STATISTICAL ESTIMATION OF THE PARAMETERS IN THE A.L.M. DISTRIBUTION AND THEIR PROPERTIES

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ABSTRACT

STATISTICAL ESTIMATION OF THE PARAMETERS IN THE A.L.M. DISTRIBUTION AND THEIR PROPERTIES

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A non-parametric approach is given to the problem of estimating the parameters of the Almost Lack of Memory distribution (A.L.M.), a new class of probability distributions.

This distribution is concerned with the random occurrence of some event when environmental conditions (or other factors) forces the failure rate into a periodic pattern.

No assumption is made on the random variable representing the first time occurrence of the event other than the periodicity of its failure rate.

A distribution for the consistent estimators of the hazard function and the failure rate is proposed. Expected values and variances are derived for the probability of occurrence in a time interval and for the cumulative distribution function, given sample observations.

This approach offers flexibility and adaptability to modelling periodic phenomena, a feature frequently met in environmental studies.

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1. INTRODUCTION

A class of probability distributions exhibiting a periodic behaviour in time and a random occurrence of some events on each period has been recently introduced by Chukova & Dimitrov (1992). Further work on this class of distributions has been done by Dimitrov et al. (1992) and Dimitrov and Khalil (1993). A non-negative random variable X is in this class if, $P(X \geq ic + x \mid X \geq ic) = P(X \geq x)$

where $i = 1,2,3,...$ and $c > 0$ is the period.

This property is called Almost Lack of Memory (ALM) and is shared by all r.v. where failure rate is periodic or constant.

The event of interest is the realization of the first event occurrence X ; therefore, the n realizations $X_1,...,X_n$ a collection of independent, identically distributed random variables having the ALM property.

The following results are given in the recent work of Dimitrov and Khalil (1993).

If X is a random variable having the ALM property, then

$$F_X(x) = 1 - \alpha^{\left[\frac{x}{c}\right]}[1 - (1 - \alpha) F_Y(x - \left[\frac{x}{c}\right] c)] \quad ,$$

where $\left[\frac{x}{c}\right]$ means the integer value of $\frac{x}{c}$.

and equivalently, X can be represented as the sum of two independent components $X = Y + cZ$,

where Y is a r.v. taking values on [0,c) with probability 1 and Z having the geometric distribution

$$P(Z = m) = \alpha^m (1 - \alpha) \quad , \quad m = 0, 1, 2, ...$$

This condition states that the r.v. belongs to the class $ALM(\alpha, F_Y, c)$.

The aim of the present investigation is the statistical estimation of the following parameters:

(1) α, the probability of non-occurrence of the event under consideration in each interval $[i,(i+1)c)$;

(2) $F_X(x)$, the c.d.f. of the r.v. X;

(3) The failure rate $\lambda(t)$ defined as,

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} . P[t < x \le t + \Delta t \,|\, X \ge t] \quad \text{or,}$$

$$\lambda(t) = \frac{F_X{}'(t)}{1 - F_X(t)} \quad ;$$

(4) The hazard function, $\Lambda(t) = \int_0^t \lambda(u)\,du$;

It is assumed that the distribution of X is continuous and that the period c is a known constant. The following information is known:

- the observed time of occurrence of each r.v. $X_j = x_j$; $j = 1 \ldots N$;
- the total number of observations over the time T = rc is n, T or r and n are known meaning that N-n random variables did not occur during the observation time.

The smallest σ-field containing all the above information is

$\mathcal{F}_r$, $\mathcal{F}_r = \sigma(X_1, X_2, \ldots, X_n, T, N)$.

From a Bayesian point of view, we would regard $\hat{\alpha}_r$ (the estimator of α) as a random variable having distribution Beta(b+1,n+1) where b is the observed number of entire intervals on which the event did not occur.

The estimator $\hat{\alpha}_r$ considered for α is a biased estimator but converges to α almost surely and in $\mathcal{L}^2$ when $N \to \infty$. This choice makes it possible to derive a probability distribution for the sequence of estimated parameters based on the sample data and their ALM properties.

Let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)} \leq c$ be the order statistics corresponding to $Y_1,\ldots,Y_n$ constructed according to the following rule

$$Y_i = X_i - \left[\frac{X_i}{c}\right]c \quad ,i = 1,\ldots,n$$

Then the following results hold:

(1) $(1-\hat{\alpha}_r)F_Y(Y_{(k)})$ is Beta(k,n+b-k+2), **(Theorem 8).**

(2) $\hat{P}(X \in [Y_{(i)},Y_{(j)})) = (1-\hat{\alpha}_r)(F(Y_{(i)}) - F(Y_{(j)}))$ where $\hat{P}$ estimates the probability that $X \in [Y_{(i)},Y_{(j)})$, is Beta(j-i,b+n-j+i+2), **(Corollary 1).**

(3) $1-(1-\hat{\alpha}_r)F_Y(Y_{(k)})$ is Beta(n+b-k+2,k);

(4) $\hat{\Lambda}_X(Y_{(k)}) = -\ln(1-(1-\hat{\alpha})F(Y_{(k)})) = u_1 + \ldots + u_k$, **(Corollary 2).**

where $\{u_i\}$ are independent exponential r.v. with parameters $(n+b-i+2)^{-1}$;

The independence of the u_i's will then allow access to the estimation of

$\Lambda_X(t)$ and $\lambda_X(t)$, **(Theorem 9).**

(5) the estimator of $\alpha^{\left[\frac{t}{c}\right]}(1-\alpha) = \int_0^1 \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,d(F_{\alpha|\mathcal{F}_r})$ is then derived, and by using the independence of $\hat{\alpha}^{\left[\frac{t}{c}\right]}$ and $\hat{F}_Y(Y_{(k)})$, $\hat{F}_X(x)$ is obtained.

The expected values and variances of the distribution of the estimators are used as the estimate of the corresponding random variables and their variances.

Throughout this investigation we will assume that the distribution $F_Y(y)$ is arbitrary $y \in [0,c)$; in other words, the only properties assumed for the random variable X are its ALM properties.

2. ESTIMATION OF THE PARAMETER α AND SOME PROPER FUNCTIONS OF α

2.1 THE MINIMUM VARIANCE UNBIASED ESTIMATOR

Let $X_1,...,X_n$ be the realizations of the first event occurrence in N independent copies of the model with known period c>0. Suppose the observations have been obtained during a time T=rc, and N-n copies of the event did not occur.

First, a larger set of events is described. Consider the hypothetical situation where the event of interest is allowed to occur on any interval

$[(i-1)c, ic)$, $i=1...r$ with a probability of occurrence $(1-\alpha)$.

We denote the 1st, 2nd,... time occurrences by $X_{j1}, X_{j2},...$

We then have N independent copies of the sequence of times of occurrences $\{X_{jk}$, $j=1...N$, $k \leq r\}$.

Let $\Im_r$ be the σ-field generated by X_{jk} , $j=1...N$, $k \leq r$

and let $\Delta_i(X_j) = \begin{cases} 1 \ , \ if \ X_{jk} \notin [(i-1)c, ic) \\ 0 \ , \ if \ X_{jk} \in [(i-1)c, ic) \end{cases}$;

$\Delta_i(X_j)$ are independent identically distributed Bernoulli variables

$P(\Delta_i(X_j)=1)=\alpha$, $E(\Delta_i(X_j))=\alpha$.

Consider the stopping times

$\gamma_j(\omega) = \{inf \ i \ : \ \Delta_i(X_j)=0\} \wedge r$ or equivalently

$$\gamma_j(\omega) = \left\{ inf \ i \ : \ i \geq \left[\frac{X_j}{c} \right] \right\} \wedge r \ ;$$

where X_j denotes the first time occurrence X_{j1} .

Then $\gamma_j(\omega)$ is $\Im_r$ measurable, it is finite if r is finite.

Define $\Im_\infty$ = σ-field generated by $\bigcup_{r \geq 0} \Im_r$,

$$\Im_{\gamma_j} = \{B \in \Im_\infty : B \cap \{\gamma_j(\omega)\} \in \Im_r \ , \ \forall \ r \geq 0\} \ .$$

And $\Im_\gamma = \bigcap_{j=1}^{N} \Im_{\gamma_j}$, $\Im_\gamma$ is a σ-field and $\Im_\gamma \subset \Im_r$

Then $\Im_R$ is a σ-field and $\gamma(\omega) \leq \{T(\omega) = r\}$,

which implies $\Im_\gamma \subset \Im_r$, e.g. Rao (1984), ch. 7.

$\{\Delta_i(X_j) \ , \ i = 1 \ldots \gamma_j \ , \ j = 1 \ldots N\}$ are iid r.v.s. on $(r, \Im_r, P)$

with a finite common mean $\mathrm{E}(\Delta_i(X_j)) = \alpha$, then according to Wald (1947),

$$\mathrm{E}(S_{\gamma j}) = \mathrm{E}(\gamma_j)\mathrm{E}(\Delta_i(X_j)) \quad , \ j = 1 \ldots N$$

where $S_{\gamma j} = \sum_{i=1}^{\gamma_j} \Delta_i(X_j)$.

then $\mathrm{E}\sum_{j=1}^{N} S_{\gamma_j} = \alpha \, \mathrm{E}\sum_{j=1}^{N} \gamma_j$,

and $\gamma_j = \begin{cases} \left[\dfrac{X_j}{c}\right] + 1 & , \ \textit{if } X_j \textit{ occurs in } [0, rc) \ ; \\ r & , \ \textit{otherwise} \ . \end{cases}$

For the η variables $X_1, X_2, \ldots, X_n$ that occur in $[0, rc)$

$$\sum_{i=1}^{\gamma_i} \Delta_i(X_j) = \sum_{i=1}^{\left[\frac{X_j}{c}\right]+1} \Delta_i(X_j) \ ,$$

and because in $[0, \gamma_j)$, $\Delta_i(X_j) = \begin{cases} 1 \ , \ \textit{if } i \leq \left[\dfrac{X_i}{c}\right] \ ; \\ 0 \ , \ \textit{if } i = \left[\dfrac{X_i}{c}\right] + 1 \ . \end{cases}$

we obtain $\sum_{i=1}^{\left[\frac{X_j}{c}\right]+1} \Delta_i(X_j) = \sum_{i=1}^{\left[\frac{X_j}{c}\right]} 1 = \left[\frac{X_i}{c}\right]$.

For the $N-\eta$ variables, that do not occur in $[0,rc)$,

$$\sum_{i=1}^{\gamma_j} \Delta_i(X_j) = r \ .$$

Therefore, $\sum_{j=1}^{N} S_{\gamma_j} = \sum_{j=1}^{N} \sum_{i=1}^{\gamma_i} \Delta_i(X_j) = \sum_{j=1}^{\eta} \left[\frac{X_j}{c}\right] + (N-\eta)r$.

