SEQUENTIAL SELECTING PROCEDURES FOR THE LARGEST MEAN OF k INDEPENDENT NORMAL POPULATIONS WITH PREFIXED-WIDTH CONFIDENCE INTERVAL

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ABSTRACT

SEQUENTIAL SELECTING PROCEDURES FOR THE LARGEST MEAN OF K INDEPENDENT NORMAL POPULATIONS WITH PREFIXED-WIDTH CONFIDENCE INTERVAL

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The main purpose of this expository article is to construct a prefixed width confidence interval for the largest mean of $k \geq 1$ independent normal populations by sequential procedures, in the case of a common but unknown variance. The appropriate tools are obtained through a preliminary consideration of the same problem in the case of a common and known variance. A study of the case of unequal variances is provided and a comprehensive analysis of the performance of the sequential procedures is also given. The work consists of four chapters. In Chapter I the problem of selection of the population with the largest mean is examined and several related concepts are introduced. In Chapter II, a prefixed width symmetric confidence interval for the largest mean is constructed by sequential procedures, while in Chapter III the same task is carried out by unsymmetric confidence intervals possessing some optimal properties. Finally, in Chapter IV, the problem of a prefixed width confidence estimation of the largest mean with simultaneous selection of the best population is considered, from a sequential point of view, combining informations mainly from Chapters I and II.

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DEDICATION

To my wife Eliza

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c.d.f.	cumulative distribution function
i.i.d.	independent and identically distributed
i.o.	infinitely often
L.F.C.	least favourable configuration
M.L.R.	monotone likelihood ratio
p.d.f.	probability density function
r.v.	random variable

CHAPTER I

INTRODUCTION

THE PROBLEM OF SELECTION

A usual problem in statistical decision making is the selection of the best among $K(\geq 2)$ given populations ([12]). Each of these populations, under consideration, is associated with a parameter θ which serves as a measure of goodness of the corresponding population. Random observations are taken independently, from each of these k populations, which are distributed as,

$$(I.1) \qquad F(x,\theta_1),\ F(x,\theta_2),\ \ldots.,\ F(x,\theta_k),$$

under the assumptions that there is no prior knowledge about the θ values and that they are not all the same.

As best population, throughout this work, is considered that one associated with the largest θ value. Denoting, however, by $\theta_{[1]} \leq \theta_{[2]} \leq \ldots. \leq \theta_{[k]}$ the ordered θ values, (so that $\theta_{[j]}$ is the jth smallest θ value) the problem of selection of the best population becomes a problem of identification, through the collected data, of that population which corresponds to $\theta_{[k]}$. This goal is called Correct Selection (CS).

Definition I.1: As a distance function we define the difference $\delta_i = \theta_{[k]} - \theta_{[i]}$, $i = 1,2,\ldots,k-1$, which serves as a qualitative measure between the best and the remaining populations.

<u>Definition I.2</u>: A parameter space Ω is defined to be a k-dimensional region containing all possible configurations or parameter vectors of the form $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$.

<u>Definition I.3</u>: Indifference Zone (IZ) [Preference Zone (PZ)] is defined as that part of Ω which contains all vectors $\underset{\sim}{\theta}$ characterized by small [large] distances between the individual parameters.

As a result, it follows that $\Omega = (PZ) \cup (IZ)$. We also note that in the case when $\underset{\sim}{\theta} \in IZ$ we are, more or less, indifferent between two or more different selections, while our strong interest for making a Correct Selection lies in the case where $\underset{\sim}{\theta} \in PZ$. In fact, our primary concern lies mostly on the largest and next-to-largest values of θ's, and in most applications the separation between IZ and PZ is based on the magnitude of the distance $\delta = \theta_{[k]} - \theta_{[k-1]}$ and depends mainly on a threshold value $\delta^* > 0$, of that distance, properly prespecified in each experiment. However, we say that a configuration $\underset{\sim}{\theta} \in PZ$ whenever $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$, no matter about the differences between the remaining k-2 parameters.

<u>Definition I.4</u>: Least Favourable Configuration (LFC) is defined to be a special configuration in the PZ for which the Probability of Correct Selection attains a minimum value over all configurations in the PZ.

Letting $\underset{\sim}{\theta}_{LFC} = (\theta_{1LFC}, \theta_{2LFC}, \ldots, \theta_{kLFC})$ be the LFC, by the above definition clearly,

$$(I.2) \qquad P(CS|\underset{\sim}{\theta}) \geq P(CS|\underset{\sim}{\theta}_{LFC}) = P^* \quad , \text{ over all } \underset{\sim}{\theta} \in PZ,$$

where the value P^* provides a conservative lower bound to the $P(CS|\underset{\sim}{\theta})$

for all $\underset{\sim}{\theta} \in PZ$. This P^* is also prespecified at a required level and plays a crucial role to control the $P(CS|\underset{\sim}{\theta})$, for all $\underset{\sim}{\theta} \notin PZ$, throughout the selection procedures.

SELECTING THE BEST NORMAL POPULATION

Let $\pi_1,\ldots,\pi_k$ be $k(\geq 2)$ given independent normal populations with distributions $N(\mu_1,\sigma^2)$, ..., $N(\mu_k,\sigma^2)$ respectively, where μ_i's are unknown and the variance σ^2, for the time being, assumed to be known. Letting,

$$\text{(I.3)} \qquad \mu_{[1]} \leq \mu_{[2]} \leq \ldots \leq \mu_{[k]}$$

be the ordered values of μ_i's, our goal is to select that population with mean $\mu_{[k]}$. For that purpose from n random observations within each population π_i, $i = 1, \ldots, k$, we construct a sufficient estimate $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ for μ_i, $i = 1, \ldots k$, and order them accordingly as,

$$\text{(I.4)} \qquad \overline{X}_{[1]} \leq \overline{X}_{[2]} \leq \ldots \leq \overline{X}_{[k]} .$$

Then we select as best that population from which $\overline{X}_{[k]}$ is achieved.

Remarks:

(1) Clearly each $\overline{X}_i$ is distributed as $N(\mu_i,\sigma^2/n)$, $i = 1,\ldots,k$.

(2) The selected mean $\overline{X}_{[k]}$ does not necessarily come from the population with parameter $\mu_{[k]}$.

Denoting by $\overline{X}_{(i)}$ the sample mean obtained from the population with mean $\mu_{[i]}$, $i = 1, \ldots, k$, then, obviously $\overline{X}_{(k)}$ is the actual sample mean from the population with mean $\mu_{[k]}$. We let also,

(I.5) $$\delta_i = \mu_{[k]} - \mu_{[i]}\ , \quad i = 1, \ldots, k-1\ ,$$

be the distance function between the corresponding parameters.

<u>Definition I.5</u>: We say that we have a Correct Selection (CS) whenever the event $\overline{X}_{[k]} = \overline{X}_{(k)}$ occurs for $\mu_{[k]} - \mu_{[k-1]} \geq \delta^*$, i.e. for all configurations $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_k)$ in the PZ.

<u>Theorem I.1</u>:

(I.6) $$P(CS/\underset{\sim}{\mu} \in PZ) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi(y + \tau_i)\, \varphi(y)\,dy$$

where $\tau_i = \dfrac{\mu_{[k]} - \mu_{[i]}}{\sigma/\sqrt{n}} = \dfrac{\delta_i\sqrt{n}}{\sigma}$, and $\varphi(.)$, $\Phi(.)$ are the p.d.f. and c.d.f. of $N(0,1)$, respectively.

<u>Proof</u>: By definition I.5 we may have,

$$P(CS/\underset{\sim}{\mu} \in PZ) = P\left(\overline{X}_{(k)} = \overline{X}_{[k]}\right) = P\left[\overline{X}_{(k)} = \max(\overline{X}_{(1)}, \ldots, \overline{X}_{(k)})\right]$$

$$= P\left(\frac{\overline{X}_{(i)} - \mu_{[i]}}{\sigma/\sqrt{n}} < \frac{\overline{X}_{(k)} - \mu_{[k]}}{\sigma/\sqrt{n}} + \frac{\mu_{[k]} - \mu_{[i]}}{\sigma/\sqrt{n}}\ , \quad i = 1,2,\ldots,k-1\right)$$

$$= P\left(Y_i < Y_k + \tau_i\ , \ i = 1,2,\ldots,k-1\right)$$ where Y_i's are i.i.d. r.v.'s from $N(0,1)$ distribution and $\tau_i = \dfrac{\mu_{[k]} - \mu_{[i]}}{\sigma/\sqrt{n}} = \dfrac{\delta_i\sqrt{n}}{\sigma}$. Thus,

(I.7) $$P\left(CS/\underset{\sim}{\mu} \in PZ\right) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(Y_i < Y_k + \tau_i / Y_k = y\right) \varphi(y)dy$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi(y+\tau_i)\, \varphi(y)dy$$ Q.E.D.

Theorem I.2. The $P\left(CS/\underset{\sim}{\mu} \in PZ\right)$ is minimized under the LCF

(I.8) $$\mu_{[1]} = \mu_{[2]} = \ldots\ldots = \mu_{[k-1]} = \mu,\ \mu_{[k]} - \mu_{[k-1]} = \delta^* .$$

In the LCF case,

(I.9) $$P\left(CS/\underset{\sim}{\mu} \in PZ\right) \geq P\left(CS/\underset{\sim}{\mu}_{LFC}\right) = P^* ,$$ where

(I.10) $$P^* = \int_{-\infty}^{\infty} \Phi^{k-1}\left(y + \frac{\delta^*\sqrt{n}}{\sigma}\right) \varphi(y)dy, \text{ and } \underset{\sim}{\mu}_{LFC} = \left(\mu,\ldots,\mu,\mu+\delta^*\right) .$$

Proof: Since $\mu_{[1]} \leq \ldots \leq \mu_{[k]}$ and $\mu_{[k]} - \mu_{[k-1]} \geq \delta^*$

we get $\mu_{[k]} - \mu_{[1]} \geq \mu_{[k]} - \mu_{[2]} \geq \ldots \geq \mu_{[k]} - \mu_{[k-1]} \geq \delta^*$; this by (I.5)

implies $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_k$ which by $\tau_i = \frac{\delta_i\sqrt{n}}{\sigma}$ yields,

(I.11) $$\tau_1 \geq \tau_2 \geq \ldots \geq \tau_{k-1}\ (= \tau) .$$

If (I.8) is true, we obtain,

(I.12) $$\tau_1 = \tau_2 = \ldots = \tau_{k-1} = \tau = \frac{\delta^*\sqrt{n}}{\sigma} .$$

Since $\Phi(x)$ is increasing in x, from (I.6) it follows that $P(CS/\underset{\sim}{\mu} \in PZ)$ decreases as τ_i, $i = 1,\ldots,k-1$, decreases. By the restriction $\mu_{[k]} - \mu_{[k-1]} \geq \delta^*$, clearly the $P(CS/\underset{\sim}{\mu} \in PZ)$ is minimized whenever (I.12) holds i.e. whenever (I.8) holds. Then we achieve

$$(I.13) \qquad P^* = P^*(n,\delta^*) = \int_{-\infty}^{\infty} \Phi^{k-1}\left(y + \frac{\delta^*\sqrt{n}}{\sigma}\right)\varphi(y)dy \ , \quad \text{and}$$

obviously,

$$(I.14) \qquad P(CS/\underset{\sim}{\mu} \in PZ) \geq P(CS/\underset{\sim}{\mu}_{LFC}) = P^* \ .$$

Q.E.D.

Remarks:

(3) From (I.13) it follows that $P^* = P^*(n,\delta^*)$, i.e. P^* is a function of n and δ^* .

(4) Since, $\lim_{\delta^*\to 0} \Phi\left(y + \frac{\delta^*\sqrt{n}}{\sigma}\right) = \Phi(y)$, we achieve

$$\lim_{\delta^*\to 0} P^*(n,\delta^*) = \int_{-\infty}^{\infty} \Phi^{k-1}(y)d\Phi(y) = \lim_{t\to\infty} \frac{1}{k}\int_{-t}^{t} d\Phi^k(y)$$

$\frac{1}{k}\lim_{t\to\infty}\left[\Phi^k(t) - \Phi^k(-t)\right] = \frac{1}{k}$. The latter implies that the choice of the best population is randomized whenever the populations have all the same means.

(5) Since $\lim_{\delta^*\to\infty} \Phi\left(y + \frac{\delta^*\sqrt{n}}{\sigma}\right) = 1$, clearly then, $\lim_{\delta^*\to\infty} P^*(n,\delta^*) = 1$, which means that very large difference between $\mu_{[k]}$ and $\mu_{[k-1]}$ implies large probability of correct selection.

(6) For δ^* fixed, with arguments similar to those of Remark (4) we observe $\lim_{n\to 0} P^*(n,\delta^*) = \frac{1}{k}$, hence, without any sampling the choice of the best population is again randomized.

(7) Similarly as in Remark (5) we obtain, $\lim_{n\to\infty} P^*(n,\delta^*) = 1$, which means that for large sample size the $P(CS)$ is getting also large.

Corollary I.1. $1/k < P^* < 1$. (Clearly from Remarks 3-7).

Let $h_k(P^*)$ be the solution of the equation

$$P^* = \int_{-\infty}^{\infty} \Phi^{k-1}(x + h_k(P^*))d\Phi(x). \tag{I.15}$$

Corollary I.2. There exists a unique smallest sample size n_0 per population, satisfying the requirement (I.14), where this n_0 is the smallest integer $n \geq \frac{h_k^2(P^*)\sigma^2}{(\delta^*)^2}$ (P^* prefixed).

Proof: From Remark (7) we have $\lim_{n\to\infty}\left[P^*(n,\delta^*) - 1\right] = 0$, therefore, for every $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that

$$\left|P^*(n,\delta^*) - 1\right| \leq \epsilon \quad \text{for every } n \geq N(\epsilon)\ , \text{ i.e.,} \tag{I.16}$$

$$1 - \epsilon \leq P^*(n,\delta^*) \leq 1 < 1 + \epsilon \quad \text{for every } n \geq N(\epsilon)\ . \tag{I.17}$$

If P^*, however, is prefixed at a required level, say P_0^* , we choose an ϵ s.t. $1 - \epsilon = P_0^*$, i.e. $\epsilon = 1 - P_0^*$. Denoting this ϵ by ϵ_0 and the corresponding $N(\epsilon)$ by $N(\epsilon_0) = n_\alpha$, from (I.17) we get $1 - \epsilon_0 = P_0^*$, and that

(I.18) $$P_0^* = \int_{-\infty}^{\infty} \Phi^{k-1}\left[x + h_k(P_0^*)\right] d\Phi(x) \le \int_{-\infty}^{\infty} \Phi^{k-1}\left(x + \frac{\delta^*\sqrt{n}}{\sigma}\right) d\Phi(x) = P^*(n,\delta^*)$$

is satisfied for every $n \ge N(\epsilon_0) = n_0$:

Hence, $\frac{\delta^*\sqrt{n}}{\sigma} \ge h_k(P_0^*)$ is satisfied for every $n \ge n_0$, i.e.,

(I.19) n_0 is the smallest integer s.t. $n \ge \frac{\sigma^2 h_k^2(P_0^*)}{(\delta^*)^2}$.

Clearly then, by (I.18), it follows that the requirement (I.14) is satisfied for every $n \ge n_0$. Q.E.D.

The solutions $h_k(P^*)$ of the equation (I.15) are given in Table 1 ([3]) for P^* prefixed. Then for k, δ^* and σ^2 prespecified, from (I.19) we may compute n_0 numerically.

In the case where the variance σ^2 is common but unknown, instead of σ^2 we use the pooled sample variance

(I.20) $$S_\nu^2 = \sum_{j=1}^{k} \sum_{i=1}^{n} \left(X_{ij} - \overline{X}_j\right)^2 / \nu , \quad \text{where} \quad \nu = k(n-1).$$

<u>Theorem I.3</u>. For σ^2 common but unknown,

(I.21) $$P(CS/\underset{\sim}{\mu} \in PZ) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_\nu(y + \tau_i) dF_\nu(y)$$

where, $\tau_i = \frac{\mu_{[k]} - \mu_{[i]}}{S_\nu/\sqrt{n}} = \frac{\delta_i\sqrt{n}}{S_\nu}$ and $F_\nu(.)$ is the c.d.f. of the t_ν-distribution with ν-degrees of freedom.

Proof: Following the lines of Theorem I.1 we have,

$$P(CS/\underset{\sim}{\mu} \in PZ) = P(\overline{X}_{(i)} < \overline{X}_{(k)},\ i = 1,\dots,k-1)$$

$$= P\left[\frac{\sqrt{n}(\overline{X}_{(i)} - \mu_{[i]})}{S_\nu} < \frac{\sqrt{n}(\overline{X}_{(k)} - \mu_{[k]})}{S_\nu} + \frac{\mu_{[k]} - \mu_{[i]}}{S_\nu/\sqrt{n}},\ i=1,\dots,k-1\right]$$

$= P\Big(Y_i < Y_k + \tau_i,\ i=1,\dots,k-1\Big)$, where Y_i's are i.i.d. r.v.'s

with t_ν-distribution and $\tau_i = \dfrac{\mu_{[k]} - \mu_{[i]}}{S_\nu/\sqrt{n}} = \dfrac{\delta_i\sqrt{n}}{S_\nu}$. Then similarly

as in (I.7) we end up with (I.21).

Theorem I.4. For σ^2 common but unknown,

(I.22) $\inf_{\underset{\sim}{\mu}} P(CS/\underset{\sim}{\mu} \in PZ) = P(CS/\underset{\sim}{\mu}_{LFC} \in PZ) = P^*$, where

(I.23) $P^* = \int_{-\infty}^{\infty} F_\nu^{k-1}\left(y + \dfrac{\delta^*\sqrt{n}}{S_\nu}\right) dF_\nu(y)$ and $\underset{\sim}{\mu}_{LFC} = (\mu,\dots,\mu\ \mu+\delta^*)$.

Proof: Using the same arguments as in Theorem I.2.

For P^* prefixed and k,δ^* also given, the solution $\tau_\nu^{(k)}(P^*)$ of the equation

(I.24) $P^* = \int_{-\infty}^{\infty} F_\nu^{k-1}\left(y + \tau_\nu^{(k)}(P^*)\right) dF_\nu(y)$

is given in Table 2 ([5], [8]).

By (I.23) and (I.24) we obtain $\tau_{\nu}^{(k)}(P^*) = \frac{\delta^* \sqrt{n}}{S_{\nu}}$, $\nu = k(n-1)$, which indicates that there is no fixed sample size solution, so that, the requirement (I.14) is satisfied for every $\underline{\mu} \in PZ$, unless a sequential procedure is adopted ([18]). This problem will be studied in Chapter IV simultaneously with a prefixed-width interval estimation of $\mu_{[k]}$.

Closing remark: Most of the material in this introductory chapter is based on [3], [12], where various topics on ranking and selection procedures are discussed and numerous examples and applications are given.

CHAPTER II

INTERVAL ESTIMATION OF THE LARGEST NORMAL MEAN BY PREFIXED-WIDTH SYMMETRIC CONFIDENCE INTERVALS

PRELIMINARIES

A Single-Stage Procedure R Under General Assumptions

In the present chapter a solution to the problem of a prefixed-width interval estimation of $\mu_{[k]}$ is given by a symmetric confidence interval. Our problem, at first, is studied in a more general way and the conclusions are applied to the normal family. However, after a preliminary consideration of the case where the k normal populations have a common known variance σ^2, via a single-stage precedure, ([10], [20]), we obtain the appropriate tools for establishing a sequential procedure in the case where σ^2 is unknown ([22]). Single stage and sequential procedures are also provided in the case of different variances.

Let $\gamma \in (0,1)$ be a preassigned confidence level of a random interval I_d, of prefixed length $2d$, ($d > o$ arbitrary but preassigned) and $\theta^* = \theta_{[k]} = \max\limits_{1 \le i \le k} \theta_i$, where θ_i's are as in (I.1).

<u>Definition II.1</u>: The event of Correct Decision (CD) is defined to be equivalent to the event $\theta^* \in I_d$.

<u>Note</u>: Since the latter means that I_d covers θ^*, the terms Probability of Correct Decision and Coverage Probability are used indiscriminately.

Through a single-stage procedure R we construct the interval I_d, so that, the requirement

$$\text{(II.1)} \qquad P(CD/\underset{\sim}{\theta},\sigma^2,R) \geq \gamma$$

is satisfied for every $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k) \in \Omega$ and every $\sigma^2 > 0$. Notice that no indifference zone in Ω is assumed since the interval estimation of $\theta_{[k]}$ (i.e. of $\mu_{[k]}$) is carried out without any selection.

Let $\underset{\sim}{X}_i = (X_{i1},\ldots,X_{in})$, $i = 1,\ldots,k$, be mutually independent r.v.'s where $\underset{\sim}{X}_i$ denotes a random sample of size n from the π_i population. If $T_{in} = T(X_{i1},\ldots,X_{in})$ is a consistent estimate for θ_i and $g_n(t,\theta_i)$, $G_n(t,\theta_i)$ are the p.d.f. and c.d.f. of T_{in} respectively, we choose the random interval I_d to be

$$\text{(II.2)} \qquad I_d = (T_n^*-d,\ T_n^*+d), \quad \text{where } T_n^* = \max_{1\leq i\leq k} T_{in} .$$

Notice that T_n^* does not necessarily come from the population with parameter θ^*. Then for every $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k) \in \Omega$ we have,

$$\text{(II.3)} \qquad P(CD/\underset{\sim}{\theta},\sigma,R) = P(\theta^* \in I_d) = P(T_n^* < \theta^* + d) - P(T_n^* < \theta^* - d)$$

$$= P(T_{in} < \theta^* + d,\ i = 1,\ldots,k) - P(T_{in} < \theta^* - d,\ i = 1,\ldots,k)$$

$$= \prod_{i=1}^{k} G_n(\theta^* + d,\ \theta_i) - \prod_{i=1}^{k} G_n(\theta^* - d,\ \theta_i) .$$

<u>Theorem II.1</u>: Assume that for every n the p.d.f. $g_n(t,\theta_1)$ of the consistent estimate T_{1n} satisfies,

(a) $g_n(t,\theta) = g_n(t-\theta)$, i.e. $g_n(t,\theta)$ is a location parameter p.d.f.

(b) $g_n(t) = g_n(-t) > 0$, for all t, (symmetry).

(c) the family $\{g_n(t-\theta):\theta\}$ possesses the monotone likelihood ratio (M.L.R.) property.

Then, for $\underset{\sim}{\theta}_{LFC} = (\theta,\theta,\ldots,\theta)$, where θ arbitrary, we have,

$$\text{(II.4)} \qquad \inf_{\underset{\sim}{\theta}} P(CD/\underset{\sim}{\theta},\sigma,R) = P(CD/\underset{\sim}{\theta}_{LFC},\sigma,R) = G_n^k(d) - G_n^k(-d) \; .$$

<u>Proof</u>: See Appendix A.

<u>Theorem II.2</u>: Under the conditions of Theorem II.1 there exists a smallest sample size N_0 s.t.

$$\text{(II.5)} \qquad G_n^k(d) - G_n^k(-d) \geq \gamma \; , \text{ for every, } n \geq N_0 \text{ and } \underset{\sim}{\theta} \in \Omega \; .$$

<u>Proof:</u> See Appendix A.

