STRUCTURAL BEHAVIOUR OF SANDWICH PANELS

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ABSTRACT

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Sandwich panels subjected to different loading types and boundary conditions are investigated. The subjects covered include : laterally loaded simply supported plates; simply supported two-span continuous plates; effects of interlayer elastic deformations on the response of sandwich plates and beam-columns; local failure of sandwich panels; and hygrothermal effects. This study forms a phase in the overall research program at the Centre for Building Studies aiming at developing a comprehensive design manual for light-weight sandwich panels and their applications in buildings.

Navier's solutions for the deflection and stresses in simply supported sandwich plates are developed for several loading patterns. These results are further utilized in the analysis of simply supported continuous plates of two spans. It was found that both the redundant moment at the middle support and its effect on the responses of continuous sandwich plates are functions of the plate shear rigidity.

An approach based on the theory of elasticity is developed for the analysis of simply supported sandwich plates and beam-columns taking into account the possible interlayer "elastic" deformation. In contradiction with an existing theory, the results show that imperfect bonding reduces significantly the beam buckling load and increases the deflection. The analytical solutions are corroborated by test data carried out on two sandwich beams having different characteristics.

Local instability problems of sandwich panels are also studied. By using the finite difference method a solution for the dimpling stress is obtained. Expressions to define the transition state between the symmetric and antisymmetric wrinkling modes are derived by adopting an existing theory. Analytical solutions for the symmetric wrinkling stress and the crimpling load are obtained by using an elasticity approach. The effect of an initial imperfection on the wrinkling phenomena is also investigated. Results from the solutions developed for local instability problems are found comparable with experimental and analytical data in the literature.

Elasticity solutions for a sandwich plate subjected to a uniform temperature change are developed and checked against finite element solutions. The stiffness matrix of an orthotropic rectangular prism finite element is derived and used in the finite element computer program written based on the direct stiffness method. Results from the two approaches are found in close agreement. Approximate expressions for the maximum normal stress in the facings of a sandwich panel subjected to a linear moisture change across its thickness are derived based on engineering theory.

To facilitate the use of the solutions developed, simple formulas for the responses of sandwich panels under different loading types and boundary conditions are presented. Numerical values for the factors in these formulas are tabulated for a wide range of material properties, aspect ratios, loading type and position.

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In memory of my beloved mother.

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NOTATIONS

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$a, 2a$	Length in x-direction
a	Finite difference mesh width
asy	Subscript denoting antisymmetry
A	Shear rigidity
AR	Constant argument
A_f	Coss-sectional area of a face of a sandwich beam
A_m, A_{mn}, A'_m, A''_m, A'''_m	Fourier coefficient, coefficient of a hyperbolic function, constant
$[A]$	Coefficient matrix in finite difference equation
b	Length in y-direction, subscript denoting bending component, width of a sandwich beam
b_i	Span of ith sandwich panel
b^ℓ, b^r	Spans of left and right sandwich panels, respectively
B_m, B_{mn}	Fourier coefficient, coefficient of a hyperbolic or trigonometric function
B_1 to B_8	Submatrix in strain-displacement matrix
$[B]$	Coefficient matrix in finite difference equation, strain-displacement matrix

c	Length of a partially loaded area, subscript denoting core
cm	Subscript denoting connecting moment
$C, C', C'', C''', C_m, C_m^{\ell}, C_m^{r}, C_m', C_{mn}', C_m''$	Fourier coefficient, coefficient of hyperbolic or trigonometric function
$[C]$	Elasticity matrix
d	Width of a partially loaded area, core thickness
D	Flexural rigidity of a Sandwich Plate
D_m, D_{mu}', D_n	Coefficient of a hyperbolic or trigonometric function
D_{xy}	Twisting rigidity of a sandwich plate
$\{d\}$	Nodal degrees of freedom vector
$\{D\}$	Structural degrees of freedom vector
E	Young's modulus
E_c, E_f	Young's modulus of core and faces materials, respectively
E_{cx}, E_{cy}	Young's modulus of an orthotropic core material
E_m, E_m'	Coefficient of a hyperbolic function
E_t	Tangent modulus of a faces material

E_x, E_y, E_z	Young's modulus of an orthotropic material
[E]	Elasticity matrix
f	Subscript denoting face
F_m, F_{nc}, F_{nu}, F_p, F_{qc}, F_{qu}, F_{wc}, F_{wu}	Constant, coefficient of a hyperbolic or trigonometric function
G_c	Shear modulus
G_{cx}, G_{cy}	Shear modulus of an orthotropic material
G_{ij}	Shear modulus which characterizes the change in the right angle between i and j axes
h	Distance between centroids of faces
H_m, H_m'	Coefficient of a hyperbolic function
i	Superscript or subscript denoting r, ℓ, x, and y
I	Moment of inertia, identity matrix
j	Superscript denoting loading position (j = r or ℓ)
k	Constant defined by b^r/b^ℓ
K	Stiffness of an isotropic adhesive, constant, coefficient of a trigonometric function
K_{c1}, K_d, K_f	Integration constant, constant

K_i	Stiffness of an orthotropic adhesive
K_{ij}	Submatrix in stiffness matrix
K_m, K'_m, K_{mx}, K^{ij}_{mx}, $K^{\ell j}_{mx}$, K^{rj}_{mx}, K_{mxsy}, K_{mxy}, K_{my}, K^{ij}_{my}, $K^{\ell j}_{my}$, K^{rj}_{my}, K_{mysy}, K_{Qx}, K^{ij}_{Qx}, $K^{\ell j}_{Qx}$, K^{rj}_{Qx}, K_{Qxsy}, K_{Qy}, K^{ij}_{Qyf}, $K^{\ell j}_{Qyf}$, $K^{\ell j}_{Qyn}$, K^{rj}_{Qyf}, K^{rj}_{Qyn}, K_{Qysy}, K_w, K_{wb}, K^i_w, $K^{\ell j}_w$, K^{rj}_w, K_{ws}	Constant, coefficient of a hyperbolic function
K_x, K_y	Adhesive stiffness
K'	Slope of P-Δ curve
K_1, K_2	Coefficient of a trigonometric function
[K]	Stiffness matrix
ℓ	Superscript denoting left panel
L	Wrinkling wave length
L_{cr}	Critical length
m	Integer, summation counter
M	Bending moment per unit length, moisture content distribution in a core
M_c	Moment resisted by core, redundant moment in continuous plate

M_f	Moment resisted by faces
M_m, M_{masy}, M_m^{ℓ}, M_m^r, M_{msy}	Coefficient of a hyperbolic function
M_o	Intensity of a uniform distributed applied moment
M_{om}	Fourier coefficient
M_{sy}, M_{asy}	Moment distribution along a plate edges
M_x, M_y, M_{xy}	Moment per unit length
M_1, M_2	Moment distribution along a plate edges
n	Integer, summation counter
N	Normal force
N_m, N_m'	Coefficient of a trigonometric function
N_1 to N_8	Shape function
p, p_o, p_o^f, p_o^{ℓ}, p_o^r	Load intensity
P	Concentrated load, axial compression
P_{cr}	Critical load
P_E	Euler load

P^j, P^ℓ	Concentrated load
P_m, P'_m, P_{mn}, P^ℓ_{mn}, P^r_{mn}	Fourier coefficient, coefficient of a hyperbolic function
P^r	Concentrated load
P_x	Compressive load intensity
$\{P_b\}$, $\{P_s\}$	Load vector
q, q_x, q_y	Interlayer horizontal shear
Q	Concentrated load, shear force per unit length
Q_x, Q_{xasy}, Q^i_x, Q^ℓ_x, Q^r_x, Q_{xsy}, Q_y, Q_{yasy}, Q^i_{yf}, Q^i_{yn}, Q^ℓ_y, Q^r_y, Q_{ysy}	Shear force per unit length
$\{Q\}$	Load vector
r	Superscript denoting right panel
R, R^ℓ, R^r	Aspect ratio
$\{r\}$	Load vector
s	Honeycomb cell size, subscript denoting shear
ss	Subscript denoting simply supported plate
sy	Subscript denoting symmetry

S	Shear rigidity, shear strength of core material, shrinkage per cent due to a moisture movement
SP, SPL, SPR	Shear parameter
t_c	Core thickness
t_f	Face thickness
T	Extensional strength of a core material, temperature change at a sandwich plate surfaces
u, u_c, u_f	Displacement component in x-direction
U_i	Strain energy of ith finite element
v, v_c, v_f	Displacement component in y-direction
V_i	Potential energy
w	Displacement component in z-direction
w_a, w_{asy}, w_b, w_b^ℓ, w_b^r, w^i	Vertical deflection
w_o	Initial imperfection
w_o', w_o'', w_o'''	Wrinkling amplitude
w_s	Face wrinkling

w_s', w_s^{ℓ}, w_s^r, w_{sy}	Vertical deflection
W	Thickness of a distorted marginal zone in core due to wrinkling
W_m	Coefficient of a trigonometric function
W_{mn}	Fourrier coefficient
$\{w\}$	Displacement vector
X, Y	Coordinate axes
Y_m	Hyperbolic function of y
Z	Coordinate axis
α	Constant, parameter
α_c, α_f	Thermal expansion coefficient
α_m, α_m^{ℓ}, α_m^r, β_m, β_n, ξ, ξ', ξ_m, ξ_m^{ℓ}, ξ_{mn}, ξ_m^r, ϕ, ϕ', ϕ_m, ϕ_{mn}, ϕ_{mc}', $\phi_{mc}^{i'}$, ϕ_{mu}', $\phi_{mu}^{\ell'}$, $\phi_{mu}^{r'}$, Ψ_m, Ψ_m', Ψ_{mn}, μ, μ_x, μ_y, θ, θ', θ_f, θ_f'	Constant argument, coordinate of a point non dimensional parameter
θ_x^{ℓ}, θ_x^r, θ_y^{ℓ}, θ_y^r, $\theta_y^{\ell''}$, $\theta_y^{r''}$	Normal Slope
Φ, Φ^{ℓ}, Φ^r	Shear parameter

ε_c, ε_{cx}, ε_{cy}, ε_f, ε_{fx}, ε_{fy}, ε_x, ε_y, ε_z	Normal strain
$\{\varepsilon\}$, $\{\varepsilon_o\}$	Strain vector
σ	Normal stress
σ_{cr}	Critical stress
σ_{cx}, σ_{cy}, σ_f	Normal stress
σ_{fn}, σ_{fs}	Failing stress of core material
σ_{fx}, σ_{fy}, σ_s, σ_x, σ_y, σ_z	Normal stress
$\{\sigma\}$	Stress vector
γ_x, γ_{xy}	Shear strain
γ_{xmn}	Fourier coefficient
γ_y, γ_{yz}, γ_{zx}	Shear strain
γ_{ymn}	Fourier coefficient
τ_{cxy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}	Shear stress
ν, ν_f	Poisson's ratio

ν_{ij}	Poisson's ration which characterizes the decrease in j-direction during tension applied in i-direction
η	Plasticity factor
λ	Width of a finite difference mesh
ρ	Radius of curvature
δ	Amplitude of an initial imperfection
Δ, Δ_m	Relative normal strain
Δ_x, Δ_y	Interlayer horizontal displacement
∇	Laplace operator
∂	Operator used in partial differentiation

CHAPTER I

INTRODUCTION

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INTRODUCTION

1.1 GENERAL

The investigation reported in this thesis is part of a continuing research program aimed at the development of sandwich construction for application in buildings. Sandwich construction as a special form of composite materials has been extensively used in the aerospace industry, and its application in building construction is rapidly increasing. This great interest in composite materials is explained by the following quote from Dietz [21] :

"Demands on materials imposed by today's advanced technologies have become so diverse and severe that they often cannot be met by simple single-component materials acting alone. It is frequently necessary to combine several materials into a composite to which each constituent not only contributes its share, but whose combined action transcends the sum of the individual properties, and provides new performance unattainable by the constituents acting alone. Space vehicles, heat shields, rocket propellants, deep submergence vessels, buildings, vehicles for water and land transport, aircraft, pressure tanks, and many others impose requirements that are best met, and in many instances met only, by composite materials."

Combining several materials into one composite unit is not a new idea. A piece of Egyptian laminated wood stored at the Metropolitan Museum of Art, New York, was found at Thebes and shown to belong to the Eighteenth Dynasty (about 1500 B.C.) (Fig. 1.1). At its bottom are five pieces glued-laminated to the heavier piece made of glued veneer. Another example is the Egyptian Sarcophagus of wood veneers laminated to heavier wood substrat and now in the Boston Museum of Fine Arts (Fig. 1.2). The subtle principles of a laminated system were understood intuitively and by experience, and are now the subject of major research by present-day investigators.

Sandwich construction is characterized by the use of two thin layers of strong material, denoted as faces, between which a thick layer of light-weight and comparatively weak core is sandwiched. Such construction has found many applications in the aerospace and the building industries. It is used in the construction of helicopter rotor blades, helicopter flooring, aircraft wings, deck panels, fire walls, access doors, and baggage racks. Movable objects, as tank trailers that transport bulk milk and fruit juices, ladders for use in telephone line service, and truck trailer panels and doors are being built using sandwich panels. Architects are recognizing the fact that the sandwich concept is well suited for curtain wall applications. In addition, it is employed widely in structural walls, roofs [13, 22, 50, 93], and floorings for house trailers, small boat hulls, shipboard doors and bulkheads, table tops, and furniture. Folded plate roofs can also be built using sandwich panels [25]. Single storey homes, motels and similar buildings have been successfully built with sandwich panels. As an example, a motel and a restaurant were fabricated in Canada and shipped for erection in the Caribbean Islands [64]. Another example is the widespread use of sandwich panel in southern United States and in northern Canada [83], (Figs. 1.3 and 1.4), and in Europe [65]. In multi-storey buildings, sandwich panels have a place for the total system up to perhaps four storeys, but only two storey buildings have so far been built [64].

Answers to many problems related to buildings may be found in sandwich panels which offer several advantages. Panel virtues related to environmental requirements may include structural integrity, durable finishes, weather-tightness, dimensional stability, sound or microwave absorption. From the point of view of structural performance, sandwich construction is an efficient structural design due to the large stiffness achieved by spacing apart the most highly stressed elements, namely, the faces. The basic principle is much the same as that of an I-beam. Other advantages of sandwich-type construction may include high strength-to-weight ratios, increased fatigue life, endurance, low moisture permeability, electrical insulation, color processability, reduced shop labour, simplified materials handling, and potentially lower costs to the house manufacture and transportation.

As any other component, there are some disadvantages or factors to be considered. When the core or facing materials may lead to corrosion problems, special pretreatments are required. No foam is fireproof, but many of them can be made nonflammable [21]. Polyester sandwich panels were tested, both painted and unpainted [21], and it was found that the unpainted panels deteriorated the most in three years' weathering: the edgewise compressive strength reduced by 40 percent and the flexural strength by 30 percent. Painting the other polyester panel reduced the loss in edgewise compressive strength to 22 per cent, but the reduction in flexural strength was still 30 per cent. Similar effects on sandwich panel strengths were observed due to the effect of moisture contents [21]. Although several advantages can be attributed to most plastic foams, their resistance to chemical agents is low and a special process in manufacturing is required for better results.

1.2 LITERATURE SURVEY

A significant number of theoretical and experimental studies covering many aspects of sandwich construction are reported in the literature. In the present work, only those related to structural aspects are cited. Because of the variety of topics being investigated in this thesis, the literature survey of each subject considered is mentioned in the introduction of the related chapter. However, the following general remarks are noted :

(1) Existing solutions to simply supported sandwich plates are either limited to the case of uniform loading [4, 24, 29, 84] or to square plate when central concentrated load is considered [110]. Thus there is a need for more general solutions. Furthermore, it is important that such solutions be presented in a form simple enough for practical, routine use.

(2) Solutions for simply supported continuous sandwich plates are not found in the literature. However, a small number of studies on continuous sandwich beams are available [13, 26]. Again, their use is limited because of the complexity of the solutions.

(3) Interlayer deformations due to non-rigid adhesive were ignored in all the theories available for sandwich construction. An analytical approach to evaluate the effects of such deformations on the behaviour of sandwich plates and beams is required.

(4) The dimpling of sandwich plates with honeycomb cores was investigated experimentally and a semi-empirical formula is available [95]. How-

ever, its validity is not based on any analytical study.

(5) Theoretical and experimental studies were carried out to determine the wrinkling coefficient [14, 32, 42, 43, 46, 71, 95, 111, 112], and there is great discrepancies between the results of the two. The wrinkling of sandwich panel is usually classified into symmetrical and antisymmetrical wrinkling. Although each class was studied separately, the transition state has not been defined analytically.

(6) A general linear thermal gradient across a sandwich panel could be split into uniform and antisymmetric, with respect to the middle plane, distributions. Whereas an analytic solution is available for the latter case [8, 11, 23, 48, 53, 69, 70, 76], the former problem was not investigated in the literature. The behavior of sandwich panels subjected to moisture content movements has not been investigated.

(7) Some design aids are available in the literature, however, they are not sufficiently general in terms of panel configuration, loadings, and materials. Some of them were restricted to panels made of wood [19, 20], and others to sandwich beams [21, 61, 62, 93]. In addition, some aspects related to the performance requirements are found scattered in the literature [21, 50, 66, 67, 85]. A comprehensive compilation of these aspects has been presented elsewhere by the writer [49].

1.3 SCOPE AND OBJECTIVES

Although many types of sandwich panels have been invented, developed, tested, and marketed for use in the construction trade, there exist no design codes or specifications to assist designers. Thus, a designer must determine on his own the criteria for selection and design which are most appropriate for his particular application. Normally, he may gain valuable assistance out of the wealth of information accumulated so far since the beginning of the Second World War. However, in the final analysis he makes his own determination of the suitability of a specific design for a particular application under investigation, together with, in many cases, a prototype test program to verify the adequacy of the design developed.

For sandwich construction, because of the wide variety of available options for panels configuration, type of materials, and different fabrication processes, it is vital that the designer carefully establishes his desired functional and performance requirements as accurately and in as much quantified detail as possible. A substantial part of the process involves the determination of the structural response and the proportioning of the elements which make up the component. At this stage in the design process, a mathematical model is required to idealize the actual structure enabling the determination of its response to loads.

The development of new materials, a majority of which are plastics, has made an impact on the field of sandwich construction. The overall research program on sandwich construction at the Centre for Building Studies is aimed at assisting the designers in selecting suitable materials, structural design criteria, configurations, analyses of effects of loads and restraints, and in evaluating a proposed design by testing or by other suitable means. The present work forms one phase in this research program and is restricted in scope to the static response of sandwich panels to loads and environmental influences. Its objective is to develop simplified formulas and design aids related to sandwich plates subjected to statical loads, lateral and inplane, and hygrothermal effects.

Rectangular plate is the only configuration considered throughout this work. Structural response is determined in terms of stress, support reactions, deflection, and stability against failure. It should be emphasized that only structural sandwich panels are considered here. The term "structural sandwich" has been defined by ASTM C274-53: "Structural Sandwich Construction. A laminar construction comprising a combination of alternating dissimilar simple or composite materials assembled and intimately fixed in relation to each other so as to use the properties of each to attain specific structural advantages for the whole assembly."

1.4 ORGANIZATION OF THE THESIS

Chapter II of the thesis deals with simply supported sandwich plates. A simple approach for developing the three governing equations for the deflection and shears is presented. A general analytic solution is found using Navier's method. For practical purposes, simple formulas are derived to calculate the deflection and stresses in sandwich plates subjected to transverse load of five types: uniformly distributed load, hydrostatic pressure, partial load, strip load, and concentrated load. A simple approach for the application of the finite difference method is also presented.

The preceeding solutions for simply supported plates are further utilized in chapter III, which deals with simply supported sandwich plates continuous over two spans. Analytical solutions for sandwich plates subjected to edge moments are presented. The method of consistent deformation is employed to determine the redundant moment at the middle support. For practical applications, simple formulas are derived to calculate the deflection and stresses in both spans.

The traditional assumptions of perfect bonding between layers is analytically examined in chapter IV, which investigates the effect of non-rigid adhesives on the behaviour of sandwich struts, beams, and plates. A parametric study demonstrating the effect of the adhesive stiffness on the values of maximum deflection and stresses of such configurations is elaborated. For practical use, simple formulas are derived to calculate the maximum response of sandwich beam-columns considering interlayer deformations.

Chapter V deals with local failure modes of sandwich panels. A numerical solution for the dimpling stress of sandwich panel with honeycomb core is found using the finite difference method. With regard to the wrinkling of sandwich panels, an expression for defining the transition state between symmetrical and antisymmetrical wrinklings is developed. New coefficients for the wrinkling load of facings and failing stresses of core material are also presented. A simple approach, utilizing geometrical relations noted on the behaviour of sandwich struts, is used to derive an expression for the crimpling load.

Hygrothermal stresses in sandwich panels are treated in chapter VI of the thesis. Analytical solution for panels subjected to uniform temperature change is presented. For practical use simple expressions to calculate the maximum deflection and stresses in sandwich plates exposed to thermal gradient (with zero value at the middle plane) are reformulated from existing theories in the literature. Formulas are also derived for the maximum stresses in sandwich plates subjected to uniform temperature change and moisture content movements.

Finally, the conclusion and recommendations for further study are presented in chapter VII.

Fig. 1.1 FRAGMENT OF EGYPTIAN LAMINATED WOOD OF THE EIGHTEENTH DYNASTY [21]

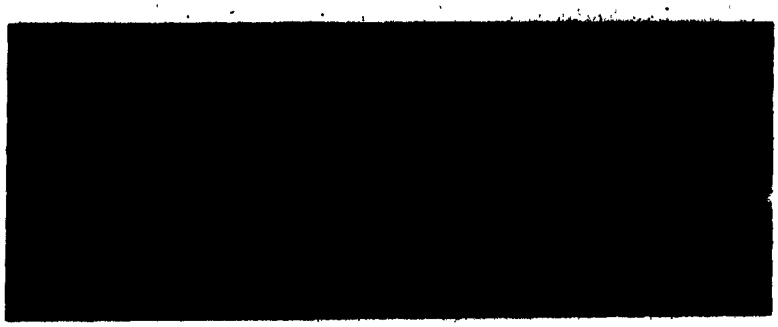


Fig. 1.2 EGYPTIAN SARCOPHAGUS OF FINE WOOD VENEERS LAMINATED TO HEAVIER WOOD SUBSTRATE [21]

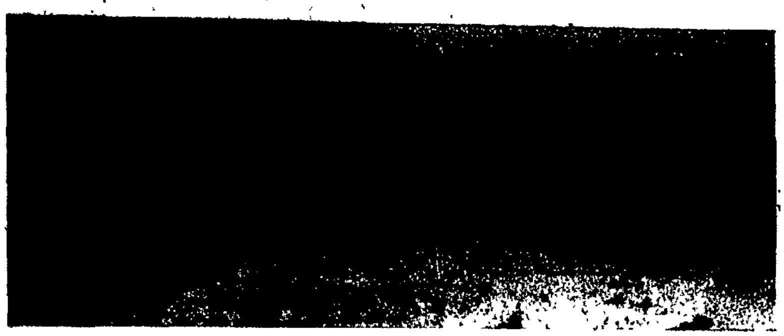


Fig. 1.3 NORTHERN SANDWICH PANEL SCHOOL [85]

Fig. 1.4 CLOSING IN NORTHERN HOUSE [85]

CHAPTER II

SIMPLY SUPPORTED SANDWICH PLATES

CHAPTER II

SIMPLY SUPPORTED SANDWICH PLATES

2.1 INTRODUCTION

In this chapter the governing equations of sandwich plates are presented. The developments are based on energy methods and classical plates theory [101]. Navier's solutions [96, 101] for simply supported sandwich plates are obtained for the following loading types: (i) uniform distributed load; (ii) concentrated load; (iii) hydrostatic pressure; (iv) partial load; and (v) strip load. For the last three loading types, solutions have not been presented in the literature.

Existing techniques for the analysis of sandwich plates include: (i) variational method; (ii) the partial deflection theory, and (iii) numerical methods such as the finite element and the finite difference methods. A survey for each of the three methods is given next.

First, Reissner [88, 89, 90] by using the variational theorems, studied the effect of transverse shear deformation on the bending of elastic plates. Taking into consideration the deformability of such plates due to transverse shear, a system of differential equations for deformations were obtained based on Castigliano's theorem of least work combined with the Lagrangian multiplier method of the calculus of variations. The principal contribution of Reissner, in addition to the equations obtained, was to furnish three boundary conditions to be satisfied by a solution at each edge of the plate instead of two as Kirchoff established. By similar variational theorems other analyses were conducted. For instance, Chong [15] in his approach based on the principle of minimum complementary energy combined with the Lagrangian multiplier method achieved similar results to Reissner's. Furthermore, a governing linear differential equation of the sixth-order for the deflection was derived, from which a solution can be obtained through the use of techniques developed for classical plates [96, 101]. Another example of solving a sandwich plate by applying the variational theorems, is the analysis carried out by Ueng [103]. In this study, Reissner's approach [88] was extended to solve sandwich plates with dissimilar facings.

Second, since the development of the partial deflection theory by March [65], its application was adopted extensively by many others [4, 24, 84]. According to this theory, any line in the core that is initially straight and normal to the middle surface will remain straight after the deformation of the plate but not necessarily normal to the deformed middle surface. Consequently, the plate deflection can be considered to compose of two components: bending and shear deflections. The former corresponds to the deflection of a sandwich plate rigid in shear while the latter to that of a sandwich plate infinitely stiff in bending. Two distinct methods can be used to determine the partial deflections: (i) Plantema's approach [84] where a governing differential equation was developed for each component of the deflection, and (ii) Allen's approach [4] where the shear strain is related to the total slope by unknown parameters which can be determined by using the principle of minimum total potential energy.

Finally, to avoid the complex mathematical operations of the two previous methods, numerical techniques have been widely utilized. Among them, the finite element method has enjoyed wide acceptance. Both the assumed displacement and hybrid approaches have been applied in deriving the stiffness matrix of finite elements of sandwich plate [1, 10, 18, 38, 40, 55, 56, 57, 72, 73, 76, 77, 92]. Several finite element computer programs to solve sandwich plates have been written [1, 10, 38, 76]. The other popular numerical technique is the finite difference method [2, 96]. This method has a long tradition and is a well-known technique for solving boundary value problems.

In summary, theories of bending of sandwich plates are well established by a number of authors. However, these theories show differences with regard to the method of analysis, the equations developed, and the solution methods.

2.2 ASSUMPTIONS

Sandwich plates under transverse load present a problem in three dimensional elasticity. However, by introducing assumptions regarding the stress or displacement distribution across the plate thickness, the problem can be reduced to a two dimensional one.

The relevant assumptions forming the basis for deriving the governing equations which are solved in this chapter are as follows:

(1) The faces are thin in comparison with the core depth. This implies that the flexural rigidity of each face about its own middle surface is negligible, and consequently, the inplane stresses resisted by each skin are uniformly distributed across its thickness.

(2) The inplane stresses in the core are negligible. This means that the core material is soft relative to the face material. Thus, the

transverse shear stresses in the core are constant across its thickness. And since the stress normal to the plate surface has an insignificant effect on the bending stresses [91], it follows that the normal to the middle plane of the plate remains straight and inextensible after deformations, as was proposed by March [65]. The general solutions developed in this chapter are applicable to sandwich plates whose cores experience warping. A modified shear modulus must be used in this particular case, and can be calculated, however, from an expression developed by Allen [4].

(3) The sandwich plate is assembled such that full interaction at the interfaces is provided through a rigid adhesive. The effects of interlayer deformations on the response of sandwich plates will be studied in chapter IV.

(4) The two facings are of equal thickness and made of the same material.

(5) Materials are homogeneous, isotropic and linearly elastic.

(6) Deformations are small.

It should be noted that these assumptions are also adopted in the rest of the present work unless stated otherwise.

2.3 GOVERNING DIFFERENTIAL EQUATIONS FOR DEFLECTION AND SHEARS

The moment expressions, equilibrium and governing equations for orthotropic sandwich plates with dissimilar facings are well established [4, 15, 84, 88, 103]. Despite this, and for completeness of the present analysis these equations are derived in this section by following a simple approach, and reduced to represent an isotropic sandwich plate for which the assumptions in section 2.2 are applicable.

Consider now a simply supported sandwich plate composed of two thin faces and a light-weight core as shown in Fig. 2.1. Bending stiffness of the facings can be ignored and transverse shear stresses in the facings are neglected. This means that the faces behave as solid membranes [38]. The core resists only transverse shear stresses, and normal stresses in directions parallel to the faces are neglected. This means that the transverse shear stresses do not vary across the core thickness.

The non-zero stress components, thus, in the faces are σ_x, σ_y, τ_{xy} and in the core τ_{xz}, τ_{yz}. The strain energy in the sandwich plate, considering small deformations and rigid adhesives, is equal to the strain energy in the two faces plus that in the core [4, 15, 38, 88, 103].

$$U = \frac{1}{2} \int_f \sum_{i=1}^{2} [t_i (\frac{\sigma_{xi}^2}{E_{xi}} + \frac{\sigma_{yi}^2}{E_{yi}} - \frac{2\nu_{xyi}}{E_{xi}} \sigma_{xi} \sigma_{yi} + \frac{\tau_{xyi}^2}{G_i})] dA +$$

$$\frac{1}{2} \int_c (\frac{\tau_{xz}^2}{G_{xz}} + \frac{\tau_{yz}^2}{G_{yz}}) dv$$

in which

f and c denote the facing and core, respectively

ν_{xy} = Poisson's ratio of face material

E = Young modulus of face material

G = shear modulus

t = face thickness

σ = normal stress

τ = shear stress

1,2 subscripts denote faces 1 and 2 , respectively (Fig. 2.1).

The resultant moments per unit length can be expressed in terms of the uniform stresses across the facings thickness, as

$$M_x = t_1 h \, \sigma_{x1} = -t_2 h \, \sigma_{x2}$$

$$M_y = t_1 h \, \sigma_{y1} = -t_2 h \, \sigma_{y2}$$

$$M_{xy} = -t_1 h \, \tau_{xy1} = t_2 h \, \tau_{xy2}$$

in which $h = t_c + \frac{1}{2}(t_1 + t_2)$, as shown in Fig. 2.2 (a). Shearing forces per unit length can also be expressed in terms of transverse shearing stresses as [4]

$$Q_x = h \, \tau_{xz} = G_{xz} h \gamma_x' = G_{xz} \frac{h^2}{t_c} \gamma_x$$

$$Q_y = h \, \tau_{yz} = G_{yz} h \gamma_y' = G_{yz} \frac{h^2}{t_c} \gamma_y$$

in which

γ_x' , γ_y' = shearing strains

γ_x , γ_y = slope of the shearing deflection surface, as shown in Fig. 2.2 (b).

Substituting the resultant forces-stresses relations in the strain energy equation and simplifying its terms yield

$$U = \frac{1}{2} \int_A \left(\frac{M_x^2}{D'_x} + \frac{M_y^2}{D'_y} - 2 \frac{M_x M_y}{D_\nu} + \frac{M_{xy}^2}{D_{xy}} + \frac{Q_x^2}{S_x} + \frac{Q_y^2}{S_y} \right) dA$$

in which

$$D'_x = h^2 \left(\frac{1}{E_{x1} t_1} + \frac{1}{E_{x2} t_2} \right)^{-1}$$

$$D'_y = h^2 \left(\frac{1}{E_{y1} t_1} + \frac{1}{E_{y2} t_2} \right)^{-1}$$

$$D_{xy} = h^2 \left(\frac{1}{G_{xy1} t_1} + \frac{1}{G_{xy2} t_2} \right)^{-1}$$

$$D_\nu = h^2 \left(\frac{\nu_{xy1}}{E_{x1} t_1} + \frac{\nu_{xy2}}{E_{x2} t_2} \right)^{-1}$$

$$S_x = G_{xz}\, h^2/t_c$$

$$S_y = G_{yz}\, h^2/t_c$$

According to assumption (5) in section 2.2, the sandwich constituents are linearly elastic. Thus from Castigliano's second theorem it follows that

$$\frac{\partial U}{\partial M_x} = - \frac{\partial \theta_x}{\partial x}$$

$$\frac{\partial U}{\partial M_y} = - \frac{\partial \theta_y}{\partial x}$$

in which

$$\theta_x = \frac{\partial w}{\partial x} - \gamma_x$$

$$\theta_y = \frac{\partial w}{\partial y} - \gamma_y$$

hence

$$\frac{M_x}{D'_x} - \frac{M_y}{D_\nu} = -\frac{\partial \theta_x}{\partial x}$$

$$\frac{M_y}{D'_y} - \frac{M_x}{D_\nu} = -\frac{\partial \theta_y}{\partial y}$$

Solving these equations in M_x and M_y yields

$$M_x = -D_x \left(\frac{\partial \theta_x}{\partial x} + \frac{D'_y}{D_\nu}\frac{\partial \theta_y}{\partial y}\right) \tag{2.1}$$

$$M_y = -D_y \left(\frac{\partial \theta_y}{\partial y} + \frac{D'_x}{D_\nu}\frac{\partial \theta_x}{\partial x}\right) \tag{2.2}$$

in which

$$D_x = \left(\frac{1}{D'_x} - \frac{D'_y}{D_\nu^2}\right)^{-1}$$

$$D_y = \left(\frac{1}{D'_y} - \frac{D'_x}{D_\nu^2}\right)^{-1}$$

In addition, the twisting moment can be obtained using the same theory as follows

$$\frac{\partial U}{\partial M_{xy}} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}$$

from which

$$M_{xy} = D_{xy} \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \tag{2.3}$$

Expressions for the shearing forces are given before. The same results can be derived using Castigliano's theorem. In the moment and shear forces expressions, θ_x and θ_y represent the normal slopes of the sandwich plate due to moment effects. D_x and D_y may be regarded as flexural rigidities of the plate, and D_{xy} as its twisting stiffness. In the literature S_x and S_y are known as shear rigidities. In the particular case of isotropic sandwich plates with similar facings, according to assumption (4) and (5) in section 2.2

$$E_f = E_{x1} = E_{x2} = E_{y1} = E_{y2}$$

$$\nu = \nu_{xy1} = \nu_{xy2}$$

$$t_f = t_1 = t_2$$

$$h = t_c + t_f$$

$$G_f = G_{xy1} = G_{xy2} = \frac{E_f}{2(1+\nu)}$$

$$G_c = G_{xz} = G_{yz}$$

Consequently, the stiffness constants become

$$D = D_x = D_y$$

$$= \frac{E_f t_f h^2}{2(1-\nu^2)} \qquad (2.4)$$

$$D_{xy} = \frac{1}{2} D\,(1-\nu) \qquad (2.5)$$

$$S = S_x = S_y$$

$$= G_c \frac{h^2}{t_c} \qquad (2.6)$$

The equilibrium conditions of the sandwich element in Fig. 2.2 (a) are the same as those for homogeneous plates. Consider, for example, the equilibrium of moments in x-direction

$$\frac{\partial M_x}{\partial x}\,dy\,dx - Q_x\,dy\,dx - \frac{\partial M_{xy}}{\partial y}\,dx\,dy = 0$$

from which

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x \tag{2.7}$$

Similarly, the equilibrium equation of the moments in y-direction yield

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \tag{2.8}$$

The equilibrium equation of the vertical forces in z-direction is

$$p\, dx\, dy + \frac{\partial Q_x}{\partial x}\, dy\, dx + \frac{\partial Q_y}{\partial y}\, dx\, dy = 0$$

from which

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -p \tag{2.9}$$

in which

p = the load intensity.

The governing differential equations for the deflection and shearing forces can now be obtained by substituting the moment expressions (2.1) to (2.3) into the equilibrium equations (2.7) to (2.9). For an orthotropic sandwich plate having isotropic facings, these equations are derived by many others [15, 38, 88, 103] as

$$\left(1 - A_y \frac{\partial^2}{\partial x^2} - A_x \frac{\partial^2}{\partial y^2}\right) \nabla\nabla w = \frac{1}{D}\left(1 - D_2 \frac{\partial^2}{\partial x^2} - D_1 \frac{\partial^2}{\partial y^2} + \frac{2A_x A_y}{1-\nu} \nabla\nabla\right) p$$

$$\left(\frac{2A_x}{1-\nu} - \frac{1+\nu}{1-\nu} A_y\right) \frac{\partial^2 Q_x}{\partial x^2} + A_x \frac{\partial^2 Q_x}{\partial y^2} - Q_x =$$

$$D \frac{\partial}{\partial x} \nabla w + \frac{1+\nu}{1-\nu} A_y \frac{\partial p}{\partial x}$$

$$\left(\frac{2A_y}{1-\nu} - \frac{1+\nu}{1-\nu} A_x\right) \frac{\partial^2 Q_y}{\partial y^2} + A_y \frac{\partial^2 Q_y}{\partial x^2} - Q_y =$$

$$D \frac{\partial}{\partial y} \nabla w + \frac{1+\nu}{1-\nu} A_x \frac{\partial p}{\partial y}$$

in which

$$\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$D' = D'_x = D'_y$$

$$= E_f h^2 \left(\frac{1}{t_1} + \frac{1}{t_2}\right)^{-1}$$

$$D_{xy} = \frac{E_f h^2}{2(1+\nu)} \left(\frac{1}{t_1} + \frac{1}{t_2}\right)^{-1}$$

$$D_\nu = \frac{E_f h^2}{\nu} \left(\frac{1}{t_1} + \frac{1}{t_2}\right)^{-1}$$

$$D = D' \left[1 - \left(\frac{D'}{D_\nu}\right)^2\right]^{-1}$$

$$A_x = \frac{D_{xy}}{S_x}$$

$$A_y = \frac{D_{xy}^*}{S_y}$$

$$D_1 = A_x + \frac{2A_y}{1-\nu}$$

$$D_2 = A_y + \frac{2A_x}{1-\nu}$$

For the particular case of an isotropic sandwich panel with similar facings the governing equations become

$$\nabla\nabla w = \frac{1}{D} p - \frac{1}{S} \nabla p \tag{2.10}$$

$$\frac{1-\nu}{2} \nabla\gamma_x - \frac{S}{D} \gamma_x = \frac{\partial}{\partial x} \nabla w + \frac{1+\nu}{2S} \frac{\partial p}{\partial x} \tag{2.11}$$

$$\frac{1-\nu}{2} \nabla\gamma_y - \frac{S}{D} \gamma_y = \frac{\partial}{\partial y} \nabla w + \frac{1+\nu}{2S} \frac{\partial p}{\partial y} \tag{2.12}$$

If a solution of these equations is found that satisfies the boundary conditions of the plate, the moments can be readily obtained from Eqs. (2.1) to (2.3). It is of interest to note that, when $G_c = \infty$, Equ. (2.10) reduces to the governing differential equation $\nabla\nabla w = p/D$ in the theory of homogeneous plates.

2.4 SOLUTIONS FOR THE GOVERNING DIFFERENTIAL EQUATIONS

Mathematically, the governing equations of sandwich plates are classified as linear partial differential equations with constant coefficients. Equation (2.10) are of fourth order. Analytical solutions for these equations are often of Navier's or Levy's types. Navier's method, in terms of double trigonometric series, is adopted throughout this chapter, even though Levy's solution, in the form of a single series, converges faster. Many reasons justify this decision. First, it provides a means to solve Equs. (2.11) and (2.12); second, sufficiently accurate results can still be obtained by using more terms, and finally considerable mathematical simplicity is achieved in the solution of simply supported sandwich plates.

Consider now the simply supported sandwich plate in Fig. 2.1. The plate composed of two faces each of thickness t_f and core of thickness t_c. The general assumptions in section 2.2 are considered. Three boundary conditions at each edge of the plate were admitted by the sixth order governing differential equations for the deflection and are commonly found in practice [4, 15, 38, 84, 88, 103]. These conditions are

$$w = o, \quad M_x = o, \quad \gamma_y = o \quad \text{for} \quad x = o \text{ and } x = a$$

and (2.13)

$$w = o, \quad M_y = o, \quad \gamma_x = o \quad \text{for} \quad y = o \text{ and } y = b$$

in which

a, b = the sandwich plate dimensions parallel to x- and y-directions, respectively.

The deflected surface of the sandwich plate satisfying the condition w = o in Equ. (2.13) is expressed in double trigonometric series as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha_m x \sin \beta_n y \tag{2.14}$$

in which

$$\alpha_m = \frac{m\pi}{a}$$

$$\beta_n = \frac{n\pi}{b}$$

W_{mn} = the coefficient of the double Fourier expansion of the deflection.

The applied load can be similarly expressed in double series as

$$p(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin \alpha_m x \sin \beta_n y \tag{2.15}$$

in which

P_{mn} = same meaning as W_{mn} but for the applied load.

Substituting Equs. (2.14) and (2.15) in the governing equation (2.10), an algebraic equation is obtained, from which the unknown W_{mn} is determined as

$$W_{mn} = P_{mn} \frac{(1/D) + [(\alpha_m^2 + \beta_n^2)/S]}{(\alpha_m^2 + \beta_n^2)^2} \tag{2.16}$$

The expression for the plate deflection is now obtained by summing the individual terms in Equ. (2.14) and by making use of Equ. (2.16).

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \frac{(1/D) + [(\alpha_m^2 + \beta_n^2)/S]}{(\alpha_m^2 + \beta_n^2)^2} \sin \alpha_m x \sin \beta_n y \tag{2.17}$$

This expression reveals that the deflection of a sandwich plate is composed of two components: the bending deflection which occurs in a plate rigid in shear, and the shear deflection which occurs in a plate infinitely stiff in bending. These two components were separately obtained by Plantema [84] based on the partial deflection theory.

Returning to the sandwich plate in Fig. 2.1, expressions for the shear deformations can be obtained now from Equ. (2.14) and the results are in agreement with those given in [28, 78, 87]. Thus,

$$\gamma_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{xmn} \cos \alpha_m x \sin \beta_n y \tag{2.18}$$

$$\gamma_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{ymn} \sin \alpha_m x \cos \beta_n y \tag{2.19}$$

in which

γ_{xmn}, γ_{ymn} = the coefficients of the double Fourier expansion of the shearing deformations.

It should be noted that both Equs. (2.18) and (2.19) satisfy the conditions $\gamma = o$ in Equ. (2.13); and together with Equ. (2.14), they satisfy the conditions M = o along the edges.

Making use of Equs. (2.14), (2.18), and (2.19), the coefficients γ_{xmn} and γ_{ymn} can be readily determined from Equs. (2.11) and (2.12) respectively. The results are

$$\gamma_{xmn} = \frac{m\pi \; P_{mn}}{aS(\alpha_m^2 + \beta_n^2)} \tag{2.20}$$

and

$$\gamma_{ymn} = \frac{n\pi \; P_{mn}}{bS(\alpha_m^2 + \beta_n^2)} \tag{2.21}$$

The final expressions for the shear forces in a simply supported sandwich plate subjected to transverse load are obtained as

$$Q_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\pi \; P_{mn}}{a(\alpha_m^2 + \beta_n^2)} \cos \alpha_m x \sin \beta_n y \tag{2.22}$$

$$Q_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n\pi \; P_{mn}}{b(\alpha_m^2 + \beta_n^2)} \sin \alpha_m x \cos \beta_n y \tag{2.23}$$

2.5 MOMENTS IN SIMPLY SUPPORTED SANDWICH PLATES

Having solved the governing differential equations (2.10) to (2.12), the bending and twisting moments at any point of a simply supported sandwich plate can be obtained now from Equs. (2.1) to (2.3). This is by direct substitution of Equs. (2.17), (2.22), and (2.23) into the moment expressions.

$$M_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \frac{(\alpha_m^2 + \nu\beta_n^2)}{(\alpha_m^2 + \beta_n^2)^2} \sin \alpha_m x \sin \beta_n y \qquad (2.24)$$

$$M_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \frac{(\beta_n^2 + \nu\alpha_m^2)}{(\alpha_m^2 + \beta_n^2)^2} \sin \alpha_m x \sin \beta_n y \qquad (2.25)$$

$$M_{xy} = (1-\nu) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi^2 P_{mn}}{ab} \cdot \frac{mn}{(\alpha_m^2 + \beta_n^2)^2} \cos \alpha_m x \cos \beta_n y \qquad (2.26)$$

2.6 PRACTICAL FORMULAS FOR FORCES AND STRESSES

Existing expressions [4, 84, 110] for evaluating the deflection and stresses in simply supported sandwich plates are neither general nor simple. For example, Allen's formulas [4] involve many complexe calculations, whereas Yen's results [110] are restricted to square sandwich plates. In addition, only uniform loading, in the former study, and concentrated loading, in the latter one, have been considered.

In the following, formulas are developed for simply supported sandwich plates under transverse load of five types: uniformly distributed, hydrostatic pressure, partial load on rectangular area, concentrated load, and strip load uniformly distributed across the plate width. The formulas developed are simpler than previous ones [4, 84, 110]. For this purpose, consider the simply supported sandwich plate in Fig. 2.1. The deflection and bending moments at the plate centre are obtained from Equs. (2.17), (2.24), and (2.25) as follows

$$(w)_{\substack{x=a/2\\y=b/2}} = - \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{P_{mn}}{C} \left[\frac{1}{D} + \frac{1}{S}\left(\alpha_m^2 + \beta_n^2\right)\right] \quad (2.27)$$

$$(M_x)_{\substack{x=a/2\\y=b/2}} = - \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{P_{mn}}{C} \left(\alpha_m^2 + \nu\beta_n^2\right) \quad (2.28)$$

$$(M_y)_{\substack{x=a/2\\y=b/2}} = - \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{P_{mn}}{C} \left(\beta_n^2 + \nu\alpha_m^2\right) \quad (2.29)$$

The twisting moment is maximum at the plate corners and is obtained from Equ. (2.26) as

$$(M_{xy})_{\substack{x=0\\y=0}} = (1-\nu) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi^2}{ab} \frac{P_{mn}}{C} mn \quad (2.30)$$

The concentrated shear force at the plate corners can be obtained from Equ. (2.30) [52,96,101]. The shear forces are calculated at the mid edges from Equs. (2.22) and (2.23) as

$$(Q_x)_{\substack{x=0\\y=b/2}} = -\sum_{m=1}^{\infty}\sum_{n=1,3,\ldots}^{\infty} (-1)^{(n+1)/2} \frac{P_{mn}}{\sqrt{C}} \frac{m\pi}{a} \tag{2.31}$$

$$(Q_y)_{\substack{x=a/2\\y=0}} = -\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1}^{\infty} (-1)^{(m+1)/2} \frac{P_{mn}}{\sqrt{C}} \frac{n\pi}{b} \tag{2.32}$$

in which

$$C = (\alpha_m^2 + \beta_n^2)^2$$

where the coefficient P_{mn} of Fourier expansion of the applied load, may be obtained from the following equation [96, 101]

$$P_{mn} = \frac{4}{ab} \int_o^a \int_o^b p\,(x,y) \sin \alpha_m x \sin \alpha_n y \, dxdy \tag{2.33}$$

in which

$p\,(x,y)$ = function representing the external applied load.

Equations (2.27) to (2.32) furnish the analytical formulas from which the results corresponding to individual loading are derived. Detailed derivation is presented for the case of uniform loading, and only the final results are given for the other four cases.

2.6.1 UNIFORM DISTRIBUTED LOAD

The applied load in this case is uniformly distributed of intensity p_o (Fig. 2.3 (a)) :

$$p\ (x,y) \quad = \quad p_o \tag{2.34}$$

Substituting of Equ. (2.34) into Equ. (2.33) and performing the double integration yields :

$$p_{mn} = \begin{cases} \dfrac{16\ p_o}{\pi^2 mn} & \text{for odd m and n} \\ 0 & \text{otherwise} \end{cases} \tag{2.35}$$

The central deflection is obtained now by substituting Equ. (2.35) into Equ. (2.27)

$$(w)_{\substack{x=a/2 \\ y=b/2}} = - \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{16\ p_o}{\pi^2 Cmn} \left(\frac{1}{D} + \frac{\alpha_m^2 + \beta_n^2}{S}\right) \tag{2.36}$$

In a more compact form the central deflection equation can be written as

$$(w)_{\substack{x=a/2 \\ y=b/2}} = \frac{p_o a^4}{D}\ K_{wb} + \frac{p_o a^2}{S}\ K_{ws} \tag{2.37}$$

in which

$$\left.\begin{aligned} K_{wb} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{-16\,(-1)^{(m+n)/2}}{\pi^6 mnC'^2} \\ K_{ws} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{-16\,(-1)^{(m+n)/2}}{\pi^4 mnC'} \end{aligned}\right\} \qquad (2.38)$$

$$C' = m^2 + R^2n^2$$

$$R = \frac{a}{b}$$

It can be seen from this equation that two terms are to be added: the bending and shear components of the deflection, and each involves two factors. The first contains parameters reflecting the material properties and panel geometry and the second the aspect ratio of the plate. To facilitate the use of Equ. (2.37), numerical results are calculated for K_{wb} and K_{ws} and presented in the second and third columns of Table A.1. A range of one to five for the plate aspect ratio was considered.

Concerning the bending and twisting moments produced in a simply supported sandwich plate due to a uniformly distributed load, a similar procedure as before is used. This is by substituting Equ. (2.35) in Equs. (2.28), (2.29) and (2.30) from which it follows that

$$(M_x)_{\substack{x=a/2\\y=b/2}} = p_o a^2 K_{mx} \qquad (2.39)$$

$$(M_y)_{\substack{x=a/2\\y=b/2}} = p_o a^2 K_{my} \tag{.240}$$

$$(M_{xy})_{\substack{x=0\\y=0}} = - p_o a^2 R\, K_{mxy} \tag{2.41}$$

in which

$$\left.\begin{aligned}
K_{mx} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16}{\pi^4 mnC'^2} (m^2 + \nu R^2 n^2) \\
K_{my} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16}{\pi^4 mnC'^2} (R^2 n^2 + \nu m^2) \\
K_{mxy} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16(\nu-1)}{\pi^4 C'^2}
\end{aligned}\right\} \tag{2.42}$$

Detailed investigations of the convergence of the series in Equ. (2.42) are reported in Ref. [4, 52, 96, 101]. Values for the factors K_{mx}, K_{my} and K_{mxy} are obtained by taking the sums of the series in Equ. (2.42) up to and including the term m=n=100. The results are tabulated in Table A.1.

Stresses are sometimes of interest in many problems. The inplane stresses in the faces may be obtained from the results (2.39), (2.40) and (2.41). For example, the normal stress σ_x and σ_y are calculated by

$$\sigma_x = \frac{M_x}{ht_f}$$

$$= \frac{p_o a^2}{ht_f} K_{mx} \tag{2.43}$$

and

$$\sigma_y = \frac{M_y}{ht_f}$$

$$= \frac{p_o a^2}{ht_f} K_{my} \tag{2.44}$$

The shear stress τ_{xy} is obtained from Equ. (2.41) as

$$\tau_{xy} = \frac{M_{xy}}{ht_f}$$

$$= -\frac{p_o a^2 R}{ht_f} K_{mxy} \tag{2.45}$$

Equs. (2.43), (2.44), and (2.45) represent the peak in plane stresses in the faces of a simply supported sandwich plate subjected to a uniformly distributed load.

It is of interest to observe that the expressions for the stresses do not include G_c, nor any other term which refers to the shear stiffness of the plate. Indeed, not only are the stresses in the faces independent of the shear stiffness, but also the results in Equs. (2.43), (2.44), and (2.45) are identical to those determined by the classical theory of homogeneous plates [96,101]. This is true due to the manner in which a simply supported sandwich plate deforms. Under the applied transverse load, the thin faces of a simply supported sandwich plate undergo uniform extension or contractions as they bend about the middle plane of the whole sandwich plate. In this way, the inplane stresses are uniformly distributed across the thicknesses of the faces. In addition, the sandwich plate undergoes shear strains which correspond to an additional transverse deflection. The faces accomodate this extra deflection by bending about their own middle planes, as well as by displacing vertically only. This implies that, while shear deformations are taking place, the thin faces of a simply supported sandwich plate do not undergo stretching or contraction, thus they retain their bending stresses unaltered.

Returning to the plate in Fig. 2.3 (a), the shear forces in the simply supported sandwich plate under uniformly distributed load are obtained from Equs. (2.31) and (2.32) as

$$(Q_x)_{\substack{x=0\\ y=b/2}} = p_o a\, K_{Qx} \tag{2.46}$$

$$(Q_y)_{\substack{x=a/2\\ y=0}} = p_o a\, R\, K_{Qy} \tag{2.47}$$

in which

$$\left.\begin{aligned} K_{Qx} &= \sum_{m=1,3,\ldots}^{\infty} \; \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n+1)/2} \frac{-16}{\pi^3 n\, C'} \\ K_{Qy} &= \sum_{m=1,3,\ldots}^{\infty} \; \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+1)/2} \frac{-16}{\pi^3 m\, C'} \end{aligned}\right\} \tag{2.48}$$

Numerical values for the coefficients K_{Qx} and K_{Qy} are evaluated and shown in Table A.1 The shear stresses can be obtained from

$$\tau_{xz} = \frac{Q_x}{h}$$

$$= \frac{p_o a}{h} K_{Qx} \tag{2.49}$$

$$\tau_{yz} = \frac{p_o a R}{h} K_{Qy} \tag{2.50}$$

While the transverse shear stress τ_{xz} is greatest at the middle of the sides of length b (e.g. at x=0,a and $y=\frac{b}{2}$), the transverse shear stress τ_{yz} is greatest at the middle of the sides of length a (e.g. $x=\frac{a}{2}$, y=0 and b).

In summary, a uniformly distributed load can be expressed mathematically by Equ. (2.34) which is together with Equ. (2.33) leads to an expression for the coefficient of Fourier expansion of the load. Having determined this coefficient, the deflection, shears, moments, and stresses produced due to the applied load are calculated readily from Equs. (2.27) to (2.32). The formulas developed are simplified and numerical values are also tabulated for a wide range of parameters.

2.6.2 HYDROSTATIC PRESSURE

The hydrostatic pressure shown in Fig. 2.3 (b) can be represented by the equation

$$p\,(x,y) = \frac{P_o\, x}{a} \tag{2.51}$$

in which

p_o = the load intensity at $x = a$.

In this case, the coefficient of Fourier expansion for the load is obtained as

$$P_{mn} = \frac{8\,(-1)^{m+1}\, p_o}{\pi^2 mn} \tag{2.52}$$

where $m=1,2,3,\ldots$ and $n=1,3,5,\ldots$

Similar to the case of uniform distributed load, the expressions for the deflection, bending and twisting moments, and shears produced in a simply supported sandwich plate due to the hydrostatic pressure, can also be represented by Equs. (2.37), (2.39), (2.40), (2.41), (2.46), and (2.47). However, the factors in these equations are given by

$$\left.\begin{aligned}
K_{wb} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(3m+n)/2} \frac{8}{\pi^6 mnC'^2} \\
K_{ws} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(3m+n)/2} \frac{8}{\pi^4 mnC'} \\
K_{mx} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(3m+n)/2} \frac{8}{\pi^4 mnC'^2} (m^2 + \nu R^2 n^2)
\end{aligned}\right\} \tag{2.53}$$

$$K_{my} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(3m+n)/2} \frac{8}{\pi^4 mnC'^2} (R^2n^2 + \nu m^2)$$

$$K_{mxy} = \sum_{m=1}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^m \frac{8\,(1-\nu)}{\pi^4\, C'^2}$$

$$K_{Qx} = \sum_{m=1}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(2m+n+1)/2} \frac{8}{\pi^3\, n\, C'}$$

$$K_{Qy} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(3m+1)/2} \frac{8}{\pi^3\, m\, C'}$$

Numerical values for these factors are tabulated in Table A.2.

2.6.3 PARTIAL LOAD ON A RECTANGULAR AREA

In the case of partial load on a rectangular area, as shown in Fig. 2.3 (c), the loading function is

$$p\,(x,y) = \begin{cases} p_o & \text{for } x = (\xi - \frac{c}{2}) \text{ to } x = (\xi + \frac{c}{2}) \\ & \text{and } y = (\eta - \frac{d}{2}) \text{ to } y = (\eta + \frac{d}{2}) \\ 0 & \text{otherwise} \end{cases} \qquad (2.54)$$

in which

p_o = the load intensity

$c \times d$ = the loaded area

ξ, η = x- and y- coordinates of the loaded area's center, respectively.

The coefficient of Fourier expansion of the partial load is obtained from Equ. (2.33) as

$$P_{mn} = \frac{16\, p_o}{\pi^2\, mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi c}{2a} \sin\frac{n\pi d}{2b} \qquad (2.55)$$

for all integers of m and n.

The factors in Equs. (2.37), (2.39), (2.40), (2.41), (2.46), and (2.47) for the present loading case are given by

$$\left.\begin{aligned}
K_{wb} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16\, C''}{\pi^6 mnC'^2} \\
K_{ws} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16\, C''}{\pi^4 mnC'} \\
K_{mx} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16\, C''}{\pi^4 mnC'^2} (\\
&\qquad m^2 + \nu R^2 n^2) \\
K_{my} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-16\, C''}{\pi^4 mnC'^2} (\\
&\qquad R^2 n^2 + \nu m^2)
\end{aligned}\right\} \qquad (2.56)$$

$$K_{mxy} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16(\nu-1)C''}{\pi^4 C'^2}$$

$$K_{Qx} = \sum_{m=1}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n+1)/2} \frac{-16\, C''}{\pi^3\, n\, C'}$$

$$K_{Qy} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+1)/2} \frac{-16\, C''}{\pi^3\, m\, C'}$$

in which

$$C'' = \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi c}{2a} \sin\frac{n\pi d}{2b}$$

In addition to the aspect ratio, the position of the load on the plate is another parameter in Equ. (2.56); but due to symmetry only one quarter of the plate surface is considered. This quarter is divided into 3 x 3 rectangular mesh, each of size $\frac{a}{6} \times \frac{b}{6}$; and the loaded area's center is fixed at the mesh points. The ratio of loaded area to plate surface is chosen as follows

$$\frac{c}{a} = \frac{d}{b} = \frac{1}{4}$$

hence,

$$\frac{c \times d}{a \times b} = \frac{1}{16}$$

Based on this choice, numerical values for the factors in Equ. (2.56) are evaluated for each position of the partial load and the results are presented in Tables A.3 to A.11. Since the tabulated values represent the plate response due to moving load, the tables are titled "Influence Coefficients for Simply Supported Sandwich Plate Under Partial Load".

2.6.4 CONCENTRATED LOAD

Concentrated load is a particular case of the partial load in which the loaded area is diminishingly small. Thus, the coefficient of Fourier's expansion for the concentrated load is obtained from Equ. (2.55) by substituting $p_o = P/cd$, where P is the concentrated load shown in Fig. 2.3 (d). Letting the loaded area approach zero by permitting $c \to o$ and $d \to o$, yields

$$P_{mn} = \frac{4P}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \tag{2.57}$$

for all integers of m and n.

The procedure to follow in developing the practical formulas for a simply supported sandwich plate subjected to a concentrated load is not different from that in the previous section; however, the formulas obtained are given by

$$(w)_{\substack{x=a/2\\y=b/2}} = \frac{P\,a^2\,R}{D} K_{wb} + \frac{P\,R}{S} K_{ws} \tag{2.58}$$

$$(M_x)_{\substack{x=a/2\\y=b/2}} = P\,R\,K_{mx} \tag{2.59}$$

$$(M_y)_{\substack{x=a/2\\y=b/2}} = P\,R\,K_{my} \tag{2.60}$$

$$(M_{xy})_{\substack{x=0\\y=0}} = -P\,R^2\,K_{mxy} \tag{2.61}$$

$$(Q_x)_{\substack{x=a/2\\y=0}} = \frac{P\,R}{a}\,K_{Qx} \tag{2.62}$$

$$(Q_y)_{\substack{x=0\\y=b/2}} = \frac{P\,R^2}{a}\,K_{Qy} \tag{2.63}$$

in which

$$\left.\begin{aligned}
K_{wb} &= \sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2}\,\frac{-4\,C'''}{\pi^4\,C'^2}\\
K_{ws} &= \sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2}\,\frac{-4\,C'''}{\pi^2\,C'}\\
K_{mx} &= \sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2}\,\frac{-4\,C'''}{\pi^2\,C'^2}\,(m^2+\nu R^2 n^2)\\
K_{my} &= \sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2}\,\frac{-4\,C'''}{\pi^2\,C'^2}\,(R^2 n^2+\nu m^2)
\end{aligned}\right\} \tag{2.64}$$

$$K_{mxy} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(\nu-1)mnC'''}{\pi^2 \; C'^2}$$

$$K_{Qx} = \sum_{m=1}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n+1)/2} \; \frac{-4m \; C'''}{\pi \quad C'}$$

$$K_{Qy} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+1)/2} \; \frac{-4n \; C'''}{\pi \quad C'}$$

where

$$C''' = \sin \frac{m\pi\xi}{a} \; \sin \frac{n\pi\eta}{b}$$

ξ, η = x- and y- coordinates of the concentrated load's position on the plate respectively, as shown in Figl 2.3 (d).

Numerical values for the factors in Equ. (2.64) are given in Tables A.12 to A.20.

2.6.5 LINE LOAD

Line load is another particular case of the partial load which was considered in section 2.6.3. As shown in Fig. 2.3 (e), the load is uniformly distributed across the plate width with a constant intensity. In this case, the coefficient of Fourier expansion of the load is obtained by substituting

$$\eta = \frac{b}{2}$$

and

$$c = b$$

into Equ. (2.55) and then by permitting $c \rightarrow o$, it follows that

$$P_{mn} = \frac{8\,p_o}{\pi a n} \sin \frac{m\pi\xi}{a} \qquad (2.65)$$

in which

p_o = the load intensity per unit length of the side b

ξ = the x- coordinate of the line load (Fig. 2.3 (e)).

The formulas for the deflection, moments, and shears of a simply supported sandwich plate subjected to the line load are obtained as

$$(w)_{\substack{x=a/2\\y=b/2}} = \frac{2\,p_o a^3}{D} K_{wb} + \frac{2\,p_o a}{S} K_{ws} \qquad (2.66)$$

$$(M_x)_{\substack{x=a/2\\y=b/2}} = 2\,p_o a\,K_{mx} \qquad (2.67)$$

$$(M_y)_{\substack{x=a/2\\y=b/2}} = 2\,p_o a\,K_{my} \qquad (2.68)$$

$$(M_{xy})_{\substack{x=0\\y=0}} = -2\,p_o R\,a\,K_{mxy} \qquad (2.69)$$

$$(Q_x)_{\substack{x=a/2\\ y=0}} = 2\ p_o\ K_{Qx} \tag{2.70}$$

$$(Q_y)_{\substack{x=0\\ y=b/2}} = 2\ p_o R\ K_{Qy} \tag{2.71}$$

in which

$$\left.\begin{aligned}
K_{wb} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-4}{\pi^5 n C'^2} \sin \frac{m\pi\xi}{a} \\
K_{ws} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-4}{\pi^3 n C'} \sin \frac{m\pi\xi}{a} \\
K_{mx} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-4}{\pi^3 n C'^2} (\\
&\qquad m^2 + \nu R^2 n^2) \sin \frac{m\pi\xi}{a} \\
K_{my} &= \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(m+n)/2} \frac{-4}{\pi^3 n C'^2} (\\
&\qquad R^2 n^2 + \nu m^2) \sin \frac{m\pi\xi}{a} \\
K_{mxy} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(\nu-1)m}{\pi^3 C'^2} \sin \frac{m\pi\xi}{a} \\
K_{Qx} &= \sum_{m=1}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n+1)/2} \frac{-4m}{\pi^2 n C'} \sin \frac{m\pi\xi}{a}
\end{aligned}\right\} \tag{2.72}$$

$$K_{Qy} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+1)/2} \frac{-4}{\pi^2 C'} \sin \frac{m\pi\xi}{a}$$

These factors are to be evaluated for different loading positions, however, due to the symmetry about the central axis $x = \frac{a}{2}$, only one half of the plate surface is considered. This half is divided into five strips each of a width equal to one-tenth of the side length a. The line load is applied on the lines where each two adjacent strips are intersected, and the corresponding numerical values for the factors in Equ. (2.72) are given Tables A.21 to A.25.

In summary, simply supported sandwich plates subjected to various types of lateral loads are solved by applying Navier's method. To facilitate the use of the solutions developed and to reduce the amount of work involved, practical formulas are obtained from which the deflection, moments, and shears can be calculated. Two factors are involved in the formulas. The first contains parameters reflecting the material properties and the panel geometry and the second the aspect ratio and the load position. Numerical values for the latter factor can be obtained from Tables A.1 to A.25.

2.7 SIMPLY SUPPORTED SANDWICH PLATES BY THE FINITE DIFFERENCE METHOD

Analytic solutions to the governing differential equations of many plate problems cannot be easily found [96]. However, for most practical purposes, acceptable results can be obtained by numerical treatment of the governing differential equations. Among the numerical techniques presently available, the finite difference method is one of the most general. In applying this method, the derivatives in the differential equation under consideration are replaced, in general, by difference quantities at some located points that form a reference network called finite difference mesh, and consequently, the governing differential equations are transformed into a set of algebraic simultaneous equations. Solving these equations yields approximate values for the parameter being described by the governing equation under consideration, for instance the plate deflection.

Although in the field of structural mechanics, there are many other numerical techniques for solving plate problems, the finite difference method was chosen because of several advantages. First, it is suitable for programmable desk calculators and even for hand calculations. Second, it provides simple and reusable formulas that are known as finite difference stencils. Third, the accuracy of the results is acceptable for most practical purposes. Finally, it is a versatile technique. On the other hand, the finite difference method is characterized, unfortunately, by slow convergence to exact solutions since the accuracy deteriorates progressively when the order of derivatives is increased. However, to overcome this drawback in the present application, improved finite difference methods were used. Specifically, two methods are adopted here [96], the higher approximation method, and the funicular polygon method. The stencil of the former method is shown in Fig. 2.4 whereas for the latter method, it is shown in Fig. 2.5.

The finite difference method is used now in conjunction with the partial deflection theory to determine the deflection of a simply supported square sandwich plate subjected to three loading types: a uniformly distributed load, a hydrostatic pressure, and a central partial load. The governing differential equations for the bending and shear deflections are obtained from Equ. (2.10) as

$$\nabla \nabla w_b = \frac{p}{D} \tag{2.73}$$

$$\nabla w_s = -\frac{p}{S} \tag{2.74}$$

in which

$$\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

w_b, w_s = the bending and shear deflections, respectively

p = the intensity of the applied load on the sandwich plate.

The bending deflection w_b is obtained from Equ. (2.73) by applying the funicular polygon method stencil in Fig. 2.5 to the finite difference mesh points. The resulting equations are written in matrix form as

$$[A] \{w_b\} = \{P_b\} \tag{2.75}$$

in which

[A] = a matrix containing the coefficients of the bending deflection ordinates

$\{P_b\}$ = the joint load vector

$\{w_b\}$ = a vector containing the bending deflection ordinates at the mesh points

from Equ. (2.75) it follows that

$$\{w_b\} = [A]^{-1} \{P_b\} \qquad (2.76)$$

In a similar manner, the shear deflection w_s is obtained from Equ. (2.74) by applying the higher order stencil in Fig. 2.4 to the finite difference mesh points, and in matrix form the result is

$$[B] \{w_s\} = \{P_s\} \qquad (2.77)$$

in which

$[B]$, $\{P_s\}$, $\{w_s\}$ = same meaning as $[A]$, $\{P_b\}$, and $\{w_b\}$ respectively, but for the shear deflection

from which

$$\{w_s\} = [B]^{-1} \{P_s\} \qquad (2.78)$$

Consider now, the simply supported square sandwich plates shown in Figs. 2.6, 2.7, and 2.8 which show the finite difference meshes and the boundary conditions. The coefficient matrices, the load vectors, and the deflections obtained are given in the next sections for each of the loading types.

2.7.1 UNIFORMLY LOADED SQUARE SANDWICH PLATE, Fig. 2.6

The coefficient matrices [A], [B] and the load vectors $\{P_b\}$, $\{P_s\}$, in this case, are presented in Appendix B. From Equs. (2.76) and (2.78) the deflection coordinates are obtained as

$$\begin{Bmatrix} w_{b1} \\ w_{b2} \\ w_{b3} \\ w_{b4} \end{Bmatrix} = \frac{p_o\ a^4}{36864.\ D} \begin{Bmatrix} 70.43 \\ 97.75 \\ 97.75 \\ 135.90 \end{Bmatrix}$$

$$\begin{Bmatrix} w_{s1} \\ w_{s2} \\ w_{s3} \\ w_{s4} \end{Bmatrix} = \frac{3\ p_o\ a^4}{4\ S} \begin{Bmatrix} 0.054 \\ 0.070 \\ 0.070 \\ 0.091 \end{Bmatrix}$$

2.7.2 SANDWICH PLATE UNDER HYDROSTATIC PRESSURE, Fig. 2.7

Similar to the case of hydrostatic pressure, the finite difference mesh for the present loading type is symmetrical in one direction as shown in Fig. 2.7. The coefficient matrices and load vectors are presented in Appendix B, and the deflection ordinates obtained as

$$\begin{Bmatrix} w_{b1} \\ w_{b2} \\ w_{b3} \\ w_{b4} \\ w_{b5} \\ w_{b6} \end{Bmatrix} = \frac{p_o \ a^4}{147456.\ D} \begin{Bmatrix} 128.4 \\ 197.7 \\ 156.2 \\ 179.2 \\ 275.9 \\ 216.3 \end{Bmatrix}$$

$$\begin{Bmatrix} w_{s1} \\ w_{s2} \\ w_{s3} \\ w_{s4} \\ w_{s5} \\ w_{s6} \end{Bmatrix} = \frac{3\ p_o \ a^2}{4\ S} \begin{Bmatrix} 0.020 \\ 0.035 \\ 0.033 \\ 0.027 \\ 0.045 \\ 0.043 \end{Bmatrix}$$

2.7.3 PARTIALLY LOADED SANDWICH PLATE, Fig. 2.8

A simply supported square sandwich plate subjected to a partial load applied at its center is shown in Fig. 2.8. In the same figure, the finite difference mesh and the boundary conditions are shown. The coefficient matrices and load vectors are presented in Appendix B, and the deflection ordinates obtained as

$$\{w_b\} = \frac{p_o\ a^2}{4096.\ D} \begin{Bmatrix} 0.399 \\ 0.756 \\ 1.016 \\ 1.112 \\ 1.443 \\ 1.957 \\ 2.148 \\ 2.695 \\ 2.975 \\ 3.302 \end{Bmatrix}$$

$$\{w_s'\} = \frac{3\ p_o\ a^2}{16\ S} \begin{Bmatrix} 0.007 \\ 0.014 \\ 0.020 \\ 0.022 \\ 0.029 \\ 0.044 \\ 0.050 \\ 0.074 \\ 0.086 \\ 0.105 \end{Bmatrix}$$

2.8 COMPARISON OF THE ANALYTIC AND FINITE DIFFERENCE SOLUTIONS

The results obtained by the analytical expressions presented and by the finite difference method are shown in Table 2.1. It is seen in this table that the numerical and analytical results are in reasonable agreement. Better agreement can be obtained by using finer finite difference meshes.

Effort has been paid to the verification of the correctness of the formulas presented. Agreement between the present theory and the results by Timoshenko [101] has been verified for the special case of infinite shear stiffness (Table 2.1).

To check the convergence of the finite difference method, a 5 x 5 finite difference mesh is used instead of the one shown in Fig. 2.6. By applying the funicular polygon method, the maximum bending deflection at the center of the plate is found as

$$\text{max. } w_b = 0.0039 \frac{p_o a^4}{D} \quad \text{(Error in max. deflection = 5\%)}$$

From the result obtained by using the 3 x 3 finite difference mesh in Fig. 2.6, the maximum deflection is

$$\text{max. } w_b = 0.0037 \frac{p_o a^4}{D} \quad \text{(Error in max. deflection = 10\%)}$$

Since the analytic solution is

$$\text{max. } w_b = 0.0041 \frac{p_o a^4}{D}$$

it is seen that the finite difference results converge to the analytic solution as the number of mesh points increases.

TABLE 2.1 - COMPARISON OF THE ANALYTIC AND THE FINITE DIFFERENCE SOLUTIONS

LOAD CASE	PRESENT SOLUTION		FINITE DIFFERENCE		TIMOSHENKO [84]
	K_{wb}*	K_{ws}**	K_{wb}	K_{ws}	K_{wb}
UNIFORM	.0041	.0737	.0037	.0682	.0041
HYDROSTATIC	.0020	.0368	.0019	.0341	.0020
PARTIAL	.0007	.0182	.0008	.0197	—

* $w_b = K_{wb} \dfrac{p_o a^4}{D}$

** $w_s = K_{ws} \dfrac{p_o a^2}{D}$

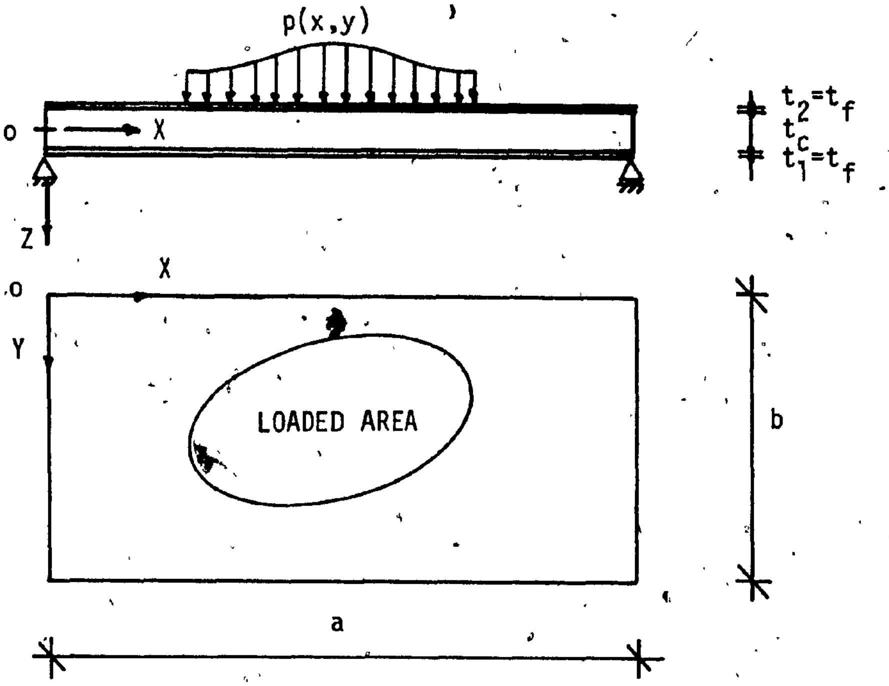
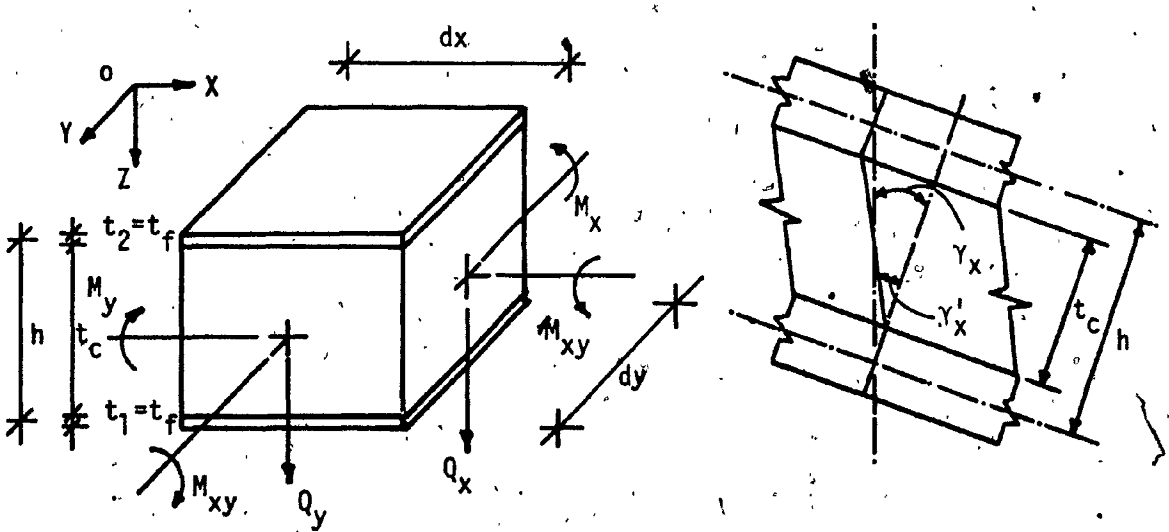


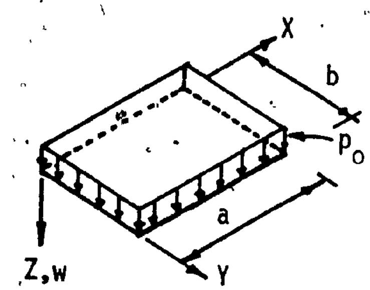
Fig. 2.1 - SANDWICH PLATE



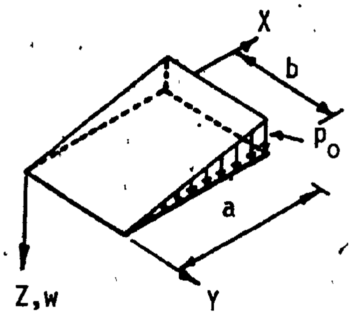
(a) SANDWICH ELEMENT

(b) DEFORMATION IN ZX PLANE

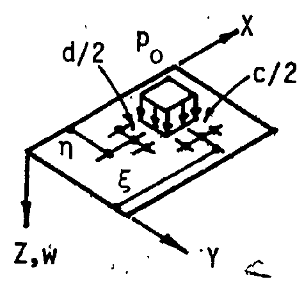
Fig. 2.2 - SIGN CONVENTIONS AND SHEAR DEFORMATION OF SANDWICH ELEMENT



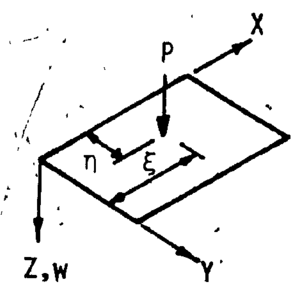
(a) UNIFORM LOAD



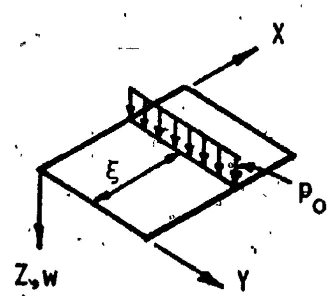
(b) HYDROSTATIC LOAD



(c) PARTIAL LOAD



(d) CONCENTRATED LOAD



(e) LINE LOAD

Fig. 2.3 - LOAD TYPES ON SIMPLY SUPPORTED SANDWICH PLATE

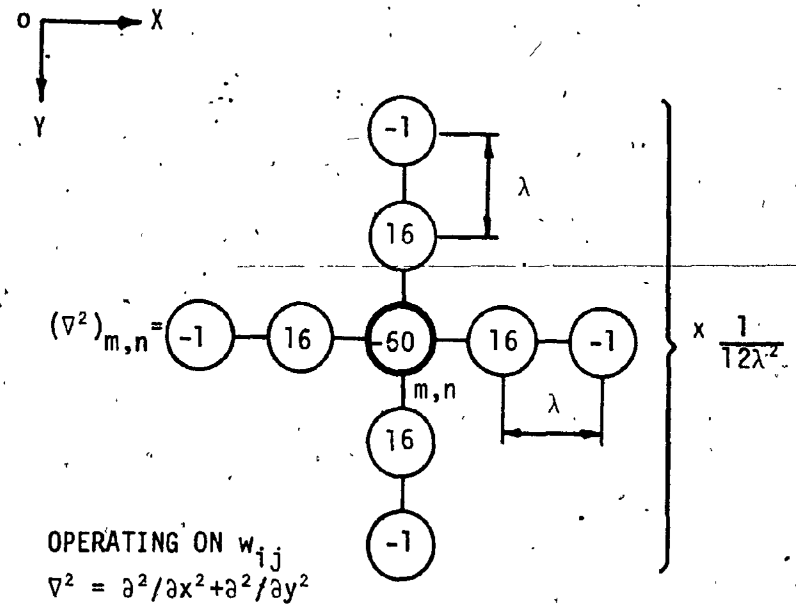
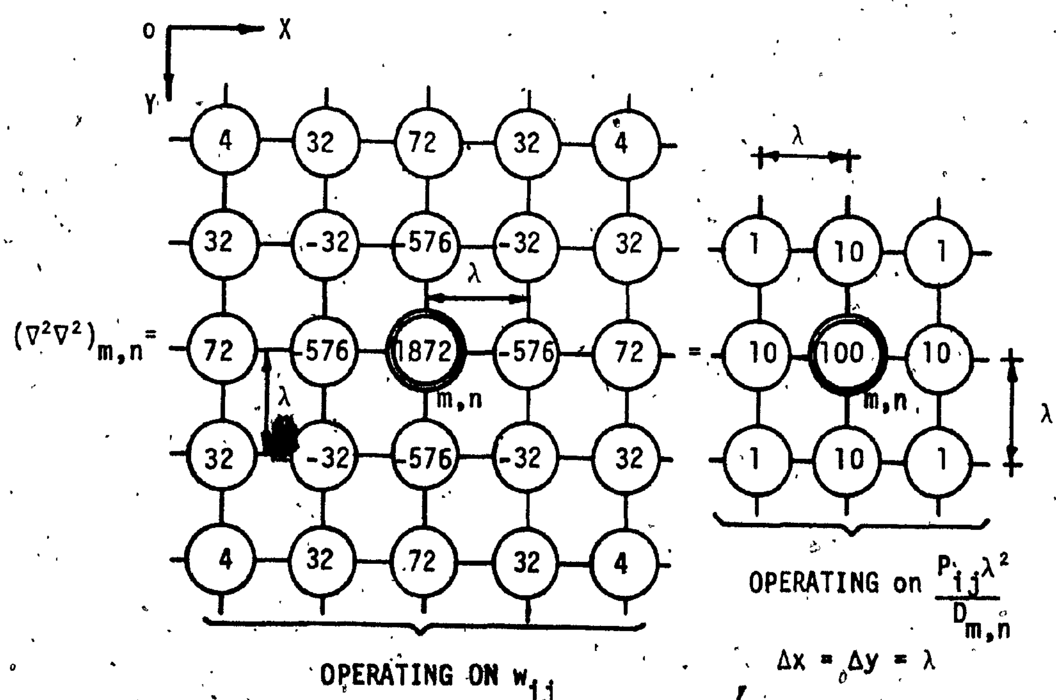
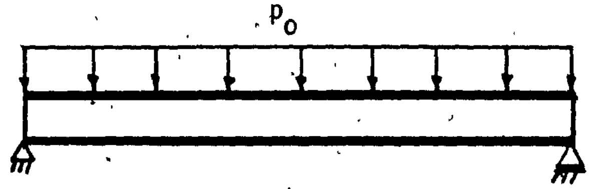


Fig. 2.4 - STENCIL FOR HIGHER APPROXIMATION METHOD [96]

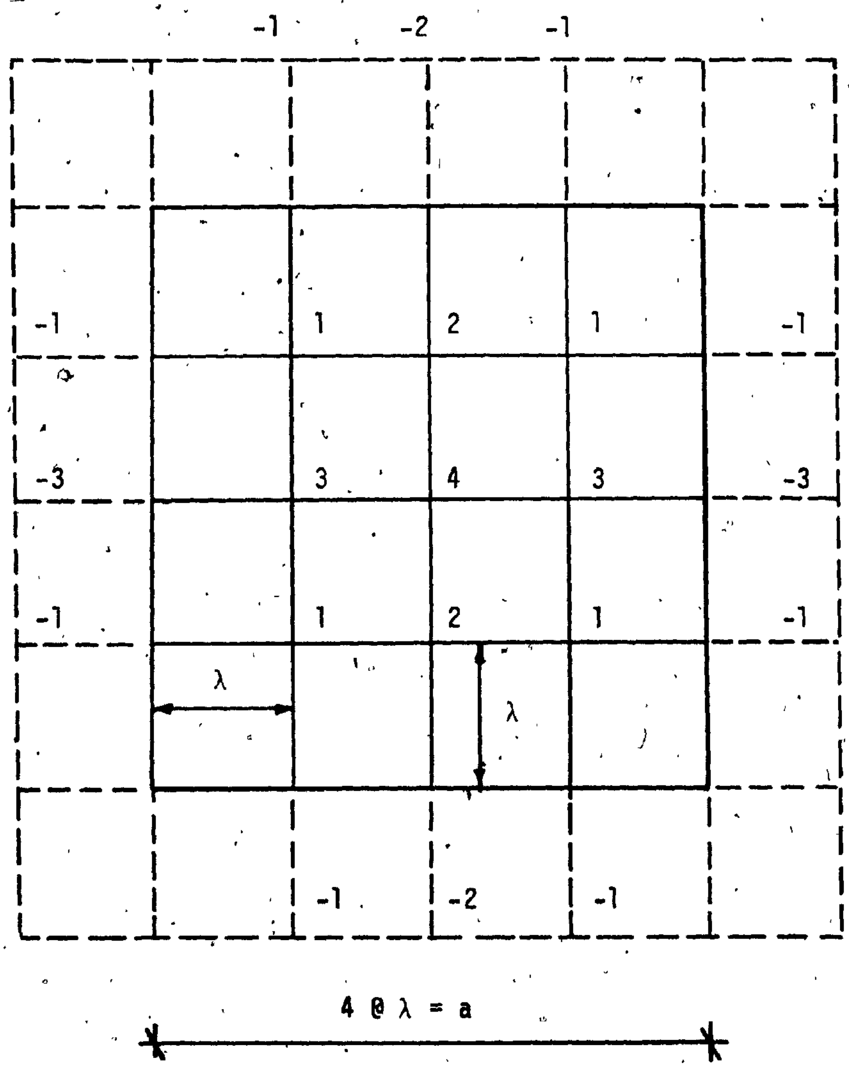


$$P_{m,n} = \frac{(\Delta x)(\Delta y)}{144}\left\{\begin{array}{l} P_{m-1,n-1} + 10P_{m,n-1} + P_{m+1,n-1} \\ +10P_{m-1,n} + 100P_{m,n} + 10P_{m+1,n} \\ +P_{m-1,n+1} + 10P_{m,n+1} + P_{m+1,n+1} \end{array}\right\}$$

Fig. 2.5 - STENCIL FOR FUNICULAR POLYGON METHOD [96]



(a) LATERAL LOAD



(b) NUMBERING OF MESH POINTS

Fig. 2.6 - UNIFORMLY LOADED SQUARE SANDWICH PLATE

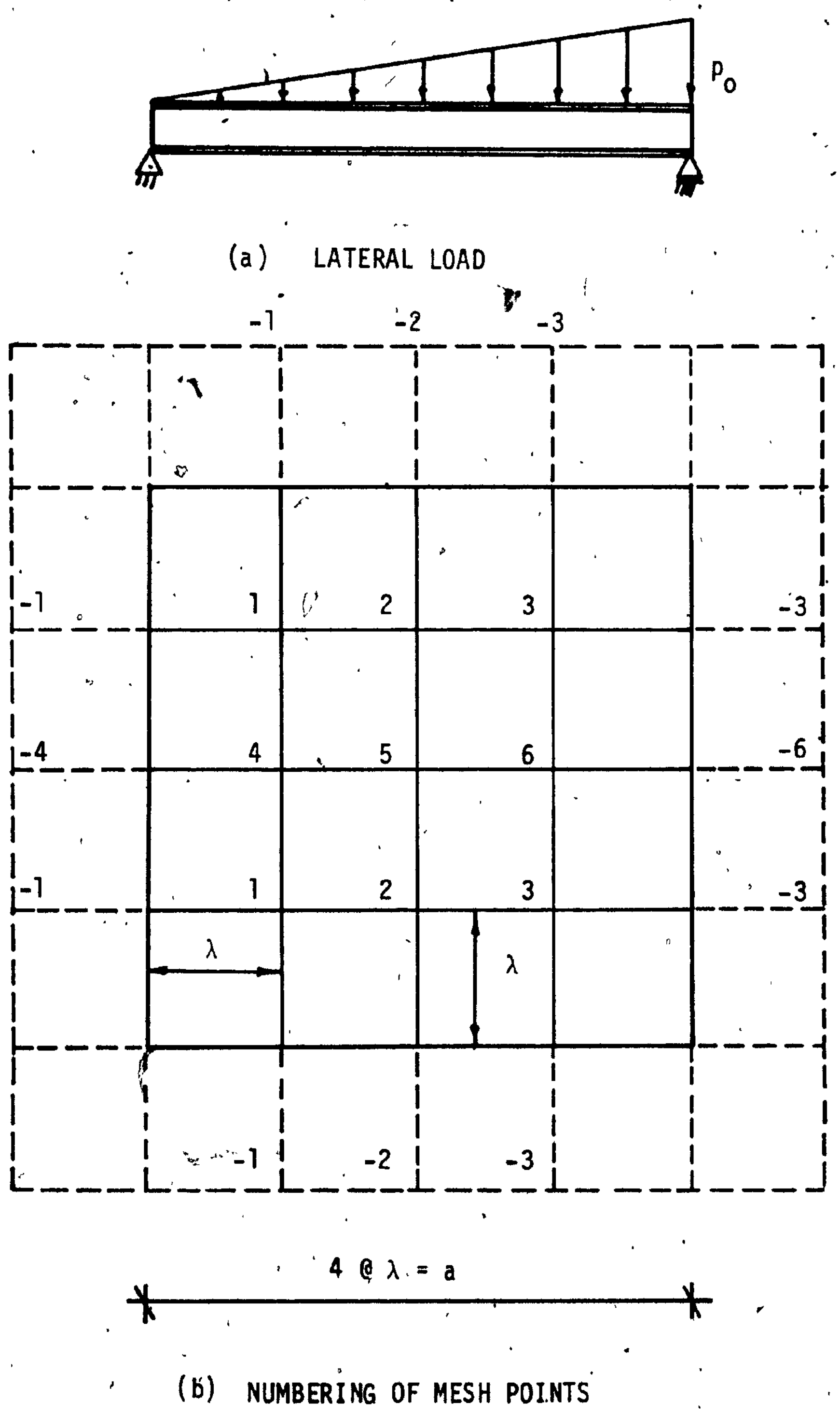
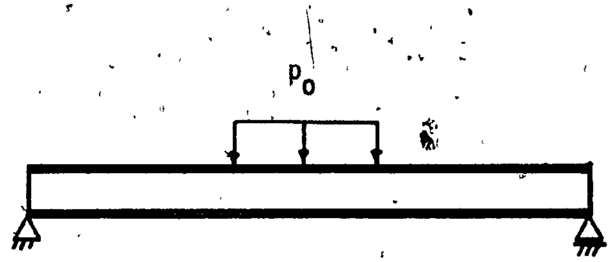
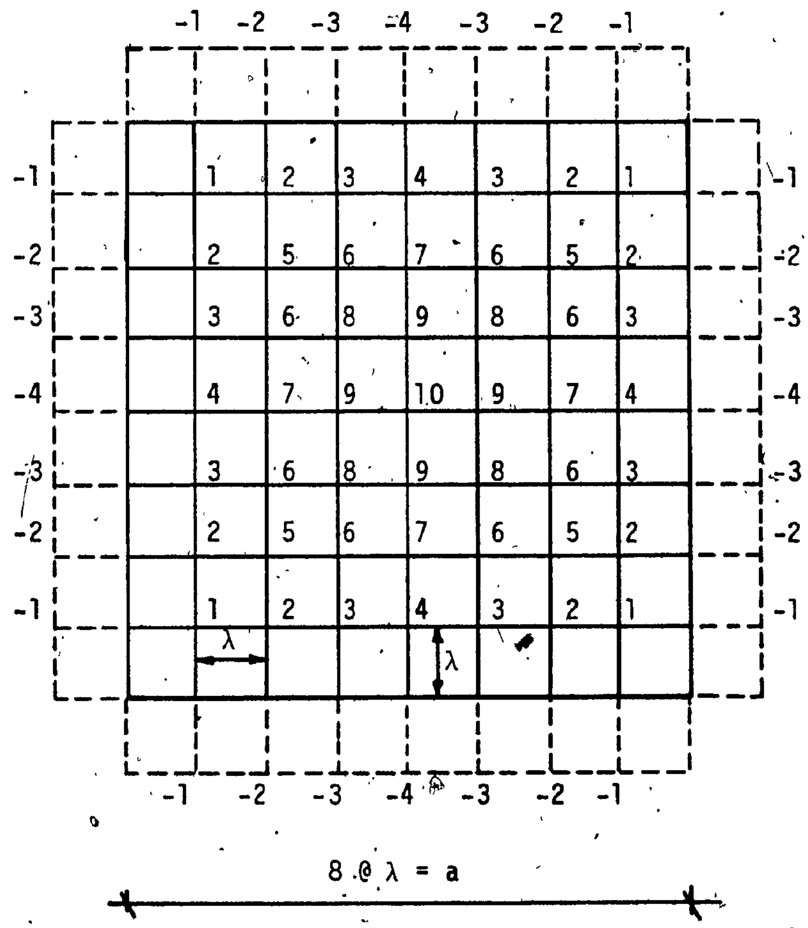


Fig. 2.7 - SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO HYDROSTATIC PRESSURE



(a) LATERAL LOAD



(b) NUMBERING OF MESH POINTS

Fig. 2.8 - PARTIALLY LOADED SANDWICH PLATE

CHAPTER III

CONTINUOUS SANDWICH PLATES

CHAPTER III

CONTINUOUS SANDWICH PLATES

3.1 INTRODUCTION

The problem of continuous sandwich plates is of considerable practical interest, since a floor or roof is usually subdivided by its supports into several panels not all of them are simply supported. In this chapter, analytic solutions for the deflection, shears, and moments in continuous sandwich plates subjected to lateral loads are developed. The plate considered is continuous over one intermediate support, and its panels have the same width while their spans are not necessarily equal or subjected to the same loading type.

