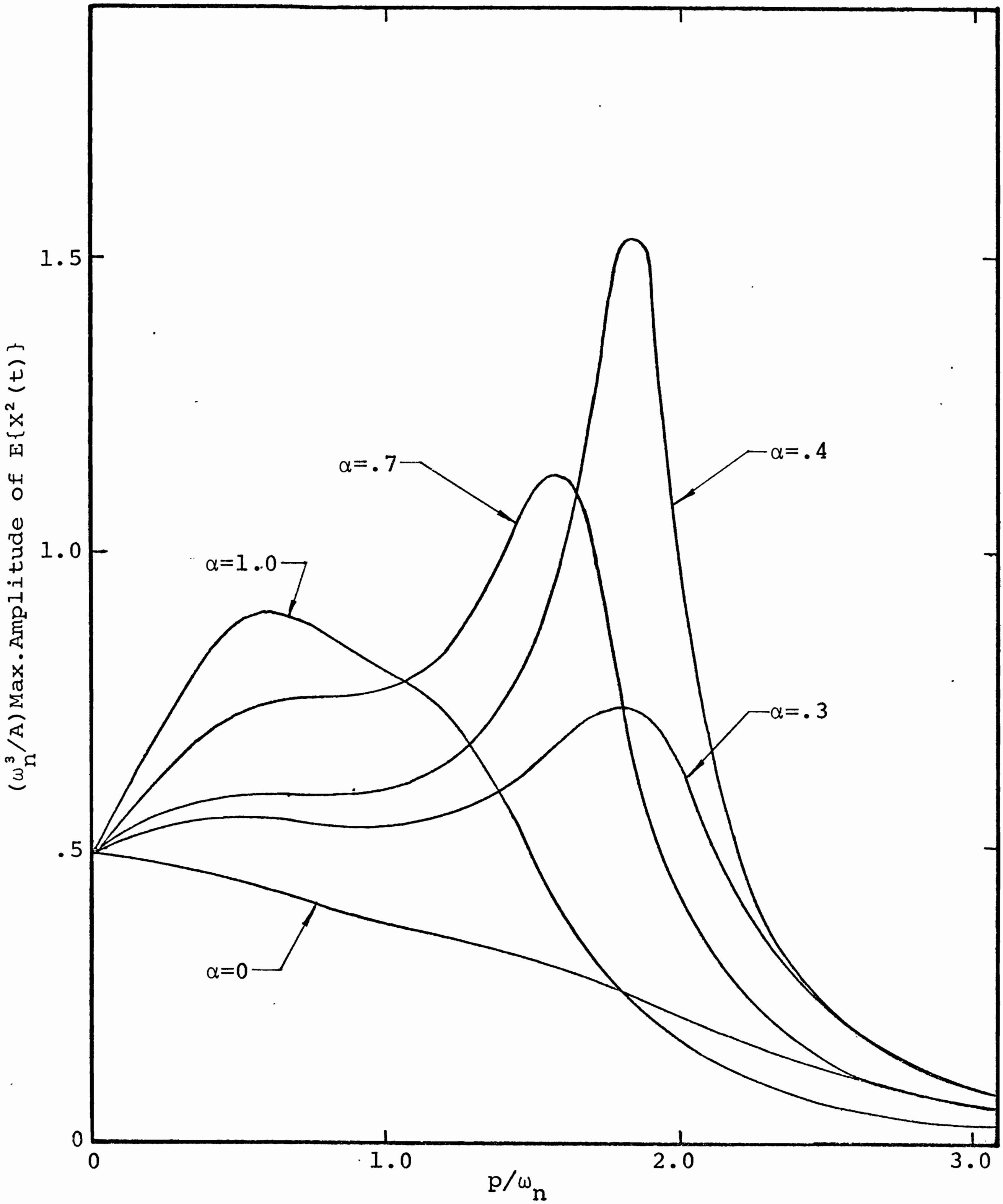

Fig. 3.14. Maximum Amplitude of $E\{X^2(t)\}$ against p/ω_n for $\zeta=.45$

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-\alpha p|t_1|}\cos pt_1$
Governing Equation : (3.19)

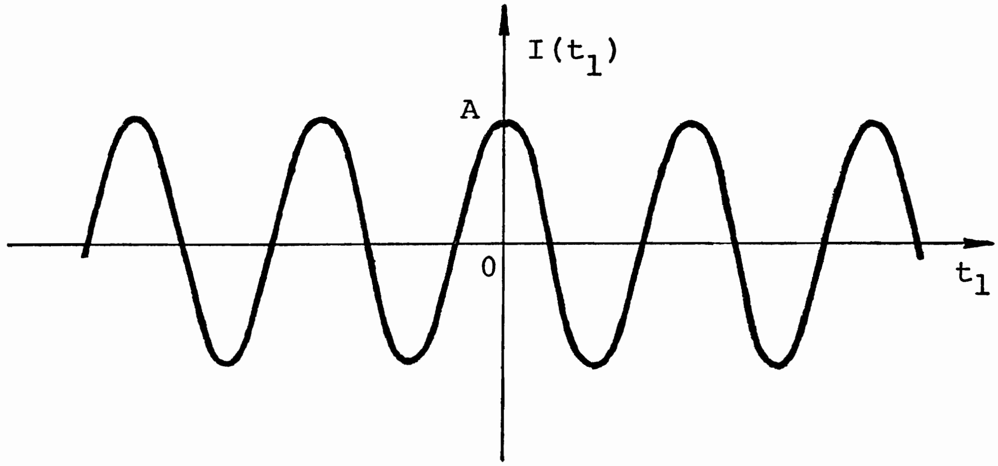

Fig. 3.15. Strength Function $I(t_1) = A\cos pt_1$

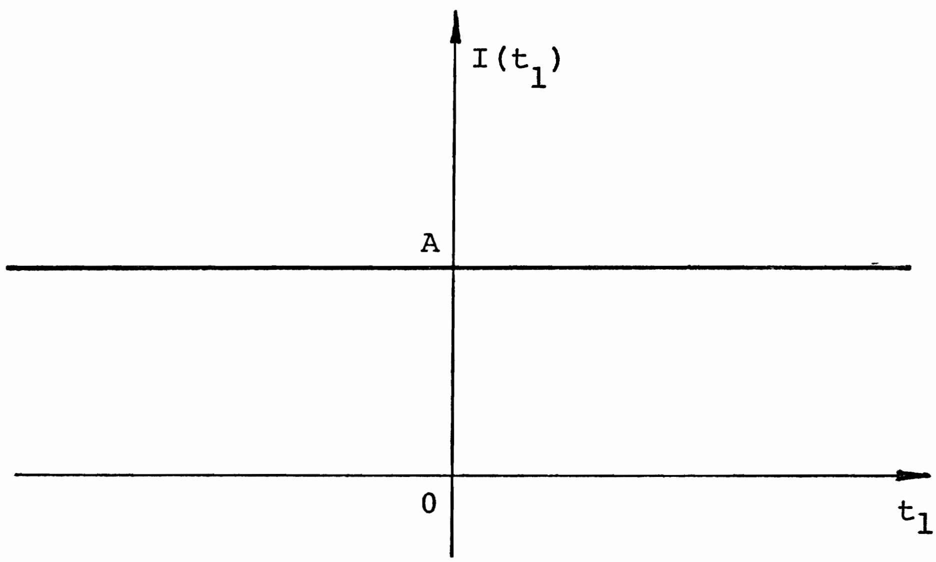

Fig. 3.16. Strength Function $I(t_1) = A$

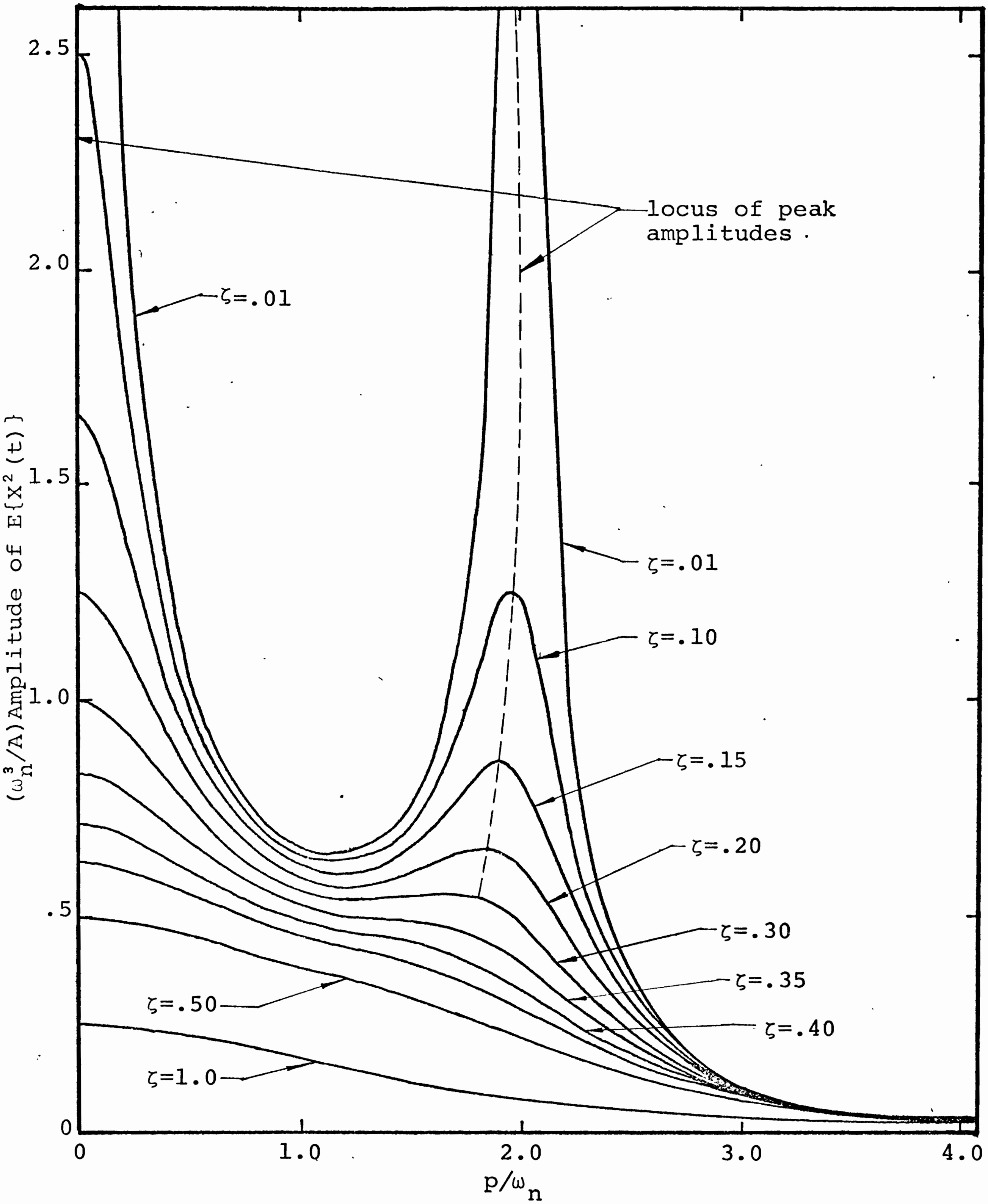

Fig. 3.17. Amplitude of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation = A cos pt_1
Governing Equation : (3.36)

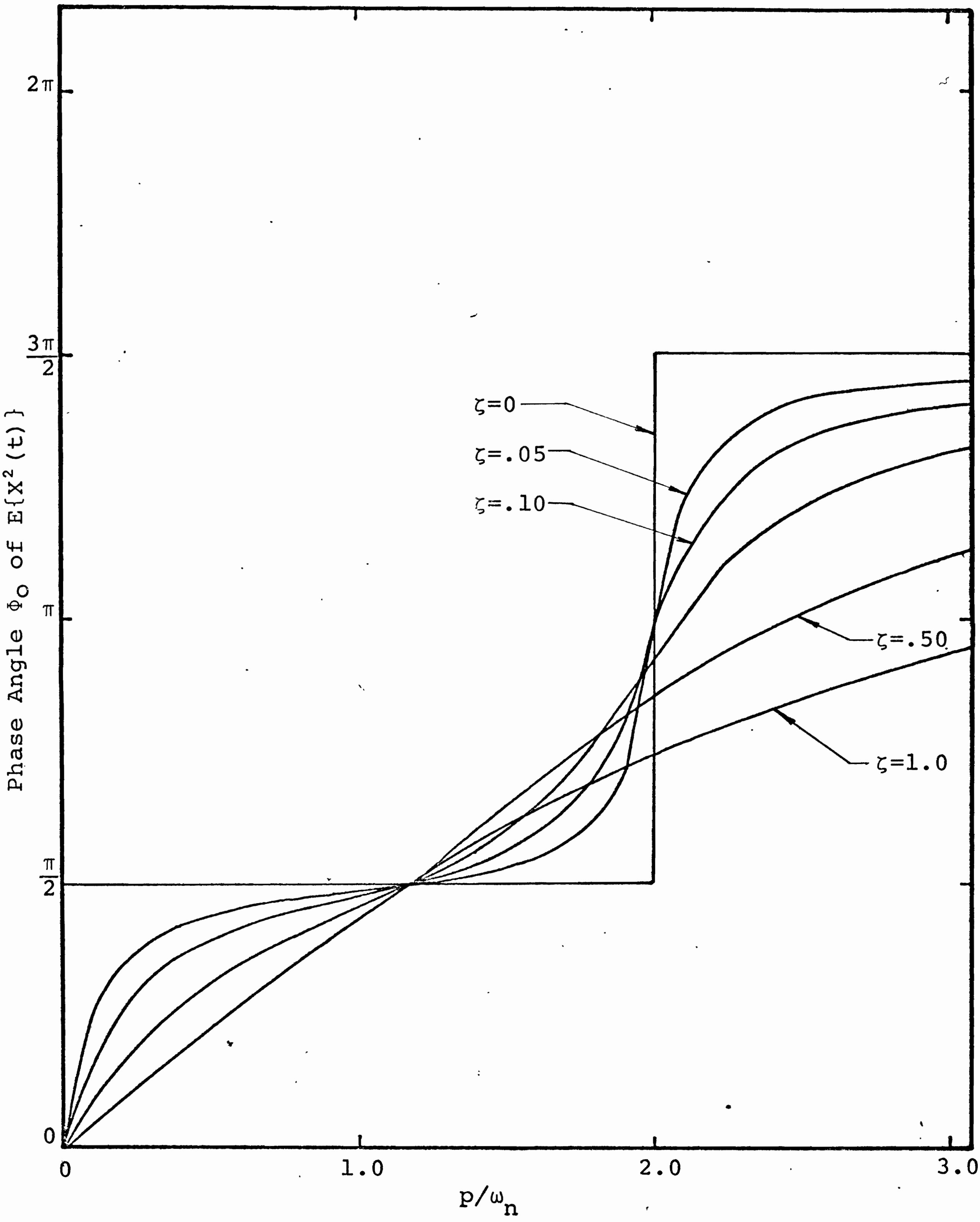

Fig. 3.18. Phase Angle of $E\{X^2(t)\}$ against p/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitxtion = A cos pt_1
Governing Equation : (3.36)

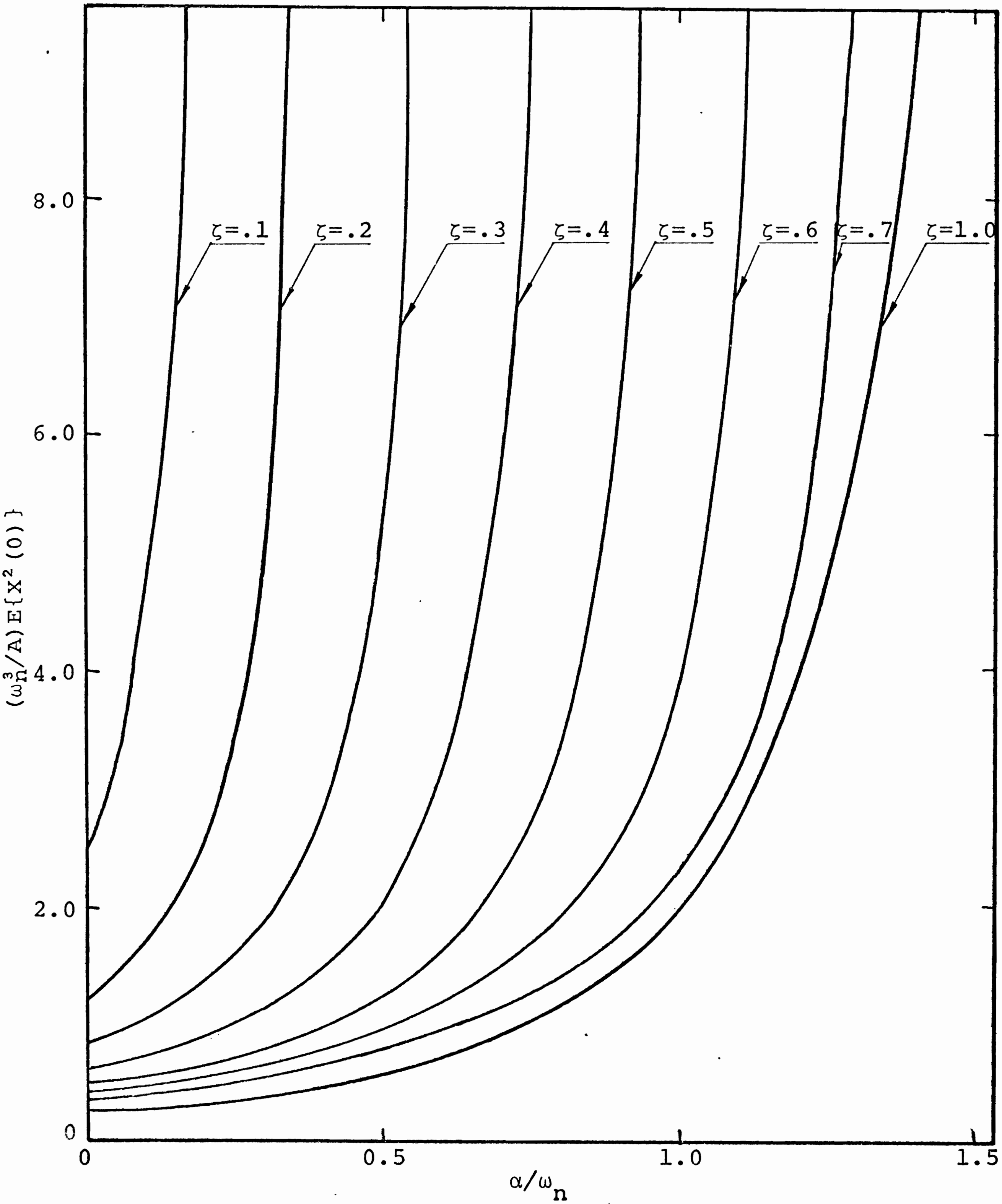

Fig. 3.19. Maximum Amplitude of $E\{X^2(t)\}$ against α/ω_n

One-Degree-Of-Freedom System
Strength Function of Excitation = $Ae^{-\alpha|t_1|}$
Governing Equation : (3.48).

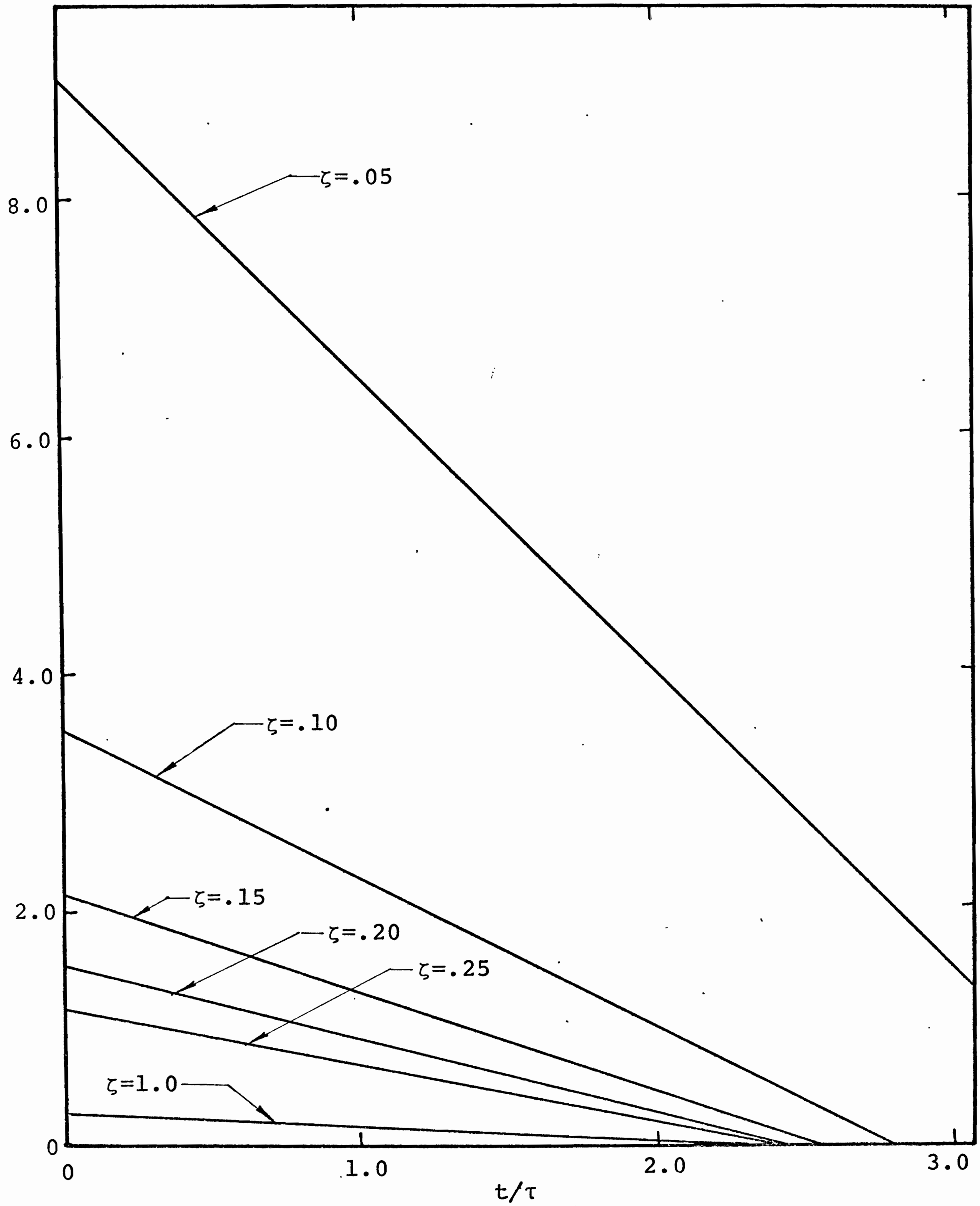

Fig. 3.20. $E\{X^2(t)\}$ against t/τ

One-Degree-Of-Freedom System
Strength Function of Excitation = $A\ (1-0.5\ \frac{|t_1|}{\tau})$
Governing Equation : (3.53)

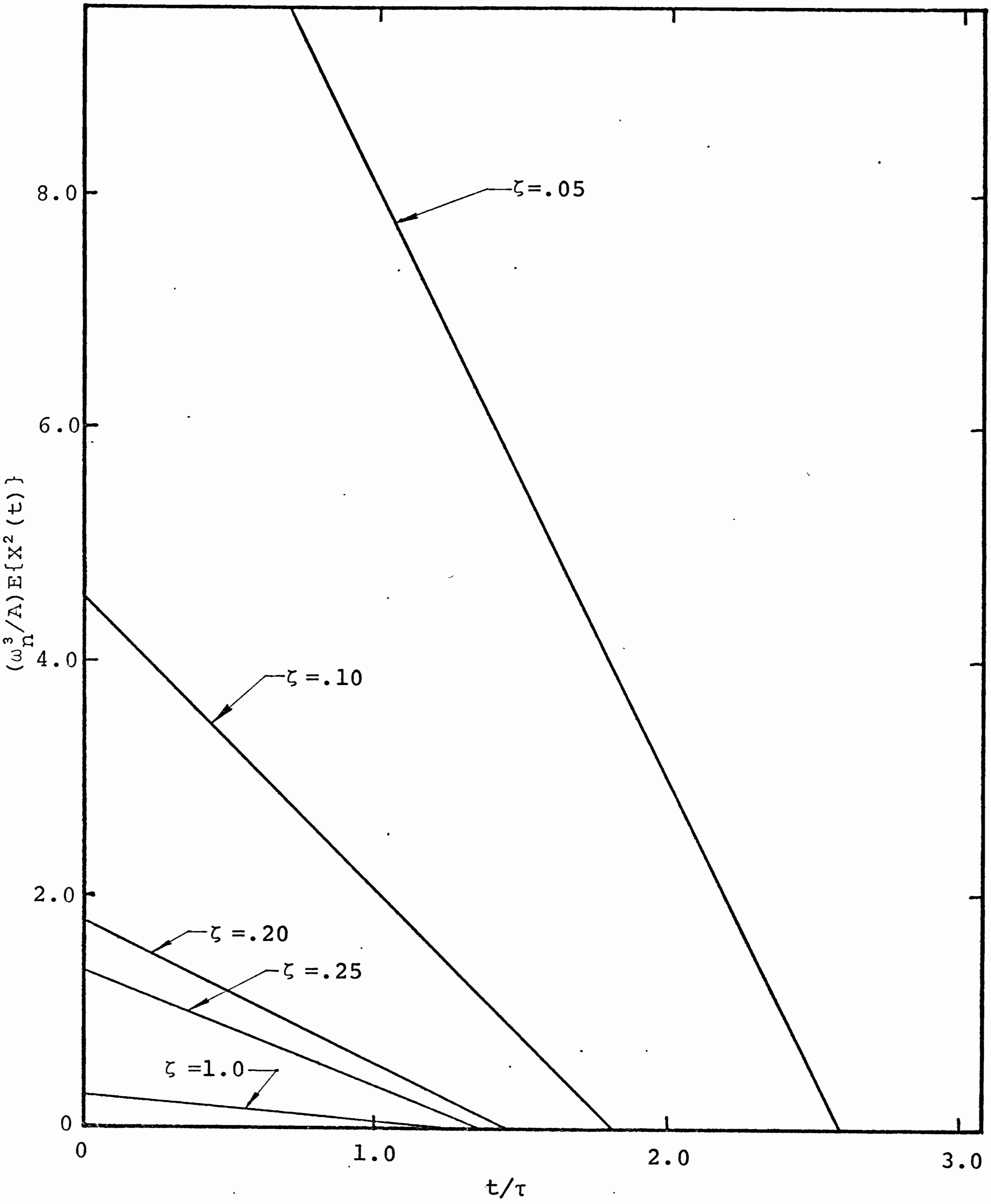

Fig. 3.21. $E\{X^2(t)\}$ against t/τ

One-Degree-Of-Freedom System
Strength Function of Excitation = $A\ (1-\dfrac{|t_1|}{\tau})$
Governing Equation : (3.53)

CHAPTER 4

TWO-DEGREE-OF-FREEDOM LINEAR MECHANICAL SYSTEMS UNDER NONSTATIONARY RANDOM EXCITATIONS

4.1 Introduction

In the two previous chapters, a single-degree-of-freedom system under nonstationary random excitations was investigated. The input-output relations obtained from these chapters are extended and applied here to study the behaviour of a mechanical system with two degrees of freedom subjected to similar types of excitation. Crandall and Mark [2] have studied such systems but only under a white noise type of stationary forces. The autocorrelation of the nonstationary excitation force is taken in this chapter as a Dirac delta function with a harmonically varying strength function of the form

$$I(t_1) = A \cos pt_1 \tag{4.1}$$

It may be noted that when $p = 0$, the excitation becomes a stationary white noise process and the results should conform to those in [2].