Clearly from theorems II.1 and II.2 it follows that

$$\text{(II.6)} \qquad P(CD/\underset{\sim}{\theta},\sigma,R) \geq \inf_{\underset{\sim}{\theta}} P(CD/\underset{\sim}{\theta},\sigma,R) = G_n^k(d) - G_n^k(-d) \geq \gamma$$

for every $n \geq N_0$, hence the requirement (II.1) is satisfied for every $n \geq N_0$.

Application of R to the Normal Family with a Common Known Variance

If σ^2 is the common known variance, letting $T_{in} = \overline{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$, $i = 1,\ldots,k$, $T_n^* = \overline{X}_{[k]}$ and $\theta^* = \mu_{[k]}$, under the light of the previous results we have,

(II.7) $$I_d = \left(\overline{X}_{[k]} - d \, , \, \overline{X}_{[k]} + d\right)$$

and since the conditions of Theorem II.1 in the normal case are satisfied, then

(II.8) $$\inf_{\underset{\sim}{\mu}} P\left(CD/\underset{\sim}{\mu},\sigma,R\right) = \Phi^k\left(\frac{d\sqrt{n}}{\sigma}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma}\right) = \Phi^k\left(\frac{\sqrt{n}}{\lambda}\right) - \Phi^k\left(-\frac{\sqrt{n}}{\lambda}\right)$$

where, $\lambda = \sigma/d$ and $\Phi(.)$ is the c.d.f. of $N(0,1)$ distribution. For a preassigned $\gamma \in (0,1)$, by Theorem II.2, we also get,

(II.9) $$\Phi^k\left(\frac{d\sqrt{n}}{\sigma}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma}\right) \geq \gamma \quad \text{for every } n \geq N_0 .$$

If z_γ denotes the unique solution of the equation

(II.10) $$\beta_k(z_\gamma) = \Phi^k(z_\gamma) - \Phi^k(-z_\gamma) = \gamma$$

then by (II.9) and (II.8) it follows that,

(II.11) "N_0 is the smallest integer n s.t. $\sigma^2 \leq \frac{nd^2}{z_\gamma^2}$ " i.e.,

(II.12) $N_0 = \left[\frac{z_\gamma^2 \sigma}{d^2}\right] = \left[z_\gamma^2 \lambda^2\right]$, (where $[x]$ denotes the smallest integer $\geq x$), is the smallest sample size required to satisfy the requirement (II.1) for every, $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k) \in \Omega$, $\sigma^2 > 0$.

The values z_γ, for k and $\gamma \in (0,1)$ prefixed are given in Table 3 (see [6]).

INTERVAL ESTIMATION OF THE LARGEST NORMAL MEAN BY A SEQUENTIAL PROCEDURE R_S

The Performance of the Procedure R in Case of a Common but Unknown Variance

Letting the pooled sample variance

$$S_\nu^2 = \frac{1}{\nu} \sum_{i=1}^{k} \sum_{j=1}^{n} \left(X_{ij} - \overline{X}_i\right)^2, \quad \nu = k(n-1) \tag{II.13}$$

be an estimate for the unknown σ^2, we establish now a new expression for the coverage probability under R. However,

$$P(CD/\underline{\mu},\sigma,R) = P\left(\mu_{[k]} \in I_d\right) = \prod_{i=1}^{k} P\left(\overline{X}_{(i)} \le \mu_{[k]} + d\right) - \prod_{i=1}^{k} P\left(\overline{X}_{(i)} \le \mu_{[k]} - d\right) \tag{II.14}$$

$$= \prod_{i=1}^{k} P\left[\frac{\sqrt{n}(\overline{X}_{(i)} - \mu_{[i]})}{S_\nu} \le \frac{\mu_{[k]} - \mu_{[i]} + d}{S_\nu/\sqrt{n}}\right] - \prod_{i=1}^{k} P\left[\frac{\sqrt{n}(\overline{X}_{(i)} - \mu_{[i]})}{S_\nu} \le \frac{\mu_{[k]} - \mu_{[i]} - d}{S_\nu/\sqrt{n}}\right]$$

$$= \prod_{i=1}^{k} P\left(T_i \le \Delta_i + D\right) - \prod_{i=1}^{k} P\left(T_i \le \Delta_i - D\right),$$

where T_i's are i.i.d. r.v.'s from t_ν-distribution and

$$D = \frac{d\sqrt{n}}{S_\nu}, \quad \Delta_i = \frac{\mu_{[k]} - \mu_{[i]}}{S_\nu/\sqrt{n}} = \frac{\delta_i \sqrt{n}}{S_\nu}, \quad i = 1,\dots,k . \tag{II.15}$$

We have also assumed independence between $\overline{X}_{(i)}$ and S_ν^2 for $i=1,\dots,k$ (see below Lemma II.2). Thus, (II.16) $$P(CD/\underline{\mu},\sigma,R) = \prod_{i=1}^{k} F_\nu(\Delta_i+D) - \prod_{i=1}^{k} F_\nu(\Delta_i-D)$$

where $F_\nu(.)$ denotes the c.d.f. of t_ν-distribution; obviously $\Delta_i = 0$ for some $i = 1,\ldots,k$.

Since the t_ν-distribution does not possess a Monotone Likelihood Ratio (MLR) property, in order to derive an expression for the $\inf_{\underline{\mu}} P(CD/\underline{\mu},\sigma,R)$, in analogy with that of Theorem II.1, we consider a conditional argument on S_ν^2 as follows.

From the fact $\overline{X}_{(i)} \sim N(\mu_{[i]}, \sigma^2/n)$, we have that

$$\left.\frac{\sqrt{n}\,(\overline{X}_{(i)} - \mu_{[i]})}{S_\nu}\right| S_\nu^2 \sim N(0, \sigma^2/S_\nu^2), \quad \text{i.e.} \quad T_i \left| S_\nu^2 \sim N(0, \sigma^2/S_\nu^2), \right.$$ where

$T_i | S_\nu^2$ denotes T_i conditioned on S_ν^2. Then of course,

(II.17) $$(S_\nu/\sigma)T_i \,|\, S_\nu^2 \sim N(0,1), \quad \text{hence,} \quad UT_i \,|\, S_\nu^2 \sim N(0,1),$$ where

(II.18) $$U^2 = S_\nu^2/\sigma^2 \text{ is a r.v. s.t. } \nu U^2 \sim \chi_\nu^2 .$$

Now we state and prove the following Lemma.

<u>Lemma II.1</u>:

(a). If T is a r.v. from t_ν-distribution, then

$$F_\nu(c) = P(T \le c) = E[\Phi(cU)]$$

where c is a constant and the expectation is over the distribution of the r.v. U defined in (II.18).

(b) If T_i, $i = 1,\ldots,k$, are i.i.d. r.v.'s from t_ν-distribution,

then $F_\nu^k(c) = P(T_i \le c,\ i = 1,\ldots,k) = \prod_{i=1}^{k} P(T_i \le c) = E[\Phi^k(cU)]$.

<u>Proof</u>:

(a) Since from (II.17) we get that $UT \mid S_\nu^2 \sim N(0,1)$, therefore,

$$F_\nu(c) = P(T \le c) = \int_0^\infty P(UT \le Uc \mid U=u) f(u)\,du = \int_0^\infty \Phi(cu) f(u)\,du = E[\Phi(cU)].$$

(b) $$F_\nu^k(c) = \int_0^\infty \prod_{i=1}^{k} P(UT_i \le Uc \mid U = u) f(u)\,du = \int_0^\infty \Phi^k(cu) f(u)\,du = E[\Phi^k(cU)]\ .$$

Q.E.D.

Applying now Lemma II.1 into (II.16) we achieve,

$$\text{(II.19)} \qquad P(CD \mid \underline{\mu},\sigma,R) = E\left\{ \prod_{i=1}^{k} \Phi\left[U(\Delta_i + D)\right] - \prod_{i=1}^{k} \Phi\left[U(\Delta_i - D)\right]\right\},$$

therefore,

$$\text{(II.20)} \qquad \inf_{\underline{\mu}} P(CD \mid \underline{\mu},\sigma,R) = E\left\{\inf_{\underline{\mu}}\left[\prod_{i=1}^{k}\Phi\left(U(\Delta_i + D)\right) - \prod_{i=1}^{k}\Phi\left(U(\Delta_i - D)\right)\right]\right\}.$$

Since the normal family possesses a MLR property, by Theorem II.1, the expression inside the curly brackets yields,

$$\text{(II.21)} \qquad \inf_{\underline{\mu}}\left[\prod_{i=1}^{k}\Phi\left(U(\Delta_i + D)\right) - \prod_{i=1}^{k}\Phi\left(U(\Delta_i - D)\right)\right] = \Phi^k(UD) - \Phi^k(-UD)$$

which by Lemma II.1 implies that,

$$\text{(II.22)} \qquad \inf_{\underline{\mu}} P(CD \mid \underline{\mu},\sigma,R) = E\left[\Phi^k(UD) - \Phi^k(UD)\right] = F_\nu^k(D) - F_\nu^k(-D)\ .$$

From (II.21), clearly this infimum is achieved for $\Delta_i = \dfrac{\delta_i\sqrt{n}}{S_\nu} = 0$

for $i=1,\ldots,k-1$, i.e. for $\delta_i=0$, $i=1,\ldots,k$, therefore, the L.F.C. of the vector $\underset{\sim}{\mu}$ is $\underset{\sim}{\mu}_{LFC} = (\mu,\mu,\ldots,\mu)$. Thus, in analogy with Theorem II.1, we have the following Theorem.

Theorem II.3. In the case where σ^2 is common but unknown,

$$\text{(II.23)} \qquad \inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\sigma,R) = P(CD|\underset{\sim}{\mu}_{LFC},\sigma,R) = F_\nu^k(D) - F_\nu^k(-D)$$

where $\underset{\sim}{\mu}_{LFC} = (\mu,\mu,\ldots,\mu)$ and $F_\nu(.)$ is the c.d.f. of the t_ν-distribution.

If $\gamma \in (0,1)$ is a preassigned confidence level the unique solution a_ν of the equation

$$\text{(II.24)} \qquad F_\nu^k(a_\nu) - F_\nu^k(-a_\nu) = \gamma$$

is given in Table 4 (see [12] p. 395).

From (II.23) and (II.24) we observe, in analogy with (II.12), that,

$$\text{(II.25)} \qquad D = \frac{d\sqrt{n}}{S_\nu} = a_\nu \ , \quad \text{i.e.} \quad n = \left[\frac{S_\nu^2 a_\nu}{d^2}\right] , \quad (\nu=k(n-1), \text{ and}$$

$[x]$ is as in (II.12)). The latter shows that the single-stage procedure R fails to give a solution to our problem when σ^2 is unknown. For that reason two-stage, ([4], [22]), and multistage procedures, ([22]), have been developed by several authors. An alternative solution to the same problem is exposed here by a sequential procedure R_s ([22]).

The Sequential Procedure R_s

This consists of the following steps.

(a) If $n_0 \geq 2$ is a preassigned integer, let $\underset{\sim}{X}_j = (X_{1j},\ldots,X_{kj})$, $j = 1,2,\ldots,$ be a sequence of observations taken one vector at a time and each time we compute S_ν^2, as in (II.13).

(b) Stop sampling when,

(II.26) "N is the 1st integer $n \geq n_0$ s.t. $S_\nu^2 \leq nd^2/a_\nu^2$ ".

(c) After sampling is terminated we take $\overline{X}_{[k]_N}$ and $S^2_{\nu_N}$, $\nu_N = k(N-1)$, and estimate $\mu_{[k]}$ by the interval,

(II.27) $$I_d = (\overline{X}_{[k]_N} - d\ ,\ \overline{X}_{[k]_N} + d)\ .$$

Notice that, for $\nu = k(n-1)$

(II.28) $$\lim_{n\to\infty} a_\nu = z_\gamma$$

where a_ν and z_γ are as in (II.24) and (II.10) respectively. The latter follows since the (multivariate) t_ν-distribution converges to the corresponding (multivariate) normal distribution for n large enough.

INVESTIGATION OF THE PERFORMANCE OF R_s

The Stopping Variable N and Related Topics

At first we derive an expression for the p.d.f. of the stopping variable N defined as in (II.26). By Helmert's orthogonal transformations we construct the r.v., (II.29) $$W_{ij} = \frac{1}{\sigma\sqrt{j(j+1)}}\left(\sum_{p=1}^{j} X_{ip} - jX_{i,j+1}\right), \quad j=1,\ldots,n-1,$$

where W_{ij}'s are i.i.d. r.v.'s distributed as $N(0,1)$. Then we have,

(II.30) $$\sum_{j=1}^{n-1} W_{ij}^2 = \frac{1}{\sigma^2}\sum_{j=1}^{n}\left(X_{ij} - \overline{X}_j\right)^2, \quad i = 1,\ldots,k,$$

which is distributed as a χ^2_{n-1} r.v., for every $n \geq 2$. Hence,

(II.31) $$\frac{\nu S_\nu^2}{\sigma^2} = \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{(X_{ij} - \overline{X}_i)^2}{\sigma^2} = \sum_{i=1}^{k}\sum_{j=1}^{n-1} W^2_{ij}$$

which is a r.v. from χ^2_ν-distribution, $\nu = k(n-1)$.

Let,

(II.32) $$U_j = \sum_{i=1}^{k} W_{ij}^2, \quad j = 1,\ldots,n-1 ;$$

then U_j's are i.i.d. r.v.'s from χ^2_k-distribution. Observing that

(II.33) $$\frac{\nu S_\nu^2}{\sigma^2} = \sum_{j=1}^{n-1} U_j ,$$

we provide ourselves a new version of the stopping rule as follows;

(II.34) "N is the 1st integer $n \geq n_0$ s.t.

$$\sum_{j=1}^{n-1} U_j \leq \frac{n\nu d^2}{a_\nu^2\,\sigma^2} = \frac{kn(n-1)}{a_\nu^2\,\lambda^2}, \quad \text{where} \quad \lambda = \sigma/d" .$$

Setting,

$$Z_t = \sum_{j=1}^{t} U_j \quad \text{and} \quad \frac{k(t+1)t}{a^2_{\nu_t}\,\lambda^2} = D_{t+1} ,$$

where $\nu_t = kt$ for $t=1,\ldots,n-1$, we obtain an expression for the p.d.f. of the r.v. N as follows ;

(II.35) $$P_n(\lambda) = P(N=n) = P(Z_1 > D_2, Z_2 > D_3, \ldots, Z_{n-2} > D_{n-1}, Z_{n-1} \le D_n)$$

$$= P(Z_{n-1} \le D_n \mid Z_{n-2} > D_{n-1}, \ldots, Z_1 > D_2)\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots P(Z_2 > D_3 \mid Z_1 > D_2)P(Z_1 > D_2) .$$

A further inquire into the evaluation of (II.35) by extending the method used by Ray ([15]), derives expressions not immediately applicable since they involve a lot of truncated gamma, beta as well as hyper-geometric functions. However, it is possible to compute the exact distribution of N following the lines of a numerical method proposed by Wang ([25]). (See also [17], [18], [21]).

Some properties of N are given by the following theorem.

Theorem II.4. For the r.v. $N = N(d)$, as it was defined in (II.26), we have,

(a) $N(d)$ is a proper stopping variable, i.e. $P[N(d)<\infty] = 1$.

(b) If $d_1 < d_2$ then $N(d_1) \ge N(d_2)$ a.s., and $\lim_{d\to 0} N(d) = \infty$ a.s.

(c) $E[N(d)]<\infty$ for all $d > 0$ and $\sigma^2 > 0$.

Proof: For $n \ge n_0$ and d, σ^2 as above, we have,

(II.36) $$P[N(d)<\infty] = 1 - P[N(d)=\infty] .$$

By (II.26) clearly,

(II.37) $P[N(d)=\infty] = P[S_\nu^2 > \frac{nd^2}{a_\nu^2}$, for all $n \geq n_o]$

$$= P\left[S_\nu^2 > \frac{nd^2}{a_\nu^2} \text{ i.o.}\right] = P\left[\lim_{n\to\infty} \sup\left(S_\nu^2 > \frac{nd^2}{a_\nu^2}\right)\right] .$$

Let now the sequences of intervals $A_n = \left(nd^2/a_\nu^2 , \infty\right)$ and $B_n = \left(-\infty, S_\nu^2\right)$. Obviously then, for some n,

(II.38) "$S_\nu^2 > nd^2/a_\nu^2$ iff $C_n = A_n \cap B_n \neq \phi$ " .

Notice that for $\nu = k(n-1)$ then,

(i) $\lim_{n\to\infty} S_\nu^2 = \sigma^2$ a.s., implies, $\lim_{n\to\infty} \sup B_n = B_\sigma$,

where $B_\sigma = (-\infty, \sigma^2)$ for fixed $\sigma^2 < \infty$, and

(ii) $\lim_{n\to\infty} a_\nu = z_\gamma$, implies, $\lim_{n\to\infty} (nd^2/a_\nu^2) = \infty$, i.e. $\lim_{n\to\infty} \sup A_n = \phi$.

By the above observations and (II.37), (II.38), we achieve,

$$P[N(d)=\infty] = P\left[\lim_{n\to\infty} \sup C_n\right] = P\left[\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \left(A_j \cap B_j\right)\right]$$

$$= P\left[\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j\right) \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B_j\right)\right] = P\left[\lim_{n\to\infty} \sup A_n \cap \lim_{n\to\infty} \sup B_n\right]$$

$= P\left[\phi \cap B_\sigma\right] = P(\phi) = 0.$ Thus, via (II.36), part (a) follows.

(b) Letting $d_1 < d_2$, by (II.26) clearly, $\frac{a_\nu^2 S_\nu^2}{d_1^2} \geq \frac{a_\nu^2 S_\nu^2}{d_2^2}$ a.s.,

for some $n \geq 2$, hence $N(d_1) \geq N(d_2)$ a.s. Moreover, since

$\lim_{d\to 0} \frac{a_\nu^2 S_\nu^2}{d^2} = \infty$ a.s., then $\lim_{d\to\infty} N(d) = \infty$ a.s.

(c) By arguments as in [27] we can show $E[N(d)] = \sum_{n=1}^{\infty} P[N(d) \geq n] < \infty$

Lemma II.2: $\overline{X}_{[k]_n} = \max(\overline{X}_{1n}, \ldots, \overline{X}_{kn})$ is independent of the pooled sample variance S_ν^2.

Proof: Robbins ([17]), has shown that the sample mean $\overline{X}_{in}$ of $n(\geq 2)$ observations, $X_{i1}, \ldots, X_{in}$, within the ith population, $i=1,\ldots,k$, is independent of the Borel σ-field $B_i^{(n)} = B(S_{i2}^2, \ldots, S_{in}^2)$ generated by $S_{i2}^2, \ldots, S_{in}^2$. Consequently, each sample mean $\overline{X}_{1n}, \ldots, \overline{X}_{kn}$ is independent of the sum $\sum_{i=1}^{k} S_{in}^2$, i.e. is also independent of the pooled sample variance $S_\nu^2 = \frac{1}{k} \sum_{i=1}^{k} S_{in}^2 = \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n-1} \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2$, and the lemma follows.

Lemma II.3: The event $\{N = n\}$ is independent of $\overline{X}_{[k]_n}$ and depends only on the σ-field $B^{(n)} = B(S_k^2, \ldots, S_{k(n-1)}^2)$.

Proof: From the stopping rule (II.26), it clearly follows that the event $\{N = n\}$ depends only on $S_\nu^2 = S_{k(n-1)}^2$, $n \geq 2$. This fact combined with Lemma II.2 completes the proof.

The Coverage Probability Under R_s

Under the Sequential Procedure R_s the coverage probability is expressed as follows.

$$\text{(II.39)} \quad P(CD|\underline{\mu},\sigma,R_s) = \sum_{n=n_0}^{\infty} P(CD;\underline{\mu},\sigma,R|N=n)\, P(N=n)$$

$$= \sum_{n=n_0}^{\infty} P(\mu_{[k]} \in I_d | N=n)\, P(N=n) .$$

Denoting by $C_k(\lambda)$ the infimum of the coverage probability under R_S, from (II.22) we obtain,

$$\text{(II.40)} \qquad C_k(\lambda) = \inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\sigma,R_s) = \sum_{n=n_0} \inf_{\underset{\sim}{\mu}} P(CD;\, \underset{\sim}{\mu},\sigma,R|N=n)\; P(N=n)$$

$$= \sum_{n=n_0}^{\infty} E\left\{\Phi^k(UD) - \Phi^k(-UD)\,\middle|\, N=n\right\} P(N=n) = \sum_{n=n_0}^{\infty} E\left\{\beta_k(UD)\,\middle|\, B^{(n)}\right\} P(N=n)$$

where $\beta_k(.)$ is as in (II.10), $B^{(n)}$ as in Lemma II.3, and the expectation is over the distribution of the r.v. $U = S_\nu/\sigma$ defined in (II.18). The latter by Lemma II.3 yields,

$$\text{(II.41)} \qquad C_k(\lambda) = E\left\{ E\left[\beta_k(UD)\,\middle|\, B^{(n)}\right]\right\} = E\left[\beta_k(UD)\right] ,$$

which for $D = d\sqrt{n}/S_\nu$, (see (II.15)) and $\lambda = \sigma/d$ gives,

$$\text{(II.42)} \qquad C_k(\lambda) = E\left[\beta_k\left(\frac{d\sqrt{N}}{\sigma}\right)\right] = E\left[\beta_k\left(\frac{\sqrt{N}}{\lambda}\right)\right] ,$$

where the expectation is over the distribution of N.

In what follows we try to reach a final conclusion about the behaviour of $C_k(\lambda)$ in general.

<u>Lemma II.4</u>: The function,

$$\beta_k(y) = \Phi^k(y) - \Phi^k(-y) , \quad y > 0$$

(a) is monotonically increasing in y for every $k \geq 1$.

(b) if $k \leq 2$, it is concave for every $y > 0$, and

(c) if $k > 2$, it is concave only for $y > y_k$, where y_k is the inflection point associated with each k.

(A rough computation gives $y_3 = 0.5$, $y_4 = 0.9$).

Proof: See Appendix A.

Lemma II.5: If f is a concave positive function on a real interval and finite on it, and X, $f(X)$ are r.v.'s so that $E(X)<\infty$, $E(f(X)<\infty$, then

$$f(E(X)) \geq E(f(X)) .$$

Proof: See Appendix A.

Remarks:

(1) $\beta_k(y) > 0$, for every $y > 0$.

(2) $0 \leq \beta_k(y) \leq 1$, for every $y > 0$.

(3) $0 \leq E[\beta_k(Y)] \leq 1$.

(4) letting $Y = \frac{\sqrt{N}}{\lambda}$ by Theorem II.4(c) and Lemma II.5 we get $E\sqrt{N} \leq \sqrt{EN} < \infty$, therefore, $E(Y) < \infty$.

Theorem II.5: For the infimum of the coverage probability it can be shown that,

$$C_k(\lambda) \leq \beta_k\left(\frac{\sqrt{EN}}{\lambda}\right) = \Phi^k\left(\frac{\sqrt{EN}}{\lambda}\right) - \Phi^k\left(-\frac{\sqrt{EN}}{\lambda}\right) \tag{II.43}$$

is true,

(a) if $k \leq 2$, for every $\lambda > 0$.

(b) if $k > 2$, for $y_k \leq \frac{\sqrt{N}}{\lambda}$, i.e. for every $\lambda \leq \frac{\sqrt{N}}{y_k}$.