This structure is externally as well as internally statically indeterminate, and the available methods of analysis fall into two distinct categories: the force and deformation methods [96]. In either approach, the plate is divided into simple-span panels and the analysis is based on (i) the equilibrium conditions of the individual spans and (ii) the compatibility of displacements or forces at the commond support. In accordance with the force method, which is adopted in the present analysis, the redundant distributed moment is first eliminated by introducing a fictitious hinge at the intermediate support as shown in Fig. 3.1 (b). A means to determine this moment is provided by the compatibility condition which ensures the continuity of deformations of the adjacent panels. Once the redundant moment has been found, the deflection, shears and moments at any point on either panel can be determined readily by combining the solutions of laterally loaded, simply supported sandwich plates established in the previous chapter with the effect of the edge moment as developed here. This general procedure is illustrated by the detailed development of the analytical solutions for two particular loading types: uniformly distributed and centrally concentrated loads acting on either panel. Correctness of the derived formulas is verified by comparison with the results obtained by the finite difference method.

In addition to the general assumptions indicated in Chapter II, it is further assumed for the present development that the continuous plate is simply supported along its edges, and both of its panels are of the same materials and cross-sectional properties.

Concerning the previous work in this area, continuous sandwich beams have been analysed by Chong [13], Fazio and Salahuddin [26]. In the former work, experimental and theoretical analyses were conducted to investigate continuous sandwich beams composed of two equal spans and subjected to a uniformly distributed load throughout. A fourth order governing differential equation for the deflection was developed based on the displacement method, and for which a solution was found by superimposing the particular solution upon the homogeneous one. Fazio and Salahuddin [26] have studied the effect of settlement of a support of continuous sandwich beams on their redundant moments and shears. The analysis is based on the three-moment equation and considering the settlement of one support only, however, the combined effect of more than one support settlement can be treated by applying the principle of superposition. The theory was substantiated by experimental results obtained from tests on sandwich beams continuous over five supports and subjected to concentrated loads at their mid spans. On the other hand, solutions for homogeneous, isotropic continuous plates were reported by Timoshenko [101].

3.2 RECTANGULAR SANDWICH PLATES SUBJECTED TO MOMENTS DISTRIBUTED ALONG THE EDGES

Consider now a rectangular sandwich plate subjected to moments distributed along the edges $y = \pm b/2$, as shown in Fig. 3.2. The governing differential equation for the deflection in this case is obtained from Equ. (2.10) as

$$\nabla \nabla w = 0 \qquad (3.1)$$

A solution for the plate deflection, w, must satisfy Equ. (3.1) as well as the following boundary conditions

$$\text{at } x = 0, a \quad w = 0, \quad M_x = 0, \quad \text{and } \gamma_y = 0 \qquad (3.2)$$

$$\text{at } y = \pm b/2 \quad w = 0, \quad \left.\begin{array}{llll} \text{and} & M_y = M_1 & \text{for} & y = b/2 \\ \text{or} & M_y = M_2 & \text{for} & y = -b/2 \end{array}\right\} \qquad (3.3)$$

in which

M_1 and M_2 = the distributed moments along the edges $y = \pm b/2$, respectively.

a, b = plate dimensions in x- and y- directions, respectively.

A solution for Equ. (3.1) is taken in the form of the series [96, 101]

$$w = \sum_{m=1}^{\infty} Y_m(y) \sin \beta_m x \qquad (3.4)$$

in which

$$\beta_m = m\pi/a$$

Y is a function defined by

$$Y_m(y) = A_m \sinh \beta_m y + B_m \cosh \beta_m y$$
$$+ C_m \beta_m y \sinh \beta_m y + D_m \beta_m y \cosh \beta_m y \qquad (3.5)$$

where A_m, B_m, C_m, and D_m are unknown coefficients to be determined to satisfy the boundary conditions in Equs. (3.2) and (3.3).

To simplify the following analysis, three particular cases of moment distributions will be considered: (i) the symmetrical case in which $(M_y)_{y=b/2} = (M_y)_{y=-b/2}$ as shown in Fig. 3.3 (a); (ii) the antisymmetrical case in which $(M_y)_{y=b/2} = -(M_y)_{y=-b/2}$ as shown in Fig. 3.3 (b); and (iii) the antisymmetrical case in which $-(M_y)_{y=b/2} = (M_y)_{y=-b/2}$. The general case (Fig. 3.2) can be obtained from these cases by applying the principle of superposition.

In the symmetrical case Y_m must be an even function [52, 101]. Thus in Equ. (3.5), $A_m = D_m = 0$ and consequently

$$w_{sy} = \sum_{m=1}^{\infty} (B_m \cosh \beta_m y + C_m \beta_m y \sinh \beta_m y) \sin \beta_m x$$

in which

sy = subscript denotes the symmetrical case in Fig. 3.3 (a).

By satisfying w = 0 in Equ. (3.3), this expression becomes

$$w_{sy} = \sum_{m=1}^{\infty} C_m \left(\beta_m y \sinh \beta_m y - \alpha_m \tanh \alpha_m \cosh \beta_m y\right) \sin \beta_m x \quad (3.6)$$

in which

$$\alpha_m = \frac{m\pi}{2R}$$

R = the plate aspect ration = a/b.

Expressions for the shear forces can be determined from Equ. (3.6) and are taken as

$$Q_{xsy} = \sum_{m=1}^{\infty} \left(E_m \cosh \beta_m y + H_m \beta_m y \sinh \beta_m y\right) \cos \beta_m x$$

$$Q_{ysy} = \sum_{m=1}^{\infty} \left(F_m \sinh \beta_m y + P_m \beta_m y \cosh \beta_m y\right) \sin \beta_m x \quad (3.7)$$

in which

E_m, H_m, F_m, P_m = unknown coefficients.

By substituting Equs. (3.6) and (3.7) in the governing differential equations for shears, i.e. Equs. (2.11) and (2.12), it is found that

$$H_m = P_m = 0$$

$$E_m = F_m = K_m C_m$$

in which

$$K_m = -2 D \beta_m^3$$

Hence, Equ. (3.7) becomes

$$Q_{xsy} = \sum_{m=1}^{\infty} K_m C_m \cosh \beta_m y \cos \beta_m x \tag{3.8}$$

$$Q_{ysy} = \sum_{m=1}^{\infty} K_m C_m \sinh \beta_m y \sin \beta_m x \tag{3.9}$$

The second boundary condition in Equ. (3.3) can be used now to determine the coefficient C_m. Representing the moment distribution along the edges $y = \pm b/2$ by a trigonometric series, as

$$M_{sy} = \sum_{m=1}^{\infty} M_{msy} \sin \beta_m x \tag{3.10}$$

in which

M_{sy} = the symmetrical moment distribution (Fig. 3.3 (a)).

M_{msy} = the coefficient of Fourier expansion of the symmetrical moment, M_{sy}.

An expression for the moment M_y is obtained from Equs. (2.2), (3.6), (3.8), and (3.9). This expression and Equ. (3.10) together with the second condition in Equ. (3.3) result in

$$C_m = \frac{-b^2 M_{msy}}{8 \alpha_m^2 D \xi_m \cosh \alpha_m} \tag{3.11}$$

in which

$$\xi_m = 1 + \frac{4D}{b^2 S} (1 - \nu) \alpha_m^2 .$$

Having obtained an expression for C_m, the deflection and shears of a sandwich plate subjected to symmetrical moment can be calculated readily from Equs. (3.6), (3.8), and (3.9). By substituting Equ. (3.11) in (3.6), the deflection surface is obtained as

$$w_{sy} = \sum_{m=1}^{\infty} \frac{b^2 M_{msy}}{8 \alpha_m^2 D \xi_m \cosh \alpha_m} (\alpha_m \tanh \alpha_m \cosh \beta_m y - \beta_m y \sinh \beta_m y) \sin \beta_m x \qquad (3.12)$$

In the particular case of uniform distribution of the edge moment

$$M_{msy} = \frac{4 M_o}{m \pi} , \qquad m=1,3,5 \ldots$$

in which

M_o = the intensity of the uniform moment

and consequently the deflection along the axis of symmetry $y = 0$ is

$$w_{sy} = \frac{2 M_o a^2}{\pi^3 D} \sum_{m=1,3,..}^{\infty} \frac{1}{m^3 \xi_m} \frac{\alpha_m \tanh \alpha_m}{\cosh \alpha_m} \sin \beta_m x \qquad (3.13)$$

When a is very large in comparison to b : $\tanh \alpha_m \approx \alpha_m$ and $\cosh \alpha_m \approx \xi_m \approx 1$, and the previous expression becomes

$$w_{sy} = \frac{M_o b^2}{8 D}$$

which is the deflection at the middle of a sandwich strip of length b, as to be expected.

For the antisymmetrical case shown in Fig. 3.3 (b), the deflection surface must be an odd function of y [52, 101]. Hence $B_m = C_m = 0$ in Equ. (3.5), and Equ. (3.4) becomes

$$w_{asy} = \sum_{m=1}^{\infty} (A_m \sinh \beta_m y + D_m \beta_m y \cosh \beta_m y) \sin \beta_m x$$

in which

asy = subscript denoting the antisymmetrical case in Fig. 3.3 (b).

By satisfying w = 0 in Equ. (3.3), it is found that

$$w_{asy} = \sum_{m=1}^{\infty} A_m \left(\sinh \beta_m y - \frac{\beta_m y}{\alpha_m} \tanh \alpha_m \cosh \beta_m y\right) \sin \beta_m x \quad (3.14)$$

The shear forces in this case take the form

$$\left.\begin{aligned} Q_{xasy} &= \sum_{m=1}^{\infty} (E'_m \sinh \beta_m y + H'_m \beta_m y \cosh \beta_m y) \cos \beta_m x \end{aligned}\right\} \quad (3.15)$$

$$Q_{yasy} = \sum_{m=1}^{\infty} (F'_m \cosh \beta_m y + P'_m \beta_m y \sinh \beta_m y) \sin \beta_m x$$

in which

$$E'_m, H'_m, F'_m, P'_m = \text{unknown coefficients.}$$

From Equ. (3.15) together with the governing differential equations for shears, i.e. Equs. (2.11) and (2.12), it is found that

$$H'_m = P'_m = 0$$

and

$$E'_m = F'_m = K'_m A_m$$

in which

$$K'_m = 2 D \beta_m^3 \frac{\tanh \alpha_m}{\alpha_m}$$

Consequently, Equ. (3.15) becomes

$$Q_{xasy} = \sum_{m=1}^{\infty} K'_m A_m \sinh \beta_m y \cos \beta_m x \tag{3.16}$$

$$Q_{yasy} = \sum_{m=1}^{\infty} K'_m A_m \cosh \beta_m y \sin \beta_m x \tag{3.17}$$

Having obtained Equs.(3.14), (3.16), and (3.17), the coefficient A_m can be easily determined. Representing the antisymmetrical moment distribution along the edges $y = \pm b/2$ by a trigonometric series :

$$M_{asy} = \sum_{m=1}^{\infty} M_{masy} \sin \beta_m x \tag{3.18}$$

in which

M_{asy} = the antisymmetrical moment distribution, Fig. 3.3 (b)

M_{masy} = the coefficient of Fourier expansion of the edges antisymmetrical moment, M_{asy}.

From which and Equs. (2.2), (3.14), (3.16) and (3.17), together with the second boundary condition in Equ. (3.3), it follows that

$$A_m = \frac{b^2 M_{masy}}{8 \alpha_m D \xi_m \cosh \alpha_m}$$

For the antisymmetrical case in which $(M_y)_{y=b/2} = (M_y)_{y=-b/2}$, the deflection surface and the shear forces take the same forms as in Equs. (3.14), (3.16), and (3.17). The coefficient A_m in this case is determined by following the same procedure and the expression obtained is

$$A'_m = \frac{-b^2 M_{masy}}{8 \alpha_m D \xi_m \cosh \alpha_m},$$

where A'_m is the coefficient which replaces A_m in Equs. (3.14), (3.16), and (3.17).

Finally, for the general case, represented by the second and third boundary conditions in Equ. (3.3), the deflection surface and shears can be obtained from the solutions developed for the symmetrical and antisymmetrical cases. For this purpose, the moment distribution M_1 and M_2 can be split into a symmetrical moment distribution M'_y and an antisymmetrical distribution M_y'' as follows

$$(M_y')_{y=b/2} = (M_y')_{y=-b/2} = [M_1(x) + M_2(x)]/2$$

$$(M_y'')_{y=b/2} = - (M_y'')_{y=-b/2} = [M_1(x) - M_2(x)]/2$$

By representing M_y' and M_y'' by trigonometric series, the general solution can be easily determined. When a moment M_c is distributed only along one edge (Fig. 3.4), either $M_1(x) = 0$ or $M_2(x) = 0$, and from the previous equations $M_{msy} = M_{masy} = M_m/2$ where M_m is the coefficient of the trigonometric expansion of M_c :

$$M_c = \sum_{m=1}^{\infty} M_m \sin \beta_m x \qquad (3.19)$$

Solutions for two particular cases are presented next. When $M_c = (M_y)_{y=b^r/2}$, as shown in Fig. 3.4 (a), the deflection surface and shear forces are obtained as

$$w^r = \sum_{m=1}^{\infty} \frac{b^{r2}\, M_m^r}{16\, \alpha_m^{r2}\, D\, \xi_m^r} \left[\frac{1}{\cosh \alpha_m^r} (\alpha_m^r \tanh \alpha_m^r \cosh \beta_m y\right.$$

$$- \beta_m y \sinh \beta_m y) + \frac{1}{\sinh \alpha_m^r} (\alpha_m^r \coth \alpha_m^r \sinh \beta_m y$$

$$\left.- \beta_m y \cosh \beta_m y)\right] \sin \beta_m x \qquad (3.20)$$

$$Q_x^r = \sum_{m=1}^{\infty} \frac{\alpha_m^r M_m^r}{b^r \xi_m^r} \left(\frac{\cosh \beta_m y}{\cosh \alpha_m^r} + \frac{\sinh \beta_m y}{\sinh \alpha_m^r}\right) \cos \beta_m x \tag{3.21}$$

$$Q_y^r = \sum_{m=1}^{\infty} \frac{\alpha_m^r M_m^r}{b^r \xi_m^r} \left(\frac{\sinh \beta_m y}{\cosh \alpha_m^r} + \frac{\cosh \beta_m y}{\sinh \alpha_m^r}\right) \sin \beta_m x \tag{3.22}$$

in which

b^r = the plate dimension parallel to y- axis

R^r = the plate aspect ratio

= a/b^r

M_m^r = the coefficient of the series in Equ. (3.19)

$$\alpha_m^r = \frac{m\,\pi}{2\,R^r}$$

$$\xi_m^r = 1 + \frac{4\,D}{b^{r2}\,S}(1 - \nu)\,\alpha_m^{r2}$$

The moments of the sandwich plate in this case are determined from Equs. (3.20), (3.21), and (3.22) in conjunction with Equs. (2.1), (2.2), and (2.3). In abreviated forms, the results are

$$M_x^r = -D\left(\frac{\partial \theta_x^r}{\partial x} + \nu \frac{\partial \theta_y^r}{\partial y}\right)$$

$$M_y^r = -D\left(\frac{\partial \theta_y^r}{\partial y} + \nu \frac{\partial \theta_x^r}{\partial x}\right) \tag{3.23}$$

$$M_{xy}^r = D_{xy}\left(\frac{\partial \theta_x^r}{\partial y} + \frac{\partial \theta_y^r}{\partial x}\right)$$

in which

$$\theta_x^r = \frac{\partial w^r}{\partial x} - \frac{Q_x^r}{S}$$

$$= \sum_{m=1}^{\infty} \frac{b^r M_m^r}{8 \alpha_m^r D \xi_m^r} \left[\frac{1}{\cosh \alpha_m^r} \left(- \beta_m y \sinh \beta_m y\right.\right.$$

$$\left. + \alpha_m^r \tanh \alpha_m^r \cosh \beta_m y - 8 \frac{\alpha_m^{r2} D}{b^{r2} S} \cosh \beta_m y\right)$$

$$+ \frac{1}{\sinh \alpha_m^r} \left(\alpha_m^r \coth \alpha_m^r \sinh \beta_m y - \beta_m y \cosh \beta_m y\right.$$

$$\left.\left. - 8 \frac{\alpha_m^{r2} D}{b^{r2} S} \sinh \beta_m y\right)\right] \cos \beta_m x$$

$$\theta_y^r = \frac{\partial w^r}{\partial y} - \frac{Q_y^r}{S}$$

$$= \sum_{m=1}^{\infty} \frac{b^r M_m^r}{8 \alpha_m^r D \xi_m^r} \left[\frac{1}{\cosh \alpha_m^r} \left(- \sinh \beta_m y - \beta_m y \cosh \beta_m y\right.\right.$$

$$\left. + \alpha_m^r \tanh \alpha_m^r \sinh \beta_m y - 8 \frac{\alpha_m^{r2} D}{b^{r2} S} \sinh \beta_m y\right)$$

$$+ \frac{1}{\sinh \alpha_m^r} \left(\alpha_m^r \coth \alpha_m^r \cosh \beta_m y - \cosh \beta_m y\right.$$

$$\left.\left. - \beta_m y \sinh \beta_m y - 8 \frac{\alpha_m^{r2} D}{b^{r2} S} \cosh \beta_m y\right)\right] \sin \beta_m x \qquad (3.24)$$

When the moment M_c is acting on the edge $y = -b^\ell/2$ as shown in Fig. 3.4 (b), the deflection and shears are obtained as

$$w^\ell = \sum_{m=1}^{\infty} \frac{b^{\ell 2} M_m^\ell}{16\,\alpha_m^{\ell 2}\, D\, \xi_m^\ell} \left[\frac{1}{\cosh \alpha_m^\ell} (-\beta_m y \sinh \beta_m y + \alpha_m^\ell \tanh \alpha_m^\ell \cosh \beta_m y) - \frac{1}{\sinh \alpha_m^\ell} (\alpha_m^\ell \coth \alpha_m^\ell \sinh \beta_m y - \beta_m y \cosh \beta_m y)\right] \sin \beta_m x \tag{3.25}$$

$$Q_x^\ell = \sum_{m=1}^{\infty} \frac{\alpha_m^\ell M_m^\ell}{b^\ell \xi_m^\ell} \left(\frac{\cosh \beta_m y}{\cosh \alpha_m^\ell} - \frac{\sinh \beta_m y}{\sinh \alpha_m^\ell}\right) \cos \beta_m x \tag{3.26}$$

$$Q_y^\ell = \sum_{m=1}^{\infty} \frac{\alpha_m^\ell M_m^\ell}{b^\ell \xi_m^\ell} \left(\frac{\sinh \beta_m y}{\cosh \alpha_m^\ell} - \frac{\cosh \beta_m y}{\sinh \alpha_m^\ell}\right) \sin \beta_m x \tag{3.27}$$

in which

b^ℓ = the plate dimension parallel to y- axis

$\alpha_m^\ell = \dfrac{m\pi}{2 R^\ell}$

R^ℓ = the plate aspect ratio

$= a/b^\ell$

M_m^ℓ = the coefficient of the series in Equ. (3.19).

$\xi_m^\ell = 1 + \dfrac{4 D}{b^{\ell 2} S} (1-\nu)\, \alpha_m^{\ell 2}$

The moments in this case can be obtained from Equ. (3.23), noting that θ_x^ℓ and θ_y^ℓ should replace θ_x^r and θ_y^r , respectively, where

$$\theta_x^\ell = \sum_{m=1}^{\infty} \frac{b^\ell M_m^\ell}{8\,\alpha_m^\ell\, D\, \xi_m^\ell} \Big[\frac{1}{\cosh \alpha_m^\ell} (-\beta_m y \sinh \beta_m y$$

$$+ \alpha_m^\ell \tanh \alpha_m^\ell \cosh \beta_m y - 8\,\frac{\alpha_m^{\ell 2}\, D}{b^{\ell 2}\, S} \cosh \beta_m y)$$

$$- \frac{1}{\sinh \alpha_m^\ell} (\alpha_m^\ell \coth \alpha_m^\ell \sinh \beta_m y - \beta_m y \cosh \beta_m y$$

$$- 8\,\frac{\alpha_m^{\ell 2}\, D}{b^{\ell 2}\, S} \sinh \beta_m y)\Big] \cos \beta_m x$$

$$\theta_y^\ell = \sum_{m=1}^{\infty} \frac{b^\ell M_m^\ell}{8\,\alpha_m^\ell\, D\, \xi_m^\ell} \Big[\frac{1}{\cosh \alpha_m^\ell} (-\sinh \beta_m y - \beta_m y \cosh \beta_m y$$

$$+ \alpha_m^\ell \tanh \alpha_m^\ell \sinh \beta_m y - 8\,\frac{\alpha_m^{\ell 2}\, D}{b^{\ell 2}\, S} \sinh \beta_m y)$$

$$- \frac{1}{\sinh \alpha_m^\ell} (\alpha_m^\ell \coth \alpha_m^\ell \cosh \beta_m y - \cosh \beta_m y$$

$$- \beta_m y \sinh \beta_m y - 8\,\frac{\alpha_m^{\ell 2}\, D}{b^{\ell 2}\, S} \cosh \beta_m y)\Big] \sin \beta_m x \qquad (3.28)$$

3.3 CONTINUOUS SANDWICH PLATES

A rectangular sandwich plate of width a and length $(b^{\ell} + b^{r})$, supported along the edges and also along the intermediate line bb (Fig. 3.1 (a)) forms a simply supported continuous sandwich plate over two spans [101]. Solutions for its deflected surface, shears, and moments can be obtained by combining those obtained for laterally loaded, simply supported plates (treated in chapter II), with those developed for plates subjected to distributed moments along their edges (section 3.2). This procedure requires the knowledge of the redundant moment which is of such a magnitude as to ensure compatibility of the deformations of the two adjacent panels at the commond support.

Consider now the simply supported continuous sandwich plate which is subjected to a lateral load on both of its spans as shown in Fig. 3.1 (a). The slope of the bending deflection surface along the edge $y = b$ of the laterally loaded, simply supported right span (Fig. 3.5 (a)) is obtained from Equ. (2.14) as

$$\left(\frac{\partial w_b^{r'}}{\partial y^r}\right)_{y^r=b^r} = \sum_{m=1}^{\infty} C_m^r \sin \beta_m x \tag{3.29}$$

in which

$w_b^{r'}$ = the bending component of the total deflection of the right span

X^r, Y^r = the coordinate axes for the right panel (Fig. 3.5 (a))

$$C_m^r = \sum_{n=1}^{\infty} \frac{n\pi}{Db^r} \frac{(-1)^{(n+2)} P_{mn}^r}{[(m\pi/a)^2 + (n\pi/b^r)^2]^2}$$

where P_{mn}^r is the coefficient of Fourier expansion for the applied load on the right panel. Let M_c be the redundant moment distributed along the edge $y^r = b^r$ (Fig. 3.5 (a)), and be represented by the sine series in Equ. (3.19). The deflection surface due to M_c is represented by Equ. (3.20), and its bending slope θ_y^r along the edge $y = b^r/2$ (Fig. 3.5(b)) is determined from Equ. (3.24) as

$$\theta_y^{r''} = \sum_{m=1}^{\infty} \frac{b^r M_m}{8 \alpha_m^r D \xi_m^r} (- \tanh \alpha_m^r - 2 \alpha_m^r + \alpha_m^r \tanh^2 \alpha_m^r$$

$$- 8 \alpha_m^{r2} \phi^r \tanh \alpha_m^r + \alpha_m^r \coth^2 \alpha_m^r - \coth \alpha_m^r$$

$$- 8 \alpha_m^{r2} \phi^r \coth \alpha_m^r) \sin \beta_m x \qquad (3.30)$$

in which

ϕ^r = the shear parameter of the right panel

$= \dfrac{D}{b^{r2} S}$

$\theta_y^{r''}$ = the slope of the bending deflection surface of the plate due to the moment M_c distributed along the edge $y^r = b^r/2$.

The slope of the bending deflection surface along the edge $y^\ell = 0$ of the laterally loaded simply supported left panel (Fig. 3.5 (a)) is obtained from Equ. (2.14) as

$$\left(\frac{\partial w_b^{\ell'}}{\partial y^\ell}\right)_{y^\ell=0} = \sum_{m=1}^{\infty} C_m^\ell \sin \beta_m x \tag{3.31}$$

in which

$w_b^{\ell'}$ = the same as $w_b^{r'}$ but for the left panel

X^ℓ, Y^ℓ = coordinate axes for the left panel (Fig. 3.5 (a))

$$C_m^\ell = \sum_{n=1}^{\infty} \frac{n\pi}{Db^\ell} \frac{p_{mn}^\ell}{[(m\pi/a)^2 + (n\pi/b^\ell)^2]^2}$$

where p_{mn}^ℓ is the coefficient of Fourier expansion of the load acting on the left panel. Due to the moment M_c (Equ. 3.19) which is distributed along the edge $y^\ell = 0$ (Fig. 3.5 (a)), the deflection can be determined from Equ. (3.25) and its bending slope θ_y^ℓ along the edge $y^\ell = -b^\ell/2$ (Fig. 3.5 (b)) is obtained from Equ. (3.28) as

$$\begin{aligned}\theta_y^{\ell''} = \sum_{m=1}^{\infty} \frac{b^\ell M_m}{8\alpha_m^\ell D \xi_m^\ell} & (\tanh \alpha_m^\ell + 2\alpha_m^\ell - \alpha_m^\ell \tanh^2\alpha_m^\ell \\ & + 8\alpha_m^{\ell 2} \phi^\ell \tanh \alpha_m^\ell - \alpha_m^\ell \coth^2\alpha_m^\ell \\ & + \coth \alpha_m^\ell + 8\alpha_m^{\ell 2} \phi^\ell \coth \alpha_m^\ell) \sin \beta_m x\end{aligned} \tag{3.32}$$

From the condition of continuity it is concluded that the sum of the expressions in the right hand side of Equs. (3.29) and (3.30), which represents the bending slope of the right sandwich panel, of the continuous plate in Fig. 3.1 (a), along the intermediate support must be equal to that in the left panel. Thus

$$\left(\frac{\partial w_b^{r'}}{\partial y^r}\right)_{y^r=b^r} + \theta_y^{r''} = \left(\frac{\partial w_b^{\ell'}}{\partial y^\ell}\right)_{y^\ell=0} + \theta_y^{\ell''}$$

By substituting Equs. (3.29), (3.30), (3.31), and (3.32) in this equation, and by equating the coefficients of the sine terms in both sides it is found that

$$M_m = \frac{\phi_m}{\Psi_m} \tag{3.33}$$

in which

$$\phi_m = C_m^r - C_m^\ell$$

$$\Psi_m = \frac{b^\ell}{8\,D}\,\Psi_m' \tag{3.34}$$

where

$$\Psi_m' = \frac{1}{\alpha_m^\ell\,\xi_m^\ell}\,[(\tanh\alpha_m^\ell + \coth\alpha_m^\ell)$$

$$(1 + 8\,\alpha_m^{\ell 2}\,\phi^\ell) + \alpha_m^\ell\,(2 - \tanh^2\alpha_m^\ell - \coth^2\alpha_m^\ell)]$$

$$+ \frac{b^r}{b^\ell} \frac{1}{\alpha_m^r \xi_m^r} [(\tanh \alpha_m^r + \coth \alpha_m^r)(1 + 8 \alpha_m^{r2} \phi^r)$$

$$+ \alpha_m^r (2 - \tanh^2\alpha_m^r - \coth^2\alpha_m^r)]$$

Two particular loading types are now considered. First, when the applied load is uniformly distributed:

$$\phi_m = \frac{16 a^4}{\pi^5 D b^r} \phi'_{mu} \tag{3.35}$$

in which

$$\phi'_{mu} = \sum_{n=1,3,...}^{\infty} \left[\frac{(-1)^{(n+2)} p_o^r}{m(m^2 + n^2 R^{r2})^2} - \frac{b^r}{b^\ell} \frac{p_o^\ell}{m(m^2 + n^2 R^{\ell 2})^2}\right]$$

where p_o^r and p_o^ℓ are the intensities of the uniform loads acting on the right and left panels respectively. Substituting Equs. (3.34) and (3.35) in Equ. (3.33) gives

$$M_m = \frac{128 a^4}{\pi^5 b^r b^\ell} \frac{\phi'_{mu}}{\Psi'_m} \tag{3.36}$$

The right hand side of this equation represents Fourier expansion coefficient of the redundant moment of a simply supported continuous sandwich plate (Fig. 3.1 (a)) subjected to unfiorm loads on both panels. Knowing the coefficient M_m from Equ. (3.36), the value of the redundant moment M_c, at any point along the common support can be obtained from Equ. (3.19).

The value of this moment at x = a/2, that is at the middle of the width of the plate is obtained as

$$(M_c)_{\substack{y^r=b^r/2 \\ x^r=a/2}} = b^{r2} \sum_{m=1,3,\ldots}^{\infty} \frac{-128}{\pi^5} (-1)^{(m+1)/2} \frac{R^{\ell 2} R^{r2}}{k} \frac{\phi'_{mu}}{\psi'_m} \tag{3.37}$$

in which

$$k = \frac{b^r}{b^\ell} = \frac{R^\ell}{R^r}$$

Second, when the external load is a single concentrated load at the center of each panel :

$$\phi_m = \frac{4 a^3}{\pi^3 D b^{r2}} \phi'_{mc} \tag{3.38}$$

in which

$$\phi'_{mc} = \sum_{n=1}^{\infty} n (-1)^{(m+n)/2} \left[\frac{(-1)^{(n+2)} P^r}{(m^2 + n^2 R^{r2})^2} - \frac{b^{r2}}{b^{\ell 2}} \frac{P^\ell}{(m^2 + n^2 R^{\ell 2})^2}\right],$$

where P^r and P^ℓ are the central concentrated loads on the right and left panels respectively.

The value of the redundant moment at x = a/2, in this case is obtained from Equs. (3.33), (3.34), (3.38) together with Equ. (3.19) as

$$(M_c)_{\substack{y^r=b^r/2 \\ x^r=a/2}} = \sum_{m=1,3,\ldots}^{\infty} \frac{-32}{\pi^3} (-1)^{(m+1)/2} k R^{r3} \frac{\phi'_{mc}}{\Psi'_m} \qquad (3.39)$$

3.4 PRACTICAL FORMULAS FOR SANDWICH PLATES SUBJECTED TO EDGES MOMENT

To reduce the amount of work involved in the analysis of continuous sandwich plates, practical formulas are devised from which the deflection w, moments M_x and M_y, and shear forces Q_x and Q_y can be determined. The simple expressions developed contain two main factors:

The first one reflects the material properties and the plate geometry, whereas the second one reflects the aspect ratio of the panels, the ratio of the right to left spans, the loading type and the shear parameter which is defined as

$$\text{shear parameter} = \phi^i$$

$$= \frac{D}{S\, b^{i2}}$$

in which

$$i = r \text{ or } \ell$$

where r and ℓ denote the right and left panels respectively.

Numerical results are tabulated over a certain range for each of these parameters as follows :

(i) the ratio of the right to left spans, $k = b^r/b^\ell$, varies from 0.25 to 1.

(ii) the apsect ratio of the right panel, $R^r = a/b^r$, varies from 1. to 5.

(iii) the shear parameter of the right panel, ϕ^r, varies from 0.0 to 0.5.

(iv) Poisson's ratio of facing materials is fixed at 0.30.

Three loading types are considered: (i) a uniformly distributed moment along the edges $y = \pm b/2$ (Fig. 3.6), (ii) a uniformly distributed load (Fig. 3.6), and (iii) a central concentrated load (Fig. 3.6). To extend further the possible application of the present formulas, the uniform and concentrated loads are considered acting on either the right or left panel. The combined effect of more than one loading type can be obtained by applying the principle of superposition. For each loading type, the numerical values of the factors in the practical formulas are tabulated in Appendix C.

In these tables, the following notations are used

SPR = ϕ^r = the shear parameter for the right panel (Fig. 3.1)

SPL = ϕ^ℓ = the shear parameter for the left panel

$= k^2 \cdot \phi^r$

POR, PR $= p_o^r$, p^r, respectively.

POL, PL $= p_o^\ell$, p^ℓ, respectively.

where p_o^i and $\bar{p}_o^i$ are the intensity of a uniformly distributed load and the value of a central concentrated load acting on panel i, respectively.

k $= \dfrac{b^r}{b^\ell} = \dfrac{R^\ell}{R^r}$

where b^i and R^i are the span and aspect ratio of panel i.

POISSR = Poisson's ratio of facing materials.

N, F = Denote the intermediate support and the one parallel to it, respectively. Thus, for example, K_{Qyn}^r means that it is the factor of the shear force Q_y at the middle of the intermediate support in the right sandwich panel (Fig. 3.6). Additional notations are shown in Fig. 3.6.

RR, RL $= R^r$, R^ℓ, respectively.

KWR, KWL $= K_w^r$, K_w^ℓ, respectively.

KMC $= K_{mc}$

KQXR, KQXL $= K_{Qx}^r$, K_{Qx}^ℓ, respectively.

KQYRN, KQYLN $= K_{Qyn}^r$, K_{Qyn}^ℓ, respectively.

KQYRF, KQYLF $= K_{Qyf}^r$, K_{Qyf}^ℓ, respectively.

KMXR, KMXL $= K_{mx}^r$, K_{mx}^ℓ, respectively.

KMYR, KMYL $= K_{my}^r$, K_{my}^ℓ, respectively.

It should be emphasized that all the tabulated values as well as formulas presented in the following are the response of continuous sandwich plates due to only the redundant moment. The principle of superposition must be applied to determine the total effect of the applied loads.

3.4.1 SANDWICH PLATE SUBJECTED TO UNIFORM MOMENTS AT $y = \pm b/2$ (Fig. 3.6)

The central deflection in this case can be determined from Equ. (3.13) as

$$w_{sy} = \frac{M_o a^2}{D} K_{wsy} \tag{3.40}$$

in which

$$K_{wsy} = \sum_{m=1,3,\ldots}^{\infty} \frac{-(-1)^{(m+1)/2}}{\pi^2 R \xi_m m^2} \frac{\tanh \alpha_m}{\cosh \alpha_m}$$

The factor K_{wsy} is evaluated for a wide range of aspect ratios and shear parameters and the results are shown in Fig. 3.7.

For the particular case when the sandwich plate is rigid in shear : $\xi_m = 1$, and Equ. (3.40) becomes

$$w_{sy} = \frac{M_o b^2}{D} \sum_{m=1,3,\ldots}^{\infty} \frac{-(-1)^{(m+1)/2} R}{\pi^2 \xi_m m^2} \frac{\tanh \alpha_m}{\cosh \alpha_m}$$

Some numerical results are obtained from this expression and shown in Table C.1. These results are in agreement with those obtained by Timoshenko [101] for homogeneous plates.

The bending moments M_x and M_y at the plate center, are obtained From Equs. (3.13), (3.16), (3.17) in conjunction with Equs. (2.1) and (2.2), as

$$M_x = M_o K_{mxsy} \qquad (3.41)$$

$$M_y = M_o K_{mysy} \qquad (3.42)$$

in which

$$K_{mxsy} = \sum_{m=1,3,...}^{\infty} \frac{-2\ (-1)^{(m+1)/2}\ R}{\pi^2\ \xi_m\ m^2\ \cosh \alpha_m}\ [2\ \alpha_m^2\ (1-\nu)\ \tanh \alpha_m$$

$$+\ 4\ \alpha_m\ \nu + 16\ \alpha_m^3\ \phi\ (\nu-1)]$$

$$K_{mysy} = \sum_{m=1,3,...}^{\infty} \frac{-2\ (-1)^{(m+1)/2}\ R}{\pi^2\ \xi_m\ m^2\ \cosh \alpha_m}\ [4\ \alpha_m + 2\ \alpha_m^2\ (\nu-1)\ \tanh \alpha_m$$

$$+\ 16\ \alpha_m^3\ \phi\ (1-\nu)$$

The higher normal stress in the faces corresponds to the bending moment M_y, and consequently the numerical values of only K_{mysy} are shown in Fig. 3.7.

When $\xi_m = 1$, some numerical results of M_x and M_y are shown in Table C.1. Observation of this table shows that the entire applied edge moment M_0 is transmitted by the transverse section at the middle of a strip whereas the bending moment M_y at the center of a plate decreases, as compared with M_0, with decreasing ratio a/b. This is due to a damping effect of the edges x = o and x = a which are not exposed to couples [101].

The shear forces at the middle of the edges x = 0 and y = b/2 are determined from Equs. (3.8) and (3.9) as

$$Q_{xsy} = \frac{4\,M_0}{a}\,K_{Qxsy} \tag{3.43}$$

$$Q_{ysy} = \frac{4\,M_0}{a}\,K_{Qysy} \tag{3.44}$$

in which

$$K_{Qxsy} = \sum_{m=1,3,\ldots}^{\infty} \frac{1}{\xi_m \cosh \alpha_m}$$

$$K_{Qysy} = \sum_{m=1,3,\ldots}^{\infty} \frac{-\,(-1)^{(m+1)/2}\tanh \alpha_m}{\xi_m}$$

The numerical values for only the factor K_{Qysy} are shown in Fig. 3.7.

When a is very large in comparison to b : $\tanh \alpha_m \approx \alpha_m$ and $\cosh \alpha_m \approx \xi_m \approx 1$, and consequently, Q_{ysy} becomes diminishingly small which is the case of a sandwich strip problem.

3.4.2 CONTINUOUS SANDWICH PLATE SUBJECTED TO CONCENTRATED LOADS AT THE CENTER OF ITS PANELS

When a central concentrated load is applied on either panel of the continuous sandwich plate (Fig. 3.6), the deflection, shear forces and bending moments can be determined from the following formulas

$$w^i = \frac{b^{i2} P^j}{D} K_w^{ij} \tag{3.45}$$

$$Q_x^i = \frac{P^j}{b^i} K_{Qx}^{ij} \tag{3.46}$$

$$Q_{yn}^i = \frac{P^j}{b^i} K_{Qyn}^{ij} \tag{3.47}$$

$$Q_{yf}^i = \frac{P^j}{b^i} K_{Qyf}^{ij} \tag{3.48}$$

$$M_x^i = P^j K_{mx}^{ij} \tag{3.49}$$

$$M_y^i = P^j K_{my}^{ij} \tag{3.50}$$

$$M_c = P^j K_{mc}^j$$

in which

$$j = r \text{ means } P^r = 1 \text{ and } P^\ell = 0, \text{ thus } \phi_{mc}^{j'} = (\phi'_{mc})_{\substack{P^r=1 \\ P^\ell=0}}$$

$$= \ell \text{ means } P^r = 0 \text{ and } P^\ell = 0, \text{ thus } \phi_{mc}^{j'} = (\phi'_{mc})_{\substack{P^r=0 \\ P^\ell=1}}$$

$$K_w^{ij} = \sum_{m=1,3,\ldots}^{\infty} \frac{-2}{\pi^3} \frac{(-1)^{(m+1)/2}}{\alpha_m^i \; \xi_m^i} \frac{\phi_{mc}^{j'}}{\Psi'_m} R^{r2} R^\ell \frac{\tanh \alpha_m^i}{\cosh \alpha_m^i}$$

$$K_{Qx}^{ij} = \sum_{m=1,3,\ldots}^{\infty} \frac{32}{\pi^3} \frac{\alpha_m^i}{\xi_m^i} \frac{\phi_{mc}^{j'}}{\Psi'_m} k R^{r3} \frac{1}{\cosh \alpha_m^i}$$

$$K_{Qyn}^{ij} = \pm \sum_{m=1,3,\ldots}^{\infty} \frac{-32}{\pi^3} (-1)^{(m+1)/2} \frac{\alpha_m^i}{\xi_m^i} \frac{\phi_{mc}^{j'}}{\Psi'_m} k R^{r3}$$

$$(\tanh \alpha_m^i + \coth \alpha_m^i)$$

$$K_{Qyf}^{ij} = \pm \sum_{m=1,3,\ldots}^{\infty} \frac{-32}{\pi^3} (-1)^{(m+1)/2} \frac{\alpha_m^i}{\xi_m^i} \frac{\phi_{mc}^{j'}}{\Psi'_m} k R^{r3}$$

$$(-\tanh \alpha_m^i + \coth \alpha_m^i)$$

$$K_{mx}^{ij} = \sum_{m=1,3,\ldots}^{\infty} \frac{-8}{\pi^3} (-1)^{(m+1)/2} \frac{R^{\ell 3}}{\xi_m^i k^2} \frac{\phi_{mc}^{j'}}{\Psi'_m} \frac{1}{\cosh \alpha_m^i} [$$

$$(1-\nu) \; \alpha_m^i \; \tanh \alpha_m^i + 2\nu + 8 \alpha_m^{i2} \phi^i (\nu-1)]$$

$$K_{my}^{ij} = \sum_{m=1,3,...}^{\infty} \frac{-8}{3} (-1)^{(m+1)/2} \frac{R^{\ell 3}}{\xi_m^i k^2} \frac{\phi_{mc}^{j'}}{\Psi_m'} \frac{1}{\cosh \alpha_m^i} [$$

$$(\nu-1) \alpha_m^i \tanh \alpha_m^i + 2 + 8 \alpha_m^{i2} \phi^i (1-\nu)]$$

$$K_{mc}^{j} = \sum_{m=1,3,...}^{\infty} \frac{-32}{\pi^3} (-1)^{(m+1)/2} k R^{r3} \frac{\phi_{mc}^{j'}}{\Psi_m'}$$

where the + ve sign of ± is for the right panel while the - ve sign is for the left panel. Numerical values for these factors are tabulated in Tables C.2 to C.43.

3.4.3 CONTINUOUS SANDWICH PLATE SUBJECTED TO UNIFORMLY DISTRIBUTED LOADS

The practical formulas in this case are given by

$$w^i = \frac{b^{i4}}{D} p_o^j K_w^{ij} \tag{3.51}$$

$$Q_x^i = b^i p_o^j K_{Qx}^{ij} \tag{3.52}$$

$$Q_{yn}^i = b^i p_o^j K_{Qyn}^{ij} \tag{3.53}$$

$$Q_{yf}^i = b^i p_o^j K_{Qyf}^{ij} \tag{3.54}$$

$$M_x^i = b^{i2} p_o^j K_{mx}^{ij} \tag{3.55}$$

$$M_y^i = b^{i2}\, p_o^j\, K_{my}^{ij} \tag{3.56}$$

$$M_c = b^{r2}\, p_o^j\, K_{mc}^j$$

in which

$j = r$ means $p_o^r = 1$ and $p_o^\ell = 0$

ℓ means $p_o^r = 0$ and $p_o^\ell = 1$

$$K_w^{rj} = \sum_{m=1,3,\ldots}^{\infty} \frac{-8}{\pi^5} (-1)^{(m+1)/2} \frac{1}{\alpha_m^r\, \xi_m^r} \frac{\theta_{mu}^{j\,'}}{\Psi_m'}\, k\, R^{r4}\, \frac{\tanh \alpha_m^r}{\cosh \alpha_m^r}$$

$$K_w^{\ell j} = \sum_{m=1,3,\ldots}^{\infty} \frac{-8}{\pi^5} (-1)^{(m+1)/2} \frac{1}{\alpha_m^\ell\, \xi_m^\ell} \frac{\phi_{mu}^{j\,'}}{\Psi_m'} \frac{R^{\ell 4}}{k} \frac{\tanh \alpha_m^\ell}{\cosh \alpha_m^\ell}$$

$$K_{Qx}^{rj} = \sum_{m=1,3,\ldots}^{\infty} \frac{128}{\pi^5} \frac{\alpha_m^r}{\xi_m^r} \frac{\phi_{mu}^{j\,'}}{\Psi_m'}\, k\, R^{r4}\, \frac{1}{\cosh \alpha_m^r}$$

$$K_{Qx}^{\ell j} = \sum_{m=1,3,\ldots}^{\infty} \frac{128}{\pi^5} \frac{\alpha_m^\ell}{\xi_m^\ell} \frac{\phi_{mu}^{j\,'}}{\Psi_m'} \frac{R^{\ell 4}}{k} \frac{1}{\cosh \alpha_m^\ell}$$

$$K_{Qyn}^{rj} = \sum_{m=1,3,\ldots}^{\infty} \frac{-128}{\pi^5} (-1)^{(m+1)/2} \frac{\alpha_m^r}{\xi_m^r} \frac{\phi_{mu}^{j\,'}}{\Psi_m'}\, k\, R^{r4} \left(\tanh \alpha_m^r + \coth \alpha_m^r \right)$$

$$K^{\ell j}_{Qyn} = \sum_{m=1,3,...}^{\infty} \frac{128}{\pi^5} (-1)^{(m+1)/2} \frac{\alpha^{\ell}_m}{\xi^{\ell}_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{R^{\ell 4}}{k} ($$

$$\tanh \alpha^{\ell}_m + \coth \alpha^{\ell}_m)$$

$$K^{rj}_{Qyf} = \sum_{m=1,3,...}^{\infty} \frac{-128}{\pi^5} (-1)^{(m+1)/2} \frac{\alpha^{r}_m}{\xi^{r}_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} k\, R^{r4} ($$

$$\coth \alpha^{r}_m - \tanh \alpha^{r}_m)$$

$$K^{\ell j}_{Qyf} = \sum_{m=1,3,...}^{\infty} \frac{-128}{\pi^5} (-1)^{(m+1)/2} \frac{\alpha^{\ell}_m}{\xi^{\ell}_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{R^{\ell 4}}{k} ($$

$$\tanh \alpha^{\ell}_m - \coth \alpha^{\ell}_m)$$

$$K^{rj}_{mx} = \sum_{m=1,3,...}^{\infty} \frac{-32}{\pi^5} (-1)^{(m+1)/2} \frac{k\, R^{r4}}{\xi^{r}_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{1}{\cosh \alpha^{r}_m} [$$

$$(1-\nu)\ \alpha^{r}_m \tanh \alpha^{r}_m + 2\nu + 8\, \alpha^{r2}_m\, \phi^{r} (\nu-1)]$$

$$K^{\ell j}_{mx} = \sum_{m=1,3,...}^{\infty} \frac{-32}{\pi^5} (-1)^{(m+1)/2} \frac{R^{\ell 4}}{k\, \xi^{\ell}_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{1}{\cosh \alpha^{\ell}_m} [$$

$$(1-\nu)\ \alpha^{\ell}_m \tanh \alpha^{\ell}_m + 2\nu + 8\, \alpha^{\ell 2}_m\, \phi^{\ell} (\nu-1)]$$

$$K^{rj}_{my} = \sum_{m=1,3,\ldots}^{\infty} \frac{-32}{\pi^5} (-1)^{(m+1)/2} \frac{k\ R^{r4}}{\xi^r_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{1}{\cosh \alpha^r_m} [$$

$$(\nu-1)\ \alpha^r_m \tanh \alpha^r_m + 2 + 8\ \alpha^{r2}_m \Phi^r (1-\nu)]$$

$$K^{\ell j}_{my} = \sum_{m=1,3,\ldots}^{\infty} \frac{-32}{\pi^5} (-1)^{(m+1)/2} \frac{R^{\ell 4}}{k\ \xi^\ell_m} \frac{\phi^{j'}_{mu}}{\Psi'_m} \frac{1}{\cosh \alpha^\ell_m} [$$

$$(\nu-1)\ \alpha^\ell_m \tanh \alpha^\ell_m + 2 + 8\ \alpha^{\ell 2}_m \Phi^\ell (1-\nu)]$$

$$K^j_{mc} = \sum_{m=1,3,\ldots}^{\infty} \frac{-128}{\pi^5} (-1)^{(m+1)/2} \frac{R^{\ell 2}\ R^{r2}}{k} \frac{\phi^{j'}_{mu}}{\Psi'_m}$$

Numerical values for these factors are tabulated in Tables C.44 to C.85.

3.5 CONTINUOUS SANDWICH PLATES BY THE FINITE DIFFERENCE METHOD

In the previous chapter, the finite difference technique was effectively applied in conjunction with the partial deflection theory to solve simply supported sandwich plates. The same approach may be used to determine an approximate solution for the deflection of continuous sandwich plates subjected to lateral loads.

For this purpose, consider for example the continuous plate in Fig. 3.8 (a) which consists of two square panels of equal spans, and subjected to a uniformly distributed load on the left one. The finite difference mesh used is shown in Fig. 3.8 (b).

By applying the funicular polygon's stencil (Fig. 2.5) on the partial deflection equation (2.73), the bending deflection ordinates at the finite difference mesh points is determined from Equ. (2.76) as

$$\begin{Bmatrix} w_{b1} \\ w_{b2} \\ w_{b3} \\ w_{b4} \\ w_{b5} \\ w_{b6} \\ -w_{b7} \\ -w_{b8} \\ -w_{b9} \\ -w_{b10} \end{Bmatrix} = \frac{p_o^{\ell} \lambda^4}{D} \begin{Bmatrix} 0.488 \\ 0.677 \\ 0.488 \\ 0.677 \\ 0.941 \\ 0.677 \\ 0.407 \\ 0.208 \\ 0.573 \\ 0.294 \end{Bmatrix} \tag{3.57}$$

in which

λ = the finite difference mesh width

$= \frac{a}{4}$

where the coefficient and load matrices in Equ. (2.76) are given in Appendix D.

In a similar manner, by applying the higher approximation stencil (Fig. 2.4) on the partial deflection equation (2.74), the shear deflection ordinates at the finite difference mesh points are determined from Equ. (2.78) as

$$\begin{Bmatrix} w_{s1} \\ w_{s2} \\ w_{s3} \\ w_{s4} \\ w_{s5} \\ w_{s6} \end{Bmatrix} = \frac{12\,\lambda^2\; p_o^{\ell}}{S} \begin{Bmatrix} 0.0538 \\ 0.0696 \\ 0.0538 \\ 0.0696 \\ 0.0909 \\ 0.0696 \end{Bmatrix} \qquad (3.58)$$

where the coefficient and load matrices in Equ. (2.78) are given in Appendix D.

The approximate total deflection at point 5 of the finite difference mesh (Fig. 2.8 (b)) can be determined by combining the bending and shear components. For a shear parameter equal to 0.1, it is found from Equs. (3.57) and (3.58) that

$$w_5 = 104.9 \times 10^{-4} \frac{p_o^{\ell}\, a^4}{D} \qquad (3.59)$$

3.6 COMPARISON OF THE ANALYTIC AND FINITE DIFFERENCE RESULTS

The analytical solution for the central deflection of the left panel of the continuous sandwich plate in Fig. 3.8 (a) is determined by applying the principle of superposition :

$$w_5 = w_{5ss} + w_{5cm}$$

in which

w_{5ss} = the central deflection of a simply supported sandwich plate having the same properties as the left panel of the continuous plate in Fig. 3.8 (a), and subjected to the same load.

w_{5cm} = the central deflection of the left panel of the continuous sandwich plate in Fig. 3.8 (a) due to the redundant moment.

Thus, by using the practical formulas in Equs. (2.37) and (3.51), an expression for w_5 is obtained as

$$w_5 = (K_{wb} + \phi^{\ell} K_{ws} + K_w^{\ell\ell}) \frac{p_o^{\ell} a^4}{D}$$

With the numerical values for K_{wb} and K_{ws} from Table A.1, and for $K_w^{\ell\ell}$ from Table C.81, noting that $\phi^{\ell} = 0.1$, it is found that

$$w_5 = 112.7 \times 10^{-4} \frac{p_o^{\ell} a^4}{D} \tag{3.60}$$

Comparison of this result with that in Equ. (3.59) shows a reasonable agreement.

On the other hand, if the continuous plate (Fig. 3.8 (a)) is rigid in shear, the deflection w_5 is determined from the finite difference results in Equ. (3.57) as

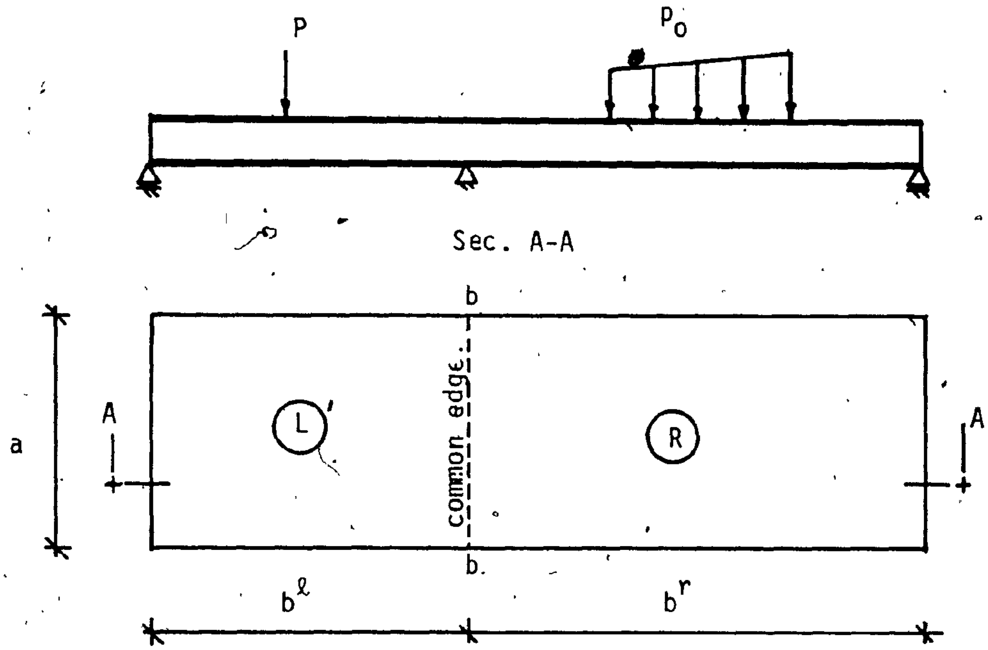
$$w_5 = 36.8 \times 10^{-4} \frac{p_o^{\ell} a^4}{D}$$

and from the analytic solution in Equs. (2.37) and (3.51) as

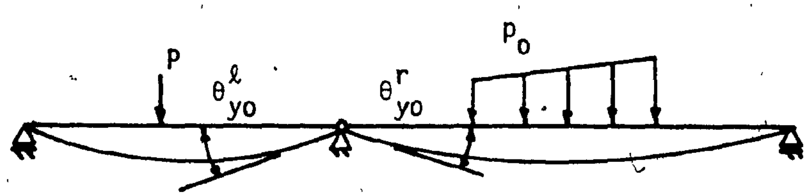
$$w_5 = (K_{wb} + K_w^{\ell\ell}) \frac{p_o^{\ell} a^4}{D}$$

$$= 35.0 \times 10^{-4} \frac{p_o^{\ell} a^4}{D}$$

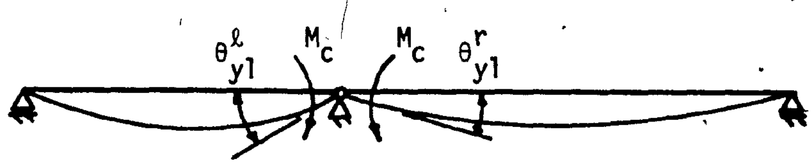
where $K_w^{\ell\ell}$ is obtained from Table C.80. The agreement in this case is also good.



(a) CONTINUOUS SANDWICH PLATE



(b) HINGED PANELS SUBJECTED TO EXTERNAL LOADS



(c) HINGED PANELS SUBJECTED TO REDUNDANT MOMENT

Fig. 3.1 - THE FORCE METHOD OF ANALYSIS

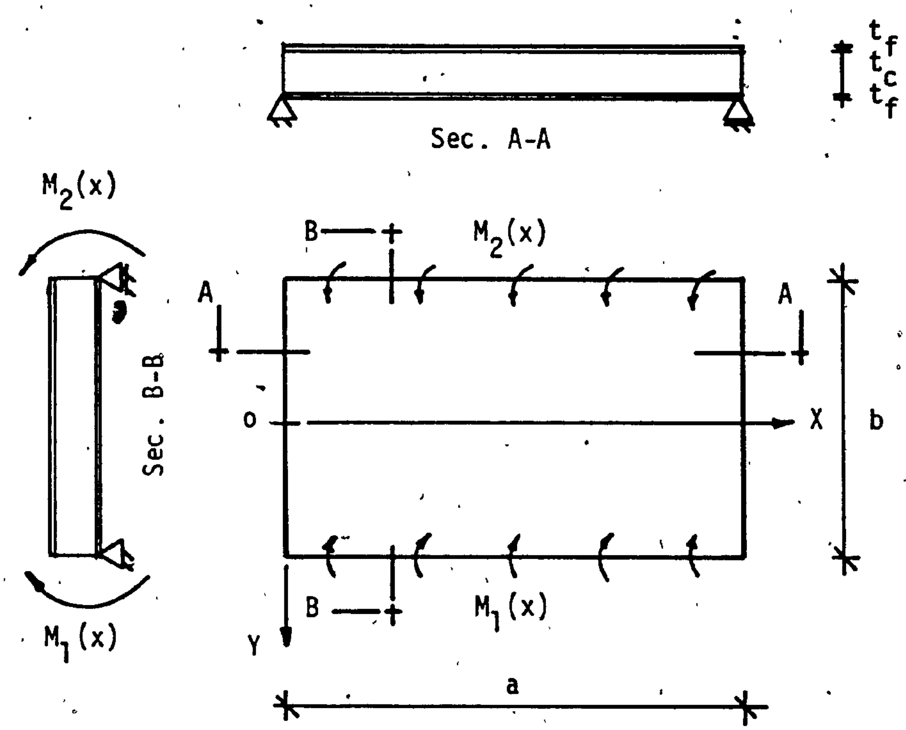
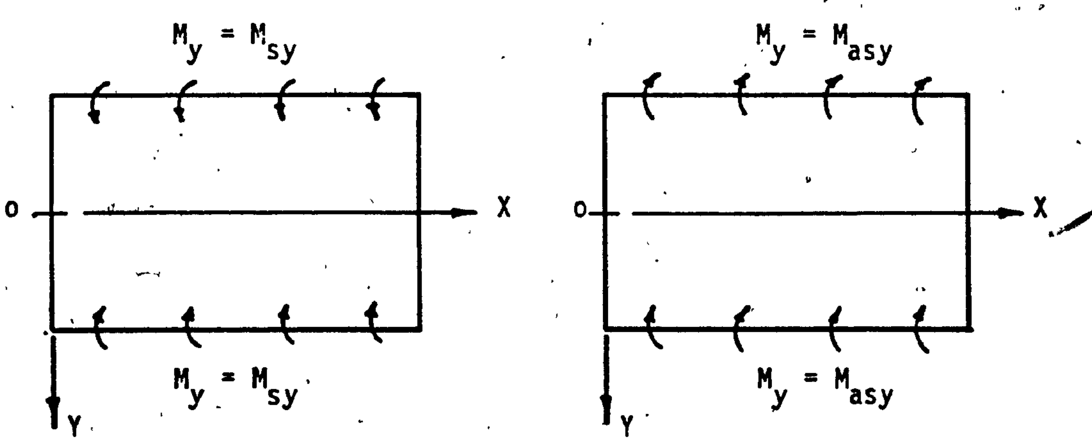


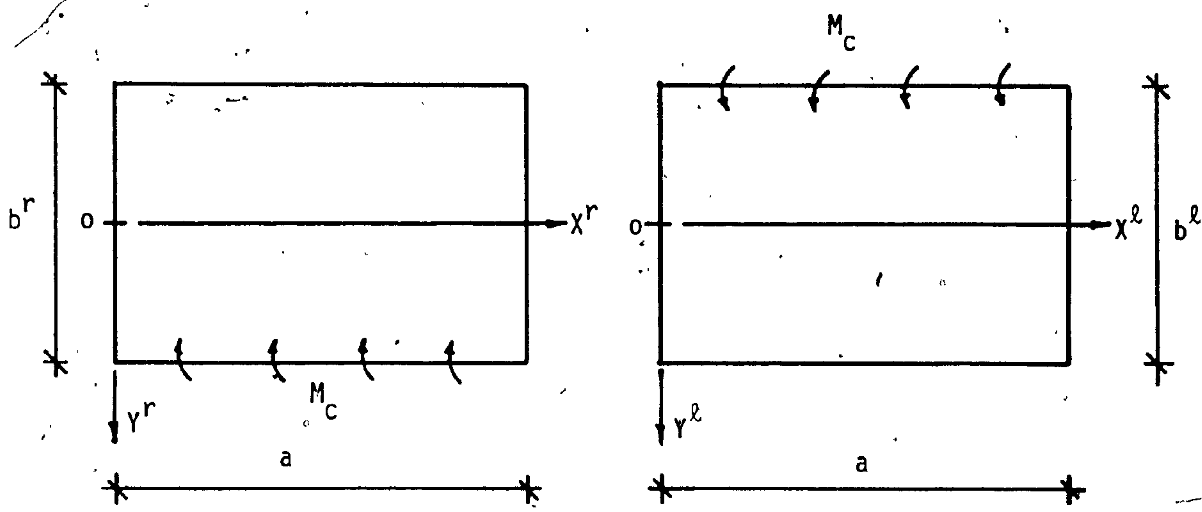
Fig. 3.2 - RECTANGULAR SANDWICH PLATES UNDER MOMENTS DISTRIBUTED ALONG PARALLEL EDGES



(a) SYMMETRICAL EDGE MOMENTS

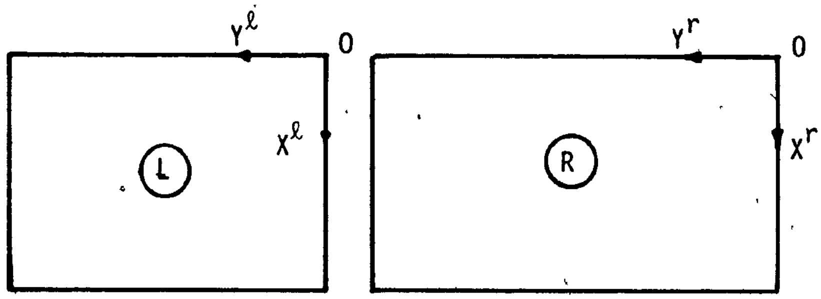
(b) ANTISYMMETRICAL EDGE MOMENTS

Fig. 3.3 - TWO CASES OF MOMENTS DISTRIBUTION ALONG SANDWICH PLATE EDGES

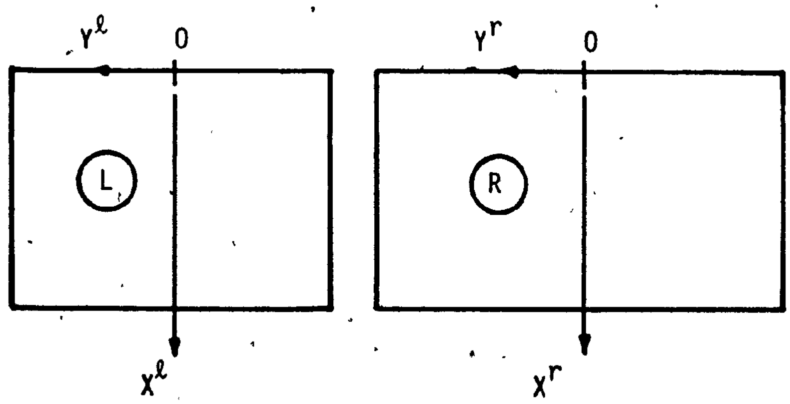


(a) $(M_y)_{y=b^r/2} = M_c$ (b) $(M_y)_{y=-b^\ell/2} = M_c$

Fig. 3.4 - SANDWICH PLATE UNDER EDGE MOMENTS



(a) SIMPLY SUPPORTED PLATE UNDER LATERAL LOAD



(b) SANDWICH PLATE UNDER EDGE MOMENT

Fig. 3.5 - COORDINATE AXES FOR SANDWICH PANELS

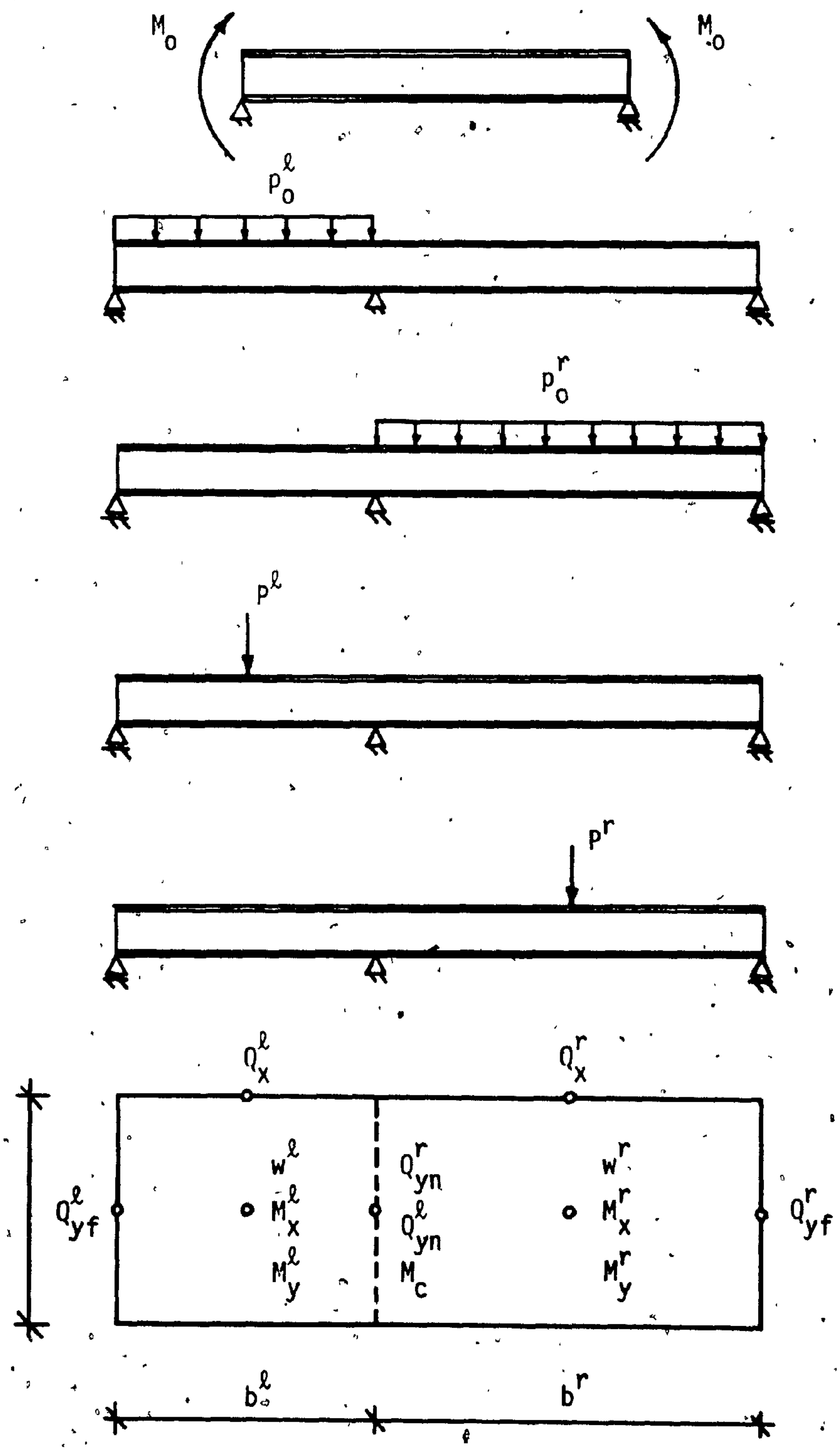


Fig. 3.6 - LOADING TYPES AND THE NOTATIONS USED IN THE PRACTICAL FORMULAS

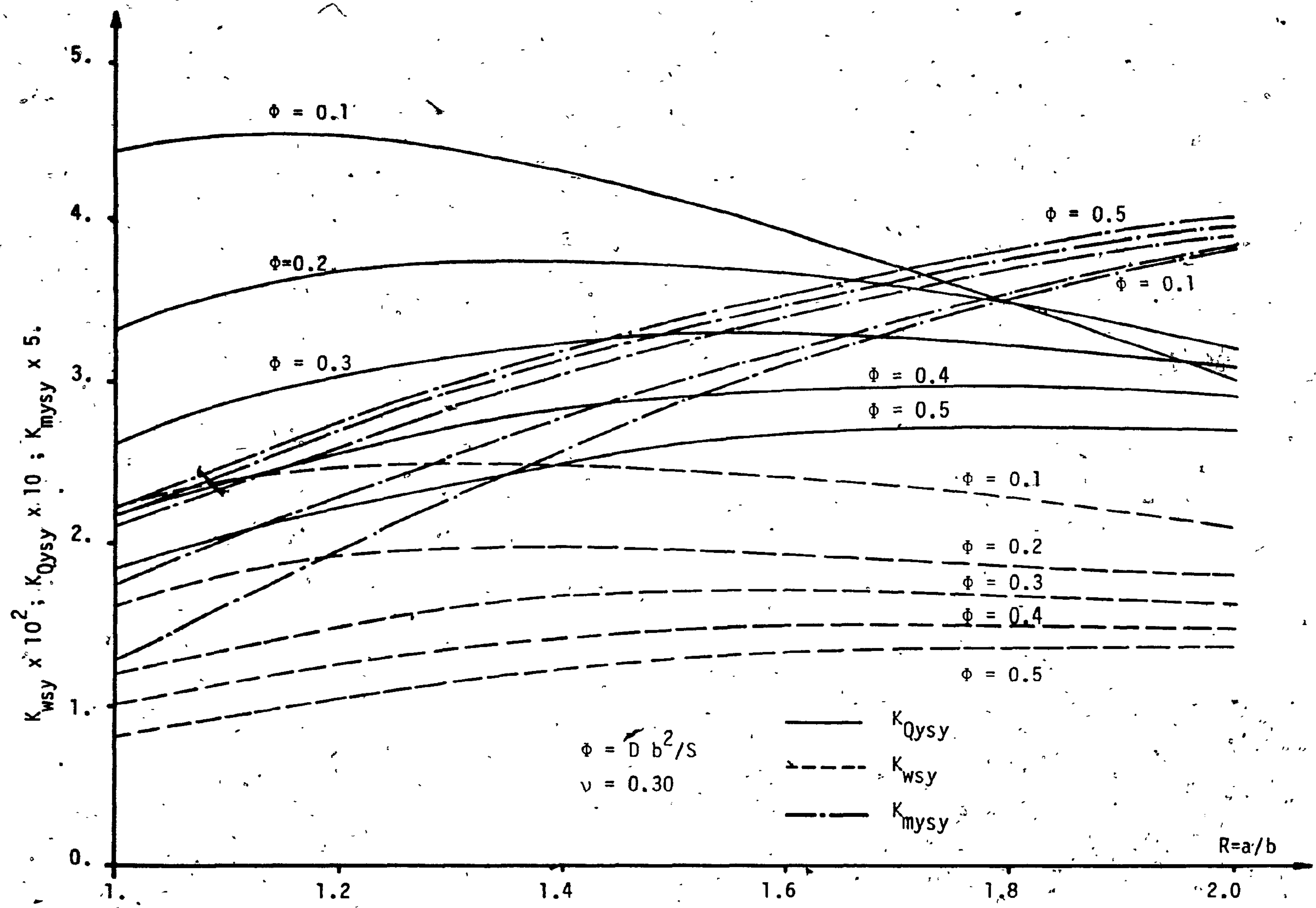
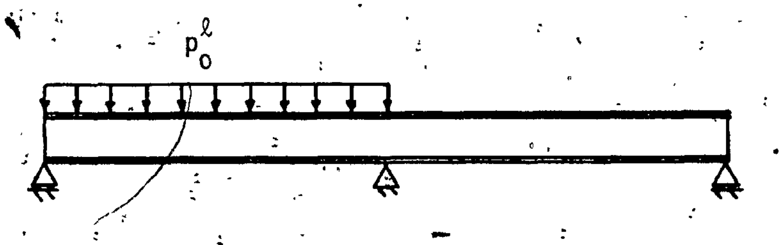
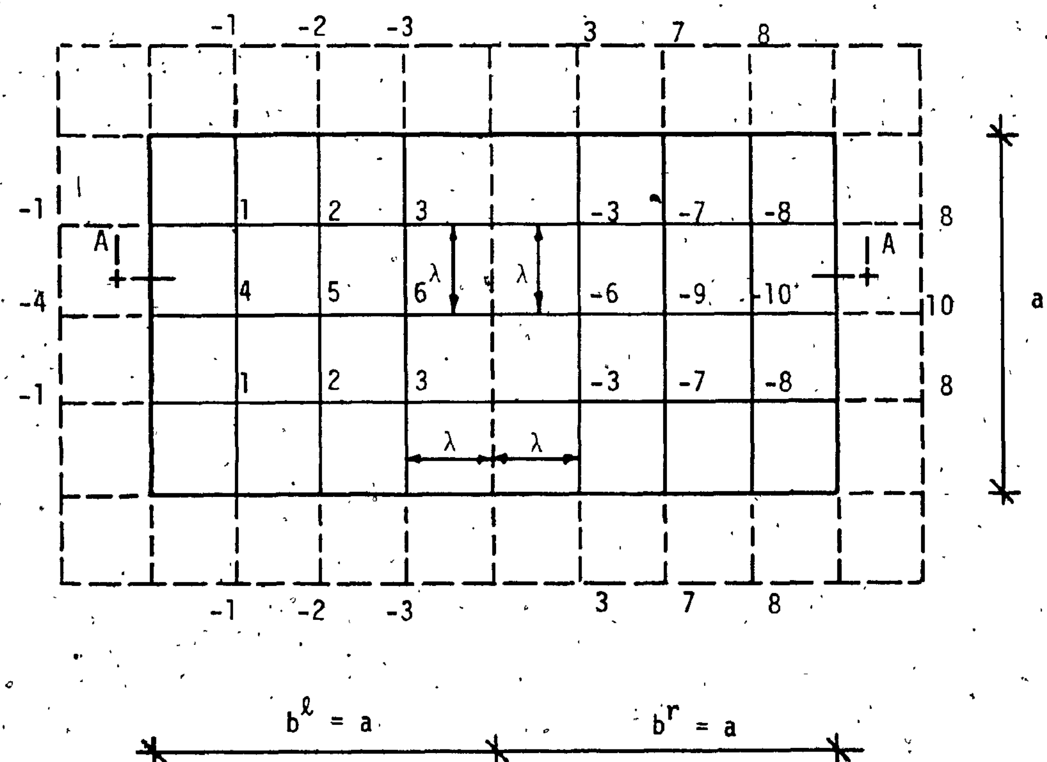


Fig. 3.7 - NUMERICAL VALUES FOR THE FACTORS IN EQUS. (3.40), (3.42), and (3.44)



Sec. A-A

(a) LATERAL LOAD



(b) NUMERING OF FINITE DIFFERENCE MESH POINTS

Fig. 3.8 - CONTINUOUS SANDWICH PLATE BY THE FINITE DIFFERENCE METHOD

CHAPTER IV

SANDWICH PLATES AND BEAM-COLUMNS WITH INTERLAYER ELASTIC DEFORMATIONS

CHAPTER IV

SANDWICH PLATES AND BEAM-COLUMNS WITH INTERLAYER ELASTIC DEFORMATIONS

4.1 INTRODUCTION

Sandwich plates were treated in the literature as well as in the preceding chapters with the view to determine their responses to different types of loadings and boundary conditions. The basically three-dimensional problem was reduced to a two-dimensional one by introducing a number of assumptions concerning stress or deformation distribution. One of the basic assumptions was that the constituent material layers are assembled with adhesives whose stiffnesses are such as to prevent any interlayer deformations. This ideal behaviour of bending materials may not be achieved in practice, since adhesives have a certain degree of flexibility. As the response of a sandwich construction depends on the stiffnesses of its constituents, the finite stiffness of the bonding material should thus be included in the analysis.