4.2 Frequency Response of a Two-Degree-Of-Freedom System

An idealized two-degree-of-freedom system as shown in Fig. 4.1 is considered. The equation of motion may be written using d'Alembert principle as

$$\left.\begin{aligned} m_1\ddot{X}_1(t)+(c_1+c_2)\dot{X}_1(t)-c_2\dot{X}_2(t)+(k_1+k_2)X_1(t)-k_2X_2(t) &= F_1(t) \\ m_2\ddot{X}_2(t)+c_2\dot{X}_2(t)-c_2\dot{X}_1(t)+k_2X_2(t)-k_2X_1(t) &= F_2(t) \end{aligned}\right\} \tag{4.2}$$

where m_1, m_2 : masses of oscillators

c_1, c_2 : constants of viscous dampers

$F_1(t), F_2(t)$: random excitations on m_1 and m_2

$X_1(t), X_2(t)$: displacements of m_1 and m_2

Let

$$\left.\begin{aligned}
&\frac{k_1}{m_1} = \omega_{n1}^2 \;; \qquad \frac{k_2}{m_2} = \omega_{n2}^2 \\
&\frac{c_1}{\sqrt{k_1 m_1}} = 2\zeta_1 \;; \qquad \frac{c_2}{\sqrt{k_2 m_2}} = 2\zeta_2 \\
&\frac{F_1(t)}{m_1} = Z_1(t) \;; \qquad \frac{F_2(t)}{m_2} = Z_2(t) \\
&\frac{m_2}{m_1} = \mu
\end{aligned}\right\} \tag{4.3}$$

where ω_{n1}, ω_{n2} are the uncoupled natural frequencies of the system; ζ_1, ζ_2 are the uncoupled damping ratios; and μ is the mass ratio. Eqs.(4.2a) and (4.2b) now take the form

$$\ddot{X}_1(t) + (2\zeta_1\omega_{n1} + 2\mu\zeta_2\omega_{n2})\dot{X}_1(t) - 2\mu\zeta_2\omega_{n2}\dot{X}_2(t)$$
$$+ (\omega_{n1}^2 + \mu\omega_{n2}^2)X_1(t) - \mu\omega_{n2}^2X_2(t) = Z_1(t) \tag{4.4a}$$

$$\ddot{X}_2(t) + 2\zeta_2\omega_{n2}\dot{X}_2(t) - 2\zeta_2\omega_{n2}\dot{X}_1(t) + \omega_{n2}^2X_2(t)$$
$$- \omega_{n2}^2X_1(t) = Z_2(t) \tag{4.4b}$$

In chapter 2, the analysis for the response of an one-degree-of-freedom linear mechanical system was considered in the time domain through the impulse response function. For two-degree-of-freedom systems, a similar approach becomes tedious due to the difficulty of obtaining the necessary impulse response functions of the system and hence the problem is considered here in the frequency domain. Supposing that the system is infinite operating and using the Laplace tranform with initial conditions

$$\begin{aligned} x_1(0) &= \dot{x}_1(0) = \ddot{x}_1(0) = 0 \\ x_2(0) &= \dot{x}_2(0) = \ddot{x}_2(0) = 0 \end{aligned} \tag{4.5}$$

Eqs.(4.4a) and (4.4b) may be written as

$$(j\omega)^2 X_1(j\omega) + (2\zeta_1\omega_{n1} + 2\mu\zeta_2\omega_{n2})j\omega X_1(j\omega) + (2\mu\zeta_2\omega_{n2})j\omega X_2(j\omega)$$

$$+ (\omega_{n1}^2 + \mu\omega_{n2}^2)X_1(j\omega) - \mu\omega_{n2}^2 X_2(j\omega) = Z_1(j\omega) \tag{4.6a}$$

$$(j\omega)^2 X_2(j\omega) + (2\zeta_2\omega_{n2})j\omega X_2(j\omega) - (2\zeta_2\omega_{n2})j\omega X_1(j\omega)$$

$$+ \omega_{n2}^2 X_2(j\omega) - \omega_{n2}^2 X_1(j\omega) = Z_2(j\omega) \tag{4.6b}$$

Here, $X(j\omega)$, $Z(j\omega)$ are the Laplace transforms of $X(t)$, $Z(t)$.

Rearranging and solving Eqs.(4.6a) and (4.6b), one can obtain

$$X_1(j\omega) = \frac{[\omega_{n2}^2-\omega^2+j(2\zeta_2\omega_{n2})\omega]Z_1(j\omega)+[\mu\omega_{n2}^2+j(2\mu\zeta_2\omega_{n2})\omega]Z_2(j\omega)}{\Delta(j\omega)} \tag{4.7a}$$

$X_2(j\omega) =$

$$\frac{[\omega_{n2}^2+j(2\zeta_2\omega_{n2})\omega]Z_1(j\omega)+[\omega_{n1}^2+\mu\omega_{n2}^2-\omega^2+j(2\zeta_1\omega_{n1}+2\mu\zeta_2\omega_{n2})\omega]Z_2(j\omega)}{\Delta(j\omega)} \tag{4.7b}$$

where the determinant

$$\begin{aligned}\Delta(j\omega) = &\left[\omega^4-[\omega_{n1}^2+\omega_{n2}^2(1+\mu)+4\zeta_1\zeta_2\omega_{n1}\omega_{n2}]\omega^2+\omega_{n1}^2\omega_{n2}^2\right]\\ &+j\left[[2\zeta_2(1+\mu)\omega_{n2}-2\zeta_1\omega_{n1}]\omega^3+[2\zeta_1\omega_{n1}\omega_{n2}^2+2\zeta_2\omega_{n1}^2\omega_{n2}]\omega\right]\end{aligned} \tag{4.8}$$

If the input $Z_2(t)$ is absent and let $Z_1(t) = Z(t)$, Eqs. (4.7a) and (4.7b) become

$$X_1(j\omega) = \frac{[\omega_{n2}^2-\omega^2+j(2\zeta_2\omega_{n2})\omega]Z(j\omega)}{\Delta(j\omega)} \tag{4.9a}$$

$$X_2(j\omega) = \frac{[\omega_{n2}^2+j(2\zeta_2\omega_{n2})\omega]Z(j\omega)}{\Delta(j\omega)} \tag{4.9b}$$

Further, Eqs.(4.9a) and (4.9b) may be represented in the form

$$X_1(j\omega) = H_1(j\omega)Z(j\omega) \tag{4.10a}$$

$$X_2(j\omega) = H_2(j\omega)Z(j\omega) \tag{4.10b}$$

where

$$H_1(j\omega) = \frac{\omega_{n2}^2-\omega^2+j(2\zeta_2\omega_{n2})\omega}{\Delta(j\omega)} \tag{4.11a}$$

$$H_2(j\omega) = \frac{\omega_{n2}^2+j(2\zeta_2\omega_{n2})\omega}{\Delta(j\omega)} \tag{4.11b}$$

are the receptance functions of the system when input $Z_2(t)=0$.

On substituting these receptance functions into Eq.(2.44), the power spectral densities of the responses $X_1(t)$ and $X_2(t)$ may be obtained in terms of the power spectral density of the excitation force $Z(t)$. From these spectral densities, the autocorrelations, mean square values and other required statistical properties of $X_1(t)$ and $X_2(t)$ can be determined by using Eqs.(2.39), (2.32) and other suitable equations given in Chapter 2.

4.3 Response Under Nonstationary Force Z(t) With Harmonic Strength Function

As a specific application of the result in the previous section, a two-degree-of-freedom linear system with only the input $Z(t)=Z_1(t)$ to the system is considered. Further as mentioned earlier, the autocorrelation of the excitation $Z(t)$ is taken to be delta-correlated with harmonically varying strength function in the form

$$R_{zz}(t_1,t_2) = A\cos pt_1 \delta(t_1-t_2) \tag{4.12}$$

The strength function $I(t_1)$ of this input autocorrelation is shown in Fig. 3.15 and the corresponding generalized spectral density of the excitation for this case may be determined by putting $\alpha = 0$ in Eq.(3.6)

$$S_{zz}(\omega_1,\omega_2) = \lim_{t\to\infty} A\left[\frac{[p-(\omega_1-\omega_2)]\sin[p-(\omega_1-\omega_2)]t}{[p-(\omega_1-\omega_2)]^2} + \frac{[p+(\omega_1-\omega_2)]\sin[p-(\omega_1-\omega_2)]t}{[p+(\omega_1-\omega_2)]^2}\right] \tag{4.13}$$

This is illustrated in Fig. 4.3 using data taken from Program 0.A in Appendix.

In the case $p = 0$, the strength function of the input $Z(t)$ in Eq.(4.1) becomes a constant, the process $Z(t)$ is therefore stationary, and the generalized spectral density of $Z(t)$ in Eq.(4.13) takes the form of Eq.(3.9), it is

$$S_{zz}(\omega_1,\omega_2) = \lim_{t\to\infty} 2A\left[\frac{\sin(\omega_1-\omega_2)t}{(\omega_1-\omega_2)}\right] = 2A\delta(\omega_1-\omega_2) \tag{4.14}$$

4.3.1 Spectral Densities of the Response

The autocorrelations of the responses $X_1(t)$ and $X_2(t)$ may be evaluated if their spectral densities are known. These spectral densities are determined from the spectral density of the input process $Z(t)$ which is given by Eq.(2.38) as the following

$$S_{zz}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{zz}(t_1,t_2)e^{-j(\omega_1 t_1-\omega_2 t_2)}dt_1 dt_2 \qquad (4.15)$$

Substituting the input autocorrelation from Eq.(4.12) into Eq.(4.15)

$$S_{zz}(\omega_1,\omega_2) = A\int_{\infty}^{\infty}\int_{\infty}^{\infty} \cos pt_1\delta(t_1-t_2)e^{-j(\omega_1 t_1-\omega_2 t_2)}dt_1 dt_2$$

$$= A\int_{-\infty}^{\infty} \cos pt_2 e^{-j(\omega_1-\omega_2)t_2}dt_2$$

$$= A\int_{-\infty}^{\infty} \cos pt_2 \cos(\omega_1-\omega_2)t_2 dt_2 \qquad (4.16)$$

since $\int_{-\infty}^{\infty} \cos pt_2 \sin(\omega_1-\omega_2)t_2 dt_2 = 0$ and in addition only the real part of $S_{zz}(\omega_1,\omega_2)$ is considered here .

Utilizing the fact that [3]

$$\lim_{t\to\infty}\int_{-t}^{t} \cos(ax)\cos(bx)dx = \lim_{t\to\infty}\left[\frac{\sin(a-b)t}{a-b} + \frac{\sin(a+b)t}{a+b}\right]$$

$$= \pi[\delta(a-b) + \delta(a+b)] \qquad (4.17)$$

Eq.(4.16) now becomes

$$S_{zz}(\omega_1,\omega_2) = A\pi \; \delta[p-(\omega_1-\omega_2)] + \delta[p+(\omega_1-\omega_2)] \tag{4.18}$$

From this equation, it may be seen [3] that the spectral de density of input process Z(t) is zero except along the planes where [see Fig. 4.2]

$$\left.\begin{aligned} p &= \omega_1-\omega_2 \\ p &= -(\omega_1-\omega_2) \end{aligned}\right\} \tag{4.19}$$

From Eq.(2.44), the spectral densities of the responses $X_1(t)$ and $X_2(t)$ are given by

$$S_{x_1x_1}(\omega_1,\omega_2) = H_1(j\omega_1)H_1^*(j\omega_2)S_{zz}(\omega_1,\omega_2) \tag{4.20a}$$

$$S_{x_2x_2}(\omega_1,\omega_2) = H_2(j\omega_1)H_2^*(j\omega_2)S_{zz}(\omega_1,\omega_2) \tag{4.20b}$$

where $H_1(j\omega)$ and $H_2(j\omega)$ are the receptances of the system with $Z_2(t) = 0$, and are given in Eqs.(4.11a) and (4.11b). Substituting Eq.(4.18) into Eqs.(4.20a) and (4.20b),

$$S_{x_1x_1}(\omega_1,\omega_2) = A\pi H_1(j\omega_1)H_1^*(j\omega_2)\Big[\delta[p-(\omega_1-\omega_2)]+\delta[p+(\omega_1-\omega_2)]\Big] \tag{4.21a}$$

$$S_{x_2x_2}(\omega_1,\omega_2) = A\pi H_2(j\omega_1)H_2^*(j\omega_2)\Big[\delta[p-(\omega_1-\omega_2)]+\delta[p+(\omega_1,\omega_2)]\Big] \tag{4.21b}$$

Case of Stationary Input

When p=0, the input Z(t) becomes a stationary white noise process and from Eq.(4.19), $\omega_1 = \omega_2 = \omega$. The spectral densities of the responses $X_1(t)$ and $X_2(t)$ in this case, Eqs.(4.21a) and (4.21b), become [8]

$$S_{x_1x_1}(\omega) = A\pi \left| H_1(j\omega) \right|^2 \tag{4.22a}$$

$$S_{x_2x_2}(\omega) = A\pi \left| H_2(j\omega) \right|^2 \tag{4.22b}$$

These spectral densities are plotted in Figs. 4.4, 4.5, and 4.6 against the frequency ratio ω/ω_{n1} for a system with mass ratio $\mu = m_2/m_1 = 0.1$, ratio of natural frequencies $\omega_{n2}/\omega_{n1} = 2.0$ and different values of damping ratio $\zeta_1=\zeta_2= 0$ (Fig.4.4), $\zeta_1=\zeta_2= 0.01$ (Fig. 4.5), and $\zeta_1=\zeta_2 =0.2$ (Fig.4.6). The values indicated in the plots were obtained using Program 3 in Appendix.

In Fig. 4.4 ($\zeta_1=\zeta_2= 0$), it is noted that the quantity $\frac{\omega_{n1}^4}{A\pi} S_{x_2x_2}(\omega)$ representing the power spectral density of the response $X_2(t)$ starts at a value of 1.0 when $\omega/\omega_{n1} = 0$, attains very large values in the neighbourhood of $\omega/\omega_{n1} = 1.0$ and $\omega/\omega_{n1} = 2.0$, and then decreases asymptotically to zero value when $\omega/\omega_{n1} >> 3.0$. Same characteristics hold good for the spectral density of the response $X_1(t)$, except in the region $\omega/\omega_{n1} \simeq 2.0$, the quantity $\frac{\omega_{n1}^4}{A} S_{x_1x_1}(\omega)$ has both a maximum and a minimum value. It must be noted that $S_{x_2x_2}(\omega)$

is always numerically larger than $S_{x_1x_1}(\omega)$.

From Figs. 4.5 ($\zeta_1=\zeta_2=0.01$) and 4.5 ($\zeta_1=\zeta_2=0.2$), it may be seen that the essential features of the spectral densities remain the same. Because of the presence of damping, the numerical values for $S_{x_1x_1}(\omega)$ and $S_{x_2x_2}(\omega)$ are bounded as compared to the previous case. The value of spectral density in the region $\omega/\omega_{n1}\simeq 1.0$ is always larger than that corresponding to the region $\omega/\omega_{n1}\simeq 2.0$ for positive values of damping ratio ζ.

These characteristics are similar to those obtained by Crandall and Mark [2].

4.3.2 Autocorrelations of the Response

The autocorrelations of the responses $X_1(t)$ and $X_2(t)$ are determined from their power spectral densities using Eq.(2.39)

$$R_{x_1x_1}(t_1,t_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{x_1x_1}(\omega_1,\omega_2)e^{j(\omega_1t_1-\omega_2t_2)}d\omega_1d\omega_2 \tag{4.23a}$$

$$R_{x_2x_2}(t_1,t_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{x_2x_2}(\omega_1,\omega_2)e^{j(\omega_1t_1-\omega_2t_2)}d\omega_1d\omega_2 \tag{4.23b}$$

Substituting for $S_{x_1x_1}(\omega_1,\omega_2)$ and $S_{x_2x_2}(\omega_1,\omega_2)$ from Eqs.(4.21a) and (4.21b),

$$R_{x_1x_1}(t_1,t_2) = \frac{A}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_1(j\omega_1)H_1^*(j\omega_2)\times$$

$$\left[\delta[p-(\omega_1-\omega_2)]+\delta[p+(\omega_1-\omega_2)]\right]e^{j(\omega_1t_1-\omega_2t_2)}d\omega_1d\omega_2 \qquad (4.24a)$$

$$R_{x_2x_2}(t_1,t_2) = \frac{A}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_2(j\omega_1)H_2^*(j\omega_2)\times$$

$$\left[\delta[p-(\omega_1-\omega_2)]+\delta[p+(\omega_1-\omega_2)]\right]e^{j(\omega_1t_1-\omega_2t_2)}d\omega_1d\omega_2 \qquad (4.24b)$$

and on integration, the autocorrelation of the responses $X_1(t)$ and $X_2(t)$ become

$$R_{x_1x_1}(t_1,t_2) = \frac{A}{4\pi}\Bigg\{e^{jpt_2}\int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega-p)]e^{j\omega(t_1-t_2)}d\omega$$

$$+e^{-jpt_1}\int_{-\infty}^{\infty} H_1[j(\omega-p)]H_1^*(j\omega)e^{j\omega(t_1-t_2)}d\omega\Bigg\} \qquad (4.25a)$$

$$R_{x_2x_2}(t_1,t_2) = \frac{A}{4\pi}\Bigg\{e^{jpt_2}\int_{-\infty}^{\infty} H_2(j\omega)H_2^*[j(\omega-p)]e^{j\omega(t_1-t_2)}d\omega$$

$$+e^{-jpt_2}\int_{-\infty}^{\infty} H_2[j(\omega-p)]H_2^*(j\omega)e^{j\omega(t_1-t_2)}d\omega\Bigg\} \qquad (4.25b)$$

where $H_1(j\omega)$ and $H_2(j\omega)$ are given in Eqs.(4.11a) and (4.11b).

4.4 Mean Square Values of the Responses $X_1(t)$ and $X_2(t)$

The mean square values of the outputs $X_1(t)$ and $X_2(t)$ are obtained by setting $t_1 = t_2 = t$ in the expressions (4.25a) and (4.25b) giving their autocorrelations

$$E\{X_1^2(t)\} = \frac{A}{4\pi}\left\{e^{jpt}\int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega-p)]d\omega\right.$$

$$\left. + e^{-jpt}\int_{-\infty}^{\infty} H_1[j(\omega-p)]H_1^*(j\omega)d\omega\right\} \quad (4.26a)$$

$$E\{X_2^2(t)\} = \frac{A}{4\pi}\left\{e^{jpt}\int_{-\infty}^{\infty} H_2(j\omega)H_2^*[j(\omega-p)]d\omega\right.$$

$$\left. + e^{-jpt}\int_{-\infty}^{\infty} H_2[j(\omega-p)]H_2^*(j\omega)d\omega\right\} \quad (4.26b)$$

Eqs.(4.26a) and (4.26b) may also be written as

$$E\{X_1^2(t)\} = \frac{A}{4\pi}\left\{e^{jpt}\int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega-p)]d\omega\right.$$

$$\left. + e^{-jpt}\int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega+p)]d\omega\right\} \quad (4.27a)$$

$$E\{X_2^2(t)\} = \frac{A}{4\pi}\left\{e^{jpt}\int_{-\infty}^{\infty} H_2(j\omega)H_2^*[j(\omega-p)]d\omega\right.$$

$$\left. + e^{-jpt}\int_{-\infty}^{\infty} H_2(j\omega)H_2^*[j(\omega+p)]d\omega\right\} \quad (4.27b)$$

In the first equation, let

$$I_1 = \int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega+p)]d\omega \quad (4.28a)$$

$$I_1^* = \int_{-\infty}^{\infty} H_1(j\omega)H_1^*[j(\omega-p)]d\omega \quad (4.28b)$$

Writing $(\omega - \frac{p}{2})$ in place of ω in Eq.(4.28a) and $(\omega + \frac{p}{2})$ instead of ω in Eq.(4.28b),

$$I_1 = \int_{-\infty}^{\infty} H_1[j(\omega - \frac{p}{2})]H_1^*[j(\omega + \frac{p}{2})]d\omega \quad (4.29a)$$

$$I_1^* = \int_{-\infty}^{\infty} H_1[j(\omega + \frac{p}{2})]H_1^*[j(\omega - \frac{p}{2})]d\omega \quad (4.29b)$$

Now, I_1 and I_1^* may be recognized as complex conjugates. Then one can write

$$I_1 = P_1 + jQ_1 \quad (4.30a)$$

$$I_1^* = P_1 - jQ_1 \quad (4.30b)$$

Eq.(4.27a) for the mean square value of the response $X_1(t)$ now becomes

$$E\{X_1^2(t)\} = \frac{A}{4\pi}[\,e^{jpt}(P_1 - jQ_1) + e^{-jpt}(P_1 + jQ_1)\,]$$

$$= \frac{A}{4\pi}[\,P_1 \cos pt + Q_1 \sin pt\,] \quad (4.31)$$

or,

$$E\{X_1^2(t)\} = \frac{A}{4\pi} R_1 \cos(pt + \Phi_1) \tag{4.32}$$

where

$$\left.\begin{aligned} R_1 &= (P_1^2 + Q_1^2)^{\frac{1}{2}} \\ \Phi_1 &= \tan^{-1} \left(\frac{Q_1}{P_1} \right) \end{aligned}\right\} \tag{4.33}$$

Similarly, by defining

$$I_2 = \int_{-\infty}^{\infty} H_2(j\omega) H_2^*[j(\omega+p)]d\omega \tag{4.34a}$$

$$I_2^* = \int_{-\infty}^{\infty} H_2(j\omega) H_2^*[j(\omega-p)]d\omega \tag{4.34b}$$

or

$$I_2 = P_2 + jQ_2 \tag{4.35a}$$

$$I_2^* = P_2 - jQ_2 \tag{4.35b}$$

Eq.(4.27b) for the mean square value of the response $X_2(t)$ takes the form

$$E\{X_2^2(t)\} = \frac{A}{4\pi} R_2 \cos(pt + \Phi_2) \tag{4.36}$$

where

$$\left.\begin{aligned} R_2 &= (P_2^2 + Q_2^2)^{\frac{1}{2}} \\ \Phi_2 &= \tan^{-1} \left(\frac{Q_2}{P_2} \right) \end{aligned}\right\} \tag{4.37}$$

4.4.1 Study of the Receptance Product $H(j\omega)H^*[j(\omega+p)]$

It may be seen from Eqs.(4.27a) and (4.27b) that the mean square values of the responses $X_1(t)$ and $X_2(t)$ are determined by the integrals I_1 and I_2 of the products of the receptances and their corresponding complex conjugates given in Eqs.(4.28a) and (4.34a). For the case $p = 0$, when the excitation is a stationary white noise process, these two integrals can be evaluated analytically [2]. But for non-stationary, i.e. $p \neq 0$, they are not integrable readily and may be evaluated only by numerical procedures.

