A NOTE ON NOTATION

In what follows we only assume the reader has some knowledge of the notation of classical tensor calculus. We shall use the sign convention found in Ohanian's Gravitation and Spacetime. All constants are taken to be 1 unless otherwise stated.

x^0 shall always denote co-ordinate time t.

Latin indices take values 0, 1, 2, 3.

Greek indices take values 1, 2, 3.

The Einstein summation convention is used.

The Kronecker Delta will be written as:

$$\delta_j^{\ i} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The Minkowski metric of flat four-dimen-space-time is denoted

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The proper-time interval on flat space-time is denoted.

$$\begin{aligned} ds^2 &= \eta_{ij}\, dx^i\, dx^j \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \end{aligned}$$

The proper-time interval on curved space time is denoted

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

with $g_{00} > 0$ (time-like sign convention).

ABSTRACT

STATIC BLACK HOLE

Frederic Mayer

We consider one of the consequences of the General Theory of Relativity, the static black hole. The position of this theory in the history of science and its relevance to current research is first examined. The motivation for the field equations is considered at length. The main tests of General Relativity are considered in terms of one of few exact solutions available; the Schwarzschild solution. Examining the Schwarzschild solution in standard form we observe a co-ordinate (removable) singularity at $r_s = 2m$. We then show that a source of gravitation lying inside radius $r = r_s$ is enveloped within an event horizon, at $r = r_s$, making communication with the rest of the universe impossible; such a source is called a black hole. The possibility of black holes existing is then explored.

INTRODUCTION

Our object in this thesis is with A. Einstein's General Theory of Relativity and one of the consequences of this theory, the static black hole. In order to present a coherent account of static black holes we first consider some of the reasons why the General Theory of Relativity (of which the theory of black holes is a part) is an important branch of scientific research today; this is followed by a critical look at the trend of thought which led Einstein to his field equations. This approach will give us insight into the theory without going into all the details necessary in a logical construction. We shall also mention some of the other standard approaches to the field equations. We then consider the Schwarzschild solution (one of the few exact solutions) and this solution leads us to re-examine some of the standard tests of this theory, then we finally embark on our main topic the spherically symmetric black hole and its significance.

Is relativity (the general theory) a part of mathematics or physics? There is no definitive answer to this question and in reading the literature one is startled by the strengths of the opinions advanced without as it seems firm evidence. What is perhaps significant is that here alone (and in no other field of mathematical physics) does the question arise. A typical opinion is that of A. Trautman: "One of the many unsolved problems connected with the general theory of relativity is whether the theory belongs to physics or rather to mathematics. One of my colleagues at this summer school said that those who work in the theory of relativity do so because of its mathematical beauty rather than because they want to make predictions which could be checked against experiment. I think there is some truth in this statement, and probably I am no exception to it."[1]

[1] A. Trautman, Brandeis Lectures, 1964; Prentice Hall Inc., 1965.

CHAPTER I

HISTORICAL SURVEY

1. HISTORICAL BACKGROUND

Why does one take up the study of General Relativity? Well, if one is interested in applied mathematics (the construction of mathematical models as an aid to understanding and hopefully to making predictions), General Relativity or Geometrodynamics as some people prefer to call it, is perhaps of all the mathematical models ever created by the human mind, the most logically complete and satisfying. In the last twenty years General Relativity has become an active and still growing branch of mathematics. This subject has attracted to it many people specialized in various other field such as Quantum Mechanics, Group Theory, and of course Topologists. One cannot foresee this field soon running out of interesting problems.

Activity in General Relativity till late 1950's was confined to a small group of physicists and mathematicians, who were very much out of the main streams of either physics or mathematics. These people were mostly interested in either cosmology, constructing models of the universe, or in finding field laws for the electromagnetic field as purely geometrical restrictions on the structure of space-time, like those of the gravitational field, a unified field theory. A.S. Eddington thought that these two problems were intimately related.

Many people have tried to solve the problem of finding a unified field theory. A. Einstein spent a good deal of time and effort between 1915 when he discovered the field equations for this gravitational model until his death in April 1955, working on the generalization of his gravitational model to a unified field theory.[1] Unfortunately, Einstein's efforts were never completely rewarded.

Nowadays interest in a unified field theory of the gravitational and electromagnetic fields seems to have faded, with every little research done on this topic. C. Misner and J.A. Wheeler have expressed the opinion that the theory of General Relativity is an already unified field theory, this is based on the combined field equations of Einstein and Maxwell.[2]

Quite clearly a unified field theory should include the gravitational field, the electromagnetic field along with the strong and weak fields of Quamtum Mechanics. Recently a highly successful theory that unifies the electromagnetic field to the fields of Quantum Mechanics was in the news, when it was announced that S.L. Glashow, A. Salam and S. Weinberg shared the 1979 Nobel Prize in physics, for their respective work on this theory. Yet one still cannot call this theory a unified field theory since the gravitational field is not included. S. Weinberg has expressed the opinion that one would understand how the gravitational force fits in with the other forces had one been around at the big bang, when the universe began to expand. Thus the search for a unified field theory would seem to bring us to cosmological consideration.

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[1] See for example, Einstein's Meaning of Relativity (2) Appendix II.

[2] R. Adler, M. Bazin, M. Schiffer, Introduction to General Relativity, McGraw Hill, New York, 1965.

In the words of S. Chandrasekhar:

"In a real sense astronomy is the natural home of general relativity. It has to be so for the general theory of relativity is a theory of gravity; and in what context except in the context of astronomy, can one hope to observe manifestations of phenomena derived from strong gravitational fields. In other words, the right physical questions one can ask of general relativity are necessarily in the context of astronomical possibilities."[1]

This is true either whether one is considering some model of the universe when working in cosmology or the structure of a neutron star, in astrophysics.

It is interesting to note in passing that the first two applications (tests) of General Relativity, the calculation for the advance of perihelia of the planet Mercury, and the deflection of starlight by the sun, are both astronomical in nature. These two applications constitute two fundamental tests of the theory (see page 26).

It was in 1917 that relativistic cosmology was born when W. de Sitter and A. Einstein proposed their respective models of the universe. It was found that cosmology based on General Relativitty led to a startling consequence, the universe is expanding. This prediction seemed to be too fantastic to be real, Einstein could not accept this consequence of his theory; every astronomer Einstein had talked to had told him that the universe was static. It was thus that Einstein was led to alter his field equations by the addition of a quantity that would

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1 S. Chandrasekhar: Nature, Vol. 252, Nov. 1, 1974, p. 15-17.

become known as the cosmological term.* This Einstein believed would remedy the defect of expansion of the universe although he felt the elegance of the field equations suffered.

At the time (1915-1917) Willem de Sitter was director of the Leiden Observatory in the Netherlands. He was an astronomer noted, by his colleagues, for both his handsome white beard as well as his absent minded manner, but more than this he was what would be called nowadays an applied mathematician, given to constructing mathematical models. From the day he received in the mail, a copy, from A. Einstein, of the General Theory of Relativity, he began to speculate as to the consequences of this theory in cosmology. He constructed a model of the universe (which would appear in print in November 1917) by assuming it was empty. At first glance this model seems to bear absolutely no relationship to nature; after all the universe is not empty.

We must ask the question: is it legitimate to consider such a mathematical model as an approximation to reality? The answer is that a mathematical model is a construct, an application of a general physical law to a very specific circumstance. The laws are taken as our initial assumptions, to which we may add some additional constraints, such as in cosmology, the absence of any matter. Then we can construct a mathematical model based on these assumptions, and then compare the behaviour of the model with what we conceive to be the behaviour of reality; thereby the model is tested. But what can a model of an empty universe tell us about the real universe which we know is not empty, is the model totally nonsensical?

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* λg_{ij} where λ is called the cosmological constant.

The answer here is that the real universe is empty! At least it is approximately empty in the large scale. The average matter density of the universe is believed to be about 10^{-30} grams per cubic centimeter, or equivalently on the average one hydrogen atom per cubic meter.

The de Sitter universe was a mathematical model of an expanding universe, at least so far as its geometry was concerned. Since in 1917 the universe was considered to be static, de Sitter remarked that his model could also be considered static, since there was nothing in it that could expand.* This was not strictly so, since the absence of matter can be interpreted as matter, on the average, in the real universe. Thus we should not rank the de Sitter universe as a static universe which the Einstein universe is.

By 1922 the astronomer V.M. Slipher had unpublished data which showed the notion of an expanding universe in a more favourable light.[1] Also at about this time the Russian mathematician Alexander Friedmann, at Petrograd, and later (1927) the Belgian Abbé Georges Lemaître, independently constructed more realistic models of the universe, and showed that with the cosmological term included in the field equations one might have models of the universe that expand, contract, or even oscillate.

Lemaître (1927) anticipated that the universe would be found to be expanding and that the prediction could be confirmed by looking for a red shift in the spectra of galaxies. Edwin Hubble (1929) discovered that light from distant galaxies was indeed shifted towards the red,

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* Published in The Mathematical Theory of Relativity: A.S. Eddington (1923), p. 162, Chelsea Publishing Company, N.Y., 1975.

the farther the galaxy, the greater the red shift. This was, reluctantly, interpreted to mean that the galaxies are receding from us in all directions, with velocity proportion to their distance.

In the words of Eddington we summarize:

> "the choice between Einstein's and de Sitter's models is no longer urgent because we are not now restricted to these two extremes; we have available a whole chain of intermediate solutions between motionless matter and matterless motion, from which we can pick out the solution with right proportion of matter and motion to correspond with what we observe. These solutions were not sought earlier, because their appropriateness was not realized; it was the preconceived idea that a static solution was a necessity in order that everything might be referred to unchanging background of space. We have seen that this requirement should strictly have barred out de Sitter's solution, but by a fortunate piece of gate crashing it gained admission it was the precursor of the other non-static solutions to which attention is now mainly directed."[1]

In the original model of the universe proposed in 1927 by Abbé G. Lemaître, the universe had been static for an infinite period of time before it has begun to expand. For this model to work, the field equations of General Relativity has to include the cosmological term. Friedmann had shown (1922) that the universe, in a relativistic setting, even with the cosmological term, could still expand. This

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1 The Expanding Universe: A.S. Eddington (1932), reprint, Ann Arbor Paperbacks (1958).

1927 Lemaitre model was always the one favoured by Eddington.

The results obtained by Hubble suggested that the cosmological constant was unnecessary. Einstein in a joint paper with de Sitter finally discarded the cosmological constant in 1931. Einstein who had introduced the cosmological constant in 1917, was glad to be rid of it.

"If Hubble's expansion had been discovered at the time of the creation of the general theory of relativity, the cosmological member would never have been introduced. It seems now so much less justified to introduce such a member into the field equations since its introduction loses its sole original justification, that of leading to a natural solution of the cosmological problem."[1]

However not all of Einstein's colleagues were willing to drop the cosmological constant, least of all Eddington who needed the constant in his favoured cosmology.

"And if even the theory of relativity falls into disrepute the cosmological constant will be the last stronghold to collapse. To drop the cosmological constant would knock the bottom out of space."[2]

"To set $\lambda = 0$ implies a reversion to the imperfectly relativistic theory a step which is no more to be thought of than a return to the Newtonian theory."[3]

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1 The Meaning of Relativity: A. Einstein; Princeton U. Press (1956), p.124.

2 The Expanding Universe, A.S. Eddington, p. 104.

3 New Pathways in Science, A.S. Eddington, p. 215 (1934), reprint Ann Arbor Paperpacks (1959).

In order to drop the cosmological constant Lemaître (1933) had to change his account of genesis. In dropping the cosmological constant, the universe could not have had a static phase in its past, as there was in the original Lemaître universe of 1927. Lemaître proposed that the universe was born from a primeval atom and that the universe began violently, in a "fireworks."

