SOME STUDIES ON NETWORK SENSITIVITIES AND NETWORK OPERATIONS

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ABSTRACT

This thesis is concerned with a study of sensitivities of different classes of networks as well as with the development and applications of some "network operations".

Tellegen's theorem in the frequency domain, in conjunction with the concept of adjoint networks, is used to establish a number of sensitivity invariants of any order, for different classes of linear time-invariant networks. Expressions are also derived for sensitivity functions of any order in terms of the immittance parameters of an augmented network. Bounds are obtained for the sum of the sensitivity magnitudes of different network functions for an RLC-gyrator network. For two-element kind networks, these bounds are expressed in terms of relevant driving point functions and sensitivities of network functions with respect to frequency. The first order sensitivity invariants for the different classes of linear time-invariant networks are used to obtain lower bounds on Schoeffler's quadratic sensitivity index. It is shown that this lower bound decreases as the number of elements in the network increases, thereby strengthening the conjecture of Leeds and Ugron.

For a general class of nonlinear networks, Tellegen's theorem is used in the time domain to obtain explicit expressions for sensitivities of a response due to nonlinear elements. These are used to esstablish some invariant relationships for the sum of sensitivities over different sets of parameters, for two different classes of nonlinear networks.

The definitions of Mitra, et al., for generalized duals and inverses are extended and planarity defined for a three-terminal two-port network, consisting as subnetworks one-ports and three-terminal two-ports. A simple method of directly obtaining the generalized dual, of such a planar three-terminal two-port network, as well as its applications are considered. A new theorem, similar to the Tellegen's theorem, but for networks having dual topologies is developed. This theorem is used in the time domain to define a new network called "Dual Adjoint" and it is shown how the network and its dual adjoint may be used in sensitivity calculations. Finally, this new theorem is used in the frequency domain to define a new network operation called "Generalized Dual Transposition", whose applications in sensitivity calculations and network synthesis are discussed.

LIST OF IMPORTANT ABBREVIATIONS AND SYMBOLS

CCG	Current controlled gyrator
CCR	Current controlled resistor
CCL	Current controlled inductor
CCT	Current controlled current source
CNIC	Current type negative impedance converter ?
CTF	Current transfer function
CVT	Current controlled voltage source
DPA	Driving point admittance
DPF	Driving point function
DPI	Driving point impedance
FCL	Flux controlled inductor
GD	Generalized dual
GDT	Generalized dual transpose
GI	Generalized inverse
HG	Hybrid gyrator
IC	Impedance converter
LTI	Linear time-invariant
NIC	Negative impedance converter
NUTL	Non-uniform transmission line
QCC	Charge controlled capacitor
UCNIC	Unity current inversion type negative impedance converter
UVNIC	Unity voltage inversion type negative impedance converter

VCC	Voltage controlled capacitor
VCG	Voltage controlled gyrator
VCR	Voltage controlled resistor
VCT	Voltage controlled current source
VNIC	Voltage type negative impedance converter
VTF	Voltage transfer function
VVT	Voltage controlled voltage source
$[a]$	Chain matrix of a two-port network
$[a]_D$	Chain matrix of the generalized dual of a two-port network
a_i	Coefficients of the polynomial $D(s)$
b_j	Coefficients of the polynomial $N(s)$
B_{ii}	Imaginary part of $Y_{ii}(\omega)$
$\overline{B}_{ii}$	Imaginary part of $\overline{Y}_{ii}(\omega)$
C_i, C	Lumped capacitance
$\overline{C}_i$	$C_{oi}\ g(x)$, the capacitance per unit length of a tapered $\overline{RC}$- or $\overline{LC}$-line
D_i	$(1/C_i)$
D_{oi}	$(1/C_{oi})$
$F(s)$, $F_1(s)$, $F_2(s)$	Arbitrary functions of s
g_i	$(1/r_i)$
$\underline{G}$	Vectror of sensitivity components
G	Dirceted graph of a given network
G_D	The dual of the graph G
G_i	$(1/R_i)$

G_{ii}	Real part of $Y_{ii}(\omega)$
$\overline{G}_{ii}$	Real part of $\overline{Y}_{ii}(\omega)$
i_e, I_e	Current through the element port e of a given network
$\underline{I}_e$	Vector of currents through the ports of an internal element of a given network
$\mathbf{I}_e$	Vector of currents through the element ports of a given network
i_p, I_p	Current through the port p of a given network
$\mathbf{I}_p$	Vector of currents through the ports of a given network
i''_e, I''_e	Current through the element port e in the network " or N"
i''_p, I''_p	Current through the port p in the network " or N"
$\mathbf{I}_{pI}$	Vector of currents through the ports of the generalized inverse network
$\mathbf{I}_{pD}$	Vector of currents through the ports of the generalized dual network
$\begin{bmatrix} k_1 & 0 \\ 0 & -1/k_2 \end{bmatrix}$	The chain matrix of an NIC (which includes VNIC, CNIC or an ideal transformer)
L_i, L	Lumped inductance
$\overline{L}_i$	$L_{oi}\ f(x)$, the series inductance per unit length of a tapered $\overline{LC}$-line
$\{m_k\}$	$\{L_i, C_i, L_{oi}, C_{oi}\}$, where C_{oi} includes the capacitances of both $\overline{RC}$- and $\overline{LC}$- lines

$\{m_{k1}\}$	$\{L_i, L_{oi}\}$
$\{m_{k2}\}$	$\{C_i, C_{oi}\}$, where C_{oi} includes the capacitances of $\overline{\text{LC}}$-lines only
$\{m_{k3}\}$	$\{m_{k1}\} \cup \{m_{k2}\}$
$\{m_{k4}\}$	$\{L_i, C_i\}$
N	An arbitrary planar LTI network;also stands for an arbitrary planar LTI two-port consisting of one-ports and two-ports as internal elements
N_D	Generalized dual of N
N_I	Generalized inverse of N
N''	Generalized dual transpose of N
N^T	Transpose of network N
N_0	An arbitrary n-port network consisting of lumped resistors, inductors and capacitors, gyrators, $\overline{\text{RC}}$- and $\overline{\text{LC}}$-tapered lines, CCTs, VVTs, CVTs, VCTs and NICs
N_0^A	Adjoint of N_0
N_0'	The n-port network N_0 considered as an (n + E)-port network, where the E internal ports of N_0 are also treated as external ports
N_1	An arbitrary n-port network consisting of lumped resistors and capacitors, gyrators, $\overline{\text{RC}}$-tapered lines, CCTs, VVTs, CVTs, VCTs and NICs
N_2	An arbitrary n-port network consisting of lumped inductors and capacitors, $\overline{\text{LC}}$-tapered lines, CCTs, VVTs, CVTs, VCTs and NICs

N_3	An arbitrary n-port network consisting of lumped resistors, inductors and capacitors, gyrators, CCTs, VVTs, CVTs, VCTs and NICs
N_4	An arbitrary n-port network consisting of lumped resistors and capacitors, gyrators, CCTs, VVTs, CVTs, VCTs and NICs
N_5	An arbitrary n-port network consisting of lumped inductors and capacitors, CCTs, VVTs, VCTs and NICs.
N_6	An arbitrary RLC-gyrator n-port network
$N(s)$	Numerator polynomial of a network function
$\mathcal{N}$	An arbitrary planar network
$\mathcal{N}^A$	Adjoint of $\mathcal{N}$
$\mathcal{N}_D$	Dual of $\mathcal{N}$
$\mathcal{N}_I$	Inverse of $\mathcal{N}$
$\mathcal{N}''$	Dual adjoint of $\mathcal{N}$
$\mathcal{N}_o$	An arbitrary planar n-port network consisting of CCRs, VCRs, VCCs, QCCs, CCLs, FCLs, CCTs, VVTs, CVTs, VCTs, VCGs, CCGs, HGs and ICs
$\mathcal{N}_1$	Same as $\mathcal{N}_o$, but with the requirement that either the QCCs and CCLs are absent, or if present their initial conditions $q_c(0)$ and $i_L(0)$ be zero
$\mathcal{N}_2$	Same as $\mathcal{N}_o$, but with the requirement that either the VCCs and FCLs are absent, or if

	present, their initial conditions $v_c(0)$ and $\eta_L(0)$ be zero
$\underline{p}$	Vector of the component parameters of a network
$\{p_k\}$	$\{R_i, L_i, D_i, (\alpha_{i1}, \alpha_{i2}), (R_{oi},D_{oi}), Z_{\overline{LC}i}, r_i, \gamma_i\}$
$\{p'_k\}$	$\{p_k\} \cup \{\rho_i\}$
P_i	The ith pole of a network function of an active lumped network
$P_R(P_R^A)$	Total average power dissipated in a given network (its adjoint)
$\{q_k\}$	$\{q_{k1}\} \cup \{q_{k2}\}$
$\{q'_k\}$	$\{q_k\} \cup \{\rho_i\}$
$\{q_{k1}\}$	$\{R_i, (\alpha_{i1},\alpha_{i2}), R_{oi}, r_i, \gamma_i\}$
$\{q_{k2}\}$	$\{C_i, C_{oi}$, where C_{oi} includes the capacitances $\overline{RC}$-lines only
$\{q_{k3}\}$	$\{R_i, (\alpha_{i1},\alpha_{i2}), r_i, \gamma_i\}$
$\{q_{k4}\}$	$\{q_{k3}\} \cup \{C_i\}$
Q	The pole-Q of a second order network function
r_i	Transfer resistance of a linear CVT
R_i,R	Lumped resistances
$\overline{R}_i$	$R_{oi}f(x)$, the series resistance per unit length of a tapered $\overline{RC}$-line
R_n	nth common factor between the polynomials N(s) D(s)
R_{ii}	Real part of $Z_{ii}(\omega)$

$\overline{R}_{ii}$	Real part of $\overline{Z}_{ii}(\omega)$
s	Complex frequency
S	Scattering matrix of an n-port network
S_x^F	$(x/F)(\partial F/\partial x)$, the normalized sensitivity of F with respect to x
T	Transfer voltage or current ratio
T_v	Transfer voltage ratio
T_c	Transfer current ratio
v_e, V_e	Voltage across the element port e of a given network
$\underline{V}_e$	Vector of voltages across the ports of an internal element of a given network
$\mathbf{V}_e$	Vector of voltages across the element ports of a given network
v_p, V_p	Voltage across the port p of a given n-port network
$\mathbf{V}_p$	Vector of voltages across the ports of a given n-port network
v''_e, V''_e	Voltage across the element port e in the network $\mathcal{N}''$ or N"
v''_p, V''_p	Voltage across the port p in the network $\mathcal{N}''$ or N"
$\mathbf{V}_{pI}$	Vector of currents across the ports of the generalized inverse network
$\mathbf{V}_{pD}$	Vector of voltages across the ports of the generalized dual network
W	$W_C + W_L$
W^A	$W_C^A + W_L^A$

$W_C (W_C^A)$	Total average energy stored in the capacitors of a given network (its adjoint)
$W_L (W_L^A)$	Total average energy stored in the inductors of a given network (its adjoint)
x_i	Parameter of a network
x_{ii}	Imaginary part of $Z_{ii}(\omega)$
X_{ii}	Imaginary part of $\overline{Z}_{ii}(\omega)$
y_e	The admittance matrix of an internal element
Y	The admittance matrix of an LTI n-port network
Y_D	The admittance matrix of the generalized dual network
Y''	The admittance matrix of the network N"
Y_{ij}	Elements of Y
$Y_i(s,x)$	$y_{oi}(s)\ g(x)$, the shunt admittance per unit length of an arbitrary NUTL
$Y_{\overline{LCi}}$	$(1/Z_{\overline{LCi}})$
$Y_{\overline{RCi}}$	$(1/Z_{\overline{RCi}})$
$\overline{Y}_{11}$ and $\overline{Y}_{22}$	$(1/\overline{Z}_{11})$ and $(1/\overline{Z}_{22})$ respectively
z_e	The impedance matrix of an internal element
Z	The impedance matrix of an LTI n-port network
Z_D	The impedance matrix of the generalized dual network
Z''	The impedance matrix of the network N"
z_j	The jth zero of a network function of an active lumped network
Z_{ij}	Elements of Z

Z'_{ij}	Transfer impedance between ports i and j of the network N'_0
$\overline{Z}_i(s,x)$	$Z_{oi}(s)\ f(x)$, the series impedance per unit length of an arbitrary NUTL
$Z_{\overline{LC}i}$	$\sqrt{L_{oi}/C_{oi}}$, the characteristic impedance of a tapered $\overline{LC}$-line
$Z_{\overline{RC}i}$	$\sqrt{R_{oi}/C_{oi}}$, the characteristic impedance of a tapered $\overline{RC}$-line
$\overline{Z}_{ii}$ and $\overline{Z}_{22}$	$(1/\overline{Y}_{11})$ and $(i/\overline{Y}_{22})$ respectively
$\begin{bmatrix} 0 & \alpha_{i1} \\ \alpha_{i2} & 0 \end{bmatrix}$	Impedance matrix of a gyrator
	Gain of a linear CCT
$\begin{bmatrix} 0 & \beta_{i2} \\ \beta_{i1} & 0 \end{bmatrix}$	Admittance matrix of a gyrator
γ_i	$(1/\delta_i)$
Γ_i	$(1/L_i)$
δ_i	Transfer conductance of a linear VCT
ϕ_e	Current through the element port e in the adjoint network
$\underline{\phi}_e$	Vector of currents through the ports of an internal element of the adjoint network
Φ_e	Vector of currents through the element ports of the adjoint network

ϕ_p	Current through the port p of the adjoint network
Φ_p	Vector of currents through the ports of the adjoint network
ϕ^F	$\sum_i \lvert S^F_{x_i} \rvert^2$, the Schoeffler's quadratic sensitivity index
ψ_e	Voltage across the element port e in the adjoint network
$\underline{\psi}_e$	Vector of voltages across the ports of an internal element in the adjoint network
Ψ_e	Vector of voltages across the element ports of the adjoint network
ψ_p	Voltage across the port p of the adjoint network
Ψ_p	Vector of voltages across the ports of the adjoint network
$\lambda, \lambda', \lambda'', \lambda'''$	Linear operators
μ	Gain at a linear VVT
$\{\rho_i\}$	The set of normalizing resistors
$\boldsymbol{\rho}$	Diagonal matrix with ρ_i as the elements
ω	Angular frequency
ω_0	Undamped natural frequency of a second order network function
3-T Network	Three-terminal network

CHAPTER 1

INTRODUCTION

1.1 General:

Two important topics within the domain of electric network theory are network analysis and network synthesis. In analysis, there is a unique solution although it may be difficult to find. In synthesis, however, solutions are not unique and there may exist no solution at all. If there is any solution to a given problem, there are an indefinite number of other solutions from which a choice may be made. The synthesis itself involves two major porblems, namely, approximation and realization. The former deals with the approximation of a specified transfer characteristic to a realizable transfer characteristic. The latter deals with the realization of the approximated transfer characteristic using linear, lumped, passive or RC-active, or distributed or non-linear networks or a combination of these. Some of the major problems encountered in the realization are the sensitivity, stability and tuning of the realized circuit, its size and cost of its implementation. These will be discussed briefly in the following paragraphs.

The sensitivity of a network function is a quantitative measure of change in the response, caused by variations in network components from their nominal design values. The variation of component values can occur due to many reasons, some of them being manufacturing tolerances and environmental conditions such as change in temperature or bias point etc.

The stability of a passive RLC-circuit does not pose a problem. A passive RLC-circuit can never become unstable with the change of element values or due to the presence of parasitics, since the poles of the network function are restricted to lie in the left half s-plane. On the other hand, an active network is susceptible to oscillations due to variations of the network components.

The tuning or adjusting the components of a practical active filter is an other important consideration. Since passive integrated network components are available only with certain amount of tolerances, the performance of a filter may be quite different from the desired one. ? Thus the post design adjustments of the circuit parameters are essential. In order to minimise the production cost, an attempt should be made to minimise the number of such adjustments, which make it necessary to know the sensitivities with respect to different network parameters.

The cost of implementation of a circuit is directly related to the size of the circuit. In any integrated circuit, a capacitor requires more area than a resistor, which, in turn, uses more space than a transistor. As a result, for an economic production of a circuit in an integrated form, the circuit must be designed with minimum number of passive components, especially the capacitors. In addition, an attempt should be made to minimise the total resistance and capacitance of the circuit in order to minimise the area.

Thus, it is clear that a study of sensitivity is very important in network design. This study is also useful in optimal synthesis of networks where it is necessary to calculate the gradients of network functions (which are essentially the unnormalised sensitivities) with respect to various parameters.

The thesis is concerned with a study of sensitivities of different classes of networks as well as with the development and applications of "Network operations", more emphasis being placed on the former aspect. The following two sections deal with a brief survey of literature on the sensitivity and the "network operations".

1.2 Network Sensitivities:

Network sensitivity may be considered as a quantitative measure of the change in some performance characteristic of a network resulting from changes in the values of the network parameters. A commonly used definition of the sensitivity is the ratio of the normalized change in the network function to a normalized change in a network parameter, even though unnormalized sensitivity has also been used by some authors. Knowledge of network sensitivity with respect to network elements helps the circuit designer in selecting the proper network configuration and its element values for a specified application. Sensitivity calculation also enables one to determine the critical elements of the network and their allowable tolerances.

In performing a sensitivity analysis or during the course of an optimal synthesis, it is often necessary to calculate the sensitivities of a network function with respect to a large number of parameters.

Such calculations are invariably tedious and time-consuming. Considerable effort has been made in devising efficient methods and algorithms for their evaluation(1,2,3). A collection of sensitivity formulas is available in literature(4). Further minimization of sensitivity with respect to one parameter generally increases the sensitivities with respect to some of the other parameters, thus leading to a problem of minimising simultaneously the sensitivities with respect to several parameters, or in other words to a multiparameter sensitivity problem. In such multiparameter sensitivity analyses, the methods mentioned earlier(1,2,3) require atleast as many network analyses as there are variable parameters. Obviously, these approaches become very inefficient for networks containing a large number of parameters.

An efficient method for calculating sensitivities for networks containing R, L, C elements and dependent current sources was given by Leeds(5). His method depends on the analysis of the original network and a related network called the "auxiliary network". This method was simplified, for the case of reciprocal networks, by Leeds and Ugron(6). Parker extended these results to nonlinear networks(7). Recently, Director and Rohrer(8,9,10) introduced the concept of "adjoint network" and showed how the sensitivity for a given network with respect to all its parameters may be computed in an elegant manner through the analyses of the given network and its adjoint. Subsequently, it was shown by Director(11,12) that for linear time-invariant networks (LTI), the computational effort could be considerably reduced through LU factorization.

Schoeffler(13) introduced the concept of continuously equivalent networks to obtain minimum sensitivity networks. It consists of expressing the elements of a passive LTI network as a function of a continuous real dummy variable x, while keeping the specified network function invariant, and at the same time, optimising the continuously equivalent network with respect to a sensitivity criteria defined as the sum of the squares of the magnitudes of sensitivities (quadratic sensitivity index). Leeds and Ugron(6) in their sensitivity studies postulated three hypotheses concerning continuously equivalent networks. The first is that the optimum network resulting from minimising Schoeffler's sensitivity index at a given frequency is also optimal in the same sense at all frequencies. The second conjecture is that the sum of the magnitude squared of the sensitivity functions decreases as the number of elements increases. The third is that the sum of sensitivities is invariant.

The first two hypotheses are yet to be proven even for the class of continuously equivalent networks. The third hypothesis has not only been established for the class of continuously equivalent networks(14,15), but also for general RLC-networks(16-18, 52,53). These results have also been extended to some classes of lumped active networks(19-23, 54-56). Recently, Sablatash and Seviora(24) have established the invariance of summed sensitivity for a class of networks containing uniformly distributed RC and LC lines.

The theoretical limitations of the continuously equivalent network approach for passive RLC-networks were considered by Schmidt and

Kasper(15). They used the invariance of first order sensitivity to obtain the theoretical minimum for Schoeffler's quadratic performance index and concluded that networks with substantially lower sensitivity index can be obtained only if the number of nodes, that is, the number of network elements is allowed to increase sufficiently, thus strengthening the second hypothesis of Leeds and Ugron(6). The requirement for an LC-network to be potentially optimally insensitive, according to Schoeffler's criterion, were investigated by Holt and Fidler(25). An excellent survey of literature along with references on sensitivity is given in(26,27).

Another multiparameter sensitivity index commonly used in optimal network synthesis is the sum of the magnitudes of sensitivities. The bounds on such a performance index for networks consisting of R, L, C elements and gyrators have been established by Smith(28). It should be pointed out that the upper bound on the sum of the magnitudes of the sensitivities for any network function would directly give an upper bound on Schoeffler's quadratic performance index also.

1.3 Network Operations:

As mentioned earlier, there may be many realizations for a given specification. One or the other may be desirable from the point of view of low sensitivity or low cost etc. Having selected a network with desirable properties, it would be useful in synthesis to be able to translate these properties to other networks through simple "network operations". For example, the well known low-pass - high-pass transformation of lossless networks may be considered to be a network operation; in this case, not only are the elements of the two networks interrelated, but also the optimal properties of the low-pass are carried over to the high-pass. Similarly, other transformations such as $s \rightarrow (s + \frac{1}{s})$, RC-CR, LC-RC etc.$^{(23)}$ may be considered as network operations.

A network operation is defined as one which transforms a given network η_1 into another network η_2, whose input-output characteristics are related to those of η_1, in such a way that

(i) the elements of η_2 may directly be obtained from those of η_1;

(ii) the properties of η_2 may readily be determined from those of η_1.

Recently, "network transposition" has been introduced and its applications in synthesis considered$^{(29)}$. In fact, according to the above definition, transposition may be considered as a network operation, which transforms a given network into another, whose topology is identical to that of the original, but having its immittance matrix as the transpose of that of the original. Thus, both networks have the same driving point immittances, while the voltage transfer function of one is the reverse current transfer function of the other. The elements of the transposed network, whether they be active, lumped or distributed, may directly be obtained from those of the original$^{(29)}$. Transposition also ensures the sensitivities of the original and the transposed network to be the same with respect to the corresponding network parameters(29). Further, the other properties of the original network such as margin of stability etc. are transferred to the transposed network. The conventional dual as well as the generalized dual defined by Mitra et al(30) may also be considered as network operations.

It may be of interest to note that transposition as well as the adjoint defined by Director and Rohrer(9,10) for the calculation of network sensitivities, are the same for the case of linear time invariant networks.

1.4 Scope of the Thesis:

This thesis may be divided into two sections. The first section deals with the computation of sensitivities of any order, the sensitivity invariances for different types of networks and establishing the bounds on two multiparameter sensitivity indices, namely Schoeffler's quadratic sensitivity index and the sum of magnitudes of sensitivities. The second section deals with the development and application of some network operations such as dual and dual adjoint.

In Chapter 2, expressions for sensitivity functions of any order are obtained in terms of the immittance parameters of an augmented network. The first order sensitivity invariance of network functions for different classes of networks including non-uniform transmission lines, and different types of active elements is established. These results are also extended to higher order sensitivities.

For networks consisting of R, L, C elements and gyrators, bounds on the sum of the magnitudes of the sensitivities for different network functions are established in Chapter 3. For two-element kind networks, these bounds are given in terms of relevant driving point functions and the sensitivities of network functions with respect to frequency.

In Chapter 4, lower bounds on Schoeffler's quadratic performance index are obtained for the different classes of networks considered in Chapter 2, and it is shown that this lower bound decreases as the number of elements in the network increases. For two-element kind networks, the upper bounds are also obtained for the performance index.

For a general class of nonlinear networks, explicit expressions for sensitivities of responses due to nonlinear elements have been derived in Chapter 5. These are used to establish the invariant relationships for the sum of sensitivities over different sets of parameters for two different classes of nonlinear networks.

Chapter 6 deals with the development and applications of some network operations. The definition of the generalised inverse(30) has been extended and planarity defined for a network N consisting as subnetworks, one-ports, and three-terminal two-ports. A simple method of obtaining the generalised dual of a planar network is given. A new theorem, similar to the Tellegen's theorem, but for networks having dual topology is developed and used to define a new network called "Dual Adjoint" and a new network operation called "Generalized Dual Transposition." Applications of generalized dual, dual adjoint, and generalized dual transposition have also been considered.

Chapter 7 contains a summary of the major contributions of the thesis and some suggestions for further work.

CHAPTER 2

SENSITIVITY EVALUATION AND INVARIANCE

2.1 Introduction:

The conjecture of Leeds and Ugron(6) that the sum of the sensitivity functions (over all components of each continuously equivalent network) is invariant with respect to various equivalent networks, has led to a number of investigations on the invariant nature of the sum of the sensitivities of network functions. These were discussed in Section 1.2. It was pointed out, in Section 1.2, that in performing a multiparameter sensitivity analysis or during the course of an optimal synthesis, evaluation of sensitivities with respect to a large number of parameters becomes necessary and that considerable effort has been made in devising efficient methods and algorithms to evaluate the same(1-3,5-12). In most methods of optimization, only the first order terms in the Taylor's series expansion of the network function about the nominal component values are used. It has been pointed out(31) that such an expansion may not be adequate, as the higher order terms may not be so small as to be neglected. The inclusion of higher order terms necessitates the evaluation of higher order sensitivities. Formulas exist in the literature to evaluate second order sensitivities(32, 57-62). However, these formulas are such that computation of higher order sensitivities is not easy.

In this Chapter, expressions for first order sensitivities are obtained in terms of the immittance parameters of an "augmented network", so that, higher order sensitivities may be obtained directly by repeated applications of the first order sensitivity formulas(33). Also, a number of sensitivity invariant relationships are established for different classes of LTI networks(34,35). Since the derivation of all these results is based on adjoint network(9), some of the results given in(9) will first be derived in Section 2.2.

2.2 Sensitivity Expressions in terms of Network and its Adjoint:

Consider a general linear time-invariant n-port network consisting of lumped resistors, inductors, capacitors, gyrators, current controlled current sources (CCTs), voltage controlled voltage sources (VVTs), current controlled voltage sources (CVTs), voltage controlled current sources (VCTs), voltage and current inversion type negative impedance converters (VNICs and CNICs), lossless tapered lines and $\overline{RC}$-tapered lines. Such a network will hereafter, in this thesis, be referred to as N_0. Let N_0^A denote the adjoint (transpose) of N(9,29). It is known from Tellegen's theorem(36,37)

$$\sum_p(\Delta V_p\phi_p - \Delta I_p\psi_p) = \sum_e(\Delta V_e\phi_e - \Delta I_e\psi_e) \qquad ..(2.1)$$

where

V_e and V_p are respectively the voltages across interior elements or ports of interior elements and external ports for the original network.

ψ_e and ψ_p are respectively the voltages across interior elements or ports of interior elements and external ports for the adjoint network.

I_e and I_p are respectively the currents through interior elements or ports of interior elements and external ports for the original network.

ϕ_e and ϕ_p are respectively the currents through interior elements or ports of interior elements and external ports for the adjoint network.

ΔV_e, ΔV_p, ΔI_e and ΔI_p are respectively the changes in V_e, V_p, I_e and I_p due to a small change in one of the parameters p_k of the network N_0.

Suppose we are interested in the sensitivity with respect to p_k of the element Z_{ij} of the impedance matrix $\mathbf{Z}$ of the network N_0. Let jth port of N_0 and the ith port of N_0^A be excited by unit currents, all other ports of N_0 and N_0^A being kept open. Then (2.1) reduces to

$$\Delta V_i = \sum_e (\Delta V_e \phi_e - \Delta I_e \psi_e) \qquad .. \text{(2.2)}$$

Since $V_i = Z_{ij} I_j = Z_{ij}$ as I_j is unity, it follows from (2.2) that

$$\Delta Z_{ij} = \sum_e (\Delta V_e \phi_e - \Delta I_e \psi_e) = \Delta \mathbf{V}_e^t \, \mathbf{\Phi}_e - \Delta \mathbf{I}_e^t \, \mathbf{\Psi}_e \qquad .. \text{(2.3)}$$

where $\Phi_e(I_e)$ is the column vector of currents through the element port(s) in the adjoint (original) network and $\Psi_e(V_e)$ is the column vector of voltage across the element port(s) in the adjoint (original) network.

Consider one of the above elements for which the impedance matrix $\mathbf{z}_e$ (1x 1 or 2x 2 matrix) exists. Let the parameter p_k be contained in $\mathbf{z}_e$, then

$$\underline{V}_e = \mathbf{z}_e \underline{I}_e \quad , \quad \underline{\psi}_e = \mathbf{z}_e^t \underline{\phi}_e \qquad .. (2.4a)$$

and

$$\Delta\underline{V}_e = \Delta\mathbf{z}_e \underline{I}_e + \mathbf{z}_e \Delta\underline{I}_e \qquad .. (2.4b)$$

where $\underline{V}_e(\underline{\psi}_e)$ and $\underline{I}_e(\underline{\phi}_e)$ are the column vectors of the element port voltages and currents in $N_0(\hat{N}_0)$.

The contribution to the right hand side of (2.3) due to the terms corresponding to this element is given by

$$\underline{I}_e^t \Delta\mathbf{z}_e^t \underline{\phi}_e + \Delta\underline{I}_e^t \mathbf{z}_e^t \underline{\phi}_e - \Delta\underline{I}_e^t \underline{\psi}_e = \underline{\phi}_e^t \Delta\mathbf{z}_e \underline{I}_e \qquad .. (2.5)$$

Consider next an element for which $\mathbf{z}_e$ does not exist but $\mathbf{y}_e$ exists. Let p_k be contained in admittance matrix $\mathbf{y}_e$ of this element, then the term in the right hand side of (2.3) due to this element is given by

$$-\underline{V}_e^t \Delta\mathbf{y}_e \underline{\psi}_e = -\underline{\psi}_e^t \Delta\mathbf{y}_e \underline{V}_e \qquad .. (2.6)$$

For VVTs, CCTs and NICs, $\mathbf{z}_e$ as well as $\mathbf{y}_e$ do not exist; in such a case, sensitivity component is derived as follows. Consider a VVT described by

$$V_\lambda = \mu V_\mu \quad , \quad I_\mu = 0 \qquad .. (2.7)$$

$$\Delta V_\lambda = \Delta\mu \, V_\mu + \mu\Delta \, V_\mu \quad , \quad \Delta I_\mu = 0 \qquad .. (2.8)$$

Using (2.8), the contribution of the VVT to the right hand side of (2.3) can be obtained as

$$\Delta V_\mu(\mu\phi_\lambda + \phi_\mu) - \Delta I_\mu\psi_\mu - \Delta I_\lambda\psi_\lambda + \Delta\mu V_\mu\phi_\lambda \quad .. (2.9)$$

Since the adjoint of a VVT is described by$^{(9,29)}$

$$\phi_\mu = -\mu\phi_\lambda \quad ; \quad \psi_\lambda = 0 \quad .. (2.10)$$

Expression (2.9) reduces to $\Delta\mu V_\mu\phi_\lambda$ as $\Delta I_\mu = 0$.

Consider next a CCT described by

$$I_\lambda = \beta I_\mu \quad , \quad V_\mu = 0 \quad .. (2.11)$$

$$\Delta I_\lambda = \Delta\beta I_\mu + \beta\Delta I_\mu \quad , \quad \Delta V_\mu = 0 \quad .. (2.12)$$

Using (2.12), the contribution of the VVT to the right hand side of (2.3) can be obtained as

$$-\Delta I_\mu(\beta\psi_\lambda + \psi_\mu) + \Delta V_\lambda\phi_\lambda + \Delta V_\mu\phi_\mu - \Delta\beta I_\mu\psi_\lambda \quad .. (2.13)$$

Since the adjoint of a CCT is described by$^{(9,29)}$

$$\psi_\mu = -\beta\psi_\lambda \quad , \quad \phi_\lambda = 0 \quad ... (2.14)$$

Expression (2.14) reduces to $-\Delta\beta I_\mu\psi_\lambda$ as $\Delta V_\mu = 0$.

Consider next a NIC described by

$$V_\mu = k_1V_\lambda \quad , \quad I_\lambda = k_2I_\mu \quad .. (2.15)$$

$$\Delta V_\mu = \Delta k_1V_\lambda + k_1\Delta V_\lambda$$

$$\Delta I_\lambda = \Delta k_2I_\mu + k_2\Delta I_\mu$$

Using above equations, the contribution of NIC to the right hand side of (2.3) can be obtained as

$$\Delta V_\lambda(\phi_\lambda + k_1\phi_\mu) - \Delta I_\mu(\psi_\mu + k_2\psi_\lambda) + \Delta k_1 V_\lambda \phi_\mu - \Delta k_2 I_\mu \psi_\lambda \qquad .. (2.16)$$

If adjoint of NIC is described as

$$\phi_\lambda = -k_1\phi_\mu \quad , \quad \psi_\mu = -k_2\psi_\lambda \qquad .. (2.17)$$

then (2.16) reduces to $\Delta k_1\phi_\mu V_\lambda - \Delta k_2 I_\mu \psi_\lambda$

It should be noted that NIC considered above is of voltage inversion type if k_1 and k_2 are negative and current inversion type if k_1 and k_2 are positive. It should also be noted that if $k_2 = -k_1$, equation (2.15) represents an ideal transformer of ratio k_1.

From the above discussion, it is clear that (2.3) can be written as

$$\Delta Z_{ij} = \sum_{I} \underline{\phi}_e^t \, \Delta \mathbf{z}_e \, \underline{I}_e + \sum_{II} -\underline{\psi}_e^t \, \Delta \mathbf{y}_e \, \underline{V}_e + \sum_{III} \Delta\mu V_\mu \phi_\lambda + \sum_{IV} -\Delta\beta I_\mu \psi_\lambda$$

$$+ \sum_{V} (\Delta k_1 \phi_\mu V_\lambda - \Delta k_2 I_\mu \psi_\lambda) \qquad .. (2.18)$$

where

$\sum_{I}$ gives the summation over all the elements which can be described by impedance matrix $\mathbf{z}_e$,

$\sum_{II}$ gives the summation over all the elements for which $\mathbf{y}_e$ exists,

$\sum_{III}$ gives the summation over all the VVTs,

$\sum\limits_{IV}$ gives the summation over all the CCTs,

$\sum\limits_{V}$ gives the summation over all the NICs.

It should be observed that the elements which have both $\boldsymbol{z}_e$ and $\boldsymbol{y}_e$, will be included either in $\sum\limits_{I}$ or $\sum\limits_{II}$, but not in both. Dividing both sides of (3.13) by Δp_k and letting $\Delta p_k \to 0$, we have

$$\frac{\partial Z_{ij}}{\partial p_k} = \sum_{I} \underline{\phi}_e{}^t \frac{\partial \boldsymbol{z}_e}{\partial p_k} \underline{I}_e + \sum_{II} -\underline{\psi}_e{}^t \frac{\partial \boldsymbol{y}_e}{\partial p_k} \underline{V}_e + \sum_{III} \frac{\partial \mu}{\partial p_k} V_\mu \phi_\lambda + \sum_{IV} - \frac{\partial \beta}{\partial p_k} I_\mu \psi_\lambda$$
$$+ \sum_{V} \left(\frac{\partial k_1}{\partial p_k} \phi_\mu V_\lambda - \frac{\partial k_2}{\partial p_k} I_\mu \psi_\lambda \right) \qquad .. (2.19)$$

It may be noted that if the parameter p_k is not associated with an element, then the corresponding term in the right hand side of (2.19) is zero.

Following the same procedure as in deriving (2.19), it can easily be shown that for transfer admittance Y_{ij} of the short circuit admittance matrix $\mathbf{Y}$, we have

$$\frac{\partial Y_{ij}}{\partial p_k} = - \sum_{I} \underline{\phi}_e^t \frac{\partial \boldsymbol{z}_e}{p_k} \underline{I}_e + \sum_{II} \underline{\psi}_e^t \frac{\partial \boldsymbol{y}_e}{\partial p_k} \underline{V}_e - \sum_{III} \frac{\partial \mu}{\partial p_k} V_\mu \phi_\lambda + \sum_{IV} \frac{\partial \beta}{\partial p_k} I_\mu \psi_\lambda$$
$$- \sum_{V} \left(\frac{\partial k_1}{\partial p_k} \phi_\mu V_\lambda - \frac{\partial k_2}{\partial p_k} I_\mu \psi_\lambda \right) \qquad .. (2.20)$$

In case of Y_{ij}, the jth (ith) port of $N_0 (N_0^A)$ is excited by unit voltage, all other ports being short-circuited.

Similarly, for the voltage transfer function T_v between jth and ith ports, it can easily be shown that

$$\frac{\partial T_v}{\partial p_k} = \sum_{I} \underline{\phi}_e^t \frac{\partial \mathbf{z}_e}{\partial p_k} \underline{I}_e + \sum_{II} -\underline{\psi}_e^t \frac{\partial \mathbf{y}_e}{\partial p_k} \underline{V}_e + \sum_{III} \frac{\partial \mu}{\partial p_k} V_\mu \phi_\lambda + \sum_{IV} - \frac{\partial \beta}{\partial p_k} I_\mu \psi_\lambda$$

$$+ \sum_{V} \frac{\partial k_1}{\partial p_k} (\phi_\mu V_\lambda - \frac{\partial k_2}{\partial p_k} I_\mu \psi_\lambda) \qquad .. (2.21)$$

It should be noted that, in this case, the jth port of N_0 is excited by a unit voltage with all other ports open, while in N_0^A, the ith port is excited by a unit current with the jth port short-circuited and all other ports open-circuited.

Let T_c be the current transfer function between jth and ith ports, then it can easily be shown that

$$\frac{\partial T_c}{\partial p_k} = - \sum_{I} \underline{\phi}_e^t \frac{\partial \mathbf{z}_e}{\partial p_k} \underline{I}_e + \sum_{II} \underline{\psi}_e^t \frac{\partial \mathbf{y}_e}{\partial p_k} \underline{V}_e - \sum_{III} \frac{\partial \mu}{\partial p_k} V_\mu \phi_\mu + \sum_{IV} \frac{\partial \beta}{\partial p_k} I_\mu \psi_\lambda$$

$$- \sum_{V} (\frac{\partial k_1}{\partial p_k} \phi_\mu V_\lambda - \frac{\partial k_2}{\partial p_k} I_\mu \psi_\lambda) \qquad .. (2.22)$$

In this case, the jth (ith) port of $N_0(N_0^A)$ is excited by a unit current (unit voltage), all other ports of $N_0(N_0^A)$ being short-circuited, excepting the jth port of N_0^A which is open.

2.3 Sensitivity Formulas in terms of Impedance Parameters:

In this Section, we first obtain formulas for the first order sensitivities in terms of the immittance parameters of an augmented network and then give a procedure to obtain formulas for higher order sensitivites(33).

Consider an LTI n-port network N_0. Let each internal element be a one-port or a two-port. Let there be E internal ports in N_0. If the E internal ports are also treated as external ports, then an (n + E)-port network can be obtained from N_0. Let such an augmented network be denoted by N_0'. Let the first n-ports of N_0' correspond to the external ports of N_0.

Consider any internal element of N_0 having an admittance matrix $\mathbf{y}_e$. Let the two ports of this element correspond to the ports of λ and μ of N_0'. Let

$$\mathbf{y}_e = \begin{bmatrix} y_{\lambda\lambda} & y_{\lambda\mu} \\ y_{\mu\lambda} & y_{\mu\mu} \end{bmatrix} \qquad .. \ (2.23)$$

If p_k is identified with $y_{\lambda\mu}$, then it follows from (2.19) that,

$$\frac{\partial Z_{ij}}{\partial y_{\lambda\mu}} = - \underline{\psi}_e^t \frac{\partial \mathbf{y}_e}{\partial y_{\lambda\mu}} \underline{V}_e$$

$$= - \psi_\lambda V_\mu \qquad .. \ (2.24)$$

Now

$$V_\mu = Z'_{\mu j} \ I_j = Z'_{\mu j} \ , \quad \text{since } I_j \text{ is unity} \qquad .. \ (2.25)$$

$$\psi_\lambda = Z'_{i\lambda} \ \phi_i = Z'_{i\lambda} \ , \quad \text{since } \phi_i \text{ is unity} \qquad .. \ (2.26)$$

where the prime denotes the parameters of N'_0. In obtaining (2.26), the fact that the impedance matrix of the adjoint of a network is the transpose of the impedance matrix of the network is utilized.