Proof: From Remarks (1) - (4) and Lemma II.4 we observe that $\beta_k(y)$ satisfies the requirements of Lemma II.5 in the following two cases:

(a) if $k \le 2$, for every $y > 0$, i.e. for every $\lambda > 0$ (since $\sqrt{N} > 0$).

(b) if $k > 2$, for every $y = \frac{\sqrt{N}}{\lambda} \ge y_k$, i.e., for every $\lambda \le \frac{\sqrt{N}}{y_k}$.

As a result it follows that,

(II.44) $$C_k(\lambda) = E\left[\beta_k\left(\frac{\sqrt{N}}{\lambda}\right)\right] \le \beta_k\left(\frac{E\sqrt{N}}{\lambda}\right)$$

only on intervals as in the above cases (a) and (b). Since $\beta_k(y)$ is increasing in y and $E\sqrt{N} \le \sqrt{EN}$ (see Remark (4)) we end up with,

(II.45) $$\beta_k\left(\frac{E\sqrt{N}}{\lambda}\right) \le \beta_k\left(\frac{\sqrt{EN}}{\lambda}\right) ,$$

which combined with (II.44) proves (II.43) for intervals as in the cases (a) and (b). Q.E.D.

If in addition we consider the ratio

(II.46) $$\eta_k(\lambda) = \frac{E(N)}{N_o} , \quad \lambda > 0 \quad \text{(efficiency)} ,$$

where N_o is the optimal sample size for σ^2 known (see (II.12)), and also if

(II.47) $$E(N) > N_o , \text{ i.e., } \quad \eta_k(\lambda) > 1$$

then, by (II.9), for $\lambda = \sigma/d$, we achieve

(II.48) $$\beta_k\left(\frac{\sqrt{EN}}{\lambda}\right) > \beta_k\left(\frac{\sqrt{N_o}}{\lambda}\right) = \Phi^k\left(\frac{\sqrt{N_o}}{\lambda}\right) - \Phi^k\left(-\frac{\sqrt{N_o}}{\lambda}\right) \ge \gamma .$$

This together with the main result of Theorem II.5, i.e.

(II.49) $$" C_k(\lambda) \le \beta_k\left(\frac{\sqrt{EN}}{\lambda}\right), \text{ for some } \lambda\text{'s} ",$$

suggests that the standard requirement $C_k(\lambda) \ge \gamma$ may be satisfied, for these λ's, only if (II.47) is true. Summarizing we state the following corollary.

Corollary II.1: For a $\lambda > 0$ whenever $k \le 2$, and a $\lambda \le \frac{\sqrt{N}}{y_k}$ whenever $k > 2$, the requirement $C_k(\lambda) \ge \gamma$ can be satisfied only if $\eta_k(\lambda) > 1$.

As a consequence of the previous conclusions we turn immediately our attention to the asymptotic properties of R_s in order to establish that it solves always our problem only asymptotically.

Asymptotic Behaviour of R_s

At first we state and prove two important Lemmas.

Lemma II.6 (Chow and Robbins [7]).

Let $\{V_n;\ n=1,2,\ldots\}$ be any sequence of r.v.'s so that,

(II.50) $$V_n > 0 \text{ a.s. and } \lim_{n\to\infty} V_n = 1 \text{ a.s.}$$

Let also $f(n)$ be a sequence of constants so that,

(II.51) $$\text{(i) } f(n) > 0\,,\ \text{(ii) } \lim_{n\to\infty} f(n) = \infty\,,\ \text{(iii) } \lim_{n\to\infty} \frac{f(n)}{f(n-1)} = 1\,,$$

and for each $t > 0$ we define,

(II.52) $$" N = N(t) \text{ is the smallest integer } k \ge 1 \text{ s.t. } f(k) \ge tV_k ".$$

Then,

(a) $N(t)$ is a proper, nondecreasing in t stopping variable.

(b) $\lim_{t\to\infty} N(t) = \infty$ a.s.

(c) $\lim_{t\to\infty} E(N(t)) = \infty$.

(d) $\lim_{t\to\infty} \frac{f(N(t))}{t} = 1$ a.s.

Proof:

(a) From (II.50) and (II.52) by arguments similar to those of Theorem II.4 we can show that $N(t)$ is a proper variable nondecreasing in t.

(b) Let $k < \infty$; then (II.51) (i), (ii) imply $f(k) < \infty$. From (II.52) and (II.51) (ii), if $\lim_{t\to\infty} tV_k = \infty$ then $\lim_{t\to\infty} f(k) = \infty$,

i.e. $\lim_{t\to\infty} k = \infty$, however, $\lim_{t\to\infty} N(t) = \infty$ a.s.

(c) Combining parts (a) and (b) via monotone convergence theorem we achieve, $\lim_{t\to\infty} E(N(t)) = \infty$.

(d) From (II.52) we get $f(N(t)-1) < tV_{N(t)-1}$, however,

$$\frac{1}{t} < \frac{V_{N(t)-1}}{f(N(t)-1)} \quad \text{and by (II.52)} \quad V_{N(t)} \le \frac{f(N(t))}{f(N(t)-1)} V_{N(t)-1} .$$

Hence,

$$\text{(II.53)} \quad \lim_{t\to\infty} V_{N(t)} \le \lim_{t\to\infty} \frac{f(N(t))}{t} < \lim_{t\to\infty} \frac{f(N(t))}{f(N(t)-1)} \lim_{t\to\infty} V_{N(t)-1} .$$

Since $\lim_{t\to\infty} N(t) = \infty$ a.s., by (II.50) we get $\lim_{t\to\infty} V_{N(t)} = 1$ a.s.

which applied to (II.53) yields $1 \le \lim_{t\to\infty} \frac{f(N(t))}{t} < 1$ a.s., and (d) follows.

Lemma II.7: (Chow and Robbins [7]).

If the conditions of Lemma II.6 hold and also if

(II.54) $$E\left\{\sup_{n\geq 1} V_n\right\} < \infty , \quad \text{then}$$

(II.55) $$\lim_{t\to\infty} E\left\{\frac{f(N(t))}{t}\right\} = 1 .$$

Proof: Letting $W = \sup_{n\geq 1} V_n$ we choose a k (see (II.52)) so large that $f(n)/f(n-1) \leq 3/2$ for all $n \geq k$. If $N(t) \geq k$ we have,

(II.56) $$\frac{f(N(t))}{t} = \frac{f(N(t))}{f(N(t)-1)} \frac{f(N(t)-1)}{t} < \frac{3}{2} V_{N(t)-1} \leq \frac{3}{2} W ,$$

and if $N(t) < k$, $k > 1$ and $t \geq 1$, then

(II.57) $$\frac{f(N(t))}{t} \leq \max_{1\leq n<k} \frac{f(n)}{t} \leq \frac{1}{t} (f(1)+\ldots+f(k)) \leq \sum_{i=1}^{k} f(i) .$$

Hence, for all $t \geq 1$ putting (II.56) and (II.57) together we get

$$\frac{f(N(t))}{t} < \frac{3}{2} W I_{[N(t)\geq k]} + \sum_{i=1}^{k} f(i) I_{[N(t)<k]} \leq 3W + \sum_{i=1}^{k} f(i)$$

which combined with $E(W) < \infty$ (see (II.54)) and (d) of Lemma II.6, via Lebesque's dominated convergence theorem results in,

$$\lim_{t\to\infty} E\left\{\frac{f(N(t))}{t}\right\} = E\left\{\lim_{t\to\infty} \frac{f(N(t))}{t}\right\} = 1. \qquad \text{Q.E.D.}$$

Returning now to our problem we set,

(II.58) $V_n = S_\nu^2/\sigma^2$ where S_ν^2 as in (II.13) and

(II.59) (i) $f(n) = n/a_\nu^2$, (ii) $t = \sigma^2/d^2 = \lambda^2$.

Remarks:

(5) The stopping variable $N = N(t)$ defined in (II.52), via arrangements (II.58), (II.59), for $k = n$, coincides with that one defined in (II.26); thus $N(t) = N(d) = N$ and

(II.60) $$\lim_{t\to\infty} N(t) = \lim_{d\to o} N(d) = \lim_{\lambda\to\infty} N = \infty \quad \text{a.s.}$$

(6) From (II.58) and (II.33) we get that,

(II.61) $$V_n = \frac{S_\nu^2}{\sigma^2} = \frac{1}{\nu}\sum_{j=1}^{n-1} U_j \text{ , } U_j\text{'s are as in (II.32) .}$$

(7) By Wiener's ergodic theorem, ([26]) we may see that the condition $E\left\{\sup_{n\geq 1} V_n\right\} < \infty$ of Lemma II.7 is satisfied if $E(U_1) < \infty$; this in turn requires $E(X_{ij}^4) < \infty$, which is true for the normal distribution with $\sigma^2 < \infty$.

Theorem II.6: Under R_s for $k \geq 1$ finite and $\sigma^2 < \infty$ we have,

(a) $\lim_{\lambda\to\infty} \frac{N}{N_o} = 1$ a.s., where N_o is as in (II.12).

(b) $\lim_{\lambda\to\infty} \eta_k(\lambda) = 1$, (asymptotic efficiency).

(c) $\lim_{\lambda\to\infty} C_k(\lambda) = \gamma$, (asymptotic consistency).

Proof:

(a) If $\nu_N = k(N-1)$, by (II.59)(i), clearly $f(N) = N/a_{\nu_N}^2$, hence $\frac{f(N(t))}{t} = \frac{N}{\lambda^2 a_{\nu_N}^2}$. On the otherhand by (II.28) and (II.60) we obtain, (II.62) $$\lim_{\lambda\to\infty} a_{\nu_N} = z_\gamma \ .$$

From Lemma II.6 (d) and (II.60) we achieve,

$$\lim_{t\to\infty} \frac{f(N(t))}{t} = \lim_{\lambda\to\infty} \frac{N}{\lambda^2 a^2_{\nu_N}} = \lim_{\lambda\to\infty} \frac{N}{\lambda^2 z^2_\gamma} = \lim_{\lambda\to\infty} \frac{N}{N_0} = 1 \quad \text{a.s.} \ ,$$

where the next to the last equality comes from (II.12).

(b) Since, by Remark (7), the requirement of Lemma II.7 is satisfied, from the same lemma it follows that,

$$\lim_{\lambda\to\infty} E\left\{\frac{f(N(t))}{t}\right\} = \lim_{\lambda\to\infty} E\left\{\frac{N}{a^2_{\nu_N}\lambda^2}\right\} = \lim_{\lambda\to\infty} \frac{E(N)}{z^2_\gamma \lambda^2}$$

$$= \lim_{\lambda\to\infty} \frac{E(N)}{N_0} = \lim_{\lambda\to\infty} \eta_k(\lambda) = 1.$$ Q.E.D.

(c) If $\{\lambda_j\}$ is an arbitrary but fixed monotonically increasing sequence, so that $\lim_{j\to\infty} \lambda_j = \infty$, from part (a) we get that $\lim_{j\to\infty} \frac{\sqrt{N}}{\lambda_j} = z_\gamma$ a.s., and by continuity of $\beta_k\left(\frac{\sqrt{N}}{\lambda_j}\right)$ we obtain,

(II.63) $$\lim_{j\to\infty} \beta_k\left(\frac{\sqrt{N}}{\lambda_j}\right) = \beta_k(z_\gamma) \quad \text{a.s.}$$

Let F_j, $(j=1,\ldots,2)$, F be respectively the c.d.f.'s of

$\beta_k\left(\frac{\sqrt{N}}{\lambda_j}\right) = W_j$ and $\beta_k(z_\gamma) = W$, say .

We observe that $\beta_k(z_\gamma)$ is a degenerate variable, since $P[\beta_k(z_\gamma) = \gamma] = 1$ (see (II.10)), however,

$$\lim_{j\to\infty} F_j(w) = F(w) = \begin{cases} 0 & \text{for } w < \gamma \\ 1 & \text{for } w \geq \gamma \end{cases} \quad , \text{ where } \gamma \in (0,1) .$$

Now from (II.63) and by boundedness of $\beta_k(.)$ (see Remark (2)), via bounded convergence theorem and Helly-Bray theorem (see [14]), we achieve,

$$\lim_{j\to\infty} C_k(\lambda_j) = \lim_{j\to\infty} E\left[\beta_k\left(\frac{\sqrt{N}}{\lambda_j}\right)\right] = \lim_{j\to\infty} \int_0^1 wdF_j(w)$$

$$= \int_0^1 wdF(w) = E\left(WI_{[0,\gamma)} + WI_{[\gamma,1]}\right) = \gamma E\left(I_{w\in[\gamma,1]}\right)$$

$$= \gamma P(W \geq \gamma) = \gamma \; ; \text{ the latter follows since } P\left[\beta_k(z_\gamma) > \gamma\right] = 0.$$

Choosing $\{\lambda_j\}$ to be arbitrary the proof is completed.

SINGLE-STAGE AND SEQUENTIAL PROCEDURES IN THE CASE OF UNEQUAL VARIANCES

Single-Stage Procedures when the Variances are known

In the case where the populations $\pi_1,\ldots\pi_k$ have different variances we modify our procedures considered in previous sections, so that, the standard requirement (II.1) is again satisfied for all vectors, $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k)$ and $\underset{\sim}{\sigma}^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ in Ω. Investigating at first the case where the vector $\underset{\sim}{\sigma}^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ is known, we present two single-stage procedures R_1 and R_2 as follows.

The Procedure R_1 consists of choosing the individual sample sizes, $n_1,\ldots,n_k$, so that,

(II.64) $$\frac{\sigma_1^2}{n_1} = \frac{\sigma_2^2}{n_2} = \ldots = \frac{\sigma_k^2}{n_k} = \frac{1}{n'} \quad , \text{ therefore,}$$

(II.65) $n_i = \sigma_i^2 n'$, for every $i = 1,\ldots,k$.

Using these sample sizes we compute the sample means and order them as,

(II.66) $\bar{X}_{[1]} \leq \ldots \leq \bar{X}_{[k]}$.

Then the coverage probability under R_1 is,

(II.67) $$P\left(CD\,\middle|\,\underset{\sim}{\mu},\underset{\sim}{\sigma},R_1\right) = P\left(\bar{X}_{[k]} \leq \mu_{[k]} + d\right) - P\left(\bar{X}_{[k]} \leq \mu_{[k]} - d\right)$$

$$= P\left(\frac{\bar{X}_{(i)} - \mu_{[i]}}{\sigma_i/\sqrt{n_i}} \leq \frac{\mu_{[k]} - \mu_{[i]} + d}{\sigma_i/\sqrt{n_i}},\quad i = 1,\ldots,k\right)$$

$$- P\left(\frac{\bar{X}_{(i)} - \mu_{[i]}}{\sigma_i/\sqrt{n_i}} \leq \frac{\mu_{[k]} - \mu_{[i]} - d}{\sigma_i/\sqrt{n_i}},\ i = 1,\ldots,k\right)$$

$$= \prod_{i=1}^{k} \Phi\left[\frac{(\delta_i + d)\sqrt{n_i}}{\sigma_i}\right] - \prod_{i=1}^{k} \Phi\left[\frac{(\delta_i - d)\sqrt{n_i}}{\sigma_i}\right]$$, where δ_i's

are as in (I.5).

According to Theorem II.1 for $\delta_i = 0$, $i = 1,\ldots,k$, clearly

(II.68) $$\inf_{\underset{\sim}{\mu}} P\left(CD\,\middle|\,\underset{\sim}{\mu},\underset{\sim}{\sigma},R_1\right) = \prod_{i=1}^{k} \Phi\left(\frac{d\sqrt{n_i}}{\sigma_i}\right) - \prod_{i=1}^{k} \Phi\left(-\frac{d\sqrt{n_i}}{\sigma_i}\right)$$

which by condition (II.64) implies that,

(II.69) $$\inf_{\underset{\sim}{\mu}} P\left(CD\,\middle|\,\underset{\sim}{\mu},\underset{\sim}{\sigma},R_1\right) = \Phi^k(d\sqrt{n'}) - \Phi^k(-d\sqrt{n'}) .$$

Hence, for a given $\gamma \in (0,1)$ and a z_γ, such that,

(II.70) $$\beta_k(z_\gamma) = \Phi^k(z_\gamma) - \Phi^k(-z_\gamma) = \gamma$$

by Theorem II.2 we get, $\Phi^k(d\sqrt{n'}) - \Phi^k(-d\sqrt{n'}) \geq \gamma$

for every $n' \geq N'_o$, say, i.e.,

(II.71) " N'_o is the smallest integer n' s.t. $n' \geq \frac{z_\gamma^2}{d^2}$ " .

Thus,

(II.72) $N'_o = [z_\gamma^2/d^2]$, (the notation [x] is as in (II.12)),

is the smallest sample size, under R_1, needed to satisfy the requirement (II.1). By (II.65), however, the smallest individual sample sizes must be chosen, so that,

(II.73) $$n_i = \sigma_i^2 N'_o , \quad i = 1,\ldots,k .$$

The Procedure R_2 consists of taking a common sample size n per population and ordering the values of σ_i's as,

(II.74) $$\sigma^2_{[1]} \leq \sigma^2_{[2]} \leq \ldots . \leq \sigma^2_{[k]} .$$

We note that $\sigma^2_{[k]}$ and $\bar{X}_{[k]}$ may not come from the same population and also that,

(II.75) $$\frac{\sigma^2_{[1]}}{n} \leq \ldots . \leq \frac{\sigma^2_{[k]}}{n} .$$

The coverage probability under R_2 now is,

$$P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = \prod_{i=1}^{k} \Phi\left[\frac{(\delta_i+d)\sqrt{n}}{\sigma_i}\right] - \prod_{i=1}^{k} \Phi\left[\frac{(\delta_i-d)\sqrt{n}}{\sigma_i}\right]$$

and also by Theorem II.1 we achieve,

(II.76) $$\inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = \prod_{i=1}^{k} \Phi\left(\frac{d\sqrt{n}}{\sigma_i}\right) - \prod_{i=1}^{k} \Phi\left(-\frac{d\sqrt{n}}{\sigma_i}\right)$$

$$= \prod_{i=1}^{k} \Phi\left(\frac{d\sqrt{n}}{\sigma_{[i]}}\right) - \prod_{i=1}^{k} \Phi\left(-\frac{d\sqrt{n}}{\sigma_{[i]}}\right) .$$

<u>Lemma II.8</u> : The function $\beta(\underset{\sim}{y}) = \prod_{i=1}^{k} \Phi(y_i) - \prod_{i=1}^{k} \Phi(-y_i)$ is decreasing as y_i, $i=1,\ldots,k$ decreases, where $k \geq 1$ and $\underset{\sim}{y} = (y_1,\ldots,y_k)$.

<u>Proof</u>: See Appendix A.

This lemma combined with (II.75) and (II.76) yields,

(II.77) $$\inf_{\underset{\sim}{\mu}} P(CD \mid \underset{\sim}{\mu}, \underset{\sim}{\sigma}, R_2) \geq \inf_{\underset{\sim}{\mu}, \underset{\sim}{\sigma}} P(CD \mid \underset{\sim}{\mu}, \underset{\sim}{\sigma}, R_2)$$

$$= \Phi^k\left(\frac{d\sqrt{n}}{\sigma_{[k]}}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma_{[k]}}\right)$$

which by Theorem II.2 and γ, z_γ as in (II.70), implies that,

(II.78) $$\Phi^k\left(\frac{d\sqrt{n}}{\sigma_{[k]}}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma_{[k]}}\right) \geq \gamma \text{ , for every } n \geq N_0 \text{ , say.}$$

However,

(II.79) " N_0 is the smallest integer n s.t. $n \geq \sigma^2_{[k]} \frac{z^2_\gamma}{d^2}$ " i.e. $N_0 = [\lambda^2 z^2_\gamma]$, where $\lambda = \sigma^2_{[k]}/d^2$ and the notation $[x]$ as in (II.12).

<u>Remark</u>:

(8) From (II.72) and (II.73) it is obvious that the smallest possible sample sizes under R_1, are

(II.80) $$n_{(i)} = \left[\sigma^2_{[i]} z^2_\gamma / d^2\right] \text{ , for } i = 1,2,\ldots,k.$$

where $n(.)$ denotes the sample size of the population with variance $\sigma^2_{[.]}$. Under R_2 the smallest possible sample size per population is, (II.81) $$n = \left[\sigma^2_{[k]}\ z^2_\gamma/d^2\right] ,$$

which by (II.74) indicates that R_2 requires a larger sample size per population. Nevertheless, the inequality in (II.77) implies that R_2 guarantees a higher coverage probability in most of the cases.

A Sequential Procedure when the Variances are Unknown

Extending the Procedure R_2 to the case where the vector $\sigma^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ is unknown, we reach the point where the sequential procedure R_s' is introduced. Using, however, a common sample size n per population we compute, $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ and $S_i^2 = \sum_{j=1}^{n}(X_{ij} - \overline{X}_i)^2/(n-1)$,

$i = 1,\ldots,k$

and order them accordingly as,

$$\overline{X}_{[1]} \leq \ldots \leq \overline{X}_{[k]} \quad , \quad S^2_{[1]} \leq \ldots \leq S^2_{[k]} \ .$$

Notice that $\overline{X}_{[k]}$ and $S^2_{[k]}$ may not come from the same population.

Theorem II.7: For $\sigma^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ unknown,

$$P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = \prod_{i=1}^{k} P(T_i \leq \Delta_i + D_i) - \prod_{i=1}^{k} P(T_i \leq \Delta_i - D_i)$$

(II.82) $$= \prod_{i=1}^{k} F_{n-1}(\Delta_i + D_i) - \prod_{i=1}^{k} F_{n-1}(\Delta_i - D_i) ,$$

where T_i's are i.i.d. r.v.'s from t_{n-1}-distribution with $F_{n-1}(.)$ as a c.d.f., and $T_i = \dfrac{\sqrt{n}(\overline{X}_{(i)} - \mu_{[i]})}{S_{(i)}}$, $D_i = \dfrac{d\sqrt{n}}{S_{(i)}}$, $\Delta_i = \dfrac{\delta_i\sqrt{n}}{S_{(i)}}$ for $i = 1,\ldots,k$.

Proof: Replacing in (II.14) S_ν by $S_{(i)}$, $i=1,\ldots,k$ and by independence of $\bar{X}_{(i)}$ and $S_{(i)}$, (see Lemma II.2), the proof follows.

By a conditioning argument on $S_{(i)}$, $i=1,\ldots,k$, we have again

$$T_i \Big| S_{(i)} = \frac{\sqrt{n}(\bar{X}_{(i)} - \mu_{[i]})}{S_{(i)}} \Big| S_{(i)} \sim N\left(0, \frac{\sigma^2_{(i)}}{S^2_{(i)}}\right)$$

and if $U_i^2 = S^2_{(i)}/\sigma^2_{(i)}$ is a r.v. s.t. $(n-1)U_i^2 \sim \chi^2_{n-1}$, for each $i=1,\ldots,k$, it follows that U_i's are i.i.d. r.v.'s and $U_i T_i \Big| S_{(i)} \sim N(0,1)$. Then by Lemma II.1(a) and independence of U_i's we obtain,

$$\text{(II.83)} \qquad \prod_{i=1}^{k} P\left(T_i \le \Delta_i + D_i\right) = \prod_{i=1}^{k} E_{U_i}\left[\Phi\left(U_i(\Delta_i+D_i)\right)\right]$$

$$= E_{U_1,\ldots,U_k}\left[\prod_{i=1}^{k} \Phi\left(U_i(\Delta_i+D_i)\right)\right] = E_{\underset{\sim}{U}}\left[\prod_{i=1}^{k} \Phi\left(U_i(\Delta_i+D_i)\right)\right]$$

where the expectation is over the joint distribution of $\underset{\sim}{U} = (U_1,\ldots,U_k)$.