For all the respect and affection Eddington had for Lemaitre, he found this idea of an abrupt beginning to the universe to be extremely distasteful. Eddington could not or would not change from his preconceived idea what nature should be like.

George Gamow coined the term "big bang" theory in (1948); Gamow who had studied under Friedmann, in the early twenties, proposed in a joint paper with Ralph Alpher (1948) a cosmological model, similar to the Lemaître model (1933) in which the universe began a highly compressed state. Unlike the Lemaître model, where the primeval atom was made of densely packed matter, Gamow's model was composed mostly of pure energy with only trace element of matter. He called the stuff of his primeval atom Ylem, after the Greek term for chaos which gave birth to the world.

Later work on Gamow's "big bang" theory suggested that if the universe had once been very hot, and cooling ever since, it ought to have some energy left over from the original flash. This residual energy would take the form of low-level background radiation coming in from all directions at about equal strength.

1948 was also the year Herman Bondi and Thomas Gold proposed their steady-state cosmology. This cosmological model was an attempt at describing a universe that had existed forever and will exist forever.

This model used the de Sitter model as a basis*, to account for the red shift, to which was added an interesting notion suggested by Gold, the spontaneous creation of matter in space; thus the universe could expand and yet never thin out. The steady-state cosmology had the epistemologic advantage in that it did not suppose a beginning or an end of the universe, as do the Friedmann-Lemaître models. It had the disadvantage of violating the law of conservation of energy** with its unexpected spontaneous creation of matter. Atoms popping out of empty space did not seem as impossible, to some cosmologists, as did the "big bang" theory.

The original steady-state theory of Bondi, Gold (1948) made no attempt to relate the global gravitational field with the matter in the universe, which is a defect if one assumes Mach's principle to be true. An improved version of the steady-state theory due to Fred Hoyle (1949) was able to remedy this situation somewhat by introducing an additional field into the theory which was responsible for the creation of matter.

The discovery of Quasars in 1960, cosmic black-body radiation in 1965, and Pulsars in 1968, have largely been responsible for the number of research articles in the field of relativistic cosmology tripling in the last two decades (1960-1980).

The discovery in 1965 of residual radiation made by Arno Penzias and Robert Wilson is considered to be the residual black-body radiation,

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* The metric is the same for both the de Sitter universe and the steady-state theory, the interpretation is different.

** The de Sitter metric plus the field equations of General Relativity yield an empty de Sitter universe, but when $T^{ij}{}_{;i} \neq 0$ we have a non-empty universe as was shown by Fred Hoyle.

the "Primeval Fireball" which was left over from the "big bang". Because of this discovery the "big bang" model is now called the "standard model". Hoyle (1965)[1] abandoned the steady-state model in favour of the evolutionary Friedmann-Lemaître model because of this new observational data.

Another fascinating discovery was made in 1968 by A. Hewish the Pulsar. In 1968 T. Gold made the conjecture that Pulsars are rotating neutron stars; this was subsequently confirmed. Originally the concept of a neutron star had been invented by Baade and Zwiky in 1933-34, the neutron had only been discovered in 1932 by Chadwick. At this time Baade and Zwiky had identified a new class of astronomical objects which they called "Supernovas". They made the hypothesis, that supernova might be created by the collapse of a normal star to form a neutron star. In 1939 Oppenheimer and Volkoff performed the first detailed calculations for the structure of neutron stars. This laid the foundations of the General Relativistic theory of stellar structures. It was also in 1939 that Oppenheimer and Snyder calculated the collapse of a homogenous sphere of pressure-free fluid, using General Relativity and discovered that the sphere cuts itself off from communication with the rest of the universe. This type of collapsed star is now called a black hole.

From the last two paragraphs we see why the study of singularities in space-time is of such importance. With the discovery of neutron stars the search for black hole has been intensified. Before considering black holes we shall look as the trend of thought which led Einstein to the gravitational model we shall use.

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1 F. Hoyle, Nature, 208, 111, (1965).

CHAPTER II

THE FIELD EQUATIONS

The geometric model of gravitation we are about to consider is usually called "The General Theory of Relativity" the name given it by its creator A. Einstein; but even this name has become controversial. Both V. Frock and J.D. Synge have been very critical of Einstein for describing his gravitational model as General Relativity. The confusion stems from Einstein's mistaking the principle of general covariance with the principle of General Relativity. The principle of general covariance states that equations should remain unchanged under any transformation of co-ordinates. To Einstein this seemed (at least naively) to be a generalization of Lorentz invariance . The principle of general covariance is not a relativity principle, it is a dynamical principle that imposes restrictions on the possible interactions of geometry and matter. We shall consider this principle later.

Several names have been proposed as replacements for The General Theory of Relativity. Chronogeometry by A.D. Fokker, Chronometry by J.L. Synge, and Geometrodynamics by J.A. Wheeler, are possibly the best known. Geometrodynamics is the name most of then used in place of the General Theory of Relativity. The term Geometrodynamics was originally introduced by J.A. Wheeler to describe a theory, that attempts to interpret matter as an excited state of geometry. According to this view:

"There is nothing in the world except empty curved space. Matter, charge, electromagnetism, and other fields are manifestations of the bending of space. Physics is geometry."[1]

Here we shall refer to A. Einstein's geometric model for gravitation by the name originally given by its creator the Theory of General Relativity, or equivalently by the General Theory of Relativity, since doing so now can cause no confusion, and heuristically at least, there is much to be said for the name: (in particular it is almost a truism that the laws of physics should be independent of frames.)

Einstein in his search for the ultimate world law found a geometrical description for gravity. It is important to understand the General Relativity was formulated from very general considerations[2], not in the search for a correction to Newton's Theory of Universal Gravitation.

In the construction of a gravitation model one must assume that Special Relativity will have some sort of role locally. This is to be expected since Special Relativity lies at the foundation of macroscopic physics and is basic to much of microscopic physics.

Granted that Special Relativity must have some sort of role in our model we must also insist that our model shall reduce to Newton's model in the weak field approximation. The reason for this is the success of Newton's Theory of Universal Gravitation. With the Newtonian model, celestial mechanics can predict the position of the major planets which agree with observation to within a few seconds of

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1 Geometrodynamics: J.A. Wheeler, Academic Press, New York, 1962.

2 Based on the phylosophy of E. Mach.

arc for time intervals of many years. Again the accuracy of the theory is responsible for Le Verrier's discovery of the planet Neptune in 1946. One must note that it was the same Le Verrier who found flaws in Newton's most successful model; in 1845 he had calculated that the observed precession of the perihelion of the planet Mercury was 35 seconds of arc per century faster than expected in the Newtonian model.

The discrepancy was confirmed by Newcomb in 1882, who gave the value 43 seconds of arc per century. Some astronomers thought that this excess was due to a small planet, Vulcan, between Mercury and the sun, but Vulcan was never found. Other suggestions were made, none satisfactory. Newcomb concluded in 1895 "to prefer provisionally the hypothesis that the sun's gravitation is not exactly as the inverse square."[1]

Although the Newtonian model is not perfect, it is a very good approximation in the limiting case of motion at a low velocity in a weak gravitational field. Any other mathematical model of gravitation must agree with Newtonian model in this limiting case.

Einstein in his first attempts to construct a gravitational model 1907-11 formulated his equivalence principles. The first or weak equivalence principle states that inertial mass is equivalent to gravitational mass. This principle is known to hold experimentally with very great accuracy. (In 1890 Eötvos using a torsion balance showed that $\alpha \triangleq \frac{|m_i - m_g|}{m_i} < 5 \times 10^{-8}$ here m_i and m_g denote respectively the inertial and gravitational mass. This experiment was repeated again by Eötvos in 1905 with the result that $\alpha < 3 \times 10^{-9}$. In a somewhat

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[1] Gravitation and Cosmology: Steven Weinberg, John Wiley & Sons Inc., 1972.

similar experiment Dicke in 1964 gave $\alpha < 3 \times 10^{-11}$. In 1971 Braginski gave $\alpha < 9 \times 10^{-13}$.

Einstein stated the strong equivalence principle in the 1911 paper titled "On the Influence of Gravitation on the Propagation of Light." This principle has caused much confusion and many arguments among the devotees of this field. V. Fock and J.L. Synge are the strongest opponents of the strong equivalence principle, (please forgive an unintentional pun). We quote from J.L. Synge "I have never been able to understand his principles.... Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none; according to the Riemann tensor does or does not vanish.... The principle of equivalence performed the essential office of midwife at the birth of general relativity. I suggest that the midwife be now buried with appropriate honours and the facts of absolute space time be faced."[1]

In order to understand the opinions of Fock and Synge, concerning the strong equivalence principle, we consider the statement of the principle as given by H. Weyl " A closed box, such as a lift, whose suspension wire has snapped and which descends without friction in the gravitational field of the earth, is a striking example of such a system of reference. All bodies that are falling freely will appear to be at rest to an observer in the box and physical events will happen in the box in just the same way as if the box were at rest and there were no

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[1] Relativity: The General Theory, J.L. Synge, North Holland, Amsterdam, 1971.

gravitational field in spite of the fact that the gravitational force is acting."[1] We have underlined the offending part of the quotation from Weyl. To show that this statement can lead to a misunderstanding, we consider our lift L_0 high above the earth say in space and an identical lift in intergalactic space not subject to any noticeable gravitational effects (call it L_1).

An observer in L_0 would find himself in a zero-g environment, he would feel no gravitational force, similarly to an observer L_1. This might lead the observer in L_0 to conclude that the gravitational effects of the earth are completely eliminated by free fall. So our observers might ask, is there any local experiment that can distinguish L_0 from L_1? The answer is yes. If we place a drop of the same liquid in the centre of each lift, the drop in L_1 will be perfectly spherical, but the drop in L_0 will have two bulges one pointing towards the earth the other to the opposite side. Thus we may detect a gravitational field by the tidal effects it produces. The bulges result from the inhomogenity of the earth's gravitational field, one side of the drop is pulled too much, the other is not pulled enough.

As stated by Weyl, the strong principle of equivalence is wrong as we showed in the last paragraph. When the strong principle of equivalence holds, it is point wise.[2]

Steven Weinberg's approach to general Relativity is anti-geometric and can be summarized as follows:

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[1] Space-Time-Matter, H. Weyl, Dover, 1952.

[2] Here we have a uniform gravitational field, which of course does not exists in nature.

"I believe that the geometrical approach has driven a wedge between general relativity and the theory of elementary particles. As long as it could be hoped, as Einstein did hope, that matter would eventually be understood in geometrical terms it made sense to give Riemannian geometry a primary role in describing the theory of gravitation. But now the passage of time has taught us not to expect that the strong, weak, and electromagnetic interactions can be understood in geometrical terms, and a great emphasis on geometry can only obscure the deep connection between gravitation and the rest of physics.

"In place of Riemannian geometry, I have based the discussion of general relativity on a principle derived from experiment: the Principle of the Equivalence of Gravitation and Inertia."

"This approach naturally leads us to ask why gravitation should obey the Principle of Equivalence. In my opinion the answer is not to be found in the realm of classical physics, and certainly not in Riemannian geometry, but in the constraints imposed by the quantum theory of gravitation. It seems to be impossible to construct any Lorentz-invariant quantum theory of particles of mass zero and spin two, unless the corresponding classical field theory obeys the Principle of Equivalence."[1]

Here the point of view is quite the opposite to that expressed by Wheeler in Geometrodynamics where everything is geometry; for Weinberg there is no geometry. By assuming a "uniform " gravitational

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[1] Gravitation and Cosmology, S. Weinberg, John Wiley & Sons, 1972.

field, the strong equivalence principle can be very useful in understanding why space is curved. It motivates the metric $ds^2 = g_{ij}dx^i dx^j$ where the g_{ij} are not constants like the η_{ij} of Special Relativity. When one is learning a subject, heuristic principles like the strong equivalence principle should not be underestimated. One can always clear up one's cluttered thinking once there is some thinking.