Thus, from (2.24), (2.25) and (2.26), we have

$$\frac{\partial Z_{ij}}{\partial y_{\lambda\mu}} = - Z'_{i\lambda} \; Z'_{\mu j} \qquad ..(2.27)$$

Using (2.27), we may find the sensitivity components $(\partial Z_{ij}/\partial y_{\lambda\mu})$ corresponding to lumped and distributed passive elements as well as gyrators and VCTs. These are given in Table 2.1. Now, for a CVT, $\mathbf{y}_e$ does not exist and hence (2.24) cannot be used. However, $\mathbf{z}_e$ exists with $z_{\lambda\lambda} = z_{\lambda\mu} = z_{\mu\mu} = 0$ and $z_{\mu\lambda} = r_{\mu\lambda}$. If p_k is identified to be $r_{\mu\lambda}$, then it follows from (2.19) that,

$$\frac{\partial Z_{ij}}{\partial r_{\mu\lambda}} = \phi_\mu I_\lambda = \frac{\psi_\lambda}{r_{\mu\lambda}} \cdot \frac{V_\mu}{r_{\mu\lambda}} = \frac{Z'_{i\lambda} \, Z'_{\mu j}}{r^2_{\mu\lambda}}$$

or

$$\frac{\partial Z_{ij}}{\partial g_{\mu\lambda}} = - Z'_{i\lambda} \; Z'_{\mu j} \qquad ..(2.28)$$

where $g_{\mu\lambda} = 1/r_{\mu\lambda}$

Consider next the voltage transfer function T_v between the jth and ith ports. Then

$$T_v = \frac{Z_{ji}}{Z_{ii}} \qquad ..(2.29)$$

TABLE 2.1

SENSITIVITY COMPONENTS DUE TO ACTIVE LUMPED/DISTRIBUTED ELEMENTS

Elements	Parameters	Sensitivity Components
1. Conductance	G_k	$\frac{\partial Z_{ij}}{\partial G_k} = -Z'_{ik}Z'_{kj}$
2. Elastance	$\Gamma_k = \frac{1}{L_k}$	$\frac{\partial Z_{ij}}{\partial \Gamma_k} = -\frac{1}{s} Z'_{ik}Z'_{kj}$
3. Capacitance	C_k	$\frac{\partial Z_{ij}}{\partial C_k} = -s\, Z'_{ik}Z'_{kj}$
4. Gyrator	$\mathbf{y}_e = \begin{bmatrix} 0 & \beta_{\lambda\mu} \\ \beta_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial \beta_{\lambda\mu}} = -Z'_{i\lambda}Z'_{\mu j}$; $\frac{\partial Z_{ij}}{\partial \beta_{\mu\lambda}} = -Z'_{i\mu}Z'_{\lambda j}$
5. VCT	$\mathbf{y}_e = \begin{bmatrix} 0 & 0 \\ \delta_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial \delta_{\mu\lambda}} = -Z'_{i\mu}Z'_{\lambda j}$
6. CVT	$\mathbf{z}_e = \begin{bmatrix} 0 & 0 \\ r_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial g_{\mu\lambda}} = -Z'_{i\lambda}Z'_{\mu j}$
7. $\overline{\text{RC}}$ Tapered Lines	$\mathbf{y}_e = \begin{bmatrix} Y_{\lambda\lambda} & Y_{\lambda\mu} \\ Y_{\mu\lambda} & Y_{\mu\mu} \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial G_{oi}} = -\{\frac{\partial Y_{\lambda\lambda}}{\partial G_{oi}} Z'_{i\lambda}Z'_{\lambda j} + \frac{\partial Y_{\lambda\mu}}{\partial G_{oi}} Z'_{i\lambda}Z'_{\mu j} + \frac{\partial Y_{\mu\lambda}}{\partial G_{oi}} Z'_{i\mu}Z'_{\lambda j} + \frac{\partial Y_{\mu\mu}}{\partial G_{oi}} Z'_{i\mu}Z'_{\mu j}\}$ $\frac{\partial Z_{ij}}{\partial C_{oi}} = -\{\frac{\partial Y_{\lambda\lambda}}{\partial C_{oi}} Z'_{i\lambda}Z'_{\lambda j} + \frac{\partial Y_{\lambda\mu}}{\partial C_{oi}} Z'_{i\lambda}Z'_{\mu j} + \frac{\partial Y_{\mu\lambda}}{\partial C_{oi}} Z'_{i\mu}Z'_{\lambda j} + \frac{\partial Y_{\mu\mu}}{\partial C_{oi}} Z'_{i\mu}Z'_{\mu j}\}$
8. Lossless tapered lines	$\mathbf{y}_e = Y_{\overline{LCi}} = \begin{bmatrix} \psi_{\lambda\lambda} & \psi_{\lambda\mu} \\ \psi_{\mu\lambda} & \psi_{\mu\mu} \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial Y_{\overline{LCi}}} = -\{\psi_{\lambda\lambda}Z'_{i\lambda}Z'_{\lambda j} + \psi_{\lambda\mu}Z'_{i\lambda}Z'_{\mu j} + \psi_{\mu\lambda}Z'_{i\mu}Z'_{\lambda j} + \psi_{\mu\mu}Z'_{i\mu}Z'_{\mu j}\}$

$$\frac{\partial T_v}{\partial y_{\lambda\mu}} = \frac{\partial}{\partial y_{\lambda\mu}} \left(\frac{Z_{ji}}{Z_{ii}}\right)$$

$$= \frac{1}{Z_{ii}} \frac{\partial Z_{ji}}{\partial y_{\lambda\mu}} + \frac{Z_{ji}}{Z^2_{ii}} \left(- \frac{\partial Z_{ii}}{\partial y_{\lambda\mu}}\right)$$

$$= \frac{1}{Z_{ii}} (-Z'_{\mu i}\ Z'_{j\lambda}) + \frac{Z_{ji}}{Z^2_{ii}} (Z'_{\mu i} Z'_{i\lambda})$$

$$= -Z'_{j\lambda}\ T'_{\mu i} + Z'_{i\lambda}\ T'_{ji}\ T'_{\mu i} \qquad .. (2.30)$$

Dual expressions may be obtained for the sensitivities of admittance and current transfer functions in terms of the admittance matrix parameters of the augmented network.

Formulas for second order sensitivity components can now be obtained in a straight forward manner with the help of the formulas for first order ones. For example, if p_1 and p_2 correspond to the admittances of R, L or C, then

$$\frac{\partial^2 Z_{ij}}{\partial p_2 \partial p_1} = \frac{\partial}{\partial p_2} \left(\frac{\partial Z_{ij}}{\partial p_1}\right) = \frac{\partial}{\partial p_2}(-Z'_{i1}\ Z'_{1j})$$

$$= Z'_{i1}\ Z'_{12}\ Z'_{2j} + Z'_{1j}\ Z'_{21}\ Z'_{i2} \qquad .. (2.31)$$

Higher order sensitivity components may be similarly determined by repeated use of Table 2.1. We illustrate the application of these formulas with the help of an example.

Example:

Consider the three-terminal two-port network shown in Fig. 2.1. The second order sensitivity of transfer impedance Z_{12} with respect to the admittances of branch 2 and 4 is evaluated using (2.31)

$$\frac{\partial^2 Z_{12}}{\partial y_2 \partial y_4} = Z'_{14}\, Z'_{42}\, Z'_{22} + Z'_{12}\, Z'_{24}\, Z'_{42} \qquad .. (2.32)$$

where Z'_{14}, Z'_{42}, Z'_{22}, Z'_{12}, Z'_{24}, Z'_{42} are the open-circuit impedance parameters of the augmented five-port network. These parameters can be easily evaluated and are given below

$$Z'_{14} = \frac{y_2 + y_5}{y_4 y_5 + y_5 y_2 + y_2 y_4} \qquad .. (2.33)$$

$$Z'_{42} = Z'_{24} = \frac{y_5}{y_4 y_5 + y_5 y_2 + y_2 y_4} \qquad .. (2.34)$$

$$Z'_{12} = \frac{y_5}{y_4 y_5 + y_5 y_2 + y_2 y_4} \qquad .. (2.35)$$

$$Z'_{22} = \frac{y_4 + y_5}{y_4 y_5 + y_5 y_2 + y_2 y_4} \qquad .. (2.36)$$

where

$$y_2 = sC$$

$$y_4 = \frac{1}{sL} = \frac{\Gamma}{s}$$

$$y_5 = G_2 \qquad .. (2.37)$$

Substituting (2.33), (2.34), (2.35) and (2.36) in (2.32), we get

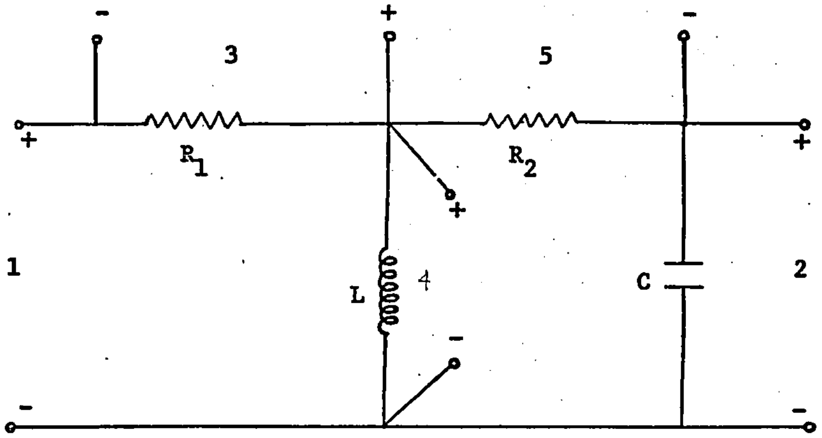


Fig. 2.1: The Network for the Example in Section 2.3

$$\frac{\partial^2 Z_{12}}{\partial \Gamma \partial C} = \frac{2G_2 s^3 + G_2 s^2 (s^2 CG_2 + sC\Gamma + \Gamma G_2)}{(s^2 CG_2 + sC\Gamma + \Gamma G_2)^3} \qquad .. (2.38)$$

2.4 Sensitivity Invariants for Network Functions:

In this Section, we establish the invariant nature of the sum of the sensitivities of the network functions for the network N_0 (as defined in the Section 2.2) over a set of parameters to be given later in this Section(34).

Let the parameter p_k be identified with the value of a resistor ($R_i = 1/G_i$), inductance ($L_i = 1/\Gamma_i$), inverse capacitance ($D_i = 1/C_i$), or transfer resistance of a CVT ($r_i = 1/g_i$), it is seen that

$$p_k \frac{\partial \mathbf{z}_e}{\partial p_k} = \mathbf{z}_e \qquad \text{if } p_k \in \mathbf{z}_e$$

$$= 0 \qquad \text{otherwise.}$$

Hence, if $p_k \in \mathbf{z}_e$ alone, then it follows from (2.19) that

$$p_k \frac{\partial Z_{ij}}{\partial p_k} = \underline{\phi}_e^t \, \underline{V}_e \qquad .. (2.39)$$

Consider the gyrator, whose impedance matrix is given by

$$\mathbf{z}_e = \begin{bmatrix} 0 & \alpha_{i1} \\ \alpha_{i2} & 0 \end{bmatrix} \qquad .. (2.40)$$

From (2.40), it is seen that

$$\alpha_{i1} \frac{\partial \mathbf{z}_e}{\partial \alpha_{i1}} + \alpha_{i2} \frac{\partial \mathbf{z}_e}{\partial \alpha_{i2}} = \mathbf{z}_e$$

Hence from (2.19)

$$\alpha_{i1} \frac{\partial Z_{ij}}{\partial \alpha_{i1}} + \alpha_{i2} \frac{\partial Z_{ij}}{\partial \alpha_{i2}} = \underline{\phi}_e^t \, \underline{V}_e \qquad .. \; (2.41)$$

A non-uniform transmission line (NUTL) is described as

$$\begin{aligned} \overline{Z}_i(s,x) &= Z_{oi}(s)\; f(x) \\ \overline{Y}_i(s,x) &= Y_{oi}(s)\; g(x) \end{aligned} \qquad a \le x \le b \qquad .. \; (2.42)$$

where $\overline{Z}_i(s,x)$ and $\overline{Y}_i(s,x)$ are the series impedance per unit length and shunt admittance per unit length of NUTL. The parameters of lossless line are given by

$$\overline{L}_i = L_{oi}\, f(x)$$

$$\text{and} \qquad a \le x \le b \qquad .. \; (2.43a)$$

$$\overline{C}_i = C_{oi}\; g(x)$$

and those of an $\overline{RC}$-tapered line are given by

$$\begin{aligned} \overline{R}_i &= R_{oi}\, f(x) \\ \overline{C}_i &= C_{oi}\; g(x) \end{aligned} \qquad a \le x \le b \qquad .. \; (2.43b)$$

Let

$$D_{oi} = (1/C_{oi}) \qquad .. \; (2.44)$$

For an $\overline{RC}$-tapered line, it is known that$^{(38)}$

$$\mathbf{z}_e = R_{oi}\; F \qquad .. \; (2.45)$$

where each element of the (2 x 2) matrix F is a function of the product $(s\ R_{oi}\ C_{oi}) = (s\ R_{oi}\ \frac{1}{D_{oi}})$, then

$$R_{oi}\frac{\partial z_e}{\partial R_{oi}} + D_{oi}\frac{\partial z_e}{\partial D_{oi}} = R_{oi}\frac{\partial z_e}{\partial R_{oi}} - C_{oi}\frac{\partial z_e}{\partial C_{oi}}$$

$$= R_{oi}(F + R_{oi}\ \dot{F}s\ C_{oi}) - C_{oi}(R_{oi}\ \dot{F}s\ R_{oi})$$

$$= R_{oi}F$$

$$= z_e$$

where dots denote differentiation with respect to $(s\ R_{oi}\ C_{oi})$.

Hence, from (2.19)

$$R_{oi}\frac{\partial Z_{ij}}{\partial R_{oi}} + D_{oi}\frac{\partial Z_{ij}}{\partial D_{oi}} = \underline{\phi}_e^t\ \underline{V}_e$$

Let $Z_{\overline{RC}i} = \sqrt{(R_{oi}/C_{oi})} = (1/Y_{\overline{RC}i})$ denote the characteristic impedance of a tapered $\overline{RC}$-line, then (2.45) can be written as

$$z_e = Z_{\overline{RC}i}\ \sqrt{R_{oi}\ C_{oi}}\ F$$

and

$$Z_{\overline{RC}i}\ \frac{\partial z_e}{\partial Z_{\overline{RC}i}} = Z_{\overline{RC}i}(\sqrt{R_{oi}\ C_{oi}}\ F) = z_e$$

Hence, from (2.19)

$$Z_{\overline{RC}i}\ \frac{\partial Z_{ij}}{\partial Z_{\overline{RC}i}} = \underline{\phi}_e^t\ \underline{V}_e$$

Similarly for an $\overline{LC}$-tapered line

$$L_{oi} \frac{\partial Z_{ij}}{\partial L_{oi}} + D_{oi} \frac{\partial Z_{ij}}{\partial D_{oi}} = \underline{\phi}_e^t \underline{V}_e$$

and

$$Z_{\overline{LC}i} \frac{\partial Z_{ij}}{\partial Z_{\overline{LC}i}} = \underline{\phi}_e^t \underline{V}_e$$

where $Z_{\overline{LC}i} = \sqrt{L_{oi}/C_{oi}} = (1/Y_{\overline{LC}i})$ is the characteristic impedance of an $\overline{LC}$-line.

Consider next a VCT for which $\mathbf{z}_e$ does not exist but $\mathbf{y}_e$ exists. Let $\delta_i = (1/\gamma_i)$ be the transfer conductance of VCT, it is easily seen that

$$\delta_i \frac{\partial \mathbf{y}_e}{\partial \delta_i} = \mathbf{y}_e$$

Hence, from (2.19)

$$\delta_i \frac{\partial Z_{ij}}{\partial \delta_i} = - \underline{\psi}_e^t(\delta_i \frac{\partial \mathbf{y}_e}{\partial \delta_i})\underline{V}_e = - \underline{\phi}_e^t \underline{V}_e$$

or $$\gamma_i \frac{\partial Z_{ij}}{\partial \gamma_i} = \underline{\phi}_e^t \underline{V}_e \qquad .. (2.46)$$

It is seen from (2.7), (2.10), (2.11), (2.14), (2.15) and (2.17) that for VVTs, CCTs and NICs,

$$\underline{\phi}_e^t \underline{V}_e = 0 \qquad .. (2.47)$$

Thus, from the above discussion, it is seen that for the network N_0, if p_k is a member of the set

$$\{p_k\} = \{R_i,\ L_i,\ D_i,\ (\alpha_{i1},\alpha_{i2}),\ (R_{oi},D_{oi}),\ Z_{\overline{LC}i},\ r_i,\ \gamma_i\} \quad .. (2.48)$$

{where the subset (R_{oi},D_{oi}) may be replaced by $Z_{\overline{RC}i}$, while the subset $Z_{\overline{LC}i}$ by (L_{oi},D_{oi})}, then,

$$\sum_{p_k} p_k \frac{\partial Z_{ij}}{\partial p_k} = \sum_{e} V_e \phi_e \quad .. (2.49)$$

where $\sum_{p_k}$ stands for the sum over all the elements of the set (2.48).

From Tellegen's theorem

$$\sum_{e} V_e \phi_e = \sum_{p} V_p \phi_p \quad .. (2.50)$$

Hence

$$\sum_{p_k} p_k \frac{\partial Z_{ij}}{\partial p_k} = \sum_{p} V_p \phi_p \quad .. (2.51)$$

On the right hand side of (2.51), only ϕ_i is non-zero, and since

$$V_i = Z_{ij}\ I_j = Z_{ij} \qquad \text{as } I_j = 1$$

we have

$$\sum_{p_k} p_k \frac{\partial Z_{ij}}{\partial p_k} = Z_{ij} \quad .. (2.52)$$

Thus

$$\sum_{p_k} \frac{p_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial p_k} = 1 \quad .. (2.53)$$

That is, the sum of the sensitivities of Z_{ij} is a constant independent of Z_{ij}. Because of the arbitrary choice of the indices i and j in (2.19), we get

$$\sum_{p_k} p_k \frac{\partial Z}{\partial p_k} = Z \qquad .. \ (2.54)$$

where Z is the impedance matrix of N.

Similarly, it can be shown that

$$\sum_{p_k} \frac{p_k}{Y_{ij}} \frac{\partial Y_{ij}}{\partial p_k} = -1 \qquad .. \ (2.55)$$

That is, the sum of the sensitivities of Y_{ij} is invariant; also

$$\sum_{p_k} p_k \frac{\partial Y}{\partial p_k} = -Y \qquad .. \ (2.56)$$

If T is a voltage or current transfer function, then, it can be shown similarly that

$$\sum_{p_k} p_k \frac{\partial T}{\partial p_k} = 0 \qquad .. \ (2.57)$$

The basis-free normalized scattering matrix for a linear time-invariant network with a resistive reference is$^{(39)}$

$$S = U - 2\rho^{\frac{1}{2}}[Z + \rho]^{-1}\rho^{\frac{1}{2}} \qquad .. \ (2.58)$$

where $\rho = \text{dia}\{\rho_i\}$ and ρ_i are the normalising resistors.

Differentiating (2.58) with respect to p_k and summing over all p_k, we have

$$\sum_{p_k} p_k \frac{\partial S}{\partial p_k} = 2\rho^{\frac{1}{2}}(Z + \rho)^{-1} \sum_{p_k} p_k \frac{\partial Z}{\partial p_k} (Z + \rho)^{-1}\rho^{\frac{1}{2}} \qquad .. \ (2.59)$$

and using (2.54)

$$\sum_{p_k} p_k \frac{\partial S}{\partial p_k} = 2\rho^{\frac{1}{2}}(Z + \rho)^{-1} Z(Z + \rho)^{-1}\rho^{\frac{1}{2}} = \frac{1}{2}(U - S^2) \qquad .. \ (2.60)$$

where U is a unit matrix. That is, for two different realizations of S, the sum of the sensitivities of S over all $\{p_k\}$ is same. Further, following the approach used in$^{(24)}$, it can be shown that the sensitivities of group delay over p_k for different realizations are same. However, if normalizing resistors $\{\rho_i\}$ are also included in the set p_k to obtain the set $P' = \{p_k'\}$, then, it can be shown that the sum of the sensitivities of any scattering parameter or group delay is zero. It should be observed that in the above discussion, even though N_0 consists of VVTs, CCTs, VNICs and CNICs, the parameters of these devices are not included in the set $\{p_k\}$ or $\{p_k'\}$.

We illustrate the results of this section by two examples.

Example 1:

Consider the circuit of Fig.2.2. This circuit has been used to obtain a low-pass filter$^{(40)}$. The transfer function $T_v = (V_2/V_1)$ is given by

$$T_v = \frac{GG_aD_aD_b}{s^2 + s(G_aD_a+G_bD_b) + (G_aG_b+G^2)D_aD_b} \qquad .. (2.61)$$

Let

$$X = s^2 + s(G_aD_a + G_bD_b) + (G_aG_b + G^2)D_aD_b \qquad .. (2.62)$$

then

$$R_a \frac{\partial T_v}{\partial R_a} = -G_a \frac{\partial T_v}{\partial G_a} = - \frac{G_a\{GD_aD_bX - GG_aD_aD_b(sD_a + G_bD_aD_b)\}}{X^2} \qquad .. (2.63)$$

$$R_b \frac{\partial T_v}{\partial R_b} = -G_b \frac{\partial T_v}{\partial G_b} = - \frac{G_b\{-GG_aD_aD_b(sD_b + G_aD_aD_b)\}}{X^2} \qquad .. (2.64)$$

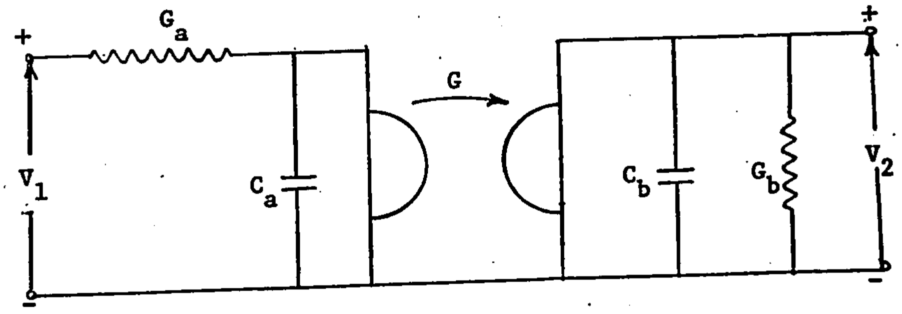


Fig. 2.2: The Network for the Example 1 in Section 2.4

$$D_a \frac{\partial T_v}{\partial D_a} = \frac{D_a\{XGG_aD_b - GG_aD_aD_b(sG_a + D_b(G_aG_b + G^2)\}}{X^2} \quad .. (2.65)$$

$$D_b \frac{\partial T_v}{\partial D_b} = \frac{D_b\{XGG_aD_a - GG_aD_aD_b(sG_b + D_a(G_aG_b + G^2)\}}{X^2} \quad .. (2.66)$$

and

$$R \frac{\partial T_v}{\partial R} - G \frac{\partial T_v}{\partial G} = -\frac{G\{G_aD_aD_bX - GG_aD_aD_b(2GD_aD_b)\}}{X^2} \quad .. (2.67)$$

From (2.63), (2.64), (2.65), (2.66) and (2.67), it is seen that

$$R_a \frac{\partial T_v}{\partial R_a} + R_b \frac{\partial T_v}{\partial R_b} + D_a \frac{\partial T_v}{\partial D_a} + D_b \frac{\partial T_v}{\partial D_b} + R \frac{\partial T_v}{\partial R} = 0 \quad .. (2.68)$$

Thus the sum of the sensitivities of T_v over the set $\{R_a, R_b, D_a, D_b, R\}$ is zero.

Example 2:

In this example, we illustrate the invariant nature of the sum of the sensitivities of the network function of a lumped distributed RC-network. Consider the circuit of Fig.2.3. This has been used by Kerwin(41) to realize a band pass filter. The voltage transfer function $T_v = (V_2/V_1)$ can be evaluated as

$$T_v = \frac{V_2}{V_1} = \frac{K[\frac{\theta}{R} + G_2 \text{ cosech}\theta]}{[\frac{R}{\theta} G_1G_2 + (\coth\theta - \text{cosech}\theta)(2G_1 - KG_2) + G_2\coth\theta + \frac{\theta}{R}]} \quad .. (2.69)$$

where

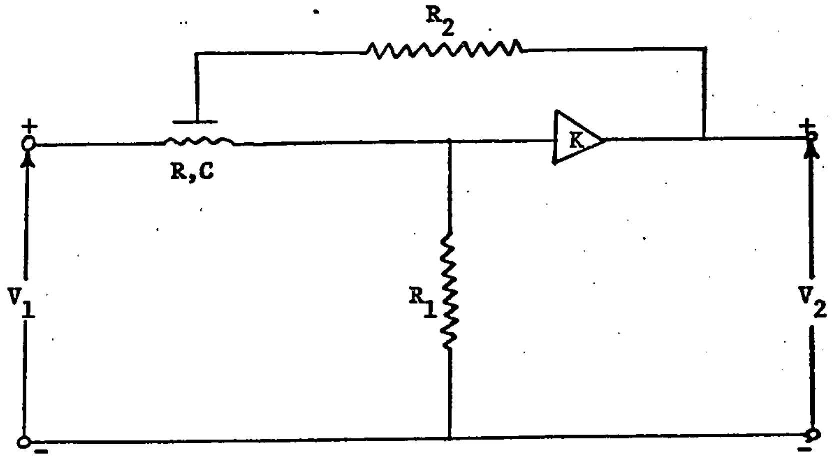


Fig. 2.3: The Network for the Example 2 in Section 2.4

K is the gain of operational amplifier, and $\theta = \sqrt{sRC}$,

R is the total series resistance of uniform $\overline{RC}$-line, and

C is the total shunt capacitance of the same line.

Let

$$X = \frac{R}{\theta} G_1G_2 + (\coth\theta - \mathrm{cosech}\theta)(2G_1 - KG_2) + G_2\coth\theta + \frac{\theta}{R} \quad .. (2.70)$$

then, we get

$$R_1 \frac{\partial T_v}{\partial R_1} = -G_1 \frac{\partial T_v}{\partial G_1} = - \frac{G_1K[\frac{\theta}{R} + G_2\mathrm{cosech}\theta]- [-\{\frac{R}{\theta} G_2 + (\coth\theta - \mathrm{cosech}\theta)^2\}]}{X^2} \quad .. (2.71)$$

$$R_2\frac{\partial T_v}{\partial R_2} = -G_2\frac{\partial T_v}{\partial G_2} = - \frac{G_2K\{X\mathrm{cosech}\theta -(\frac{\theta}{R} +G_2\mathrm{cosech}\theta)[\frac{R}{\theta} G_1-K(\coth\theta-\mathrm{cosech}\theta)+\coth\theta)]\}}{X^2} \quad .. (2.72)$$

$$\begin{aligned} R \frac{\partial T_v}{\partial R} = \frac{1}{X^2} \cdot RK\{X(- \frac{\theta}{R^2} + \frac{1}{R} \frac{\partial\theta}{\partial R} - G_2\mathrm{cosech}\theta \coth\theta \frac{\partial\theta}{\partial R}) \\ -(\frac{\theta}{R} + G_2\mathrm{cosech}\theta)[(\frac{1}{\theta} - \frac{R}{\theta^2} \frac{\partial\theta}{\partial R}) G_1G_2 \\ +(2G_1-KG_2)(-\mathrm{cosech}^2\theta + \mathrm{cosech}\theta\coth\theta)\frac{\partial\theta}{\partial R} - G_2\mathrm{cosech}^2\theta\frac{\partial\theta}{\partial R} - \frac{\theta}{R2} \\ + \frac{1}{R} \frac{\partial\theta}{\partial R}]\} \end{aligned} \quad .. (2.73)$$

Noting $R \frac{\partial\theta}{\partial R} = C \frac{\partial\theta}{\partial C}$, it can be seen from the above relations that the sum of the sensitivities of T_v over the set $\{R_1, R_2, R, \frac{1}{C}\}$ is equal to zero.

2.5 Summed Sensitivity Invariants:

In the sensitivity functions considered in the previous section, sensitivity with respect to inverse capacitance is used. Since it is more usual to calculate sensitivity with respect to capacitance, rather than with respect to inverse capacitance, Holt and Fidler(19) have defined what is called the "summed sensitivity" of a network function.

The sum of the sensitivities of a network function F over a set of parameters is called the summed sensitivity of F, if the member of the set corresponding to a capacitor is the capacitance. Holt and Fidler have obtained certain results on the invariant nature of the summed sensitivity of a network function for a class of active lumped networks. In this section, we establish the invariant nature of the summed sensitivity for a general active lumped/distributed network(34). These will be expressed in terms of the sensitivity of the network function under consideration with respect to frequency. It will be shown further in Chapter 4 that the summed sensitivity gives better lower bounds on Schoeffler's quadratic sensitivity index(13) as compared to those given by sensitivity invariants considered in Section 2.4.

2.5.1 Summed Sensitivity Invariants for Immittance Parameters:

Consider first a network consisting of lumped resistors, capacitors, gyrators, $\overline{RC}$-tapered lines, controlled sources (CVTs, VCTs, CCTs and VVTs) and NICs (VNICs and CNICs). Such a network will hereafter be referred to as N_1 in this thesis. Let the set $\{q_k\}$ of the parameters of N_1 be

$$\{R_i, C_i, (\alpha_{i1},\alpha_{i2}), (R_{oi},C_{oi}), r_i,\gamma_i\} \quad .. (2.74)$$

If Z_{ij} is the (i,j)th element of the impedance matrix $\mathbf{Z}$ of N_1, then the summed sensitivity of Z_{ij} with respect to the set $\{q_k\}$ is defined as equal to(6)

$$\sum_{q_k} \frac{q_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_k} \quad .. (2.75)$$

Let

$$\{q_k\} = \{q_{k1}\} \cup \{q_{k2}\}$$

where

$$\{q_{k1}\} = \{R_i, (\alpha_{i1},\alpha_{i2}), R_{oi}, r_i, \gamma_i\}$$

$$\{q_{k2}\} = \{C_i, C_{oi}\}$$

then

$$\sum_{q_k} \frac{q_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_k} = \sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} + \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} \quad .. (2.76)$$

It should be noted that the sensitivity sum considered in Section 2.4 is

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} - \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} \quad .. (2.77)$$

If q_{k2} is identified with a lumped capacitor C_i, then it is easily seen that

$$q_{k2}\frac{\partial z_e}{\partial q_{k2}} = -z_e, \qquad \text{if } q_{k2} \varepsilon z_e$$

$$= 0 \qquad \text{otherwise}$$

Thus

$$q_{k2}\frac{\partial Z_{ij}}{\partial q_{k2}} = -\underline{\phi}_e^t \, z_e \, \underline{I}_e, \qquad \text{if } q_{k2} \varepsilon z_e \qquad .. (2.78)$$

For an $\overline{RC}$-tapered line, using (2.45), we have

$$C_{oi}\frac{\partial z_e}{\partial C_{oi}} = s \, R_{oi}^2 \, C_{oi} \, \dot{F} \qquad .. (2.79)$$

where the 'dot' denotes differentiation with respect to $(s \, R_{oi} \, C_{oi})$. Thus, from (2.78) and (2.79)

$$\sum_{q_{k2}} q_{k2}\frac{\partial Z_{ij}}{\partial q_{k2}} = -\sum_{VI} \underline{\phi}_e^t \, z_e \, \underline{I}_e + \sum_{VII} \underline{\phi}_e^t \, s \, R_{oi}^2 \, C_{oi} \, \dot{F} \, \underline{I}_e \qquad .. (2.80)$$

where the sum $\sum_{VI}$ consists of all terms due to lumped capacitors, and the sum $\sum_{VII}$ consists of all terms due to the capacitance of $\overline{RC}$-tapered lines.

If the complex frequency variable s is treated as a parameter, then (2.19) reduces to

$$s\frac{\partial Z_{ij}}{\partial s} = \sum_{I} \underline{\phi}_e^t \, s \frac{\partial z_e}{\partial s} \underline{I}_e \qquad .. (2.81)$$

as the contribution to the right hand side of (2.19) due to a VCT, VVT, CCT and NIC is zero.

The contribution to the right hand side of (2.81) due to all resistors, gyrators, CVTs is zero. The contribution from the lumped capacitors is

$$-\sum_{VI} \underline{\phi}_e^t \, \mathbf{z}_e \, \underline{I}_e \qquad .. (2.82)$$

while, for an $\overline{RC}$-line, we have from (2.45)

$$s \frac{\partial \mathbf{z}_e}{\partial s} = s \, R_{oi}^2 \, C_{oi} \, \dot{F} \qquad .. (2.83)$$

Thus from (2.80), (2.81), (2.82) and (2.83)

$$\sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} = \frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s} \qquad .. (2.84)$$

Thus the summed sensitivity of Z_{ij} is invariant over q_{k2} and is independent of realization of Z_{ij}.

It is known from (2.53) that

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} - \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} = 1 \qquad .. (2.85)$$

Thus from (2.84) and (2.85), we have

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} = 1 + \frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s} \qquad .. (2.86)$$

From (2.84) and (2.86)

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} + \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} = \frac{2s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s} + 1 \qquad .. (2.87)$$

Hence the summed sensitivity of Z_{ij} over $\{q_{k1}\}$, $\{q_{k2}\}$ and $\{q_k\}$ is invariant and is independent of the realization of Z_{ij}. Since (2.87) holds for arbitrary i and j, we get

$$\sum_{q_k} q_k \frac{\partial Z}{\partial q_k} = 2s \frac{\partial Z}{\partial s} + Z \qquad .. (2.88)$$

Similarly, it can be shown that

$$\sum_{q_{k1}} \frac{q_{k1}}{Y_{ij}} \frac{\partial Y_{ij}}{\partial q_{k1}} = \frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s} - 1 \qquad .. (2.89a)$$

$$\sum_{q_{k2}} \frac{q_{k2}}{Y_{ij}} \frac{\partial Y_{ij}}{\partial q_{k2}} = \frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s} \qquad .. (2.89b)$$

$$\sum_{q_k} \frac{q_k}{Y_{ij}} \frac{\partial Y_{ij}}{\partial q_k} = \frac{2s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s} - 1 \qquad .. (2.89c)$$

and

$$\sum_{q_k} q_k \frac{\partial Y}{\partial q_k} = 2s \frac{\partial Y}{\partial s} - Y \qquad .. (2.90)$$

Also, for a current or voltage transfer function T, it can be shown that

$$\sum_{q_{k1}} \frac{q_{k1}}{T} \frac{\partial T}{\partial q_{k1}} = \sum_{q_{k2}} \frac{q_{k2}}{T} \frac{\partial T}{\partial q_{k2}} = \frac{s}{T} \frac{\partial T}{\partial s} \qquad .. (2.91a)$$

and

$$\sum_{q_k} \frac{q_k}{T} \frac{\partial T}{\partial q_k} = \frac{2s}{T} \frac{\partial T}{\partial s} \qquad .. (2.91b)$$

That is, the summed sensitivity of T is also an invariant and is independent of the realization.

For the network N_0, if the set $m = \{m_k\}$ be defined by $\{L_i, C_i, L_{oi}, C_{oi}\}$, then the summed sensitivity of a network function F over the set $\{m_k\}$ can be shown to be

$$\sum_{m_k} \frac{m_k}{F} \frac{\partial F}{\partial m_k} = \frac{s}{F} \frac{\partial F}{\partial s} \qquad .. (2.92)$$

where F can be an impedance or an admittance parameter, or a voltage or current transfer function.

Consider now a network consisting of only inductors, capacitors, LC-tapered lines, VVTs, CCTs and NICs;such a network will hereafter be denoted by N_2. Let the set $\{m_{k3}\}$ be

$$\{m_{k1}\} \cup \{m_{k2}\} = \{L_i, L_{oi}\} \cup \{C_i, C_{oi}\}$$

Then, it follows from (2.53), (2.55), (2.57) and (2.92) that

$$\sum_{m_{k1}} \frac{m_{k1}}{F} \frac{\partial F}{\partial m_{k1}} = \frac{1}{2}\left(\frac{s}{F} \frac{\partial F}{\partial s} + \zeta\right) \qquad .. (2.93a)$$

$$\sum_{m_{k2}} \frac{m_{k2}}{F} \frac{\partial F}{\partial m_{k2}} = \frac{1}{2}\left(\frac{s}{F} \frac{\partial F}{\partial s} - \zeta\right) \qquad .. (2.93b)$$

$$\sum_{m_{k3}} \frac{m_{k3}}{F} \frac{\partial F}{\partial m_{k3}} = \frac{s}{F} \frac{\partial F}{\partial s} \qquad .. (2.93c)$$

where

$$\zeta = \begin{cases} +1 & \text{if F is an impedance parameter,} \\ -1 & \text{if F is an admittance parameter,} \\ 0 & \text{if F is voltage or a current transfer function.} \end{cases} \qquad .. (2.94)$$

2.5.2 Summed Sensitivity Invariants for Scattering Matrix:

The basis-free normalized scattering matrix S for a linear time-invariant network, with a resistive reference is given by (2.58).

For network N_1, the summed sensitivity of S over the set $\{q_k\}$ is given by

$$\sum_{q_k} q_k \frac{\partial S}{\partial q_k} = 2\rho^{\frac{1}{2}}[Z+\rho]^{-1}\,[\sum_{q_k} q_k \frac{\partial Z}{\partial q_k}]\,[Z+\rho]^{-1}\,\rho^{\frac{1}{2}} \qquad .. \ (2.95)$$

Using (2.88) in (2.95), we get

$$\sum_{q_k} q_k \frac{\partial S}{\partial q_k} = 2s\,\frac{\partial S}{\partial s} + \frac{1}{2}(U - S^2) \qquad .. \ (2.96)$$

If the set $\{\rho_i\}$ of normalizing resistors is also included, then

$$\sum_{q_{k1}} \frac{\partial S}{\partial q_{k1}} q_{k1} - \sum_{q_{k2}} q_{k2} \frac{\partial S}{\partial q_{k2}} + \sum_{\rho_i} \rho_i \frac{\partial S}{\partial \rho_i} = 0 \qquad .. \ (2.97)$$

From (2.60) and (2.97), it is seen that

$$\sum_{\rho_i} \rho_i \frac{\partial S}{\partial \rho_i} = -\frac{1}{2}(U - S^2) \qquad .. \ (2.98)$$

Hence, from (2.96) and (2.98)

$$\sum_{q_k'} q_k' \frac{\partial S}{\partial q_k'} = 2s\,\frac{\partial S}{\partial s} \qquad .. \ (2.99)$$

where

$$\{q_k'\} = \{q_{k1}\} \cup \{q_{k2}\} \cup \{\rho_i\} = \{q_k\} \cup \{\rho_i\}$$

Thus, the summed sensitivity of the scattering matrix over the set $Q' = \{q_k'\}$ is an invariant, independent of the realization. It could also be shown that the summed sensitivity of group delay over the set $\{q_k\}$ or $\{q_k'\}$ for two different realizations is the same.

For the network N_0 over the set $\{m_k\}$, it can easily be shown following the same procedure as above, that

$$\sum_{m_k} m_k \frac{\partial S}{\partial m_k} = s \frac{\partial S}{\partial s}$$

Thus, the summed sensitivity of the scattering matrix over the set $\{m_k\}$ is invariant.

2.6 Root and Coefficient Sensitivity Invariants for Active Lumped Networks:

In a recent paper, Kumpel$^{(42)}$ has established the invariance of the summed root sensitivities for linear two element kind networks. Such results have also been obtained for RLC-networks$^{(43)}$, and for a class of active networks containing RLC elements and controlled sources, using frequency and magnitude scalings$^{(23)}$. These results will now be extended to more general classes of active networks by using the results of previous sections. The nature of summed coefficient sensitivities will also be investigated.

For a lumped active network, any of the network function F is rational. Let

$$F = \frac{N(s)}{D(s)} = A\frac{s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots. + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots. + a_{n-1}s + a_n} \quad .. (2.100)$$

Eqn. (2.100) can be rewritten in the following form

$$F = \frac{A \prod_{i=1}^{M} (s - Z_i)^{\ell_i} \prod_{n=1}^{NI} (s - R_n)^{\ell_n'}}{\prod_{j=1}^{N} (s - P_j)^{\ell_j} \prod_{n=1}^{NI} (s - R_n)^{\ell_n}} \qquad \text{.. (2.101)}$$

where

A is the constant multiplier,

Z_i are the zeros, $i = 1,2, \ldots,M$

P_j are the poles, $j = 1,2, \ldots,N$

R_n are the common roots, $n = 1,2, \ldots,NI$

ℓ_i, ℓ_j, ℓ_n and ℓ_n' are the corresponding multiplicities.

It should be noted that even though R_n are the common roots between N(s) and D(s), they are, in general, not the same functions of network parameters.