Since $H_1(j\omega)H_1^*[j(\omega+p)]$ and $H_2(j\omega)H_2^*[j(\omega+p)]$ in Eqs.(4.28a) and (4.34a) are symmetrical with respect to the axis $\omega = -p/2$, these two equations may be written as

$$I_1 = 2\int_{-\frac{p}{2}}^{\infty} H_1(j\omega)H_1^*[j(\omega+p)]d\omega \tag{4.38}$$

$$I_2 = 2\int_{-\frac{p}{2}}^{\infty} H_2(j\omega)H_2^*[j(\omega+p)]d\omega \tag{4.39}$$

Thus, I_1 and I_2 are equal to twice the area under the curve obtained by plotting $H_1(j\omega)H_1^*[j(\omega+p)]$ and $H_2(j\omega)H_2^*[j(\omega+p)]$ respectively against ω . Since $H_1(j\omega)H_1^*[j(\omega+p)]$ and $H_2(j\omega)H_2^*[j(\omega+p)]$ are complex functions, I_1 and I_2 will have real and imaginary parts. Real part of I is given by the area under the curve $\mathrm{Re}\{H(j\omega)H^*[j(\omega+p)]\}$ against ω. Similarly, imaginary part of I is given by the area under the curve $\mathrm{Im}\{H(j\omega)H^*[j(\omega+p)]\}$ against ω.

The real and imaginary parts of the products $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ and $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$ for a two-degree-of-freedom linear system with a damping ratio $\zeta_1=\zeta_2=0.2$, mass ratio $m_2/m_1 = 0.1$, natural frequency ratio $\omega_{n2}/\omega_{n1}=2.0$, are plotted against ω/ω_{n1} in Fig. 4.7 (for $p/\omega_{n1}=0.1$), Fig. 4.8 (for $p/\omega_{n1}=0.5$), Fig. 4.9 (for $p/\omega_{n1}=1.0$), Fig. 4.10 (for $p/\omega_{n1}=2.0$), and Fig. 4.11 (for $p/\omega_{n1}=3.0$). Data for these curves are taken from Program 3 in Appendix.

From these five figures, it may be seen that for any fixed value of p/ω_{n1}, all the real and imaginary curves of $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ and $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$ tend asymptotically to zero when $\omega/\omega_{n1}>>2.0$, hence the integrals of Eqs. (4.38) and (4.39) converge rapidly. It is also noted that for any given value p/ω_{n1}, the shapes of the curves representing the imaginary and real parts of $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ are similar to those representing the imaginary and real parts of $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$, but the absolute values for $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ are numerically smaller compared to those for $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$. Therefore, for a two-degree-of-freedom system with $\zeta_1=\zeta_2=0.2$, $m_2/m_1=0.1$, $\omega_{n2}/\omega_{n1}=2.0$ and for any value of p/ω_{n1}, the amplitude of the mean square response $X_2(t)$ is greater than that for $X_1(t)$ and the phase angles of these two mean square values being approximately the same.

It may be also seen from these figures that the shapes of the plots for real and imaginary parts of the two products $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$ change with different values of frequency ratio p/ω_{n1}. The influence of this ratio p/ω_{n1} on the shapes of the curves may be more easily seen if the quantities $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$ are plotted vectorially. Figs. 4.12 and 4.13 present the family of vector plots of $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ and $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$ respectively for the same system with a damping ratio $\zeta_1=\zeta_2=.2$, mass ratio $m_2/m_1=.1$ and natural frequency ratio $\omega_{n2}/\omega_{n1}=2.0$. Similar vectorial plots were obtained by Roberts [3] when he considered one-degree-of-freedom linear systems subjected to nonstationary forces.

From Fig.4.12, it may be seen for different values of p/ω_{n1}, the vectorial plots of $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ have different shapes. When $p/\omega_{n1} \leq 1.5$, the curves lie above the real axis, whereas for $p/\omega_{n1} > 1.5$, most of them lie below the real axis. As expected, for large values of $p/\omega_{n1} >> 2.0$, the real and imaginary parts of the product $\omega_{n1}^4 H(j\omega)H_1^*[j(\omega+p)]$ approach the origin passing through the first quadrant of the complex plane. Further, the curve starts in the first quadrant for $p/\omega_{n1} \leq 1.5$, in the third quadrant when $p/\omega_{n1} = 2.0$, and in the fourth quadrant when $p/\omega_{n1} \geq 2.5$.

Same remarks are valid for Fig.4.13, except that here the magnetudes of the vector plots are greater than those in Fig.4.11. This means that the mean square values of the response $X_2(t)$ are larger than those of $X_1(t)$ for the given system.

4.4.2 Study of the Mean Square Values of the Responses $X_1(t)$ and $X_2(t)$

The mean square responses of $X_1(t)$ and $X_2(t)$ are given in Eqs.(4.32) and (4.36) as

$$E\{X_1^2(t)\} = \frac{A}{4\pi} R_1 \cos(pt+\Phi_1) \tag{4.40}$$

$$E\{X_2^2(t)\} = \frac{A}{4\pi} R_2 \cos(pt+\Phi_2) \tag{4.41}$$

with

$$\left.\begin{aligned} R_1 &= (P_1^2 + Q_1^2)^{\frac{1}{2}} \\ \Phi_1 &= \tan^{-1}\left(\frac{Q1}{P1}\right) \end{aligned}\right\} \tag{4.42}$$

$$\left.\begin{aligned} R_2 &= (P_2^2 + Q_2^2)^{\frac{1}{2}} \\ \Phi_2 &= \tan^{-1}\left(\frac{Q2}{P2}\right) \end{aligned}\right\} \tag{4.43}$$

where P_1, Q_1 and P_2, Q_2 are determined using Eqs.(4.38) and (4.39) as

$$P_1 + jQ_1 = 2 \int_{-\frac{p}{2}}^{\infty} H_1(j\omega) H_1^*[j(\omega+p)] \tag{4.44}$$

$$P_2 + jQ_2 = 2 \int_{-\frac{p}{2}}^{\infty} H_2(j\omega) H_2^*[j(\omega+p)] \tag{4.45}$$

Substituting for $H(j\omega)H^*[j(\omega+p)]$ the values computed in Section 4.4.1, the integrals of Eqs.(4.44) and (4.45) may be evaluated numerically using Simpson's rule method. Once P and Q are obtained, the mean square values of the responses $X_1(t)$ and $X_2(t)$ may be derived.

Figs. 4.14, 4.15, 4.16, and 4.17 show the variation of amplitudes and phase angles of the mean square responses of $X_1(t)$ and $X_2(t)$ with respect to p/ω_{n1} for different values of damping ratio ζ. In all these cases, mass ratio $m_2/m_1=0.1$, natural frequency ratio $\omega_{n2}/\omega_{n1}=2.0$, the excitation $Z(t)$ on mass m_1 is a nonstationary white noise with an autocorrelation $R_{zz}(t_1,t_2)= A \cos pt_1 \delta(t_1-t_2)$. Program 4 in Appendix is used to obtain the numerical results.

From Fig. 4.14, it is noted that large amplitudes of the mean square response of $X_1(t)$ occur when $p/\omega_{n1}=0$ and $p/\omega_{n1}\simeq 2.0$. The peaks are more pronounced when $\zeta_1=\zeta_2\leq 0.2$. For $p/\omega_{n1}>>3.0$, all these amplitudes decrease asymptotically to zero.

Similar characteristics are observed in Fig. 4.16 for the mean square amplitudes of $X_2(t)$. Here, when damping ratio $\zeta_1=\zeta_2\leq 0.1$, the amplitudes slightly deviate to higher values around the regions $p/\omega_{n1}\simeq 1.0$ and $p/\omega_{n1}\simeq 3.0$; this discrepancy may be due to the fact that the peaks are too narrow in the plot $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)]$ in Eq. (4.39) for very small values of ζ_1 and ζ_2 and therefore the area under the curve cannot be found exactly. For example, when $\zeta_1=\zeta_2=0$, the quantities $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$ and $\omega_{n1}^4 H_2(j\omega)H_2^*[j(\omega+p)$ vs ω/ω_{n1} will have discontinuity at their poles and so accurate values for P_1, Q_1 and P_2, Q_2 in Eqs.(4.44) and (4.45) are difficult to obtain through Simpson's numerical procedure.

In Fig. 4.15, all the phase angles of the mean square responses of $X_1(t)$ start at 0, pass approximatly through $\pi/2$ when $p/\omega_{n1}=1.0$ and increase with p/ω_{n1} values. For small values of ζ_1 and ζ_2, the curves are not stable as can be seen from the figure. The reason may be due to non accurate evaluation of the integrals in Eq.(4.44) as explained earlier.

Fig. 4.17 shows the variation of phase angle of $E\{X_2^2(t)\}$ against p/ω_{n1}. This has same characteristics as Fig. 4.15 but there is no common phase angle for $p/\omega_{n1}=1.0$. As expected, from these four figures, it can be seen that for the same frequency ratio p/ω_{n1}, amplitudes of $E\{X_2^2(t)\}$ are larger than those of $E\{X_1^2(t)\}$, but both of them exhibit approximately the same phase angle.

It may be interesting to know how the mean square responses of the system vary with respect to the natural frequency ratio ω_{n2}/ω_{n1} for a given frequency p in the auto-correlation function of Z(t). For this, the ratio p/ω_{n1} is fixed at 0.5 and Figs. 4.18, 4.19, 4.20, 4.21 showing the variation of amplitudes and phases of $E\{X_1^2(t)\}$ and $E\{X_2^2(t)\}$ against the natural frequency ratio ω_{n2}/ω_{n1} are plotted. Damping ratio of the system is $\zeta_1=\zeta_2=0.2$. Similar plots for amplitudes and phase angles of the mean square responses are shown in Figs. 4.22, 4.23, 4.24, and 4.25 for the same system with frequency ratio $p/\omega_{n1}=1.0$. Data for these curves is obtained using Program 5 in Appendix.

In Fig. 4.18, the curves of $(P_1^2+Q_1^2)^{\frac{1}{2}}$ in Eq.(4.42) giving the amplitude of $E\{X_1^2(t)\}$ as function of natural frequency ratio ω_{n2}/ω_{n1} are shown for different values of mass ratio m_2/m_1. All these curves start approximately at the same point when $\omega_{n2}/\omega_{n1}= 0$, decrease to a minimum when $1.0 < \omega_{n2}/\omega_{n1} < 2.0$, and then increase asymptotically to a certain value depending on the mass ratio m_2/m_1. The amplitude of $E\{X_1^2(t)\}$ decreases with an increase in mass ratio for a given value of ω_{n2}/ω_{n1}.

Similar characteristics are exhibited in Fig. 4.20 where the amplitude of $E\{X_2^2(t)\}$ is plotted against ω_{n2}/ω_{n1} for different mass ratio m_2/m_1. Here the amplitude increases to a maximum in the region $1.0 < \omega_{n2}/\omega_{n1} < 2.0$ and then decreases asymptotically to a constant value. Therefore it is interesting to note here that in the interval $1.0 < \omega_{n2}/\omega_{n1} < 2.0$, the amplitude of $E\{X_2^2(t)\}$ is a maximum, whereas the amplitude of $E\{X_1^2(t)\}$ is a minimum. Hence, in this interval the mass m_2 may be used as an absorber for the system to counteract large amplitudes of m_1.

The phase angles of $E\{X_1^2(t)\}$ are shown in Fig. 4.19 for excitation frequency $p/\omega_{n1}=0.5$. All the curves start approximately at the same value when $\omega_{n2}/\omega_{n1} \simeq 0$, decrease to a minimum when ω_{n2}/ω_{n1} approaches 1.0, and then increase asymptotically to a maximum value at $\omega_{n2}/\omega_{n1} \simeq 3.0$ depending on the mass ratio m_2/m_1. It is also noted that the phase angles decrease with increasing of mass ratio around the region $\omega_{n2}/\omega_{n1} \leq 1.0$, and increase with an increase in m_2/m_1 for $\omega_{n2}/\omega_{n1} > 1.0$.

Similarly, Fig. 4.21 showns the phase angle of $E\{X_2^2(t)\}$ as function of ω_{n2}/ω_{n1} for different values of m_2/m_1. These curves also start at the same value when $\omega_{n2}/\omega_{n1} \simeq 0$, reach maximum values at $\omega_{n2}/\omega_{n1} = 0.2$ and at $\omega_{n2}/\omega_{n1} \simeq 1.0$, and then asymptotically decrease to constant values for large values of ω_{n2}/ω_{n1}. In the region $\omega_{n2}/\omega_{n1} \simeq 1.0$, the phase angles increase with a decrease in the mass ratio, whereas the opposite effect is resulted in the region $\omega_{n2}/\omega_{n1} > 2.0$. It is also noted from these two figures that for any given frequency ratio p/ω_{n1}, the phase angles of $E\{X_2^2(t)\}$ for the considered system are always larger than those of $E\{X_1^2(t)\}$.

For the sake of completeness, Figs. 4.22, 4.23, 4.24, and 4.25 are given when $p/\omega_{n1} = 1.0$. The behaviour of the system essentially remains the same as before.

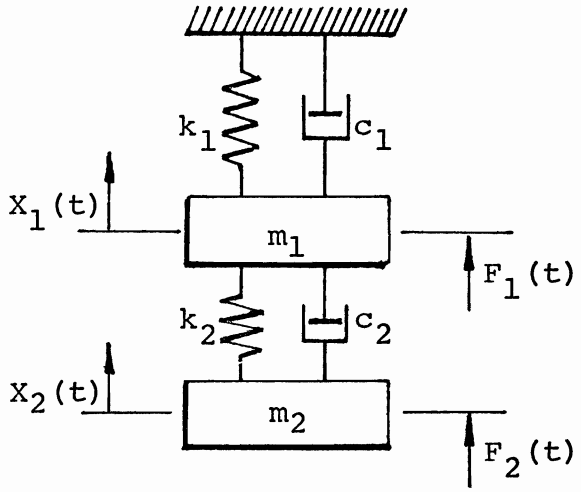

Fig. 4.1. Two-Degree-Of-Freedom System

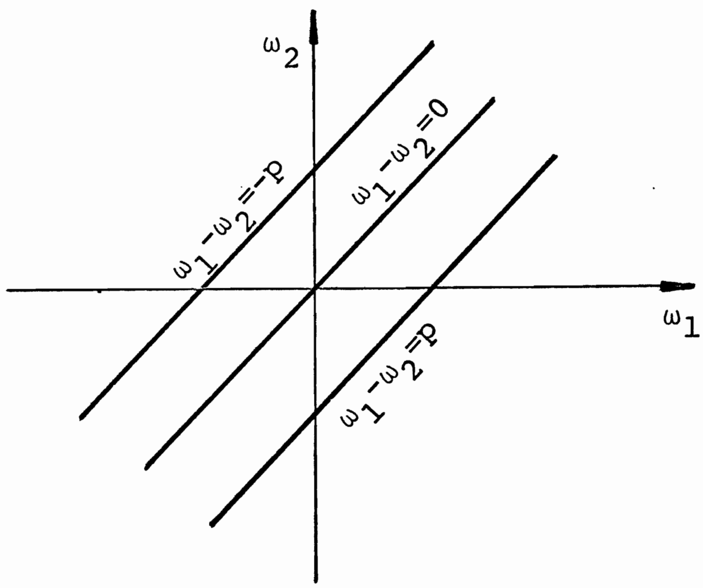

Fig. 4.2. Planes $p = \omega_1-\omega_2$ and $p = -(\omega_1-\omega_2)$ where $S_{zz}(\omega_1,\omega_2)$ is nonzero

Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.19)

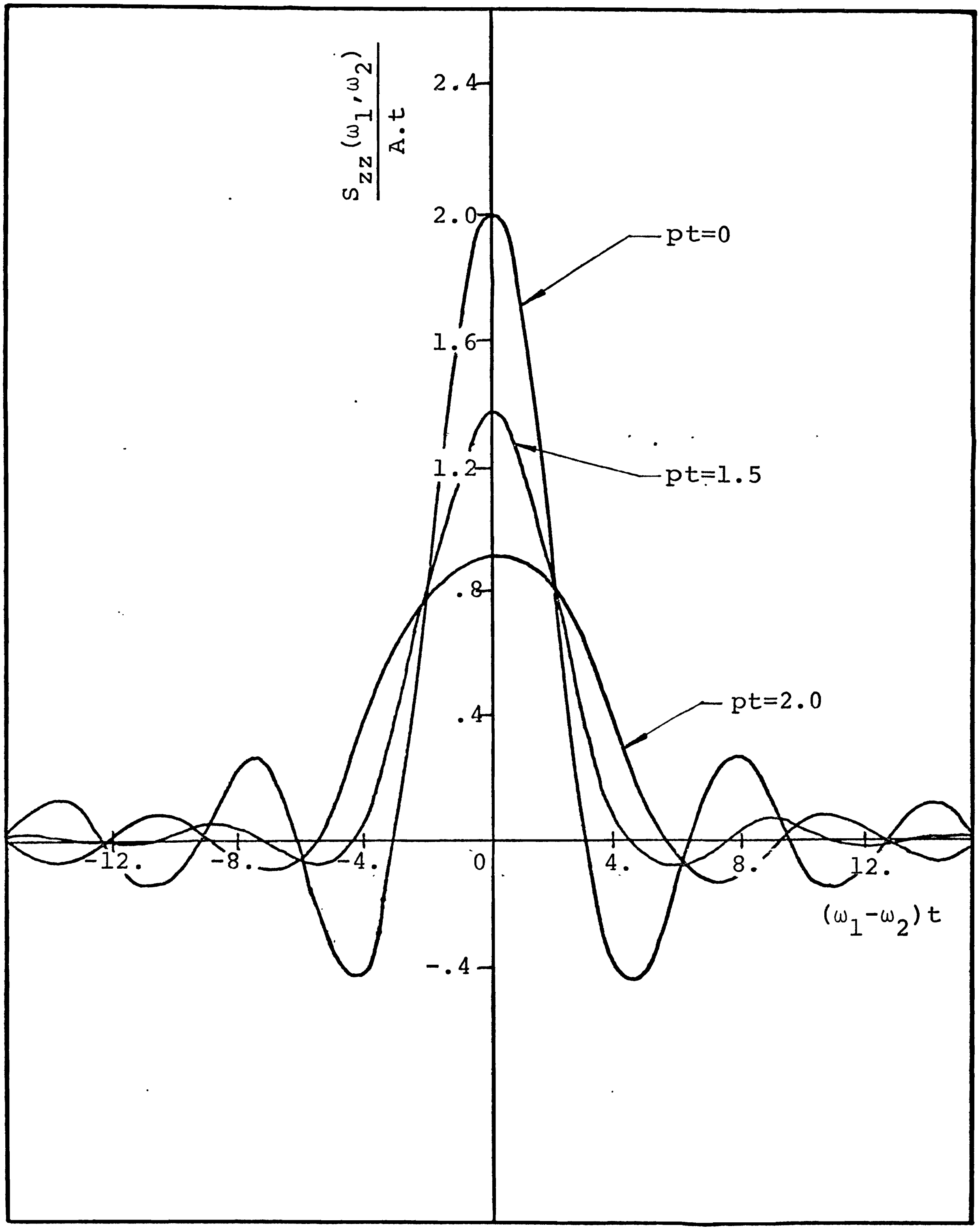

Fig. 4.3. Generalized Power Spectral Density of Excitation Z(t)

Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.13)

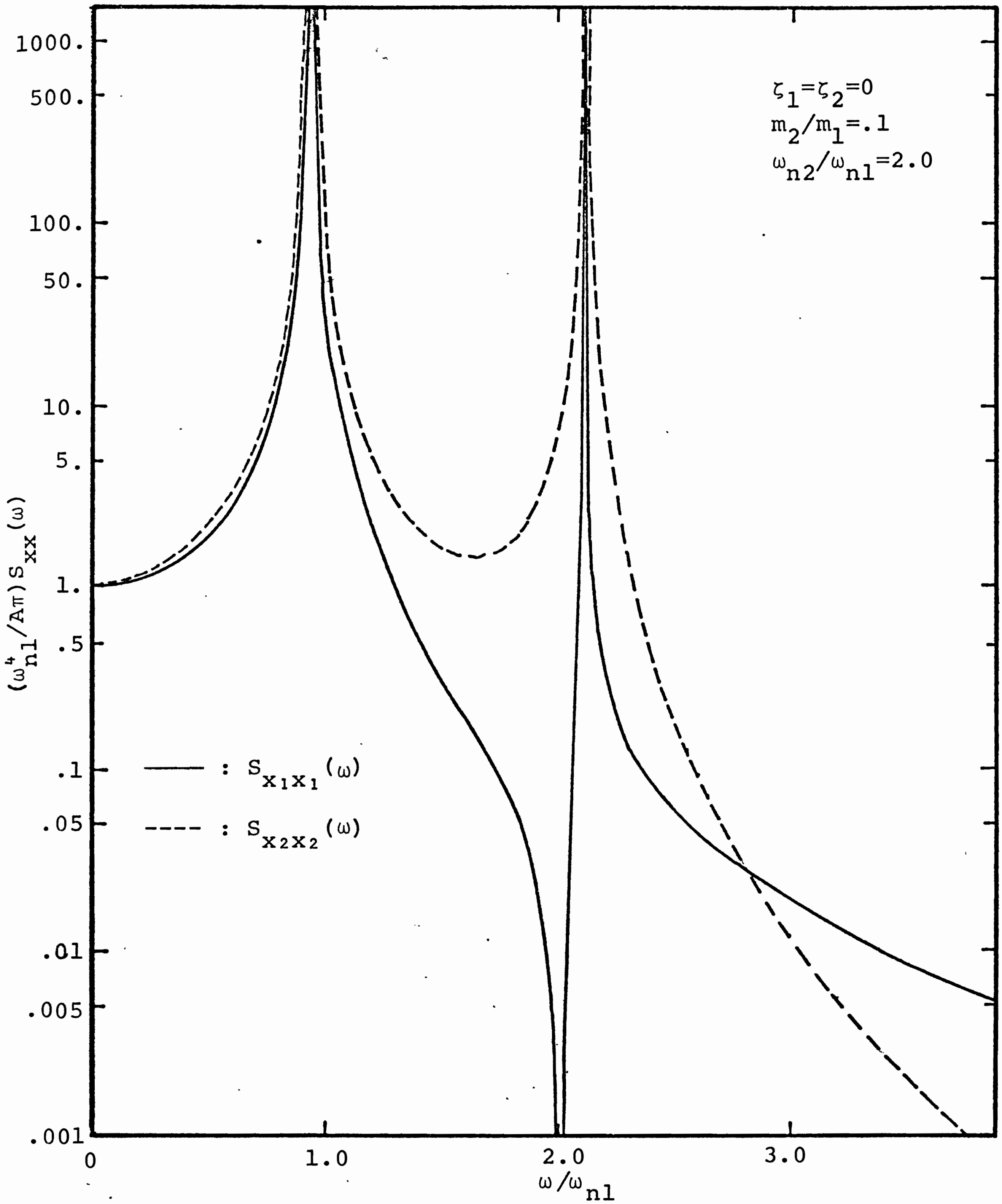

Fig. 4.4. Power Spectral Densities of Responses $X_1(t)$ & $X_2(t)$

Two-Degree-Of-Freedom System
Strength Function of Excitation = A
Governing Equation : (4.22)