We define

$$\beta = \sum_{j=1}^{\eta} \left[\frac{X_j}{c}\right] + (N-\eta)r \ , \tag{1}$$

where β is the total number of entire intervals on which the event did not occur. Then the following relation holds:

$$\sum_{j=1}^{N} \gamma_j = \sum_{j=1}^{\eta} \left(\left[\frac{X_j}{c}\right] + 1\right) + (N-\eta)r = \eta + \sum_{j=1}^{\eta} \left[\frac{X_j}{c}\right] + (N-\eta)r \ ;$$

$$\sum_{j=1}^{N} \gamma_j = \eta + \beta \ .$$

If X_j has the ALM property, then

$$X_j = \left[\frac{X_j}{c}\right]c + \left(X_j - \left[\frac{X_j}{c}\right]c\right) \ .$$

It has been proven by Dimitrov et al (1992) that

$$\left[\frac{X_j}{c}\right] c \perp X_j - \left[\frac{X_j}{c}\right]c \ , \quad X_j - \left[\frac{X_j}{c}\right]c \in [0,1) \ .$$

Therefore, $(\gamma_j - 1)c \perp X_j - \left[\frac{x_j}{c}\right]c$

ie, the stopping time γ_j is independent from its distribution within an interval of length c. This fact is denoted by $\perp$. We can therefore assume that the random variables in $\mathfrak{F}_r$, stopped at their first time of occurrence, have a periodic failure rate, in which case the σ-field $\mathfrak{F}_v$ is the same as $\mathcal{F}_r = \sigma(X_1 ... X_n, T, N)$.

We use the special notations:

$$X_j = x_j \ , \ \Delta_i(X_j) = \delta_{ij} \ , \ b = \sum_{j=1}^{N} \sum_{i=1}^{Y_j} \delta_{ij} \ , \ \eta = n \ , \ j = 1, ..., N \ , \ i = 1, ..., r \qquad (2)$$

for the observed values of the r.v.s in $\mathcal{F}_r$.

Therefore in $\mathcal{F}_r$, Wald's equation is $E(\beta) = (n+b)\alpha$ or, $\alpha = \frac{E(\beta)}{n+b}$

Thus $\frac{\beta}{n+b}$ is an unbiased estimator of α in $\mathcal{F}_r$. In this way we have proved:

Theorem 1.

An unbiased estimator of the parameter α of an ALM distribution is $\hat{\alpha} = \frac{\beta}{n+b}$,

where β is given (1), n and b in (2).

2.2 THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATOR

In Theorem 1 it is shown that $\frac{\beta}{n+b}$ is an unbiased estimator of α in $\mathcal{F}_r$. The unbiasedness of this estimator is a desirable feature because, if it can be shown that β is a complete and sufficient statistic for α, then according to Bahadur (1957), $\frac{\beta}{n+b}$ is the unique uniformly minimum variance unbiased estimator (UMVU) of α.

To show sufficiency, we first observe that:

the distribution of $\beta = \sum_{J=1}^{N} \sum_{i=1}^{Y_j} \Delta_i(X_j)$ is binomial, ie. we have

$$P(\Delta_1(X_1) = \delta_{11}, ..., \Delta_r(X_n) = \delta_{rn}; \alpha) = \alpha^b (1-\alpha)^n$$

and $P(\beta = b \mid \eta + \beta = n + b; \alpha) = \binom{n+b}{b} \alpha^b (1-\alpha)^n$.

where $\binom{n+b}{b}$ is the number of ways $\{\beta = b\}$ can occur in a sample of size n+b observed time periods of length c each. Moreover,

$$P(\Delta_1(X_1)=\delta_{11},\ldots,\delta_r(X_n)=\delta_{rn}\mid\beta=b,\eta=n)$$

$$=\frac{P(\Delta_1(X_1)=\delta_{11},\ldots,\delta_r(X_n)=\delta_{rn};\alpha)}{P(\beta=b,\eta=n;\alpha)}$$

$$=\frac{\alpha^b(1-\alpha)^n}{\binom{n+b}{b}\alpha^b(1-\alpha)^n}=\binom{n+b}{b}^{-1},$$

which does not depend on the value of the parameter α.

As for completeness, by definition, a statistic T is said to be complete if, satisfying $\mathbf{E}[f(T)]=0$ for all $\beta\in r$ implies $f(T)=0$ a.e. For a fixed sample size b+n, we have

$$\mathbf{E}_b[f(\beta)]=\sum_{i=0}^{n+b}f(i)\binom{n+b}{i}\alpha^i(1-\alpha)^n$$

$$=(1-\alpha)^n\sum_{i=0}^{n+b}\binom{n+b}{i}\alpha^i f(i)\ .$$

If $\mathbf{E}_b[f(\beta)]=0$ then $\sum_{i=1}^{n+b}\binom{n+b}{i}f(i)\alpha^i=0$.

This means that all the coefficients of α are zero, or $f(i)=0$ for all b which proves the completeness of β for a fixed n+b. Thus we have proven:

Theorem 2.

The unbiased estimator $\hat{\alpha}=\frac{\beta}{n+b}$, found in Theorem 1, is the UMVU estimator of α in $\mathcal{F}_r$.

2.3 THE VARIANCE OF THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATOR

In section 2.1, we have seen that $\Delta_i(X_j)$ is a sequence of iid Bernoulli r.v.s with parameter α.

Therefore its variance is $E(\Delta_i(X_j) - \alpha)^2 = \alpha(1-\alpha)$

Let $S^o_{\nu_j} = \sum_{i=1}^{\nu_j} (\Delta_i(X_j) - \alpha)$

then $\sum_{j=1}^{N} S^o_{\nu_j} = \sum_{j=1}^{N} \sum_{i=1}^{\nu_j} \Delta_i(X_j) - \sum_{j=1}^{N} \nu_j \alpha$

Moreover, in $\mathcal{F}_r$, $\sum_{j=1}^{N} \nu_j = n+b$, then $\sum_{j=1}^{N} S^o_{\nu_j} = \beta - (n+b)\alpha$

and $\left(\sum_{j=1}^{N} S^o_{\nu_j}\right)^2 = (n+b)^2\left[\frac{\beta}{n+b} - \alpha\right]^2$,

$$E\left(\sum_{j=1}^{N} S^o_{\nu_j}\right)^2 = (n+b)^2 E\left[\frac{\beta}{n+b} - \alpha\right]^2 = (n+b)^2 Var\left(\frac{\beta}{n+b}\right) .$$

Because $\Delta_i(X_j)$ and ν_j depend on X_j alone, the independence of X_j's implies the independence of $S^o_{\nu_j}$, $j = 1 \ldots N$.

Thus, $E\sum_{j=1}^{N} \left(S^o_{\nu_j}\right)^2 = E\sum_{j=1}^{N} \left(S^o_{\nu_j}\right) = (n+b)^2\, Var\left(\frac{\beta}{n+b}\right)$

According to the second part of Wald's theorem we have:

$$E(S^o_{\nu_j})^2 = E\,\nu_j\, Var\,\Delta_i(X_j) ,$$

when the process is stopped at the optional time ν_j .

Then $E\sum_{j=1}^{N} \left(S^o_{\nu_j}\right)^2 = E\sum_{j=1}^{N} \nu_j\, Var(\Delta_i(X_j))$ (3)

i.e. $(n+b)^2 \, Var\left(\frac{\beta}{n+b}\right) = (n+b)\, Var\, \Delta_i(X_j)$, by equation (3)

Therefore $Var\left(\frac{\beta}{n+b}\right) = \frac{1}{n+b} Var\, \Delta_i(X_j) = \frac{\alpha(1-\alpha)}{n+b}$. (4)

Lehman (1983) shows that the unbiased estimator of $\alpha(1-\alpha)$ given a sample size n+b is

$$\frac{\beta(n+b-\beta)}{(n+b-1)(n+b)} .$$

A slightly modified version of the proof is given below:

Let $g(\beta)$ be the unbiased estimator for $var\,\Delta_i(X_j)$, Then it is true that:

$$E(g(\beta)) = \alpha(1-\alpha) ,$$

i.e. $\sum_{b=0}^{n+b} \binom{n+b}{b} g(b)\, \alpha^b (1-\alpha)^n = \alpha(1-\alpha)$

with $\rho = \frac{\alpha}{1-\alpha}$ so that $\alpha = \frac{\rho}{1+\rho}$ and $1-\alpha = \frac{1}{1+\rho}$.

Then it will be true that

$$\sum_{b=0}^{n+b} \binom{n+b}{b} g(b)\, \rho^b (1+\rho)^{-(b+n)} = \rho(1+\rho)^{-2} ;$$

$$\sum_{b=0}^{n+b} \binom{n+b}{b} g(b)\, \rho^b = \rho(1+\rho)^{n+b-2} = \rho \sum_{b=0}^{n+b-2} \binom{n+b-2}{b} \rho^b = \sum_{b=1}^{n+b-1} \binom{n+b-2}{b-1} \rho^b .$$

Equating the coefficients of ρ^b we arrive at the equation

$$g(b) = \frac{bn}{(n+b)(n+b-1)} ,$$

for $(n+b)$ being fixed and for a given $\beta = b$, $g(b)$ is an unbiased estimator of $\alpha(1-\alpha)$.

Therefore, by equation (4)

$$\hat{Var}\left(\frac{\beta}{n+b}\right)=\frac{bn}{(n+b)^2(n+b-1)}$$

which is the unbiased estimate of $Var\frac{\beta}{n+b}$. Thus, the following result holds:

Theorem 3.

For the observations $X_1 \dots X_N$ on a random variable $X \in ALM(\alpha, F_Y, c)$, the unbiased estimator of $Var\,\hat{\alpha}$ is $\frac{bn}{(n+b)^2(n+b-1)}$. Here, $\hat{\alpha}=\frac{\beta}{n+b}$ is the uniformly minimum variance unbiased estimator of the parameter α, n is the number of actual observations on X, and b is the sum of all intervals of length c on which each X_i , $i=1 \dots N$ fails to occur during the time T=rc.

2.4 A BIASED ESTIMATOR OF α

As the value of α is unknown, it can be considered as a r.v. having the a priori uniform density distribution. That is,the assumption $\alpha \sim U[0,1]$ merely reflecting our ignorance of its value and expressing the fact that all numbers in [0,1] are equally likely to be the true value of the unknown parameter.

We know that in a fixed sample size $\eta+\beta=n+b$, β is complete and sufficient for the estimation of α so that the only useful information in $\mathcal{F}_r$ for this purpose are n and b .

For a given sample size n+b, the joint density of β and α is

$$g(\beta=b\mid\alpha)=g(\beta=b,\alpha)=\binom{n+b}{b}\alpha^b(1-\alpha)^n \ .$$

The density of β is then

$$f(\beta=b)=\int_0^1\binom{n+b}{b}\alpha^b(1-\alpha)^n d\alpha$$

$$=\binom{n+b}{b}\int_0^1\alpha^b(1-\alpha)^n d\alpha=\binom{n+b}{b}\frac{\Gamma(b+1)\Gamma(n+1)}{\Gamma(b+n+2)} \ .$$

Hence $f(\alpha\mid\mathcal{F}_r)=f(\alpha\mid\beta=b)=\frac{g(\beta=b,\alpha)}{f(\beta=b)}$, i.e.

$$f(\alpha \mid \mathcal{F}_r) = \frac{\Gamma(n+b+2)}{\Gamma(n+1)\Gamma(b+1)} \alpha^b (1-\alpha)^n \ .$$

Therefore, the conditional probability distribution of α is Beta $(b+1, n+1)$, thus we have derived the following:

Theorem 4.