The problem of interlayer deformations has been investigated by a number of authors. With regard to layered wood systems, there are numerous theories well established and available in the literature. Amana and Booth [5], Goodman et al. [33, 34, 35, 104], Kuenzi and Wilkinson [60] conducted analytical and experimental studies on layered systems of various materials. In the first study, three types of plywood components were considered: T-beam type; double skin, double rib type; double skin, multiple rib type of constructions. In the second study, two types of wood components were considered: layered beams of three layers, and wood joist floor systems. In the third study, composite beams were considered. In all these studies, the material layers were assembled with nails or by gluing their ends, and although the interlayer slip in these systems was accounted for in the theoretical model, transverse shear deformations were neglected. The finite element method was used by Thompson et al. [98, 99] to study layered wood systems with interlayer slip. In this study, the total strain energy of a layered beam was obtained from the contribution of bending and axial deformations in each layer and slip deformation at the interfaces. On the other hand, interlaminar shear in composites under plane stress was investigated analytically by Puppo and Evensen [86], and with the finite element method by Isakson and Levy [51].

In this chapter, the effect of interlayer elastic deformations on the responses of sandwich plates and beam columns is investigated analytically and experimentally. The objective is to provide a means for a more rational and efficient design in terms of structural performance.

4.2 SIMPLY SUPPORTED SANDWICH PLATES UNDER LATERAL LOADS

Consider a sandwich plate composed of two thin faces each of thickness t_f and a core of thickness t_c shown in Fig. 4.1. The stress state in the faces and core elements is shown in Fig. 4.2.

As the need to consider interlayer deformations necessitates the consideration of compatibility of strains and stresses at the interfaces, the problem is irreducibly three dimensional and therefore is more complex than before. Several assumptions mentioned in Chapter II can still be retained, and new ones are introduced. These are :

(1) The faces are thin in comparison with the core depth. This implies that the flexural rigidity of each face about its own middle surface is negligible, and consequently, the inplane stresses resisted by each skin are uniformly distributed across its thickness. However, the facing material is still much stiffer than the core material, thus any inplane applied loading is resisted mainly by the skins.

(2) The two facings are of equal thickness and made of the same material.

(3) Materials are homogeous, isotropic, and linearly elastic.

(4) Deformations are small. The deformations in the direction of layer thicknesses are negligible.

(5) Poisson's ratio of the core material, and the inplane shear stress τ_{cxy} in the core can be ignored.

(6) Interlayer deformations are proportional to the interlayer shears. Thus

$$\Delta_x = \frac{q_x}{K_x} \qquad \Delta_y = \frac{q_y}{K_y} \tag{4.1}$$

in which

Δ_i = interlayer deformation in i- direction

K_i = stiffness of the employed adhesive in the i- direction. This value can be determined experimentally from the shear test described by Kuenzi and Wilkinson [60].

q_i = interlayer shear stress in the i- direction

i = subscript denoting x or y.

It is noted that the last two assumptions are new. Assumption (5) effectively implies the existence of a simplified three dimensional stress state in the core. The non-zero stress components are σ_{cx}, σ_{cy}, τ_{cxz}, τ_{cyz} and each need not be uniform across the core thickness. Thus, the associated displacement components u_c along x and v_c along y become functions of x, y and z. The effect of ignoring the Poisson's ratio is not at all serious in the case of light weight core materials [27, 32, 43, 46, 68, 74, 111]. Assumption (6) represents an idealized form of behaviour of the adhesive layer. This form has been used by other investigators whose works have been discussed in the introduction of this chapter.

4.2.1 GENERAL

When a sandwich plate or beam is subjected to applied loads, the different layers tend to slide over one another, and thus interlayer shear stresses are developed. The magnitude of these shear stresses at the interfaces is such as to maintain the compatibility of stresses and deformations.

Now consider the facings, which are assumed to be in a plane stress condition with the non-zero stress components σ_{fx}, σ_{fy}, and τ_{xy}. The standard differential equations of equilibrium for the facings (Fig. 4.2) are

$$\frac{\partial \sigma^j_{fx}}{\partial x} + \frac{\partial \tau^j_{yx}}{\partial y} \mp \frac{q_x}{t^j_f} = 0 \qquad (4.2)$$

$$\frac{\partial \sigma^j_{fy}}{\partial y} + \frac{\partial \tau^j_{xy}}{\partial x} \mp \frac{q_x}{t^j_f} = 0 \qquad (4.3)$$

in which

j = superscript denoting stresses in the face j, while t^j_f is the thickness of this face

f = subscript denoting face.

The negative and positive signs correspond to j=1 for the bottom face, and j=2 for the upper one.

The state of stresses in the core must satisfy the following equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{cx}}{\partial x} + \frac{\partial \tau_{czx}}{\partial z} &= 0 \\ \frac{\partial \sigma_{cy}}{\partial y} + \frac{\partial \tau_{czy}}{\partial z} &= 0 \end{aligned} \qquad (4.4)$$

in which

σ_{ci} = normal stress in the core in the i- direction

τ_{czi} = shear stress in the core

or in terms of the displacement components

$$E_{cx} \frac{\partial^2 u_c}{\partial x^2} + G_{cx} \frac{\partial^2 u_c}{\partial z^2} = 0 \tag{4.5}$$

$$E_{cy} \frac{\partial^2 v_c}{\partial y^2} + G_{cy} \frac{\partial^2 v_c}{\partial z^2} = 0 \tag{4.6}$$

in which

u_c, v_c = displacement in the core along the x- and y-directions respectively

E, G = the elastic and shear modulii of the core material

c = subscript denoting core.

The stress components in the facings and core must also satisfy the equilibrium equation in terms of the applied load, which is

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \, \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - p \tag{4.7}$$

in which

M_x, M_y and M_{xy} = the applied moments

p = the applied load intensity.

At the interfaces between the core and the skins, the stresses and strains must be compatible. The compatibility equations in terms of stresses are

$$q_x = G_{cx} \left(\frac{\partial u_c}{\partial z} + \frac{\partial w}{\partial x}\right)_{z=t_c/2}$$

$$q_y = G_{cy} \left(\frac{\partial v_c}{\partial z} + \frac{\partial w}{\partial y}\right)_{z=t_c/2} \qquad (4.8)$$

in which

t_c = the core depth

w = the lateral deflection.

In terms of strains, the compatibility equations are written as [5, 33, 34, 104]

$$\frac{\partial \Delta_x}{\partial x} = \varepsilon_{fx} - (\varepsilon_{cx})_{z=t_c/2} \qquad (4.9)$$

$$\frac{\partial \Delta_y}{\partial x} = \varepsilon_{fy} - (\varepsilon_{cy})_{z=t_c/2} \qquad (4.10)$$

in which

ε_{fi} = normal strain in the face along the i- direction

= $\frac{\partial u_f}{\partial x}$ along x- direction, and $\frac{\partial v_f}{\partial y}$ along y- direction

ε_{ci} = normal strain in the core along the i- direction

= $\frac{\partial u_c}{\partial x}$ along x- direction, and $\frac{\partial v_c}{\partial y}$ along y- direction

Solutions to the problem must also satisfy the prescribed boundary conditions. With respect to the simply supported sandwich plate subjected to transverse loads (Fig. 4.1), the relevant boundary conditions are :

(i) No deflection or normal stresses should exist at the plate edges, thus

$$\begin{aligned} &\text{at } x = 0,\ 2a \qquad \sigma_{fx} = 0 \text{ and } \sigma_{cx} = \gamma_y = 0 \quad \text{and } w = 0 \\ &\text{at } y = 0,\ 2b \qquad \sigma_{fy} = 0 \text{ and } \sigma_{cy} = \gamma_x = 0 \quad \text{and } w = 0 \end{aligned} \tag{4.11}$$

in which

2a, 2b = the plate dimensions in x- and y- directions, respectively

γ_i = the shear strain in the plane zi.

(ii) As the two facings are identical, the middle plane of the plate remains unstrained under loads, thus

$$\text{at } z = 0 \qquad u_c = v_c = 0 \tag{4.12}$$

(iii) For symmetrical loading about the plate center lines, the in-plane displacement components vanish along those lines, thus

$$\begin{aligned} &\text{at } x = a \qquad u_f = u_c = 0 \\ &\text{at } y = b \qquad v_f = v_c = 0 \end{aligned} \tag{4.13}$$

in which

u_f, v_f = the displacement in the facings along x- and y- directions, respectively.

4.2.2 SIMPLY SUPPORTED SANDWICH PLATES

Consider the simply supported sandwich plate in Fig. 4.1, made of two faces each of thickness t_f and a core of thickness t_c. The displacement components u_f and v_f which satisfy $u = 0$ and $v = 0$ in Equ. (4.13) are taken as [11, 82, 83, 105, 107]

$$u_f = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} A_{mn} \cos \alpha_m x \sin \beta_n y \tag{4.14}$$

$$v_f = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} B_{mn} \sin \alpha_m x \cos \beta_n y \tag{4.15}$$

in which

$$\alpha_m = \frac{m}{2}\frac{\pi}{a}$$

$$\beta_n = \frac{n}{2}\frac{\pi}{b}$$

A_{mn}, B_{mn} = unknown coefficients.

From Equs. (4.14) and (4.15), expressions for the stresses in the facings can be written as

$$\sigma_{fx} = \frac{E_f}{\nu^2-1} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (A_{mn}\alpha_m + \nu B_{mn}\beta_n) \sin \alpha_m x \sin \beta_n y \tag{4.16}$$

$$\sigma_{fy} = -\frac{E_f}{\nu^2-1} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (B_{mn}\,\beta_n + \nu A_{mn}\,\alpha_m) \sin\alpha_m x \;\sin\beta_n y \qquad (4.17)$$

$$\tau_{xy} = \frac{E_f}{2(1+\nu)} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} (A_{mn}\,\beta_n + B_{mn}\,\alpha_m) \cos\alpha_m x \;\cos\beta_n y \qquad (4.18)$$

in which

E_f = elastic modulus of the face material

ν = Poisson's ratio of the facings material

It is noted that the expressions in Equs. (4.16), (4.17) and (4.18) satisfy the boundary conditions $\sigma_{fx} = 0$ and $\sigma_{fy} = 0$ in Equ. (4.11).

Expressions for the interlayer shear stresses q_x and q_y are obtained by differentiating the above stresses in accordance with the equilibrium equations (4.2) and (4.3) as

$$q_x = \frac{E_f\,t_f}{1+\nu} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \left(\frac{A_{mn}\,\alpha_m^2 + \nu B_{mn}\,\beta_n\,\alpha_m}{\nu-1} - \frac{A_{mn}\,\beta_n^2 + B_{mn}\,\alpha_m\,\beta_n}{2}\right) \cos\alpha_m x \;\sin\beta_n y \qquad (4.19)$$

$$q_y = \frac{E_f\,t_f}{1+\nu} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \left(\frac{B_{mn}\,\beta_n^2 + \nu A_{mn}\,\alpha_m\,\beta_n}{\nu-1} - \frac{A_{mn}\,\beta_n\,\alpha_m + B_{mn}\,\alpha_m^2}{2}\right) \sin\alpha_m x \;\cos\beta_n y \qquad (4.20)$$

On the other hand, a solution for the core displacement satisfying the equilibrium equations (4.5) and (4.6) as well as the boundary conditions $u_c = 0$ and $v_c = 0$ in Equs. (4.12) and (4.13), $\sigma_{cx} = 0$ and $\sigma_{cy} = 0$ in Equ. (4.11) is found as [48]

$$u_c = \sum_{m=1,3,\ldots}^{\infty} C_m \cos \alpha_m x \sinh \mu_x \alpha_m z \qquad (4.21)$$

$$v_c = \sum_{n=1,3,\ldots}^{\infty} D_n \cos \beta_n y \sinh \mu_y \beta_n z \qquad (4.22)$$

in which

$$\mu_x = \left(\frac{E_{cx}}{G_{cx}}\right)^{1/2}$$

$$\mu_y = \left(\frac{E_{cy}}{G_{cy}}\right)^{1/2}$$

C_m, D_n = unknown functions of y and x, respectively.

The functions C_m and D_n can be expressed in terms of A_{mn} and B_{mn} by using the compatibility equations (4.9) and (4.10). By substituting Equs. (4.1), (4.14), (4.19) and (4.21) in Equ. (4.9) an expression for C_m is obtained as

$$C_m = \sum_{n=1,3,\ldots}^{\infty} \frac{1}{\sinh (\mu_x \alpha_m t_c/2)} \left[\frac{E_f t_f}{K_x (1+\nu)} \left(\frac{A_{mn} \alpha_m^2 + \nu B_{mn} \beta_n \alpha_m}{1 - \nu}\right.\right.$$

$$+ \frac{A_{mn}\beta_n^2 + B_{mn}\alpha_m\beta_n}{2}) + A'_{mn}] \sin \beta_n y \qquad (4.23)$$

In a similar manner, from Equs. (4.1), (4.15), (4.20), (4.22) and (4.10) it follows that

$$D_n = \sum_{m=1,3,...}^{\infty} \frac{1}{\sinh(\mu_y \beta_n t_c/2)} [\frac{E_f t_f}{K_y (1+\nu)} (\frac{B_{mn}\beta_n^2 + \nu A_{mn}\alpha_m\beta_n}{1-\nu}$$

$$+ \frac{A_{mn}\beta_n\alpha_m + B_{mn}\alpha_m^2}{2}) + B_{mn}] \sin \alpha_m x \qquad (4.24)$$

The coefficient B_{mn} can be also expressed in terms of A_{mn} by using the compatibility equations of stresses (4.8). The deflection function for simply supported sandwich plates which satisfies $w = 0$ in Equ. (4.11) is taken as

$$w = \sum_{m=1,3,...}^{\infty} W_m \sin \alpha_m x \sin \beta_n y$$

By proper substitution of w, q_x and u_c in the first equation of (4.8), and by equating the coefficients of $\cos \alpha_m x \sin \beta_n y$ in both of its sides, it is found that

$$W_m = \frac{E_f t_f}{G_{cx}(1+\nu)} (\frac{A_{mn}\alpha_m + \nu B_{mn}\beta_n}{\nu - 1} - \frac{A_{mn}\beta_n^2 + B_{mn}\alpha_m\beta_n}{2\alpha_m}$$

$$- \mu_x (\coth \frac{1}{2} \mu_x \alpha_m t_c)[\frac{E_f t_f}{K_x (1+\nu)} (\frac{A_{mn}\alpha_m^2 + \nu B_{mn}\beta_n\alpha_m}{1-\nu}$$

$$+ \frac{A_{mn} \beta_n^2 + B_{mn} \alpha_m \beta_n}{2}) + A_{mn}] \tag{4.25a}$$

In a similar manner, from Equs. (4.20), (4.22), and the second equation in (4.8), it is found that

$$w_m = \frac{E_f t_f}{G_{cy} (1+\nu)} \left(\frac{B_{mn} \beta_n + \nu A_{mn} \alpha_m}{\nu - 1} - \frac{A_{mn} \alpha_m \beta_n + B_{mn} \alpha_m^2}{2 \beta_n} \right)$$

$$- \mu_y (\coth \frac{1}{2} \mu_y \beta_n t_c) [\frac{E_f t_f}{K_y (1+\nu)} (\frac{B_{mn} \beta_m^2 + \nu B_{mn} \alpha_m \beta_n}{1 - \nu}$$

$$+ \frac{A_{mn} \beta_n \alpha_m + B_{mn} \alpha_m^2}{2}) + B_{mn}] \tag{4.25b}$$

By equating the right hand side of Equs. (4.25a) and (4.25b), it follows that

$$B_{mn} = \frac{\phi_{mn}}{\psi_{mn}} A_{mn} \tag{4.26}$$

in which

$$\phi_{mn} = \frac{E_f t_f}{G_{cy} (1+\nu)} [- \frac{1}{2} \alpha_m + \frac{G_{cy}}{2 G_{cx}} \frac{\beta_n^2}{\alpha_m}$$

$$+ \frac{G_{cy} \mu_y}{2 K_y} \alpha_m \beta_n \frac{\nu + 1}{\nu - 1} \coth \frac{1}{2} \mu_y \beta_n t_c$$

$$+ \frac{G_{cy} \mu_x}{2 K_x} (\frac{2 \alpha_m^2}{1-\nu} + \beta_n^2) \coth \frac{1}{2} \mu_x \alpha_m t_c$$

$$+ \frac{G_{cy}}{G_{cx}} \frac{\alpha_m}{1-\nu} + \frac{\nu\, \alpha_m}{\nu - 1}$$

$$+ \frac{G_{cy}\,(1+\nu)}{E_f\, t_f} \mu_x \coth \frac{1}{2} \mu_x \alpha_m t_c]$$

$$\Psi_{mn} = \frac{E_f\, t_f}{G_{cx}\,(1+\nu)} [- \frac{1}{2} \beta_n \frac{G_{cx}}{2\, G_{cy}} \frac{\alpha_m^2}{\beta}$$

$$+ \frac{G_{cx}\, \mu_x}{2\, K_x} \alpha_m \beta_n \frac{\nu + 1}{\nu - 1} \coth \frac{1}{2} \mu_x \alpha_m t_c$$

$$+ \frac{G_{cx}\, \mu_y}{2\, K_y} (\frac{2\, \beta_n^2}{1-\nu} + \alpha_m^2) \coth \frac{1}{2} \mu_y \beta_n t_c$$

$$+ \frac{G_{cx}}{G_{cy}} \frac{\beta_n}{1-\nu} + \frac{\nu\, \beta_n}{\nu - 1}$$

$$+ \frac{G_{cx}\,(1+\nu)}{E_f\, t_f} \mu_y \coth \frac{1}{2} \mu_y \beta_n t_c]$$

The only unknown coefficient now is A_{mn}. This can be determined by using the equilibrium equation (4.7). The moments M_x, M_y and M_{xy} can be expressed in terms of the stresses components as

$$M_x = \sigma_{fx}\, t_f\, h + \int_{-t_c/2}^{t_c/2} \sigma_{cx}\, z\, dz$$

$$M_y = \sigma_{fy}\, t_f\, h + \int_{-t_c/2}^{t_c/2} \sigma_{cy}\, z\, dz$$

$$M_{xy} = -\tau_{xy}\, t_f\, h$$

in which

$$\left.\begin{aligned} h &= t_c + t_f \\ \sigma_{cx} &= E_{cx}\frac{\partial u_c}{\partial x} \\ \sigma_{cy} &= E_{cy}\frac{\partial v_c}{\partial y} \end{aligned}\right\}$$

In terms of these stresses, the equilibrium equation (4.7) becomes

$$t_f\, h\left(\frac{\partial^2\sigma_{fx}}{\partial x^2} + 2\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} + \frac{\partial^2\sigma_{fy}}{\partial y^2}\right)$$

$$+ E_{cx}\frac{\partial^2}{\partial x^2}\int_{-t_c/2}^{t_c/2}\frac{\partial u_c}{\partial x}\, z\, dz + E_{cy}\frac{\partial^2}{\partial y^2}\int_{-t_c/2}^{t_c/2}\frac{\partial v_c}{\partial y}\, z\, dz$$

$$= -p$$

By expanding the applied load in double trigonometric series, and by substituting Equs. (4.16), (4.17), (4.18), (4.21), (4.22), and by considering Equs. (4.23), (4.24), and (4.26) in the above equation, an expression for A_{mn} is obtained as

$$A_{mn} = \frac{P_{mn}}{\xi_{mn}} \qquad (4.27)$$

in which

P_{mn} = the coefficient of Fourier expansion of the applied load.

$$\xi_{mn} = \frac{E_f\ t_f\ h}{\nu^2 - 1} \left(\alpha_m^3 + \nu\beta_n\ \alpha_m^2 \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$- \frac{E_f\ t_f\ h}{1 + \nu} \left(\alpha_m\ \beta_n^2 + \alpha_m^2\ \beta_n \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$+ \frac{E_f\ t_f\ h}{\nu^2 - 1} \left(\nu\alpha_m\ \beta_n^2 + \beta_n^3 \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$+ \frac{2\ E_{cx}\ \mu_m\ C'_{mn}}{\mu_x^2} \left(\frac{1}{2}\mu_x\ \alpha_m\ t_c \cosh \frac{1}{2}\mu_x\ \alpha_m\ t_c\right.$$

$$\left.- \sinh \frac{1}{2}\mu_x\ \alpha_m\ t_c\right) + \frac{2\ E_{cy}\ \beta\ D'_{mn}}{\mu_y^2} \Big($$

$$\left.\frac{1}{2}\mu_y\ \beta_n\ t_c \cosh \frac{1}{2}\mu_y\ \beta_n\ t_c - \sinh \frac{1}{2}\mu_y\ \beta_n\ t_c\right)$$

where

$$C'_{mn} = \frac{1}{\sinh \frac{1}{2}\mu_x\ \alpha_m\ t_c} \left[\frac{E_f\ t_f}{K_x\ (1+\nu)} \left(\frac{\alpha_m^2 + \nu\alpha_m\ \beta_n\ \phi_{mn}/\Psi_{mn}}{\nu - 1}\right.\right.$$

$$\left.\left.- \frac{\beta_n^2 + \alpha_m\ \beta_n\ \phi_{mn}/\Psi_{mn}}{2}\right) - 1\right]$$

$$D'_{mn} = \frac{1}{\sinh \frac{1}{2}\mu_y\ \beta_n\ t_c} \left[\frac{E_f\ t_f}{K_y\ (1+\nu)} \left(\frac{\nu\alpha_m\ \beta_n + \beta_n^2\ \phi_{mn}/\Psi_{mn}}{\nu - 1}\right.\right.$$

$$-\frac{\alpha_m \beta_n + \alpha_m^2 \phi_{mn}/\Psi_{mn}}{2}) - \frac{\phi_{mn}}{\Psi_{mn}}]$$

In the particular case of isotropic core and adhesive materials :

$$G_{cx} = G_{cy} = G_c$$

$$E_{cx} = E_{cy} = E_c$$

$$K_x = K_y = K$$

hence,

$$\mu_x = \mu_y = (E_c/G_c)^{1/2} = \mu$$

and simplified formulas can be obtained as will be shown in subsequent sections.

4.3 PARAMETRIC STUDY OF ADHESIVE EFFECTS ON THE BEHAVIOUR OF SANDWICH PLATES

The complexity of the above solution makes it difficult to see the effects of adhesive on the plate responses. To illustrate these effects, a simply supported square sandwich plate is considered (Fig. 4.3). The plate is made of two aluminum faces, a plastic foam core, and assembled with an isotropic adhesive. The facings and core properties are :

(i) for facings

t_f = .04 in.

E_f = 10^7 psi

ν = .33

(ii) for core

$t_c = 2.0$ in.

$E_c = 2 \times 10^4$ psi

$G_c = 10$ psi

(iii) for the plate

$2a = 2b = 40$ in.

Thus

$$\mu = (E_c/G_c)^{1/2} = \sqrt{2}$$

In this particular case the expressions developed for ϕ_{mn}, Ψ_{mn}, and ξ_{mn} become

$$\phi_{mn} = \frac{E_f\ t_f}{G_c\ (1+\nu)} \left[\frac{1}{2}\ \alpha_m + \frac{1}{2}\left(\frac{\beta_n^2}{\alpha_m}\right)\right.$$

$$+ \frac{G_c\ \mu}{2\ K}\ \alpha_m\ \beta_n\ \frac{\nu+1}{\nu-1}\ \coth\frac{1}{2}\ \mu\ \beta_n\ t_c$$

$$+ \frac{G_c\ \mu}{2\ K}\ \left(\frac{2\ \alpha_m^2}{1-\nu} + \beta_n^2\right)\ \coth\frac{1}{2}\ \mu\ \alpha_m\ t_c$$

$$\left. + \frac{G_c\ (1+\nu)}{E_f\ t_f}\ \mu\ \coth\frac{1}{2}\ \mu\ \alpha_m\ t_c\right]$$

$$\Psi_{mn} = \frac{E_f\ t_f}{G_c\ (1+\nu)}\ \left[\frac{1}{2}\ \beta_n + \frac{1}{2}\ \frac{\alpha_m^2}{\beta_n}\right.$$

$$+ \frac{G_c\ \mu}{2\ K}\ \alpha_m\ \beta_n\ \frac{\nu+1}{\nu-1}\ \coth\frac{1}{2}\ \mu\ \alpha_m\ t_c$$

$$+ \frac{G_c \mu}{2K} \left(\frac{2\beta_n^2}{1-\nu} + \alpha_m^2\right) \coth \frac{1}{2} \mu \beta_n t_c$$

$$+ \frac{G_c (1+\nu)}{E_f t_f} \mu \coth \frac{1}{2} \mu \beta_n t_c$$

$$\xi'_{mn} = \frac{E_f t_f h}{\nu^2 - 1} \left(\alpha_m^3 + \nu\beta_n \alpha_m^2 \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$- \frac{E_f t_f h}{1+\nu} \left(\alpha_m \beta_n^2 + \alpha_m^2 \beta_n \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$+ \frac{E_f t_f h}{\nu^2 - 1} \left(\nu\alpha_m \beta_n^2 + \beta_n^3 \frac{\phi_{mn}}{\Psi_{mn}}\right)$$

$$+ 2 G_c \alpha_m C'_m \left(\frac{1}{2} \mu \alpha_m t_c \cosh \frac{1}{2} \mu \alpha_m t_c - \sinh \frac{1}{2} \mu \alpha_m t_c\right)$$

$$+ 2 G_c \beta_n D'_n \left(\frac{1}{2} \mu \beta_n t_c \cosh \frac{1}{2} \mu \beta_n t_c - \sinh \frac{1}{2} \mu \beta_n t_c\right)$$

in which

$$C'_{mn} = \frac{1}{\sinh \frac{1}{2} \mu \alpha_m t_c} \left[\frac{E_f t_f}{K(1+\nu)} \left(\frac{\alpha_m^2 + \nu\alpha_m \beta_n \phi_{mn}/\Psi_{mn}}{\nu - 1}\right.\right.$$

$$\left.\left.- \frac{\beta_n^2 + \alpha_m \beta_n \phi_{mn}/\Psi_{mn}}{2}\right) - 1\right]$$

$$D'_{mn} = \frac{1}{\sinh \frac{1}{2} \mu \beta_n t_c} \left[\frac{E_f t_f}{K(1+\nu)} \left(\frac{\nu\alpha_m \beta_n + \beta_n^2 \phi_{mn}/\Psi_{mn}}{\nu - 1}\right.\right.$$

$$- \frac{\alpha_m \beta_n + \alpha_m^2 \phi_{mn}/\Psi_{mn}}{2}) - \frac{\phi_{mn}}{\Psi_{mn}}]$$

Three loading types are considered : uniformly distributed load of intensity p_0 (Fig. 4.3 (a)), partial load on a central square area (Fig. 4.3 (b)), and a central concentrated load P (Fig. 4.3 (c)). Because of the symmetry in the plate geometry and the applied loads :

$$\left.\begin{array}{lllll}
(\sigma_{fx})_{\substack{x=a\\y=a}} & = & (\sigma_{fy})_{\substack{x=a\\y=a}} & = & \sigma_f \\[2ex]
(\sigma_{cx})_{\substack{x=a\\y=a\\z=t_c/2}} & = & (\sigma_{cy})_{\substack{x=a\\y=a\\z=t_c/2}} & = & \sigma_c \\[2ex]
(q_x)_{\substack{x=0,2a\\y=a\\z=t_c/2}} & = & (q_y)_{\substack{x=a\\y=0,2a\\z=t_c/2}} & = & q \\[2ex]
(\tau_{xz})_{\substack{x=0,2a\\y=a\\z=0}} & = & (\tau_{yz})_{\substack{x=a\\y=0,2a\\z=0}} & = & \tau_c \\[2ex]
(\tau_{xy})_{\substack{x=0,2a\\y=0,2b}} & & & = & \tau_{xy}
\end{array}\right\} \qquad (4.28)$$

in which

σ_f, σ_c = the normal stress in the facings and core, respectively, at the plate center

q, τ_c = the interlayer shear stress and transverse shear stress in the core, respectively, at the mid edges of the plate

τ_{xy} = the shear stress in the facings at the plate corners.

These stresses, in addition to the central deflection, are calculated in each load case for a chosen range of adhesive stiffnesses from 10^3 to 10^4 psi/in. The results are shown graphically in Figs. 4.4, 4.5, 4.6 and 4.7. It is seen that the deflection shows greater sensitivity to the variation of the K value when the latter is in the lower range; and beyond a certain level of adhesive stiffness, the bonding can be practically considered as rigid. In the case of uniform loading (Fig. 4.4), for example, a change in K value from 3000 to 2000 psi/in. induces a deflection increase fourteen times greater than that when K changes from 10 000 to 9000 psi/in. On the other hand, an increase in the adhesive stiffness is accompanied by a decrease in the normal stress of the core. The resulting loss in the resisting moment is compensated by a slight increase in the faces normal stress. In all cases, the interlaminar shear stress is practically independent of the adhesive stiffness. The effects on the normal and shear stress distributions in the core are shown in Fig. 4.8. It is seen that by increasing the K value the normal stress, σ_c, is reduced, and the shear stress, τ_c, changes to linear distribution.

The above discussion has yet to bring an important point. By using the existing theories [4, 84], normal and shear stresses in the facings, and the transverse shear stress in the core of a sandwich plate can be determined with a small margin of error. The adhesive stiffness should be included in the analysis whenever the deflection is the quantity of interest. To illustrate this important point, consider the sandwhich plate in Fig. 4.3 under a uniformly distributed load of unit intensity. For an adhesive stiffness of 4000 psi/in., the deflection and stresses in the plate are obtained with the present theory as

$$w = 0.0313 \text{ in.}$$
$$\sigma_f = 950. \text{ psi.}$$
$$\tau_{xy} = 592. \text{ psi.}$$
$$\tau_c = 6.7 \text{ psi.}$$

The corresponding values as obtained from an existing theory of sandwich plates [84] are

$$w = 0.0168 \text{ in.}$$
$$\sigma_f = 939. \text{ psi.}$$
$$\tau_{xy} = 610. \text{ psi.}$$
$$\tau_c = 6.6 \text{ psi.}$$

Comparison of these two sets of results indicates that the deflection determined by the existing theory is under estimated by 46%, whereas the stresses of the two are comparable.

As an indication of the correctness of the results obtained with the present theory, two particular cases where solutions exist are considered: (i) square sandwich plate with rigid adhesive, and (ii) square sandwich plate with rigid core and adhesive. In both cases, the plate is subjected to the three loading types as shown in Fig. 4.3, where unit value is assigned to both p_0 and P. The K value is taken to be 10^9 psi/in., so is the core rigidity. The normal stress σ_f, shear stresses τ_{xy} and τ_c, and the deflection w are calculated in each case. The results are shown in Table 4.1 for case (i), and in Table 4.2 for case (ii). It is seen that there is close agreement between the results by the present theory and by others.

4.4 ADHESIVE EFFECTS ON THE RESPONSE OF SANDWICH BEAM-COLUMNS

4.4.1 GENERAL

A simply supported beam of span 2a is subjected to a uniform load of intensity p_0, a concentrated load Q at mid span and axial forces P, as shown in Fig. 4.9 (a). Beam imperfection is assumed in the form of a parabolic distribution as

$$w_0 = 2\,\delta\,\frac{x}{a}\left(1 - \frac{x}{a}\right) \tag{4.29}$$

in which

w_0 = initial imperfection

δ = amplitude of w_0 at mid span (Fig. 4.9 (b)).

The assumptions mentioned in section 4.2 can still be retained here. The behaviour of the sandwich beam in Fig. 4.9 (a) is a special case of that for the plate in Fig. 4.1, where the stresses and deformations are considered only in the transverse plane zx. Therefore, it does not seem necessary to represent the conditions governing the behaviour of the beam in a separate section, and the solution procedure is presented next.

4.4.2 SANDWICH BEAM-COLUMN

The equilibrium of a core element requires that

$$\frac{\partial \sigma_c}{\partial x} + \frac{\partial \tau}{\partial z} = 0 \tag{4.30}$$

in which

σ_c = normal stress in the core

τ = transverse shear stress in the core

x,z = coordinate axes as shown in Fig. 4.9 (a).

The stresses are related to the core deformations by

$$\sigma_c = E_c \frac{\partial u}{\partial x} \tag{4.31}$$

$$\tau = G \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{4.32}$$

in which

u, w = displacement of a point in the core along x- and z- axes, respectively

E_c, G = elastic and shear modulii of the core material, respectively.

Substituting Equs. (4.31) and (4.32) into Equ. (4.30) and by making use of assumption (4) in section 4.2 results in the following equilibrium equation in terms of displacement

$$E_c \frac{\partial^2 u}{\partial x^2} + G_c \frac{\partial^2 u}{\partial z^2} = 0 \tag{4.33}$$

It is noted that Equs. (4.30) and (4.33) have the same forms as the first equation in Equ. (4.4) and Equ. (4.5), respectively. Thus, a solution which satisfies the following boundary conditions

$$\text{at } x = a \text{ and } z = 0; \qquad u = 0 \tag{4.34}$$

$$\text{at } x = 0 \text{ and } x = 2a; \qquad \frac{\partial u}{\partial x} = 0 \tag{4.35}$$

and Equ. (4.33) would have the form of Equ. (4.21) that is [48]

$$u = \sum_{m=1,3,\ldots}^{\infty} C_m \cos \alpha_m x \sinh \mu \alpha_m z \tag{4.36}$$

in which

$$\alpha_m = \frac{m\pi}{2a}$$

$$\mu = (E_c/G)^{1/2}$$

C_m = unknown coefficient.

At the adhesive layer, compatibility of deformation and shear stress must be satisfied. The compatibility equation in terms of normal strains is

$$\frac{d\Delta}{dx} = \varepsilon_f - (\varepsilon_c)_{z=t_c/2} \tag{4.37}$$

in which

Δ = interlayer deformation

ε_i = normal strain of the i- layer

f, c = subscripts denoting face and core, respectively.

and in terms of shear stress is

$$\frac{q}{b} = G \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_{z=t_c/2} \tag{4.38}$$

in which

q = interlayer shear flux

b = beam width.

Equations (4.37) and (4.38) are similar to the compatibility equations (4.9) and (4.8), respectively.

The equilibrium equation of a face element (Fig. 4.9 (c)) requires that

$$q = \frac{dN}{dx} \tag{4.39}$$

in which

N = normal force in the facing.

According to assumption (6) in section 4.2, the shear flux q would introduce an interlayer deformation of amount

$$\Delta = \frac{q}{K} \tag{4.40}$$

in which

K = adhesive stiffness.

Equations (4.39) and (4.40) give

$$\frac{d\Delta}{dx} = \frac{1}{K}\frac{d^2N}{dx^2} \quad (4.41)$$

Substitution of Equs. (4.36) and (4.41) into Equ. (4.37), noting that $\epsilon_a = N/E_f A_f$, yields

$$\frac{d^2N}{dx^2} - \frac{K N}{E_f A_f} - K \sum_{m=1,3,...}^{\infty} \alpha_m C_m \sin \alpha_m x \sinh \frac{1}{2} \mu \alpha_m t_c = 0 \quad (4.42)$$

in which

A_f = the face area

= t_f b, where t_f is the face thickness.

The boundary conditions to be satisfied by N are

$$\begin{aligned} &\text{at } x = 0,\ 2a \qquad && N = \frac{P}{2} \\ &\text{at } x = a && \frac{dN}{dx} = 0 \end{aligned} \quad (4.43)$$

Thus, the solution for N may be written as

$$N = \frac{1}{2} P (\cosh \theta x - \tanh \theta a \sin \theta x) - \sum_{m=1,3,...}^{\infty} N_m C_m \sin \alpha_m x \quad (4.44)$$

in which

$$N_m = \frac{\alpha_m \sinh \frac{1}{2} \mu \alpha_m t_c}{\frac{1}{E_f A_f} + \frac{\alpha_m^2}{K}}$$

$$\theta = \left(\frac{K}{E_f A_f}\right)^{1/2}$$

For the symmetrical loading shown in Fig. 4.9 (a), the deflection curve takes the form :

$$w = \sum_{m=1,3,...}^{\infty} W_m \sin \alpha_m x \qquad (4.45)$$

in which

W_m = unknown coefficients.

The primary unknown now are C_m and W_m. The applied bending moment, M, at any section of the beam can be expressed as

$$M = \frac{Q}{2} x + p_o ax \left(1 - \frac{x}{2a}\right) + P (w + w_o) \qquad (4.46)$$

Substituting Equs. (4.29) and (4.45) and expressing the result in the form of sine series :

$$M = \sum_{m=1,3,...}^{\infty} \left[(-1)^{(m+3)/2} \frac{Q}{a \alpha_m^2} + \frac{2 p_o}{a \alpha_m^3} + P \left(\frac{4 \delta}{a^3 \alpha_m^3} + W_m\right)\right] \sin \alpha_m x \qquad ..... (4.47)$$

The resisting moment contributed by the core and facings is

$$M_r = \int_{-t_c/2}^{t_c/2} \sigma_c \, z \, b \, dz + Nh$$

$$= \int_{-t_c/2}^{t_c/2} E_c \frac{\partial u}{\partial x} \, z \, b \, dz + Nh \qquad (4.48)$$

in which

$$h = t_c + t_f$$

Equations (4.48) in conjunction with Equs. (4.36) and (4.44) give

$$M_r = \sum_{m=1,3,\ldots}^{\infty} \left[\frac{h P \alpha_m}{a (\alpha_m^2 + \theta^2)} - h N_m C_m - M_m C_m\right] \sin \alpha_m x \qquad (4.49)$$

in which

$$M_m = E_c b \left(\frac{t_c}{\mu} \cosh \frac{1}{2} \mu \alpha_m t_c - \frac{2}{\mu^2 \alpha_m} \sinh \frac{1}{2} \mu \alpha_m t_c\right)$$

Now equate the coefficients of $\sin \alpha_m x$ in Equs. (4.47) and (4.49) yielding the first equation for the determination of W_m and C_m :

$$(-1)^{(m+3)/2} \frac{Q}{a \alpha_m^2} + \frac{2 p_o}{a \alpha_m^3} + P \left(\frac{4 \delta}{a^3 \alpha_m^3} + W_m\right) =$$

$$\frac{h\,P\,\alpha_m}{a\,(\alpha_m^2+\theta^2)} - h\,N_m\,C_m - M_m\,C_m \tag{4.50}$$

The second equation required is obtained from the stress compatibility relation expressed by Equ. (4.38), with the appropriate substitution for q, N, u, and w. For the m^{th} term :

$$\frac{P\,\alpha_m}{G\,a\,b\,(\alpha_m^2+\theta^2)} = C_m\left(\frac{N_m}{b\,G} + \mu\cosh\frac{1}{2}\mu\,\alpha_m\,t_c\right) + W_m \tag{4.51}$$

Solving for C_m from Equs. (4.50) and (4.51) yields :

$$C_m = \frac{A'_m}{A''_m} \tag{4.52}$$

in which

$$A'_m = (-1)^{(m+3)/2}\,\frac{Q}{a\,\alpha_m^2} + \frac{2\,p_o}{a\,\alpha_m^3} + \frac{4\,P\,\delta}{a^3\,\alpha_m^3} +$$

$$\frac{P\,\alpha_m}{a\,(\alpha_m^2+\theta^2)}\left(\frac{P}{b\,G} - h\right)$$

$$A''_m = P\left(\frac{N_m}{b\,G} + \mu\cosh\frac{1}{2}\mu\,\alpha_m\,t_c\right) - (M_m + h\,N_m)$$

The deflection amplitude W_m can thus be determined from either Equ. (4.50) or (4.51). Knowing C_m and W_m, the quantities of interest such as stress distribution, interlayer shear, and deflection can readily be obtained.

Particular cases can now be obtained from the previous general analysis. In specific, solutions for a sandwich beam subjected to three loading types are presented next.

4.4.2.1 UNIFORM LOAD OF INTENSITY p_0

In this case $Q = P = \delta = 0$, and the Equs. (4.50) and (4.51) yield :

$$C_m = \frac{-2 p_0}{a \alpha_m^3 (M_m + h N_m)} \tag{4.53}$$

$$W_m = -C_m (\mu \cosh \frac{1}{2} \mu \alpha_m t_c + \frac{N_m}{b G}) \tag{4.54}$$

4.4.2.2 MIDSPAN CONCENTRATED LOAD

In this case $p_0 = P = \delta = 0$, and Equ. (4.52) yields :

$$C_m = \frac{(-1)^{(m+3)/2} Q}{a \alpha_m^2 (M_m + h N_m)} \tag{4.55}$$

while W_m is as given by Equ. (4.54).

4.4.2.3 BUCKLING LOAD FOR SANDWICH PLATES

In this case $p_0 = Q = 0$, and Equs. (4.51) and (4.52) yield :

$$C_m = \frac{A'''_m}{A''_m} \tag{4.56}$$

$$W_m = \frac{W'_m}{A''_m} \tag{4.57}$$

in which

$$A'''_m = \frac{4\ P\ \delta}{a^3\quad \alpha_m^3} + \frac{P\ \alpha_m}{a\ (\alpha_m^2 + \theta^2)}\ (\frac{P}{b\ G} - h)$$

$$W'_m = \frac{P\ \alpha_m\ A''_m}{G\ a\ b\ (\alpha_m^2 + \theta^2)} - A'''_m\ (\frac{N_m}{b\ G} + \mu\ \cosh\frac{1}{2}\ \mu\ \alpha_m\ t_c)$$

The deflection w becomes infinitely large when the denominator in the right hand side of Equ. (4.57) becomes zero. Thus an equation for determining the critical condition is [102]

$$P\ (\frac{N_m}{b\ G} + \mu\ \cosh\frac{1}{2}\ \mu\ \alpha_m\ t_c) - (M_m + h\ N_m) = 0$$

from which

$$P_{cr} = \frac{M_m + h\ N_m}{(N_m/b\ G) + \mu\ \cosh\frac{1}{2}\ \mu\ \alpha_m\ t_c} \tag{4.58}$$

The smallest value of the critical load can be obtained by taking m=1.

4.5 PARAMETRIC STUDY OF ADHESIVE EFFECTS ON THE BEHAVIOUR OF SANDWICH BEAM-COLUMNS

The complexity of the results obtained in the previous section makes it difficult to see the effects of adhesive on the response of sandwich beams. For the purpose of illustrating these effects, a particular sandwich beam is considered. The beam has the following properties :

$$a = 20.\text{ in.}$$
$$t_f = 0.04\text{ in.}$$
$$t_c = 2.\text{ in.}$$
$$b = 1.\text{ in.}$$
$$E_f = 10^7\text{ psi.}$$
$$E_c = 2 \times 10^4\text{ psi.}$$
$$G = 10^4\text{ psi.}$$

The main parameter of interest is the adhesive stiffness K. Its effects on the following quantities are shown in Fig. 4.10, 4.11, and 4.14.

$$(w)_{x=a} = w$$
$$(N)_{x=a} = N$$
$$(\sigma_c)_{\substack{x=a \\ z=t_c/2}} = \sigma_c$$
$$(q)_{\substack{x=0 \\ z=t_c/2}} = q$$

in which

w = the central deflection

N = the normal force in the faces at mid span

σ_c = the maximum normal stress in the core at mid span

q = the interlayer shear at the beam end.

It is seen in these figures that the deflection shows greater sensitivity to variation of the K value when the latter is in the lower range, and beyond a certain level of adhesive stiffness, the bonding can be practically considered as rigid. For instance, in the case of uniform loading (Fig. 4.10), a change in K value from 3000 to 2000 psi. induces a deflection increase fifteen times greater than that when K changes from 10 000 to 9000 psi. It is noted that the level of K for a rigid adhesive varies little with the types of lateral loading, however, it is substantially lower than that in the case of buckling due to axial compressive load.

An increase in the adhesive stiffness is accompanied by a decrease in the normal stress of the core. The resulting loss in the resisting moment is compensated by a slight increase in the face normal stress. For example, in Fig. 4.10, a change in K value from 8000 to 9000 psi. induces a decrease in the bending moment contributed by the core by 0.087 p_0, whereas the increase in the moment contributed by the facings is 0.088 p_0. The interlayer shear flux, q, is practically independent of the adhesive stiffness (Figs. 4.10 and 4.11). The adhesive effects on the normal stress, σ_c, and shear stress, τ, are shown in Figs. 4.12 and 4.13. It is seen that by increasing the K value the normal stress is reduced and the shear stress approaches a linear distribution with constant value.

As an indication of the correctness of the results obtained with the present analyses, two particular cases where solutions exist are considered : (i) sandwich beam with $G_c = 10^4$ psi, and (ii) sandwich beam with $G_c = 10^9$ psi. In both cases the adhesive is assumed to be rigid with $K = 10^9$ psi. The normal force, N, interlayer shear, q, the deflection, w, of the beam under uniform distributed load of intensity p_o and mid span concentrated load Q, are calculated and shown in Table 4.3. In the same table, the critical load of the beam is presented. The values obtained by Allen's theory [4] are shown in the table, where it is seen that there is close agreement between the results by the present analysis and by others.

4.6 PRACTICAL FORMULAS FOR SANDWICH BEAM-COLUMNS

To facilitate the application of the present theory, simple formulas are derived from which the maximum deflection, normal force in facings and interlayer flux can be determined. Three loading types are considered separately : (i) uniformly distributed load, (ii) mid span concentrated load, and (iii) axial compression.

4.6.1 UNIFORM LOAD OF INTENSITY p_o

The quantities of primary interest are : the maximum normal force in facings, the maximum interlayer shear flux, and the maximum deflection. These quantities are calculated from Equs. (4.44), (4.39) and (4.45) in conjunction with Equs. (4.53) and (4.54). The results are written in abbreviated forms as

$$N = (N)_{x=a} = \frac{1}{2} p_0 t_c F_{nu} \qquad (4.59)$$

$$q = (q)_{x=0} = p_0 F_{qu} \qquad (4.60)$$

$$w = (w)_{x=a} = \frac{p_0 t_c}{2 E_c b} F_{wu} \qquad (4.61)$$

in which

N = the maximum normal force in facings

q = the maximum interlayer shear flux

w = the maximum deflection

$$F_{nu} = \sum_{m=1,3,\ldots}^{\infty} \frac{-16 \mu a}{\pi^2 m^2 t_c} (-1)^{(m+1)/2} F_m$$

$$F_{qu} = \sum_{m=1,3,\ldots}^{\infty} \frac{4 \mu}{m \pi} F_m$$

$$F_{wu} = \sum_{m=1,3,\ldots}^{\infty} \frac{-64 (-1)^{(m+1)/2} \mu^2 a^2}{\pi^3 m^3 t_c^2} F_m \left[\left(\xi + \frac{m^2 \pi^2 t_c^2 \Phi}{16 a^2}\right) \coth \frac{m \pi t_c \mu}{4 a} + \frac{m \pi t_c \mu}{4 a}\right]$$

where

$$\xi = \frac{E_c t_c}{2 E_f t_f}$$

$$\Phi = 2\,\frac{E_c\,b}{K\,t_c}$$

$$F_m = \frac{\sinh\,(m\,\pi\,t_c\,\mu/4\,a)}{F'_m}$$

$$F'_m = \left(\xi + \frac{m^2\,\pi^2\,t_c^2\,\Phi}{16\,a^2}\right)\left(2\cosh\frac{m\,\pi\,t_c\,\mu}{4\,a} - \frac{8\,a}{m\,\pi\,t_c\,\mu}\sinh\frac{m\,\pi\,t_c\,\mu}{4\,a}\right) + \frac{m\,\pi\,t_c\,\mu}{2\,a}\,\sinh\frac{m\,\pi\,t_c\,\mu}{4\,a}$$

Numerical values for F_{nu}, F_{qu}, and F_{wu} are obtained for the following range of ξ, Φ, and $(2a/t_c)$:

(i) ξ varies from 0.05 to 0.30.

(ii) Φ varies from 0.0 to 10.

(iii) $\frac{2a}{t_c}$ varies from 20. to 50.

The results are presented in Figs. 4.15 to 4.18.

4.6.2 MID SPAN CONCENTRATED LOAD

In this case, the quantities N, q, and w are obtained from Equs. (4.44), (4.39) and (4.45) in conjunction with Equs. (4.55) and (4.54) as

$$N = Q\,F_{nc} \tag{4.62}$$

$$q = \frac{Q}{a}\,F_{qc} \tag{4.63}$$

$$W = \frac{Q}{E_c\, b} F_{wc} \tag{4.64}$$

in which

$$F_{nc} = \sum_{m=1,3,\ldots}^{\infty} \frac{2\mu}{m\pi} F_m$$

$$F_{qc} = \sum_{m=1,3,\ldots}^{\infty} -(-1)^{(m+1)/2} \mu F_m$$

$$F_{wc} = \sum_{m=1,3,\ldots}^{\infty} \frac{-8\,(-1)^{(m+1)/2}\,\mu^2\, a}{m^2\,\pi^2\, t_c} F_m \left[\left(\xi + \frac{m^2\,\pi^2\, t_c^2\, \phi}{16\, a^2}\right)\right.$$

$$\left. \coth \frac{m\,\pi\, t_c\, \mu}{4\, a} + \frac{m\,\pi\, t_c\, \mu}{4\, a}\right]$$

Numerical values for F_{nc}, F_{qc}, and F_{wc} are obtained for the same range of ξ, ϕ and $(2a/t_c)$ considered in the previous section. The results are presented in Figs. 4.19 to 4.22.

4.6.3 BUCKLING LOAD FOR SANDWICH STRUTS

The buckling load in this case is determined from Equ. (4.58) as

$$P_{cr} = \frac{E_c\, t_c\, b}{2} F_p \tag{4.65}$$

in which

$$F_p = \frac{F'_p}{F''_p}$$

where

$$F'_p = (\xi + \frac{m^2 t_c^2 \phi}{16 a^2}) (2 \cosh \frac{\pi t_c \mu}{4 a} -$$

$$\frac{8 a}{\pi t_c \mu} \sinh \frac{\pi t_c \mu}{4 a}) + \frac{\pi t_c \mu}{2 a} \sinh \frac{\pi t_c \mu}{4 a}$$

$$F''_p = \mu^2 [\frac{\pi t_c \mu}{4 a} \sinh \frac{\pi t_c \mu}{4 a} + (\xi + \frac{\pi^2 t_c^2 \phi}{16 a^2}) \cosh \frac{\pi t_c \mu}{4 a}]$$

Numerical values for F_p can be obtained from Figs. 4.23 and 4.24 for the same range of ξ, ϕ and $(2a/t_c)$ considered in section 4.6.1.

4.7 EXPERIMENTAL MEASUREMENT ON SANDWICH BEAMS UNDER MID SPAN CONCENTRATED LOADS

Sandwich beams subjected to mid span concentrated loads are used to measure the normal strain in facings and the deflection at their mid spans. The experiments included are :

(i) flexural tests to determine the elastic modulus of the core materials;

(ii) shear tests to determine the K value for adhesives;

(iii) experiments on sandwich beams.

A description of each of these tests is given next.

4.7.1 FLEXURAL TESTS TO DETERMINE E_c (Fig. 4.25 (a))

A straight beam of rectangular cross section, simply supported near its ends and subjected to concentrated loads at its mid span is used to determine E_c. A dial gage permits measurements to the nearest 0.001" is used to measure the deflection at the center of the span. After setting the gage, the specimen loaded using weights and the dial gage readings were recorded when there was no change in the deflection in five minutes.

The deflection of a simply supported beam subjected to mid span concentrated load is given by

$$(w)_{x=a} = \frac{P\,a^3}{6\,E_c\,I} = \Delta$$

from which

$$E_c = \frac{P}{\Delta}\,\frac{a^3}{6\,I} \tag{4.66}$$

in which

$2a$ = beam span (Fig. 4.25 (a))

E_c = elastic modulus of specimen material

I = moment of inertial of beam cross section

= $\frac{b\,t_c^3}{12}$, where b and t_c are beam width and thickness, respectively

$\frac{P}{\Delta}$ = the slope of P-Δ curve obtained experimentally, where P and Δ are the applied load and its corresponding maximum deflection, respectively.

Two materials were used to prepare the specimen in the flexural tests: pine wood and rigid urethane. The wood specimen was 2" x 4" x 47", while the urethane one was 2" x 4" x 29". By conducting the flexural test on each specimen and by using Equ. (4.66) it is found that

$$E_c = 10^6 \text{ psi.} \quad \text{for the pine wood}$$

$$= 1.85 \times 10^3 \text{ psi.} \quad \text{for the rigid urethane.}$$

4.7.2 SHEAR TESTS TO DETERMINE K (Fig. 4.25 (b) and (c))

The adhesive stiffness, K, is determined by conducting a shear test in accordance with the description by Kuenzi and Wilkinson [60]. A sketch for the test is down in Fig. 4.25 (b). Two wood blocks of dimensions 2.5" x 2.5" x 3"-3.5" are overlapped with a thin layer of adhesive in between. The contact area is 2" x 2.5". By setting one of the blocks on the testing machine platform and by applying compressive load on the other, the P-Δ curve is obtained, where Δ is the deformation induced in the adhesive because of the applied load P.

The K value can be determined from Equ. (4.40) as

$$K = \frac{P}{\Delta} \cdot \frac{1}{t_a} \cdot \frac{b}{b_a} \qquad (4.67)$$

t_a, b_a = the dimension of the adhesive area along the loading direction and perpendicular to it, respectively (Fig. 4.25 (b))

$\frac{P}{\Delta}$ = the slope of P-Δ curve obtained from the shear test (Fig. 4.25 (c))

Two adhesive materials were used to prepare the specimen in the shear tests : epoxy and contact cement. By conducting the shear test using each adhesive and by using Equ. (4.67) it is found that

K = 19728 psi. for the contact cement

= 100786 psi. for the epoxy.

4.7.3 SANDWICH BEAMS SUBJECTED TO MID SPAN CONCENTRATED LOADS (Fig. 4.25 (d))

Sandwich beams made of different materials and subjected to mid span concentrated loads are used to measure the maximum deflection and normal strain in facings. In specific, two beams are prepared. The first one is composed of two aluminum facings, a core of pine wood, and assembled with contact cement. The beam properties are

t_f = 0.025 in.

a = 23.5 in.

t_c = 1.575 in.

b = 3.6 in.

E_f = 10^7 psi

E_c = 10^6 [illegible]

K = 19728 psi.

The second beam is made of aluminum facings, a core of rigid urethane, and assembled with adhesive. The beam properties are

$$t_f = 0.025 \text{ in.}$$
$$a = 23.5 \text{ in.}$$
$$t_c = 1.65 \text{ in.}$$
$$b = 3.6 \text{ in.}$$
$$E_f = 10^7 \text{ psi.}$$
$$E_c = 1.85 \times 10^3 \text{ psi.}$$
$$K = 100786 \text{ psi.}$$

After twenty-four hours of assembling each beam, a strain gage is installed on each face at mid span and in the longitudinal direction. The central deflection is measured using a dial gage which permits measurement to the nearest 0.001", and the strain is measured using a strain indicator of a sensitivity of 2×10^{-6} in./in. After setting the dial and strain gages, the beams loaded at the center using weights, and measurements taken when there was no change in both the deflection and strains in five minutes. The results are shown in Figs. 4.26 and 4.27.

4.8 COMPARISON OF ANALYTIC AND EXPERIMENTAL RESULTS

The analytic solutions for the sandwich beams in the previous section are determined from Equs. (4.62) and (4.64), and the results are shown in Figs. 4.26 and 4.27. It is seen that the analytic and experimental results are in good agreement. In the same figures, the results of sandwich beams with rigid adhesives are presented, where it is seen that the deflections of the beams are higher when flexible adhesive is used.

TABLE 4.1 - MAXIMUM DEFLECTION AND STRESSES IN THE SANDWICH PLATE IN Fig. 4.3.

	LOAD CASE No. 1			LOAD CASE No. 2			LOAD CASE No. 3		
	A*	B**	Ref.	A	B	Ref.	A	B	Ref.
w (in)	.017	.0168	[84]	.0033	.0033	Chap. II	$5x10^{-5}$	$5.4x10^{-5}$	Chap. II
σ_f (psi)	954.	939.	[84]	236.	237.	Chap. II	4.85	5.75	Chap. II
τ_{xy} (psi)	604.	610.	[84]	68.	69.	Chap. II	--	--	--
τ_c (psi)	6.51	6.76	[84]	.506	.52	Chap. II	$82x10^{-4}$	$83x10^{-4}$	Chap. II

* A the results in this column are obtained from the present analysis.

** B the results in this column are obtained from the references shown in the table.

TABLE 4.2 - MAXIMUM DEFLECTION AND STRESSES IN THE SANDWICH PLATE IN Fig. 4.3

	LOAD CASE No. 1			LOAD CASE No. 2			LOAD CASE No. 3		
	A*	B**	Ref.	A	B	Ref.	A	B	Ref.
w (in)	.0116	.0114	[101]	18.8×10^{-4}	19.6×10^{-4}	Chap. II	2.1×10^{-5}	2.03×10^{-5}	[101]
σ_f (psi)	933.	939.	[101]	230.	233.	[101]	4.73	5.62	Chap. II
τ_{xy} (psi)	631.	637.	[101]	--	--	--	--	--	--
τ_c (psi)	6.51	6.76	[101]	--	--	--	--	--	--

* A the results in this column are obtained from the present analysis.

** B the results in this column are obtained from the references shown in the table.

	PRESENT ANALYSIS	ALLEN [4]	NOTE
max. w (in)	0.050 p_o	0.051 p_o	$G = 10^4$ psi.
	0.0021 Q	0.00214 Q	$G = 10^4$ psi.
max. N (lb)	96.5 p_o	101.9 p_o	$G = 10^4$ psi.
	4.80 Q	4.99 Q	$G = 10^4$ psi.
	96.5 p_o	96.5 p_o	$G = 10^9$ psi.
	4.8 Q	4.8 Q	$G = 10^9$ psi.
max. q (lb/in)	9.9 p_o	9.4 p_o	$G = 10^4$ psi.
	0.25 Q	0.25 Q	$G = 10^4$ psi.
P_{cr} (lb)	4090.	4118.	$G = 10^4$ psi.
	5114.	5134.	$G = 10^9$ psi.

TABLE 4.3 - MAXIMUM DEFLECTION, INTERLAYER SHEAR, NORMAL FORCE IN THE FACES, AND CRITICAL LOAD OF THE SANDWICH BEAM IN Fig. 4.9 (a).

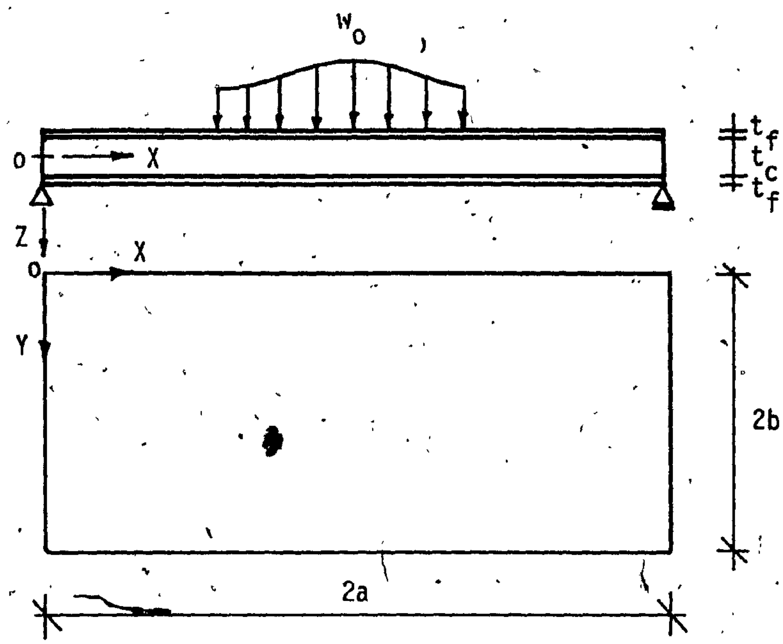


Fig. 4.1 - SANDWICH PLATE

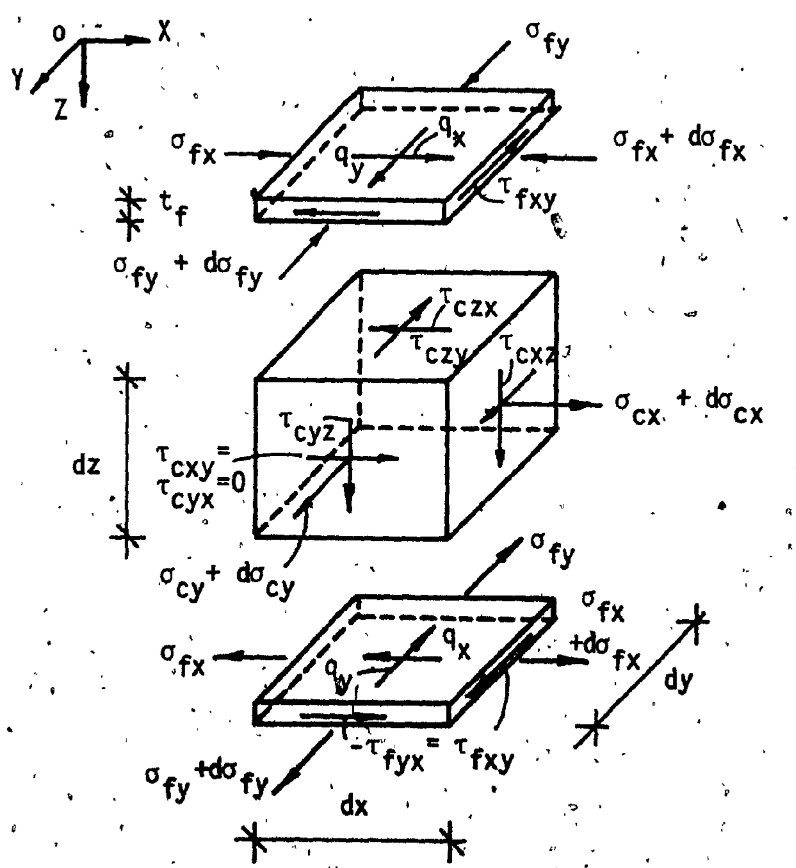
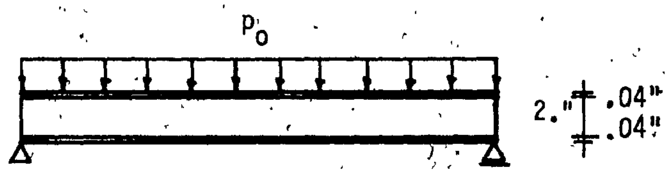
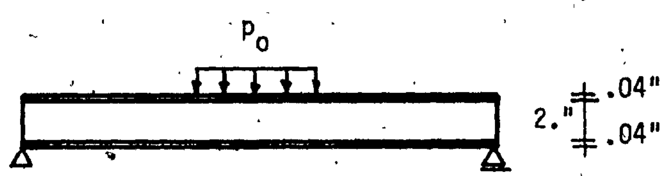


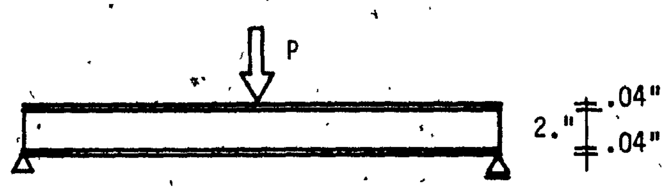
Fig. 4.2 - STRESS STATE IN SANDWICH ELEMENT



(a) LOAD CASE NO. 1



(b) LOAD CASE NO. 2



(c) LOAD CASE NO. 3

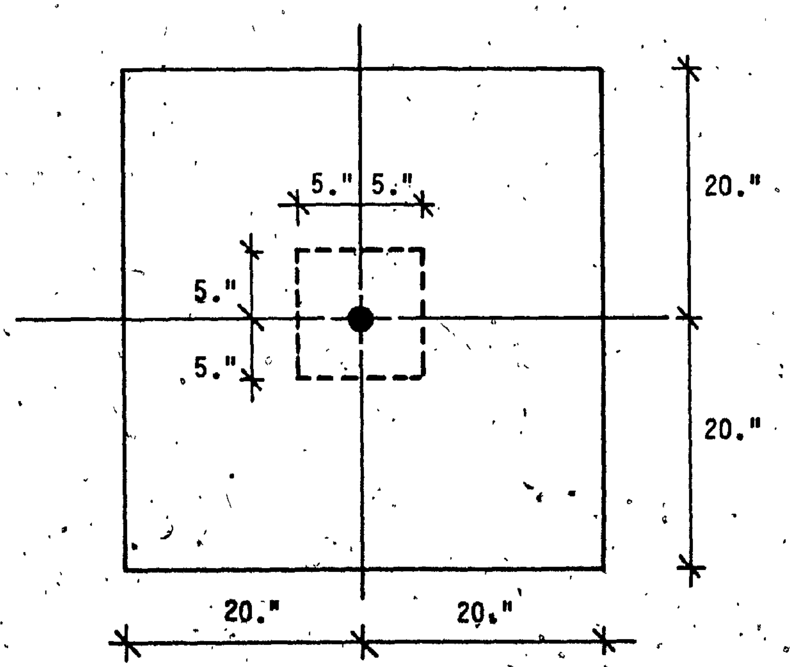


Fig. 4.3 - SANDWICH PLATE IN THE PARAMETRIC STUDY

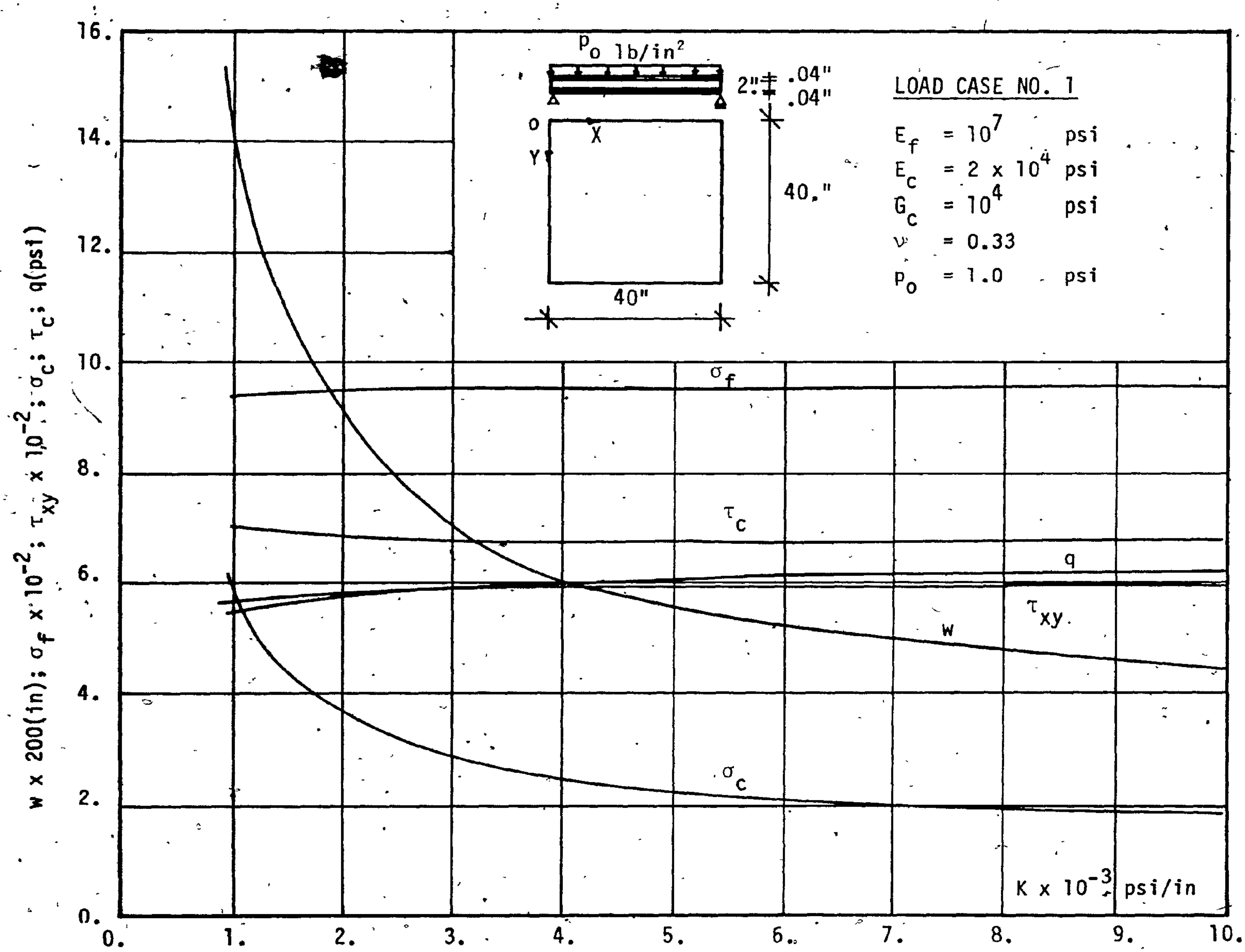


Fig. 4.4 - ADHESIVE EFFECTS ON THE DEFLECTION AND STRESSES IN THE SANDWICH PLATE IN Fig. 4.3

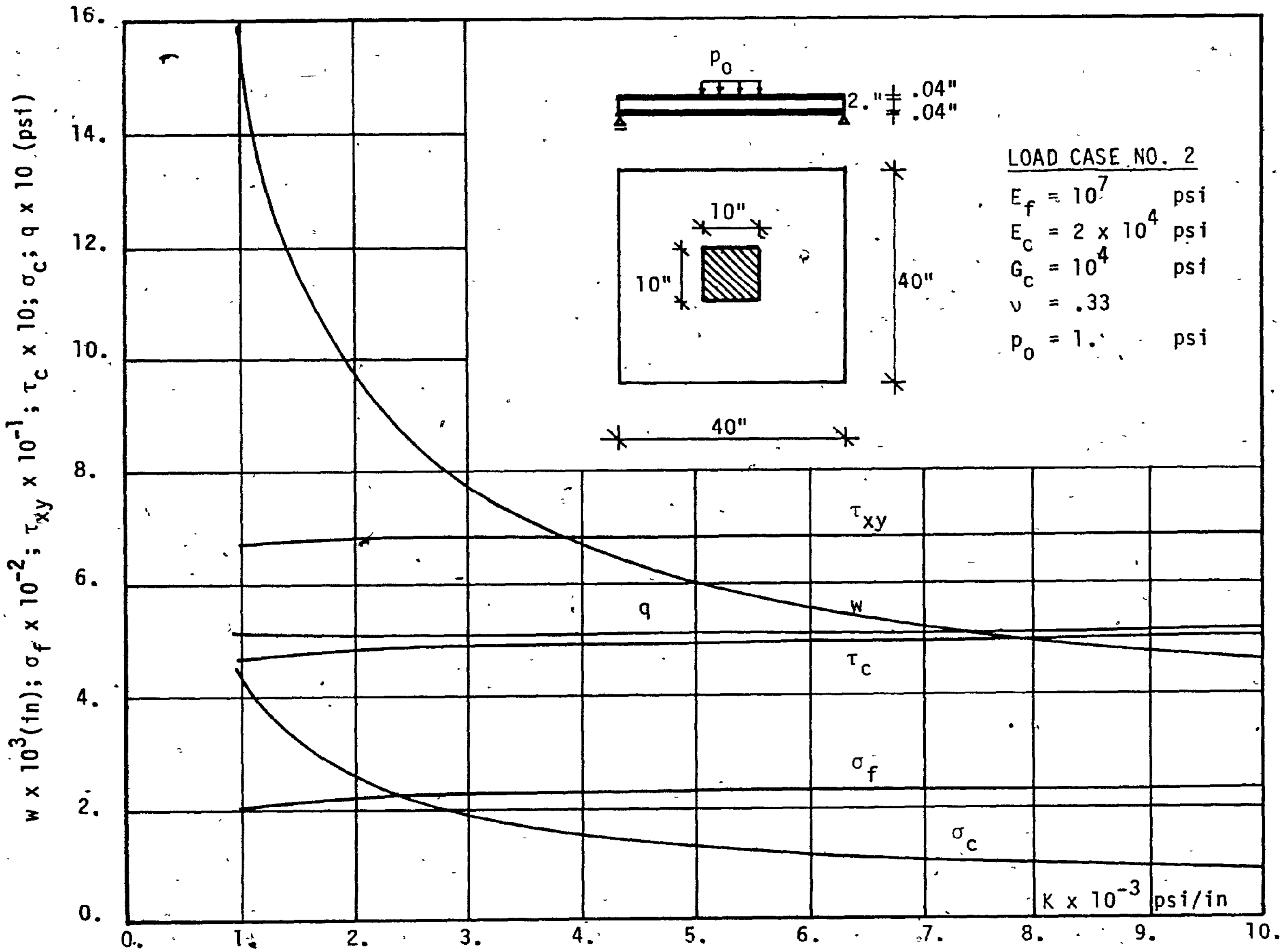


Fig. 4.5 - ADHESIVE EFFECTS ON THE DEFLECTION AND STRESSES IN THE SANDWICH PLATE IN Fig. 4.3

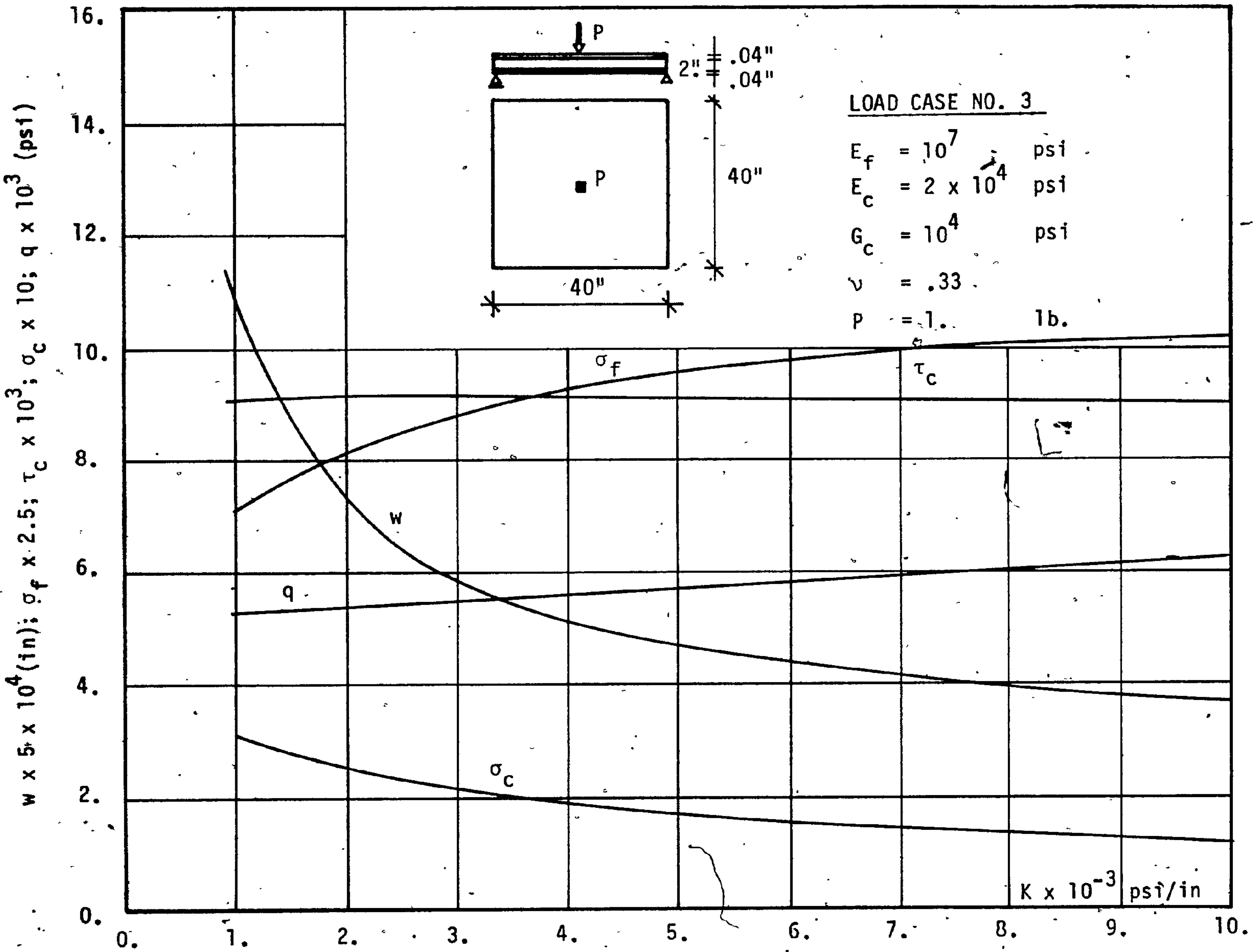


Fig. 4.6 - ADHESIVE EFFECTS ON THE DEFLECTION AND STRESSES IN THE SANDWICH PLATE IN Fig. 4.3

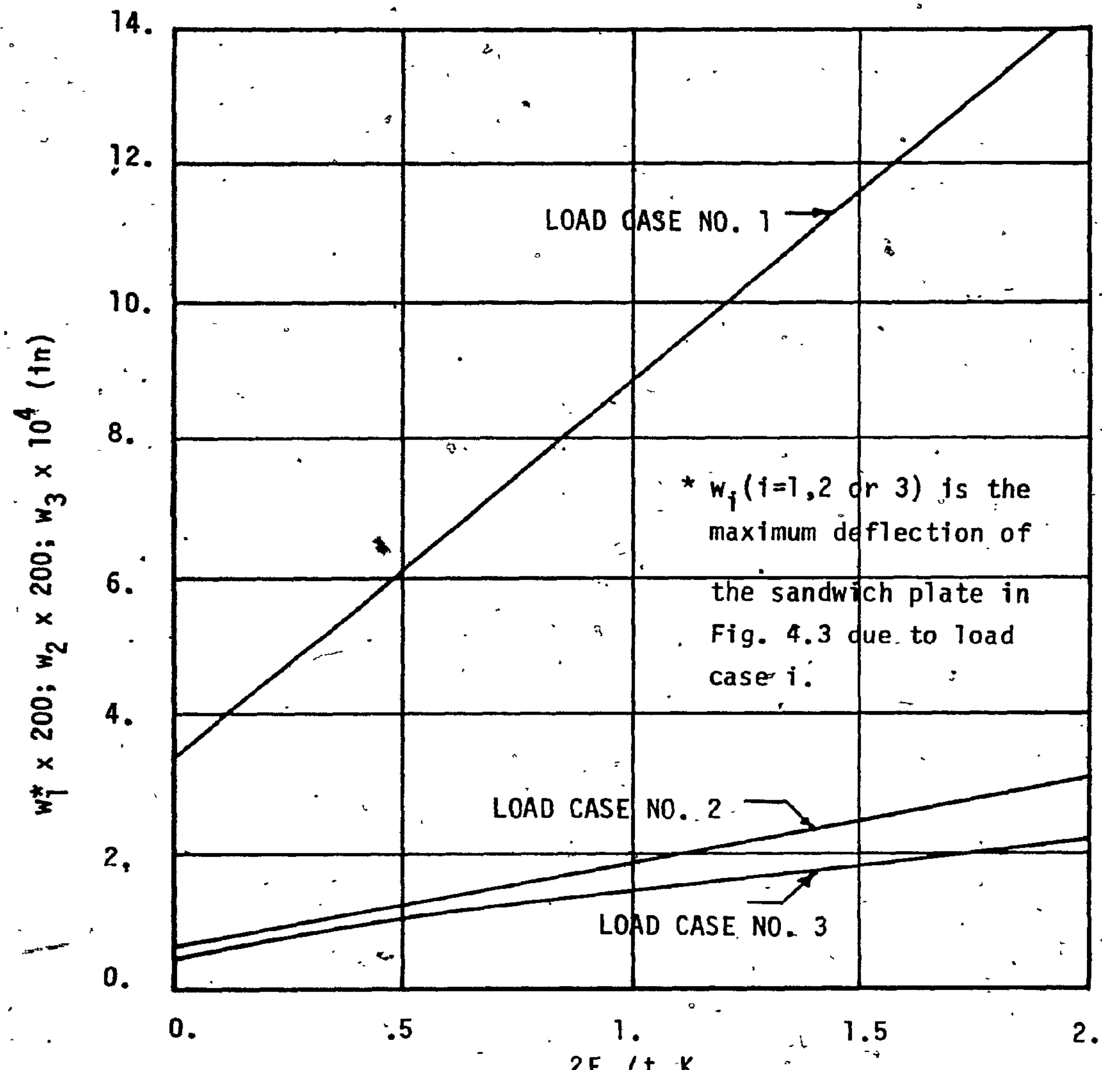


Fig. 4.7 - THE EFFECT OF $(2E_c/t_cK)$ ON THE MAXIMUM DEFLECTION

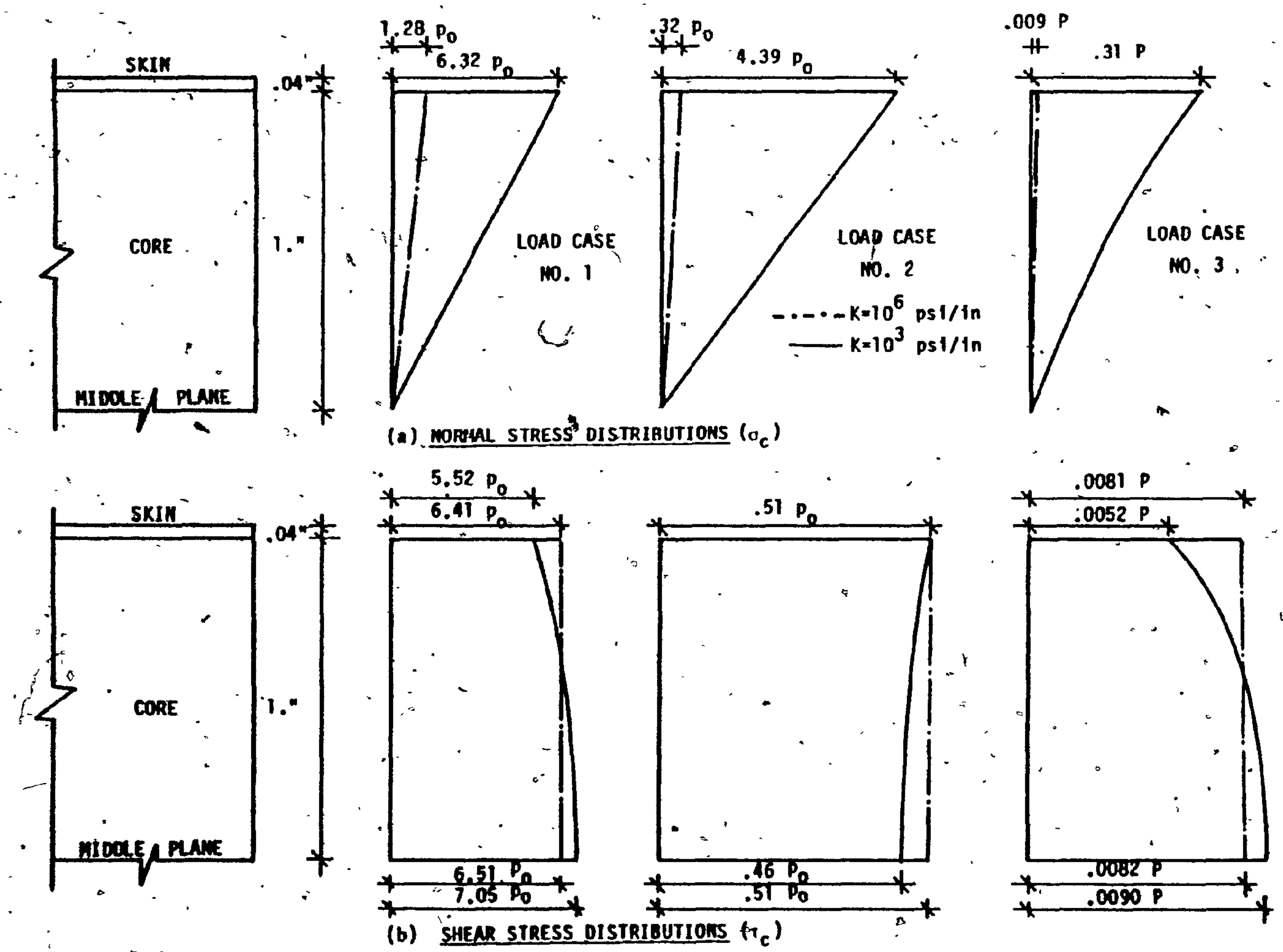
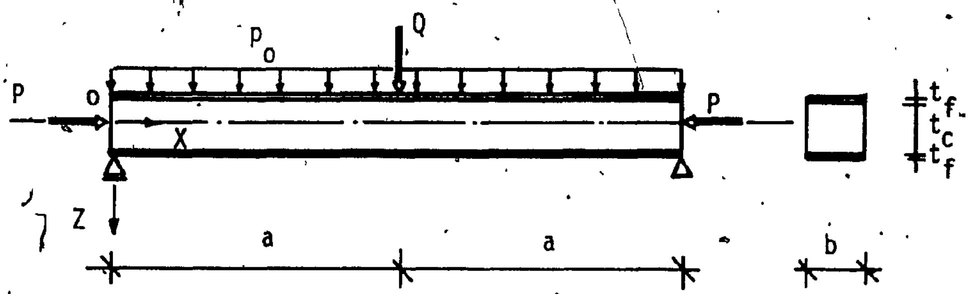
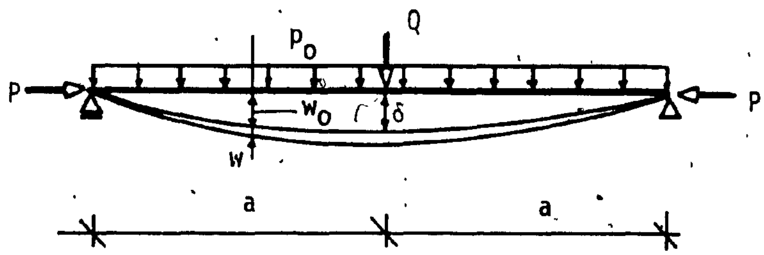