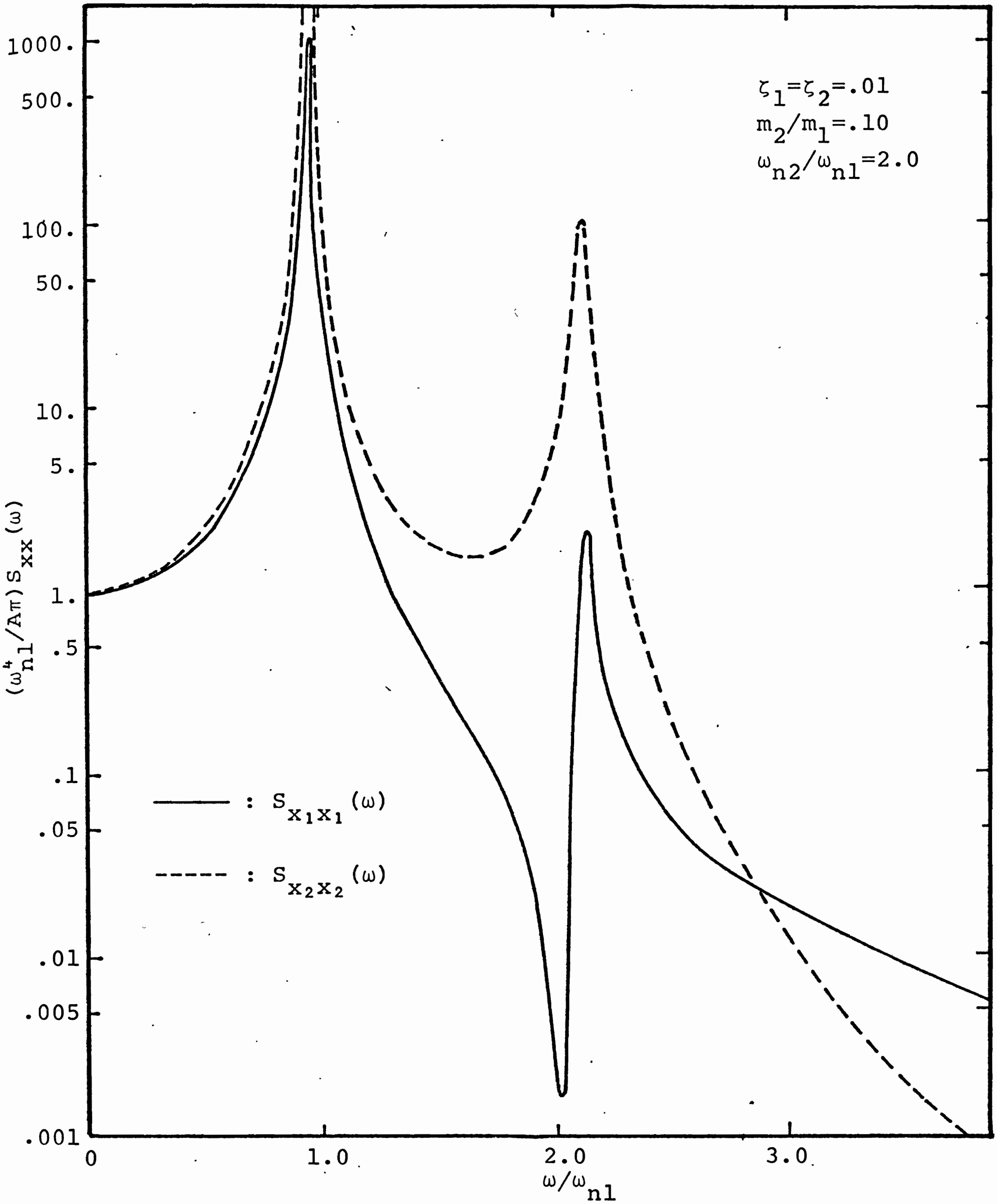

Fig. 4.5. Power Spectral Densities of Responses $X_1(t)$ & $X_2(t)$

Two-Degree-Of-Freedom System
Strength Function of Excitation = A
Governing Equation : (4.22)

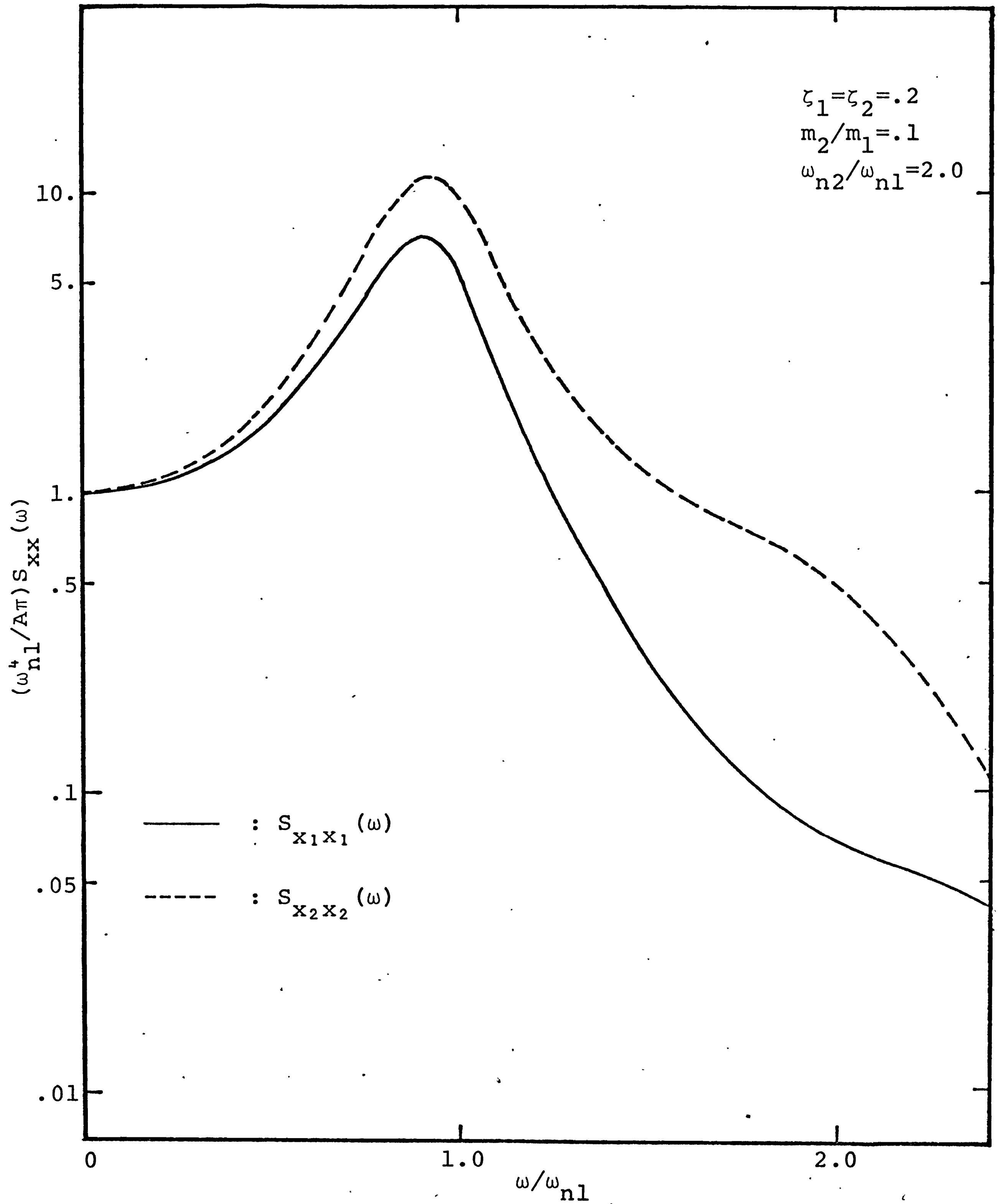

Fig. 4.6. Power Spectral Densities of Responses $X_1(t)$ & $X_2(t)$

Two-Degree-Of-Freedom System
Strength Function of Excitation = A
Governing Equation : (4.22)

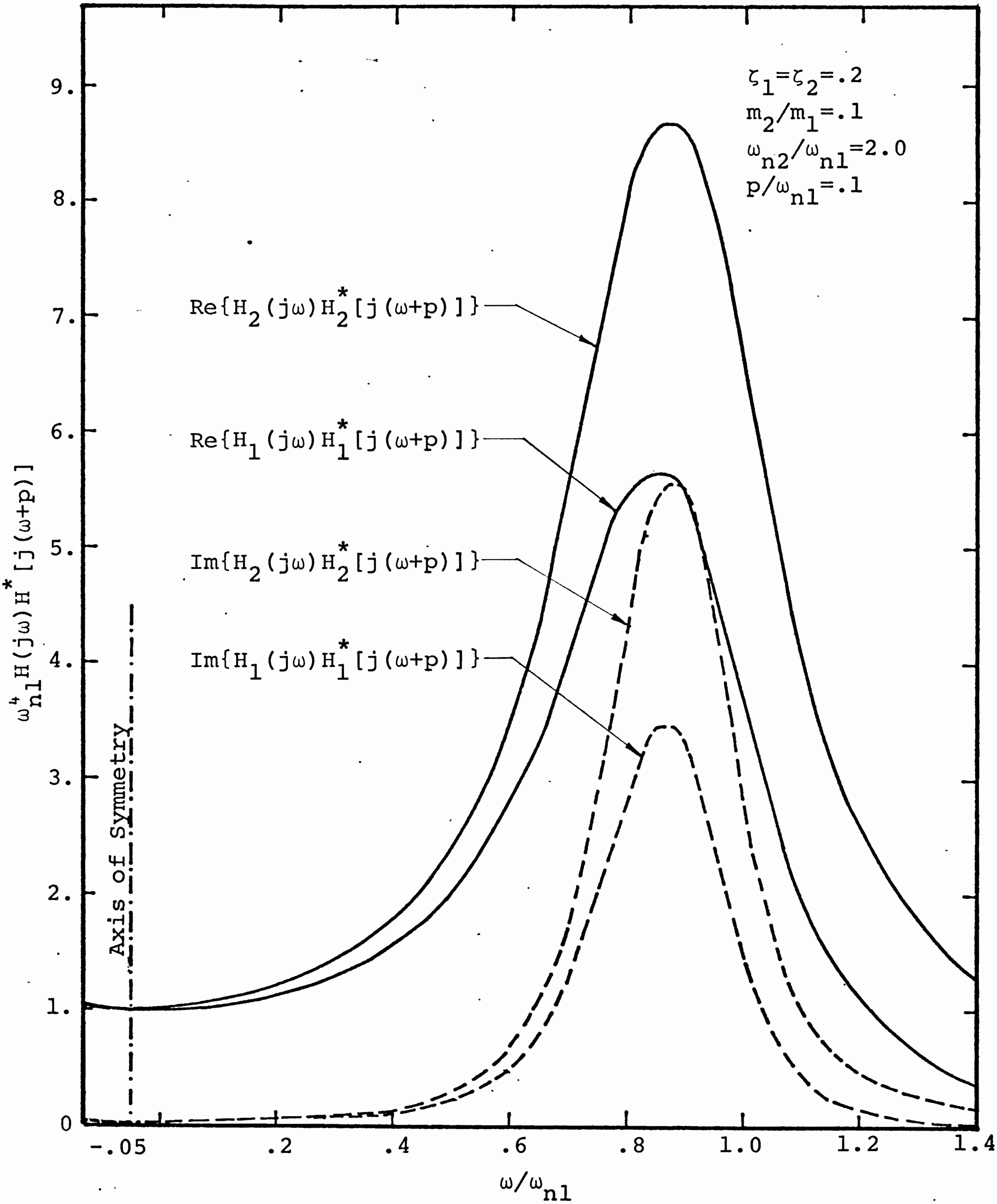

Fig. 4.7. Real and Imaginary Parts of Products $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=.1\omega_{n1}$
Governing Equations : (4.38) & (4.39)

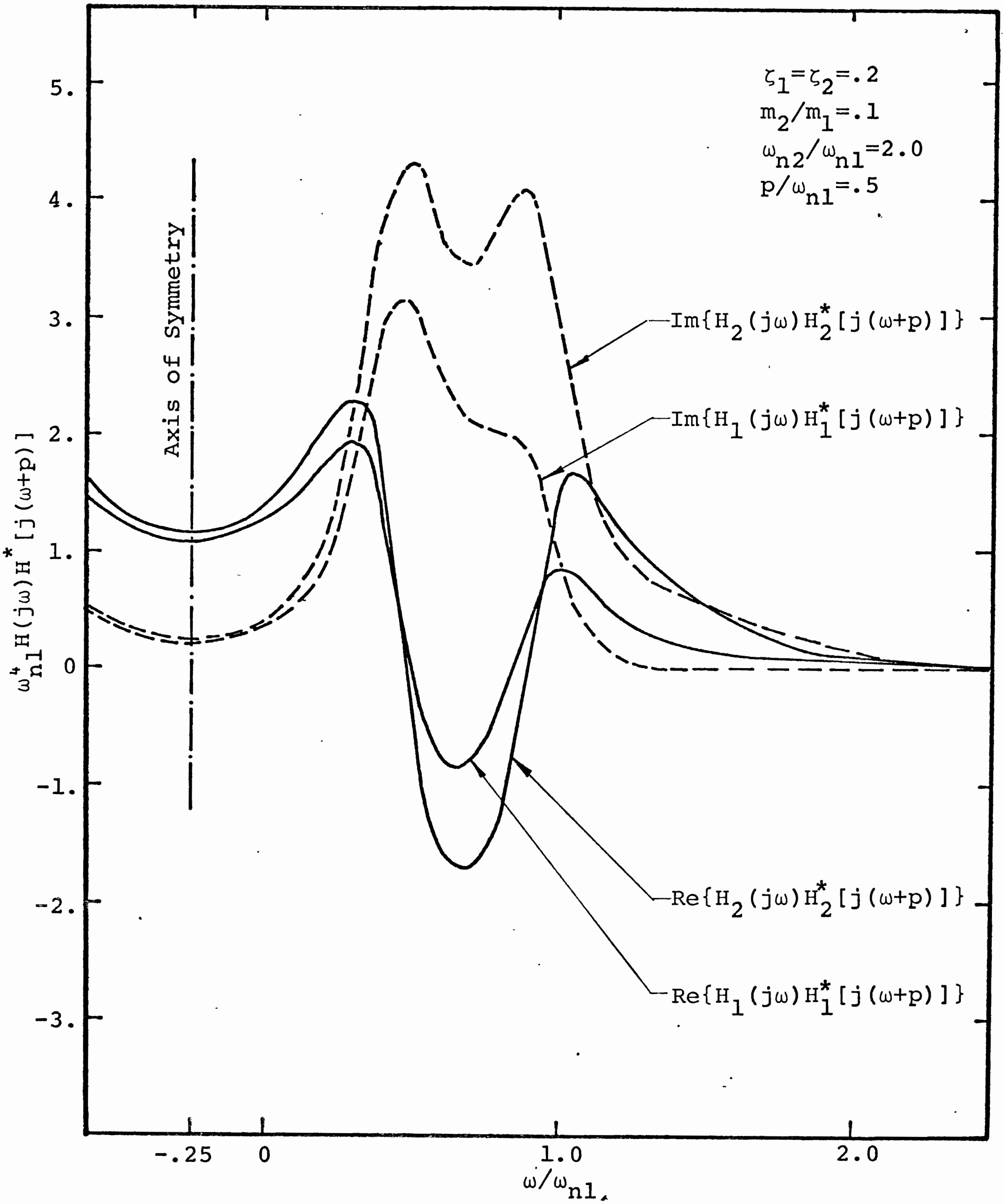

Fig. 4.8. Real and Imaginary Parts of Products $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=.5\omega_{n1}$
Governing Equations : (4.38) & (4.39)

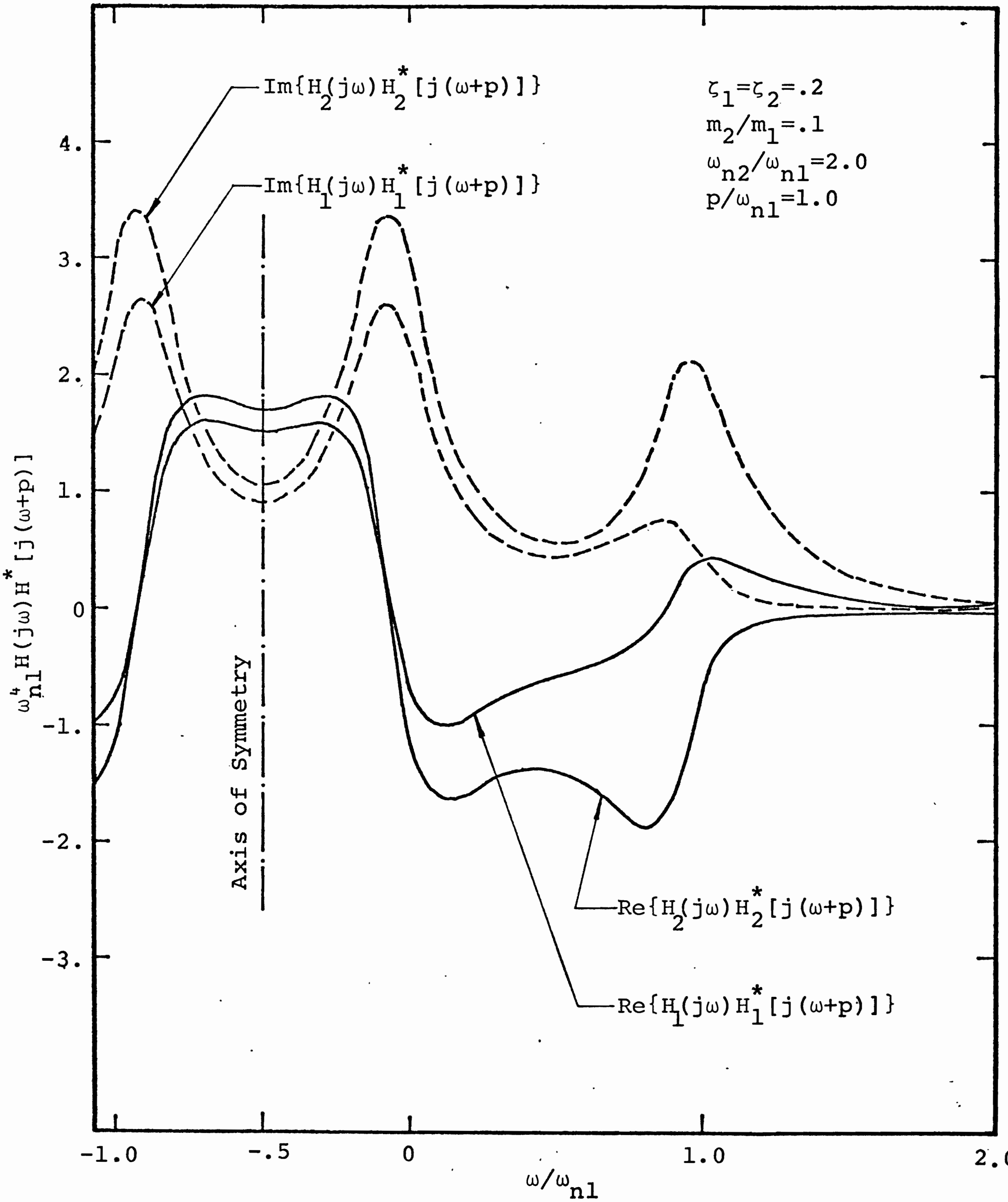

Fig. 4.9. Real and Imaginary Parts of Products $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = Acospt$_1$; p=ω_{n1}
Governing Equations : (4.38) & (4.39)

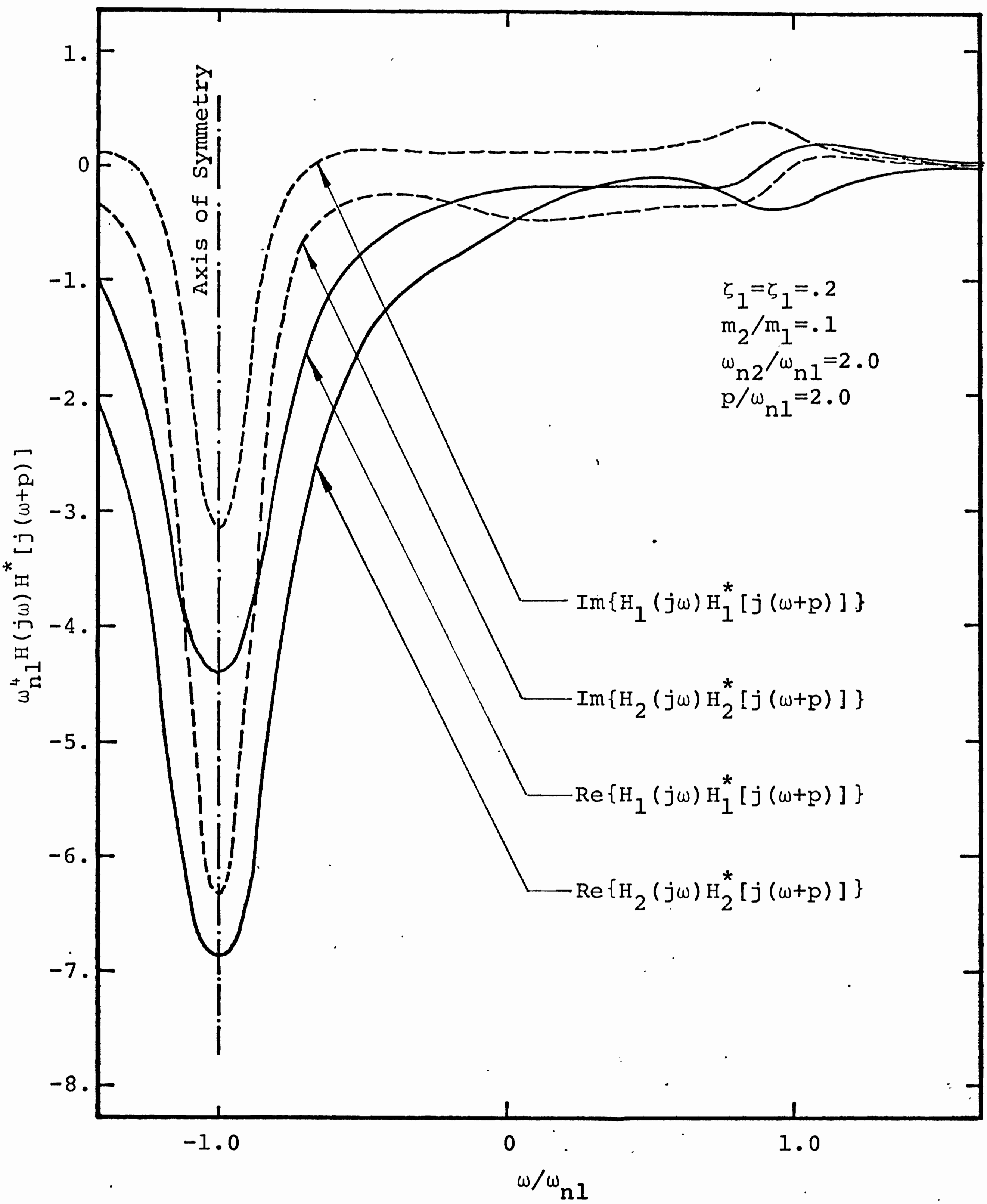

Fig. 4.10. Real and Imaginary Parts of Products $\omega^4_{n1}H(j\omega)H^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=2\omega_{n1}$
Governing Equations : (4.38) & (4.39)