If the a priori distribution of the parameter α is uniform, then the conditional expectation and variance of the posterior distribution, given the observations $X_1 \ldots X_n$, rc are:

$$E(\alpha \mid \mathcal{F}_r) = \frac{b+1}{b+n+2} \ ;$$

$$Var(\alpha \mid \mathcal{F}_r) = \frac{(b+1)(n+1)}{(b+n+3)(b+n+2)^2} \ .$$

We now ask the question: Is there a prior Beta(k,s) p.d.f. of α such that the posterior distribution of α satisfies $E(\alpha \mid b,n) = \frac{b}{n+b}$?

Let α be Beta(k,s), i.e.

$$f(\alpha) = \frac{\Gamma(k+s)}{\Gamma(k)\Gamma(s)} \alpha^{k-1} (1-\alpha)^{s-1} \ , \tag{5}$$

Then the joint distribution of β and α as defined in (4) is

$$\gamma(\beta = b, \alpha) = g(\beta = b \mid \alpha) f(\alpha)$$

$$= \binom{n+b}{b} \frac{\Gamma(k+s)}{\Gamma(k)\Gamma(s)} \alpha^{b+k-1} (1-\alpha)^{s+n-1}$$

The marginal density of β is

$$\int_0^1 \binom{n+b}{b} \alpha^{b+k-1} (i-\alpha)^{s+n-1} d\alpha$$

$$= \binom{n+b}{b} \frac{\Gamma(k+s)}{\Gamma(k)\Gamma(s)} \frac{\Gamma(b+k)\Gamma(n+s)}{\Gamma(b+k+n+s)} \ .$$

Thus we obtain the posterior distribution of equation (5)

$$f(\alpha \mid \mathcal{F}_r) = \frac{\Gamma(b+k+n+s)}{\Gamma(b+k)\Gamma(n+s)} \alpha^{b+k-1}(1-\alpha)^{s+n-1}$$

is Beta (b+k,n+s) and therefore,

$$E(\alpha \mid \mathcal{F}_r) = \frac{b+k}{b+n+s+k} \ .$$

Hence, $E(\alpha \mid \mathcal{F}_n) = \dfrac{b}{b+n}$ is true when k=0 and s=0

The function Beta(0,0) is not a proper probability distribution since

$$\int_0^1 \frac{1}{\alpha(1-\alpha)} d\alpha = \infty$$

We have thus proved the following result:

Theorem 5.

There is no proper a priori Beta distribution for the parameter α, that would generate an unbiased estimator of the same parameter.

2.5 COMPARISON OF THE $\mathcal{L}^2$ DISTANCES BETWEEN THE ESTIMATED PARAMETER α AND ITS BIASED AND UNBIASED ESTIMATOR

Since α does not belong to $\mathcal{F}_r$ for any integer r, we conclude that independence holds:

$$\left(\frac{\beta+1}{n+b+2} - \alpha)\right) \perp \sigma(\beta, \eta + \beta = n+b)$$

Therefore, we have

$$E\left(\frac{\beta+1}{n+b+2} - \alpha\right)^2 = E\left[\frac{\beta+1}{n+b+2} - E\left(\frac{\beta+1}{n+b+2}\right)\right]^2 + E\left[E\left(\frac{\beta+1}{n+b+2}\right) - \alpha\right]^2 ,$$

and

$$E\left(\frac{\beta+1}{n+b+2} - \alpha\right)^2 = \frac{(b+1)(n+1)}{(b+n+3)(b+n+2)^2} + E\left[E\left(\frac{\beta+1}{n+b+2}\right) - \alpha\right]^2 \ .$$

By Wald's equation $E(\beta)=(n+b)\alpha$ we obtain

$$E\left(\frac{\beta+1}{n+b+2}\right)=\frac{(n+b)\alpha+1}{n+b+2} \quad .$$

Hence

$$E\left(\frac{\beta+1}{n+b+2}\right)-\alpha=\frac{(n+b)\alpha+1}{n+b+2}-\alpha \quad ;$$

$$E\left(\frac{\beta+1}{n+b+2}\right)-\alpha=\frac{1-2\alpha}{n+b+2} \quad .$$

then

$$\left[E\frac{\beta+1}{n+b+2}-\alpha\right]^2=\frac{1-4\alpha(1-\alpha)}{(n+b+2)^2}$$ is a constant, and

$$E\left[E\frac{\beta+1}{n+b+2}-\alpha\right]^2=\frac{1-4\alpha(1-\alpha)}{(n+b+2)^2}$$

By Theorem 3, the unbiased estimator of $E\left[\frac{\beta}{n+b}-\alpha\right]^2$ given $(\beta=b,\eta=n)$ is

$$V_1=\frac{bn}{(n+b-1)(n+b)^2} \quad .$$

Let V_2 be the unbiased estimator of $E\left[\frac{\beta-1}{n+b+2}-\alpha\right]^2$, given by

$$V_2=\frac{(b+1)(n+1)}{(n+b+3)(n+b+2)^2}+\frac{(n+b)(n+b-1)-4bn}{(n+b+2)^2(n+b)(n+b-1)}$$

Figures. 1-3 illustrate the graph of V_1 and V_2 versus β , where

β varies from 0 to n+b for n+b=10, 50 and 100. It can be seen that if β is close to 0 or close to n+b, then $V_1<V_2$.

Therefore $E(V_1)=E\left(\frac{\beta}{n+b}-\alpha\right)^2<E\left(\frac{\beta+1}{n+b+2}-\alpha\right)^2=E(V_2)$.

However, for most values of β we have the reverse inequality;

i.e. $E\left(\frac{\beta+1}{n+b+2}-\alpha\right)^2 < E\left(\frac{\beta}{n+b}-\alpha\right)^2$ leading to the following conclusion: The biased estimator $\frac{\beta+1}{n+b+2}$ can be considered a better estimator than the unbiased one $\frac{\beta}{n+b}$ because its mean square distance to α is less than that of the unbiased estimator in the range (0.2,0.8). In the range close to the bounds [0,0.2] and [0.8,1], the unbiased estimator $\frac{\beta}{n+b}$ is preferable, being closer to the unknown value of the parameter α.

Fig. 1:

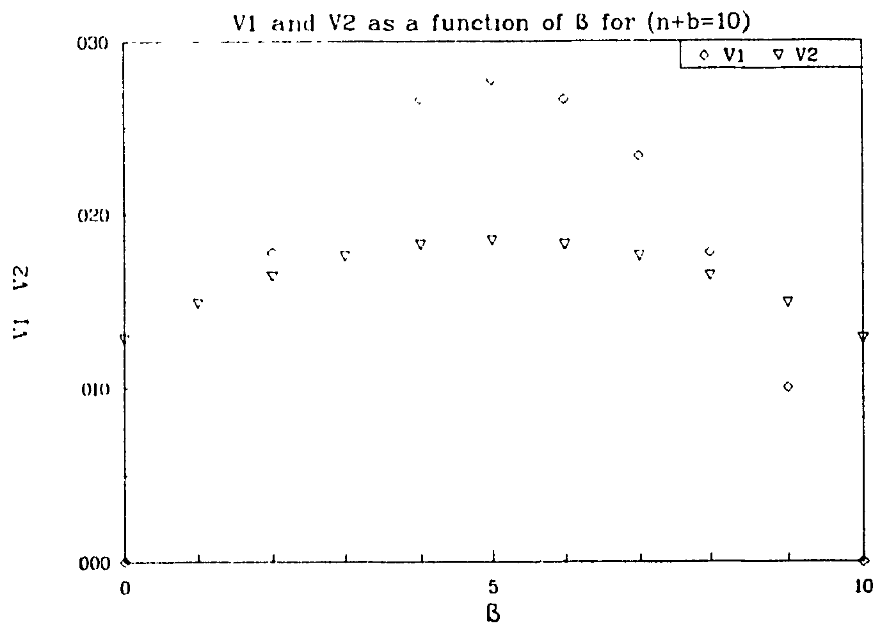


Fig. 2:

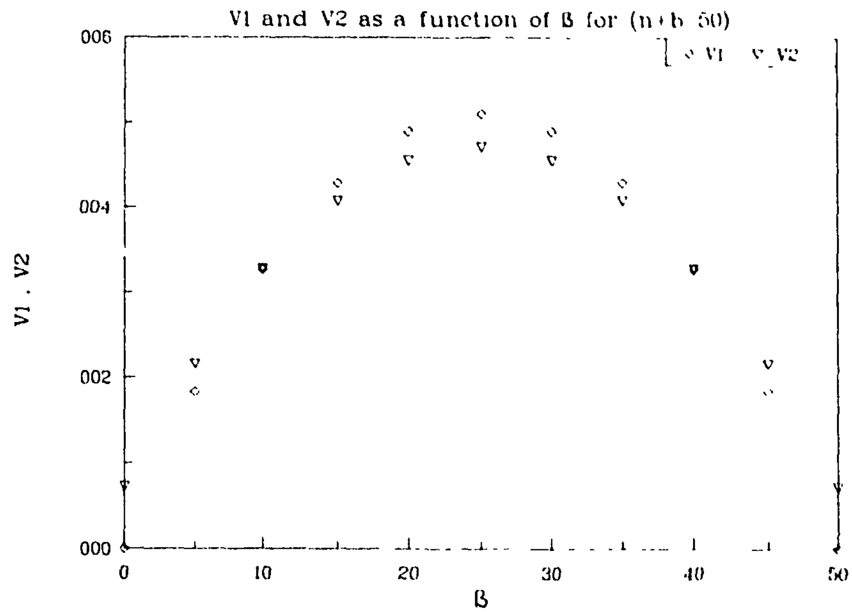
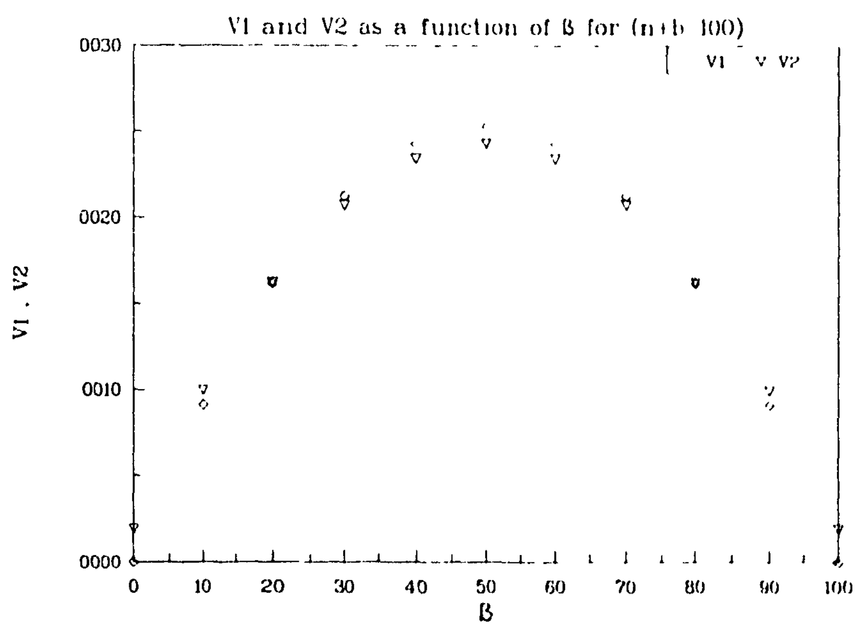

V1 and V2 as a function of ß for (n+b 50)
V1
V2
V1 . V2
006
004
002
000
0
10
20
30
40
50
ß

Fig. 3:


V1 and V2 as a function of ß for (n+b 100)
V1
V2
V1 . V2
0030
0020
0010
0000
0
10
20
30
40
50
60
70
80
90
100
ß

2.6 CONVERGENCE OF THE ESTIMATORS

We consider now the properties of these estimators of α in each σ-field $\mathscr{F}_r$, $r=1,2\ldots$

Let $b_r=\sum_{j=1}^{N}\sum_{i=1}^{\gamma_i}\delta_{ij}$, $n_r+b_r=\sum_{j=1}^{N}\gamma_j$.