In a 1911[2] paper, Einstein was able, by using the strong equivalence principle, to calculate the red shift, and found half the actual value for the deflection of light. L.I. Schiff in 1960[2] showed that the total value for the deflection of light may be calculated using only Special Relativity and the strong equivalence principle. It should be mentioned however that some scientists do not agree with Schiff's "heuristics" and insist that the deflection of light is a significant test for the field equations of General Relativity.

In 1912 Einstein went from a mainly physical approach, where the thought experiment was basic to the conclusion drawn, to a mainly mathematical approach, where tensor calculus would reign. This change was possible due to his friendship and colaboration with the mathematician Marczel Grossman at the Zurich Polytechnicum. So with a strong physical intuition for what gravity is, based on his famous thought experiments, Einstein set about constructing his geometrical model. Here were the ingredients at his disposal:

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1 The Principle of Relativity: A Collection of Original Papers, Dover, 1952.

2 On Experimental Tests of the General Theory of Relativity, L.J. Schiff (1960), Ann. J. Phys., 28, 340-343.

(1) The Newtonian model must be included in the weak field approximation of any other model;

(2) Special Relativity;

(3) The weak and strong equivalence principle.

To these ingredients he would add the principle of general covariance.

In his first papers on General Relativity, Einstein held the view that the principle of general covariance has a physical meaning. Krestchmann in 1917 published a paper in which he showed that any physical law could be formulated in a generally covariant way. Einstein came to accept this point of view, as do most of the General Relativists of today.

However not everyone shares this view, C. Lanczos states "Certain antagonists of Einstein tried to point out that the principle of general covariance is in fact an empty principle since any equation whatever can be transformed to arbitrary coordinates and thus be made generally covariant.

Even the Newtonian theory of gravitation can be put into generally covariant form, by transformation to arbitrary curvilinear coordinates. The misunderstanding here involved affected even Einstein himself. He saw the value of absolute calculus in providing us with the simplest possible generally covariant field equations. In actual fact, the transformation of the Newtonian equations to, let us say, rotating coordinates is not covariant, because the equations then contain quantities such as the angular velocity of rotation which cannot be interpreted indigenously, i.e. without reference to another priviledged reference system namely the system of Cartesian coordinates "General Covariance" means that every reference system can exist on its own merits, and there is no need to go outside that system for help."[1]

J.L. Anderson in his careful treatment[2] of space-time theories also argues:

"That this principle is not devoid of physical content as some authors have claimed but properly interpreted, can serve as the sole basis for the theory."[2]

Ohanian in his very readable textbook[3] follows (after a fashion) Anderson's treatment.

For Einstein the ingredients required for setting up the Theory of General Relativity were:

(1) The Newtonian model whereby to check his own model in the weak field approximation;

(2) Special Relativity;

(3) The Equivalence Principles;

(as we have already mentioned), to which was added

(4) General Covariance.

From (2) using Minkowski's geometric approach we have the space-time invariant $ds^2 = \eta_{ij}dx^i dx^j = dt^2 - dx^2 - dy^2 - dz^2$. From (3) we see that (heuristically at least) that $g_{oo} \neq \eta_{oo}$ in fact g_{oo} is not constant, where the space-time invariant is now $ds^2 = g_{ij}dx^i dx^j$. This is what led Einstein to consider the g_{ij} as physical quantities (dynamical quantities than can be chosen to account for the presence of

1 Einstein's path from Special to General Relativity; C. Lanczos General Relativity, papers in honour of J.L. Synge, Clarendon Press, Oxford, 1972.

2. Anderson, J.L., Principles of Relativity Physics, Academic Press, New York, 1967.

3 Ohanian, H.C., Gravitation and Spacetime, W.W. Norton & Company, Inc., New York, 1976.

a gravitational field) depending on the distribution of matter and energy in space. What Einstein had to do next was to obtain field equations for the g_{ij}.

It was known at this time that the Riemann-Christoffel curvature tensor denoted by $R_{ijk\ell}$ was the simplest generally covariant tensor formed from the g_{ij} and their first two derivatives. Now when we have $R_{ijk\ell} = 0$ we must have $g_{ij} = \eta_{ij}$ and our space is flat. In the presence of a gravitational field we have $R_{ijk\ell} \neq 0$ also since we have only, at most, ten independent g_{ij} and in 4 dimensions we have twenty equations for the $R_{ijk\ell}$ thus the $R_{ijk\ell}$ are overdetermined. If we contract the Riemann-Christoffel curvature tensor we may obtain the Ricci tensor thus

$$R_{jk} = g^{i\ell} R_{ijk\ell}$$

To Einstein the Ricci tensor appeared to be the simplest characterization of the metric $ds^2 = g_{ij}dx^i dx^j$ that gives us more than just flat space-time. So in empty space the field equations where

$$R_{ij} = 0$$

they first appeared in 1913 in a paper of Einstein and Grossman. They were at first rejected by Einstein who could not reduce them to the Newtonian model at this time but they were to be resurrected 2 years later.

In the presence of matter, Einstein at first assumed that the field equations had the form

$$R_{ij} = -K\, T_{ij} \qquad (1)$$

where T_{ij} is the energy momentum tensor, (normally first encountered in Special Relativity) and K is some constant. From conservation of energy consideration, we must have zero-divergence of the energy-momentum tensor. Thus the field equations had to be amended to

$$R_{ij} - \frac{1}{2} g_{ij}R = K T_{ij} \qquad (2)$$

since the divergence of the left hand side of (1) is now zero.

Thus it was in 1915 ten years after his first publication on Special Relativity, that Einstein introduced the General Theory of Relativity. This theory contained in a consistent way the empirically supported structural ingredients of both Newtonian mechanics including gravity theory (in the weak field approximation) and Special Relativity (in the absence of gravity). All the forces found in Special Relativity (electromagnetic, elastic, etc.) have their counterparts in General Relativity, only gravity of all the classical forces has no counterpart in General Relativity. To paraphrase E.T. Whittaker, instead of being one of the players, gravity became part of the stage. This has been a very brief summary of a rather tortured road, that led Einstein to the field equations governing his gravitational model. The important thing was obtaining the field equations, whatever road is taken, via say for example, a variational principle.

In 1917 Einstein amended the field equations to

$$R_{ij} - \frac{1}{2} g_{ij}R + \lambda g_{ij} = -K T_{ij}$$

where λ is called the cosmological constant. This was done again heuristically from cosmological considerations, that we have already discussed. Einstein would later describe the introduction of the cosmological constant into the field equations as "the biggest blunder of my life."[1] Here we take $\lambda = 0$.

Of the many other approaches to the field equations of General Relativity, one of the most important, is usually called the linearized theory of gravity, or spin 2 approach. This is the approach taken by Gupa, Feinman Thiring, Weinberg and Ohanian.[2] Here gravitation is taken as a field theory analogous to electrodynamics, whereas electrodynamics is described by a vector field, gravitation, even in the Newtonian model, is described by a tensor field. Here one easily obtains a "linear approximation" then the nonlinear field equations of Einstein emerge as a "natural" generalization of the equations of the linear approximation. Here one gains a clear understanding of why we have the field equations of Einstein. In this approach we deal with many interesting problems such as light deflection, retardation, gravitational red-shift, without the mathematics of pseudo-Reimannian space-time.

Another interesting view, first popularized by A.L. Trautman[3], is that general covariance, in any of its forms, is not essential for setting up General Relativity. This involves putting Newton's gravitational

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1 Gravitation: C. Misne, K. Thorne and J.A. Wheeler, Freeman, 1973.

2 Gravitation and Spacetime, H.C. Ohanian, W.W. Norton & Co., Inc., New York, 1976.

3 Trautman, A., Brandeis Lectures 1964, Prentice-Hall Inc., 1965.

model in a geometrical setting, here the affine connection and curvature tensor, (but not the metric) make their appearance, this was first done by E. Cartan. Then General Relativity emerges as the most "natural" generalization of Special Relativity (at this stage the metric appears) to include Cartan's setting of the Newtonian model in the non-relativistic limit. This approach is gone into in great detail in a rather massive book by Misner, Thorne and Wheeler.[2]

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1 Misner, C.W., K.S. Thorne, and J.A. Wheeler, Gravitation, W.H. Freeman, San Francisco, 1973.

CHAPTER III

1. THE TESTS OF THE GENERAL THEORY OF RELATIVITY

How are we to judge a particular mathematical model (theory) to be a fitting description of nature?

To be taken seriously a theory must pass certain tests. The tests are usually based on certain quantitatively verifiable physical phenomena. The theory must predict each of these required phenomena, and do so as accurately as is verifiable experimentally. If the theory cannot do this, it should be discarded.

It is important to make our point of view clear as to what is a significant test for a theory. Our experiment (test) is significant when it helps to distinguish between several competing models, so that we are able to choose the one which gives the best description of nature (at least some particular aspect of nature). When a model is successful, in the above sense, we can sometimes fall into the danger of thinking of the model as reality itself. This is especially true when the model is is as attractive as is General Relativity. The correct or healthy attitude to have for a mathematical model might be that of J.L. Synge:

> "Do I believe in the theory of relativity, special and general? Much as I dislike the name (I would much prefer to follow Minkowski, but it is now too late) my immediate answer is "yes" but there are second thoughts, and I count as my seventh milestone the fact that I permit myself to have second thoughts. In saying this, I do not wish to associate myself with that small but persistent group of people convinced that the theory contains

some essential error against which they protest but which they seem unable to make clear. I take the view, probably shared by many today that no mathematical theory can possibly account accurately for all physical phenomena. Improvements in the techniques of experiment and observation are bound to reveal inadequacies in all theories which, when patched up to fit the facts, lose their intellectual appeal. We are concerned with mathematical structures which have in the words of Leopold Infeld, links with reality. Without those links, all so called physical theories would merely be elegant pieces of pure mathematics with suggestive physical words thrown in to catch the attention of physicists."[1]

Einstein did not feel the need to test his theory, but he did suggest these tests: The precession of the perihelion of the orbit of the planet Mercury, the data on this test was available at the time General Relativity was formulated (1915); the deflection of starlight by the sun, confirmed in 1919 by A.S. Eddington; the gravitational red shift of spectral lines, confirmed by Adams in 1924.

It is now sixty five years that General Relativity is considered to be the model for gravity of best fit with the data available, that is, it has passed all tests people have been thinking up for it. Here we consider the test that led to the acceptance of General Relativity as a replacement for the Newtonian gravitational model. We also consider a few of the competing models, those of historical interest,

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1 My Relativistic Milestones: J.L. Synge, in Albert Einstein's Theory of General Relativity, (ed. G. Tauber), Crown Pub. Co., New York, 1979.

then we mention one of the competing models that is of interest today.[1] Recently (1970's) there have sprung up many competing models and even Theories of theories, but we shall not mention these again.

Most basic of all tests, for any gravitational model, are the Eötvös-type experiments. These experiments test the weak principle of equivalence to a very great accuracy, as we have seen. This principle is built into General Relativity. There is nothing like this in the Newtonian model: there, nothing forbids the existence of say a particle which is "gravity neutral".

The most accurate after the Eötvös type experiments are those dealing with the gravitational red shift of spectral lines. This test was accurately predicted by Einstein in 1911 using the strong equivalence principle. The displacement of spectral lines towards the red end of the spectrum was definitely established by W.S. Adams in 1924 by astronomical observations of the dense companion of Sirius. This effect on Sirius' dense companion is about thirty times greater than for the sun. But this test, like other astronomical test of the same type that followed lacked convincing quantitative verification until 1959. This was due in part because of the difference between the gravitational potential at the observer and that at the source depends on the inverse of an uncertain solar radius. In 1959 R.V. Pound and G.A. Rebka were able to calculate this effect on earth in a tower 22.5 meters high, at Harvard. The certainty in this test is now very high, the experiment having been repeated many times. Any gravitational model unable to describe this experiment accurately must be rejected. As it can be

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[1] The Brans-Dicke Theory.

obtained using strong equivalence, it is not considered a test of the field equation of General Relativity, although it is easily obtained from them.