From (II.82) via (II.83) we now achieve,

$$\text{(II.84)} \qquad P(CD\,|\,\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = E_{\underset{\sim}{U}}\left\{\prod_{i=1}^{k} \Phi\left(U_i(\Delta_i+D_i)\right) - \prod_{i=1}^{k} \Phi\left(U_i(\Delta_i-D_i)\right)\right\}$$

which by Theorem II.1 results in,

$$\text{(II.85)} \qquad \inf_{\underset{\sim}{\mu}} P\left(CD\,|\,\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2\right) = E_{\underset{\sim}{U}}\left\{\prod_{i=1}^{k} \Phi(U_iD_i) - \prod_{i=1}^{k} \Phi(-U_iD_i)\right\}$$

$$= E_{\underset{\sim}{U}}\left\{\beta(\underset{\sim}{U},\underset{\sim}{D})\right\}, \text{ say, where,}$$

$$\text{(II.86)} \qquad \beta(\underset{\sim}{U},\underset{\sim}{D}) = \prod_{i=1}^{k} \Phi(U_i D_i) - \prod_{i=1}^{k} \Phi(-U_i D_i) \quad \text{and}$$

$$\underset{\sim}{U} = (U_1,\ldots,U_k) \;, \quad \underset{\sim}{D} = (D_1,\ldots,D_k) \;.$$

Since for every $i = 1,\ldots,k$ obviously

$$\text{(II.87)} \qquad U_i D_i = \frac{S_{(i)}}{\sigma_{(i)}} \frac{d\sqrt{n}}{S_{(i)}} = \frac{d\sqrt{n}}{\sigma_{(i)}} \geq \frac{d\sqrt{n}}{\sigma_{[k]}} = \frac{S_{[k]}}{\sigma_{[k]}} \frac{d\sqrt{n}}{S_{[k]}} = U_{[k]} \, D_{[k]} \;, \text{ say}$$

from Lemma II.8 it follows that,

$$\text{(II.88)} \qquad \beta(\underset{\sim}{U},\underset{\sim}{D}) \geq \beta\left(U_{[k]},D_{[k]}\right) = \Phi^k\left(U_{[k]} \, D_{[k]}\right) - \Phi^k\left(-U_{[k]} \, D_{[k]}\right)$$

where $U_{[k]}$ does not indicate any ordering of U's.

<u>Theorem II.8</u>: For $\underset{\sim}{\sigma}^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ unknown,

$$\text{(II.89)} \qquad \inf_{\underset{\sim}{\mu}} P(CD \,|\, \underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) \geq F_{n-1}^k \left(\frac{d\sqrt{n}}{S_{[k]}}\right) - F_{n-1}^k \left(-\frac{d\sqrt{n}}{S_{[k]}}\right)$$

<u>Proof</u>: Combining, at first, (II.85) and (II.88) we obtain,

$$\text{(II.90)} \qquad \inf_{\underset{\sim}{\mu}} P(CD \,|\, \underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = E_{\underset{\sim}{U}} \left\{\beta(\underset{\sim}{U},\underset{\sim}{D})\right\} \geq E_{\underset{\sim}{U}} \left\{\beta(U_{[k]},D_{[k]})\right\}$$

$$= E_{U_{[k]}}\left\{\beta\left(U_{[k]},D_{[k]}\right)\right\} = E_{U_{[k]}}\left\{\Phi^k\left(\frac{d\sqrt{n}}{\sigma_{[k]}}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma_{[k]}}\right)\right\}$$

$$= F_{n-1}^k\left(\frac{d\sqrt{n}}{S_{[k]}}\right) - F_{n-1}^k\left(-\frac{d\sqrt{n}}{S_{[k]}}\right) ,$$ where the last equality

follows from Lemma II.1. Q.E.D.

Let now $\gamma \in (0,1)$ be preassigned and a_{n-1} be such that,

$F_{n-1}^k(\alpha_{n-1}) - F_{n-1}^k(-\alpha_{n-1}) = \gamma$; then, for a z_γ as in (II.10), clearly

$$\text{(II.91)} \qquad \lim_{n\to\infty} a_{n-1} = z_\gamma \ .$$

At this stage we introduce the sequential procedure R_s' as follows:

(a) If $n_0 \geq 2$ is a preassigned integer, let $\underset{\sim}{X}_j = (X_{1j},\ldots,X_{kj})$, $j = 1,2,\ldots,$ be a sequence of observations taken one vector at a time and each time we compute $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$, $i = 1,\ldots,k$, $S_i^2 = \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2/(n-1)$ and order them accordingly as,

$$\overline{X}_{[1]_n} \leq \ldots \leq \overline{X}_{[k]_n} \ , \quad S^2_{[1]_n} \leq \ldots \leq S^2_{[k]_n}$$

(b) Stop sampling when,

$$\text{(II.92)} \quad \text{"N is the 1st integer } n \geq n_0 \text{ s.t. } S^2_{[k]_n} \leq \frac{nd^2}{a^2_{n-1}} \text{"} .$$

(c) After sampling is terminated we estimate $\mu_{[k]}$ by the interval $I_d = (\overline{X}_{[k]_N} - d \ , \ \overline{X}_{[k]_N} + d)$.

Considering the performance of R_s' let,

$$\lambda = \frac{\sigma_{[k]}}{d} \ , \ \beta_k\left(\frac{\sqrt{n}}{\lambda}\right) = \Phi^k\left(\frac{\sqrt{n}}{\lambda}\right) - \Phi^k\left(-\frac{\sqrt{n}}{\lambda}\right) , \quad B^{(n)} = B\left(S^2_{[k]_2},\ldots,S^2_{[k]_n}\right) .$$

Then by (II.90) and arguments similar to those used in (II.39) - (II.42), replacing also U by $U_{[k]} = S_{[k]_n}/\sigma_{[k]}$, we can determine the infimum of the coverage probability under R_s' as follows.

$$\text{(II.93)} \qquad C_K(\lambda) = \sum_{n=n_0}^{\infty} \inf_{\underset{\sim}{\mu}} P(CD;\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2/N=n)\ P(N=n)$$

$$\geq \sum_{n=n_0}^{\infty} E_{U_{[k]}}\left\{\beta_k\left(\frac{\sqrt{N}}{\lambda}\right) \middle| N=n\right\} P(N=n) = E\left\{\beta_k\left(\frac{\sqrt{N}}{\lambda}\right)\right\} .$$

At this point we observe that the inequality in (II.93) implies better chances for the requirement $C_k(\lambda) \geq \gamma$ to be satisfied under R_s', than under R_s (see Theorem II.5 and Corollary II.1), in the corresponding cases.

Theorem II.9. Under R_s' for every $k \geq 1$ finite and every n_0, with $\sigma_1^2, \ldots, \sigma_k^2$ all finite, we have,

(a) $\lim\limits_{\lambda \to \infty} \frac{N}{N_0} = 1$ a.s. ($N_0 = [\lambda^2 z_\gamma^2]$ as in (II.79)).

(b) $\lim\limits_{\lambda \to \infty} \frac{E(N)}{N_0} = 1$ (asymptotic efficiency).

(c) $\lim\limits_{\lambda \to \infty} C_k(\lambda) \geq \gamma$ (asymptotic consistency).

Proof: Letting $V_n = S^2_{[k]_n}/\sigma^2_{[k]}$, $f(n) = n/a^2_{n-1}$, $t = \sigma^2_{[k]}/d^2 = \lambda^2$, by Lemmas II.6 and II.7 (Chow and Robbins [7]), and following the lines of the proof of Theorem II.6 this proof also follows.

Remark:

(9) An extension of the procedure R_1 to the case where the vector $\underset{\sim}{\sigma}^2$ is unknown, yields a two-stage procedure, say R_1', (see [4], [8]). Comparing R_1' and R_s', it is apparent (see Remark (8)) that, while R_1' needs sometimes a smaller sample size than R_s', the latter procedure seems to be more reliable since it handles many more sample informations and its behaviour, in terms of coverage probability, is not bad at all.

CHAPTER III

OPTIMAL INTERVAL ESTIMATION OF THE LARGEST NORMAL MEAN BY PREFIXED WIDTH UNSYMMETRIC CONFIDENCE INTERVALS

PRELIMINARIES

A Single-Stage Procedure R Under General Assumptions

In Chapter II a solution is provided to the problem of interval estimation of the largest location parameter $\theta^* = \theta_{[k]} = \max_{1 \le i \le k} \theta_i$, letting $T_{in} = T(X_{i1},\ldots,X_{in})$ be a consistent estimate for θ_i, $i = 1,\ldots,k$, and considering a prefixed-width, $L = 2d$, symmetric confidence interval for θ^* of the form,

$$\text{(III.1)} \qquad I_d = (T_n^* - d\ ,\ T_n^* + d) = (T_n^* - L/2\ ,\ T_n^* + L/2), \quad \text{for } T_n^* = \max_{1 \le i \le k} T_{in}.$$

In [9] Dudewicz has shown that if T_n^* is a consistent estimate for θ^* and $E_{\theta^*}(T_n^*) \xrightarrow[n\to\infty]{} \theta^*$ then T_n^* overestimates θ^* for $k \ge 2$, and the bias increases as k increases. This result suggests that a modified approach to our problem must be considered, by unsymmetric about T_n^* confidence intervals, of the form,

$$\text{(III.2)} \qquad I_{d_1,d_2} = (T_n^* - d_1\ ,\ T_n^* + d_2)\ , \quad \text{with } d_1 + d_2 = L\ .$$

In this case we expect I_{d_1,d_2} to perform better than I_d, in terms of coverage probability, for an optimal choice of d_2(or d_1). This task will be carried out in this chapter considering, at first, some general results and applying them, later on, to the normal family ([2], [11]).

A sequential procedure then is presented in the case of a common unknown variance, ([23]), and single-stage as well as sequential procedures are also provided in the case of different variances.

Definition III.1: The event of Correct Decision (CD) is defined now to be equivalent to the event $\theta^* \in I_{d_1,d_2}$.

If $g_n(t,\theta_i)$, $G_n(t,\theta_i)$ are again the p.d.f. and c.d.f. of T_{in} respectively and d_1, d_2 are assumed, for the time being, to be arbitrary but fixed, so that $d_1 + d_2 > 0$, then for a single-stage procedure R we may have (as in (II.3)) ,

$$\text{(III.3)} \qquad P(CD|\underset{\sim}{\theta},\sigma,R) = \prod_{i=1}^{k} G_n(\theta^*+d_1,\theta_i) - \prod_{i=1}^{k} G_n(\theta^* - d_2,\theta_i)$$

$$= b_{\underset{\sim}{\theta}}(d_1,d_2) \text{ , say, for every } \underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k) \in \Omega .$$

Note: In what follows the index n is dropped from the above notation.

The least favorable configuration, $\underset{\sim}{\theta}_{LFC}$, satisfying

$$\text{(III.4)} \qquad \inf_{\underset{\sim}{\theta}} b_{\underset{\sim}{\theta}}(d_1,d_2) = b_{\underset{\sim}{\theta}_{LFC}}(d_1,d_2)$$

is determined by the following generalized form of Theorem II.1.

Theorem III.1: Suppose that,

(i) $g(t,\theta) = g(t-\theta)$ ($g(t,\theta)$ is a location parameter p.d.f.) .

(ii) $g(t) = g(-t) > 0$, for all t, (symmetric) .

(iii) the family $\{g(t-\theta); \theta\}$ has a M.L.R. property.

Then, for arbitrary but fixed d_1, d_2 with $d_1 + d_2 > 0$, we have,

(a) If $G(d_1) - G(-d_2) < G^k(d_1) - G^k(-d_2)$, then,

$\underset{\sim}{\theta}_{LFC} = (-\infty,\ldots,-\infty,\theta)$ and $b_{\underset{\sim}{\theta}_{LFC}}(d_1,d_2) = G(d_1) - G(-d_2)$

(b) If $G(d_1) - G(-d_2) > G^k(d_1) - G^k(-d_2)$, then,

$\underset{\sim}{\theta}_{LFC} = (\theta,\ldots,\theta,\theta)$ and $b_{\underset{\sim}{\theta}_{LFC}}(d_1,d_2) = G^k(d_1) - G^k(-d_2)$

where θ is an arbitrary real number.

<u>Proof</u>: It is carried out in Appendix B utilizing the following lemma.

<u>Lemma III.1</u>: Define

(III.5) $f(r) = G^r(d_1) - G^k(-d_2)$, $r = 1,\ldots,k$;

then, under the conditions of Theorem III.1, $\min_{1\le r\le k} f(r)$ is either $f(1)$ of $f(k)$ or both, (but not $f(r)$ if $r \neq 1,k$) .

<u>Proof</u>: See Appendix B.

<u>Remark</u>:

(1) From Theorem III.1 it is clear that for $d_1 = d_2$ we obtain $\underset{\sim}{\theta}_{LFC} = (\theta,\ldots,\theta,\theta)$, as in Theorem II.1.

<u>Corollary III.1</u>: If $d_1 + d_2 = L$, L prefixed and $d_2 = d$, then the infimum of the coverage probability is

(III.6) $$b(d) = b_{\underset{\sim}{\theta}_{LFC}}(d) = \min\left\{\left[G(L-d) - G(-d)\right],\left[G^k(L-d) - G^k(-d)\right]\right\}.$$

<u>Proof</u>: It trivially follows from Theorem III.1.

Theorem III.2: Under the conditions of Theorem III.1

(a) for every $k \geq 2$ there exists a $d' = d'(k,L)$ s.t.

$$b(d) = \begin{cases} G(L-d) - G(-d) & \text{if } d < d' \\ G^k(L-d) - G^k(-d) & \text{if } d > d' . \end{cases}$$

~~(b)~~ for $k = 2$ then $d' = L/2$, and for $k > 2$ then $d' < L/2$.

Proof: We write,

(III.7) $$G^k(L-d)-G^k(-d) = [G(L-d) - G(-d)]\, Q(d) \text{ , where}$$

$$Q(d) = \sum_{j=0}^{k-1} G^{(k-1)-j}(L-d)G^j(-d) .$$

Then d' is that value of d such that $Q(d') = 1$. A detailed proof is provided in Appendix B.

Remark:

(2) From (III.7) it is obvious that

(III.8) $$b(d) = G^k(L-d) - G^k(-d) = G(L-d) - G(-d) \text{ , for } d = d' .$$

If $d_1 + d_2 = L$, $d_2 = d$, the confidence interval I_{d_1,d_2}, as it was defined in (III.2), becomes,

(III.9) $$I_{L,d} = (T_n^* - (L-d),\ T_n^* + d)$$

where $d \in (-\infty, \infty)$ and $L(> 0)$ is prefixed.

Let now $d_o = d_o(k,L)$ be a value of d s.t.

(III.10) $$b(d_o) = \sup_d b(d) ,$$

i.e., d_o is that optimal choice of d which maximizes the infimum of the coverage probability under the procedure R.

Theorem III.3: Under the conditions of Theorem III.1 we have $d_0 = L/2$ for $k = 1,2$, and $d_0 < L/2$ for $k > 2$.

Proof: We use the fact that the density g, possessing the M.L.R. property, is strongly unimodal ([13]). For $k = 1$, then $b(d) = G(L-d) - G(-d)$, hence by symmetry and unimodality of g we achieve,

$$\frac{db(d)}{dd} = -g(L-d) + g(-d) = -g(L-d) + g(d) = 0, \text{ i.e., } d_0 = L/2 .$$

For $k = 2$, Theorem III.1 implies that,

$$b(d) = \begin{cases} G(L-d) - G(-d), & \text{if } d < d' = L/2 , \\ G^2(L-d) - G^2(-d) = [G(L-d)-G(-d)][G(L-d)+G(-d)], & \text{if } d > d' = L/2 . \end{cases}$$

Clearly then, the 1st factor is maximized (as previously was shown) at $d_0 = L/2$, and the 2nd one (being ≤ 1 for $d \geq L/2$), attains its maximum value 1 also at $d_0 = L/2$. In both cases, however, $d_0 = L/2$.

For $k > 2$, by Theorem III.2(b), it must be

(III.11) $$d' < L/2.$$

(A) Consider $d' < d$; by the same theorem, $b(d) = G^k(L-d) - G^k(-d)$. Then either (i) $d' < L/2 < d$, or, (ii) $d' < d < L/2$.

(i) Letting $d' < L/2 < d$ we obtain,

$$\frac{db(d)}{dd} = k\left[G^{k-1}(-d)g(-d) - G^{k-1}(L-d)g(L-d)\right] = 0 , \quad \text{iff,}$$

(III.12) $$\frac{G(L-d)}{G(-d)} = \left[\frac{g(-d)}{g(L-d)}\right]^{1/k-1}.$$

We now observe that by M.L.R. property, the ratio

$$\text{(III.13)} \qquad \frac{g(-d)}{g(L-d)} = \frac{g(d)}{g(d-L)} = \frac{g(d;0)}{g(d;L)}$$

is decreasing in d and it is ≤ 1 for $d \geq L/2 > 0$ (since then $-d \leq d-L \leq d$). On the other hand, the ratio on the lefthand side of (III.12) is monotonically increasing in d (see [10]) and always >1. Therefore, there exists a unique d'' satisfying (III.12) but this d'' must be $<L/2$, since only then the ratio in (III.13) becomes >1. Remarking that $b(d) = G^k(L-d) + G^k(-d)$ is monotonically increasing for $d < d''$ and monotonically decreasing for $d'' < d$, we end up with $d_o = d''$, hence $d_o < L/2$.

(ii) Letting $d' < d < L/2$, the ratio in (III.13) is again decreasing in d and >1 for $d < L/2$ (since then $0 < d < L-d$, or $d < -d < L-d$ if $d < 0$). Observing also that the lefthand side ratio in (III.12) behaves as previously, the existence of a $d_o < L/2$, satisfying (III.12), immediately follows.

(B) Consider $d \leq d'$; clearly, $b(d) = G(L-d) - G(-d)$. Then by (III.11) obviously $d < L/2$, hence; either $-d < d < L-d$, or, $d < -d < L-d$ (if $d < 0$). Therefore,

$$\text{(III.14)} \qquad \frac{db(d)}{dd} = -g(L-d) + g(-d) \gtrless 0 \text{ , whenever, } d \lesseqgtr \frac{L}{2} = d'' \text{ .}$$

The latter, by (III.11), gives $d' < d''$ which in our case $(d \leq d')$ implies that d'' is not an admissible value for a d_o. But the case $d \leq d'$, from (III.14), it follows that we may consider $d_o = d'$, i.e.,

$$\text{(III.15)} \qquad d_o = d' < L/2 \quad \text{(by Theorem III.2(b))} \qquad \text{Q.E.D.}$$

Remarks:

(3) By (III.15) and Remark (2) we see that d_0 is either the root of the equation (III.8), or d_0 is the value of d maximizing $b(d) = G^k(L-d) - G^k(-d)$ (see part A of the previous proof).

(4) We may have $d' \to -\infty$ as $k \to \infty$ (see Appendix B).

(5) For L fixed, $d'(k+1) < d'(k)$ iff $G^k(L-d'(k)) < G(d'(k))$ (see Appendix B).

(6) Theorem III.3 asserts that the symmetric interval is optimal for $k \le 2$, while for $k > 2$ the unsymmetric interval, with $d < L/2$, should be considered.

(7) For given k, L and G the optimal value d_0 can be computed numerically. This d_0 in the normal case can be negative for large k's indicating that the entire interval is to the left of T_n^*.

Application of R to the Normal Family with a Common Known Variance

Letting σ^2 be the common variance, $\theta^* = \mu_{[k]}$ and $T_n^* = \bar{X}_{[k]}$, the interval (III.9) takes now the form,

(III.16) $$I_{L,d} = \left(\bar{X}_{[k]} - (L-d), \bar{X}_{[k]} + d\right),$$

with L prefixed (>0) and $d \in (-\infty, \infty)$. By (III.6) the infimum of the coverage probability under R now becomes

(III.17) $$b(d) = \min\left\{\left[\Phi\left(\frac{\sqrt{n}(L-d)}{\sigma}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right)\right], \left[\Phi^k\left(\frac{\sqrt{n}(L-d)}{\sigma}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma}\right)\right]\right\}.$$

If d_0 is the optimal value for d then,

(III.18) $$b(d_0) = \sup_d b(d)$$

and from Theorem III.3 and Remark (3) it follows that d_0 is, either (i), the root of the equation

$$\Phi\left(\frac{\sqrt{n}(L-d)}{\sigma}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right) = \Phi^k\left(\frac{\sqrt{n}(L-d)}{\sigma}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma}\right),$$

or (ii), the value where $\Phi^k\left(\frac{\sqrt{n}(L-d)}{\sigma}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma}\right)$ achieves its maximum. Letting also,

$$\text{(III.19)} \qquad c = L\sqrt{n}/\sigma\ ,\ x = d\sqrt{n}/\sigma \text{ and } x_0 \text{ be s.t. } b(x_0) = \sup_x b(x),$$

then x_0 is (i) either the root of the equation

$$\text{(III.20)} \qquad \Phi(c-x) - \Phi(-x) = \Phi^k(c-x) - \Phi^k(-x)\ ,$$

or (ii) the value maximizing $\Phi^k(c-x) - \Phi^k(-x)$.

The x_0 values satisfying (III.20) are tabulated in Table 5, ([11]), for $k = 3(1)6(2)14$ and $c = 1.0(0.1)4.0$. Then for given k and L we compute $c = L\sqrt{n}/\sigma$ and for this pair (k,c) we select the optimal value x_0. Thus by (III.19) the interval now becomes

$$\text{(III.21)} \qquad I_{L,d} = \left(\bar{X}_{[k]} - \left(L - \frac{\sigma}{\sqrt{n}}x_0\right), \bar{X}_{[k]} + \frac{\sigma}{\sqrt{n}}x_0\right).$$

We realize that under this choice of x_0 the configurations $\underset{\sim}{\mu}_{LFC} = (-\infty,\ldots,-\infty,\mu)$ and $\underset{\sim}{\mu}_{LFC} = (\mu,\mu,\ldots,\mu)$ are simultaneously least favourable and the coverage probability corresponding to them is as in (III.20). This is tabulated in Table 6 ([11]).

Suppose that for a preassigned $\gamma \in (0,1)$ it is required that,

$$\text{(III.22)} \qquad b(d) = \inf_{\underset{\sim}{\mu}} P(\mu_{[k]} \in I_{L,d}) \geq \gamma\ .$$

Then by (III.17) and (III.19) for every L,d,σ,n,k, we have

$$\text{(III.23)} \qquad \inf_{\underline{\mu}} P(\mu_{[k]} \in I_{L,d}) = \min\left\{\Phi^k(c-x) - \Phi^k(-x),\ \Phi(c-x) - \Phi(-x)\right\}$$

$$= b_k(c,x), \text{ say,}$$

and for the pair (c,k) let x_o be the optimal value of x, such that,

$$\text{(III.24)} \qquad b_k(c,x_o) = \sup_x b_k(c,x).$$

Consider now a c_o s.t.

$$\text{(III.25)} \qquad c_o = \inf\left\{c:\ b_k(c,x_o) \geq \gamma\right\}$$

which by (III.19) ends up with,

$$\text{(III.26)} \qquad \text{"}N_o \text{ is the smallest integer } n \text{ s.t. } n \geq \frac{c_o^2\,\sigma^2}{L^2}\text{"}.$$

Then with this smallest sample size N_o we compute $\overline{X}_{[k]_{N_o}}$ and construct the optimal interval as,

$$\text{(III.27)} \qquad I_{L,d_o} = (\overline{X}_{[k]_{N_o}} - (L-d_o),\ \overline{X}_{[k]_{N_o}} + d_o)\ ,\ d_o = x_o \frac{\sigma}{\sqrt{N_o}}\ ,$$

in the case where σ is known.