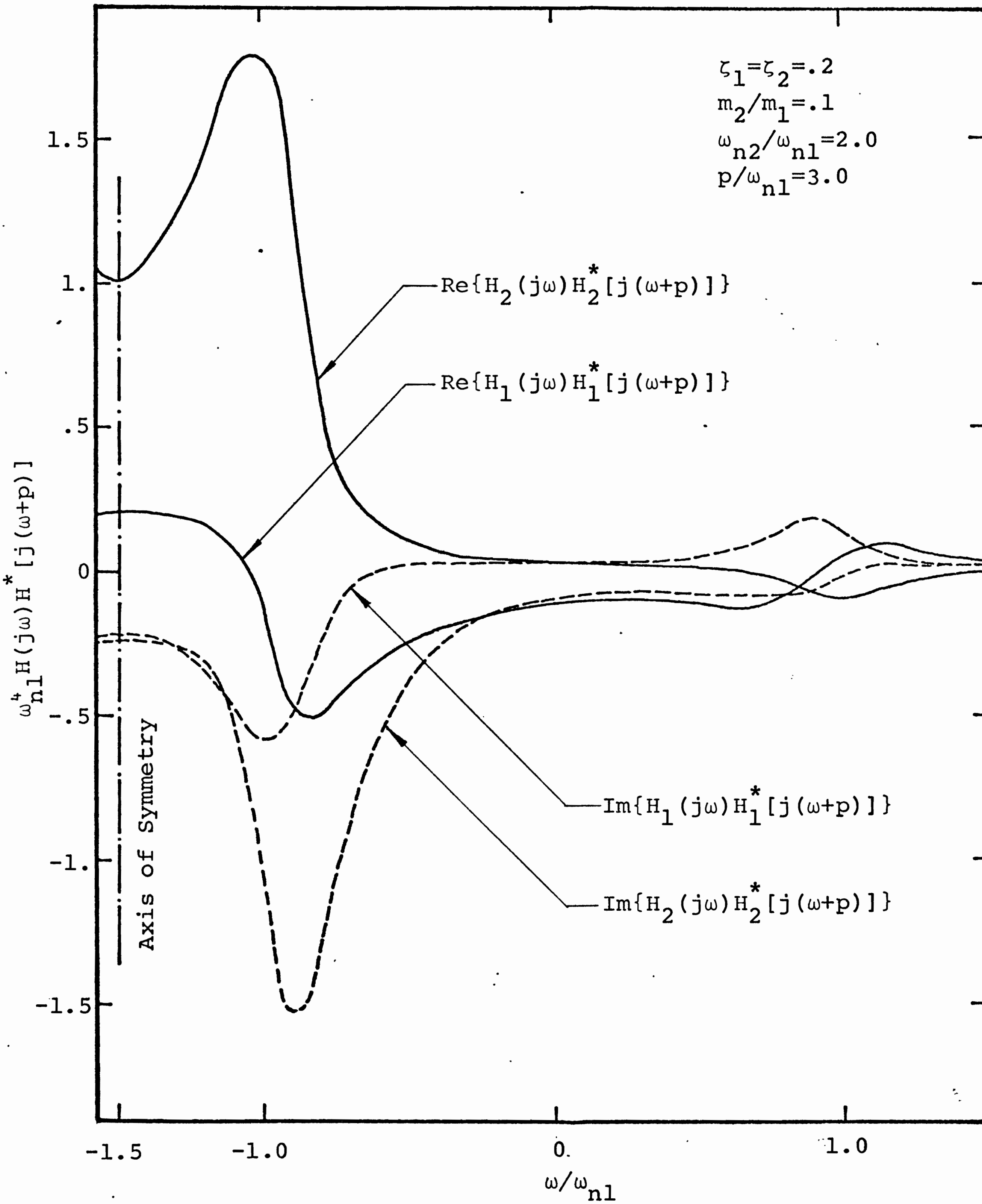

Fig.4.11. Real and Imaginary Parts of Products $\omega_{n1}^4 H(j\omega)H^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=3\omega_{n1}$
Governing Equations : (4.38) & (4.39)

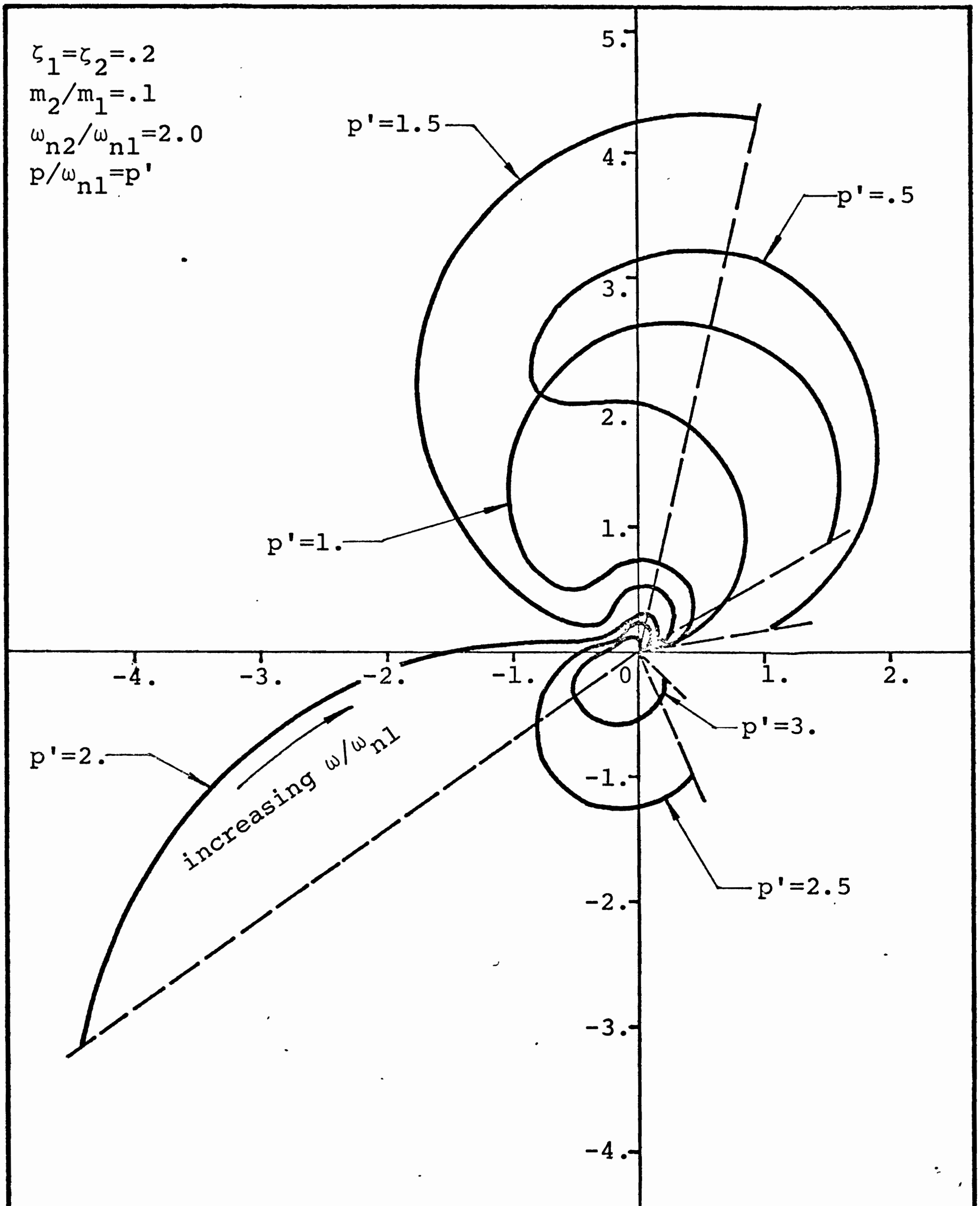

Fig. 4.12. Vectorial Plots of $\omega_{n1}^4 H_1(j\omega)H_1^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.38)

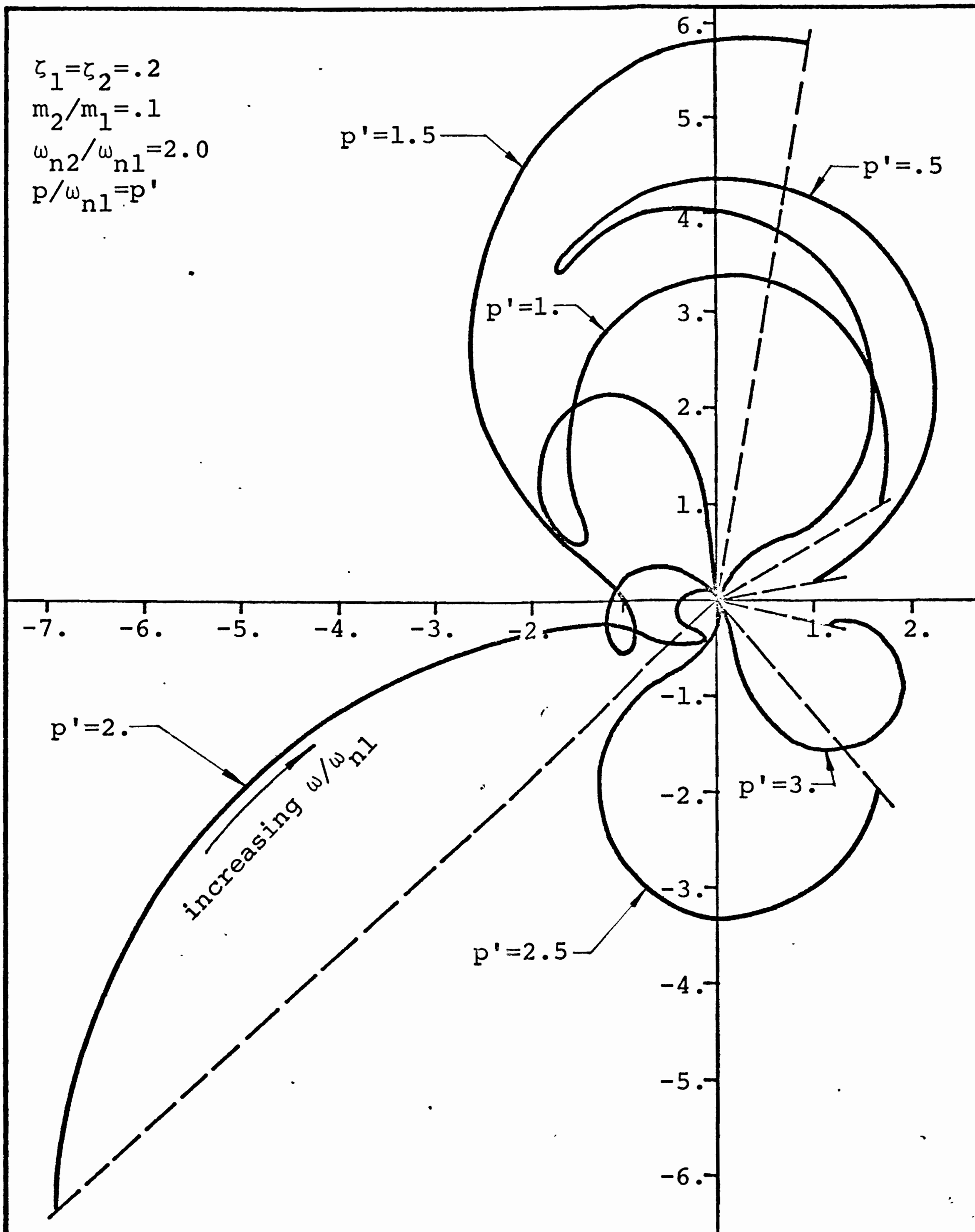

Fig. 4.13. Vectorial Plots of $\omega_{n1}^4 H_2(j\omega) H_2^*[j(\omega+p)]$

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.39)

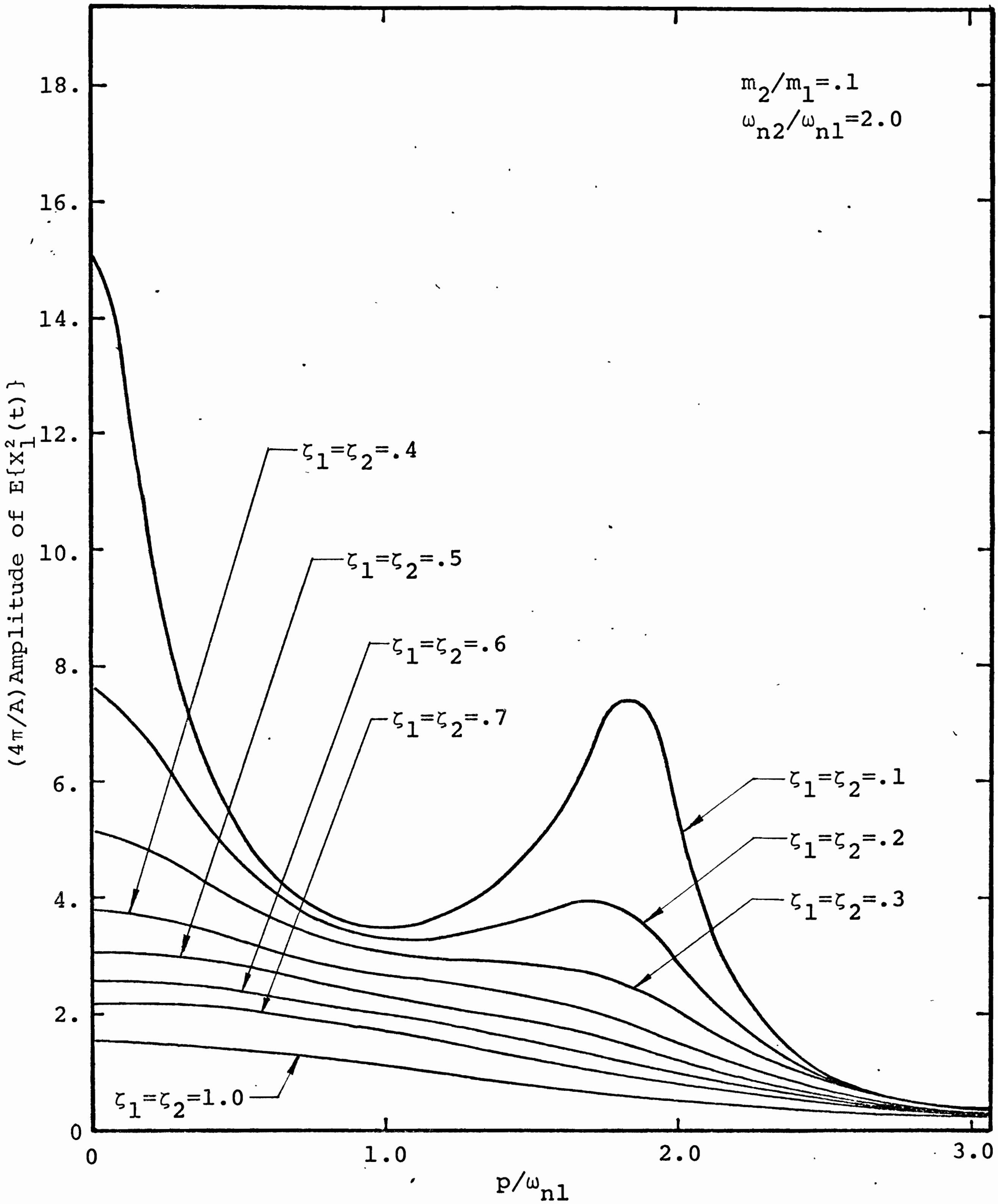

Fig. 4.14. Amplitude of $E\{X_1^2(t)\}$ against p/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.40)

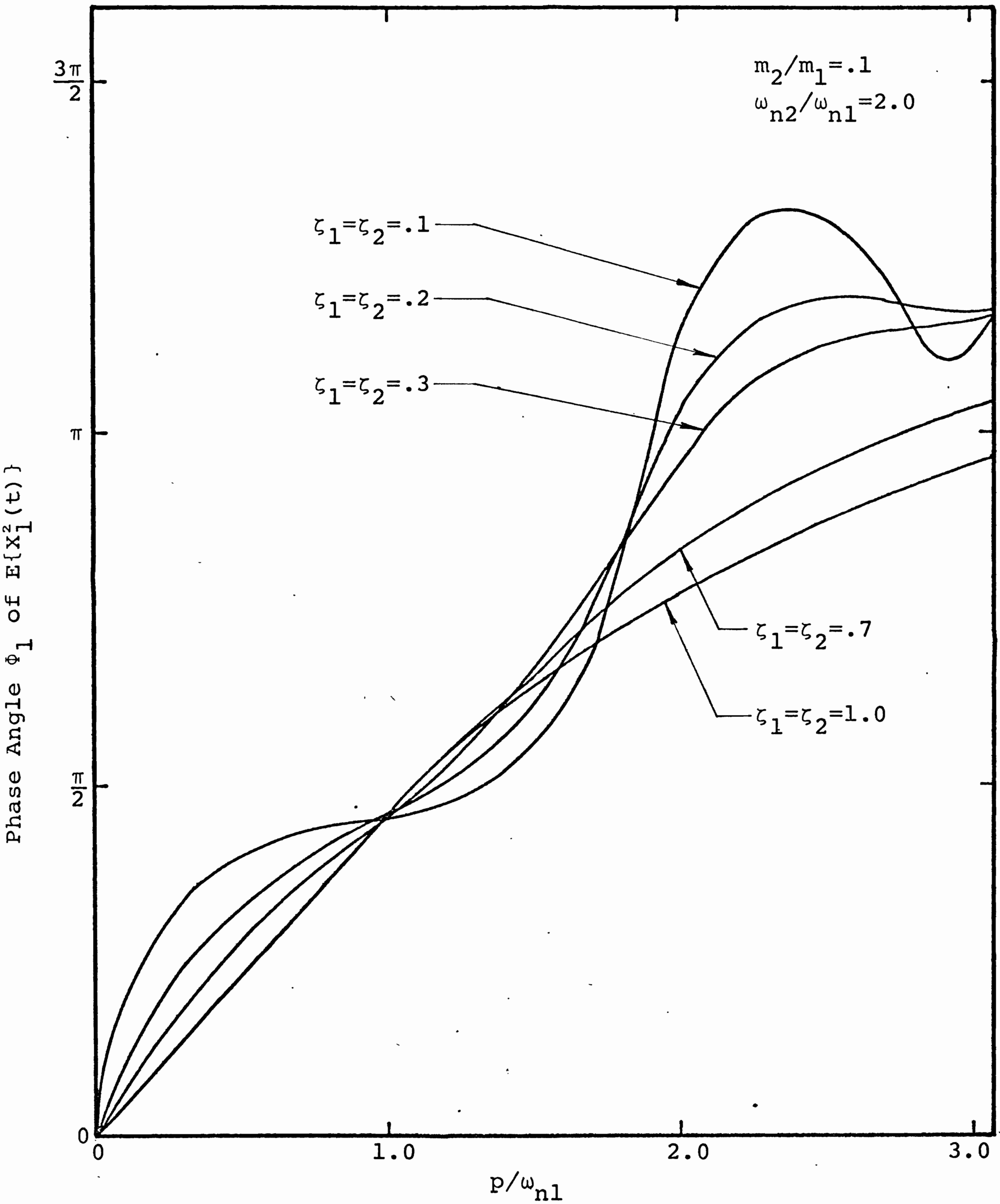

Fig. 4.15. Phase Angle of $E\{X_1^2(t)\}$ against p/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.40)

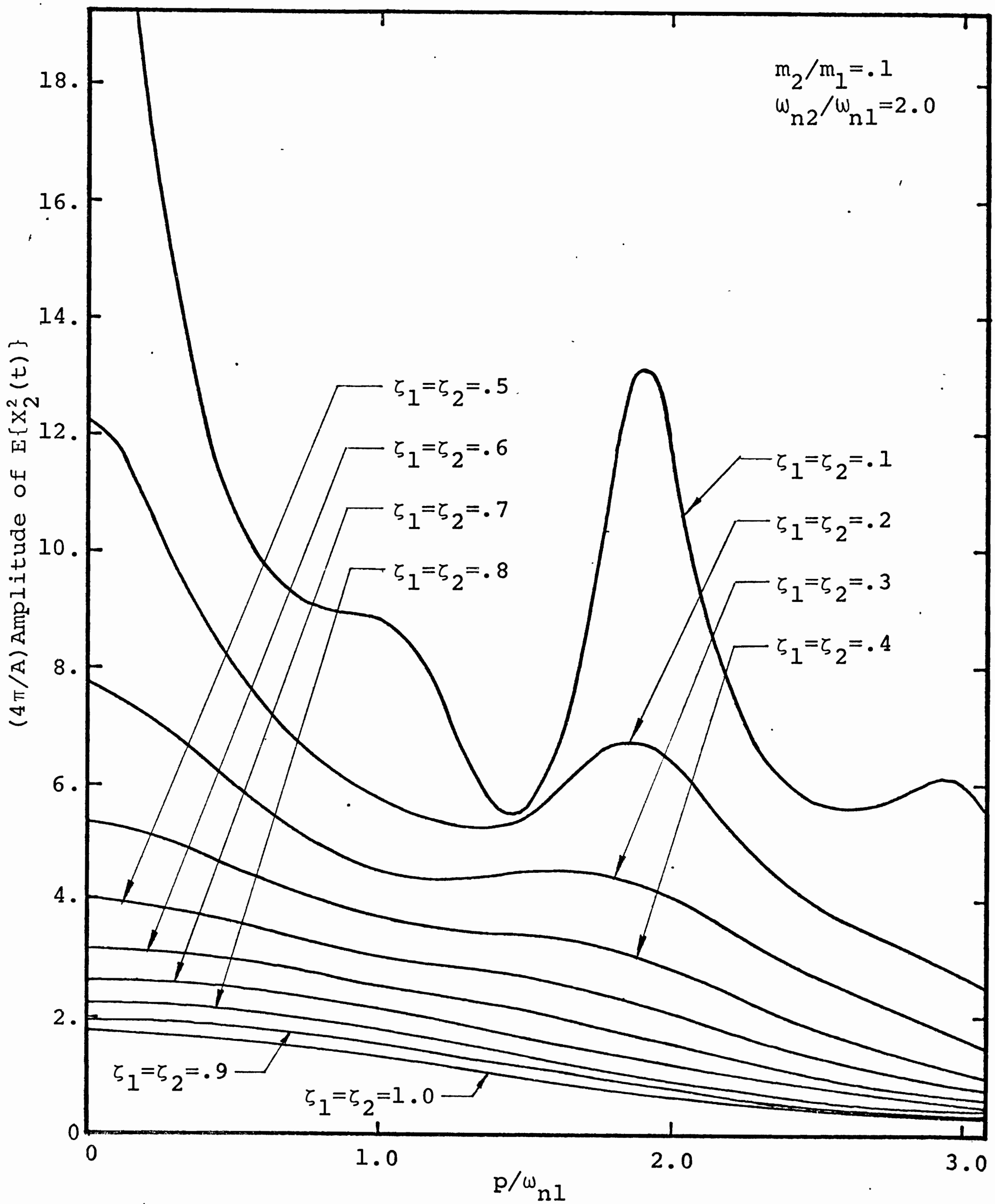

Fig. 4.16. Amplitude of $E\{X_2^2(t)\}$ against p/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$
Governing Equation : (4.41)