Let $K_1,K_2, \ldots ,K_n$ be the roots of a polynomial

$$C(s) = s^n + c_1 s^{n-1} + c_2 s^{n-2} + \ldots + c_{n-1} s + c_n$$

The ℓth coefficient is related to the roots as follows

$$c_\ell = (-1)^\ell \sum_{\substack{i,\, i+1,\, ..\, ,i+\ell = 1 \\ i \neq i+1,\, .. \neq i+\ell}} (K_i \, K_{i+1} \cdots K_{i+\ell}) \qquad \text{.. (2.102)}$$

The coefficient of N(s) or D(s) is an implicit function of the network parameters. Let e_k be the network parameter, then it follows from (2.102) that

$$e_k \frac{\partial c_\ell}{\partial e_k} = (-1)^\ell [\sum_{\substack{i,i+1,\ldots,i+\ell=1 \\ i\neq i+1 \,\ldots\, \neq i+\ell}}^{n} (\frac{e_k}{K_i}\frac{\partial K_i}{\partial e_k} + \frac{e_k}{K_{i+1}}\frac{\partial K_{i+1}}{\partial e_k} + \ldots + \frac{e_k}{K_{i+\ell}}\frac{\partial K_{i+\ell}}{\partial e_k})$$

$$(K_i \, K_{i+1} \ldots K_{i+\ell}) \quad \ldots (2.103)$$

If the root sensitivity is defined as

$$\bar{S}^{K_i}_{e_k} = e_k \frac{\partial K_i}{\partial e_k} \quad \ldots (2.104)$$

then it follows from (2.103) that

$$\bar{S}^{c_\ell}_{e_k} = (-1)^\ell \sum_{\substack{i,i+1,\ldots,i+\ell=1 \\ i\neq i+1\neq \ldots \neq i+\ell}}^{n} (K_i \, K_{i+1} \ldots K_{i+\ell}) (\frac{1}{K_i} \bar{S}^{K_i}_{e_k} + \frac{1}{K_{i+1}} \bar{S}^{K_{i+1}}_{e_k} + \ldots$$

$$+ \frac{1}{K_{i+\ell}} \bar{S}^{K_{i+\ell}}_{e_k}) \quad \ldots (2.105)$$

Taking logarithms on both sides of (2.101), we have

$$\ell nF = \ell nA + \sum_{i=1}^{M} \ell_i (s - Z_i) - \sum_{j=1}^{N} \ell_j \ell n(s - P_j) + \sum_{n=1}^{NI} \ell'_n \ell n(s - R_n)$$

$$- \sum_{n=1}^{NI} \ell_n \, \ell n(s - R_n) \quad \ldots (2.106)$$

Considering F as a function of element values, it follows from (2.106) that

$$\sum_{e_k} S^F_{e_k} = \sum_{e_k} \frac{e_k}{F}\frac{\partial F}{\partial e_k} = \sum_{e_k}\frac{e_k}{A}\frac{\partial A}{\partial e_k} - \sum_{i=1}^{M}\frac{\ell_i}{(s-Z_i)}\sum_{e_k}\overline{S}^{Z_i}_{e_k} + \sum_{j=1}^{N}\frac{\ell_j}{(s-P_j)}\sum_{e_k}\overline{S}^{P_j}_{e_k}$$

$$+ \sum_{n=1}^{NI}\frac{\ell_n}{(s-R_n)}\sum_{e_k}\overline{S}^{R_n}_{e_k} - \sum_{n=1}^{NI}\frac{\ell'_n}{(s-R_n)}\sum_{e_k}\overline{S}^{R_n}_{e_k} \quad .. (2.107)$$

$\overline{S}^{R_n}_{e_k}$ in the fourth term of the right hand side of (2.107) is not the same as $\overline{S}^{R_n}_{e_k}$ in the fifth term of the right hand side of (2.107), as R_n in N(s) and D(s) are different functions of the network elements. These functions assume the same values only at nominal element values. To distinguish between the two, let $\overline{S}^{R_n}_{e_k}$ in the fourth term of the right hand side of (2.107) be denoted by $\overline{S}^{P = R_n}_{e_k}$ and $\overline{S}^{R_n}_{e_k}$ in the fifth term of the right hand side of (2.107) by $\overline{S}^{Z=R_n}_{e_k}$. Thus, (2.107) becomes

$$\sum_{e_k} S^F_{e_k} = \sum_{e_k} \frac{e_k}{F}\frac{\partial F}{\partial e_k} = \sum_{e_k}\frac{e_k}{A}\frac{\partial A}{\partial e_k} - \sum_{i=1}^{M}\frac{\ell_i}{(s-Z_i)}\sum_{e_k}\overline{S}^{Z_i}_{e_k} + \sum_{j=1}^{N}\frac{\ell_i}{(s-P_j)}\sum_{e_k}\overline{S}^{P_j}_{e_k}$$

$$+ \sum_{n=1}^{NI}\frac{\ell_n}{(s-R_n)}\sum_{e_k}\overline{S}^{P=R_n}_{e_k} - \sum_{n=1}^{NI}\frac{\ell'_n}{(s-R_n)}\sum_{e_k}\overline{S}^{Z=R_n}_{e_k}$$

$$.. (2.108)$$

Now we will investigate the summed root and coefficient sensitivities for different classes of active lumped networks.

2.6.1 Summed Root and Coefficient Sensitivities for Network N_3:

Consider a network consisting of all the elements included in network N_0 of Section 2.2, excepting the $\overline{RC}$ and $\overline{LC}$-tapered lines. This network will hereafter be referred to as N_3. From (2.53), (2.55) and (2.57), it is seen that the right hand side of (2.108), for an impedance, or an admittance or transfer current or voltage ratio, is identically a constant over the parameter set $\{p_{k1}\} = \{R_i, L_i, D_i, (\alpha_{i1}, \alpha_{i2}), r_i \; \gamma_i\}$. Hence, it follows from (2.108) that

$$\sum_{p_{k1}} \bar{S}^{Z_i}_{p_{k1}} = 0, \qquad \text{for all } i$$

$$\sum_{p_{k1}} \bar{S}^{P_j}_{p_{k1}} = 0, \qquad \text{for all } j$$

and

$$\ell_n \sum_{p_{k1}} \bar{S}^{P=R_n}_{p_{k1}} - \ell'_n \sum_{p_{k1}} \bar{S}^{Z=R_n}_{p_{k1}} = 0, \quad \text{for all } n \qquad .. \ (2.109)$$

For the same network N_3 over the parameter set $\{m_{k4}\} = \{L_i, C_i\}$, it follows from (2.92) that

$$\sum_{m_{k4}} S^F_{m_{k4}} = \frac{s}{F}\frac{\partial F}{\partial s} = S^F_s$$

From (2.106), it is seen that

$$S^F_s = \sum_{i=1}^{M} \frac{s\ell_i}{(s - Z_i)} - \sum_{j=1}^{N} \frac{s\ell_j}{(s - P_j)} + \sum_{n=1}^{NI} \frac{s\ell'_n}{(s - R_n)} - \sum_{n=1}^{NI} \frac{s\ell_n}{(s - R_n)} \ .. \quad (2.110)$$

Thus, from (2.107) and (2.110), we have

$$\sum_{m_{k4}} \bar{S}^{Z_i}_{m_{k4}} = -Z_i, \qquad \text{for all } i$$

$$\sum_{m_{k4}} \bar{S}^{P_j}_{m_{k4}} = -P_j, \qquad \text{for all } j$$

$$\ell_n \sum_{m_{k4}} \bar{S}^{P=R_n}_{m_{k4}} - \ell'_n \sum_{m_{k4}} \bar{S}^{Z=R_n}_{m_{k4}} = -\ell_n R_n + \ell'_n R_n, \quad \text{for all } n \qquad .. (2.111)$$

Now we will investigate the summed coefficient sensitivities for network N_3 over the sets $\{p_{k1}\}$ and $\{m_{k4}\}$. If the network functions does not have any common roots between N(s) and D(s), then it follows from (2.105) and (2.109) for the set $\{p_{k1}\}$ that,

$$\sum_{p_{k1}} \bar{S}^{b_j}_{p_{k1}} = 0, \qquad \text{for } j = 1,2, .. ,m$$

and .. (2.112)

$$\sum_{p_{k1}} \bar{S}^{a_i}_{p_{k1}} = 0, \qquad \text{for } i = 1,2, .. ,n \quad .$$

For the set $\{m_{k4}\}$, it follows from (2.102), (2.105) and (2.111) that

$$\sum_{m_{k4}} \bar{S}^{b_j}_{m_{k4}} = (-1)^j \sum_{\substack{i,i+1,.,i+j \\ i\neq i+1\neq ..\neq i+j}} - (Z_i\, Z_{i+1} .. Z_{i+j})(j)$$

$$= (-j)b_j, \qquad \text{for } j = 1,2, .. ,m \qquad .. (2.113)$$

Similarly

$$\sum_{m_{k4}} \bar{S}^{a_i}_{m_{k4}} = -i\, a_i, \qquad \text{for } i = 1,2, .. ,n \qquad .. (2.114)$$

Thus, the summed coefficient sensitivities of network N_3 over the sets $\{p_{k1}\}$ and $\{m_{k4}\}$ are invariant.

2.6.2 Summed Root and Coefficient Sensitivities for Networks N_4 and N_5:

Consider a network consisting of all the elements included in network N_1 of Section 2.5, excepting the $\overline{RC}$-tapered lines. This network will hereafter be referred to as N_4. It is seen from (2.84), (2.86), (2.89) and (2.91) that the summed sensitivity of the network function over the parameter sets $\{q_{k3}\} = \{R_1\ (\alpha_{i1}, \alpha_{i2}), r_i, \gamma_i\}$, $\{C_i\}$ and $\{q_{k4}\} = \{q_{k3}\} \cup \{C_i\}$ may be written as

$$\sum_{q_{k3}} S^F_{q_{k3}} = S^F_s + \zeta \qquad .. (2.115)$$

$$\sum_{C_i} S^F_{C_i} = s^F_s \qquad .. (2.116)$$

and

$$\sum_{q_{k4}} S^F_{q_{k4}} = 2s^F_s + \zeta \qquad .. (2.117)$$

where ζ is ±1 or 0 depending on whether F is an impedance, admittance or voltage (current) transfer function. Using an approach similar to the one used in Section 2.6.1, it is easily shown that for the network N_4 over the sets $\{q_{k3}\}$, $\{C_i\}$ and $\{q_{k4}\}$, the following relations hold:

$$\sum_{q_{k3}} \overline{S}^{Z_i}_{q_{k3}} = - Z_i, \qquad \text{for all } i$$

$$\sum_{q_{k3}} \overline{S}^{P_j}_{q_{k3}} = - P_j, \qquad \text{for all } j \qquad .. (2.118)$$

$$\ell_n \sum_{q_{k3}} \overline{S}^{P=R_n}_{q_{k3}} - \ell'_n \sum_{q_{k3}} \overline{S}^{Z=R_n}_{q_{k3}} = -\ell_n R_n + \ell'_n R_n, \qquad \text{for all } n$$

$$\sum_{C_i} \overline{S}^{Z_i}_{C_i} = -Z_i, \qquad \text{for all } i$$

$$\sum_{C_i} \overline{S}^{P_j}_{C_i} = -P_j, \qquad \text{for all } j \qquad .. \text{ (2.119)}$$

$$\ell_n \sum_{C_i} \overline{S}^{P=R_n}_{C_i} - \ell'_n \sum_{C_i} \overline{S}^{Z=R_n}_{C_i} = -\ell_n R_n + \ell'_n R_n, \qquad \text{for all } n$$

and

$$\sum_{q_{k4}} \overline{S}^{Z_i}_{q_{k4}} = -2Z_i, \qquad \text{for all } i$$

$$\sum_{q_{k4}} \overline{S}^{P_j}_{q_{k4}} = -2P_j, \qquad \text{for all } j \qquad .. \text{ (2.120)}$$

$$\ell_n \sum_{q_{k4}} \overline{S}^{P=R_n}_{q_{k4}} - \ell'_n \sum_{q_{k4}} \overline{S}^{Z=R_n}_{q_{k4}} = -2\ell_n R_n + 2\ell'_n R_n, \text{ for all } n$$

Now we will investigate the summed coefficient sensitivities. If the network function does not have any common roots in N(s) and D(s), then it is easily shown by following an approach similar to one used in deriving (2.113) and (2.114), that for network N_4 over the sets $\{q_{k3}\}$, $\{C_i\}$ and $\{q_{k4}\}$, the following relations hold:

$$\sum_{q_{k3}} \overline{S}^{b_j}_{q_{k3}} = -jb_j, \qquad \text{for } j = 1,2, .. ,m$$

$$.. \text{ (2.121)}$$

$$\sum_{q_{k3}} \overline{S}^{a_i}_{q_{k3}} = -ia_i, \qquad \text{for } i = 1,2, .. ,n$$

$$\sum_{C_i} \bar{S}^{b_j}_{C_i} = -jb_j, \qquad \text{for } j = 1,2, \ldots ,m$$

$$\sum_{C_i} \bar{S}^{a_i}_{C_i} = -ia_i, \qquad \text{for } i = 1,2, \ldots ,n \qquad \ldots (2.122)$$

and

$$\sum_{q_{k4}} \bar{S}^{b_j}_{q_{k4}} = -2jb_j, \qquad \text{for } j = 1,2, \ldots ,m$$

$$\sum_{q_{k4}} \bar{S}^{a_i}_{q_{k4}} = -2ia_i, \qquad \text{for } i = 1,2, \ldots ,n \qquad \ldots (2.123)$$

Finally, it can be shown that for the network N_5, (N_5 is defined as a network consisting of lumped inductors, capacitors, VVTs, CCTS and NICs), the following invariant relations hold.

(i) over the set $\{L_i\}$ and $\{C_i\}$

$$\sum_{C_i} \bar{S}^{Z_i}_{C_i} = \sum_{L_i} \bar{S}^{Z_i}_{L_i} = -\frac{1}{2} Z_i, \qquad \text{for all } i$$

$$\sum_{C_i} \bar{S}^{P_j}_{C_i} = \sum_{L_i} \bar{S}^{P_j}_{L_i} = -\frac{1}{2} P_j, \qquad \text{for all } j$$

$$\ell_n \sum_{C_i} \bar{S}^{P=R_n}_{C_i} - \ell'_n \sum_{C_i} \bar{S}^{Z=R_n}_{C_i} = \ell_n \sum_{L_i} \bar{S}^{P=R_n}_{L_i} - \ell'_n \sum_{L_i} \bar{S}^{Z=R_n}_{L_i} = \frac{1}{2}(-\ell_n R_n + \ell'_n R_n),$$

for all n

(ii) over the set $\{m_{k4}\} = \{L_i, C_i\}$

$$\sum_{m_{k4}} \overline{S}^{Z_i}_{m_{k4}} = -Z_i, \qquad \text{for all } i$$

$$\sum_{m_{k4}} \overline{S}^{P_j}_{m_{k4}} = -P_j, \qquad \text{for all } j$$

$$\ell_n \sum_{m_{k4}} \overline{S}^{P=R_n}_{m_{k4}} - \ell'_n \sum_{m_{k4}} \overline{S}^{Z=R_n}_{m_{k4}} = -\ell_n R_n + \ell'_n R_n, \qquad \text{for all } n$$

If the network function does not have any common roots between N(s) and D(s), then the following invariant relations for the coefficient sensitivities hold;

(i) over the set $\{L_i\}$ and $\{C_i\}$

$$\sum_{C_i} \overline{S}^{b_j}_{C_i} = \sum_{L_i} \overline{S}^{b_j}_{L_i} = -\frac{1}{2} j b_j, \qquad \text{for } j = 1,2, \ldots ,m$$

$$\sum_{C_i} \overline{S}^{a_i} = \sum_{L_i} \overline{S}^{a_i}_{L_i} = -\frac{1}{2} i a_i, \qquad \text{for } i = 1,2, \ldots ,n$$

(ii) over the set $\{m_{k4}\} = \{L_i, C_i\}$

$$\sum_{m_{k4}} \overline{S}^{b_j}_{m_{k4}} = - j b_j, \qquad \text{for } j = 1,2, \ldots ,m$$

$$\sum_{m_{k4}} \overline{S}^{a_i}_{m_{k4}} = - i a_i, \qquad \text{for } i = 1,2, \ldots ,n$$

2.7 Q and ω_0 Sensitivity Invariants for Active Lumped Networks:

It has been shown in(23) that from the sensitivity point of view, it is definitely better to realize a higher order function by cascading second order networks. We will now investigate the summed sensitivities of pole Q and ω_0 (the undamped natural frequency). For RC-operational amplifier networks, summed pole Q sensitivity and summed ω_0 sensitivity invariances have been established in(44). We will extend these invariances for different classes of active lumped networks over different sets of parameters.

Consider a second order transfer function

$$T(s) = \frac{N(s)}{s^2 + a_1 s + a_2}$$

Depending on the nature of N(s), T(s) will exhibit different types of characteristics, such as low-pass, band pass etc.

Now, the Q of the network is given by

$$Q = \sqrt{a_2}/a_1 \qquad .. (2.124)$$

and undamped natural frequency ω_0 by

$$\omega_0 = \sqrt{a_2} \qquad .. (2.125)$$

From (2.124) and (2.125), it is clear that Q and ω_0 sensitivities are related to coefficient sensitivities as follows:

$$\bar{S}^{Q}_{e_k} = e_k \frac{\partial Q}{\partial e_k} = [\frac{1}{2a_1\sqrt{a_2}} \; e_k \frac{\partial a_2}{\partial e_k} - \frac{\sqrt{a_2}}{{a_1}^2} e_k \frac{\partial a_1}{\partial e_k}]$$

$$= Q[\frac{1}{2a_2} \bar{S}^{a_2}_{e_k} - \frac{1}{a_1} \bar{S}^{a_1}_{e_k}] \qquad \text{.. (2.126)}$$

and

$$\bar{S}^{\omega_0}_{e_k} = e_k \frac{\partial \omega_0}{\partial e_k} = \frac{1}{2\sqrt{a_2}} e_k \frac{\partial a_2}{\partial e_k} = \frac{1}{2\omega_o} \bar{S}^{a_2}_{e_k} \qquad \text{.. (2.127)}$$

Now, the summed Q and summed ω_0 sensitivities will be investigated for different classes of active lumped networks.

2.7.1 Summed Q and ω_0 Sensitivities for Network N_3:

Consider the network N_3 defined in subsection 2.6.1. For the sets $\{p_{k1}\}$ and $\{m_{k4}\}$, it follows from (2.112), (2.114) and (2.126) that

$$\sum_{p_{k1}} \bar{S}^{Q}_{p_{k1}} = 0$$

and .. (2.128)

$$\sum_{m_{k4}} \bar{S}^{Q}_{m_{k4}} = 0$$

Similarly, it can be shown from (2.112), (2.114) and (2.127) that

$$\sum_{p_{k1}} \bar{S}^{\omega_0}_{p_{k1}} = 0$$

and .. (2.129)

$$\sum_{m_{k4}} \bar{S}^{\omega_0}_{m_{k4}} = -\omega_0$$

Thus, the summed Q and ω_0 sensitivities for network N_3 over sets $\{p_{k1}\}$ and $\{m_{k4}\}$ are invariant.

2.7.2 Summed Q and ω_0 Sensitivities for Networks N_4 and N_5:

Consider the network N_4 defined in Section 2.6.2. Let us examine the summed Q sensitivities of network N_4 over the sets $\{q_{k3}\}$, $\{C_i\}$ and $\{q_{k4}\}$. From (2.121), (2.122), (2.123) and (2.126), it follows that

$$\begin{aligned}
&\sum_{q_{k3}} \overline{S}^{Q}_{q_{k3}} = 0 \\
&\sum_{C_i} \overline{S}^{Q}_{C_i} = 0 \qquad\qquad .. (2.130) \\
&\sum_{q_{k4}} \overline{S}^{Q}_{q_{k4}} = 0
\end{aligned}$$

For summed ω_0 sensitivity, it follows from (2.121), (2.122), (2.123) and (2.127), that

$$\begin{aligned}
&\sum_{q_{k3}} \overline{S}^{\omega_0}_{q_{k3}} = -\omega_0 \\
&\sum_{C_i} \overline{S}^{\omega_0}_{C_i} = -\omega_0 \qquad\qquad .. (2.131) \\
&\sum_{q_{k4}} \overline{S}^{\omega_0}_{q_{k4}} = -2\omega_0
\end{aligned}$$

Thus, the summed Q and ω_0 sensitivities are invariant over the sets $\{q_{k3}\}$, $\{C_i\}$ and $\{q_{k4}\}$

Similarly, for the network N_5, it can be shown that

(i) for summed Q sensitivity

$$\sum_{C_i} \overline{S}^{Q}_{Ci} = \sum_{L_i} \overline{S}^{Q}_{Li} = 0$$

and .. (2.132)

$$\sum_{m_{k4}} \overline{S}^{Q}_{m_{k4}} = 0$$

(ii) for summed ω_0 sensitivity

$$\sum_{C_i} \overline{S}^{\omega_0}_{C_i} = \sum_{L_i} \overline{S}^{\omega_0}_{L_i} = - \frac{\omega_0}{2}$$

and .. (2.133)

$$\sum_{m_{k4}} \overline{S}^{\omega_0}_{m_{k4}} = -\omega_o$$

Thus, the summed Q and ω_0 sensitivities are invariant over the sets $\{L_i\}$, $\{C_i\}$ and $\{m_{k4}\}$.

2.8 Second Order Sensitivity Invariants:

In this Section, we establish the invariant nature of the sum of the second order sensitivities for the different network functions, for the classes of networks considered in the previous Sections(35).

It has been shown in Section 2.4, that for the network N_0, we have over the set $\{p_k\}$,

$$Z = \sum_{k=1}^{n} p_k \frac{\partial Z}{\partial p_k} \qquad .. (2.134)$$

In writing the above expression, it is assumed that the set $\{p_k\}$ consists of n elements p_k, $k = 1,2, \dots ,n$.

From (2.134)

$$\sum_{i=1}^{n} p_i \frac{\partial Z}{\partial p_i} = \sum_{i=1}^{n} \sum_{k=1}^{n} p_i p_k \frac{\partial^2 Z}{\partial p_i \partial p_k} + \sum_{i=1}^{n} \sum_{k=1}^{n} p_i \frac{\partial}{\partial p_i}(p_k) \frac{\partial Z}{\partial p_k} \qquad .. (2.135)$$

Since

$$\frac{\partial p_k}{\partial p_i} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$$

it is readily seen that the second term on the right hand side of (2.135) is equal to $\sum_{i=1}^{n} p_i \frac{\partial Z}{\partial p_k}$. Thus, it follows from (2.135) that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} p_i p_k \frac{\partial^2 Z}{\partial p_i \partial p_k} = 0 \qquad .. (2.136)$$

That is, the sum of second order sensitivities of the impedance matrix Z of the network N_0 is invariant over the set $\{p_k\}$.

It can similarly be shown that for the admittance matrix

$$\sum_{i=1}^{n} \sum_{k=1}^{n} p_i p_k \frac{\partial^2 Y}{\partial p_i \partial p_k} = 2Y \qquad .. (2.137)$$

If T is a current or voltage transfer function, then

$$\sum_{i=1}^{n} \sum_{k=1}^{n} p_i p_k \frac{\partial^2 T}{\partial p_i \partial p_k} = 0 \qquad .. (2.138)$$

Following the same procedure, it can also be shown that for a scattering matrix

$$\sum_{i=1}^{n} \sum_{i=1}^{n} p_i p_k \frac{\partial^2 S}{\partial p_i \partial p_k} = -\frac{1}{2} (U + S)(U - S^2) \qquad .. (2.139)$$

and

$$\sum_{p_i' \in \{p_k'\}} \sum_{p_k' \in \{p_k'\}} p_i' p_k' \frac{\partial^2 S}{\partial p_i' \partial p_k'} = 0 \qquad .. (2.140)$$

where $\{p_k'\} = \{p_k\} \cup \{p_i\}$, includes the set of normalising resistors.

Consider next the network N_1 over the set $\{q_k\}$. Let the set $\{q_k\}$ consists of n elements q_k, $k = 1,2,..,n$. It has been shown in Section 2.5, that for N_1 over the set $\{q_k\}$

$$\sum_{k=1}^{n} q_k \frac{\partial Z}{\partial q_k} = 2s \frac{\partial Z}{\partial s} + Z \qquad .. (2.141)$$

Following the same procedure as outlined earlier, the following can be proved.

$$\sum_{i=1}^{n} \sum_{k=1}^{n} q_i q_k \frac{\partial^2 Z}{\partial q_i \partial q_k} = 6s \frac{\partial Z}{\partial s} + 4s^2 \frac{\partial^2 Z}{\partial s^2} \qquad .. (2.142)$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} q_i q_k \frac{\partial^2 Y}{q_i \; q_k} = 4s^2 \frac{\partial^2 Y}{\partial s^2} - 2s \frac{\partial Y}{\partial s} + 2Y \qquad .. (2.143)$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} q_i q_k \frac{\partial^2 T}{\partial q_i \partial q_k} = 2s \frac{\partial T}{\partial s} + 4s^2 \frac{\partial^2 T}{\partial s^2} \qquad .. (2.144)$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} q_i q_k \frac{\partial^2 S}{\partial q_i \partial q_k} = 4s^2 \frac{\partial^2 S}{\partial s^2} + 2s \frac{\partial S}{\partial s} - 4sS \frac{\partial S}{\partial s} - \frac{1}{2}(U-S^2) - \frac{S}{2}(U-S^2) \qquad .. (2.145)$$

$$\sum_{q_i' \in \{q_k'\}} \sum_{q_k' \in \{q_k'\}} q_i' q_k' \frac{\partial^2 S}{\partial q_i' \partial q_k'} = 4s^2 \frac{\partial^2 S}{\partial s^2} + 2s \frac{\partial S}{\partial s} \qquad .. (2.146)$$

A similar expression can be obtained for the summed second order sensitivity for lossless networks, the expression being

$$\sum_{i=1}^{n} \sum_{k=1}^{n} m_i m_k \frac{\partial^2 F}{\partial m_i \partial m_k} = \frac{s^2 \partial^2 F}{\partial s^2} \quad .. (2.147)$$

where m_i and m_k are the members of the set $\{m_k\}$ and F is an impedance or admittance parameter or voltage or current transfer function. The same expression holds good even if F is a Z or Y matrix.

We now give a general result which will help in obtaining the sum of the (k+1)th order sensitivities of a network function from the sum of kth order sensitivities.

Consider any network function $F(p_1, p_2. .. ,p_n)$. Let X_k be the sum of kth order sensitivities and X_{k+1}, the sum of (k+1)th order sensitivities, then

$$X_k = \sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} p_{i_1} p_{i_2} \cdots p_{i_k} \frac{\partial^k F}{\partial p_{i_1} \partial p_{i_2} \cdots \partial p_{i_k}} \quad .. (2.148)$$

and

$$X_{k+1} = \sum_{i_{k+1}=1}^{n} \sum_{i_k=1}^{n} \cdots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} p_{i_1} p_{i_2} \cdots p_{i_{k+1}} \frac{\partial^{k+1} F}{\partial p_{i_1} \partial p_{i_2} \cdots \partial p_{i_{k+1}}} \quad .. (2.149)$$

From (2.148)

$$\sum_{i_{k+1}=1}^{n} p_{i_{k+1}} \frac{\partial X_k}{\partial p_{i_{k+1}}} = X_{k+1} + \sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} \frac{\partial^k F}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_k}} \cdot$$

$$\sum_{i_{k+1}=1}^{n} p_{i_{k+1}} \frac{\partial (p_{i_1} p_{i_2} \cdots p_{i_k})}{\partial p_{i_{k+1}}} \quad .. (2.150)$$

In the second term of right hand side of (2.150), k parameters are taken at a time, but some of the parameters may occur more than once. Let a typical term of $(p_{i_1} \, p_{i_2} \, .. \, p_{i_k})$ be

$$p_{i_1}^{x_{i_1}} \, p_{i_2}^{x_{i_2}} \, ... \, p_{i_k}^{x_{i_k}} \triangleq A \qquad i_j \, \varepsilon \, (1,2, \, ... \, ,n) \qquad .. \, (2.151)$$

It should be noted that $\sum_{j=1}^{k} x_{i_j} = k$

Then,

$$p_{i_j} \frac{\partial}{\partial p_{i_j}}(A) = \begin{cases} 0, & \text{if } x_{i_j} = 0 \\ x_{i_j} A, & \text{if } x_{i_j} \neq 0 \end{cases} \qquad .. \, (2.152)$$

Hence

$$\sum_{j=1}^{n} p_{i_j} \frac{\partial (A)}{\partial p_{i_j}} = A \sum_{j=1}^{k} x_{i_j} = kA \qquad .. \, (2.153)$$

Using (2.153) in (2.150), we have

$$\sum_{i_{k+1}=1}^{n} p_{i_{k+1}} \frac{\partial X_k}{\partial p_{i_{k+1}}} = X_{k+1} + kX_k$$

or

$$X_{k+1} = \sum_{i=1}^{n} p_i \frac{\partial X_k}{\partial p_i} - kX_k \qquad .. \, (2.154)$$

Eq. (2.154) expresses (k+1)th order summed sensitivity in terms of summed kth order sensitivity.

Using (2.154), higher order sensitivity invariants may be established for root, coefficient, Q and ω_0 sensitivities for the different classes of networks, N_3, N_4 and N_5, considered in Sections 2.6 and 2.7.

2.9 Conclusions:

Tellegen's theorem has been used in conjunction with the concept of the adjoint or the transpose of a network in establishing a number of results concerning the sensitivities of network functions of a general linear time-invariant network. In particular, the following results are established.

(1) Expressions for the first order sensitivities have been obtained, in terms of the immittance parameters of an augmented network, with a view to easily compute higher order sensitivities.

(2) For the network N_0

  (a) The sum of the sensitivities of Z_{ij}, Y_{ij}, T_v and T_c over the set $\{p_k\}$ is ±1 or 0, and hence an invariant.

  (b) The sum of the sensitivities of S over the set $\{p_k\}$ depends only on S and not on the realization, while this sum over the set $\{p_k'\}$ is zero and hence an invariant.

  (c) The sum of sensitivities of Z_{ij}, Y_{ij}, T_v, T_c and S matrix over the set $\{m_k\}$ is related to corresponding sensitivities with respect to complex frequency s, and hence is independent of the realization.

(3) For the network N_1, the sum of the sensitivities of Z_{ij}, Y_{ij}, T_v and T_c over the set $\{q_k\}$ is related to corresponding sensitivities with respect to complex frequency s, and hence is independent of the realization; the same result has been shown to be true for the S matrix of network N_1 over the set $\{q_k\}$ or $\{q_k'\}$.

(4) Invariant nature for the sum of root sensitivities and coefficient sensitivities has been established for the networks N_3, N_4 and N_5.

(5) Invariant nature for the sum of the Q and ω_0 sensitivities have also been established for a second order transfer functions realized by networks N_3, N_4 or N_5.

(6) Invariant nature for the sum of the higher order sensitivities has also been established.

The first order invariant relations obtained in this Chapter are summarized in Tables 2.2 and 2.3.

Many of these relations will be used in Chapter 4, to obtain lower bounds on the Schoeffler's quadratic performance index(13).

TABLE 2.2

SUMMED SENSITIVITY INVARIANTS OF NETWORK FUNCTIONS FOR DIFFERENT NETWORKS

Network	Parameter Set $\{x_k\}$	$\sum_{x_k} \frac{x_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial x_k}$	$\sum_{x_k} \frac{x_k}{Y_{ij}} \frac{\partial Y_{ij}}{\partial x_k}$	$\sum_{x_k} \frac{x_k}{T} \frac{\partial T}{\partial x_k}$	$\sum_{x_k} x_k \frac{\partial S}{\partial x_k}$
N_0	$\{p_k\}$	1	-1	0	$\frac{1}{2}(U - S^2)$
	$\{m_k\}$	$\frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$\frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$\frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$s \frac{\partial S}{\partial s}$
N_1	$\{q_{k1}\}$	$1 + \frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$-1 + \frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s}$	$\frac{s}{T} \frac{\partial T}{\partial s}$	$s \frac{\partial S}{\partial s} + \frac{1}{2}(U - S^2)$
	$\{q_{k2}\}$	$\frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$\frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s}$	$\frac{s}{T} \frac{\partial T}{\partial s}$	$s \frac{\partial S}{\partial s}$
	$\{q_k\}$	$1 + \frac{2s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$-1 + \frac{2s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s}$	$2 \frac{s}{T} \frac{\partial T}{\partial s}$	$2s \frac{\partial S}{\partial s} + \frac{1}{2}(U - S^2)$
N_2	$\{m_{k1}\}$	$\frac{1}{2}(1 + \frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s})$	$\frac{1}{2}(-1 + \frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s})$	$\frac{1}{2} \frac{s}{T} \frac{\partial T}{\partial s}$	$\frac{1}{2}s \frac{\partial S}{\partial s} + \frac{1}{4}(U - S^2)$
	$\{m_{k2}\}$	$\frac{1}{2}(-1 + \frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s})$	$\frac{1}{2}(1 + \frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s})$	$\frac{1}{2} \frac{s}{T} \frac{\partial T}{\partial s}$	$\frac{1}{2}s \frac{\partial S}{\partial s} - \frac{1}{4}(U - S^2)$
	$\{m_{k3}\}$	$\frac{s}{Z_{ij}} \frac{\partial Z_{ij}}{\partial s}$	$\frac{s}{Y_{ij}} \frac{\partial Y_{ij}}{\partial s}$	$\frac{s}{T} \frac{\partial T}{\partial s}$	$s \frac{\partial S}{\partial s}$

T is a voltage or current transfer function

TABLE 2.3

SUMMED SENSITIVITY INVARIANTS FOR ROOTS, COEFFICIENTS, Q AND ω_0 FOR DIFFERENT NETWORKS

Network	Parameter Set $\{x_k\}$	$\sum_{x_k} \bar{S}^{s_i}_{x_k}$	$\sum_{x_k} \bar{S}^{c_i}_{x_k}$	$\sum_{x_k} \bar{S}^{Q}_{x_k}$	$\sum_{x_k} \bar{S}^{\omega_0}_{x_k}$
N_3	$\{p_{k1}\}$	0	0	0	0
	$\{m_{k4}\}$	$-s_i$	$-ic_i$	0	0
N_4	$\{q_{k3}\}$	$-s_i$	$-ic_i$	0	$-\omega_0$
	$\{C_i\}$	$-s_i$	$-ic_i$	0	$-\omega_0$
	$\{q_{k4}\}$	$-2s_i$	$-2ic_i$	0	$-2\omega_0$
N_5	$\{L_i\}$	$-\frac{1}{2}s_i$	$-\frac{1}{2}ic_i$	0	$-(\omega_0/2)$
	$\{C_i\}$	$-\frac{1}{2}s_i$	$-\frac{1}{2}ic_i$	0	$-(\omega_o/2)$
	$\{m_{k4}\}$	$-s_i$	$-ic_i$	0	$-\omega_0$

where

s_i may be a pole or zero,

c_i may be the coefficient of N(s) or D(s).

CHAPTER 3

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR NETWORK FUNCTIONS.

3.1 Introduction:

In the previous Chapter, it was established that the sum of sensitivities of network functions for different classes of networks over different sets of parameters is invariant. In this chapter, the sum of the magnitudes of the sensitivities for network functions for different types of passive networks is considered. Smith(28) has obtained using Vratsano's theorem, bounds on the sum of the magnitudes of the sensitivities for driving point impedance. Here, we use the adjoint network approach to establish bounds on the sum of the magnitudes of the sensitivities for transfer immittances and transfer voltage and current ratios(45).

In computer aided design of networks, a commonly used measure of performance is the Schoeffler's quadratic sensitivity index(13). It will be shown later in Chapter 4, that the bounds established in this chapter, will be useful in establishing upper bounds on Schoeffler's quadratic sensitivity index.

We will first derive some results, which will be used in establishing bounds on the sum of element sensitivity magnitudes for different network functions.

Consider a general two-port network N_0. Let Z_{12} be the transfer impedance between ports (1) and (2) of the network. Let the 2nd port of the network and 1st port of the adjoint network be excited by a current source of value I_0, the other ports being kept open; then it follows from Eq.(2.1) that

$$\Delta Z_{12} I_0^2 = \sum_e (\Delta V_e \phi_e - \Delta I_e \psi_e) \qquad ..(3.1)$$

where all the quantities in (3.1) are defined in Chapter 2.

Let us consider an internal one-port element having an impedance Z_i. Let there be a small change in Z_i. Then,

$$V_i = Z_i I_i, \qquad \psi_i = Z_i \phi_i \qquad ..(3.2)$$

and

$$\Delta V_i = \Delta Z_i I_i + Z_i \Delta I_i$$

From (3.1) and (3.2), it follows that

$$\Delta Z_{12} I_0^2 = \phi_i \Delta Z_i I_i$$

or

$$\frac{\Delta Z_{12}}{\Delta Z_i} = \frac{I_i \phi_i}{I_0^2} \qquad ..(3.3)$$

Letting $\Delta Z_i \to 0$, we have

$$\frac{\partial Z_{12}}{\partial Z_i} = \frac{I_i \phi_i}{I_0^2} \qquad ..(3.4)$$

Similarly, for a transfer admittance Y_{12} between ports (1) and (2) of a general two-port network N_0, it can be easily be shown that,

$$\frac{\partial Y_{12}}{\partial Z_i} = \frac{-I_i \phi_i}{V_0^2} \qquad ..(3.5)$$

where I_i = the current through the element Z_i when port (2) of the network is excited with a voltage source of value V_0 and port (1) is short-circuited, and

ϕ_i = is the current through the element Z_i when port (1) of the adjoint network is excited with a voltage source of value V_0, port (2) being short-circuited.

For a voltage transfer function, it can be shown that

$$\frac{\partial T_v}{\partial Z_i} = \frac{I_i \phi_i}{V_0 I_0} \qquad ..(3.6)$$

where I_i = the current through the element Z_i when port (1) of the network is excited with a voltage source of value V_0, port (2) being kept open,

ϕ_i = the current through the element Z_i, when port (2) of the adjoint network is excited with a current source of value I_0 and port (1) is short-circuited.

3.2 Bounds on the sum of Element sensitivity magnitudes for transfer impedances:

In this Section, we will establish bounds on the element sensitivity magnitudes of a transfer impedance for different types of passive non-reciprocal and reciprocal networks.

It follows from (3.4) that

$$|S_{Z_i}^{Z_{12}}| = \frac{|I_i||\phi_i|}{|I_0|^2} \frac{|Z_i|}{|Z_{12}|} \qquad .. (3.7)$$

where

$S_{Z_i}^{Z_{12}}$ is the sensitivity of Z_{12} with respect to Z_i,

I_i is the current through Z_i when port (2) of N_0 is excited by a current source of value I_0, with port (1) open, and

ϕ_i is the current through Z_i when port (1) of N_0^A is excited by a current source of value I_0, with port (2) open.

3.2.1 RLC-Gyrator Network:

Let us consider a network consisting of resistors, capacitors, inductors and passive gyrators. We shall denote such a network by N_6 and its adjoint by N_6^A.

If Z_i corresponds to a resistor R, then from (3.7), we get

$$|S_R^{Z_{12}}| = \frac{|I_R||\phi_R|}{|I_0|^2} \frac{R}{|Z_{12}|}$$

$$= \frac{\sqrt{w_R \, w_R^A}}{|I_0|^2 |Z_{12}|} \qquad .. (3.8)$$

where $w_R = |I_R|^2R$ and $w_R^A = |\phi_R|^2R$ denote the average power dissipated in the resistor R, with port conditions as described earlier. It may be noted that I_0 refers to the effective value of I_0.

If Z_i represents an inductor L or a capacitor C, then from (3.7), we get the following

$$|S_L^{Z_{12}}| = \frac{2\omega\sqrt{w_L w_L^A}}{|I_0|^2 |z_{12}|} \quad .. (3.9)$$

and

$$|S_C^{Z_{12}}| = \frac{2\omega\sqrt{w_C w_C^A}}{|I_0|^2 |z_{12}|} \quad .. (3.10)$$

where

$$w_L = \frac{1}{2} L \; |I_L|^2 \qquad \text{and} \qquad w_L^A = \frac{1}{2} L|\phi_L|^2$$

denote the average energy stored in the inductor in the network N_6 and N_6^A, while

$$w_C = \frac{\frac{1}{2}|I_C|^2}{\omega^2 C} \qquad \text{and} \qquad w_C^A = \frac{\frac{1}{2}|\phi_C|^2}{\omega^2 C}$$

denote the average energy stored in the capacitor in the networks N_6 and N_6^A.

Let P_R (P_R^A) denote the total average power dissipated in $N_6(N_6^A)$. Then

$$\begin{aligned} P_R &= \sum_R w_R \\ P_R^A &= \sum_R w_R^A \end{aligned} \quad .. (3.11)$$

Let at frequency ω

$$Z_{22} = R_{22} + jX_{22}$$

and

$$Z^A_{11} = R^A_{11} + jX^A_{11} \qquad .. (3.12)$$

where

$Z_{22}(Z^A_{11})$ is the open-circuit driving point impedance at port 2(1) of $N_6(N^A_6)$.