We note that, for given k and prefixed L and γ, we can determine c_o from Table 6, using γ as an optimal coverage probability and then from Table 5 we select the optimal value x_o. The pair (c_o,x_o) is tabulated in Table 7 for exact γ values ([23]).

A SEQUENTIAL PROCEDURE R_s IN THE CASE OF A COMMON BUT UNKNOWN VARIANCE

For k given and L,γ prespecified, from Table 7 we choose the corresponding pair (c_0,x_0) and proceed as follows.

(a) If $n_0 \geq 2$ is a preassigned integer, let $\underset{\sim}{X}_j = (X_{1j},\ldots,X_{kj})$, $j=1,2,\ldots$, be a sequence of observations taken one vector at a time and each time we compute S_ν^2, as in (II.13).

(b) Stop sampling when,

$$\text{(III.28)} \qquad \text{"}N \text{ is the 1st integer } n \geq n_0 \text{ s.t. } n \geq \frac{c_0^2 S_\nu^2}{L^2}\text{"}.$$

(c) After sampling is terminated we select $\overline{X}_{[k]_N}$ and $S^2_{\nu_N}$, $\nu_N = k(N-1)$, and estimate $\mu_{[k]}$ by the interval,

$$\text{(III.29)} \qquad I_{L,d_0} = \left(\overline{X}_{[k]_N} - (L-d_0), \overline{X}_{[k]_N} + d_0\right), \quad d_0 = x_0\frac{S_{\nu_N}}{\sqrt{N}}$$

Let $\lambda = \sigma/L$ and $C(\lambda,d_0)$ be the infimum of the coverage probability under R_s. Then,

$$\text{(III.30)} \qquad C(\lambda,d_0) = \sum_{n=n_0}^{\infty} \inf_{\underset{\sim}{\mu}} P\left(\mu_{[k]} \in I_{L,d_0} \middle| N=n\right) P(N=n).$$

If $\Lambda = L\sqrt{n}/S_\nu$, $U = S_\nu/\sigma$ and since $x_0 = d_0\sqrt{n}/S_\nu$, by Theorem III.1, Lemma II.1 and following the lines of the proof of Theorem II.3 we can show that,

$$\text{(III.31)} \qquad \inf_{\underset{\sim}{\mu}} P(CD/\underset{\sim}{\mu},\sigma,R) = \inf_{\underset{\sim}{\mu}} P(\mu_{[k]} \in I_{L,d_0})$$

$$= E\left\{\min\left[\Phi(U(\Lambda - x_0)) - \Phi(-Ux_0), \Phi^k(U(\Lambda - x_0)) - \Phi^k(-Ux_0)\right]\right\}$$

$$= E\left[g_k(\lambda,n,S_\nu,d_0)\right], \text{ where,}$$

$$\text{(III.32)} \qquad g_k(\lambda,n,S_\nu,d_o) = \min\left[\Phi\left(\frac{S_\nu}{\sigma}\,\frac{\sqrt{n}(L-d_o)}{S_\nu}\right) - \Phi\left(-\frac{S_\nu}{\sigma}\,\frac{d_o\sqrt{n}}{S_\nu}\right),\right.$$

$$\left.\Phi^k\left(\frac{S_\nu}{\sigma}\,\frac{\sqrt{n}(L-d_o)}{S_\nu}\right) - \Phi^k\left(-\frac{S_\nu}{\sigma}\,\frac{d_o\sqrt{n}}{S_\nu}\right)\right]$$

$$= \min\left[\Phi\left(\frac{\sqrt{n}}{\lambda} - x_o\,\frac{S_\nu}{\sigma}\right) - \Phi\left(-x_o\,\frac{S_\nu}{\sigma}\right),\ \Phi^k\left(\frac{\sqrt{n}}{\lambda} - x_o\,\frac{S_\nu}{\sigma}\right) - \Phi^k\left(-x_o\,\frac{S_\nu}{\sigma}\right)\right].$$

Therefore, by (III.30), (III.31) and Lemma II.3, we obtain,

$$\text{(III.33)} \qquad C(\lambda,d_o) = \sum_{n=n_o}^{\infty} E\left[g_k(\lambda,N,S_{\nu_N},d_o)/N=n\right] P(N=n)$$

$$= E\left\{E\left[g_k(\lambda,N,S_{\nu_N},d_o)/N=n\right]\right\} = E\left[g_k(\lambda,N,S_{\nu_N},d_o)\right].$$

Moreover, for every $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k)$ and σ, clearly

$$\text{(III.34)} \qquad P(CD\,|\,\underset{\sim}{\mu},\sigma,R_s) \geq E\left[g_k(\lambda,N,S_{\nu_N},d_o)\right].$$

ASYMPTOTIC BEHAVIOUR OF R_s

Theorem III.4: Under the procedure R_s we have,

(a) $P(N<\infty) = 1$, for every $\underset{\sim}{\mu}$ and σ^2.

(b) $\lim_{L\to 0} N = \infty$ a.s.

(c) $\lim_{L\to 0} \frac{N}{(c_o\lambda)^2} = \lim_{L\to 0} \frac{N}{N_o} = 1$ a.s. (N_o as in (III.26)).

(d) $\lim_{L\to 0} \frac{E(N)}{N_o} = 1$ (asymptotic efficiency).

(e) $\lim_{L\to 0} C(\lambda,d_o) = \gamma$ (asymptotic consistency).

<u>Proof</u>: From (III.28) obviously N is a function of L, i.e. $N = N(L)$.

(a) follows directly from Theorem II.4(a), since, $\lim_{n\to\infty} S_\nu^2 = \sigma^2$ a.s.

(b) and (c) come out as a consequence of Lemma II.6 letting,

$$V_n = \frac{S_\nu^2}{\sigma^2}, \quad f(n) = n \ , \quad t = \left(\frac{c_0 \sigma}{L}\right)^2 .$$

(d) follows from Lemma II.7, since $E\left\{\sup_{n>1} V_n\right\} < \infty$.

(see Remark (7) of Chapter II).

(e) From part (c) we get $\lim_{L\to 0} \frac{\sqrt{N}}{\lambda} = c_0$ a.s. and by part (b),

$\lim_{L\to 0} S_{\nu_N} = \lim_{N\to\infty} S_{\nu_N} = \sigma$ a.s. Since also a.s. convergence is

preserved by continuous mappings, by (III.32) and (III.23) we achieve,

$$\lim_{L\to 0} g_k(\lambda,N,S_{\nu_N},d_0) = \lim_{L\to 0} \left\{ \min\left[\Phi\left(\frac{\sqrt{N}}{\lambda} - x_0 \frac{S_{\nu_N}}{\sigma}\right) - \Phi\left(- x_0 \frac{S_{\nu_N}}{\sigma}\right) ,\right.\right.$$

$$\left.\left.\Phi^k\left(\frac{\sqrt{N}}{\lambda} - x_0 \frac{S_{\nu_N}}{\sigma}\right) - \Phi^k\left(- x_0 \frac{S_{\nu_N}}{\sigma}\right)\right]\right\}$$

$$= \min\left\{\Phi(c_0 - x_0) - \Phi(-x_0), \Phi^k(c_0 - x_0) - \Phi^k(-x_0)\right\} = b_k(c_0,x_0) \quad \text{a.s.}$$

From (III.25) clearly $b_k(c_0,x_0) = \gamma$ and since $g_k(\lambda,N,S_{\nu_N},d_0)$ is uniformly bounded $(0 \le g_k(.) \le 1)$, via bounded convergence theorem we end up with,

$$\text{(III.35)} \qquad \lim_{L\to 0} C(\lambda,d_0) = \lim_{L\to 0} E\left[g_k(\lambda,N,S_{\nu_N},d_0)\right]$$

$$= E\left[\lim_{L\to 0} g_k(\lambda,N,S_{\nu_N},d_0)\right] = b_k(c_0,x_0) = \gamma$$

Q.E.D.

We complete this section comparing now the sequential procedure referred to symmetric confidence intervals, say R_{s1}, with the sequential procedure considered just before, say R_{s2}. In case of R_{s1} we denote the stopping variable N by N_1 and λ by $\lambda_1(=\sigma/d)$, where $d = L/2$. Hence, $\lambda_1 = 2\sigma/L$. If in the case of R_{s2} N is denoted by N_2 and λ by $\lambda_2(=\sigma/L)$, clearly, $2\lambda_2 = \lambda_1$. By theorems II.6(b), III.4(c) we have $\lim_{L\to 0} E(N_1) = (\lambda_1 z_\gamma)^2$ and $\lim_{L\to 0} E(N_2) = (c_0\lambda_2)^2$ respectively; however, $\lim_{L\to 0} E(N_1) = (2\lambda_2 z_\gamma)^2$,

$$\text{and} \quad \lim_{L\to 0} \frac{E(N_1)}{E(N_2)} = \left(\frac{2\lambda_2 z_\gamma}{c_0\lambda_2}\right)^2 = \left(\frac{2z_\gamma}{c_0}\right)^2 = \alpha_k(\gamma) \ , \quad \text{say.}$$

Some values of $\alpha_k(\gamma)$, given below, reveal the improvement achieved through the unsymmetric intervals approach to our problem, especially for k large (see [23]).

Values of $\alpha_k(\gamma)$

K \ γ	0.75	0.9	0.95
3	1.297	1.196	1.147
8	1.941	1.535	1.406
12	2.098	1.647	1.484

SINGLE STAGE AND SEQUENTIAL PROCEDURES IN THE CASE OF UNEQUAL VARIANCES

Single-Stage Procedures when the Variances are Known

When the vector $\underset{\sim}{\sigma}^2 = (\sigma_1^2,\ldots,\sigma_k^2)$ is known we present again two single-stage procedures, R_1 and R_2, as follows.

The Procedure R_1 consists of the same steps as in (II.64), (II.65), (II.66) of Chapter II. Moreover, we let $c = L\sqrt{n_i}/\sigma_i = L\sqrt{n'}$ and $x = d\sqrt{n_i}/\sigma_i = d\sqrt{n'}$, $i = 1,\ldots,k$. Then, as in (III.17), we obtain,

$$\text{(III.36)} \qquad \inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_1) = \min\Big\{\Phi\left(\sqrt{n'}(L-d)\right) - \Phi(-d\sqrt{n'}),$$

$$\Phi^k\left(\sqrt{n'}(L-d)\right) - \Phi^k(-d\sqrt{n'})\Big\} = \min\left\{\Phi(c-x)-\Phi(-x), \Phi^k(c-x)-\Phi^k(-x)\right\} = b_k(c,x).$$

If x_0 is s.t. $b_k(c,x_0) = \sup_x b_k(c,x)$, this $x_0 (= d_0\sqrt{n'})$ is as in (III.20), and for $\gamma \in (0,1)$ prefixed, let c_0 be as in (III.25). Then $N_0' = [c_0^2/L^2]$ ([x] is the smallest integer $\geq x$), and the smallest individual sample sizes determined by (II.65) are $n_i = \sigma_i^2 N_0'$, $i = 1,\ldots,k$.

The procedure R_2 consists of taking a common sample size, n per population, and ordering the values of σ_i's as in (II.74). If $\sigma_{(i)}^2$ denotes the variance of the population with mean $\mu_{[i]}$, $i=1,\ldots,k$, by Theorem III.1 and extending Lemma II.8 (see Appendix B), we obtain[1],

---

[1] Procedures R_2, R_s' in Chapters II and III are adaptations of existing methods.

(III.37) $$\inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2)$$

$$= \min\left\{\left[\Phi\left(\frac{\sqrt{n}(L-d)}{\sigma_{(k)}}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma_{(k)}}\right)\right], \left[\prod_{i=1}^{k} \Phi\left(\frac{\sqrt{n}(L-d)}{\sigma_{(i)}}\right) - \prod_{i=1}^{k} \Phi\left(-\frac{d\sqrt{n}}{\sigma_{(i)}}\right)\right]\right\}$$

$$\geq \min\left\{\Phi\left(\frac{\sqrt{n}(L-d)}{\sigma_{[k]}}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma_{[k]}}\right), \Phi^k\left(\frac{\sqrt{n}(L-d)}{\sigma_{[k]}}\right) - \Phi^k\left(-\frac{d\sqrt{n}}{\sigma_{[k]}}\right)\right\}$$

$$= \min\left\{\Phi(c-x) - \Phi(-x), \Phi^k(c-x) - \Phi^k(-x)\right\} = b_k(c,x)$$

where $c = L\sqrt{n}/\sigma_{[k]}$ and $x = d\sqrt{n}/\sigma_{[k]}$. Letting x_o and c_o be s.t. $b_k(c;x_o) = \sup_x b_k(c,x)$, and

(III.38) $$c_o = \inf\{c : b_k(c,x_o) \geq \gamma\}$$

then this $x_o\left(= d_o\sqrt{n}/\sigma_{[k]}\right)$ is as in (III.20) and the smallest common sample size is going to be

(III.39) $$N_o = [c_o^2\,\sigma_{[k]}^2/L^2], \quad ([x] \text{ is the smallest integer } \geq x).$$

Comparing the sample sizes of the procedures R_1 and R_2, we may verify that Remark (8) of Chapter II is again valid.

A Sequential Procedure when the Variances are Unknown

Using a common sample size n we compute,

(III.40) $$\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n, \quad S_i^2 = \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2/n-1, \; i = 1,\ldots,k,$$

and let the ordered values of them be,

(III.41) $$\overline{X}_{[1]} \leq \ldots \leq \overline{X}_{[k]}, \quad S_{[1]}^2 \leq \ldots \leq S_{[k]}^2 .$$

Then, as in (II.82), the coverage probability under R_2, for $\underset{\sim}{\sigma}^2$ unknown is,

$$P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = P\left(\bar{X}_{[k]} \le \mu_{[k]} + (L-d)\right) - P\left(\bar{X}_{[k]} \le \mu_{[k]} - d\right)$$

$$= \prod_{i=1}^{k} P\left(T_i \le \frac{\mu_{[k]} - \mu_{[i]} + L-d}{S_{(i)}/\sqrt{n}}\right) - \prod_{i=1}^{k} P\left(T_i \le \frac{\mu_{[k]} - \mu_{[i]} - d}{S_{(i)}/\sqrt{n}}\right)$$

$$= \prod_{i=1}^{k} F_{n-1}\left(\frac{\delta_i + L-d}{S_{(i)}/\sqrt{n}}\right) - \prod_{i=1}^{k} F_{n-1}\left(\frac{\delta_i - d}{S_{(i)}/\sqrt{n}}\right),$$ where F_{n-1} is the c.d.f. of t_{n-1}-distribution.

By Theorem III.1 and Lemma II.1 we achieve,

$$\inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) = \min\left\{\left[F_{n-1}\left(\frac{\sqrt{n}(L-d)}{S_{(k)}}\right) - F_{n-1}\left(-\frac{d\sqrt{n}}{S_{(k)}}\right)\right],\right.$$

$$\left.\left[\prod_{i=1}^{k} F_{n-1}\left(\frac{\sqrt{n}(L-d)}{S_{(i)}}\right) - \prod_{i=1}^{k} F_{n-1}\left(-\frac{d\sqrt{n}}{S_{(i)}}\right)\right]\right\}$$

$$= E_{\underset{\sim}{U}}\left[\min\left\{\Phi\left(\frac{S_{(k)}}{\sigma_{(k)}}\frac{\sqrt{n}(L-d)}{S_{(k)}}\right) - \Phi\left(-\frac{S_{(k)}}{\sigma_{(k)}}\frac{d\sqrt{n}}{S_{(k)}}\right),\right.\right.$$

$$\left.\left.\prod_{i=1}^{k}\Phi\left(\frac{S_{(i)}}{\sigma_{(i)}}\frac{\sqrt{n}(L-d)}{S_{(i)}}\right) - \prod_{i=1}^{k}\Phi\left(-\frac{S_{(i)}}{\sigma_{(i)}}\frac{d\sqrt{n}}{S_{(i)}}\right)\right\}\right],$$

where $\underset{\sim}{U} = (U_1,\ldots,U_k) = \left(\frac{S_{(1)}}{\sigma_{(1)}},\ldots,\frac{S_{(k)}}{\sigma_{(k)}}\right)$.

Observing that for every $i = 1,\ldots,k$,

(a) $\frac{S_{(i)}}{\sigma_{(i)}}\frac{d\sqrt{n}}{S_{(i)}} = \frac{d\sqrt{n}}{\sigma_{(i)}} \ge \frac{d\sqrt{n}}{\sigma_{[k]}} = \frac{S_{[k]}}{\sigma_{[k]}}\frac{d\sqrt{n}}{S_{[k]}}$ and

(b) $\frac{S_{(i)}}{\sigma_{(i)}}\frac{\sqrt{n}(L-d)}{S_{(i)}} = \frac{\sqrt{n}(L-d)}{\sigma_{(i)}} \ge \frac{\sqrt{n}(L-d)}{\sigma_{[k]}} = \frac{S_{[k]}}{\sigma_{[k]}}\frac{\sqrt{n}(L-d)}{S_{[k]}}$

and letting $x = \frac{d\sqrt{n}}{S_{[k]}}$, via the extension of Lemma II.8 (see Appendix B), we now obtain,

$$\text{(III.42)} \quad \inf_{\underset{\sim}{\mu}} P(CD|\underset{\sim}{\mu},\underset{\sim}{\sigma},R_2) \geq E_{\underset{\sim}{U}}\left[\min\left\{\Phi\left(\frac{S_{[k]}}{\sigma_{[k]}}\,\frac{\sqrt{n}(L-d)}{S_{[k]}}\right)-\Phi\left(-\frac{S_{[k]}}{\sigma_{[k]}}\,\frac{d\sqrt{n}}{S_{[k]}}\right),\right.\right.$$

$$\left.\left.\Phi^k\left(\frac{S_{[k]}}{\sigma_{[k]}}\,\frac{\sqrt{n}(L-d)}{S_{[k]}}\right)-\Phi^k\left(-\frac{S_{[k]}}{\sigma_{[k]}}\,\frac{d\sqrt{n}}{S_{[k]}}\right)\right\}\right]$$

$$= E_{U_{[k]}}\left\{g_k(\lambda,n,S_{[k]},d)\right\}, \quad \text{where } U_{[k]} = \frac{S_{[k]}}{\sigma_{[k]}},\ \lambda = \frac{\sigma_{[k]}}{L} \quad \text{and}$$

$$\text{(III.43)} \quad g_k(\lambda,n,S_{[k]},d) = \min\left\{\Phi\left(\frac{\sqrt{n}}{\lambda} - x\,\frac{S_{[k]}}{\sigma_{[k]}}\right) - \Phi\left(-x\,\frac{S_{[k]}}{\sigma_{[k]}}\right),\right.$$

$$\left.\Phi^k\left(\frac{\sqrt{n}}{\lambda} - x\,\frac{S_{[k]}}{\sigma_{[k]}}\right) - \Phi^k\left(-x\,\frac{S_{[k]}}{\sigma_{[k]}}\right)\right\}.$$

For given k and prefixed L and γ from Table 5 we select the corresponding values c_o, x_o and the sequential procedure R_S' is the following:

(a) If $n_o \geq 2$ is a preassigned integer, let $\underset{\sim}{X}_j = (X_{1j},\ldots,X_{kj})$, $j = 1,2,\ldots,$ be a sequence of observations taken one vector at a time and each time we compute $\overline{X}_i$'s and S_i^2's as in (III.40) and order them as in (III.41).

(b) Stop sampling when,

$$\text{(III.44)} \quad \text{"}N \text{ is the 1st integer } n \geq n_o \text{ s.t. } n \geq \frac{c_o^2\, S_{[k]_n}^2}{L^2}\text{"}.$$

(c) After sampling is terminated we estimate $\mu_{[k]}$ by the interval,

$$I_{L,d_o} = \left(\overline{X}_{[k]_N} - (L-d_o),\, \overline{X}_{[k]_N} + d_o\right),\quad d_o = x_o\,\frac{S_{[k]_N}}{\sqrt{N}}.$$

If $C(\lambda, d_o)$ is the infinum of the coverage probability under R_s', then by (III.42) we achieve,

(III.45) $$C(\lambda,d_o) = \sum_{n=n_o}^{\infty} \inf_{\underset{\sim}{\mu}} P(CD;\underset{\sim}{\mu},\sigma,R_2/N=n)\; P(N=n)$$

$$\geq E\left[E_{U_{[k]}}\left(g_k(\lambda,N,S_{[k]_N},d_o)/N=n\right)\right] = E\left[g_k\left(\lambda,N,S_{[k]_N},d_o\right)\right] .$$

Theorem III.5 : Under the procedure R_s' we have,

(a) $P(N<\infty) = 1$, for every vector $\underset{\sim}{\mu}$ and $\underset{\sim}{\sigma}^2$.

(b) $\lim_{L\to o} N = \infty$ a.s.

(c) $\lim_{L\to o} \frac{N}{(c_o\lambda)^2} = \lim_{L\to o} \frac{N}{N_o} = 1$ a.s. (N_o as in (III.39)).

(d) $\lim_{L\to o} \frac{E(N)}{N_o} = 1$, (asymptotic efficiency).

(e) $\lim_{L\to o} C(\lambda,d_o) \geq \gamma$, (asymptotic consistency).

Proof:

(a) follows, as in Theorem II.4(a), since $\lim_{n\to\infty} S_{[k]_n} = \sigma_{[k]}$ a.s.

(b), (c) and (d) come out as a consequence of Lemmas II.6 and II.7 letting, $V_n = S^2_{[k]_n}/\sigma^2_{[k]}$, $f(n) = n$ and $t = (c_o\sigma_{[k]}/L)^2 = (c_o\lambda)^2$.

(e) Since $\lim_{L\to o} (\sqrt{N}/\lambda) = c_o$ a.s. and $\lim_{L\to o} S_{[k]_N} = \lim_{N\to\infty} S_{[k]_N} = \sigma_{[k]}$ a.s. (from (a) and (b)), by arguments similar to those of Theorem III.4(e), via (III.43) and (III.38), we obtain

$\lim_{L\to o} g_k(\lambda,N,S_{[k]_N},d_o) = b_k(c_o,x_o) = \gamma$ a.s.; however, by (III.45) and bounded convergence theorem we end up with,

$$\lim_{L\to o} C(\lambda,d_o) \geq \lim_{L\to o} E\left[g_k(\lambda,N,S_{[k]_N},d_o)\right]$$

$$= E\left[\lim_{L\to o} g_k(\lambda,N,S_{[k]_N},d_o)\right] = b_k(c_o,x_o) = \gamma \; .$$

CHAPTER IV

PREFIXED-WIDTH INTERVAL ESTIMATION OF THE LARGEST NORMAL MEAN WITH SIMULTANEOUS SELECTION OF THE BEST POPULATION

PRELIMINARIES

In Chapters II and III we have studied the construction of a prefixed-width confidence interval for $\mu_{[k]}$ without any prior selection procedure. In the present chapter the selection of the best among $k(\geq 2)$ independent normal populations, will be combined with a prefixed-width interval estimation for the unknown $\mu_{[k]}$. According, however, to the reasonable demand of the selection procedure, the least favourable configuration (LFC) of the mean vector $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k)$ must lie in the preference zone (PZ). The present study[1] will be carried out assuming the existence of a common variance σ^2, introducing, at first, a single-stage procedure when σ^2 is known and afterwards a sequential procedure when σ^2 is unknown. We note that all concepts and notations will be used as they were defined in Chapters I and II unless otherwise denoted.