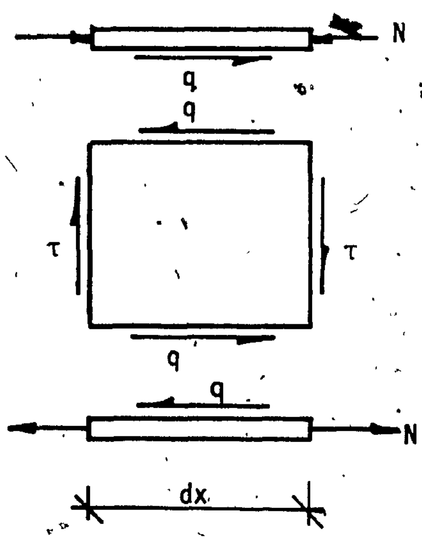
Fig. 4.8 - ADHESIVE EFFECTS ON STRESS DISTRIBUTIONS IN THE CORE OF THE PLATE IN Fig. 4.3



(a) LOADS AND GEOMETRY



(b) DEFLECTION AND INITIAL IMPERFECTION



(c) INTERNAL STRESSES

Fig. 4.9 - SANDWICH BEAM-COLUMN

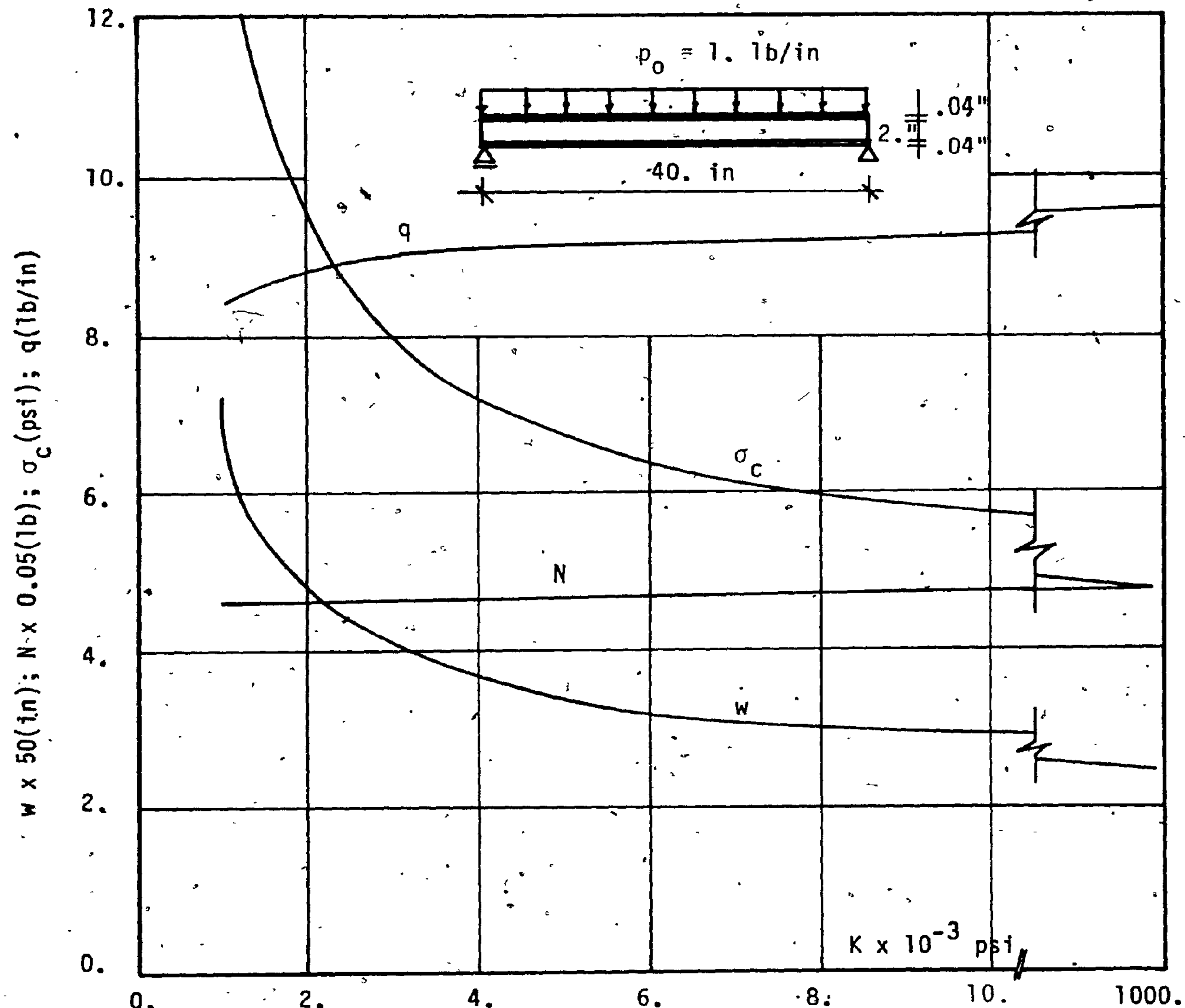


Fig. 4.10 - ADHESIVE EFFECTS ON SANDWICH BEAM BEHAVIOUR

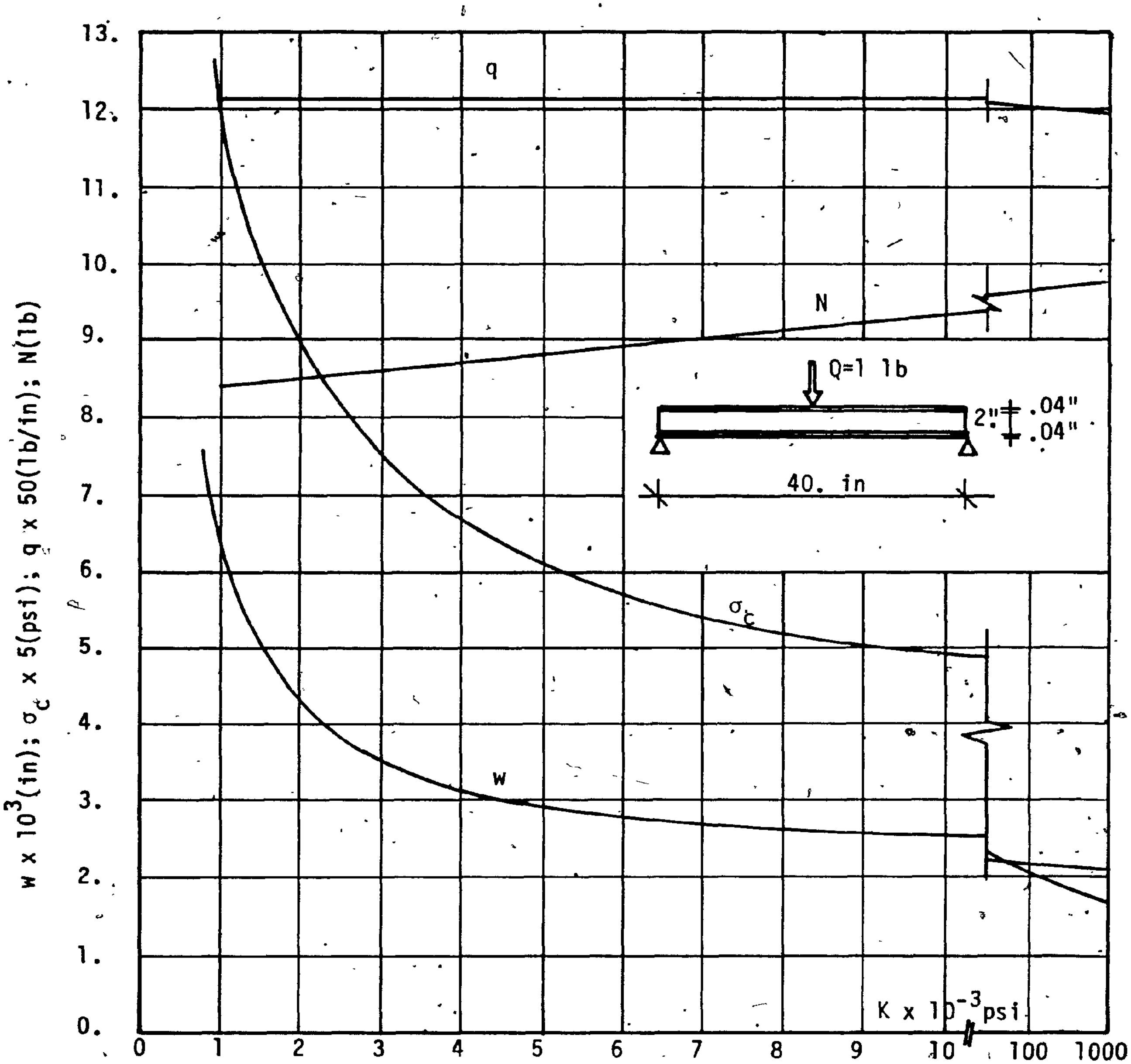
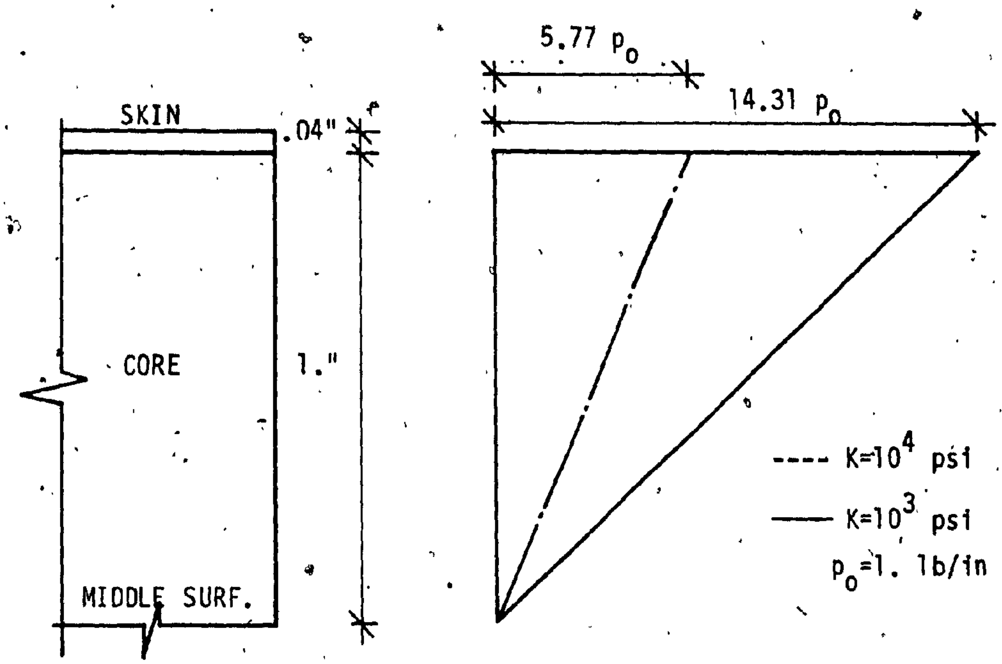
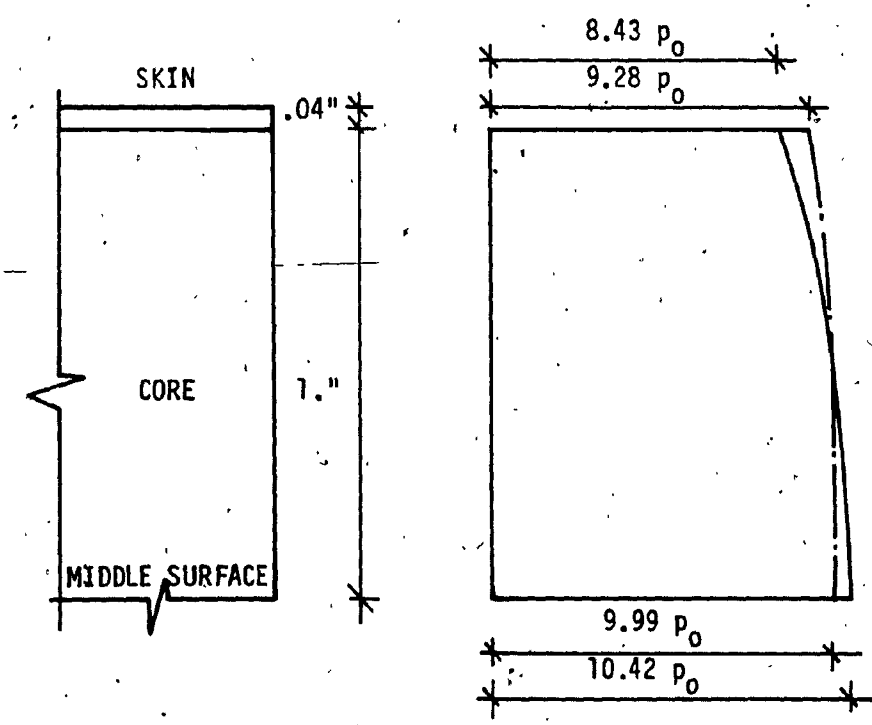


Fig. 4.11 - ADHESIVE EFFECTS ON SANDWICH BEAM BEHAVIOUR

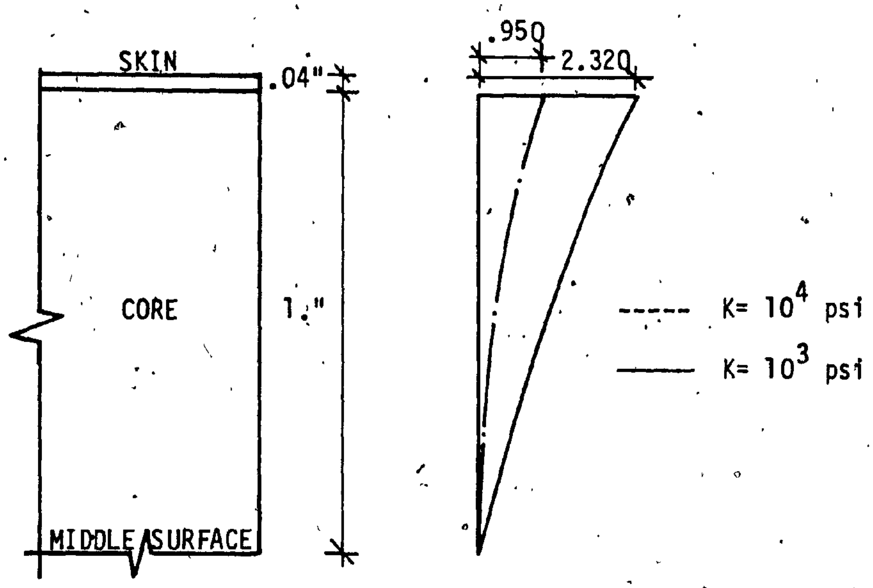


(a) NORMAL STRESS DISTRIBUTION (σ_c)

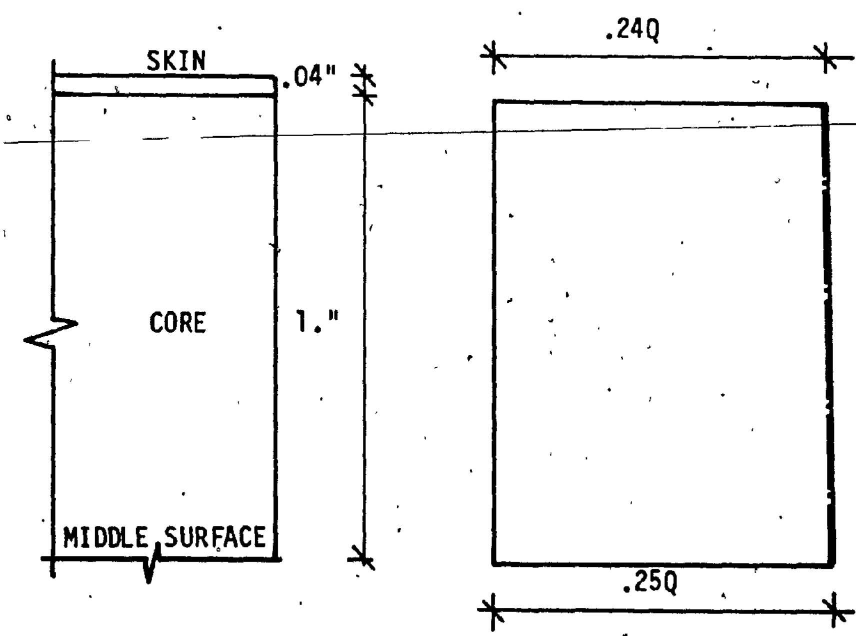


(b) SHEAR STRESS DISTRIBUTION (τ)

Fig. 4.12.- ADHESIVE EFFECTS ON STRESS DISTRIBUTION IN THE CORE



(a) NORMAL STRESS DISTRIBUTION (σ_c)



(b) SHEAR STRESS DISTRIBUTION (τ)

Fig. 4.13 - ADHESIVE EFFECT ON STRESS DISTRIBUTIONS IN THE CORE

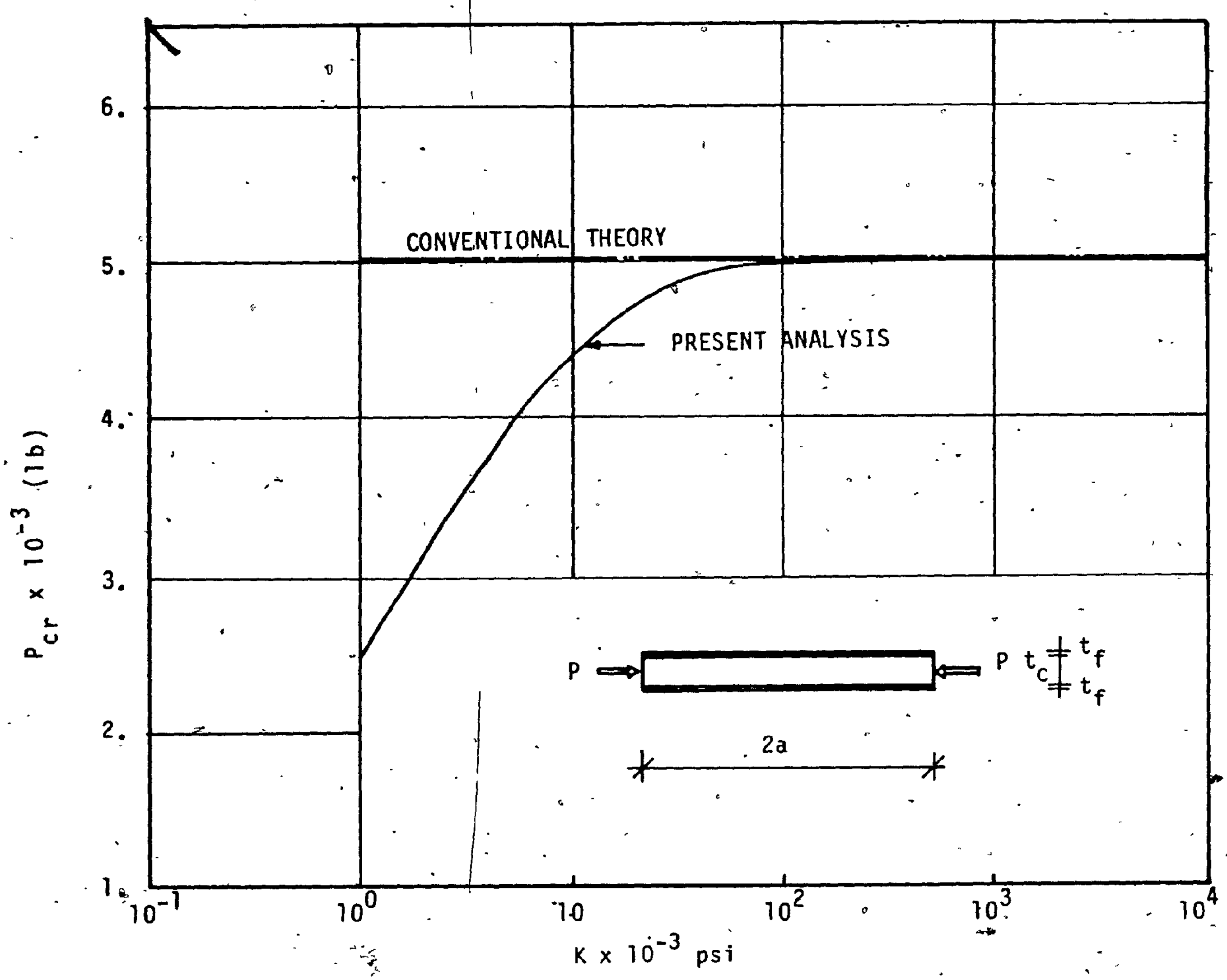


Fig. 4.14. - ADHESIVE EFFECT ON THE CRITICAL LOAD OF A SANDWICH STRUT

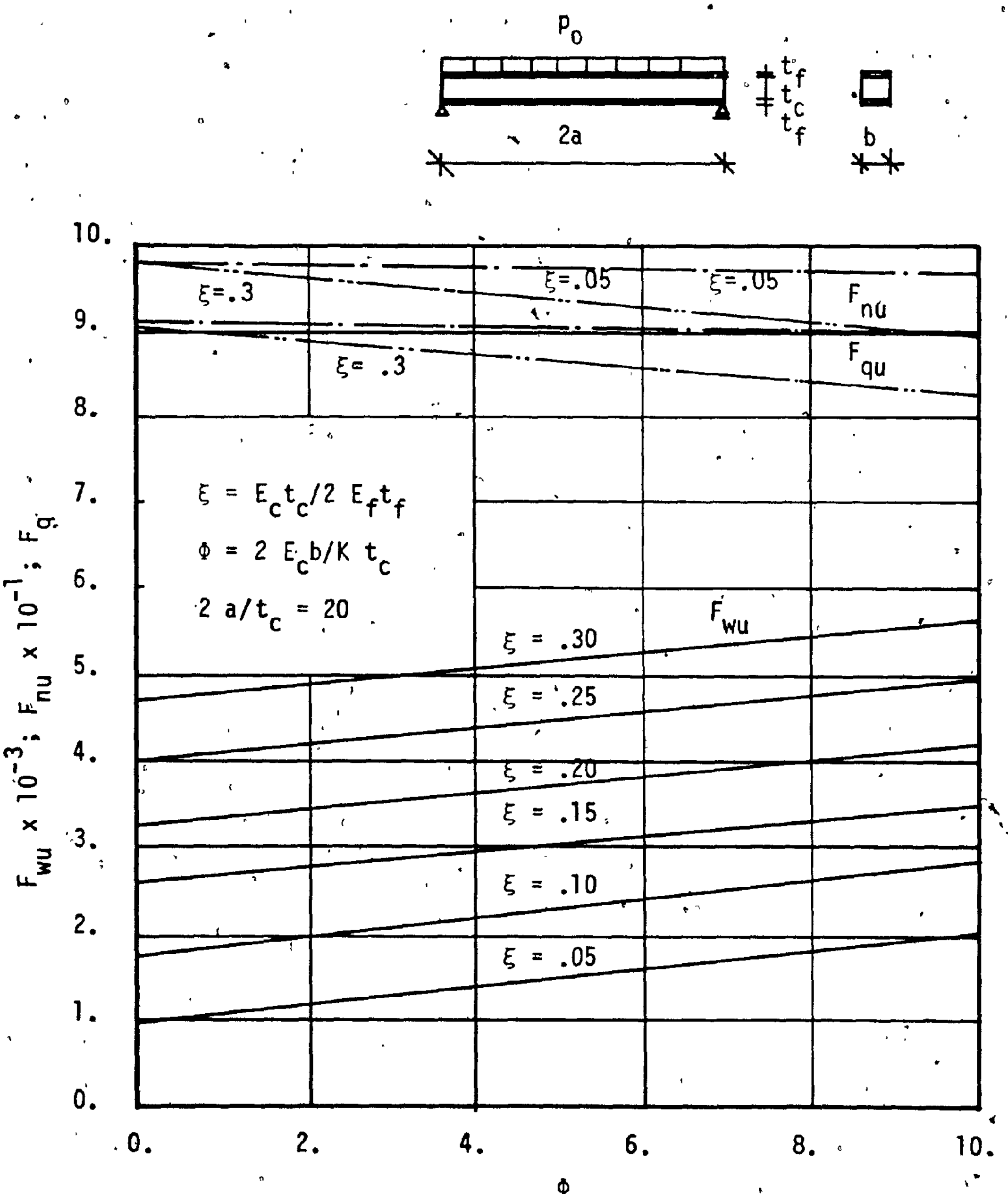


Fig. 4.15 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.44),(4.45), AND (4.46)

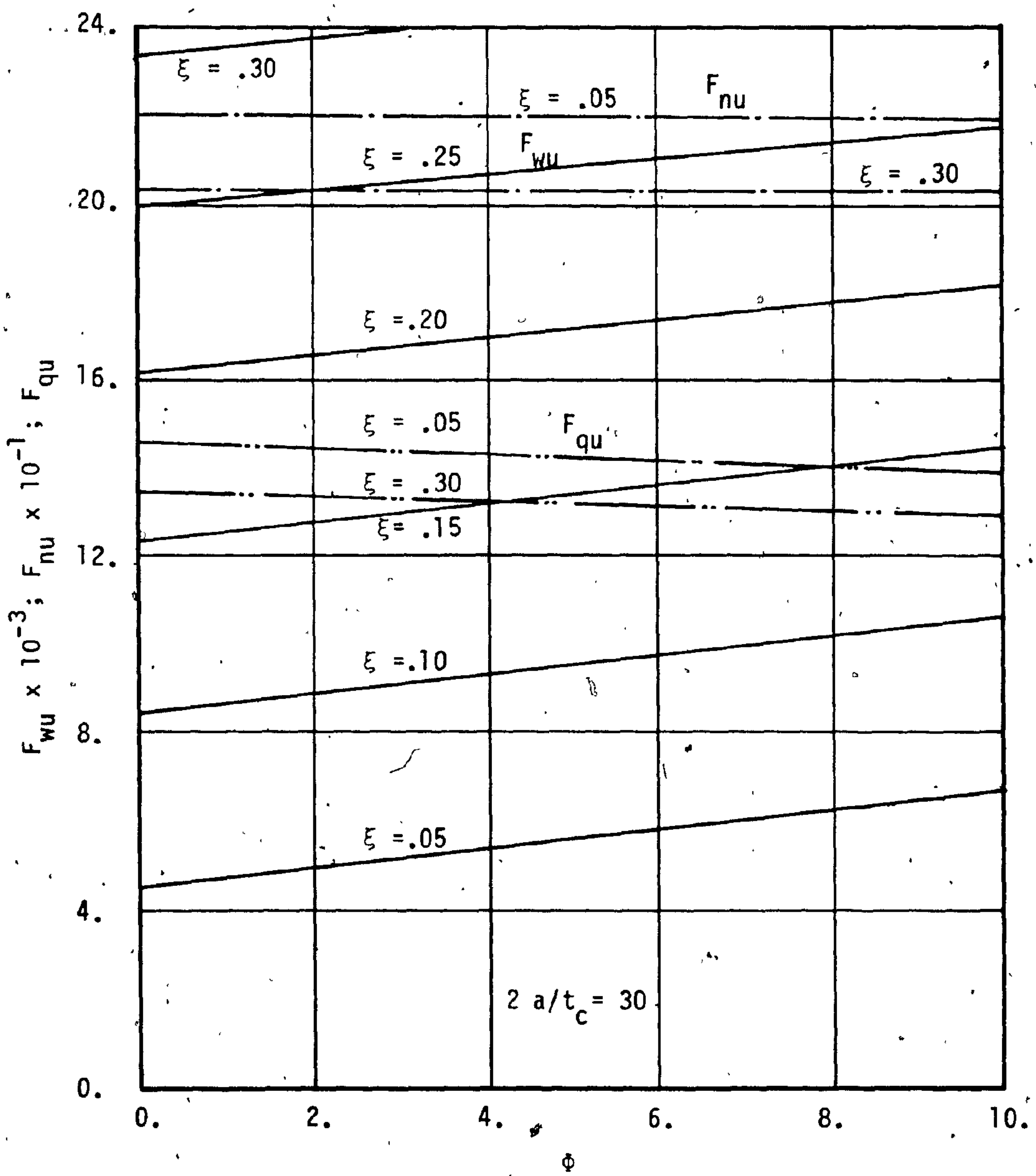


Fig. 4.16 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.44), (4.45), AND (4.46)

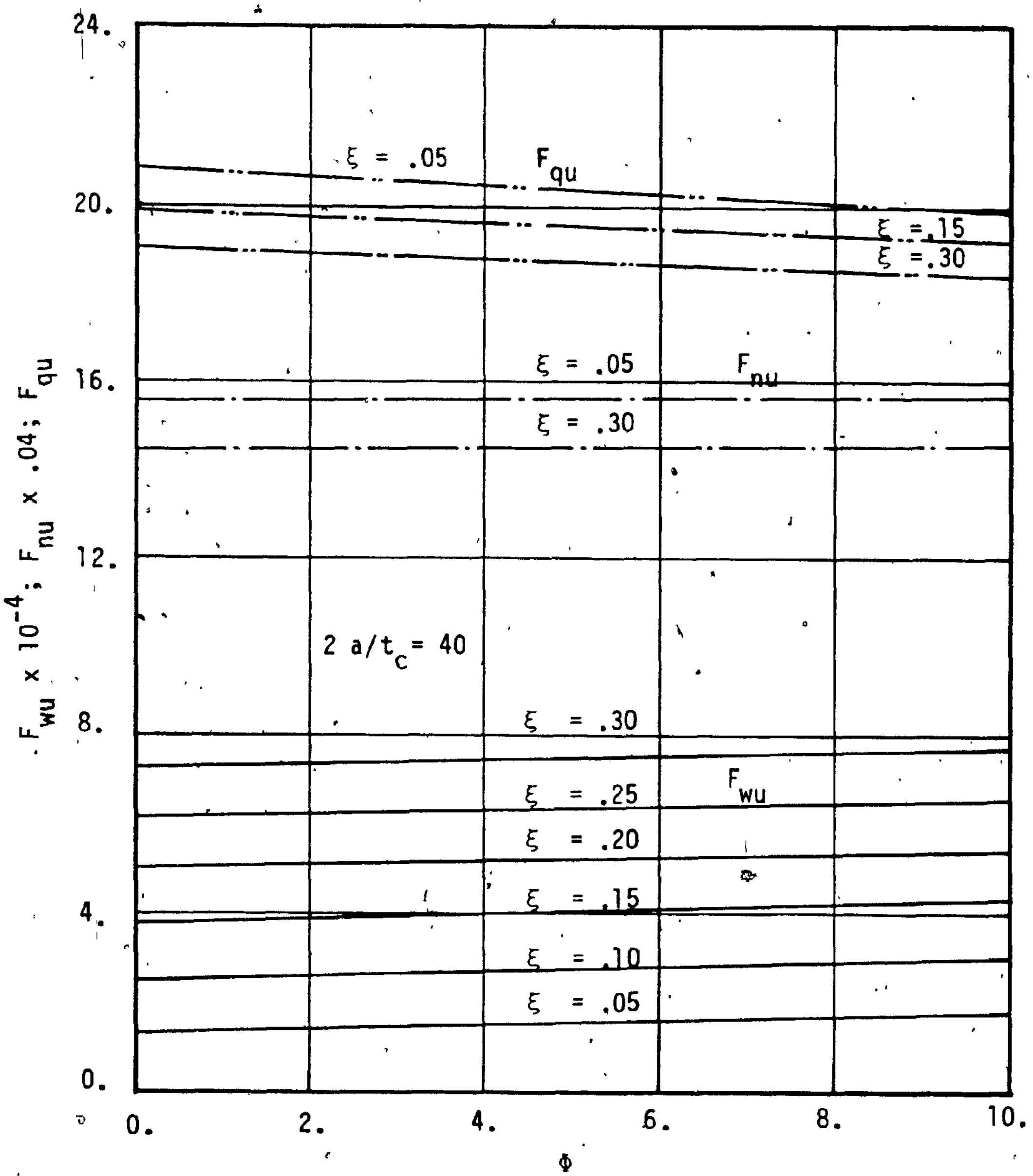


Fig. 4.17 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.44), (4.45), AND (4.46)

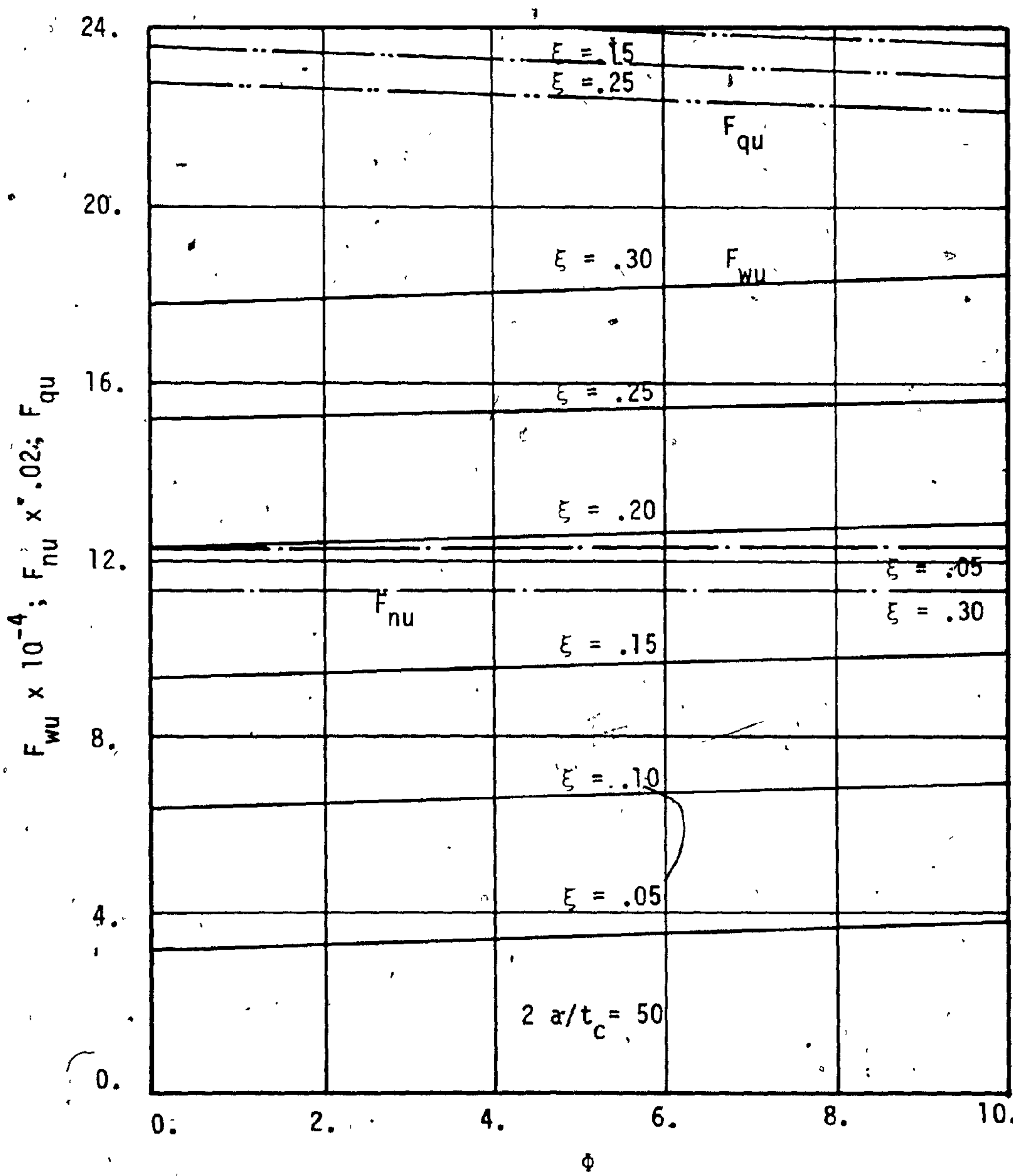


Fig. 4.18 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.44), (4.45), AND (4.46)

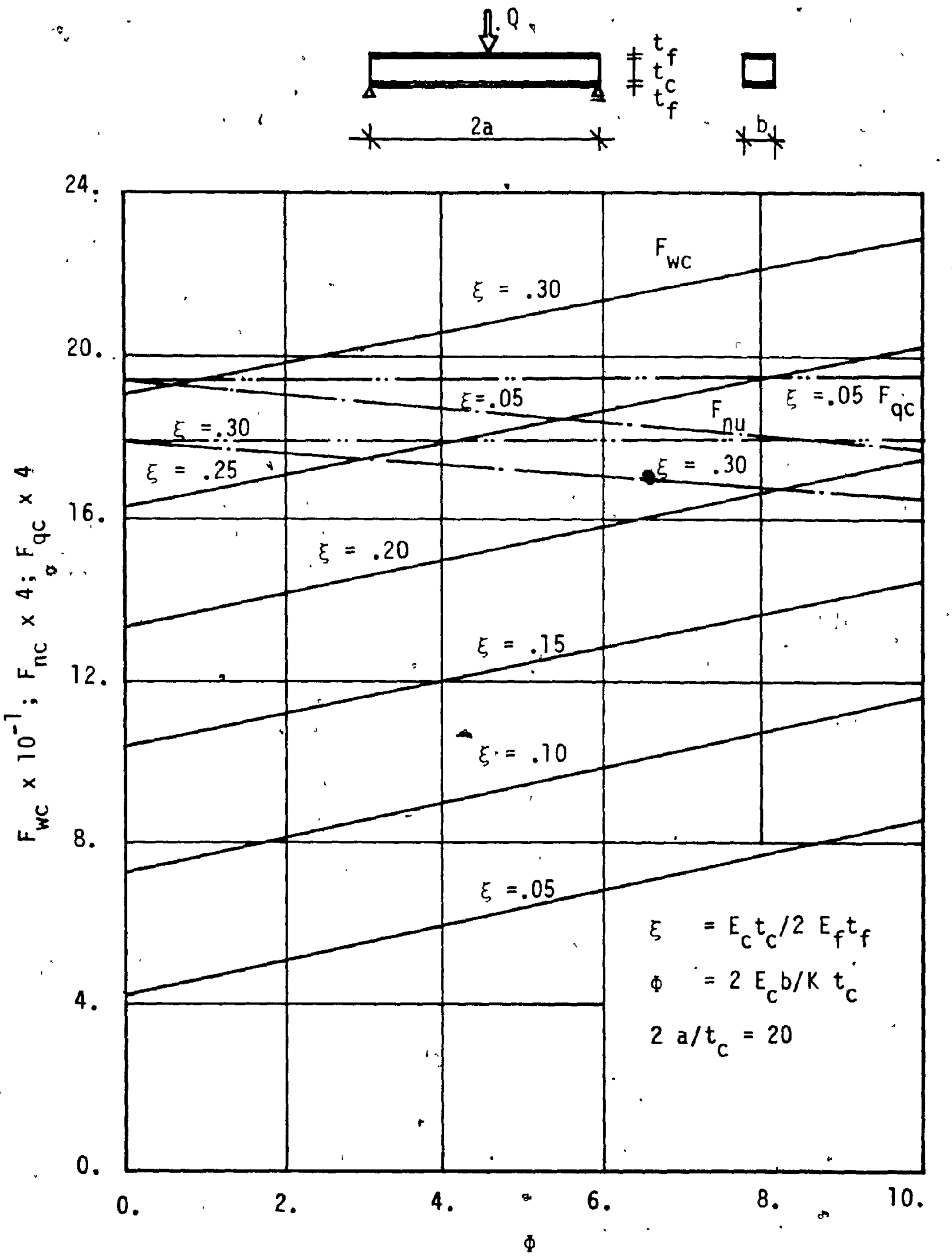


Fig. 4.19 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.47),(4.48), AND (4.49)

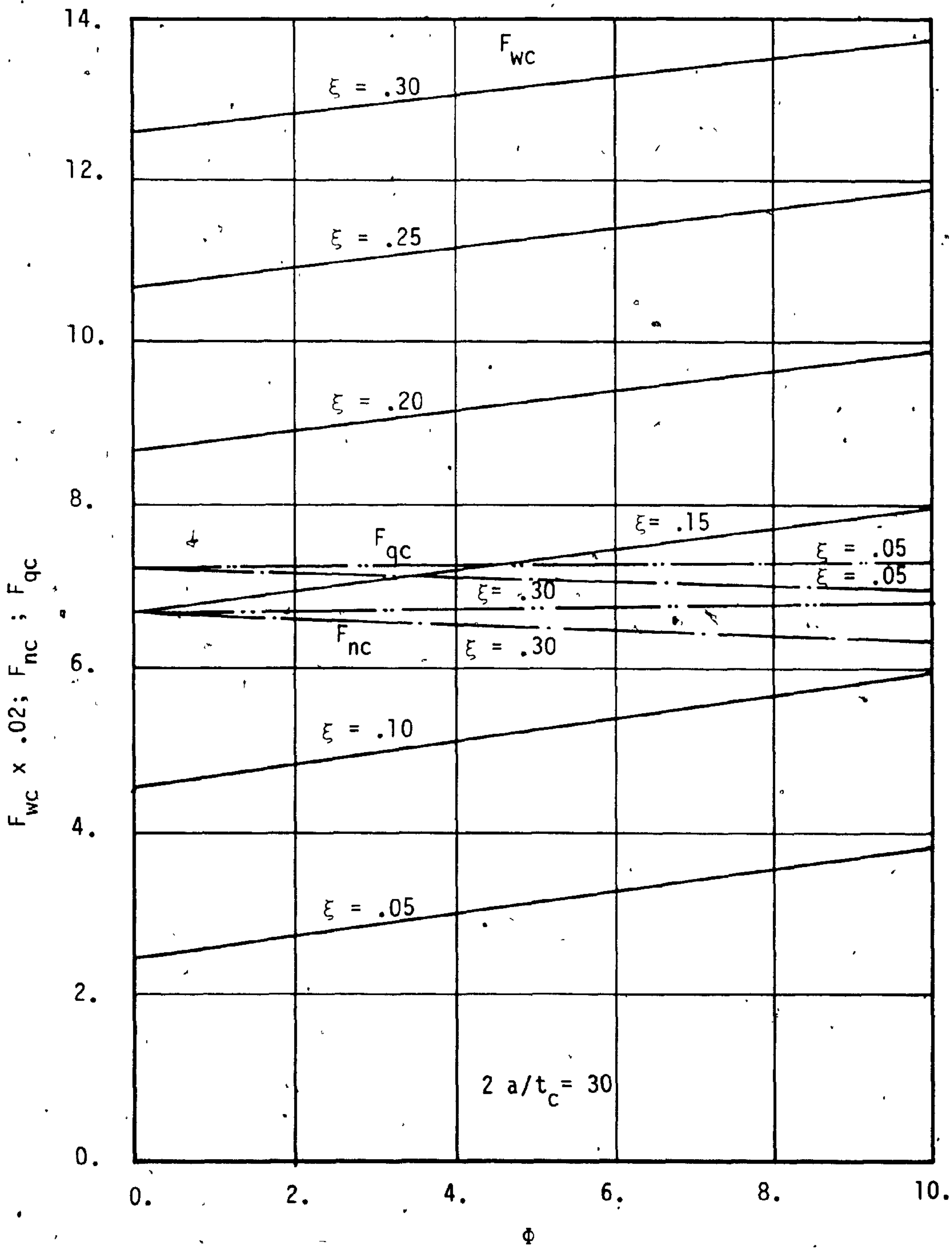


Fig. 4.20 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.47), (4.48), AND (4.49)

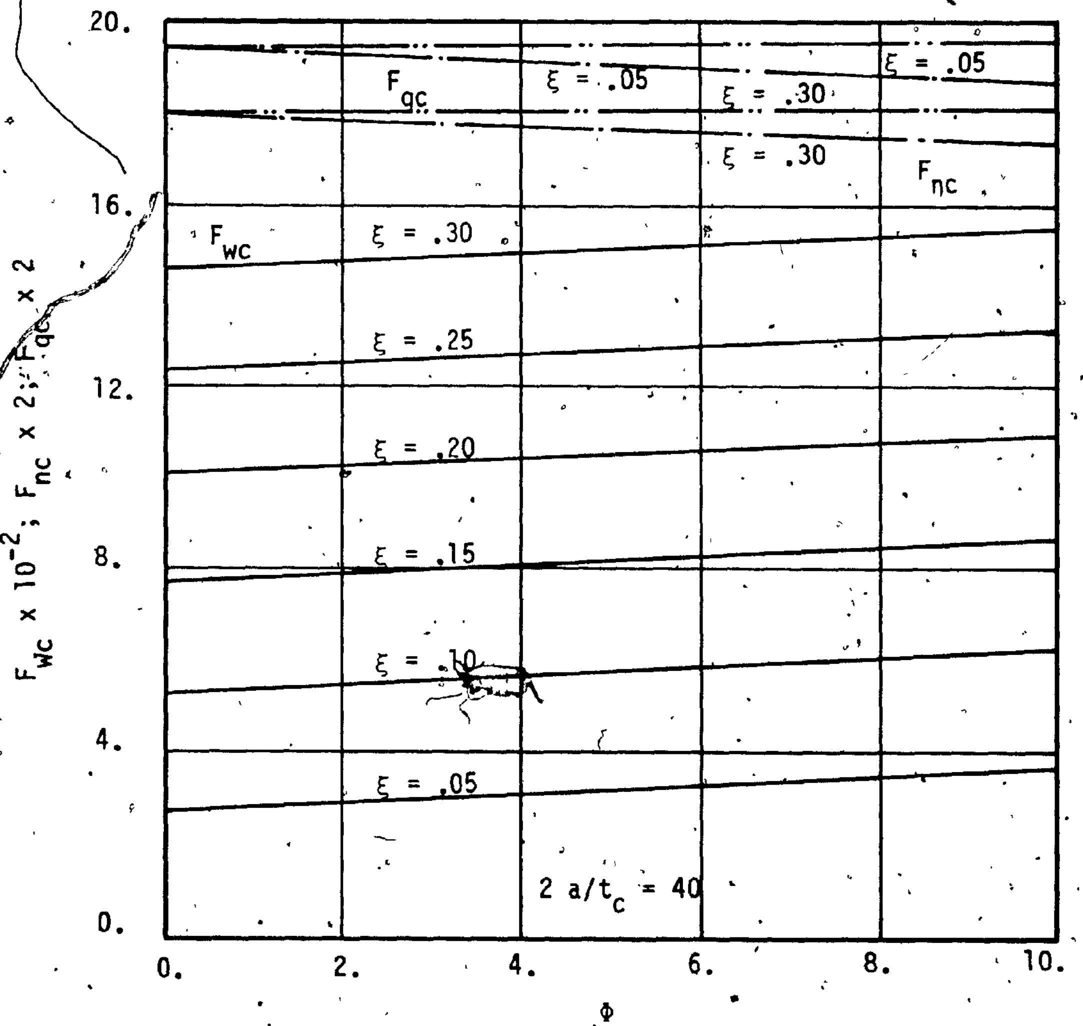


Fig. 4.21 - NUMERICAL VALUES FOR THE FACTORS IN EQUS. (4.47), (4.48), AND (4.49)

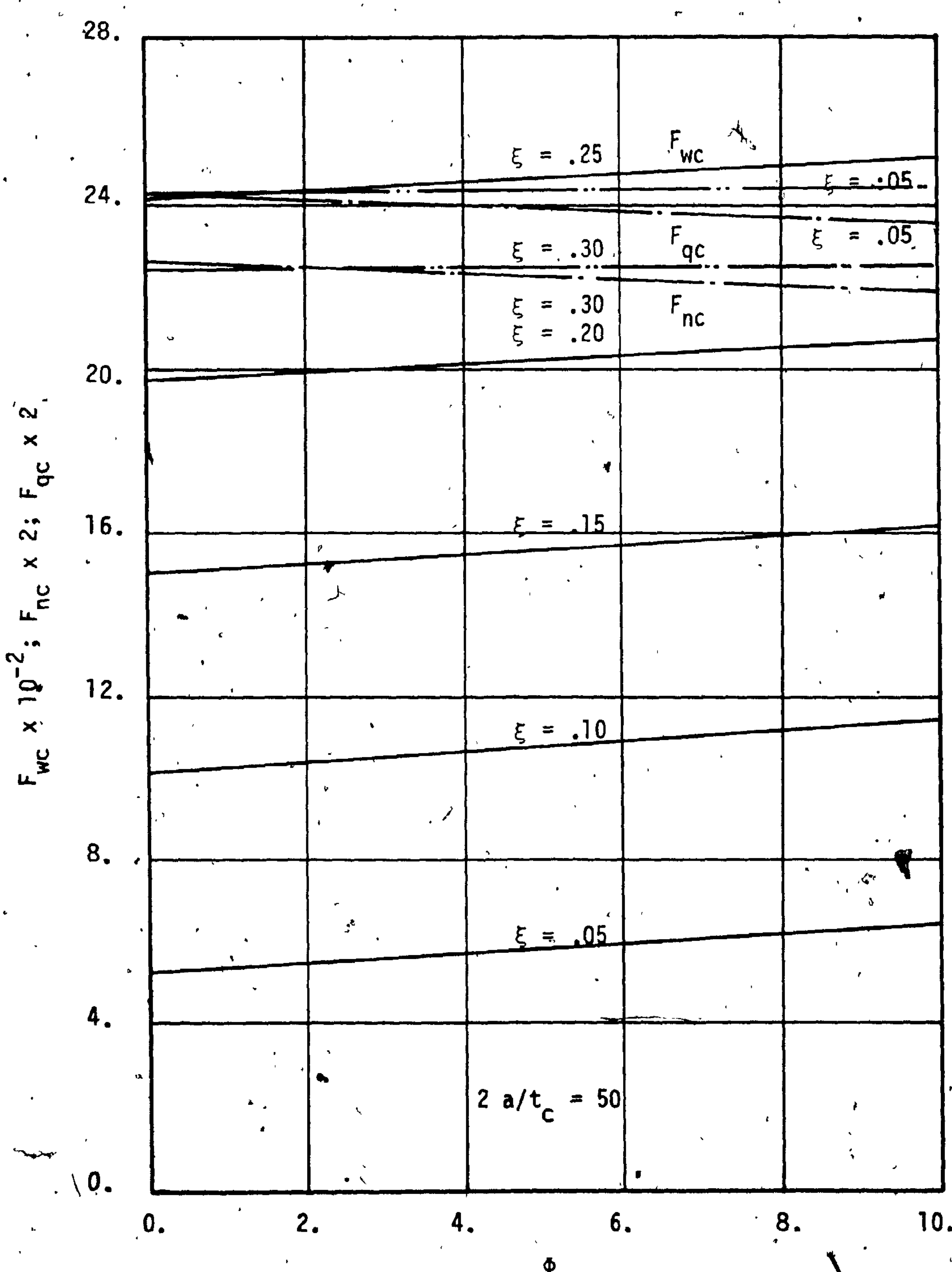


Fig. 4.22 - NUMERICAL VALUES FOR FACTORS IN EQUS.(4.47), (4.48), AND (4.49)

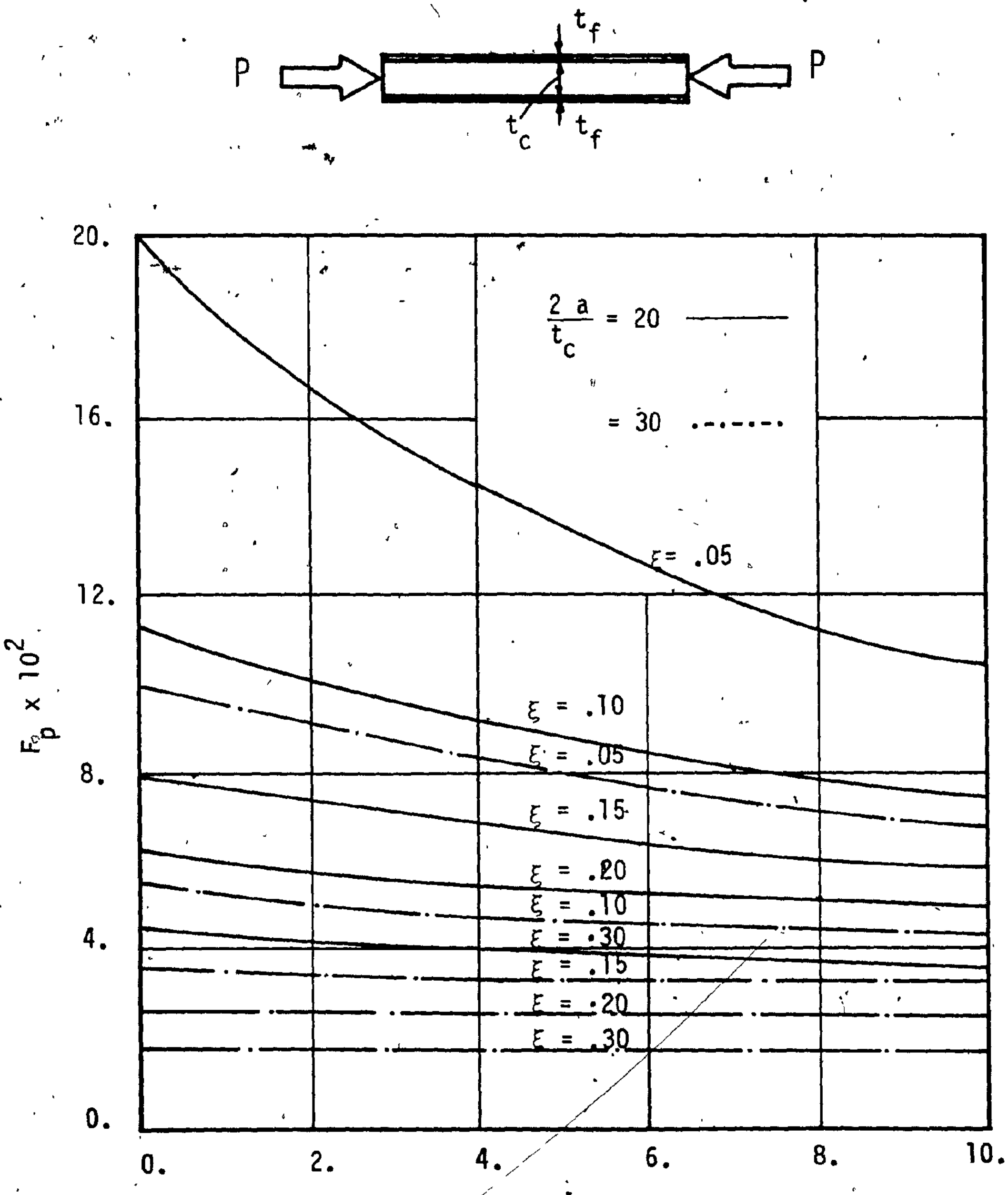


Fig. 4.23 - NUMERICAL VALUES FOR FACTOR IN EQU.(4.50)

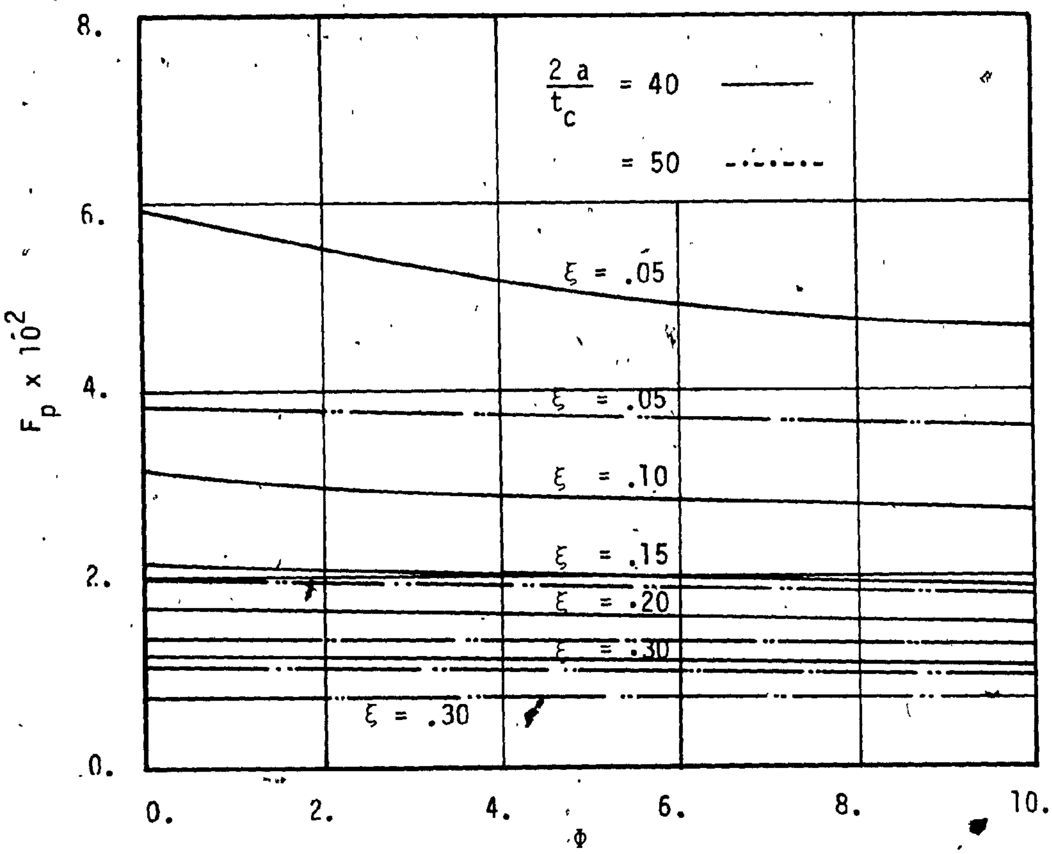
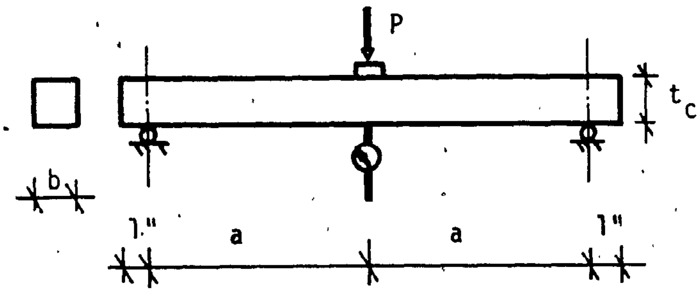
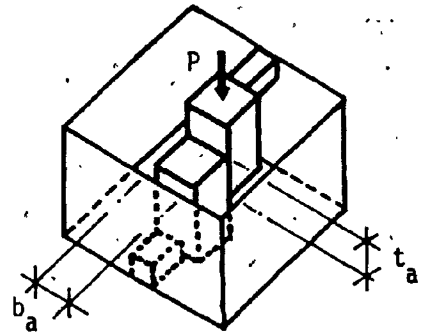


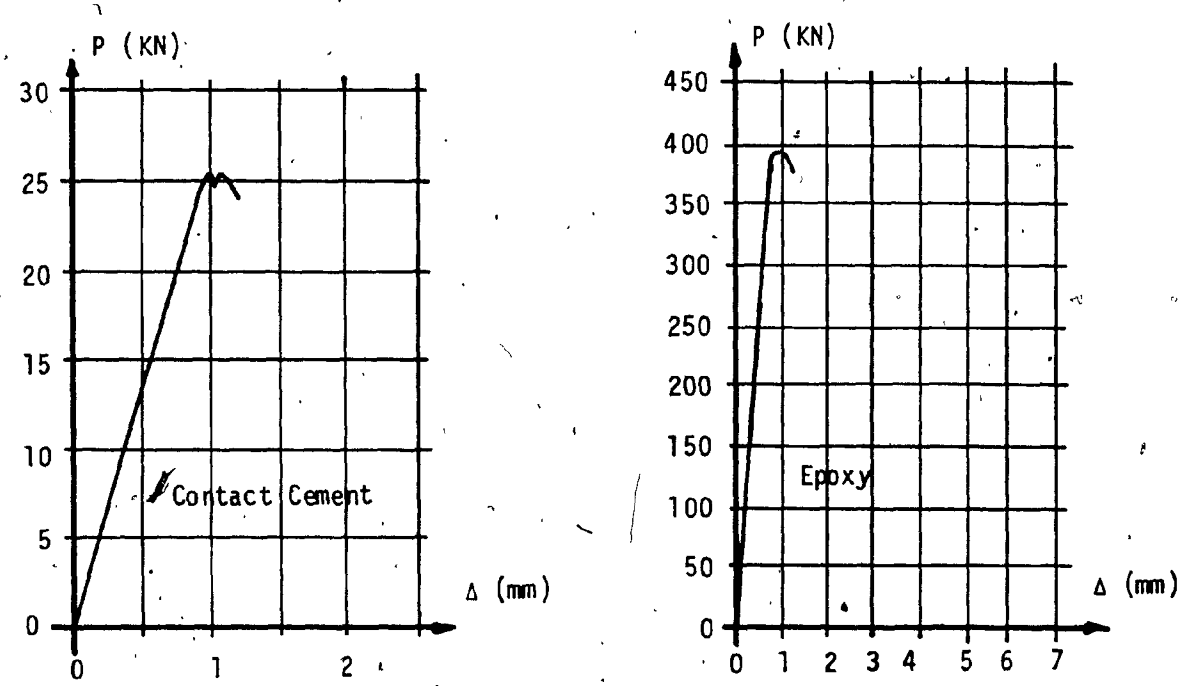
Fig. 4.24 - NUMERICAL VALUES FOR FACTOR IN EQU. (4.50)



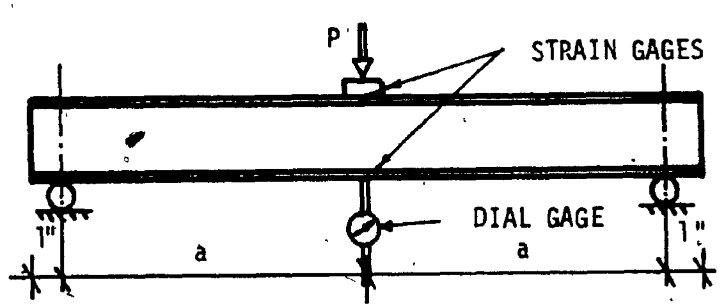
(a) TEST USED TO DETERMINE E_c



(b) SHEAR TEST



(c) TYPICAL P-Δ CURVES FOR ADHESIVES USED IN SHEAR TESTS



(d) TEST USED TO MEASURE MAXIMUM DEFLECTION AND NORMAL STRESS IN SANDWICH BEAMS

Fig. 4.25 - EXPERIMENTAL MODELS

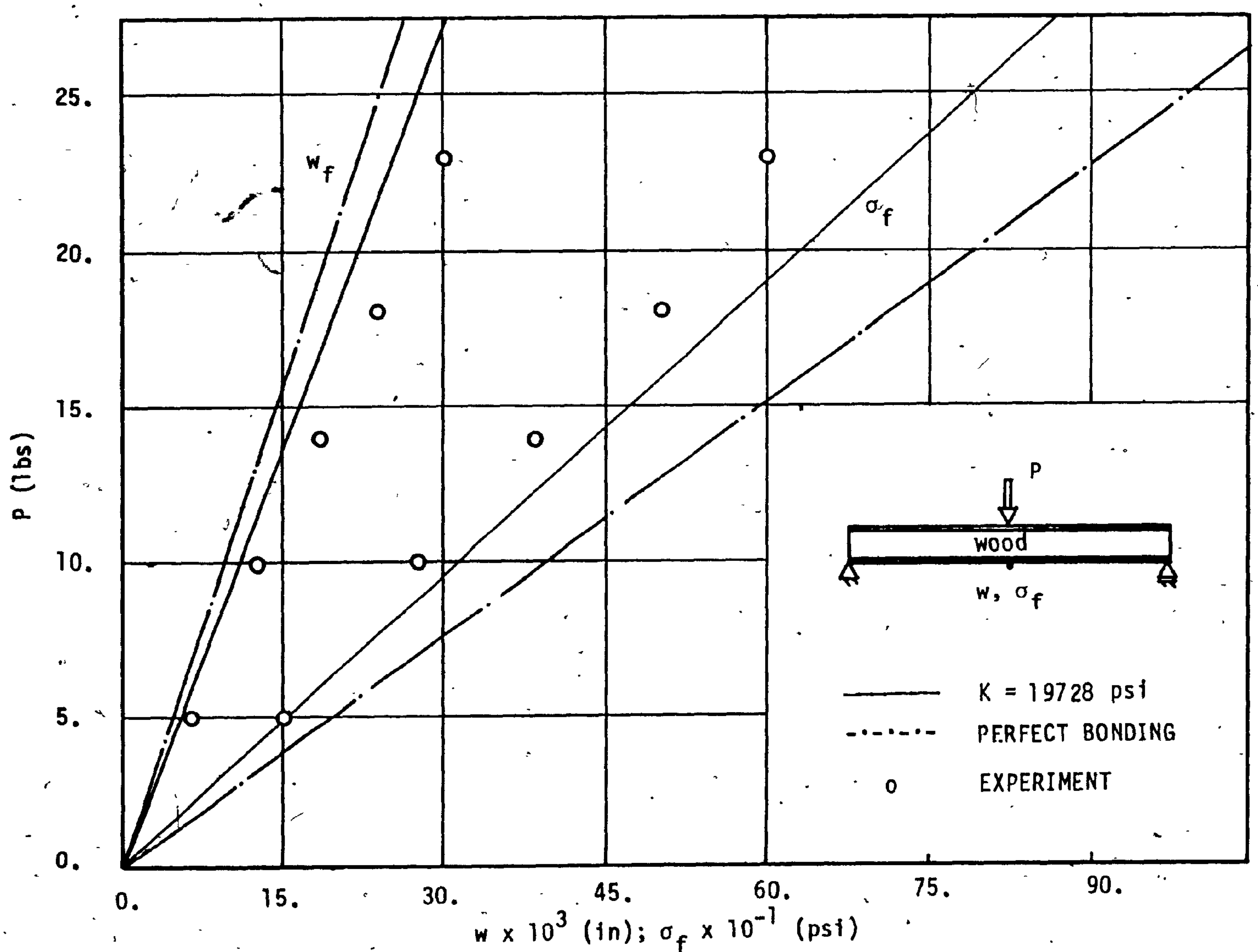


Fig. 4.26 - EXPERIMENTAL AND ANALYTICAL RESULTS FOR A SANDWICH BEAM

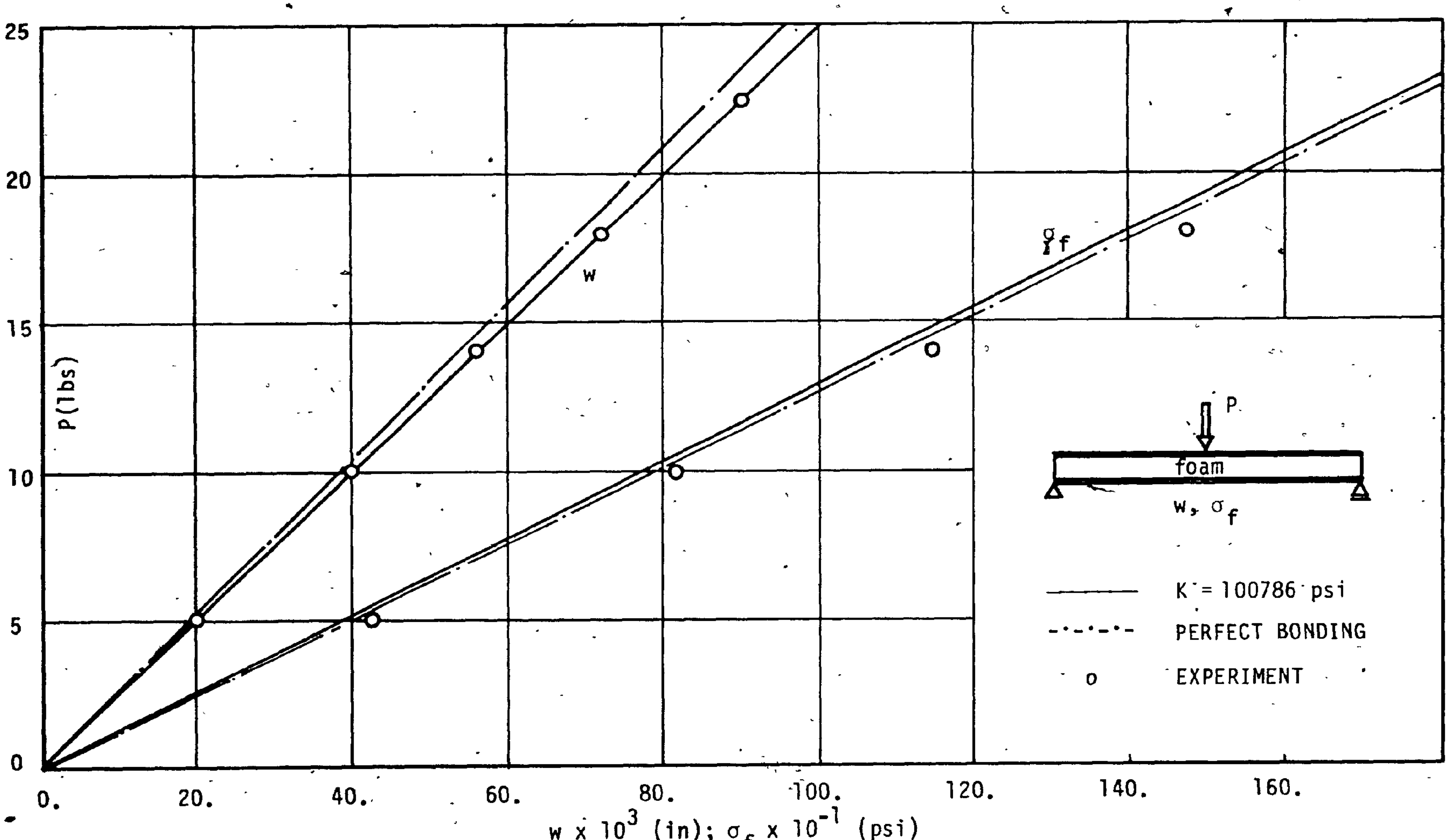


Fig. 4.27 - EXPERIMENTAL AND ANALYTICAL RESULTS FOR A SANDWICH BEAM

CHAPTER V

INSTABILITY OF SANDWICH PANELS

CHAPTER V

INSTABILITY OF SANDWICH PANELS

5.1 INTRODUCTION

The elastic instability problem of sandwich panels is investigated in this chapter. The two major failure modes, in which instability may manifest itself are : overall buckling and local instability.

Overall buckling occurs when a sandwich panel becomes elastically unstable under the application of inplane loads with the buckling mode characterized by long waves. This problem has been extensively investigated by many investigators both theoretically and analytically. Analytic solutions for the critical loads, in correlation with experimental results are found in References [4, 46, 59, 64, 71, 84]. Existing work covers a large variety of loading types, boundary conditions, and panel fabrications. A comprehensive coverage of this topic is found in References [16, 104]. For the present work, only local instability of sandwich panels is considered.

Local instability of sandwich panels may be classified into three categories ; dimpling, wrinkling, and crimping. Dimpling occurs in sandwich construction with a honeycomb core, where in the region above the cell, the face buckles in a plate-like fashion with the cell walls acting as edge supports. Failure by wrinkling occurs in the form of short wave lengths in the facings, and involves straining of the core material. Such failure mode could be symmetric or antisymmetric with respect to the middle plane of the sandwich panel. Finally, crimping is a shear failure mode.

In this chapter, the three local failure modes are investigated analytically. By applying the finite difference method, a numerical value for the dimpling coefficient is found. Comparison is made with existing semi-empirically derived coefficient. Analytic solutions for the wrinkling stress and failing stresses of the core material are also developed. By adopting the available theory by Höff and Mautner [46], a solution is found to define the transition state between the symmetric and antisymmetric wrinkling modes. Although a solution for the crimping load has been developed, the approach presented here is believed to be simpler.

Previous work related to the instability phenomena of dimpling was carried out at the Forest Products Laboratory [95] where paper honeycomb sandwich panels and panels having a solid spruce core through which circular holes were drilled to simulate a core cell were used in the tests. Based on the experimental results, a semi-empirical formula for the dimpling stress was developed [95]. With regard to the wrinkling load and failing stress of core materials two approaches were used in the past. Hoff and Mautner [46] made use of the principle of strain energy with a linear distribution of the transverse displacement through the core. On the other hand, Chong and Hartsock [14]; Harris et al. [42, 43]; and Yusuff [111, 112] considered the faces as plates on an elastic foundation, for which an equivalent foundation modulus was obtained by equating the strain energy stored in the foundation to the sum of the extensional and shear strain energies stored in the core. In all these investigations, experimental works were conducted. Hoff and Mautner tested fifty-one flat rectangular sandwich specimens in edgewise compression, and observed both modes of the wrinkling.

5.2 DIMPLING OF SANDWICH PANELS

The critical stress causing elastic buckling of a homogeneous plate subjected to compressive forces can be put in the following form [102]

$$\sigma_{cr} = \frac{K \pi^2 E_f}{12 (1 - \nu_f^2)} \left(\frac{t_f}{S}\right)^2 \tag{5.1}$$

in which

- K = coefficient which depends on the plate boundary conditions, and loading type.
- E_f = elastic modulus of the plate material
- ν_f = Poisson's ratio of the plate material
- t_f = the plate thickness
- S = the plate width.

Equation (5.1) can be written as

$$\left(\frac{t_f}{S}\right)^2 = \frac{\sigma_{cr} (1 - \nu_f^2)}{K_d \cdot E_f} \tag{5.2}$$

in which

$$K_d = \frac{\pi^2 K}{12}$$

Since dimpling occurs in a plate-like fashion, where in the region above a cell in the core the face buckles with the cell walls acting as edge supports, Equation (5.2) may be used to estimate the dimpling stress. The symbol S is now interpreted to be the diameter of the largest circle that can be inscribed within a cell of the core [95] (Fig. 5.1 (a)); E_f, ν_f, t_f are the facing properties. With this procedure in conjunction with experimental results (Fig. 5.1 (b)), the Forest Products Laboratory [95] has obtained a coefficient of 2 for K_d.

As an attempt to give some theoretical justification to this semi-empirical coefficient, the finite difference method is now used to determine the buckling stress for a plate supported by honeycombed core cells. For this purpose, consider the sandwich plate in Figure 5.2. The governing differential equation for the buckling of a thin plate is [96, 101, 102]

$$\nabla^4 w + \frac{p_x}{D} \frac{\partial^2 w}{\partial x^2} = 0 \qquad (5.3)$$

in which

p_x = the applied compressive force, per unit length, of the plate in X- direction (Fig. 5.2)

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

D = the flexural rigidity of the plate

$$= \frac{E_f \quad t_f^3}{12\ (1 - \nu_f^2)}$$

w = plate deflection.

By applying the ordinary finite difference stencil [96, 101] to the finite difference mesh points (Fig. 5.2), the resulting system of linear simultaneous equation is

$$[A]\ \{w\} = \alpha\ [B]\ \{w\}$$

or

$$[C]\ \{w\} = \alpha\ \{w\} \tag{5.4}$$

in which

$$\alpha = \frac{p_x\ \lambda^2}{D}$$

where λ is the mesh width, $\lambda = \frac{a}{3}$ and a is the side-length of a square within which the cell in Fig. 5.2 is inscribed

{w} = the vector of the deflection ordinates at the finite difference mesh points

[C] = $[B]^{-1}$ [A], where [A] and [B] are coefficient matrices obtained as

$$[A] = \begin{bmatrix} 18 & -8 & -8 & 2 \\ & 18 & 2 & -8 \\ \text{Sym.} & & 18 & -8 \\ & & & 18 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ & 2 & 0 & 0 \\ \text{Sym.} & & 2 & -1 \\ & & & 2 \end{bmatrix}$$

Hence, the governing equation (5.3) is reduced to an eigenvalue problem in Equation (5.4). Of the four eigenvalues obtained from a computerized solution [41], the lowest one is

$$\alpha = 4 .$$

Thus, from the definition of α, the critical stress is found as

$$\sigma_{cr} = 2.25 \frac{E_f}{(1 - \nu_f^2)} \left(\frac{t_f}{S}\right)^2 \qquad (5.5)$$

The coefficient for K_d is then 2.25, which is slightly higher than the value recommended by the Forest Products Laboratory.

For the case when the facings are subjected to biaxial compression, the interaction formula developed by Timoshenko [102] for homogeneous plates may be used

$$\sigma_x + \sigma_y = \sigma_{cr} \qquad (5.6)$$

in which

σ_x, σ_y = the applied compressive stresses in X- and Y- directions, respectively.

This formula was recommended by Sullins [95] for dimpling analysis. When one of the applied stresses is tensile, a conservative solution could be obtained by assuming that the compressive stress is acting alone. For more general loading types, the principal stresses should be used in Equation (5.6) [95] in the place of σ_x and σ_y.

5.3 WRINKLING OF SANDWICH PANELS

As has been mentioned, wrinkling is characterized by its short waves involving bending of the skins and compression or elongation of the core material in the transverse direction. This type of local failure occurs when the core thickness is such that the overall buckling is not likely to happen.

In the following, formulas to define the transition state between the symmetric and antisymmetric wrinklings are presented. In addition, solutions for the symmetrical wrinkling stress, and the failing stresses of the core material are developed.

5.3.1 TRANSITION STATE BETWEEN THE TWO WRINKLING MODES

In the symmetrical wrinkling mode (Fig. 5.3 (a)), the core material is called upon to experience normal and shear stresses within a marginal zone adjacent to the faces. In the middle part of the core, little deformations take place. By applying the strain energy method, Hoff and Mautner obtained the wrinkling stress for this failure mode as [46]

$$\sigma_{crs} = 0.817 \left(\frac{E_f\ E_c\ t_f}{t_c}\right)^{1/2} + 0.166\ \frac{G_c\ t_c}{t_f} \quad \text{when } w = \frac{t_c}{2} \quad (5.7)$$

$$= 0.91\ (E_f\ E_c\ G_c)^{1/3} \quad \text{when } w < \frac{t_c}{2} \quad (5.8)$$

in which

σ_{crs} = the symmetrical wrinkling stress

w = the width of the marginal zone (Fig. 5.3 (a))

= $1.44\ t_f\ (E_f/E_c)^{1/3}$

t_c = the core thickness.

When the core is neither thin enough to cause overall buckling, nor thick enough to cause symmetric wrinkling, the possible wrinkling mode is a skew ripple as shown in Fig. 5.3 (b). By using the strain energy method, the wrinkling stress in this case was obtained as [46]

$$\sigma_{cra} = 0.59 \left(\frac{E_f\ E_c\ t_f}{t_c}\right)^{1/2} + 0.387\ \frac{G_c\ t_c}{t_f} \quad \text{when } w = \frac{t_c}{2} \quad (5.9)$$

$$= 0.51\ (E_f\ E_c\ G_c)^{1/3} + 0.33\ \frac{G_c\ t_c}{t_f} \quad \text{when } w < \frac{t_c}{2} \quad (5.10)$$

in which

σ_{cra} = the antisymmetrical wrinkling stress

$$w = 2.38\, t_f\, (E_f/E_c)^{1/3}$$

By equating the right hand side of Equs. (5.8) and (5.10), and of Equs. (5.7) and (5.9), it is found that

$$\frac{t_c}{t_f} = 1.924\, (E_f/E_c)^{1/3} \qquad \text{when } w < \frac{t_c}{2} \qquad (5.11)$$

and

$$\frac{t_c}{t_f} = 1.616\, (E_f/E_c)^{1/3} \qquad \text{when } w = \frac{t_c}{2} \qquad (5.12)$$

Equs. (5.11) and (5.12) can be used to predict the correct mode of wrinkling. If the ratio of the core to face thicknesses is less than that given by the expression (5.11) or (5.12) an antisymmetric wrinkling mode is to be expected.