From Schwartz inequality

$$\sum_i \sqrt{X_i Y_i} \le \sqrt{\sum_i X_i \sum_i Y_i} \qquad .. (3.13)$$

Using (3.8), (3.11), (3.12) and (3.13), we get the following

$$\sum_R |S_R^{Z_{12}}| = \frac{\sum_R \sqrt{w_R w_R^A}}{|I_0|^2 |Z_{12}|} \le \frac{\sqrt{P_R P_R^A}}{|I_0|^2 |Z_{12}|}$$

$$= \frac{\sqrt{R_{22} R^A_{11}}}{|Z_{12}|} = \frac{\sqrt{R_{22} R_{11}}}{|Z_{12}|} \qquad .. (3.14)$$

The last step in (3.14) follows from the relation $Z^A_{11} = Z_{11}$.

Let

$$W_C = \sum_C w_C \quad ; \quad W^A_C = \sum_C w^A_C$$

$$W_L = \sum_L w_L \quad ; \quad W^A_L = \sum_L w^A_L \qquad .. (3.15)$$

$$W = W_C + W_L \quad \text{and} \quad W^A = W^A_C + W^A_L$$

Using (3.9), (3.10), (3.13) and (3.15), we get the following

$$\sum_L |S_L^{Z_{12}}| + \sum_C |S_C^{Z_{12}}| = \frac{2\omega \sum_L \sqrt{w_L w_L^A} + \sum_C \sqrt{w_C w_C^A}}{|I_0|^2 |Z_{12}|} \le \frac{2\omega \sqrt{WW^A}}{|I_0|^2 |Z_{12}|} \qquad .. (3.16)$$

In Chapter 2, for network under consideration, it has been shown that

$$\sum_L S_L^F + \sum_C S_C^F = S_\omega^F \qquad .. (3.17)$$

where F is any network function. Hence, we get

$$\sum_L |S_L^F| + \sum_C |S_C^F| \ge |S_\omega^F| \qquad .. (3.18)$$

From (3.16) and (3.18), we get the following

$$|S_\omega^{Z_{12}}| \le \sum_L |S_L^{Z_{12}}| + \sum_C |S_C^{Z_{12}}| \le \frac{2\omega \sqrt{WW^A}}{|I_0|^2 |Z_{12}|} \qquad .. (3.19)$$

3.2.2 RC-Networks:

Let us now consider an RC-network. Then, it follows from the results of Chapter 2, that

$$\sum_R S_R^{Z_{12}} = 1 + S_\omega^{Z_{12}} \qquad .. (3.20a)$$

$$\sum_C S_C^{Z_{12}} = S_\omega^{Z_{12}} \qquad .. (3.20b)$$

$$\sum_R S_R^{Z_{12}} + \sum_C S_C^{Z_{12}} = 1 + 2S_\omega^{Z_{12}} \qquad .. (3.20c)$$

Hence, on taking the magnitudes of both sides of (3.20), we get

$$\sum_R |S_R^{Z_{12}}| \geq |1 + S_\omega^{Z_{12}}| \qquad \text{.. (3.21a)}$$

$$\sum_C |S_C^{Z_{12}}| \geq |S_\omega^{Z_{12}}| \qquad \text{.. (3.21b)}$$

$$\sum_R |S_R^{Z_{12}}| + \sum_C |S_C^{Z_{12}}| \geq |1 + 2\, S_\omega^{Z_{12}}| \qquad \text{.. (3.21c)}$$

Further, from (3.10) and Schwartz inequality

$$\sum_C |S_C^{Z_{12}}| \leq \frac{2\omega\sqrt{W_C W_C^A}}{|I_0|^2 |Z_{12}|} \qquad \text{.. (3.22)}$$

For the network under consideration

$$W_C = \frac{|I_0|^2 |X_{22}|}{2\omega} \qquad \text{.. (3.23)}$$

$$W_C^A = \frac{|I_0|^2 |X_{11}|}{2\omega} \qquad \text{.. (3.24)}$$

Thus, it follows from (3.22), (3.23) and (3.24) that

$$\sum_C |S_C^{Z_{12}}| \leq \frac{\sqrt{|X_{11}|\,|X_{22}|}}{|Z_{12}|} \qquad \text{.. (3.25)}$$

Thus from (3.14), (3.21) and (3.25), we obtain the following inequalities

$$|1 + S_\omega^{Z_{12}}| \leq \sum_R |S_R^{Z_{12}}| \leq \frac{\sqrt{R_{11} R_{22}}}{|Z_{12}|} \qquad \text{.. (3.26a)}$$

$$|S_\omega^{Z_{12}}| \leq \sum_C |S_C^{Z_{12}}| \leq \frac{\sqrt{|X_{11}|\,|X_{22}|}}{|Z_{12}|} \qquad \text{.. (3.26b)}$$

$$|1 + 2S_{\omega}^{Z_{12}}| \leq \sum_{R}|S_{R}^{Z_{12}}| + \sum_{C}|S_{C}^{Z_{12}}| \leq \frac{\sqrt{R_{11}R_{22}} + \sqrt{|X_{11}\ \ X_{22}|}}{|Z_{12}|} \qquad .. \ (3.26c)$$

Inequalities similar to (3.26) can easily be obtained for an RL-network.

3.2.3 LC-Networks:

Let the network consist of only inductors and capacitors. It is known that(46)

$$W = W_C + W_L = \frac{1}{2}\,|I_0|^2\,\frac{\partial X_{22}}{\partial\omega}$$

and .. (3.27)

$$W^A = W_C^A + W_L^A = \frac{1}{2}|I_0|^2\,\frac{\partial X_{11}}{\partial\omega}$$

Hence, from (3.19) and (3.27), we get the following inequality

$$\sum_{L}|S_{L}^{Z_{12}}| + \sum_{C}|S_{C}^{Z_{12}}| \leq \frac{\omega}{|Z_{12}|}\sqrt{\frac{\partial X_{22}}{\partial\omega}\,\frac{\partial X_{11}}{\partial\omega}} = \frac{\sqrt{|S_{\omega}^{Z_{22}}||S_{\omega}^{Z_{11}}||Z_{11}||Z_{22}|}}{|Z_{12}|} \qquad .. \ (3.28)$$

Thus, from (3.18) and (3.28), for the network under consideration, we get the following inequality

$$|S_{\omega}^{Z_{12}}| \leq \sum_{L}|S_{L}^{Z_{12}}| + \sum_{C}|S_{C}^{Z_{12}}|$$

$$\leq \frac{\sqrt{|S_{\omega}^{Z_{22}}||S_{\omega}^{Z_{11}}||Z_{11}||Z_{22}|}}{|Z_{12}|} \qquad .. \ (3.29)$$

This completes our discussion of the bounds on the sum of magnitudes of sensitivities of transfer impedances. The results of these discussions are summarized in Table 3.1.

3.3 Bounds on the Sum of Element Sensitivity Magnitudes for Transfer Admittance:

In this Section, we will establish bounds on the element sensitivity magnitudes of a transfer admittance for different types of passive non-reciprocal and reciprocal networks.

It follows from (3.5) that

$$|S_{Z_i}^{Y_{12}}| = \frac{|I_i||\phi_i|}{|V_0|^2} \frac{|Z_i|}{|Y_{12}|} \qquad .. (3.30)$$

where

$S_{Z_i}^{Y_{12}}$ is the sensitivity of Y_{12} with respect to Z_i,

I_i is the current through the element Z_i, when port (2) of the network N_0 is excited with a voltage source of value V_0, port (1) being short-circuited,

ϕ_i is the current through the element Z_i when port (1) of N_0^A is excited by voltage source of value V_0, port (2) being short-circuited.

3.3.1 RLC-Gyrator Network:

Let us now consider the network N_6. Let the quantities w_R,

TABLE 3.1

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR Z_{12}

Network Type	Bounds
RLC-gyrator	(i) $\sum_R \lvert S_R^{Z_{12}} \rvert \le \frac{\sqrt{R_{22}R_{11}}}{\lvert Z_{12} \rvert}$ (ii) $\lvert S_\omega^{Z_{12}} \rvert \le \sum_L \lvert S_L^{Z_{12}} \rvert + \sum_C \lvert S_C^{Z_{12}} \rvert \le \frac{2\omega\sqrt{WW^A}}{\lvert I_0 \rvert^2 \lvert Z_{12} \rvert}$
RC-network	$\lvert 1 + S_\omega^{Z_{12}} \rvert \le \sum_R \lvert S_R^{Z_{12}} \rvert \le \frac{\sqrt{R_{11}R_{22}}}{\lvert Z_{12} \rvert}$ $\lvert S_\omega^{Z_{12}} \rvert \le \sum_C \lvert S_C^{Z_{12}} \rvert \le \frac{\sqrt{\lvert X_{11} \rvert \lvert X_{22} \rvert}}{\lvert Z_{12} \rvert}$ $\lvert 1 + 2S_\omega^{Z_{12}} \rvert \le \sum_R \lvert S_R^{Z_{12}} \rvert + \sum_C \lvert S_C^{Z_{12}} \rvert \le \frac{\sqrt{R_{11}R_{22}} + \sqrt{\lvert X_{11} \rvert \lvert X_{22} \rvert}}{\lvert Z_{12} \rvert}$
LC-network	$\lvert S_\omega^{Z_{12}} \rvert \le \sum_L \lvert S_L^{Z_{12}} \rvert + \sum_C \lvert S_C^{Z_{12}} \rvert \le \frac{\sqrt{\lvert S_\omega^{Z_{22}} \rvert \lvert S_\omega^{Z_{11}} \rvert \lvert Z_{11} \rvert \lvert Z_{22} \rvert}}{\lvert Z_{12} \rvert}$

w_C and w_L be the average power dissipated in the resistor R, average energy stored in the capacitor C, and average energy stored in the inductor L respectively of the network N_6, with port conditions as described above. Let w_R^A, w_C^A and w_L^A be the corresponding quantities of N_6^A with port conditions as described above.

If Z_i corresponds to a resistor R, an inductor L or a capacitor C, then it follows from (3.30) that

$$|S_R^{Y_{12}}| = \frac{\sqrt{w_R w_R^A}}{|V_0|^2 |Y_{12}|} \quad .. \text{ (3.31)}$$

$$|S_L^{Y_{12}}| = \frac{2\omega\sqrt{w_L w_L^A}}{|V_0|^2 |Y_{12}|} \quad .. \text{ (3.32)}$$

and

$$|S_C^{Y_{12}}| = \frac{2\omega\sqrt{w_C w_C^A}}{|V_0|^2 |Y_{12}|} \quad .. \text{ (3.33)}$$

Let at frequency ω

$$\begin{aligned} Y_{22} &= G_{22} + jB_{22} \\ Y_{11}^A &= G_{11}^A + jB_{11}^A \end{aligned} \quad .. \text{ (3.34)}$$

where

$Y_{22}(Y_{11}^A)$ is the short-circuit driving point admittance (D.P.A.) across port 2(1) of $N_6(N_6^A)$.

Using (3.11), (3.31) along with Schwartz inequality, we get the following

$$\sum_R |S_R^{Y_{12}}| \le \frac{\sqrt{P_R P_R^A}}{|V_0|^2 |Y_{12}|} = \frac{\sqrt{G_{22}G_{11}}}{|Y_{12}|} \qquad .. (3.35)$$

Using (3.15), (3.32) and (3.33) along with the Schwartz inequality, we have

$$\sum_L |S_L^{Y_{12}}| + \sum_C |S_C^{Y_{12}}| = \frac{2\omega \sum_L \sqrt{w_L w_L^A} + \sum_C \sqrt{w_C w_C^A}}{|V_0|^2 |Y_{12}|} \le \frac{2\omega \sqrt{WW^A}}{|V_0|^2 |Y_{12}|} \qquad .. (3.36)$$

From (3.18) and (3.36), we get the following

$$|S_\omega^{Y_{12}}| \le \sum_L |S_L^{Y_{12}}| + \sum_C |S_C^{Y_{12}}| \le \frac{2\omega\sqrt{WW^A}}{|V_0|^2 |Y_{12}|} \qquad .. (3.37)$$

3.3.2 RC-Networks:

If the network consists of only resistors and capacitors, then from the results of Chapter 2, it is known that,

$$\sum_R S_R^{Y_{12}} = -1 + S_\omega^{Y_{12}} \qquad .. (3.38a)$$

$$\sum_C S_C^{Y_{12}} = S_\omega^{Y_{12}} \qquad .. (3.38b)$$

$$\sum_R S_R^{Y_{12}} + \sum_C S_C^{Y_{12}} = -1 + 2S_\omega^{Y_{12}} \qquad .. (3.38c)$$

Hence on taking the magnitudes of both sides of (3.38), we get

$$\sum_{R}|S_{R}^{Y_{12}}| \geq |-1 + S_{\omega}^{Y_{12}}| \qquad .. \text{(3.39a)}$$

$$\sum_{C}|S_{C}^{Y_{12}}| \geq |S_{\omega}^{Y_{12}}| \qquad .. \text{(3.39b)}$$

$$\sum_{R}|S_{R}^{Y_{12}}| + \sum_{C}|S_{C}^{Y_{12}}| \geq |-1 + 2S_{\omega}^{Y_{12}}| \qquad .. \text{(3.39c)}$$

From (3.33) and Schwartz inequality

$$\sum_{C}|S_{C}^{Y_{12}}| \leq \frac{2\omega\sqrt{W_{C}W_{C}^{A}}}{|V_{0}|^{2}|Y_{12}|} \qquad .. \text{(3.40)}$$

For the network under consideration

$$W_{C} = \frac{|V_{0}|^{2}|B_{22}|}{2\omega} \qquad .. \text{(3.41)}$$

$$W_{C}^{A} = \frac{|V_{0}|^{2}|B_{11}|}{2\omega} \qquad .. \text{(3.42)}$$

Then, from (3.40), (3.41) and (3.42), we get the following

$$\sum_{C}|S_{C}^{Y_{12}}| \leq \frac{\sqrt{|B_{22}||B_{11}|}}{|Y_{12}|} \qquad .. \text{(3.43)}$$

Thus, from (3.35), (3.39) and (3.43), we obtain the following inequality

$$|-1 + S_{\omega}^{Y_{12}}| \leq \sum_{R}|S_{R}^{Y_{12}}| \leq \frac{\sqrt{G_{11}G_{22}}}{|Y_{12}|} \qquad .. \text{(3.44a)}$$

$$|S_{\omega}^{Y_{12}}| \le \sum_{C}|S_{C}^{Y_{12}}| \le \frac{\sqrt{|B_{22}||B_{11}|}}{|Y_{12}|} \qquad .. (3.44b)$$

and

$$|2S_{\omega}^{Y_{12}} - 1| \le \sum_{R}|S_{R}^{Y_{12}}| + \sum_{C}|S_{C}^{Y_{12}}| \le \frac{\sqrt{G_{11}G_{22}} + \sqrt{|B_{11}||B_{22}|}}{|Y_{12}|} \qquad .. (3.44c)$$

Inequalities similar to (3.44) can be obtained easily for an RL-network.

3.3.3 LC-Networks:

Let the network consist of only inductors and capacitors. Then, it is known that$^{(46)}$

$$\begin{aligned} W &= W_C + W_L = \frac{1}{2}|V_0|^2 \frac{\partial B_{22}}{\partial \omega} \\ W^A &= W_C^A + W_L^A = \frac{1}{2}|V_0|^2 \frac{\partial B_{11}}{\partial \omega} \end{aligned} \qquad .. (3.45)$$

Hence, from (3.37) and (3.45), we get the following inequality

$$\sum_{L}|S_{L}^{Y_{12}}| + \sum_{C}|S_{C}^{Y_{12}}| \le \frac{\omega}{Y_{12}}\sqrt{\frac{\partial B_{22}}{\partial \omega}\,\frac{\partial B_{11}}{\partial \omega}} = \frac{\sqrt{|S_{\omega}^{Y_{22}}||S_{\omega}^{Y_{11}}||Y_{11}||Y_{22}|}}{|Y_{12}|} \qquad .. (3.46)$$

Thus, from (3.18) and (3.45), for the network under consideration, we have the following inequality

$$|S_{\omega}^{Y_{12}}| \le \sum_{L}|S_{L}^{Y_{12}}| + \sum_{C}|S_{C}^{Y_{12}}| \le \frac{\sqrt{|S_{\omega}^{Y_{22}}||S_{\omega}^{Y_{11}}||Y_{11}||Y_{22}|}}{|Y_{12}|} \qquad .. (3.47)$$

This completes our discussion of the bounds on the sum of magnitudes of sensitivities of transfer admittances. The results of these discussions are summarized in Table 3.2.

We will illustrate the results of this Section with the help of an example.

Example:

Consider the RC-network shown in Fig.3.1. The network function Y_{12} is given by

$$Y_{12} = -\frac{1}{(r_1 + r_2) + sr_1r_2c}$$

Letting

$$X = (r_1 + r_2) + sr_1r_2c$$

and

$$|X| = \sqrt{(r_1 + r_2)^2 + \omega^2 r_1^2 r_2^2 c^2},$$

we get the following

$$S_{r_1}^{Y_{12}} = \frac{r_1 + sr_1r_2c}{X^2} \cdot \frac{1}{Y_{12}}$$

$$|S_{r_1}^{Y_{12}}| = \frac{\sqrt{r_1^2 + \omega^2 r_1^2 r_2^2 c^2}}{|X^2|} \cdot \frac{1}{Y_{12}}$$

$$S_{r_2}^{Y_{12}} = \frac{r_2 + sr_1r_2c}{X^2} \cdot \frac{1}{Y_{12}}$$

TABLE 3.2

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR Y_{12}

Network Type	Bounds
RLC-gyrator	$\sum_R \lvert S_R^{Y_{12}} \rvert \le \dfrac{\sqrt{G_{22}G_{11}}}{\lvert Y_{12} \rvert}$ $\lvert S_\omega^{Y_{12}} \rvert \le \sum_L \lvert S_L^{Y_{12}} \rvert + \sum_C \lvert S_C^{Y_{12}} \rvert \le \dfrac{2\omega\sqrt{WW^A}}{\lvert V_0 \rvert^2 \lvert Y_{12} \rvert}$
RC-network	$\lvert -1 + S_\omega^{Y_{12}} \rvert \le \sum_R \lvert S_R^{Y_{12}} \rvert \le \dfrac{\sqrt{G_{11}G_{22}}}{\lvert Y_{12} \rvert}$ $\lvert S_\omega^{Y_{12}} \rvert \le \sum_C \lvert S_C^{Y_{12}} \rvert \le \dfrac{\sqrt{\lvert B_{11} \rvert \lvert B_{22} \rvert}}{\lvert Y_{12} \rvert}$ $\lvert -1 + 2S_\omega^{Y_{12}} \rvert \le \sum_R \lvert S_R^{Y_{12}} \rvert + \sum_C \lvert S_C^{Y_{12}} \rvert \le \dfrac{\sqrt{G_{11}G_{22}} + \sqrt{\lvert B_{11} \rvert \lvert B_{22} \rvert}}{\lvert Y_{12} \rvert}$
LC-network	$\lvert S_\omega^{Y_{12}} \rvert \le \sum_L \lvert S_L^{Y_{12}} \rvert + \sum_C \lvert S_C^{Y_{12}} \rvert \le \dfrac{\sqrt{\lvert S_\omega^{Y_{22}} \rvert \lvert S_\omega^{Y_{11}} \rvert \lvert Y_{11} \rvert \lvert Y_{22} \rvert}}{\lvert Y_{12} \rvert}$

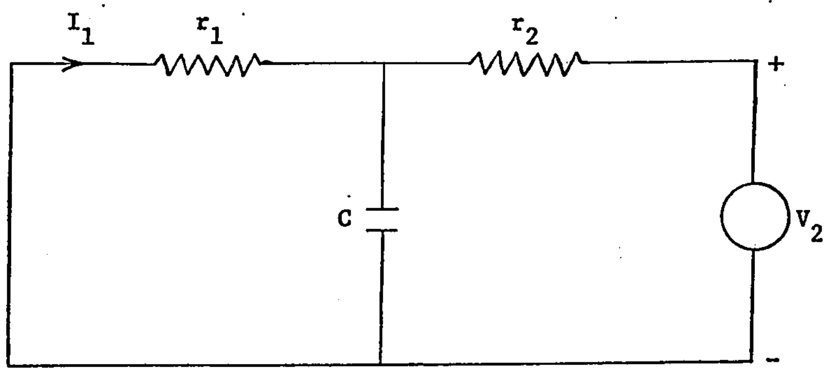


Fig.3.1: The Network for the Example in Section 3.3

$$|S_{r_2}^{Y_{12}}| = \frac{\sqrt{r_2^{2} + \omega^{2} r_1^{2} r_2^{2} c^{2}}}{|X^{2}|} \cdot \frac{1}{Y_{12}}$$

$$S_{c}^{Y_{12}} = \frac{s r_1 r_2 c}{X^{2}} \cdot \frac{1}{Y_{12}}$$

$$|S_{c}^{Y_{12}}| = \frac{\omega r_1 r_2 c}{|X|^{2}} \cdot \frac{1}{|Y_{12}|}$$

$$Y_{11} = \frac{1 + s r_2 c}{X}$$

$$Y_{22} = \frac{1 + s r_1 c}{X}$$

$$G_{11} = \frac{(r_1 + r_2) + \omega^{2} c^{2} r_1 r_2^{2}}{|X|^{2}} \qquad B_{11} = \frac{\omega r_2^{2} c}{|X|^{2}}$$

$$G_{22} = \frac{(r_1 + r_2) + \omega^{2} c^{2} r_1^{2} r_2}{|X|^{2}} \qquad B_{22} = \frac{\omega r_1^{2} c}{|X|^{2}}$$

From the above, it can be verified that the following inequality is true for all values of the parameters

$$|S_{r_1}^{Y_{12}}| + |S_{r_2}^{Y_{12}}| + |S_{c}^{Y_{12}}| \leq \frac{1}{Y_{12}}(\sqrt{G_{11}G_{22}} + \sqrt{B_{11}B_{22}})$$

3.4 Bounds on the Sum of Element Sensitivity Magnitudes for Voltage Transfer Ratio:

In this Section, bounds on the sum of magnitudes of sensitivities of transfer voltage ratio T_v are established for the types of networks considered in Section 3.2. It follows from (3.6) that

$$|S_{Z_i}^{T_v}| = \frac{|I_i||\phi_i|}{|V_0||I_0|} \frac{|Z_i|}{|T_v|} \qquad .. (3.48)$$

where

$S_{Z_i}^{T_v}$ is the sensitivity of T_v with respect to Z_i,

I_i is the current through the element Z_i when port (1) of the network is excited by a voltage source of value V_0, port (2) being open,

ϕ_i is the current through the element Z_i, when port (2) of the adjoint network is excited by a current source of value I_0, and port (1) is short-circuited.

3.4.1 RLC-Gyrator Network:

Let us consider the network N_6. If Z_i corresponds to a resistor R, then from (3.48), it follows that

$$|S_R^{T_v}| = \frac{\sqrt{w_R w_R^A}}{|V_0||I_0|} \frac{1}{|T_v|} \qquad .. (3.49)$$

where $w_R(w_R^A)$ denotes the average power dissipated in the resistor R, in the network $N_6(N_6^A)$, when its ports are terminated as described above.

If Z_i represents an inductor L, or a capacitor C, then from (3.48), we get

$$|S_L^{T_v}| = \frac{2\omega\sqrt{w_L w_L^A}}{|V_0||I_0|} \; \frac{1}{|T_v|} \qquad .. \; (3.50)$$

$$|S_C^{T_c}| = \frac{2\omega\sqrt{w_C w_C^A}}{|V_0||I_0|} \; \frac{1}{|T_v|} \qquad .. \; (3.51)$$

where

> $w_L(w_L^A)$ and $w_C(w_C^A)$ denote the average energy stored in the inductor and the capacitor respectively in the network $N_6(N_6^A)$, with the ports terminated as described earlier.

Let $P_R(P_R^A)$ denote the total average power dissipated in $N_6(N_6^A)$. Let at the frequency ω

$$\bar{Y}_{11} = \frac{1}{Z_{11}} = \bar{G}_{11} + j\bar{B}_{11}$$

and

$$\bar{Z}_{22}^A = \frac{1}{Y_{22}^A} = \bar{R}_{22}^A + j\bar{X}_{22}^A \qquad .. \; (3.52)$$

Then, using (3.49) with Schwartz inequality, we get the following

$$\sum_R |S_R^{T_v}| = \frac{\sum_R \sqrt{w_R w_R^A}}{|V_0||I_0|} \; \frac{1}{|T_v|} \le \frac{1}{|T_v|} \; \frac{\sqrt{P_R P_R^A}}{|V_0||I_0|} = \frac{1}{|T_v|}\sqrt{\bar{G}_{11}\bar{R}_{22}} \qquad .. \; (3.53)$$

The last step in (3.53) follows from the relation $\bar{Z}_{22}^A = \bar{Z}_{22}$.

$$W = \sum_C w_C + \sum_L w_L$$

$$W^A = \sum_C w_C^A + \sum_L w_L^A$$

then, we obtain from (3.50), (3.51) and Schwartz inequality that

$$\sum_L |S_L^{T_v}| + \sum_C |S_C^{T_v}| \le \frac{2\omega\sqrt{WW^A}}{|V_0||I_0||T_v|} \qquad .. \ (3.54)$$

From (3.18) and (3.54), we get

$$|S_\omega^{T_v}| \le \sum_L |S_L^{T_v}| + \sum_C |S_C^{T_v}| \le \frac{2\omega\sqrt{WW^A}}{|V_0||I_0||T_v|} \qquad .. \ (3.55)$$

3.4.2 <u>RC-Networks</u>:

Consider an RC-network. Then, it follows from (3.51) and Schwartz inequality that

$$\sum_C |S_C^{T_v}| = \frac{2\omega \sum_C \sqrt{w_C w_C^A}}{|V_0||I_0|} \frac{1}{|T_v|} \le \frac{2\omega\sqrt{W_C W_C^A}}{|V_0||I_0|} \frac{1}{|T_v|} \qquad .. \ (3.56)$$

For an RC-network, it is known that

$$W_C = \frac{|V_0|^2|\bar{B}_{11}|}{2\omega}$$

$$W_C^A = \frac{|I_0|^2|\bar{X}_{22}|}{2\omega} \qquad .. \ (3.57)$$

Thus, from (3.56) and (3.57), it follows that

$$\sum_C |S_C^T| \le \sqrt{|\overline{B}_{11}|\,|\overline{X}_{22}|}\;\frac{1}{|T_v|} \qquad .. (3.58)$$

For RC-networks it is known from the results of Chapter 2 that

$$\sum_R S_R^{T_v} = S_\omega^{T_v} \qquad .. (3.59a)$$

$$\sum_C S_C^{T_v} = S_\omega^{T_v} \qquad .. (3.59b)$$

$$\sum_R S_R^{T_v} + \sum_C S_C^{T_v} = 2S_\omega^{T_v} \qquad .. (3.59c)$$

Taking magnitudes of both sides of (3.59), we obtain the following inequalities.

$$\sum_R |S_R^{T_v}| \ge |S_\omega^{T_v}| \qquad .. (3.60a)$$

$$\sum_C |S_C^{T_v}| \ge |S_\omega^{T_v}| \qquad .. (3.60b)$$

$$\sum_R |S_C^{T_v}| + \sum_C |S_C^{T_v}| \ge |2S_\omega^{T_v}| \qquad .. (3.60c)$$

Thus, it follows from (3.53), (3.58) and (3.60)

$$|S_\omega^{T_v}| \le \sum_R |S_R^{T_v}| \le \frac{\sqrt{\overline{G}_{11}\overline{R}_{22}}}{|T_v|} \qquad .. (3.61a)$$

$$|S_\omega^{T_v}| \le \sum_C |S_C^{T_v}| \le \frac{\sqrt{|\overline{B}_{11}|\,|\overline{X}_{22}|}}{|T_v|} \qquad .. (3.61b)$$

$$|2S_{\omega}^{T_v}| \leq \sum_{R}|S_{R}^{T_v}| + \sum_{C}|S_{C}^{T_v}| \leq \frac{\sqrt{\overline{G}_{11}\overline{R}_{22}} + \sqrt{|\overline{B}_{11}||\overline{X}_{22}|}}{|T_v|} \qquad .. (3.61c)$$

Inequalities similar to (3.61) can be established in a similar manner for RL-networks.

3.4.3 LC-Networks:

Let the network N consist of reactive elements only. It can then be shown[(46)] that

$$W = \frac{1}{2}|V_0|^2 \frac{\partial \overline{B}_{11}}{\partial \omega}$$

$$W^A = \frac{1}{2}|I_0|^2 \frac{\partial \overline{X}_{22}}{\partial \omega} \qquad .. (3.62)$$

From (3.18), (3.50), (3.51) and (3.62), we can then get an inequality for an LC-network.

$$|S_{\omega}^{T_v}| \leq \sum_{L}|S_{L}^{T_v}| + \sum_{C}|S_{C}^{T_v}| \leq \frac{1}{|T_v|}\sqrt{|S_{\omega}^{\overline{Y}_{11}}||S_{\omega}^{\overline{Z}_{22}}||\overline{Y}_{11}||\overline{Z}_{22}|} \qquad .. (3.63)$$

This completes our discussion on the sum of sensitivity magnitudes for a transfer voltage ratio. The results of these discussions are summerized in Table 3.3.

We will illustrate the results of this Section with the help of examples.

TABLE 3.3

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR T_v

Network Type	Bounds
RLC-gyrator	(i) $\sum_R \lvert S_R^{T_v} \rvert \le \frac{\sqrt{\bar{G}_{11}\bar{R}_{22}}}{\lvert T_v \rvert}$ (ii) $\lvert S_\omega^{T_v} \rvert \le \sum_L \lvert S_L^{T_v} \rvert + \sum_C \lvert S_C^{T_v} \rvert \le \frac{2\omega\sqrt{WW^A}}{\lvert V_0 \rvert \lvert I_0 \rvert \lvert T_v \rvert}$
RC-network	$\lvert S_\omega^{T_v} \rvert \le \sum_R \lvert S_R^{T_v} \rvert \le \frac{\sqrt{\bar{G}_{11}\bar{R}_{22}}}{\lvert T_v \rvert}$ $\lvert S_\omega^{T_v} \rvert \le \sum_C \lvert S_C^{T_v} \rvert \le \frac{\sqrt{\lvert \bar{B}_{11} \rvert \lvert \bar{X}_{22} \rvert}}{\lvert T_v \rvert}$ $\lvert 2S_\omega^{T_v} \rvert \le \sum_R \lvert S_R^{T_v} \rvert + \sum_C \lvert S_C^{T_v} \rvert \le \frac{\sqrt{\bar{G}_{11}\bar{R}_{22}} + \sqrt{\lvert \bar{B}_{11} \rvert \lvert \bar{X}_{22} \rvert}}{\lvert T_v \rvert}$
LC-network	$\lvert S_\omega^{T_v} \rvert \le \sum_L \lvert S_L^{T_v} \rvert + \sum_C \lvert S_C^{T_v} \rvert \le \frac{\sqrt{\lvert S_\omega^{\bar{Y}_{11}} \rvert \lvert S_\omega^{\bar{Z}_{22}} \rvert \lvert \bar{Y}_{11} \rvert \lvert \bar{Z}_{22} \rvert}}{\lvert T_v \rvert}$

7.

Example 1:

Consider the RC-network shown in Fig.3.2. The transfer function $T_v = \frac{V_2}{V_1}$ is given by

$$T_v = \frac{sr_2c}{1 + sr_2c}$$

Letting

$$X = 1 + sr_2c \quad \text{and} \quad |X| = \sqrt{1 + \omega^2 r_2^2 c^2} ,$$

the following quantities can be easily obtained

$$S_{r_2}^{T_v} = \frac{sr_2c}{X^2}\,\frac{1}{T_v} \quad ; \quad |S_{r_2}^{T_v}| = \frac{\omega r_2 c}{|X|^2}\,\frac{1}{|T_v|}$$

$$S_{r_1}^{T_v} = 0$$

$$S_{c}^{T_v} = \frac{sr_2c}{X^2}\,\frac{1}{T_v} \quad ; \quad |S_{c}^{T_v}| = \frac{\omega\, r_2 c}{|X|^2}\,\frac{1}{|T_v|}$$

$$\overline{Y}_{11} = \frac{1 + sc(r_1+r_2)}{r_1(1+sr_2c)}$$

$$\overline{G}_{11} = \frac{1 + \omega^2 c^2 r_2(r_1+r_2)}{r_1|X|^2}$$

$$|\overline{B}_{11}| = \frac{\omega c}{|X|^2}$$

$$\overline{Z}_{22} = \frac{r_2}{1 + sr_2c}$$

$$\overline{R}_{22} = \frac{r_2}{|X|^2} \quad ; \quad |\overline{X}_{22}| = \frac{\omega c r_2^2}{|X|^2}$$

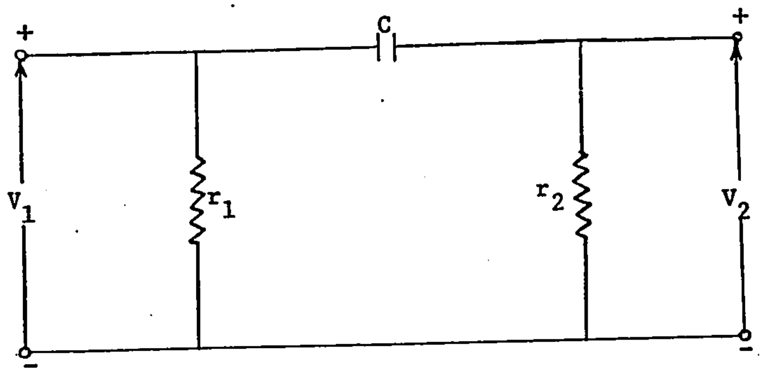


Fig.3.2: The Network for the Example 1 in Section 3.4

From the above quantities, it may be easily verified that the following inequality is true for all values of the circuit parameters

$$|S_{r_2}^{T_v}| + |S_c^{T_v}| \leq \frac{\sqrt{\overline{R}_{22}\overline{G}_{11}} + \sqrt{|\overline{X}_{22}|\,|\overline{B}_{11}|}}{|T_v|}$$

<u>Example 2</u>:

Consider the RC-gyrator circuit shown in Fig.3.3. The transfer function $T_v = \frac{V_2}{V_1}$ is given by

$$T_v = \frac{GG_aD}{sG_a + G_aG_bD + G^2D} .$$

Letting

$$X = sG_a + G_aG_bD + G^2D$$

$$|X| = \sqrt{(G_aG_bD + G^2D)^2 + \omega^2G_a^2} ,$$

the following quantities can be easily obtained

$$S_{G_a}^{T_v} = \frac{G_aD^2G^3}{X^2}\,\frac{1}{T_v}$$

$$S_{G_b}^{T_v} = -\frac{GG_a^2D^2G_b}{X^2}\,\frac{1}{T_v}$$

$$|S_{G_a}^{T_v}| = \frac{G_aG^3D^2}{|X|^2}\,\frac{1}{|T_v|}$$

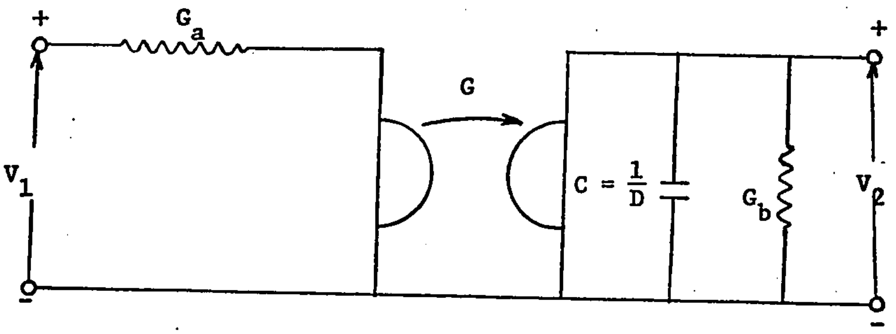


Fig.3.3: The Network for the Example 2 in Section 3.4

$$|S_{G_b}^{T_v}| = \frac{G_b G_a^2 D^2 G}{|X|^2} \frac{1}{|T_v|}$$

$$|\overline{Y}_{11}| = \frac{G^2 G_a D}{|X|}$$

$$|\overline{Z}_{22}| = \frac{G_a D}{|X|}$$

$$\overline{G}_{11} = \frac{G^2 G_a D(G^2 D + G_b D G_a)}{|X|^2}$$

$$\overline{R}_{22} = \frac{G_a D(G^2 D + G_a D G_b)}{|X|^2}$$

Using the above quantities, it may be verified easily that the following inequality is true for all values of the parameters.

$$|S_{G_a}^{T_v}| + |S_{G_b}^{T_v}| \leq \frac{1}{|T_v|} \sqrt{\overline{G}_{11} \overline{R}_{22}}$$

3.5 Bounds on the Sum of Element Sensitivity Magnitudes for current Transfer Ratio:

Let us next consider the transfer current ratio T_c between ports (2) and (1) of the different types of networks considered in Section 3.4. Let $W(W^A)$ denote the total average energy stored in the inductors and capacitors of the given network and its adjoint, when port 1(2) of the given network (its adjoint) is excited with a current (voltage) source of value $I_0(V_0)$ when port 2(1) of the given network (its adjoint) is short-circuited (open-circuited). Further, let for the original and adjoint networks

$$\overline{Z}_{11} = \frac{1}{\overline{Y}_{11}} = \overline{R}_{11} + j\,\overline{X}_{11}$$

and
$$\overline{Y}^A_{22} = \frac{1}{\overline{Z}^A_{22}} = \overline{G}^A_{22} + j\,\overline{B}^A_{22} \qquad ..(3.64)$$

Observing that the transfer current ratio between ports (2) and (1) of the original network is the same as the transfer voltage between ports (1) and (2) of its adjoint, it is then very easily seen that all the equations and inequalities from (3.48) through (3.63) hold true if $\overline{G}_{11}$, $\overline{B}_{11}$, $\overline{R}_{22}$ and $\overline{X}_{22}$ are replaced respectively by $\overline{G}_{22}$, $\overline{B}_{22}$ $\overline{R}_{11}$ and $\overline{X}_{11}$. The results for T_c are summarised in Table 3.4.

3.6 Conclusions:

Bounds on the sum of magnitudes of element sensitivities of network functions are established for networks containing resistors, inductors, capacitors and gyrators. It is shown that the magnitude of sensitivity with respect to a resistor (a reactive element) is related to the average power dissipated (average energy stored) in that resistor (that reactive element) in the original and the adjoint networks, under proper port conditions.

In particular, for an RC-network, an upper bound, in terms of the real parts of driving point functions, is given for the sum of the magnitudes of sensitivities with respect to all resistances. In the case of RC(LC) networks, lower bounds, in terms of real and imaginary parts of the relevant driving point functions and the

TABLE 3.4

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR T_c

Network Type	Bounds
RLC-gyrator	$\sum_R \lvert S_C^{T_c}\rvert \le \frac{\sqrt{\overline{G}_{22}\overline{R}_{11}}}{\lvert T_c\rvert}$ $\lvert S_\omega^{T_c}\rvert \le \sum_L \lvert S_L^{T_c}\rvert + \sum_C \lvert S_C^{T_c}\rvert \le \frac{2\omega\sqrt{WW^{A}}}{\lvert I_0\rvert \lvert V_0\rvert \lvert T_c\rvert}$
RC-network	$\lvert S_\omega^{T_c}\rvert \le \sum_R \lvert S_R^{T_c}\rvert \le \frac{\sqrt{\overline{G}_{22}\overline{R}_{11}}}{\lvert T_c\rvert}$ $\lvert S_\omega^{T_c}\rvert \le \sum_C \lvert S_C^{T_c}\rvert \le \frac{\sqrt{\lvert\overline{B}_{22}\rvert \lvert\overline{X}_{11}\rvert}}{\lvert T_c\rvert}$ $\lvert 2S_\omega^{T_c}\rvert \le \sum_R \lvert S_R^{T_c}\rvert + \sum_C \lvert S_C^{T_c}\rvert \le \frac{\sqrt{\overline{G}_{22}\overline{R}_{11}} + \sqrt{\lvert\overline{B}_{22}\rvert \lvert\overline{X}_{11}\rvert}}{\lvert T_c\rvert}$
LC-network	$\lvert S_\omega^{T_c}\rvert \le \sum_L \lvert S_L^{T_c}\rvert + \sum_C \lvert S_C^{T_c}\rvert \le \frac{\sqrt{\lvert S_\omega^{\overline{Z}_{11}}\rvert \lvert S_\omega^{\overline{Y}_{22}}\rvert \lvert\overline{Z}_{11}\rvert \lvert\overline{Y}_{22}\rvert}}{\lvert T_c\rvert}$

sensitivities of network functions with respect to frequency, are given for the sum of magnitudes of sensitivities with respect to all resistances and capacitances (inductances and capacitances).