Letting $d > 0$ and $\delta^* > 0$ be arbitrary but preassigned, we consider the random interval

(IV.1) $$I_d = (\overline{X}_{[k]} - d, \overline{X}_{[k]} + d) .$$

[1] Modification of similar existing methods.

Definition IV.1: We say that we have a Correct Decision (CD) whenever the event $\mu_{[k]} \in I_d$ occurs simultaneously with the event $\{\bar{X}_{[k]} = \bar{X}_{(k)}\}$, for every $\mu_{[k]} - \mu_{[i]} \geq \delta^*$, $i = 1,\dots,k-1$. i.e., for all configurations $\underset{\sim}{\mu} = (\mu_1,\dots,\mu_k)$ in the preference zone.

A SINGLE-STAGE PROCEDURE R_0 IN CASE OF A COMMON KNOWN VARIANCE

Formulating the previous definition in terms of our notation, assuming σ^2 to be known, we obtain

$$\text{(IV.2)} \qquad P(CD \mid \underset{\sim}{\mu} \in PZ, \sigma, R_0) = P\left[\left(\mu_{[k]} \in I_d\right) \cap \left(\bar{X}_{[k]} = \bar{X}_{(k)}\right)\right]$$

$$= P\left(-d < \bar{X}_{[k]} - \mu_{[k]} < d \,\middle|\, \bar{X}_{[k]} = \bar{X}_{(k)}\right) P\left(\bar{X}_{[k]} = \bar{X}_{(k)}\right)$$

$$= P\left(-\frac{d\sqrt{n}}{\sigma} < \frac{\sqrt{n}\left(\bar{X}_{(k)} - \mu_{[k]}\right)}{\sigma} < \frac{d\sqrt{n}}{\sigma}\right) P\left(\bar{X}_{[k]} = \bar{X}_{(k)}\right)$$

$$= \left[\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right)\right] \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi\left(y + \frac{\delta_i \sqrt{n}}{\sigma}\right) d\Phi(y) \quad .$$

The latter is a consequence of Theorem I.1, where $\delta_i = \mu_{[k]} - \mu_{[i]}$, $i = 1,\dots,k-1$, and $\Phi(.)$ denotes the standard normal c.d.f. Furthermore via Theorem I.2 we achieve,

$$\text{(IV.3)} \qquad \inf_{\underset{\sim}{\mu}} P(CD \mid \underset{\sim}{\mu} \in PZ, \sigma, R_0) = P(CD \mid \underset{\sim}{\mu}_{LFC}, \sigma, R_0)$$

$$= \left[\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right)\right] \int_{\infty}^{\infty} \Phi^{k-1}\left(y + \frac{\delta^*\sqrt{n}}{\sigma}\right) d\Phi(y) = \beta\left(\frac{d\sqrt{n}}{\sigma}\right) P^*(n,\delta^*),$$

where, $\underset{\sim}{\mu}_{LFC} = (\mu,\dots,\mu,\mu+\delta^*)$ and $P^*(n,\delta^*)$ as in (I.13), and

$$\text{(IV.4)} \qquad \beta(x) = \Phi(x) - \Phi(-x) \quad , \quad x > 0 \, .$$

Suppose that for an arbitrary but preassigned real number $\xi \in (0,1)$ we require,

(IV.5) $$\inf_{\underset{\sim}{\mu}} P(CD \mid \underset{\sim}{\mu} \in PZ, \sigma, R_o) = \beta\left(\frac{d\sqrt{n}}{\sigma}\right) P^*(n,\delta^*) \geq \xi \ .$$

If P_o^* is a prefixed level of $P^* = P^*(n,\delta^*)$ recall that $h_k(P_o^*)$ denotes the solution of the equation

(IV.6) $$P_o^* = \int_{-\infty}^{\infty} \Phi^{k-1}(y+h_k(P_o^*))d\Phi(y) \ .$$

Theorem IV.1: Under R_o, for a preassigned $\xi \in (0,1)$ and a common known variance σ^2; the smallest sample size per population satisfying (IV.5) is,

(IV.7) $$N_o = \max\left\{N_1 , N_2\right\} \ , \text{ where}$$

(IV.8) "N_1 is the smallest integer $\geq \dfrac{h_k^2(1-\epsilon_o)\,\sigma^2}{(\delta^*)^2}$" ,

(IV.9) "N_2 is the smallest integer $\geq \dfrac{\sigma^2}{d^2}\left[\Phi^{-1}(1-\epsilon_o)\right]^2$"

for an ϵ_o, $(0 < \epsilon_o < 1/2)$, representing the root of the equation $2\epsilon^2 - 3\epsilon + 1 - \xi = 0$.

Proof: By Remark (7) of Chapter I we have $\lim_{n\to\infty} P^*(n,\delta^*) = 1$, hence, as in Corollary I.2,

(IV.10) "for every $\epsilon > 0$ there exits an integer $N_1(\epsilon)$ s.t. $1-\epsilon \leq P^*(n,\delta^*) \leq 1$, for every $n \geq N_1(\epsilon)$".

Since also $\overline{X}_{[k]_n} \left(= \overline{X}_{(k)_n}\right)$ is a consistent estimate for $\mu_{[k]}$, then, $\lim_{n\to\infty} P\left(-d < \bar{X}_{(k)_n} - \mu_{[k]} < d\right) = \lim_{n\to\infty}\left[\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right)\right] = 1$,

for every $d > 0$. However, $\lim_{n\to\infty} \Phi\left(-\frac{d\sqrt{n}}{\sigma}\right) = \lim_{n\to\infty}\left[1-\Phi\left(\frac{d\sqrt{n}}{\sigma}\right)\right] = 0$.

Thus, "for every $\epsilon > 0$ there exists an integer $N_2(\epsilon)$ s.t.

(IV.11) $$\Phi\left(-\frac{d\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{d\sqrt{n}}{\sigma}\right) \le \epsilon , \quad \text{for every } n \ge N_2(\epsilon)".$$

The latter, by (IV.4), implies that,

(IV.12) $$"\beta\left(\frac{d\sqrt{n}}{\sigma}\right) \ge 1 - 2\epsilon , \text{ for every } n \ge N_2(\epsilon)" .$$

Combining now (IV.10) and (IV.12) we end up with,

(IV.13) $$"\beta\left(\frac{d\sqrt{n}}{\sigma}\right) P^*(n,\delta^*) \ge (1-2\epsilon)(1-\epsilon) = 2\epsilon^2 - 3\epsilon + 1$$
$$\text{for every } n \ge \max\left\{N_1(\epsilon),\ N_2(\epsilon)\right\}" .$$

From (IV.10) and (IV.11) clearly this ϵ must be $0 < \epsilon < 1$, and the requirement (IV.5) is satisfied when this ϵ is s.t.

(IV.14) $$2\epsilon^2 - 3\epsilon + 1 = \xi , \quad \text{for } \xi\in(0,1) \text{ preassigned.}$$

Since for every $\xi\in(0,1)$ there exists a unique[1] $\epsilon_0\in(0,1/2)$ satisfying (IV.14), considering this ϵ_0 from (IV.10) we obtain,

" $1 - \epsilon_0 \le P^*(n,\delta^*)$, for every $n \ge N_1(\epsilon_0) = N_1$, say ", which as in Corollary I.2, results in,

(IV.15) $$"N_1 \text{ is the smallest integer } \ge \frac{\sigma^2 h_k^2 (1-\epsilon_0)}{(\delta^*)^2}" .$$

[1]A rough computation yields $\epsilon_0 = 1/2$ for $\xi = 0$, $\epsilon_0 = 0.19$ for $\xi = 0.5$, $\epsilon_0 = 0$ for $\xi = 1$ e.t.c.

On the otherhand, by (IV.11), for this ϵ_0 we get, $\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) \geq 1-\epsilon_0$, for every $n \geq N_2(\epsilon_0) = N_2$, say. Thus, N_2 is the smallest integer $\geq \frac{\sigma^2}{d^2}\left[\Phi^{-1}(1-\epsilon_0)\right]^2$. The latter together with (IV.15) and (IV.13) completes the proof.

<u>Corollary IV.1</u>: Under the assumptions of Theorem IV.1 if in addition $P^* = P^*(n,\delta^*)$ is prefixed at a level, say, $P_0^*(\geq\xi)$, then,

(IV.16) $N_0 = \max\left\{N_1 , N_2\right\}$, where,

(IV.17) "N_1 is the smallest integer $\geq \frac{h_k^2(P_0^*)\sigma^2}{(\delta^*)^2}$" and

(IV.18) "N_2 is the smallest integer $\geq \frac{\sigma^2}{d^2}\left[\Phi^{-1}\left(\frac{1+\gamma}{2}\right)\right]^2$"

with $\gamma = \xi/P_0^*$.

<u>Proof</u>: For that value P_0^*, by Corollary I.2, we get (IV.17). On the other hand (IV.5) yields,

(IV.19) $\beta\left(\frac{d\sqrt{n}}{\sigma}\right) \geq \xi/P_0^* = \gamma$, say.

Notice that P_0^* and ξ must be properly chosen, so that, $\xi \leq P_0^*$, since $\beta(x) \leq 1$ for every x. By (IV.19) and symmetry of $\Phi(x)$, clearly, $\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) \geq (\gamma+1)/2$. This implies (IV.18) and by Theorem IV.1 the proof is completed.

<u>Corollary IV.2</u>: (Criterion of choice).

With assumptions as in Theorem IV.1,

(a) $N_0 = N_1$ iff $\frac{\delta^*}{d} < \frac{h_k(1-\epsilon_0)}{\Phi^{-1}(1-\epsilon_0)}$

(b) $N_0 = N_2$ iff $\frac{\delta^*}{d} > \frac{h_k(1-\epsilon_0)}{\Phi^{-1}(1-\epsilon_0)}$

Proof: Evident from (IV.8) and (IV.9).

SEQUENTIAL INTERVAL ESTIMATION OF THE LARGEST NORMAL MEAN WITH SIMULTANEOUS SELECTION OF THE BEST POPULATION

The Performance of R_0 in Case of a Common but Unknown Variance

Instead of the unknown σ^2 we use again the pooled sample variance $S_\nu^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} \left(X_{ij} - \overline{X}_i\right)^2 / \nu$, $\nu = k(n-1)$. Then, by Theorem I.3 and replacing in (IV.2) $\Phi(.)$ by $F_\nu(.)$, (the c.d.f. of t_ν-distribution), we get,

$$\text{(IV.20)} \quad P\left(CD \middle| \underline{\mu} \in PZ, \sigma, R_0\right) = \left[F_\nu\left(\frac{d\sqrt{n}}{S_\nu}\right) - F_\nu\left(-\frac{d\sqrt{n}}{S_\nu}\right)\right] \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_\nu\left(y + \frac{\delta_i\sqrt{n}}{S_\nu}\right) dF_\nu(y)$$

which by Theorem I.4 results in,

$$\text{(IV.21)} \quad \inf_{\underline{\mu}} P\left(CD \middle| \underline{\mu} \in PZ, \sigma, R_0\right) = P\left(CD \middle| \underline{\mu}_{LFC}, \sigma, R_0\right)$$

$$= \left[F_\nu\left(\frac{d\sqrt{n}}{S_\nu}\right) - F_\nu\left(-\frac{d\sqrt{n}}{S_\nu}\right)\right] \int_{-\infty}^{\infty} F_\nu^{k-1}\left(y + \frac{\delta^*\sqrt{n}}{S_\nu}\right) dF_\nu(y)$$

where, $\underline{\mu}_{LFC} = (\mu, \ldots, \mu, \mu + \delta^*)$.

Remarks:

(1) If $\tau_\nu^{(k)}(1-\epsilon_0)$ and $h_k(1-\epsilon_0)$ are s.t.,

(IV.22) $$1-\epsilon_0 = \int_{-\infty}^{\infty} F_\nu^{k-1}\left(y+\tau_\nu^{(k)}(1-\epsilon_0)\right)dF_\nu(y) = \int_{-\infty}^{\infty} \Phi^{k-1}\left(y+h_k(1-\epsilon_0)\right)d\Phi(y)$$

then, for k fixed, clearly,

(IV.23) $$\lim_{n\to\infty} \tau_\nu^{(k)}(1-\epsilon_0) = h_k(1-\epsilon_0) , \quad \nu = k(n-1) .$$

(2) Letting a_ν and $z_{1-\epsilon_0}$ be s.t. $F_\nu(a_\nu) = 1-\epsilon_0$, $\Phi\left(z_{1-\epsilon_0}\right) = 1-\epsilon_0$,

then,

(IV.24) $$\lim_{n\to\infty} a_\nu = z_{1-\epsilon_0} \left(= \Phi^{-1}(1-\epsilon_0)\right) .$$

The Sequential Procedure R_s and its Asymptotic Behaviour

Prespecifying the quantities $d>0$, $\delta^*>0$ and $\xi\in(0,1)$, from (IV.14) we obtain the value ϵ_0 . Then for a $k\geq 2$ the sequential procedure R_s is established as follows.

(a) If $n_0 \geq 2$ is a preassigned integer, let $\underset{\sim}{X}_j = \left(X_{1j},\ldots,X_{kj}\right)$, $j = 1,2,\ldots,$ be a sequence of observations taken one vector at a time and each time we compute S_ν^2 .

(b) Continue sampling until we find out two integers N_1' and N_2' , such that, (IV.25) " N_1' is the 1st integer $n \geq n_0$ s.t. $n \geq \dfrac{\left[\tau_\nu^{(k)}(1-\epsilon_0)\right]^2 S_\nu^2}{(\delta^*)^2}$

and

N_2' is the 1st integer $n \geq n_0$ s.t. $n \geq \dfrac{S_\nu^2 \, a_\nu^2}{d^2}$ " .

(c) After sampling is terminated we select $N = \max\left\{N_1', N_2'\right\}$.

Then we observe $\overline{X}_{[k]_N}$ and $S_{\nu_N}^2$, $\nu_N = k(N-1)$, and choose as best the population giving rise to $\overline{X}_{[k]_N}$, while the confidence interval for $\mu_{[k]}$ is going to be $I_d = \left(\overline{X}_{[k]_N} - d \, , \, \overline{X}_{[k]_N} + d\right)$.

Remarks:

(3) The stopping rule (IV.25) can also be expressed as,

" N is the 1st integer $n \geq n_0$ s.t.

$$n \geq \max\left\{\frac{S_\nu^2\left[\tau_\nu^{(k)}(1-\epsilon_0)\right]^2}{(\delta^*)^2} \, , \, \frac{S_\nu^2 \, a_\nu^2}{d^2}\right\} \text{ " .}$$

(4) The r.v. N is a function of δ^* and d, i.e. $N = N(\delta^*, d)$ and also $\lim N = \infty$, as, either $\delta^* \to 0$, or, $d \to 0$ (or both).

The probability of CD under R_s, by Lemma II.3, is expressed as,

$$P\left(CD \,\middle|\, \underline{\mu} \in PZ, \sigma, R_s\right) = \sum_{n=n_0}^{\infty} P\left[\left(\mu_{[k]} \in I_d\right) \cap \left(\overline{X}_{[k]_N} = \overline{X}_{(k)}\right) \,\middle|\, N = n\right] P(N=n)$$

$$= \sum_{n=n_0}^{\infty} P\left(CD \,\middle|\, \underline{\mu} \in PZ, \sigma, R_0\right) P(N=n).$$ Hence, by (IV.21) we achieve,

(IV.26) $$C(\delta^*,d) = \inf_{\underset{\sim}{\mu}} P\left(CD \middle| \underset{\sim}{\mu} \in PZ,\sigma,R_s\right) = \sum_{n=n_0}^{\infty} \inf_{\underset{\sim}{\mu}} P\left(CD \middle| \underset{\sim}{\mu} \in PZ,\sigma,R_0\right) P(N=n)$$

$$= E\left\{\left[F_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right) - F_{\nu_N}\left(-\frac{d\sqrt{N}}{S_{\nu_N}}\right)\right] \int_{-\infty}^{\infty} F_{\nu_N}^{k-1}\left(y + \frac{\delta^*\sqrt{N}}{S_{\nu_N}}\right) dF_{\nu_N}(y)\right\}$$

$$= E\left\{f_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right) \int_{-\infty}^{\infty} F_{\nu_N}^{k-1}\left(y + \frac{\delta^*\sqrt{N}}{S_{\nu_N}}\right) dF_{\nu_N}(y)\right\}, \quad \text{where,}$$

the expectation is over the distribution of N and

(IV.27) $$f_{\nu_N}(x) = F_{\nu_N}(x) - F_{\nu_N}(-x), \quad \text{for every } x > 0.$$

<u>Remarks</u>:

(5) $f_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right)$ is a continuous and bounded function of the r.v. N since $0 \le f_{\nu_N}(.) \le 1$.

(6) $F_{\nu_N}^{k-1}\left(y + \frac{\delta^*\sqrt{N}}{S_{\nu_N}}\right)$ is also a continuous and bounded function of N.

(7) By Remarks (5) and (6) it follows that the expression inside the curly bracket in (IV.26) is a continuous and bounded function of N.

<u>Theorem IV.2</u>: For a moderate fixed value of d we have

(a) $\lim_{\delta^* \to 0} \frac{N}{N_o} = 1$ a.s. (N_o is as in (IV.7)).

(b) $\lim_{\delta^* \to 0} \frac{E(N)}{N_o} = 1$ (asymptotic efficiency).

(c) $\lim_{\delta^* \to 0} C(\delta^*,d) \ge \xi$ (asymptotic consistency).

Proof: Letting $V_n = \frac{S_\nu^2}{\sigma^2}$, $f(n) = \frac{n}{\left[\tau_\nu^{(k)}(1-\epsilon_o)\right]^2}$, $t = \frac{\sigma^2}{(\delta^*)^2}$

by (IV.23) and Remark (4), for $\nu_N = k(N-1)$, we have,

(IV.28) $$\lim_{\delta^*\to 0} \tau_{\nu_N}^{(k)}(1-\epsilon_o) = \lim_{N\to\infty} \tau_{\nu_N}^{(k)}(1-\epsilon_o) = h_k(1-\epsilon_o) .$$

(a) By Lemma II.6 we obtain $\lim_{\delta^*\to 0} \frac{N(\delta^*)^2}{\sigma^2\left[\tau_{\nu_N}^{(k)}(1-\epsilon_o)\right]^2}$

$$= \lim_{\delta^*\to 0} \frac{N(\delta^*)^2}{\sigma^2\, h_k^2(1-\epsilon_o)} = \lim_{\delta^*\to 0} \frac{N}{N_1} = \lim_{\delta^*\to 0} \frac{N}{N_o} = 1 \text{ a.s., since}$$

for $\delta^*\to 0$ then $N_1 = N_o$, in case of σ^2 known (see (IV.8)).

(b) From (IV.28) and Lemma II.7 we get,

$$\lim_{\delta^*\to 0} E\left\{\frac{N(\delta^*)^2}{\sigma^2\left[\tau_{\nu_N}^{(k)}(1-\epsilon_o)\right]^2}\right\} = \lim_{\delta^*\to 0} \frac{E(N)(\delta^*)^2}{\sigma^2\, h_k^2(1-\epsilon_o)} = \lim_{\delta^*\to 0} \frac{E(N)}{N_o} = 1$$

by arguments as in part (a).

(c) We note that $\lim_{\delta^*\to 0} S_{\nu_N} = \lim_{N\to\infty} S_{\nu_N} = \sigma$ a.s., and

(IV.29) $$\mathcal{L} \lim_{\delta^*\to 0} F_{\nu_N}(.) = \lim_{N\to\infty} F_{\nu_N}(.) = \Phi(.) .$$

Since by part (a) $\lim_{\delta^*\to 0} N = N_o$ a.s., and a.s. convergence is preserved by continuous mappings, we obtain,

(IV.30) $$\lim_{\delta^*\to 0} f_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right) = \lim_{\delta^*\to 0} f_{\nu_N}\left(\frac{d\sqrt{N}}{\sigma}\right) = \lim_{\delta^*\to 0} \beta\left(\frac{d\sqrt{N}}{\sigma}\right) = \beta\left(\frac{d\sqrt{N_o}}{\sigma}\right) \text{ a.s.}$$

where $\beta(.)$ as in (IV.4). Observing that $N_o \geq N_2$ (as $\delta^*\to 0$), then by

(IV.9) and monotonicity of $\beta(.)$ (see Lemma II.4(a)), it follows that,

$\beta\left(\frac{d\sqrt{N_0}}{\sigma}\right) \geq \beta\left(\Phi^{-1}(1-\epsilon_0)\right) = 1-2\epsilon_0$. This by (IV.30) implies,

$$\text{(IV.31)} \qquad \lim_{\delta^*\to 0} f_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right) \geq 1 - 2\epsilon_0 \quad \text{a.s.}$$

On the other hand, by (IV.29), Remark 6, bounded convergence theorem and Helly-Bray theorem, we get,

$$\text{(IV.32)} \qquad \lim_{\delta^*\to 0} \int_{-\infty}^{\infty} F_{\nu_N}^{k-1}\left(y+\frac{\delta^*\sqrt{N}}{S_{\nu_N}}\right) dF_{\nu_N}(y) = \lim_{\delta^*\to 0} \int_{-\infty}^{\infty} \Phi^{k-1}\left(y+\frac{\delta^*\sqrt{N}}{\sigma}\right) dF_{\nu_N}(y)$$

$$= \lim_{\delta^*\to 0} \int_{-\infty}^{\infty} \Phi^{k-1}\left(y+\frac{\delta^*\sqrt{N}}{\sigma}\right) d\Phi(y) = \int_{-\infty}^{\infty} \lim_{\delta^*\to 0} \Phi^{k-1}\left(y+\frac{\delta^*\sqrt{N}}{\sigma}\right) d\Phi(y)$$

$$= \int_{-\infty}^{\infty} \Phi^{k-1}\left(y+h_k(1-\epsilon_0)\right) d\Phi(y) = 1-\epsilon_0 \quad \text{a.s.}$$

The two latter equalities follow from the facts

(i) $\lim_{\delta^*\to 0} \frac{N}{N_1} = \lim_{\delta^*\to 0} \frac{N(\delta^*)^2}{h_k^2(1-\epsilon_0)\,\sigma^2} = 1$ a.s. (see proof of part (a))

(ii) continuity of $\Phi(.)$ combined with the condition (IV.22).