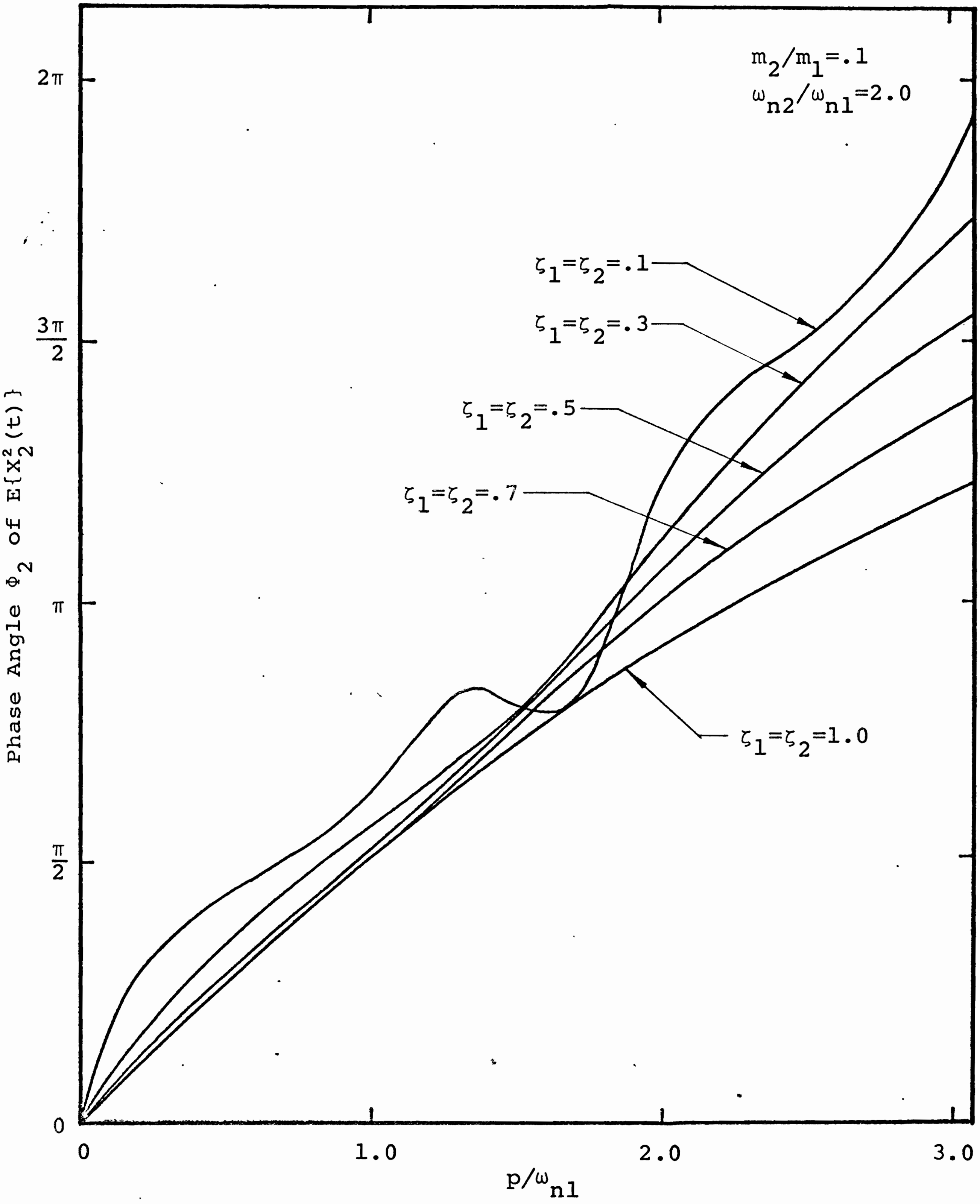

Fig. 4.17. Phase Angle of $E\{X_2^2(t)\}$ against p/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $Acospt_1$
Governing Equation : (4.41)

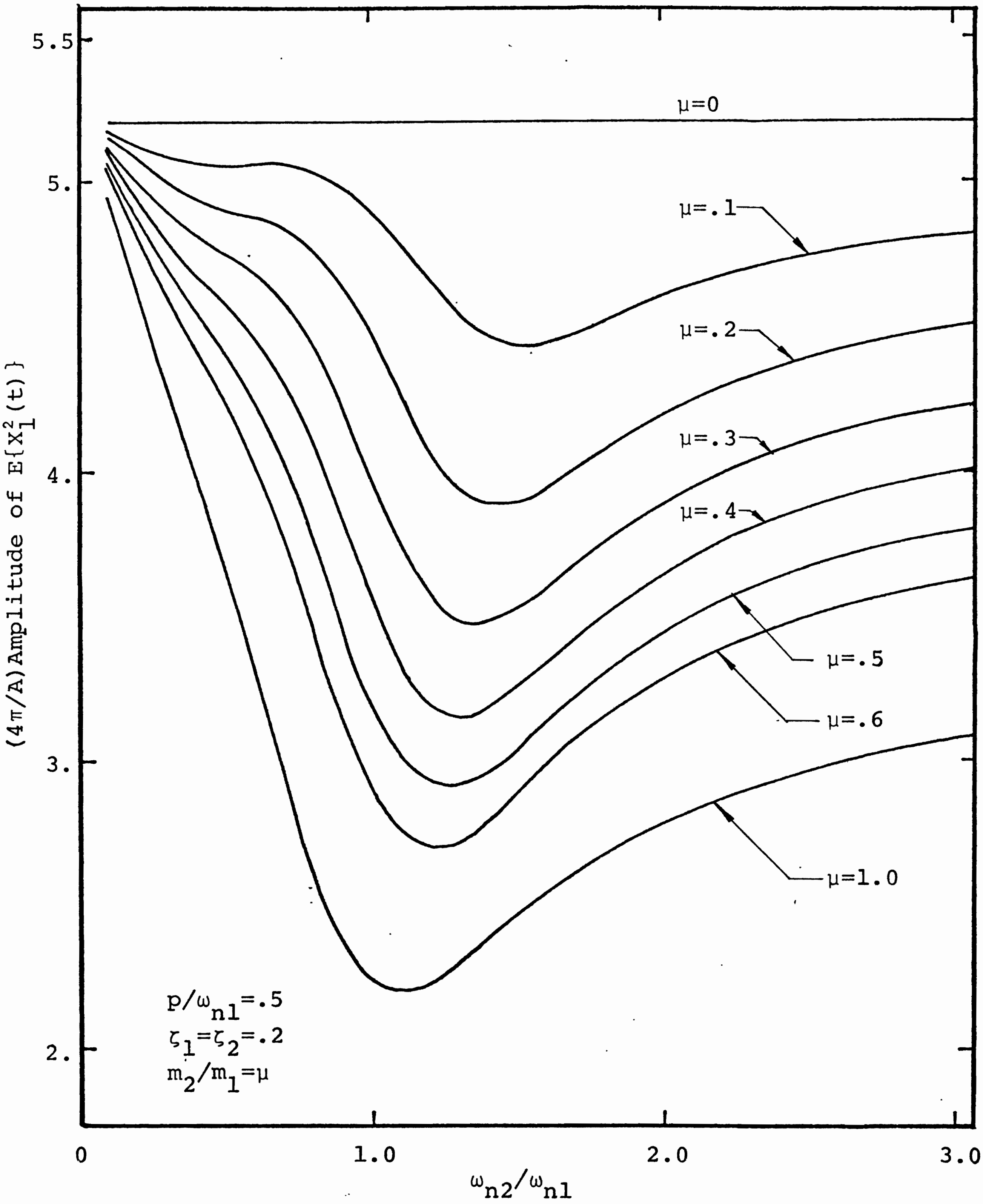

Fig. 4.18. Amplitude of $E\{X_1^2(t)\}$ against ω_{n2}/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=.5\omega_{n1}$
Governing Equation : (4.40)

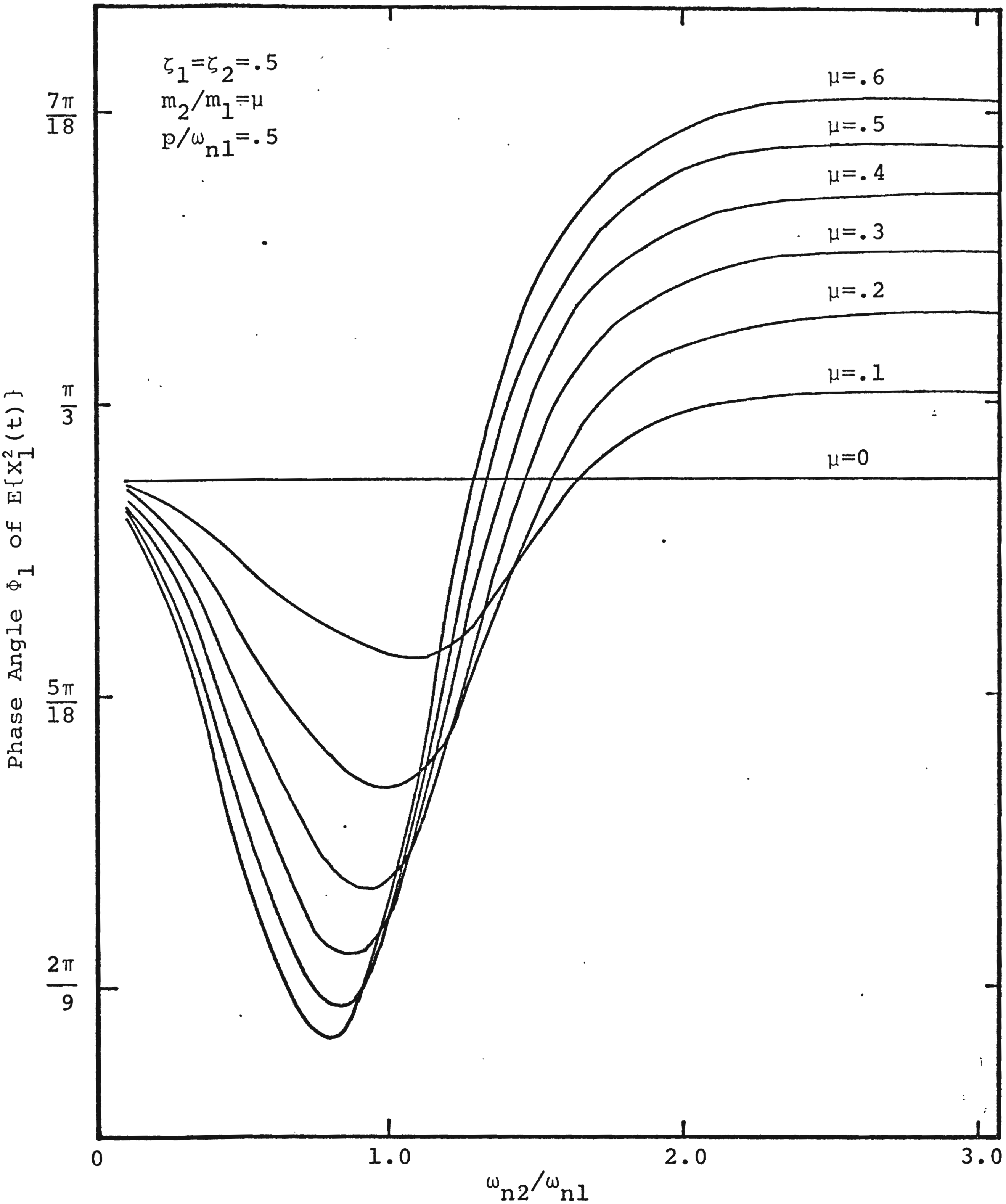

Fig. 4.19. Phase Angle of $E\{X_1^2(t)\}$ against ω_{n2}/ω_{n1}

Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=.5\omega_{n1}$
Governing Equation : (4.40)

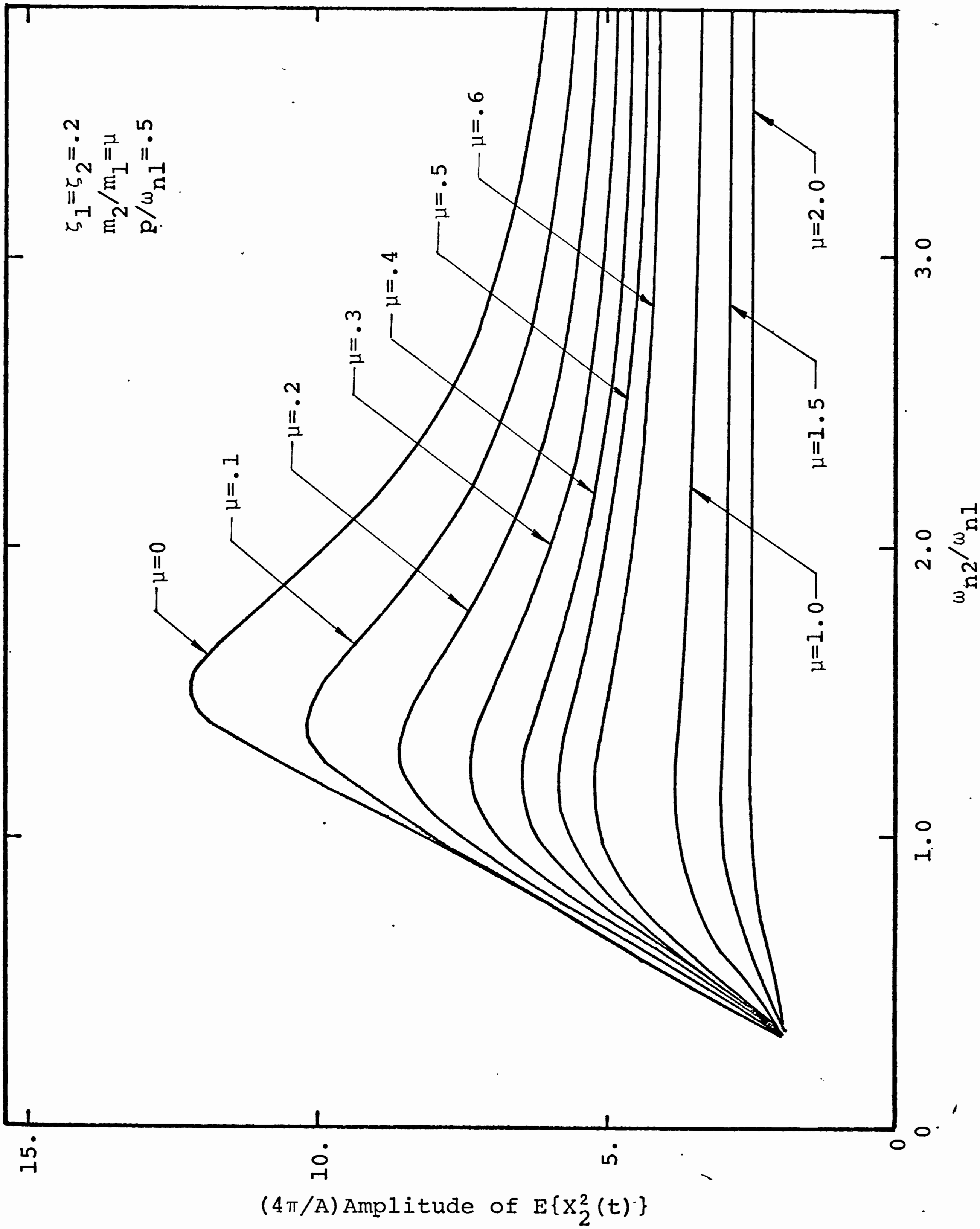

Fig.4.20. Amplitude of $E\{X_2^2(t)\}$ against ω_{n2}/ω_{n1}

Two-Degree-Of-Freedom System

Strength Function Of Excitation = $A\cos pt_1$; $p=.5\omega_{n1}$

Governing Equation : (4.41)

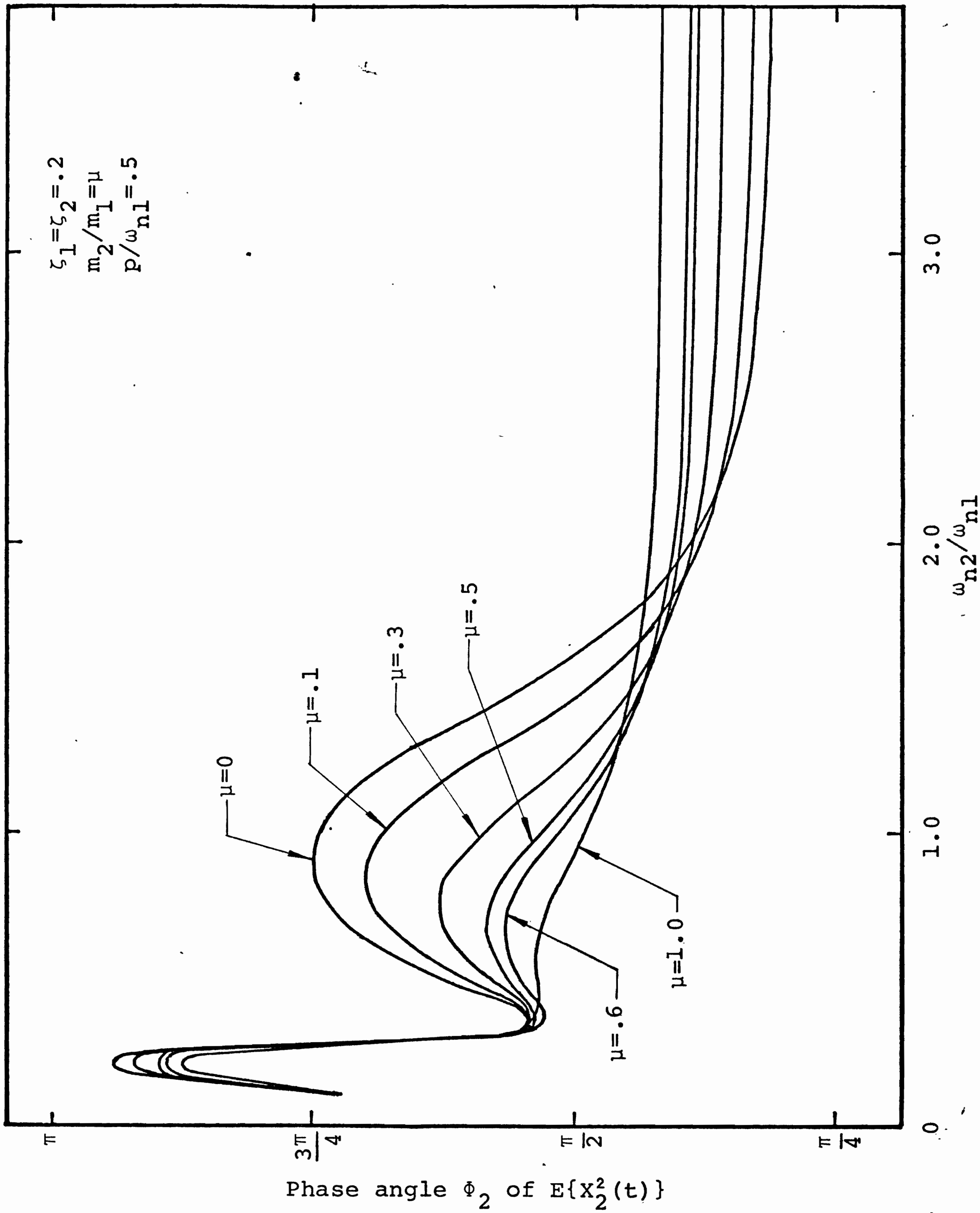

Fig. 4.21. Phase Angle of $E\{X_2^2(t)\}$ against ω_{n2}/ω_{n1}

Two-Degree-Of-Freedom System

Strength Function of Excitation = $A\cos pt_1$; $p=.5\omega_{n1}$

Governing Equation : (4.41)

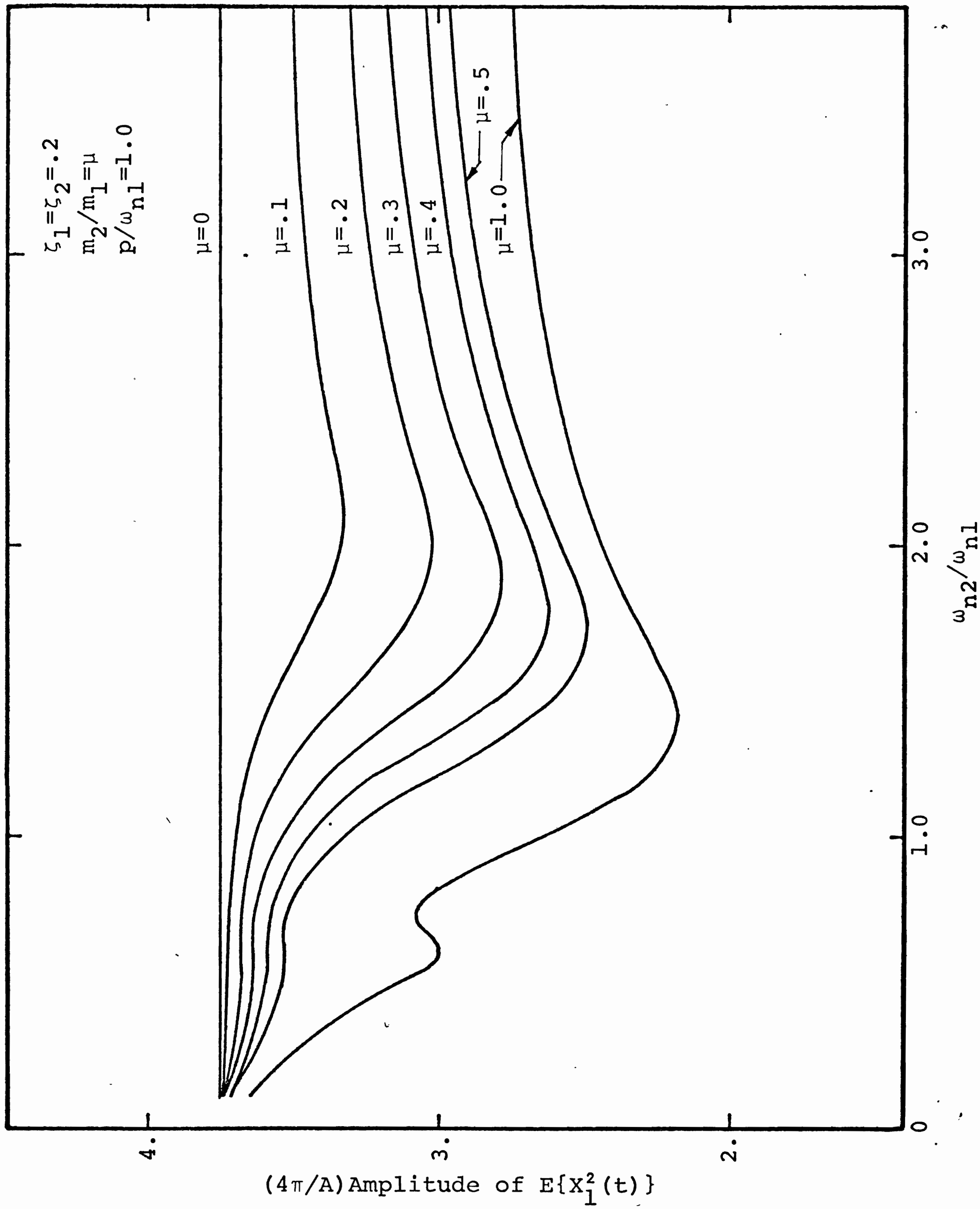

Fig. 4.22. Amplitude of $E\{X_1^2(t)\}$ against ω_{n2}/ω_{n1}
Two-Degree-Of-Freedom System
Strength Function of Excitztion = $A\cos pt_1$; $p=\omega_{n1}$
Governing Equation : (4.40)