For driving point impedances or admittances, the bounds obtained by Smith(28) can be obtained from the results of Sections 3.2 and 3.3, by observing that Schwartz inequality reduces to equality.

It should be mentioned that the upper bound on the sum of magnitudes of sensitivities for any network function would directly given an upper bound on Schoeffler's quadratic performance index(13) also. The bounds on the quadratic performance index for different kinds of networks will be obtained in the next chapter.

CHAPTER 4

BOUNDS ON QUADRATIC SENSITIVITY INDEX FOR ACTIVE LUMPED/DISTRIBUTED NETWORKS

4.1 Introduction:

In the area of optimal synthesis (optimal in the sense that some multiparameter sensitivity performance of the network has been minimised), the concept of sensitivity invariance is of great importance. The synthesis of optimal networks has been considered by several authors. Schoeffler(13) used the theory of continuously equivalent networks to generate a series of networks whose transfer functions are identical, but whose elements differ from one network to another by an incremental amount. The index ϕ, which has been used to find the optimum network from the series of networks that have been generated, is the sum of the squares of the magnitudes of sensitivities.

Leeds and Ugron(6) in the course of their minimisation procedure, based on Schoeffler's approach(13) found that the optimum network, selected from the series of continuously equivalent networks, tends to have the value of the summed sensitivities distributed uniformly with respect to network elements. The theoretical limitations of the continuously equivalent network approach (passive networks) were considered by Schmidt and Kasper(15), who concluded that networks with substantially lower sensitivity index can be obtained only if the number of nodes, that is, the number of network elements, is allowed to increase sufficiently.

In this chapter, we first obtain the lower and upper bounds on the sensitivity index ϕ for two-element kind networks(45). The lower bound gives the theoretical limit up to which the sensitivity index may be reduced (Section 4.2), but does not necessarily mean that the network with that index is always realizable. The lower bounds on the sensitivity index ϕ for different network functions, for different networks are obtained in Section 4.3(35), using Lagrange's multiplier technique.

4.2 Bounds on Quadratic Sensitivity Index for Two-Element Kind Networks:

In this Section, we will establish for RC, RL and LC-networks bounds on the quadratic sensitivity index ϕ^F which is defined as

$$\phi^F = \sum_i |S^F_{p_i}|^2 \qquad .. (4.1)$$

where (i = 1,2, .. ,n) are the number of elements in the network, p_i is the parameter associated with the ith element and F is any network function.

4.2.1 RC-Networks:

In Chapter 2, it was shown that the summed sensitivity of a network function for RC-network, could be expressed in terms of the frequency sensitivity of the function. For the sake of convenience, these equations are repeated below.

For a transfer or driving point impedance, we have

$$\sum_{C} S_{C}^{Z_{12}} = S_{\omega}^{Z_{12}} \qquad .. (4.3)$$

$$\sum_{R} S_{R}^{Z_{12}} = S_{\omega}^{Z_{12}} + 1 \qquad .. (4.4)$$

and $$\sum_{R} S_{R}^{Z_{12}} + \sum_{C} S_{C}^{Z_{12}} = 2S_{\omega}^{Z_{12}} + 1$$

where ω is the frequency.

For a transfer or driving point admittance, we have

$$\sum_{C} S_{C}^{Y_{12}} = S_{\omega}^{Y_{12}} \qquad .. (4.5)$$

$$\sum_{R} S_{R}^{Y_{12}} = S_{\omega}^{Y_{12}} - 1 \qquad .. (4.6)$$

and $$\sum_{R} S_{R}^{Y_{12}} + \sum_{C} S_{C}^{Y_{12}} = 2S_{\omega}^{Y_{12}} - 1$$

For a voltage or current transfer function T, we have

$$\sum_{C} S_{C}^{T} = S_{\omega}^{T} \qquad .. (4.7)$$

$$\sum_{R} S_{R}^{T} = S_{\omega}^{T} \qquad .. (4.8)$$

and $$\sum_{R} S_{R}^{T} + \sum_{C} S_{C}^{T} = 2S_{\omega}^{T}$$

From Chebyshev's inequality, we have

$$\sum_{i=1}^{n} |a_i|^2 \geq \frac{1}{n} |\sum_{i=1}^{n} a_i|^2 \qquad .. (4.9)$$

Using (4.3) and (4.9), we have for a transfer or driving point impedance

$$\phi^{Z_{12}} = \sum_{R} |S_R^{Z_{12}}|^2 + \sum_{C} |S_C^{Z_{12}}|^2 \geq \frac{1}{n_1}|S_\omega^{Z_{12}} + 1|^2 + \frac{1}{n_2}|S_\omega^{Z_{12}}|^2 \qquad .. \ (4.10)$$

where n_1 and n_2 are respectively the number of resistive and capacitive elements in the RC-network.

Eq. (4.10) gives the lower bound for $\phi^{Z_{12}}$. The upper bound could be obtained from the results of Chapter 3. For RC-networks, from (3.26), we have

$$\sum_{R}|S_R^{Z_{12}}| \leq \frac{\sqrt{R_{11}R_{22}}}{|Z_{12}|}$$

$$\sum_{C}|S_C^{Z_{12}}| \leq \frac{\sqrt{|X_{11}||X_{22}|}}{|Z_{12}|}$$

where R_{11}, R_{22}, X_{11} and X_{22} are defined in Section 3.2.

Thus

$$\phi^{Z_{12}} = \sum_{R} |S_R^{Z_{12}}|^2 + \sum_{C} |S_C^{Z_{12}}|^2 \leq \frac{(\sqrt{R_{11}R_{22}})^2 + (\sqrt{|X_{11}||X_{22}|})^2}{|Z_{12}|^2}$$

or

$$\phi^{Z_{12}} \leq \frac{R_{11}R_{22} + |X_{11}||X_{22}|}{|Z_{12}|^2} \qquad .. \ (4.11)$$

From (4.10) and (4.11), we have

$$\frac{1}{n_1}\left|S_\omega^{Z_{12}} + 1\right|^2 + \frac{1}{n_2}\left|S_\omega^{Z_{12}}\right|^2 \le \phi^{Z_{12}} \le \frac{R_{11}R_{22} + |X_{11}||X_{22}|}{|Z_{12}|^2} \qquad .. \ (4.12)$$

Similarly, using Chebyshev's inequality and the results of Chapter 2, lower bounds for transfer admittance function Y_{12} and transfer voltage or current ratio T could be found out and are given by

$$\phi^{Y_{12}} = \sum_R |S_R^{Y_{12}}|^2 + \sum_C |S_C^{Y_{12}}|^2 \ge \frac{1}{n_1}|S_\omega^{Y_{12}} - 1|^2 + \frac{1}{n_2}|S_\omega^{Y_{12}}|^2 \qquad .. \ (4.13)$$

and

$$\phi^T = \sum_R |S_R^T|^2 + \sum_C |S_C^T|^2 \ge \left(\frac{1}{n_1} + \frac{1}{n_2}\right)|S_\omega^T|^2 \qquad .. \ (4.14)$$

The upper bounds could be obtained using the results of Chapter 3 and are given by

$$\sum_R |S_R^{Y_{12}}|^2 + \sum_C |S_C^{Y_{12}}|^2 \le \frac{G_{11}G_{22} + |B_{11}||B_{22}|}{|Y_{12}|^2} \qquad .. \ (4.15)$$

$$\sum_R |S_R^{T_v}|^2 + \sum_C |S_C^{T_v}|^2 \le \frac{\overline{G}_{11}\overline{R}_{22} + |\overline{B}_{11}||\overline{X}_{22}|}{|T_v|^2} \qquad .. \ (4.16)$$

and

$$\sum_R |S_R^{T_c}|^2 + \sum_C |S_C^{T_c}|^2 \le \frac{\overline{G}_{22}\overline{R}_{11} + |\overline{B}_{22}||\overline{X}_{11}|}{|T_c|^2} \qquad .. \ (4.17)$$

where T_v and T_c are respectively the voltage and current transfer functions, and G_{11}, G_{22}, B_{11}, B_{22}, $\overline{G}_{11}$, $\overline{R}_{22}$, $\overline{B}_{11}$, $\overline{X}_{22}$, $\overline{G}_{22}$, $\overline{R}_{11}$, $\overline{B}_{22}$ and $\overline{X}_{11}$ are as defined in Sections 3.3, 3.4 and 3.5. Hence, using (4.13)-(4.17), we have

$$\frac{1}{n_1}\,|S_\omega^{Y_{12}} - 1|^2 + \frac{1}{n_2}|S_\omega^{Y_{12}}|^2 \le \phi^{Y_{12}} \le \frac{G_{11}G_{22} + |B_{11}||B_{22}|}{|Y_{12}|^2} \quad .. \ (4.18)$$

$$(\frac{1}{n_1} + \frac{1}{n_2})|S_\omega^{T_v}|^2 \le \phi^{T_v} \le \frac{\overline{G}_{11}\overline{R}_{22} + |\overline{B}_{11}||\overline{X}_{22}|}{|T_v|^2} \quad .. \ (4.19)$$

and

$$(\frac{1}{n_1} + \frac{1}{n_2})|S_\omega^{T_c}|^2 \le \phi^{T_c} \le \frac{\overline{G}_{22}\overline{R}_{11} + |\overline{B}_{22}||\overline{X}_{11}|}{|T_c|^2} \quad .. \ (4.20)$$

4.2.2 LC-Networks:

The invariance relationships obtained in Chapter 2 for an LC-network could be used along with Chebyshev's inequality to obtain the lower bounds on ϕ for different network functions. The upper bounds could be obtained using the results of Chapter 3. The upper and lower bounds on ϕ for different network functions are given by

$$\frac{1}{4n_1}\,|S_\omega^{Z_{12}} - 1|^2 + \frac{1}{4n_2}\,|S_\omega^{Z_{12}} + 1|^2 \le \phi^{Z_{12}} \le \frac{|S_\omega^{Z_{22}}||S_\omega^{Z_{11}}||Z_{11}||Z_{22}|}{|Z_{12}|^2}$$

$$\frac{1}{4n_1}\,|S_\omega^{Y_{12}} + 1|^2 + \frac{1}{4n_2}\,|S_\omega^{Y_{12}} - 1|^2 \le \phi^{Y_{12}} \le \frac{|S_\omega^{Y_{22}}||S_\omega^{Y_{11}}||Y_{11}||Y_{22}|}{|Y_{12}|^2}$$

$$\frac{1}{4}[\frac{1}{n_1} + \frac{1}{n_2}]|S_\omega^{T_v}|^2 \le \phi^{T_v} \le \frac{|S_\omega^{\overline{Y}_{11}}||S_\omega^{\overline{Z}_{22}}||\overline{Y}_{11}||\overline{Z}_{22}|}{|T_v|^2}$$

and

$$\frac{1}{4}\left[\frac{1}{n_1}+\frac{1}{n_2}\right]\left|S_\omega^{T_c}\right|^2 \leq \phi^{T_c} \leq \frac{|S_\omega^{\bar{Y}_{22}}||S_\omega^{\bar{Z}_{11}}||\bar{Y}_{22}||\bar{Z}_{11}|}{|T_c|^2}$$

where n_1 and n_2 are respectively the number of inductors and capacitors in the LC-network, and Z_{11}, Z_{22}, Y_{11}, Y_{22} etc, are defined in Sections (3.2-3.5).

Similar inequalities could also be found out for RL-networks.

4.3 Lower Bounds on Quadratic Sensitivity Index for Active Lumped/Distributed Networks:

Lower bounds on quadratic sensitivity index could be obtained as shown in Section 4.2 for active lumped/distributed networks by using the invariance relations obtained in Chapter 2 along with Chebyshev's inequality. However, in this section, these lower bounds will be obtained using Lagrange multiplier technique.

4.3.1 Lower Bounds for ϕ^F for the Network N_1:

Consider the network N_1, consisting of lumped resistors, capacitors, gyrators, $\overline{\text{RC}}$-tapered lines, controlled sources (CVT's, VCT's, CCT's and VVT's) and NIC's (VNIC's and CNIC's). Let the set $Q = \{q_k\} = \{q_{k1}\} \cup \{q_{k2}\}$ where $\{q_{k1}\} = \{R_i, (\alpha_{i1}, \alpha_{i2}), (R_{0i}), r_i, \gamma_i\}$ and $\{q_{k2}\} = \{C_i, C_{0i}\}$. It has been shown in Chapter 2 that for a transfer or driving point impedance Z_{12}

$$\sum_{q_{k2}} S_{q_{k1}}^{Z_{12}} = S_\omega^{Z_{12}} \qquad ..(4.21)$$

and

$$\sum_{q_{k1}} S_{q_{k1}}^{Z_{12}} = S_{\omega}^{Z_{12}} + 1 \qquad .. (4.22)$$

For a transfer admittance Y_{12}

$$\sum_{q_{k2}} S_{q_{k2}}^{Y_{12}} = S_{\omega}^{Y_{12}} \qquad .. (4.23)$$

and

$$\sum_{q_{k1}} S_{q_{k1}}^{Y_{12}} = S_{\omega}^{Y_{12}} - 1 \qquad .. (4.24)$$

For a current or voltage transfer function T

$$\sum_{q_{k2}} S_{q_{k2}}^{T} = \sum_{q_{k1}} S_{q_{k1}}^{T} = S_{\omega}^{T} \qquad .. (4.25)$$

We will first find the lower bound for $\phi^{Z_{12}}$. Hence, we are interested in minimising $\phi^{Z_{12}}$, given by

$$\phi^{Z_{12}} = (\sum_{q_{k1}} |S_{q_{k1}}^{Z_{12}}|^2 + \sum_{q_{k2}} |S_{q_{k2}}^{Z_{12}}|^2) \qquad .. (4.26)$$

with the constraints given by (4.21) and (4.22).

For the sake of convenience, we drop the superscript Z_{12}. Let at frequency ω

$$\begin{aligned} S_{q_{k1}} &= S'_{q_{k1}} + j\, S''_{q_{k1}} \\ S_{q_{k2}} &= S'_{q_{k2}} + j\, S''_{q_{k2}} \end{aligned} \qquad .. (4.27)$$

then

$$\phi = [\sum_{q_{k1}} (S'^2_{q_{k1}} + S''^2_{q_{k1}}) + \sum_{q_{k2}} (S'^2_{q_{k2}} + S''^2_{q_{k2}})] \qquad .. (4.28)$$

Since the right hand sides of (4.21) and (4.22) are also complex, these equations can be written as

$$\sum_{q_{k1}} S'_{q_{k1}} = (S'_\omega + 1)$$

$$\sum_{q_{k1}} S''_{q_{k1}} = S''_\omega$$

and

$$\sum_{q_{k2}} S'_{q_{k2}} = S'_\omega$$

$$\sum_{q_{k2}} S''_{q_{k2}} = S''_\omega$$

where S' and S'' denote the real and imaginary parts of S respectively.

Using Lagrange multipliers, a new sensitivity measure can be written as

$$\hat{\phi} = \sum_{q_{k1}} (S'^2_{q_{k1}} + S''^2_{q_{k1}}) + \sum_{q_{k2}} (S'^2_{q_{k2}} + S''^2_{q_{k2}}) + \lambda_1[\sum_{q_{k1}} S'_{q_{k1}} - (S'_\omega + 1)]$$

$$+ \lambda_2(\sum_{q_{k1}} S''_{q_{k1}} - S''_\omega) + \lambda_3(\sum_{q_{k2}} S'_{q_{k2}} - S'_\omega) + \lambda_4(\sum_{q_{k2}} S''_{q_{k2}} - S''_\omega) \qquad .. (4.29)$$

The critical points of $\hat{\phi}$ are obtained by differentiating (4.29) with respect to $S'_{q_{k1}}$, $S''_{q_{k1}}$, $S'_{q_{k2}}$, $S''_{q_{k2}}$ and setting the corresponding terms to zero.

$$-2\ S'_{q_{k1}} = \lambda_1 \quad \text{for all} \quad q_{k1} \in \{q_{k1}\} \qquad .. (4.30)$$

$$-2\ S''_{q_{k1}} = \lambda_2 \quad \text{for all} \quad q_{k1} \in \{q_{k1}\} \qquad .. (4.31)$$

$$-2\ S'_{q_{k2}} = \lambda_3 \quad \text{for all} \quad q_{k2} \in \{q_{k2}\} \qquad .. (4.32)$$

$$-2\ S''_{q_{k2}} = \lambda_4 \quad \text{for all} \quad q_{k2} \in \{q_{k2}\} \qquad .. (4.33)$$

Summing (4.30) yields

$$-2 \sum_{q_{k1}} S'_{q_{k1}} = n_1\lambda_1 \qquad .. (4.34)$$

where n_1 is the number of elements in the set $\{q_{k1}\}$.

Then

$$-2(S'_\omega + 1) = n_1\lambda_1$$

or

$$\lambda_1 = -\frac{2}{n_1}(S'_\omega + 1) \qquad .. (4.35)$$

Similarly

$$\lambda_2 = -\frac{2}{n_1}\ S''_\omega$$

$$\lambda_3 = -\frac{2}{n_2}\ S'_\omega \qquad .. (4.36)$$

$$\lambda_4 = -\frac{2}{n_2}\ S''_\omega$$

where n_2 is the number of elements in the set $\{q_{k2}\}$.

The minimum value ϕ_{min} of ϕ is then obtained by using (4.30)-(4.33),

$$\phi_{min} = \frac{n_1}{4}(\lambda_1^2 + \lambda_2^2) + \frac{n_2}{4}(\lambda_3^2 + \lambda_4^2)$$

or

$$\phi_{min} = \frac{1}{n_1}[(S'_\omega+1)^2 + (S''_\omega)^2] + \frac{1}{n_2}[(S'_\omega)^2 + (S''_\omega)^2]$$

$$= \frac{1}{n_1}|S_\omega^{Z_{12}}+1|^2 + \frac{1}{n_2}|S_\omega^{Z_{12}}|^2 \qquad .. (4.37)$$

ϕ_{min} is the smallest value of ϕ that can be obtained by any network realizing Z_{12} and having n_1 elements in the set $\{q_{k1}\}$ and n_2 in the set $\{q_{k2}\}$. It is seen that in order for the network to be "optimally sensitive" (that is, to have $\phi = \phi_{min}$), the sensitivities over the different elements of the set $\{q_{k1}\}$ must be equal, with a similar statement holding true over the different elements of the set $\{q_{k2}\}$.

Thus, in general,

$$\phi^{Z_{12}} \geq \frac{1}{n_1}|1 + S_\omega^{Z_{12}}|^2 + \frac{1}{n_2}|S_\omega^{Z_{12}}|^2 \qquad .. (4.38)$$

In a similar manner, the lower bounds for the transfer admittance Y_{12} and current or voltage transfer function T could be found out and, are given by

$$\phi^{Y_{12}} = \sum_{q_{k1}} |S_{q_{k1}}^{Y_{12}}|^2 + \sum_{q_{k2}} |S_{q_{k2}}^{Y_{12}}|^2 \geq \frac{1}{n_1}|S_\omega^{Y_{12}} - 1|^2 + \frac{1}{n_2}|S_\omega^{Y_{12}}|^2 \qquad .. (4.39)$$

$$\phi^T = \sum_{q_{k1}} |S^T_{q_{k1}}|^2 + \sum_{q_{k2}} |S^T_{q_{k2}}|^2 \geq (\frac{1}{n_1} + \frac{1}{n_2}) |S^T_\omega|^2 \qquad ..(4.40)$$

4.3.2 Lower Bounds for ϕ^F for the Network N_0:

Consider the network N_0, consisting of lumped resistors, inductors, capacitors, gyrators, controlled sources (CVT's, VCT's, VVT's and CCT's), negative impedance converters (VNIC's and CNIC's), $\overline{LC}$ and $\overline{RC}$ tapered lines. Let the set $P = \{p_k\} = \{R_i, L_i, D_i, (\alpha_{i1}, \alpha_{i2}), (R_{oi}, D_{oi}), \dot{Z}_{\overline{LC}i}, r_i, \gamma_i\}$. It has been shown in Chapter 2 that

$$\sum_{p_k} S^F_{p_k} = \zeta$$

where ζ is 1, -1 or 0 depending on whether F is Z_{12}, Y_{12} or T. Using the same procedure as used in the previous section, it can be shown that

$$\phi^{Z_{12}} = \sum_{p_k} |S^{Z_{12}}_{p_k}|^2 \geq \frac{1}{n} \qquad ..(4.41)$$

$$\phi^{Y_{12}} = \sum_{p_k} |S^{Y_{12}}_{p_k}|^2 \geq \frac{1}{n} \qquad ..(4.42)$$

$$\phi^T = \sum_{p_k} |S^T_{p_k}|^2 \geq 0 \qquad ..(4.43)$$

where n is the number of elements in the set $\{p_k\}$.
It should be observed that Eq.(4.43) does not give any new information as ϕ is the sum of nonnegative quantities.

Also, for the network N_0, over the set $\{m_k\} = \{L_i, C_i, (C_{oi}),$

TABLE 4.1

BOUNDS ON ϕ^F FOR DIFFERENT CLASSES OF NETWORKS

Network	Number of elements	ϕ^F	Bounds on ϕ^F
Lumped RC-network	n_1 = number of resistors n_2 = number of capacitors	$\phi^F = \sum_R \lvert S_R^F \rvert^2 + \sum_C \lvert S_C^F \rvert^2$	$\frac{1}{n_1}\lvert S_\omega^{Z_{12}}+1\rvert^2 + \frac{1}{n_2}\lvert S_\omega^{Z_{12}}\rvert^2 \le \phi^{Z_{12}} \le \frac{R_{11}R_{22} + \lvert X_{11}\rvert\lvert X_{22}\rvert}{\lvert Z_{12}\rvert^2}$ $\frac{1}{n_1}\lvert S_\omega^{Y_{12}}-1\rvert^2 + \frac{1}{n_2}\lvert S_\omega^{Y_{12}}\rvert^2 \le \phi^{Y_{12}} \le \frac{G_{11}G_{22} + \lvert B_{11}\rvert\lvert B_{22}\rvert}{\lvert Y_{12}\rvert^2}$ $(\frac{1}{n_1} + \frac{1}{n_2})\lvert S_\omega^{T_v}\rvert^2 \le \phi^{T_v} \le \frac{\overline{G}_{11}\overline{R}_{22} + \lvert\overline{B}_{11}\rvert\lvert\overline{X}_{22}\rvert}{\lvert T_v\rvert^2}$ $(\frac{1}{n_1} + \frac{1}{n_2})\lvert S_\omega^{T_c}\rvert^2 \le \phi^{T_c} \le \frac{\overline{G}_{22}\overline{R}_{11} + \lvert\overline{B}_{22}\rvert\lvert\overline{X}_{11}\rvert}{\lvert T_c\rvert^2}$

(Cont'd)

Lumped LC-network	n_1 = number of inductors n_2 = number of capacitors	$\phi^F = \sum_L \lvert S_L^F \rvert^2 + \sum_C \lvert S_C^F \rvert^2$	$\frac{1}{4n_1}\lvert S_\omega^{Z_{12}}-1\rvert^2 + \frac{1}{4n_2}\lvert S_\omega^{Z_{12}}+1\rvert^2 \le \phi^{Z_{12}} \le \frac{\lvert S_\omega^{Z_{22}}\rvert \lvert S_\omega^{Z_{11}}\rvert \lvert Z_{11}\rvert \lvert Z_{22}\rvert}{\lvert Z_{12}\rvert^2}$ $\frac{1}{4n_1}\lvert S_\omega^{Y_{12}}+1\rvert^2 + \frac{1}{4n_2}\lvert S_\omega^{Y_{12}}-1\rvert^2 \le \phi^{Y_{12}} \le \frac{\lvert S_\omega^{Y_{22}}\rvert \lvert S_\omega^{Y_{11}}\rvert \lvert Y_{11}\rvert \lvert Y_{22}\rvert}{\lvert Y_{12}\rvert^2}$ $\frac{1}{4}(\frac{1}{n_1} + \frac{1}{n_2})\lvert S_\omega^{T_v}\rvert^2 \le \phi^{T_v} \le \frac{\lvert S_\omega^{\overline{Y}_{11}}\rvert \lvert S_\omega^{\overline{Z}_{22}}\rvert \lvert \overline{Y}_{11}\rvert \lvert \overline{Z}_{22}\rvert}{\lvert T_v\rvert^2}$ $\frac{1}{4}(\frac{1}{n_1} + \frac{1}{n_2})\lvert S_\omega^{T_c}\rvert^2 \le \phi^{T_c} \le \frac{\lvert S_\omega^{\overline{Y}_{22}}\rvert \lvert S_\omega^{\overline{Z}_{11}}\rvert \lvert \overline{Y}_{22}\rvert \lvert \overline{Z}_{11}\rvert}{\lvert T_c\rvert^2}$
N_0	n = number of elements in the set $\{p_k\}$	$\phi^F = \sum_{p_k} \lvert S_{p_k}^F \rvert^2$	$\phi^{Z_{12}}, \phi^{Y_{12}} \ge \frac{1}{n}$; $\phi^T \ge 0$
	n = number of elements in the set $\{m_k\}$	$\phi^F = \sum_{m_k} \lvert S_{m_k}^F \rvert^2$	$\phi^{Z_{12}}, \phi^{Y_{12}}, \phi^T \ge \frac{1}{n}\lvert S_\omega^F\rvert^2$

(Cont'd)

N_1	n_1 = number of elements in the set $\{q_{k1}\}$ n_2 = number of elements in the set $\{q_{k2}\}$	$\phi^F = \sum_{q_k} \lvert S^F_{q_k} \rvert^2$	$\phi^{Z_{12}} \geq \frac{1}{n_1} \lvert S_\omega^{Z_{12}} + 1 \rvert^2 + \frac{1}{n_2} \lvert S_\omega^{Z_{12}} \rvert^2$ $\phi^{Y_{12}} \geq \frac{1}{n_1} \lvert S_\omega^{Y_{12}} - 1 \rvert^2 + \frac{1}{n_2} \lvert S_\omega^{Y_{12}} \rvert^2$ $\phi^T \geq (\frac{1}{n_1} + \frac{1}{n_2}) \lvert S_\omega^T \rvert^2$
N_2	n_1 = number of elements in the set $\{m_{k1}\}$ n_2 = number of elements in the set $\{m_{k2}\}$	$\phi^F = \sum_{m_{k3}} \lvert S^F_{m_{k3}} \rvert^2$	$\phi^{Z_{12}} \geq \frac{1}{4n_1} \lvert S_\omega^{Z_{12}} - 1 \rvert^2 + \frac{1}{4n_2} \lvert S_\omega^{Z_{12}} + 1 \rvert^2$ $\phi^{Y_{12}} \geq \frac{1}{4n_1} \lvert S_\omega^{Y_{12}} + 1 \rvert^2 + \frac{1}{4n_2} \lvert S_\omega^{Y_{12}} - 1 \rvert^2$ $\phi^T \geq \frac{1}{4}(\frac{1}{n_1} + \frac{1}{n_2}) \lvert S_\omega^T \rvert^2$

$(L_{0i}, C_{0i})\}$, it can be readily be shown that

$$\phi^F = \sum_{m_k} |S^F_{m_k}|^2 \geq \frac{1}{n}|S^F|^2$$

where F is any network function Z_{12}, Y_{12} or T.

4.3.3 Lower Bounds for ϕ^F for the Network N_2:

Consider the network N_2 consisting of lumped capacitors, inductors, lossless tapered lines, VVT's, CCT's, CNIC's and VNIC's. Let the set $\{m_{k3}\} = \{m_{k1}\} \cup \{m_{k2}\} = \{C_i, C_{0i}\} \cup \{L_i, L_{0i}\}$, then it can readily be shown that

$$\phi^{Z_{12}} = \sum_{m_{k3}} |S^{Z_{12}}_{m_{k3}}|^2 \geq \frac{1}{4n_1}|S^{Z_{12}}_{\omega} - 1|^2 + \frac{1}{4n_2}|S^{Z_{12}}_{\omega} + 1|^2$$

$$\phi^{Y_{12}} = \sum_{m_{k3}} |S^{Y_{12}}_{m_{k3}}|^2 \geq \frac{1}{4n_1}|S^{Y_{12}}_{\omega} + 1|^2 + \frac{1}{4n_2}|S^{Y_{12}}_{\omega} - 1|^2$$

$$\phi^T = \sum_{m_{k3}} |S^T_{m_{k3}}|^2 \geq \frac{1}{4}(\frac{1}{n_1} + \frac{1}{n2})|S^T_{\omega}|^2$$

where n_1 and n_2 are respectively the number of elements in $\{m_{k1}\}$ and $\{m_{k2}\}$.

4.4 Conclusions:

Bounds have been obtained in the quadratic performance index ϕ^F for different kinds of networks. In particular, lower bounds on ϕ^F have been obtained for the networks N_0, N_1 and N_2, while both upper and lower bounds have been obtained for two element kind lumped networks. These have all been tabulated in Table 4.1.

It should be observed that for the network N_1, a better lower bound on ϕ^F is obtained by considering the parameter set $\{q_k\}$ rather than $\{p_k\}$. A similar statement may be made for the network N_2. It is also to be noted that for any of the networks considered in this Chapter, the lower bound on ϕ^F decreases as the number of elements in the network is increased, thereby, strengthening the second conjecture of Leeds and Ugron(6).

CHAPTER 5

SENSITIVITY INVARIANTS FOR NONLINEAR NETWORKS

5.1 Introduction:

In the previous Chapters, several sensitivity invariant relations were established for different classes of linear time-invariant networks; also the use of these invariants in establishing lower bounds for quadratic sensitivity index was discussed. No such invariant relations exist in the literature for nonlinear networks. With the increasing use of nonlinear elements in circuit design$^{(47)}$, such in invariant relationships for nonlinear networks should be useful in computer aided circuit design. In this Chapter, sensitivity invariants which offer a promising means of determining lower bounds for quadratic sensitivity index for nonlinear networks, are established$^{(48)}$. In Section 5.2, explicit formulas for sensitivities with respect to various parameters are established. These results are used in Section 5.3 to establish sensitivity invariant relationships for different combinations of excitations and responses. Sensitivity of the response due to changes in the excitation amplitudes are also taken into consideration in establishing the invariant relationships.

5.2 Sensitivity Components due to Nonlinear Elements:

In this Section, we will calculate the sensitivity of a response due to nonlinear elements.

Consider a general linear/nonlinear time-invariant network $\mathfrak{N}_0$, consisting of current controlled and voltage controlled resistors (CCRs and VCRs), voltage controlled and charge controlled capacitors (VCCs and QCCs), current controlled and flux controlled inductors (CCLs and FCLs)(47), current controlled current sources (CCTs), current controlled voltage sources (CVTs), voltage controlled current sources (VCTs), voltage controlled voltage sources (VVTs), voltage and current controlled gyrators, "hybrid gyrators" and impedance converters. Let $\mathcal{N}_0^A$ denote the adjoint of $\mathcal{N}_0$. We will use $\mathcal{N}_0$ and $\mathcal{N}_0^A$ and follow the procedure of Seth and Singhal(49) to obtain explicit formulas for sensitivities due to above nonlinear elements. Let the network $\mathcal{N}_0$ be considered at time ξ and $\mathcal{N}_0^A$ at time $\tau = t-\xi$. We know by Tellegen's theorem that

$$\sum_p \Delta v_p(\xi)\phi_p(t-\xi) - \Delta i_p(\xi)\psi_p(t-\xi) = \sum_e \Delta v_e(\xi)\phi_e(t-\xi) - \Delta i_e(\xi)\psi_e(t-\xi) \qquad .. (5.1)$$

where

i_p denotes current through the ports of $\mathcal{N}_0$.

v_p denotes the voltage across the ports of $\mathcal{N}_0$.

i_e denotes the current through the internal elements of $\mathcal{N}_0$.

v_e denotes the voltage across the internal elements of $\mathcal{N}_0$.

ϕ_p denotes the current through the ports of $\mathcal{N}_0^A$.

ψ_p denotes the voltage across the ports of $\mathcal{N}_0^A$.

ϕ_e denotes the current through the internal elements of $\mathcal{N}_0^A$.

ψ_e denotes the voltage across the internal elements of $\mathcal{N}_0^A$.

Δv_p, Δi_p, Δv_e and Δi_e are respectively the changes in v_p, i_p, v_e and i_e due to a small change in the parameters of network $\mathcal{N}_o^A$.

Integrating (5.1) from 0 to t, we get

$$\sum_p \int_0^t [\Delta v_p(\xi)\phi_p(t-\xi) - \Delta i_p(\xi)\psi_p(t-\xi)]d\xi$$
$$= \sum_e \int_0^t [\Delta v_e(\xi)\phi_e(t-\xi) - \Delta i_e(\xi)\psi_e(t-\xi)]d\xi \qquad .. (5.2)$$

Denoting $x*y = \int_0^t x(\xi)y(t-\xi)d\xi$, we may rewrite (5.2) in the form

$$\sum_p (\Delta v_p * \phi_p - \Delta i_p * \psi_p) = \sum_e (\Delta v_e * \phi_e - \Delta i_e * \psi_e) \qquad .. (5.3)$$

Let us consider a voltage controlled resistor (VCR) branch h described by

$$i_G = a_G f_{vG}(\beta_G v_G)$$

where

f_{vG} is assumed to be a single valued function of v_G.†

---

† For example, in the case of a junction diode governed by $i = I_s[\exp \frac{qv}{KT} - 1]$, I_s being the reverse saturation constant, q and K being constants and T the temperature, we may identify I_s and $\frac{q}{KT}$ with a and β respectively in (5.4). A variation in I_s corresponds to a variation in the parameter a, while a change in T would cause a change in β.

For small variations in the parameters a_G and β_G, we have

$$\Delta i_G = \Delta a_G f_{vG}(\beta_G v_G) + a_G f'_{vG}(\beta_G v_G)\beta_G \Delta v_G$$
$$+ a_G f'_{vG}(\beta_G v_G) v_G \Delta\beta_G$$

where the prime denotes differentiation with respect to $(\beta_G v_G)$, which is assumed to exist.

The contribution to the right hand side of (5.3) due to the terms corresponding to this element is given by

$$\int_0^t \Delta v_G[\phi_G(t-\xi) - a_G\beta_G f'_{vG}\{\beta_G v_G(\xi)\}\psi_G(t-\xi)]d\xi - \frac{\Delta a_G}{a_G}(i_G * \psi_G)$$
$$- \Delta\beta_G \int_0^t a_G f'_{vG}\{\beta_G v_G(\xi)\} v_G(\xi)\psi_G(t-\xi)d\xi \qquad .. (5.5)$$

If the corresponding branch in $\mathfrak{N}_0^A$ is defined by

$$\phi_G(t-\xi) = a_G\beta_G f'_{vG}\{\beta_G v_G(\xi)\}\psi_G(t-\xi) \qquad .. (5.6)$$

then (5.5) reduces to

$$- \frac{\Delta a_G}{a_G}(i_G * \psi_G) - \frac{\Delta\beta_G}{\beta_G}(v_G * \phi_G) \qquad .. (5.7)$$

The first term in (5.7) gives the sensitivity component due to a_G, while the second term that due to β_G.

Similarly, for a current controlled resistor (CCR) described by

$$v_R = \alpha_R f_{iR}(b_R i_R) \qquad .. (5.8)$$

it may be shown that the sensitivity components due to α_R and b_R are respectively

$$\frac{\Delta\alpha_R}{\alpha_R}(v_R * \phi_R) \qquad .. (5.9)$$

and

$$\frac{\Delta b_R}{b_R}(i_R * \psi_R) \qquad .. (5.10)$$

by choosing the adjoint element to be

$$\psi_R(t-\xi) = \alpha_R b_R f'_{iR}\{b_R i_R(\xi)\}\phi_R(t-\xi) \qquad .. (5.11)$$

If the resistor happens to be linear

$$v_R = R\ i_R \quad \text{or} \quad i_G = G\ v_G$$

we may associate G with either a_G or β_G, or R with b_G or α_G.

Let us next consider a voltage controlled capacitor (VCC) branch described by

$$\begin{aligned} q_C &= a_C f_{vC}(\beta_C v_C) \\ i_C &= \frac{d}{d\xi}\ (q_C) \end{aligned} \qquad .. (5.12)$$

a small variation in parameters gives

$$\Delta i_C = \frac{d}{d\xi}[\Delta a_C f_{vC}(\beta_C v_C) + a_C f'_{vC}(\beta_C v_C)\beta_C \Delta v_C + a_C f'_{vC}(\beta_C v_C)v_C \Delta\beta_C]$$

where the prime denotes differentiation with respect to $(\beta_C v_C)$. The contribution to the right hand side of (5.2) due to terms corresponding to this element is given by

$$\int_0^t \{\Delta v_C(\xi)\phi_C(t-\xi) - \frac{d}{d\xi}[\Delta a_C f_{vC}(\beta_C v_C) + a_C f'_{vC}(\beta_C v_C)\beta_C \Delta v_C + a_C f'_{vC}(\beta_C v_C) v_C \Delta\beta_C]\psi_C(t-\xi)\}d\xi$$

$$= \int_0^t \{\Delta v_C(\xi)[\phi_C(t-\xi) + a_C\beta_C f'_{vC}(\beta_C v_C)\frac{d}{d\xi}\psi_C(t-\xi)]\}d\xi$$

$$- [a_C\beta_C f'_{vC}(\beta_C v_C)\Delta v_C \psi_C(t-\xi)]_0^t - \frac{\Delta a_C}{a_C}(i_C * \psi_C)$$

$$- [a_C f'_{vC}(\beta_C v_C) v_C \Delta\beta_C \psi_C(t-\xi)\ ]_0^t + \int_0^t a_C f'_{vC}(\beta_C v_C) v_C \Delta\beta_C \frac{d}{d\xi}\psi_C(t-\xi)d\xi \qquad ..\ (5.13)$$

If the corresponding branch in $\mathfrak{N}_0^A$ is defined by

$$\phi_C(\tau) = a_C\beta_C f'_{vC}(\beta_C v_C)\frac{d}{d\tau}\psi_C(\tau),$$
$$\tau = (t-\xi) \qquad ..\ (5.14)$$

the first term on the right hand side of (5.13) is zero. The second term on the right hand side of (5.13) is

$$a_C\beta_C[f'_{vC}\{\beta_C v_C(0)\}\Delta v_C(0)\psi_C(t) - f'_{vC}\{\beta_C v_C(t)\}\Delta v_C(t)\psi_C(0)]$$

The second term in the above equation is set to zero by choosing the initial condition $\psi_C(0) = 0$ in $\mathfrak{N}_0^A$. The first term is either equal to zero for a given initial condition in $\mathfrak{N}$, or gives the sensitivity component with respect to the initial condition.

The third term on the right hand side of (5.13) gives the sensitivity component with respect to parameter a_C. The fourth and fifth terms on the right hand side of (5.13) can be written by using (5.14) and noting that $\psi_C(0) = 0$, as

$$- \frac{\Delta\beta_C}{\beta_C}[-a_C\beta_C f'_{vC}\{\beta_C v_C(0)\}v_C(0)\psi_C(t) + (v_C * \phi_C)] \quad .. (5.15)$$

which gives the sensitivity component due to β_C.