5.3.2 ANALYTIC SOLUTION FOR THE SYMMETRICAL WRINKLING STRESS

The problem of symmetrical wrinkling of sandwich panels was studied by many investigators [14, 42, 43, 46, 111, 112]. In all these studies, a linear distribution of the transverse displacement through the core was considered, and the faces were treated as plates on elastic foundation.

In this section, an analytic solution for the symmetrical wrinkling stress is obtained by using an elasticity approach. The assumptions commonly accepted for this type of analysis are :

(1) The inplane stresses in the core are neglected. That is, with X-Y axes in the plane of a sandwich plate, and Z- axis perpendicular to it :

$$\sigma_{cx} = \sigma_{cy} = \tau_{cxy} = 0$$

in which

σ, τ = normal and shear stresses, respectively

c = subscript denoting the core.

Thus, the relevant deformations in the core are in the transverse direction and shear deformations in XZ and YZ planes.

(2) The wrinkling consists of a plane deformation. Thus, if a sandwich panel is compressed in X- direction (Fig. 5.3 (a)), the lateral deflection w is independent of y [32, 42, 43, 46, 71, 111, 112].

(3) The core can be treated as a semi-infinite medium in which the displacement decreases exponentially with maximum value at the interface with the skin [36, 71, 106, 111].

(4) The faces are thin in comparison with the core thickness. This implies that the deflection of each face is identified with the displacement of the core at its surface.

(5) The effect of Poisson's ratio of the core material is neglected [27, 32, 43, 46, 68, 74, 111].

Consider now an element of the core in Fig. 3.5 (a). The equilibrium equations of such element are

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \qquad \text{in X- direction}$$

$$\frac{\partial \tau_{zy}}{\partial z} = 0 \qquad \text{in Y- direction}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad \text{in Z- direction}$$

With u, v, w as the core displacement components in X-, Y-, and Z-directions respectively, the stress-strain relations are

$$\tau_{xz} = G_c \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = G_{cyz} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$= G_{cyz} \frac{\partial v}{\partial z} \qquad \text{since w is independent of y}$$

$$\sigma_z = E_c \frac{\partial w}{\partial z} \qquad \text{since the effect of Poisson's ratio of the core material is neglected}$$

and the third equilibrium equation becomes

$$G_c \frac{\partial^2 w}{\partial x^2} + E_c \frac{\partial^2 w}{\partial z^2} = 0 \qquad (5.13)$$

Because of the symmetry about the middle plane of the core (Fig. 5.3 (a)), only the left half is considered here, and a solution for Equ. (5.13) is taken as [71]

$$w = w_o' e^{kz} \sin \frac{\pi x}{L}$$

in which

L = the wave length

w_o' = the deflection at $x = \frac{L}{2}$ and $z = 0$.

Substituting of this equation into Equ. (5.13) yields

$$K = \frac{\pi}{L} \left(\frac{G_c}{E_c}\right)^{1/2}$$

Consequently,

$$w_s = (w)_{z=0} = w_o' \sin \frac{\pi x}{L}$$

$$\sigma_s = (\sigma_z)_{z=0} = E_c K w_o' \sin \frac{\pi x}{L}$$

The equilibrium equations of a face element have the form of beam-column equations [102]

$$\frac{dM}{dx} - Q = 0$$

$$-\frac{dQ}{dx} + P \frac{d^2 w_s}{dx^2} + \sigma_s = 0$$

$$M = -D \frac{d^2 w_s}{dx^2}$$

in which

M, Q = the bending moment and shearing force per unit width of the face

P = the axial force on the face, Fig. 5.3 (a)

σ_s = the interlayer normal stress

D = flexural rigidity of the face

$$= \frac{1}{12} E_f t_f^3$$

By eliminating M and Q in the equilibrium equations, the governing equation for the face deflection is obtained as

$$D \frac{d^4 w_s}{dx^4} + P \frac{d^2 w_s}{dx^2} = -\sigma_s \qquad (5.14)$$

Substituting the expressions for w_s and σ_s into this equation and rearranging the terms gives

$$P = D \left(\frac{\pi}{L}\right)^2 + E_c K \left(\frac{L}{\pi}\right)^2$$

The above represents the axial load required to keep a face in its wrinkling pattern. The critical wave length is the one which makes P a minimum

$$\frac{dP}{dL} = 0$$

This yields

$$L_{cr} = \left(\frac{2\pi^3 D}{\sqrt{E_c G_c}}\right)^{1/3} \quad (5.15)$$

consequently,

$$P_{cr} = 0.825\, t_f\, (E_f\, E_c\, G_c)^{1/3}$$

and the critical stress is

$$\sigma_{cr} = 0.825\, (E_f\, E_c\, G_c)^{1/3} \quad (5.16)$$

In a more general form, Equ. (5.16) can be written as

$$\sigma_{cr} = K_w\, (E_f\, E_c\, G_c)^{1/3}$$

where K_w is called the wrinkling coefficient and has been found to be equal to 0.961, 0.91, 0.85, and 0.78 respectively in the investigations carried out by Yusuff [111, 112], Hoff [46], Williams [105], and Gough [36]. In practice, an experimental value of 0.5 is commonly used for K [46] (Fig. 5.4).

As an indication of the correctness of Equ. (5.16) some results are compared to the experimental and analytical data from Reference [111] as shown in Table 5.1 It is seen that the present solutions are in better agreement with the experimental results.

5.3.3 EFFECT OF INITIAL IMPERFECTION ON WRINKLING PHENOMENA

In the foregoing derivations of wrinkling stress, a perfectly flat sandwich panel is assumed. To study the effect of an initial imperfection in the form

$$w_o = w_o'' \sin \frac{\pi x}{L}$$

in which

w_o = the initial shape of an irregularity in the face

w_o'' = the initial amplitude at $x = \frac{L}{2}$

on the wrinkling phenomena, the method of equivalent lateral load, as was used by Timoshenko [102], is adopted. According to this method, the governing equation (5.14) becomes

$$D \frac{d^4 w_a}{dx^4} + P \frac{d^2 w_a}{dx^2} = - P \frac{d^2 w_o}{dx^2} \tag{5.17}$$

in which

w_a = an additional deflection to be added to the initial deflection when the sandwich panel is submitted to the action of the compressive force P.

The deflection w_a is assumed in the form

$$w_a = w_o''' \sin \frac{\pi x}{L}$$

in which

$$w_o''' = \text{the amplitude of the additional deflection at } x = \frac{L}{2}$$

It follows from Equ. (5.17) that

$$w_o''' = \frac{\alpha\, w_o''}{1 - \alpha} \tag{5.18}$$

in which

$$\alpha = \frac{P\, L^2}{\pi^2\, E_f\, I_f} = \frac{P}{P_m}$$

$$I = \frac{t_f^3}{12}$$

$$P_m = \frac{\pi^2\, E_f\, I_f}{L^2}$$

and consequently, the total deflection is

$$w = w_o + w_a$$

$$= \frac{w_o''}{1 - \alpha} \sin \frac{\pi x}{L}$$

The deflection amplitude seems to be a product of two terms, the first term being the initial amplitude without axial load and the second term being an amplification factor which depends upon the value of α.

It is to be expected that w_o''' is maximum at the critical state of stress, but since only the initial irregularities, whose wave lengths are the same as the critical wave length of wrinkling, are considered in the present analysis, the most critical condition is produced when α becomes (P/P_{cr}), where P_{cr} is the wrinkling load [42, 43, 112]. Thus, Equ. (5.18) can be written as

$$w_o''' = \frac{w_o''}{\frac{\sigma_{cr}}{\sigma} - 1} \tag{5.19}$$

As an increase in the applied compressive load will give rise to additional core deformations, loading to critical situation where the ultimate strength of the core material is reached before wrinkling can occur, and thus results in an internal core failure. An analysis to determine the failing stress is given next.

Consider a sandwich panel subjected to compressive load. The maximum normal and shear strains in the core are

$$\varepsilon_z = \left(\frac{\partial w_a}{\partial z}\right)_{\substack{z=0 \\ x=L/2}} = K\, w_o'''$$

$$\gamma_{xz} = \left(\frac{\partial w_a}{\partial x}\right)_{\substack{z=0 \\ x=0}} = \frac{\pi}{L}\, w_o'''$$

From Hooke's law, assuming that the stress-strain relations for the core material is linear up to failure point :

$$w_o''' = \frac{T}{E_c K}$$

or

$$w_o''' = \frac{L \cdot S}{\pi G_c}$$

in which

T = the core compressive or tensile strength

S = the core shear strength.

By substituting these expressions into Equ. (5.19), it is found that

$$\sigma_{fn} = \frac{\sigma_{cr}}{1 + \dfrac{E_c K w_o''}{T}}$$

$$\sigma_{fs} = \frac{\sigma_{cr}}{1 + \dfrac{\pi G_c w_o''}{L S}} \qquad (5.20)$$

in which

σ_{fn} = the failing stress of the core material based on its strength in compression or tension

σ_{fs} = same meaning as σ_{fn} but for shear stress.

By means of Equ. (5.20), for any given initial imperfection and core strength, the failing stress can be determined, or conversely, for any given initial imperfection, the core strength required to sustain a specific axial load can be computed.

As an indication of the correctness of the present formulas, some results are compared to the experimental and analytical data from Reference [112] as shown in Table 5.2. It is seen that the present simple theory yields slightly less accurate results than Yusuff's theory [112].

It should be noted that, since the core material is stressed beyond its proportional limit, the so-called plasticity factor should be used in calculating the wrinkling stress in Equs. (5.20). Thus

$$\sigma_{cr} = K_w (\eta E_f E_c G_c)^{1/3} \qquad (5.21)$$

in which

η = the plasticity factor for facing material, when subjected to edge compression. A formula for this factor can be obtained from References [42, 95, 102, 112].

5.4 CRIMPING LOAD OF SANDWICH PANELS

Consider a sandwich strut (Fig. 5.6) under an axial load P. The total deflection w is composed of two superimposed components: w_b due to flexure and w_s due to shear, as shown in Fig. 5.6 (a). The slope of the deflected surface can be obtained from

$$\frac{dw}{dx} = \frac{dw_b}{dx} + \frac{dw_s}{dx}$$

$$= \left(\int_0^x -\frac{M}{D}\,dx \right) + \frac{Q}{S} \tag{5.22}$$

in which

M = the applied bending moment on a section at a distance x from the origin (Fig. 5.6 (b))

$= P\,w$

Q = the shear force at the same section.

$= P\,\frac{dw}{dx}$

S = the shear stiffness given in chapter II.

Differentiating Equ. (5.22) once with respect to x and substituting the expressions for M and Q resulted in

$$\frac{d^2w}{dx^2} = \frac{P/D}{\frac{P}{S} - 1}\,w \tag{5.23}$$

A solution for this equation must satisfy the following boundary conditions

$$w = 0 \qquad \text{at } x = 0 \text{ and } x = a$$

$$\frac{d^2w_b}{dx^2} = 0$$

and is taken in the form

$$w = \sum_{m=1}^{\infty} C_1 \sin \alpha_m x + C_2 \cos \alpha_m x + C_3$$

in which

$$\alpha_m = \frac{m\pi}{a} \qquad m = 1,2,\ldots$$

C_1, C_2, C_3 = coefficients to be determined by satisfying the boundary conditions.

By satisfying the boundary conditions, it is found that

$$w = \sum_{m=1}^{\infty} C_1 \sin \alpha_m x \qquad (5.24)$$

Substituting this solution into Equ. (5.23) and simplifying the terms gives

$$P = \frac{m^2 P_E}{1 + m^2 \frac{P_E}{S}}$$

in which

P_E = Euler load

$= \frac{\pi^2 D}{a^2}$

D = the flexural rigidity of the strut

a = the length of the strut (Fig. 5.6 (a)).

When $m \to \infty$ this formula becomes

$$P = S \tag{5.25}$$

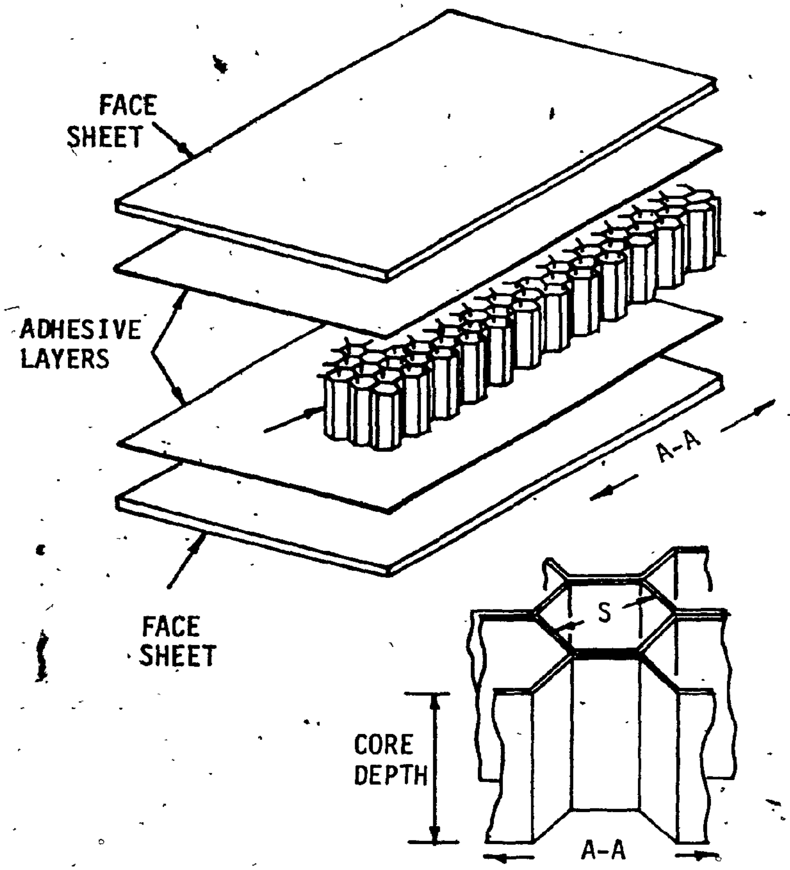
which is conventionally known as the crimping load of sandwich panels.

E_f (psi)	E_c (psi)	G_c (psi)	WRINKLING STRESS (psi) PRESENT	DATA IN REF. [111] THEORY	TEST
30×10^6	356	137	9366	13850	10500
	1135	437	20292	32600	24500
	1410	542	23437	37800	24000
	2575	990	35019	33200	38000
	2580	992	35065	33200	36400
	3250	1250	40903	51600	44000
3×10^6	1500	750	12376	11720	12590
10^7	1400	400	14650	17060	18840
	1200	250	11899	13860	14370

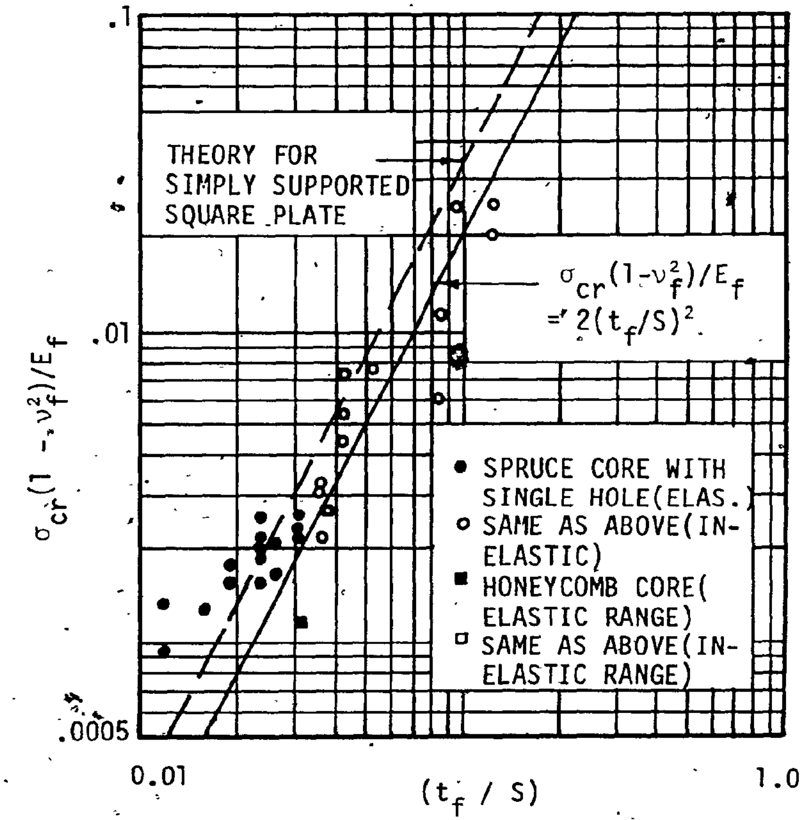
TABLE 5.1 - COMPARISON OF ANALYTICAL AND EXPERIMENTAL WRINKLING STRESSES

E_f (psi)	E_c (psi)	G_c (psi)	T (psi)	η w_o'' (in)	FAILING STRESS (psi) PRESENT	DATA IN REF. [112] THEORY	 TEST
10^7	23000	8200	270	.04 .00	35000	41000	40300 42100 46300
10^7	23000	8200	270	.095 .000	47000	54000	51800 53400 56800 59500

TABLE 5.2 - COMPARISON OF ANALYTICAL AND EXPERIMENTAL FAILING STRESS



(a) HONEYCOMB SANDWICH CONSTRUCTION



(b) DIMPLING STRESS DUE TO UNIAXIAL COMPRESSION[95]

Fig. 5.1 - HONEYCOMB SANDWICH PLATE AND EXPERIMENTAL DATA FOR DIMPLING

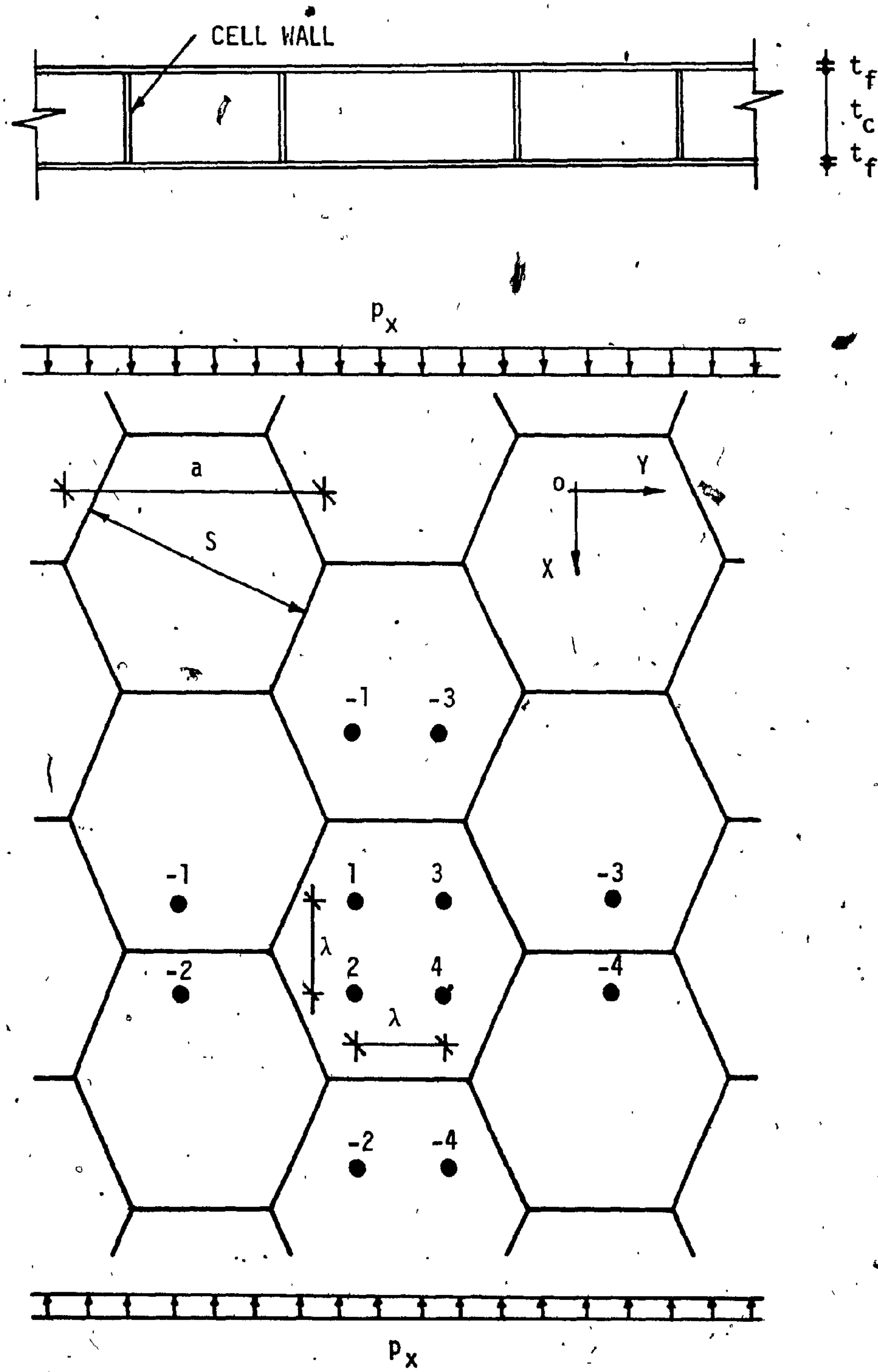


Fig. 5.2 - FINITE DIFFERENCE MESH FOR HONEYCOMB CORE

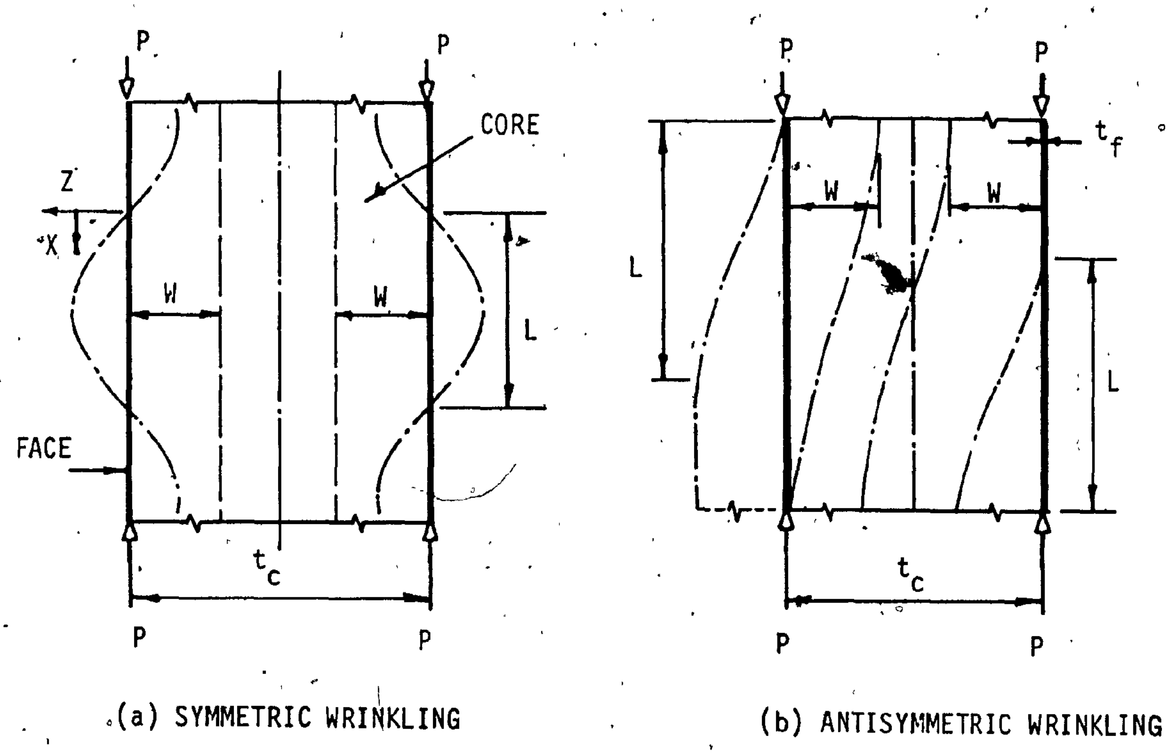


Fig. 5.3 - FACE WRINKLING

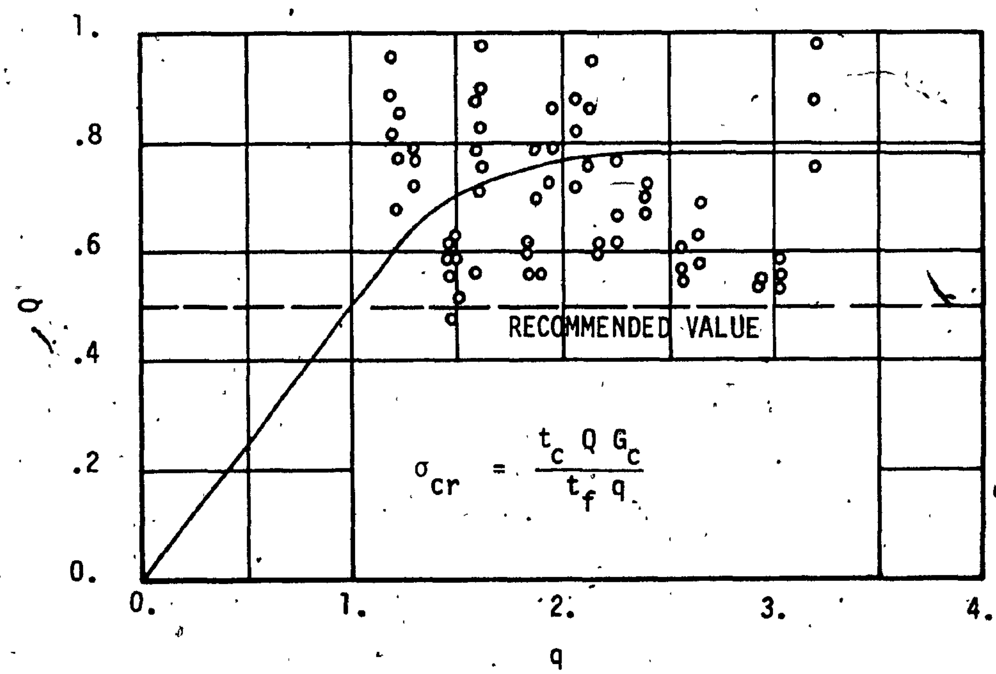


Fig. 5.4 - RECOMMENDED VALUE FOR WRINKLING COEFFICIENT [95]

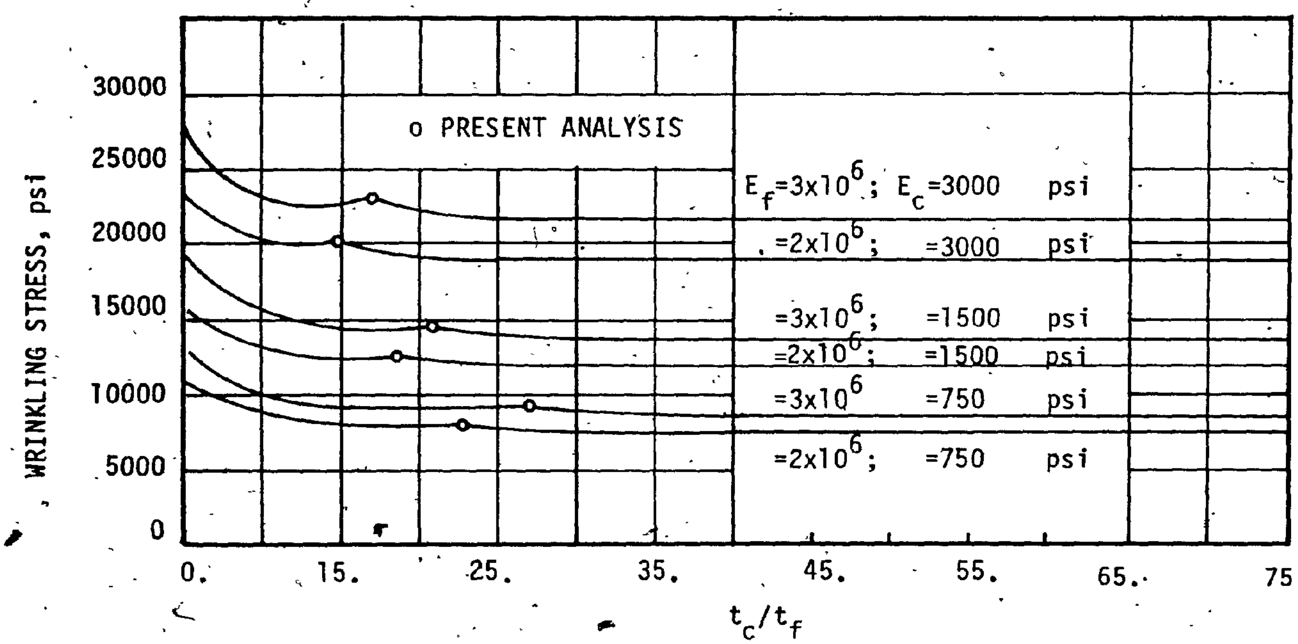
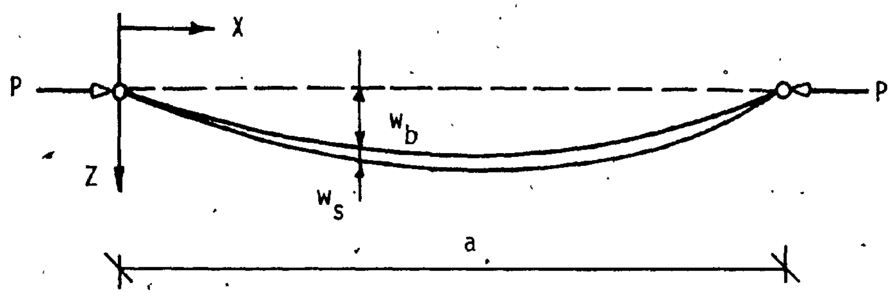
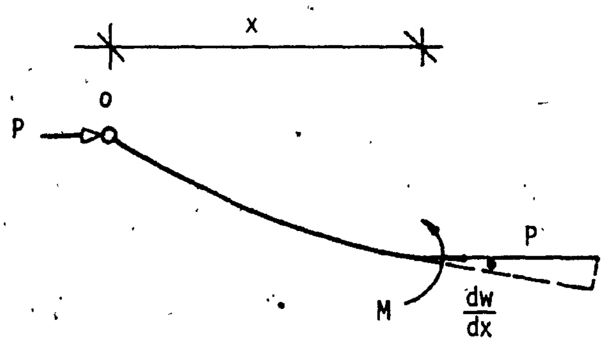


Fig. 5.5 - COMPARISON OF RESULTS OF THE EXPRESSIONS OBTAINED TO DEFINE WRINKLING TRANSITION STATE WITH HOFF' THEORY[46]



(a) SANDWICH STRUT UNDER COMPRESSIVE LOAD



(b) FREE BODY DIAGRAM OF A SANDWICH STRUT

Fig. 5.6 - CRIMPING LOAD OF SANDWICH PANELS

CHAPTER VI

HYGROTHERMAL STRESSES IN SANDWICH PANELS

CHAPTER VI

HYGROTHERMAL STRESSES IN SANDWICH PANELS

6.1 INTRODUCTION

Structural systems are usually called upon to experience thermal stresses when they are subjected to a temperature change. There are three causes for the thermally induced stresses. First of all, when the temperature change in a homogeneous plate, for instance, is not uniform, the different elements of its body tend to deform differently, but since the restriction of displacement continuity will prevent such free deformations, it follows that various elements exert upon each other a restraining action which produces a system of strains to cancel out a part of the free deformations at every point so as to ensure continuity of the displacement. This system of strains must be accompanied by a corresponding system of stresses which are known as thermal stresses. Second, when the temperature change in a homogeneous plate is uniform, but the free displacements are limited by certain boundary conditions, thermal stresses will be imposed. Finally, when a plate is heterogeneous or made of dissimilar materials, then it may be unable to deform freely in a manner compatible with the temperature distribution through it, and hence thermal stresses will be exhibited, although the plate may be free to move at its boundaries [9, 31, 44, 96]. Based on this discussion, sandwich components will undergo thermal stresses due to uniform and gradient temperature changes, whether the boundaries are free or not.

In this chapter a solution for thermally induced stresses in sandwich plates subjected to a uniform temperature change is developed using an elasticity approach. The finite element method is used for comparison. The explicit form of 24 x 24 stiffness matrix of an orthotropic rectangular prism finite element is derived and used in the finite element computer program written based on the direct stiffness method. To facilitate the use of this solution, simple formulas are derived to calculate the maximum normal stress in facings and interlayer shear in sandwich panels subjected to uniform temperature change. Based on engineering theory, simple formulas are derived to calculate the maximum normal stresses in facings of sandwich panels subjected to a moisture content change. The results of existing theories in the literature for thermal stresses in sandwich plates subjected to thermal gradient are reformulated in this chapter and presented in simple forms for practical applications.

With regard to the previous work in this area, several theories are available in the literature for thermal stresses in sandwich plates subjected to thermal gradient. Bijlaard [8] and Jerzy [53] used the theory of elasticity to solve sandwich plates subjected to thermal gradient and with the following boundary conditions : plate with clamped edges, plate with simply supported edges, plate with two opposite edges simply supported and remaining edges free, and plate with two opposite edges simply supported and remaining edges free but with edge stiffeners. According to the results by Bijlaard and Jerzy, the states of deformation and stresses for a sandwich plate with simply supported or clamped edges and subjected to thermal gradient are the same as for homogeneous plates. By using the principle of minimum total potential energy, Ebcioglu [11, 23] solved simply supported sandwich plates subjected to transverse load, edge compression and thermal gradient.

The problem of thermal stresses has also been investigated experimentally and by using the finite element method. Sandwich panels with cold-formed steel facings and subjected to a thermal gradient across the thickness were used in a series of experiments conducted by Chong [12] to measure the deflection and stresses induced. The results were compared with analytical expressions derived based on a fourth order governing differential equation established for the deflection. March [69, 70] investigated experimentally and by using an elasticity approach the thermal stresses in sandwich strips subjected to uniform temperature change and thermal gradient. By using the finite element, the transverse displacement of an axisymmetric heated sandwich cylinder was determined by Monforton [76]. On the other hand, the hybrid finite element is used by Hoa [45, 97] to investigate the thermal stresses in homogeneous plates.

6.2 THERMAL STRESSES IN SANDWICH PLATES SUBJECTED TO A UNIFORM TEMPERATURE CHANGE

6.2.1 ASSUMPTIONS

Consider a sandwich plate composed of two facings each of thickness t_f, a core of thickness t_c, and subjected to a uniform temperature change as shown in Fig. 6.1 As the need to consider the thermal deformations of different materials necessitates the consideration of compatibility of deformations and stresses at the interfaces, the problem is irreducibly three dimensional. Several assumptions mentioned in Chapter IV can still be retained, and new ones are introduced. These are :

(1) The faces are thin in comparison with the core depth. This implies

that the flexural rigidity of each face about its own middle surface is negligible, and consequently, the inplane stresses resisted by each skin are uniformly distributed across its thickness.

(2) The two facings are of equal thickness and made of the same material.

(3) Materials are homogeneous, isotropic, and linearly elastic.

(4) Deformations are small. The deflection along X- and Y- axes can be ignored.

(5) Poisson's ratio of the core material, and the inplane shear stress in the core can be ignored.

(6) Rigid adhesives are used.

(7) Inplane shear stresses in the facing can be ignored.

(8) The temperature distribution is constant in the plane of each face and across the plate thickness.

(9) The sandwich plate is free to expand along its edges.

(10) The temperature level does not affect the material properties.

It is noted that five assumptions are new. Assumption (6) implies that the interlayer shear stresses do not induce elastic deformations in the rigid adhesive layers. Assumption (7) effectively implies the existence of a simplified two dimensional stress state in the facings. Assumption (9) represents an idealized form of behaviour of free edges sandwich plate. This assumption was considered by Marsh in his analysis

[69, 70]. Considering constant temperature distribution in the plane of the plate, as in assumption (8), is commonly used in practice [8, 53].

6.2.2 GENERAL

When a sandwich plate is subjected to a temperature change, the different layers tend to deform differently. Since the restriction of displacement continuity will prevent such free deformations, it follows that various layers exert upon each other a restraining action which induces a system of strains accompanied by a system of thermal stresses.

Consider now the facings, which are assumed to be in a plane stress condition with the non zero stress components σ_{fx}, σ_{fy} and τ_{xy}. The standard differential equations of equilibrium for the facings are

$$\frac{\partial \sigma^j_{fx}}{\partial x} + \frac{\partial \tau^j_{yx}}{\partial y} \mp \frac{q_x}{t^j_f} = 0 \qquad (4.2)$$

$$\frac{\partial \sigma^j_{fy}}{\partial y} + \frac{\partial \tau^j_{xy}}{\partial x} \mp \frac{q_y}{t^j_f} = 0 \qquad (4.3)$$

in which

j = subscript denoting stresses in the face j, while t^j_f is the thickness of this face

f = subscript denoting face

q_x, q_y = interlayer shear stresses.

The negative and positive signs correspond to j=1 for the bottom face, and j=2 for the upper one, respectively.

The state of stresses in the core must satisfy the following equilibrium equations :

$$\frac{\partial\sigma_{cx}}{\partial x} + \frac{\partial\tau_{czx}}{\partial z} = 0$$

$$\frac{\partial\sigma_{cy}}{\partial y} + \frac{\partial\tau_{czy}}{\partial z} = 0 \qquad (4.4)$$

in which

σ_{ci} = normal stress in the core in the i- direction

τ_{czi} = shear stress in the core

i = subscript denoting x or y.

The equilibrium equations in terms of displacement components are

$$E_{cx}\frac{\partial^2 u_c}{\partial x^2} + G_{cx}\frac{\partial^2 u_c}{\partial z^2} + G_{cx}\frac{\partial^2 w}{\partial z\,\partial x} = 0$$

$$E_{cy}\frac{\partial^2 v_c}{\partial y^2} + G_{cy}\frac{\partial^2 v_c}{\partial z^2} + G_{cy}\frac{\partial^2 w}{\partial z\,\partial y} = 0 \qquad (6.1)$$

in which

u_c, v_c = displacement components in the core along X- and Y-directions, respectively

E, G = the elastic and shear moduli of the core material, respectively

c = subscript denoting core

w = the lateral deflection.

At the interfaces between the core and the skins, the stresses and deformations must be compatible. The compatibility equations in terms of stresses are

$$q_x = G_{cx} \left(\frac{\partial u_c}{\partial z} + \frac{\partial w}{\partial x}\right)_{z=t_c/2}$$

$$q_y = G_{cy} \left(\frac{\partial v_c}{\partial z} + \frac{\partial w}{\partial y}\right)_{z=t_c/2} \qquad (4.8)$$

In terms of displacement components, the compatibility equations are written as [48, 69, 70]

$$u_f - (u_c)_{z=t_c/2} = \Delta x$$

$$v_f - (v_c)_{z=t_c/2} = \Delta y \qquad (6.2)$$

in which

Δ = $(\alpha_c - \alpha_f)\,T$

where α_c and α_f are the thermal expansion coefficients of the core and faces materials, respectively. T is the temperature change at the sandwich plate surfaces.

u_f, v_f = displacement components in facings along X- and Y-directions, respectively.

Solutions to the problem must also satisfy the boundary conditions. With regard to the sandwich plate in Fig. 6.1, the relevant boundary conditions are :

$$\left.\begin{array}{llll} \text{at } x = 0 & u_f = u_c = 0 & \text{and} & \tau_{xz} = 0 \\ \quad x = \pm a & \sigma_{fx} = 0 & \text{and} & \sigma_{cx} = 0 \\ \text{at } y = 0 & v_f = v_c = 0 & \text{and} & \tau_{yz} = 0 \\ \quad y = \pm b & \sigma_{fy} = 0 & \text{and} & \sigma_{cy} = 0 \\ \text{at } z = 0 & & & \tau_{zx} = \tau_{zy} = 0 \end{array}\right\} \quad (6.3)$$

These equations imply that no normal stress exists in the core or facings along the boundaries. Due to the symmetry in the panel geometry and temperature distributions, the transverse shear stress τ_{xz}, τ_{yz} and the displacement components u_c, v_c, u_f and v_f vanish at the transverse central planes. Because the two facings are identical, no horizontal shear stresses exist on the middle plane.

6.2.3 THERMAL STRESSES

Consider the sandwich plate in Fig. 6.1 subjected to a uniform temperature change. By considering assumption (4), the equilibrium equations (6.1) become

$$E_{cx} \frac{\partial^2 u_c}{\partial x^2} + G_{cx} \frac{\partial^2 u_c}{\partial z^2} = 0 \qquad (4.4)$$

$$E_{cy} \frac{\partial^2 v_c}{\partial y^2} + G_{cy} \frac{\partial^2 v_c}{\partial z^2} = 0 \qquad (4.5)$$

The solution satisfying the boundary conditions of the core layer in Equ. (6.3) is found as [48, 69, 70]

$$\left.\begin{aligned} u_c &= \sum_{m=1,3,\ldots}^{\infty} C_m \sin \alpha_m x \ \cosh \mu_x \ \alpha_m z \\ v_c &= \sum_{n=1,3,\ldots}^{\infty} D_n \sin \beta_n y \ \cosh \mu_y \ \beta_n z \end{aligned}\right\} \qquad (6.4)$$

in which

$$\alpha_m = \frac{m \pi}{2 a}$$

$$\beta_n = \frac{n \pi}{2 b}$$

$$\mu_x = \left(\frac{E_{cx}}{G_{cx}}\right)^{1/2}$$

$$\mu_y = \left(\frac{E_{cy}}{G_{cy}}\right)^{1/2}$$

C_m, D_n = unknown functions of y and x, respectively

where

2a, 2b = plate dimensions along X- and Y- axes, respectively.

By considering assumption (4), the transverse shear stresses in the core are related to u_c and v_c by the following equations

$$\tau_{xz} = G_{cx} \frac{\partial u_c}{\partial z}$$

$$\tau_{yz} = G_{cy} \frac{\partial v_c}{\partial z}$$

Thus, substituting Equ. (6.4) in these relations gives

$$\tau_{xz} = G_{cx} \sum_{m=1,3,\dots}^{\infty} \mu_x \alpha_m C_m \sin \alpha_m x \sinh \mu_x \alpha_m z \qquad (6.5)$$

$$\tau_{yz} = G_{cy} \sum_{n=1,3,\dots}^{\infty} \mu_y \beta_n D_n \sin \beta_n y \sinh \mu_y \beta_n z \qquad (6.6)$$

from which, the interlayer shear stresses are

$$q_x = G_{cx} \sum_{m=1,3,\dots}^{\infty} \mu_x \alpha_m C_m \sin \alpha_m x \sinh \frac{1}{2} \mu_x \alpha_m t_c$$

$$q_y = G_{cy} \sum_{n=1,3,\dots}^{\infty} \mu_y \beta_n D_n \sin \beta_n y \sinh \frac{1}{2} \mu_y \beta_n t_c$$

These equations together with Equs. (4.2) and (4.3); noting that $\tau_{xy} = \tau_{yx} = 0$, $\sigma_{fx} = 0$ at $x = \pm a$ and $\sigma_{fy} = 0$ at $y = \pm b$; give

$$\sigma_{fx} = - \mu_x \frac{G_{cx}}{t_f} \sum_{m=1,3,\dots}^{\infty} C_m \cos \alpha_m x \sinh \frac{1}{2} \mu_x \alpha_m t_c \qquad (6.7)$$

$$\sigma_{fy} = -\mu_y \frac{G_{cy}}{t_f} \sum_{n=1,3,...}^{\infty} D_n \cos \beta_n y \sinh \frac{1}{2} \mu_y \beta_n t_c \quad (6.8)$$

The displacement components in the facings can be obtained from Equs. (6.7) and (6.8) together with the two dimensional Hook's law as

$$u_f = \sum_{m=1,3,...}^{\infty} - C_m \frac{\psi}{\alpha_m} \sinh \frac{1}{2} \mu_x \alpha_m t_c \sin \alpha_m x + \nu \psi' \xi' x$$

$$v_f = \sum_{n=1,3,...}^{\infty} - D_n \frac{\psi'}{\beta_n} \sinh \frac{1}{2} \mu_y \beta_n t_c \sin \beta_n y + \nu \psi \xi y$$

in which

$$\psi = \frac{\mu_x G_{cx}}{E_f t_f}$$

$$\psi' = \frac{\mu_y G_{cy}}{E_f t_f}$$

$$\xi' = \sum_{n=1,3,...}^{\infty} D_n \cos \beta_n y \sinh \frac{1}{2} \mu_y \beta_n t_c$$

$$\xi = \sum_{m=1,3,...}^{\infty} C_m \cos \alpha_m x \sinh \frac{1}{2} \mu_x \alpha_m t_c$$

By proper substitution of u_c, v_c, u_f and v_f in the compatibility equations (6.2), it is found that

$$\sum_{m=1,3,\ldots}^{\infty} C_m \left[\left(\frac{\Psi}{\alpha_m} \sinh \frac{1}{2} \mu_x \alpha_m t_c\right) + \cosh \frac{1}{2} \mu_x \alpha_m t_c\right] \sin \alpha_m x = x\,(\nu\,\Psi'\,\xi' - \Delta)$$

$$\sum_{n=1,3,\ldots}^{\infty} D_n \left[\left(\frac{\Psi'}{\beta_n} \sinh \frac{1}{2} \mu_y \beta_n t_c\right) + \cosh \frac{1}{2} \mu_y \beta_n t_c\right] \sin \beta_n y = y\,(\nu\,\Psi\,\xi - \Delta)$$

Expanding x and y in the right hand side of these equations by single Fourier series in sine terms, and equating the coefficients in both sides, expressions for C_m and D_n are determined as

$$\left.\begin{aligned} C_m &= \frac{2\,(-1)^{(m-1)/2}}{a\,\alpha_m^2} \; \frac{(\nu\ \Psi'\,\xi' - \Delta)}{\frac{\Psi}{\alpha_m} \sinh \frac{1}{2} \mu_x \alpha_m t_c + \cosh \frac{1}{2} \mu_x \alpha_m t_c} \\ D_n &= \frac{2\,(-1)^{(n-1)/2}}{b\,\beta_n^2} \; \frac{(\nu\,\Psi\,\xi - \Delta)}{\frac{\Psi'}{\beta_n} \sinh \frac{1}{2} \mu_y \beta_n t_c + \cosh \frac{1}{2} \mu_y \beta_n t_c} \end{aligned}\right\} \quad (6.9)$$

Equivalent expressions can be obtained for ξ and ξ' by multiplying the first equation in (6.9) by $\cos \alpha_m x \sinh \frac{1}{2} \mu_x \alpha_m t_c$, hence

$$\xi = \theta\,(\nu\,\Psi'\,\xi' - \Delta) \tag{6.10}$$

in which

$$\theta = \sum_{m=1,3,\ldots}^{\infty} \frac{2\,(-1)^{(m-1)/2}}{a\,\alpha_m^2} \frac{\cos \alpha_m x \sinh \frac{1}{2}\mu_x \alpha_m t_c}{\frac{\Psi}{\alpha_m} \sinh \frac{1}{2}\mu_x \alpha_m t_c + \cosh \frac{1}{2}\mu_x \alpha_m t_c}$$

in a similar manner, from the second equation in (6.9), it is found that

$$\xi' = \theta' \,(\nu\,\Psi\,\xi - \Delta) \tag{6.11}$$

in which

$$\theta' = \sum_{n=1,3,\ldots}^{\infty} \frac{2\,(-1)^{(n-1)/2}}{b\,\beta_n^2} \frac{\cos \beta_n y \sinh \frac{1}{2}\mu_y \beta_n t_c}{\frac{\Psi'}{\beta_n} \sinh \frac{1}{2}\mu_y \beta_n t_c + \cosh \frac{1}{2}\mu_y \beta_n t_c}$$

Solving for ξ and ξ' from Equs. (6.10) and (6.11) yields

$$\xi = \Delta\,\theta \, \frac{1 + \nu\,\Psi'\,\theta'}{\nu^2\,\Psi\,\Psi'\,\theta\,\theta' - 1}$$

$$\xi' = \Delta\,\theta' \, \frac{1 + \nu\,\Psi\,\theta}{\nu^2\,\Psi\,\Psi'\,\theta\,\theta' - 1}$$

The complexity of the above analysis justifies now examining the simplified case of sandwich strips, that is when $2b \to \infty$. In this case, $\beta_n \to 0$. Thus

$$V_c = \tau_{yz} = 0$$

$$\xi' = \theta' = 0$$

and the first equation in (6.9) becomes

$$C_m = -\frac{2\,(-1)^{(m-1)/2}}{a\,\alpha_m^2}\;\frac{\Delta}{\frac{\Psi}{\alpha_m}\sinh\frac{1}{2}\mu_x\,\alpha_m\,t_c + \cosh\frac{1}{2}\mu_x\,\alpha_m\,t_c}$$

which is in exact agreement with the result obtained by Marsh [69, 70].

6.3 THERMAL STRESSES BY THE FINITE ELEMENT METHOD

6.3.1 INTRODUCTION

In the analysis of structures, the finite element method has wide acceptance by engineers. However, the basic concept of this method is that a continuum can be modeled analytically through its subdivision into elements, the behaviour of each is described with the aid of either a stress or a displacement function. The method has a theoretical basis within the framework of the classical theory and is related to the Ritz's method applied in solving structural problems.

In the conventional Ritz procedure, the displacement field in the entire continuum is described by one set of functions, nevertheless, in the finite element method, the displacement field for each element is defined individually in terms of the displacements at its nodes or points of connections, and hence the continuity is assured within the separate elements whereas the whole domain can be regarded as a piece-wise continuous. However, the continuity is preserved at the nodes by definition, while a careful choice of the displacement field of the elements may assure the continuity across element boundaries as well.

Research in the field of the finite element idealization has produced a large number of basic elements to fit a particular purpose or to be applied in solving a specific problem [1, 6, 7, 10, 17, 18, 30, 38, 45, 47, 51, 55, 57, 72, 73, 75, 76, 77, 80, 92, 96, 108, 109, 113]. As the present work is concerned with the thermal stresses in a sandwich plate subjected to a temperature change, the search will be directed towards the orthotropic three dimensional finite elements. In this category, the rectangular prism with a linear variation of the displacements along its edges seems to be reasonably acceptable, since it ensures the displacement compatibility at the element interfaces and was proved to produce better results than others [75].

6.3.2 FINITE ELEMENT ANALYSIS

To solve a structural problem by applying the finite element method, three conditions are to be satisfied : geometrical, mechanical, and static conditions. First, once the displacement field is described within an element in terms of its nodal displacements, the strain components can be determined readily by the proper differentiation of the assumed displacement function, that is

$$\{\varepsilon\} = [B]\{d\} \tag{6.12}$$

in which

$\{\varepsilon\}$ = the strain components vector

$\{d\}$ = the nodal displacements vector

$[B]$ = the strain-displacement matrix

and this equation represents the geometric relation of compatibility.

Second, the constitutive equation which relates stresses to strains is used to express the second condition, hence

$$\{\sigma\} = [E]\{\varepsilon\} \tag{6.13}$$

in which

$\{\sigma\}$ = the stress components vector

$[E]$ = the elasticity matrix.

Finally, the third condition is satisfied now through the use of the theorem of stationary total potential energy which states [39] : "Among the numerous sets of compatible displacements, the one which also satisfies equilibrium will extremize the total potential energy of the elastic system".

Thus

$$\frac{\partial \Pi}{\partial \{D\}} = 0 \tag{6.14}$$

in which

Π = the total potential energy function of the structural system,

$$= \sum_{i=1}^{n} \Pi_i$$

where, Π_i is the total potential energy of the i^{th} finite element

$\{D\}$ = the structural nodal degrees of freedom.

By following the conventional procedure of the assumed displacement finite element method [18, 30, 47, 75, 113], it is concluded that

$$\Pi_i = U_i + V_i$$

in which

U_i = the element strain energy,

$$= \frac{1}{2} \int_{vol.} \{d\}_i^T [B]_i^T [E]_i [B]_i \{d\}_i \, dv$$

where dv is the element volume

V_i = the potential of the nodal loads $\{r\}_i$

$$= - \{r\}_i^T \{d\}_i$$

consequently, it follows from Equ. (6.14) that

$$\sum_{i=1}^{n} [K]_i \{d\}_i = \sum_{i=1}^{n} \{r\}_i$$

or in an abbreviated form

$$[K] \{D\} = \{Q\} \tag{6.15}$$

in which

$[K]_i$ = the element stiffness matrix

$$= \int_{vol.} [B]_i^T [E]_i [B]_i \, dv \tag{6.16}$$

$[K]$ = the structural stiffness matrix

$$= \sum_{i=1}^{n} [K]_i$$

$\{Q\}$ = the structural nodal forces

$$= \sum_{i=1}^{n} \{r\}_i$$

The preceding formulation is the general description step by step of the finite element method. For the particular case of the rectangular prism in Fig. 6.2, the details are given next. The displacement field within the element is defined by [113]

$$\{f\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

in which

u, v, w = the displacement components in X-, Y- and Z-directions, respectively

{f} = the displacement within the element

which can be written in terms of the nodal degrees of freedom as

$$\{f\} = [IN_1 \;\; IN_2 \;\; IN_3 \;\; IN_4 \;\; IN_5 \;\; IN_6 \;\; IN_7 \;\; IN_8] \{d\} \tag{6.17}$$

in which

I = identity matrix of order three

$\{d\}$ = the element nodal degrees of freedom

= $\{u_1\ v_1\ w_1\ \ldots\ldots\ u_\ell\ v_\ell\ w_\ell\ \ldots\ldots\ u_8\ v_8\ w_8\}^T$

where ℓ=1 to 8 is a node's number of the element (Fig. 6.2)

N_ℓ = shape function at node ℓ (ℓ=1 to 8)

= $\frac{1}{8}(1 + \xi\ \xi_\ell)\ (1 + \eta\ \eta_\ell)\ (1 + \zeta\ \zeta_\ell)$

where

$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \zeta = \frac{z}{c}$

ξ_ℓ, η_ℓ, ζ_ℓ take the values $\pm$ 1 depending on the node position relative to the coordinate axis as shown in Fig. 6.2, and 2a, 2b, and 2c = the element dimensions parallel to X-, Y- and Z- axes, respectively.

By choosing the displacement function, the strain components can be determined according to

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{Bmatrix} \qquad (6.18)$$

in which

ε, γ = normal and shear strains, respectively

and consequently the strain-displacement matrix is obtained from Equs. (6.17) and (6.18) as

$$[B] = [B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \quad B_6 \quad B_7 \quad B_8] \tag{6.19}$$

in which

B_h = a six by three matrix. The explicit forms of B_1 to B_8 are given in Appendix E (h=1 to 8).

The general Hook's law for orthotropic material is written in matrix form as [63]

$$\{\varepsilon\} = [C] \{\sigma\}$$

in which

$[C]$ = a square matrix of order six which includes the elastic properties of the finite element material

$\{\sigma\}$ = stress components corresponding to the strain $\{\varepsilon\}$.

The elasticity matrix [E] in Equ. (6.13), thus, is given by

$$[E] = [C]^{-1}$$

$$= \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix} \tag{6.20}$$

in which

$$[E_{11}] = \frac{1}{e}\begin{bmatrix} \frac{1}{E_yE_z} - \frac{\nu_{zy}}{E_z}\frac{\nu_{yz}}{E_y} & \frac{1}{E_z}\frac{\nu_{yx}}{E_y} + \frac{\nu_{zx}}{E_z}\frac{\nu_{yz}}{E_y} & \frac{1}{E_y}\frac{\nu_{zx}}{E_z} + \frac{\nu_{yx}}{E_y}\frac{\nu_{zy}}{E_z} \\ \frac{1}{E_z}\frac{\nu_{xy}}{E_x} + \frac{\nu_{xz}}{E_x}\frac{\nu_{zy}}{E_z} & \frac{1}{E_xE_z} - \frac{\nu_{zx}}{E_z}\frac{\nu_{xz}}{E_x} & \frac{1}{E_x}\frac{\nu_{zy}}{E_z} + \frac{\nu_{zx}}{E_z}\frac{\nu_{xy}}{E_x} \\ \frac{1}{E_y}\frac{\nu_{xz}}{E_x} + \frac{\nu_{xy}}{E_x}\frac{\nu_{yz}}{E_y} & \frac{1}{E_x}\frac{\nu_{yz}}{E_y} + \frac{\nu_{yx}}{E_y}\frac{\nu_{yz}}{E_x} & \frac{1}{E_xE_y} - \frac{\nu_{yx}}{E_y}\frac{\nu_{xy}}{E_x} \end{bmatrix}$$

$$[E_{22}] = \begin{bmatrix} G_{xy} & 0 & 0 \\ 0 & G_{yz} & 0 \\ 0 & 0 & G_{zx} \end{bmatrix}$$

where

$$e = \frac{1}{E_xE_yE_z} - \frac{1}{E_x}\frac{\nu_{zy}}{E_z}\frac{\nu_{yz}}{E_y} - \frac{1}{E_z}\frac{\nu_{xy}}{E_x}\frac{\nu_{yx}}{E_y}$$

$$- \frac{\nu_{xy}}{E_x}\frac{\nu_{zx}}{E_z}\frac{\nu_{yz}}{E_y} - \frac{\nu_{xz}}{E_x}\frac{\nu_{yx}}{E_y}\frac{\nu_{zy}}{E_z}$$

$$- \frac{1}{E_y}\frac{\nu_{xz}}{E_x}\frac{\nu_{zx}}{E_z}$$

E_x, E_y, E_z = elastic modulii along X-, Y- and Z- axes, respectively

ν_{ab} = Poisson's ratio, which characterizes the decrease in b- direction during tension applied in a- direction; where a and b can take x, y or z.

G_{ab} = the shear modulus which characterizes the change in the angle between a and b axes.

Substituting Equs. (6.19) and (6.20) in Equ. (6.16) and performing the triple matrix product $[B]^T [E] [B]$ resulted in a 24 x 24 matrix populated by expression which are function of ξ, η, and ζ. Thus, integration of these expressions will lead to subsequent evaluation of the K_{cd} terms of the element stiffness matrix $[K]_i$, and after proper simplification it is found that

$$[K]_i = \begin{bmatrix} K_{11} & & \text{Symmetric} & \\ K_{21} & K_{22} & & \\ K_{31} & K_{32} & K_{33} & \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \qquad (6.21)$$

in which

K_{cd} = a square matrix of order six. The explicit forms of the matrices K_{cd} are given in Appendix E (c, d = 1 to 4).

By obtaining the element stiffness matrix, the structural stiffness matrix can be generated by simple summation of the individual element stiffness coefficients for common degrees of freedom at any node. This approach is often referred to as the direct stiffness method, and by which a set of simultaneous linear algebraic equations will be obtained, relating loads to displacement through the structural matrix as presented in Equ. (6.15). The temperature effect in this analysis is included in the load vector $\{r\}_i$. The explicit form of $\{r\}_i$ is given in Appendix E and is calculated from [18]

$$\{r\}_i = \int_{vol.} [B]_i^T [E] \{\varepsilon_o\} dv \tag{6.22}$$

in which

$\{\varepsilon_o\}$ = initial strain vector

= $\{\alpha T \;\; \alpha T \;\; \alpha T \;\; o \;\; o \;\; o\}^T$

where T is the average change of temperature at the element nodes, i.e.

$$T = \frac{1}{8} \sum_{\ell=1}^{8} T_\ell$$

(ℓ denoting a node number)

It should be noted that, since the element stiffness matrix is derived with respect to the selected natural coordinates which are the same as the global coordinates, no transformation is needed. The next step is to account for the appropriate boundary conditions in the structural stiffness matrix. In the present problem these are the symmetry conditions at the central planes x = 0, y = 0, and z = 0 of the sandwich plate.

in Fig. 6.1. The modified equation (6.15) can now be solved for the unknown displacements. Among the widely used solution methods, the Gaussian elimination is applied in this work [18]. Its main features being a forward elimination of the unknowns to obtain a triangular coefficient matrix, after which the unknowns are determined by back-substitution starting with the last modified equation. When the nodal displacements have been determined, the element stresses follow from Hook's law :

$$\{\sigma\} = [E]\,(\{\varepsilon\} - \{\varepsilon_0\})$$

$$= [E]\,([B]\,\{d\} - \{\varepsilon_0\}) \tag{6.23}$$

6.3.3 COMPUTER PROGRAM

Based on the method of analysis previously discussed, a computer program has been developed in Fortran language to evaluate the thermal stresses in sandwich plates subjected to a temperature change. The program is a direct application of the assumed displacement finite element method.

6.4 PRACTICAL FORMULAS FOR THERMAL STRESSES IN SANDWICH PANELS

The analytic solutions developed for thermal stresses in sandwich panels with different boundary conditions [8, 11, 12, 23, 48, 53, 69, 70] are characterized by complex mathematical expressions which could limit their practical application. Consequently, to facilitate the use of these solutions, simple expressions are derived for the maximum deflection and stresses in sandwich plates subjected to temperature changes.

6.4.1 SIMPLY SUPPORTED SANDWICH PLATES SUBJECTED TO A THERMAL GRADIENT $\pm$ T

The analytic solution in this case was obtained by Bijlaard [8] and Jerzy [53]. As stated in their works the maximum thermal moments occur at the boundary of the sandwich plate and are given by

$$M = (M_x)_{max.} = (M_y)_{max.}$$

$$= \frac{\alpha\ T\ D\ (1-\nu^2)}{(t_c + t_f)} \qquad (6.24)$$

in which

α = thermal expansion coefficient of the faces material

T = temperature at the plate surfaces

ν = Poisson's ration of faces material

D = flexural rigidity of the sandwich plate

$$= \frac{E_f\ t_f\ (t_c + t_f)^2}{2\ (1-\nu^2)}$$

Substituting the expression for D into Equ. (6.24) gives

$$M = \frac{1}{2}\alpha\ T\ E_f\ t_f\ (t_c + t_f)$$

from which the maximum normal stress in the faces is determined from

$$(\sigma_f)_{max.} = \frac{M}{t_f\ (t_c + t_f)}$$

as

$$(\sigma_f)_{max.} = \frac{1}{2}\alpha\ T\ E_f \qquad (6.25)$$

The maximum deflection in this case was determined as

$$(w)_{max.} = \frac{4\, \alpha\, T\, a^2\, (1+\nu)}{\pi^3\, (t_c + t_f)}\, K_w \tag{6.26}$$

in which

$$K_w = \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^3} (1 - \operatorname{sech} \alpha_m)$$

where

$$\alpha_m = \frac{m\, \pi\, b}{2\, a}$$

a = the plate length

b = the plate width

Numerical values for K_w are obtained by the writer for a range of (b/a) = .5 to 4. The results are presented in Fig. 6.3.

The maximum shear stress in this case occurs at the plate corners and has an infinite value. Bijlaard [8] used an approximate method to determine the maximum transverse shear force in the plate. He made use of the twisting moment distributions along the edges in his calculations.

6.4.2 CLAMPED EDGES SANDWICH PLATES SUBJECTED TO A THERMAL GRADIENT ± T

The analytic solution in this case was found by Bijlaard [8] and Jerzy [53]. The maximum thermal moments were determined as

$$M = (M_x)_{max.} = (M_y)_{max.}$$

$$= \frac{\alpha\ T\ D\ (1+\nu)}{(t_c + t_f)} \tag{6.27}$$

and by following the same procedure adopted in the previous section, the maximum normal stress in the faces is obtained as

$$(\sigma_f)_{max.} = \frac{\alpha\ T\ E_f}{2\ (1-\nu)} \tag{6.28}$$

A clamped sandwich plate will remain flat when it is exposed to a thermal gradient, thus

$$w\ (x,\ y) = 0$$

The shear forces are also vanished in this case.

6.4.3 A SANDWICH PANEL WITH ISOTROPIC CORE AND SUBJECTED TO A UNIFORM TEMPERATURE CHANGE (Fig. 6.1)

Normal and interlayer shear stresses induced in sandwich plates subjected to uniform temperature change can be obtained from Equs. (6.7), (6.8) and (4.8) as

$$\sigma_{fx} = (\sigma_{fx})_{\substack{x=0\\y=0}}$$

$$= -\mu_x \frac{G_{cx}}{t_f} \sum_{m=1,3,\ldots}^{\infty} C_m \sinh \frac{1}{2} \mu_x\ \alpha_m\ t_c \tag{6.29}$$

$$\sigma_{fy} = (\sigma_{fy})_{\substack{x=0\\y=0}}$$

$$= -\mu_y \frac{G_{cy}}{t_f} \sum_{n=1,3,\ldots}^{\infty} D_n \sinh \frac{1}{2}\mu_y \beta_n t_c \qquad (6.30)$$

$$q_x = (\tau_{xz})_{\substack{x=a\\y=0\\z=t_c/2}}$$

$$= G_{cx} \sum_{m=1,3,\ldots}^{\infty} (-1)^{(m-1)/2} \mu_x \alpha_m C_m \sinh \frac{1}{2}\mu_x \alpha_m t_c \qquad (6.31)$$

$$q_y = (\tau_{yz})_{\substack{x=0\\y=b\\z=t_c/2}}$$

$$= G_{cy} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n-1)/2} \mu_y \beta_n D_n \sinh \frac{1}{2}\mu_y \beta_n t_c \qquad (6.32)$$

in which

σ_{fx}, σ_{fy} = normal stresses in facings at the plate center

q_x, q_y = transverse shear stresses at the plate edges.

In the particular case of isotropic core material :

$$E_c = E_{cx} = E_{cy}$$

$$G_c = G_{cx} = G_{cy}$$

$$\mu = \mu_x = \mu_y = \left(\frac{E_c}{G_c}\right)^{1/2}$$

$$\Psi = \Psi' = \frac{\mu\, G_c}{E_f\, t_f}$$

and consequently Equs. (6.9) becomes

$$C_m = \frac{2\,(-1)^{(m-1)/2}}{a\,\alpha_m^2}\;\frac{(\nu\,\Psi\,\xi' - \Delta)}{\frac{\Psi}{\alpha_m}\sinh\frac{1}{2}\,\mu\,\alpha_m\,t_c + \cosh\frac{1}{2}\,\mu\,\alpha_m\,t_c}$$

$$D_n = \frac{2\,(-1)^{(n-1)/2}}{b\,\beta_n^2}\;\frac{(\nu\,\Psi\,\xi - \Delta)}{\frac{\Psi}{\beta_n}\sinh\frac{1}{2}\,\mu\,\beta_n\,t_c + \cosh\frac{1}{2}\,\mu\,\beta_n\,t_c}$$

With these expressions, Equs. (6.29) to (6.32) can be written in abbreviated form as

$$\sigma_f = \sigma_{fx} = \sigma_{fy}$$

$$= -\frac{4}{\pi^2}\,\Delta\,\frac{t_c}{t_f}\,(E_c\,G_c)^{1/2}\,K_f \qquad (6.33)$$

$$\tau = \tau_{yz}$$

$$= -\frac{5.2}{\pi}\,\Delta\,(E_c\,G_c)^{1/2}\,K_{c1} \qquad (6.34)$$

in which

σ_f = maximum normal stresses in facings

τ = maximum interlayer shear stresses

$$K_f = \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^2} \frac{K_2}{\frac{2}{m\pi} \cdot \frac{K_2}{K_1} + \coth\mu \frac{m\pi}{2K_2}} \frac{\theta_f' + 1}{\theta_f' \theta_f - 1}$$

$$K_{c1} = \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \frac{1}{\frac{2}{n\pi R} \frac{K_2}{K_1} + \coth\mu \frac{n\pi R}{2K_2}}$$

where

$$K_1 = \frac{2 E_f t_f}{t_c (E_c G_c)^{1/2}}$$

$$K_2 = \frac{2a}{t_c}$$

$$R = \frac{a}{b}$$

$$\theta_f' = \frac{8\nu}{\pi^2} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \frac{K_2}{R K_1} \frac{1}{\frac{2}{n\pi R} \frac{K_2}{K_1} + \coth\mu \frac{n\pi R}{2K_2}}$$

$$\theta_f = \frac{8\nu}{\pi^2} \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^2} \frac{K_2}{K_1} \frac{1}{\frac{2}{m\pi} \frac{K_2}{K_1} + \coth\mu \frac{m\pi}{2K_2}}$$

Numerical values for the factors in Equs. (6.33) and (6.34) are calculated for the following range of K_2, K_1, μ and R.

K_2	changes from 10. to 100.
K_1	changes from 0.025 to 0.25
μ	takes the value 1. or 1.5
R	changes from 1. to 2.

The results are presented in Figs. 6.4 to 6.10.

6.5 MOISTURE EFFECT IN SANDWICH PANELS

To determine the maximum normal stresses in the faces of a sandwich panel due to a moisture content change an approximate analysis based on engineering mechanics is conducted, from which simple formulas are derived. Consider the sandwich panel in Figs. 6.1. A moisture change in its core induces S_1 and S_2 dimensional change percent at its upper and lower faces, respectively. To simplify the following analysis two particular cases are considered: (i) the symmetrical case in which the panel exposed to a uniform moisture change in the core, and (ii) the antisymmetric case as shown in Fig. 6.1. The general case can be obtained by combining these two cases.

6.5.1 SANDWICH PANEL SUBJECTED TO A UNIFORM DIMENSIONAL CHANGE

In this case, the moisture-induced strains at the facings center are determined in accordance to Hooke's law as

$$\varepsilon_{fx} = \frac{1}{E_f}(\sigma_{fx} - \nu\,\sigma_{fy})$$
$$\varepsilon_{fy} = \frac{1}{E_f}(\sigma_{fy} - \nu\,\sigma_{fx}) \tag{6.35}$$

It should be noted that the moisture change produces stresses in the sandwich panel in an analogous manner to that in which the temperature affects it. That is, when a sandwich plate is exposed to a temperature change at its surfaces, the compatibility condition of the displacements at the interfaces will restrict the free state deformations and hence induce thermal stresses. At the same time, because the free expansion of the faces is in some practical applications bigger than that of the core, it is understood that the core is restricting the faces. In the case of a moisture change in the core, the outer layers are the constraints.

Based on numerical values for the thermal stresses obtained in the preceding sections, it is found that normal stresses in facings at the plate center are equal in X- and Y- directions. Thus, Equ. (6.35) becomes

$$\varepsilon_f = \varepsilon_{fx} = \varepsilon_{fy}$$
$$= \frac{(1-\nu)}{E_f}\,\sigma_f \tag{6.36}$$

in which

$$\sigma_f = (\sigma_{fx}')_{\substack{x=0\\y=0}} = (\sigma_{fy})_{\substack{x=0\\y=0}}$$

ε_f = normal strain in facings at x=y=0.

In a similar manner

$$\varepsilon_c = \frac{\sigma_c}{E_c}$$

in which

ε_c = normal strain in the core at x=y=0.

The equilibrium equation of a sandwich element at the panel center is

$$2\,\sigma_f\,t_f + \sigma_c\,t_c = 0$$

from which

$$\sigma_c = -\frac{2\,\sigma_f\,t_f}{t_c}$$

and consequently

$$\varepsilon_c = -\frac{2\,\sigma_f\,t_f}{E_c\,t_c}$$

By proper substitution of ε_f and ε_c in the compatibility equation of strains at the interfaces between the core and the faces, which is

$$\varepsilon_f - \varepsilon_c = \Delta_m \tag{6.37}$$

it is found that

$$\sigma_f = \frac{E_f \ \Delta_m}{1 - \nu + \frac{2 \ E_f \ t_f}{E_c \ t_c}} \tag{6.38}$$

in which

Δ_m = the relative free strain at the interface between facings and core due to a uniform moisture change.

In the particular case of a sandwich strip, the normal stress σ_f at its mid span can be determined from Equ. (6.38) by neglecting the effect of Poisson's ration, i.e. ν.

6.5.2 SANDWICH PANEL SUBJECTED TO AN ANTISYMMETRIC DIMENSIONAL CHANGE ACROSS ITS THICKNESS

In this case, the normal strain in the core at x=y=0 can be obtained from,

$$\varepsilon_c = \frac{z}{\rho}$$

in which

ρ = the radius of curvature of the core at x=y=0.

z = vertical coordinate in z- direction (Fig. 6.1).

The equilibrium equation of moments on a sandwich element in this case is

$$M_f + M_c = 0$$

in which

M_f, M_c = the moment contributed by facings and core, respectively.

The moments M_f and M_c can be expressed as

$$M_f = \sigma_f \, t_f \, h$$

$$M_c = E_c \, I_c/\rho, \text{ where } I_c = t_c^3/12$$

Thus, from the equilibrium equation, it is found that

$$\frac{1}{\rho} = -12 \frac{\sigma_f \, t_f \, h}{E_c \, t_c^3}$$

in which

$$h = t_c + t_f$$

and consequently

$$(\varepsilon_f)_{z=t_c/2} = -6 \frac{\sigma_f t_f h}{E_c t_c^2}$$

Substituting this equation and Equ. (6.36) into the compatibility equation (6.37) gives

$$\sigma_f = \frac{E_f \Delta_m}{1 - \nu + \frac{6 E_f t_f h}{E_c t_c^2}} \qquad (6.39)$$

It should be noted that Equs. (6.38) and (6.39) can be used to determine the maximum normal stress in the faces of a sandwich plate subjected to a uniform temperature change and thermal gradient, respectively. In this case Δ should be used instead of Δ_m.

6.6 COMPARISON OF THE ANALYTIC AND FINITE ELEMENT RESULTS

Consider a sandwich plate made of two aluminum faces and a wood core, as shown in Fig. 6.11. The plate has the following properties

t_f = 0.025 in.

t_c = 1.74 in.

2a = 36. in.

2b = 24. in.

E_f = 10^7 psi.

$\nu = 0.33$

$E_c = 1.2 \times 10^6$ psi.

$G_c = .16 \times 10^6$ psi.

$\alpha_f = 13 \times 10^{-6}$ in./in. per °F

$\alpha_c = 3 \times 10^{-6}$ in./in. per °F

The plate is subjected to a uniform temperature change of 50°F.

The normal stress distributions in the facings along lines parallel to X and Y axes are obtained using the finite element method. Because of the symmetry in the plate geometry and temperature distribution, only one-eigth of the plate is considered. The finite element mesh for this portion of the plate is shown in Fig. 6.12. The normal stress, σ_f, is calculated from Equ. (6.23) and the results are presented in Fig. 6.13.

The normal stress distributions are also determined from the analytic solution in Equs. (6.6) and (6.7), (Fig. 6.13). The discrepancies between the results are due to the coarse size of the finite element mesh used. However, the accuracy improved when a finer mesh is used as will be shown later.

It should be noted that it is found from the finite element solution that the maximum inplane shear stress in the facings is 1.8% of the maximum normal stress. This result is in agreement with assumption (7) in section 6.2.1.

Another example solved is a sandwich strip having the following properties (Fig. 6.14 (a))

t_f = 0.025 in.

t_c = 1.74 in.

$2a$ = 36. in.

E_f = 10^7

ν = 0.33

E_c = 1.2×10^6 psi.

G_c = $.16 \times 10^6$ psi.

α_f = 13×10^{-6} in./in. per °F

α_c = 3×10^{-6} in./in. per °F

The strip is subjected to a uniform temperature change of 50°F. Because of the symmetry in the geometry and temperature distribution only one quarter is considered in the analysis as showin in Fig. 6.14 (b).

The normal and interlayer shear stress distributions along half the strip length are determined using the finite element method (Equ. (6.23)), and the analytic solution in Equs.(6.6) and (6.4). The results are presented in Fig. 6.15. It is seen that the results of the two are in good agreement.

As an indication of the correctness of the approximate formula (6.38), the maximum normal stress in the facings of the sandwich plate in Fig. 6.13 and in the facings of the strip in Fig. 6.14 (a) are determined. The results are shown in Figs. 6.13 and 6.15. It is seen that both the analytic and approximate solutions are in excellent agreement.