The most important test to those interested in black holes is the one on the gravitational deflection of light. Using the strong equivalence principle Einstein (1911) gave the first scientific explanation of light deflection by gravitational fields. Laplace (1798) had assumed light to be influenced by gravity, in his black-hole model, but Laplace had no scientific basis for this assumption. The value obtained by Einstein (1911) for star light deflection by the sun is half of that obtained later (1915-16) using General Relativity.

In 1919 Eddington and Dyson organized two expeditions to observe the eclipse of May 29, 1919, on the island of Sobral (off Brazil) and Principe (in the Gulf of Guinea). During the eclipse, using an especially constructed apparatus, they were able to observe the stars near the sun, which are only visible in a total eclipse, and by taking photographs of these stars, they then compared these photographs with other photographs of the same stars taken when the sun was not in this star field.
The values obtained for star light deflection by the sun were:

at Sobral $(1.98 \pm 0.16)''$;

at Principe $(1.61 \pm 0.40)''$;

which are of the right magnitude when compared with the predicted value of General Relativity, which is 1.75".

In telling about the expedition later, Eddington, described how he on the island of Principe, hastened to measure his eclipse photographs and on finding that the observed deflection seemed to agree with the predicted value of General Relativity, experienced what he called the greatest moment of his life.

These results, at the time (1919), made a sensation in the press, and Einstein became a world celebrity. Light deflection was considered the confirming test of General Relativity. Ironically, since 1919, many such measurements have been made and the result range from 1.5" to nearly 3". Today many relativist consider light deflection a poor test of General Relativity. But this view is not shared by everyone:

> "No method for determining the gravitational deflection of light has as simple a concept or as little dependence on secondary parameters as the classical method that makes use of photography during a solar eclipse. The older eclipse observations can be criticized on several grounds, including failure to use identical optics for eclipse and reference exposures, failure to obtain night plates with exactly the same instrumental setup as used for day plates, absence of temperature control, and the unavailability of modern microdensitometric reduction techniques."[1]

To date the best publicized eclipse expedition, using modern techniques, was led by Bryce S. De Witt, known as "The Texas Mauritanian Eclipse Expedition" of 1973. For the details on this expedition apparatus and problems one should consult 1 and 2

A note of De Witt's summarizes the results of this expedition:

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[1] Gravitational Deflection of Light, Solar Eclipse of 30 June 1973, B.A. De Witt; Albert Einstein's Theory of General Relativity (ed. G. Tauber) Crown Pub. Inc., New York, (1979).

[2] The Texas Mauratanian Eclipse Expedition, B.S. De Witt, General Relativity and Gravitation (ed. G. Shaviv and I. Rosen), John Wiley & Sons, 1975.

"The analysis reveals that there was much more "noise" on the plates than we had hoped, resulting directly from bad atmospheric conditions at eclipse time.... In consequence our precautions and innovations e.g., artificial grid and sistometric standards .... have proved irrelevant. The final result of our measurements is $\gamma = 0.95 \pm 1.0$ where $\gamma = 1.0$ corresponds to Einsteins prediction."[1]

A recent (new) test of General Relativity is described here in the words of C.W.F. Everitt:[2]

"A new test of Einstein's theory, related to the star-light deflection, was suggested in 1964 by I.I. Shapiro, who pointed out that a relativistic delay may be expected in the round-trip travel time of radar ranging signals to planets or spacecraft passing behind the sun. The delay adds a bump to the apparent orbit. The effect was first observed to about 10% accuracy in radar ranging of Venus during 1967. It has since been studied in data from Mariner 6 and 7 spacecraft and most recently in ranging to the Viking orbiters and landers on Mars. Many corrections have to be applied to take out subsidiary effects. The best data are from the Viking landers, which agree with the relativistic prediction to 0.3%, the most accurate of all the

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1 Gravitational Deflection of Light, Solar Eclipse of 30 June 1973, B.A. De Witt; Albert Einstein's Theory of General Relativity (ed. G. Tauber) Crown Pub. Inc., New York (1979).

2 Experimental Tests of General Relativity: Past, Present and future, C.W.F. Everitt; Albert Einstein's Theory of General Relativity (ed. G. Tauber) Crown Pub., Inc., New York (1979).

tests of General Relativity to date except for the gravitational red shift."

The importance of this test is in that it distinguishes between General Relativity and a competing theory of Brans-Dicke. We shall discuss the Brans-Dicke theory later. This data is available in (1).

The earliest test of General Relativity that we have mentioned is the one dealing with the perihelion of the planet Mercury. The observational data for the motion of the perihelion of Mercury was available at the time Einstein formulated his General Theory of Relativity. It was seen that General Relativity explained the previously inaccountable discrepancy between the theoretical motion predicted by Newton's Theory of Universal Gravitation and the observational motion.

Early critics of General Relativity claimed the observational data to be wrong; this was possible since "the determination of the precessional motion is one of the most difficult problems of positional astronomy if not the most difficult."[1]

The observational data on the motion of Mercury came to be regarded as satisfactory with the work of G.M. Clemence.[2] This test is again controversial, Dicke has claimed the sun to be an "oblate spheroid" with the centre of the sun rotating faster than the surface. However, this view is not commonly shared.

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1 Test of Theories of Gravity in the Solar System; J.P. Richard, General Relativity and Gravitation (ed. G. Shaviv & S. Rosen), John Wiley & Sons, Inc., New York (1975).

2 The Relativity Effects in Planetary Motions; G.M. Clemence, Reviews of Modern Physics, 19, No. 4, (1947), 361-364.

At the moment this is being written, the sun is probably as controversial an astronomical object as is Cygnus X-1 , so we shall disregard this controversy for the time being and accept this original test as still valid.

A.N. Whitehead proposed an alternate model for gravity in 1922. This model yielded the same predictions as General Relativity, it was able to give the same results for the perihelion advance by admitting an additional adjustable parameter. In this way he could account for any amount of perihelion advance or retardation. Up until 1971 both theories were thought to give equivalent result and general relativists preferred their theory on the grounds of simplicity and elegance. In 1971 the work of C.M. Will showed that the Whitehead model predicts a time dependence for the ebb and flow of ocean tides, which is quite unrealistic.

Another model which satisfied the same first three tests was Birkhoff's theory formulated in 1943. Unfortunately it required that sound waves travel with the speed of light. This defect was due to pressure inside gravitating bodies being equal to the total density of mass-energy. Today the most notable competitor to General Relativity is the Brans-Dicke gravitational model. In the Brans-Dicke model the gravitational fields is described by a tensor field (as is General Relativity) plus an extra gravitational scalar field, (unlike General Relativity). The qualitative effect of this extra scalar field is that it makes the gravitational constant depend on position. It is often claimed in the literature that experimental evidence is against the Brans-Dicke model.

Whatever point of view one takes as to the competing gravitational models either as foils against which to test General Relativity, or as viable alternatives to General Relativity, one must admit that so far as solar system experiments are concerned, the competing models can only match General Relativity which has passed all of these tests with flying colours. Thus to find the best model that describes gravity, one must look outside the solar system where strong gravitational fields may be found.

There are many alternate models to General Relativity, many of the recent ones like the earlier ones, have been disproved and those that remain as reasonable competitors to General Relativity are invariably more complicated and so are rejected on the ground of elegance and simplicity.

2. THE SCHWARZSCHILD SOLUTION

To date most of the key experiments that have been carried out to test the difference between Einstein's Theory of General Relativity and its competitors have been based on predictions obtained with the use of the Schwarzschild solution. The best known of these predictions are:

(1) The gravitational red shift of spectral lines;

(2) The deflection of star light by the sun;

(3) The precession of the perihelion of the orbit of the planet Mercury;

(4) The time delay of radar echoes passing the sun.

Historically the second and third predictions were obtained by A. Einstein in 1915 by the use of the method of linear approximation; when he had investigated the static, spherically symmetric gravitational field outside a massive spherically symmetric body; at this time there were no exact solutions to the field equations. Karl Schwarzschild gave the first exact solution to the field equations of General Relativity when he considered the same problem in 1916.

We would like to digress before considering the Schwarzschild solution in detail, and mention the circumstances under which Karl Schwarzschild derived his famous solution. This is not only a great piece of mathematics, but must also stand out as an outstanding example of human courage. In 1915, Karl Schwarzschild was with the German army on the Eastern Front, there at this time he contracted an infectious disease. Given by army doctors, only a few weeks to live, he returned to Germany in the late fall of 1915; he was to die on the 11th May 1916. During this period, when he was dying, he wrote his two most famous papers. One was on the theory of the Stark effect and the foundations for the interpretation of band spectra on the Bhor-Sommerfeld theory; in the other paper he gave the first exact solution to Einstein's field equations. This solution would become known as the Schwarzschild solution. In this second paper Schwarzschild used his solution to calculate the predicted values for both the perihelion of Mercury and light deflection by the sun. Eddington would write in a notice published later in 1916:

> "Schwarzschild's end is a sad story of long suffering from a terrible illness contracted in the field borne with courage and patience."[1]

Here we list the assumptions that we shall use, as did Karl Schwarzschild, to find a mathematical model for the gravitational field surrounding a spherically symmetric, electrically neutral, mass distribution at rest in a vacuum. These are:

(1) The gravitational field shall be static;

(2) The gravitational field is itself spherically symmetric;

(3) The space becomes flat at infinity.

The static character of the gravitational field imposes restrictions on the form of the metric

$$ds^2 = g_{ij} dx^i dx^j$$

By static we mean that the field is both time independent (i.e., $\frac{\partial g_{ij}}{\partial x^0} = 0$ for $ij = 0,1,2,3$), and time symmetric (terms of the

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[1] Development of General Relativity, S. Chendroseker, Nature, Vol. 252, Nov. 1, 1974.

form $g_{0\alpha}$, $\alpha = 1,2,3$ in the metric, must be zero otherwise the terms of the form $dx^0 dx^\alpha$ change sign under the time reversal $x^0 \to -x^0$). So the static requirement means that out metric takes the form

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta$$

The demand for spherical symmetry of the gravitational field means that, in this problem at least, the spherical co-ordinates (t,r,θ,φ) are favoured. Radial symmetry implies that there is no preferred angular direction, thus the metric is independent of the terms involving $d\theta\, d\varphi$, $dr\, d\theta$, and $dr\, d\varphi$, which would change sign under the transformations $\theta \to -\theta$ and $\varphi \to -\varphi$.

The metric becomes entirely diagonal, that is, has the form

$$ds^2 = g_{ii}(dx^i)^2$$

Also since the only rotational invariants are

$$r$$

$$rdr$$

$$dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

the metric must take the form

$$ds^2 = A(r)dt^2 - B(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) - C(r)dr^2 \qquad (1)$$

where A B and C are functions of r only by our assumption of radial symmetry.

The metric (1) can be simplified if we let

$$t' = t$$
$$r' = r\sqrt{B(r)}$$
$$\theta' = \theta$$
$$\varphi' = \varphi$$

so that (1) becomes

$$ds^2 = A'(r')(dt')^2 - B'(r)(dr')^2 - (r')^2(d\theta')^2 - (r')^2 \sin^2\theta'(d\varphi')^2$$

where

$$A' = A \quad , \quad B' = B + C \; .$$

If we omit the primes we have

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In order to exhibit clearly the signature of g_{ij} and the sign of the determinant $\det g_{ij} = g$ we write $A(r)$ and $B(r)$ as the intrinsically positive functions:

Define $\quad 2a(r) = \ell n|A(r)|$,

$\quad\quad\quad\quad 2b(r) = \ell n|B(r)|$,

so the metric becomes

$$ds^2 = e^{2a(r)}dt^2 - e^{2b(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

The boundary condition (3), will be satisfied if we have both $a(r)$ and $b(r)$ go to zero when r approaches infinity.