Let us next consider a nonlinear voltage controlled voltage source (VVT) described as in Table 5.1. A variation in the parameters α_{VV} and β_{VV} gives

$$\Delta v_2 = \Delta\alpha_{VV} f_{VV}(\beta_{VV}v_1) + \alpha_{VV}f'_{VV}(\beta_{VV}v_1)\beta_{VV}\Delta v_1$$

$$+ \alpha_{VV}f'_{VV}(\beta_{VV}v_1)v_1\Delta\beta_{VV}$$

$$\Delta i_1 = 0$$

where the prime denotes differentiation with respect to $(\beta_{VV}v_1)$. The contribution to the right hand side of (5.3) due to terms corresponding to this element is given by

$$\int_0^t \{\Delta v_1[\phi_1(t-\xi) + \alpha_{VV}\beta_{VV}f'_{VV}\{\beta_{VV}v_1(\xi)\}\phi_2(t-\xi)]$$

$$- \Delta i_2(\xi)\psi_2(t-\xi)\}d\xi + \frac{\Delta\alpha_{VV}}{\alpha_{VV}}(v_2 * \phi_2) \quad .. (5.16)$$

$$+ \frac{\Delta\beta_{VV}}{\beta_{VV}} \int_0^t \alpha_{VV}\beta_{VV}f'_{VV}\{\beta_{VV}v_1(\xi)\}v_1(\xi)\phi_2(t-\xi)d\xi$$

If the corresponding element in $\mathcal{M}_0^A$ is given by

TABLE 5.1

ADJOINT DESCRIPTIONS OF NONLINEAR ELEMENTS AND SENSITIVITY COMPONENTS

Element Classification	Description of the Element	Adjoint Description for Sensitivity Computation at time $t(\tau=t-\xi)$	Contribution to the right hand side of (2) due to a change in the parameter	
			a or b	α or β
Voltage controlled resistance (VCR)	$i_G=a_G f_{vG}(\beta_G v_G)$	$\phi_G(\tau) = a_G\beta_G f'_{vG}\{\beta_G v_G(\xi)\}\psi_G(\tau)$	$-\frac{\Delta a_G}{a_G}(i_G * \psi_G)$	$-\frac{\Delta\beta_G}{\beta_G}(v_G * \phi_G)$
Current controlled resistance (CCR)	$v_R=\alpha_R f_{iR}(b_R i_R)$	$\psi_R(\tau)=\alpha_R b_R f'_{iR}\{b_R i_R(\xi)\}\phi_R(\tau)$	$\frac{\Delta b_R}{b_R}(i_R * \psi_R)$	$\frac{\Delta\alpha_R}{\alpha_R}(v_R * \phi_R)$
Voltage controlled capacitance (VCC)	$q_C=a_C f_{vC}(\beta_C v_C)$ $i_c = \frac{d}{d\xi} q_C$	$\phi_C(\tau)=a_C\beta_C f'_{vC}\{\beta_C v_C(\xi)\}$ $\cdot \frac{d}{d\tau} \psi_G (\tau)$	$-\frac{\Delta a_C}{a_C}(i_C * \psi_C)$	$-\frac{\Delta\beta_C}{\beta_C}[v_C * \phi_C -$ $\beta_C a_C v_C(0)\psi_C(t)$ $\cdot f'_{vC}\{\beta_C v_C(0)\}]$
Charge controlled capacitance (QCC)	$v_C=\alpha_C f_{qC}(b_C q_C)$ $i_C= \frac{d}{d\xi} q_C$	$\frac{d}{d\tau}\psi_C(\tau)=\alpha_C b_C f'_{qC}\{b_C q_C(\xi)\}$ $\cdot \phi_C(\tau)$	$\frac{\Delta b_C}{b_C}[i_C * \psi_C + q_C(0) \cdot$ $\psi_{C(t)}]$	$\frac{\Delta\alpha_C}{\alpha_C}(v_C * \phi_C)$
Flux controlled inductor (FCL)	$i_L=a_L f_{\eta L}(\beta_L \eta_L)$ $v_L = \frac{d\eta_L}{d\xi}$	$\frac{d}{d\tau} \phi_L(\tau)= a_L\beta_L f'_{\eta L}\{\beta_L \eta_L(\xi)\}$ $\cdot \psi_L(\tau)$	$-\frac{\Delta a_L}{a_L}(i_L * \psi_L)$	$-\frac{\Delta\beta_L}{\beta_L}[v_L * \phi_L$ $+ \eta_L(0)\phi_L(t)]$

Current controlled inductor (CCL)	$\eta_L=\alpha_L f_{iL}(b_L i_L)$ $v_L = \frac{d\eta_L}{d\xi}$	$\psi_L(\tau) = \alpha_L b_L f'_{iL}\{b_L i_L(\xi)\}$ $\cdot \frac{d}{d\tau}\phi_L(\tau)$	$\frac{\Delta b_L}{b_L}[i_L * \psi_L)$ $-\alpha_L b_L i_L(0)\phi_L(t)$ $\cdot f'_{iL}\{b_L i_L(0)\}]$	$\frac{\Delta\alpha_L}{\alpha_L}(v_L * \phi_L)$
Current to current transducer (CCT)	$i_2=a_{CC}f_{CC}(b_{CC}i_1)$ $v_1 = 0$	$\psi_1(\tau)=-a_{CC}b_{CC}f'_{CC}\ \{b_{CC}i_1(\xi)\}$ $\cdot \psi_2(\tau)$ $\phi_2(\tau) = 0$	$-\frac{\Delta a_{CC}}{a_{CC}}(i_2 * \psi_2)$ $\frac{\Delta b_{CC}}{b_{CC}}(i_1 * \psi_1)$	
Voltage to voltage transducer (VVT)	$v_2=\alpha_{VV}f_{VV}(\beta_{VV}v_1)$ $i_1 = 0$	$\phi_1(\tau)=-\alpha_{VV}\beta_{VV}f'\{\beta_{VV}v_1(\xi)\}$ $\cdot \phi_2(\tau)$ $\psi_2(\tau) = 0$		$\frac{\Delta\alpha_{VV}}{\alpha_{VV}}(v_2 * \phi_2)$ $-\frac{\Delta\beta_{VV}}{\beta_{VV}}(v_1 * \phi_1)$
Voltage to current transducer (VCT)	$i_2=a_{VC}f_{VC}(\beta_{VC}v_1)$ $i_1 = 0$	$\phi_1(\tau)=a_{VC}\beta_{VC}f'\{\beta_{VC}v_1(\xi)\}$ $\cdot \psi_2(\tau)$ $\phi_2(\tau) = 0$	$-\frac{\Delta a_{VC}}{a_{VC}}(i_2 * \psi_2)$	$-\frac{\Delta\beta_{VC}}{\beta_{VC}}(v_1 * \phi_1)$
Current to voltage transducer (CVT)	$v_2=\alpha_{CV}f_{CV}(b_{CV}i_1)$ $v_1 = 0$	$\psi_1(\tau)=\alpha_{CV}b_{CV}f'\{b_{CV}i_1(\xi)\}$ $\cdot \phi_2(\tau)$ $\psi_2(\tau) = 0$	$\frac{\Delta b_{CV}}{b_{CV}}(i_1 * \psi_1)$	$\frac{\Delta\alpha_{CV}}{\alpha_{CV}}(v_2 * \phi_2)$

Voltage controlled gyrator (VCG)	$i_1 = a_{VG1} \cdot$ $f_{VG1}(\beta_{VG1} v_2)$ $i_2 = a_{VG2} \cdot$ $f_{VG2}(\beta_{VG2} v_1)$	$\phi_2(\tau) = a_{VG1} \beta_{VG1} \cdot$ $f'_{VG1}\{\beta_{VG1} v_2(\xi)\} \cdot \psi_1(\tau)$ $\phi_1(\tau) = a_{VG2} \beta_{VG2} \cdot$ $f'_{VG2}\{\beta_{VG2} v_1(\xi)\} \cdot \psi_2(\tau)$	$-\frac{\Delta a_{VG1}}{a_{VG1}}(i_1 * \psi_1)$ $-\frac{\Delta a_{VG2}}{a_{VG2}}(i_2 * \psi_2)$	$-\frac{\Delta \beta_{VG1}}{\beta_{VG1}}(v_2 * \phi_2)$ $-\frac{\Delta \beta_{VG2}}{\beta_{VG2}}(v_1 * \phi_1)$
Current controlled gyrator (CCG)	$v_1 = \alpha_{CG1} f_{CG1}(b_{CG1} i_2)$ $v_2 = \alpha_{CG2} f_{CG2}(b_{CG2} i_2)$	$\psi_1(\tau) = \alpha_{CG2} b_{CG2} f'_{CG2}\{b_{CG2} i_1(\xi)\}$ $\cdot \phi 2(\tau)$ $\psi_2(\tau) = \alpha_{CG1} b_{CG1} f'_{CG1}\{b_{CG1} i_2(\xi)\}$ $\cdot \phi_1(\tau)$	$\frac{\Delta b_{CG1}}{b_{CG1}}(i_2 * \psi_2)$ $\frac{\Delta b_{CG2}}{b_{CG2}}(i_1 * \psi_1)$	$\frac{\Delta \alpha_{CG1}}{\alpha_{CG1}}(v_1 * \phi_1)$ $\frac{\Delta \alpha_{CG2}}{\alpha_{CG2}}(v_2 * \phi_2)$
Hybrid gyrator (HG)	$v_1 = \alpha_{HG} f_{HG1}(b_{HG} i_2)$ $i_1 = a_{HG} f_{HG2}(\beta_{HG} v_2)$	$\psi_2(\tau) = \alpha_{HG} b_{HG} f'_{HG1}\{b_{HG} i_2(\xi)\}$ $\cdot \phi_1(\tau)$ $\phi_2(\tau) = a_{HG} \beta_{HG} f'_{HG2}\{\beta_{HG} v_2(\xi)\}$ $\cdot \psi_1(\tau)$	$-\frac{\Delta a_{HG}}{a_{HG}}(i_1 * \psi_1)$ $\frac{\Delta b_{HG}}{b_{HG}}(i_2 * \psi_2)$	$\frac{\Delta \alpha_{HG}}{\alpha_{HG}}(v_1 * \phi_1)$ $-\frac{\Delta \beta_{HG}}{\beta_{HG}}(v_2 * \phi_2)$
Impedance converter (IC)	$v_1 = \alpha_{IC} f_{IC1}(\beta_{IC} v_2)$ $i_1 = a_{IC} f_{IC2}(b_{IC} i_2)$	$\phi_2(\tau) = -\alpha_{IC} \beta_{IC} f'_{IC1}\{\beta_{IC} v_2(\xi)\}$ $\cdot \phi_1(\tau)$ $\psi_2(\tau) = -a_{IC} b_{IC} f'_{IC2}\{b_{IC} i_2(\xi)\}$ $\cdot \psi_1(\tau)$	$-\frac{\Delta a_{IC}}{a_{IC}}(i_1 * \psi_1)$ $\frac{\Delta b_{IC}}{b_{IC}}(i_2 * \psi_2)$	$\frac{\Delta \alpha_{IC}}{\alpha_{IC}}(v_1 * \phi_1)$ $-\frac{\Delta \beta_{IC}}{\beta_{IC}}(v_2 * \phi_2)$

where $f'(x) = \frac{\partial f(x)}{\partial x}$; $x * y = \int_0^t x(\xi) y(t-\xi) d\xi$

$$\phi_1(t-\xi) = -\alpha_{VV}\beta_{VV}f'_{VV}\{\beta_{VV}v_1(\xi)\}\phi_2(t-\xi) \qquad .. (5.17)$$

$$\psi_2(t-\xi) = 0$$

then the first term in (5.16) is zero, and the second term gives the sensitivity component with respect to α_{VV}. The last term in (5.16) can be rewritten with the help of (5.17) as

$$-\frac{\Delta\beta_{VV}}{\beta_{VV}}(v_1 * \phi_1) \qquad .. (5.18)$$

which gives the sensitivity component due to β_{VV}.

In a similar manner, the adjoint descriptions and the sensitivity components due to all the other elements could be found out and they are listed in Table 5.1. It is to be noted that all the different functions "f" are assumed to be single valued and differentiable. It should also be pointed out (Table 5.1) that the parameter symbols a and b are always associated with the current variables, while α and β with the voltage variables;further, the symbol b is always associated with an independent current variable, while the symbol β with an independent voltage variable. These symbols have been chosen in this manner so that the invariant relationships to be derived in Section 5.3 may be written in a compact way.

5.3 Sensitivity Invariants for Nonlinear Networks:

Consider the general n-port network $\mathcal{N}_0$ defined in the previous Section. Let $\mathcal{N}_0^A$ be its adjoint. Let the network $\mathcal{N}_0$ be considered at time ξ and $\mathcal{N}_0^A$ at time $\tau = t-\xi$. We know by Tellegen's theorem that

$$\sum_p\{i_p(\xi)\psi_p(t-\xi)\} = \sum_e\{i_e(\xi)\psi_e(t-\xi)\} \qquad .. (5.19)$$

and

$$\sum_p\{v_p(\xi)\phi_p(t-\xi) = \sum_e\{v_e(\xi)\phi_e(t-\xi) \qquad .. (5.20)$$

where

i_e and i_p are respectively the currents through internal elements and external ports for the network $\mathcal{N}_0$.

ψ_e and ψ_p are respectively the voltages across internal elements and external ports for the network $\mathcal{N}_0^A$.

v_e and v_p are the voltages across the internal element and external ports for the network $\mathcal{N}_0$.

ϕ_e and ϕ_p are the currents through the internal element and external ports for the network $\mathcal{N}_0^A$.

Integrating (5.19) and (5.20) from 0 to t, we get

$$\sum_p(i_p * \psi_p) = \sum_e (i_e * \psi_e) \qquad .. (5.21)$$

and

$$\sum_p(v_p * \phi_p) = \sum_e (v_e * \phi_e) \qquad .. (5.22)$$

Let the sensitivity of any response U, with respect to a parameter p_k be defined as

$$\bar{S}^{U}_{p_k} = p_k \frac{\partial U}{\partial p_k} \qquad .. \ (5.23)$$

We shall now establish some invariant relations, for two classes of nonlinear networks with different types of responses and excitations. We shall denote by $\mathcal{N}_1$ that class of nonlinear networks $\mathcal{N}_0$, where either the charge controlled capacitors (QCCs) and current controlled inductors (CCLs) are absent, or if present, their initial conditions $q_C(0) = i_L(0) = 0$. Also, we shall denote by $\mathcal{N}_2$ that class of nonlinear networks $\mathcal{N}_0$, where either the voltage controlled capacitors (VCCs) and flux controlled inductors (FCLs) are absent, or if present , their initial conditions $v_C(0) = \eta_L(0) = 0$. We will first obtain sensitivity invariant relations, with different types of excitations and responses, assuming changes only in the parameters a, b, α and β in the different network elements.

Case 1. Voltage Response due to Voltage Excitation:

Consider the network $\mathcal{N}_1$ whose adjoint is denoted by $\mathcal{N}_1^A$. Let v_2 denotes the open-circuit voltage response at port (2) of $\mathcal{N}_1$, due to a voltage excitation at port (1), all other ports being open. Let port (2) of $\mathcal{N}_1^A$ be excited by an impulse current function $\delta(\tau)$ with port (1) short-circuited, and all other ports open-circuited. Then the left hand side of (5.2) can be written as

$$\int_0^t \Delta v_2(\xi)\delta(t-\xi)d\xi = \Delta v_2(t) \qquad .. \ (5.24)$$

The right hand side of (5.2) gives the contributions to this change

$\Delta v_2(t)$ due to the variations in the different parameters a's, α's, b's, β's of the network elements in $\mathcal{N}_1$. These have already been tabulated in Table 5.1. For example, the contribution due to the a_G of a VCR is given by

$$-\frac{\Delta a_G}{a_G}(i_G * \psi_G) \quad .. (5.25)$$

Thus, from (5.24) and (5.25), we have

$$\bar{S}^{v_2}_{a_G} = -a_G \frac{\partial v_2}{\partial a_G} = -(i_G * \psi_G) \quad .. (5.26)$$

Similarly, with respect to β_G of a VCR, we have from Table 5.1 and (5.24)

$$\bar{S}^{v_2}_{\beta_G} = -(v_G * \phi_G) \quad .. (5.27)$$

The contributions due to the parameters of the other one-port nonlinear elements namely VCC, QCC, FCL and CCL may be obtained using Table 5.1 in a similar manner.

It may also be seen from Table 5.1 and (5.24) that for a CCT

$$-\bar{S}^{v_2}_{a_{CC}} + \bar{S}^{v_2}_{b_{CC}} = \sum_e (i_e * \psi_e) \quad .. (5.28)$$

where the summation on the right hand side of (5.28) extends over the two ports of the CCT in $\mathcal{N}_1$ and the corresponding adjoint element in $\mathcal{N}_1^A$. Similarly, we may obtain the sensitivity components for

CVT, VCT, VCG, CCG, HG and IC.

Thus, using (5.24) and Table 5.1, we get

$$- \Sigma \, \overline{S}^{v_2}_{a_i} + \Sigma \, \overline{S}^{v_2}_{b_j} = \Sigma'_e (i_e * \psi_e) \qquad .. (5.29)$$

It should be pointed out that on the left hand side of (5.29), only those elements in whose descriptions a and/or b are present, are included, further, the prime on the right hand side indicates that the summation is only over such elements. Thus the VVT is not included. However, for a VVT, it may be verified directly using Table 5.1, that

$$\Sigma_e (i_e * \psi_e) = 0 \qquad .. (5.30)$$

Hence, (5.29) may be written as

$$-\Sigma \, \overline{S}^{v_2}_{a_i} + \Sigma \, \overline{S}^{v_2}_{b_j} = \Sigma_e (i_e * \psi_e) \qquad .. (5.31)$$

where the right hand side now includes all the elements of $\mathfrak{N}_1$, even though the sensitivities due to the parameters of a VVT are not included in the left hand side. Now, using (5.22) in (5.31), we have

$$-\Sigma \, \overline{S}^{v_2}_{a_i} + \overline{S}^{v_2}_{b_j} = \Sigma_p (i_p * \psi_p) \qquad .. (5.32)$$

The right hand side of (5.32) is zero, in view of the excitations and terminations of $\mathfrak{N}_1$ and $\mathfrak{N}^A_1$. Hence

$$\Sigma \bar{S}_{a_i}^{v_2} - \Sigma \bar{S}_{b_j}^{v_2} = 0 \qquad .. (5.33)$$

We may also obtain using (5.24) and Table 5.1, that

$$\Sigma \bar{S}_{\alpha_i}^{v_2} - \Sigma \bar{S}_{\beta_j}^{v_2} = \Sigma_e' (v_e * \psi_e) \qquad .. (5.34)$$

where now the left hand side includes all the elements whose descriptions contain α and/or β. Hence, the CCT parameters cannot be included on the left hand side of (5.34) and thus Σ_e' does not contain the CCT elements. However, it may again be verified directly from Table 5.1 that for a CCT

$$\Sigma_e (v_e * \phi_e) = 0 \qquad .. (5.35)$$

Thus

$$\Sigma \bar{S}_{\alpha_i}^{v_2} - \Sigma \bar{S}_{\beta_j}^{v_2} = \Sigma_e (v_e * \phi_e) \qquad .. (5.36)$$

where Σ_e, now includes all the elements of $\mathfrak{N}_1$. By using (5.21) in (5.36), we get

$$\Sigma \bar{S}_{\alpha_i}^{v_2} - \Sigma \bar{S}_{\beta_j}^{v_2} = \Sigma_p (v_p * \phi_p) \qquad .. (5.37)$$

In view of terminations and excitations in $\mathfrak{N}_1$ and $\mathfrak{N}_1^A$, we have

$$\Sigma_p (v_p * \phi_p) = (v_1 * \phi_1) + v_2 \qquad .. (5.38)$$

which need not be an invariant in general. Hence, we have no invariance in general for the expression

$$\Sigma \bar{S}_{\alpha_1}^{v_2} - \Sigma \bar{S}_{\beta_j}^{v_2} \qquad .. (5.39)$$

Case 2. Current Response due to Voltage Excitation:

Let i_2 denote the short circuit current response at port (2) of $\mathcal{N}_1$ due to voltage excitation v_1 at port (1), all other ports being short-circuited. Let port (2) of $\mathcal{N}_1^A$ be excited by an impulse voltage function $\delta(\tau)$, all other ports being short-circuited, then the left hand side of (5.3) can be written as

$$- \int_0^t \Delta i_2(\xi)\delta(t-\xi)d\xi = - \Delta i_2(t) \qquad .. (5.40)$$

Following the same procedure as used in Case 1, we have from the right hand side of (5.3), (5.40) and Table 5.1

$$\Sigma \bar{S}_{a_i}^{i_2} - \Sigma \bar{S}_{b_j}^{i_2} = \sum_e (i_e * \psi_e) = \sum_p (i_p * \psi_p) \qquad .. (5.41)$$

But, in view of the excitations and terminations in $\mathcal{N}_1$ and $\mathcal{N}_1^A$, we have

$$\sum_p (i_p * \psi_p) = i_2 \qquad .. (5.42)$$

Thus, from (5.41) and (5.42), we get

$$\Sigma \bar{S}_{a_i}^{i_2} - \Sigma \bar{S}_{b_j}^{i_2} = i_2 \qquad .. (5.43)$$

Case 3. Voltage response due to current excitation:

Let us now consider the open-circuit voltage response v_2 at port (2) due to current excitation i_1 at port (1) of $\mathcal{N}_2$, all other ports being open. Let port (2) of $\mathcal{N}_2^A$, the adjoint of $\mathcal{N}_2$, be excited by an impulse current function $\delta(\tau)$, all other ports of $\mathcal{N}_2^A$ being open. Then the left hand side of (5.2) can be written as

$$\int_0^t \Delta v_2(\xi)\delta(t-\xi)d\xi = \Delta v_2(t) \qquad .. \ (5.44)$$

Following the same procedure as in Case 1, and using the right hand side of (5.3), (5.44) and Table 5.1, we can show that

$$\Sigma \, \bar{S}_{\alpha_i}^{v_2} - \Sigma \, \bar{S}_{\beta_j}^{v_2} = \sum_e (v_e * \phi_e) \qquad .. \ (5.45)$$

Again using (5.22), we have

$$\sum_e (v_e * \phi_e) = \sum_p (v_p * \phi_p) = v_2$$

in view of the excitations and terminations in $\mathcal{N}_2$ and $\mathcal{N}_2^A$. Hence (5.45) reduces to

$$\Sigma \, \bar{S}_{\alpha_i}^{v_2} - \Sigma \, \bar{S}_{\beta_j}^{v_2} = v_2 \qquad .. \ (5.46)$$

Case 4. Current response due to current excitation:

Let us consider the short-circuit current response i_2 at port (2), due to a current excitation i_1 at port (1) of $\mathcal{N}_2$, all other ports being short-circuited. Let port (2) of $\mathcal{N}_2^A$ be excited by an impulse voltage function $\delta(\tau)$, with port (1) open, and all other ports short-circuited. Then it follows from the left hand side of (5.3) that

$$- \int_0^t \Delta i_2(\xi)\delta(t-\xi)d\xi = -\Delta i_2(t) \qquad .. (5.47)$$

Following the same procedure as in Case 1 and using the right hand side of (5.3), (5.47), Table 5.1, and the excitations as well as terminating conditions in $\mathcal{N}_2$ and $\mathcal{N}_2^A$, we can show that

$$\Sigma \bar{S}_{\alpha_i}^{i_2} {}^2 - \Sigma \bar{S}_{\beta_j}^{i_2} {}^2 = 0 \qquad .. (5.48)$$

We shall next establish similar invariant relations, with different types of excitations and responses, when the change in the response due to change in the amplitude of excitation is also considered.

Case 5. Voltage response due to voltage excitation:

Let us consider the open-circuit voltage response v_2 at port (2), due to voltage excitation v_1 at port (1) of $\mathcal{N}_2$, all other ports being open-circuited. Let the second port of $\mathcal{N}_2^A$ be excited by an impulse current function $\delta(\tau)$ with port (1) short-circuited and all other ports open.

Let us first consider the variation in the response due to a variation in the amplitude of excitation only. Since the variation in the response due to the parameters a's, b's, α's and β's of the network elements have been assumed to be zero, equation (5.2) can be written as

$$\int_0^t \Delta v_1(\xi)\phi_1(t-\xi)d\xi + \Delta v_2(t) = 0 \qquad .. (5.49)$$

Let

$$v_1 = V_1\ f(t) \qquad .. (5.50)$$

then the variation in v_1 due to a change in amplitude V_1 of v_1 is

$$\Delta v_1 = \Delta V_1\ f(t)$$

Hence, (5.49) may be written as

$$\Delta v_2 = -\frac{\Delta V_1}{V_1}(v_1 * \phi_1) \qquad .. (5.51)$$

Thus

$$\bar{S}_{V_1}^{v_2} = -(v_1 * \phi_1)$$

We also have, in view of the terminating and excitations conditions in $\mathcal{N}_2$ and $\mathcal{N}_2^A$, from (5.22) that

$$(v_1 * \phi_1) + v_2 = \sum_e (v_e * \phi_e) \qquad .. (5.52)$$

Thus, from (5.51) and (5.52), we get

$$-\overline{S}^{v_2}_{V_1} + v_2 = \sum_e (v_e * \phi_e) \qquad .. (5.53)$$

We have already shown in Case 1 that

$$\Sigma \overline{S}^{v_2}_{\alpha_1} - \Sigma \overline{S}^{v_2}_{\beta_j} = \sum_e (v_e * \phi_e) \qquad .. (5.36)$$

Hence, from (5.53) and (5.36), we have

$$\Sigma \overline{S}^{v_2}_{V_1} + \Sigma \overline{S}^{v_2}_{\alpha_i} - \Sigma \overline{S}^{v_2}_{\beta_j} = v_2 \qquad .. (5.54)$$

Case 6. Voltage response due to current excitation:

Consider the open-circuit voltage response v_2 at port (2) due to a current excitation i_1 at port (1) of $\mathcal{N}_1$, all other ports being open-circuited. Let port (2) of $\mathcal{N}_1^A$ be excited by an impulse current function $\delta(\tau)$, all other ports being open. If there is a small variation in the amplitude I_1 of the excitation $i_1 = I_1 f(t)$, then it can be shown following the same procedure as used in Case 5 that

$$\overline{S}^{v_2}_{I_1} + \Sigma \overline{S}^{v_2}_{a_i} - \Sigma \overline{S}^{v_2}_{b_j} = 0$$

Case 7. Current Response due to Voltage Excitation:

Consider the short-circuit current response i_2 at port (2) due to a voltage excitation v_1 at port (1) of network $\mathcal{N}_2$, all other ports being short-circuited. Let port (2) of $\mathcal{N}_2^A$ be excited by an impulse voltage function $\delta(\tau)$, all other ports being short-circuited. If there is a small change in the amplitude V_1 of excitation $v_1 = V_1 f(t)$, then it can readily be shown that

$$\bar{S}_{V_1}^{i_2} + \Sigma\, \bar{S}_{\alpha_1}^{i_2} - \Sigma\, \bar{S}_{\beta_j}^{i_2} = 0 \qquad .. \ (5.56)$$

Case 8. Current response due to current excitation:

Consider a short-circuit current response i_2 at port (2), due to a current excitation i_1 at port (1) of $\mathcal{N}_1$, all other ports being short-circuited. Let port (2) of $\mathcal{N}_1^A$ be excited by an impulse voltage function $\delta(\tau)$, port (1) being open and all other ports short-circuited. If there is a slight variation in the amplitude I_1 of excitation $i_1 = I_1 f(t)$, then it can be readily shown that

$$\bar{S}_{I_1}^{i_2} + \Sigma\, \bar{S}_{a_i}^{i_2} - \Sigma\, \bar{S}_{b_j}^{i_2} = i_2 \qquad .. \ (5.57)$$

The different sensitivity invariant relationships derived above for networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are summarized below:

I. For network $\mathcal{N}_1$:

(a) For voltage excitation

$$\text{(i)}\ \Sigma\, \bar{S}_{a_i}^{v_2} - \Sigma\, \bar{S}_{b_j}^{v_2} = 0 \qquad .. \ (5.58)$$

$$\text{(ii)}\ \Sigma\, \bar{S}_{a_i}^{i_2} - \Sigma\, \bar{S}_{b_j}^{i_2} = i_2 \qquad .. \ (5.59)$$

(b) For current excitation $i_1 = I_1 f(t)$

$$\text{(i)}\quad \overline{S}_{I_1}^{v_2} + \Sigma\, \overline{S}_{a_i}^{v_2} - \Sigma\, \overline{S}_{b_j}^{v_2} = 0 \qquad \text{.. (5.60)}$$

$$\text{(ii)}\quad \overline{S}_{I_1}^{i_2} + \Sigma\, \overline{S}_{a_i}^{i_2} - \Sigma\, \overline{S}_{b_j}^{i_2} = i_2 \qquad \text{.. (5.61)}$$

II. <u>For network</u> $\mathcal{N}_2$:

(a) For voltage excitation $v_1 = V_1 f(t)$

$$\text{(i)}\quad \overline{S}_{V_1}^{v_2} + \Sigma\, \overline{S}_{\alpha_i}^{v_2} - \Sigma\, \overline{S}_{\beta_j}^{v_2} = v_2 \qquad \text{.. (5.62)}$$

$$\text{(ii)}\quad \overline{S}_{V_1}^{i_2} + \Sigma\, \overline{S}_{\alpha_i}^{v_2} - \Sigma\, \overline{S}_{\beta_j}^{i_2} = 0 \qquad \text{.. (5.63)}$$

(b) For current excitation

$$\text{(i)}\quad \Sigma\, \overline{S}_{\alpha_i}^{i_2} - \Sigma\, \overline{S}_{\beta_j}^{i_2} = 0 \qquad \text{.. (5.64)}$$

$$\text{(ii)}\quad \Sigma\, \overline{S}_{\alpha_i}^{v_2} - \Sigma\, \overline{S}_{\beta_j}^{v_2} = v_2 \qquad \text{.. (5.65)}$$

It should be pointed that if in the network $\mathcal{N}_0$, all the initial conditions for VCCs, QCCs, CCLs and FCLs are zero, that is,

$$V_C(0) = q_C(0) = i_L(0) = \eta_L(0) = 0$$

then the two classes $\mathcal{N}_1$ and $\mathcal{N}_2$ are indistinguishable. In such a case, all the relations (5.58)-(5.65) apply.

It should also be observed that if the networks $\mathcal{N}_1$ and $\mathcal{N}_2$ contain linear CCTs and VVTs, then it may be verified directly from Table 5.1 that for both CCT and VVT

$$\sum_e (v_e * \phi_e) = 0$$

and

$$\sum_e (i_e * \psi_e) = 0$$

Thus, the equations (5.58)-(5.65) are true provided the sensitivity terms corresponding to the VVTs and CCTs are not included, though the networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are allowed to contain linear VVTs and CCTs.

The results of this Section will now be verified with the help of an example.

Example:

Consider the circuit given in Fig.5.1, where the diode is described by

$$i_1 = a(e^{\beta v_2} - 1) \qquad .. (5.66)$$

Let V_1 be a d.c. excitation and v_2 the response. Since the network may be considered as belonging to either class $\mathcal{N}_1$ or $\mathcal{N}_2$, the relations (5.58) as well as (5.62) apply.

The linear resistors may be considered either as VCRs

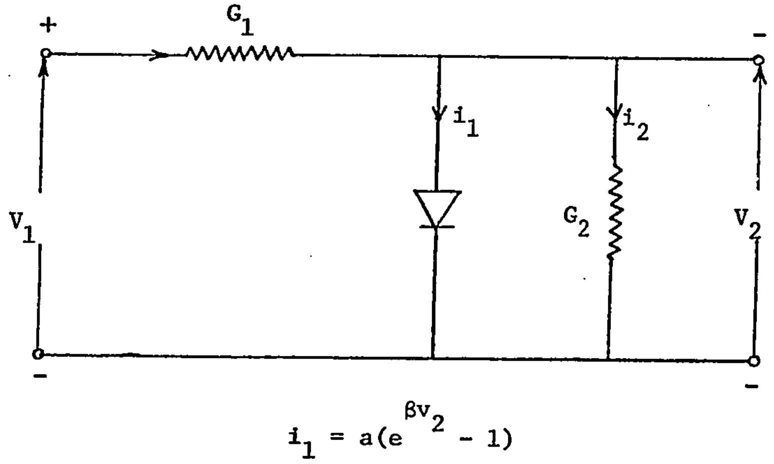


Fig. 5.1: The Network for the Example in Section 5.3

$$i = G_1 v = G_1(V_1 - v_2)$$

$$i_2 = G_2 v_2$$

or as CCRs

$$v = (V_1 - v_2) = R_1 i$$

$$v_2 = R_2 i_2$$

First let us consider the linear resistors as VCRs. Then, we may identify G_1 and G_2 as a_1 and a_2 or β_1 and β_2. In order to use (5.58), G_1 and G_2 will have to be identified with a_1 and a_2 not with β_1 and β_2, since otherwise the left hand side of (5.58) will not include the resistors (but at the same time, the summation should be over all the elements of network). Also, in order to use (5.62), G_1 and G_2 will have to be identified with β_1 and β_2 and not with a_1 and a_2, since otherwise left hand side of (5.62) will not include the linear resistors.

Thus, from (5.58), we get

$$\bar{S}_{G_1}^{v_2} + \bar{S}_{G_2}^{v_2} + \bar{S}_{a}^{v_2} = 0 \qquad .. (5.67)$$

while from (5.62), we have

$$\bar{S}_{V_1}^{v_2} - \bar{S}_{G_1}^{v_2} - \bar{S}_{G_2}^{v_2} - \bar{S}_{\beta}^{v_2} = v_2 \qquad .. (5.68)$$

If we now consider the linear resistors as CCRs, then we may identify R_1 and R_2 with b_1 or b_2 or α_1 and α_2. For reasons similar to the one given above, R_1 and R_2 have to be identified with b_1 and b_2 in (5.58), and with α_1 and α_2 in (5.62). Then, we get

$$- \bar{S}^{v_2}_{R_1} - \bar{S}^{v_2}_{R_2} + \bar{S}^{v_2}_{a} = 0 \qquad .. (5.69)$$

$$\bar{S}^{v_2}_{V_1} + \bar{S}^{v_2}_{R_1} + \bar{S}^{v_2}_{R_2} - \bar{S}^{v_2}_{\beta} = v_2 \qquad .. (5.70)$$

Of course (5.67) and (5.69) as well as (5.70) and (5.68) are the same.

If now i_2 is the response, then (5.59) and (5.63) apply. Whether we consider the linear resistors to be VCRs or CCRs, they will give rise to

$$\bar{S}^{i_2}_{G_1} + \bar{S}^{i_2}_{G_2} + \bar{S}^{i_2}_{a} = i_2 \qquad .. (5.71)$$

$$\bar{S}^{i_2}_{V_1} - \bar{S}^{i_2}_{G_1} - \bar{S}^{i_2}_{G_2} - \bar{S}^{i_2}_{\beta} = 0 \qquad .. (5.72)$$

These relations will now be verified directly by writing Kirchoff's equation for the network and calculating the sensitivities directly Now we have for the network of Fig. 5.1,

$$G_1V_1 = G_1v_2 + a(e^{\beta v_2} - 1) + G_2v_2 \qquad .. (5.73)$$

Differentiating (5.73) with respect to G_1, G_2 and a respectively, we have

$$(V_1 - v_2) = G_1 \frac{\partial v_2}{\partial G_1} + a\beta e^{\beta v_2} \frac{\partial v_2}{\partial G_1} + G_2 \frac{\partial v_2}{\partial G_1}$$

or

$$G_1 (V_1 - v_2) = (G_1 + a\beta e^{\beta v_2} + G_2)\, \overline{S}_{G_1}^{v_2} \qquad .. (5.74)$$

$$0 = G_1 \frac{\partial v_2}{\partial G_2} + a\beta e^{\beta v_2} \frac{\partial v_2}{\partial G_2} + v_2 + G_2 \frac{\partial v_2}{\partial G_2}$$

or

$$0 = (G_1 + a\beta e^{\beta v_2} + G_2)\, \overline{S}_{G_2}^{v_2} + G_2 v_2 \qquad .. (5.75)$$

and

$$0 = G_1 \frac{\partial v_2}{\partial a} + a\beta e^{\beta v_2} \frac{\partial v_2}{\partial a} + (e^{\beta v_2} - 1) + G_2 \frac{\partial v_2}{\partial a}$$

or

$$(G_1 + a\beta e^{\beta v_2} + G_2)\, \overline{S}_{a}^{v_2} + a(e^{\beta v_2} - 1) \qquad .. (5.76)$$

Adding (5.74), (5.75) and (5.76), and bearing in mind (5.73), we have

$$(G_1 + G_2 + a\beta e^{\beta v_2})(\overline{S}_{G_1}^{v_2} + \overline{S}_{G_2}^{v_2} + \overline{S}_{a}^{v_2}) = 0 \qquad .. (5.77)$$

Since the first factor in left hand side of (5.77) is not zero, the second term must be zero. Hence

$$\overline{S}_{G_1}^{v_2} + \overline{S}_{G_2}^{v_2} + \overline{S}_{a}^{v_2} = 0 \qquad .. (5.78)$$

thus verifying the relation (5.67).

Similarly, by differentiating (5.73) with respect to β and v_1 respectively, we have

$$0 = (G_1 + a\beta e^{\beta v_2} + G_2)\,\bar{S}_{\beta}^{v_2} + v_2 a\beta e^{\beta v_2} \qquad ..\ (5.79)$$

and

$$GV_1 = (G_1 + G_2 + a\beta e^{\beta v_2})\,\bar{S}_{V_1}^{v_2} \qquad ..\ (5.80)$$

From (5.74), (5.75), (5.79) and (5.80), we have

$$\bar{S}_{V_1}^{v_2} - \bar{S}_{G_1}^{v_2} - \bar{S}_{G_2}^{v_2} - \bar{S}_{\beta}^{v_2} = v_2 \qquad ..\ (5.81)$$

thus verifying (5.68). Equations (5.71) and (5.72) will now be verified by noting that

$$v_2 = \frac{i_2}{G_2} \qquad ..\ (5.82)$$

and using (5.78) and (5.81).

From (5.82), for any parameter x of the network, it follows that

$$\begin{aligned} \bar{S}_x^{v_2} &= \frac{1}{G_2}\,\bar{S}_x^{i_2} - v_2 & x &= G_2 \\ &= \frac{1}{G_2}\,\bar{S}_x^{i_2} & x &\neq G_2 \end{aligned} \qquad ..\ (5.83)$$

Using (5.83) in (5.78), we have

$$\frac{i}{G_2}\bar{S}_{G_1}^{i_2} + \frac{1}{G_2}\bar{S}_{G_2}^{i_2} - v_2 + \frac{1}{G_2}\bar{S}_{a}^{i_2} = 0$$

or

$$\bar{S}_{G_1}^{i_2} + \bar{S}_{G_2}^{i_2} + \bar{S}_{a}^{i_2} = i_2 \qquad ..\ (5.84)$$

which verifies (5.71).

Similarly, using (5.83) in (5.81), we have

$$\frac{1}{G_2}\bar{S}_{V_1}^{i_2} - \frac{1}{G_2}\bar{S}_{G_1}^{i_2} - \frac{1}{G_2}\bar{S}_{G_2}^{i_2} + v_2 - \frac{1}{G_2}\bar{S}_{\beta}^{i_2} = v_2$$

or

$$\bar{S}_{V_1}^{i_2} - \bar{S}_{G_1}^{i_2} - \bar{S}_{G_2}^{i_2} - \bar{S}_{\beta}^{i_2} = 0 \qquad ..\ (5.85)$$

thus verifying (5.72).

5.4 Conclusions:

In this Chapter, we have established some sensitivity invariant relationships for certain classes of nonlinear networks with different types of excitations and responses.

We note that higher order sensitivity invariants for networks $\mathcal{N}_1$ and $\mathcal{N}_2$ may be established using the following relation (2.154) proved in Chapter 2;

$$X_{k+1} = \sum_{i=1}^{n} p_i \frac{\partial X_k}{\partial p_i} - kX_k$$

where X_j denotes the sum of jth order sensitivities over the set of parameters (p_i, $i = 1,2, \ldots ,n$).

Further, if the quadratic sensitivity index ϕ is defined for nonlinear networks in the same manner as it is done for linear networks(13), then the sensitivity invariant relationships established in this Chapter, can be used to obtain lower bounds for ϕ, following the approach used in Chapter 4.

CHAPTER 6

GENERALIZED DUALS, DUAL ADJOINTS, GENERALIZED DUAL TRANSPOSES AND THEIR APPLICATIONS

6.1 Introduction:

For LTI networks, Mitra et al(30) have defined generalized duals and generalized inverses. They define the generalized inverse of an n-port network to be a network for which the impedance matrix Z is F(s) times the admittance matrix of the given n-port. Depending on the nature of F(s), they also define resistive, inductive and capacitive inverses. They also define the generalized dual of an n-port network consisting of one-ports and controlled sources to be a network for which the topology is dual of that of the given network and each element is the generalized inverse of the corresponding element in the given network. They also show that the generalized dual is always a generalized inverse, while the converse is not true. They also consider some applications of capacitive inverses and duals in active lumped RC-network design.

In Section 6.2, we first extend the definition of generalized inverse for a general n-port consisting of linear/nonlinear, time-varying/time-invariant, lumped/distributed elements. We then give a simple method of obtaining the generalized dual of a general planar three-terminal (3-T) two-port network N, consisting as subnetworks not only one-ports but also 3-T two-ports. In this method, not only one-ports but also the internal 3-T two-ports of N are treated as blocks and replaced in N_D, the generalized dual network by suitable one-ports and 3-T two-ports. We will also consider some applications of the generalized duals in synthesis.

It was pointed out in Section 1.2 that the Tellegen's theorem for networks having the same topology has been used by Director and Rohrer in defining the adjoint network, and making use of it in the calculations of sensitivities of a given network. This theorem has also been widely used in this thesis in obtaining the results of Chapters 2 and 5 concerning sensitivity invariance. It was also pointed out in Section 1.3 that, in the case of LTI networks, the adjoint reduces to the network transpose defined by Bhattacharyya and Swamy(29), who have used it as a network operation in obtaining structures for realizing a voltage transfer function from that of a current transfer function and vice-versa, and in obtaining structures for realizing driving point functions.