Then $E(\beta\,|\mathscr{F}_r)=b_r$, $E\left(\sum_{j=1}^{N}\gamma_j\,|\mathscr{F}_r\right)=n_r+b_r$

The Wald equation $E(\beta)=\alpha\,E\left(\sum_{j=1}^{N}\gamma_j\right)$ implies

$$E(\beta\,|\mathscr{F}_r)=E\left(\sum_{j=1}^{N}\gamma_j\,|\mathscr{F}_R\right)E(\alpha\,|\mathscr{F}_r) \ .$$

Denote by $E(\alpha\,|\mathscr{F}_r)$, the unbiased point estimator of α in $\mathscr{F}_r$

We obtain a sequence $\hat{\alpha}_r=\dfrac{b_r}{n_r+b_r}$, $r=1,2,\ldots$ of estimators of α.

Theorem 6.

The sequence has the following convergence properties

(a) almost sure convergence i.e.

$$P\left(\lim_{r\to\infty}\frac{b_r}{n_r+b_r}=\alpha\right)=1$$

(b) $\mathscr{L}^2$ convergence i.e.

$$E\left(\frac{b_r}{n_r+b_r}-\alpha\right)^2\to 0 \quad as\ r\to\infty$$

Proof:

Let $M_r=E(\alpha\,|\mathscr{F}_r)$

the process M is a martingale relative to $\{\mathcal{F}_r\}, P$ because the following conditions are satisfied:

$E(\alpha \mid \mathcal{F}_r)$ is adapted

$$E(M_r \mid \mathcal{F}_{r-1}) = E(E(\alpha \mid \mathcal{F}_r) \mid \mathcal{F}_{r-1}) = E(\alpha \mid \mathcal{F}_{r-1}) = M_{r-1}$$

$$0 \leq \frac{b_r}{n_r + b_r} \leq 1 \text{ implies } E\left|\frac{br}{n_r + b_r}\right| < \infty$$

Therefore, M is a martingale bounded in $\mathcal{L}^2$ because for all r, $0 \leq M_r \leq 1$ implies $0 \leq M_r^2 \leq 1$, therefore $EM_r^2 < \infty$

then according to Williams (1992), p.111,

$M_r \to M_\infty$ almost surely and in $\mathcal{L}^2$.

Moreover, because $\alpha \in \mathcal{F}_\infty$ then

$E(\alpha \mid \mathcal{F}_\infty) = \alpha$, Therefore, for the sequence $\hat{\alpha}_r = \dfrac{b_r}{n_r + b_r}$, the assertions of the theorem are true.

Theorem 7.

The biased estimators sequence $\dfrac{b_r + 1}{b_r + n_r + 2}$ achieve the almost sure convergence and the $\mathcal{L}^2$ convergence under the condition that $r \to \infty$ and $N \to \infty$.

<u>Proof</u>:

We represent $$\frac{b_r + 1}{b_r + n_r + 2} = \frac{n_r + b_r}{n_r + b_r + 2}\left(\frac{b_r}{n_r + b_r}\right) + \frac{1}{b_r + n_r + 2}$$

as $n_r + b_r \to \infty$,then obviously

$$\lim_{r \to \infty} \frac{b_r + 1}{b_r + n_r + 2} = \lim_{r \to \infty} \frac{b_r}{n_r + b_r} = \alpha \quad \text{a.s.}$$

since

$$b_r + n_r = \sum_{j=1}^{N} \gamma_j$$

$$P(\gamma_j = m) = \alpha^{m-1}(1-\alpha)$$

therefore, $P(\gamma_j = \infty) = 1 - \sum_{m=1}^{\infty} P\{\gamma_j = m\} = 0$

Therefore, in order to have $b_r + n_r \to \infty$ we should have $N \to \infty$ in which case the sequence converges almost surely to α.

The family $\left\{\frac{br+1}{n_r + b_r + 2}\right\}$ is bounded, and

$$0 < \frac{b_r + 1}{n_r + b_r + 2} \leq 1 \text{ implies } \left(\frac{b_r + 1}{n_r + b_r + 2}\right)^2 < 1 \quad \text{for all } r \geq 1$$

By the bounded convergence theorem for martingales e.g. Williams (1991), p.130,we have

$$\mathbf{E}\left[\frac{b_r + 1}{n_r + b_r + 2} - \alpha\right]^2 \to 0 \text{ as } n \to \infty \ , \ r \to \infty$$

which proves $\mathcal{L}^2$ convergence.

3. ESTIMATORS BASED ON THE ORDERED STATISTICS $y_{(k)}$ IN $(0,c]$

3.1 INDEPENDENCE OF y_i's AND CONTINUITY OF THEIR DISTRIBUTION

Let $X_1,\ldots,X_n$ be continuous independent and identically distributed random variables having the ALM property. Then their common density function is continuous.

Consider the random sample $Y_1,\ldots,Y_n$ where $Y_i = X_i - \left[\frac{X_i}{c}\right]c$

If $X_i \in [0,c)$ then $X_i = Y_i$,

therefore, $Y_i = X_i \mid X_i \in [0,c)$, and

$$f_Y(y_i) = f_X(X = x_i \mid x_i \in [0,c)) = \frac{f_X(x_i)}{1-\alpha} \quad \text{where } x_i \in [0,c)$$

Therefore the continuity of $f_X(x)$ implies the continuity of $f_Y(y)$, and Y_i 's are identically distributed. Also the joint density of $Y_1,\ldots,Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n \mid x_1,\ldots,x_n \in [0,c))$$

$$= \frac{1}{(1-\alpha)^n} f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{1-\alpha} f(x_i) \ ,$$

and the $Y_1,\ldots,Y_n$ are also independent.

3.2 ESTIMATION OF $F(Y_{(k)})$

Consider the order statistics of the random sample

$$Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(n)} \le c \ .$$

Since F_Y is an increasing function, then

$$F_Y(Y_{(1)}) \le F_Y(Y_{(2)}) \le \ldots \le F_Y(Y_{(n)}) \le F_Y(c) = 1$$

Moreover, since F_Y is continuous, then $F_Y(Y)$ is uniformly distributed over $[0,1]$.

Also, $F(Y_{(k)})$ is the kth order statistics of a sample of size n. Its density function is $f_k(F(Y_{(k)}) = \frac{n!}{(k-1)!(n-k)!} F(Y_{(k)})^{k-1}(1-F(Y_{(k)}))^{n-k}$.

Therefore, $F_y(y_{(k)})$ has a Beta distribution with parameters (k,n-k+1) and

$$E(F_y(y_{(k)}) = \frac{k}{n+1} \;;$$

$$Var(F_y(y_{(k)}) = \frac{k(n-k+1)}{(n+2)(n+1)^2} \;.$$

We refer to the following result (Rao 1965):

If X and Y are independent Beta variables with parameters (γ_1,δ_1) and (γ_2,δ_2) if $\gamma_1=\gamma_2+\delta_2$,the distribution of $U=X\cdot Y$ is also a Beta variable with parameter $(\gamma_2,\delta_1+\delta_2)$.

$F_y(Y_{(k)})$ is a function of $(Y_1 \ldots Y_n)$ each $y_i \in [0,c)$ $i=1 \ldots n$ with probability 1. As in section 2.1, we again introduce the r.v.

$$\beta = \sum_{j=1}^{N} \sum_{i=1}^{v_j} \Delta_i(X_j)$$

where $\Delta_i(X_j)$ depend only on α and γ_j is independent of Y_i .

Then we conclude that $\hat{\alpha}_r = \frac{\beta_r + 1}{n_r + b_r + 2}$ is independent of $F_Y(Y_{(k)})$. The proof of a more generalized result follows.

Lemma 1.

Let $g(\alpha)\in\mathcal{F}_r$ and $F_Y(Y_{(k)})\in\mathcal{F}_r$, $r<\infty$.

Then $E[g(\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_r]=E(F_Y(Y_{(k)}))E[g(\alpha)\,|\,\mathcal{F}_r]$

Proof: On the basis of conditional expectation properties we have

$$E[g(\alpha)F_X(Y_{(k)})\,|\,\mathcal{F}_r]=E[(E(g(\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_r]$$

$$=E[(g(\alpha)E(F_Y(Y_{(k)})\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_r]$$

$$=E[(g(\alpha)E(F_Y(Y_{(k)})\,|\,\mathcal{F}_r)\,|\,\mathcal{F}_r]$$

$$=E(F_Y(Y_{(k)})\,|\,\mathcal{F}_r)E(g(\alpha)\,|\,\mathcal{F}_r)=E(F_Y(Y_{(k)}))E[g(\alpha)\,|\,\mathcal{F}_r]\ .$$

Since α is an unknown constant in $\mathcal{F}_\infty$, then

$1-\alpha$, $\alpha^{\left[\frac{t}{c}\right]}$, *and* $(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha))^2$ are also in $\mathcal{F}_\infty$.

Because of $F_Y(Y_{(k)})\in\sigma(Y_1,\dots,Y_n)\in\mathcal{F}_r$, we conclude that:

1. $E[(1-\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_r]=E\,F_Y(Y_{(k)})E[(1-\alpha)\,|\,\mathcal{F}_r]$;

2. $E[\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_r]=E\,F_Y(Y_{(k)})E[\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathcal{F}_r]$;

3. $E[(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)}))^2\,|\,\mathcal{F}_r]$
$=E[F_Y(Y_{(k)})]^2E[(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha))^2\,|\,\mathcal{F}_r]$.