Since

$$g_{ij} = \begin{matrix} e^{2a} & 0 & 0 & 0 \\ 0 & -e^{2b} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{matrix}$$

so we have

$$g = -e^{2(a+b)}\ r^4 \sin^2\theta\ .$$

To find the Christoffel symbols of the second kind

$$\left\{ {\ell \atop m\ n} \right\} = \frac{1}{2}\, g^{\ell k}(g_{mk,n} + g_{nk,m} - g_{mn,k})$$

we first have to find g^{ij} where $g^{ij}g_{ik} = \delta^i{}_k$.

Since

$$g_{ij} = \begin{matrix} e^{2a(r)} & 0 & 0 & 0 \\ 0 & -e^{2b(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{matrix}$$

we must have

$$g^{ij} = \begin{matrix} e^{-2a(r)} & 0 & 0 & 0 \\ 0 & -e^{-2b(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{matrix}$$

with

$$x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi\ .$$

Clearly the only non-zero Christoffel symbols of the second kind are:

$$\left\{{0 \atop 0\ 1}\right\} = \left\{{0 \atop 1\ 0}\right\} = a' \quad , \quad \left\{{1 \atop 0\ 0}\right\} = a' \, e^{2(a-b)}$$

$$\left\{{1 \atop 1\ 1}\right\} = b' \quad , \quad \left\{{1 \atop 2\ 2}\right\} = -r \, e^{-2b} \quad , \quad \left\{{1 \atop 3\ 3}\right\} = -r \sin^2\theta \, e^{-2b}$$

$$\left\{{2 \atop 2\ 1}\right\} = \left\{{2 \atop 1\ 2}\right\} = \left\{{3 \atop 3\ 1}\right\} = \left\{{3 \atop 1\ 3}\right\} = \frac{1}{r} \quad ,$$

$$\left\{{2 \atop 3\ 3}\right\} = -\sin\theta\cos\theta \quad , \quad \left\{{3 \atop 3\ 2}\right\} = \left\{{3 \atop 2\ 3}\right\} = \cot\theta \; .$$

Now we calculate the Ricci tensors

$$R_{mn} = R^{\ell}_{mn\ell} = \left\{{\ell \atop m\ \ell}\right\}_{,n} - \left\{{\ell \atop m\ n}\right\}_{,\ell} + \left\{{\ell \atop n\ s}\right\}\left\{{s \atop m\ \ell}\right\} - \left\{{\ell \atop \ell\ s}\right\}\left\{{s \atop m\ n}\right\}$$

and obtain

$$R_{00} = e^{2(a-b)}\left(-a'' - (a')^2 + a'b' - \frac{2a'}{r}\right) \tag{2}$$

$$R_{11} = a'' + (a')^2 - a'b' - \frac{2b'}{r} \tag{3}$$

$$R_{22} = (1 + ra' - rb')e^{-2b} - 1 \tag{4}$$

$$R_{33} = R_{22} \sin^2\theta \tag{5}$$

In a vacuum we must have

$$R_{00} = R_{11} = R_{22} = 0$$

Thus $R_{00} + R_{11} = 0$ gives $a' + b' = 0$ which implies that $a + b$ is constant. The boundary conditions on a and b as $r \to \infty$ yield $a + b = 0$. Next on replacing $a' = -b'$ in (4) and setting equal to zero we have $(1+2ra')e^{2a} = 1$ or $(re^{2a})' = 1$.

Then on integrating we have

$$r\, e^{2a} = r - C$$

where C is some constant. Thus

$$e^{2a} = 1 - \frac{C}{r}$$

and

$$e^{2b} = e^{-2a} = \frac{1}{1 - \frac{C}{r}}$$

The metric for a static spherically symmetric field takes the form

$$ds^2 = (1 - \frac{C}{r})dt^2 - \frac{1}{1 - \frac{C}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

It is easy to show that the constant C is $\frac{2Gm}{c^2}$ in c.g.s. units. If we choose $c = G = I$, the Schwarzschild metric is then

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{1 - \frac{2m}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

This is the Schwarzschild solution in standard form.

3. GRAVITATIONAL RED SHIFT

Consider a light emitted at the surface of the sun at the point (r_1, θ, φ) and received on earth at the point (r_2, θ, φ). Assume that in both these points there is a standard clock. The proper-time intervals at the sun is related to the co-ordinate-time intervals by

$$ds_1^2 = (1 - \frac{2m}{r_1})dt^2 \quad (1)$$

Similarly, on earth, the proper-time intervals are related to the coordinate-time intervals by

$$ds_2^2 = (1 - \frac{2m}{r_2})dt^2 \quad (2)$$

Suppose that n waves of frequency ν_2 are emitted in proper time Δs_1 from an atom on the sum. Then

$$n = \nu_1 \Delta s_1 \quad (3)$$

On earth one must receive n waves, but the frequency and time duration of the wave train have changed. So we have

$$n = \nu_2 \Delta s_2 \quad (4)$$

Thus (3) and (4) imply that $\frac{\nu_2}{\nu_1} = \frac{\Delta s_1}{\Delta s_2}$ and from (1) and (2) we obtain

$$\frac{\nu_2}{\nu_1} = \left(\frac{1 - \frac{2m}{r_1}}{1 - \frac{2m}{r_2}} \right)^{\frac{1}{2}}$$

$$= \left(1 - \frac{2m}{r_1}\right)^{\frac{1}{2}} \left(1 - \frac{2m}{r_2}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{m}{r_1} - \frac{1}{8}\frac{4m^2}{r_1^{\,2}} + \ldots\right)\left(1 + \frac{m}{r_2} + \frac{3}{8}\frac{4m^2}{r_1^{\,2}} + \ldots\right) = 1 - \frac{m}{r_1} + \frac{m}{r_2} \quad (5)$$

to first order in $\frac{m}{r_1}$ and $\frac{m}{r_2}$. On simplifying equation (5) we have

$$\frac{\nu_2 - \nu_1}{\nu_1} = m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \quad (6)$$

Now since $r_2 > r_1$ our change of potential is negative and we have the frequency of light decreasing as it leaves the sun and when it is received on earth, we see a shift toward the red end of the spectrum.

4. THE PERIHELION OF MERCURY

In what follows we shall be concerned with a central spherically symmetric gravitating body and a test particle; by a test particle we mean a particle that does not influence the gravitational field. Henceforth when we write particle we shall mean test particle, unless otherwise stated. We postulate that the motion of a particle in a gravitational field, not acted upon by any other forces, is governed by the geodesic equations

$$\frac{d^2x^k}{ds^2} + \begin{Bmatrix} k \\ m\ n \end{Bmatrix} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0 \tag{1}$$

and that the motion of a photon, under similar conditions, is governed by the equations

$$\frac{d^2x^k}{d\sigma^2} + \begin{Bmatrix} k \\ m\ n \end{Bmatrix} \frac{dx^m}{d\sigma} \frac{dx^n}{d\sigma} = 0 \tag{2}$$

since for a photon the proper time s is zero, where we have to choose some other parameter σ. It follows from (1) that for a particle we have the constant of motion

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1 \tag{3}$$

and from (2) that for a photon we have the constant of motion

$$g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = 0 \tag{4}$$

The motion of a particle moving about in the gravitational field of a spherically symmetric body shall be taken as the model for the motion of the planet Mercury about the sun. In this model it is assumed that the other planets or stars do not significantly perturb the motion of Mercury, so none of them are included in the model. (i.e. in the whole universe there is only the Sun and Mercury) so then the world line of Mercury is a geodesic in the four dimensional manifold associated with the Schwarzschild metric.

From

$$ds^2 = (1-\frac{2m}{r})dt^2 - \frac{1}{1-\frac{2m}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

we have

$$\left\{ {1 \atop 0\ 0} \right\} = \frac{m}{r^2}(1 - \frac{2m}{r}) ,$$

$$\left\{ {1 \atop 1\ 0} \right\} = \left\{ {0 \atop 0\ 1} \right\} = \frac{m}{r^2}(1 - \frac{2m}{r})^{-1} ,$$

$$\left\{ {1 \atop 1\ 1} \right\} = -\frac{m}{r^2}(1 - \frac{2m}{r})^{-1} ,$$

$$\left\{ {2 \atop 2\ 1} \right\} = \left\{ {2 \atop 1\ 2} \right\} = \left\{ {3 \atop 3\ 1} \right\} = \left\{ {3 \atop 1\ 3} \right\} = \frac{1}{r} ,$$

$$\left\{ {1 \atop 2\ 2} \right\} = -r(1 - \frac{2m}{r}) ,$$

$$\left\{ {1 \atop 3\ 3} \right\} = -r\sin^2\theta\,(1 - \frac{2m}{r}) ,$$

$$\left\{ {2 \atop 3\ 3} \right\} = -\sin\theta\cos\theta ,$$

$$\left\{ {3 \atop 3\ 2} \right\} = \cot\theta ,$$

as the only non-zero Christoffel symbols of the second kind.

Replacing the non-zero Christoffel symbols of the second kind into equation (1) we obtain

$$\frac{d^2x^0}{ds^2} + 2\left\{ {0 \atop 0\ 1} \right\} \frac{dx^0}{ds} \frac{dx^1}{ds} = 0 \tag{5}$$

$$\frac{d^2x^1}{ds^2} + \left\{ {1 \atop 0\ 0} \right\} \left(\frac{dx^0}{ds}\right)^2 + \left\{ {1 \atop 1\ 1} \right\} \left(\frac{dx^1}{ds}\right)^2 + \left\{ {1 \atop 2\ 2} \right\} \left(\frac{dx^2}{ds}\right)^2 + \left\{ {1 \atop 3\ 3} \right\} \left(\frac{dx^3}{ds}\right)^2 = 0 \tag{6}$$

$$\frac{d^2x^1}{ds^2} + 2\left\{ {2 \atop 1\ 2} \right\} \frac{dx^1}{ds} \frac{dx^2}{ds} + \left\{ {2 \atop 3\ 3} \right\} \left(\frac{dx^3}{ds}\right)^2 = 0 \tag{7}$$

$$\frac{d^2x^3}{ds^2} + 2\left\{ {3 \atop 1\ 3} \right\} \frac{dx^1}{ds} \frac{dx^3}{ds} + 2\left\{ {3 \atop 2\ 3} \right\} \frac{dx^2}{ds} \frac{dx^3}{ds} = 0 \tag{8}$$

or equivalently we have on substitution

$$\frac{d^2t}{ds^2} + \frac{2m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} \frac{dt}{ds} \frac{dr}{ds} = 0 \tag{5}$$

$$\frac{d^2r}{ds^2} + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r\left(1 - \frac{2m}{r}\right)\left(\frac{d\theta}{ds}\right)^2$$

$$- r^2\sin^2\theta\left(1 - \frac{2m}{r}\right)\left(\frac{d\varphi}{ds}\right)^2 = 0 \tag{6}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = 0 \tag{7}$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\varphi}{ds} = 0 \tag{8}$$

for obvious reasons we do not want to use equation (6) so we replace it with the constant of motion given by equation (3) which is

$$\left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{1-\frac{2m}{r}}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\varphi}{ds}\right) - r^2\sin^2\varphi\left(\frac{d\phi}{ds}\right)^2 = 1 \qquad (9)$$

Let us assume that our particle moves in the plane $\theta = \frac{\pi}{2}$ that is

$$\left.\frac{d\theta}{ds}\right|_{t=0} = 0 \qquad \text{and} \qquad \cos\theta = 0$$

from equation (7) we have

$$\left.\frac{d^2\theta}{ds^2}\right|_{t=0} = 0$$

hence $\left.\frac{d^n\theta}{ds^n}\right|_{t=0} = 0$ for all n.