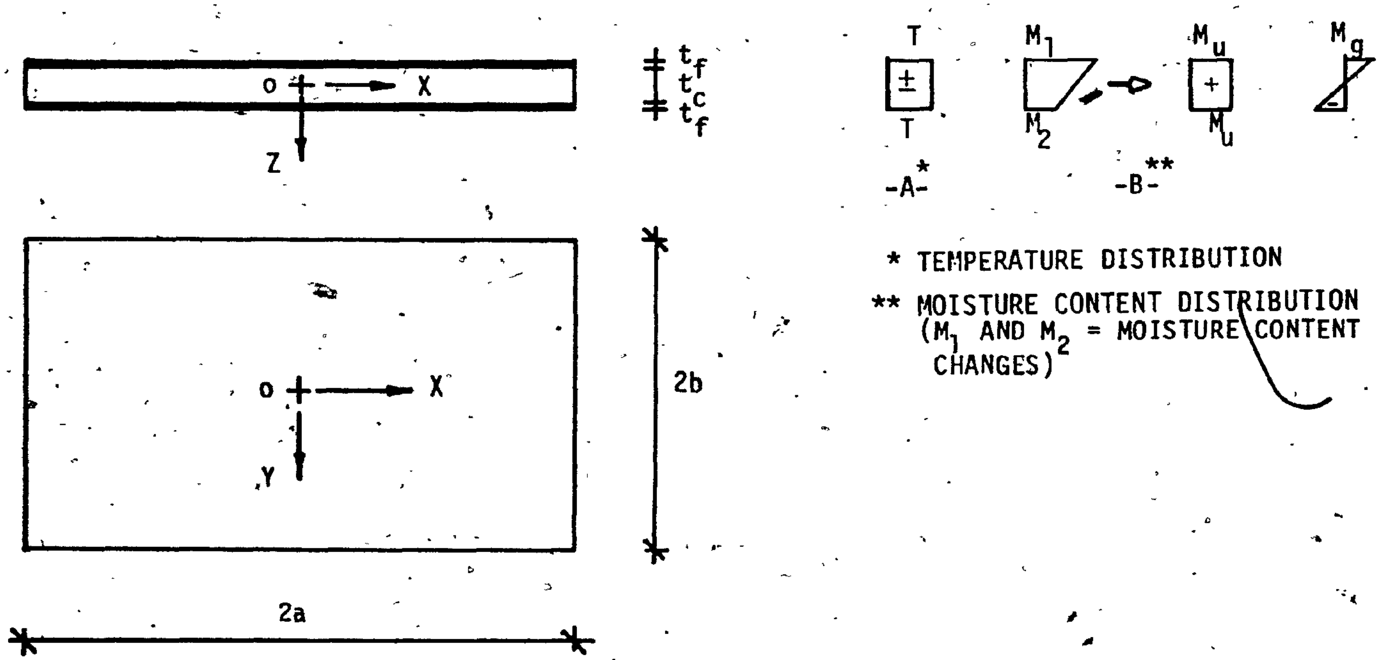


Fig. 6.1 - SANDWICH PANEL UNDER HYGROTHERMAL EFFECT

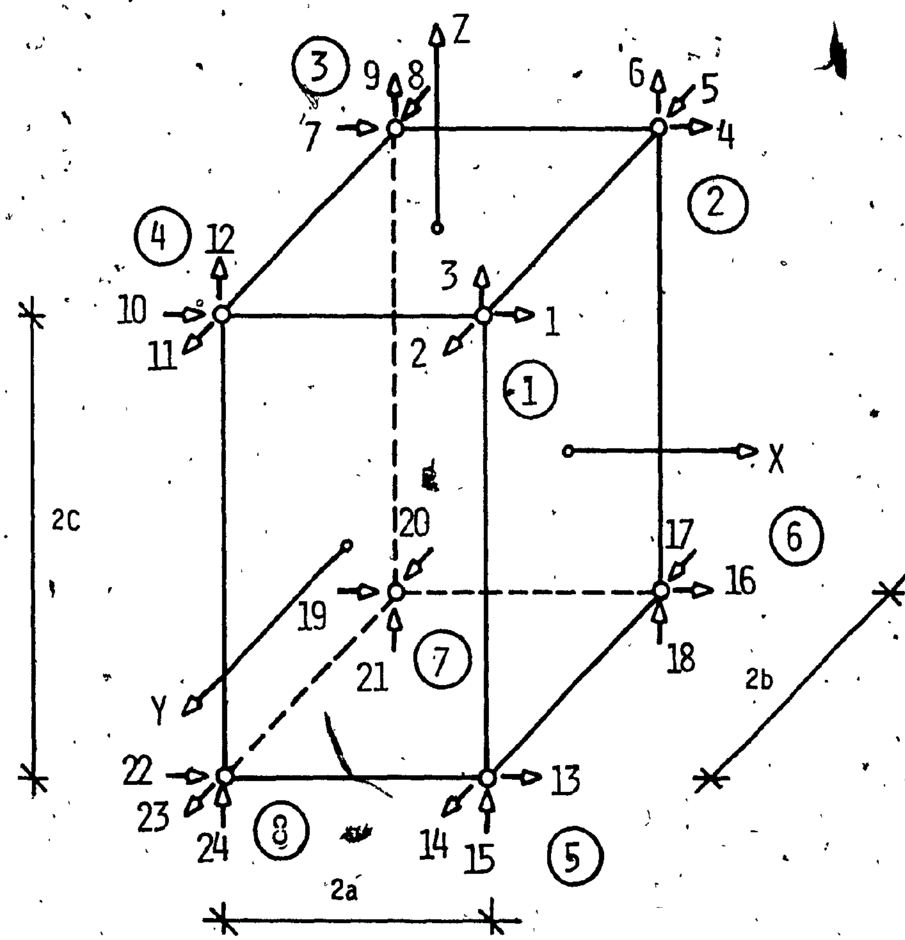


Fig. 6.2 - THE RECTANGULAR PRISM FINITE ELEMENT

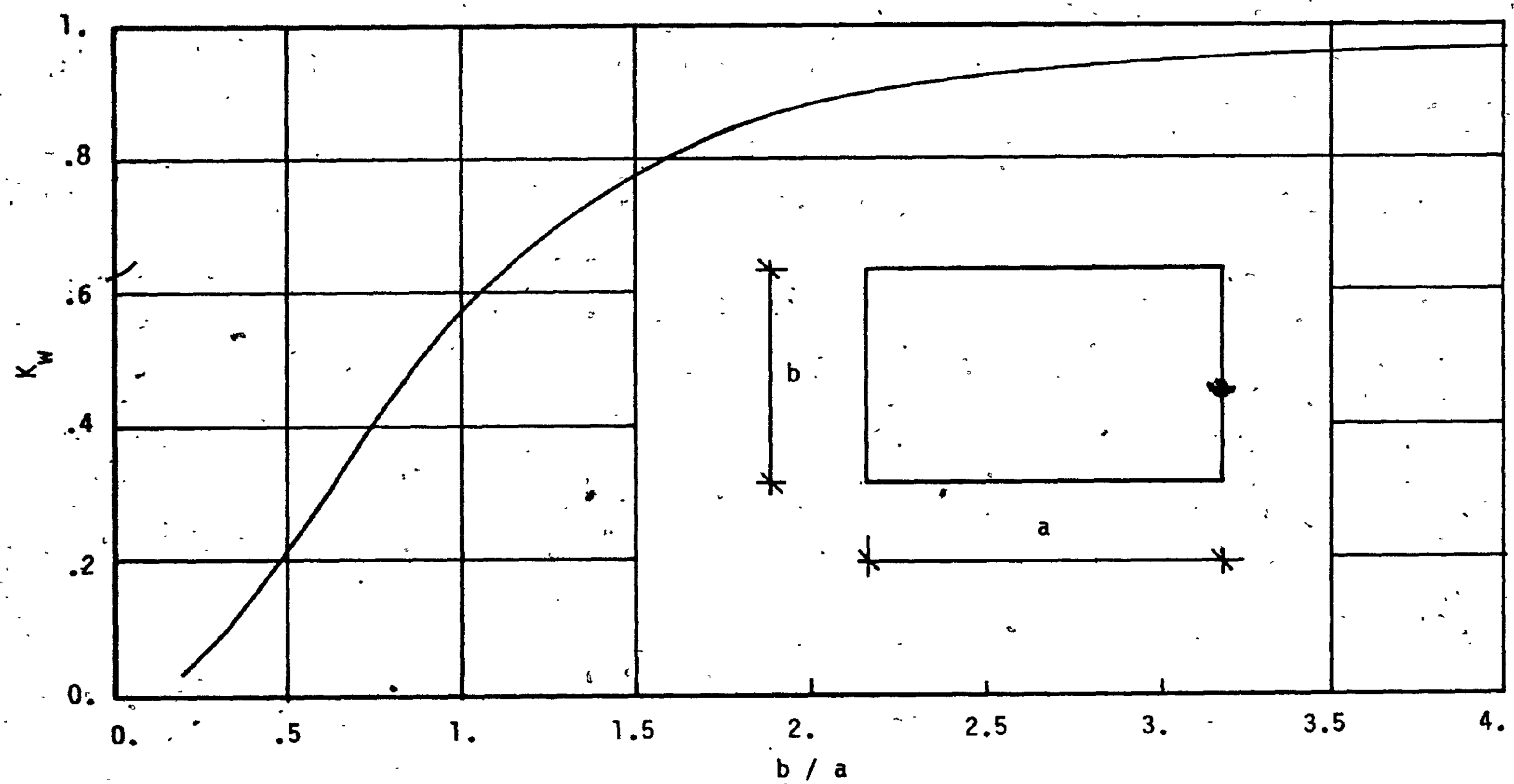


Fig.6.3 - VALUES FOR K_w^c IN EQU.(6.27)

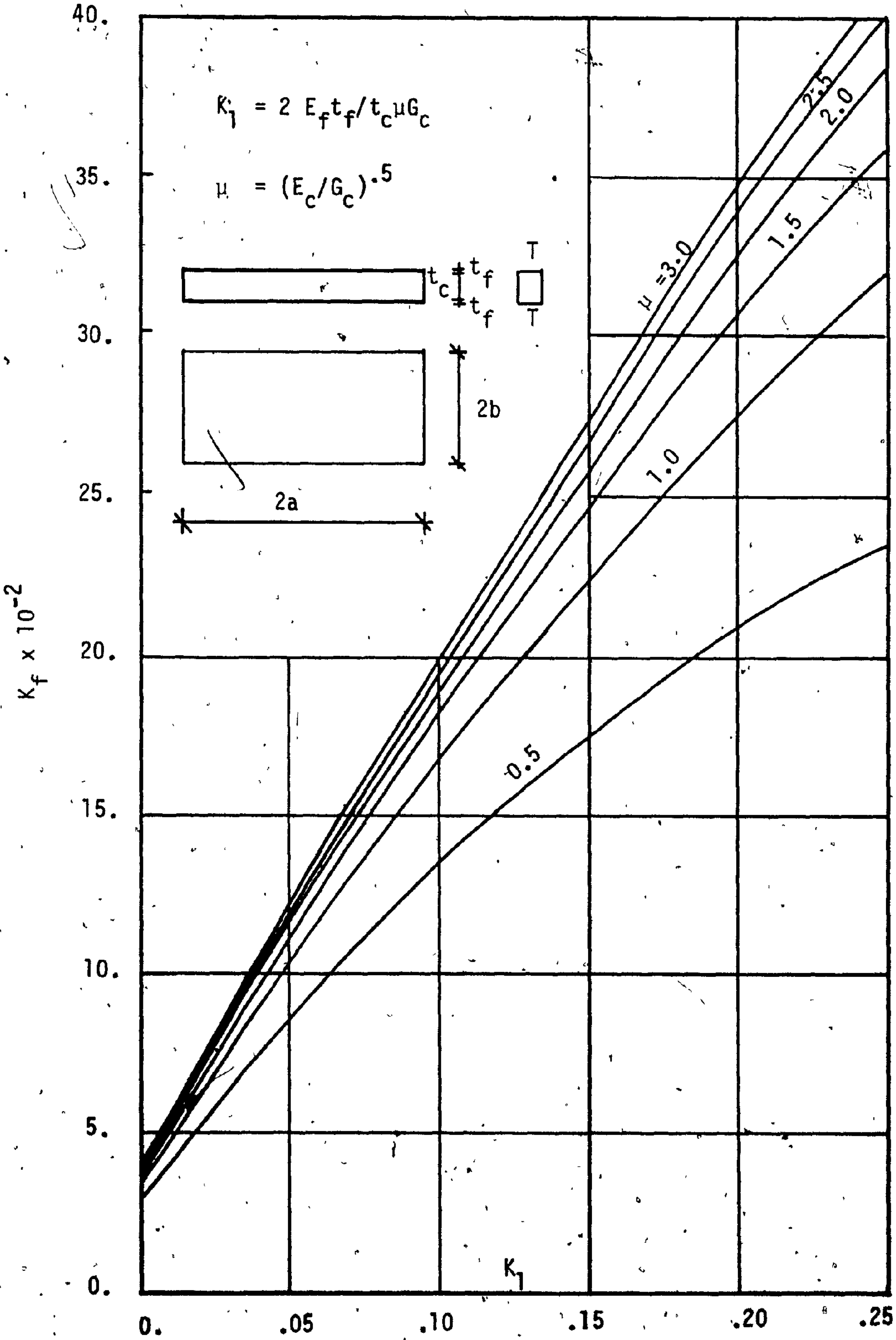


Fig. 6.4- NUMERICAL VALUES FOR FACTOR IN EQU.(6.33)

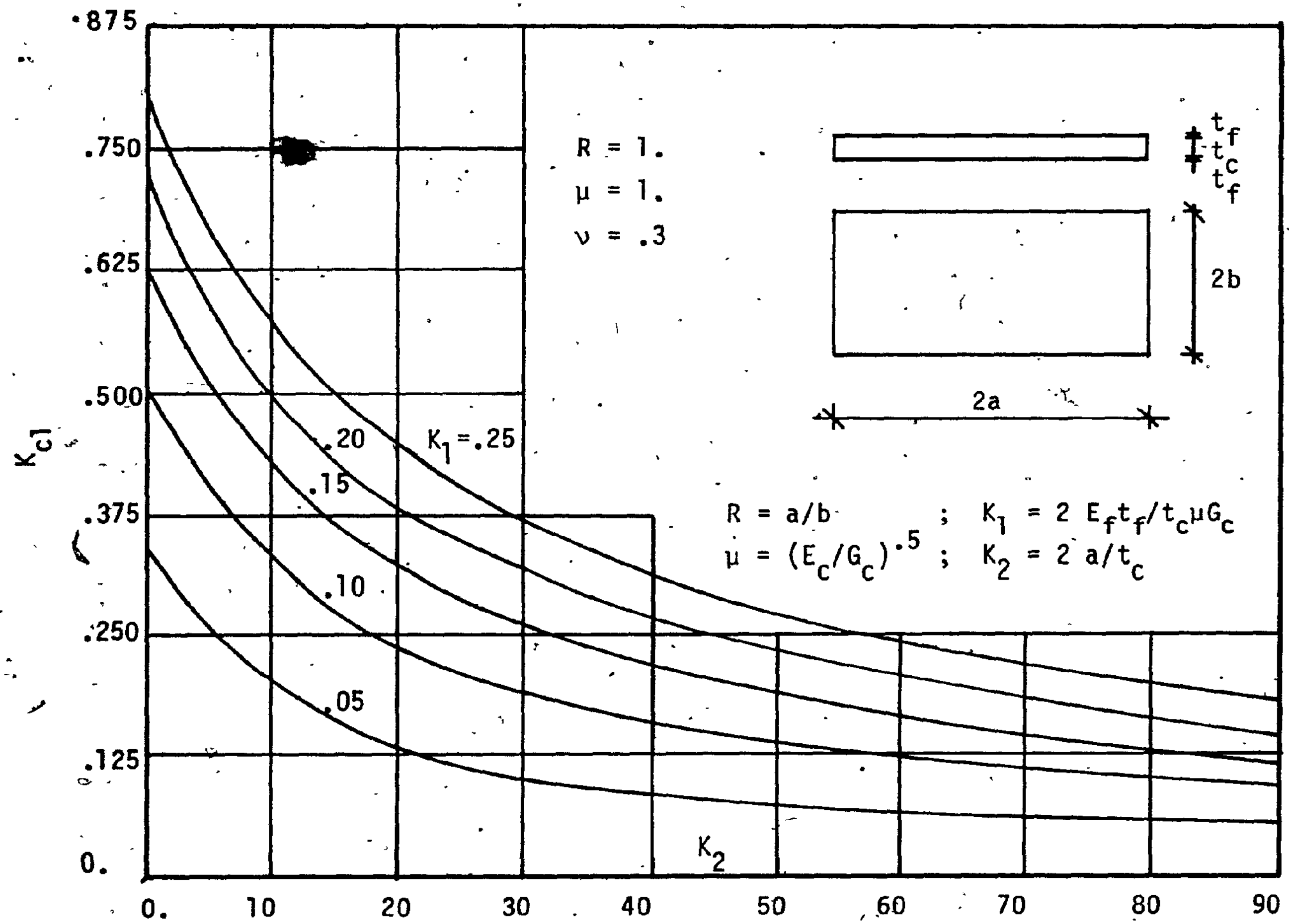


Fig. 6.5 - NUMERICAL VALUES FOR FACTOR IN EQU. (6.34)

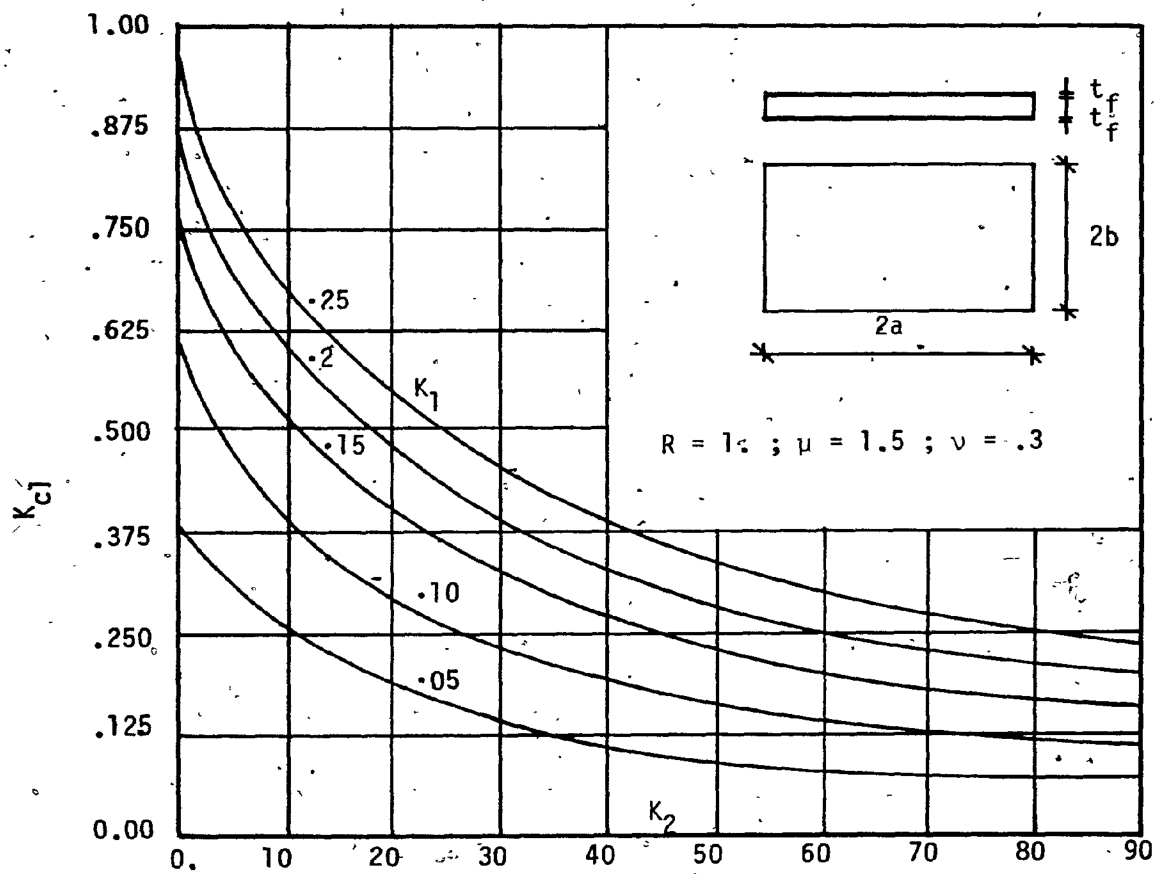


Fig. 6.6 - NUMERICAL VALUES FOR FACTOR IN EQU. (6.34)

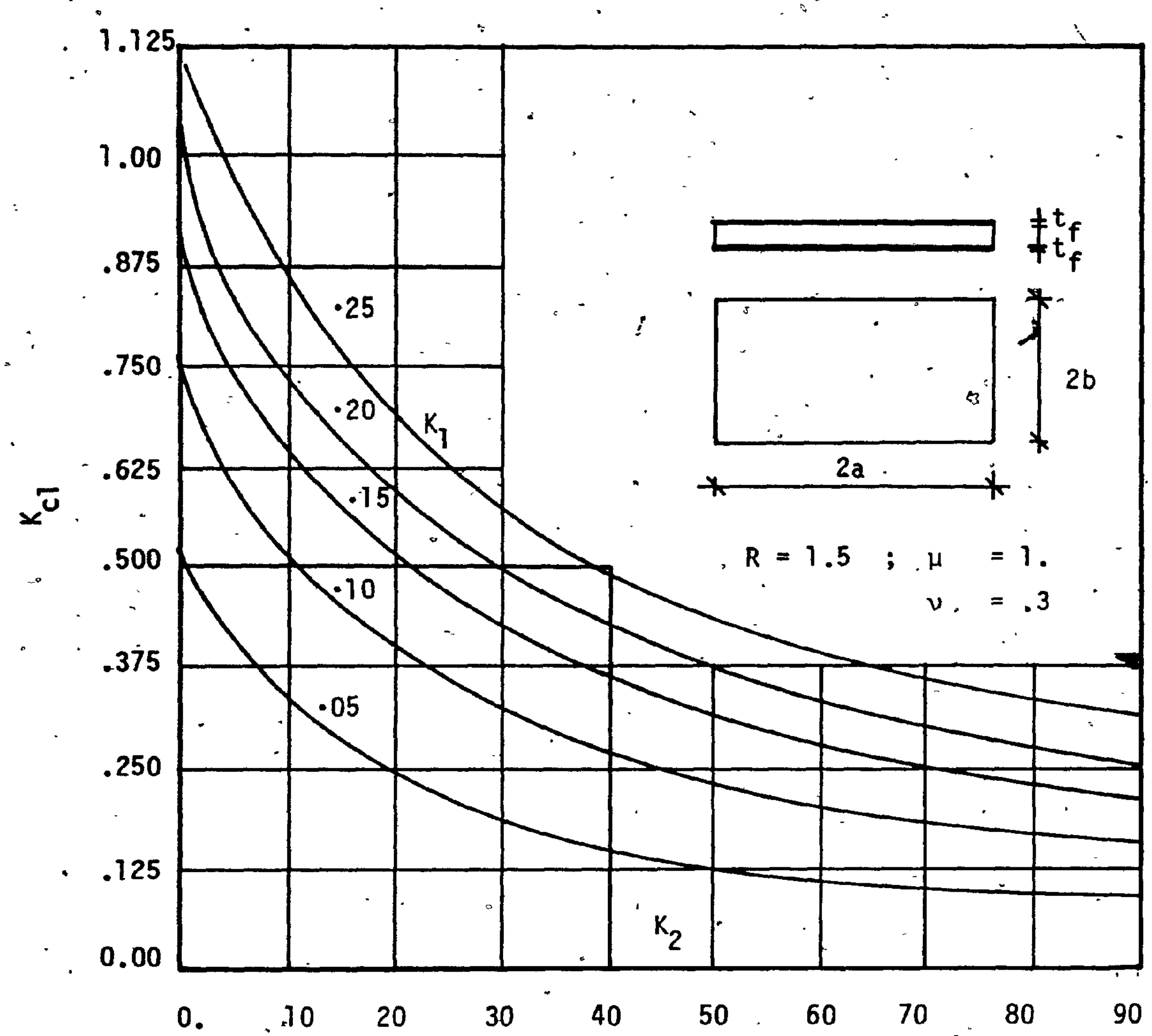


Fig. 6.7 - NUMERICAL VALUES FOR FACTOR IN EQU. (6.34)

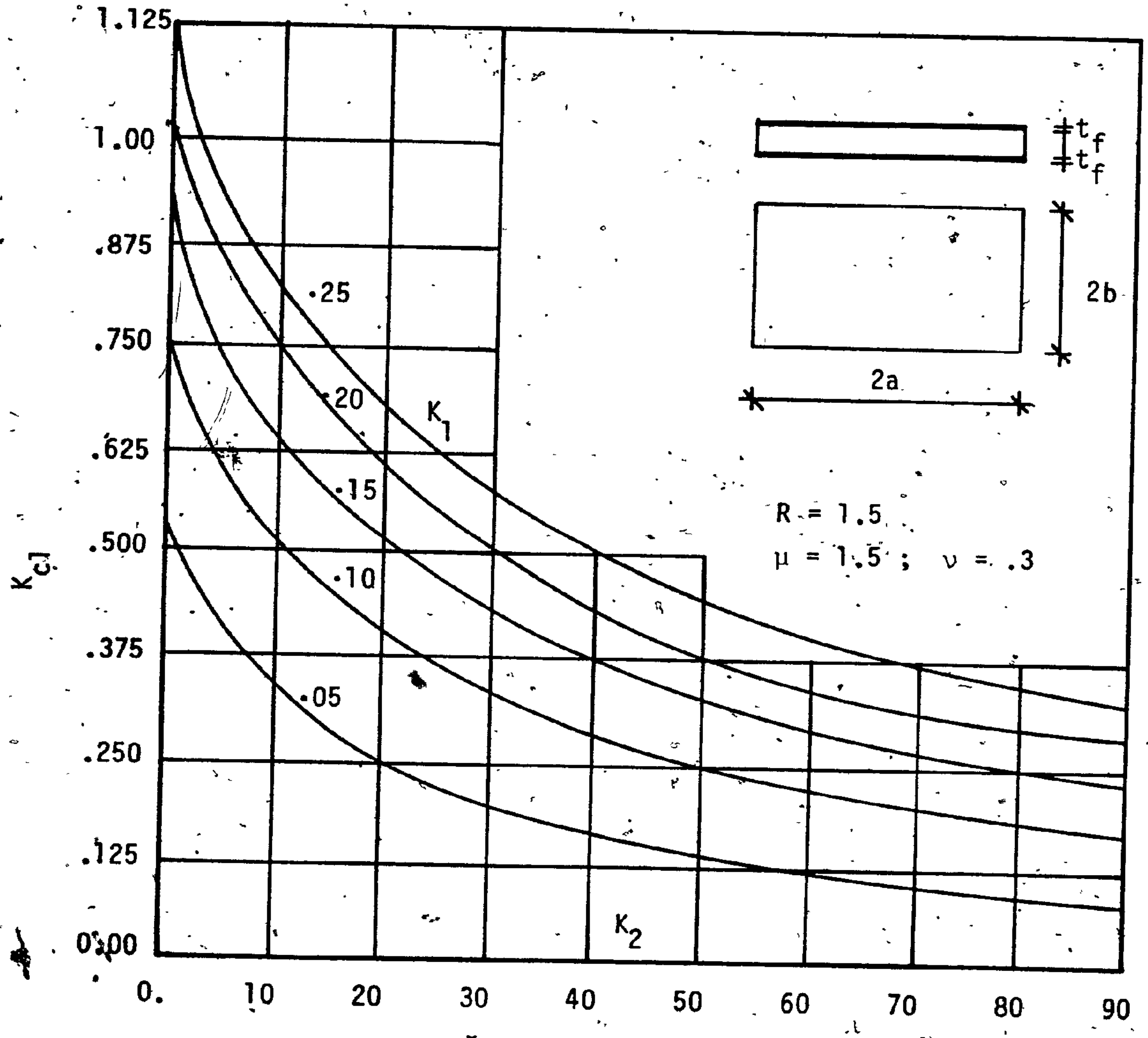


Fig. 6.8 - NUMERICAL VALUES FOR FACTOR IN EQU. (6.34)

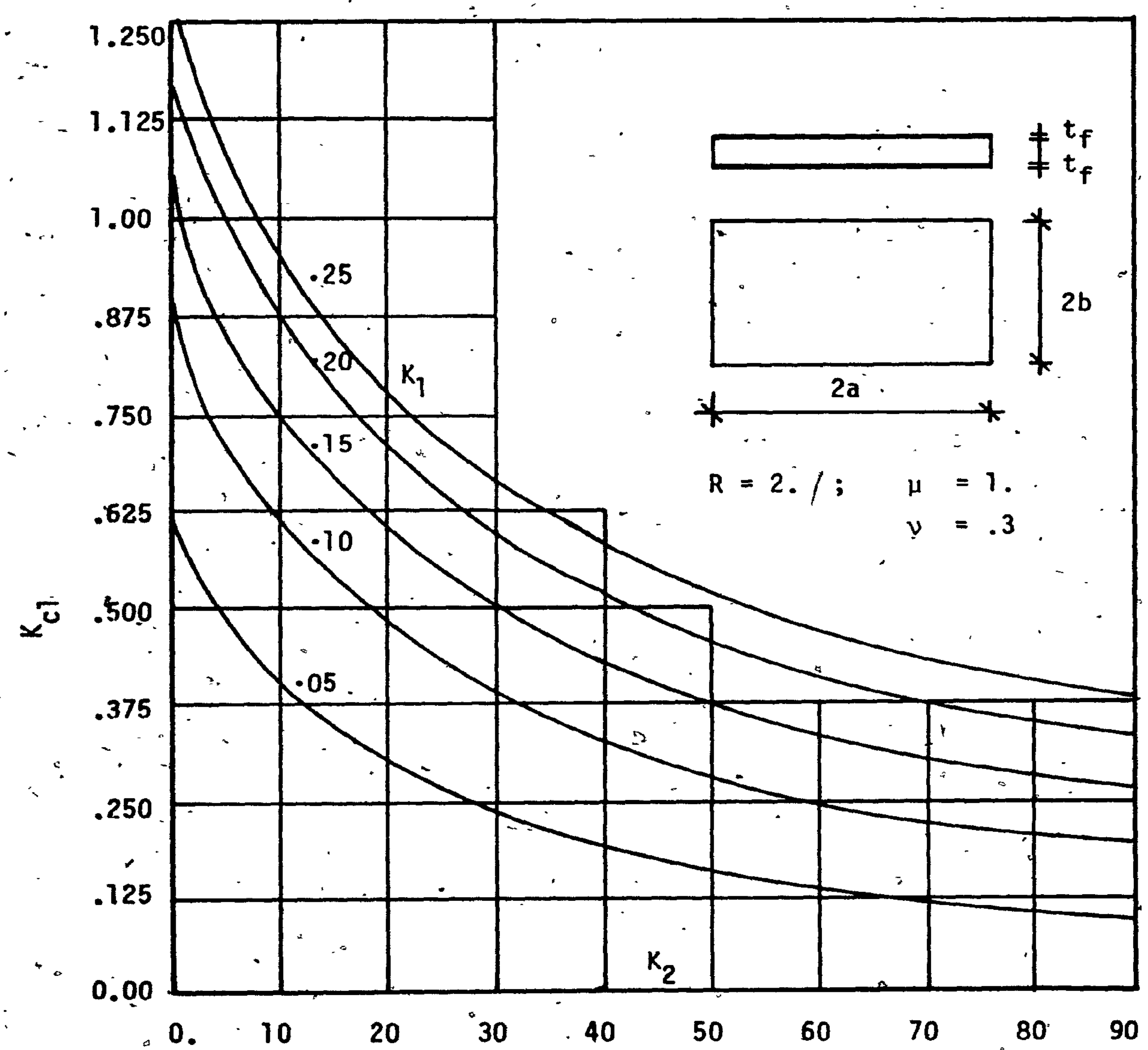


Fig. 6.9 - NUMERICAL VALUES FOR FACTOR IN EQU. (6.34)

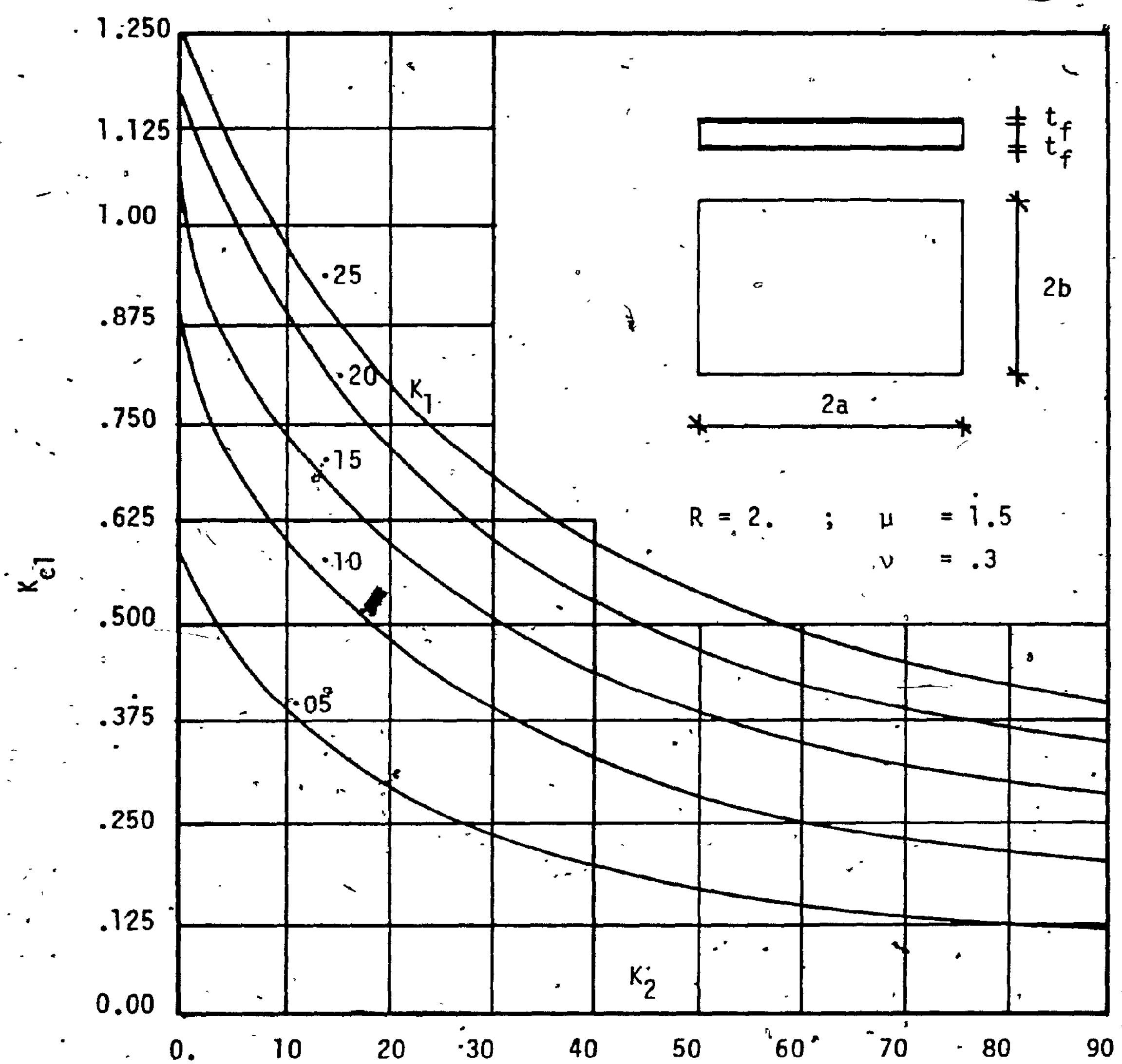


Fig. 6.10 - NUMERICAL VALUES FOR FACTOR IN EQU.(6.34)

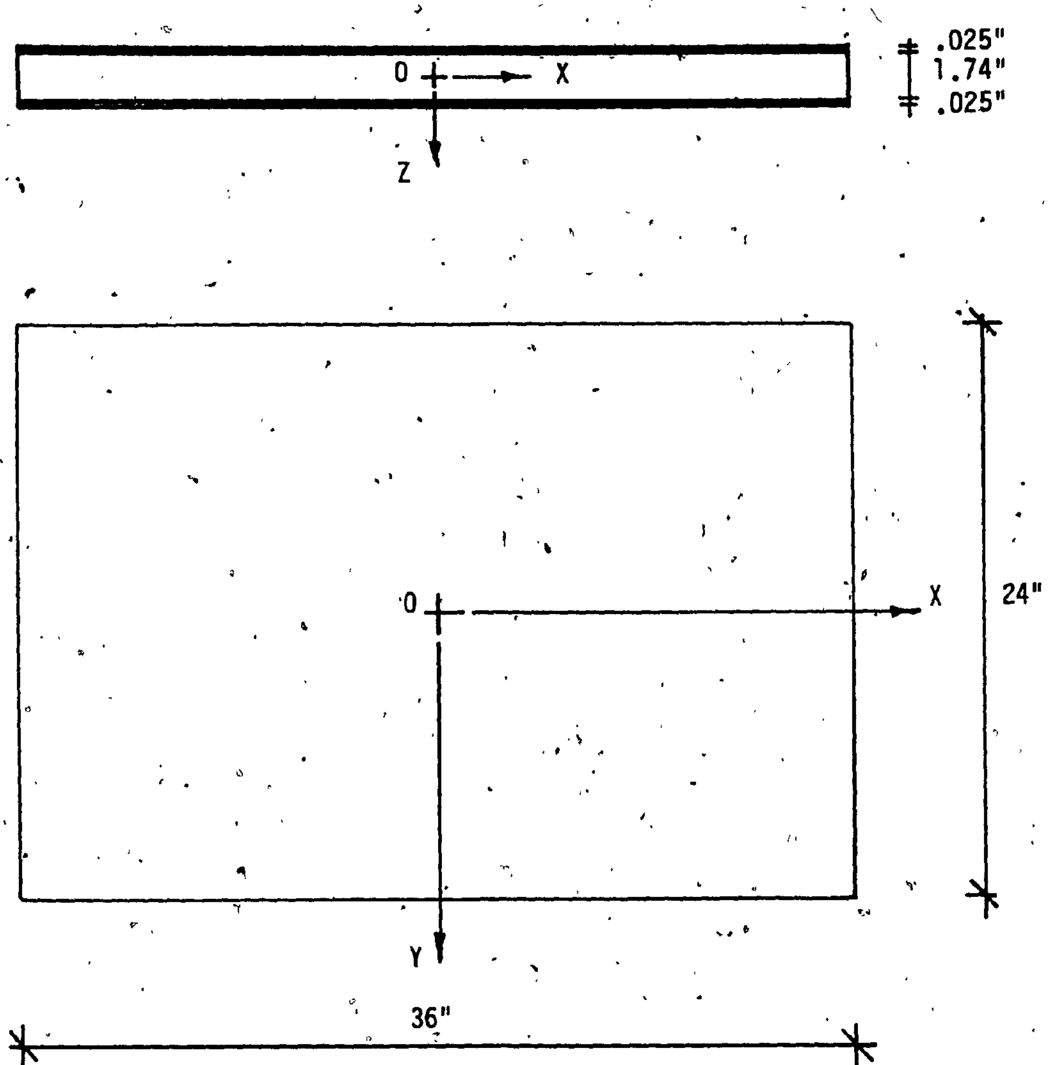


Fig. 6.11 - SANDWICH PLATE USED IN FINITE ELEMENT ANALYSIS OF THERMAL STRESSES

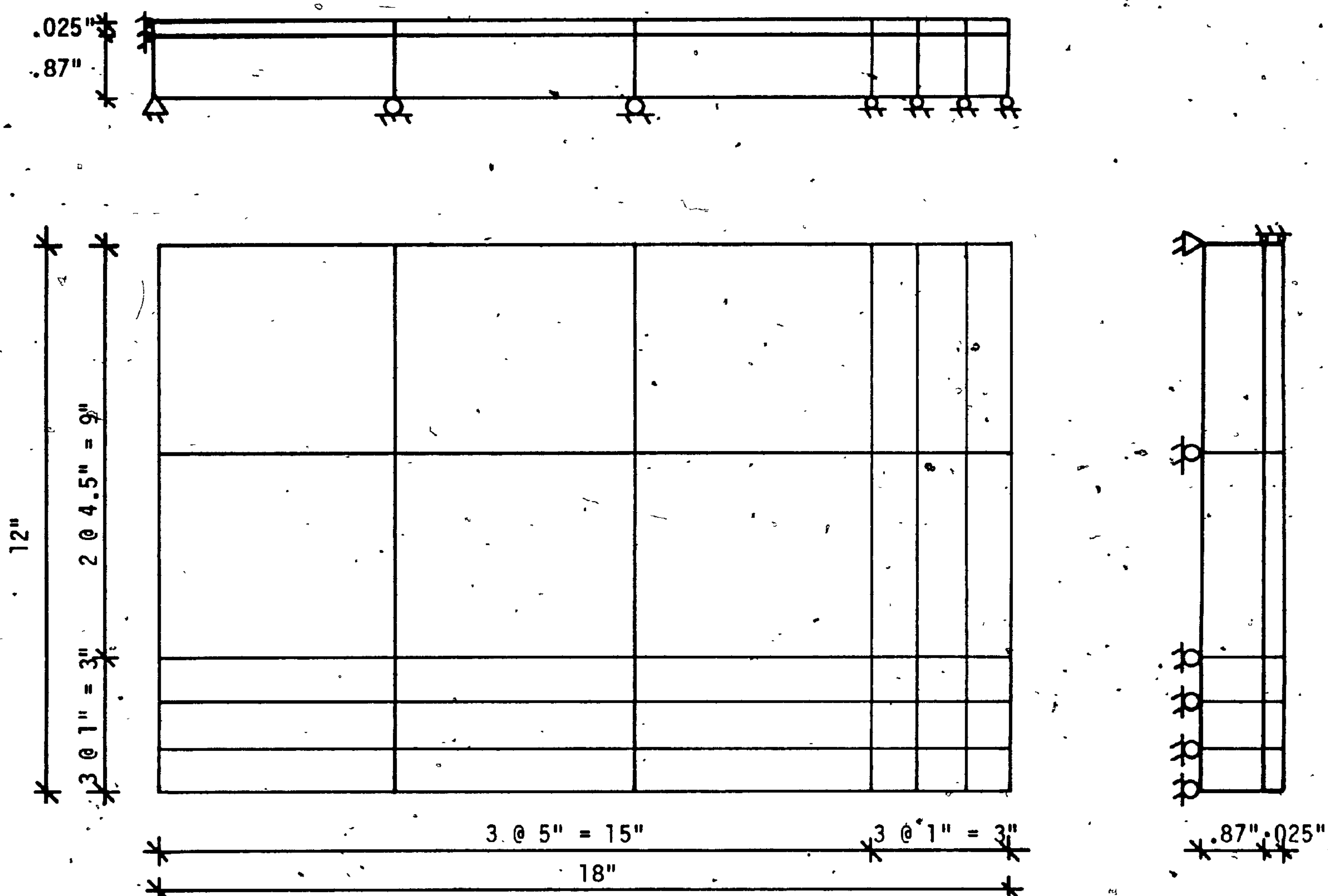
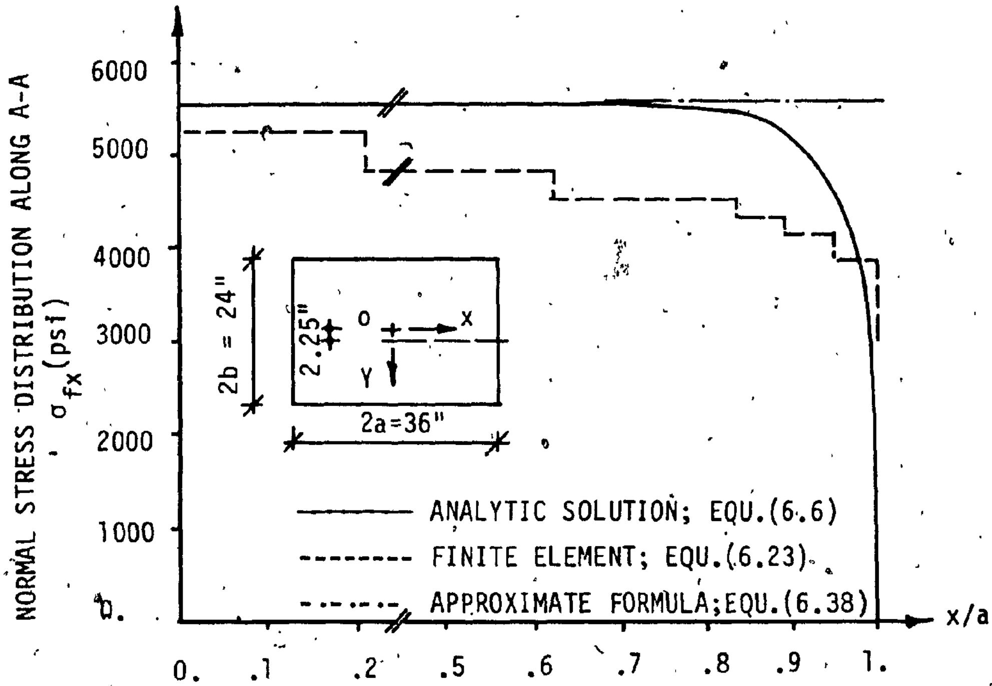


Fig. 6.12 - FINITE ELEMENT MESH FOR THE SANDWICH PLATE IN Fig. 6.11



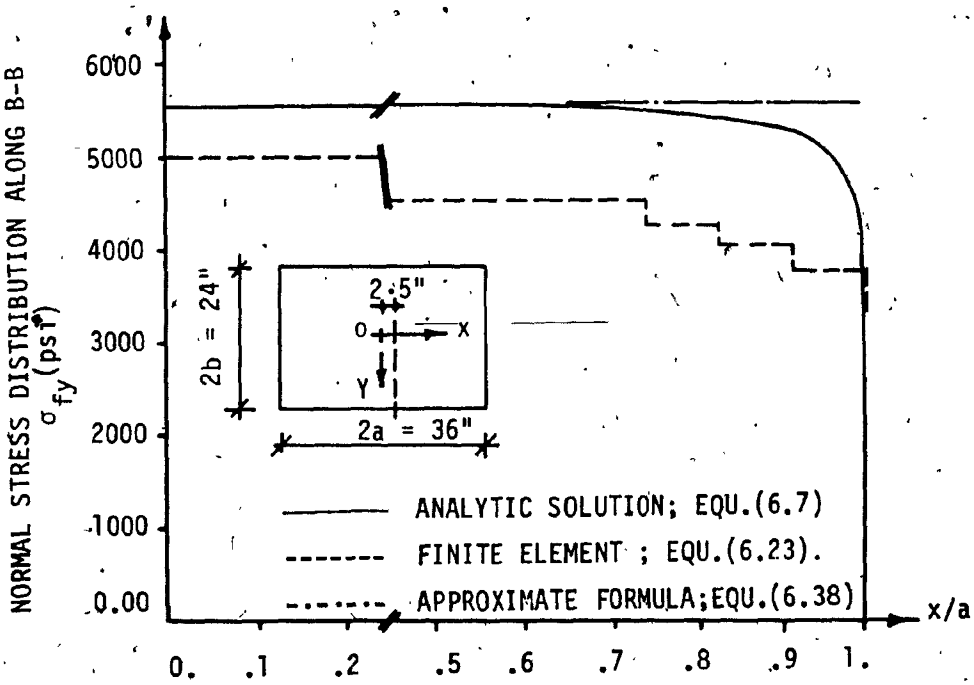
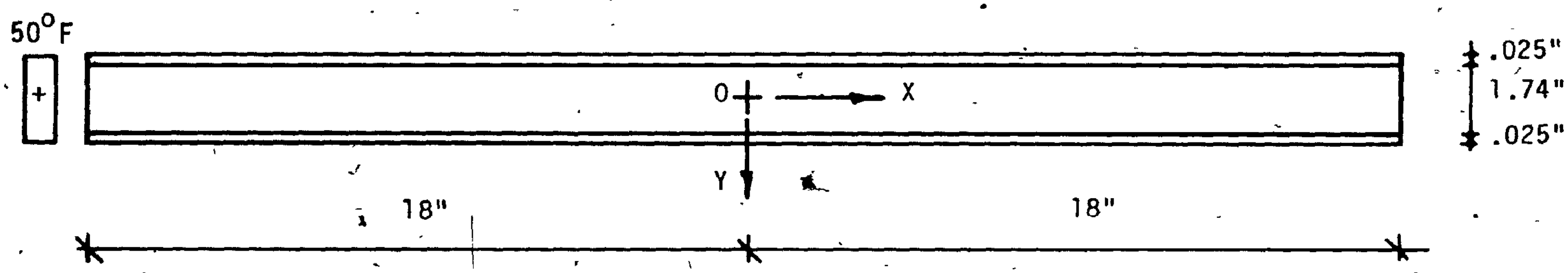
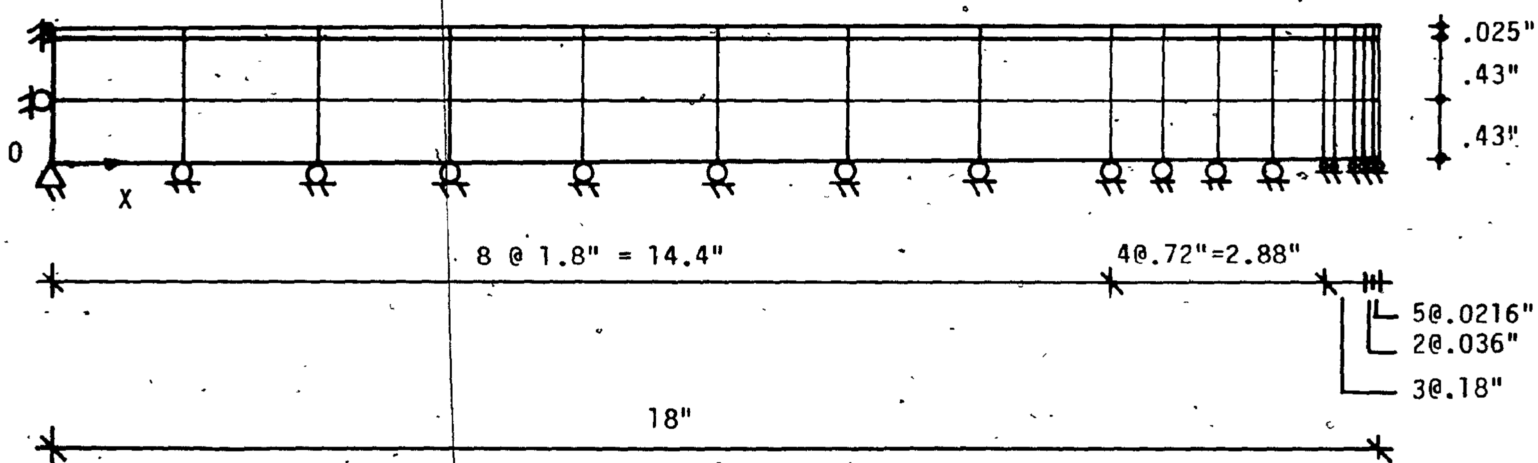


Fig. 6.13 - NORMAL SRESS DISRIBUTIONS IN FACINGS OF A SANDWICH PLATE UNDER UNIFORM TEMPERATURE CHANGE



(a) SANDWICH STRIP



(b) FINITE ELEMENT MESH FOR THE STRIP IN (a)

Fig. 6.14 - SANDWICH STRIP UNDER UNIFORM TEMERATURE CHANGE

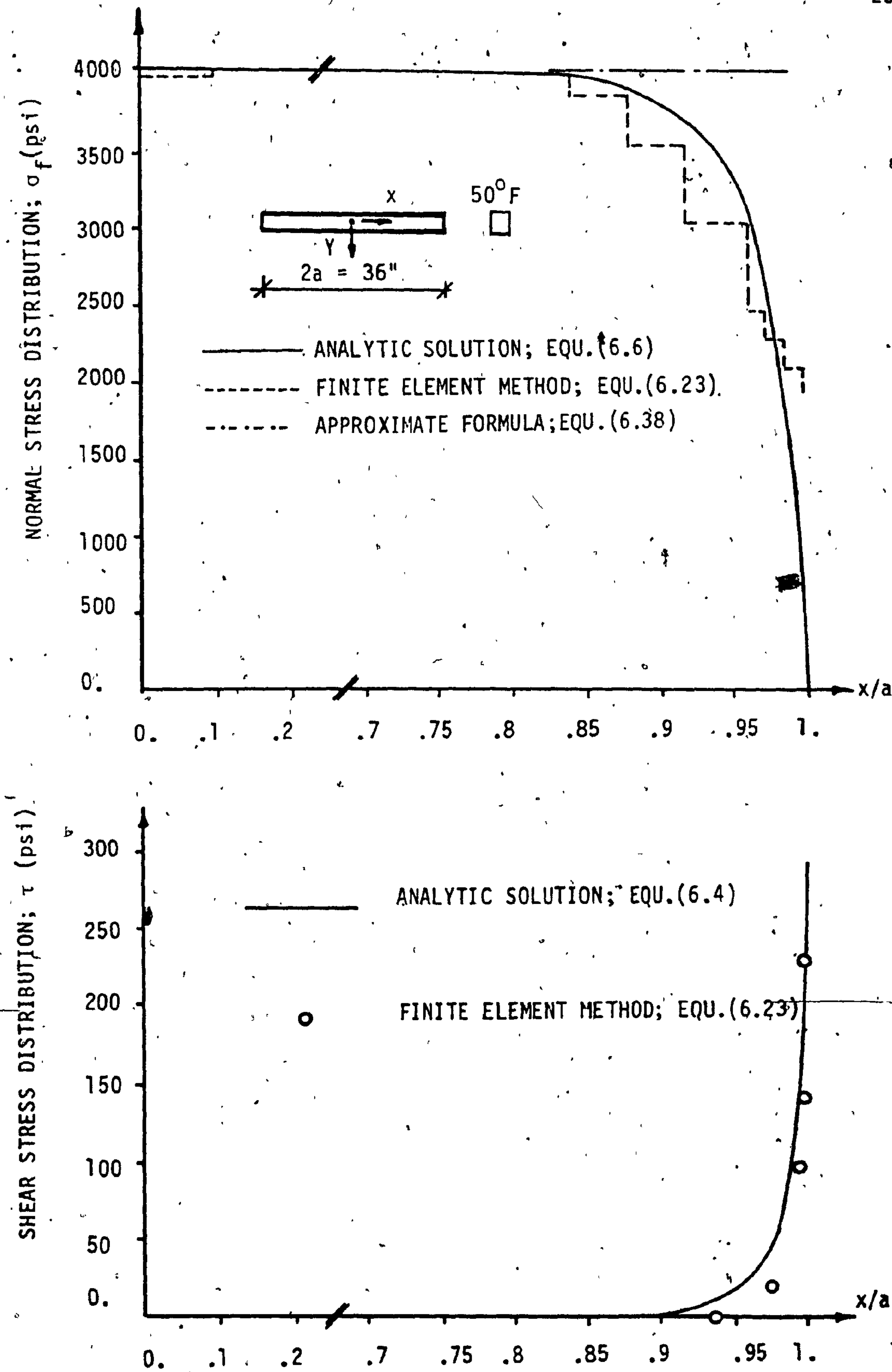


Fig. 6.15 - NORMAL AND SHEAR STRESS DISTRIBUTIONS IN SANDWICH STRIP UNDER UNIFORM TEMP.

CHAPTER VII

SUMMARY, CONCLUSIONS AND RECOMMENDATION

CHAPTER VII

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

7.1 SUMMARY AND CONCLUSIONS

Sandwich plates subjected to different loading types and boundary conditions have been investigated in the present work. The subjects covered include : laterally loaded, simply supported plates; simply supported two-span continuous plates; effects of interlayer elastic deformations on the response of sandwich plates and beam-columns; local failure of sandwich panels; and hygrothermal effects.

General solutions of Navier type for simply supported plates subjected to several loadings were developed. It is observed that the deflection of a such plate is composed of two components : one associated with bending deformations and the other with transverse shear deformations. This agrees with the conventional partial deflection theory. Simple formulas were also derived to determine the central deflection and relevant internal forces.

The preceding solutions for simply supported sandwich plates were further utilized in the analysis of continuous plates over one intermediate support. Solutions for single panels subjected to distributed edge moments were also developed. To facilitate the use of the solutions developed, simple formulas and tables were presented for a range of the ratio of right to left spans, loading types, and shear parameters. These results were verified by comparison with those obtained by the finite difference method and by the classical plate theory.

The effects of interlayer elastic deformations on the response of laterally loaded simply supported sandwich plates and beam-columns were investigated analytically and experimentally. It was observed that the deflection of sandwich plates and beams show greater sensitivity to variation of the adhesive stiffness when the latter is in the lower range, and beyond a certain level of stiffness, the bonding can be practically considered as rigid. In the case of beams, this level is lower than that in the case of buckling of sandwich struts due to axial compressive load. An increase in the adhesive stiffness is accompanied by a decrease in the normal stress of the core. The resulting loss in the resisting moment is compensated by a slight increase in the face normal stress. In all cases, the interlayer shear stress is practically independent of the adhesive stiffness. It is concluded that the ratio of core stiffness to adhesive stiffness is one of the main parameters influencing the behaviour of sandwich construction: a very stiff adhesive would be wasteful if the core is too soft, and the converse would be unwise. The theory was verified by tests conducted on two sandwich beams made of different adhesives and core materials, and subjected to mid-span concentrated loads. The maximum deflection and normal strains in the facings were measured, and the results are in reasonable agreement with the analytical values.

Local instability of sandwich panels is classified into dimpling, wrinkling, and crimpling. A numerical value for the dimpling coefficient was obtained by using the finite difference method, and found to be in good agreement with the semi-empirical one available in the literature. With respect to instability by wrinkling, three distinct analyses were conducted. First, formulas to define the transition state between symmetric and antisymmetric wrinkling were obtained by adopting an available theory in the literature. Second, an analytical solution for the symmetric wrinkling load was developed which yields better agreement with available data. And third, formulas for the failing stresses of the core material of a sandwich panel with initial imperfection were derived.

With respect to the hygrothermal effects on sandwich panels, two simple cases were considered : temperature effect and moisture content change. Existing solutions for sandwich plates subjected to thermal gradient across the thickness were reformulated and presented in simple forms to calculate the maximum deflection and stresses in the plates. For plates subjected to uniform temperature change, analytic solutions for the stresses in the facings and core were developed using an elasticity approach. The results are in good agreement with those obtained by finite element analysis. For the case of a sandwich panel subjected to a linear moisture content change across its thickness, approximate formulas were developed for the maximum normal stresses in the facings.

In conclusion, the present work, as a phase in the overall project about sandwich constructions at the Centre for Building Studies, provided design aids for sandwich panels subjected to different loading types and boundary conditions. The subjects considered related only to structural aspects. The practical formulas developed can be used to ensure that a sandwich panel will perform adequately when constructed in a structural system.

7.2 RECOMMENDATIONS FOR FURTHER STUDIES

The present work forms a phase in the overall project at the Centre for Building Studies which aims at developing a comprehensive design manual for light-weight sandwich construction. This project has been limited to the structural aspect of sandwich plates. Subjects for further study in this area may include :

(1) Performance of sandwich panels subjected to dynamic loads.

(2) Effects of creep of adhesive and constituent materials on the structural performance.

(3) Optimum design of sandwich panels for both structural and non-structural performances.

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APPENDIX A

NUMERICAL VALUES FOR THE FACTORS IN THE PRACTICAL FORMULAS FOR SIMPLY SUPPORTED SANDWICH PLATES

(CHAPTER II)

$R = \frac{a}{b}$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0041	.0737	.3357	.3357	.0479	.0479	-.0325
1.1000	.0033	.0666	.3129	.2958	.0408	.0459	-.0267
1.2000	.0027	.0602	.2921	.2621	.0348	.0435	-.0220
1.3000	.0022	.0545	.2732	.2333	.0298	.0411	-.0182
1.4000	.0018	.0494	.2561	.2067	.0256	.0385	-.0152
1.5000	.0015	.0448	.2407	.1875	.0222	.0361	-.0127
1.6000	.0013	.0407	.2267	.1690	.0193	.0337	-.0107
1.7000	.0011	.0371	.2141	.1530	.0158	.0314	-.0091
1.8000	.0009	.0339	.2027	.1389	.0148	.0293	-.0078
1.9000	.0007	.0310	.1923	.1266	.0131	.0273	-.0067
2.0000	.0006	.0285	.1828	.1158	.0116	.0254	-.0058
3.0000	.0002	.0136	.1217	.0545	.0045	.0132	-.0018
4.0000	.0001	.0078	.0908	.0310	.0024	.0077	-.0007
5.0000	.0000	.0050	.0722	.0199	.0015	.0050	-.0004

TABLE A.1 - NUMERICAL VALUES FOR THE FACTORS IN EQUS. (2.38), (2.42), and (2.48).

$R = \frac{a}{b}$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0020	.0368	.1678	.1678	.0239	.0239	-.0162
1.1000	.0017	.0333	.1565	.1479	.0204	.0229	-.0133
1.2000	.0014	.0301	.1461	.1310	.0174	.0218	-.0110
1.3000	.0011	.0273	.1366	.1167	.0149	.0205	-.0091
1.4000	.0009	.0247	.1281	.1044	.0128	.0193	-.0076
1.5000	.0008	.0224	.1203	.0937	.0111	.0180	-.0064
1.6000	.0006	.0204	.1134	.0845	.0096	.0168	-.0054
1.7000	.0005	.0186	.1070	.0765	.0084	.0157	-.0046
1.8000	.0004	.0169	.1013	.0695	.0074	.0146	-.0039
1.9000	.0004	.0155	.0961	.0633	.0065	.0136	-.0033
2.0000	.0003	.0142	.0914	.0579	.0058	.0127	-.0029
3.0000	.0001	.0068	.0609	.0273	.0023	.0066	-.0009
4.0000	.0000	.0039	.0454	.0155	.0012	.0039	-.0004
5.0000	.0000	.0025	.0361	.0100	.0008	.0025	-.0002

TABLE A.2 - NUMERICAL VALUES FOR THE FACTORS IN EQU. (2.53).

$R = \frac{a}{b}$
$\xi' = .17 \quad (\xi' = \xi/a)$
$\eta' = .17 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0001	.0019	.0160	.0160	.0012	.0012	-.0054
1.1000	.0001	.0017	.0164	.0128	.0010	.0012	-.0044
1.2000	.0001	.0015	.0167	.0104	.0008	.0012	-.0037
1.3000	.0001	.0014	.0168	.0084	.0006	.0012	-.0031
1.4000	.0001	.0012	.0168	.0069	.0005	.0011	-.0026
1.5000	.0000	.0011	.0167	.0058	.0004	.0010	-.0022
1.6000	.0000	.0009	.0165	.0048	.0003	.0009	-.0019
1.7000	.0000	.0008	.0163	.0040	.0002	.0009	-.0017
1.8000	.0000	.0007	.0160	.0034	.0001	.0008	-.0015
1.9000	.0000	.0006	.0157	.0029	.0001	.0007	-.0013
2.0000	.0000	.0006	.0154	.0025	.0001	.0007	-.0011
3.0000	.0000	.0002	.0119	.0006	-.0000	.0003	-.0004
4.0000	.0000	.0001	.0090	.0002	-.0000	.0001	-.0002
5.0000	.0000	.0000	.0068	.0001	-.0000	.0000	-.0001

TABLE A.3 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATES UNDER PARTIAL LOAD, EQU. (2.56).

R = $\frac{a}{b}$
ξ' = .33 $(\xi' = \xi/a)$
η' = .17 $(\eta' = \eta/b)$
ν = .30

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0002	.0037	.0166	.0551	.0028	.0021	-.0044
1.1000	.0002	.0035	.0158	.0465	.0024	.0021	-.0034
1.2000	.0002	.0032	.0149	.0396	.0021	.0021	-.0027
1.3000	.0001	.0029	.0139	.0340	.0018	.0020	-.0022
1.4000	.0001	.0027	.0130	.0294	.0016	.0020	-.0017
1.5000	.0001	.0025	.0120	.0256	.0013	.0019	-.0014
1.6000	.0001	.0023	.0111	.0224	.0012	.0018	-.0012
1.7000	.0001	.0021	.0102	.0197	.0010	.0017	-.00[illegible]
1.8000	.0001	.0019	.0094	.0174	.0009	.0016	-.0008
1.9000	.0000	.0018	.0086	.0155	.0008	.0015	-.0007
2.0000	.0000	.0016	.0079	.0138	.0007	.0014	-.0006
3.0000	.0000	.0008	.0032	.0052	.0002	.0008	-.0001
4.0000	.0000	.0004	.0013	.0024	.0001	.0005	-.0000
5.0000	.0000	.0002	.0006	.0013	.0000	.0003	-.0000

TABLE A.4 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$
$\xi' = .50 \quad (\xi' = \xi/a)$
$\eta' = .17 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0003	.0046	.0114	.1064	.0037	.0023	-.0029
1.1000	.0002	.0044	.0101	.0937	.0033	.0023	-.0022
1.2000	.0002	.0041	.0089	.0832	.0030	.0023	-.0017
1.3000	.0002	.0038	.0078	.0743	.0027	.0023	-.0013
1.4000	.0001	.0036	.0068	.0668	.0024	.0022	-.0010
1.5000	.0001	.0034	.0059	.0603	.0022	.0022	-.0008
1.6000	.0001	.0032	.0051	.0548	.0020	.0021	-.0006
1.7000	.0001	.0030	.0045	.0499	.0019	.0020	-.0005
1.8000	.0001	.0028	.0039	.0457	.0017	.0019	-.0004
1.9000	.0001	.0026	.0033	.0420	.0016	.0018	-.0003
2.0000	.0001	.0025	.0029	.0387	.0015	.0018	-.0003
3.0000	.0000	.0015	.0007	.0197	.0008	.0012	-.0000
4.0000	.0000	.0010	.0002	.0118	.0005	.0008	-.0000
5.0000	.0000	.0007	.0000	.0078	.0003	.0006	-.0000

TABLE A.5 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$

$\xi' = .17 \quad (\xi' = \xi/a)$

$\eta' = .33 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0002	.0037	.0551	.0166	.0021	.0028	-.0044
1.1000	.0002	.0033	.0536	.0142	.0015	.0026	-.0038
1.2000	.0002	.0029	.0520	.0122	.0013	.0025	-.0033
1.3000	.0001	.0025	.0503	.0104	.0010	.0023	-.0029
1.4000	.0001	.0022	.0486	.0080	.0008	.0021	-.0025
1.5000	.0001	.0019	.0469	.0076	.0006	.0019	-.0022
1.6000	.0001	.0017	.0452	.0066	.0004	.0018	-.0020
1.7000	.0001	.0015	.0435	.0057	.0003	.0016	-.0017
1.8000	.0000	.0013	.0419	.0049	.0002	.0015	-.0015
1.9000	.0000	.0011	.0404	.0042	.0002	.0013	-.0014
2.0000	.0000	.0010	.0389	.0037	.0001	.0012	-.0012
3.0000	.0000	.0003	.0267	.0010	-.0001	.0004	-.0005
4.0000	.0000	.0001	.0188	.0003	-.0000	.0002	-.0002
5.0000	.0000	.0000	.0136	.0001	-.0000	.0001	-.0001

TABLE A.6 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$

$\xi' = .33 \quad (\xi' = \xi/a)$

$\eta' = .33 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0005	.0083	.0370	.0370	.0054	.0054	-.0051
1.1000	.0004	.0075	.0337	.0331	.0046	.0052	-.0042
1.2000	.0003	.0068	.0307	.0296	.0039	.0049	-.0035
1.3000	.0003	.0062	.0280	.0266	.0033	.0047	-.0029
1.4000	.0002	.0056	.0254	.0239	.0029	.0044	-.0024
1.5000	.0002	.0050	.0231	.0215	.0025	.0041	-.0020
1.6000	.0001	.0046	.0210	.0194	.0021	.0038	-.0017
1.7000	.0001	.0042	.0191	.0175	.0018	.0036	-.0014
1.8000	.0001	.0038	.0174	.0159	.0016	.0033	-.0012
1.9000	.0001	.0035	.0158	.0144	.0014	.0031	-.0010
2.0000	.0001	.0032	.0144	.0131	.0012	.0029	-.0009
3.0000	.0000	.0014	.0057	.0058	.0004	.0015	-.0002
4.0000	.0000	.0008	.0024	.0029	.0001	.0009	-.0001
5.0000	.0000	.0004	.0010	.0017	.0000	.0005	-.0000

TABLE A.7 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$

$\xi' = .50 \quad (\xi' = \xi/a)$

$\eta' = .33 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0005	.0115	.0215	.0493	.0083	.0066	-.0041
1.1000	.0004	.0106	.0187	.0454	.0074	.0064	-.0032
1.2000	.0004	.0098	.0162	.0418	.0065	.0062	-.0025
1.3000	.0003	.0091	.0141	.0386	.0059	.0059	-.0020
1.4000	.0003	.0084	.0121	.0357	.0053	.0057	-.0016
1.5000	.0002	.0078	.0105	.0331	.0048	.0054	-.0013
1.6000	.0002	.0072	.0091	.0307	.0043	.0051	-.0010
1.7000	.0002	.0067	.0078	.0286	.0039	.0048	-.0008
1.8000	.0001	.0063	.0068	.0267	.0036	.0046	-.0007
1.9000	.0001	.0059	.0058	.0249	.0033	.0043	-.0005
2.0000	.0001	.0055	.0050	.0234	.0030	.0041	-.0004
3.0000	.0000	.0032	.0012	.0133	.0016	.0026	-.0001
4.0000	.0000	.0020	.0003	.0085	.0009	.0017	-.0000
5.0000	.0000	.0014	.0001	.0059	.0006	.0013	-.0000

TABLE A.8 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56.)

$R = \frac{a}{b}$

$\xi' = .17 \quad (\xi' = \xi/a)$

$\eta' = .50 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0003	.0046	.1064	.0114	.0023	.0037	-.0029
1.1000	.0002	.0040	.0994	.0103	.0018	.0034	-.0026
1.2000	.0002	.0035	.0931	.0093	.0014	.0031	-.0023
1.3000	.0002	.0030	.0874	.0083	.0011	.0028	-.0021
1.4000	.0001	.0026	.0822	.0074	.0008	.0026	-.0018
1.5000	.0001	.0023	.0774	.0066	.0006	.0023	-.0016
1.6000	.0001	.0020	.0730	.0058	.0005	.0021	-.0015
1.7000	.0001	.0017	.0690	.0051	.0003	.0019	-.0013
1.8000	.0001	.0015	.0653	.0045	.0002	.0017	-.0012
1.9000	.0000	.0013	.0619	.0040	.0002	.0016	-.0011
2.0000	.0000	.0012	.0587	.0035	.0001	.0014	-.0010
3.0000	.0000	.0003	.0366	.0010	-.0001	.0005	-.0004
4.0000	.0000	.0001	.0244	.0003	-.0001	.0002	-.0002
5.0000	.0000	.0000	.0171	.0001	-.0000	.0001	-.0001

TABLE A.9 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56)

$R = \frac{a}{b}$
$\xi' = .33 \quad (\xi' = \xi/a)$
$\eta' = .50 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0005	.0115	.0493	.0215	.0066	.0083	-.0041
1.1000	.0004	.0102	.0439	.0201	.0056	.0077	-.0035
1.2000	.0004	.0091	.0392	.0187	.0047	.0071	-.0030
1.3000	.0003	.0081	.0352	.0173	.0040	.0066	-.0025
1.4000	.0002	.0073	.0316	.0159	.0034	.0061	-.0021
1.5000	.0002	.0065	.0284	.0147	.0029	.0056	-.0018
1.6000	.0002	.0059	.0256	.0135	.0025	.0052	-.0016
1.7000	.0001	.0053	.0231	.0124	.0021	.0048	-.0013
1.8000	.0001	.0048	.0209	.0114	.0019	.0044	-.0012
1.9000	.0001	.0044	.0189	.0105	.0016	.0041	-.0010
2.0000	.0001	.0040	.0172	.0097	.0014	.0038	-.0009
3.0000	.0000	.0018	.0068	.0047	.0004	.0019	-.0002
4.0000	.0000	.0009	.0029	.0025	.0001	.0010	-.0001
5.0000	.0000	.0005	.0013	.0015	.0000	.0006	-.0000

TABLE A.10 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$

$\xi' = .50 \quad (\xi' = \xi/a)$

$\eta' = .50 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0007	.0182	.0261	.0261	.0118	.0118	-.0037
1.1000	.0005	.0165	.0224	.0248	.0104	.0111	-.0030
1.2000	.0004	.0150	.0193	.0234	.0091	.0104	-.0024
1.3000	.0004	.0137	.0166	.0220	.0081	.0097	-.0020
1.4000	.0003	.0125	.0143	.0207	.0072	.0091	-.0016
1.5000	.0003	.0115	.0123	.0194	.0065	.0085	-.0013
1.6000	.0002	.0106	.0106	.0182	.0059	.0079	-.0011
1.7000	.0002	.0098	.0092	.0171	.0053	.0074	-.0009
1.8000	.0002	.0091	.0079	.0161	.0049	.0070	-.0007
1.9000	.0001	.0085	.0068	.0152	.0044	.0066	-.0006
2.0000	.0001	.0079	.0059	.0144	.0041	.0062	-.0005
3.0000	.0000	.0044	.0014	.0087	.0020	.0037	-.0001
4.0000	.0000	.0028	.0004	.0058	.0012	.0024	-.0000
5.0000	.0000	.0019	.0001	.0041	.0008	.0017	-.0000

TABLE A.11 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER PARTIAL LOAD, EQU. (2.56).

$R = \frac{a}{b}$
$\xi' = .17 \quad (\xi' = \xi/a)$
$\eta' = .17 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0023	.0306	.2556	.2556	.0199	.0199	-.1052
1.1000	.0019	.0275	.2675	.1997	.0157	.0201	-.0864
1.2000	.0015	.0245	.2762	.1561	.0122	.0197	-.0716
1.3000	.0012	.0216	.2821	.1265	.0093	.0189	-.0597
1.4000	.0010	.0190	.2855	.1022	.0070	.0178	-.0501
1.5000	.0008	.0167	.2867	.0833	.0051	.0166	-.0424
1.6000	.0007	.0146	.2859	.0684	.0036	.0154	-.0360
1.7000	.0005	.0128	.2836	.0566	.0025	.0141	-.0308
1.8000	.0004	.0112	.2799	.0471	.0016	.0129	-.0265
1.9000	.0004	.0097	.2750	.0394	.0009	.0118	-.0229
2.0000	.0003	.0085	.2691	.0331	.0004	.0107	-.0199
3.0000	.0000	.0022	.1897	.0068	-.0009	.0037	-.0059
4.0000	.0000	.0006	.1191	.0016	-.0005	.0012	-.0022
5.0000	.0000	.0002	.0720	.0005	-.0002	.0004	-.0010

TABLE A.12 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$
$\xi' = .33 \quad (\xi' = \xi/a)$
$\eta' = .17 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KOX	KQY	KMX	KMY	KMXY
1.0000	.0041	.0599	.2712	.8535	.0457	.0321	-.0768
1.1000	.0034	.0557	.2585	.7061	.0393	.0331	-.0592
1.2000	.0028	.0515	.2434	.5878	.0337	.0332	-.0462
1.3000	.0023	.0473	.2269	.4924	.0288	.0327	-.0365
1.4000	.0019	.0434	.2101	.4150	.0246	.0318	-.0292
1.5000	.0016	.0397	.1934	.3518	.0210	.0306	-.0235
1.6000	.0013	.0362	.1773	.2999	.0179	.0292	-.0191
1.7000	.0011	.0331	.1619	.2570	.0153	.0278	-.0157
1.8000	.0009	.0302	.1475	.2213	.0130	.0263	-.0129
1.9000	.0008	.0276	.1341	.1915	.0111	.0248	-.0107
2.0000	.0007	.0253	.1217	.1664	.0094	.0234	-.0089
3.0000	.0002	.0107	.0442	.0495	.0015	.0124	-.0018
4.0000	.0000	.0049	.0158	.0186	-.0003	.0066	-.0005
5.0000	.0000	.0023	.0058	.0081	-.0006	.0036	-.0001

TABLE A.13 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .50 \quad (\xi' = \xi/a)$

$\eta' = .17 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KOX	KOY	KMX	KMY	KMXY
1.0000	.0049	.0759	.1849	1.5505	.0628	.0358	-.0488
1.1000	.0041	.0720	.1641	1.4348	.0568	.0366	-.0365
1.2000	.0034	.0681	.1441	1.3343	.0517	.0368	-.0276
1.3000	.0028	.0642	.1257	1.2465	.0472	.0363	-.0212
1.4000	.0024	.0606	.1090	1.1692	.0433	.0354	-.0164
1.5000	.0020	.0572	.0942	1.1007	.0400	.0343	-.0128
1.6000	.0017	.0541	.0812	1.0396	.0372	.0331	-.0101
1.7000	.0014	.0512	.0699	.9848	.0347	.0318	-.0080
1.8000	.0012	.0485	.0601	.9355	.0326	.0305	-.0063
1.9000	.0011	.0461	.0516	.8908	.0307	.0292	-.0051
2.0000	.0009	.0439	.0443	.8502	.0290	.0280	-.0041
3.0000	.0003	.0294	.0096	.5837	.0190	.0191	-.0005
4.0000	.0001	.0220	.0022	.4440	.0142	.0143	-.0001
5.0000	.0001	.0176	.0006	.3571	.0114	.0114	-.0000

TABLE A.14 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .17 \quad (\xi' = \xi/a)$

$\eta' = .33 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0041	.0599	.8535	.2712	.0321	.0457	-.0768
1.1000	.0033	.0524	.8428	.2308	.0249	.0431	-.0674
1.2000	.0027	.0457	.8245	.1960	.0191	.0402	-.0592
1.3000	.0022	.0397	.8012	.1663	.0145	.0372	-.0521
1.4000	.0018	.0345	.7744	.1413	.0108	.0341	-.0460
1.5000	.0014	.0300	.7455	.1202	.0078	.0312	-.0407
1.6000	.0012	.0261	.7155	.1024	.0055	.0284	-.0362
1.7000	.0009	.0226	.6851	.0875	.0037	.0257	-.0322
1.8000	.0008	.0197	.6547	.0749	.0023	.0233	-.0287
1.9000	.0006	.0171	.6248	.0642	.0012	.0213	-.0256
2.0000	.0005	.0149	.5955	.0552	.0004	.0190	-.0230
3.0000	.0001	.0038	.3584	.0136	-.0015	.0065	-.0084
4.0000	.0000	.0010	.2124	.0039	-.0008	.0022	-.0035
5.0000	.0000	.0003	.1252	.0013	-.0003	.0007	-.0016

TABLE A.15 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64.).

$R = \frac{a}{b}$

$\xi' = .33 \quad (\xi' = \xi/a)$

$\eta' = .33 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0076	.1324	.5988	.5988	.0861	.0861	-.0874
1.1000	.0062	.1196	.5450	.5365	.0721	.0834	-.0716
1.2000	.0051	.1078	.4942	.4797	.0604	.0797	-.0588
1.3000	.0042	.0970	.4471	.4285	.0506	.0756	-.0485
1.4000	.0035	.0874	.4039	.3827	.0424	.0711	-.0401
1.5000	.0029	.0787	.3645	.3420	.0356	.0667	-.0333
1.6000	.0024	.0709	.3287	.3059	.0299	.0623	-.0278
1.7000	.0020	.0640	.2964	.2738	.0251	.0581	-.0232
1.8000	.0017	.0578	.2671	.2454	.0211	.0540	-.0195
1.9000	.0014	.0523	.2407	.2202	.0178	.0502	-.0165
2.0000	.0012	.0474	.2169	.1979	.0150	.0467	-.0139
3.0000	.0003	.0191	.0768	.0736	.0020	.0228	-.0030
4.0000	.0001	.0085	.0276	.0310	-.0007	.0117	-.0008
5.0000	.0000	.0040	.0108	.0143	-.0011	.0063	-.0002

TABLE A.16 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$
$\xi' = .50 \quad (\xi' = \xi/a)$
$\eta' = .33 \quad (\eta' = \eta/b)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0092	.1894	.3499	1.2182	.1466	.0997	-.0678
1.1000	.0076	.1775	.3029	1.1021	.1328	.0979	-.0529
1.2000	.0063	.1663	.2614	1.0028	.1209	.0952	-.0415
1.3000	.0052	.1559	.2250	.9175	.1108	.0919	-.0326
1.4000	.0044	.1464	.1934	.8437	.1021	.0883	-.0258
1.5000	.0037	.1378	.1660	.7795	.0945	.0846	-.0204
1.6000	.0031	.1299	.1424	.7234	.0881	.0809	-.0163
1.7000	.0026	.1228	.1221	.6740	.0824	.0773	-.0130
1.8000	.0023	.1163	.1046	.6303	.0774	.0738	-.0104
1.9000	.0019	.1104	.0897	.5915	.0731	.0705	-.0084
2.0000	.0017	.1051	.0768	.5569	.0692	.0674	-.0068
3.0000	.0005	.0702	.0166	.3466	.0456	.0457	-.0009
4.0000	.0002	.0526	.0040	.2496	.0341	.0343	-.0001
5.0000	.0001	.0420	.0016	.1945	.0272	.0274	-.0000

TABLE A.17 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .17 \quad (\xi' = \xi/a)$

$\eta' = .50 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0049	.0759	1.5505	.1849	.0358	.0628	-.0488
1.1000	.0039	.0653	1.3815	.1682	.0280	.0569	-.0440
1.2000	.0031	.0564	1.2405	.1512	.0217	.0516	-.0397
1.3000	.0025	.0488	1.1210	.1347	.0167	.0467	-.0358
1.4000	.0020	.0422	1.0164	.1192	.0126	.0423	-.0323
1.5000	.0017	.0366	.9293	.1051	.0094	.0382	-.0292
1.6000	.0013	.0318	.8512	.0924	.0069	.0345	-.0265
1.7000	.0011	.0277	.7821	.0810	.0049	.0311	-.0241
1.8000	.0009	.0241	.7206	.0709	.0033	.0281	-.0219
1.9000	.0007	.0210	.6654	.0620	.0021	.0253	-.0200
2.0000	.0006	.0184	.6155	.0543	.0011	.0228	-.0182
3.0000	.0001	.0052	.2976	.0144	-.0011	.0060	-.0077
4.0000	.0000	.0019	.1419	.0041	-.0005	.0029	-.0035
5.0000	.0000	.0009	.0574	.0013	-.0000	.0012	-.0017

TABLE A.18 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .33 \quad (\xi' = \xi/a)$

$\eta' = .50 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0092	.1894	1.2182	.3499	.0997	.1466	-.0678
1.1000	.0075	.1659	1.1087	.3283	.0826	.1330	-.0581
1.2000	.0061	.1459	1.0136	.3054	.0587	.1210	-.0496
1.3000	.0050	.1289	.9303	.2825	.0573	.1103	-.0423
1.4000	.0041	.1143	.8567	.2603	.0479	.1007	-.0360
1.5000	.0034	.1017	.7914	.2393	.0402	.0920	-.0307
1.6000	.0028	.0907	.7330	.2197	.0338	.0842	-.0262
1.7000	.0023	.0612	.6807	.2015	.0285	.0771	-.0224
1.8000	.0019	.0720	.6336	.1848	.0241	.0706	-.0192
1.9000	.0016	.0657	.5912	.1695	.0204	.0650	-.0165
2.0000	.0014	.0593	.5528	.1554	.0172	.0598	-.0141
3.0000	.0003	.0238	.3149	.0674	.0030	.0279	-.0034
4.0000	.0001	.0110	.2123	.0310	-.0001	.0144	-.0009
5.0000	.0000	.0056	.1615	.0149	-.0006	.0079	-.0002

TABLE A.19 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .50 \quad (\xi' = \xi/a)$

$\eta' = .50 \quad (\eta' = \eta/b)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0116	.7057	.6679	.6679	.4587	.4587	-.0610
1.1000	.0095	.6407	.5977	.6130	.4104	.4225	-.0497
1.2000	.0078	.5853	.5369	.5639	.3700	.3909	-.0403
1.3000	.0065	.5376	.4843	.5201	.3359	.3630	-.0326
1.4000	.0054	.4962	.4385	.4811	.3069	.3382	-.0264
1.5000	.0045	.4600	.3987	.4465	.2819	.3161	-.0213
1.6000	.0038	.4281	.3639	.4157	.2604	.2962	-.0173
1.7000	.0033	.3999	.3335	.3883	.2416	.2783	-.0140
1.8000	.0028	.3747	.3069	.3638	.2251	.2621	-.0113
1.9000	.0024	.3522	.2836	.3418	.2105	.2475	-.0092
2.0000	.0021	.3320	.2631	.3221	.1975	.2341	-.0075
3.0000	.0006	.2055	.1508	.2008	.1189	.1463	-.0010
4.0000	.0003	.1448	.1095	.1445	.0823	.1058	-.0002
5.0000	.0001	.1097	.0884	.1126	.0616	.0810	-.0000

TABLE A.20 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE UNDER CONCENTRATED LOAD, EQU. (2.64).

$R = \frac{a}{b}$

$\xi' = .10 \quad (\xi' = \xi/a)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0009	.0127	.4210	.0524	.0067	.0098	.0288
1.1000	.0007	.0110	.4118	.0436	.0051	.0093	.0247
1.2000	.0006	.0076	.4026	.0365	.0038	.0086	.0214
1.3000	.0005	.0083	.3934	.0306	.0028	.0080	.0187
1.4000	.0004	.0072	.3843	.0258	.0020	.0073	.0165
1.5000	.0003	.0062	.3753	.0218	.0014	.0067	.0146
1.6000	.0003	.0053	.3664	.0185	.0009	.0060	.0130
1.7000	.0002	.0046	.3576	.0157	.0005	.0055	.0116
1.8000	.0002	.0040	.3489	.0133	.0002	.0049	.0104
1.9000	.0001	.0034	.3404	.0114	.0000	.0044	.0094
2.0000	.0001	.0029	.3320	.0097	-.0001	.0039	.0085
3.0000	.0000	.0007	.2565	.0021	-.0004	.0012	.0037
4.0000	.0000	.0002	.1965	.0005	-.0002	.0004	.0019
5.0000	.0000	.0000	.1504	.0001	-.0001	.0001	.0011

TABLE A.21 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO STRIP LOAD, EQU. (2.72)

$R = \frac{a}{b}$

$\xi' = .20 \quad (\xi' = \xi/a)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0018	.0265	.3006	.1161	.0148	.0196	.0330
1.1000	.0014	.0233	.2849	.0975	.0117	.0186	.0276
1.2000	.0012	.0204	.2696	.0823	.0091	.0174	.0233
1.3000	.0010	.0179	.2547	.0698	.0071	.0162	.0198
1.4000	.0008	.0157	.2404	.0595	.0054	.0150	.0170
1.5000	.0006	.0137	.2266	.0508	.0041	.0138	.0146
1.6000	.0005	.0120	.2135	.0436	.0030	.0126	.0126
1.7000	.0004	.0105	.2009	.0375	.0021	.0115	.0110
1.8000	.0003	.0092	.1889	.0324	.0015	.0105	.0096
1.9000	.0003	.0080	.1775	.0260	.0009	.0095	.0084
2.0000	.0002	.0070	.1667	.0243	.0005	.0086	.0074
3.0000	.0000	.0020	.0861	.0064	-.0006	.0031	.0024
4.0000	.0000	.0006	.0417	.001[illegible]	-.0004	.0011	.0009
5.0000	.0000	.0002	.0177	.0006	-.0002	.0004	.0004

TABLE A.22 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO STRIP LOAD, EQU. (2.72).

$R = \frac{a}{b}$

$\xi' = .30 \quad (\xi' = \xi/a)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0026	.0424	.2403	.2095	.0260	.0291	.0316
1.1000	.0021	.0379	.2217	.1782	.0215	.0278	.0258
1.2000	.0017	.0339	.2041	.1526	.0178	.0263	.0213
1.3000	.0014	.0302	.1876	.1314	.0146	.0247	.0177
1.4000	.0012	.0270	.1722	.1138	.0120	.0231	.0147
1.5000	.0009	.0241	.1580	.0990	.0099	.0215	.0124
1.6000	.0008	.0216	.1449	.0864	.0081	.0199	.0104
1.7000	.0007	.0193	.1328	.0758	.0066	.0185	.0088
1.8000	.0005	.0173	.1217	.0666	.0054	.0171	.0075
1.9000	.0005	.0155	.1115	.0588	.0044	.0158	.0064
2.0000	.0004	.0139	.1022	.0520	.0035	.0146	.0055
3.0000	.0001	.0051	.0439	.0173	.0000	.0066	.0014
4.0000	.0000	.0020	.0209	.0066	-.0005	.0031	.0004
5.0000	.0000	.0009	.0120	.0028	-.0004	.0015	.0001

TABLE A.23 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO STRIP LOAD, EQU. (2.72).

$R = \frac{a}{b}$
$\xi' = .40 \quad (\xi' = \xi/a)$
$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0031	.0615	.1683	.3899	.0419	.0380	.0278
1.1000	.0026	.0561	.1498	.3379	.0364	.0366	.0222
1.2000	.0021	.0512	.1328	.2951	.0317	.0349	.0179
1.3000	.0018	.0468	.1175	.2595	.0277	.0332	.0145
1.4000	.0015	.0428	.1036	.2296	.0243	.0314	.0118
1.5000	.0012	.0392	.0912	.2042	.0214	.0296	.0097
1.6000	.0010	.0360	.0802	.1825	.0189	.0279	.0079
1.7000	.0009	.0331	.0703	.1639	.0168	.0263	.0065
1.8000	.0007	.0305	.0616	.1477	.0149	.0247	.0054
1.9000	.0006	.0282	.0539	.1336	.0133	.0233	.0045
2.0000	.0005	.0260	.0470	.1212	.0119	.0219	.0037
3.0000	.0001	.0130	.0103	.0528	.0043	.0125	.0007
4.0000	.0001	.0072	-.0062	.0269	.0016	.0077	.0001
5.0000	.0000	.0042	-.0032	.0151	.0004	.0050	.0000

TABLE A.24 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO STRIP LOAD, EQU. (2.72).