It has been shown in Theorem 1 that $\hat{\alpha}_r$ can be considered as a r.v. having a Beta(b+1,n+1) distribution. Then $(1-\hat{\alpha}_r)$ has a Beta (γ_1,δ_1) distribution with $\gamma_1=n+1$ $\delta_1=b+1$

The ordered statistics $F_Y(Y_{(k)})$ has a Beta (γ_2,δ_2) distribution with parameters $\gamma_2=k$, $\delta_2=n-k+1$.

Since $\gamma_2 + \delta_2 = k + n - k + 1 = n + 1 = \gamma_1$ then according to the above Theorem, we conclude that $(1 - \hat{\alpha}_2)F_Y(Y_{(k)})$ is a Beta(k,n+b-k+2) r.v.,

$F_X(Y_{(k)}) = (1 - \alpha)F_Y(Y_{(k)})$. Thus by Lemma 1,

$$E(F_X(Y_{(k)}) \mid \mathcal{F}_r) = E(F_Y(Y_{(k)}))E((1 - \alpha) \mid \mathcal{F}_r)$$

The r.v. $\hat{\alpha}_r = E(\alpha \mid \beta, \eta + \beta = n + b)$ is the estimator of α in $\mathcal{F}_r$, and the above equation implies that $(1 - \hat{\alpha}_r)F_Y(Y_{(k)})$ is the corresponding estimator of $F_X(Y_{(k)})$ and that it is a Beta(k,n+b-k+2) r.v. which proves the following:

Theorem 8.

The r.v. $\hat{\alpha}_r$, the estimator of α in $\mathcal{F}_r$ has an a posteri Beta(n+1,b+1) distribution implying that $\hat{F}_X(Y_{(k)}) = (1 - \hat{\alpha}_r)F_X(Y_{(k)})$ is the corresponding estimator of $F_X(Y_{(k)})$ and is Beta(k,n+b-k+1).

$$E((1 - \alpha)F_Y(Y_{(k)}) \mid \mathcal{F}_r) = \frac{k}{n + b + 2} ,$$

$$Var(\hat{F}_X(Y_{(k)})) = \frac{k(n + b - k + 2)}{(n + b + 3)(n + b + 2)^2} .$$

3.3 ESTIMATION OF THE PROBABILITY THAT $X \in (Y_{(i)}, Y_{(j)})$

The probability that the r.v. X lies in the interval between two ordered statistics satisfies the equation

$$P(X \in (Y_{(i)}, Y_{(j)}) = (1 - \alpha)F_Y(Y_{(j)}) - (1 - \alpha)F_Y(Y_{(i)})$$
$$= (1 - \alpha)(F_Y(Y_{(j)}) - F_Y(Y_{(i)}) \quad , \; i<j \; .$$

We will use the following results e.g. (Reiss 1989): If F_Y is a continuous c.d.f., then it is true that

$$F_Y(Y_{(j)}) - F_Y(Y_{(i)}) = F_Y(Y_{(j-i)}) \; .$$

therefore $F_Y(Y_{(j)}) - F_Y(Y_{(i)})$ has a Beta (γ_2, δ_2) p.d.f. with $\gamma_2 = j - i$,

and $\delta_2 = n - (j-1) + 1$.

The estimator of the probability $P(X \in (Y_{(i)} - Y_{(j)}))$ is $(1 - \hat{\alpha}_r)(F_Y(Y_{(j)} - F_Y(Y_{(i)})$, and it is a product of Beta variables since $(1 - \hat{\alpha}_r)$ has a Beta (γ_1, δ_1) p.d.f. with $\gamma_1 = n + 1$, $\delta_2 = b + 1$ (based on Theorem 1);

In accordance with Rao's theorem cited above, we conclude that the parameters of the β distributed factors in $(1 - \hat{\alpha}_r)[F_Y(Y_{(j)}) - F_Y(Y_{(i)})]$ satisfy

$\gamma_2 + \delta_2 = j - i - n - (j - i) + 1 = n + 1 = \gamma_1$.

Corollary 1.

Therefore, it is true that the estimator of the probability $P\{X \in (Y_{(i)}, Y_{(j)})\}$ has a Beta distribution with parameter (j-i,b+n-j+i+2).

$$E(P(X \in (Y_{(i)} - Y_{(j)}) = E((1 - \alpha)(F_Y(Y_{(j)}) - F_Y(Y_{(i)}) \mid \mathscr{F}_r) = \frac{j-i}{b+n+2} \ .$$

$$Var(P(X \in (Y_{(i)}, Y_{(j)})) = \frac{(j-i)(b+n-j+i+2)}{(b+n+2)^2(b+n+3)}$$

The proof is now a simple consequence from the previous considerations.

3.4 ESTIMATION OF $\Lambda_X(Y_{(k)})$

Corollary 2.

On the basis of section 3.2 we conclude that the estimator of the function $\exp - \Lambda_X(Y_{(k)})$ is $1 - (1 - \hat{\alpha}_r)F(y_{(k)})$. Moreover, it will then have a Beta(n+b-k+2,k) distribution. Therefore,

$$E((1 - \alpha)F(Y_{(k)}) \mid \mathscr{F}_r) = \frac{n+b-k+2}{n+b+2} \ ;$$

$$Var(1 - (1 - \hat{\alpha}_r)F(Y_{(k)}) \mid \mathscr{F}_r) = \frac{k(n+b-k+2)}{(n+b+3)(n+b+2)^2} \ .$$

3.4.1 DENSITY FUNCTION OF THE ESTIMATOR OF $\Lambda_X(Y_{(k)})$

Let $\hat{\Lambda}_X(Y_{(k)})$ be the estimator of $\Lambda_X(Y_{(k)})$, and let us set

$S = 1 - (1 - \hat{\alpha}_r)F(Y_{(k)})$.

Then S has a Beta(n+b-k+2,k) distribution. Hence, for

$\hat{\Lambda}_X(y_{(k)}) = -\ln S$ or $S = e^{-\hat{\Lambda}_X(y_{(k)})}$ we obtain

$$\left|\frac{dS}{d\hat{\Lambda}_X(y_{(k)})}\right| = e^{-\Lambda_X(y_{(k)})} ,$$

i.e.

$$f(S) = \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)\Gamma(k)} S^{n+b-k+1}(1-S)^{k-1} .$$

Thus

$$f(\hat{\Lambda}_X(y_{(k)}) = \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)\Gamma(K)}\left[e^{-\hat{\Lambda}_X(y_{(k)})}\right]^{n+b-k+1}\left[1-e^{-\hat{\Lambda}_X(y_{(k)})}\right]^{k-1} e^{-\Lambda_X(y_{(k)})}$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)\Gamma(K)}\left[e^{-\hat{\Lambda}_X(Y_{(k)})}\right]^{n+b-k+2}\left[1-e^{-\hat{\Lambda}_X(Y_{(k)})}\right]^{k-1} .$$

is the conditional p.d.f. of $\Lambda_X(Y_{(k)})$.

3.4.2 MOMENT GENERATING FUNCTION OF $\hat{\Lambda}_X(Y_{(k)})$

Theorem 9.

If $X_1,\ldots,X_n$ are iid observations on a r.v. $X \in ALM(\alpha, F_Y, c)$, then the estimate of the hazard function $\Lambda_X(t)$ at the order statistics $Y_{(k)}$ has the distribution of the sum of k independent exponential random variables $u_1,\ldots,u_k$ with parameter

$\nu_i = (n+b-i+2)^{-1}$, $i = 1,2\ldots,k$

i.e $\Lambda_X(Y_{(k)}) = \sum_{i=1}^{k} u_i$, $u_i \in \exp(\nu_i)$, $\nu_i = (n+b-i+2)^{-1}$

Proof: In order to simplify the equations obtained in 3.4.1, we set

$$a = n + b - k + 2 \quad , \quad q = \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(k)} \quad , \quad Z = \hat{\Lambda}_X(Y_{(k)})$$

The density function of $\hat{\Lambda}_X(y_{(k)}) = Z$ can then be written as

$$f(Z) = q\left[e^{-Z}\right]^a\left[1 - e^{-Z}\right]^{k-1}$$

Its moment generating function is

$$M_Z(t) = q\int_0^\infty e^{tZ}e^{-aZ}\left[1 - e^{-Z}\right]^{k-1}dZ = q\int_0^\infty e^{-(a-t)Z}\left[1 - e^{-Z}\right]^{k-1}dZ$$

Integrating by parts after setting

$$u = (1 - e^{-Z})^{k-1} \; ; \qquad dv = e^{-(a-t)Z}dZ \; ;$$

$$du = (k-1)(1 - e^{-Z})^{k-2}e^{-Z}dZ \; ; \qquad v = \frac{e^{-(a-t)Z}}{a-t}$$

in $\int_0^\infty u\,dv = uv|_0^\infty - \int_0^\infty v\,du$

we obtain

$$M_Z(t) = qe^{-(a-t)Z}(1 - e^{-Z})^{k-1}\Big|_0^\infty + q\int_0^\infty \frac{k-1}{a-t}e^{-(a-t)Z}(1 - e^{-Z})^{k-2}e^{-Z}dZ \;.$$

Now we observe that when Z is zero, $1 - e^{-Z}\big|_{z=0} = 0$; when Z is infinite, $e^{-(a-t)Z} = 0$ so that $e^{-(a-t)Z}(1 - e^{-Z})^{k-1}\big|_0^\infty = 0$, and

$$M_Z(t) = \int_0^\infty q\frac{k-1}{a-t}e^{-(a-t-1)Z}(1 - e^{-Z})^{k-2}dZ \;.$$

Proceeding the same way with $u = (1 - e^{-Z})^{k-2}$, $dv = \frac{k-1}{a-1}e^{-(a-t-1)Z}dZ$ we obtain

$$M_Z(t) = \frac{q(k-1)}{(a-t)(a-t-1)}e^{-(a-t-1)Z}(1 - e^{-Z})^{k-2}\Big|_0^\infty$$

$$+ q\int_0^\infty \frac{(k-1)(k-2)}{(a-t)(a-t-1)}(1 - e^{-Z})^{k-3}e^{-(a-t-2)Z}dZ$$

The first term is zero, hence

$$M_Z(t) = q\int_0^\infty \frac{(k-1)(k-2)}{(a-t)(a-t-1)}(1-e^{-Z})^{k-3}e^{-(a-t-2)Z}dZ$$

Repetitive integration by parts gives

$$M_Z(t) = \int_0^\infty \frac{q(k-1)!}{(a-t)(a-t-1)\ldots(a+k-t-2)}e^{-(a+k-1)Z}dZ$$

$$= \frac{q(k-1)!}{(a-t)(a-t-1)\ldots(a+k-t-1)}e^{-(a+k-1)Z}\Big|_0^\infty$$

$$= \frac{q(k-1)!}{(a-t)(a-t-1)\ldots(a+k-t-1)} \quad \text{,where } q = \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(k)}$$

$$M_Z(t) = \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(k)}\frac{\Gamma(k)}{(a-t)(a-t-1)\ldots(a+k-t-1)} \quad .$$