Thus for $\theta = \frac{\pi}{2}$ at any time the equation (7) vanishes and equations (8) and (9) simplify to

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = 0 \qquad (10)$$

and

$$\left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{1-\frac{2m}{r}}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\varphi}{ds}\right)^2 = 1 \qquad (11)$$

Now equation (5) is equivalent to

$$\left(1-\frac{2m}{r}\right)^{-1}\frac{d}{ds}\left(\left(1-\frac{2m}{r}\right)\frac{dt}{ds}\right) = 0$$

from which we must have

$$\left(1-\frac{2m}{r}\right)\frac{dt}{ds} = k$$

for some constant k. Also equation (10) can be put in the form

$$\frac{1}{r^2}\frac{d}{ds}\left(r^2\frac{d\varphi}{ds}\right) = 0 \qquad (12)$$

which implies that

$$r^2 \frac{d\varphi}{ds} = h \tag{13}$$

for some constant h. With proper use of (12) and (13) we can eliminate t and s from (11) and obtain

$$\left(1 - \frac{2m}{r}\right)\left(\frac{h}{1 - \frac{2m}{r}}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{h}{r^2}\frac{dr}{d\varphi}\right)^2 - r^2\left(\frac{h}{r^2}\right)^2 = 1$$

or

$$\left(\frac{dr}{d\varphi}\right)^2 = -\frac{r^4}{h^2}\left[\left(1 + \frac{h^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) - k^2\right] \tag{14}$$

on substituting $r = \frac{1}{u}$ into (14) we have

$$\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)^2 = -\frac{1}{h^2u^4}\left[(1 + h^2u^2)(1 - 2mu) - k^2\right]$$

hence

$$\left(\frac{du}{d\varphi}\right)^2 = -\frac{1}{h^2}\left[(1 + h^2u^2)(1 - 2mu) - k^2\right] \tag{15}$$

now differentiate (15) with respect to φ

$$2\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} = -\frac{1}{h^2}\left[2h^2u(1 - 2mu) + (1 + h^2u^2)(-2m)\right]\frac{du}{d\varphi} . \tag{16}$$

We can reduce equation (16) to

$$\frac{d^2u}{d\varphi^2} + u = 3mu^2 + \frac{m}{h^2} \tag{17}$$

For planets in the solar system $3mu^2$ is much smaller than m/h^2, so if we neglect the term $3mu^2$ we then have the orbital equation of the Newtonian theory. Neglecting the term $3mu^2$, the solution to equation

(17) is

$$u = \frac{m}{h^2}\left[1 + e\cos(\varphi - \varphi_0)\right] \qquad (18)$$

where e is the eccentricity of the elliptic orbit and φ_0 is the longitude of the perihelion.

If we now consider equation (17) with the term $3mu^2$ with u replaced by the value given in (18) we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + \frac{3m^3}{h^4}\left[1 + 2e\cos(\varphi - \varphi_0) + e^2\cos^2(\varphi - \varphi_0)\right]$$

now for a nearly circular orbit e^2 is very small; so we may neglect the term containing it. The term $\frac{3m^3}{h^4}$ can also be ignored*, so we are left with

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 6e\,\frac{m^3}{h^4}\cos(\varphi - \varphi_0)$$

The solution of this differential equation is

$$u = \frac{m}{h^2}\left[1 + e\cos(\varphi - \varphi_0) + 3\,\frac{em^3}{h^4}\,\varphi\sin(\varphi - \varphi_0)\right]$$

which is approximately

$$u \simeq \frac{m}{h^2}\left[1 + e\cos\left(\varphi - \varphi_0 - \frac{3m^2}{h^2}\varphi\right)\right] \qquad (19)$$

if $\frac{m^2}{h^2}\varphi$ is small. We shall denote $\frac{3m^2\varphi}{h^2}$ by $\delta\varphi$.

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* This is equivalent to a change in the constant $\frac{m}{h^2}$ and produces no interesting observable effects.

Thus the major axis of the elliptic orbit is slowly rotating about its focus, and there will be an advance of perihelion of the orbit per revolution which is given by $\delta\varphi = \frac{6\pi m^2}{h^2}$ * or in cgs unit we have

$$\delta\varphi = 6\pi \left(\frac{Gm}{hc}\right)^2$$

From equation (18) we have

$$r = \frac{h^2}{m} \frac{1}{1 + e\cos(\varphi - \varphi_0)}$$

also it is known that the equation of an ellipse has the form

$$r = \frac{a(1 - e^2)}{1 + e\cos(\varphi - \varphi_0)}$$

where a is the semi-major axis. Thus we have

$$h^2 = m\,a(1 - e^2)$$

and

$$\delta\varphi = \frac{6\pi G^2 m}{ac(1-e^2)} \tag{20}$$

So when we know G, the universal gravitational constant, the mass m of the sun, c the speed of light, the semi-major axis of the orbit of the planet Mercury, and the eccentricity of this planet we have $\delta\varphi$ its advance of perihelion. The predicted value is $\delta\varphi = 43.03''$ per century and the observe perihelion advance is $(43.11 \pm .45)''$ per century, hence we can see very good agreement between the predicted value and the observed value.

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* $\varphi = 2\pi$.

To finish with the advance of perihelion of the orbit of the planet Mercury we quote a letter Einstein wrote to Paul Ehrenfest in 1916. This is more in order to show the frame of mind of this man than as to the importance he attached to the experimental verification of his theory.

> "Imagine my joy at the feasibility of the general covariance and at the result that the equations yield the correct perihelion motion of Mercury. I was beside myself with ecstacy for days."

5. LIGHT DEFLECTION

The equations of motion of a photon are like those of a particle except that since proper-time s is zero we must use some other parameter σ. The equations of motion that we choose are

$$\frac{d^2t}{d\sigma^2} + \frac{2m}{r^2}\frac{1}{1-\frac{2m}{r}}\frac{dt}{d\sigma}\frac{dr}{d\sigma} = 0 \qquad (1)$$

$$\frac{d^2\theta}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0 \qquad (2)$$

$$\frac{d^2\varphi}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma}\frac{d\varphi}{d\sigma} + 2\cot\theta\frac{d\theta}{d\sigma}\frac{d\varphi}{d\sigma} = 0 \qquad (3)$$

we omit the equation corresponding to equation (6) for the perihelion, and take in its place the constant of motion for a photon

$$\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\theta}{d\sigma}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0 \qquad (4)$$

As we did for the perihelion we take $\theta = \frac{\pi}{2}$ so that equation (2) vanishes and (1) and (3) become

$$r^2\frac{d\varphi}{d\sigma} = h$$

h some constant,

$$\left(1-\frac{2m}{r}\right)\frac{dt}{d\sigma} = k$$

k some constant, respectively, now equation (4) becomes

$$\left(1-\frac{2m}{r}\right)\left(\frac{k}{1-\frac{2m}{r}}\right)^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{h}{r^2}\frac{dr}{d\varphi}\right)^2 - r^2\left(\frac{h}{r^2}\right)^2 = 0$$

upon the appropriate substitutions. Now we solve for $\left(\frac{dr}{d\varphi}\right)^2$ and obtain

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{h^2}\left[k^2 - \frac{h^2}{r^2}\left(1 - \frac{2m}{r}\right)\right] \quad ;$$

now on replacing r with $\frac{1}{u}$ in the above we get

$$\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)^2 = \frac{1}{u^4h^2}\left[k^2 - h^2u^2(1 - 2mu)\right]$$

we next differentiate the above with respect to φ thus,

$$2\,\frac{du}{d\varphi}\,\frac{d^2u}{d\varphi^2} = \frac{1}{h^2}\left[-2h^2u + 6mh^2u^2\right]\frac{du}{d\varphi}$$

and this reduces to

$$\frac{d^2u}{d\varphi^2} + u = 3mu^2 \quad . \qquad (5)$$

If we neglect $3mu^2$ in equation (5) the solution is

$$U_0 = A\cos(\varphi - \varphi_0) \qquad (6)$$

by an appropriate orientation of axes (6) may be written

$$U_0 = A\cos\varphi$$

now $\frac{1}{r} = U_0$ so that $\frac{1}{A} = r\cos\varphi$ but $r\cos\varphi$ is the cartesian co-ordinate x, which is the equation of a straight line parallel to the y-axis. (The light ray on first approximation is not deflected by the sun's gravitational field). Let $a = \frac{1}{A}$ then $a = r\cos\varphi$ and $r_{min} = a$ (i.e. the minimum distance between the sun and the light ray).

Replacing

$$U = \frac{1}{a} \cos\varphi$$

into equation (5) we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{3m}{a^2} \cos^2\varphi$$

$$= \frac{3}{2} \frac{m}{a^2} (1 + \cos 2\varphi) \qquad (7)$$

Clearly the solution of equation (7) must be of the form $u_1 = \alpha + \beta \cos 2\varphi$ and since

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \alpha - 3\beta \cos 2\varphi$$

we have

$$\alpha = \frac{3}{2} \frac{m}{a^2} \quad \text{and} \quad \beta = -\frac{m}{2a^2} ,$$

and $u_1 = \frac{m}{2a^2} (3 - \cos 2\varphi)$. The complete solution is

$$u = u_0 + u_1$$

$$= \frac{1}{a} \cos\varphi + \frac{m}{2a^2} (3 - \cos 2\varphi) \qquad (8)$$

Equation (8) is the perturbation of the equation of a straight line, this perturbation produces a small deflection in a light ray passing near the sun.

Now as $r \to \infty$, $u \to 0$ and

$$\frac{1}{a} \cos\varphi + \frac{m}{2a^2} (3 - \cos 2\varphi) = 0$$

or

$$\cos^2\varphi - \frac{a}{m}\cos\varphi - 2 = 0$$

which has solutions

$$\begin{aligned} \cos\varphi &= \frac{1}{2}\left(\frac{a}{m} \pm \sqrt{\frac{a^2}{m^2} + 8}\right) \\ &= \frac{a}{2m}\left(1 \pm \sqrt{1 + \frac{8(3m)^2}{9\,a^2}}\right) \\ &= \frac{a}{2m}\left(-\frac{4(3m)^2}{9\,a^2} + \frac{8(3m)^4}{81\,a^4} + \ldots\right)^* \\ &\simeq -\frac{2m}{a} \end{aligned} \qquad (9)$$

to first order in 3m.

Clearly $\varphi = \frac{\pi}{2} + \delta$ or $\varphi = -\frac{\pi}{2} - \delta$ where δ is small** and positive and equation (9) becomes

$$\sin\delta = \frac{2m}{a}$$

or

$$\delta = \frac{2m}{a}$$

to first order in δ, for either $\frac{\pi}{2} + \delta$ or $\frac{\pi}{2} - \delta$. Thus the total deflection of the light ray is $\Delta = \frac{4m}{a}$ (or $\Delta = \frac{4GM}{ac^2}$ in cgs. units).

For a light ray which just grazes the sun we have $R_\odot = 6.96 \times 10^6$ cm. and $M_\odot = 1.99 \times 10^{35}$ gm, which gives a predicted deflection of $\Delta = 1.75''$.

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* Choosing the negative sign.

** To first order.

CHAPTER IV

BLACK HOLES

Now that we have seen how one can use the Schwarzschild solution to obtain the theoretical values of the three classical tests for the Theory of General Relativity, (the gravitational red shift, the perihelion of the planet Mercury, and the deflection of star light by the sun), we now investigate this solution in its own right and especially with respect to strong gravitational fields.