Finally by (IV.26), Remark (7), and bounded convergence theorem we achieve,

$$\lim_{\delta^*\to 0} C(\delta^*,d) = E\left\{\lim_{\delta^*\to 0} f_{\nu_N}\left(\frac{d\sqrt{N}}{S_{\nu_N}}\right) \lim_{\delta^*\to 0} \int_{-\infty}^{\infty} F_{\nu_N}^{k-1}\left(y+\frac{\delta^*\sqrt{N}}{S_{\nu_N}}\right) dF_{\nu_N}(y)\right\}$$

$$\geq E\left\{\left(1-2\epsilon_0\right)\left(1-\epsilon_0\right)\right\} = E\left\{2\epsilon_0^2 - 3\epsilon_0 + 1\right\} = E(\xi) = \xi ,$$

as a result of (IV.31), (IV.32) and (IV.14).

Theorem IV.3: For a moderate fixed value of δ^* we have,

(a) $\lim_{d\to 0} \frac{N}{N_2} = \lim_{d\to 0} \frac{N}{N_o} = 1$ a.s. (N_o, N_2 as in Theorem IV.1).

(b) $\lim_{d\to 0} \frac{E(N)}{N_o} = 1$.

(c) $\lim_{d\to 0} C(\delta^*,d) \geq \xi$.

Proof: Letting $V_n = \frac{S_\nu^2}{\sigma^2}$, $f(n) = \frac{n}{a_\nu}$, $t = \frac{\sigma^2}{d^2}$ and

noting that, (i) $\lim_{d\to 0} a_{\nu_N} = \lim_{N\to\infty} a_{\nu_N} = z_{1-\epsilon_o}$

(ii) $\lim_{d\to 0} S_{\nu_N} = \lim_{N\to\infty} S_{\nu_N} = \sigma$ a.s. , (iii) $\lim_{d\to\infty} F_{\nu_N}(.) = \lim_{N\to\infty} F_{\nu_N}(.) = \Phi(.)$,

(iv) $N_o = N_2$ for $d\to 0$, we can carry out this proof following the pattern of the proof of Theorem IV.2.

Corollary IV.3:

(a) $\lim \frac{N}{N_o} = 1$ a.s., as both $\delta^*\to 0$, $d\to 0$, where now $N_o = N_1 = N_2$.

(b) $\lim \frac{E(N)}{N_o} = 1$ as both $\delta^*\to 0$, $d\to 0$.

(c) $\lim C(\delta^*,d) = \xi$, as both $\delta^*\to 0$, $d\to 0$.

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APPENDIX A

SUPPLEMENTARIES TO CHAPTER II

Proof of Theorem II.1

Letting $\delta_i = \theta^* - \theta_i \geq 0$, $i = 1, \ldots, k$, and $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_k)$ by (II.3) and condition (a) we get,

$$\text{(A1)} \qquad P(CD|\underset{\sim}{\theta}, \sigma, R) = \prod_{i=1}^{k} G_n(\theta^* - \theta_i + d) - \prod_{i=1}^{k} G_n(\theta^* - \theta_i - d)$$

$$= \prod_{i=1}^{k} G_n(\delta_i + d) - \prod_{i=1}^{k} G_n(\delta_i - d) = H_n(\underset{\sim}{\delta}, d), \text{ say.}$$

Since at least one $\delta_i = 0$, assume without loss of generality that $\delta_k = 0$. Take now any $j \leq k-1$ and assume $\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_{k-1}$, fixed. Then,

$$\text{(A2)} \qquad H_n(\underset{\sim}{\delta}, d) = AG_n(d)G_n(\delta_j + d) - BG_n(-d)G_n(\delta_j - d), \text{ where,}$$

$$A = \prod_{\substack{i=1 \\ i \neq j}}^{k-1} G_n(\delta_i + d), \qquad B = \prod_{\substack{i=1 \\ i \neq j}}^{k-1} G_n(\delta_i - d). \text{ with } A > B.$$

However, $\frac{\partial}{\partial \delta_j} H_n(\underset{\sim}{\delta}, d) = g_n(\delta_j + d)\left[AG_n(d) - BG_n(-d)\,\frac{g_n(\delta_j - d)}{g_n(\delta_j + d)}\right]$.

Since for $\delta_j = 0$ the ratio $H(\delta_j) = \frac{g_n(\delta_j - d)}{g_n(\delta_j + d)}$ attains its largest value[1] and $g_n(\delta_j + d)$ is kept at a lowest level, while for $\delta_j \to \infty$ $\lim g_n(\delta_j + d) = 0$ and $\lim H(\delta_j) = 1$, we assert that $\inf_{\delta_j} H_n(\underset{\sim}{\delta}, d)$ is achieved at either $\delta_j = 0$ or $\delta_j \to \infty$. Then, for fixed $\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_{k-1}$, by (A2) we have, $H_n(\underset{\sim}{\delta}, d)\Big|_{\delta_j = 0} = AG_n^2(d) - BG_n^2(-d)$ and $H_n(\underset{\sim}{\delta}, d)\Big|_{\delta_j = \infty} = AG_n(d) - BG_n(-d)$ since $\lim_{\delta_j \to \infty} G_n(\delta_j + d) = \lim_{\delta_j \to \infty} G_n(\delta_j - d) = 1$. Clearly then,

$H_n(\underset{\sim}{\delta}, d)\Big|_{\delta_j = 0} \le H_n(\underset{\sim}{\delta}, d)\Big|_{\delta_j = \infty}$, for every fixed $\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_{k-1}$. Repeating the same process for each δ_j, iteratively, by (A1) we finally obtain,

$\inf_{\underset{\sim}{\theta}} P(CD|\underset{\sim}{\theta}, \sigma, R) = P(CD|\theta_{LFC}, \sigma, R) = G_n^k(d) - G_n^k(-d)$, where, $\underset{\sim}{\theta}_{LFC} = (\theta, \theta, \ldots, \theta)$.

Q.E.D.

---

[1]By conditions (b), (a) of Theorem II.1 we may have

$H(\delta_j) = \dfrac{g_n(-\delta_j + d)}{g_n(-\delta_j - d)} = \dfrac{g_n(\theta_j - \theta^* + d)}{g_n(\theta_j - \theta^* - d)} = \dfrac{g_n(\theta_j, \theta^* - d)}{g_n(\theta_j, \theta^* + d)}$. This, by M.L.R., is decreasing as θ_j (i.e., $\theta_j - \theta^*$) is increasing, however, $H(\delta_j)$ increases as δ_j decreases (note, $\delta_j \ge 0$).

Proof of Theorem II.2

Since T_{in} is a consistent estimate for θ_i, $i = 1,\ldots,k$, we have $P(-d \le T_{in}-\theta_i \le d) \to 1$ as $n\to\infty$ for every $d>0$, which implies that $G_n(d) - G_n(-d) \to 1$ as $n\to\infty$. Thus, $\lim G_n(-d) = \lim[1-G_n(d)] = 0$ as $n\to\infty$. Hence, for every $\epsilon>0$ there exists an integer N_0 s.t. $G_n(-d) = 1 - G_n(d) \le \epsilon$, for every $n \ge N_0$. The latter implies that $G_n^k(d) - G_n^k(-d) \ge (1-\epsilon)^k - \epsilon^k$ for every $n \ge N_0$. Choosing an ϵ, $(0<\epsilon<1)$, s.t. $(1-\epsilon)^k - \epsilon^k = \gamma$ the proof is completed, where N_0 is the smallest sample size to satisfy the requirement (II.1).

Proof of Lemma II.4

(a) Let $y_1 < y_2$ then $\Phi^k(y_1) < \Phi^k(y_2)$, for every $k \ge 1$.

Since $-y_2 < -y_1$ then $\Phi^k(-y_2) < \Phi^k(-y_1)$ for every $k \ge 1$.

Therefore, $\Phi^k(y_1) + \Phi^k(-y_2) < \Phi^k(y_2) + \Phi^k(-y_1)$,

however, $\beta_k(y_1) = \Phi^k(y_1) - \Phi^k(-y_1) < \Phi^k(y_2) - \Phi^k(-y_2) = \beta_k(y_2)$

for every $k \ge 1$.

(b) If $k=1$ we get $\beta''_k(y) = -y\varphi(y)$, and if $k \ge 2$ we get

$$\beta''(y) = k\varphi(y)\Big\{(k-1)\varphi(y)\Big[\Phi^{k-2}(y)-\Phi^{k-2}(-y)\Big] - y\Big[\Phi^{k-1}(y) - \Phi^{k-1}(-y)\Big]\Big\},$$

where, $\varphi(.)$ is the p.d.f. of $N(0,1)$ distribution.

Now we can verify, (i) if $k = 1,2$, then $\beta_k''(y) < 0$ for every $y>0$ and (ii) if $k>2$, then $\beta_k''(y) < 0$ only for $y > y_k$, while for $y \le y_k$ we have $\beta_k''(y) \ge 0$.

Proof of Lemma II.5

Since f is concave on an interval of R, the negative of f is convex on that interval, so that $E(X) < \infty$ and $-\infty < E(-f(X)) < E(f(X)) < \infty$. By Jensen's inequality we now obtain $-f(E(X)) \leq E(-f(X))$ i.e. $f(E(X)) \geq E(f(X))$.

Q.E.D.

Proof of Lemma II.8

Consider a $1 \leq j \leq k$. If $A = \prod_{i=1}^{k} \Phi(y_i)$, $B = \prod_{i=1}^{k} \Phi(-y_i)$ for $i \neq j$, then $\beta(\underset{\sim}{y}) = A\Phi(y_j) - B\Phi(-y_j) = \Phi(y_j)(A+B) - B$ and obviously $\beta(\underset{\sim}{y})$ is decreasing as y_j decreases. Repeating the same process for every $j = 1,\ldots,k$, where $k \geq 1$, the proof is completed.

APPENDIX B

SUPPLEMENTARIES TO CHAPTER III

Proof of Lemma III.1

By symmetry condition of Theorem III.1, for every $r = 1,2,\ldots,k-1$, we obtain the following expression;

$$f(r+1) - f(r) = G^{r+1}(d_1) - G^{r+1}(-d_2) - G^r(d_1) + G^r(-d_2)$$

$$= G^r(d_1)\left[G(d_1)-1\right] - G^r(-d_2)\left[G(-d_2)-1\right] = G^r(-d_2)G(d_2) - G^r(d_1)G(-d_1).$$

However,

(i) $\dfrac{G^r(-d_2)}{G^r(d_1)} > \dfrac{G(-d_1)}{G(d_2)}$ iff $f(r+1) - f(r) > 0$

(ii) $\dfrac{G^r(-d_2)}{G^r(d_1)} = \dfrac{G(-d_1)}{G(d_2)}$ iff $f(r+1) - f(r) = 0$.

(iii) $\dfrac{G^r(-d_2)}{G^r(d_1)} < \dfrac{G(-d_1)}{G(d_2)}$ iff $f(r+1) - f(r) < 0$.

(iv) $\dfrac{G(-d_2)}{G(d_1)} < 1$, since $f(1) = G(d_1) - G(-d_2) > 0$.

From (ii) and (iii) we get that, $f(r+1) \leq f(r)$ iff $\dfrac{G^r(-d_2)}{G^r(d_1)} \leq \dfrac{G(-d_1)}{G(d_2)}$,

which combined with (iv) yields, $\dfrac{G^{r+1}(-d_2)}{G^{r+1}(d_1)} < \dfrac{G(-d_1)}{G(d_2)}$, hence

$f(r+2) < f(r+1)$. Repeating the same process we end up with,

(B1) $f(k) < \ldots < f(r+2) < f(r+1) \leq f(r)$ for every $r = 1,\ldots,k-1$.

On the other hand (i) and (ii) imply that,

$f(r+1) \geq f(r)$ iff $\dfrac{G^r(-d_2)}{G^r(d_1)} \geq \dfrac{G(-d_1)}{G(d_2)}$ which also by (iv) gives

$\dfrac{G^{r-1}(-d_2)}{G^{r-1}(d_1)} > \dfrac{G(-d_1)}{G(d_2)}$, hence $f(r) > f(r-1)$. Similarly as previously

we obtain, $f(r+1) \geq f(r) > f(r-1) > \ldots > f(1)$ for $r = 2,\ldots,k$. This and (B1) complete the proof.

Proof of Theorem III.1

By condition (i) we rewrite (III.3) as,

$$P(CD|\underset{\sim}{\theta},\sigma,R) = \prod_{i=1}^{k} G(\delta_i+d_1) - \prod_{i=1}^{k} G(\delta_i-d_2) = b_{\underset{\sim}{\theta}}(d_1,d_2),$$

where $\delta_i = \theta^* - \theta_i \geq 0$, for every $i = 1,\ldots,k$. Then $\delta_i = 0$ for some $i = 1,\ldots,k$; without loss of generality assume $\delta_k = 0$, i.e. $\theta_k = \theta^*$.

Then, $b_{\underset{\sim}{\theta}}(d_1,d_2) = \prod_{i=1}^{k-1} G(\delta_i+d_1)G(d_1) - \prod_{i=1}^{k-1} G(\delta_i-d_2)G(-d_2)$.

Let a $1 \leq j \leq k-1$ assuming $\delta_1,\ldots,\delta_{j-1}, \delta_{j+1},\ldots,\delta_{k-1}$ fixed. Then,

(B2) $\qquad b_{\underset{\sim}{\theta}}(d_1,d_2) = AG(d_1)G(\delta_j+d_1) - BG(-d_2)G(\delta_j-d_2)$,

where, $A = \prod_{i=1}^{k-1} G(\delta_i+d_1)$, $B = \prod_{i=1}^{k-1} G(\delta_i-d_2)$ with $i \neq j$. Thus,

$$\frac{\partial}{\partial\delta_j} b_{\underset{\sim}{\theta}}(d_1,d_2) = AG(d_1)g(\delta_j+d_1) - BG(-d_2)g(\delta_j-d_2)$$

$$= g(\delta_j+d_1)\left[AG(d_1) - BG(-d_2)\frac{g(\delta_j-d_2)}{g(\delta_j+d_1)}\right].$$

Since $d_1+d_2 > 0$, i.e. $-d_2 < d_1$ we have $A > B$ and as in the proof of Theorem II.1 (see Appendix A) we also observe that for $\delta_j = 0$ the ratio $H(\delta_j) = \frac{g(\delta_j - d_2)}{g(\delta_j + d_1)}$ attains its largest value while $g(\delta_j+d_1)$ is kept at a lowest level. Since also $\lim g(\delta_j+d_1) = 0$ and $\lim H(\delta_j) = 1$, as $\delta_j \to \infty$ it follows that $b_{\underset{\sim}{\theta}}(d_1,d_2)$ is minimized at either $\delta_j = 0$ $(\theta^* = \theta_j)$ or $\delta_j = \infty$ $(\theta_j = -\infty)$ for fixed $\delta_1,\ldots,\delta_{j-1}, \delta_{j+1},\ldots\delta_{k-1}$. Then from (B2) we obtain,

$$b_{\underset{\sim}{\theta}}(d_1,d_2)\Big|_{\delta_j=0} = AG^2(d_1) - BG^2(-d_2),\quad b_{\underset{\sim}{\theta}}(d_1,d_2)\Big|_{\delta_j=\infty} = AG(d_1)-BG(-d_2) .$$

Repeating the above process, iteratively, for each δ_j, $j = 1,\ldots,k-1$, we see that,

$$b_{\underset{\sim}{\theta}}(d_1,d_2)\Big|_{\delta_j=0,\ j=1,\ldots,k-1} = G^k(d_1)-G^k(-d_2), \quad \text{thus} \quad \underset{\sim}{\theta}_{LFC} = (\theta,\ldots,\theta)$$

$$b_{\underset{\sim}{\theta}}(d_1,d_2)\Big|_{\delta_j=\infty,\ j=1,\ldots,k-1} = G(d_1)-G(-d_2), \quad \text{thus} \quad \underset{\sim}{\theta}_{LFC} = (-\infty,\ldots,-\infty,\theta),$$

and the rest of the proof follows from Lemma III.1.

Proof of Theorem III.2

(a) Since $G^k(L-d) - G^k(-d) = \left[G(L-d) - G(-d)\right] Q(d)$, where

$Q(d) = \sum_{j=0}^{k-1} G^{(k-1)-j}(L-d)G^j(-d)$, we have,

$$\text{(B3)}\quad \begin{array}{lll} \text{(i)} & Q(d) > 1 \text{ iff } & G^k(L-d) - G^k(-d) > G(L-d) - G(-d) \\ \text{(ii)} & Q(d) = 1 \text{ iff } & \quad -\text{''}- \quad = \quad -\text{''}- \\ \text{(iii)} & Q(d) < 1 \text{ iff } & \quad -\text{''}- \quad < \quad -\text{''}- \end{array}$$

Observing that $Q(-\infty) = k$, $Q(\infty) = 0$,[1] and $Q(d)$ is monotonically decreasing in d, clearly then the equation $Q(d) = 1$ has a unique solution, say d', so that,

(i) If $d < d'$, then $Q(d) > Q(d') = 1$, however (B3)(i) is true, i.e., $b(d) = G\ (L-d) - G(-d)$.

(ii) If $d > d'$, then $Q(d) < Q(d') = 1$, however (B3)(iii) is true, i.e., $b(d) = G^k(L-d) - G^k(-d)$,

and the proof of part (a) is completed.

(b) Let $d < L/2$, then $2d < L$, i.e. $d < L-d$, hence $G(L-d) > G(d)$ and $G(L-d) + G(-d) > 1$. Thus,

$$\text{(B4)}\quad \begin{array}{llll} \text{(i)} & \text{If } d < L/2 & \text{then } G(L-d) + G(-d) & > 1 \\ \text{(ii)} & \text{If } d = L/2 & -\text{''}- \quad -\text{''}- & = 1 \\ \text{(iii)} & \text{If } d > L/2 & -\text{''}- \quad -\text{''}- & < 1 . \end{array}$$

---

[1] Since $\lim G(L-d) = \lim G(-d) = 1$ as $d \to -\infty$ and $\lim G(L-d) = \lim G(-d) = 0$ as $d \to \infty$.

For $k = 2$ we now have,

$$G^2(L-d) = G^2(-d) = \left[G(L-d) - G(-d)\right]\left[G(L-d) + G(-d)\right], \text{ with}$$

$Q(d) = G(L-d) + G(-d)$. This by (B4)(ii) implies $Q(d') = 1$ with $d' = L/2$.

For $k > 2$ we have that,

$$Q(L/2) = \sum_{j=0}^{k-1} G^{(k-1)-j}(L-L/2)G^j(-L/2) = \sum_{j=0}^{k-1} G^{(k-1)-j}(L/2)G^j(-L/2)$$

$$< \sum_{j=0}^{k-1} \binom{k-1}{j} G^{(k-1)-j}(L/2)G^j(-L/2) = \left[G(L/2) + G(-L/2)\right]^{k-1}$$

$$= 1 = Q(d') \quad \text{i.e.} \quad Q(L/2) < Q(d'), \quad \text{hence} \quad d' < L/2.$$

Proof of Remark (4)

Since $Q_k(d') = \sum_{j=0}^{k-1} G^{(k-1)-j}(L-d')G^j(-d') = 1$, for every $k = 1,2,\ldots,$

then $G^{(k-1)-j}(L-d')G^j(-d') = c_j(k)$, a constant, so that,

(B5) $\qquad 0 < c_j(k) < 1$, for every $j = 0,1,\ldots,k-1$.

Letting $k \to \infty$, clearly $G^{(k-1)-j}(L-d')G^j(-d') \to 0$, which by condition (B5), implies that $-d' \to \infty$, i.e. $d' \to -\infty$.

Proof of Remark (5)

Consider

(B6) $$Q_k(d'(k)) = \sum_{j=0}^{k-1} G^{(k-1)-j}(L-d'(k))G^j(-d'(k)) = 1 \text{ , and}$$

(B7) $$Q_{k+1}(d'(k+1)) = \sum_{j=0}^{k-1} G^{k-j}(L-d'(k+1))G^j(-d'(k+1)) = 1 .$$

From (B6), by symmetry, we may also get,

$$Q_{k+1}(d'(k)) = \sum_{j=0}^{k} G^{k-j}(L-d'(k))G^j(-d'(k)) = G^k(L-d'(k)) + G(-d'(k))Q_k(d'(k))$$

$$= G^k(L-d'(k)) - G(d'(k)) + 1 .$$ Comparing this and (B7)

we see that there exists a possibility to have

$Q_{k+1}(d'(k)) < Q_{k+1}(d'(k+1))$, i.e. $d'(k+1) < d'(k)$, iff

$G^k(L-d'(k)) - G(d'(k)) < 0$. Q.E.D.

Extension of Lemma II.8

If $\underset{\sim}{y} = (y_1,\ldots,y_k)$ and $\underset{\sim}{w} = (w_1,\ldots,w_k)$ the function

$\beta(\underset{\sim}{y},\underset{\sim}{w}) = \prod_{i=1}^{k} \Phi(y_i) - \prod_{i=1}^{k} \Phi(-w_i)$ decreases as y_i and w_i decrease

for each $i = 1,\ldots,k$ and $k \geq 1$.

<u>Proof</u>: Let a $1 \leq j \leq k$. Letting $A = \prod_{i=1}^{k} \Phi(y_i)$ and $B = \prod_{i=1}^{k} \Phi(-w_i)$

for $i \neq j$. Then $\beta(y,w) = \Phi(y_j)A - \Phi(-w_j)B = \Phi(y_j)A + B\Phi(w_j) - B$.

Hence for y_j and w_j decreasing $\beta(\underset{\sim}{y},\underset{\sim}{w})$ decreases too. Repeating the same process for all j's, $1 \leq j \leq k$, the proof is completed.