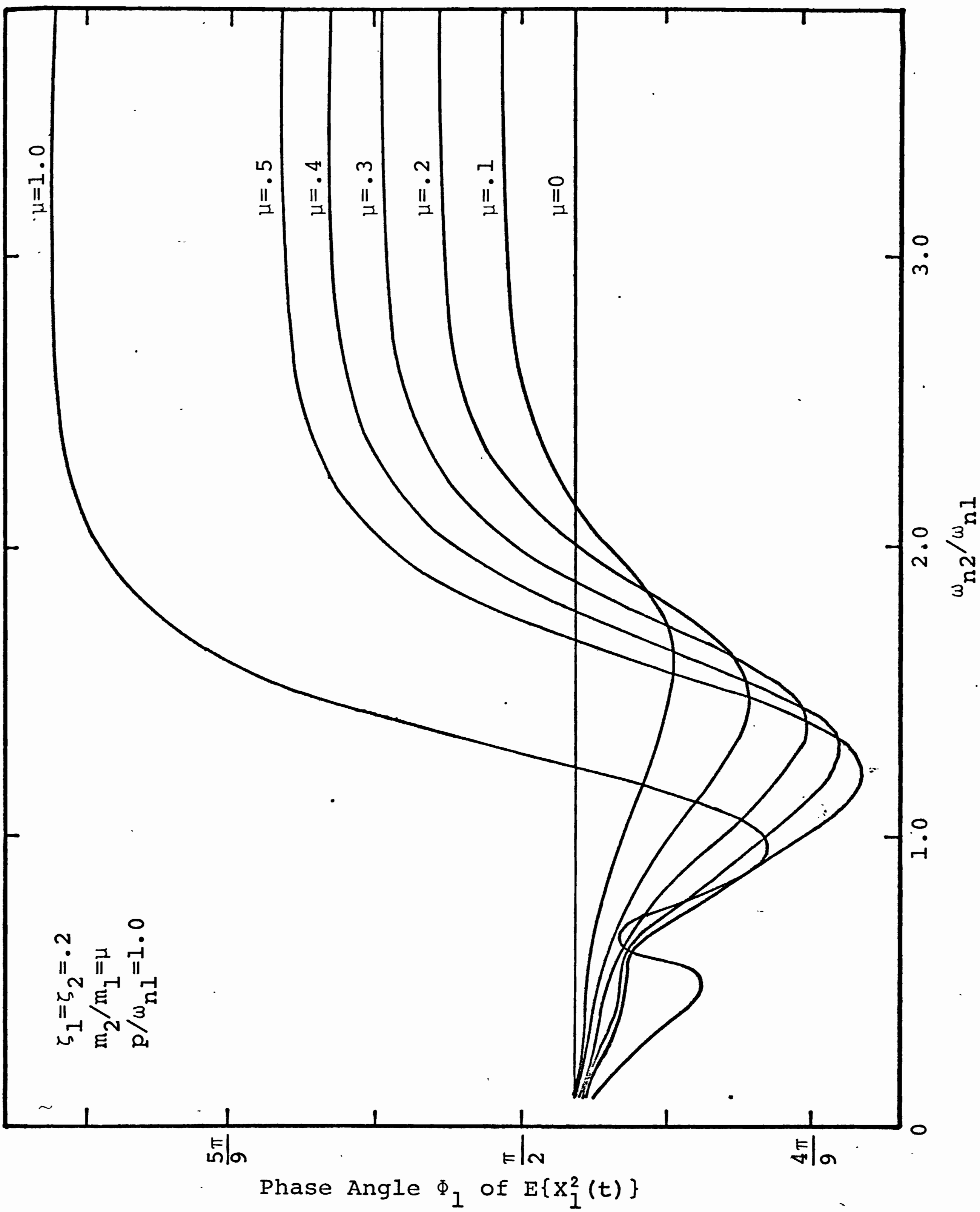

Fig. 4.23. Phase Angle of $E\{X_1^2(t)\}$ against ω_{n2}/ω_{n1}

Two-Degree-Of-Freedom System

Strength Function of Excitation = $A\cos pt_1$; $p=\omega_{n1}$

Governing Equation : (4.40)

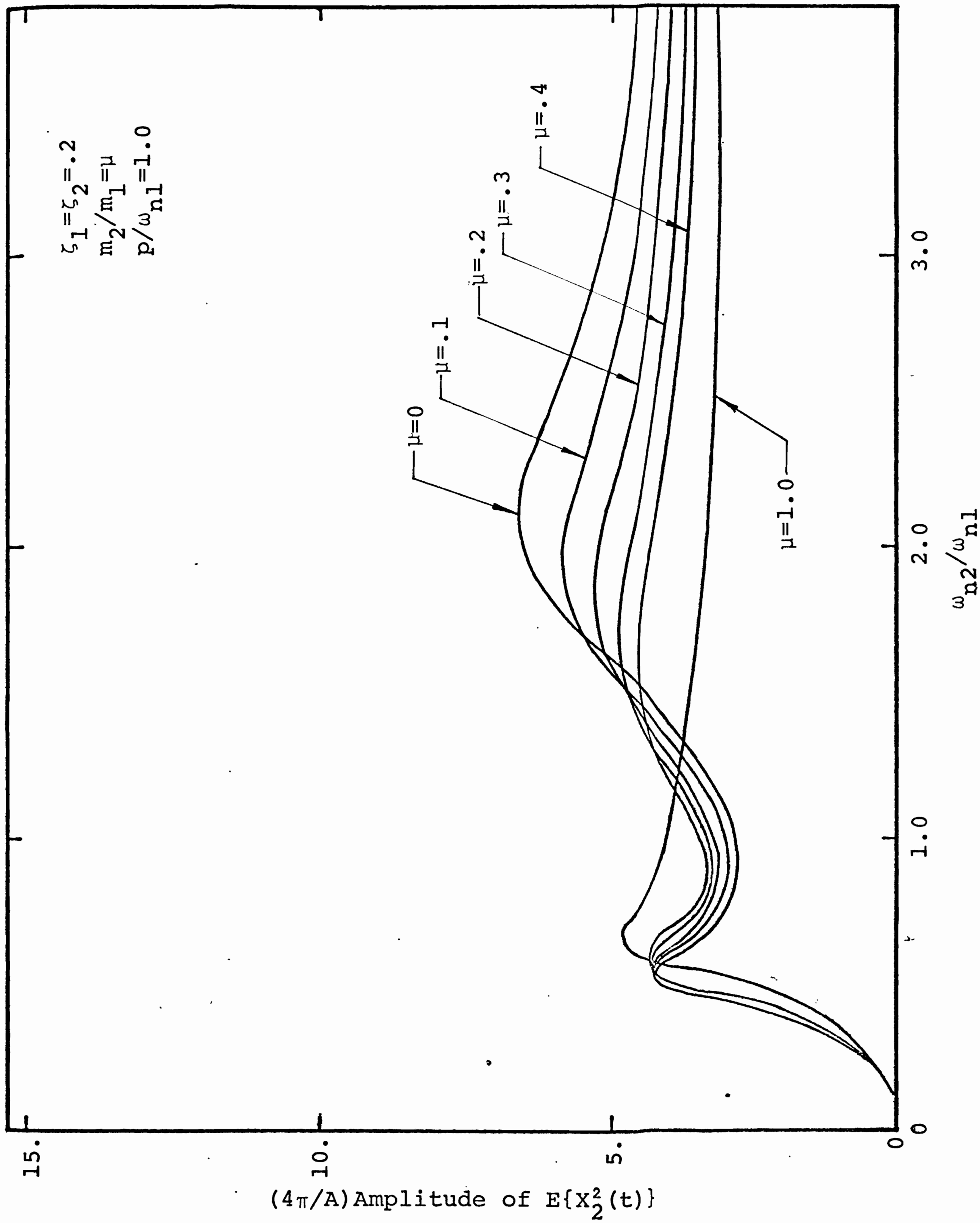

Fig. 4.24. Amplitude of $E\{X_2^2(t)\}$ against ω_{n2}/ω_{n1}
Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=\omega_{n1}$
Governing Equation : (4.41)

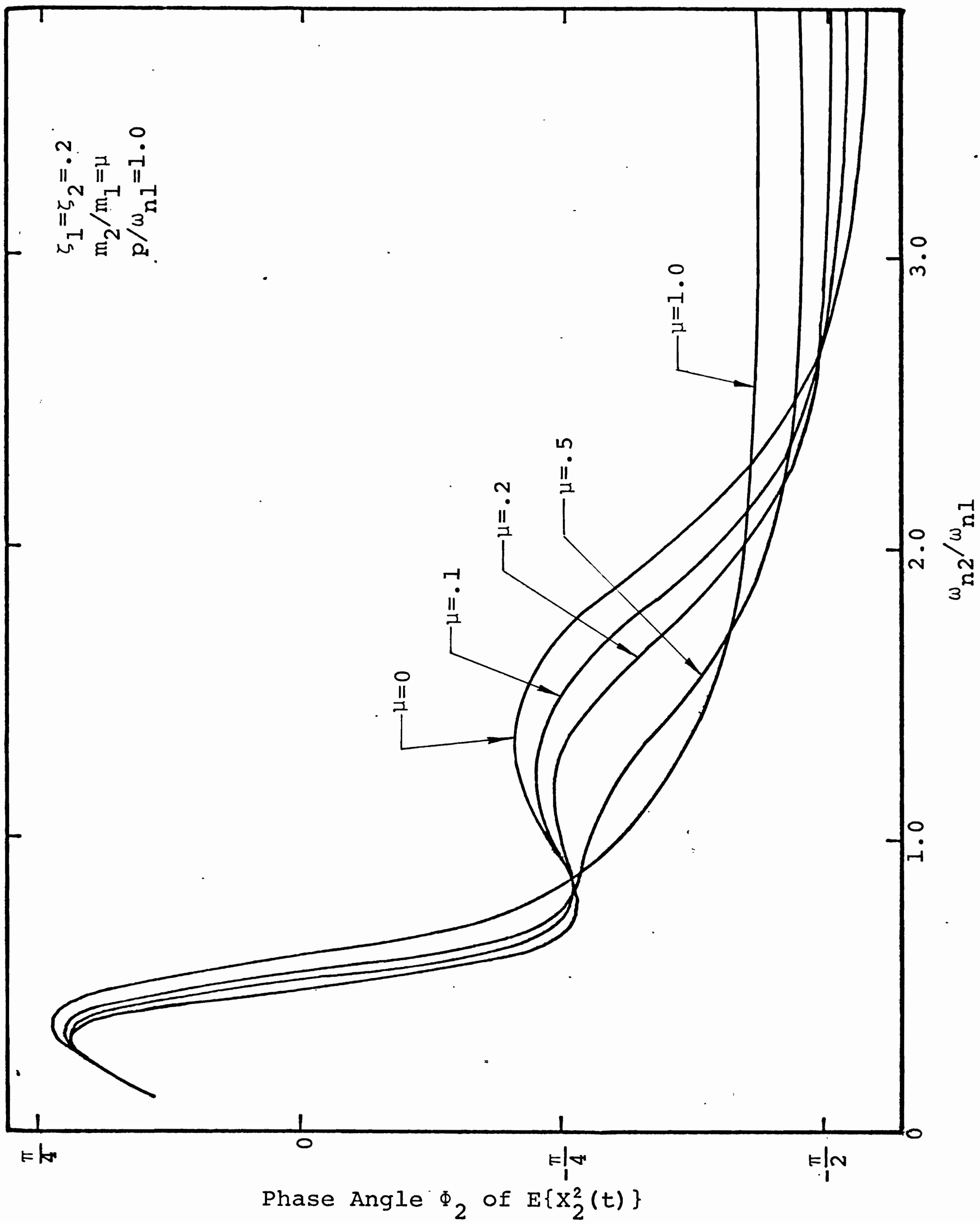

Fig. 4.25. Phase Angle of $E\{X_2^2(t)\}$ against ω_{n2}/ω_{n1}
Two-Degree-Of-Freedom System
Strength Function of Excitation = $A\cos pt_1$; $p=\omega_{n1}$
Governing Equation : (4.41)

CHAPTER 5

CONCLUSIONS

A detailed investigation on the response of linear mechanical systems subjected to nonstationary type of random excitations is presented. The external forces are modeled in terms of a modulating part and a white noise stationary part with a zero mean. From the autocorrelation of the input excitation and the impulse or frequency response function of the system, the statistics of the response are evaluated. Analytical results are described in the form of graphs showing the mean square responses and their variations with respect to the basic parameters of the system.

From the knowledge of the system response under nonstationary excitation with different strength functions $I(t_1)$, it is found that the time variation of the mean square response depends directly on the functional nature of the strength $I(t_1)$. When $I(t_1)$ is harmonic, the maximum mean square amplitude $E\{X^2(t)\}$ varies rapidly with the frequency ratio depending on the amount of damping. If $I(t_1)$ is an exponentially decaying function of time, maximum $E\{X^2(t)\}$ changes monotonically to large values depending on the value of the decay factor.

For one-degree-of-freedom systems, the mean square response is very sensitive to the damping of the system as well as to the frequency ratio p' and the decay parameter α. As expected, for low damping values, peak responses occur at one or two "resonance regions" , defined by $p \simeq 0$ and $p \simeq 2\omega_n$, where p is the frequency of the strength function $I(t_1)$. For exponentially decaying strength function, the maximum mean square response increases with the value of the decay parameter α and is asymptotic at $\alpha = 2\zeta\omega_n$. Under linearly varying strength function, the value of $E\{X^2(t)\}$ has a maximum value at $t=0$ and decreases rapidly as time increases. The variations of the responses with regard to parameters ζ, α, p/ω_n etc... are clearly indicated and the "regions of resonance" in each case are identified in Chapter 3.

Similar characteristics for maximum mean square values of the response were obtained for the case of two-degree-of-freedom systems. Here, the system is sensitive not only to the type of parameters mentioned above, but also to the ratio of the masses and the two natural frequencies of the system. Information on the variation of the phase angle is also provided at the end of Chapter 4. It is found that the second mass of the two-degree-of-freedom system acts as an absorber to the main system. In all these above cases, the results were checked by setting the strength function to a constant and comparing the results with the previous investigations [2] on the response of linear systems to stationary excitations.

The importance of the result presented in this thesis can be related to the fact that, to a large extent, random excitations on mechanical systems are of nonstationary type. The reason for this is, in most of the systems, the external forces include random disturbances in addition to dynamically fluctuating deterministic forces. For example, mechanical systems such as connecting rods,suspensions, gear trains, etc... do experience random excitations, over and above dynamic forces, that change rapidly in magnitude during specific operating time intervals. Another justification for nonstationary excitation arises from the consideration that any measured excitation process may not be long enough to provide statistical properties independent of time. On such occasions, a type of analysis presented in this thesis must be used in evaluating responses.

The following suggestions are made as future extension to the investigation presented here.

(i) Experimental verifications of the results must be made on the real mechanical system even though simulation of nonstationary forces and measurements of responses in such cases are very difficult.

(ii) The present analysis may be extended to many-degree-of-freedom systems. Here, the mathematical calculations become cumbersome and physical visualization of results is difficult.

(iii) The random component of the excitation is assumed as a white noise, stationary process in this investigation. For some cases, it may be necessary to consider this part as a correlated process [11,12], in which case, the nonstationary forces will not have a delta-correlation.

(iv) Investigations to include nonlinear systems are necessary. In this case, the technique of Fokker-Planck is to be employed which, in turn, may be difficult to solve.