For $a = n+b-k+2$ we have

$$M_{\Lambda_x}(y_{(k)}) = \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)}\left(\frac{1}{n+b+1-t}\right)\left(\frac{1}{n+b-t}\right)\ldots\left(\frac{1}{n+b-k+2-t}\right)$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)}\left(\frac{1/(n+b+1)}{(n+b+1-t)/(n+b+1)}\right)\left(\frac{1/(n+b}{(n+b-t)/(n+b)}\right)\ldots$$

$$\ldots\left(\frac{1/(n+b-k+2)}{(n+b-k+2-t)/(n+b-k+2)}\right)$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)}\left(\frac{1}{n+b+1}\left(1-\frac{t}{n+b+1}\right)^{-1}\frac{1}{n+b}\left(1-\frac{t}{n+b}\right)^{-1}\ldots\right.$$

$$\left.\ldots\frac{1}{n+b-k+2}\left(1-\frac{t}{n+b-k+2}\right)^{-1}\right)$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)}\frac{1}{(n+b+1)(n+b)\ldots(n+b-k+2)}\left(1-\frac{t}{n+b+1}\right)^{-1}\left(1-\frac{t}{n+b}\right)^{-1}\ldots$$

$$\ldots\left(1-\frac{t}{n+b-k+2}\right)^{-1}$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(n+b-k+2)}\frac{\Gamma(n+b-k+2)}{\Gamma(n+b+2)}\left(1-\frac{t}{n+b+1}\right)^{-1}\left(1-\frac{t}{n+b}\right)^{-1}\ldots\left(1-\frac{t}{n+b-k+2}\right)^{-1}$$

$$M_{\hat{\Lambda}_X}(y_{(k)}) = \left(1 - \frac{t}{n+b+1}\right)^{-1}\left(1 - \frac{t}{n+b}\right)^{-1} \cdots \left(1 - \frac{t}{n+b-k+2}\right)^{-1}$$

$$M_{\hat{\Lambda}_X}(y_{(k)}) = (1 - \nu_1 t)^{-1}(1 - \nu_2 t)^{-1} \ldots (1 - \nu_k t)^{-1} \quad ,$$

where

$\nu_1 = (n+b+1)^{-1};$

$\nu_2 = (n+b)^{-1};$

.

.

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$\nu_k = (n+b-k+2)^{-1}.$

This is the moment generating function of the sum of k independent exponential random variables $u_1 \ldots u_k$ with parameters $\nu_1 \ldots \nu_k$ as above. Hence,

$$M_{\hat{\Lambda}_X(Y_{(k)})}(t) = M_{u_1+u_2+\ldots+u_k}(t) \quad .$$

By the uniqueness theorem we have

$$\hat{\Lambda}_X(Y_{(k)}) = \sum_{i=1}^{k} u_i \quad ,$$

where u_i has an exponential distribution with parameter $(n+b-i+2)^{-1}$ as stated in the theorem.

Corollary 2.

The expected value and the variance of $\hat{\Lambda}_X(Y_{(k)})$ are:

$$\mathrm{E}\left[\hat{\Lambda}_X(Y_{(k)})\right] = \sum_{i=1}^{k} \frac{1}{n+b-i+2}$$

$$Var\left[\hat{\Lambda}_X(Y_{(k)})\right] = \sum_{i=1}^{k} \frac{1}{(n+b-i+2)^2}$$

Proof: $\mathrm{E}(\hat{\Lambda}_X(Y_{(k)}) = \sum_{i=1}^{k} \mathrm{E}(u_i)$

and since u_i 's are independent it implies

$$Var\left[\hat{\Lambda}_X(Y_{(k)})\right] = \sum_{i=1}^{k} Var(u_i)$$

and the corollary is obvious.

3.5 ESTIMATION OF THE FAILURE RATE

Using the result of Theorem 9, it is easy to see that

$$\hat{\Lambda}_X(Y_{(k-1)}) - \hat{\Lambda}_X(Y_{(k)}) = u_{k+1}$$

where u_{k+1} is an exponential r.v. with parameter $(n+b-k+1)^{-1}$

By definition $\Lambda_X(y) = \int_0^y \lambda_X(t)\,dt$;

i.e. $\Lambda_X(s) - \Lambda_X(u) = \int_u^s \lambda_X(t)\,dt$.

Therefore, the estimator $\hat{\lambda}_X(y)$ of $\lambda_X(y)$ should satisfy:

$$\hat{\Lambda}_X(Y_{(k+1)}) - \hat{\Lambda}_X(Y_{(k)}) = \int_{Y_{(k)}}^{Y_{(k+1)}} \hat{\lambda}_X(t)\,dt$$

or equivalently

$$u_{k+1} = \int_{Y_{(k)}}^{Y_{(k+1)}} \hat{\lambda}_X(t)\,dt \ .$$

The mean value theorem for integrals states that there exists a

$y \in (Y_{(k)}, Y_{(k+1)})$ such that $\hat{\lambda}_X(y)$ satisfies $u_{k+1} = \hat{\lambda}_X(y)(Y_{(k+1)} - Y_{(k)})$.

If $\hat{\lambda}_x(y)$ is an estimator of $\lambda_X(y)$ for all $y \in [Y_{(k)}, Y_{(k+1)})$, then

$$\mathrm{E}(\hat{\lambda}_X(y) \mid \mathscr{F}_r) = \frac{1}{(Y_{(k+1)} - Y_{(k)})(n+b-k+1)} \ ,$$

and $Var(\hat{\lambda}_X(y) \mid \mathscr{F}_r) = \dfrac{Var(u_{k+1})}{Y_{(k+1)} - Y_{(k)}} = \dfrac{1}{(n+b-k+1)^2 (Y_{(k+1)} - Y_{(k)})^2}$, if $y \in [Y_{(k)}, Y_{(k+1)})$.

Theorem 10.

The function $\hat{\lambda}(y)=\dfrac{u_{k+1}}{Y_{(k+1)}-Y_{(k)}}$,for $y\in[Y_{(k)},Y_{(k+1)})$ is a consistent estimator

of $\lambda_X(y)$ i.e.it satisfies $\lim_{N\to\infty}\dfrac{u_{k+1}}{Y_{(k+1)}-Y_{(k)}}=\lambda_X(y)$, if and only if $b\to\infty$ and $n\to\infty$ as

$N\to\infty$. This condition is fulfilled when $\alpha\neq 0$ and $\alpha\neq 1$

We will consider two possibilities:

(1) Assume $b=E(\beta\,|\mathcal{F}_r)\to\infty$ and $n=E(\eta\,|\mathcal{F}_r)\to\infty$ as $N\to\infty$

Fix a point y , $y\in[Y_k,Y_{k+1})$, k will increase as $N\to\infty$ but stays smaller than n.

(a) Suppose $Y_{(k+1)}-Y_{(k)}$ does not go to zero as $b\to\infty$ and $n\to\infty$

then $E(\hat{\lambda}_X(y))=\dfrac{1}{(Y_{(k+1)}-Y_{(k)})(n+b-k+1)}$

Since we have $k<n+1$, then $\lim_{b\to\infty}E(\lambda_X(y))=0$

Also, in accordance with Corollary 1, we have

$P(Y\in[Y_{(k)},Y_{(k+1)}))$ is a r.v. having a Beta(1,n) distribution.

Then, $\lim_{n\to\infty}E(P(Y\in[Y_{(k)},Y_{(k+1)})))=\lim_{n\to\infty}\dfrac{1}{n+1}=0$.

$P(.)$ is a non-negative function. We conclude that

$P(Y\in[Y_{(k)},Y_{(k+1)}))=0$ a.e.

Therefore, the density function of any $Y\in[Y_{(k)},Y_{(k+1)})$ is

$f_Y(y)=0$, hence $\lambda_X(y)=\dfrac{f_Y(y)}{1-F_Y(y)}=0$.

thus $E(\hat{\lambda}_X(y))\to\lambda_X(y)$. (4)

i.e. $\hat{\lambda}_X(y) = \dfrac{u_{k+1}}{Y_{(k+1)} - Y_{(k)}}$ for $y \in (Y_{(k)}, Y_{(k+1)})$ is a consistent estimator of $\lambda_X(y)$

We observe that $b \to \infty$ and $n \to \infty$ as $N \to \infty$ is a necessary condition for convergence, since $N \to \infty$ implies $b + n \to \infty$. If either b or n is finite, the convergence in (4) is not achieved.

(b) Suppose $Y_{(k+1)} - Y_{(k)} \to 0$ as $b \to \infty$ and $n \to \infty$, (then $N \to \infty$),

Then it is fulfilled that $\displaystyle\lim_{N \to \infty} \frac{\hat{\Lambda}_X(Y_{(k+1)}) - \hat{\Lambda}_X(Y_{(k)})}{Y_{(k+1)} - Y_{(k)}}$

$$= \lim_{N \to \infty} \frac{\Lambda_X(Y_{(k+1)}) - \Lambda_X(Y_{(k)})}{Y_{(k+1)} - Y_{(k)}} \quad .$$

Denote $\Delta t = Y_{(k+1)} - Y_{(k)}$,then the above expression is

$$\lim_{\Delta t \to 0} \frac{\Lambda_X(Y_{(k)} + \Delta t) - \Lambda_X(Y_{(k)})}{\Delta t} = \lambda_X(Y_{(k)}) \text{ by definition.}$$

We have $Y_{(k)} \le y < Y_{(k+1)}$, and $\Delta t \to 0$ implies $Y_{(k)} \to y$

therefore $\displaystyle\lim_{N \to \infty} \hat{\lambda}_X(y) \to \lambda_X(y)$

Thus, as $N \to \infty$

$$\hat{\lambda}_X(y) = \frac{u_{k+1}}{Y_{(k+1)} - Y_{(k)}} \text{ if } y \in [Y_{(k)}, Y_{(k+1)}) \text{ is a consistent estimator of } \lambda_X(y) \quad .$$

(2) Suppose that $b = \mathrm{E}(\beta \mid \mathscr{F}_r) < \infty$ as $N \to \infty$

then $\mathrm{E}(\beta \mid \mathscr{F}_1) < \infty$

For r=1 we have $\gamma_j = \left\{ \left[\dfrac{X_j}{c} \right] + 1 \right\} \wedge 1 = 1$.

since $\beta = \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{\gamma_j} \Delta_1(x_j) = \sum_{j=1}^{N} \Delta_1(x_j)$, we have $E(\beta \mid \mathscr{F}_1) = \sum_{j=1}^{N} E\,\Delta_i(x_j) = N\alpha$.

If $E(\beta \mid \mathscr{F}_1) < \infty$ as $N \to \infty$, then it implies $\lim_{N \to \infty} N\alpha < \infty$;

this is possible iff $\alpha = 0$.

Therefore, $b < \infty$ as $N \to \infty$ iff $\alpha = 0$.