$R = \frac{a}{b}$

$\xi' = .50 \quad (\xi' = \xi/a)$

$\nu = .30$

R	KWB	KWS	KQX	KQY	KMX	KMY	KMXY
1.0000	.0034	.0834	.1282	1.2201	.0627	.0457	.0232
1.1000	.0028	.0777	.1117	1.1016	.0568	.0443	.0182
1.2000	.0023	.0725	.0970	1.0017	.0517	.0426	.0144
1.3000	.0019	.0678	.0840	.9166	.0473	.0408	.0114
1.4000	.0016	.0635	.0727	.8432	.0436	.0390	.0090
1.5000	.0014	.0597	.0628	.7795	.0404	.0372	.0072
1.6000	.0011	.0562	.0543	.7237	.0376	.0354	.0058
1.7000	.0010	.0530	.0470	.6744	.0352	.0338	.0046
1.8000	.0008	.0502	.0407	.6307	.0330	.0322	.0037
1.9000	.0007	.0476	.0353	.5917	.0311	.0307	.0030
2.0000	.0006	.0452	.0306	.5567	.0294	.0293	.0025
3.0000	.0002	.0299	.0089	.3398	.0191	.0198	.0003
4.0000	.0001	.0222	.0044	.2365	.0141	.0148	.0001
5.0000	.0000	.0176	.0034	.1774	.0111	.0118	.0000

TABLE A.25 - INFLUENCE COEFFICIENTS FOR SIMPLY SUPPORTED SANDWICH PLATE SUBJECTED TO STRIP LOAD, EQU. (2.72).

APPENDIX B

THE COEFFICIENT MATRICES AND LOAD VECTORS IN THE FINITE DIFFERENCE ANALYSIS IN SECTION 2.7

APPENDIX B

THE COEFFICIENT MATRICES AND LOAD VECTORS IN THE FINITE DIFFERENCE ANALYSIS IN SECTION 2.7

Simply supported sandwich plates subjected to lateral loads were analysed in Section 2.7 by the finite difference method. Three loading types were considered: (i) uniform distributed load, (ii) hydrostatic pressure, and (iii) central partial load. For each loading type, the coefficient matrices [A], [B], and the load vectors $\{P_b\}$, $\{P_s\}$ in Equs. (2.76) and (2.78), are given here.

(i) In the case of uniformly distributed load (Fig. 2.6)

$$[A] = \begin{array}{c} \begin{array}{cccc} w_{b1} & w_{b2} & w_{b3} & w_{b4} \end{array} \\ \begin{bmatrix} 1872. & -576. & -576. & -32. \\ -1152. & 1872. & -64. & -576. \\ -1152. & -64. & 1872. & -576. \\ -128. & -1152. & -1152. & 1872. \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$[B] = \begin{array}{c} \begin{array}{cccc} w_{s1} & w_{s2} & w_{s3} & w_{s4} \end{array} \\ \begin{bmatrix} -60. & 16. & 16. & 0. \\ 32. & -60. & 0. & 16. \\ 32. & 0. & -60. & 16. \\ 0. & 32. & 32. & -60. \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\{P_b\} = \frac{\lambda^4 \, p_o}{144 \, D} \begin{Bmatrix} 14880. \\ 17320. \\ 17320. \\ 20160. \end{Bmatrix}$$

$$\{P_s\} = \frac{-12 \, \lambda^2 \, p_o}{S} \begin{Bmatrix} 1. \\ 1. \\ 1. \\ 1. \end{Bmatrix}$$

in which

λ = the mesh width

= $\frac{a}{4}$ (where a is the side length of the plate)

p_o = the uniform load intensity.

(ii) In the case of hydrostatic pressure (Fig. 2.7)

$$[A] = \begin{array}{c} \begin{array}{cccccc} w_{b1} & w_{b2} & w_{b3} & w_{b4} & w_{b5} & w_{b6} \end{array} \\ \begin{bmatrix} 1800. & -576. & 72. & -608. & -32. & 32. \\ -576. & 1872. & -576. & -32. & -576. & -32. \\ 72. & -576. & 1800. & 32. & -32. & -608. \\ -1216. & -64. & 64. & 1800. & -576. & 72. \\ -64. & -1152. & -64. & -576. & 1872. & -576. \\ 64. & -64. & -1216. & 72. & -576. & 1800. \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} w_{s1} & w_{s2} & w_{s3} & w_{s4} & w_{s5} & w_{s6} \end{array} \\
[B] = & \left[\begin{array}{cccccc}
-59. & 16. & -1. & 16. & 0. & 0. \\
16. & -60. & 16. & 0. & 16. & 0. \\
-1. & 16. & -59. & 0. & 0. & 16. \\
32. & 0. & 0. & -59. & 16. & -1. \\
0. & 32. & 0. & 16. & -60. & 16. \\
0. & 0. & 32. & -1. & 16. & -59.
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

$$
\{P_b\} = \frac{p_o \lambda^4}{576\ D} \begin{Bmatrix} 17570. \\ 34650. \\ 41970. \\ 20450. \\ 42730. \\ 48850. \end{Bmatrix}
$$

$$
\{P_s\} = -\frac{12 \lambda^2 p_o}{S} \begin{Bmatrix} .25 \\ .50 \\ .70 \\ .25 \\ .50 \\ .75 \end{Bmatrix}
$$

in which

p_o = the load intensity at the plate edge (Fig. 2.7)

λ = the finite difference mesh width

$= \frac{a}{4}$

(iii) In the case of partial load (Fig. 2.8)

[A] =

	w_{b1}	w_{b2}	w_{b3}	w_{b4}	w_{b5}	w_{b6}	w_{b7}	w_{b8}	w_{b9}	w_{b10}
1	18.	-16.	2.	0.	2.	0.	0.	0.	0.	0.
2	-8.	21.	-8.	1.	-8.	3.	0.	0.	0.	0.
3	1.	-8.	20.	-8.	2.	-8.	2.	1.	0.	0.
4	0.	2.	-16.	19.	0.	4.	-8.	0.	1.	0.
5	2.	-16.	4.	0.	20.	-16.	2.	2.	0.	0.
6	0.	3.	-8.	2.	-8.	23.	-8.	-8.	3.	0.
7	0.	0.	4.	-8.	2.	-16.	20.	4.	-8.	1.
8	0.	0.	2.	0.	2.	-16.	4.	22.	-16.	2.
9	0.	0.	0.	1.	0.	6.	-8.	-16.	25.	-8.
10	0.	0.	0.	0.	0.	0.	4.	8.	-32.	20.

$$
[B] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} w_{s1} & w_{s2} & w_{s3} & w_{s4} & w_{s5} & w_{s6} & w_{s7} & w_{s8} & w_{s9} & w_{s10} \end{array} \\
\left[\begin{array}{cccccccccc}
-58. & 32. & -2. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
16. & -59. & 16. & -1. & 16. & -1. & 0. & 0. & 0. & 0. \\
-1. & 16. & -60. & 16. & 0. & 16. & 0. & -1. & 0. & 0. \\
0. & -2. & 32. & -59. & 0. & 0. & 16. & 0. & -1. & 0. \\
0. & 32. & 0. & 0. & -60. & 32. & -2. & 0. & 0. & 0. \\
0. & -1. & 16. & 0. & 16. & -61. & 16. & 16. & -1. & 0. \\
0. & 0. & 0. & 16. & -2. & 32. & -60. & 0. & 16. & -1. \\
0. & 0. & -2. & 0. & 0. & 32. & 0. & -62. & 32. & 0. \\
0. & 0. & 0. & -1. & 0. & -2. & 16. & 32. & -61. & 16. \\
0. & 0. & 0. & 0. & 0. & 0. & -4. & 0. & 64. & -60.
\end{array}\right]
\end{array}
$$

$$\{P_b\} = \frac{p_o \lambda^4}{D} \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.5 \\ 0.5 \\ 1.0 \end{Bmatrix}$$

$$\{P_s\} = \frac{-12\ p_o\ \lambda^{-2}}{S} \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.5 \\ 0.5 \\ 1.0 \end{Bmatrix}$$

in which

p_o = the partial load intensity

λ = the finite difference mesh width

$= \frac{a}{8}$

APPENDIX C

NUMERICAL VALUES FOR THE FACTORS IN EQUS. (3.45) TO (3.50) AND (3.51) TO (3.56)

APPENDIX C

NUMERICAL VALUES FOR THE FACTORS IN EQUS. (3.45) TO (3.50) AND (3.51) TO (3.56)

$R = \frac{a}{b}$	$w \times \frac{D}{M_o b^2}$	$\frac{M_x}{M_o}$	$\frac{M_y}{M_o}$
1.0	0.037	0.394	0.256
1.2	0.052	0.420	0.393
1.4	0.067	0.424	0.515
1.6	0.079	0.415	0.619
1.8	0.087	0.402	0.703
2.0	0.096	0.387	0.770
3.0	0.117	0.331	0.939
4.0	0.122	0.309	0.985
5.0	0.125	0.302	0.996

TABLE C.1 - DEFLECTIONS AND BENDING MOMENTS AT THE CENTRE OF RECTANGULAR SANDWICH PLATES SUBJECTED TO UNIFORMLY DISTRIBUTED MOMENTS ALONG THE EDGES $y = \pm \frac{b}{2}$, (Fig. 3.6)

SPR......= 0.000
SPL......= 0.000
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0024	-.0553	-.0011	-.0006	-.0004
.2500	-.0000	-.006	-.0001	.2210	.0000	-.0000	.0000
1.2000	-.0002		-.0060	-.0640	-.0035	-.0016	-.0014
.3000	-.0000	-.011	-.0005	.2549	.0000	-.0001	.0000
1.4000	-.0004		-.0111	-.0751	-.0081	-.0031	-.0034
.3500	-.0000	-.018	-.0017	.2970	.0000	-.0004	.0001
1.6000	-.0008		-.0169	-.0872	-.0148	-.0048	-.0064
.4000	-.0000	-.028	-.0041	.3404	.0002	-.0009	.0002
1.8000	-.0013		-.0228	-.0992	-.0232	-.0068	-.0103
.4500	-.0000	-.038	-.0078	.3806	.0005	-.0017	.0002
2.0000	-.0019		-.0284	-.1106	-.0328	-.0087	-.0148
.5000	-.0001	-.050	-.0130	.4151	.0011	-.0029	.0002
3.0000	-.0049		-.0441	-.1529	-.0822	-.0169	-.0392
.7500	-.0007	-.100	-.0496	.4966	.0124	-.0120	-.0034
4.0000	-.0070		-.0425	-.1728	-.1150	-.0212	-.0558
1.0000	-.0018	-.131	-.0778	.4860	.0340	-.0201	-.0123
5.0000	-.0080		-.0340	-.1799	-.1309	-.0231	-.0639
1.2500	-.0030	-.145	-.0864	.4539	.0547	-.0240	-.0219

TABLE C.2 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .100
SPL......= .006
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0000		-.0008	-.0046	-.0004	-.0000	-.0003
.2500	-.0000	-.003	-.0000	.0184	.0000	-.0000	.0000
1.2000	-.0001		-.0025	-.0106	-.0015	-.0002	-.0010
.3000	-.0000	-.006	-.0002	.0419	.0000	-.0000	.0000
1.4000	-.0002		-.0054	-.0192	-.0039	-.0006	-.0025
.3500	-.0000	-.012	-.0008	.0753	.0000	-.0001	.0000
1.6000	-.0005		-.0092	-.0297	-.0080	-.0014	-.0048
.4000	-.0000	-.019	-.0022	.1144	.0001	-.0004	.0000
1.8000	-.0008		-.0136	-.0411	-.0138	-.0023	-.0078
.4500	-.0000	-.027	-.0047	.1549	.0003	-.0009	.0000
2.0000	-.0012		-.0180	-.0528	-.0208	-.0035	-.0114
.5000	-.0001	-.036	-.0083	.1938	.0007	-.0016	-.0001
3.0000	-.0038		-.0339	-.1018	-.0626	-.0102	-.0324
.7500	-.0005	-.081	-.0378	.3196	.0095	-.0084	-.0033
4.0000	-.0057		-.0361	-.1291	-.0940	-.0149	-.0480
1.0000	-.0015	-.111	-.0642	.3474	.0280	-.0156	-.0110
5.0000	-.0068		-.0308	-.1412	-.1105	-.0172	-.0561
1.2500	-.0025	-.125	-.0747	.3380	.0469	-.0196	-.0197

TABLE C.3 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .200
SPL.......= .013
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0000		-.0005	-.0027	-.0002	.0001	-.0003
.2500	-.0000	-.002	-.0000	.0108	.0000	-.0000	-.0000
1.2000	-.0001		-.0016	-.0066	-.0009	.0002	-.0010
.3000	-.0000	-.005	-.0001	.0262	.0000	-.0000	-.0000
1.4000	-.0001		-.0036	-.0126	-.0026	.0001	-.0022
.3500	-.0000	-.010	-.0005	.0494	.0000	-.0001	-.0000
1.6000	-.0003		-.0063	-.0203	-.0055	-.0001	-.0041
.4000	-.0000	-.016	-.0015	.0782	.0001	-.0002	-.0000
1.8000	-.0006		-.0097	-.0291	-.0098	-.0005	-.0067
.4500	-.0000	-.023	-.0033	.1097	.0002	-.0005	-.0001
2.0000	-.0009		-.0132	-.0385	-.0152	-.0011	-.0098
.5000	-.0000	-.031	-.0060	.1414	.0005	-.0010	-.0002
3.0000	-.0030		-.0275	-.0818	-.0505	-.0062	-.0282
.7500	-.0004	-.070	-.0306	.2565	.0076	-.0063	-.0032
4.0000	-.0049		-.0310	-.1085	-.0798	-.0107	-.0426
1.0000	-.0013	-.098	-.0547	.2906	.0238	-.0125	-.0102
5.0000	-.0059		-.0276	-.1213	-.0962	-.0133	-.0505
1.2500	-.0022	-.112	-.0656	.2870	.0411	-.0164	-.0180

TABLE C.4 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .300
SPL......= .019
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0000		-.0004	-.0019	-.0002	.0001	-.0003
.2500	-.0000	-.002	-.0000	.0077	.0000	-.0000	-.0000
1.2000	-.0000		-.0012	-.0048	-.0007	.0003	-.0009
.3000	-.0000	-.005	-.0001	.0191	.0000	-.0000	-.0000
1.4000	-.0001		-.0027	-.0094	-.0019	.0005	-.0021
.3500	-.0000	-.009	-.0034	.0368	.0000	-.0000	-.0000
1.6000	-.0002		-.0048	-.0155	-.0042	.0006	-.0038
.4000	-.0000	-.014	-.0012	-.0595	.0000	-.0001	-.0001
1.8000	-.0004		-.0075	-.0226	-.0076	.0005	-.0061
.4500	-.0000	-.020	-.0026	.0850	.0002	-.0003	-.0002
2.0000	-.0007		-.0104	-.0304	-.0120	.0003	-.0089
.5000	-.0000	-.027	-.0048	.1114	.0004	-.0007	-.0003
3.0000	-.0025		-.0231	-.0685	-.0424	-.0034	-.0254
.7500	-.0004	-.063	-.0257	.2147	.0064	-.0048	-.0032
4.0000	-.0042		-.0271	-.0939	-.0693	-.0077	-.0386
1.0000	-.0011	-.089	-.0476	.2509	.0207	-.0102	-.0095
5.0000	-.0052		-.0248	-.1069	-.0853	-.0104	-.0461
1.2500	-.0020	-.102	-.0584	.2512	.0365	-.0139	-.0167

TABLE C.5 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .400
SPL.......= .025
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0000		-.0003	-.0015	-.0001	.0002	-.0063
.2500	-.0000	-.002	-.0000	.0060	.0000	.0000	-.0000
1.2000	-.0000		-.0009	-.0038	-.0005	.0004	-.0009
.3000	-.0000	-.005	-.0001	.0150	.0000	-.0000	-.0000
1.4000	-.0001		-.0021	-.0075	-.0015	.0008	-.0020
.3500	-.0000	-.008	-.0063	.0293	.0000	-.0000	-.0000
1.6000	-.0002		-.0039	-.0125	-.0034	.0010	-.0036
.4000	-.0000	-.013	-.0009	.0480	.0000	-.0001	-.0001
1.8000	-.0004		-.0061	-.0184	-.0062	.0012	-.0058
.4500	-.0000	-.019	-.0021	.0694	.0001	-.0002	-.0002
2.0000	-.0006		-.0086	-.0251	-.0099	.0012	-.0083
.5000	-.0000	-.025	-.0039	.0920	.0003	-.0004	-.0004
3.0000	-.0022		-.0199	-.0589	-.0365	-.0015	-.0234
.7500	-.0003	-.058	-.0221	.1846	.0055	-.0037	-.0031
4.0000	-.0037		-.0240	-.0829	-.0613	-.0054	-.0356
1.0000	-.0010	-.082	-.0421	.2210	.0183	-.0085	-.0090
5.0000	-.0047		-.0225	-.0957	-.0766	-.0081	-.0427
1.2500	-.0018	-.095	-.0527	.2240	.0329	-.0119	-.0156

TABLE C.6 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .500
SPL.......= .031
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0000		-.0002	-.0012	-.0001	.0002	-.0003
.2500	-.0000	-.002	-.0000	.0049	.0000	.0000	-.0000
1.2000	-.0000		-.0008	-.0031	-.0004	.0005	-.0009
.3000	-.0000	-.004	-.0001	.0124	.0000	.0000	-.0000
1.4000	-.0001		-.0018	-.0062	-.0013	.0009	-.0019
.3500	-.0000	-.008	-.0003	.0244	.0000	-.0000	-.0000
1.6000	-.0002		-.0033	-.0105	-.0028	.0013	-.0035
.4000	-.0000	-.012	-.0008	.0402	.0000	-.0000	-.0001
1.8000	-.0003		-.0052	-.0156	-.0052	.0016	-.0055
.4500	-.0000	-.018	-.0018	.0586	.0001	-.0001	-.0002
2.0000	-.0005		-.0073	-.0213	-.0084	.0018	-.0079
.5000	-.0000	-.024	-.0034	.0783	.0003	-.0003	-.0004
3.0000	-.0019		-.0175	-.0517	-.0321	-.0000	-.0218
.7500	-.0003	-.054	-.0194	.1620	.0049	-.0029	-.0031
4.0000	-.0034		-.0216	-.0742	-.0550	-.0035	-.0332
1.0000	-.0009	-.076	-.0378	.1976	.0164	-.0071	-.0086
5.0000	-.0043		-.0206	-.0867	-.0696	-.0063	-.0398
1.2500	-.0016	-.088	-.0480	.2024	.0299	-.0103	-.0147

TABLE C.7 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= 0.000
SPL.......= 0.000
PL........= 0.000
PR........= 1.000
POIS.R...= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0013		-.0574	-.3819	-.0251	-.0149	-.0091
.2500	-.0000	-.098	-.0022	1.5233	.0000	-.0004	.0002
1.2000	-.0019		-.0599	-.3507	-.0357	-.0165	-.0140
.3000	-.0000	-.102	-.0052	1.3924	.0001	-.0011	.0004
1.4000	-.0024		-.0557	-.3265	-.0428	-.0167	-.0177
.3500	-.0000	-.103	-.0089	1.2878	.0002	-.0019	.0006
1.6000	-.0026		-.0483	-.3096	-.0469	-.0163	-.0200
.4000	-.0001	-.102	-.0127	1.2124	.0005	-.0027	.0006
1.8000	-.0028		-.0397	-.2984	-.0487	-.0157	-.0212
.4500	-.0001	-.100	-.0161	1.1602	.0010	-.0035	.0005
2.0000	-.0028		-.0314	-.2910	-.0491	-.0151	-.0217
.5000	-.0001	-.099	-.0187	1.1246	.0016	-.0042	.0003
3.0000	-.0027		-.0047	-.2773	-.0457	-.0136	-.0206
.7500	-.0003	-.093	-.0215	1.0564	.0056	-.0055	-.0015
4.0000	-.0025		.0024	-.2738	-.0433	-.0133	-.0194
1.0000	-.0004	-.091	-.0161	1.0414	.0081	-.0053	-.0027
5.0000	-.0025		.0027	-.2718	-.0423	-.0132	-.0188
1.2500	-.0005	-.090	-.0100	1.0346	.0091	-.0049	-.0033

TABLE C.8 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .100
SPL......= .006
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0004		-.0193	-.0996	-.0084	-.0007	-.0073
.2500	-.0000	-.054	-.0017	.3968	.0000	-.0001	.0000
1.2000	-.0008		-.0253	-.1070	-.0148	-.0021	-.0105
.3000	-.0000	-.061	-.0021	.4237	.0000	-.0003	.0001
1.4000	-.0011		-.0281	-.1075	-.0207	-.0033	-.0132
.3500	-.0000	-.064	-.0043	.4214	.0001	-.0007	.0001
1.6000	-.0014		-.0283	-.1050	-.0253	-.0041	-.0152
.4000	-.0000	-.065	-.0069	.4059	.0003	-.0012	.0001
1.8000	-.0016		-.0267	-.1017	-.0284	-.0046	-.0164
.4500	-.0000	-.065	-.0095	.3870	.0006	-.0018	.0000
2.0000	-.0017		-.0242	-.0985	-.0303	-.0049	-.0171
.5000	-.0001	-.065	-.0119	.3693	.0010	-.0023	-.0001
3.0000	-.0018		-.0095	-.0905	-.0308	-.0047	-.0172
.7500	-.0002	-.061	-.0167	.3225	.0042	-.0038	-.0015
4.0000	-.0017		-.0015	-.0886	-.0290	-.0043	-.0163
1.0000	-.0004	-.059	-.0141	.3120	.0067	-.0039	-.0026
5.0000	-.0017		.0010	-.0880	-.0281	-.0042	-.0159
1.2500	-.0004	-.059	-.0098	.3098	.0077	-.0036	-.0032

TABLE C.9 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .200
SPL......= .013
PL.......= 0.000
PR.......= 1.000
POIS.R...= -.300
K........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0003		-.0116	-.0595	-.0050	.0021	-.0069
.2500	-.0000	-.046	-.0004	.2373	.0000	-.0000	-.0060
1.2000	-.0005		-.0159	-.0669	-.0093	.0017	-.0096
.3000	-.0000	-.051	-.0014	.2650	.0000	-.0002	-.0000
1.4000	-.0008		-.0186	-.0698	-.0137	.0008	-.0117
.3500	-.0000	-.054	-.0029	.2733	.0001	-.0004	-.0000
1.6000	-.0010		-.0195	-.0701	-.0174	.0000	-.0132
.4000	-.0000	-.055	-.0048	.2705	.0002	-.0007	-.0001
1.8000	-.0011		-.0192	-.0691	-.0202	-.0007	-.0143
.4500	-.0000	-.055	-.0068	.2625	.0004	-.0011	-.0002
2.0000	-.0013		-.0180	-.0678	-.0221	-.0012	-.0149
.5000	-.0001	-.055	-.0087	.2529	.0008	-.0015	-.0003
3.0000	-.0014		-.0089	-.0628	-.0243	-.0018	-.0152
.7500	-.0002	-.053	-.0135	.2191	.0034	-.0028	-.0014
4.0000	-.0014		-.0026	-.0610	-.0232	-.0017	-.0146
1.0000	-.0003	-.052	-.0122	.2084	.0057	-.0031	-.0024
5.0000	-.0013		-.0000	-.0604	-.0225	-.0015	-.0143
1.2500	-.0004	-.051	-.0090	.2657	.0067	-.0029	-.0030

TABLE C.10 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .300
SPL.......= .019
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0083	-.0425	-.0036	.0033	-.0068
.2500	-.0000	-.043	-.0003	.1693	.0000	-.0000	-.0000
1.2000	-.0004		-.0117	-.0487	-.0068	.0034	-.0092
.3000	-.0000	-.047	-.0010	.1929	.0000	-.0001	-.0001
1.4000	-.0006		-.0139	-.0518	-.0102	.0029	-.0110
.3500	-.0000	-.049	-.0021	.2027	.0000	-.0002	-.0001
1.6000	-.0007		-.0149	-.0528	-.0132	.0022	-.0122
.4000	-.0000	-.050	-.0036	.2037	.0001	-.0004	-.0002
1.8000	-.0009		-.0150	-.0527	-.0156	.0015	-.0131
.4500	-.0000	-.050	-.0053	.2000	.0003	-.0007	-.0003
2.0000	-.0010		-.0143	-.0522	-.0174	.0009	-.0136
.5000	-.0000	-.050	-.0069	.1944	.0006	-.0010	-.0005
3.0000	-.0012		-.0079	-.0491	-.0202	-.0002	-.0139
.7500	-.0002	-.048	-.0114	.1697	.0029	-.0021	-.0014
4.0000	-.0012		-.0029	-.0477	-.0196	-.0002	-.0135
1.0000	-.0003	-.047	-.0107	.1601	.0049	-.0025	-.0023
5.0000	-.0011		-.0005	-.0471	-.0190	-.0001	-.0132
1.2500	-.0003	-.047	-.0082	.1572	.0060	-.0024	-.0028

TABLE C.11 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .400
SPL......= .025
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0064	-.0330	-.0028	.0040	-.0067
.2500	-.0000	-.041	-.0072	.1316	.0000	.0000	-.0000
1.2000	-.0003		-.0092	-.0383	-.0054	.0044	-.0089
.3000	-.0000	-.044	-.0008	.1517	.0000	-.0000	-.0001
1.4000	-.0004		-.0111	-.0412	-.0081	.0041	-.0105
.3500	-.0000	-.046	-.0017	.1612	.0000	-.0001	-.0002
1.6000	-.0006		-.0120	-.0424	-.0107	.0035	-.0116
.4000	-.0000	-.047	-.0029	.1635	.0001	-.0002	-.0003
1.8000	-.0007		-.0122	-.0427	-.0128	.0029	-.0123
.4500	-.0000	-.047	-.0043	.1619	.0003	-.0004	-.0004
2.0000	-.0008		-.0119	-.0425	-.0144	.0023	-.0127
.5000	-.0000	-.047	-.0057	.1583	.0005	-.0006	-.0005
3.0000	-.0010		-.0070	-.0406	-.0173	.0009	-.0130
.7500	-.0001	-.045	-.0098	.1395	.0025	-.0017	-.0014
4.0000	-.0010		-.0029	-.0396	-.0171	.0008	-.0126
1.0000	-.0002	-.044	-.0096	.1312	.0044	-.0020	-.0022
5.0000	-.0010		-.0008	-.0391	-.0166	.0009	-.0124
1.2500	-.0003	-.044	-.0075	.1283	.0054	-.0020	-.0026

TABLE C.12 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .500
SPL.......= .031
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0053	-.0270	-.0023	.0044	-.0066
.2500	-.0000	-.039	-.0002	.1076	.0000	.0000	-.0000
1.2000	-.0002		-.0076	-.0316	-.0044	.0050	-.0088
.3000	-.0000	-.043	-.0006	.1250	.0000	.0000	-.0001
1.4000	-.0004		-.0092	-.0342	-.0068	.0049	-.0103
.3500	-.0000	-.044	-.0014	.1338	.0000	-.0000	-.0002
1.6000	-.0005		-.0101	-.0354	-.0089	.0044	-.0112
.4000	-.0000	-.044	-.0025	.1366	.0001	-.0001	-.0003
1.8000	-.0006		-.0104	-.0359	-.0108	.0038	-.0118
.4500	-.0000	-.044	-.0036	.1360	.0002	-.0002	-.0004
2.0000	-.0007		-.0101	-.0359	-.0122	.0032	-.0121
.5000	-.0000	-.044	-.0048	.1336	.0004	-.0004	-.0006
3.0000	-.0009		-.0062	-.0348	-.0151	.0017	-.0123
.7500	-.0001	-.043	-.0066	.1189	.0022	-.0013	-.0014
4.0000	-.0009		-.0028	-.0340	-.0151	.0015	-.0120
1.0000	-.0002	-.042	-.0086	.1116	.0039	-.0017	-.0021
5.0000	-.0009		-.0009	-.0335	-.0148	.0016	-.0118
1.2500	-.0003	-.042	-.0069	.1089	.0049	-.0017	-.0025

TABLE C.13 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= 0.000
SPL.......= 0.000
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .500

RR RL	KHR KHL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0006		-.0272	-.1939	-.0118	-.0070	-.0043
.5000	-.0001	-.045	-.0118	.3868	.0010	-.0026	.0002
1.2000	-.0013		-.0403	-.2115	-.0235	-.0107	-.0093
.6000	-.0002	-.063	-.0233	.4195	.0034	-.0054	-.0004
1.4000	-.0020		-.0503	-.2201	-.0368	-.0139	-.0153
.7000	-.0004	-.079	-.0359	.4325	.0077	-.0086	-.0019
1.6000	-.0028		-.0564	-.2233	-.0497	-.0163	-.0215
.8000	-.0007	-.091	-.0477	.4329	.0137	-.0117	-.0041
1.8000	-.0035		-.0591	-.2234	-.0612	-.0180	-.0271
.9000	-.0011	-.101	-.0572	.4261	.0207	-.0144	-.0070
2.0000	-.0041		-.0591	-.2219	-.0707	-.0192	-.0320
1.0000	-.0015	-.109	-.0641	.4158	.0280	-.0166	-.0102
3.0000	-.0056		-.0410	-.2113	-.0943	-.0208	-.0445
1.5000	-.0032	-.125	-.0667	.3658	.0568	-.0206	-.0240
4.0000	-.0060		-.0218	-.2054	-.0992	-.0208	-.0473
2.0000	-.0039	-.127	-.0476	.3426	.0682	-.0202	-.0304
5.0000	-.0060		-.0103	-.2028	-.0999	-.0208	-.0477
2.5000	-.0042	-.128	-.0292	.3342	.0714	-.0197	-.0325

TABLE C.14 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .100
SPL.......= .025
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0091	-.0460	-.0039	-.0003	-.0034
.5000	-.0000	-.025	-.0039	.0917	.0003	-.0004	-.0004
1.2000	-.0005		-.0167	-.0677	-.0098	-.0014	-.0069
.6000	-.0001	-.038	-.0097	.1340	.0014	-.0013	-.0011
1.4000	-.0010		-.0245	-.0859	-.0178	-.0029	-.0113
.7000	-.0002	-.051	-.0174	.1680	.0037	-.0028	-.0023
1.6000	-.0015		-.0309	-.0996	-.0270	-.0045	-.0160
.8000	-.0004	-.062	-.0259	.1918	.0074	-.0046	-.0040
1.8000	-.0021		-.0355	-.1093	-.0361	-.0061	-.0205
.9000	-.0007	-.071	-.0340	.2063	.0122	-.0064	-.0062
2.0000	-.0026		-.0382	-.1158	-.0445	-.0075	-.0245
1.0000	-.0009	-.079	-.0407	.2138	.0177	-.0082	-.0087
3.0000	-.0041		-.0341	-.1252	-.0689	-.0111	-.0361
1.5000	-.0024	-.096	-.0515	.2080	.0423	-.0130	-.0200
4.0000	-.0046		-.0214	-.1250	-.0751	-.0118	-.0391
2.0000	-.0031	-.099	-.0411	.1963	.0537	-.0135	-.0258
5.0000	-.0046		-.0116	-.1246	-.0762	-.0119	-.0396
2.5000	-.0034	-.100	-.0275	.1920	.0573	-.0133	-.0279

TABLE C.15 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .200
SPL.......= .050
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0054	-.0276	-.0024	.0010	-.0033
.5000	-.0000	-.021	-.0024	.0550	.0002	-.0000	-.0005
1.2000	-.0003		-.0106	-.0427	-.0062	.0011	-.0063
.6000	-.0001	-.032	-.0061	.0844	.0009	-.0003	-.0012
1.4000	-.0006		-.0161	-.0566	-.0118	.0006	-.0100
.7000	-.0001	-.042	-.0115	.1106	.0025	-.0009	-.0024
1.6000	-.0010		-.0212	-.0681	-.0185	-.0002	-.0139
.8000	-.0003	-.052	-.0178	.1312	.0051	-.0019	-.0040
1.8000	-.0015		-.0252	-.0771	-.0257	-.0012	-.0177
.9000	-.0005	-.060	-.0241	.1456	.0087	-.0031	-.0059
2.0000	-.0019		-.0280	-.0838	-.0325	-.0023	-.0211
1.0000	-.0007	-.066	-.0298	.1547	.0129	-.0043	-.0080
3.0000	-.0033		-.0278	-.0965	-.0548	-.0063	-.0312
1.5000	-.0019	-.082	-.0414	.1592	.0338	-.0087	-.0176
4.0000	-.0037		-.0190	-.0976	-.0615	-.0075	-.0341
2.0000	-.0026	-.086	-.0351	.1507	.0445	-.0098	-.0227
5.0000	-.0038		-.0111	-.0975	-.0630	-.0078	-.0346
2.5000	-.0028	-.086	-.0247	.1467	.0482	-.0098	-.0246

TABLE C.16 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .300
SPL......= .075
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0039	-.0197	-.0017	.0016	-.0032
.5000	-.0000	-.020	-.0017	.0393	.0001	.0002	-.0005
1.2000	-.0002		-.0077	-.0312	-.0045	.0022	-.0060
.6000	-.0000	-.029	-.0044	.0616	.0007	.0002	-.0013
1.4000	-.0005		-.0120	-.0422	-.0088	.0024	-.0094
.7000	-.0001	-.038	-.0086	.0825	.0018	-.0000	-.0025
1.6000	-.0008		-.0162	-.0518	-.0141	.0021	-.0128
.8000	-.0002	-.046	-.0136	.0997	.0039	-.0005	-.0040
1.8000	-.0011		-.0196	-.0597	-.0199	.0015	-.0161
.9000	-.0004	-.053	-.0167	.1126	.0067	-.0013	-.0057
2.0000	-.0015		-.0221	-.0658	-.0256	.0006	-.0191
1.0000	-.0005	-.059	-.0234	.1215	.0102	-.0021	-.0076
3.0000	-.0027		-.0234	-.0792	-.0455	-.0032	-.0280
1.5000	-.0016	-.073	-.0346	.1302	.0282	-.0059	-.0160
4.0000	-.0032		-.0168	-.0811	-.0523	-.0047	-.0306
2.0000	-.0022	-.076	-.0304	.1241	.0381	-.0073	-.0205
5.0000	-.0033		-.0103	-.0812	-.0540	-.0051	-.0311
2.5000	-.0025	-.077	-.0221	.1206	.0417	-.0075	-.0222

TABLE C.17 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR....... = .400
SPL....... = .100
PL........ = 1.000
PR........ = 0.000
POIS.R.... = .300
K......... = .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0030	-.0153	-.0013	.0019	-.0031
.5000	-.0000	-.019	-.0013	.0305	.0001	.0003	-.0006
1.2000	-.0002		-.0061	-.0245	-.0035	.0028	-.0059
.6000	-.0000	-.028	-.0035	.0485	.0005	.0005	-.0013
1.4000	-.0004		-.0096	-.0336	-.0070	.0034	-.0090
.7000	-.0001	-.036	-.0068	.0658	.0015	.0005	-.0025
1.6000	-.0006		-.0130	-.0418	-.0114	.0035	-.0121
.8000	-.0002	-.043	-.0109	.0805	.0031	.0003	-.0040
1.8000	-.0009		-.0160	-.0487	-.0162	.0032	-.0151
.9000	-.0003	-.049	-.0153	.0919	.0055	-.0001	-.0056
2.0000	-.0012		-.0182	-.0542	-.0211	.0026	-.0178
1.0000	-.0005	-.054	-.0193	.1000	.0084	-.0007	-.0073
3.0000	-.0023		-.0201	-.0672	-.0390	-.0010	-.0257
1.5000	-.0013	-.067	-.0296	.1104	.0241	-.0039	-.0149
4.0000	-.0028		-.0149	-.0696	-.0455	-.0027	-.0280
2.0000	-.0019	-.070	-.0268	.1060	.0333	-.0054	-.0188
5.0000	-.0029		-.0095	-.0699	-.0473	-.0032	-.0285
2.5000	-.0022	-.070	-.0199	.1030	.0368	-.0058	-.0203

TABLE C.18 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .500
SPL......= .125
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0025	-.0125	-.0011	.0021	-.0031
.5000	-.0000	-.018	-.0011	.0250	.0001	.0004	-.0006
1.2000	-.0002		-.0050	-.0202	-.0029	.0033	-.0058
.6000	-.0000	-.026	-.0029	.0400	.0004	.0007	-.0014
1.4000	-.0003		-.0080	-.0283	-.0058	.0041	-.0087
.7000	-.0001	-.034	-.0057	.0547	.0012	.0009	-.0025
1.6000	-.0005		-.0109	-.0350	-.0096	.0044	-.0117
.8000	-.0001	-.041	-.0092	.0674	.0026	.0009	-.0039
1.8000	-.0008		-.0135	-.0411	-.0137	.0043	-.0145
.9000	-.0002	-.047	-.0129	.0776	.0046	.0007	-.0055
2.0000	-.0010		-.0155	-.0461	-.0180	.0039	-.0169
1.0000	-.0004	-.051	-.0165	.0853	.0072	.0003	-.0071
3.0000	-.0020		-.0177	-.0585	-.0341	.0007	-.0239
1.5000	-.0012	-.062	-.0260	.0959	.0211	-.0024	-.0140
4.0000	-.0025		-.0134	-.0611	-.0403	-.0012	-.0260
2.0000	-.0017	-.065	-.0239	.0927	.0295	-.0040	-.0175
5.0000	-.0026		-.0087	-.0615	-.0421	-.0018	-.0264
2.5000	-.0019	-.065	-.0180	.0902	.0329	-.0045	-.0188

TABLE C.19 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= 0.000
SPL.......= 0.000
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0013		-.0574	-.3819	-.0251	-.0149	-.0091
.5000	-.0002	-.098	-.0250	.7617	.0022	-.0056	.0004
1.2000	-.0019		-.0599	-.3508	-.0357	-.0165	-.0140
.6000	-.0003	-.102	-.0352	.6964	.0052	-.0081	-.0006
1.4000	-.0024		-.0558	-.3267	-.0429	-.0168	-.0177
.7000	-.0005	-.103	-.0415	.6444	.0089	-.0100	-.0021
1.6000	-.0026		-.0484	-.3101	-.0470	-.0163	-.0261
.8000	-.0007	-.102	-.0441	.6074	.0128	-.0110	-.0038
1.8000	-.0028		-.0400	-.2992	-.0489	-.0158	-.0214
.9000	-.0009	-.101	-.0439	.5822	.0162	-.0115	-.0054
2.0000	-.0029		-.0319	-.2923	-.0496	-.0152	-.0220
1.0000	-.0010	-.099	-.0418	.5655	.0190	-.0116	-.0068
3.0000	-.0028		-.0060	-.2811	-.0481	-.0141	-.0217
1.5000	-.0014	-.096	-.0227	.5371	.0251	-.0105	-.0102
4.0000	-.0028		.0008	-.2790	-.0472	-.0140	-.0213
2.0000	-.0015	-.095	-.0090	.5322	.0258	-.0099	-.0109
5.0000	-.0028		.0013	-.2777	-.0470	-.0140	-.0212
2.5000	-.0015	-.095	-.0029	.5295	.0258	-.0098	-.0110

TABLE C.20 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .100
SPL......= .025
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0004		-.0193	-.0996	-.0084	-.0007	-.0073
.5000	-.0001	-.054	-.0083	.1984	.0007	-.0009	-.0008
1.2000	-.0008		-.0253	-.1070	-.0148	-.0021	-.0105
.6000	-.0001	-.061	-.0146	.2119	.0021	-.0020	-.0016
1.4000	-.0011		-.0281	-.1076	-.0207	-.0033	-.0132
.7000	-.0002	-.064	-.0202	.2108	.0043	-.0032	-.0026
1.6000	-.0014		-.0283	-.1051	-.0253	-.0041	-.0152
.8000	-.0004	-.065	-.0242	.2033	.0070	-.0043	-.0038
1.8000	-.0016		-.0268	-.1020	-.0285	-.0047	-.0165
.9000	-.0005	-.065	-.0265	.1943	.0096	-.0051	-.0049
2.0000	-.0018		-.0243	-.0990	-.0305	-.0049	-.0172
1.0000	-.0006	-.065	-.0272	.1860	.0120	-.0056	-.0059
3.0000	-.0019		-.0102	-.0925	-.0321	-.0049	-.0178
1.5000	-.0010	-.063	-.0201	.1666	.0185	-.0059	-.0089
4.0000	-.0019		-.0025	-.0918	-.0314	-.0047	-.0175
2.0000	-.0011	-.062	-.0105	.1640	.0197	-.0055	-.0096
5.0000	-.0019		.0000	-.0918	-.0312	-.0047	-.0174
2.5000	-.0012	-.062	-.0044	.1638	.0198	-.0053	-.0098

TABLE C.21 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .200
SPL......= .050
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0003		-.0116	-.0595	-.0050	.0021	-.0069
.5000	-.0000	-.046	-.0050	.1186	.0004	-.0000	-.0010
1.2000	-.0005		-.0159	-.0669	-.0093	.0017	-.0096
.6000	-.0001	-.051	-.0092	.1325	.0014	-.0004	-.0019
1.4000	-.0008		-.0166	-.0698	-.0137	.0008	-.0117
.7000	-.0002	-.054	-.0133	.1367	.0029	-.0011	-.0028
1.6000	-.0010		-.0196	-.0701	-.0174	.0000	-.0133
.8000	-.0003	-.055	-.0166	.1354	.0048	-.0018	-.0037
1.8000	-.0012		-.0192	-.0692	-.0202	-.0007	-.0143
.9000	-.0004	-.055	-.0188	.1317	.0068	-.0024	-.0046
2.0000	-.0013		-.0181	-.0680	-.0222	-.0012	-.0149
1.0000	-.0005	-.055	-.0200	.1273	.0088	-.0029	-.0055
3.0000	-.0015		-.0093	-.0640	-.0251	-.0019	-.0156
1.5000	-.0008	-.054	-.0166	.1132	.0147	-.0038	-.0079
4.0000	-.0015		-.0033	-.0631	-.0248	-.0019	-.0155
2.0000	-.0009	-.054	-.0097	.1101	.0162	-.0036	-.0086
5.0000	-.0015		-.0007	-.0630	-.0247	-.0019	-.0154
2.5000	-.0010	-.053	-.0047	.1097	.0164	-.0035	-.0088

TABLE C.22 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .300
SPL......= .075
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0083	-.0425	-.0036	.0033	-.0068
.5000	-.0000	-.043	-.0036	.0846	.0003	.0004	-.0011
1.2000	-.0004		-.0117	-.0487	-.0068	.0034	-.0092
.6000	-.0001	-.047	-.0067	.0965	.0010	.0003	-.0020
1.4000	-.0006		-.0139	-.0518	-.0102	.0029	-.0110
.7000	-.0001	-.049	-.0100	.1014	.0021	-.0000	-.0029
1.6000	-.0007		-.0149	-.0528	-.0132	.0022	-.0122
.8000	-.0002	-.050	-.0127	.1020	.0036	-.0005	-.0037
1.8000	-.0009		-.0150	-.0528	-.0157	.0015	-.0131
.9000	-.0003	-.050	-.0146	.1003	.0053	-.0010	-.0045
2.0000	-.0010		-.0144	-.0523	-.0175	.0009	-.0136
1.0000	-.0004	-.050	-.0158	.0977	.0069	-.0014	-.0052
3.0000	-.0012		-.0081	-.0499	-.0207	-.0002	-.0142
1.5000	-.0007	-.049	-.0140	.0875	.0122	-.0024	-.0072
4.0000	-.0012		-.0033	-.0492	-.0207	-.0003	-.0141
2.0000	-.0008	-.049	-.0087	.0846	.0137	-.0025	-.0079
5.0000	-.0012		-.0010	-.0491	-.0206	-.0003	-.0140
2.5000	-.0008	-.049	-.0046	.0841	.0140	-.0024	-.0080

TABLE C.23 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .400
SPL.......= .100
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0064	-.0330	-.0028	.0040	-.0067
.5000	-.0000	-.041	-.0028	.0658	.0002	.0006	-.0012
1.2000	-.0003		-.0092	-.0383	-.0054	.0044	-.0089
.6000	-.0000	-.044	-.0053	.0759	.0008	.0007	-.0020
1.4000	-.0004		-.0111	-.0412	-.0081	.0041	-.0105
.7000	-.0001	-.046	-.0079	.0806	.0017	.0006	-.0029
1.6000	-.0006		-.0121	-.0424	-.0107	.0035	-.0116
.8000	-.0002	-.047	-.0102	.0818	.0029	.0003	-.0037
1.8000	-.0007		-.0123	-.0427	-.0128	.0029	-.0123
.9000	-.0002	-.047	-.0119	.0811	.0043	-.0000	-.0044
2.0000	-.0008		-.0119	-.0426	-.0144	.0023	-.0127
1.0000	-.0003	-.047	-.0130	.0795	.0057	-.0004	-.0050
3.0000	-.0011		-.0072	-.0412	-.0176	.0009	-.0132
1.5000	-.0006	-.046	-.0121	.0718	.0105	-.0015	-.0067
4.0000	-.0011		-.0032	-.0406	-.0179	.0007	-.0131
2.0900	-.0007	-.045	-.0078	.0693	.0120	-.0017	-.0073
5.0000	-.0011		-.0012	-.0405	-.0178	.0007	-.0131
2.5000	-.0007	-.045	-.0043	.0688	.0123	-.0016	-.0075

TABLE C.24 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .500
SPL.......= .125
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0053	-.0270	-.0023	.0044	-.0066
.5000	-.0000	-.039	-.0023	.0538	.0002	.0007	-.0012
1.2000	-.0002		-.0076	-.0316	-.0044	.0050	-.0088
.6000	-.0000	-.043	-.0044	.0625	.0006	.0010	-.0021
1.4000	-.0004		-.0092	-.0342	-.0068	.0049	-.0103
.7000	-.0001	-.044	-.0066	.0669	.0014	.0010	-.0029
1.6000	-.0005		-.0101	-.0354	-.0089	.0044	-.0112
.8000	-.0001	-.044	-.0086	.0684	.0025	.0009	-.0037
1.8000	-.0006		-.0104	-.0359	-.0108	.0038	-.0118
.9000	-.0002	-.044	-.0101	.0682	.0036	.0006	-.0044
2.0000	-.0007		-.0101	-.0360	-.0123	.0032	-.0121
1.0000	-.0003	-.044	-.0111	.0671	.0049	.0003	-.0049
3.0000	-.0009		-.0064	-.0351	-.0153	.0017	-.0124
1.5000	-.0005	-.043	-.0106	.0611	.0092	-.0008	-.0064
4.0000	-.0009		-.0030	-.0347	-.0157	.0015	-.0123
2.0000	-.0006	-.043	-.0071	.0589	.0106	-.0011	-.0069
5.0000	-.0009		-.0012	-.0346	-.0157	.0015	-.0123
2.5000	-.0006	-.043	-.0040	.0583	.0109	-.0011	-.0070

TABLE C.25 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= 0.000
SPL......= 0.000
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0012		-.0496	-.3157	-.0216	-.0127	-.0078
.7500	-.0006	-.082	-.0403	.4198	.0101	-.0097	-.0028
1.2000	-.0019		-.0596	-.3062	-.0350	-.0161	-.0138
.9000	-.0010	-.095	-.0537	.4055	.0194	-.0135	-.0066
1.4000	-.0026		-.0629	-.2930	-.0468	-.0179	-.0195
1.0500	-.0016	-.104	-.0614	.3856	.0292	-.0161	-.0109
1.6000	-.0031		-.0613	-.2809	-.0559	-.0187	-.0241
1.2000	-.0021	-.108	-.0639	.3669	.0380	-.0176	-.0150
1.8000	-.0036		-.0568	-.2711	-.0624	-.0190	-.0275
1.3500	-.0025	-.111	-.0627	.3515	.0452	-.0183	-.0185
2.0000	-.0039		-.0509	-.2638	-.0668	-.0190	-.0300
1.5000	-.0028	-.113	-.0590	.3398	.0506	-.0185	-.0214
3.0000	-.0044		-.0220	-.2483	-.0739	-.0184	-.0343
2.2500	-.0036	-.115	-.0319	.3141	.0617	-.0177	-.0278
4.0000	-.0044		-.0075	-.2447	-.0745	-.0182	-.0347
3.0000	-.0037	-.115	-.0134	.3084	.0632	-.0173	-.0289
5.0000	-.0044		-.0023	-.2434	-.0745	-.0182	-.0347
3.7500	-.0038	-.115	-.0050	.3066	.0634	-.0172	-.0290

TABLE C.26 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .100
SPL.......= .056
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0004		-.0166	-.0842	-.0072	-.0006	-.0063
.7500	-.0002	-.046	-.0134	.1120	.0034	-.0010	-.0031
1.2000	-.0008		-.0248	-.1016	-.0145	-.0020	-.0103
.9000	-.0004	-.057	-.0223	.1344	.0080	-.0025	-.0058
1.4000	-.0012		-.0307	-.1110	-.0225	-.0036	-.0143
1.0500	-.0008	-.066	-.0297	.1455	.0141	-.0042	-.0088
1.6000	-.0017		-.0339	-.1150	-.0299	-.0050	-.0178
1.2000	-.0011	-.071	-.0348	.1492	.0204	-.0057	-.0117
1.8000	-.0021		-.0348	-.1159	-.0360	-.0060	-.0206
1.3500	-.0014	-.075	-.0375	.1487	.0262	-.0068	-.0144
2.0000	-.0024		-.0340	-.1153	-.0407	-.0068	-.0227
1.5000	-.0017	-.077	-.0381	.1464	.0311	-.0076	-.0166
3.0000	-.0030		-.0210	-.1109	-.0501	-.0079	-.0269
2.2500	-.0025	-.080	-.0274	.1358	.0430	-.0087	-.0222
4.0000	-.0031		-.0094	-.1099	-.0512	-.0079	-.0274
3.0000	-.0027	-.080	-.0142	.1333	.0450	-.0086	-.0233
5.0000	-.0031		-.0036	-.1097	-.0513	-.0079	-.0275
3.7500	-.0027	-.080	-.0062	.1330	.0453	-.0085	-.0235

TABLE C.27 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .200
SPL.......= .113
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KWC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0099	-.0505	-.0043	.0018	-.0059
.7500	-.0001	-.039	-.0061	.0671	.0020	.0007	-.0032
1.2800	-.0005		-.0156	-.0638	-.0091	.0016	-.0094
.9000	-.0003	-.048	-.0140	.0844	.0051	.0003	-.0056
1.4800	-.0008		-.0202	-.0726	-.0148	.0008	-.0126
1.0500	-.0005	-.055	-.0196	.0951	.0093	-.0004	-.0081
1.6000	-.0011		-.0232	-.0776	-.0204	-.0001	-.0154
1.2000	-.0008	-.060	-.0238	.1006	.0139	-.0014	-.0105
1.8000	-.0014		-.0247	-.0801	-.0254	-.0011	-.0176
1.3500	-.0010	-.063	-.0265	.1027	.0185	-.0023	-.0126
2.0000	-.0017		-.0248	-.0811	-.0294	-.0019	-.0193
1.5000	-.0013	-.065	-.0277	.1027	.0225	-.0031	-.0144
3.0000	-.0023		-.0173	-.0799	-.0387	-.0038	-.0229
2.2500	-.0019	-.067	-.0221	.0971	.0333	-.0049	-.0189
4.0000	-.0024		-.0088	-.0790	-.0401	-.0041	-.0234
3.0000	-.0021	-.068	-.0127	.0949	.0356	-.0051	-.0199
5.0000	-.0024		-.0038	-.0789	-.0403	-.0041	-.0235
3.7500	-.0022	-.068	-.0062	.0945	.0360	-.0051	-.0201

TABLE C.28 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .300
SPL.......= .169
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .750

RR PL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0071	-.0360	-.0031	.0029	-.0058
.7500	-.0001	-.036	-.0058	.0479	.0014	.0015	-.0032
1.2000	-.0004		-.0114	-.0465	-.0067	.0033	-.0089
.9000	-.0002	-.044	-.0102	.0615	.0037	.0017	-.0055
1.4000	-.0006		-.0151	-.0540	-.0110	.0030	-.0118
1.0500	-.0004	-.050	-.0146	.0707	.0069	.0014	-.0078
1.6000	-.0009		-.0177	-.0587	-.0155	.0024	-.0142
1.2000	-.0006	-.053	-.0181	.0761	.0106	.0008	-.0099
1.8000	-.0011		-.0191	-.0614	-.0196	.0016	-.0161
1.3500	-.0008	-.056	-.0205	.0787	.0143	.0001	-.0117
2.0000	-.0013		-.0195	-.0628	-.0230	.0008	-.0175
1.5000	-.0010	-.058	-.0217	.0795	.0176	-.0006	-.0132
3.0000	-.0019		-.0145	-.0632	-.0315	-.0014	-.0203
2.2500	-.0016	-.060	-.0184	.0766	.0273	-.0026	-.0169
4.0000	-.0020		-.0079	-.0627	-.0331	-.0019	-.0208
3.0000	-.0018	-.060	-.0111	.0748	.0296	-.0030	-.0177
5.0000	-.0020		-.0037	-.0625	-.0334	-.0020	-.0208
3.7500	-.0018	-.060	-.0057	.0744	.0300	-.0031	-.0178

TABLE C.29 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .400
SPL.......= .225
PL........= 1.000
PR........= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0055	-.0280	-.0024	.0034	-.0057
.7500	-.0001	-.034	-.0045	.0372	.0011	.0019	-.0033
1.2000	-.0003		-.0090	-.0366	-.0052	.0042	-.0087
.9000	-.0002	-.041	-.0081	.0484	.0029	.0024	-.0054
1.4000	-.0005		-.0120	-.0430	-.0088	.0043	-.0113
1.0500	-.0003	-.046	-.0116	.0563	.0055	.0025	-.0076
1.6000	-.0007		-.0142	-.0472	-.0125	.0039	-.0134
1.2000	-.0005	-.050	-.0146	.0612	.0085	.0022	-.0095
1.8000	-.0009		-.0156	-.0498	-.0160	.0033	-.0151
1.3500	-.0006	-.052	-.0167	.0638	.0116	.0017	-.0111
2.0000	-.0011		-.0161	-.0513	-.0189	.0026	-.0162
1.5000	-.0008	-.053	-.0179	.0650	.0145	.0011	-.0123
3.0000	-.0016		-.0124	-.0525	-.0267	.0002	-.0186
2.2500	-.0013	-.055	-.0157	.0635	.0231	-.0010	-.0154
4.0000	-.0017		-.0070	-.0522	-.0283	-.0004	-.0189
3.0000	-.0015	-.055	-.0098	.0621	.0253	-.0016	-.0161
5.0000	-.0017		-.0035	-.0521	-.0285	-.0005	-.0189
3.7500	-.0015	-.055	-.0053	.0617	.0257	-.0017	-.0162

TABLE C.30 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR....... = .500
SPL....... = .281
PL........ = 1.000
PR........ = 0.000
POIS.R.... = .300
K......... = .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0045	-.0229	-.0020	.0038	-.0057
.7500	-.0001	-.033	-.0037	.0305	.0009	.0021	-.0033
1.2000	-.0002		-.0074	-.0302	-.0043	.0048	-.0085
.9000	-.0001	-.040	-.0067	.0399	.0024	.0029	-.0054
1.4000	-.0004		-.0100	-.0357	-.0073	.0052	-.0110
1.0500	-.0002	-.044	-.0097	.0468	.0046	.0032	-.0074
1.6000	-.0006		-.0119	-.0395	-.0105	.0050	-.0129
1.2000	-.0004	-.047	-.0122	.0512	.0071	.0031	-.0092
1.8000	-.0008		-.0131	-.0419	-.0135	.0044	-.0144
1.3500	-.0005	-.049	-.0141	.0537	.0098	.0027	-.0106
2.0000	-.0009		-.0137	-.0434	-.0161	.0038	-.0154
1.5000	-.0007	-.050	-.0152	.0549	.0123	.0022	-.0118
3.0000	-.0014		-.0159	-.0450	-.0231	.0014	-.0173
2.2500	-.0012	-.051	-.0137	.0544	.0200	.0001	-.0144
4.0000	-.0015		-.0063	-.0448	-.0247	.0007	-.0175
3.0000	-.0013	-.051	-.0088	.0532	.0221	-.0006	-.0149
5.0000	-.0015		-.0032	-.0447	-.0250	.0005	-.0175
3.7500	-.0014	-.051	-.0048	.0529	.0226	-.0007	-.0150

TABLE C.31 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= 0.008
SPL.......= 0.000
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0013		-.0575	-.3824	-.0252	-.0149	-.0091
.7500	-.0006	-.098	-.0469	.5086	.0117	-.0114	-.0032
1.2000	-.0019		-.0602	-.3521	-.0359	-.0166	-.0141
.9000	-.0011	-.103	-.0547	.4666	.0199	-.0139	-.0067
1.4000	-.0024		-.0565	-.3292	-.0434	-.0169	-.0180
1.0500	-.0015	-.104	-.0560	.4341	.0270	-.0151	-.0100
1.6000	-.0027		-.0496	-.3138	-.0480	-.0167	-.0205
1.2000	-.0018	-.104	-.0532	.4119	.0325	-.0153	-.0127
1.8000	-.0029		-.0416	-.3040	-.0506	-.0162	-.0221
1.3500	-.0020	-.103	-.0480	.3973	.0363	-.0152	-.0147
2.0000	-.0030		-.0338	-.2979	-.0519	-.0158	-.0230
1.5000	-.0022	-.103	-.0418	.3881	.0388	-.0148	-.0162
3.0000	-.0031		-.0084	-.2885	-.0526	-.0150	-.0239
2.2500	-.0025	-.101	-.0155	.3733	.0423	-.0137	-.0186
4.0000	-.0031		-.0011	-.2866	-.0525	-.0150	-.0238
3.0000	-.0025	-.101	-.0042	.3705	.0425	-.0135	-.0189
5.0000	-.0031		.0001	-.2852	-.0525	-.0150	-.0238
3.7500	-.0025	-.101	-.0009	.3686	.0425	-.0134	-.0189

TABLE C.32 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .100
SPL.......= .056
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0004		-.0193	-.0996	-.0084	-.0007	-.0073
.7500	-.0002	-.054	-.0156	.1324	.0039	-.0012	-.0037
1.2000	-.0008		-.0253	-.1072	-.0148	-.0021	-.0106
.9000	-.0004	-.061	-.0228	.1417	.0082	-.0026	-.0059
1.4000	-.0011		-.0262	-.1080	-.0208	-.0033	-.0133
1.0500	-.0007	-.064	-.0274	.1418	.0130	-.0039	-.0081
1.6000	-.0014		-.0286	-.1060	-.0256	-.0042	-.0153
1.2000	-.0009	-.066	-.0296	.1379	.0174	-.0048	-.0101
1.8000	-.0016		-.0273	-.1033	-.0289	-.0047	-.0167
1.3500	-.0012	-.066	-.0297	.1331	.0210	-.0055	-.0116
2.0000	-.0018		-.0250	-.1009	-.0312	-.0050	-.0176
1.5000	-.0013	-.066	-.0284	.1289	.0238	-.0058	-.0128
3.0000	-.0020		-.0114	-.0959	-.0342	-.0052	-.0189
2.2500	-.0017	-.066	-.0159	.1199	.0288	-.0059	-.0152
4.0000	-.0020		-.0035	-.0955	-.0342	-.0051	-.0189
3.0000	-.0017	-.065	-.0064	.1191	.0293	-.0057	-.0155
5.0000	-.0020		-.0007	-.0955	-.0342	-.0051	-.0189
3.7500	-.0017	-.065	-.0021	.1190	.0293	-.0057	-.0155

TABLE C.33 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .200
SPL......= .113
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0003		-.0116	-.0595	-.0050	.0021	-.0069
.7500	-.0001	-.046	-.0094	.0791	.0023	.0008	-.0037
1.2000	-.0005		-.0160	-.0670	-.0093	.0017	-.0096
.9000	-.0003	-.051	-.0144	.0886	.0092	.0004	-.0057
1.4000	-.0008		-.0186	-.0699	-.0137	.0008	-.0118
1.0500	-.0005	-.054	-.0181	.0918	.0086	-.0004	-.0075
1.6000	-.0010		-.0197	-.0704	-.0175	.0000	-.0133
1.2000	-.0006	-.055	-.0202	.0915	.0119	-.0011	-.0090
1.8000	-.0012		-.0194	-.0698	-.0204	-.0007	-.0144
1.3500	-.0008	-.056	-.0210	.0898	.0148	-.0018	-.0102
2.0000	-.0013		-.0184	-.0688	-.0225	-.0012	-.0151
1.5000	-.0010	-.056	-.0208	.0877	.0172	-.0023	-.0111
3.0000	-.0016		-.0099	-.0657	-.0262	-.0021	-.0162
2.2500	-.0013	-.056	-.0133	.0814	.0223	-.0031	-.0131
4.0000	-.0016		-.0039	-.0652	-.0264	-.0021	-.0163
3.0000	-.0014	-.055	-.0062	.0804	.0229	-.0030	-.0134
5.0000	-.0016		-.0012	-.0652	-.0264	-.0021	-.0163
3.7500	-.0014	-.055	-.0024	.0803	.0230	-.0030	-.0134

TABLE C.34 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .300
SPL.......= .169
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0083	-.0425	-.0036	.0033	-.0068
.7500	-.0001	-.043	-.0067	.0565	.0017	.0017	-.0038
1.2000	-.0004		-.0117	-.0487	-.0068	.0034	-.0092
.9000	-.0002	-.047	-.0165	.0644	.0038	.0017	-.0056
1.4000	-.0006		-.0139	-.0518	-.0102	.0029	-.0110
1.0500	-.0003	-.049	-.0135	.0680	.0064	.0013	-.0072
1.6000	-.0007		-.0150	-.0529	-.0133	.0022	-.0123
1.2000	-.0005	-.050	-.0154	.0688	.0090	.0008	-.0085
1.8000	-.0009		-.0151	-.0530	-.0157	.0015	-.0131
1.3500	-.0006	-.050	-.0163	.0682	.0114	.0002	-.0095
2.0000	-.0010		-.0145	-.0527	-.0176	.0009	-.0137
1.5000	-.0007	-.050	-.0163	.0670	.0134	-.0003	-.0102
3.0000	-.0013		-.0085	-.0508	-.0213	-.0003	-.0146
2.2500	-.0011	-.050	-.0112	.0627	.0182	-.0014	-.0117
4.0000	-.0013		-.0037	-.0504	-.0217	-.0004	-.0146
3.0000	-.0011	-.050	-.0057	.0617	.0190	-.0015	-.0120
5.0000	-.0013		-.0014	-.0504	-.0217	-.0004	-.0146
3.7500	-.0011	-.050	-.0024	.0616	.0190	-.0015	-.0120

TABLE C.35 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .400
SPL.......= .225
PL........= 0.000
PR........= 1.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0064	-.0330	-.0028	.0040	-.0067
.7500	-.0001	-.041	-.0052	.0439	.0013	.0022	-.0038
1.2000	-.0003		-.0092	-.0383	-.0054	.0044	-.0089
.9000	-.0002	-.044	-.0083	.0507	.0030	.0025	-.0056
1.4000	-.0004		-.0111	-.0412	-.0081	.0041	-.0106
1.0500	-.0003	-.046	-.0107	.0540	.0051	.0023	-.0070
1.6000	-.0006		-.0121	-.0424	-.0107	.0035	-.0116
1.2000	-.0004	-.047	-.0124	.0551	.0073	.0019	-.0081
1.8000	-.0007		-.0123	-.0428	-.0128	.0029	-.0123
1.3500	-.0005	-.047	-.0133	.0550	.0093	.0014	-.0090
2.0000	-.0008		-.0120	-.0428	-.0145	.0023	-.0128
1.5000	-.0006	-.047	-.0134	.0544	.0111	.0010	-.0096
3.0000	-.0011		-.0073	-.0417	-.0179	.0009	-.0134
2.2500	-.0009	-.046	-.0096	.0513	.0154	-.0003	-.0108
4.0000	-.0011		-.0034	-.0414	-.0184	.0007	-.0134
3.0000	-.0010	-.046	-.0051	.0504	.0162	-.0005	-.0110
5.0000	-.0011		-.0014	-.0413	-.0185	.0007	-.0134
3.7500	-.0010	-.046	-.0023	.0503	.0163	-.0005	-.0110

TABLE C.36 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .500
SPL......= .261
PL.......= 0.000
PR.......= 1.000
POIS.R...= .300
K........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0053	-.0270	-.0023	.0044	-.0066
.7500	-.0001	-.039	-.0043	.0359	.0011	.0025	-.0038
1.2000	-.0002		-.0076	-.0316	-.0044	.0050	-.0088
.9000	-.0001	-.043	-.0068	.0417	.0025	.0030	-.0055
1.4000	-.0004		-.0092	-.0342	-.0068	.0049	-.0103
1.0500	-.0002	-.044	-.0089	.0448	.0042	.0030	-.0069
1.6000	-.0005		-.0101	-.0354	-.0090	.0044	-.0112
1.2000	-.0003	-.044	-.0104	.0460	.0061	.0027	-.0079
1.8000	-.0006		-.0104	-.0359	-.0108	.0038	-.0118
1.3500	-.0004	-.045	-.0112	.0462	.0079	.0023	-.0086
2.0000	-.0007		-.0102	-.0360	-.0123	.0032	-.0121
1.5000	-.0005	-.044	-.0114	.0458	.0094	.0018	-.0091
3.0000	-.0009		-.0065	-.0354	-.0155	.0017	-.0125
2.2500	-.0008	-.044	-.0064	.0435	.0133	.0005	-.0101
4.0000	-.0010		-.0031	-.0352	-.0160	.0015	-.0125
3.0000	-.0008	-.043	-.0046	.0428	.0141	.0002	-.0102
5.0000	-.0010		-.0013	-.0351	-.0161	.0014	-.0125
3.7500	-.0009	-.043	-.0022	.0426	.0142	.0002	-.0102

TABLE C.37 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= 0.000
SPL......= 0.000
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0014		-.0580	-.3848	-.0254	-.0150	-.0092
1.0000	-.0014	-.099	-.0580	.3848	.0254	-.0150	-.0092
1.2000	-.0020		-.0613	-.3566	-.0365	-.0169	-.0144
1.2000	-.0020	-.105	-.0613	.3566	.0365	-.0169	-.0144
1.4000	-.0025		-.0583	-.3354	-.0447	-.0174	-.0185
1.4000	-.0025	-.107	-.0583	.3354	.0447	-.0174	-.0185
1.6000	-.0028		-.0519	-.3212	-.0500	-.0173	-.0214
1.6000	-.0028	-.108	-.0519	.3212	.0500	-.0173	-.0214
1.8000	-.0030		-.0443	-.3121	-.0533	-.0170	-.0233
1.8000	-.0030	-.108	-.0443	.3121	.0533	-.0170	-.0233
2.0000	-.0032		-.0367	-.3064	-.0552	-.0167	-.0245
2.0000	-.0032	-.108	-.0367	.3064	.0552	-.0167	-.0245
3.0000	-.0034		-.0108	-.2969	-.0574	-.0160	-.0261
3.0000	-.0034	-.107	-.0108	.2969	.0574	-.0160	-.0261
4.0000	-.0034		-.0025	-.2948	-.0575	-.0160	-.0262
4.0000	-.0034	-.107	-.0025	.2948	.0575	-.0160	-.0262
5.0000	-.0034		-.0005	-.2934	-.0575	-.0160	-.0262
5.0000	-.0034	-.107	-.0005	.2934	.0575	-.0160	-.0262

TABLE C.38 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= .100
SPL.......= .100
PL........= 1.000
PR........= 0.000
POIS.R...= .300
K.........= 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0004		-.0193	-.0998	-.0084	-.0007	-.0073
1.0000	-.0004	-.054	-.0193	.0998	.0084	-.0007	-.0073
1.2000	-.0008		-.0254	-.1077	-.0149	-.0021	-.0106
1.2000	-.0008	-.061	-.0254	.1077	.0149	-.0021	-.0106
1.4000	-.0012		-.0285	-.1090	-.0210	-.0033	-.0134
1.4000	-.0012	-.065	-.0285	.1090	.0210	-.0033	-.0134
1.6000	-.0015		-.0291	-.1075	-.0260	-.0043	-.0156
1.6000	-.0015	-.067	-.0291	.1075	.0260	-.0043	-.0156
1.8000	-.0017		-.0279	-.1052	-.0296	-.0048	-.0171
1.8000	-.0017	-.068	-.0279	.1052	.0296	-.0048	-.0171
2.0000	-.0019		-.0257	-.1031	-.0321	-.0052	-.0181
2.0000	-.0019	-.068	-.0257	.1031	.0321	-.0052	-.0181
3.0000	-.0021		-.0123	-.0988	-.0359	-.0055	-.0198
3.0000	-.0021	-.068	-.0123	.0988	.0359	-.0055	-.0198
4.0000	-.0022		-.0043	-.0984	-.0362	-.0055	-.0200
4.0000	-.0022	-.068	-.0043	.0984	.0362	-.0055	-.0200
5.0000	-.0022		-.0012	-.0984	-.0362	-.0054	-.0200
5.0000	-.0022	-.068	-.0012	.0984	.0362	-.0054	-.0200

TABLE C.39 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .200
SPL......= .200
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0003		-.0116	-.0596	-.0050	.0021	-.0069
1.0000	-.0003	-.046	-.0116	.0596	.0050	.0021	-.0069
1.2000	-.0005		-.0160	-.0671	-.0093	.0017	-.0096
1.2000	-.0005	-.051	-.0160	.0671	.0093	.0017	-.0096
1.4000	-.0008		-.0187	-.0702	-.0137	.0008	-.0118
1.4000	-.0008	-.054	-.0187	.0702	.0137	.0008	-.0118
1.6000	-.0010		-.0198	-.0708	-.0176	.0000	-.0134
1.6000	-.0010	-.056	-.0198	.0708	.0176	.0000	-.0134
1.8000	-.0012		-.0196	-.0703	-.0206	-.0007	-.0145
1.8000	-.0012	-.056	-.0196	.0703	.0206	-.0007	-.0145
2.0000	-.0013		-.0186	-.0695	-.0228	-.0012	-.0153
2.0000	-.0013	-.057	-.0186	.0695	.0228	-.0012	-.0153
3.0000	-.0016		-.0103	-.0668	-.0269	-.0021	-.0166
3.0000	-.0016	-.057	-.0103	.0668	.0269	-.0021	-.0166
4.0000	-.0016		-.0042	-.0664	-.0273	-.0022	-.0168
4.0000	-.0016	-.057	-.0042	.0664	.0273	-.0022	-.0168
5.0000	-.0016		-.0015	-.0664	-.0273	-.0022	-.0168
5.0000	-.0016	-.056	-.0015	.0664	.0273	-.0022	-.0168

TABLE C.40 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .300
SPL......= .300
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0083	-.0425	-.0036	.0033	-.0068
1.0000	-.0002	-.043	-.0083	.0425	.0036	.0033	-.0068
1.2000	-.0004		-.0117	-.0487	-.0068	.0034	-.0092
1.2000	-.0004	-.047	-.0117	.0487	.0068	.0034	-.0092
1.4000	-.0006		-.0139	-.0518	-.0102	.0029	-.0110
1.4000	-.0006	-.049	-.0139	.0518	.0102	.0029	-.0110
1.6000	-.0007		-.0150	-.0530	-.0133	.0022	-.0123
1.6000	-.0007	-.050	-.0150	.0530	.0133	.0022	-.0123
1.8000	-.0009		-.0151	-.0531	-.0158	.0015	-.0132
1.8000	-.0009	-.050	-.0151	.0531	.0158	.0015	-.0132
2.0000	-.0010		-.0145	-.0528	-.0177	.0009	-.0137
2.0000	-.0010	-.051	-.0145	.0528	.0177	.0009	-.0137
3.0000	-.0013		-.0086	-.0512	-.0215	-.0003	-.0147
3.0000	-.0013	-.050	-.0086	.0512	.0215	-.0003	-.0147
4.0000	-.0013		-.0038	-.0508	-.0220	-.0005	-.0148
4.0000	-.0013	-.050	-.0038	.0508	.0220	-.0005	-.0148
5.0000	-.0013		-.0015	-.0508	-.0220	-.0005	-.0148
5.0000	-.0013	-.050	-.0015	.0508	.0220	-.0005	-.0148

TABLE C.41 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .400
SPL......= .400
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF -KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0064	-.0339	-.0028	.0040	-.0067
1.0000	-.0001	-.041	-.0064	.0330	.0028	.0040	-.0067
1.2000	-.0003		-.0092	-.0383	-.0054	.0043	-.0089
1.2000	-.0003	-.044	-.0092	.0383	.0054	.0043	-.0089
1.4000	-.0004		-.0111	-.0411	-.0081	.0041	-.0105
1.4000	-.0004	-.046	-.0111	.0411	.0081	.0041	-.0105
1.6000	-.0006		-.0120	-.0424	-.0107	.0035	-.0116
1.6000	-.0006	-.047	-.0120	.0424	.0107	.0035	-.0116
1.8000	-.0007		-.0123	-.0427	-.0128	.0029	-.0123
1.8000	-.0007	-.047	-.0123	.0427	.0128	.0029	-.0123
2.0000	-.0008		-.0119	-.0427	-.0144	.0023	-.0128
2.0000	-.0008	-.047	-.0119	.0427	.0144	.0023	-.0128
3.0000	-.0011		-.0073	-.0417	-.0179	.0009	-.0134
3.0000	-.0011	-.046	-.0073	.0417	.0179	.0009	-.0134
4.0000	-.0011		-.0034	-.0414	-.0184	.0007	-.0134
4.0000	-.0011	-.046	-.0034	.0414	.0184	.0007	-.0134
5.0000	-.0011		-.0014	-.0413	-.0185	.0007	-.0134
5.0000	-.0011	-.046	-.0014	.0413	.0185	.0007	-.0134

TABLE C.42 -- NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR......= .500
SPL......= .500
PL.......= 1.000
PR.......= 0.000
POIS.R...= .300
K........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0053	-.0270	-.0023	.0044	-.0066
1.0000	-.0001	-.039	-.0053	.0270	.0023	.0044	-.0066
1.2000	-.0002		-.0076	-.0315	-.0044	.0050	-.0087
1.2000	-.0002	-.043	-.0076	.0315	.0044	.0050	-.0087
1.4000	-.0004		-.0092	-.0341	-.0067	.0049	-.0102
1.4000	-.0004	-.044	-.0092	.0341	.0067	.0049	-.0102
1.6000	-.0005		-.0101	-.0353	-.0089	.0044	-.0112
1.6000	-.0005	-.044	-.0101	.0353	.0089	.0044	-.0112
1.8000	-.0006		-.0103	-.0358	-.0108	.0038	-.0117
1.8000	-.0006	-.044	-.0103	.0358	.0108	.0038	-.0117
2.0000	-.0007		-.0101	-.0359	-.0122	.0032	-.0121
2.0000	-.0007	-.044	-.0101	.0359	.0122	.0032	-.0121
3.0000	-.0009		-.0064	-.0352	-.0154	.0017	-.0124
3.0000	-.0009	-.043	-.0064	.0352	.0154	.0017	-.0124
4.0000	-.0010		-.0031	-.0350	-.0159	.0015	-.0125
4.0000	-.0010	-.043	-.0031	.0350	.0159	.0015	-.0125
5.0000	-.0010		-.0013	-.0349	-.0160	.0014	-.0125
5.0000	-.0010	-.043	-.0013	.0349	.0160	.0014	-.0125

TABLE C.43 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.45) TO (3.50)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0006		-.0274	-.1200	-.0118	-.0069	-.0043
.2500	-.0000	-.041	-.0001	.0299	.0000	-.0000	.0000
1.2000	-.0010		-.0339	-.1133	-.0194	-.0088	-.0077
.3000	-.0000	-.049	-.0002	.0280	.0000	-.0000	.0000
1.4000	-.0015		-.0376	-.1032	-.0265	-.0098	-.0111
.3500	-.0000	-.051	-.0003	.0251	.0000	-.0001	.0000
1.6000	-.0018		-.0393	-.0922	-.0324	-.0103	-.0141
.4000	-.0000	-.053	-.0006	.0219	.0000	-.0001	.0000
1.8000	-.0021		-.0398	-.0817	-.0369	-.0103	-.0165
.4500	-.0000	-.053	-.0008	.0188	.0000	-.0002	.0000
2.0000	-.0023		-.0397	-.0724	-.0402	-.0100	-.0184
.5000	-.0000	-.053	-.0010	.0160	.0001	-.0002	.0000
3.0000	-.0026		-.0369	-.0439	-.0434	-.0076	-.0211
.7500	-.0000	-.045	-.0018	.0069	.0004	-.0004	-.0001
4.0000	-.0024		-.0354	-.0331	-.0385	-.0057	-.0192
1.0000	-.0000	-.037	-.0021	.0033	.0009	-.0005	-.0003
5.0000	-.0021		-.0347	-.0285	-.0336	-.0047	-.0169
1.2500	-.0001	-.032	-.0021	.0019	.0012	-.0005	-.0005

TABLE C.44 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .006
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0091	-.0447	-.0039	-.0003	-.0034
.2500	-.0000	-.024	-.0000	.0111	.0000	-.0000	.0000
1.2000	-.0004		-.0139	-.0537	-.0081	-.0012	-.0057
.3000	-.0000	-.030	-.0001	.0133	.0000	-.0000	.0000
1.4000	-.0007		-.0178	-.0588	-.0129	-.0021	-.0081
.3500	-.0000	-.035	-.0002	.0143	.0000	-.0000	.0000
1.6000	-.0010		-.0206	-.0606	-.0177	-.0030	-.0104
.4000	-.0000	-.038	-.0003	.0145	.0000	-.0001	.0000
1.8000	-.0013		-.0223	-.0603	-.0221	-.0038	-.0124
.4500	-.0000	-.040	-.0005	.0141	.0000	-.0001	.0000
2.0000	-.0015		-.0233	-.0586	-.0258	-.0045	-.0140
.5000	-.0000	-.041	-.0006	.0133	.0001	-.0001	-.0000
3.0000	-.0021		-.0230	-.0449	-.0346	-.0057	-.0175
.7500	-.0000	-.039	-.0014	.0081	.0003	-.0003	-.0001
4.0000	-.0021		-.0217	-.0347	-.0341	-.0053	-.0170
1.0000	-.0000	-.034	-.0017	.0045	.0007	-.0004	-.0003
5.0000	-.0019		-.0211	-.0293	-.0312	-.0047	-.0155
1.2500	-.0001	-.030	-.0018	.0027	.0010	-.0004	-.0004

TABLE C.45 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .013
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0055	-.0269	-.0024	.0010	-.0033
.2500	-.0000	-.020	-.0000	.0067	.0000	-.0000	-.0000
1.2000	-.0003		-.0087	-.0341	-.0051	.0009	-.0052
.3000	-.0000	-.025	-.0000	.0084	.0000	-.0000	-.0000
1.4000	-.0005		-.0117	-.0391	-.0085	.0004	-.0072
.3500	-.0000	-.029	-.0001	.0096	.0000	-.0000	-.0000
1.6000	-.0007		-.0141	-.0422	-.0122	-.0002	-.0090
.4000	-.0000	-.031	-.0002	.0101	.0000	-.0000	-.0000
1.8000	-.0009		-.0158	-.0437	-.0157	-.0009	-.0106
.4500	-.0000	-.033	-.0003	.0102	.0000	-.0001	-.0000
2.0000	-.0011		-.0169	-.0441	-.0189	-.0016	-.0120
.5000	-.0000	-.034	-.0005	.0100	.0000	-.0001	-.0000
3.0000	-.0017		-.0179	-.0388	-.0282	-.0038	-.0152
.7500	-.0000	-.034	-.0011	.0072	.0003	-.0002	-.0001
4.0000	-.0018		-.0171	-.0322	-.0297	-.0044	-.0151
1.0000	-.0000	-.031	-.0014	.0045	.0006	-.0003	-.0003
5.0000	-.0018		-.0166	-.0279	-.0283	-.0043	-.0141
1.2500	-.0000	-.028	-.0015	.0029	.0009	-.0003	-.0004

TABLE C.46 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .300
SPL.......= .019
POL.......= 0.000
POR.......= 1.000
POIS.P....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001	-.018	-.0039	-.0192	-.0017	.0016	-.0032
.2500	-.0000		-.0000	.0048	.0000	-.0000	-.0000
1.2000	-.0002	-.023	-.0064	-.0250	-.0037	.0018	-.0050
.3000	-.0000		-.0000	.0062	.0000	-.0000	-.0000
1.4000	-.0003	-.026	-.0088	-.0293	-.0064	.0017	-.0067
.3500	-.0000		-.0001	.0072	.0000	-.0000	-.0000
1.6000	-.0005	-.028	-.0107	-.0323	-.0093	.0013	-.0083
.4000	-.0000		-.0002	.0078	.0000	-.0000	-.0000
1.8000	-.0007	-.030	-.0122	-.0342	-.0122	.0007	-.0097
.4500	-.0000		-.0003	-.0080	.0000	-.0000	-.0000
2.0000	-.0009	-.031	-.0133	-.0351	-.0149	.0001	-.0108
.5000	-.0000		-.0004	.0080	.0000	-.0001	-.0000
3.0000	-.0014	-.031	-.0147	-.0334	-.0238	-.0025	-.0136
.7500	-.0000		-.0009	.0062	.0002	-.0002	-.0001
4.0000	-.0016	-.028	-.0143	-.0292	-.0261	-.0036	-.0136
1.0000	-.0000		-.0012	.0042	.0005	-.0003	-.0002
5.0000	-.0016	-.026	-.0139	-.0260	-.0257	-.0038	-.0129
1.2500	-.0000		-.0013	.0026	.0008	-.0003	-.0004

TABLE C.47 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .025
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0030	-.0150	-.0013	.0019	-.0031
.2500	-.0000	-.017	-.0000	.0037	.0000	.0000	-.0000
1.2000	-.0002		-.0050	-.0197	-.0029	.0023	-.0048
.3000	-.0000	-.021	-.0000	.0049	.0000	-.0000	-.0000
1.4000	-.0003		-.0070	-.0234	-.0051	.0024	-.0064
.3500	-.0000	-.024	-.0001	.0057	.0000	-.0000	-.0000
1.6000	-.0004		-.0087	-.0262	-.0075	.0022	-.0079
.4000	-.0000	-.026	-.0001	.0063	.0000	-.0000	-.0000
1.8000	-.0006		-.0100	-.0280	-.0100	.0018	-.0091
.4500	-.0000	-.027	-.0002	.0066	.0000	-.0000	-.0000
2.0000	-.0007		-.0110	-.0292	-.0123	.0012	-.0101
.5000	-.0000	-.028	-.0003	.0067	.0000	-.0000	-.0000
3.0000	-.0012		-.0126	-.0293	-.0205	-.0014	-.0124
.7500	-.0000	-.028	-.0008	.0055	.0002	-.0001	-.0001
4.0000	-.0014		-.0124	-.0265	-.0233	-.0029	-.0125
1.0000	-.0000	-.026	-.0011	.0039	.0005	-.0002	-.0002
5.0000	-.0015		-.0121	-.0241	-.0234	-.0034	-.0119
1.2500	-.0000	-.024	-.0012	.0027	.0007	-.0003	-.0003

TABLE C.48 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .031
PDL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0025	-.0123	-.0011	.0021	-.0031
.2500	-.0000	-.017	-.0000	.0031	.0000	.0000	-.0000
1.2000	-.0001		-.0041	-.0162	-.0024	.0027	-.0047
.3000	-.0000	-.020	-.0000	.0040	.0000	.0000	-.0000
1.4000	-.0002		-.0058	-.0195	-.0042	.0029	-.0063
.3500	-.0000	-.023	-.0001	.0048	.0000	-.0000	-.0000
1.6000	-.0004		-.0073	-.0220	-.0063	.0028	-.0076
.4000	-.0000	-.024	-.0001	.0053	.0000	-.0000	-.0000
1.8000	-.0005		-.0084	-.0237	-.0084	.0025	-.0087
.4500	-.0000	-.025	-.0002	.0056	.0000	-.0000	-.0000
2.0000	-.0006		-.0093	-.0249	-.0105	.0020	-.0096
.5000	-.0000	-.026	-.0003	.0057	.0000	-.0000	-.0000
3.0000	-.0011		-.0110	-.0259	-.0181	-.0007	-.0115
.7500	-.0000	-.026	-.0007	.0049	.0002	-.0001	-.0001
4.0000	-.0013		-.0109	-.0242	-.0210	-.0023	-.0115
1.0000	-.0000	-.024	-.0010	.0036	.0004	-.0002	-.0002
5.0000	-.0013		-.0107	-.0224	-.0214	-.0030	-.0110
1.2500	-.0000	-.022	-.0011	.0026	.0006	-.0002	-.0003