APPENDIX C

TABLES

Table 1

Solutions h_k of $\int_{-\infty}^{\infty} \Phi^{k-1}(y+h_k)\,d\Phi(y) = P^*$

P^* \ k	2	3	4	5	6	7	8	9	10
.9995	4.6535	4.9163	5.0639	5.1661	5.2439	5.3066	5.3590	5.4039	5.4432
.9990	4.3703	4.6450	4.7987	4.9049	4.9856	5.0505	5.1047	5.1511	5.1917
.9950	3.6428	3.9517	4.1224	4.2394	4.3280	4.3989	4.4579	4.5083	4.5523
.99	3.2900	3.6173	3.7970	3.9196	4.0121	4.0861	4.1475	4.1999	4.2456
.98	2.9045	3.2533	3.4432	3.5722	3.6692	3.7466	3.8107	3.8653	3.9128
.97	2.6598	3.0232	3.2198	3.3529	3.4528	3.5324	3.5982	3.6543	3.7030
.96	2.4759	2.8504	3.0522	3.1885	3.2906	3.3719	3.4390	3.4961	3.5457
.95	2.3262	2.7101	2.9162	3.0552	3.1591	3.2417	3.3099	3.3679	3.4182
.94	2.1988	2.5909	2.8007	2.9419	3.0474	3.1311	3.2002	3.2590	3.3099
.93	2.0871	2.4865	2.6996	2.8428	2.9496	3.0344	3.1043	3.1637	3.2152
.92	1.9871	2.3931	2.6092	2.7542	2.8623	2.9479	3.0186	3.0785	3.1305
.91	1.8961	2.3082	2.5271	2.6737	2.7829	2.8694	2.9407	3.0012	3.0536
.90	1.8124	2.2302	2.4516	2.5997	2.7100	2.7972	2.8691	2.9301	2.9829
.88	1.6617	2.0899	2.3159	2.4668	2.5789	2.6676	2.7406	2.8024	2.8560
.86	1.5278	1.9655	2.1956	2.3489	2.4627	2.5527	2.6266	2.6893	2.7434
.84	1.4064	1.8527	2.0867	2.2423	2.3576	2.4486	2.5235	2.5868	2.6416
.82	1.2945	1.7490	1.9865	2.1441	2.2609	2.3530	2.4286	2.4926	2.5479
.80	1.1902	1.6524	1.8932	2.0528	2.1709	2.2639	2.3403	2.4049	2.4608
.75	.9539	1.4338	1.6822	1.8463	1.9674	2.0626	2.1407	2.2067	2.2637
.70	.7416	1.2380	1.4933	1.6614	1.7852	1.8824	1.9621	2.0293	2.0873
.65	.5449	1.0568	1.3186	1.4905	1.6168	1.7159	1.7970	1.8653	1.9242
.60	.3583	.8852	1.1532	1.3287	1.4575	1.5583	1.6407	1.7102	1.7700
.55	.1777	.7194	.9936	1.1726	1.3037	1.4062	1.4899	1.5604	1.6210
.50	—	.5565	.8368	1.0193	1.1526	1.2568	1.3418	1.4133	1.4748
.45	—	.3939	.6803	.8662	1.0019	1.1078	1.1941	1.2666	1.3289
.40	—	.2289	.5215	.7111	.8491	.9567	1.0443	1.1178	1.1810
.35	—	.0585	.3578	.5510	.6915	.8008	.8897	.9643	1.0284
.30	—	—	.1855	.3827	.5257	.6369	.7272	.8030	.8679
.25	—	—	—	.2014	.3472	.4604	.5523	.6292	.6951
.20	—	—	—	—	.1489	.2643	.3579	.4361	.5032
.15	—	—	—	—	—	.0364	.1319	.2117	.2800

Table 2

Solutions $\tau_n^{(k)}$ of $\int_{-\infty}^{\infty} F_n^{k-1}\left(y+\tau_n^{(k)}\right) dF_n(y) = P^*$

k	n	P^* = .750	P^* = .900	P^* = .950	P^* = .975	P^* = .990
2	5	1.14	2.29	3.11	3.94	5.14
	10	1.03	2.00	2.61	3.18	3.89
	15	1.00	1.93	2.50	3.02	3.64
	20	0.99	1.90	2.45	2.95	3.54
	25	0.98	1.88	2.42	2.91	3.48
	30	0.98	1.87	2.41	2.88	3.45
	∞	0.96	1.82	2.33	2.78	3.29
3	5	1.74	2.90	3.75	4.63	5.91
	10	1.56	2.48	3.08	3.64	4.34
	15	1.51	2.39	2.94	3.43	4.04
	20	1.49	2.34	2.87	3.35	3.92
	25	1.48	2.32	2.84	3.30	3.85
	30	1.47	2.30	2.81	3.27	3.81
	∞	1.44	2.24	2.72	3.13	3.62
4	5	2.08	3.26	4.14	5.05	6.40
	10	1.84	2.75	3.34	3.90	4.60
	15	1.78	2.63	3.17	3.67	4.27
	20	1.75	2.58	3.10	3.57	4.13
	25	1.74	2.55	3.06	3.51	4.05
	30	1.73	2.54	3.03	3.48	4.01
	∞	1.69	2.46	2.92	3.23	3.80
5	5	2.32	3.53	4.42	5.37	6.78
	10	2.03	2.94	3.53	4.08	4.79
	15	1.96	2.81	3.34	3.83	4.43
	20	1.93	2.75	3.26	3.72	4.28
	25	1.91	2.72	3.21	3.66	4.20
	30	1.90	2.69	3.18	3.62	4.14
	∞	1.85	2.60	3.06	3.46	3.92
6	5	2.52	3.74	4.66	5.64	7.09
	10	2.18	3.08	3.67	4.22	4.93
	15	2.10	2.93	3.46	3.95	4.55
	20	2.06	2.87	3.38	3.83	4.39
	25	2.04	2.84	3.33	3.77	4.30
	30	2.03	2.81	3.30	3.73	4.25
	∞	1.97	2.71	3.16	3.56	4.02
7	5	2.67	3.92	4.85	5.86	7.35
	10	2.30	3.19	3.79	4.34	5.05
	15	2.21	3.04	3.57	4.05	4.64
	20	2.17	2.97	3.47	3.93	4.48
	25	2.14	2.93	3.42	3.86	4.39
	30	2.13	2.91	3.39	3.82	4.33
	∞	2.07	2.80	3.25	3.64	4.09

Table 2 (continued)

k	n	P* = .750	P* = .900	P* = .950	P* = .975	P* = .990
8	5	2.81	4.07	5.03	6.05	7.58
	10	2.40	3.29	3.88	4.44	5.15
	15	2.30	3.12	3.65	4.13	4.73
	20	2.25	3.05	3.55	4.00	4.55
	25	2.23	3.01	3.50	3.93	4.46
	30	2.21	2.98	3.46	3.89	4.40
	∞	2.15	2.87	3.31	3.70	4.15
9	5	2.93	4.21	5.18	6.22	7.79
	10	2.48	3.37	3.96	4.52	5.23
	15	2.37	3.20	3.72	4.20	4.80
	20	2.33	3.12	3.62	4.07	4.62
	25	2.30	3.08	3.56	4.00	4.53
	30	2.28	3.05	3.53	3.93	4.46
	∞	2.21	2.94	3.37	3.76	4.20
10	5	3.04	4.33	5.32	6.38	7.98
	10	2.55	3.45	4.04	4.60	5.31
	15	2.44	3.26	3.79	4.26	4.86
	20	2.39	3.18	3.68	4.13	4.68
	25	2.36	3.14	3.62	4.05	4.58
	30	2.35	3.11	3.58	4.01	4.51
	∞	2.27	2.99	3.42	3.80	4.25
15	5	3.46	4.82	5.87	7.01	8.75
	10	2.83	3.72	4.32	4.88	5.60
	15	2.69	3.50	4.02	4.50	5.09
	20	2.63	3.41	3.90	4.34	4.88
	25	2.59	3.35	3.83	4.26	4.78
	30	2.57	3.32	3.79	4.20	4.71
	∞	2.47	3.18	3.61	3.98	4.42
20	5	3.77	5.18	6.28	7.49	9.34
	10	3.02	3.91	4.51	5.07	5.80
	15	2.85	3.66	4.18	4.66	5.25
	20	2.78	3.56	4.04	4.49	5.03
	25	2.74	3.50	3.97	4.40	4.91
	30	2.72	3.46	3.92	4.34	4.84
	∞	2.61	3.30	3.73	4.09	4.53
25	5	4.02	5.47	6.62	7.88	9.82
	10	3.16	4.06	4.66	5.23	5.96
	15	2.98	3.78	4.30	4.78	5.37
	20	2.90	3.67	4.16	4.60	5.14
	25	2.85	3.61	4.08	4.50	5.01
	30	2.83	3.57	4.03	4.44	4.94
	∞	2.70	3.40	3.81	4.18	4.61

Table 3

Solutions z_γ of $\Phi^k(z_\gamma) - \Phi^k(-z_\gamma) = \gamma$

k \ γ	0.80	0.90	0.95	0.975	0.99
2	1.281552	1.644854	1.959964	2.241403	2.57583
3	1.464413	1.818473	2.121241	2.390901	2.71194
4	1.604910	1.943198	2.234003	2.494347	2.80582
5	1.700843	2.036469	2.318080	2.572334	2.87090
6	1.792737	2.110520	2.380170	2.634687	2.93390
7	1.860060	2.171709	2.442111	2.686406	2.98139
8	1.918757	2.223718	2.489778	2.730720	3.02201
9	1.968786	2.268859	2.531237	2.769267	3.05747
10	2.012812	2.308678	2.567875	2.803371	3.08889
11	2.052071	2.344258	2.600604	2.833930	3.11708
12	2.087456	2.370385	2.630313	2.861503	3.14263
13	2.119635	2.405648	2.657351	2.886845	3.16508
14	2.149119	2.432500	2.682180	2.910063	3.18747
15	2.176308	2.457293	2.705147	2.931640	3.20736

Table 4

Solutions a_ν of $F_\nu^k(a_\nu) - F_\nu^k(-a_\nu) = \gamma$

	$k=2$				$k=3$				$k=4$			
ν \ γ	.900	.950	.975	.990	.900	.950	.975	.990	.900	.950	.975	.990
2	2.92	4.30	6.21	9.92	3.34	4.89	7.03	11.22	3.71	5.41	7.77	12.39
3	2.35	3.18	4.18	5.84	2.66	3.57	4.65	6.48	2.93	3.89	5.06	7.02
4	2.13	2.78	3.50	4.60	2.40	3.09	3.86	5.05	2.62	3.34	4.15	5.41
5	2.02	2.57	3.16	4.03	2.26	2.84	3.47	4.39	2.46	3.06	3.71	4.67
6	1.94	2.45	2.97	3.71	2.18	2.70	3.24	4.02	2.36	2.89	3.45	4.25
7	1.89	2.36	2.84	3.50	2.12	2.60	3.09	3.78	2.29	2.78	3.28	3.98
8	1.86	2.31	2.75	3.36	2.08	2.53	2.99	3.61	2.24	2.70	3.17	3.80
9	1.83	2.26	2.68	3.25	2.04	2.48	2.91	3.49	2.21	2.64	3.08	3.66
10	1.81	2.23	2.63	3.17	2.02	2.44	2.85	3.39	2.18	2.60	3.01	3.56
12	1.78	2.18	2.56	3.05	1.98	2.38	2.76	3.26	2.14	2.53	2.91	3.41
14	1.76	2.14	2.51	2.98	1.96	2.34	2.70	3.17	2.11	2.48	2.85	3.31
16	1.75	2.12	2.47	2.92	1.94	2.31	2.66	3.11	2.08	2.45	2.80	3.24
18	1.73	2.10	2.44	2.88	1.93	2.29	2.63	3.06	2.07	2.43	2.76	3.19
20	1.72	2.09	2.42	2.85	1.91	2.27	2.60	3.02	2.06	2.41	2.73	3.15
25	1.71	2.06	2.38	2.79	1.89	2.24	2.56	2.95	2.03	2.37	2.68	3.07
30	1.70	2.04	2.36	2.75	1.88	2.22	2.53	2.91	2.02	2.35	2.65	3.03
40	1.68	2.02	2.33	2.70	1.87	2.19	2.49	2.86	2.00	2.32	2.61	2.97
50	1.68	2.01	2.31	2.68	1.86	2.18	2.47	2.83	1.99	2.30	2.59	2.93
60	1.67	2.00	2.30	2.66	1.85	2.17	2.46	2.81	1.98	2.29	2.57	2.91
∞	1.64	1.96	2.24	2.58	1.82	2.12	2.39	2.71	1.94	2.23	2.49	2.81

Table 4 (continued)

	k=5				k=6				k=7			
ν \ γ	.900	.950	.975	.990	.900	.950	.975	.990	.900	.950	.975	.990
2	4.02	5.85	8.38	13.35	4.28	6.21	8.89	14.15	4.50	6.52	9.32	14.84
3	3.14	4.15	5.38	7.46	3.32	4.37	5.66	7.83	3.47	4.56	5.89	8.14
4	2.80	3.54	4.39	5.70	2.94	3.71	4.58	5.94	3.06	3.85	4.75	6.15
5	2.62	3.23	3.90	4.90	2.74	3.38	4.06	5.08	2.85	3.50	4.20	5.24
6	2.50	3.05	3.62	4.44	2.62	3.17	3.76	4.59	2.72	3.28	3.87	4.73
7	2.43	2.93	3.44	4.15	2.54	3.04	3.56	4.28	2.63	3.14	3.66	4.40
8	2.37	2.84	3.31	3.95	2.48	2.95	3.42	4.07	2.57	3.04	3.52	4.17
9	2.33	2.77	3.21	3.80	2.43	2.88	3.32	3.91	2.52	2.96	3.41	4.01
10	2.30	2.72	3.14	3.69	2.40	2.82	3.24	3.80	2.48	2.90	3.32	3.89
12	2.25	2.65	3.03	3.53	2.35	2.74	3.12	3.63	2.43	2.82	3.20	3.71
14	2.22	2.60	2.96	3.42	2.31	2.68	3.05	3.51	2.39	2.76	3.12	3.59
16	2.20	2.56	2.90	3.35	2.28	2.64	2.99	3.43	2.36	2.72	3.06	3.50
18	2.18	2.53	2.86	3.29	2.26	2.61	2.95	3.37	2.34	2.68	3.02	3.44
20	2.16	2.51	2.83	3.24	2.25	2.59	2.91	3.32	2.32	2.66	2.98	3.39
25	2.14	2.47	2.78	3.16	2.22	2.55	2.85	3.24	2.29	2.61	2.92	3.30
30	2.12	2.44	2.74	3.11	2.20	2.52	2.81	3.18	2.27	2.58	2.88	3.24
40	2.10	2.41	2.70	3.05	2.18	2.48	2.77	3.12	2.24	2.55	2.83	3.17
50	2.09	2.39	2.67	3.01	2.16	2.46	2.74	3.08	2.23	2.53	2.80	3.13
60	2.08	2.38	2.65	2.99	2.16	2.45	2.72	3.05	2.22	2.51	2.78	3.11
∞	2.04	2.32	2.57	2.88	2.11	2.39	2.63	2.93	2.17	2.44	2.69	2.98

	k=8				k=9			
ν \ γ	.900	.950	.975	.990	.900	.950	.975	.990
2	4.69	6.78	9.69	15.43	4.85	7.01	10.02	15.94
3	3.60	4.72	6.09	8.41	3.71	4.86	6.26	8.64
4	3.17	3.97	4.89	6.33	3.26	4.08	5.02	6.48
5	2.94	3.60	4.31	5.38	3.03	3.69	4.41	5.50
6	2.81	3.37	3.97	4.84	2.88	3.46	4.06	4.94
7	2.71	3.22	3.75	4.50	2.78	3.30	3.83	4.59
8	2.64	3.12	3.60	4.27	2.71	3.18	3.67	4.34
9	2.59	3.04	3.49	4.09	2.66	3.10	3.55	4.17
10	2.55	2.98	3.40	3.96	2.61	3.04	3.46	4.03
12	2.49	2.89	3.27	3.78	2.55	2.95	3.33	3.84
14	2.45	2.82	3.18	3.65	2.51	2.88	3.24	3.71
16	2.42	2.78	3.12	3.56	2.48	2.83	3.18	3.62
18	2.40	2.74	3.07	3.50	2.45	2.80	3.13	3.55
20	2.38	2.72	3.04	3.44	2.43	2.77	3.09	3.49
25	2.35	2.67	2.97	3.35	2.40	2.72	3.02	3.40
30	2.33	2.64	2.93	3.29	2.38	2.69	2.98	3.34
40	2.30	2.60	2.88	3.22	2.35	2.65	2.92	3.26
50	2.28	2.58	2.85	3.18	2.33	2.62	2.89	3.22
60	2.27	2.56	2.83	3.15	2.32	2.61	2.87	3.19
∞	2.22	2.49	2.73	3.02	2.27	2.53	2.77	3.06

Table 5

Solutions x_0 of $\Phi(c-x) - \Phi(-x) = \Phi^k(c-x) - \Phi^k(-x)$

c \ k	3	4	5	6	8	10	12	14
1.0	0.3087	0.1767	0.0767	-0.0033	-0.1263	-0.2188	-0.2926	-0.3537
1.1	0.3596	0.2285	0.1295	0.0504	-0.0709	-0.1620	-0.2345	-0.2945
1.2	0.4105	0.2804	0.1824	0.1043	-0.0152	-0.1047	-0.1758	-0.2346
1.3	0.4614	0.3325	0.2356	0.1586	0.0416	-0.0469	-0.1167	-0.1743
1.4	0.5125	0.3848	0.2891	0.2132	0.0975	0.0113	-0.0571	-0.1136
1.5	0.5636	0.4372	0.3427	0.2680	0.1544	0.0698	0.0028	-0.0526
1.6	0.6148	0.4897	0.3966	0.3231	0.2115	0.1286	0.0629	0.0087
1.7	0.6661	0.5424	0.4507	0.3784	0.2689	0.1877	0.1234	0.0703
1.8	0.7174	0.5953	0.5049	0.4339	0.3266	0.2470	0.1839	0.1319
1.9	0.7688	0.6482	0.5593	0.4896	0.3844	0.3064	0.2447	0.1937
2.0	0.8203	0.7013	0.6139	0.5454	0.4423	0.3659	0.3054	0.2555
2.1	0.8718	0.7545	0.6686	0.6014	0.5004	0.4256	0.3663	0.3173
2.2	0.9234	0.8078	0.7234	0.6576	0.5585	0.4852	0.4271	0.3791
2.3	0.9751	0.8612	0.7783	0.7137	0.6167	0.5449	0.4879	0.4408
2.4	1.0268	0.9147	0.8333	0.7700	0.6750	0.6046	0.5487	0.5025
2.5	1.0786	0.9682	0.8884	0.8263	0.7332	0.6642	0.6094	0.5641
2.6	1.1305	1.0219	0.9435	0.8827	0.7914	0.7237	0.6700	0.6253
2.7	1.1823	1.0755	0.9987	0.9391	0.8496	0.7832	0.7305	0.6869
2.8	1.2343	1.1293	1.0539	0.9954	0.9077	0.8426	0.7909	0.7481
2.9	1.2862	1.1830	1.1091	1.0518	0.9658	0.9019	0.8512	0.8091
3.0	1.3382	1.2368	1.1643	1.1081	1.0237	0.9611	0.9113	0.8700
3.1	1.3902	1.2906	1.2194	1.1643	1.0816	1.0201	0.9712	0.9307
3.2	1.4423	1.3444	1.2746	1.2206	1.1394	1.0791	1.0310	0.9912
3.3	1.4943	1.3982	1.3297	1.2767	1.1970	1.1378	1.0907	1.0516
3.4	1.5464	1.4520	1.3848	1.3328	1.2546	1.1964	1.1501	1.1117
3.5	1.5985	1.5058	1.4398	1.3888	1.3120	1.2549	1.2094	1.1717
3.6	1.6506	1.5595	1.4948	1.4447	1.3693	1.3132	1.2686	1.2315
3.7	1.7027	1.6132	1.5497	1.5005	1.4265	1.3714	1.3275	1.2910
3.8	1.7548	1.6669	1.6045	1.5562	1.4835	1.4294	1.3863	1.3504
3.9	1.8068	1.7205	1.6592	1.6118	1.5404	1.4872	1.4448	1.4096
4.0	1.8589	1.7741	1.7139	1.6673	1.5972	1.5440	1.5032	1.4686

Table 6

The values of $\Phi(c-x_0) - \Phi(-x_0) = \Phi^k(c-x_0) - \Phi^k(-x_0)$

c \ k	3	4	5	6	8	10	12	14
1.0	.376540	.364961	.352646	.340427	.319733	.301936	.286838	.273866
1.1	.410896	.398629	.385613	.373142	.350914	.332177	.316280	.302615
1.2	.444367	.431526	.417935	.404936	.381801	.362312	.345775	.331549
1.3	.476899	.463599	.449561	.436157	.412339	.392287	.375264	.380604
1.4	.508442	.494797	.480438	.466754	.442471	.422036	.404675	.389707
1.5	.538952	.525076	.510548	.496673	.472135	.451489	.433933	.418773
1.6	.568392	.554395	.539758	.525863	.501269	.480573	.462953	.447711
1.7	.596731	.582717	.568112	.554274	.529807	.509210	.491647	.476427
1.8	.623943	.610012	.595542	.581860	.557686	.537322	.519926	.504825
1.9	.650009	.636252	.622014	.608574	.584843	.564833	.547710	.532810
2.0	.674917	.661418	.647496	.634375	.611218	.591669	.574906	.560288
2.1	.698658	.685493	.671961	.659228	.636758	.617761	.601437	.587172
2.2	.721233	.708466	.695388	.683100	.661411	.643045	.627229	.613377
2.3	.742644	.730332	.717762	.705965	.685135	.667465	.652214	.638829
2.4	.762901	.751091	.739071	.727801	.707891	.690970	.676332	.663460
2.5	.782019	.770747	.759310	.748594	.729651	.713319	.699533	.687209
2.6	.800015	.789311	.778479	.768336	.750391	.735077	.721772	.710024
2.7	.816914	.806796	.796583	.787025	.770096	.755620	.743015	.731866
2.8	.832742	.823221	.813634	.804663	.788758	.775126	.763238	.752699
2.9	.847529	.838609	.829647	.821262	.806376	.7[illegible]593	.782421	.772501
3.0	.861308	.852987	.844643	.836835	.822956	.811015	.800559	.791257
3.1	.874116	.866385	.858645	.851402	838511	.827397	.817647	.808960
3.2	.885990	.878836	.871684	.864989	.853058	.842754	.833696	.825613
3.3	.896970	.890375	.883791	.877624	.866621	.857099	.848716	.841224
3.4	.907098	.901041	.895000	.889340	.879227	.870459	.862729	.855808
3.5	.916415	.910873	.905349	.900171	.890908	.882864	.875759	.869389
3.6	.924965	.919910	.914877	.910155	.901699	.894345	.887835	.881994
3.7	.932790	.928196	.923624	.919333	.911637	.904934	.898995	.893655
3.8	.939934	.935773	.931633	.927745	.920765	.914674	.909270	.904407
3.9	.946438	.942682	.938945	.935434	.929121	.923606	.918706	.914291
4.0	.952346	.948965	.945602	.942441	.936750	.931771	.927342	.923545

Table 7

The Values of c_0 (upper entry) and x_0 (lower entry) for Exact γ Values of $\Phi(c_0-x_0)-\Phi(-x_0) = \Phi^k(c_0-x_0)-\Phi^k(-x_0)$

k \ γ	0.75	0.90	0.95	0.975	0.99
2	2.3007 1.1503	3.2897 1.6448	3.9200 1.9600	4.4830 2.2415	5.1510 2.5755
3	2.3357 0.9936	3.3290 1.5094	3.9590 1.8375	4.5204 2.1295	5.1870 2.4752
4	2.3946 0.9118	3.3899 1.4466	4.0173 1.7833	4.5760 2.0816	5.2380 2.4325
5	2.4534 0.8627	3.4473 1.4108	4.0717 1.7530	4.6270 2.0547	5.2860 2.4092
6	2.5070 0.8303	3.4983 1.3878	4.1200 1.7338	4.6722 2.0376	5.3280 2.3942
7	2.5550 0.8073	3.5435 1.3718	4.1622 1.7201	4.7121 2.0257	5.3650 2.3837
8	2.5981 0.7903	3.5837 1.3600	4.2000 1.7102	4.7480 2.0172	5.3980 2.3761
9	2.6369 0.7731	3.6197 1.3509	4.2340 1.7027	4.7800 2.0106	5.4280 2.3704
10	2.6722 0.7667	3.6524 1.3437	4.2644 1.6966	4.8083 2.0050	5.4550 2.3659
11	2.7044 0.7581	3.6821 1.3379	4.2923 1.6917	4.8350 2.0010	5.4790 2.3620
12	2.7340 0.7511	3.7094 1.3330	4.3180 1.6877	4.8590 1.9974	5.5020 2.3592
13	2.7613 0.7450	3.7346 1.3289	4.3416 1.6843	4.8810 1.9942	5.5220 2.3562
14	2.7868 0.7400	3.7580 1.3255	4.3635 1.6813	4.9020 1.9919	5.5410 2.3538