Suppose that $n = E(\eta \mid \mathscr{F}_r) < \infty$ as $N \to \infty$

We have $\eta = \sum_{j=1}^{N} \sum_{j=1}^{\gamma_j} (1 - \Delta_i(X_j))$,

then $E(\eta \mid \mathscr{F}_1) = \sum_{j=1}^{N} E(1 - \Delta_i(X_j)) = N(1-\alpha)$.

If $E(\eta \mid \mathscr{F}_1) < \infty$ as $N \to \infty$, it implies $\lim_{N \to \infty} N(1-\alpha) < \infty$.

This is possible iff $\alpha = 1$.

Therefore, $n < \infty$ as $N \to \infty$ iff $\alpha = 1$.

In either case, $\hat{\lambda}_X(y)$ does not converge to its estimated parameter.

4. ESTIMATOR FOR FUNCTIONS OF THE PARAMETERS OF THE $ALM(\alpha, F_Y, c)$ DISTRIBUTION

4.1 ESTIMATOR OF THE HAZARD FUNCTION $\Lambda_X(t)$, $t>0$

Let $X_i - \left[\frac{X_i}{c}\right]c$ be the kth ordered statistics in $[0,c)$

that is, $Y_{(k)} = X_i - \left[\frac{X_i}{c}\right]c$ is the ordered statistics of the conditional times to failure,

counted from the beginning of the period.

Since summation is a linear function, and the failure rate is periodic, then

$$\hat{\Lambda}_X(X_i) = \hat{\Lambda}_X(Y_{(k)}) + \left[\frac{X_i}{c}\right]\hat{\Lambda}_X(c)$$

The end of the period c can be considered as the y_{n+1}th ordered statistics. Then

$$\hat{\Lambda}_X(X_i) = \hat{\Lambda}_X(Y_{(k)}) + \left[\frac{X_i}{c}\right]\hat{\Lambda}_X(Y_{(n+1)})$$

It was shown in Theorem 9 that $\hat{\Lambda}_{(}Y_{(k)}) = \sum_{i=1}^{k} u_i$ where u_i's are independent

$u_i \in \exp(\nu_i)$, $\nu_i = (n+b-i+2)^{-1}$.

Let $V_1 = \sum_{i=1}^{k} u_i$ and $V_2 = \sum_{i=k+1}^{n+1} u_i$,

Then, $\hat{\Lambda}_X(Y_{(k)}) = V_1$ and $\hat{\Lambda}_X(Y_{(n+1)}) = V_1 + V_2$.

The r.v.s V_1 and V_2 are independent since the u_i's are independent.

$$\Lambda_X(X_i) = V_1 + \left[\frac{X_i}{c}\right](V_1 + V_2)$$

The relation $\hat{\Lambda}_X(Y_{(k+1)}) - \hat{\Lambda}_X(Y_{(k)}) = u_{k+1}$ where u_{k+1} is an exponential r.v. shows that all the moments of $\hat{\Lambda}_X(y)$ are constant over $[Y_{(k)}, Y_{(k+1)})$. We can replace

$X_i - \left[\frac{X_i}{c}\right]c = Y_{(k)}$ in the previous derivation by $t - \left[\frac{t}{c}\right]c \in [Y_{(k)}, Y_{(k+1)})$, $t>0$.

This gives the following result:

$$\hat{\Lambda}_X(t) = V_1 + \left[\frac{t}{c}\right](V_1 + V_2) = \left(\left[\frac{t}{c}\right] + 1\right)V_1 + \left[\frac{t}{c}\right]V_2 = \left(\left[\frac{t}{c}\right] + 1\right)V_1 + \left[\frac{t}{c}\right]V_2$$

Theorem 11.

The estimator of $\Lambda_X(t)$, $t>0$ is a time satisfying

$t - \left[\frac{t}{c}\right]c \in [Y_{(k)}, Y_{(k+1)})$, has the form

$$\Lambda_X(t) = \left(\left[\frac{t}{c}\right] + 1\right)V_1 + \left[\frac{t}{c}\right]V_2$$

its expectation and variance are given by the equations

$$\mathrm{E}(\hat{\Lambda}_X(t)) = \left(\left[\frac{t}{c}\right] + 1\right)\mathrm{E}\,V_1 + \left[\frac{t}{c}\right]\mathrm{E}\,V_2 \;;$$

$$Var(\hat{\Lambda}_X(t)) = \left(\left[\frac{t}{c}\right] + 1\right)^2 Var\,V_1 + \left[\frac{t}{c}\right]^2 Var\,V_2 \;.$$

where

$$\mathrm{E}\,V_1 = \sum_{i=1}^{k} \frac{1}{n+b-i+2} \quad ; \quad \mathrm{E}\,V_2 = \sum_{i=k+1}^{n+1} \frac{1}{n+b-i+2} \;;$$

$$Var\,V_1 = \sum_{i=1}^{k} \frac{1}{(n+b-i+2)^2} \quad ; \quad Var\,V_2 = \sum_{i=k+1}^{n+1} \frac{1}{(n+b-i+2)^2} \;.$$

4.2 ESTIMATION OF SOME FUNCTIONS OF THE PARAMETER α

The probability that the event will occur in any interval of the form

$[(i-1)c, ic)$ is $(P(\Delta_i X_j) = 1)^{i-1} P(\Delta_i(X_j) = 0)$.

Therefore $P(\lambda^{*} \in [(i-1)c, ic)) = \alpha^{i-1}(1-\alpha)$.

Here, α is an unknown constant in $\mathcal{F}_\infty$. We can however, compute an estimate of this probability by using the Beta(b+1,n+1) distribution of the estimate of α in $\mathcal{F}_r$.

The following theorem, Chung (1974) ,p.340, will be used:

For a fixed integrable r.v. Z, it is true that

$$\lim_{r\to\infty} E(Z\,|\,\mathcal{F}_r) = E(Z\,|\,\mathcal{F}_\infty) \ .$$

Since we have $\alpha^{i-1}(1-\alpha)\in\mathcal{F}_\infty$ and $[\alpha^{i-1}(1-\alpha)]^2$ are fixed integrable r.v.s, then it is true that

$$\lim_{r\to\infty} E(\alpha^{i-1}(1-\alpha)\,|\,\mathcal{F}_r) = E(\alpha^{i-1}(1-\alpha)\,|\,\mathcal{F}_\infty) = \alpha^{i-1}(1-\alpha) \ ,$$

and

$$\lim_{r\to\infty} E([\alpha^{i-1}(1-\alpha)]^2\,|\,\mathcal{F}_r) = \alpha^{2(i-1)}(1-\alpha)^2 \ .$$

Theorem 12.

A consistent estimator of $\alpha^S(1-\alpha)^k$ is

$$E(\alpha^s(1-\alpha)^k\,|\,\mathcal{F}_r) = \frac{(n+b+1)!\,(s+b)!\,(k+n)!}{b!\,n!\,(s+b+k+n)!}$$

Proof:

$E(\alpha^s(1-\alpha)^k\,|\,\mathcal{F}_r) \to \alpha^s(1-\alpha)^k$ By Chung's theorem i.e. $E(\alpha^s(1-\alpha)^k\,|\,\mathcal{F}_r)$ is a consistent estimator of $\alpha^s(1-\alpha)^k$.

We have

$$E(\alpha^s(1-\alpha)^k\,|\,\mathcal{F}_r) = \int_0^1 \alpha^s(1-\alpha)^k dF_{\alpha|\mathcal{F}_r}$$

$$= \int_0^1 \alpha^s(1-\alpha)^k \frac{\Gamma(n+b+2)}{\Gamma(b+1)\Gamma(n+1)} \alpha^b(1-\alpha)\,d\alpha$$

$$= \frac{\Gamma(n+b+2)}{\Gamma(b+1)\Gamma(n+1)} \int_0^1 \alpha^{S+b}(1-\alpha)^{k+n} d\alpha$$

$$= \frac{\Gamma(n+b+2)\Gamma(s+b+1)\Gamma(k+n+1)}{\Gamma(b+1)\Gamma(n+1)\Gamma(s+b+k+n+2)}$$

$$= \frac{(n+b+1)!\,(s+b)!\,(k+n)!}{b!\,n!\,(s+b+n+1)!}.$$

Now getting back to the estimation of $\alpha^{i-1}(1-\alpha)$, we have by Theorem 12,

$$\mathrm{E}[\alpha^{i-1}(1-\alpha)\,|\,\mathscr{F}_r] = \frac{(n+1)(n+b+1)!\,(i+b-1)!}{b!\,(i+b+n+1)!}$$

as a consistent estimator.

In the same way, we observe that the estimator of $[\alpha^{i-1}(1-\alpha)]^2$ is

$$\mathrm{E}((1-\alpha)^2\,\alpha^{2(i-1)}\,|\,\mathscr{F}_r) = \frac{(n+2)(n+1)(n+b+1)!\,(2i+b-2)!}{b!\,(2i+b+n+1)!} \quad .$$

Since

$$Var[\alpha^{i-1}(1-\alpha)\,|\,\mathscr{F}_r] = \mathrm{E}[\alpha^{2(i-1)}(1-\alpha)^2\,|\,\mathscr{F}_r] - [\mathrm{E}\,\alpha^{i-1}(1-\alpha)\,|\,\mathscr{F}_r]^2 \quad ,$$

then

$$Var[\alpha^{i-1}(1-\alpha)\,|\,\mathscr{F}_r] =$$

$$\frac{(n+1)(n+b+1)!}{b!}\left[\frac{(n+2)(2i+b-2)!}{(2i+b+n+1)!} - \frac{(n+1)(n+b+1)!\,((i+b-1)!)^2}{b!\,[(i+b+n+1)!]^2}\right] .$$

As a conclusion, we list some particular equations to be used in the following section:

By Theorem 12

$$\mathrm{E}[\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathscr{F}_r] = \frac{(n+1)(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(\left[\frac{t}{c}\right]+b+n+2)!} \quad , \tag{6}$$

and

$$\mathrm{E}[(1-\alpha)^2\,\alpha^{2\left[\frac{t}{c}\right]}\,|\,\mathscr{F}_r] = \frac{(n+2)(n+1)(n+b+1)!\,(2\left[\frac{t}{c}\right]+b)!}{b!\,(2\left[\frac{t}{c}\right]+b+n+3)!} \quad . \tag{7}$$

$$E(\alpha^{\left[\frac{t}{c}\right]} \mid \mathscr{F}_r) = \frac{(n+b+1)!\,(b+\left[\frac{t}{c}\right])!}{b!\,(b+n+\left[\frac{t}{c}\right]+1)!} \quad ; \tag{8}$$

$$E(\alpha^{2\left[\frac{t}{c}\right]} \mid \mathscr{F}_r) - \frac{(n+b+1)!\,(b+2\left[\frac{t}{c}\right])!}{b!\,(b+n+2\left[\frac{t}{c}\right]+1)!} \quad .$$

Then

$$Var(\alpha^{\left[\frac{t}{c}\right]} \mid \mathscr{F}_r) = E(\alpha^{2\left[\frac{t}{c}\right]} \mid \mathscr{F}_r) - [E(\alpha^{\left[\frac{t}{c}\right]} \mid \mathscr{F}_r)]^2$$

$$= \frac{(b+n+1)!\,(b+2\left[\frac{t}{c}\right])!}{b!\,(b+n+2\left[\frac{t}{c}\right])!} - \left[\frac{(n+b+1)!\,(b+\left[\frac{t}{c}\right])!}{(b+n+\left[\frac{t}{c}\right]+1)!\,b!}\right]^2 \quad ; \tag{9}$$

$$E(\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha) \mid \mathscr{F}_r) = \frac{(n+2)(n+1)(n+b+1)!\,(2\left[\frac{t}{c}\right]+b)!}{b!\,(2\left[\frac{t}{c}\right]+b+n+3)!} \quad . \tag{10}$$

Moreover, $E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(y_{(k)}) \mid \mathscr{F}_r) = E(F_Y(y_{(k)}))E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha) \mid \mathscr{F}_r)$ by Lemma 1,

$F_Y(y_{(k)})$ has a Beta(k,n-k+1) distribution.