Perhaps the first thing we should note about the metric of K. Schwarzschild

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

is that it is here that mass makes its appearance for the first time in General Relativity, and that it does so as the integration constant $2m$. If m is zero our metric takes the form

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

this being the metric of Special Relativity in spherical co-ordinates, as would be expected. In our calculations for the three classical tests we assumed that m was positive, and there we showed that the orbits of planets, in the first approximation, are ellipses just as they are in the Newtonian model. Here we can say that gravity is attractive. Had we assumed that m is negative, then we could have shown that the orbits of planets are hyperbolas. In this case we can say that gravity is repulsive. Observational evidence leads one to believe that mass should be positive and the gravitational fields attractive. Steven Hawking, Roger Penrose, and others in their investigations of the global structure of space-time, which resulted in the now celebrated singularity theorems, made the assumption that gravity is attractive.[1]

For about twenty years one of the major outstanding questions in General Relativity, known as the positive mass conjecture, was to show that an isolated gravitational system which has a non-negative local mass density must have a non-negative total mass as measured gravitationally at spatial infinity. The conjecture was at first shown to be true in a large number of special cases,[2] the most interesting being the spherically symmetric case; the proof here was of surprising difficulty.[3] The conjecture was finally proved true in general in 1979 by Schoen and Yau[4]; this closed one of the major open problems in General Relativity.

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1 The large scale structure; Hawking & Ellis (Chap. 8) Cambridge University Press 1973.

2 Brill, D. and Deser, S., 1968, Ann. Phys., 50, 548.
Brill, D., Deser, S. and Foddev, Le, 1968, Phys. Letters, 26A, 538.

3 Leibovitz, C., and Jareal, W., 1970, Phys. Rev., ID, 3226.
Misner, C., 1971, in Astrophysics and General Relativity, ed. M. Chretien, S. Deser and J. Goldstein (New York: Gordon & Breach).
Jang, P.S., 1976, J. Math. Phys., 1, 141.

4 R. Schoen, S. Yau, 1979, Commun. Math Phys., 65, 45.

Our next observation is that the Schwarzschild solution is asymptotically flat, that is

$$g_{ij} = \eta_{ij} + O(\frac{1}{r})$$

for large r. This is not surprising since it was a boundary condition used in obtaining the solution.

The Schwarzschild solution, in the form it now takes, is only valid for $r > 2m$ since

$$g_{00} = e^{2a(r)} = 1 - \frac{2m}{r}$$

must be positive and

$$g_{11} = - e^{2b(r)} = - \frac{1}{1 - \frac{2m}{r}}$$

is divergent $r = 2m$;

$$r = r_s = 2m$$

is called the Schwarzschild radius. The exact nature of this singularity was not understood at the time this solution was discovered. Karl Schwarzschild in a second paper, published a month after one which introduced his solution, claimed that the singularity was not relevant. Where his first paper had introduced his exterior solution, here he introduces an interior solution valid inside a "star". Considering the field equation of General Relativity appropriate to the equilibrium of a static sphere of constant density, he argued that a "star" of such configuration must always have a radius greater than or equal to $1.125r_s$.

This result was invalidated by Birkhoff's theorem (1923). George Birkhoff showed that a spherically symmetric solution satisfying the field equations in a vacuum must necessarily be static and thus Schwarzschild's (exterior) solutions is applicable even if the source (spherically symmetric) is not static, hence the lower limit of $1.125r_s$ is not applicable under non-static conditions.

Eddington showed in 1924 that there is no real singularity in the geometry of space-time at $r = r_s$. This was done with the singular mapping defined by the equations:

$$t = t' \pm 2m \ \ln\left(\frac{r'}{2m} - 1\right)$$

$$r = r'$$

$$\theta = \theta'$$

$$\varphi = \varphi'$$

which yields

$$g'_{ij} = \begin{pmatrix} 1 - \frac{2m}{r} & \pm\frac{2m}{r} & 0 & 0 \\ \pm\frac{2m}{r} & -(1 + \frac{2m}{r}) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

from which we have the metric

$$ds^2 = dt'^2 - dr^2 - \frac{2m}{r}(dt' \pm dr)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Eddington was thus able able to show that the singularity at $r = r_s$ is a co-ordinate singularity, since none of the g'_{ij} diverge $r = r_s$. But unlike the solution in standard form the Eddington solution is not static since both g'_{01} and g'_{10} are not zero.

Note that both solutions have a singularity at $r = 0$. This singularity is not removable, unlike the one at $r = r_s$. This is because the four nonvanishing curvature invariant are not singular at $r = r_s$, but are singular at the origin. That is, if one of the curvature invariants is singular at some value of $r = r_0$ in one coordinate system then this singularity is present in all coordinate systems.

Since the singularity in the Schwarzschild metric at $r = r_s$ is a coordinate singularity, we ask, what is the real significance of the surface $r = r_s$? Since the Schwarzschild field applies to the empty space outside a spherically symmetric source, we shall first assume we have a source which is concentrated within a coordinate radius r less than r_s. Later we shall consider such objects in greater detail.

If we consider the radial motion of a photon in a Schwarzschild field, then we have

$$0 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1} dr^2$$

which yields

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right)^2$$

and

$$\left.\frac{dr}{dt}\right|_{r=r_s} = 0$$

that is, the radial coordinate velocity of light is zero at $r = r_s$. This means that if we have an observer, at infinity from the source, with a clock running at the coordinate time rate of the photon, he would see the photon passing the r markers at ever increasing intervals of time stopping altogether at $r = r_s$.

One must be careful about interpreting this result, since we have not yet examined the significance of the radial coordinates. The proper length of an infinitesimal measuring rod oriented along the radius vector at rest at radius r is given by

$$d\ell^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

from which we have

$$dr = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} d\ell$$

so we see that as $r \to r_s$ the r markers get infinitely closely spaced. We now evaluate the time taken for a photon to get out from radius r, we have:

$$dt = \left(1 - \frac{2m}{r}\right)^{-1} dr$$

$$= \left(1 + \frac{2m}{r-2m}\right) dr$$

so that

$$t = \int_{r_1}^{r_2} \left(1 + \frac{2m}{r-2m}\right) dr$$

$$= (r_2 - r_1) + 2m \, \ell n \left(\frac{r_2 - 2m}{r_1 - 2m}\right) ,$$

thus as $r_2 \to r_s$, $t \to \infty$ for any $r_2 > r_1$. The conclusion here is inescapable: light (thus anything capable of carrying a signal) cannot get out past $r = r_s$.

This result might appear to be associated with the singularity in g_{11}. However we see that the rate at which a standard clock at rest is

$$ds = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} dt$$

hence at any distance greater than r_s, clocks appear to run infinitely slowly at $r = r_s$, and light is infinitely red shifted.

Thus a source of gravitation lying wholly within the critical radius $r = r_s$ is enveloped in an event horizon (the surface $r = r_s$) making communication with the outside world impossible. Such a source is called a black hole.

Consider now a material particle falling radially into a static black hole. Since $(t^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$ we have

$$\frac{dx^0}{ds} = \frac{dt}{ds} \quad , \quad \frac{dx^1}{ds} = \frac{dr}{ds} \quad , \quad \frac{dx^2}{ds} = \frac{d\theta}{ds} = 0 \quad \text{and} \quad \frac{dx^3}{ds} = \frac{d\varphi}{ds} = 0 \; .$$

The motion here is determined by the geodesic equation

$$\frac{d^2x^\ell}{ds^2} + \left\{ {\ell \atop m \; n} \right\} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0$$

so that

$$\frac{d^2x^0}{ds^2} + \frac{1}{2} g^{oo}(g_{mo,n} + g_{no,m} - g_{mn,o}) \frac{dx^m}{ds} \frac{dx^n}{ds} = 0$$

thus we have

$$\frac{d^2x^o}{ds^2} + g^{oo} \; g_{oo,1} \frac{dx^o}{ds} \frac{dx^1}{ds} = 0$$

or eventually

$$\frac{d^2x^o}{ds^2} + g^{oo} \; \frac{d(g_{oo})}{ds} \frac{dx^o}{ds} = 0$$

multiplying the above by g_{oo} we have

$$g_{oo}\ \frac{d^2x^o}{ds^2} + \frac{dg_{oo}}{ds}\frac{dx^o}{ds} = 0$$

or equivalently

$$\frac{d}{ds}\left(g_{oo}\ \frac{dx^o}{ds}\right) = 0$$

thus

$$g_{oo}\ \frac{dx^o}{ds} = k$$

where k is constant; k is the value of g_{oo} when the particle starts to fall.

Next we multiply the constant of motion

$$1 = g_{ij}\frac{dx^i}{ds}\ \frac{dx^j}{ds}$$

$$= g_{oo}\left(\frac{dx^o}{ds}\right)^2 + g_{11}\left(\frac{dx^1}{ds}\right)^2$$

by g_{oo}, so we have

$$g_{oo} = \left(g_{oo}\ \frac{dx^o}{ds}\right)^2 + g_{oo}g_{11}\left(\frac{dx^1}{ds}\right)^2$$

substituting

$$\frac{dx^o}{ds} = \frac{k}{g_{oo}} \quad \text{and} \quad g_{11} = -\frac{1}{g_{oo}}$$

we obtain

$$g_{oo} = k^2 - \left(\frac{dx^1}{ds}\right)^2 \quad ;$$

eventually

$$\left(1 - \frac{2m}{r}\right) = k^2 - \left(\frac{dr}{ds}\right)^2 \quad .$$

For a falling body we have $\frac{dr}{ds} < 0$ hence

$$\frac{dr}{ds} = -\left(k^2 - (1 - \frac{2m}{r})\right)^{\frac{1}{2}}$$

now we have

$$\frac{dr}{dt} = \frac{dr}{ds} \; \frac{ds}{dt}$$

$$= -\left(k^2 - (1 - \frac{2m}{r})\right)^{\frac{1}{2}} \; \frac{1}{k}\left(1 - \frac{2m}{r}\right) \qquad (1)$$

$$= -\left(1 - \frac{2m}{r}\right)\left(1 - \frac{1}{k^2}(1 - \frac{2m}{r})\right)^{\frac{1}{2}}$$

Suppose that the particle is now close to the surface $r = r_s$ so that we have

$$r = 2m + \xi$$

where ξ is small enough that we can neglect ξ^2. Then

$$1 - \frac{2m}{r} = 1 - \frac{1}{1 + \frac{\xi}{2m}}$$

$$= 1 - \left(1 - \frac{\xi}{2m} + (\frac{\xi}{2m})^2 - (\frac{\xi}{2m})^3 + \dots\right)$$

$$= \frac{\xi}{2m}$$

then (1) becomes

$$\frac{dr}{dt} = -\frac{\xi}{2m}\left(1 - \frac{1}{h^2} \; \frac{\xi}{2m}\right)^{\frac{1}{2}}$$

$$= -\frac{\xi}{2m}\left(1 - \frac{1}{2} \; \frac{\xi}{k^2 2m} - \frac{1}{8}\left(\frac{\xi}{k^2 2m}\right)^2 - \dots\right)$$

$$= -\frac{\xi}{2m} \, ;$$

therefore

$$dt = -\frac{2m}{r}\,dr$$

$$= -\frac{2m}{r-2m}\,dr$$

integrating this we have

$$t = -2m\,\ln(r-2m) + \text{const.}$$

Thus just as in the case of a photon, as $r \to r_s$, $t \to \infty$ that is, the particle would seem to an observer at $r >> r_s$ to take infinite time to reach the surface $r = r_s$, if the particle is emitting light of a certain spectral line, the observer sees the light being red shifted by a factor $g_{11}^{\frac{1}{2}}$, this factor becoming infinite as the particle reaches the surface $r = r_s$. The observer will see all physical processes on the particle going more and more slowly as it reaches the event horizon.