In Section 6.3 we develop a new theorem, similar to the Tellegen's theorem, but for networks having dual topologies. This new theorem is applicable to any two networks which have dual topologies and obey Kirchhoff's laws, whether they contain linear or nonlinear, time-invariant or time-varying, active or passive, lumped or distributed elements; the excitation is arbitrary, and the initial conditions immaterial. In Section 6.4, this new theorem will be used to define a new network, "dual adjoint"; the application of the dual adjoint in sensitivity calculations of a planar network will also be considered. In Section 6.5, the new theorem will be used, in the frequency domain, to develop a new network operation called "Generalized Dual Transposition". The applications of the generalized dual transposition in the calculation of the sensitivities, as well as in network synthesis will also be considered.

6.2 Generalized Duals, Generalized Inverses and Applications:

In this section, some definitions concerning generalized duals, generalized inverses, planar 3-T two-ports etc., are first given. Then a simple method of obtaining directly the generalized dual of a 3-T two-port consisting as subnetworks one-ports and 3-T two-ports will be given. Finally some applications of generalized duals in RC-active synthesis will be considered.

6.2.1 Definitions:

Generalized Inverse: Let V_p and I_p be the vectors of port voltages and currents of a general n-port network consisting of linear/nonlinear, time-varying/time-invariant, and lumped/distributed elements. Then we define the generalized inverse to be a n-port network, for which the port voltages V_{pI} and currents I_{pI} are given by $V_{pI} = \lambda_1 I_p$ and $I_{pI} = \lambda_2 V_p$, where λ_1 and λ_2 are respectively Kirchoff's current and voltage operators(37).

Generalized Dual: Consider a general planar network $\mathfrak{N}$. Let G be its directed graph, where the edge e_i corresponds to the branch whose voltage and current are V_i and I_i, and the orientation of e_i corresponds to the polarities of V_i (and the direction of I_i). Let G_D be the dual of the directed graph G, where e_i and e_{iD} are the corresponding edges in G and G_D. Then we define the generalized dual network of $\mathfrak{N}$ to be network $\mathfrak{N}_D$ if the directed graph of $\mathfrak{N}_D$ is G_D and the different branch voltages and currents (V_{iD}, I_{iD}) represented by the directed edges e_{iD} are related to the corresponding branch voltages and currents (V_i, I_i) in $\mathfrak{N}$ by $V_{iD} = \lambda_1 I_i$, $I_{iD} = \lambda_2 V_i$, where λ_1 and λ_2 are respectively Kirchoff's current and voltage operators.

It should be noted that the conventional dual$^{(50)}$ is a special case for which $V_{iD} = I_i$, $I_{iD} = V_i$.

Since this section will primarily deal with the generalized duals of 3-T two-ports, consisting as subnetworks one-ports and 3-T two-ports, we shall define the following terms concerning such networks.

Graph of a 3-T two-port: Consider a 3-T two-port as shown in Fig.6.1(a). We shall then define the graph of the two-port to be that shown in Fig.6.1(b), the edges representing the port voltages and currents.

A Planar 3-T two-port: Let N be any 3-T two-port consisting

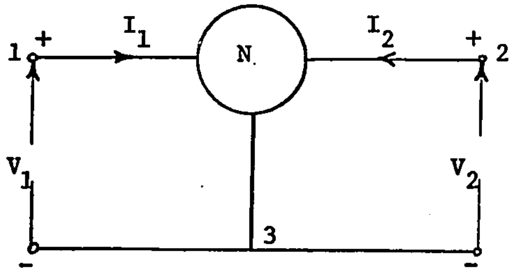


Fig. 6.1(a) 3-T Two-port Network

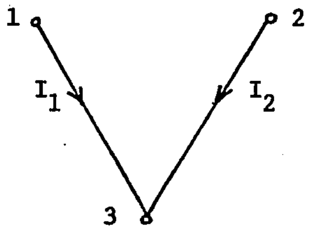


Fig. 6.1(b): Graph of Fig. 6.1(a)

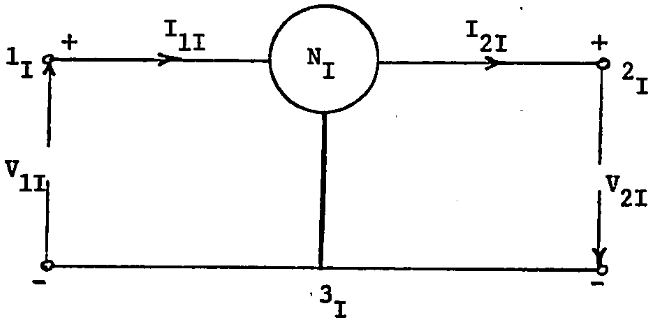


Fig. 6.1(c): Generalized Inverse of Fig. 6.1(a)

as subnetworks one-ports and 3-T two-ports. Then we define N to be planar if after (i) replacing each of the internal one-ports and 3-T two-ports by their graphs and (ii) adding external edges corresponding to the two-ports of N, the resulting graph is planar.

According to our definitions, the polarities of the port variables of the generalized inverse are arbitrary; hence, we get different realizations for the generalized inverse, depending on the choice for the polarities. However, we shall choose the polarities for the port-variables of the generalized inverse of a 3-T two-port to be as shown in Fig.6.1(c). Hereafter, in this thesis, we mean by the generalized inverse of a 3-T two-port N (Fig.6.1(a)), another 3-T two-port with polarities as shown in Fig.6.1(c), with $V_{pI} = \lambda_1 I_p$ and $I_{pI} = \lambda_2 V_p$; further the generalized inverse of N will be denoted by N_I. The reason for this particular choice of polarities will be clear later in this section.

6.2.2 Generalized Dual of a 3-T two-port:

We now give a simple procedure of directly obtaining the generalized dual N_D of a planar 3-T two-port consisting as subnetworks one-ports and 3-T two-ports. It is to be noted that the familiar technique of obtaining dual networks is not directly applicable, since it is restricted to networks containing one-ports only as elements. In the method to be discussed below, not only the one-ports but also the internal 3-T two-ports are treated as blocks, and replaced in N_D by their corresponding generalized inverses.

Consider a planar 3-T two-port which has as internal elements, one-ports and 3-T two-ports, (for example, the network of Fig.6.2(a)). Obtain the directed graph of N as shown in Fig.6.2(b). The direction of the edges in G represent the direction of the current through the corresponding port, whether it be internal or external. The tips of the arrows represent the negative polarities for the voltages across the internal ports, while they represent the positive polarities for the voltages across the two external ports. We obtain now the dual graph of G_D of the graph G in the conventional way. We assume that corresponding to an internal edge e in G, the dual edge e_D in G_D is convergent on a node n_D, if the orientation of e is in an anticlockwise direction with respect to n_D, while for an external port edge e in G, the dual edge e_D in G_D is convergent on the node n_D, if the orientation of the edge e is in a clockwise direction with respect to n_D, n_D being the node corresponding to the window in which e appears. For example, the external edge corresponding to port (1) is clockwise with respect to the node 1_D and hence the corresponding edge in G_D is convergent on node 1_D, but the internal edge 5 is clockwise with respect to node 2_D and hence the corresponding edge in G_D is divergent from the node 2_D. For the graph of Fig.6.2(b), the dual graph of G_D is shown in Fig.6.2(c). Now let

$$V_{eD} = \lambda_1 I_e, \quad I_{eD} = \lambda_2 V_e \qquad ..(6.1)$$

where the subscripts e and e_D correspond to the internal branch

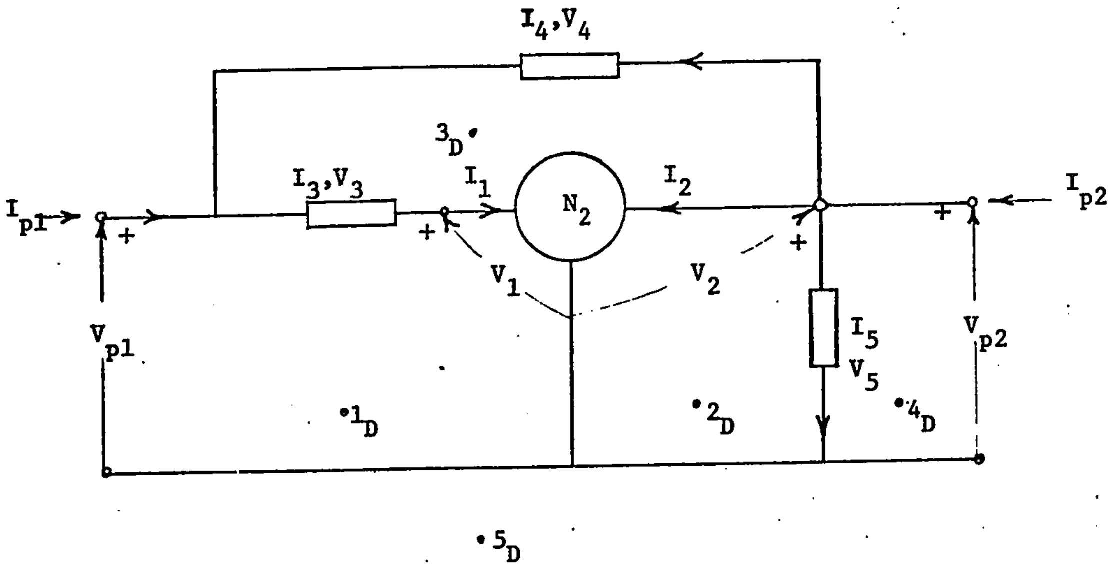


Fig. 6.2(a): 3-T Two-port Network N

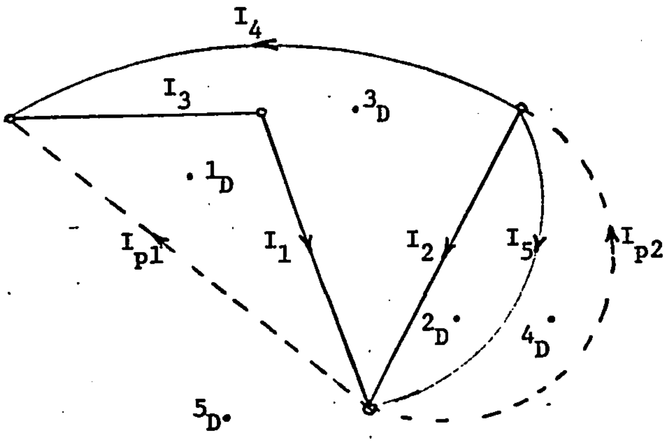


Fig. 6.2(b): Graph G of Fig. 6.2(a)

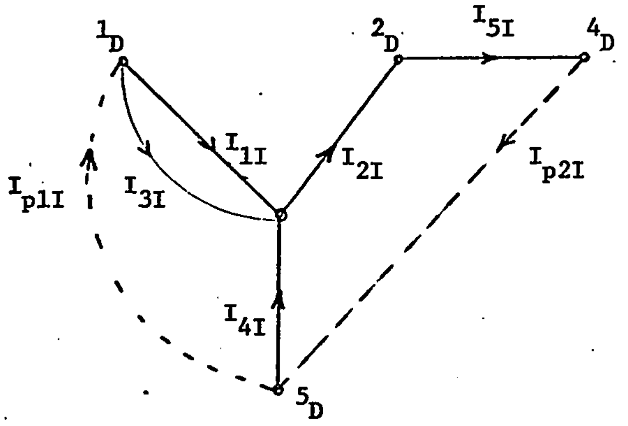


Fig. 6.2(c): Dual Graph G_D of the graph G

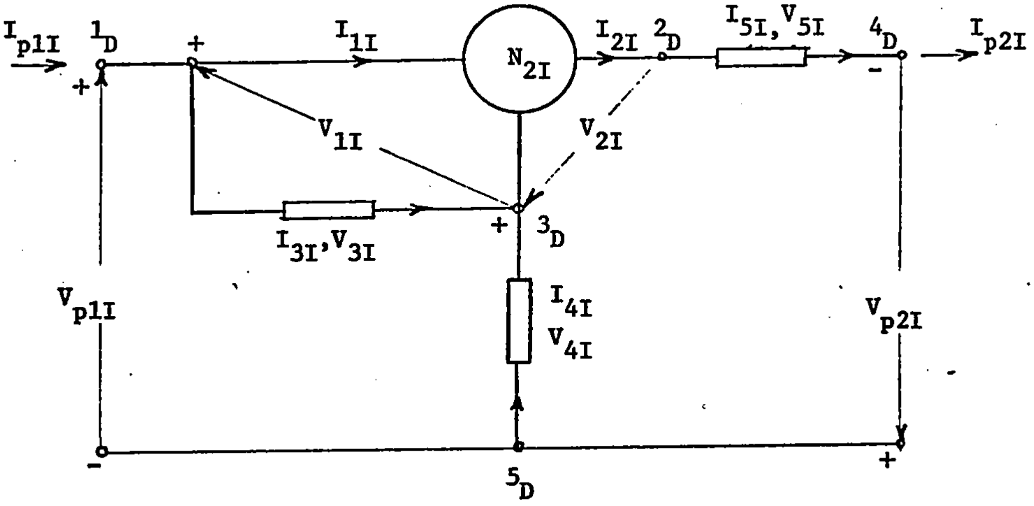


Fig. 6.2(d): Generalized Dual of Fig. 6.2(a)

variables in G and G_D respectively. Then it is seen that corresponding to an edge in G representing an internal one-port in N, the dual branch in G_D represents a one-port which is the generalized inverse of the corresponding one-port in N. Also the edges $1_D 3_D$ and $2_D 3_D$ together may be recognized as representing a 3-T two-port, which is the generalized inverse of the corresponding 3-T two-port in N.

Let A and B be the incidence and circuit matrices of G, while A_D and B_D the corresponding matrices of G_D. Then we have

$$A_D = B, \qquad B_D = A \qquad ..(6.2)$$

The matrices B and A may be suitably partitioned to write the Kirchoff's equations in the form

$$\begin{bmatrix} U & B_1 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} V_p \\ V_e \end{bmatrix} = 0 \qquad ..(6.3)$$

$$\begin{bmatrix} U & A_1 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} I_p \\ I_e \end{bmatrix} = 0 \qquad ..(6.4)$$

where U is a unit matrix of order 2,

B_1 and B_2 are respectively of order 2 x (b-2) and (b-n-1) x (b-2),

A_1 and A_2 are respectively of order 2 x (b-2) and (n-3) x (b-2),

V_p and I_p are respectively the vectors of the two external port voltages and currents,

V_e and I_e are respectively the vectors of internal port voltages and currents,

and n and b are resepctively the total number of nodes and edges in the graph G.

Using (6.1) to (6.4), the Kirchoff's equations equation for G_D may be written as

$$\begin{bmatrix} U & B_1 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I_{pD} \\ \lambda_2 V_e \end{bmatrix} = 0 \qquad ..(6.5)$$

and

$$\begin{bmatrix} U & A_1 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} V_{pD} \\ \lambda_1 I_e \end{bmatrix} = 0 \qquad ..(6.6)$$

Hence

$$V_{pD} = \lambda_1 I_p; \quad I_{pD} = \lambda_2 V_p \qquad ..(6.7)$$

Thus, from (6.1) and (6.7), it is seen that the 3-T two-port network corresponding to G_D is the generalized dual of N_D of the network N. It should be observed that the polarities of the external port variables are the same as in Fig.6.1(c). Hence the generalized dual N_D is also the generalized inverse N_I of the network N. This was the reason for the choice of the polarities of the port variables in Fig.6.1(c), for the generalized inverse of a 3-T two-port.

It is now noted that the above method of obtaining the generalized dual is quite general, and is applicable to any planar 3-T two-port, since no assumptions were made as to the exact nature of the **A** and **B** matrices.

Also, it is not necessary to draw the graph of N to obtain N_D, and hence N_I. It may be obtained directly as follows:

(i) Add external edges e_1 and e_2 corresponding to the two-ports of N.

(ii) Place nodes in each of the windows. Designate by 1_{iD}, 2_{iD} and 3_{iD}, the nodes corresponding to the windows for which the arcs $1_i\ 3_i$, $2_i\ 3_i$ and $1_i\ 2_i$ are respectively the parts of the boundaries. The other nodes may be designated in any manner.

(iii) For the generalized dual network N_D, connect between the nodes 1_{iD}, 2_{iD} and 3_{iD}, 3-T two-ports which are the generalized inverses of the corresponding 3-T two-ports in N.

(iv) The one-ports of N_D are located with respect to its nodes in the conventional way. These one-ports are the generalized inverses of the corresponding ones in N.

(v) The two external ports of N_D (and hence that of N_I) are located in the usual manner, bearing in mind the sign conventions used in this Chapter.

The procedure is illustrated by taking directly the generalized dual of the network of Fig. 6.2(a) and is shown in Fig. 6.2(d).

6.2.3 Applications to Linear Time-Invariant Networks:

In the definition of a gneralized inverse, choosing λ_1 and λ_2 to be convolution operators for LTI-networks, and taking Laplace transforms, the relationship between the currents and voltages of a given network N and its generalized inverse reduces to

$$I_I(s) = F_1(s)\ V(s) \quad , \quad V_I(s) = F_2(s)\ I(s) \qquad .. (6.8)$$

where F_1 and F_2 are arbitrary functions of s.

For a one-port element, (6.8) reduces to

$$Y_I(s) = \frac{1}{F(s)}\ Z(s) \qquad .. (6.9)$$

where

$$F(s) = \frac{F_2(s)}{F_1(s)} \qquad .. (6.10)$$

Z and Y_I respectively represent the impedance of a one-port and the corresponding admittance of the generalized inverse.

For a 3-T two-port network with chain parameters [ABCD], we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad .. (6.11)$$

Substituting (6.8) in (6.11), and rearranging, we have for the generalized inverse

$$\begin{bmatrix} V_{1I} \\ I_{1I} \end{bmatrix} = \begin{bmatrix} D & F(s)C \\ \{B/F(s)\} & A \end{bmatrix} \begin{bmatrix} -V_{2I} \\ -I_{2I} \end{bmatrix} \quad \text{.. (6.12)}$$

Bearing in mind the polarities of the port voltages and currents for the generalized inverse of a 3-T two-port [Fig. 6.1(c)], and the conventional polarities for defining the chain matrix of a 3-T two-port, it is seen from (6.12) that the chain matrix of the generalized inverse is

$$\begin{bmatrix} D & F(s)C \\ \{B/F(s)\} & A \end{bmatrix} \quad \text{..(6.13)}$$

Thus, for a given planar LTI 3-T two-port N consisting of one-ports and 3-T two-ports, if the generalized dual (and hence the generalized inverse) of N is obtained by the method given in Section 6.2.2, where each of the one-ports and 3-T two-ports in N_D is selected to the corresponding one-ports and 3-T two-ports in N by (6.9) and (6.13), then the overall chain matrix of N_D (and hence N_I) is given by

$$[a]_D = \begin{bmatrix} D & F(s)C \\ \{B/F(s)\} & A \end{bmatrix} \quad \text{..(6.14)}$$

where (A,B,C,D) are the chain parameters of N.

It should be pointed out that, if we had started with the Z matrix of the 3-T two-port N, then from (6.8), we would have obtained for N_D

$$\begin{bmatrix} I_{p1D} \\ \\ I_{p2D} \end{bmatrix} = \frac{1}{F(s)} [Z] \begin{bmatrix} V_{p1D} \\ \\ V_{p2D} \end{bmatrix} \qquad ..(6.15)$$

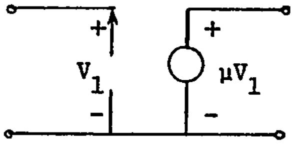
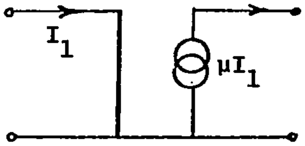
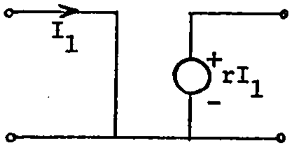
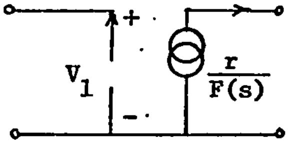
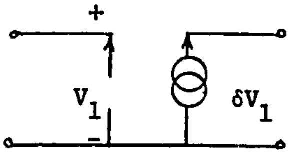
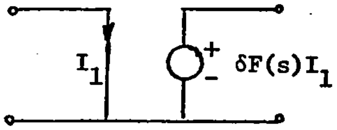
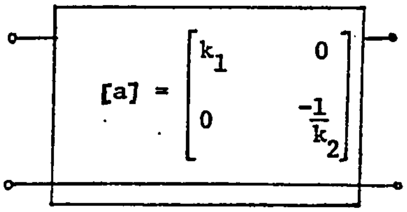
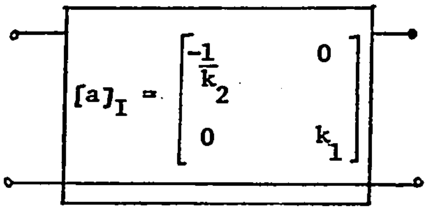
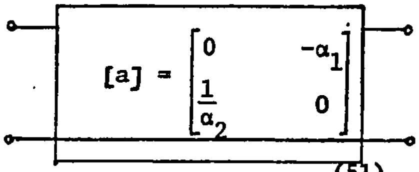
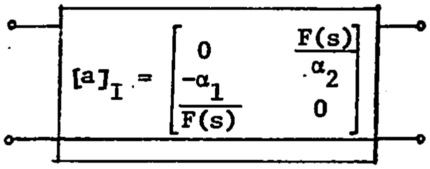
which gives the same description of the generalized dual as proposed by Mitra et al$^{(30)}$. However, the Y of N_D will not be $\frac{1}{F(s)} Z$ since the orientation of the port variables V_{p2D} and I_{p2D} are not consistent with the ones conventionally used for describing Y of a 3-T two-port. Using the usual conventions for these port variables, the Y of N_D may be obtained from (6.15) or from (6.14) as

$$Y_D = \frac{1}{F(s)} \begin{bmatrix} Z_{11} & -Z_{12} \\ \\ -Z_{21} & Z_{22} \end{bmatrix} \qquad ..(6.16)$$

Since our procedure of obtaining the generalized dual of a network N involves the replacement of the internal two-ports by their generalized inverses, these are tabulated in Table 6.1 for the different active as well as distributed elements. We note, from Table 6.1, the following interesting points:

TABLE 6.1

TWO-PORT ELEMENTS AND THEIR GENERALIZED INVERSES

Two-port Element	Corresponding Generalized Inverses
 VVT	 CCT
 CVT	 VCT
 VCT	 CVT
 $[a] = \begin{bmatrix} k_1 & 0 \\ 0 & -\frac{1}{k_2} \end{bmatrix}$ NIC	 $[a]_I = \begin{bmatrix} -\frac{1}{k_2} & 0 \\ 0 & k_1 \end{bmatrix}$ NIC
 $[a] = \begin{bmatrix} 0 & -\alpha_1 \\ \frac{1}{\alpha_2} & 0 \end{bmatrix}$ Generalized Gyrator (51)	 $[a]_I = \begin{bmatrix} 0 & \frac{F(s)}{\alpha_2} \\ \frac{-\alpha_1}{F(s)} & 0 \end{bmatrix}$ Generalized Gyrator
$Z = Z_o(s)\ f(x)$ $a \le x \le b$ $Y = Y_o(s)\ g(x)$ Tapered Line	$Z = F(s)\ y_o(s)\ g(x)$ $a \le x \le b$ $Y = \{Z_o(s)/F(s)\}\ f(x)$ Tapered Line

(i) the capacitive inverse $\{F(s) = \frac{1}{s}\}$ of an $\overline{RC}$-tapered line and the resistive inverse of an $\overline{LC}$-tapered line are respectively the dual lines defined by Bhattacharyya and Swamy(38).

(ii) The generalized inverse of a VVT is a CCT with the same amplification factor and vice-versa irrespective of the nature of F(s), a similar result being true for NICs.

From (6.14), it follows readily that,

(i) The voltage transfer function of N becomes the current transfer function of N_D and vice-versa.

(ii) The driving point functions of a one-port network N and its generalized dual are related to as $Y_D = \frac{1}{F(s)}Z$.

Thus the simple procedure of obtaining the generalized dual in conjunction with Table 6.1 can be used,

(a) to obtain directly a realization for a current transfer function given that of a voltage transfer function and vice-versa, and

(b) to obtain an alternate structure for realizing driving point functions.

Thus, obtaining the generalized dual of a given network may be considered as a network operation, according to the definition given in Chapter 1, while the generalized inverse cannot be considered as a network operation.

In RC-active synthesis, if the generalized dual operation is to be useful, F(s) must be chosen as 1/s (capacitive dual$^{(30)}$), since, in this case, a resistor is transformed into a capacitor and a capacitor into a resistor. Also, the generalized dual operation can be performed on RC-active networks which contain only VVT's, CCT's, NIC's and GIC's, since otherwise the corresponding generalized inverse elements will be frequency dependent. It is further noted that the sensitivities of corresponding network functions in N and N_D are the same with respect to the corresponding network parameters in N and N_D.

We will illustrate these points through the following example.

Example: Consider the network N of Fig.6.3(a) where the different one-ports are RC one-ports and the internal two-port is a unity current type negative impedance converter (UCNIC). This network corresponds to the Yanagisawa's first structure$^{(23)}$ for realizing a voltage transfer function, T(s). Its capacitive dual may be found directly using the procedure given in Section 6.2.2 and is shown in Fig.6.3(b), where the different one-ports are the capacitive inverses of the corresponding ones in Fig.6.3(a), and the internal two-port is a unity voltage inversion type negative impedance converter (UVNIC), the inverse of UCNIC of Fig.6.3(a).

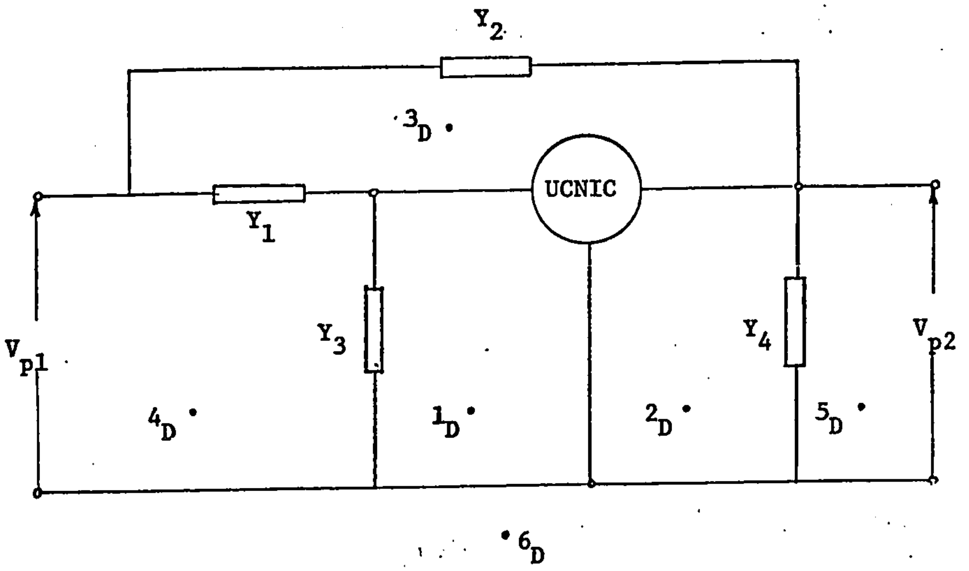


Fig. 6.3(a): Yanagisawa's Structure realizing an Arbitrary Voltage Transfer Function

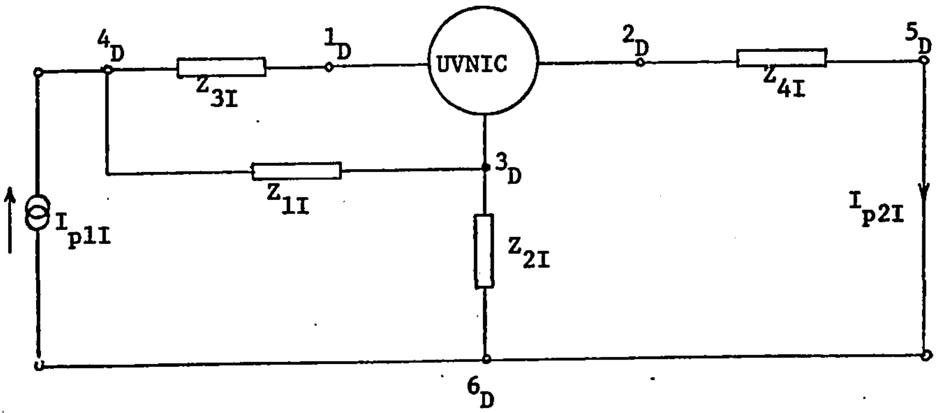


Fig. 6.3(b): Capacitive Dual of the Structure of Fig. 6.3(a)

This structure will realize T(s) as current transfer function. It may be readily recognized that the structure of Fig.6.3(b) is nothing but the one proposed by Thomas(23) for realizing an arbitrary current transfer function. It may also be verified that the sensitivities of voltage transfer function of Fig.6.3(a) and the current transfer function of Fig.6.3(b) are the same with respect to the corresponding parameters in the two structures.

Similarly it can be shown that many of the other structures proposed by different authors for realizing voltage transfer functions and current transfer functions are capacitive duals, as well as many of those proposed for driving point functions. In addition, new structures may also be obtained using the operation of generalized dual. Thus, the generalized dual as a network operation should prove useful in translating the good features (such as low sensitivity properties etc.,) of one network to another directly.

6.3 A new Theorem for Networks having Dual Topologies:

Consider an arbitrary connected planar network $\mathfrak{N}$ having b branches and n nodes. Kirchoff's current law places (n-1) constraints upon the currents, so that only (b-n+1) currents may be specified independently. All the remaining branch currents may then be found by the relation

$$I_b = B^t I_\alpha \qquad ..(6.17)$$

where

I_b is the vector of branch currents,

I_α is the vector of independent loop currents, and

B is the circuit matrix of the order (b-n+1) x b

Kirchoff's voltage law may also be expressed in terms of B, and is given by

$$BV_b = 0 \qquad \text{.. (6.18)}$$

where

V_b is the vector of branch voltages.

Now, consider a network $\mathfrak{N}''$ having a topology dual to that of $\mathfrak{N}$. It is known that the incidence matrix of $\mathfrak{N}''$ is identical to the circuit matrix B of $\mathfrak{N}$ (50). Hence, the Kirchoff's voltage and current laws for $\mathfrak{N}''$ may be written as

$$V''_b = B^t V''_\alpha \quad ; \quad BI''_b = 0 \qquad \text{.. (6.19)}$$

where

$I''_b(V''_b)$ is the vector of branch currents (voltages) of $\mathfrak{N}$, and

V''_α is the vector of independent node voltages in $\mathfrak{N}''$.

From (6.17) and (6.19), it is seen that

$$I^t_b \, I''_b = 0 \quad ; \quad V^t_b V''_b = 0 \qquad \text{.. (6.20)}$$

The above equations can be interpreted in vector space notation.

In a b-branch network, let $\mathcal{V}$ and $\mathcal{I}$ be the sets of all b-dimensional vectors of $\mathcal{N}$, that obey Krichoff's voltage and current laws respectively. If the corresponding vector spaces of $\mathcal{N}''$ are $\mathcal{V}''$ and $\mathcal{I}''$, it is known that$^{(50)}$

$$\mathcal{V} = \mathcal{I}'' \quad \text{and} \quad \mathcal{I} = \mathcal{V}'' \qquad .. \ (6.21)$$

Thus, (6.20) states that $\mathcal{I}$ and $\mathcal{I}''$, $\mathcal{V}$ and $\mathcal{V}''$ are orthogonal spaces of a b-dimensional vector space. Let $\lambda(\lambda')$ be an operator which maps a b-dimensional vector that obeys Kirchoff's current (voltage) law on to the same space $\mathcal{I}(\mathcal{V})$. Thus λ and λ' are respectively Kirchoff's current and voltabe operators$^{(37)}$. However, in view of (6.21), the same operator $\lambda(\lambda')$ also maps a b-dimensional vector of $\mathcal{N}''$ that obeys Kirchoff's voltage (current) law on to the same space $\mathcal{V}'' = \mathcal{I}$ $(\mathcal{I}'' = \mathcal{V})$ Thus, the operators λ and λ' must be both Kirchoff's current and voltage operators. Also let $\lambda''(\lambda''')$ be an operator which maps a b-dimensional vector that obeys Kirchoff's voltage or current law into the same space $\mathcal{V} = \mathcal{I}''(\mathcal{I} = \mathcal{V}'')$. Of course, λ'' and λ''' must also be both Kirchoff's voltage and current operators. Hence, we may write (6.20) in the form

$$\begin{aligned} \lambda I_b^t \, \lambda' I_b'' &= 0 \\ \lambda'' V_b^t \, \lambda''' V_b &= 0 \end{aligned} \qquad .. \ (6.22)$$

or in the form

$$\sum_b \lambda I_b \, \lambda' I_b'' = 0 \quad ; \quad \sum_b \lambda'' V_b \lambda''' V_b'' = 0 \qquad .. \ (6.23)$$

where

$I_b(V_b)$ is the current through (voltage across) the bth branch in $\mathcal{N}$, and

$I''_b(V''_b)$ is the current through (voltage across) the corresponding branch in $\mathcal{N}''$.

The relations (6.22) or (6.23) will be called as the "Generalized Tellegen's theorem for networks having dual topologies" or as "Dual Tellegen's theorem" for short. From (6.23), we may also obtain the "difference and sum forms" of the dual Tellegen's theorem as,

$$\sum_b (\lambda I_b \lambda' I''_b - \lambda'' V_b \lambda''' V''_b) = 0 \qquad .. (6.24)$$

and

$$\sum_b (\lambda I_b \lambda' I''_b + \lambda'' V_b \lambda''' V''_b) = 0 \qquad .. (6.25)$$

If some of the branches are in fact ports of the network, the terms associated with these ports can conveniently be placed on the opposite side of the equality sign to yield

$$\begin{aligned} \sum_p \lambda I_p \lambda' I''_p &= \sum_e \lambda I_e \lambda' I''_e \\ \sum_p \lambda'' V_p \lambda''' V''_p &= \sum_e \lambda'' V_p \lambda''' V''_p \end{aligned} \qquad .. (6.26)$$

or

$$\sum_p (\lambda I_p \lambda' I''_p - \lambda'' V_p \lambda''' V''_p) = \sum_e (\lambda I_e \lambda' I''_e - \lambda'' V_p \lambda''' V''_p) \qquad .. (6.27)$$

or

$$\sum_p (\lambda I_p \lambda' I''_p + \lambda'' V_p \lambda''' V''_p) = \sum_e (\lambda I_e \lambda' I''_e + \lambda'' V_p \lambda''' V''_p) \qquad .. (6.28)$$

where

p and e denote now the port and internal variables respectively.

It should again be emphasized that the above theorem applies to any two electrical networks, that obey Kirchoff's laws, as long as they have dual topologies irrespective of whether they contain linear/nonlinear, time-invariant/time-varying, active/passive, reciprocal/non-reciprocal, lumped/distributed elements. Also, the excitation is arbitrary and the initial conditions immaterial. Also, in the case of LTI-networks, (6.26)-(6.28) hold good both in time and frequency domains.

6.4 Dual Adjoint Networks:

Consider a planar network $\mathfrak{N}$ which may contain arbitrary elements, the only requirement being that each element possesses a parametric representation. Let $\mathfrak{N}''$ be another network with the requirement that the topology of $\mathfrak{N}''$ is dual of that of $\mathfrak{N}$, but not necessarily the same element types in the corresponding branches. This requirement is made so that the dual Tellegen's theorem derived in the previous Section may be employed. Let the network $\mathfrak{N}$ be considered at time ξ and $\mathfrak{N}''$ at time $(t-\xi)$. Choosing the operators λ and λ'' to be the small changes in the variables and $\lambda' = \lambda''' = 1$, it follows from the dual Tellegen's theorem (6.27) that,

$$\sum_p \Delta i_p(\xi)i''_p(t-\xi) - \Delta v_p(\xi)v''_p(t-\xi) = \sum_e i_e(\xi)i''_e(t-\xi) - \Delta v_e(\xi)v''_e(t-\xi) \quad .. (6.29)$$

Integrating (6.29) from the limits 0 to t, we have

$$\sum_p \int_0^t[\Delta i_p(\xi)i''_p(t-\xi) - \Delta v_p(\xi)v''_p(t-\xi)]d\xi$$

$$= \sum_e \int_0^t[\Delta i_e(\xi)i''_e(t-\xi) - \Delta v_e(\xi)v''_e(t-\xi)]d\xi \quad .. (6.30)$$

Let us now consider the constraints imposed by the network elements. Since the choice of the element type in $\mathcal{N}''$ is arbitrary, we may choose them in such a way so as to make the right hand side of (6.30) independent of Δv_e and Δi_e terms. The reason for this is the same as in the case of adjoint networks, namely that for sensitivity calculations, we are interested in the variation of responses with respect to element variations but not with respect to branch voltage or current variations.

A general resistive type of branch may be described by$^{(10)}$

$$\begin{aligned} v_R(\xi) &= f_R\{x_R(\xi),\ p_R, \xi\} \\ i_R(\xi) &= g_R\{x_R(\xi), p_R,\ \xi\} \end{aligned} \quad .. (6.31)$$

where

$x_R(\xi)$ is the chosen representation parameter,

the scalar p_R is a typical design parameter, and

subscript R denotes a resistive element.

Then, from (6.31), we have

$$\Delta v_R(\xi) = \frac{\partial f_R(\xi)}{\partial x_R}\,\Delta x_R(\xi) + \frac{\partial f_R(\xi)}{\partial p_R}\,\Delta p_R$$

and

$$\Delta i_R(\xi) = \frac{\partial g_R(\xi)}{\partial x_R}\,\Delta x_R(\xi) + \frac{\partial g_R(\xi)}{\partial p_R}\Delta p_R \qquad \text{.. (6.32)}$$

The contribution to the right hand side of (6.30) due to the terms corresponding to this element is given by

$$\int_0^t [\frac{\partial g_R(\xi)}{\partial x_R}\, i''_R(t-\xi) - \frac{\partial f_R(\xi)}{\partial x_R}\, v''_R(t-\xi)]\ \Delta x_R(\xi) d\xi\ .$$

$$+\int_0^t \frac{\partial g_R(\xi)}{\partial p_R}\, i''(t-\xi) - \frac{\partial f_R(\xi)}{\partial p_R}\, v''_R(t-\xi)]\Delta p_R d\xi \qquad \text{.. (6.33)}$$

Setting coefficient of Δx_R to zero, we obtain the description of the element in $\mathfrak{N}''$ corresponding to the resistive branch in $\mathfrak{N}$. This element description is,

$$\frac{\partial g_R(\xi)}{\partial x_R}\, i''(t-\xi) = \frac{\partial f_R(\xi)}{\partial x_R}\, v''_R(t-\xi) \qquad \text{.. (6.34)}$$

Also, the second term in (6.33) gives the sensitivity component with respect to p_R.

Now, consider the capacitive element described by

$$v_C(\xi) = f_C\{x_C(\xi),\ p_C, \xi\}$$

$$q_C(\xi) = g_C\{x_C(\xi),\ p_C, \xi\} \qquad \text{.. (6.35)}$$

$$i_C(\xi) = \frac{d}{d\xi}\{q_C(\xi)\}$$

From (6.35), we have

$$\Delta v_C(\xi) = \frac{\partial f_C(\xi)}{\partial x_C} \Delta x_C(\xi) + \frac{\partial f_C(\xi)}{\partial p_C} \Delta p_C$$

$$\Delta q_C(\xi) = \frac{\partial g_C(\xi)}{\partial x_C} \Delta x_C(\xi) + \frac{\partial g_C(\xi)}{\partial p_C} \Delta p_C \qquad .. \ (6.36)$$

$$\Delta i_C(\xi) = \frac{d}{d\xi} \{\Delta q_C(\xi)\}$$

The contribution to the right hand side of (6.30) due to terms corresponding to this element is given by

$$\begin{aligned}
&\int_0^t [\frac{d}{d\xi}\{\frac{\partial g_C(\xi)}{\partial x_C} \Delta x_C(\xi) + \frac{\partial g_C(\xi)}{\partial p_C} \Delta p_C\} i''_C(t-\xi) \\
&\quad -\{\frac{\partial f_C(\xi)}{\partial x_C} \Delta x_C(\xi) + \frac{\partial f_C(\xi)}{\partial p_C} \Delta p_C\} v''_C(t-\xi)]d\xi \\
&= \int_0^t \{\frac{d}{d\xi}[\frac{\partial g_C(\xi)}{\partial x_C} \Delta x_C(\xi)] i''_C(t-\xi) - \frac{\partial f_C(\xi)}{\partial x_C} \Delta x_C(\xi) v''_C(t-\xi)\}d\xi \\
&\quad + \int_0^t \{\frac{d}{d\xi}[\frac{\partial g_C(\xi)}{\partial p_C} \Delta p_C] \, i''_C \, (t-\xi) - \frac{\partial f_C(\xi)}{\partial p_C} \Delta p_C v''_C(t-\xi)\}d\xi \qquad .. \ (6.37) \\
&= -\int_0^t [\frac{\partial g_C(\xi)}{\partial x_C} . \frac{d}{d\xi}\{i''_C(t-\xi)\} + \frac{\partial f_C(\xi)}{\partial x_C} v''_C(t-\xi)] \Delta x_C(\xi) d\xi \\
&\quad + [\frac{\partial g_C(\xi)}{\partial x_C} \Delta x_C(\xi) i''_C(t-\xi)]_0^t \\
&\quad + \int_0^t [\frac{d}{d\xi} \{\frac{\partial g_C}{\partial p_C} \Delta p_C\} i''_C(t-\xi) - \frac{\partial f_C}{\partial p_C} \Delta p_C v''_C(t-\xi)]d\xi
\end{aligned}$$

If the corresponding branch in $\mathcal{N}''$ is described by

$$\frac{\partial f_C(\xi)}{\partial x_C} v_C''(t-\xi) = \frac{\partial g_C(\xi)}{\partial x_C} \frac{d}{d\tau} i_C''(\tau) \quad ; \quad \tau = (t-\xi) \qquad .. (6.38)$$

the first term on the right hand side of (6.38) is zero. The second term on the right hand side of (6.38) is

$$\frac{\partial g_C(t)}{\partial x_C} \Delta x_C(t) i_C''(0) - \frac{\partial g_C}{\partial x_C}(0) \Delta x_C(0) i_C''(t) \qquad .. (6.39)$$

The first term in the above expression is set to zero by choosing the initial condition $i_C''(0) = 0$ in $\mathfrak{N}''$. The second term is either equal to zero for a given initial condition in $\mathfrak{N}$, or gives the sensitivity component with respect to the initial condition. The last term in (6.37) gives the sensitivity component due to a variation in p_C.