Using the result in 3.2 we have

$$E(F_Y(Y_{(k)})) = \frac{k}{n+1} \quad . \tag{11}$$

By equation (6)

$$E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha) \mid \mathscr{F}_r) = \frac{(n+1)(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(\left[\frac{t}{c}\right]+b+n+2)!} \quad .$$

Therefore

$$E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)}\,|\,\mathcal{F}_r)=\frac{k(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(\left[\frac{t}{c}\right]+b+n+2)!}\;. \qquad (12)$$

We also have

$$Var(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,F_Y(Y_{(k)})\,|\,\mathcal{F}_r)$$

$$=E[(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,F_Y(Y_{(k)}))^2\,|\,\mathcal{F}_r]-[E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,F_Y(Y_{(k)}))\,|\,\mathcal{F}_r]^2\;.$$

By the conditional independence we can write

$$=E(F_Y(Y_{(k)})^2)E(\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha)^2\,|\,\mathcal{F}_r)-[E(F_Y(Y_{(k)}))E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathcal{F}_r]^2$$

or

$$E(F_Y(y_{(k)}))^2=\frac{(k+1)(k)}{(n+2)(n+1)}\;,$$

where

$E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathcal{F}_r)$ and $E(\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha)^2\,|\,\mathcal{F}_r)$ are given in equations (6) and (7).

Therefore

$$Var(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_X(Y_{(k)})\,|\,\mathcal{F}_r)$$

$$=\frac{k(k+1)(n+b+1)!\,(2\left[\frac{t}{c}\right]+b)!}{b!\,(2\left[\frac{t}{c}\right]+b+n+3)!}-\left[\frac{(n+1)(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(\left[\frac{t}{c}\right]+b+n+2)!}\right]^2\left(\frac{k}{n+1}\right)^2\;. \qquad (13)$$

4.3 ESTIMATION OF $F_X(t)$, $t>0$

Theorem 13.

If X has the ALM distribution, then the estimator of $F_X(t)$ in $\mathcal{F}_r$ will have the following properties:

(i) $E(F_X(t)\,|\,\mathcal{F}_r) = 1 + \zeta_1\left(\frac{k}{Z_1+2} - 1\right)$

(ii) $Var\,F_X(t) = \zeta_2\left(1 - \frac{2k}{Z_2+2} + \frac{k(k+1)}{(Z_2+2)(Z_2+3)}\right) - \zeta_1^2\left(1 - \frac{2k}{Z_1+2} + \frac{k^2}{(Z_1+2)^2}\right)$

k is determined by t,c and the ordered statistics $Y_{(1)},\ldots,Y_{(n)}$ as the integer which satisfies

$t - \left[\frac{t}{c}\right]c \in [Y_{(k)}, Y_{(k+1)})$; ζ_1 , ζ_2 , Z_1 *and* Z_2 are defined by the equations:

$$\zeta_1 = \frac{(n+b+1)!\,(b+\left[\frac{t}{c}\right])!}{b!\,(b+n+\left[\frac{t}{c}\right]+1)!} \;;\; \zeta_2 = \frac{(n+b+1)!\,(b+2\left[\frac{t}{c}\right])!}{b!\,(b+n+2\left[\frac{t}{c}\right]+1)!} \;;$$

$Z_1 = n + b + \left[\frac{t}{c}\right]$; $Z_2 = n + b + 2\left[\frac{t}{c}\right]$.

Proof:

Let $t - \left[\frac{t}{c}\right]c \in [Y_{(k)}, Y_{(k+1)})$, $t>0$.

then $F_X(t) = 1 - \alpha^{\left[\frac{t}{c}\right]}(1 - (1-\alpha)F_Y(Y_{(k)}))$.

Therefore $E(F_X(t)\,|\,\mathcal{F}_r) = 1 - E(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathcal{F}_r) + E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_r)$,

where $E(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathcal{F}_r)$ and $E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathcal{F}_r)$ are given by equations (6) and (11).

Hence we have

$$E(F_X(t)\,|\,\mathscr{F}_r) = 1 - \frac{(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(n+b+\left[\frac{t}{c}\right]+1)!} + \frac{k(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(\left[\frac{t}{c}\right]+b+n+2)!} .$$

Thus

$$E(F_X(t)\,|\,\mathscr{F}_r) = 1 - \frac{(n+b+1)!\,(\left[\frac{t}{c}\right]+b)!}{b!\,(n+b+\left[\frac{t}{c}\right]+1)!}\left(1 - \frac{k}{\left[\frac{t}{c}\right]+b+n+2}\right) ;$$

and hence the result (i) follows.

In a similar way we obtain (ii)

$$Var(F_X(t)\,|\,\mathscr{F}_r) = Var(\alpha^{\left[\frac{t}{c}\right]} - \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathscr{F}_r)$$

$$= Var(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathscr{F}_r) - 2\,Cov[\alpha^{\left[\frac{t}{c}\right]}, \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathscr{F}_r)$$

$$+ Var(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathscr{F}_r) , \tag{14}$$

where

$$Cov[\alpha^{\left[\frac{t}{c}\right]}, \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathscr{F}_r)$$

$$= E([\alpha^{\left[\frac{t}{c}\right]} - E(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathscr{F}_r)][\alpha^{\left[\frac{t}{c}\right]}F_Y(Y_{(k)}) - E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)})\,|\,\mathscr{F}_r)]\,|\,\mathscr{F}_r)$$

$$= E(\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathscr{F}_r)E(F_Y(Y_{(k)}) - E(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathscr{F}_r)E(\alpha^{\left[\frac{t}{c}\right]}1-\alpha)\,|\,\mathscr{F}_r)E(F_Y(Y_{(k)})\,|\,\mathscr{F}_r).$$

By Lemma 1, it is true that

$$E\,F_Y(Y_{(k)}) = [E\,\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathscr{F}_r) - E(\alpha^{\left[\frac{t}{c}\right]}\,|\,\mathscr{F}_r)E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)\,|\,\mathscr{F}_r)] ,$$

where

$$EF_Y(Y_{(k)}) \ , \ E(\alpha^{2\left[\frac{t}{c}\right]}(1-\alpha) \mid \mathcal{F}_r) \ , \ E(\alpha^{\left[\frac{t}{c}\right]} \mid \mathcal{F}_r) \text{ and } E(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha) \mid \mathcal{F}_r)$$

are given by equations (11), (7), (8), and (6) respectively. After substitution and simplification, we obtain

$$Cov(\alpha^{\left[\frac{t}{c}\right]}, \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)}) \mid \mathcal{F}_r)$$

$$= \frac{k}{n+1}\left[\frac{(n+1)(b+n+1)!\,(2\left[\frac{t}{c}\right]+b)!}{(n+b+2\left[\frac{t}{c}\right]+2)!}\right] - \left[\frac{(n+b+1)!\,(b+\left[\frac{t}{c}\right])!}{(b+n+\left[\frac{t}{c}\right]+1)!\,b!}\right]^2 \frac{k}{n+b+\left[\frac{t}{c}\right]+2} \ .$$

The $Var(\alpha^{\left[\frac{t}{c}\right]} \mid \mathcal{F}_r)$ and $Var(\alpha^{\left[\frac{t}{c}\right]} \mid \mathcal{F}_r)$ are given by equations (9) and (13). Equation (14) states

$$Var(F_X(t) \mid \mathcal{F}_r) = Var(\alpha^{\left[\frac{t}{c}\right]} \mid \mathcal{F}_r) + Var(\alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)}) \mid \mathcal{F}_r)$$

$$-2\,Cov(\alpha^{\left[\frac{t}{c}\right]}, \alpha^{\left[\frac{t}{c}\right]}(1-\alpha)F_Y(Y_{(k)}) \mid \mathcal{F}_r).$$

The result (ii) is obtained after substitution of the last expressions in (14); factorizing the terms:

$$Var(F_X(t) \mid \mathcal{F}_r)$$

$$= \frac{(n+b+1)!\,(b+2\left[\frac{t}{c}\right])!}{b!\,(b+n+2\left[\frac{t}{c}\right]+1)!}\left(1 - \frac{2k}{n+b+2\left[\frac{t}{c}\right]+2} + \frac{k(k+1)}{(n+b+2\left[\frac{t}{c}\right]+3)(n+b+2\left[\frac{t}{c}\right]+2)}\right)$$

$$- \left[\frac{(n+b+1)!(b+\left[\frac{t}{c}\right])!}{(b+n+\left[\frac{t}{c}\right]+1)!\,b!}\right]^2 \left(1 - \frac{2k}{n+b+\left[\frac{t}{c}\right]+2} + \frac{k^2}{(n+b+\left[\frac{t}{c}\right]+2)^2}\right).$$

5. CONCLUSIONS

In this work, estimators of the parameters of the distribution of a r.v. $X \in ALM(\alpha, F_Y, c)$ under the assumption that N independent copies of X are observed for a time interval rc, r>0 is a given integer have been derived.

The results are expressed as a function of:

(i) - the ordered statistics of $Y_i = X_i - \left[\frac{X_i}{c}\right]c$, $Y_i \in [0,c)$,

(ii) - the number η of intervals on which the r.v. X_i have occurred, $i=1,\ldots,N$.

(iii) - the number β of intervals with no occurrences.

A biased and an unbiased estimator of the parameter α given the sample information have been derived and their properties are investigated (section 2).

Built on these properties, it has been proven that the estimator of the hazard function is expressed as the sum of exponential variables, and this is a consistent estimator (section 3.4).

The estimator of the failure rate, expressed as a linear function of an exponential r.v. is also established and it is proved that it is a consistent estimator (section 3.5).

The estimator of the cumulative distribution function is derived along with its variance (section 4). Estimators of some functions of α are also given (section 4.2)

More work is required for constructing confidence intervals, prediction intervals, and most importantly, designing a testing procedure for these parameters.

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