An observer co-travelling with the particle would have his time scale measured by ds, proper time. Since

$$\frac{dr}{ds} = -k\left(1 - \frac{1}{k^2}\,(1 - \frac{2m}{r})\right)^{\frac{1}{2}}$$

tends to k as $r \to r_s$ the particle reaches that event horizon after a lapse of finite proper time for the co-moving observer. Also we see that the co-moving observer will reach $r = 0$ in finite proper time.

Although we have just seen that an event horizon is physical, that is, a co-ordinate independent property of space-time, they are easier to recognize and study when we have a particularly well chosen set of co-ordinates Lemaître, Rosen, Synge, Finkelstein, and Kruskal, and many others, have, like Eddington, introduce new sets of coordinates in order to remove the singularity at $r = r_2$, in the Schwarzschild metric.

However for the metric to become well behaved at $r = r_s$, we have to pay a price; in the new set of coordinate the metric no longer appears static, not even in the exterior region. This we saw with Eddington's transformation.

Here we shall only consider Kruskal's coordinates. We do this not because we wish to show, once again, that the singularity at $r = r_s$ is removable, but rather because of the unusual topology involved. The Kruskal coordinates v,u,θ,φ are defined by

$$u^2 - v^2 = T^2\left(\frac{r}{r_s} - 1\right) \exp \frac{r}{r_s} \quad ,$$

$$\frac{2uv}{u^2 + v^2} = \tanh \frac{t}{r_s} \quad ,$$

where T is an arbitrary constant. The Schwarzschild metric now takes the form

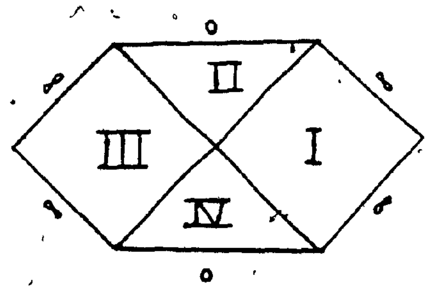
$$ds^2 = \frac{32m^3}{rT^2} \exp\left(-\frac{r}{2m}\right)\left(dv^2 - du^2\right) - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

This metric is non-singular if $r^2 > 0$, that is if $u^2 - v^2 > -T^2$. Thus during the time interval $0 < v < T$, the metric is a smooth finite function of u, for all real u. Now neither of g_{22}, g_{33} vanish when $u = 0$, so that we can approach the origin $u = 0$ from the right, and keep going on into negative u. The space described by this metric is singularity free, consisting of two identical sheets $u > 0$ and $u < 0$, joined in a smooth way by a branch point at $u = 0$. When v reaches time T the two sheets detach from each other, and the metric has a real singularity at $u = \pm\sqrt{v^2 - T^2}$ (i.e. when $r = 0$). Incidentally, the metric has no singularity at $u = v$, which corresponds to $r = r_s$.

The interpretation of some of these results are best illustrated by a few comments of M.D. Kruskal:

"The maximal extension ξ has a non-Euclidean Topology... It is remarkable that it presents just such a "bridge" between two otherwise Euclidean spaces as Einstein and Rosen sought to obtain by modifying the field equations. It may also be interpreted as describing the "throat of a wormhole" in the sense of Wheeler, connecting two distant regions in one Euclidean space...it is impossible to send a signal through the throat in such a way as to contradict the principle of causality; in effect the throat "pinches off" the light ray before it can get through."[1]

The geometry described above is most easily illustrated with the Penrose diagram below:



In the diagram, the surface $r = r_s$ divides space-time into four regions. I and III are the Euclidean regions connected by a wormhole, region II is the black hole region, region IV is a white hole

[1] M.D. Kruskal, Phys. Rev. 119, 1743 (1960).

region, this may be regarded as a time reserved black hole. Signals from region IV could go into region I and III, but no signal from these regions can enter into region IV.

It should be noted that this discussion for the maximal extension of the Schwarzschild metric (Kruskal's geometry) does not apply to the problem of gravitational collapse since for $v < T$, space is empty for all u. These arguments can only apply to black holes that were created jointly with the universe, at the time of the "big bang".

A static black hole has the interesting property that it can have a light ray going about it in a circular orbit. This is the last property we shall consider. To understand this recall that the equation for the motion of a photon in a Schwarzschild field is given by

$$\frac{d^2u}{d\varphi^2} + u = 3m\,u^2$$

where $u = \frac{1}{r}$. Clearly $u = \frac{1}{3m}$ is a solution to this equation. This means that a light ray can be in a circular orbit, of radius $r = 3m$ about a black hole. The special surface of radius $r = 3m$ is called the photosphere of the black hole. This orbit is unstable as can be seen if we consider

$$u = \frac{1}{3m} + \xi$$

where ξ is small and of first order. Then

$$\frac{d^2u}{d\varphi^2} = u\,(3mu - 1)$$

$$= \left(\frac{1}{3m} + \xi\right)(1 + 3m\xi - 1)$$

$$= \xi$$

Thus when u is slightly larger than $\frac{1}{3m}$, $\frac{d^2u}{d\varphi^2}$ is positive and when u is slightly smaller than $\frac{1}{3m}$, then $\frac{d^2u}{d\varphi^2}$ is negative.

There are a number of ways in which black holes can form; these are:

(1) As a result of fluctuations in the early universe when densities were enormous.

(2) By gravitational collapse or after the end of normal evolution of some ordinary stars.

(3) By gravitational collapse of supermassive stars, galactic nuclei or star clusters.

Here we shall consider only (2) and do so with a minimum of physical and mathematical formalism. In the early 1930's S. Chandrasekhar and L. Landau showed, respectively, that according to the Newtonian model for gravity, there is maximum mass of the order of one solar mass for a cold star. The non-linearity of the field equations of General Relativity reduce this maximum limit even more. The following two extract of S. Chandrasekhar describe the final stages in the evolution of stars:

"the life history of a star of small mass must be essentially different from the life history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass cannot go into the white-dwarf stage and one is left speculating on other possibilities."[1]

"The existence of an upper limit to white-dwarf configurations inevitably requires that in the course of evolution of at least some massive stars, black holes - as we now call them - must form."[2]

There are various mathematical models, based on General Relativity, that show a spherically symmetric cloud of dust of negligible pressure must collapse to a single point. The first such model was constructed by Oppenheimer and Snyder as was mentioned in the historical survey.

In a sense the gravitational collapse of such a "star" (dust cloud) never ends. To an outside observer the surface of the collapsing "star" asymptotically approaches its Schwarzschild radius but never reaches it. In another sense a co-moving observer would see the dimensions of the "star" shrink down to infinitely small values, in finite proper time. For the outside observer, at least, it has been suggested that the name "frozen star" is more appropriate than black hole. Since the intensity of light emitted by a collapsing surface decreases sharply as the red shift increases the gravitational and electromagnetic fields surrounding a "frozen star" asymptotically approach the configurations that correspond to a black hole. Thus the name black hole is appropriate to either observer.

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1 Chandrasekhar, S., Observatory, 57, 373 (1934).

2 Chandrasekhar, S., Nature, Vol. 252, Nov. 1, 1974 (11-15-17).

During the collapsing stage of a spherically symmetric body (electrically neutral)* the exterior solution is always the Schwarzschild solution, this is guaranteed by Birkhoff's theorem (1923). For the case of a rotating body which is collapsing there is no analog to Birkhoff's theorem, we have many different exterior solutions which depend on the shape of the mass, until the mass distribution has collapsed inside its event horizon. When everything becomes stationary, a long time after collapse, the solution is given by the Keer metric. It is not our intention to dwell on the Keer solution but would make the comment that depending on a parameter a, known as the angular momentum per unit mass, if

(1) $a^2 < m^2$ the Keer solution will have two event horizon.

(2) $a^2 = m^2$ the Keer solution has one event horizon.

(3) $a^2 > m^2$ the Keer solution has no event horizon.

Case (3) is of interest, here we have a singularity not surrounded by an horizon, such a singularity is called a naked singularity.

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* If the body is electrically charged, then the exterior solution is Reissner-Nordström (Hoffmann's theorem, 1933)

$$ds^2 = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})dt^2 - (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Roger Penrose has conjectured

"that a physically realistic collapse will not result in naked singularities."[1]

This is known as the cosmic censorship hypothesis.

"we may regard this as possibly the most important unsolved problem of classical general relativity theory."[1]

Since it is not our aim to dwell on the Keer solution we shall not mention it explicitly again.

We now consider without proof some of the so-called no-hair theorems. It is now almost accepted dogma that in the absence of external material and electrical charge sources an asymptotically flat black hole at equilibrium state is completely characterized by just three parameters: mass (m), electric charge (Q), and spin angular momentum (J). This conjecture has been proved rigorously in some important special cases:

Israel's Theorem (1967) if $J = 0$, then the stationary solutions are uniquely characterized by m and Q.[2]

Robinson's Theorem (1975) if $Q = 0$, then the stationary solutions are uniquely characterized by m and J.[2]

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1 Roger Penrose: Singularities of Space-time, in Theoretical Principles in Astrophysics and Relativity; ed. N.R. Lebovitz, W.H. Reid, P.O. Vanderoot; The University of Chicago Press (1978).

2 B. Carter: The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes, in General Relativity; ed. S.W. Hawking and W. Israel, Cambridge University Press (1979).

Thus if the conjecture is true two very different stars could form two identical black holes that would depend on m, Q and J, and no other parameters (e.g. nuclear forces). This is why we say a black hole has no hair. Of course a real black hole cannot be in an exactly stationary state.

We have chosen to describe the beginning of a vast theory which derives from the theory of static black holes. To go any further would involve us in a survey of modern astrophysics and of the paradoxes which result when quantum effects are considered. If we have been successful in describing the significance of the Schwarzschild solution in the foundation of the theory, our purpose has been achieved.

SUMMARY

The search for a unified theory has intensified in recent years. The approach now taken is no longer geometric, but rather that of the unified gauge theory of Weinberg and Salam. It is generally agreed that any new comprehensive theory should include Steven Hawking's very pretty result on black hole evaporation, this being the most successful combination of Quantum Mechanics with General Relativity to date.

Our survey of cosmology gave a brief outline of the early history of what is today a vast field of theoretical research. For many years cosmology was the main user of General Relativity, each subject contributing to the development of the other. In cosmology more than in any other scientific field the imagination of the scholar is given complete freedom; this is certainly the main attraction of cosmology. It should be noted that the scientific community does not consider all results of cosmology as equally valid.

In our survey of the literature we saw many of the different approaches taken to obtain the field equations of General Relativity. One gets the feeling each author has his own approach and does not like anyone else's. This is to be expected since General Relativity is a mathematical model of a particular aspect of nature, gravitation, and one should not expect a unique set of principles as defining such a mathematical model. If one takes the field equations as an axiom of General Relativity, one is choosing the best motivated axiom any mathematical model ever had.

We have seen that the three classical tests of General Relativity (the gravitational red shift of spectral lines, the perihelion of the planet Mercury, and the deflection of star light by the sun), can all be based on the Schwarzschild solution. In fact the Schwarzschild solution is the best experimentally verified solution of the field equations of General Relativity. The importance of this particular solution lies here. There are other solutions to the field equations, some with "miraculous" properties, but none of these is empirically supported. Here, one might draw an analogy with Maxwell's equations, it is well known that some solutions of the Maxwell equations are incompatible with observations.

Certainly the most important aspect of the Schwarzschild solution, aside from experimental verification, is the Birkhoff theorem which guarantees that the spherically symmetric vacuum solution is indeed the static solution of K. Schwarzschild. Thus we must conclude that if an object is collapsing, and is spherically symmetric (or close to it), once it has passed inside the critical radius $r_s = 2m$, the object is trapped behind an event horizon, and is cut off from communication whith the rest of the universe. This leads naturally to black-hole theory.

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