TABLE C.49 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .250

BR / RL	KWR / KWL	KMC	KOXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0009		-.0408	-.1883	-.0177	-.0104	-.0064
.2500	-.0000	-.063	-.0001	.0469	.0000	-.0000	.0000
1.2000	-.0020		-.0630	-.2308	-.0364	-.0165	-.0144
.3000	-.0000	-.092	-.0003	.0570	.0000	-.0001	.0000
1.4000	-.0035		-.0878	-.2775	-.0631	-.0236	-.0264
.3500	-.0000	-.127	-.0008	.0677	.0000	-.0002	.0001
1.6000	-.0055		-.1144	-.3290	-.0979	-.0316	-.0425
.4000	-.0000	-.169	-.0017	.0788	.0001	-.0004	.0001
1.8000	-.0080		-.1422	-.3850	-.1404	-.0404	-.0625
.4500	-.0000	-.218	-.0030	.0901	.0002	-.0006	.0001
2.0000	-.0110		-.1704	-.4450	-.1902	-.0498	-.0864
.5000	-.0000	-.272	-.0047	.1013	.0004	-.0011	.0001
3.0000	-.0305		-.2967	-.7683	-.5072	-.1010	-.2426
.7500	-.0003	-.594	-.0195	.1472	.0049	-.0047	-.0013
4.0000	-.0510		-.3748	-1.0536	-.8343	-.1471	-.4071
1.0000	-.0009	-.908	-.0370	.1675	.0160	-.0094	-.0056
5.0000	-.0675		-.4121	-1.2595	-1.0938	-.1812	-.5390
1.2500	-.0017	-1.150	-.0505	.1683	.0308	-.0132	-.0124

TABLE C.50 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .006
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0003		-.0136	-.0673	-.0059	-.0005	-.0051
.2500	-.0000	-.036	-.0000	.0166	.0000	-.0000	.0000
1.2000	-.0008		-.0260	-.1023	-.0151	-.0022	-.0107
.3000	-.0000	-.057	-.0001	.0253	.0000	-.0000	.0000
1.4000	-.0017		-.0422	-.1434	-.0306	-.0050	-.0194
.3500	-.0000	-.084	-.0004	.0350	.0000	-.0001	.0000
1.6000	-.0030		-.0615	-.1899	-.0534	-.0091	-.0314
.4000	-.0000	-.118	-.0009	.0456	.0000	-.0002	.0000
1.8000	-.0046		-.0831	-.2413	-.0836	-.0143	-.0471
.4500	-.0000	-.158	-.0018	.0567	.0001	-.0003	.0000
2.0000	-.0070		-.1061	-.2970	-.1211	-.0207	-.0663
.5000	-.0000	-.204	-.0030	.0679	.0003	-.0006	-.0000
3.0000	-.0234		-.2191	-.6060	-.3892	-.0635	-.2007
.7500	-.0002	-.492	-.0148	.1173	.0037	-.0033	-.0013
4.0000	-.0424		-.2960	-.8915	-.6935	-.1096	-.3518
1.0000	-.0007	-.787	-.0305	.1437	.0132	-.0073	-.0052
5.0000	-.0584		-.3352	-1.1058	-.9483	-.1471	-.4780
1.2500	-.0014	-1.023	-.0431	.1503	.0264	-.0109	-.0111

TABLE C.51 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .013
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .250

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0081	-.0405	-.0035	.0015	-.0049
.2500	-.0000	-.030	-.0000	.0101	.0000	-.0000	-.0000
1.2000	-.0005		-.0164	-.0648	-.0095	.0016	-.0097
.3000	-.0000	-.048	-.0001	.0160	.0000	-.0000	-.0000
1.4000	-.0011		-.0278	-.0950	-.0202	.0010	-.0171
.3500	-.0000	-.070	-.0003	.0232	.0000	-.0000	-.0000
1.6000	-.0021		-.0422	-.1311	-.0367	-.0005	-.0273
.4000	-.0000	-.098	-.0006	.0315	.0000	-.0001	-.0000
1.8000	-.0034		-.0590	-.1725	-.0594	-.0031	-.0406
.4500	-.0000	-.132	-.0013	.0405	.0001	-.0002	-.0000
2.0000	-.0051		-.0776	-.2188	-.0887	-.0068	-.0569
.5000	-.0000	-.172	-.0022	.0501	.0002	-.0004	-.0001
3.0000	-.0189		-.1760	-.4937	-.3149	-.0392	-.1746
.7500	-.0002	-.426	-.0120	.0958	.0030	-.0025	-.0013
4.0000	-.0361		-.2490	-.7654	-.5915	-.0814	-.3123
1.0000	-.0006	-.699	-.0259	.1240	.0112	-.0059	-.0048
5.0000	-.0514		-.2886	-.9786	-.8342	-.1193	-.4307
1.2500	-.0013	-.923	-.0377	.1340	.0232	-.0092	-.0101

TABLE C.52 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SRR.......= .300
SPL.......= .019
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0058	-.0289	-.0025	.0023	-.0047
.2500	-.0000	-.028	-.0000	.0072	.0000	-.0000	-.0000
1.2000	-.0004		-.0120	-.0473	-.0070	.0034	-.0093
.3000	-.0000	-.043	-.0001	.0117	.0000	-.0000	-.0000
1.4000	-.0008		-.0208	-.0710	-.0151	.0040	-.0160
.3500	-.0000	-.063	-.0002	.0174	.0000	-.0000	-.0000
1.6000	-.0016		-.0321	-.1000	-.0279	.0040	-.0252
.4000	-.0000	-.088	-.0005	.0240	.0000	-.0001	-.0000
1.8000	-.0026		-.0457	-.1341	-.0461	.0031	-.0370
.4500	-.0000	-.118	-.0010	.0315	.0001	-.0001	-.0001
2.0000	-.0040		-.0612	-.1730	-.0700	.0013	-.0516
.5000	-.0000	-.153	-.0017	.0396	.0002	-.0002	-.0001
3.0000	-.0159		-.1473	-.4157	-.2643	-.0226	-.1570
.7500	-.0001	-.381	-.0101	.0807	.0025	-.0019	-.0012
4.0000	-.0315		-.2155	-.6690	-.5153	-.0602	-.2829
1.0000	-.0005	-.632	-.0225	.1086	.0098	-.0048	-.0045
5.0000	-.0458		-.2544	-.8755	-.7438	-.0970	-.3936
1.2500	-.0011	-.843	-.0335	.1203	.0206	-.0078	-.0094

TABLE C.53 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR....... = .400
SPL....... = -.025
POL....... = 1.000
POR....... = 0.000
POIS.R.... = .300
K......... = .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0045	-.0225	-.0020	.0028	-.0047
.2500	-.0000	-.026	-.0000	.0056	.0000	.0000	-.0000
1.2000	-.0003		-.0094	-.0373	-.0055	.0044	-.0090
.3000	-.0000	-.041	-.0000	.0092	.0000	-.0000	-.0000
1.4000	-.0007		-.0166	-.0567	-.0120	.0058	-.0154
.3500	-.0000	-.059	-.0002	.0139	.0000	-.0000	-.0000
1.6000	-.0013		-.0259	-.0808	-.0225	.0068	-.0239
.4000	-.0000	-.081	-.0004	.0194	.0000	-.0000	-.0000
1.8000	-.0021		-.0374	-.1097	-.0377	.0071	-.0348
.4500	-.0000	-.109	-.0008	.0258	.0000	-.0001	-.0001
2.0000	-.0033		-.0505	-.1431	-.0578	.0065	-.0480
.5000	-.0000	-.140	-.0014	.0328	.0001	-.0002	-.0001
3.0000	-.0137		-.1267	-.3587	-.2277	-.0105	-.1442
.7500	-.0001	-.348	-.0087	.0697	.0022	-.0015	-.0012
4.0000	-.0279		-.1901	-.5937	-.4563	-.0437	-.2602
1.0000	-.0005	-.580	-.0199	.0965	.0086	-.0040	-.0042
5.0000	-.0413		-.2279	-.7913	-.6708	-.0789	-.3636
1.2500	-.0010	-.779	-.0302	.1090	.0186	-.0067	-.0088

TABLE C.54 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR....... = .500
SPL....... = .031
POL....... = 1.000
POR....... = 0.000
POIS.R.... = .300
K......... = .250

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0037	-.0184	-.0016	.0031	-.0046
.2500	-.0000	-.026	-.0000	.0046	.0000	.0000	-.0000
1.2000	-.0002		-.0078	-.0308	-.0045	.0050	-.0089
.3000	-.0000	-.039	-.0000	.0076	.0000	.0000	-.0000
1.4000	-.0006		-.0138	-.0472	-.0100	.0070	-.0149
.3500	-.0000	-.056	-.0001	.0115	.0000	-.0000	-.0000
1.6000	-.0011		-.0217	-.0678	-.0189	.0086	-.0230
.4000	-.0000	-.077	-.0003	.0163	.0000	-.0000	-.0000
1.8000	-.0018		-.0316	-.0928	-.0318	.0098	-.0332
.4500	-.0000	-.102	-.0007	.0218	.0000	-.0000	-~~.0001~~
2.0000	-.0028		-.0430	-.1219	-.0492	.0102	-.0456
.5000	-.0000	-.131	-.0012	.0279	.0001	-.0001	-.0001
3.0000	-.0120		-.1112	-.3154	-.2000	-.0014	-.1345
.7500	-.0001	-.324	-.0076	.0613	.0019	-.0011	-.0012
4.0000	-.0250		-.1701	-.5334	-.4095	-.0305	-.2422
1.0000	-.0004	-.539	-.0179	.0868	.0077	-.0033	-.0040
5.0000	-.0376		-.2065	-.7215	-.6108	-.0639	-.3391
1.2500	-.0009	-.726	-.0274	.0996	.0169	-.0058	-.0083

TABLE C.55 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0006		-.0274	-.1200	-.0118	-.0069	-.0043
.5000	-.0000	-.041	-.0029	.0597	.0003	-.0007	.0000
1.2000	-.0010		-.0339	-.1134	-.0194	-.0088	-.0077
.6000	-.0000	-.048	-.0048	.0560	.0007	-.0011	-.0001
1.4000	-.0015		-.0376	-.1033	-.0265	-.0098	-.0111
.7000	-.0001	-.051	-.0065	.0503	.0014	-.0015	-.0003
1.6000	-.0018		-.0394	-.0925	-.0325	-.0103	-.0142
.8000	-.0001	-.053	-.0079	.0440	.0023	-.0019	-.0007
1.8000	-.0021		-.0401	-.0824	-.0372	-.0103	-.0166
.9000	-.0002	-.054	-.0089	.0380	.0032	-.0022	-.0011
2.0000	-.0023		-.0401	-.0736	-.0406	-.0101	-.0186
1.0000	-.0002	-.053	-.0097	.0327	.0041	-.0024	-.0015
3.0000	-.0026		-.0386	-.0487	-.0464	-.0082	-.0225
1.5000	-.0004	-.048	-.0108	.0166	.0077	-.0025	-.0033
4.0000	-.0028		-.0380	-.0419	-.0451	-.0069	+.0224
2.0000	-.0005	-.045	-.0109	.0114	.0095	-.0022	-.0044
5.0000	-.0027		-.0380	-.0403	-.0434	-.0064	-.0217
2.5000	-.0006	-.043	-.0108	.0100	.0102	-.0019	-.0049

TABLE C.56 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .025
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0091	-.0447	-.0039	-.0003	-.0034
.5000	-.0000	-.024	-.0010	.0223	.0001	-.0001	-.0001
1.2000	-.0004		-.0139	-.0538	-.0081	-.0012	-.0057
.6000	-.0000	-.030	-.0020	.0266	.0003	-.0003	-.0002
1.4000	-.0007		-.0178	-.0588	-.0129	-.0021	-.0081
.7000	-.0000	-.035	-.0032	.0287	.0007	-.0005	-.0004
1.6000	-.0010		-.0206	-.0607	-.0177	-.0030	-.0104
.8000	-.0001	-.038	-.0043	.0291	.0012	-.0008	-.0007
1.8000	-.0013		-.0224	-.0605	-.0222	-.0038	-.0124
.9000	-.0001	-.040	-.0052	.0284	.0019	-.0010	-.0010
2.0000	-.0015		-.0234	-.0591	-.0260	-.0045	-.0141
1.0000	-.0001	-.041	-.0060	.0269	.0026	-.0012	-.0013
3.0000	-.0022		-.0239	-.0475	-.0362	-.0060	-.0183
1.5000	-.0003	-.041	-.0077	.0179	.0058	-.0017	-.0027
4.0000	-.0024		-.0234	-.0402	-.0383	-.0060	-.0191
2.0000	-.0004	-.039	-.0078	.0124	.0077	-.0018	-.0037
5.0000	-.0024		-.0233	-.0373	-.0378	-.0058	-.0188
2.5000	-.0005	-.038	-.0078	.0101	.0086	-.0016	-.0041

TABLE C.57 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .050
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0055	-.0269	-.0024	.0010	-.0033
.5000	-.0000	-.020	-.0006	.0134	.0001	-.0000	-.0001
1.2000	-.0003		-.0087	-.0341	-.0051	.0009	-.0052
.6000	-.0000	-.025	-.0013	.0169	.0002	-.0001	-.0003
1.4000	-.0005		-.0117	-.0392	-.0085	.0004	-.0072
.7000	-.0000	-.029	-.0021	.0191	.0004	-.0002	-.0004
1.6000	-.0007		-.0141	-.0423	-.0122	-.0002	-.0090
.8000	-.0000	-.031	-.0029	.0203	.0008	-.0003	-.0007
1.8000	-.0009		-.0158	-.0438	-.0158	-.0009	-.0107
.9000	-.0001	-.033	-.0037	.0206	.0013	-.0005	-.0009
2.0000	-.0011		-.0170	-.0443	-.0190	-.0016	-.0120
1.0000	-.0001	-.035	-.0044	.0203	.0019	-.0006	-.0012
3.0000	-.0018		-.0184	-.0403	-.0292	-.0040	-.0157
1.5000	-.0003	-.036	-.0061	.0155	.0047	-.0012	-.0024
4.0000	-.0020		-.0182	-.0359	-.0325	-.0048	-.0165
2.0000	-.0004	-.034	-.0063	.0115	.0064	-.0014	-.0032
5.0000	-.0020		-.0181	-.0335	-.0330	-.0050	-.0165
2.5000	-.0004	-.033	-.0064	.0094	.0074	-.0014	-.0036

TABLE C.58 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .300
SPL.......= .075
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0039	-.0192	-.0017	.0016	-.0032
.5000	-.0000	-.018	-.0004	.0096	.0000	.0000	-.0001
1.2000	-.0002		-.0064	-.0250	-.0037	.0018	-.0050
.6000	-.0000	-.023	-.0009	.0123	.0001	.0000	-.0003
1.4000	-.0003		-.0088	-.0293	-.0064	.0017	-.0067
.7000	-.0000	-.026	-.0016	.0143	.0003	-.0000	-.0004
1.6000	-.0005		-.0107	-.0323	-.0093	.0013	-.0083
.8000	-.0000	-.029	-.0022	.0155	.0006	-.0001	-.0007
1.8000	-.0007		-.0123	-.0342	-.0122	.0007	-.0097
.9000	-.0001	-.030	-.0029	.0161	.0010	-.0002	-.0009
2.0000	-.0009		-.0133	-.0353	-.0150	.0001	-.0109
1.0000	-.0001	-.031	-.0035	.0161	.0015	-.0003	-.0011
3.0000	-.0015		-.0151	-.0344	-.0244	-.0025	-.0140
1.5000	-.0002	-.032	-.0050	.0133	.0039	-.0008	-.0022
4.0000	-.0017		-.0151	-.0318	-.0281	-.0038	-.0147
2.0000	-.0003	-.030	-.0054	.0104	.0055	-.0011	-.0028
5.0000	-.0018		-.0150	-.0301	-.0291	-.0043	-.0147
2.5000	-.0004	-.030	-.0054	.0086	.0064	-.0012	-.0032

TABLE C.59 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .100
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0030	-.0150	-.0013	.0019	-.0031
.5000	-.0000	-.017	-.0003	.0075	.0000	.0001	-.0001
1.2000	-.0002		-.0050	-.0197	-.0029	.0023	-.0048
.6000	-.0000	-.021	-.0007	.0097	.0001	.0001	-.0003
1.4000	-.0003		-.0070	-.0234	-.0051	.0024	-.0064
.7000	-.0000	-.024	-.0012	.0114	.0003	.0001	-.0005
1.6000	-.0004		-.0087	-.0262	-.0075	.0022	-.0079
.8000	-.0000	-.026	-.0018	.0126	.0005	.0001	-.0007
1.8000	-.0006		-.0100	-.0281	-.0100	.0018	-.0091
.9000	-.0000	-.027	-.0024	.0132	.0008	-.0000	-.0009
2.0000	-.0007		-.0110	-.0292	-.0124	.0012	-.0101
1.0000	-.0001	-.028	-.0029	.0134	.0012	-.0001	-.0011
3.0000	-.0013		-.0128	-.0299	-.0210	-.0015	-.0127
1.5000	-.0002	-.028	-.0043	.0116	.0033	-.0006	-.0020
4.0000	-.0015		-.0129	-.0283	-.0247	-.0030	-.0133
2.0000	-.0003	-.028	-.0046	.0093	.0048	-.0009	-.0026
5.0000	-.0016		-.0129	-.0272	-.0259	-.0037	-.0133
2.5000	-.0003	-.027	-.0047	.0079	.0057	-.0010	-.0029

TABLE C.60 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .125
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0025	-.0123	-.0011	.0021	-.0031
.5000	-.0000	-.017	-.0003	.0061	.0000	.0001	-.0001
1.2000	-.0001		-.0041	-.0162	-.0024	.0027	-.0047
.6000	-.0000	-.020	-.0006	.0080	.0001	.0001	-.0003
1.4000	-.0002		-.0058	-.0195	-.0042	.0029	-.0063
.7000	-.0000	-.023	-.0010	.0095	.0002	.0002	-.0005
1.6000	-.0004		-.0073	-.0220	-.0063	.0028	-.0076
.8000	-.0000	-.024	-.0015	.0106	.0004	.0001	-.0006
1.8000	-.0005		-.0084	-.0238	-.0084	.0025	-.0087
.9000	-.0000	-.026	-.0020	.0112	.0007	.0001	-.0008
2.0000	-.0006		-.0093	-.0250	-.0105	.0020	-.0096
1.0000	-.0001	-.026	-.0024	.0114	.0011	.0000	-.0010
3.0000	-.0011		-.0111	-.0264	-.0184	-.0007	-.0117
1.5000	-.0002	-.026	-.0037	.0103	.0029	-.0004	-.0019
4.0000	-.0013		-.0113	-.0255	-.0220	-.0024	-.0122
2.0000	-.0002	-.025	-.0041	.0085	.0043	-.0007	-.0024
5.0000	-.0014		-.0113	-.0247	-.0234	-.0032	-.0121
2.5000	-.0003	-.024	-.0042	.0072	.0051	-.0008	-.0027

TABLE C.61 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR....... = 0.000
SPL....... = 0.000
PDL....... = 1.000
PDR....... = 0.000
POIS.R.... = .300
K......... = .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0009		-.0399	-.1827	-.0172	-.0101	-.0063
.5000	-.0000	-.061	-.0043	.0910	.0004	-.0010	.0001
1.2000	-.0016		-.0591	-.2144	-.0341	-.0154	-.0135
.6000	-.0001	-.086	-.0084	.1060	.0012	-.0019	-.0002
1.4000	-.0031		-.0777	-.2420	-.0557	-.0208	-.0233
.7000	-.0002	-.112	-.0137	.1181	.0029	-.0032	-.0007
1.6000	-.0045		-.0944	-.2649	-.0803	-.0259	-.0349
.8000	-.0003	-.138	-.0194	.1269	.0055	-.0047	-.0017
1.8000	-.0060		-.1084	-.2835	-.1061	-.0304	-.0473
.9000	-.0005	-.163	-.0252	.1327	.0090	-.0063	-.0031
2.0000	-.0076		-.1197	-.2982	-.1318	-.0343	-.0599
1.0000	-.0007	-.186	-.0306	.1359	.0132	-.0077	-.0048
3.0000	-.0141		-.1467	-.3326	-.2341	-.0456	-.1123
1.5000	-.0021	-.268	-.0483	.1290	.0373	-.0128	-.0160
4.0000	-.0177		-.1520	-.3382	-.2891	-.0493	-.1417
2.0000	-.0033	-.307	-.0541	.1125	.0566	-.0144	-.0258
5.0000	-.0195		-.1528	-.3368	-.3149	-.0502	-.1559
2.5000	-.0041	-.323	-.0557	.0996	.0688	-.0144	-.0325

TABLE C.62 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .025
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0003		-.0132	-.0656	-.0057	-.0005	-.0050
.5000	-.0000	-.035	-.0014	.0327	.0001	-.0002	-.0001
1.2000	-.0008		-.0243	-.0957	-.0141	-.0020	-.0100
.6000	-.0000	-.053	-.0035	.0473	.0005	-.0005	-.0004
1.4000	-.0015		-.0372	-.1262	-.0270	-.0044	-.0171
.7000	-.0001	-.074	-.0066	.0617	.0014	-.0011	-.0009
1.6000	-.0025		-.0504	-.1549	-.0437	-.0074	-.0258
.8000	-.0002	-.097	-.0105	.0745	.0030	-.0019	-.0016
1.8000	-.0036		-.0628	-.1807	-.0630	-.0108	-.0355
.9000	-.0003	-.119	-.0149	.0850	.0053	-.0028	-.0027
2.0000	-.0048		-.0736	-.2029	-.0836	-.0143	-.0457
1.0000	-.0004	-.140	-.0192	.0931	.0083	-.0039	-.0041
3.0000	-.0107		-.1030	-.2678	-.1775	-.0290	-.0912
1.5000	-.0016	-.220	-.0355	.1059	.0280	-.0085	-.0132
4.0000	-.0144		-.1100	-.2882	-.2356	-.0372	-.1190
2.0000	-.0026	-.260	-.0417	.0986	.0455	-.0108	-.0216
5.0000	-.0164		-.1112	-.2926	-.2656	-.0410	-.1333
2.5000	-.0034	-.278	-.0434	.0891	.0572	-.0116	-.0276

TABLE C.63 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .050
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0002		-.0079	-.0395	-.0035	.0015	-.0047
.5000	-.0000	-.029	-.0009	.0197	.0001	-.0000	-.0002
1.2000	-.0005		-.0153	-.0606	-.0089	.0015	-.0091
.6000	-.0000	-.044	-.0022	.0300	.0003	-.0001	-.0004
1.4000	-.0010		-.0246	-.0837	-.0178	.0009	-.0151
.7000	-.0001	-.062	-.0044	.0409	.0009	-.0004	-.0009
1.6000	-.0017		-.0346	-.1070	-.0300	-.0004	-.0224
.8000	-.0001	-.080	-.0072	.0514	.0021	-.0008	-.0016
1.8000	-.0026		-.0445	-.1292	-.0448	-.0024	-.0306
.9000	-.0002	-.099	-.0105	.0608	.0038	-.0014	-.0026
2.0000	-.0035		-.0536	-.1496	-.0611	-.0047	-.0392
1.0000	-.0003	-.117	-.0141	.0687	.0061	-.0021	-.0038
3.0000	-.0085		-.0813	-.2177	-.1422	-.0180	-.0784
1.5000	-.0012	-.189	-.0283	.0865	.0224	-.0058	-.0115
4.0000	-.0121		-.0892	-.2454	-.1974	-.0277	-.1033
2.0000	-.0022	-.226	-.0344	.0848	.0379	-.0082	-.0189
5.0000	-.0141		-.0908	-.2549	-.2280	-.0333	-.1165
2.5000	-.0029	-.244	-.0362	.0787	.0488	-.0094	-.0241

TABLE C.64 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR......= .300
SPL......= .075
PDL......= 1.000
PDR......= 0.000
PDIS.R...= .300
K........= .500

RR	KWR	KMC	KQXR	KQYRN	KQYRF	KMXR	KMYR
RL	KWL		KQXL	KQYLN	KQYLF	KMXL	KMYL
1.0000	-.0001		-.0057	-.0282	-.0025	.0023	-.0046
.5000	-.0000	-.027	-.0006	.0140	.0001	.0001	-.0002
1.2000	-.0004		-.0112	-.0443	-.0065	.0032	-.0087
.6000	-.0000	-.040	-.0016	.0219	.0002	.0001	-.0005
1.4000	-.0007		-.0183	-.0625	-.0133	.0035	-.0141
.7000	-.0000	-.055	-.0033	.0306	.0007	-.0000	-.0009
1.6000	-.0013		-.0263	-.0816	-.0229	.0033	-.0206
.8000	-.0001	-.072	-.0055	.0392	.0016	-.0002	-.0016
1.8000	-.0020		-.0345	-.1005	-.0347	.0023	-.0278
.9000	-.0002	-.088	-.0082	.0473	.0029	-.0006	-.0025
2.0000	-.0028		-.0422	-.1183	-.0482	.0008	-.0354
1.0000	-.0003	-.104	-.0111	.0543	.0048	-.0010	-.0036
3.0000	-.0071		-.0674	-.1826	-.1186	-.0105	-.0699
1.5000	-.0010	-.167	-.0235	.0727	.0186	-.0040	-.0104
4.0000	-.0104		-.0755	-.2126	-.1695	-.0206	-.0919
2.0000	-.0019	-.201	-.0293	.0738	.0325	-.0063	-.0169
5.0000	-.0123		-.0773	-.2247	-.1993	-.0271	-.1039
2.5000	-.0025	-.218	-.0312	.0699	.0425	-.0077	-.0215

TABLE C.65 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .100
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .500

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0044	-.0219	-.0019	.0027	-.0046
.5000	-.0000	-.026	-.0005	.0109	.0000	.0001	-.0002
1.2000	-.0003		-.0088	-.0349	-.0051	.0041	-.0085
.6000	-.0000	-.038	-.0013	.0173	.0002	.0002	-.0005
1.4000	-.0006		-.0146	-.0499	-.0106	.0051	-.0135
.7000	-.0000	-.052	-.0026	.0244	.0006	.0002	-.0009
1.6000	-.0010		-.0213	-.0660	-.0185	.0055	-.0195
.8000	-.0001	-.066	-.0044	.0317	.0013	.0001	-.0016
1.8000	-.0016		-.0282	-.0821	-.0283	.0053	-.0261
.9000	-.0001	-.081	-.0067	.0387	.0024	-.0000	-.0024
2.0000	-.0023		-.0348	-.0977	-.0398	.0044	-.0330
1.0000	-.0002	-.095	-.0091	.0449	.0040	-.0003	-.0034
3.0000	-.0061		-.0576	-.1571	-.1016	-.0052	-.0638
1.5000	-.0009	-.152	-.0201	.0626	.0160	-.0027	-.0097
4.0000	-.0091		-.0655	-.1872	-.1485	-.0152	-.0834
2.0000	-.0016	-.183	-.0255	.0652	.0284	-.0049	-.0154
5.0000	-.0109		-.0675	-.2805	-.1769	-.0222	-.0941
2.5000	-.0022	-.197	-.0274	.0626	.0376	-.0064	-.0195

TABLE C.66 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .125
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .500

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0036	-.0180	-.0016	.0030	-.0045
.5000	-.0000	-.025	-.0004	.0089	.0000	.0001	-.0002
1.2000	-.0002		-.0073	-.0288	-.0042	.0047	-.0083
.6000	-.0000	-.036	-.0010	.0142	.0002	.0002	-.0005
1.4000	-.0005		-.0122	-.0415	-.0088	.0061	-.0132
.7000	-.0000	-.049	-.0022	.0203	.0005	.0003	-.0010
1.6000	-.0009		-.0178	-.0553	-.0155	.0071	-.0188
.8000	-.0001	-.063	-.0037	.0266	.0011	.0004	-.0016
1.8000	-.0014		-.0238	-.0695	-.0240	.0073	-.0249
.9000	-.0001	-.076	-.0056	.0327	.0020	.0003	-.0024
2.0000	-.0020		-.0296	-.0833	-.0338	.0069	-.0312
1.0000	-.0002	-.089	-.0078	.0383	.0034	.0001	-.0034
3.0000	-.0053		-.0503	-.1378	-.0889	-.0011	-.0592
1.5000	-.0008	-.140	-.0176	.0549	.0140	-.0017	-.0091
4.0000	-.0081		-.0579	-.1670	-.1321	-.0110	-.0768
2.0000	-.0015	-.169	-.0226	.0583	.0253	-.0038	-.0143
5.0000	-.0098		-.0600	-.1808	-.1589	-.0183	-.0862
2.5000	-.0020	-.181	-.0245	.0566	.0338	-.0053	-.0180

TABLE C.67 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0006		-.0275	-.1202	-.0118	-.0069	-.0043
.7500	-.0002	-.042	-.0125	.0898	.0031	-.0030	-.0009
1.2000	-.0011		-.0341	-.1141	-.0195	-.0088	-.0078
.9000	-.0003	-.048	-.0170	.0847	.0061	-.0042	-.0021
1.4000	-.0015		-.0381	-.1049	-.0269	-.0099	-.0113
1.0500	-.0005	-.052	-.0204	.0770	.0095	-.0051	-.0036
1.6000	-.0019		-.0403	-.0951	-.0332	-.0105	-.0145
1.2000	-.0007	-.055	-.0226	.0687	.0128	-.0057	-.0051
1.8000	-.0022		-.0413	-.0862	-.0384	-.0107	-.0172
1.3500	-.0009	-.056	-.0239	.0611	.0159	-.0061	-.0066
2.0000	-.0025		-.0418	-.0785	-.0426	-.0107	-.0195
1.5000	-.0010	-.056	-.0247	.0544	.0186	-.0063	-.0080
3.0000	-.0032		-.0418	-.0580	-.0522	-.0094	-.0253
2.2500	-.0016	-.055	-.0257	.0357	.0266	-.0058	-.0125
4.0000	-.0033		-.0417	-.0531	-.0538	-.0085	-.0267
3.0000	-.0018	-.054	-.0257	.0306	.0293	-.0051	-.0142
5.0000	-.0034		-.0417	-.0522	-.0538	-.0081	-.0269
3.7500	-.0018	-.054	-.0257	.0293	.0300	-.0047	-.0148

TABLE C.68 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .050
PDL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0091	-.0447	-.0039	-.0003	-.0034
.7500	-.0001	-.024	-.0042	.0334	.0010	-.0003	-.0010
1.2000	-.0004		-.0139	-.0538	-.0081	-.0012	-.0057
.9000	-.0001	-.030	-.0070	.0400	.0025	-.0008	-.0018
1.4000	-.0007		-.0179	-.0591	-.0130	-.0021	-.0082
1.0500	-.0002	-.035	-.0097	.0435	.0046	-.0014	-.0028
1.6000	-.0010		-.0208	-.0613	-.0179	-.0031	-.0105
1.2000	-.0004	-.038	-.0119	.0446	.0069	-.0019	-.0039
1.8000	-.0013		-.0228	-.0616	-.0225	-.0039	-.0126
1.3500	-.0005	-.041	-.0135	.0441	.0093	-.0024	-.0050
2.0000	-.0015		-.0240	-.0607	-.0266	-.0046	-.0144
1.5000	-.0006	-.043	-.0147	.0427	.0115	-.0029	-.0061
3.0000	-.0023		-.0254	-.0518	-.0389	-.0064	-.0197
2.2500	-.0011	-.045	-.0165	.0332	.0195	-.0039	-.0097
4.0000	-.0026		-.0253	-.0463	-.0429	-.0067	-.0214
3.0000	-.0014	-.044	-.0166	.0275	.0230	-.0040	-.0113
5.0000	-.0027		-.0253	-.0443	-.0438	-.0067	-.0218
3.7500	-.0015	-.044	-.0166	.0254	.0242	-.0039	-.0120

TABLE C.69 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .113
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0055	-.0269	-.0024	.0010	-.0033
.7500	-.0000	-.020	-.0025	.0201	.0006	.0002	-.0010
1.2000	-.0003		-.0088	-.0341	-.0051	.0009	-.0052
.9000	-.0001	-.025	-.0044	.0254	.0016	.0001	-.0017
1.4000	-.0005		-.0118	-.0393	-.0085	.0004	-.0072
1.0500	-.0002	-.029	-.0064	.0289	.0030	-.0002	-.0026
1.6000	-.0007		-.0142	-.0425	-.0123	-.0002	-.0091
1.2000	-.0003	-.032	-.0081	.0309	.0047	-.0005	-.0035
1.8000	-.0009		-.0160	-.0442	-.0159	-.0009	-.0108
1.3500	-.0004	-.034	-.0095	.0317	.0065	-.0009	-.0044
2.0000	-.0011		-.0172	-.0450	-.0193	-.0016	-.0122
1.5000	-.0005	-.035	-.0106	.0317	.0083	-.0012	-.0052
3.0000	-.0018		-.0192	-.0426	-.0306	-.0041	-.0165
2.2500	-.0009	-.038	-.0126	.0275	.0153	-.0025	-.0081
4.0000	-.0022		-.0193	-.0394	-.0351	-.0052	-.0180
3.0000	-.0011	-.037	-.0128	.0237	.0187	-.0030	-.0095
5.0000	-.0023		-.0193	-.0378	-.0366	-.0055	-.0184
3.7500	-.0012	-.037	-.0128	.0218	.0201	-.0032	-.0101

TABLE C.70 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .300
SPL.......= .169
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0039	-.0192	-.0017	.0016	-.0032
.7500	-.0000	-.018	-.0018	.0144	.0004	.0004	-.0010
1.2000	-.0002		-.0064	-.0250	-.0037	.0018	-.0050
.9000	-.0001	-.023	-.0032	.0186	.0012	.0005	-.0017
1.4000	-.0004		-.0088	-.0294	-.0064	.0017	-.0067
1.0500	-.0001	-.026	-.0048	.0216	.0022	.0004	-.0025
1.6000	-.0005		-.0108	-.0324	-.0093	.0013	-.0083
1.2000	-.0002	-.028	-.0062	.0236	.0036	.0003	-.0033
1.8000	-.0007		-.0123	-.0344	-.0123	.0007	-.0098
1.3500	-.0003	-.030	-.0073	.0247	.0050	-.0000	-.0041
2.0000	-.0009		-.0135	-.0356	-.0151	.0001	-.0110
1.5000	-.0004	-.031	-.0083	.0251	.0065	-.0003	-.0047
3.0000	-.0015		-.0155	-.0357	-.0252	-.0026	-.0144
2.2500	-.0007	-.033	-.0102	.0232	.0126	-.0016	-.0071
4.0000	-.0018		-.0157	-.0339	-.0296	-.0040	-.0155
3.0000	-.0009	-.032	-.0105	.0205	.0158	-.0023	-.0082
5.0000	-.0019		-.0157	-.0327	-.0313	-.0046	-.0159
3.7500	-.0011	-.032	-.0106	.0190	.0172	-.0026	-.0087

TABLE C.71 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .225
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0030	-.0150	-.0013	.0019	-.0031
.7500	-.0000	-.017	-.0014	.0112	.0003	.0006	-.0010
1.2000	-.0002		-.0050	-.0197	-.0029	.0023	-.0048
.9000	-.0000	-.021	-.0025	.0146	.0009	.0008	-.0017
1.4000	-.0003		-.0070	-.0234	-.0051	.0024	-.0064
1.0500	-.0001	-.024	-.0038	.0173	.0018	.0008	-.0024
1.6000	-.0004		-.0087	-.0262	-.0075	.0022	-.0079
1.2000	-.0002	-.026	-.0050	.0191	.0029	.0007	-.0032
1.8000	-.0006		-.0100	-.0281	-.0100	.0018	-.0091
1.3500	-.0002	-.027	-.0060	.0202	.0041	.0005	-.0038
2.0000	-.0007		-.0110	-.0294	-.0124	.0012	-.0102
1.5000	-.0003	-.028	-.0068	.0208	.0054	.0003	-.0044
3.0000	-.0013		-.0130	-.0306	-.0214	-.0015	-.0130
2.2500	-.0006	-.029	-.0086	.0199	.0107	-.0010	-.0065
4.0000	-.0016		-.0133	-.0296	-.0256	-.0031	-.0138
3.0000	-.0008	-.029	-.0089	.0180	.0136	-.0018	-.0073
5.0000	-.0017		-.0133	-.0288	-.0273	-.0038	-.0140
3.7500	-.0009	-.028	-.0090	.0168	.0149	-.0022	-.0077

TABLE C.72 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .281
PDL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0025	-.0123	-.0011	.0021	-.0031
.7500	-.0000	-.017	-.0011	.0092	.0003	.0007	-.0010
1.2000	-.0001		-.0041	-.0162	-.0024	.0027	-.0047
.9000	-.0000	-.020	-.0021	.0121	.0008	.0009	-.0017
1.4000	-.0002		-.0058	-.0195	-.0042	.0029	-.0063
1.0500	-.0001	-.023	-.0031	.0144	.0015	.0010	-.0024
1.6000	-.0004		-.0073	-.0220	-.0063	.0028	-.0076
1.2000	-.0001	-.024	-.0042	.0160	.0024	.0010	-.0031
1.8000	-.0005		-.0085	-.0238	-.0084	.0025	-.0087
1.3500	-.0002	-.026	-.0050	.0171	.0035	.0009	-.0037
2.0000	-.0006		-.0094	-.0250	-.0106	.0020	-.0096
1.5000	-.0003	-.026	-.0058	.0177	.0046	.0007	-.0042
3.0000	-.0011		-.0113	-.0268	-.0186	-.0007	-.0119
2.2500	-.0005	-.027	-.0075	.0174	.0093	-.0005	-.0059
4.0000	-.0014		-.0115	-.0262	-.0225	-.0024	-.0125
3.0000	-.0007	-.026	-.0078	.0160	.0119	-.0014	-.0066
5.0000	-.0015		-.0116	-.0257	-.0242	-.0032	-.0126
3.7500	-.0008	-.025	-.0078	.0150	.0132	-.0018	-.0069

TABLE C.73 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0008		-.0348	-.1571	-.0150	-.0088	-.0055
.7500	-.0002	-.053	-.0158	.1174	.0039	-.0038	-.0011
1.2000	-.0015		-.0472	-.1666	-.0272	-.0123	-.0108
.9000	-.0005	-.068	-.0237	.1238	.0085	-.0059	-.0029
1.4000	-.0022		-.0569	-.1696	-.0405	-.0151	-.0170
1.0500	-.0008	-.080	-.0306	.1247	.0143	-.0078	-.0053
1.6000	-.0030		-.0638	-.1684	-.0537	-.0172	-.0233
1.2000	-.0011	-.091	-.0361	.1222	.0207	-.0093	-.0082
1.8000	-.0037		-.0685	-.1651	-.0658	-.0187	-.0294
1.3500	-.0015	-.099	-.0402	.1178	.0271	-.0105	-.0113
2.0000	-.0044		-.0716	-.1607	-.0765	-.0196	-.0349
1.5000	-.0018	-.106	-.0432	.1126	.0333	-.0114	-.0142
3.0000	-.0066		-.0764	-.1405	-.1097	-.0206	-.0528
2.2500	-.0032	-.122	-.0487	.0694	.0553	-.0125	-.0257
4.0000	-.0075		-.0768	-.1307	-.1217	-.0199	-.0599
3.0000	-.0039	-.126	-.0493	.0776	.0653	-.0120	-.0315
5.0000	-.0078		-.0768	-.1272	-.1255	-.0194	-.0624
3.7500	-.0043	-.127	-.0494	.0729	.0693	-.0114	-.0341

TABLE C.74 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .100
SPL.......= .056
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0003		-.0115	-.0570	-.0050	-.0004	-.0043
.7500	-.0001	-.030	-.0053	.0426	.0013	-.0004	-.0012
1.2000	-.0006		-.0193	-.0756	-.0112	-.0016	-.0080
.9000	-.0002	-.042	-.0097	.0562	.0035	-.0011	-.0025
1.4000	-.0011		-.0269	-.0904	-.0195	-.0032	-.0123
1.0500	-.0004	-.053	-.0146	.0666	.0069	-.0021	-.0043
1.6000	-.0016		-.0334	-.1011	-.0289	-.0049	-.0170
1.2000	-.0006	-.063	-.0191	.0736	.0111	-.0031	-.0064
1.8000	-.0022		-.0385	-.1083	-.0385	-.0066	-.0216
1.3500	-.0009	-.071	-.0230	.0778	.0158	-.0041	-.0086
2.0000	-.0028		-.0423	-.1128	-.0476	-.0082	-.0260
1.5000	-.0011	-.078	-.0260	.0797	.0206	-.0051	-.0108
3.0000	-.0049		-.0492	-.1152	-.0806	-.0132	-.0412
2.2500	-.0024	-.097	-.0326	.0749	.0402	-.0080	-.0201
4.0000	-.0059		-.0499	-.1097	-.0955	-.0150	-.0480
3.0000	-.0031	-.102	-.0336	.0666	.0507	-.0089	-.0251
5.0000	-.0063		-.0500	-.1060	-.1011	-.0155	-.0505
3.7500	-.0034	-.103	-.0337	.0617	.0554	-.0090	-.0275

TABLE C.75 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .200
SPL.......= .113
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0069	-.0343	-.0030	.0013	-.0041
.7500	-.0000	-.026	-.0032	.0256	.0008	.0003	-.0013
1.2000	-.0004		-.0122	-.0478	-.0071	.0012	-.0072
.9000	-.0001	-.035	-.0061	.0356	.0022	.0001	-.0024
1.4000	-.0007		-.0177	-.0599	-.0129	.0006	-.0109
1.0500	-.0002	-.044	-.0096	.0441	.0045	-.0002	-.0039
1.6000	-.0011		-.0228	-.0697	-.0198	-.0003	-.0147
1.2000	-.0004	-.052	-.0131	.0508	.0076	-.0008	-.0057
1.8000	-.0015		-.0271	-.0773	-.0272	-.0015	-.0185
1.3500	-.0006	-.059	-.0162	.0555	.0112	-.0015	-.0075
2.0000	-.0020		-.0305	-.0828	-.0345	-.0027	-.0220
1.5000	-.0008	-.065	-.0188	.0586	.0149	-.0022	-.0094
3.0000	-.0038		-.0377	-.0928	-.0632	-.0082	-.0345
2.2500	-.0018	-.081	-.0251	.0605	.0315	-.0051	-.0169
4.0000	-.0048		-.0387	-.0918	-.0778	-.0112	-.0402
3.0000	-.0025	-.086	-.0263	.0561	.0412	-.0066	-.0211
5.0000	-.0052		-.0388	-.0897	-.0840	-.0125	-.0425
3.7500	-.0028	-.087	-.0264	.0526	.0459	-.0072	-.0231

TABLE C.76 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .300
SPL.......= .169
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMG	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0049	-.0245	-.0021	.0020	-.0040
.7500	-.0000	-.024	-.0023	.0183	.0006	.0006	-.0013
1.2000	-.0003		-.0089	-.0350	-.0052	.0025	-.0069
.9000	-.0001	-.032	-.0045	.0260	.0016	.0007	-.0024
1.4000	-.0005		-.0132	-.0447	-.0096	.0025	-.0102
1.0500	-.0002	-.040	-.0072	.0330	.0034	.0007	-.0038
1.6000	-.0008		-.0173	-.0531	-.0150	.0021	-.0135
1.2000	-.0003	-.046	-.0099	.0387	.0058	.0004	-.0053
1.8000	-.0012		-.0209	-.0599	-.0210	.0013	-.0168
1.3500	-.0005	-.052	-.0125	.0430	.0086	-.0000	-.0069
2.0000	-.0016		-.0238	-.0653	-.0270	.0004	-.0198
1.5000	-.0006	-.057	-.0147	.0462	.0117	-.0005	-.0085
3.0000	-.0031		-.0306	-.0771	-.0520	-.0049	-.0302
2.2500	-.0015	-.071	-.0205	.0504	.0258	-.0032	-.0149
4.0000	-.0040		-.0318	-.0783	-.0655	-.0084	-.0349
3.0000	-.0021	-.075	-.0217	.0480	.0346	-.0050	-.0183
5.0000	-.0044		-.0319	-.0774	-.0717	-.0102	-.0368
3.7500	-.0024	-.076	-.0219	.0455	.0391	-.0058	-.0200

TABLE C.77 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .225
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0038	-.0191	-.0017	.0024	-.0040
.7500	-.0000	-.022	-.0018	.0143	.0004	.0007	-.0013
1.2000	-.0002		-.0070	-.0276	-.0041	.0032	-.0067
.9000	-.0001	-.030	-.0035	.0205	.0013	.0011	-.0024
1.4000	-.0004		-.0105	-.0357	-.0076	.0037	-.0097
1.0500	-.0001	-.037	-.0057	.0263	.0027	.0012	-.0037
1.6000	-.0007		-.0140	-.0429	-.0121	.0036	-.0128
1.2000	-.0003	-.043	-.0080	.0312	.0047	.0012	-.0051
1.8000	-.0010		-.0170	-.0489	-.0171	.0031	-.0157
1.3500	-.0004	-.048	-.0102	.0351	.0070	.0009	-.0066
2.0000	-.0013		-.0196	-.0538	-.0222	.0024	-.0183
1.5000	-.0005	-.052	-.0121	.0381	.0096	.0005	-.0080
3.0000	-.0026		-.0258	-.0658	-.0441	-.0026	-.0272
2.2500	-.0013	-.063	-.0173	.0431	.0219	-.0019	-.0135
4.0000	-.0035		-.0271	-.0681	-.0565	-.0063	-.0311
3.0000	-.0018	-.066	-.0185	.0419	.0298	-.0038	-.0163
5.0000	-.0039		-.0272	-.0679	-.0625	-.0084	-.0326
3.7500	-.0021	-.067	-.0187	.0401	.0340	-.0048	-.0177

TABLE C.78 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .281
POL.......= 1.000
POR.......= 0.000
POIS.R....= .300
K.........= .750

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0031	-.0156	-.0014	.0026	-.0039
.7500	-.0000	-.022	-.0014	.0117	.0004	.0008	-.0013
1.2000	-.0002		-.0058	-.0227	-.0034	.0037	-.0066
.9000	-.0001	-.029	-.0029	.0169	.0010	.0013	-.0023
1.4000	-.0003		-.0087	-.0297	-.0064	.0044	-.0095
1.0500	-.0001	-.035	-.0047	.0219	.0022	.0016	-.0036
1.6000	-.0006		-.0117	-.0360	-.0101	.0046	-.0123
1.2000	-.0002	-.041	-.0067	.0262	.0039	.0017	-.0050
1.8000	-.0006		-.0144	-.0413	-.0144	.0043	-.0149
1.3500	-.0003	-.045	-.0086	.0297	.0059	.0015	-.0063
2.0000	-.0011		-.0166	-.0458	-.0189	.0038	-.0173
1.5000	-.0005	-.049	-.0103	.0324	.0082	.0013	-.0076
3.0000	-.0023		-.0223	-.0574	-.0383	-.0009	-.0250
2.2500	-.0011	-.058	-.0150	.0376	.0190	-.0009	-.0124
4.0000	-.0030		-.0236	-.0602	-.0497	-.0048	-.0282
3.0000	-.0016	-.060	-.0162	.0371	.0262	-.0028	-.0148
5.0000	-.0034		-.0238	-.0604	-.0553	-.0070	-.0293
3.7500	-.0018	-.060	-.0164	.0357	.0301	-.0040	-.0159

TABLE C.79 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= 0.000
SPL.......= 0.000
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0006		-.0277	-.1214	-.0119	-.0070	-.0044
1.0000	-.0006	-.042	-.0277	.1214	.0119	-.0070	-.0044
1.2000	-.0011		-.0347	-.1165	-.0199	-.0090	-.0079
1.2000	-.0011	-.049	-.0347	.1165	.0199	-.0090	-.0079
1.4000	-.0015		-.0392	-.1087	-.0277	-.0103	-.0116
1.4000	-.0015	-.054	-.0392	.1087	.0277	-.0103	-.0116
1.6000	-.0019		-.0419	-.1003	-.0347	-.0110	-.0151
1.6000	-.0019	-.057	-.0419	.1003	.0347	-.0110	-.0151
1.8000	-.0023		-.0434	-.0925	-.0406	-.0113	-.0182
1.8000	-.0023	-.059	-.0434	.0925	.0406	-.0113	-.0182
2.0000	-.0026		-.0443	-.0857	-.0455	-.0114	-.0208
2.0000	-.0026	-.061	-.0443	.0857	.0455	-.0114	-.0208
3.0000	-.0035		-.0452	-.0673	-.0583	-.0106	-.0282
3.0000	-.0035	-.062	-.0452	.0673	.0583	-.0106	-.0282
4.0000	-.0038		-.0452	-.0626	-.0616	-.0098	-.0305
4.0000	-.0038	-.062	-.0452	.0626	.0616	-.0098	-.0305
5.0000	-.0039		-.0452	-.0614	-.0623	-.0095	-.0311
5.0000	-.0039	-.062	-.0452	.0614	.0623	-.0095	-.0311

TABLE C.80 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR....... = .100
SPL....... = .100
POL....... = 0.000
POR....... = 1.000
POIS.R.... = .300
K......... = 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0002		-.0091	-.0448	-.0040	-.0003	-.0034
1.0000	-.0002	-.024	-.0091	.0448	.0040	-.0003	-.0034
1.2000	-.0004		-.0140	-.0541	-.0081	-.0012	-.0057
1.2000	-.0004	-.030	-.0140	.0541	.0081	-.0012	-.0057
1.4000	-.0007		-.0181	-.0597	-.0131	-.0021	-.0082
1.4000	-.0007	-.035	-.0181	.0597	.0131	-.0021	-.0082
1.6000	-.0010		-.0211	-.0623	-.0182	-.0031	-.0107
1.6000	-.0010	-.039	-.0211	.0623	.0182	-.0031	-.0107
1.8000	-.0013		-.0233	-.0631	-.0230	-.0040	-.0129
1.8000	-.0013	-.042	-.0233	.0631	.0230	-.0040	-.0129
2.0000	-.0016		-.0246	-.0626	-.0274	-.0047	-.0149
2.0000	-.0016	-.044	-.0246	.0626	.0274	-.0047	-.0149
3.0000	-.0025		-.0266	-.0554	-.0412	-.0068	-.0209
3.0000	-.0025	-.048	-.0266	.0554	.0412	-.0068	-.0209
4.0000	-.0028		-.0267	-.0505	-.0462	-.0073	-.0231
4.0000	-.0026	-.048	-.0267	.0505	.0462	-.0073	-.0231
5.0000	-.0030		-.0267	-.0487	-.0476	-.0073	-.0238
5.0000	-.0030	-.048	-.0267	.0487	.0476	-.0073	-.0238

TABLE C.81 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR....... = .200
SPL....... = .200
POL....... = 0.000
POR....... = 1.000
POIS.R.... = .300
K......... = 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0055	-.0269	-.0024	.0010	-.0033
1.0000	-.0001	-.020	-.0055	.0269	.0024	.0010	-.0033
1.2000	-.0003		-.0088	-.0342	-.0051	.0009	-.0052
1.2000	-.0003	-.025	-.0088	.0342	.0051	.0009	-.0052
1.4000	-.0005		-.0118	-.0394	-.0086	.0004	-.0072
1.4000	-.0005	-.029	-.0118	.0394	.0086	.0004	-.0072
1.6000	-.0007		-.0143	-.0427	-.0123	-.0002	-.0091
1.6000	-.0007	-.032	-.0143	.0427	.0123	-.0002	-.0091
1.8000	-.0009		-.0161	-.0447	-.0161	-.0009	-.0109
1.8000	-.0009	-.034	-.0161	.0447	.0161	-.0009	-.0109
2.0000	-.0011		-.0174	-.0456	-.0195	-.0016	-.0124
2.0000	-.0011	-.036	-.0174	.0456	.0195	-.0016	-.0124
3.0000	-.0019		-.0197	-.0440	-.0315	-.0042	-.0170
3.0000	-.0019	-.039	-.0197	.0440	.0315	-.0042	-.0170
4.0000	-.0022		-.0199	-.0412	-.0365	-.0054	-.0187
4.0000	-.0022	-.039	-.0199	.0412	.0365	-.0054	-.0187
5.0000	-.0024		-.0199	-.0398	-.0383	-.0057	-.0193
5.0000	-.0024	-.039	-.0199	.0398	.0383	-.0057	-.0193

TABLE C.82 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .300
SPL.......= .300
POL.......= 0.000
POR.......= 1.000
POIS.R....= .300
K.........= 1.000

RR / RL	KWR / KWL	KMC	KQXR / KQXL	KQYRN / KQYLN	KQYRF / KQYLF	KMXR / KMXL	KMYR / KMYL
1.0000	-.0001		-.0039	-.0192	-.0017	.0016	-.0032
1.0000	-.0001	-.018	-.0039	.0192	.0017	.0016	-.0032
1.2000	-.0002		-.0064	-.0250	-.0037	.0018	-.0050
1.2000	-.0002	-.023	-.0064	.0250	.0037	.0018	-.0050
1.4000	-.0004		-.0088	-.0294	-.0064	.0017	-.0067
1.4000	-.0004	-.026	-.0088	.0294	.0064	.0017	-.0067
1.6000	-.0005		-.0108	-.0325	-.0093	.0013	-.0083
1.6000	-.0005	-.028	-.0108	.0325	.0093	.0013	-.0083
1.8000	-.0007		-.0123	-.0345	-.0123	.0007	-.0098
1.8000	-.0007	-.030	-.0123	.0345	.0123	.0007	-.0098
2.0000	-.0009		-.0135	-.0357	-.0152	.0001	-.0110
2.0000	-.0009	-.031	-.0135	.0357	.0152	.0001	-.0110
3.0000	-.0015		-.0157	-.0362	-.0255	-.0026	-.0146
3.0000	-.0015	-.033	-.0157	.0362	.0255	-.0026	-.0146
4.0000	-.0019		-.0159	-.0346	-.0302	-.0041	-.0158
4.0000	-.0019	-.033	-.0159	.0346	.0302	-.0041	-.0158
5.0000	-.0020		-.0159	-.0336	-.0320	-.0046	-.0163
5.0000	-.0020	-.033	-.0159	.0336	.0320	-.0046	-.0163

TABLE C.83 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .400
SPL.......= .400
POL.......= 0.000
PUR.......= 1.000
POIS.R....= .300
K.........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0030	-.[illegible]50	-.0013	.0019	-.0031
1.0000	-.0001	-.017	-.0030	.0150	.0013	.0019	-.0031
1.2000	-.0002		-.0050	-.0197	-.0029	.0023	-.0048
1.2000	-.0002	-.021	-.0050	.0197	.0029	.0023	-.0048
1.4000	-.0003		-.0070	-.0234	-.0051	.0024	-.0064
1.4000	-.0003	-.024	-.0070	.0234	.0051	.0024	-.0064
1.6000	-.0004		-.0087	-.0262	-.0075	.0022	-.0079
1.6000	-.0004	-.026	-.0087	.0262	.0075	.0022	-.0079
1.8000	-.0006		-.0100	-.0281	-.0100	.0018	-.0091
1.8000	-.0006	-.027	-.0100	.0281	.0100	.0018	-.0091
2.0000	-.0007		-.0110	-.0293	-.0124	.0012	-.0101
2.0000	-.0007	-.028	-.0110	.0293	.0124	.0012	-.0101
3.0000	-.0013		-.0130	-.0306	-.0214	-.0015	-.0130
3.0000	-.0013	-.029	-.0130	.0306	.0214	-.0015	-.0130
4.0000	-.0016		-.0133	-.0297	-.0257	-.0031	-.0138
4.0000	-.0016	-.029	-.0133	.0297	.0257	-.0031	-.0138
5.0000	-.0017		-.0133	-.0290	-.0274	-.0038	-.0141
5.0000	-.0017	-.029	-.0133	.0290	.0274	-.0038	-.0141

TABLE C.84 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

SPR.......= .500
SPL.......= .500
POL.......= 0.000
POR.......= 1.000
POIS.R...= .300
K.........= 1.000

RR RL	KWR KWL	KMC	KQXR KQXL	KQYRN KQYLN	KQYRF KQYLF	KMXR KMXL	KMYR KMYL
1.0000	-.0001		-.0025	-.0122	-.0011	.0021	-.0031
1.0000	-.0001	-.017	-.0025	.0122	.0011	.0021	-.0031
1.2000	-.0001		-.0041	-.0162	-.0024	.0027	-.0047
1.2000	-.0001	-.020	-.0041	.0162	.0024	.0027	-.0047
1.4000	-.0002		-.0058	-.0194	-.0042	.0029	-.0062
1.4000	-.0002	-.023	-.0058	.0194	.0042	.0029	-.0062
1.6000	-.0004		-.0072	-.0219	-.0063	.0028	-.0076
1.6000	-.0004	-.024	-.0072	.0219	.0063	.0028	-.0076
1.8000	-.0005		-.0084	-.0237	-.0084	.0025	-.0086
1.8000	-.0005	-.025	-.0084	.0237	.0084	.0025	-.0086
2.0000	-.0006		-.0093	-.0249	-.0105	.0020	-.0095
2.0000	-.0006	-.026	-.0093	.0249	.0105	.0020	-.0095
3.0000	-.0011		-.0112	-.0265	-.0184	-.0007	-.0118
3.0000	-.0011	-.026	-.0112	.0265	.0184	-.0007	-.0118
4.0000	-.0014		-.0115	-.0260	-.0223	-.0024	-.0124
4.0000	-.0014	-.026	-.0115	.0260	.0223	-.0024	-.0124
5.0000	-.0015		-.0115	-.0255	-.0240	-.0032	-.0125
5.0000	-.0015	-.025	-.0115	.0255	.0240	-.0032	-.0125

TABLE C.85 - NUMERICAL VALUES FOR FACTORS IN EQUS. (3.51) TO (3.56)

APPENDIX D

THE COEFFICIENT MATRICES AND LOAD VECTORS USED IN THE ANALYSIS OF CONTINUOUS SANDWICH PLATES BY THE FINITE DIFFERENCE METHOD

APPENDIX D

THE COEFFICIENT MATRICES AND LOAD VECTORS USED IN THE ANALYSIS OF CONTINUOUS SANDWICH PLATES BY THE FINITE DIFFERENCE METHOD

By applying the funicular polygon's stencil (Fig. 2.5) on the partial deflection equation (2.68), the coefficient matrix [A] and the load vector $\{P_b\}$ are obtained as

$$[A] = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{rrrrr} w_{b1} & w_{b2} & w_{b3} & w_{b4} & w_{b5} \end{array} \\ \left[\begin{array}{rrrrr} 18. & -576. & 72. & -608. & -32. \\ -576. & 1872. & -576. & -32. & -576. \\ 72. & -576. & 1800. & 32. & -32. \\ -1216. & -64. & 64. & 1800. & -576. \\ -64. & -1152. & -64. & -576. & 1872. \\ 64. & -64. & -1216. & 72. & -576. \\ 0. & 0. & -576. & 0. & 0. \\ 0. & 0. & 72. & 0. & 0. \\ 0. & 0. & -64. & 0. & 0. \\ 0. & 0. & 64. & 0. & 0. \end{array}\right. \end{array}$$

$$
\begin{array}{c c}
\begin{array}{ccccc} w_{b6} & w_{b7} & w_{b8} & w_{b9} & w_{b10} \end{array} & \\
\left[\begin{array}{rrrrr}
32. & 0. & 0. & 0. & 0. \\
-32. & 0. & 0. & 0. & 0. \\
-608. & 0. & 0. & 0. & 0. \\
72. & 0. & 0. & 0. & 0. \\
-576. & 0. & 0. & 0. & 0. \\
1800. & 0. & 0. & 0. & 0. \\
-32. & 1872. & -576. & -576. & -32. \\
32. & -576. & 1800. & -32. & -608. \\
-576. & -1152. & -64. & 1872. & -576. \\
72. & -64. & -1216. & -576. & 1800.
\end{array}\right] &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\end{array}
$$

$$
\{P_b\} = \frac{\lambda^4 \quad p_o^{\ell}}{D} \left\{ \begin{array}{c} 103.3 \\ 120.2 \\ 103.3 \\ 120.2 \\ 138.6 \\ 120.2 \\ 0. \\ 0. \\ 0. \\ 0. \end{array} \right\}
$$

in which

$$\lambda = \text{the finite difference mesh width (Fig. 3.8 (b))}$$
$$= \frac{a}{4}.$$

In a similar manner, by applying the higher approximation stencil (Fig. 2.4) on the partial deflection equation (2.69), the coefficient matrix [B] and load vector $\{P_s\}$ are obtained as

$$[B] = \begin{array}{c} \begin{array}{cccccc} w_{s1} & w_{s2} & w_{s3} & w_{s4} & w_{s5} & w_{s6} \end{array} \\ \left[\begin{array}{cccccc} -59. & 16. & -1. & 16. & 0. & 0. \\ 16. & -60. & 16. & 0. & 16. & 0. \\ -1. & 16. & -59. & 0. & 0. & 16. \\ 32. & 0. & 0. & -59. & 16. & -1. \\ 0. & 32. & 0. & 16. & -60. & 16. \\ 0. & 0. & 32. & -1. & 16. & -59. \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\{P_s\} = -\frac{12\,\lambda^2\,p_o^{\ell}}{S}\begin{Bmatrix} 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \end{Bmatrix}$$

APPENDIX E

EXPLICIT FORMS OF THE MATRICES USED IN THE FINITE ELEMENT ANALYSIS

APPENDIX E

EXPLICIT FORMS OF THE MATRICES USED IN THE FINITE ELEMENT ANALYSIS

The strain-displacement matrix is defined by Equ. (6.19) as

$$[B] = [B_1 \; B_2 \; B_3 \; B_4 \; B_5 \; B_6 \; B_7 \; B_8] \tag{6.19}$$

in which

$$[B_1] = \begin{bmatrix} \frac{1}{8a}(1+\eta)(1+\zeta) & 0 & 0 \\ 0 & \frac{1}{8b}(1+\xi)(1+\zeta) & 0 \\ 0 & 0 & \frac{1}{8c}(1+\xi)(1+\eta) \\ \frac{1}{8b}(1+\xi)(1+\zeta) & \frac{1}{8a}(1+\eta)(1+\zeta) & 0 \\ 0 & \frac{1}{8c}(1+\xi)(1+\eta) & \frac{1}{8b}(1+\xi)(1+\zeta) \\ \frac{1}{8c}(1+\xi)(1+\eta) & 0 & \frac{1}{8a}(1+\eta)(1+\zeta) \end{bmatrix}$$

$$[B_2] = \begin{bmatrix} \frac{1}{8a}(1-\eta)(1+\zeta) & 0 & 0 \\ 0 & -\frac{1}{8b}(1+\xi)(1+\zeta) & 0 \\ 0 & 0 & \frac{1}{8c}(1+\xi)(1-\eta) \\ -\frac{1}{8b}(1+\xi)(1+\zeta) & \frac{1}{8a}(1-\eta)(1+\zeta) & 0 \\ 0 & \frac{1}{8c}(1+\xi)(1-\eta) & -\frac{1}{8b}(1+\xi)(1+\zeta) \\ \frac{1}{8c}(1+\xi)(1-\eta) & 0 & \frac{1}{8a}(1-\eta)(1+\zeta) \end{bmatrix}$$

$$[B_3] = \begin{bmatrix} -\frac{1}{8a}(1-\eta)(1+\zeta) & 0 & 0 \\ 0 & -\frac{1}{8b}(1-\xi)(1+\zeta) & 0 \\ 0 & 0 & \frac{1}{8c}(1-\xi)(1-\eta) \\ -\frac{1}{8b}(1-\xi)(1+\zeta) & -\frac{1}{8a}(1-\eta)(1+\zeta) & 0 \\ 0 & \frac{1}{8c}(1-\xi)(1+\eta) & -\frac{1}{8b}(1-\xi)(1+\zeta) \\ \frac{1}{8c}(1-\xi)(1+\eta) & 0 & -\frac{1}{8a}(1-\eta)(1+\zeta) \end{bmatrix}$$

$$[B_4] = \begin{bmatrix} -\frac{1}{8a}(1+\eta)(1+\zeta) & 0 & 0 \\ 0 & \frac{1}{8b}(1-\xi)(1+\zeta) & 0 \\ 0 & 0 & \frac{1}{8c}(1-\xi)(1+\eta) \\ \frac{1}{8b}(1-\xi)(1+\zeta) & -\frac{1}{8a}(1+\eta)(1+\zeta) & 0 \\ 0 & \frac{1}{8c}(1-\xi)(1+\eta) & \frac{1}{8b}(1-\xi)(1+\zeta) \\ \frac{1}{8c}(1-\xi)(1+\eta) & 0 & -\frac{1}{8a}(1+\eta)(1+\zeta) \end{bmatrix}$$

$$[B_5] = \begin{bmatrix} \frac{1}{8a}(1+\eta)(1-\zeta) & 0 & 0 \\ 0 & \frac{1}{8b}(1+\xi)(1-\zeta) & 0 \\ 0 & 0 & -\frac{1}{8c}(1+\xi)(1+\eta) \\ \frac{1}{8b}(1+\xi)(1-\zeta) & \frac{1}{8a}(1+\eta)(1-\zeta) & 0 \\ 0 & -\frac{1}{8c}(1+\xi)(1+\eta) & \frac{1}{8b}(1+\xi)(1-\zeta) \\ -\frac{1}{8c}(1+\xi)(1+\eta) & 0 & \frac{1}{8a}(1+\eta)(1-\zeta) \end{bmatrix}$$

$$[B_6] = \begin{bmatrix} \frac{1}{8a}(1-\eta)(1-\zeta) & 0 & 0 \\ 0 & -\frac{1}{8b}(1+\xi)(1-\zeta) & 0 \\ 0 & 0 & -\frac{1}{8c}(1+\xi)(1-\eta) \\ -\frac{1}{8b}(1+\xi)(1-\zeta) & \frac{1}{8a}(1-\eta)(1-\zeta) & 0 \\ 0 & -\frac{1}{8c}(1+\xi)(1-\eta) & -\frac{1}{8b}(1+\xi)(1-\zeta) \\ -\frac{1}{8c}(1+\xi)(1-\eta) & 0 & \frac{1}{8a}(1-\eta)(1-\zeta) \end{bmatrix}$$

$$[B_7] = \begin{bmatrix} -\frac{1}{8a}(1-\eta)(1-\zeta) & 0 & 0 \\ 0 & -\frac{1}{8b}(1-\xi)(1-\zeta) & 0 \\ 0 & 0 & -\frac{1}{8c}(1-\xi)(1-\eta) \\ -\frac{1}{8b}(1-\xi)(1-\zeta) & -\frac{1}{8a}(1-\eta)(1-\zeta) & 0 \\ 0 & -\frac{1}{8c}(1-\xi)(1-\eta) & -\frac{1}{8b}(1-\xi)(1-\zeta) \\ -\frac{1}{8c}(1-\xi)(1-\eta) & 0 & -\frac{1}{8a}(1-\eta)(1-\zeta) \end{bmatrix}$$

$$[B_8] = \begin{bmatrix} -\frac{1}{8a}(1+\eta)(1-\zeta) & 0 & 0 \\ 0 & \frac{1}{8b}(1-\xi)(1-\zeta) & 0 \\ 0 & 0 & -\frac{1}{8c}(1-\xi)(1+\eta) \\ \frac{1}{8b}(1-\xi)(1-\zeta) & -\frac{1}{8a}(1+\eta)(1-\zeta) & 0 \\ 0 & -\frac{1}{8c}(1-\xi)(1+\eta) & \frac{1}{8b}(1-\xi)(1-\zeta) \\ -\frac{1}{8c}(1-\xi)(1+\eta) & 0 & -\frac{1}{8a}(1+\eta)(1-\zeta) \end{bmatrix}$$

The element stiffness matrix is defined by Equ. (6.21) as

$$[K]_i = \frac{1}{64}\begin{bmatrix} K'_{11} & & & \\ K'_{21} & K'_{22} & \text{Symmetric} & \\ K'_{31} & K'_{32} & K'_{33} & \\ K'_{41} & K'_{42} & K'_{43} & K'_{44} \end{bmatrix}$$

in which

K_{cd} = a square matrix of order six,

$= \frac{1}{64} K'_{cd}$

The explicit forms of the matrices K'_{cd} are given next.

The following notations are used.

V = the volume of the rectangular prism finite element in Fig. 6.2,

= $8\,a\,b\,c$

E_{ij} = a coefficient in the elasticity matrix [E] defined by Equ. (6.20).

$$
\begin{bmatrix}
\frac{16V}{9}\left(\frac{E_{11}}{a^2}+\frac{E_{44}}{b^2}+\frac{E_{66}}{c^2}\right) & & & & & \\
\frac{4V}{3ab}(E_{12}+E_{44}) & \frac{16V}{9}\left(\frac{E_{22}}{b^2}+\frac{E_{44}}{a^2}+\frac{E_{55}}{c^2}\right) & & & & \\
\frac{4V}{3ac}(E_{13}+E_{66}) & \frac{4V}{3bc}(E_{23}+E_{55}) & \frac{16V}{9}\left(\frac{E_{33}}{c^2}+\frac{E_{55}}{b^2}+\frac{E_{66}}{a^2}\right) & & \text{Symmetric} & \\
\frac{8V}{9}\left(\frac{E_{11}}{a^2}-\frac{2E_{44}}{b^2}+\frac{E_{66}}{c^2}\right) & -(K'_{11})_{51} & \frac{1}{2}(K'_{11})_{31} & (K'_{11})_{11} & & \\
\frac{4V}{3ab}(E_{44}-E_{12}) & \frac{8V}{9}\left(\frac{-2E_{22}}{b^2}+\frac{E_{44}}{a^2}+\frac{E_{55}}{c^2}\right) & -(K'_{11})_{62} & -(K'_{11})_{21} & (K'_{11})_{22} & \\
\frac{1}{2}(K'_{11})_{31} & \frac{4V}{3bc}(E_{23}-E_{55}) & \frac{8V}{9}\left(\frac{E_{33}}{c^2}-\frac{2E_{55}}{b^2}+\frac{E_{66}}{a^2}\right) & (K'_{11})_{31} & -(K'_{11})_{32} & (K'_{11})_{33}
\end{bmatrix}
$$

SUB-MATRIX $K'_{11} = K'_{44}$.

$$\begin{bmatrix}
\frac{4V}{9}\left(\frac{-2E_{11}}{a^2} - \frac{2E_{44}}{b^2} + \frac{E_{66}}{c^2}\right) & -(K'_{11})_{21} & -(K'_{21})_{31} & (K'_{21})_{41} & -(K'_{11})_{51} & -2\,(K'_{21})_{31} \\
-(K'_{11})_{21} & \frac{8V}{9}\left(\frac{-E_{22}}{b^2} - \frac{E_{44}}{a^2} + \frac{E_{55}}{2c^2}\right) & -\frac{1}{2}(K'_{11})_{62} & (K'_{11})_{51} & (K'_{21})_{52} & -\frac{1}{2}(K'_{11})_{32} \\
\frac{2V}{3ac}(E_{12} - E_{66}) & \frac{1}{2}(K'_{11})_{62} & \frac{4V}{9}\left(\frac{E_{33}}{c^2} - \frac{2E_{55}}{b^2} - \frac{2E_{66}}{a^2}\right) & 2\,(K'_{21})_{31} & -\frac{1}{2}(K'_{11})_{32} & (K'_{21})_{63} \\
\frac{8V}{9}\left(\frac{-2E_{11}}{a^2} + \frac{E_{44}}{b^2} + \frac{E_{66}}{c^2}\right) & -(K'_{11})_{51} & -2\,(K'_{11})_{31} & -(K'_{21})_{11} & (K'_{11})_{21} & -(K'_{21})_{31} \\
-(K'_{11})_{51} & \frac{8V}{9}\left(\frac{E_{22}}{b^2} - \frac{2E_{44}}{a^2} + \frac{E_{55}}{c^2}\right) & \frac{1}{2}(K'_{11})_{32} & (K'_{11})_{21} & (K'_{21})_{22} & \frac{1}{2}(K'_{11})_{62} \\
2\,(K'_{21})_{31} & \frac{1}{2}(K'_{11})_{32} & \frac{8V}{9}\left(\frac{E_{33}}{c^2} + \frac{E_{55}}{b^2} - \frac{2E_{66}}{a^2}\right) & (K'_{21})_{31} & -\frac{1}{2}(K'_{11})_{62} & (K'_{21})_{33}
\end{bmatrix}$$

SUB-MATRIX K'_{21}

$$\begin{bmatrix}
\frac{8V}{9}\left(\frac{E_{11}}{a^2}+\frac{E_{44}}{b^2}-\frac{2E_{66}}{c^2}\right) & \frac{1}{2}(K'_{11})_{21} & 2(K'_{21})_{31} & (K'_{31})_{41} & (K'_{31})_{52} & (K'_{21})_{31} \\
\frac{1}{2}(K'_{11})_{21} & \frac{8V}{9}\left(\frac{E_{22}}{b^2}+\frac{E_{44}}{a^2}-\frac{2E_{55}}{c^2}\right) & (K'_{11})_{62} & -\frac{1}{2}(K'_{11})_{51} & (K'_{11})_{32} & -(K'_{11})_{32} \\
-2(K'_{21})_{31} & -(K'_{11})_{62} & \frac{8V}{9}\left(\frac{-2E_{33}}{c^2}+\frac{E_{55}}{b^2}+\frac{E_{66}}{a^2}\right) & -(K'_{21})_{31} & -(K'_{11})_{21} & (K'_{31})_{63} \\
\frac{8V}{9}\left(\frac{E_{11}}{2a^2}-\frac{E_{44}}{b^2}-\frac{E_{66}}{c^2}\right) & -\frac{1}{2}(K'_{11})_{51} & (K'_{21})_{31} & (K'_{31})_{11} & (K'_{31})_{22} & 2(K'_{21})_{31} \\
\frac{1}{2}(K'_{11})_{51} & \frac{8V}{9}\left(\frac{-E_{22}}{b^2}+\frac{E_{44}}{2a^2}-\frac{E_{55}}{c^2}\right) & -(K'_{11})_{32} & -\frac{1}{2}(K'_{11})_{21} & (K'_{11})_{62} & -(K'_{11})_{62} \\
-(K'_{21})_{31} & -(K'_{11})_{32} & \frac{4V}{9}\left(\frac{-2E_{33}}{c^2}-\frac{2E_{55}}{b^2}+\frac{E_{66}}{a^2}\right) & -2(K'_{21})_{31} & -\frac{1}{2}(K'_{11})_{51} & (K'_{31})_{33}
\end{bmatrix}$$

SUB-MATRIX K'_{31}

$$\begin{bmatrix}
\frac{4V}{9}\left(\frac{-E_{11}}{a^2}-\frac{E_{44}}{b^2}-\frac{E_{66}}{c^2}\right) & -\frac{1}{2}(K'_{11})_{21} & -\frac{1}{2}(K'_{11})_{31} & (K'_{41})_{41} & -\frac{1}{2}(K'_{11})_{51} & -(K'_{11})_{31} \\
-\frac{1}{2}(K'_{11})_{21} & \frac{4V}{9}\left(\frac{-E_{22}}{b^2}-\frac{E_{44}}{a^2}-\frac{E_{55}}{c^2}\right) & -\frac{1}{2}(K'_{11})_{32} & \frac{1}{2}(K'_{11})_{51} & (K'_{41})_{52} & -\frac{1}{2}(K'_{11})_{62} \\
-\frac{1}{2}(K'_{11})_{31} & -\frac{1}{2}(K'_{11})_{32} & \frac{4V}{9}\left(\frac{-E_{33}}{c^2}-\frac{E_{55}}{b^2}-\frac{E_{66}}{a^2}\right) & -(K'_{11})_{31} & \frac{1}{2}(K'_{11})_{62} & (K'_{41})_{63} \\
\frac{4V}{9}\left(\frac{-2E_{11}}{a^2}+\frac{E_{44}}{b^2}-\frac{2E_{66}}{c^2}\right) & \frac{1}{2}(K'_{11})_{51} & -(K'_{11})_{31} & (K'_{41})_{11} & \frac{1}{2}(K'_{11})_{21} & -\frac{1}{2}(K'_{11})_{31} \\
-\frac{1}{2}(K'_{11})_{51} & \frac{8V}{9}\left(\frac{E_{22}}{2b^2}-\frac{E_{44}}{a^2}-\frac{E_{55}}{c^2}\right) & -(K'_{11})_{62} & \frac{1}{2}(K'_{11})_{21} & (K'_{41})_{22} & \frac{1}{2}(K'_{11})_{32} \\
-(K'_{11})_{31} & -\frac{1}{2}(K'_{11})_{62} & \frac{4V}{9}\left(\frac{-2E_{33}}{c^2}+\frac{E_{55}}{b^2}-\frac{2E_{66}}{a^2}\right) & -\frac{1}{2}(K'_{11})_{31} & \frac{1}{2}(K'_{11})_{32} & (K'_{41})_{33}
\end{bmatrix}$$

SUB-MATRIX K'_{41}

$$\begin{bmatrix} (K'_{11})_{11} & & & & & \\ (K'_{11})_{21} & (K'_{11})_{22} & & & & \\ -(K'_{11})_{31} & -(K'_{11})_{32} & (K'_{11})_{33} & & \text{Symmetric} & \\ (K'_{11})_{41} & -(K'_{11})_{51} & -\frac{1}{2}(K'_{11})_{31} & (K'_{11})_{11} & & \\ (K'_{11})_{51} & (K'_{11})_{52} & (K'_{11})_{62} & -(K'_{11})_{21} & (K'_{11})_{22} & \\ -\frac{1}{2}(K'_{11})_{31} & -(K'_{11})_{62} & (K'_{11})_{63} & -(K'_{11})_{31} & (K'_{11})_{32} & (K'_{11})_{33} \end{bmatrix}$$

SUB-MATRIX $K'_{22} = K'_{33}$

$$\begin{bmatrix} (K'_{41})_{11} & -\frac{1}{2}(K'_{11})_{21} & \frac{1}{2}(K'_{11})_{31} & (K'_{41})_{41} & -\frac{1}{2}(K'_{11})_{51} & (K'_{11})_{31} \\ -\frac{1}{2}(K'_{11})_{21} & (K'_{41})_{22} & \frac{1}{2}(K'_{11})_{32} & \frac{1}{2}(K'_{11})_{51} & (K'_{41})_{52} & \frac{1}{2}(K'_{11})_{62} \\ -\frac{1}{2}(K'_{11})_{31} & \frac{1}{2}(K'_{11})_{32} & (K'_{41})_{33} & (K'_{11})_{31} & -\frac{1}{2}(K'_{11})_{62} & (K'_{41})_{63} \\ (K'_{41})_{41} & \frac{1}{2}(K'_{11})_{51} & (K'_{11})_{31} & (K'_{41})_{11} & \frac{1}{2}(K'_{11})_{21} & \frac{1}{2}(K'_{11})_{31} \\ -\frac{1}{2}(K'_{11})_{51} & (K'_{41})_{52} & -\frac{1}{2}(K'_{11})_{62} & \frac{1}{2}(K'_{11})_{21} & (K'_{41})_{22} & -\frac{1}{2}(K'_{11})_{32} \\ (K'_{11})_{31} & \frac{1}{2}(K'_{11})_{62} & (K'_{41})_{63} & \frac{1}{2}(K'_{11})_{31} & -\frac{1}{2}(K'_{11})_{32} & (K'_{41})_{33} \end{bmatrix}$$

SUB-MATRIX K'_{32}

$$\begin{bmatrix}
(K'_{31})_{11} & \frac{1}{2}(K'_{11})_{21} & -2(K'_{21})_{31} & (K'_{31})_{41} & \frac{1}{2}(K'_{11})_{51} & -(K'_{21})_{31} \\
\frac{1}{2}(K'_{11})_{21} & (K'_{31})_{22} & -(K'_{11})_{62} & -\frac{1}{2}(K'_{11})_{51} & (K'_{31})_{51} & -(K'_{11})_{32} \\
2(K'_{21})_{31} & (K'_{11})_{62} & (K'_{31})_{33} & (K'_{21})_{31} & -(K'_{11})_{32} & (K'_{31})_{63} \\
(K'_{31})_{41} & -\frac{1}{2}(K'_{11})_{51} & -(K'_{21})_{31} & (K'_{31})_{11} & -\frac{1}{2}(K'_{11})_{21} & -2(K'_{21})_{31} \\
\frac{1}{2}(K'_{11})_{51} & (K'_{31})_{52} & -(K'_{11})_{32} & -\frac{1}{2}(K'_{11})_{21} & (K'_{31})_{22} & (K'_{11})_{62} \\
(K'_{21})_{31} & (K'_{11})_{32} & (K'_{31})_{63} & 2(K'_{21})_{31} & -(K'_{11})_{62} & (K'_{31})_{33}
\end{bmatrix}$$

SUB-MATRIX K'_{42}

$$\begin{bmatrix}
(K'_{21})_{11} & -(K'_{11})_{21} & (K'_{21})_{31} & (K'_{21})_{41} & -(K'_{11})_{51} & 2(K'_{21})_{31} \\
-(K'_{11})_{21} & (K'_{21})_{22} & -\frac{1}{2}(K'_{11})_{62} & (K'_{11})_{51} & (K'_{21})_{52} & \frac{1}{2}(K'_{11})_{32} \\
-(K'_{21})_{31} & -\frac{1}{2}(K'_{11})_{62} & (K'_{21})_{33} & -2(K'_{21})_{31} & \frac{1}{2}(K'_{11})_{32} & (K'_{21})_{63} \\
(K'_{21})_{41} & -(K'_{11})_{51} & 2(K'_{21})_{31} & (K'_{21})_{11} & (K'_{11})_{21} & (K'_{21})_{31} \\
-(K'_{11})_{51} & (K'_{21})_{52} & -\frac{1}{2}(K'_{11})_{32} & (K'_{11})_{21} & (K'_{21})_{22} & -\frac{1}{2}(K'_{11})_{62} \\
-2(K'_{21})_{31} & -\frac{1}{2}(K'_{11})_{32} & (K'_{21})_{63} & -(K'_{21})_{31} & \frac{1}{2}(K'_{11})_{62} & (K'_{21})_{35}
\end{bmatrix}$$

SUB-MATRIX K'_{43}

The nodal load vector is defined by Equ. (6.22) as

$$\{r\}_i = \left\{ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_1 \\ -r_2 \\ r_3 \\ -r_1 \\ -r_2 \\ r_3 \\ -r_1 \\ r_2 \\ r_3 \\ r_1 \\ r_2 \\ -r_3 \\ r_1 \\ -r_2 \\ -r_3 \\ -r_1 \\ -r_2 \\ -r_3 \\ -r_1 \\ r_2 \\ -r_3 \end{array} \right\}$$

in which

$$r_1 = \frac{\alpha\ T\ V}{a}(E_{11} + E_{12} + E_{13})$$

$$r_2 = \frac{\alpha\ T\ V}{b}(E_{23} + E_{22} + E_{23})$$

$$r_3 = \frac{\alpha\ T\ V}{c}(E_{31} + E_{23} + E_{33})$$

where

E_{ij} = an element in the elasticity matrix defined in Equ. (6.20)

α = the thermal expansion coefficient of the finite element material.