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APPENDIX

FORTRAN COMPUTER PROGRAMS

FOR NUMERICAL RESULTS

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 0.A
C     PROGRAM TO EVALUATE THE POWER SPECTRAL DENSITY OF THE INPUT Z(T)
C     WHEN ITS AUTOCORRELATION IS
C     A*EXP(-ALPHA*P*T1)*COS(P*T1)*DELTA(T1-T2)
C     AL=ALPHA
C     B=P*T
C     A=ALPHA*P*T1
C     SZZ=POWER SPECTRAL DENSITY OF THE INPUT Z(T)
C     X=(OMEGA1-OMEGA2)*T
      AL=0.0
    7 B=0.0
    5 PRINT 4,AL,B
    4 FORMAT(10X,9H ALPHA = ,F5.3,7H P*T = ,F5.3)
      A=AL*B
      CT3=EXP(-A)
      X=-20.5
    1 CT1=(A**2)+(B-X)**2
      CT2=(A**2)+(B+X)**2
      CT4=(B-X)*SIN(B-X)-A*COS(B-X)
      CT5=(B+X)*SIN(B+X)-A*COS(B+X)
      CT6=(A+CT3*CT4)/CT1
      CT7=(A+CT3*CT5)/CT2
      CZZ=CT6+CT7
      PRINT 3,X,CZZ
    3 FORMAT(10X,2F10.5)
      X=X+1.0
      IF(X-20.0)1,1,2
    2 B=B+0.5
      IF(B-2.0)5,5,6
    6 AL=AL+0.1
      IF(A-1.0)7,7,8
    8 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 0.8
C     PROGRAM TO EVALUATE THE POWER SPECTRAL DENSITY OF THE INPUT Z(T)
C     WHEN ITS AUTOCORRELATION IS
C     A*EXP(-ALPHA*T1)*DELTA(T1-T2)
C     A=ALPHA*T
C     CZZ=POWER SPECTRAL DENSITY OF Z(T)
C     X=(OMEGA1-OMEGA2)*T
      A=0.0
    4 PRINT 6,A
    6 FORMAT(10X,11H ALPHA*T = ,F5.2)
      CT4=EXP(-A)
      X=-20.0
    1 CT1=1.0/(A**2+X**2)
      CT2=A*COS(X)-X*SIN(X)
      CT3=A-CT4*CT2
      SZZ=CT1*CT3
      PRINT 3,X,SZZ
    3 FORMAT(10X,2F10.5)
      X=X+1.0
      IF(X-20.0)1,1,2
    2 A=A+0.5
      IF(A-3.0)4,4,5
    5 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 0.C
C     PROGRAM TO EVALUATE THE POWER SPECTRAL DENSITY OF THE INPUT Z(T)
C     WHEN ITS AUTOCORRELATION IS
      A*(1-T1/T)*DELTA(T1-T2)
C     SZZ=POWER SPECTRAL DENSITY OF THE INPUT Z(T)
C     X=(OMEGA1-OMEGA2)*T
      A=0.1
    4 PRINT 6,A
    6 FORMAT(10X,14H CONSTANT A = ,F5.2)
      X=-20.5
    1 CT1=1.0/(A*(X**2))
      CT2=A*X*SIN(X)
      CT3=1.0-COS(X)-X*SIN(X)
      CT4=CT2+CT3
      SZZ=CT1*CT4
      PRINT 3,X,SZZ
    3 FORMAT(10X,2F10.5)
      X=X+1.0
      IF(X-20.0)1,1,2
    2 A=A+0.1
      IF(A-1.0)4,4,5
    5 STOP
      END
```

DUCDOAN   CDC 6600 FTN V3.0-P296 OPT=1   72/07/2

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 1
C     ONE-DEGREE-OF-FREEDOM SYSTEM
C     PROGRAM TO COMPUTE MEAN SQUARE VALUE OF OUTPUT X(T) WHEN
C     AUTOCORRELATION OF INPUT Z(T) IS IN THE FORM =
C     A*EXP(-ALPHA*P*T1)*COS(P*T1)*DELTA(T1-T2)
C     Z = DAMPING RATIO ZETA
C     AL = ALPHA
C     P = P/NATURAL FREQUENCY OF SYSTEM
C     XS = AMPLITUDE OF MEAN SQUARE VALUE OF THE OUTPUT X(T)
C     PH = PHASE ANGLE OF MEAN SQUARE VALUE OF THE OUTPUT X(T)
      AL=0.0
   10 Z=0.05
    7 PRINT 1,AL,Z
    1 FORMAT (5X,9H ALPHA = ,F5.2,8H ZETA = ,F5.2)
      P=0.0
    5 XS1=2.0*((1.0-Z**2)**0.5)
      XS8=2.0*Z-AL*P
      XS2=XS8**2
      XS3=(XS2+P**2)*XS1
      XS4=P*(XS2+P**2-XS1**2)
      XS5=XS2+(XS1-P)**2
      XS6=XS2+(XS1+P)**2
      XS7=XS1*XS5*XS6
      XS9=XS8*(XS2+XS1**2+P**2)
      XSS=(P/XS3)-(XS4/XS7)
      XSC=(XS8/XS3)-(XS9/XS7)
      XS=((XSS**2)+(XSC**2))**0.5
      TPH=XSS/XSC
      PH1=ATAN(TPH)
      PH=(180.0/3.1416)*PH1
      IF(PH)2,3,3
    2 PH=180.0+PH
    3 PH=PH
      PRINT 4,P,XS,PH
    4 FORMAT (10X,3F15.5)
      P=P+0.1
      IF(P-5.0)5,5,6
    6 Z=Z+0.1
      IF(Z-1.0)7,7,8
    8 AL=AL+0.5
      IF(AL-1.0)10,10,9
    9 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 1,A
C     ONE-DEGREE-OF-FREEDOM SYSTEMS
C     PROGRAM TO EVALUATE THE VALUE OF P WHERE AMPLITUDE OF THE MEAN
C     SQUARE VALUE OF X(T) IS MAXIMUM,
C     NEWTON NUMERICAL METHOD IS USED TO SOLVE FOR THE DERIVATIVE
C     OF THE MAX. AMPLITUDE OF THE MEAN SQUARE VALUE OF X(T)
C     AUTOCORRELATION OF INPUT Z(T) IS =
C     A*EXP(-ALPHA*P*T1)*COS(P*T1)*DELTA(T1-T2)
C     Z=DAMPING RATIO ZETA
C     AL=ALPHA
C     P=P/NATURAL FREQUENCY OF THE SYSTEM
C     XS=AMPLITUDE OF THE MEAN SQUARE VALUE OF X(T)
C     DXS=DERIVATIVE OF XS
C     DDXC=SECOND DERIVATIVE OF XS
      AL=0,2
    7 Z=0,05
    5 PRINT 1,AL,Z
    1 FORMAT(5X,9H ALPHA = ,F5,2,8H ZETA = ,F5,2)
      P=1,6
    3 XS1=2,0*((1,0-Z**2)**0,5)
      XS8=2,0*Z-AL*P
      B=XS8**2+P**2
      CT7=2,0*XS8*AL
      DB=(2,0*P)-CT7
      CT1=(XS8**2)+(P**2)-(XS1**2)
      C=P*CT1
      DC=CT1+(P*DB)
      CT2=(XS8**2)+(XS1-P)**2
      CT3=(XS8**2)+(XS1+P)**2
      D=CT2*CT3
      CT4=-(CT7+2,0*(XS1-P))*CT3
      CT5=(-CT7+2,0*(XS1+P))*CT2
      DD=CT4+CT5
      E=XS8
      DE=-AL
      CT6=(XS8**2)+(P**2)+(XS1**2)
      F=XS8*CT6
      DF=(-AL*CT6)+XS8*(2,0*P-CT7)
      AP=(P/B)-(C/D)
      AQ=(E/B)-(F/D)
      CT8=(B-P*DB)/(B**2)
      CT9=(D*DC-C*DD)/(D**2)
      DAP=CT8-CT9
      CT10=(B*DE-E*DB)/(B**2)
      CT11=(D*DF-F*DD)/(D**2)
      DAQ=CT10-CT11
      DXS=(AP*DAP)+(AQ*DAQ)
      XP=(1,0/XS1)*AP
      XQ=(1,0/XS1)*AQ
      XS=(XP**2+XQ**2)**0,5
      ADXS=ABS(DXS)
      IF(ADXS-0,000001)4,4,11
   11 DDB=2,0*(AL**2+1,0)
      DDC=4,0*(P-(XS8*AL))+2,0*P*(AL**2+1,0)
```

```
      DDD1=8.0*(-XS8*AL-(XS1-P))
      DDD2=XS8*AL+(XS1+P)
      DDD3=2.0*(AL**2+1.0)
      DDD4=XS8**2+(XS1+P)**2
      DDD=(DDD1*DDD2)+(DDD3*DDD4)
      DDF1=-4.0*AL*(P-XS8*AL)
      DDF2=2.0*(AL**2+1.0)*XS8
      DDF=DDF1+DDF2
      D2P1=((-P*B*DDB)-2.0*DB*(B-P*DB))/(B**3)
      D2P2=(D*(D*DDC-C*DDD)-2.0*DD*(D*DC-C*DD))/(D**3)
      D2P=D2P1-D2P2
      D2Q1=((-B*E*DDB)-2.0*DB*(B*DE-E*DB))/(B**3)
      D2Q2=(D*(D*DDF-F*DDD)-2.0*DD*(D*DF-F*DD))/(D**3)
      D2Q=D2Q1-D2Q2
      D2XS=(DAP**2)+(AP*D2P)+(DAQ**2)+(AQ*D2Q)
      PRINT 2,P,DXS,XS,D2XS
    2 FORMAT(10X,15H P/NAT. FRQ. = ,F19.9,15H D(MSQ.X(T)) = ,F19.9,12H M
     1SQ.X(T) = ,F10.5,8H D2XS = ,F10.5)
      P=P-(DXS/D2XS)
      GO TO 3
    4 PRINT 2,P,DXS,XS,D2XS
      Z=Z+0.1
      IF(Z-1.0)5,5,8
    8 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 2
C     ONE-DEGREE-OF-FREEDOM SYSTEM
C     PROGRAM TO COMPUTE THE MEAN SQUARE VALUE OF THE OUTPUT X(T) AT
C     T=0 WHEN THE AUTOCORRELATION OF THE INPUT Z(T) HAS THE FORM =
C     A*EXP(-ALPHA*T1)DELTA(T1-T2)
C     Z=DAMPING RATIO
C     AL=ALPHA/NATURAL FREQUENCY
C     XMSQ = MEAN SQUARE VALUE OF X(T)
      Z=0.05
    4 PRINT 6,Z
    6 FORMAT(10X,5H Z = ,F5.2)
      AL=0.0
    2 A=2.0*Z-AL
      B=(AL**2)-(4.0*Z*AL)+4.0
      XMSQ=2.0/(A*B)
      PRINT 1,AL,XMSQ
    1 FORMAT(10X,2F10.5)
      AL=AL+0.10
      IF(AL-5.0)2,2,3
    3 Z=Z+0.05
      IF(Z-1.0)4,4,5
    5 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 2.A
C     ONE-DEGREE-OF-FREEDOM SYSTEM
C     PROGRAM TO COMPUTE THE MEAN SQUARE VALUE OF THE OUTPUT X(T)
C     IN FUNCTION OF T1/T WHEN THE AUTOCORRELATION OF THE INPUT Z(T)
C     HAS THE FORM =
C     A*(1.0-T1/T)*DELTA(T1-T2)
C     Z=DAMPING RATIO
C     CA=CONSTANT K
C     XMSQ=MEAN SQUARE VALUE OF X(T)
      CA=0.5
    7 Z=0.05
    4 PRINT 6,CA,Z
    6 FORMAT(5X,5H K = ,F5.2,5H Z = ,F5.2)
      CT1=CA*(1.0+2.0*(Z**2))
      CT2=4.0*Z*3.1416
      CT=1.0+(CT1/CT2)
      CT3=1.0/(4.0*Z)
      X=0.0
    2 XMSQ=CT3*(CT-(CA*X))
      PRINT 1,X,XMSQ
    1 FORMAT(10X,8H T1/T = ,F10.5,8H XMSQ = ,F10.5)
      X=X+0.5
      IF(X-5.0)2,2,3
    3 Z=Z+0.05
      IF(Z-1.0)4,4,5
    5 CA=CA+0.5
      IF(CA-2.0)7,7,8
    8 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 3
C     TWO-DEGREE -OF-FREEDOM SYSTEM
C     PROGRAM COMPUTES THE PRODUCTS OF RECEPTANCE FUNCTIONS OF THE
C     SYSTEM = H1(U)*H1(U+P) AND H2(U)*H2(U+P) WHEN ONLY ONE OF TWO
C     INPUTS Z1(T) AND Z2(T) EXITS, AND HAS THE FORM =
C     A*COS(P*T1)*DELTA(T1-T2)
C     P=P/NATURAL FREQUENCY OMEGA 1
C     PH1=DAMPING RATIO ZETA 1
C     PH2=DAMPING RATIO ZETA 2
C     UM=MASS RATIO M2/M1
C     OM2=NATURAL FREQUENCY 2/NATURAL FREQUENCY 1
C     OM=INPUT FREQUENCY/NATURAL FREQUENCY 1
C     H1RE=REAL PART OF RECEPTANCE H1(U), ONLY EXCITATION Z2(T)
C     H1IM=IMAGINARY PART OF RECEPTANCE H1(U), ONLY EXCITATION Z2(T)
C     HP1RE=REAL PART OF RECEPTANCE H1(U+P), ONLY EXCITATION Z2(T)
C     HP1IM=IMAGINARY PART OF RECEPTANCE H1(U+P), ONLY EXCITATION Z2(T)
C     HH1RE=REAL PART OF H1(U)*H1(U+P), ONLY EXCITATION Z2(T)
C     HH1IM=IMAGINARY PART OF H1(U)*H1(U+P), ONLY EXCITATION Z2(T)
C     HH2RE=REAL PART OF H2(U)*H2(U+P), ONLY EXCITATION Z2(T)
C     HH2IM=IMAGINARY PART OF H2(U)*H2(U+P), ONLY EXCITATION Z2(T)
C     H1RE2=REAL PART OF RECEPTANCE H1(U), ONLY EXCITATION Z1(T)
C     H1IM2=IMAGINARY PART OF RECEPTANCE H1(U), ONLY EXCITATION Z1(T)
C     HP1RE2=REAL PART OF RECEPTANCE H1(U+P), ONLY EXCITATION Z1(T)
C     HP1IM2=IMAGINARY PART OF RECEPTANCE H1(U+P), ONLY EXCITATION Z1(T)
C     HH1RE2=REAL PART OF H1(U)*H1(U+P), ONLY EXCITATION Z1(T)
C     HH1IM2=IMAGINARY PART OF H1(U)*H1(U+P), ONLY EXCITATION Z1(T)
C     HH2RE2=REAL PART OF H2(U)*H2(U+P), ONLY EXCITATION Z1(T)
C     HH2IM2=IMAGINARY PART OF H2(U)*H2(U+P), ONLY EXCITATION Z1(T)
      P=0.0
      PH1=0.2
      PH2=0.2
      UM=0.1
      OM2=2.0
    6 PRINT 5,P
    5 FORMAT(5X,4H P= ,F5.2)
C     Z1=0, COMPUTE H1(U)*H1(U+P)
      OM=-2.5
    2 A=1.0+(OM2**2)*(1.0+UM)+4.0*PH1*PH2*OM2
      B=OM2**2
      C=2.0*(PH1+PH2*(1.0+UM)*OM2)
      D=2.0*PH1*(OM2**2)+2.0*PH2*OM2
      DEL1=(OM**4)-A*(OM**2)+B
      DEL2=-C*(OM**3)+(D*OM)
      ADEL=(DEL1**2)+(DEL2**2)
      Q1=UM*(OM2**2)
      Q2=2.0*UM*PH2*OM2*OM
      H1RE=((Q1*DEL1)+(Q2*DEL2))/ADEL
      H1IM=((Q2*DEL1)-(Q1*DEL2))/ADEL
      DELP1=(OM+P)**4-A*((OM+P)**2)+B
      DELP2=-C*((OM+P)**3)+D*(OM+P)
      ADELP=(DELP1**2)+(DELP2**2)
      QP1=UM*(OM2**2)
      QP2=2.0*UM*PH2*OM2*(OM+P)
      HP1RE=((QP1*DELP1)+(QP2*DELP2))/ADELP
```

```
      HP1IM=((QP2*DELP1)-(QP1*DELP2))/ADELP
      HH1RE=(H1RE*HP1RE)+(H1IM*HP1IM)
      HH1IM=(H1IM*HP1RE)-(H1RE*HP1IM)
C     Z1=0, COMPUTE H2(U)*H2(U+P)
      P1=1.0+UM*(OM2**2)-(OM**2)
      P2=2.0*(PH1+UM*PH2*OM2)*OM
      H2RE=((P1*DEL1)+(P2*DEL2))/ADEL
      H2IM=((P2*DEL1)-(P1*DEL2))/ADEL
      PP1=1.0+UM*(OM2**2)-((OM+P)**2)
      PP2=2.0*(PH1+UM*PH2*OM2)*(OM+P)
      HP2RE=((PP1*DELP1)+(PP2*DELP2))/ADELP
      HP2IM=((PP2*DELP1)-(PP1*DELP2))/ADELP
      HH2RE=(H2RE*HP2RE)+(H2IM*HP2IM)
      HH2IM=(H2IM*HP2RE)-(H2RE*HP2IM)
C     Z2=0, COMPUTE H1(U)*H1(U+P)
      R1=(OM2**2)-(OM**2)
      R2=2.0*PH2*OM2*OM
      H1RE2=((R1*DEL1)+(R2*DEL2))/ADEL
      H1IM2=((R2*DEL1)-(R1*DEL2))/ADEL
      RP1=(OM2**2)-((OM+P)**2)
      RP2=2.0*PH2*OM2*(OM+P)
      HP1RE2=((RP1*DELP1)+(RP2*DELP2))/ADELP
      HP1IM2=((RP2*DELP1)-(RP1*DELP2))/ADELP
      HH1RE2=(H1RE2*HP1RE2)+(H1IM2*HP1IM2)
      HH1IM2=(H1IM2*HP1RE2)-(H1RE2*HP1IM2)
C     Z2=0, COMPUTE H2(U)*H2(U+P)
      S1=OM2**2
      S2=2.0*PH2*OM2*OM
      H2RE2=((S1*DEL1)+(S2*DEL2))/ADEL
      H2IM2=((S2*DEL1)-(S1*DEL2))/ADEL
      SP1=OM2**2
      SP2=2.0*PH2*OM2*(OM+P)
      HP2RE2=((SP1*DELP1)+(SP2*DELP2))/ADELP
      HP2IM2=((SP2*DELP1)-(SP1*DELP2))/ADELP
      HH2RE2=(H2RE2*HP2RE2)+(H2IM2*HP2IM2)
      HH2IM2=(H2IM2*HP2RE2)-(H2RE2*HP2IM2)
      PRINT 1,OM,HH1RE,HH1IM,HH2RE,HH2IM,HH1RE2,HH1IM2,HH2RE2,HH2IM2
    1 FORMAT(10X,9F12.5)
      OM=OM+0.01
      IF(OM-2.0)2,2,3
    3 P=P+1.0
      IF(P-5.0)6,6,4
    4 STOP
      END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 4
C     TWO-DEGREE -OF-FREEDOM SYSTEM
C     PROGRAM COMPUTES THE MEAN SQUARE VALUE OF THE OUTPUT X1(T) AND
C     X2(T) IN FUNCTION OF P/NATURAL FREQUENCY 1
C     WHEN THE AUTOCORRELATION OF THE INPUT Z1(T) HAS THE FORM =
C     A*COS(P*T1)*DELTA(T1-T2)
C     AND THE INPUT Z2(T)=0
C     P=P/NATURAL FREQUENCY OMEGA 1
C     PH1=DAMPING FACTOR ZETA 1
C     PH2=DAMPING FACTOR ZETA 2
C     UM=MASS RATIO M2/M1
C     OM2=NATURAL FREQUENCY 2/NATURAL FREQUENCY 1
C     U=INPUT FREQUENCY/NATURAL FREQUENCY 1
C     H1RE2=REAL PART OF RECEPTANCE H1(U)
C     H1IM2=IMAGINARY PART OF THE RECEPTANCE H1(U)
C     HP1RE2=REAL PART OF RECEPTANCE H1(U+P)
C     HP1IM2=IMAGINARY PART OF RECEPTANCE H1(U+P)
C     H2RE2=REAL PART OF RECEPTANCE H2(U)
C     H2IM2=IMAGINARY PART OF RECEPTANCE H2(U+P)
C     HH1RE2=REAL PART OF H1(U)*H1(U+P)
C     HH1IM2=IMAGINARY PART OF H1(U)*H1(U+P)
C     HH2RE2=REAL PART OF H2(U)*H2(U+P)
C     HH2IM2=IMAGINARY PART OF H2(U)*H2(U+P)
C     FOLLOWING SECTION IS SAME AS PROGRAM 3 TO COMPUTE THE
C     QUANTITIES H1(U)*H1(U+P) AND H2(U)*H2(U+P) WHEN THE INPUT Z1(T) IS
C     NONSTATIONARY WHITE NOISE PROCESS WITH AUTOCORRELATION
C     RZ1Z1(T1,T2)=A*COS(P*T1)*DELTA(T1-T2) AND INPUT Z2(T)=0
      DIMENSION HH1RE2(751),HH1IM2(751),HH2RE2(751),HH2IM2(751)
      PH1=0.1
      PH2=0.1
   17 PRINT 19,PH1
   19 FORMAT(10X,14H ZETA1=ZETA2= ,F5.2)
      P=0.0
   10 UM=0.1
      OM2=2.0
      I=1
    6 PRINT 5,P
    5 FORMAT(5X,4H P= ,F5.2)
      OM=-P/2.0
    2 A=1.0+(OM2**2)*(1.0+UM)+4.0*PH1*PH2*OM2
      B=OM2**2
      C=2.0*(PH1+PH2*(1.0+UM)*OM2)
      D=2.0*PH1*(OM2**2)+2.0*PH2*OM2
      DEL1=(OM**4)-A*(OM**2)+B
      DEL2=-C*(OM**3)+(D*OM)
      ADEL=(DEL1**2)+(DEL2**2)
      DELP1=(OM+P)**4-A*((OM+P)**2)+B
      DELP2=-C*((OM+P)**3)+D*(OM+P)
      ADELP=(DELP1**2)+(DELP2**2)
C     Z2=0, COMPUTE H1(U)*H1(U+P)
      R1=(OM2**2)-(OM**2)
      R2=2.0*PH2*OM2*OM
      H1RE2=((R1*DEL1)+(R2*DEL2))/ADEL
      H1IM2=((R2*DEL1)-(R1*DEL2))/ADEL
```

```
      RP1=(OM2**2)-((OM+P)**2)
      RP2=2.0*PH2*OM2*(OM+P)
      HP1RE2=((RP1*DELP1)+(RP2*DELP2))/ADELP
      HP1IM2=((RP2*DELP1)-(RP1*DELP2))/ADELP
      HH1IM2(I)=(H1IM2*HP1RE2)-(H1RE2*HP1IM2)
      HH1RE2(I)=(H1RE2*HP1RE2)+(H1IM2*HP1IM2)
C     Z2=0, COMPUTE H2(U)*H2(U+P)
      S1=OM2**2
      S2=2.0*PH2*OM2*OM
      H2RE2=((S1*DEL1)+(S2*DEL2))/ADEL
      H2IM2=((S2*DEL1)-(S1*DEL2))/ADEL
      SP1=OM2**2
      SP2=2.0*PH2*OM2*(OM+P)
      HP2RE2=((SP1*DELP1)+(SP2*DELP2))/ADELP
      HP2IM2=((SP2*DELP1)-(SP1*DELP2))/ADELP
      HH2IM2(I)=(H2IM2*HP2RE2)-(H2RE2*HP2IM2)
      HH2RE2(I)=(H2RE2*HP2RE2)+(H2IM2*HP2IM2)
      OM=OM+0.01
      I=I+1
      IF(OM-5.0)2,2,3
C     THE FOLLOWING SECTION IS THE SIMPSON@S NUMERICAL METHOD TO
C     EVALUATE THE INTEGRALS OF H1(U)*H1(U+P) AND H2(U)*H2(U+P) WITH
C     RESPECT TO U WHEN U VARIES FROM -P/2 TO INFINI
C     AX1 = AMPLITUDE OF OUPUT X1(T)
C     AX2 = AMPLITUDE OF OUTPUT X2(T)
C     PX1 = PHASE OF OUTPUT X1(T)
C     PX2 = PHASE OF OUTPUT X2(T)
    3 SU1RE2=0.0
      SU1IM2=0.0
      SU2RE2=0.0
      SU2IM2=0.0
      I=1
    4 SU1RE2=SU1RE2+(0.01/3.0)*(HH1RE2(I)+4.0*HH1RE2(I+1)+HH1RE2(I+2))
      SU1IM2=SU1IM2+(0.01/3.0)*(HH1IM2(I)+4.0*HH1IM2(I+1)+HH1IM2(I+2))
      SU2RE2=SU2RE2+(0.01/3.0)*(HH2RE2(I)+4.0*HH2RE2(I+1)+HH2RE2(I+2))
      SU2IM2=SU2IM2+(0.01/3.0)*(HH2IM2(I)+4.0*HH2IM2(I+1)+HH2IM2(I+2))
      I=I+2
      IF(I-745)4,4,7
    7 PRINT 8,SU1RE2,SU1IM2,SU2RE2,SU2IM2
    8 FORMAT(10X,4F10.5)
      AX1=2.0*((SU1RE2**2+SU1IM2**2)**0.5)
      TPX1=SU1IM2/SU1RE2
      PX11=ATAN(TPX1)
      PX1=(180.0/3.1416)*PX11
      IF(PX1)13,14,14
   13 PX1=180.0+PX1
   14 PX1=PX1
      AX2=2.0*((SU2RE2**2+SU2IM2**2)**0.5)
      TPX2=SU2IM2/SU2RE2
      PX22=ATAN(TPX2)
      PX2=(180.0/3.1416)*PX22
      IF(PX2)15,16,16
   15 PX2=180.0+PX2
   16 PX2=PX2
      PRINT 12,AX1,PX1,AX2,PX2
```

```
 12 FORMAT(50X,4F10.5)
    P=P+0.1
    IF(P-4.0)10,10,11
 11 PH1=PH1+0.1
    PH2=PH2+0.1
    IF(PH1-1.0)17,17,18
 18 STOP
    END
```

```
      PROGRAM DUCDOAN(INPUT,OUTPUT)
C     PROGRAM 5
C     TWO-DEGREE -OF-FREEDOM SYSTEM
C     PROGRAM COMPUTES THE MEAN SQUARE VALUES OF THE OUTPUTS X1(T) AND
C     X2(T) IN FUNCTION OF NATURAL FREQUENCY 2/NATURAL FREQUENCY 1
C     WHEN THE AUTOCORRELATION OF THE INPUT Z1(T) IS OF THE FORM =
C     A*COS(P*T1)*DELTA(T1-T2)
C     AND THE INPUT Z2(T)=0
C     P=P/NATURAL FREQUENCY OMEGA 1
C     PH1=DAMPING FACTOR ZETA 1
C     PH2=DAMPING FACTOR ZETA 2
C     UM=MASS RATIO M2/M1
C     OM2=NATURAL FREQUENCY 2/NATURAL FREQUENCY 1
C     U=INPUT FREQUENCY
C     OM=INPUT FREQUENCY/NATURAL FREQUENCY 1
C     H1RE2=REAL PART OF RECEPTANCE H1(U)
C     H1IM2=IMAGINARY PART OF THE RECEPTANCE H1(U)
C     HP1RE2=REAL PART OF RECEPTANCE H1(U+P)
C     HP1IM2=IMAGINARY PART OF RECEPTANCE H1(U+P)
C     H2RE2=REAL PART OF RECEPTANCE H2(U)
C     H2IM2=IMAGINARY PART OF RECEPTANCE H2(U+P)
C     HH1RE2=REAL PART OF H1(U)*H1(U+P)
C     HH1IM2=IMAGINARY PART OF H1(U)*H1(U+P)
C     HH2RE2=REAL PART OF H2(U)*H2(U+P)
C     HH2IM2=IMAGINARY PART OF H2(U)*H2(U+P)
C     THE FOLLOWING SECTION IS EXACTLY  AS PROGRAM 3 TO COMPUTE THE
C     QUANTITIES H1(U)*H1(U+P) AND H2(U)*H2(U+P) WHEN THE INPUT Z1(T) IS
C     NONSTATIONARY WHITE NOISE PROCESS WITH AUTOCORRELATION
C     RZ1Z1(T1,T2)=A*COS(P*T1)*DELTA(T1-T2) AND INPUT Z2(T)=0
      DIMENSION HH1RE2(751),HH1IM2(751),HH2RE2(751),HH2IM2(751)
      UM=0.0
   17 PRINT 19,UM
   19 FORMAT(10X,9H M2/M1 = ,F5.2)
      OM2=0.1
   10 PH1=0.2
      PH2=0.2
      P=1.0
      I=1
    6 PRINT 5,OM2
    5 FORMAT(5X,11H OM2/OM1 = ,F5.2)
      OM=-P/2.0
    2 A=1.0+(OM2**2)*(1.0+UM)+4.0*PH1*PH2*OM2
      B=OM2**2
      C=2.0*(PH1+PH2*(1.0+UM)*OM2)
      D=2.0*PH1*(OM2**2)+2.0*PH2*OM2
      DEL1=(OM**4)-A*(OM**2)+B
      DEL2=-C*(OM**3)+(D*OM)
      ADEL=(DEL1**2)+(DEL2**2)
      DELP1=(OM+P)**4-A*((OM+P)**2)+B
      DELP2=-C*((OM+P)**3)+D*(OM+P)
      ADELP=(DELP1**2)+(DELP2**2)
C     Z2=0, COMPUTE H1(U)*H1(U+P)
      R1=(OM2**2)-(OM**2)
      R2=2.0*PH2*OM2*OM
      H1RE2=((R1*DEL1)+(R2*DEL2))/ADEL
```

```
      H1IM2=((R2*DEL1)-(R1*DEL2))/ADEL
      RP1=(OM2**2)-((OM+P)**2)
      RP2=2.0*PH2*OM2*(OM+P)
      HP1RE2=((RP1*DELP1)+(RP2*DELP2))/ADELP
      HP1IM2=((RP2*DELP1)-(RP1*DELP2))/ADELP
      HH1IM2(I)=(H1IM2*HP1RE2)-(H1RE2*HP1IM2)
      HH1RE2(I)=(H1RE2*HP1RE2)+(H1IM2*HP1IM2)
C     Z2=0, COMPUTE H2(U)*H2(U+P)
      S1=OM2**2
      S2=2.0*PH2*OM2*OM
      H2RE2=((S1*DEL1)+(S2*DEL2))/ADEL
      H2IM2=((S2*DEL1)-(S1*DEL2))/ADEL
      SP1=OM2**2
      SP2=2.0*PH2*OM2*(OM+P)
      HP2RE2=((SP1*DELP1)+(SP2*DELP2))/ADELP
      HP2IM2=((SP2*DELP1)-(SP1*DELP2))/ADELP
      HH2IM2(I)=(H2IM2*HP2RE2)-(H2RE2*HP2IM2)
      HH2RE2(I)=(H2RE2*HP2RE2)+(H2IM2*HP2IM2)
      OM=OM+0.01
      I=I+1
      IF(OM-5.0)2,2,3
C     THE FOLLOWING SECTION IS THE SIMPSON≠S NUMERICAL METHOD TO
C     EVALUATE THE INTEGRALS OF H1(U)*H1(U+P) AND H2(U)*H2(U+P) WITH
C     RESPECT TO U WHEN U VARIES FROM -P/2 TO INFINI
C     AX1 = AMPLITUDE OF OUPUT X1(T)
C     AX2 = AMPLITUDE OF OUTPUT X2(T)
C     PX1 = PHASE OF OUTPUT X1(T)
C     PX2 = PHASE OF OUTPUT X2(T)
    3 SU1RE2=0.0
      SU1IM2=0.0
      SU2RE2=0.0
      SU2IM2=0.0
      I=1
    4 SU1RE2=SU1RE2+(0.01/3.0)*(HH1RE2(I)+4.0*HH1RE2(I+1)+HH1RE2(I+2))
      SU1IM2=SU1IM2+(0.01/3.0)*(HH1IM2(I)+4.0*HH1IM2(I+1)+HH1IM2(I+2))
      SU2RE2=SU2RE2+(0.01/3.0)*(HH2RE2(I)+4.0*HH2RE2(I+1)+HH2RE2(I+2))
      SU2IM2=SU2IM2+(0.01/3.0)*(HH2IM2(I)+4.0*HH2IM2(I+1)+HH2IM2(I+2))
      I=I+2
      IF(I-745)4,4,7
    7 PRINT 8,SU1RE2,SU1IM2,SU2RE2,SU2IM2
    8 FORMAT(10X,4F10.5)
      AX1=2.0*((SU1RE2**2+SU1IM2**2)**0.5)
      TPX1=SU1IM2/SU1RE2
      PX11=ATAN(TPX1)
      PX1=(180.0/3.1416)*PX11
      IF(PX1)13,14,14
   13 PX1=180.0+PX1
   14 PX1=PX1
      AX2=2.0*((SU2RE2**2+SU2IM2**2)**0.5)
      TPX2=SU2IM2/SU2RE2
      PX22=ATAN(TPX2)
      PX2=(180.0/3.1416)*PX22
      IF(PX2)15,16,16
   15 PX2=180.0+PX2
   16 PX2=PX2
```

```
      PRINT 12,AX1,PX1,AX2,PX2
   12 FORMAT(60X,4F10.5)
      OM2=OM2+0.1
      IF(OM2-5.0)10,10,11
   11 UM=UM+0.1
      IF(UM-2.0)17,17,18
   18 STOP
      END
```