In a similar manner, the description of the element in $\mathfrak{N}''$, and the sensitivity component of an inductive element in $\mathfrak{N}$ may be found. This is given in Table 6.2.

Let us consider now the class of memoryless nonlinear coupling elements described by

$$\begin{aligned} v_1(\xi) &= f_1\{x_1(\xi),\ x_2(\xi),\ p_d,\xi\} \\ i_1(\xi) &= g_1\{x_1(\xi),\ x_2(\xi),\ p_d,\xi\} \\ v_2(\xi) &= f_2\{x_1(\xi),\ x_2(\xi),\ p_d,\xi\} \\ i_2(\xi) &= g_2\{x_1(\xi),\ x_2(\xi),\ p_d,\xi\} \end{aligned} \qquad .. (6.40)$$

Due to a change in p_d, we have

$$\Delta v_1(\xi) = \frac{\partial f_1(\xi)}{\partial x_1} \Delta x_1(\xi) + \frac{\partial f_1(\xi)}{\partial x_2} \Delta x_2(\xi) + \frac{\partial f_1(\xi)}{\partial p_d} \Delta p_d$$

$$\Delta i_1(\xi) = \frac{\partial g_1(\xi)}{\partial x_1} \Delta x_1(\xi) + \frac{\partial g_1(\xi)}{\partial x_2} \Delta x_2(\xi) + \frac{\partial g_1(\xi)}{\partial p_d} \Delta p_d$$

$$\Delta v_2(\xi) = \frac{\partial f_2(\xi)}{\partial x_1} \Delta x_1(\xi) + \frac{\partial f_2(\xi)}{\partial x_2} \Delta x_2(\xi) + \frac{\partial f_2(\xi)}{\partial p_d} \Delta p_d$$

$$\Delta i_2(\xi) = \frac{\partial g_2(\xi)}{\partial x_1} \Delta x_1(\xi) + \frac{\partial g_2(\xi)}{\partial x_2} \Delta x_2(\xi) + \frac{\partial g_2(\xi)}{\partial p_d} \Delta p_d$$

Hence, the contribution to the right hand side of (6.30) corresponding to this element is given by

$$\int_0^t \Delta x_1(\xi)[\frac{\partial g_1(\xi)}{\partial x_1} i_1''(t-\xi) + \frac{\partial g_2(\xi)}{\partial x_1} i_2''(t-\xi) - \frac{\partial f_1(\xi)}{\partial x_1} v_1''(t-\xi) - \frac{\partial f_2(\xi)}{\partial x_1} v_1''(t-\xi)]d\xi$$

$$+ \int_0^t \Delta x_2(\xi)[\frac{\partial g_1(\xi)}{\partial x_2} i_1''(t-\xi) + \frac{\partial g_2(\xi)}{\partial x_2} i_2''(t-\xi) - \frac{\partial f_1(\xi)}{\partial x_2} v_1''(t-\xi) - \frac{\partial f_2(\xi)}{\partial x_2} v_2''(t-\xi)]d\xi$$

$$+ \int_0^t \Delta p_d[\frac{\partial g_1(\xi)}{\partial p_d} i_1''(t-\xi) + \frac{\partial g_2(\xi)}{\partial p_d} i_2''(t-\xi) - \frac{\partial f_1(\xi)}{\partial p_d} v_1''(t-\xi) - \frac{\partial f_2(\xi)}{\partial p_d} v_2''(t-\xi)]d\xi$$

.. (6.41)

If in $\mathfrak{N}''$, the description of the coupling element is given by

$$\frac{\partial g_1(\xi)}{\partial x_1} i_1''(\tau) + \frac{\partial g_2(\xi)}{\partial x_1} i_2''(\tau) = \frac{\partial f_1(\xi)}{\partial x_1} v_1''(\tau) + \frac{\partial f_2(\xi)}{\partial x_1} v_2''(\tau)$$

and

.. (6.42)

$$\frac{\partial g_1(\xi)}{\partial x_2} i_1''(\tau) + \frac{\partial g_2(\xi)}{\partial x_2} i_2''(\tau) = \frac{\partial f_1(\xi)}{\partial x_2} v_1''(\tau) + \frac{\partial f_2(\xi)}{\partial x_2} v_2''(\tau)$$

then, the first two terms of (6.41) are zero. The last term gives the sensitivity component due to a variation in p_d.

The description of the elements in $\mathfrak{N}''$ corresponding to all elements in $\mathfrak{N}$ considered here and the corresponding sensitivity components are listed in Table 6.2.

From the results of Seth and Singhal(49), Table 6.2, and from the definition of the conventional dual, it is seen that, for the elements listed in Table 6.2, the corresponding element descriptions in $\mathfrak{N}''$ may be obtained by first taking the dual and then the adjoint, or by taking the adjoint and then the dual. Thus the network $\mathfrak{N}''$ is the same as either the dual of the adjoint of $\mathfrak{N}$ or the adjoint of the dual of $\mathfrak{N}$; that is

$$\mathfrak{N}'' = (\mathfrak{N}^A)_D = (\mathfrak{N}_D)^A \qquad ..(6.43)$$

Hence the network $\mathfrak{N}''$ may be called as the dual adjoint or adjoint dual of the network $\mathfrak{N}$.

Also the sensitivity of any "network function" with respect to all the parameters in $\mathfrak{N}$ can be obtained by appropriate application of unity or zero-valued sources to the original and the dual adjoint networks, that is only two network analyses are required, just as in the case of adjoint networks.

6.5 Generalized Dual Transposition and its Applications:

In this section, we develop a new network called "generalized dual transpose" and consider its applications in the calculation of

TABLE 6.2

DUAL ADJOINTS OF ONE-PORTS AND TWO-PORTS AND THEIR SENSITIVITY COMPONENTS

Element classification	Parametric representation	Dual adjoint description for sensitivity computation at time t. ($\tau=t-\xi$)	Sensitivity component	parameter change
Resistive	$v_R(\xi) = f_R\{x_R(\xi),p_R,\xi\}$	$\frac{\partial g_R(\xi)}{\partial x_R} i_R''(\tau) = \frac{\partial f_R(\xi)}{\partial x_R} v_R''(\tau)$	$\int_0^t[\frac{\partial g_R(\xi)}{\partial p_R} i_R''(t-\xi) - \frac{\partial f_R(\xi)}{\partial p_R} v_R''(t-\xi)]d\xi$	Δp_k
Capacitive	$v_C(\xi) = f_C\{x_C(\xi),p_C,\xi\}$ $q_C(\xi) = g_C\{x_C(\xi),p_C,\xi\}$ $i_C(\xi) = \frac{d}{d\xi} q_C(\xi)$	$\frac{\partial f_C(\xi)}{\partial x_C} v_C''(\tau) = \frac{\partial g_C(\xi)}{\partial x_C} \frac{d}{d\tau} i_C''(\tau)$	$\int_0^t[\frac{d}{d\xi}(\frac{\partial g_C(\xi)}{\partial p_C})i_C''(t-\xi) - \frac{\partial f_C(\xi)}{\partial p_C} \cdot v_C''(t-\xi)]d\xi$	Δp_C
Inductive	$v_L(\xi) = \frac{d}{d\xi} \eta_L(\xi)$ $\eta_L(\xi) = f_L\{x_L(\xi),p_L,\xi\}$ $i_L(\xi) = g_L\{x_L(\xi),p_L,\xi\}$	$\frac{\partial g_L(\xi)}{\partial x_L} i_L''(\tau) = \frac{\partial f_L(\xi)}{x_L} \frac{d}{d\tau} v_L''(\tau)$	$\int_0^t[\frac{\partial g_L(\xi)}{\partial p_L} i_L''(t-\xi) - \frac{d}{d\xi}(\frac{\partial f_L(\xi)}{\partial p_L} \cdot v_L''(t-\xi)]d\xi$	Δp_L
Coupling	$v_1(\xi) = f_1\{x_1(\xi),x_2(\xi),p_d,\xi\}$ $i_1(\xi) = g_1\{x_1(\xi),x_2(\xi),p_d,\xi\}$ $v_2(\xi) = f_2\{x_1(\xi),x_2(\xi),p_d,\xi\}$ $i_2(\xi) = g_2\{x_1(\xi),x_2(\xi),p_d,\xi\}$	$\frac{\partial g_1(\xi)}{\partial x_1} i_1''(\tau) + \frac{\partial g_2(\xi)}{\partial x_1} i_2''(\tau)$ $= \frac{\partial f_1(\xi)}{\partial x_1} v_1''(\tau) \frac{\partial f_2(\xi)}{\partial x_1} v_2''(\tau)$ $\frac{\partial g_1(\xi)}{\partial x_2} i_1''(\tau) + \frac{\partial g_2(\xi)}{\partial x_2} i_2''(\tau)$ $\frac{\partial f_1(\xi)}{\partial x_2} v_1''(\tau) + \frac{\partial f_2(\xi)}{\partial x_2} v_2''(\tau)$	$\int_0^t[\frac{\partial g_1(\xi)}{\partial p_d} i_1''(t-\xi) + \frac{\partial g_2(\xi)}{\partial p_d} i_2''(t-\xi)$ $-\frac{\partial f_1(\xi)}{\partial p_d} v_1''(t-\xi) - \frac{\partial f_2(\xi)}{\partial p_d} v_2''(t-\xi)]d\xi$	Δp_d

sensitivities of a planar LTI network and show how it may be used in network synthesis.

6.5.1 Generalized Dual Transpose Network:

Consider a planar LTI network N, which may contain arbitrary elements. Let N" be another LTI network with the requirement that the topology of N" is the dual of that of N, but not necessarily the same element types in the corresponding branches. Since the dual Tellegen's theorem (6.27) holds both in frequency and time domain, we may apply it to N and N". Choosing the operators λ' and λ''' to be some arbitrary functions $F_2(s)$ and $F_1(s)$ respectively, and λ and λ'' to be small variations in the current and voltage variables in N, then from (6.27), we have,

$$\sum_p \{\Delta I_p(s)F_2(s)I_p''(s) - \Delta V_p(s)F_1(s)V_p''(s)\}$$

$$= \sum_e \Delta I_e(s)F_2(s)I_e''(s) - \Delta V_e(s)F_1(s)V_e''(s)\}$$

which may be rewritten as

$$\sum_p \{F(s)\Delta I_p I_p'' - \Delta V_p V_p''\} = \sum_e \{F(s)\Delta I_e I_e'' - \Delta V_e V_e''\} \quad .. (6.45)$$

where

$$F(s) = \frac{F_2(s)}{F_1(s)} \quad .. (6.46)$$

Just as in the case of dual adjoints, we may now consider the different elements in N and find the corresponding element description in N" in such a way as to render the right hand side of (6.45) independent of all ΔV_e and ΔI_e terms. Since the procedure is very similar to that given in Section 6.4, we will illustrate the procedure only for a capacitive element and a VVT.

Consider a capacitive branch in N defined by

$$I_C = sC\ V_C \qquad ..(6.47)$$

Then a variation in C gives

$$\Delta I_C = s\ \Delta C\ V_C + sC\ \Delta V_C \qquad ..(6.48)$$

The contribution to the right hand side of (6.45) due to the terms corresponding to this element is given by

$$[sF(s)C\ I''_C\ -\ V''_C]\ \Delta V_C + sF(s)\ V_C\ I''_C\ \Delta C \qquad ..(6.49)$$

If the description of the element in N" corresponding to this capacitive element in N is chosen to be

$$V''_C = sF(s)C\ I''_C \qquad ..(6.50)$$

then the first term in (6.49) becomes zero and the second term

$$sF(s)\ V_C\ I''_C = \frac{1}{C}\ V_C\ V''_C\ \Delta C \qquad ..(6.51)$$

gives the sensitivity component due to C. It should be observed that the element described by (6.50) will be a resistor, if F(s) is chosen as k/s.

In a similar manner, descriptions in N" corresponding to a resistor and an inductor in N, as well as the sensitivity components due to these elements may be found out. These are listed in Table 6.3.

Now consider an internal two-port element described by

$$\begin{aligned} V_1 &= A(p)V_2 - B(p)I_2 \\ I_1 &= C(p)V_2 - D(p)I_2 \end{aligned} \qquad ..(6.52)$$

where p is a parameter.

Due to a variation in the parameter p, we have

$$\begin{aligned} \Delta V_1 &= (\frac{\partial A}{\partial p} V_2 - \frac{\partial B}{\partial p} I_2)\Delta p + A\, \Delta V_2 - B\, \Delta I_2 \\ \Delta I_1 &= (\frac{\partial C}{\partial p} V_2 - \frac{\partial D}{\partial p} I_2)\Delta p + C\, \Delta V_2 - D\, \Delta I_2 \end{aligned} \qquad ..\ (6.53)$$

Hence, the contribution to the right hand side of (6.45) due to the terms corresponding to the two-port with descriptions (6.52) is given by

$$\begin{aligned} &\Delta I_2\{-D\, F(s)I_1'' + BV_1'' + F(s)I_2''\} + \Delta V_2\{F(s)C\, I_1'' - AV_1'' - V_2''\} \\ &\quad + \Delta p\, F(s)\{\frac{\partial C}{\partial p} V_2 - \frac{\partial D}{\partial p} I_2\}I_1'' - \{\frac{\partial A}{\partial p} V_2 - \frac{\partial B}{\partial p} I_2\}V_1'' \end{aligned} \qquad ..\ (6.54)$$

The first two terms are made zero by choosing the description of the two-port element in N" corresponding to the two-port (6.45) in N to be

TABLE 6.3

GENERALIZED DUAL TRANSPOSES OF ONE-PORTS AND TWO-PORTS AND THEIR SENSITIVITY COMPONENTS

Element Type	Branch relation	Branch relation in N"	Sensitivity component of the vector $\underline{G}$	component of the parameter vector $\Delta\underline{p}$
Resistance	$V_R = RI_R$	$V''_R = \frac{F(s)}{R} I''_R$	$-I_R V''_R$	ΔR
Capacitance	$I_C = sCV_C$	$V''_R = sC\ F(s) I''_C$	$sF(s) V_C I''_C$	ΔC
Inductance	$V_L = sLI_L$	$I''_L = \frac{sL}{F(s)} V''_L$	$-sI_L V''_L$	ΔL
Coupled two-port	$V_1 = A(p)V_2 - B(p)I_2$ $I_1 = C(p)V_2 - D(p)I_2$	$V''_2 = -A(p)V''_1 + C(p)F(s)I''_1$ $I''_2 = \frac{-B(p)}{F(s)} V''_1 + D(p)I''_1$	$-V_2 V''_1 \frac{\partial A(p)}{\partial p}$ $+I_2 V''_1 \frac{\partial B(p)}{\partial p}$ $+F(s) V_2 I''_1 \frac{\partial C(p)}{\partial p}$ $-F(s) I_2 I''_1 \frac{D(p)}{\partial p}$	Δp

Leaf 181 omitted

in page numbering.

$$V_2'' = -A\ V_1'' + F(s)C\ I_1''$$
$$I_2'' = -\frac{1}{F(s)}\ B\ V_1'' + D\ I_1'' \qquad .. (6.55)$$

then, the third term in (6.54) gives the sensitivity due to the parameter p; and is

$$(\frac{\partial C}{\partial p} V_2 - \frac{\partial D}{\partial p} I_2)F(s)\ I_1'' - (\frac{\partial A}{\partial p} V_2 - \frac{\partial B}{\partial p} I_2)V_1'' \qquad .. (6.56)$$

Of course, using (6.55) and (6.56), the network element in N'' corresponding to a VVT, or NIC etc. in N, and the corresponding sensitivity component may be obtained.

We shall call the network N'' so obtained as the "generalized dual transpose" of N; if F(s) = 1, then the corresponding network N'' will be called the dual transpose of N. The reason for these will be clear in Section 6.6. In the next two subsections, we will consider some of the applications of the generalized dual transpose.

6.5.2 Applications in Computation of Sensitivities:

Consider an arbitrary planar LTI two-port N consisting of resistors, inductors, capacitors and coupled two-port elements with descriptions as given in Table 6.3. It will now be shown how the sensitivity of any network function with respect to any of the parameters in N may be computed using N and its dual transpose N".

First, let us consider the sensitivity of the voltage transfer function of the network N. Let N be excited by a unit voltage source at port (1) with port (2) open. Then, (6.45) reduces to

$$\Delta\ I_{p1}I''_{p1} - \Delta\ V_{p2}V''_{p2} = \sum_{e}\Delta\ I_eI''_e - \Delta\ V_eV''_e \qquad .. (6.57)$$

since F(s) has been assumed to be unity.

If now we choose,

$$V''_{p2} = -1 \quad ; \quad I''_{p1} = 0$$

and use Table 6.3, then (6.57) reduces to

$$\Delta\ V_{p2} = \underline{G}^t\Delta\ \underline{p} \qquad .. (6.58)$$

where

$\underline{G}$ is the vector of sensitivity component as given by Table 6.3, and

$\Delta\underline{p}$ is the vector of variations in the parameters.

Thus, the sensitivity of the transfer function T_v with respect to any parameter of N may be obtained using Table 6.3.

In a similar manner, the sensitivities of any other network function with respect to the different parameters in N may be computed by suitably exciting the network N and its dual transpose N", and using Table 6.3.

6.5.3 Applications in Synthesis:

Consider a 3-T two-port planar network N_1 consisting as subnetworks, one-ports and 3-T two-ports. The generalized dual transpose of the network N can be obtained by first obtaining the dual topology of N, by the method given in Section 6.2, and then by replacing each one-port or internal 3-T two-port element by its corresponding generalized dual transpose element, whose descriptions are as given in Table 6.3.

Now the interrelationship between the immittance matrices of N and its generalized dual transpose N" will be obtained by using (6.27) Letting $\lambda = \lambda'' = 1$, $\lambda' = F_2(s)$ and $\lambda''' = F_1(s)$ in (6.27), it may be rewritten as

$$\sum_p I_p\ F(s)I''_p - V_p V''_p = \sum_e I_e\ F(s)I''_e - V_e V''_e \qquad .. (6.59)$$

where again

$$F(s) = \frac{F_2(s)}{F_1(s)} \qquad .. (6.60)$$

The right hand side of (6.59) is zero, since the contribution due to each element is zero as can be verified from Table 6.3. Thus,

$$I_{p1}F(s)\ I''_{p1} + I_{p2}F(s)\ I''_{p2} - V_{p1}V''_{p1} - V_{p2}V''_{p2} = 0 \qquad .. (6.61)$$

From (6.61), it is clear that

$$\left.\frac{I''_{p1}}{V''_{p1}}\right]_{V''_{p2}=0} = \frac{1}{F(s)}\left[\frac{V_{p1}}{I_{p1}}\right]_{I_{p2}=0} = \frac{1}{F(s)}\,Z_{11} \qquad .. (6.62a)$$

$$\left.\frac{I''_{p2}}{V''_{p2}}\right]_{V''_{p2}=0} = \frac{1}{F(s)}\left[\frac{V_{p1}}{I_{p2}}\right]_{I_{p1}=0} = \frac{1}{F(s)}\,Z_{21} \qquad .. (6.62b)$$

$$\left.\frac{I''_{p1}}{V''_{p2}}\right]_{V''_{p1}=0} = \frac{1}{F(s)}\left[\frac{V_{p2}}{I_{p1}}\right]_{I_{p2}=0} = \frac{1}{F(s)} Z_{12} \qquad .. (6.62c)$$

and

$$\left.\frac{I''_{p2}}{V''_{p2}}\right]_{V''_{p1}=0} = \frac{1}{F(s)}\left[\frac{V_{p2}}{I_{p2}}\right]_{I_{p1}=0} = \frac{1}{F(s)} Z_{22} \qquad .. (6.62d)$$

It was pointed out in Section 6.2 that when the dual topology is obtained, the directions of voltage and current at port (2) of any internal 3-T two-port as well as those of the overall two-port are opposite to the conventional ones used for obtaining the Z or Y etc., of a two-port network. Hence, (6.62) reduces to

$$Y''_{ii} = \frac{1}{F(s)} Z_{ii}$$

$$Y''_{ij} = -\frac{1}{F(s)} Z_{ji}$$

That is, the short-circuit admittance matrix of $[Y'']$ of N'' is related to the open-circuit impedance matrix $[Z]$ of N, as

$$[Y''] = \frac{1}{F(s)}\begin{bmatrix} Z_{11} & -Z_{21} \\ -Z_{12} & Z_{22} \end{bmatrix} \qquad .. (6.63)$$

Thus, the procedure of obtaining the generalized dual transpose network is the same as given in Section 6.2, except that the different internal one-ports and 3-T two-ports are now the generalized dual transposes of the corresponding one-ports and 3-T two-ports in N. For the sake of convenience, the 3-T two-ports in N'' corresponding to the commonly used active elements are listed in Table 6.4.

It can be easily be seen from (6.63) that,

(i) Forward voltage transfer function of the original network is identical to reverse voltage transfer function of its generalized dual transpose network.

(ii) Forward current transfer function of the original network is identical to the reverse current transfer function of its generalized dual transpose network.

(iii) The input admittance of the dual transpose of a one-port network is $\frac{1}{F(s)}$ times the input impedance of the original network.

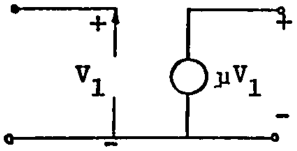
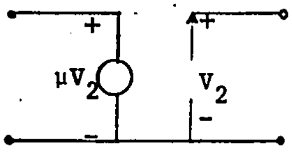
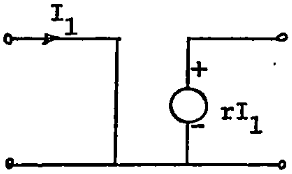
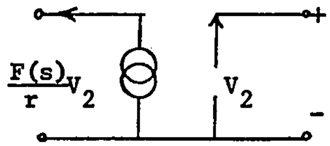
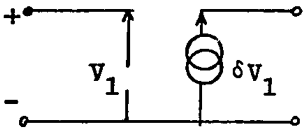
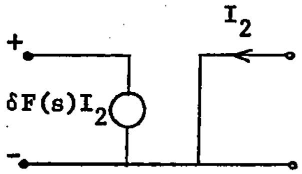
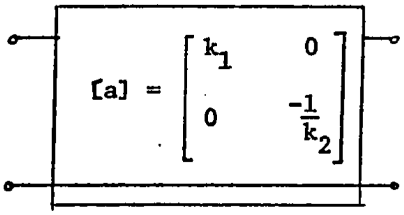
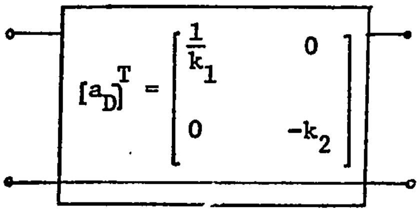
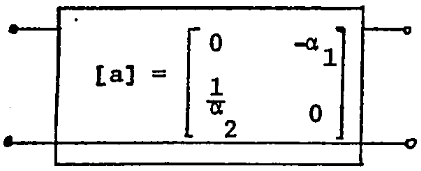
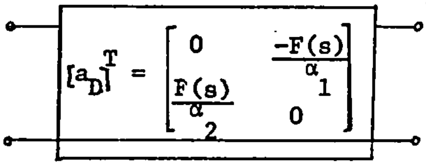
Thus, the simple procedure of obtaining the generalized dual transpose network, in conjunction with Table 6.4, can be used:

(i) to obtain an alternate structure of a voltage transfer function or a current transfer function from a given structure realizing a voltage transfer function or a current transfer function respectively, and

(ii) to get an alternate structure for driving point immittances. Since there is a one to one correspondence between the elements of N and N'', the generalized dual transposition may be considered as a network operation.

TABLE 6.4

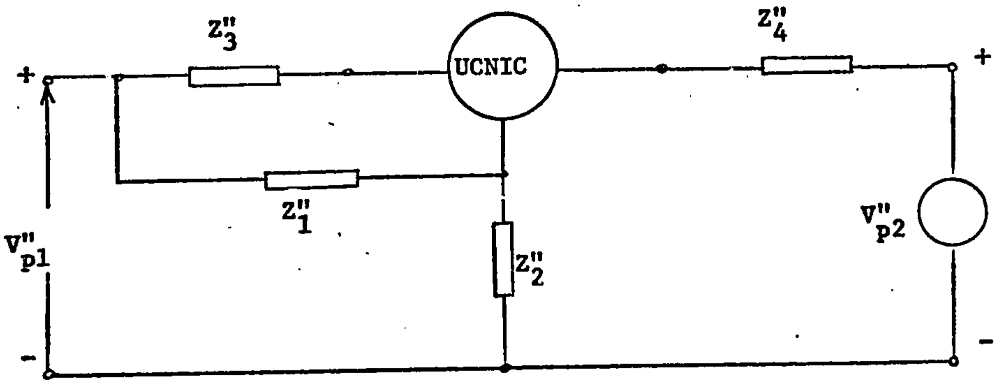
TWO-PORT ELEMENTS AND THEIR GENERALIZED DUAL TRANSPOSES

Two-port Elements	Generalized Dual Transpose
 VVT	 VVT
 CVT	 VCT
 VCT	 CVT
 $[a] = \begin{bmatrix} k_1 & 0 \\ 0 & -\frac{1}{k_2} \end{bmatrix}$ NIC	 $[a_D]^T = \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & -k_2 \end{bmatrix}$ NIC
 $[a] = \begin{bmatrix} 0 & -\alpha_1 \\ \frac{1}{\alpha_2} & 0 \end{bmatrix}$ Generalized Gyrator	 $[a_D]^T = \begin{bmatrix} 0 & -\frac{F(s)}{\alpha_1} \\ \frac{F(s)}{\alpha_2} & 0 \end{bmatrix}$ Generalized Gyrator
$Z = Z_o(s)\ f(x)$ $Y = Y_o(s)\ g(x)$ $\quad a \le x \le b$ Tapered line	$Z = F(s)\ Y_o(s)\ g(x)$ $Y = \{Z_o(s)/F(s)\}\ f(x)$ $\quad a \le x \le b$ Tapered line

It is seen from Table 6.3 that the N" descriptions of a VVT, CCT or NIC is independent of F(s), whereas the N" descriptions of all other elements is dependent on F(s). If F(s) is chosen to be $\frac{1}{s}$, then the N" descriptions of a resistive and a capacitive element are respectively a capacitive and a resistive element, while the N" descriptions of a VCT, CCT, etc., are frequency dependent. Further, the capacitive dual transpose of an $\overline{\text{RC}}$-line is the dual line itself. Thus, if the network consists of resistances, capacitances, $\overline{\text{RC}}$-lines, VVTs, CCTs and NICs, then its capacitive dual transpose is another network which contains resistances, capacitances, $\overline{\text{RC}}$-lines, VVTs, CCTs and NICs. This fact can be made use of in RC-active network synthesis. Further, it can also be verified that the sensitivity of a network function with respect to a network parameter of the original network is identical to the sensitivity of corresponding network function with respect to the corresponding parameter of its capacitive dual transpose network.

The usefulness of the capacitive dual transpose network in RC-active network synthesis will be illustrated through an example. Consider Fig. 6.3(a), which is "Yanagisawa's first structure"(23) for the realization of an arbitrary voltage transfer function, T(s). The capacitive dual transpose of Fig. 6.3(a) is as shown in Fig. 6.4. If ports of Fig. 6.4 are interchanged, it is seen that it corresponds to Yanagisawa's second structure(23).

The capacitive dual transpose can also be used to give an



$Z_i'' = F(s)\ Y_i$

Fig. 6.4: The Capacitive Dual Transpose of Fig. 6.3(a)

alternate structure for driving point functions. For example, consider Fig.6.5(a) which consists of four RC-one ports and two VVTs. This network corresponds to "Sandberg's structure"$^{(23)}$ for realization of an arbitrary driving point function. Its capacitive dual transpose may be found directly, and is shown in Fig.6.5(b), where different one-ports are capacitive dual transposes of the corresponding ones in Fig.6.5(a), and it uses two VVTs, the dual transposes of VVTs. The structure of Fig.6.5(b) gives an alternate structure for realization of a driving point function. It should be noted that this structure is different from that of Sandberg's second structure (which uses two CCTs)$^{(23)}$, and the structure proposed by Bhattacharyya and Swamy$^{(29)}$, using two VVTs. In fact, one can obtain two more equivalent structures for realization of a driving point function, using the operations of capacitive dual and capacitive dual transposition on the resulting network. Similarly, with the help of these operations, one can obtain three new equivalent structures for realization of driving point function starting with the second structure of Sandberg$^{(29,p.306,Fig.18(b))}$ for the realization of an arbitrary driving point function.

6.6 Interrelationships between Generalized Dual, Transpose and Generalized Dual Transpose Networks:

Consider a planar LTI network N consisting of one-ports and two-ports. The topologies of N and its transpose N^T are the same. Also the topologies of N_D and N" are identical, but dual of that

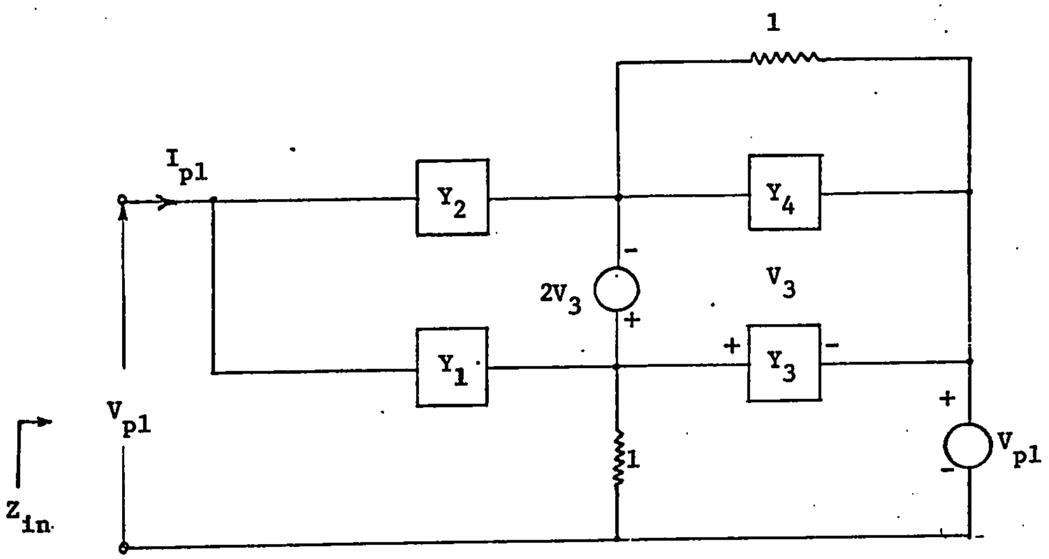
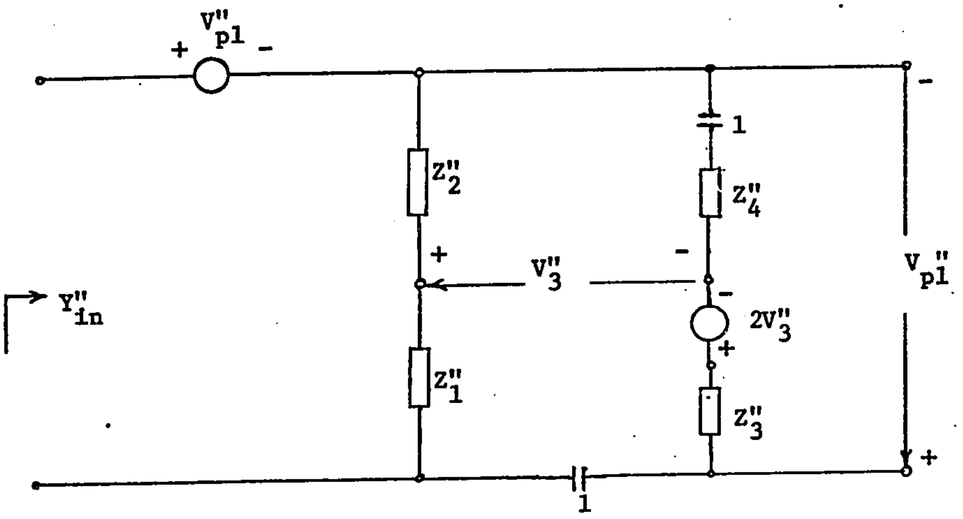


Fig 6.5(a): Sandberg's Structure realizing an arbitrary Driving Point Function



$Z_i'' = (1/s)Y_i$, $Y_{in}'' = sZ_{in}$

Fig. 6.5(b): Capacitive Dual Transpose of Fig. 6.5(a) of N. It can readily be shown that for any one-port or two-port element, the corresponding descriptions in N" as well as in $(N_D)^T$ and $(N^T)_D$ are identical. That is,

$$N'' = (N_D)^T = (N^T)_D$$

Hence, the network N" may be obtained by taking the transpose of the generalized dual of N or by taking the generalized dual of the transpose of N, or directly as discussed in Section 6.5. This is the reason for calling the network N" as generalized dual transpose of N. The different interrelationships between N, N_D, N^T and N" are pictorially shown in Fig.6.6.

6.7 Conclusions:

Generalized inverse has been defined for an n-port consisting of linear/nonlinear, time-varying/time-invariant, lumped/distributed elements. A simple method of obtaining the generalized dual of a general planar 3-T two-port, consisting as subnetworks one-ports and 3-T two-ports, has been given.

Next, a new theorem for networks having dual topologies has been established, and used to define dual adjoints and generalized dual transposes. The usefulness of these networks in the calculation of sensitivities of a planar network has been discussed. The applications of the two network operations, namely the generalized dual and generalized dual transposition, in network

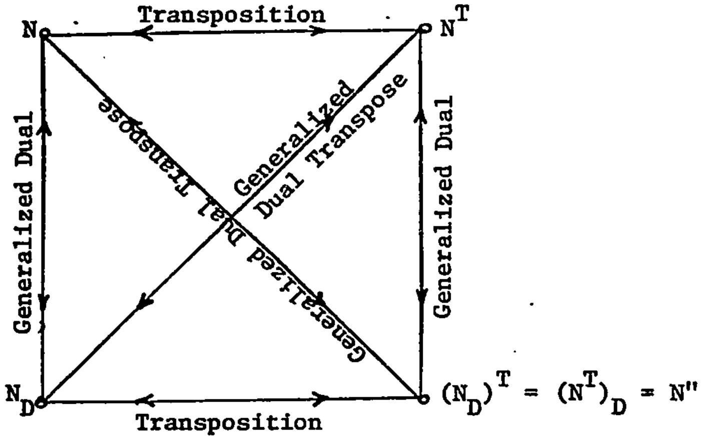


Fig. 6.6: Diagram depicting the Interrelations between a given Network N, its Transpose N^T, its Generalized Dual N_D and its Generalized Dual Transpose N".

synthesis, has been considered. It has also been established that the generalized dual transpose is nothing but $(N_D)^T$ or $(N^T)_D$. It is hoped that these operations would serve to unify the various RC-active structures, which seemingly look different, but in fact are interrelated through these operations.

CHAPTER 7

CONCLUSIONS

Tellegen's theorem, in conjunction with the concept of adjoint networks, has been used extensively in this thesis to make studies on network sensitivities. A new Tellegen's theorem applicable to networks having dual topologies has been established and used to define certain new networks and network operations, whose applications have also been considered.

In particular, Tellegen's theorem has been used to express the first order sensitivities of network functions of linear time-invariant networks in terms of the immittance parameters of an augmented network. These formulas are easily amenable to the calculation of higher order sensitivities. This theorem has also been used to establish a number of sensitivity invariant relationships for different network functions for various classes of linear time-invariant networks. It has also been shown that invariant relations exist for zeros, poles, coefficients, Q and ω_o sensitivities.

For an RLC-gyrator network, bounds on the sum of the magnitudes of the sensitivities for different network functions with respect to different network elements, have been established using Tellegen's theorem. An upper bound for the sum of the magnitudes of the sensitivities over resistive elements alone has been found in terms of the real parts of relevant driving point immittances, while lower performance indices, as in the case of linear time-invariant networks. A general expression relating the sum of (k+1)th order sensitivities in terms of that of the kth order sensitivities has also been established; this relation is applicable whether the network is linear time-invariant or not.

Generalized duals and generalized inverses have been defined for any general network consisting of linear/nonlinear, time-invariant/time-variant, lumped/distributed elements. A simple method of directly obtaining the generalized dual of a planar three-terminal two-port network, consisting of subnetworks, one-ports and three-terminal two-ports, is given. It has been shown that the generalized dual can be considered as a network operation, as defined in Section 1.3. Applications of the generalized dual in network synthesis have also been considered.

A new theorem, similar to the Tellegen's theorem, but for networks having dual topologies, is developed. This theorem is used in the time domain to define a new network called dual adjoint, and it is shown how this network along with the original may be used to compute the sensitivities of a planar network. The theorem is also applied in frequency domain to develop a new network operation called generalized dual transposition. Applications of this network operation in sensitivity computations of planar linear time-invariant networks and in RC-active synthesis have also been considered. It is hoped that these network operations would help to unify the as well as upper bounds have been obtained for the same over inductive and capacitive elements. For two-element kind networks, lower and upper bounds have been obtained in terms of the sensitivity of the network function with respect to frequency and the relevant driving point functions. It may be mentioned that in computer-aided circuit design, the sum of the magnitudes of sensitivities is a commonly used performance criterion.

Another commonly used measure in optimal synthesis is Schoeffler's quadratic performance index. It has been shown how the various invariant relations derived earlier for a general linear time-invariant network may be made use of as constraints in obtaining lower bounds for this index. It has been shown that, for any general linear time-invariant network, the lower bound on this index decreases as the number of elements in the network is increased, thereby strengthening the second conjecture of Leeds and Ugron(6). For two-element kind networks, both upper and lower bounds on the Schoeffler's index have been obtained.

Tellegen's theorem and the concept of adjoint networks have been used in the time domain to obtain explicit expressions for the sensitivities of a network response due to different types of nonlinear elements. These expressions have been utilised to establish invariant relations for the sum of sensitivities over different sets of parameters for two classes of nonlinear networks. These relationships should prove useful in establishing bounds on quadratic seemingly different RC-active structures that exist in the literature and at the same time allow the attractive features of a given structure to be retained in the other structures obtained through these operations.

The author is of the opinion that further worthwhile investigations could be carried out in the area of network sensitivities. It should be possible to compute directly, using the concept of adjoint networks, the sensitivities of zeros and poles of a network function and hence of the coefficients of the numberator and denominator polynomials, as well as those of Q and ω_o. It should also be possible to obtain formulas for the sensitivities of any order with respect to the reflection coefficients of the termination of a given network, in terms of its scattering parameters. It is also worth investigating if bounds could be obtained for the sum of the magnitudes of the Q-sensitivities, which is also used as a multiparameter sensitivity index in RC-active synthesis(23). Another criterion which might be useful in filter synthesis is the sum of the squares of the magnitudes of the Q-sensitivities. The invariances obtained in this thesis for the sum of the Q-sensitivities may, in this case, be used as constraints in obtaining bounds for the sum of magnitude squares of the Q-sensitivities. It is also worth investigating as to whether sensitivity invariant relations exist for time varying networks. The author is also of the opinion that it would be fruitful to look for new network operations both in time and frequency domains which should prove useful in synthesis of networks.

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